Flips in Low Codimension — Classification and Quantitative Theory

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Among other mathematicians who have helped me, I am especially grateful to Jan Stevens who has spent many hours explaining wonderful things.

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Declaration

Except where explicitly stated, the content of this thesis is my own research.

Summary

A *flip* is a birational map of 3-folds $X^- \dashrightarrow X^+$ which is an isomorphism away from curves $C^- \subset X^-$ and $C^+ \subset X^+$ and does not extend across these curves. Flips are the primary object of study of this thesis. I discuss their formal definition and history in Chapter 1.

Flips are well known in toric geometry. In Chapter 2, I calculate how the numbers K^3 and $\chi(nK)$ differ between X^- and X^+ for toric flips. These numbers are also related in a primary way by Riemann–Roch theorems but I keep that quiet until Chapter 5.

In Chapter 3, I describe a technique, which I learned from Miles Reid, for constructing a flip as $\mathbb{C}^*$ quotients of a local variety $0 \in A$, taken in different ways. The *codimension* of my title refers to the minimal embedding dimension of $0 \in A$. The case of codimension 0 turns out to be exactly the case of toric geometry as studied in Chapter 2. The main result of Chapter 3 classifies the cases when $A \subset \mathbb{C}^5$ is a singular hypersurface, that is, when A defines a flip in codimension 1.

Chapters 4 and 5 concern themselves with computing new examples of flips in higher codimension and studying changes in general flips. I indicate one benefit of knowing how these changes work.

The main results of Chapters 2 and 3 have been circulated informally as [2] and [3] respectively.

Chapter 1

Introduction

Before I begin, I indulge myself with a handful of pages to set the scene and to embellish the meagre abstract. I snap out of it in section 1.3.

1.1 Classification of varieties and the Minimal Model Program

I present a brief statement of the famous story of the Minimal Model Program (MMP).

The birational classification of surfaces works in two steps. Suppose S is a smooth projective surface. First is the reduction step which lets me assume that S is a 'relatively minimal model'; in other words, I can assume that S contains no -1-curves. This is easy: if I do find a -1-curve l, I apply Castelnuovo's Contractibility Criterion to construct a contraction map

$$f\colon S \longrightarrow S_1$$

where f is a projective morphism, S_1 is a smooth projective surface and the exceptional locus of f is l; this process terminates because

$$b_2(S_1) < b_2(S).$$

The classification proceeds by giving an increasingly fine case division of the relatively minimal models: first by Kodaira dimension, $\kappa = -\infty$, 0, 1 or 2, then by finer invariants. A coarse MMP-correct statement of the result is

	positivity of K_S	κ	type of S
(1)	for all curves $C \subset S$,		
	$K_SC \geq 0$	2	general type
		1	elliptic, $\kappa = 1$
		0	K3, Enriques, abelian, bielliptic
(2)	for some curve $C \subset S$,		
	$K_SC < 0$	$-\infty$	(a) ruled over a curve $f\colon S \longrightarrow B$
			(b) rational, $S = \mathbb{P}^2$, $-K_S$ is ample

Moreover, in class (1), each birational equivalence class contains a unique relatively minimal model. In class (2), each birational equivalence class contains infinitely many relatively minimal models.

That's fine and is discussed in detail in many places, for example, [1]. The problem is doing the same in higher dimensions, in particular, in 3 dimensions. The MMP is concerned with generalising the first half of the classification procedure and is the subject of this thesis; I won't remark on the second half at all.

There are several ways in which the MMP differs from the procedure of Castelnuovo. Firstly, instead of working in the category of smooth 3-folds, it is *necessary* to work in the larger category of $\mathbb{Q}$-factorial, terminal 3-folds, often called *Mori's category*; I'll avoid explaining the adjectives more thoroughly in section 1.3. In other words, I must allow singularities of a special type on my 3-folds; the vertex of the cone on the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$ is a good example of such a singularity. For comparison, the equivalent category of surfaces is simply smooth surfaces, so no change there, yet. The immediate effect of these singularities is that favourite tricks and tools of algebraic geometry become more complicated.

Next, a projective variety X in Mori's category is called a *Minimal Model* if K_X is nef; K_X is the canonical class which I'll discuss in section 1.3 and the condition *nef* holds, by definition, iff $K_X C \geq 0$ for all complete curves C lying on X.

Proposition 1 *Let S be a smooth surface of general type. Then*

$$K_S \textit{ is nef iff } S \textit{ does not contain any } -1\textit{-curves.}$$

Proof. One implication is clear by the adjunction formula. The other requires more work: a deformation calculation shows that curves of nonnegative selfintersection which are negative against the canonical class move; this contradicts S being of general type by giving a birational fibre structure to S; see [4], (3.8) for the details. Q.E.D.

This Proposition shows that for surfaces of general type nothing has changed — relatively minimal models and minimal models are the same thing. Of course, for other surfaces things are now different. For example, $\mathbb{P}^2$ has no -1-curves but $K_{\mathbb{P}^2} = \mathcal{O}(-3)$ is as far from being nef as it could be. As far as classification goes, the point is that we should now be thinking of the dichotomy general type versus non-general type as an important part of the result: this can already be seen in the case of surfaces — non-general type is the case where we get structure theorems, that is, fibrations to varieties of smaller dimension.

The question is now this: starting with a 3-fold X with K_X not nef how do we begin to transform it into one with K nef?

To understand the solution, it helps to think of the reduction step $f\colon S \longrightarrow S_1$ as one of possibly many elementary steps in the route to a minimal model. In the surface case all elementary steps look the same: as analytic germs, $(l \subset$

$S) \longrightarrow (P \in S_1)$ is the same as blowing up the origin in $\mathbb{C}^2$. In fact, S may have had several -1-curves. To get to the minimal model, though, you don't care; you simply choose one of them, contract it, and then forget about S altogether and concentrate on S_1, repeating this process a finite number of times.

The staggering leap of imagination is Mori's in [17], influenced by Hironaka; the correct generalisation of -1-curve is the notion of 'extremal ray'. (I refer to [24] for a fuller colloquial discussion of this part of the story since I will not explicitly use the setup.) The *Mori Cone* is a cone in the vector space $H^4(X, \mathbb{R})$ corresponding to the numerical classes of curves lying on X. By the intersection pairing, divisors form linear functions on the cone; an *extremal ray* is a 1–dimensional edge of this cone which is negative against the canonical class. The Cone Theorem asserts that extremal rays exist and are generated by rational curves and Kawamata's Base Point Free Theorem asserts that, given an extremal ray, there exists a projective contraction map, called an *extremal contraction*,

$$f \colon X \longrightarrow X_1$$

contracting exactly those curves whose numerical class lies in the given ray; in particular, K_X intersects contracted curves negatively just as it did in the case of -1-curves. If f contracts only a divisor then X_1 is again an element of Mori's category and I work on that. However, if f contracts only a finite number of curves then X_1 fails to be in Mori's category so f is no good as one of the elementary steps. Instead, one looks for a *flip* of f, that is, by definition, a morphism $f^+ \colon X^+ \longrightarrow X$ where X^+ is also in Mori's category, f^+ is projective and K_{X^+} is positive on the curves contracted by f^+. Now $(f^+)^{-1} \circ f$ is a birational map that is not a morphism. The whole picture is Figure 1.1.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\
& X_1 &
\end{tikzcd}$$

Figure 1.1: A flip diagram

The existence of a flip can be translated into a fg question. The following Easy Lemma is from [13], §3.

Lemma 2 *The flip $f^+ \colon X^+ \longrightarrow X_1$ exists iff the $\mathcal{O}_{X_1}$ algebra*

$$\mathcal{R}_{X_1}(K_{X_1}) := \oplus_{n \geq 0} \mathcal{O}_{X_1}(nK_{X_1})$$

is finitely generated and in that case $X^+ = \mathbf{Proj}_{X_1} \mathcal{R}_{X_1}(K_{X_1})$ and f^+ is the natural projection.

In fact, this Lemma also entails the uniqueness of flips; that is, any other variety X_1^+ which, along with some morphism f_1^+, completes a diagram like Figure 1.1 and satisfies all the conditions of a flip is isomorphic to my choice, X^+.

Theorem 3 (Mori's Flip Theorem) *If X has only terminal singularities and f^- is an extremal contraction then $\mathcal{R}_{X_1}(K_{X_1})$ is finitely generated.*

Now the MMP is up and running as an inductive procedure, the only problem is to stop it. Luckily, termination of this process is much easier than proving its existence and had been done in advance by Shokurov in [26]: if $X \longrightarrow X_1$ contracts a divisor then b^2 gets smaller as before; if $X \dashrightarrow X^+$ is a flip then b^2 remains constant while another nonnegative, integral invariant decreases. This invariant is the *difficulty*,

$$d(X) := \#\{\text{exceptional divisors in a resolution with discrepancy } < 1\}.$$

The fact that it decreases on flipping reflects the heuristic argument which holds for 3-folds that "the singularities of X_1 are milder than those of X". The precise statement is this: see [26], Theorem (2.13);

Theorem 4 (Shokurov's Flip Termination Theorem)

$$d(X) \geq d(X^+) + \# \, C^+$$

where $\# \, C^+ \geq 1$ is the number of components of C^+.

The proof is powered by the positivity conditions on $-K_X$ and K_{X^+}. This polarisation by the canonical class is also crucial to the inductive properties of flips; it is trivial to come up with bad behaviour in wrongly polarised 'flips', even when the flip exists.

1.2 Aims and results of this thesis

So the world accepts the existence of flips and the MMP. For symmetry, I relabal the 3-fold before a flip as X^-; the flip now looks like $X^- \dashrightarrow X^+$. The motivation for this thesis lies in the next phase of the study of the MMP and arises from the following 2 projects:

(A) classify flips;
(B) quantify the effects of flipping.

Miles Reid has a programme aimed at solving (A). Using $\mathbb{C}^*$ actions, this translates (A) into the problem of classifying normal affine Gorenstein 4-folds A with a $\mathbb{C}^*$ action satisfying a number of conditions. In fact, the construction of the flip is local to a fixed point $0 \in A$ of the $\mathbb{C}^*$ action. I restrict myself to cases when $0 \in A$ has low embedding dimension. The codimension of the corresponding flip is defined to be this embedding dimension.

When $A = \mathbb{C}^4$ the programme recovers the toric flips of Danilov [6] and Reid [22]. I classify the case that $A \subset \mathbb{C}^5$ is a hypersurface. This is Theorem 33 of Chapter 3. The result, modulo a little detail, is that any terminal flip determined by some $A\colon (g = 0) \subset \mathbb{C}^5$ and $\mathbb{C}^*$ action is one of the following:

	monomials in g	$\mathbb{C}^*$ action	
(1)	$x_1y_1 + g'(x_2, x_3)$	$(a_1, a_2, 1, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, b_1$
(2)	$x_1y_1 + x_3^n$	$(a_1, a_2, a_3, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, a_3, b_1$
(3)	$x_2^2 + x_1y_2^2$	$(4, 1, 1, -2, -1; 2)$	
(4)	$x_2y_1 + z^n$	$(a, 1, -1, -b, 0; 0)$	$a > b, (a, b) = 1$
(5)	$z^2 + x_1y_2^3$	$(3, 1, -2, -1, 0; 0)$	
(6)	$x_2y_1 + y_3^2$	$(4, 1, -3, -2, -1; -2)$	

where in both (1) and (2), a_2 divides $a_1 - b_1$ and all the characters are coprime except that possibly $\mathrm{hcf}(a_1, b_1) > 1$.

The condition that A can be chosen to be Gorenstein does not yet have a complete proof. In Appendix B, I recall the construction of A and show why K_A is locally free.

In low codimension, Gorenstein variety germs have the following description: codimension 0 is smooth and codimension 1 is a hypersurface point; codimension 2 is a complete intersection (Serre); codimension 3 is either a complete intersection or is given by the maximal Pfaffians of a $2k-1 \times 2k-1$ skew symmetric matrix (Buchsbaum–Eisenbud). In Chapter 4, I give examples of higher codimension flips in these forms.

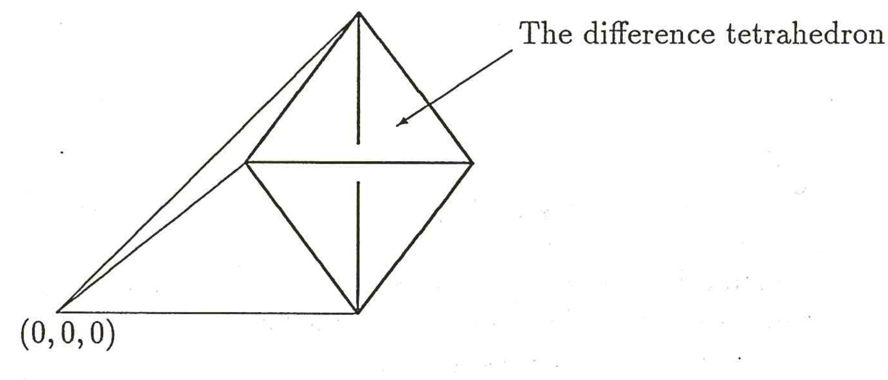


Figure 1.2: The toric pluricanonical difference

Chapter 2 is concerned with project (B). Working mainly in the toric category, it calculates how the numbers K_X^3 and $\chi(X, mK_X)$ differ before and after a flip. The results are in Theorems 17 and 20. I describe the result of the latter. A toric flip is determined by 2 different decompositions of a cone in a lattice. The 'difference' of these two decompositions is a lattice tetrahedron $\diamondsuit$ lying away from the origin; this is more clearly described by Figure 1.2. In this toric flip,

$$\chi(X^+, mK_{X^+}) - \chi(X^-, mK_{X^-}) = \#S_m$$

where S_m is the set of (suitably divisible) lattice points contained in the multiple $m\diamondsuit$ of the 'difference tetrahedron', $\diamondsuit$.

A general result of this chapter is that the difference in $\chi(X, mK_X)$ across a flip is equal to the dimension of a single cohomology group on X^-. In Chapter 5 I make the remark that this gives *a priori* information about the flipped variety X^+. For example, I get a bound on the number of singularities on X^+.

For the most effective result, I make the assumption that the pluricanonical behaviour of a flip can be represented by that of a collection of toric flips. This hypothesis will be the case if, for example, the flip deforms into a collection of toric flips. I don't know whether or not this hypothesis holds for all flips, although in section 5.1, I am able to prove the weaker statement that the pluricanonical behaviour of any flip can be represented by that of a *virtual* collection of toric flips, that is, an element of the free abelian group generated by toric flips.

1.3 Preliminaries

MMP notation and definitions

I will always denote normal, quasiprojective 3-folds by an X or a Y, often with adornments. I hope the converse also holds. The nonsingular locus of X is X^0; the inclusion is $j\colon X^0 \longrightarrow X$. Since X is normal, $X \setminus X^0$ lies in codimension 2. Actually, since it is only really this property that I need, I will also allow X^0 to be any open subset of the nonsingular locus of X that satisfies the condition that $X \setminus X^0$ must lie in codimension 2.

A basic fact, which I recall from [29], Proposition 7.1, is that if $P \in X$ and $f\colon Y \longrightarrow X$ is an isomorphism in codimension 1 which contracts a curve $C \subset Y$ to $P \in X$ then

$$f_*D \text{ is Cartier at } P \text{ iff } DC = 0$$

for any divisor $D \subset Y$. For if f_*D is Cartier at P then $(f^*f_*D)C = 0$ and $f^*f_*D = D$ because f is an isomorphism in codimension 1.

Definition 1 *The* canonical class *of X, K_X, is defined to be any Weil divisor whose divisorial sheaf is isomorphic to the divisorial sheaf $j_*\omega_{X^0}$; compare [21], Appendix to §1, for details.*

Definition 2 *Let $P \in X$ be a point of a 3-fold and $f: X^- \longrightarrow X$ be a projective morphism. I use 2 different notions of* class group*:*

$$\mathrm{Cl}(X^-/X) = \frac{\mathrm{WDiv}(X^-)}{f^*\mathrm{CDiv}(X)}$$

$$\mathrm{Cl}_P(X) = \frac{\mathrm{WDiv}(X)}{\mathrm{CDiv}_P(X)}$$

where $\mathrm{CDiv}_P(X)$ *is the group of divisors on X that are Cartier at $P \in X$.*

Note that if $P \in X$ is an isolated singularity and if $f: X^- \longrightarrow X$ is an isomorphism in codimension 1 which restricts to an isomorphism above $X \setminus P$ then

$$\mathrm{Cl}(X^-/X) \cong \mathrm{Cl}_P(X) \cong \mathrm{Cl}X$$

where $\mathrm{Cl}X$ is the usual Weil class group.

Definition 3 *A* small extremal contraction *is a projective morphism $f^-: X^- \to X$ such that*

- *(I)* X^- *has at most $\mathbb{Q}$-factorial terminal singularities,*
- *(II)* f^- *is an isomorphism away from a one dimensional subscheme $C^- \subset X^-$, and C^- is contracted to a point $P \in X$,*
- *(III)* $-K_-$ *is relatively ample for f^-; that is, if $(C^-)_{\mathrm{red}} = \bigcup_i C_i$, then $-K_-C_i > 0$, for each i,*
- *(IV)* $\mathrm{Cl}(X^-/X)$ *is of rank 1.*

The point $P \in X$ is necessarily singular by the initial remark; indeed the same argument shows that $P \in X$ is not even $\mathbb{Q}$-Gorenstein, that is, no integer multiple of K_X is Cartier. Moreover, once you know that terminal singularities live in codimension 3 it is clear that P must be an isolated singularity.

The most famous example of a small contraction is given by Zariski in [30], §3. Consider X, the cone on the quadric surface $S_2 \subset \mathbb{P}^3$,

$$X: (xy = zt) \subset \mathbb{C}^4,$$

and its vertex, P. Choose one of the two rulings of the quadric and let X^- be the correspondence between the fibres of that ruling and X; f^- is the natural projection. In this case, the exceptional locus is a generator of the other ruling, that is, a single copy of $\mathbb{P}^1$. Moreover X^- is smooth so certainly has terminal singularities. However, $P \in X$ is a finite quotient singularity. Immediately, I get that K_X is $\mathbb{Q}$-Cartier so $K_-C^- = 0$ and condition (III) fails. In fact, it has been shown by Mori in [17] that whenever all three conditions are satisfied, X^- is necessarily singular somewhere along C^-, so this was really a nonstarter as a flip example.

Incidently, the paper [30] of Zariski gives both philosophical and technical reasons for the assumption 'normal' which is always an issue in birational geometry but, in my experience, rarely discussed.

Definition 4 *Given a small extremal contraction, $f^-: X^- \longrightarrow X$, a* flip *of f^- is a normal quasiprojective 3-fold X^+ and a projective morphism $f^+: X^+ \longrightarrow X$ such that*

(II$^+$) *f^+ is an isomorphism away from a one dimensional subscheme $C^+ \subset X^+$, and C^+ is contracted to $P \in X$,*

(III$^+$) *$K_+ := K_{X^+}$ is relatively ample for f^+.*

The condition (I$^+$) is satisfied immediately by Theorem 4 and (IV$^+$) is also automatic since K_+ is $\mathbb{Q}$-Cartier and nontrivial on X^+.

The whole flip can be seen as one picture in Figure 1.3 which emphasises the point of view that a flip should really be a rational map between X^- and X^+ that is regular in codimension one; I'm thinking that C^- has been 'flipped' to C^+.

$$\begin{tikzcd} X^- \arrow[rr, dashed] \arrow[dr, "f^-"'] & & X^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

Figure 1.3: Another flip diagram

It is traditional abuse of notation to allow the word *flip* to refer also to the flipped variety, X^+, or to the rational map $X^- \dashrightarrow X^+$, or to the whole picture, $X^- \to X \leftarrow X^+$.

Take Zariski's example of a small contraction $f^-: X^- \longrightarrow X$ and construct another one, $f^+: X^+ \longrightarrow X$, by choosing the other ruling of the quadric S_2 when making the correspondence. This gives a picture like Figure 1.3 but it is not a flip picture because $KC = 0$ on both sides. In fact, it is a famous example of a flop (definition in a minute) called *Atiyah's flop* after his remark in 1954 that the cone on the Veronese has 2 distinct small resolutions. (Although I have used the example of 1942 as it was stated, Zariski chose not to observe the flop as he was taking the example only as a brief non-purity remark for singular varieties.) In this case it is clear that $X^- \cong X^+$ but not by the given birational map, $f = (f^+)^{-1} \circ f^-$.

Definition 5 *To define a* flop *I write down exactly the same words as for a flip but change the conditions (III) and (III$^+$) to*

(III$_0$) *$K_- C = 0$ for all curves C contracted by f^-, and,*

(III$_0^+$) *$K_+ C = 0$ for all curves C contracted by f^+.*

Flops have been classified by Kollár in [15].

Definition 6 *A* flip diagram *is a picture $X^- \to X \leftarrow X^+$ with morphisms f^- and f^+ respectively satisfying conditions (II) and (II$^+$) and such that $(f^+)^{-1} \circ f^-$ is not an isomorphism.*

So, for example, a flop gives a flip diagram which is not a flip.

A note on germs

Throughout this thesis I work with an affine or quasiprojective neighbourhood of a flip. That is, suppose that $X^- \to X \leftarrow X^+$ is flip of projective 3-folds and $U \subset X$ is an affine neighbourhood of the flipping point $P \in X$. Then I can consider the flip $(f^-)^{-1}U \to U \leftarrow (f^+)^{-1}U$. I call this an *affine neighbourhood of the flip*. It will be affine neighbourhoods of flips like these that I will try to classify. However, one should think of these affine flips as being representatives of analytic *germs* of flips; that is, the system of all flips as U varies among analytic neighbourhoods of $P \in X$ modulo the usual germ equivalence. This is exactly analogous to the way in which one classifies, and even defines, terminal singularities on 3-folds. The point is, of course, that one doesn't want to prejudice the birational nature of the varieties one studies, or, worse still, include birational type as part of the data one is trying to study.

Terminal 3-fold singularities

As I've indicated, singularities are important throughout the MMP. Normally I will restrict myself to terminal singularities. Rather than give the formal definition I will give the result of their classification, a list of all terminal 3-fold singularities.

These singularities have been classified by the work of a number of people; I leave it to [23] for a more complete list of references. I've taken this description from [23], §§5–6. You can also find the definition of terminal singularities there if you like but you won't need it at all here.

The notation $\frac{1}{n}(a,b,c)$ denotes the analytic germ $(X,0)$ of the quotient singularity $0 \in X = \mathbb{C}^3/\mu_n(a,b,c)$ where $\mu_n = \{\varepsilon \in \mathbb{C} \mid \varepsilon^n = 1\}$. The symbol $\frac{1}{n}(a,b,c,d;e)$ is called the *type* of the analytic germ $(X,0)$ of the hyperquotient singularity

$$0 \in X = \frac{(g=0)}{\mu_n(a,b,c,d)} \subset \frac{\mathbb{C}^4}{\mu_n(a,b,c,d)},$$

where $\mathrm{wt}(g) = e$; there can be many analytically distinct hyperquotient germs of the same type.

Theorem 5 (Mori, Reid, Morrison–Stevens, Kollár–Shepherd-Barron)

The point $P \in X$ is a terminal singularity iff $P \in X$ is a nonsingular point or $P \in X$ is analytically isomorphic to one of the following germs (allowing permutations of a, b, c, d):

- *(T) the quotient singularity* $\frac{1}{n}(a,b,c)$ *where* $n>1$, $\operatorname{hcf}(n,abc)=1$ *and* $a+b\equiv 0 \mod n$;
- *(M) a hyperquotient singularity of type* $\frac{1}{n}(a,b,c,d;e)$ *with* $0\in(g=0)\subset\mathbb{C}^4$ *an isolated singularity satisfying one of the conditions (M1–3):*

  (M1) $n>1$, $\operatorname{hcf}(n,abc)=1$, $a+b\equiv 0\mod n$, $d,e\equiv 0\mod n$ *and conditions on* g *as follows, where* $q\in g$ *is the quadratic part of* g *whose rank as a quadratic form is also denoted* q:

n	q	*type*	g
≥ 2	≥ 2	$\frac{1}{n}(a,-a,c,0;0)$	$x_1x_2+h_2(x_3,x_4)$
3	1	$\frac{1}{3}(2,1,1,0;0)$	$x_4^2+x_1^3+h$
2	2	$\frac{1}{2}(1,1,1,0;0)$	$x_1^2+x_4^2+h_2(x_2,x_3)$
2	1	$\frac{1}{2}(1,1,1,0;0)$	*several possibilities*

  where h *denotes a choice of one of the polynomials* $\{x_2^3+x_3^3,\ x_2^2x_3+x_1h_4(x_1,x_3)+h_6(x_1,x_3),\ x_2^3+x_1h_4(x_1,x_3)+h_6(x_1,x_3)\}$ *and* h_k *denotes a polynomial of degree at least* k;

  (M2) the type is $\frac{1}{4}(3,2,1,1;2)$ *or* $\frac{1}{4}(1,2,3,3;2)$ *and the 2-jet of* g *is either* $x_1^2+x_3^2$ *or* $x_1^2+x_3x_4$;

  (M3) $n=1$ *and* $0\in(g=0)$ *is a cDV singularity.*

□

I practice, I will be given quotient or hyperquotient singularities and must check whether they are terminal. They may easily be terminal without being in the form of the singularities in (T) and (M). The following lemma puts singularities into the 'minimal' form as they appear in the theorem.

Lemma 6 *If* $n=n'h$, $a=a'h$, $b=b'h$ *and* $c=c'h$ *then* $\frac{1}{n}(a,b,c)\cong\frac{1}{n'}(a',b',c')$.
If $n=n'h$, $a=a'h$ *and* $b=b'h$ *then* $\frac{1}{n}(a,b,c)\cong\frac{1}{n'}(a',b',c)$.
In particular, if $n\mid a$ *and* $n\mid b$ *then* $\frac{1}{n}(a,b,c)$ *is a nonsingular point.*
If $n>1$, $\operatorname{hcf}(n,ab)=1$ *and* $\operatorname{hcf}(n,c)>1$ *then* $\frac{1}{n}(a,b,c)$ *is a nonisolated singularity.* □

I say that the singularity $\frac{1}{n}(a,b,c)$ has *local quasireflections* if n is coprime to at most 1 of a,b,c. The lemma shows that having local quasireflections is a property of the quotient map, not the singularity germ; any quotient germ is isomorphic to some $\frac{1}{n'}(a',b',c')$ which does not have local quasireflections.

I say that the hyperquotient singularity of type $\frac{1}{n}(a,b,c,d;e)$ has *local quasireflections* if either n is coprime to at most 1 of a, b, c, d or if some $h>1$ divides exactly 3 of a, b, c, d but fails to divide e. This is a slight misnomer: it is possible for a hyperquotient singularity without local quasireflections to have genuine quasireflections if the equation is nongeneric; I can live with it, it's only a name. In this case it is not so easy to remove the quasireflections, but in practice, when I need to I will be able to.

Q-factorial singularities

In toric geometry, $\mathbb{Q}$-factoriality for a cone is the condition that it be simplicial. In general algebraic geometry, the condition is more complicated. The following Theorem of Kawamata, [13], says that, in the light of Lemma 2, if a variety with canonical singularities is not $\mathbb{Q}$-factorial then, locally, it is the base of a canonical flop.

Theorem 7 (Kawamata)
Let X be a 3-fold with canonical singularities and D be a Weil divisor on X. Then $\mathcal{R}_X(D)$ is finitely generated.

$\square$

Given this, I do not mention the question of $\mathbb{Q}$-factoriality when discussing nontoric flips; implicit in this is the removal of the $\mathbb{Q}$-factoriality condition from the definition of flip in that situation.

Kawamata–Viehweg Vanishing

I misquote the following result from [8], Corollary 5.12(d).

Theorem 8 (Kawamata, Viehweg)
Let Y be a smooth 3-fold and D be a nef and big, simple normal crossing $\mathbb{Q}$-divisor. Then

$$H^i(Y, \mathcal{O}_Y(K_Y + \lceil D \rceil)) = 0$$

for any $i > 0$.

$\square$

Conventions and conventional mistakes

In a polynomial g, I will always write the nonzero coefficients as a 1; saying the monomial xy is in g, or $xy \in g$, means that the monomial xy appears in g with a nonzero coefficient.

I find it hard thinking of $\mathbb{C}[x]$ as polynomials so I always write $k[x]$ in its place. I don't lose sleep over writing $\operatorname{Spec} k[x] = \mathbb{C}$.

Whenever the symbol $\pm$ appears more than once in a sentence, it should be read as $-$ respectively $+$ throughout. For example, $f^\pm : X^\pm \longrightarrow X$ denotes 2 maps; $f^- : X^- \longrightarrow X$ and $f^+ : X^+ \longrightarrow X$.

Chapter 2

Toric geometry

The induction steps of the MMP arise in a natural way in toric geometry. They are very explicit and easy to understand here. Equally importantly, lots of geometric tools admit simple interpretations in toric geometry so I can make calculations.

A standard reference for toric geometry is [5] but also see [22] and [23], §4, for very relevant material. As usual, $N = \mathbb{Z}^3$ is a lattice and M is the standard dual lattice. The pictures of cones I draw are in $N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathbb{R}^3$. I don't distinguish between a ray in N and a primitive vector on that ray.

In the main sections of this chapter I calculate how numbers associated to the canonical class compare on one side of a toric flip to the other. Of course, not everything changes. For example, in a flip of 3-folds, the betti numbers don't change — one way to see this is from the toric description of the complex cohomology of a quasiprojective variety in [5], §10.

2.1 Birational maps and sheaves in toric geometry

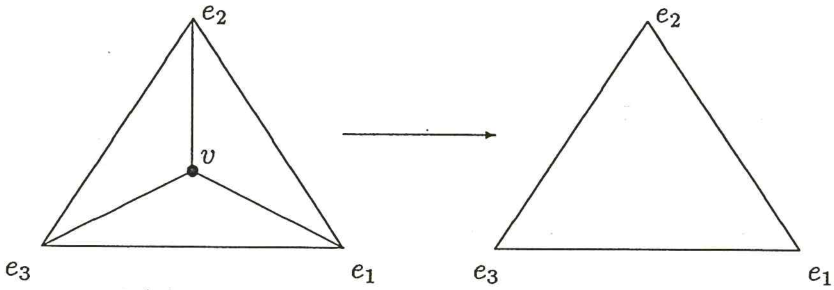
Danilov, in [6], has classified 'elementary' birational maps between toric varieties. Broadly, there are blowups of points or lines and flips; transformations of the first, second and third types, respectively, in his terminology.

Making toric blowups is the same thing as adding to the 1-skeleton of the associated fan and making the minimal compatible subdivision. So, given a fan Σ and some vector v in the interior of some (not necessarily top dimensional) cone $\sigma \in \Sigma$, I make the new fan $\Sigma(v)$. Then there is a morphism of toric varieties

$$X_{\Sigma(v)} \to X_{\Sigma}$$

induced by the inclusion of fans, $\Sigma(v) \subset \Sigma$. This map has exceptional divisor D_v corresponding to v.

For example, $\mathrm{Bl}_0\mathbb{C}^3 \to \mathbb{C}^3$ is given by the inclusion of fans in Figure 2.1; the origin is behind the page so these really are 3 dimensional fans. You can see the exceptional divisor $E \cong \mathbb{P}^2$ in the middle of the blowup.



$\Sigma(v) \subset \Sigma$ where
$e_1 = (1,0,0)$, $e_2 = (0,1,0)$ $e_3 = (0,0,1)$, $v = (1,1,1)$.

Figure 2.1: Toric blowup

I include a few remarks about blowups in section 2.4.

Toric flips

Let σ be a cone with vertices e_1, e_2, f_1, f_2 which generate it as a semigroup and which satisfy the relation

$$a_1 e_1 + a_2 e_2 = b_1 f_1 + b_2 f_2$$

for positive integers a_i, b_j. Let Σ be the fan consisting of σ and its faces and let $\Sigma^- \subset \Sigma \supset \Sigma^+$ be the subdivisions of Σ drawn in Figure 2.2; again the origin is behind the page so these are three dimensional cones. For any choice of vertices satisfying the linear relation above, I say that this is the toric flip diagram $(a_1, a_2, -b_1, -b_2)$; the resulting flip diagram is dependent on the linear relation and not on the choice of vertices.

Toric operations of this kind are particularly important in view of the following Theorem of Danilov.

Theorem 9 (Danilov, [6], §5)
Any flip that occurs in toric geometry is of the form $(a, 1, -b_1, -b_2)$ where $a > b_1$, $\mathrm{hcf}(a, b_1) = 1$ and either $b_2 = 1$ or $a = b_1 + b_2$. Moreover, I can suppose that it is given locally, as in Figure 2.2, by the decomposition of the cone σ where $e_1 = (1,0,0)$, $e_2 = (0,0,1)$, $f_1 = (0,-1,0)$, $f_2 = (b_1, a, b_2)$ in some coordinates.

□

I give a proof of this in a different context in section 3.2. The *standard model* of a toric flip is the one in particular coordinates in Danilov's Theorem.

The first example of this toric operation you try is the toric flip diagram $(1,1,-1,-1)$. By Danilov's Theorem this is not a flip. You can also see this explicitly using Corollary 11 later; the vertices of σ are coplanar therefore X is Gorenstein. In fact, this is Atiyah's flop on an excursion from the introduction.

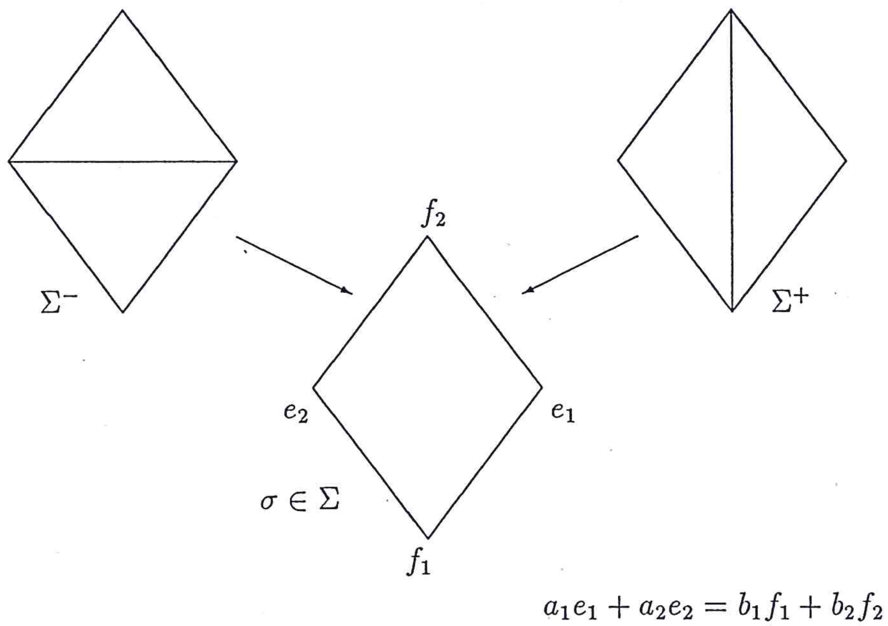


Figure 2.2: Cones in a toric flip

The Francia Flip

The simplest example of a genuine flip is given by choosing $(b_1, a, b_2) = (1, 2, 1)$ in Theorem 9 of Danilov. This is called the *Francia flip*. Recall that a cone σ is called *regular* if it is simplicial and its vertices generate it as a semigroup. Of course, this is just the condition needed to ensure that X_σ is regular — the dual cone is also simplicial and generated by its vertices m_i so

$$X_\sigma = \textbf{Spec } k[m_1, m_2, m_3] = \mathbb{C}^3$$

since there are no relations among the m_i.

In the Francia flip all the cones are regular with one exception. This exception is the cone with vertices $(1, 2, 1)$, $(1, 0, 0)$, $(0, 0, 1)$ which gives a patch on X^- isomorphic to $\mathbb{C}^3/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts by multiplication on the coordinates of $\mathbb{C}^3$; by definition, a germ around the point stratum of this patch is the quotient singularity $\frac{1}{2}(1, 1, 1)$. To see this, forget that I wanted to stay in $\mathbb{Z}^3$ and make the isomorphism of cones given in Figure 2.3; the new cone is $(\mathbb{Z}^3 + \frac{1}{2}(1, 1, 1)\mathbb{Z}) \cap \mathbb{R}^3_+$. Now write down the conditions to be in the dual cone and those to be an invariant monomial of $\mathbb{Z}_2$ acting by reflections on $\mathbb{C}^3$ — they are exactly the same. You will also notice that this patch is isomorphic to the cone on the Veronese surface, $\mathbb{P}^2 \subset \mathbb{P}^5$.

Schematically, the picture of the Francia Flip is shown in Figure 2.4.

In the original paper, [11], of Francia he constructs this flip (in reverse) on a pair of projective varieties. His point is that he has constructed two distinct birational relatively minimal models of general type in the smooth category so

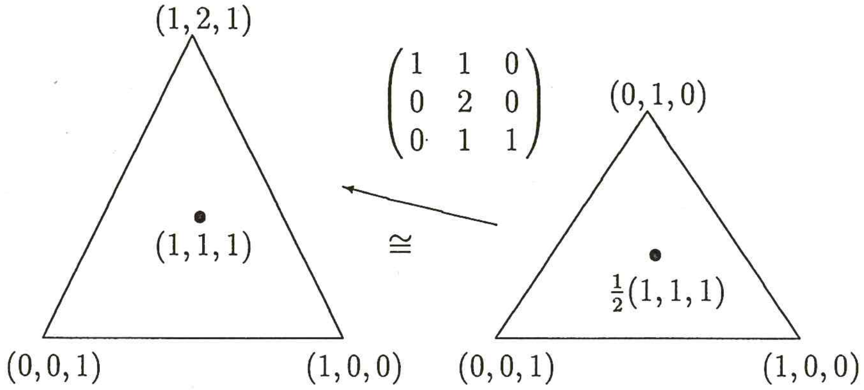


Figure 2.3: The cone on the Veronese

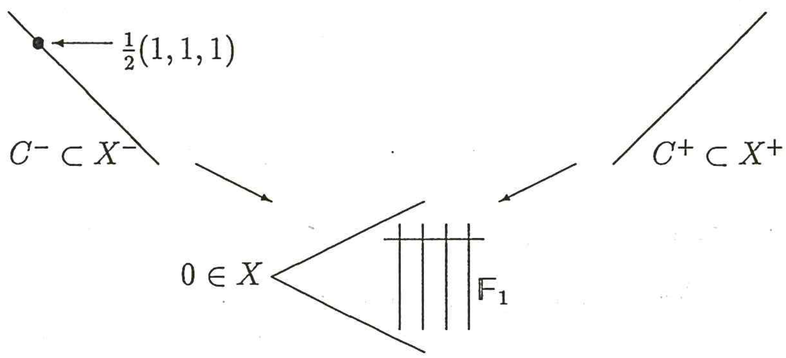


Figure 2.4: Picture of the Francia flip

classical minimal model theory won't work for 3-folds. Of course, to stay in the smooth category he doesn't allow himself to contract the exceptional $\mathbb{P}^2$ with normal bundle -2 to the Veronese cone point on X^-; blowup that cone point and then you have exactly Francia's example.

The canonical class in toric geometry

Let $X = X_\Sigma$ be any toric variety. If $\mathcal{E}$ is an invertible sheaf then, as in [5], the sheaf $\mathcal{E}$ is represented by a collection of functions $\{g_\sigma\}$ corresponding to the collection of cones σ in the fan, each of which is a linear function on the corresponding cone and is integer valued on the lattice points. These functions are also required to agree on the boundaries of the cones so that they glue to give a continuous piecewise linear function on the fan. I prefer to represent g_σ by elements of the dual space $M_\mathbb{Q} = M \otimes \mathbb{Q}$ which I call m_σ. For the basic open X_σ (see [5], §6.3),

$$H^0(X_\sigma, \mathcal{E}) = (m_\sigma + \check{\sigma}) \cap M;$$

this should be taken to mean that $H^0(X_\sigma, \mathcal{E})$ is the k vector subspace of $k(X)$ with basis $(m_\sigma + \check{\sigma}) \cap M$.

The sheaves I am interested in are rK_X. If X is smooth then they are invertible and I can use Danilov's description. Indeed, since the description is local, I can restrict myself to a *regular* cone, σ of Σ. Danilov shows, in [5] 6.6, that on the cone σ the canonical class is represented by the function that is 1 on the vertices of σ. Correspondingly, the divisor K_X is linearly equivalent to the divisor $-\sum D_v$ where the D_v are the codimension 1 strata. Let k_σ represent this function; if σ is a top dimensional cone then k_σ is unique, otherwise it is determined only modulo the rational cospan of σ.

Tensor product of invertible sheaves is given by addition of their representing functions, so, in particular, the sheaf rK_{X_σ} is represented by rk_σ. This calculation works for nonregular cones whenever rK is invertible. In particular, it is clear that for any cone σ, regular or not,

$$X_\sigma \text{ is } \mathbb{Q}\text{-Gorenstein} \quad \text{iff} \quad \text{the vertices of } \sigma \text{ are coplanar}$$

and moreover

$$X_\sigma \text{ is Gorenstein} \quad \text{iff} \quad k_\sigma \in M,$$

that is, iff the function that is 1 on the vertices of σ extends linearly to a function that is integral on the lattice points of σ.

Let $X = X_\Sigma$ be a $\mathbb{Q}$-Gorenstein toric variety of any dimension, regular in codimension 1, and let $\mathcal{V}(\sigma)$ be the set of vertices of $\sigma \subset N$. The next lemma is presumably well-known, but not by me.

Lemma 10 *For such X,*

$$rK_X = \{g_\sigma \mid g_\sigma \textit{ is linear on } \sigma \textit{ and } g(v) = r \textit{ for all } v \in \mathcal{V}(\sigma)\}_{\sigma \in \Sigma}$$

in the sense that if $k_\sigma \in M_\mathbb{Q} = M \otimes \mathbb{Q}$ satisfies $k_\sigma(v) = 1$ for all vertices v of σ then

$$H^0(X_\sigma, rK_{X_\sigma}) = (rk_\sigma + \check{\sigma}) \cap M.$$

Proof. If σ is a regular cone then this is true by the description of rk_σ above, so suppose σ is not regular. Let $S \subset X_\sigma$ be the codimension 2 singular stratum of X_σ. By definition, $K_X = i_* K_{X \setminus S}$ so $H^0(X_\sigma, rK) = H^0(X_\sigma \setminus S, rK)$. But since rK_X is a reflexive sheaf, I can remove *any* set in codimension 2 so $H^0(X_\sigma \setminus S, rK) = H^0(\bigcup_{\tau \in \mathcal{V}(\sigma)} X_\tau, rK)$. But

$$H^0\left(\bigcup_{\tau \in \mathcal{V}(\sigma)} X_\tau, rK\right) = \bigcap_{\tau \in \mathcal{V}(\sigma)} H^0(X_\tau, rK)$$

where the intersection takes place in M. Finally, since by assumption all $\tau \in \mathcal{V}(\sigma)$ are regular, this gives

$$\begin{aligned} H^0(X_\sigma, rK) &= \left(\bigcap (rk_\tau + \check{\tau})\right) \cap M \\ &= \left(rk_\sigma + \bigcap \check{\tau}\right) \cap M \\ &= (rk_\sigma + \check{\sigma}) \cap M. \end{aligned}$$

Notice that $rk_\sigma = \bigcap_{\tau \in \mathcal{V}(\sigma)} (rk_\tau + \mathbf{cospan}(\tau) \otimes \mathbb{Q})$ which makes sense by the $\mathbb{Q}$-Gorenstein assumption. Q.E.D.

That's all I need at the moment, but for later I want this corollary. I keep the notation $k_\sigma \in M_{\mathbb{Q}}$ for cones σ when X_σ is $\mathbb{Q}$-Gorenstein.

Corollary 11 *Fix $X_\sigma \subset X$ as above.*
(a) $H^0(X_\sigma, K) = (\check{\sigma})^\circ \cap M$; in particular,
(1) K is Cartier on X_σ iff this semigroup is generated by just one element;
(2) if the generators of σ are coplanar then X_σ is Gorenstein.
(b) If X_σ is Gorenstein then k_σ is the generator of the ideal $(\check{\sigma})^\circ \cap M$.
(c) If X_σ is Gorenstein then in any minimal set of generators of the semigroup $\check{\sigma} \cap M$ either 0 or 1 of them lie in the interior of $\check{\sigma}$. Moreover, if a minimal set of generators contains 1 internal point, then this point must be k_σ.

Proof. (a) This is immediate:

$$m \in (\check{\sigma})^\circ \cap M \Longrightarrow m(v) \geq 1$$

for every $v \in \mathcal{V}(\sigma)$. So, by definition, $m - k_\sigma \in (\sigma \cap M)_{\mathbb{Q}}$ and so by the lemma, $m \in H^0(X_\sigma, K)$. Since $k_\sigma \in (\check{\sigma})^\circ_{\mathbb{Q}}$ the converse is also clear.

(b) This follows from the lemma, and the fact that since $k_\sigma \in M$,

$$K_X = (k_\sigma + \check{\sigma}) \cap M = k_\sigma + (\check{\sigma} \cap M).$$

(c) Suppose that $p_1, \ldots, p_r$, $u_1, \ldots, u_s$ is a minimal set of generators of $\check{\sigma} \cap M$ with the p_i on the boundary of $\check{\sigma}$ and the u_j in its interior. Since X_σ is Gorenstein, and using multiplicative notation, the interior of $\check{\sigma}$ is $k_\sigma \cdot \check{\sigma}$ and so

$$k_\sigma = \prod p_i^{a_i} \prod u_j^{b_j}$$

with each $b_j = 0$ or 1. Moreover, if $b_j = 1$ then $k_\sigma = u_j$ otherwise $u_j \notin k_\sigma \cdot \check{\sigma}$.

So either $k_\sigma = \prod p_i^{a_i}$ or $k_\sigma = u_1$. With that in mind, if $m \geq 1$ (respectively 2) then since u_m is a generator,

$$u_m = k_\sigma \prod p_i^{c_i} \prod u_j^{d_j}$$

with c_i, $d_j \geq 0$ and $d_m \geq 1$. But then $-k_\sigma \in \check{\sigma}$ which is a contradiction unless $\check{\sigma} \cap M = M$ and $X_\sigma = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ (but this is a silly case not relevant to what I want to use this lemma for so I'll leave it and hope it goes away).

Q.E.D.

Remark This whole business of being Gorenstein is confusing looked at from the point of view of the generators of $\check{\sigma}$. For example, the cone on the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$, the quotient singularity $\frac{1}{2}(1,1,1)$ if you prefer, is an example of a toric variety whose monomial cone $\check{\sigma}$ has no internal generators but which is nonetheless not Gorenstein.

A more complicated example is the quotient singularity $\frac{1}{7}(6,5,4)$. This is a toric variety whose monomial cone has 11 generators, exactly one of which is internal, but which is patently not Gorenstein. Indeed, it takes 5 monomials to generate the ideal $(\check{\sigma})^\circ \cap M$.

For the record, the numbers are these. I use the standard coordinates and denote lattice points (a,b,c) simply by abc. The generators of $\check{\sigma}$ are

$$211,\ 102,\ 021,\ 130,\ 014,\ 401,\ 320,\ 510,\ 700,\ 070,\ 007;$$

clearly only the first one is internal. On the other hand, it takes the monomials

$$211,\ 123,\ 116,\ 151,\ 144$$

to generate the ideal of internal points.

While I'm on the subject of the canonical class, here is the statement of [23], Proposition 4.8, which I'll need in a minute.

Proposition 12 *Let v be an interior point of some $\sigma \in \Sigma$. Then $X_{\Sigma(v)} \to X_\Sigma$ has discrepancy given by*

$$K_{X_{\Sigma(v)}} = K_{X_\Sigma} + aD_v$$

with $a = g_\sigma(v) - 1$, where D_v is the divisor in $X_{\Sigma(v)}$ corresponding to v and g_σ is the canonical linear function defining K_{X_Σ} at σ.

2.2 The change in K^3 across a flip

I will start by doing an explicit calculation for the Francia flip. Danilov shows how to factorise this flip in [6], section 5. The answer is Figure 2.5 in terms of the cones and, just once for fun, in Figure 2.6 in terms of the varieties.

Since X^+ is smooth and $Y^+ \to X^+$ is the ordinary blowup of C^+,

$$K_{Y^+} = K_+ + F.$$

I use Proposition 12 for the discrepancy calculation of K_{Y^-}. Suppose $K_{Y^-} = K_- + a_- E$. A supporting function on σ_1 is $g_1(x,y,z) = x + z - y/2$ so

$$a_- = g_1(1,1,1) - 1 = 1/2$$

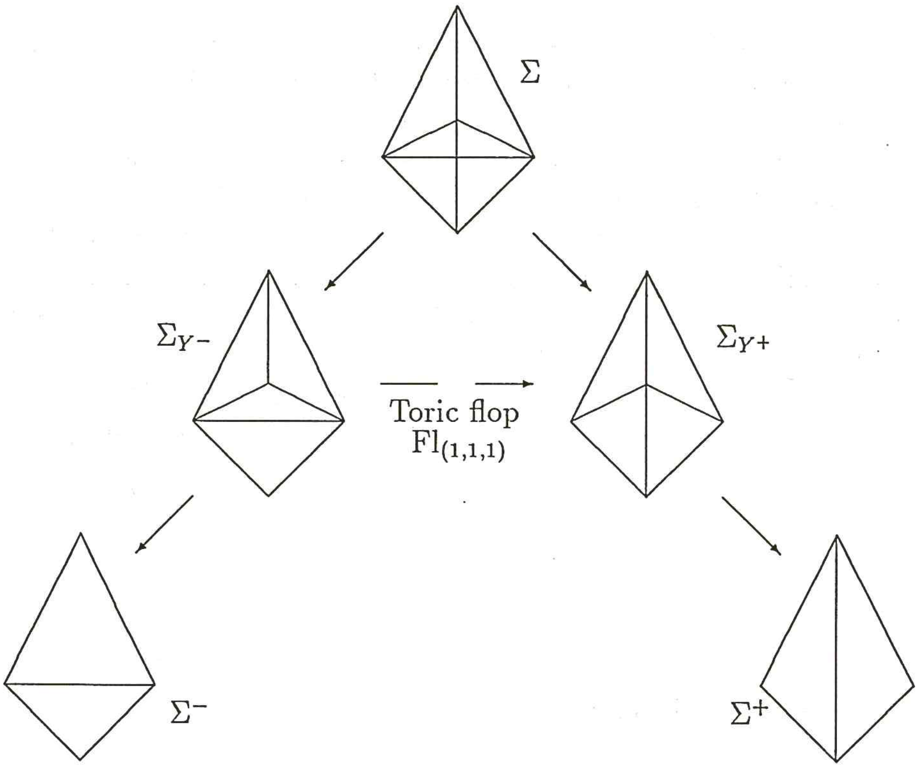


Figure 2.5: Factorising the Francia cones

and

$$K_{Y^-} = K_- + \frac{1}{2}E.$$

Multiplying out, ignoring zeros I get from $K_\pm$ being $\mathbb{Q}$-Cartier, I see that

$$K_{Y^-}^3 = K_-^3 + \frac{1}{8}E^3$$

and

$$K_{Y^+}^3 = K_+^3 + 3K_+F^2 + F^3.$$

Now calculate K_Y by adjunction in two ways:

$$K_{Y^-} + D = K_Y = K_{Y^+} + D.$$

So $K_{Y^-} = K_{Y^+}$ on Y and so $K_{Y^-}^3 = K_{Y^+}^3$. Putting this together gives

$$\delta K^3 = \frac{1}{8}E^3 - 3K_+F^2 - F^3.$$

Lemma 13 *(a)* $E^3 = 4$
(b) $F^3 = 3$, $K_+F^2 = 1$.

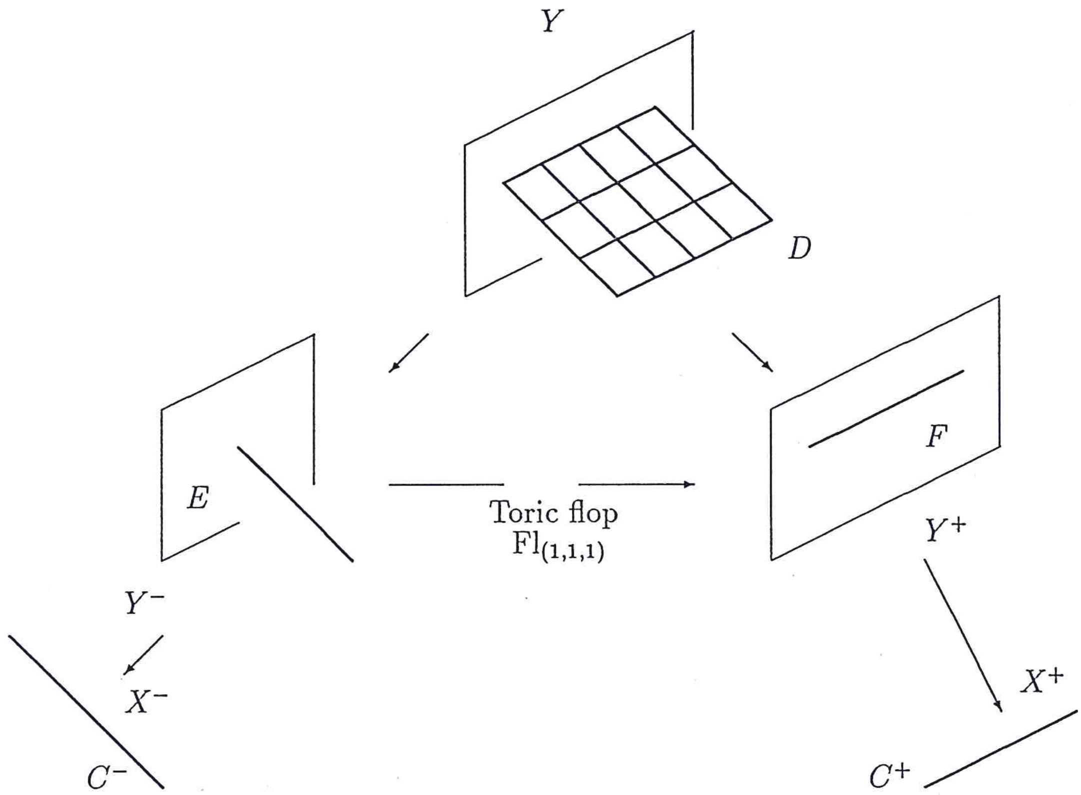


Figure 2.6: Factorising the Francia flip

Proof. (a) I can do this calculation on E because $E^3 = \left(E_{|E}\right)^2$. Clearly, $E = \mathbb{P}^2$ so $\mathrm{Pic}E = \mathbb{Z}$ and, since Y^- is smooth, in $\mathrm{Pic}E$,

$$\mathcal{O}(-3) = K_E = (K_{Y^-} + E)_{|E} = (K_- + \frac{3}{2}E)_{|E}.$$

But now, $K_-E = 0$ since K_- is $\mathbb{Q}$-Cartier, so $\mathcal{O}(-3) = \frac{3}{2}E_{|E}$ in $\mathrm{Pic}E$ and I must have

$$K_{Y^-|E} = \mathcal{O}(-1),\ E_{|E} = \mathcal{O}(-2).$$

(b) I can do this calculation in F; $F^3 = \left(F_{|F}\right)^2$ and $K_{Y^+}F^2 = K_{Y^+|F}F_{|F}$. By making an isomorphism of lattices you can see that $F = \mathbb{F}_1$ so $\mathrm{Pic}F = \mathbb{Z}f \oplus \mathbb{Z}h$ where f is the fibre of $F \to C^+ = \mathbb{P}^1$ and h is the negative section; of course, $f^2 = 0$, $fh = 1$ and $h^2 = -1$. Now, using the genus formula in F, you can see that

$$(-3,-2) = K_F = (K_{Y^+} + F)_{|F}.$$

Now $K_{Y^+}F = K_+F + F^2$ and $K_+F = (K_+C^+)f$. An elementary calculation in toric geometry given in Lemma 16 shows that $K_+C^+ = 1$. So I must have

$K_{Y^+}|_F = (-1,-1)$ and $F|_F = (-2,-1)$ in Pic$F$ and the rest of the calculation is easy. Q.E.D.

So the example is complete and

$$\delta K^3 = 1/2.$$

Reid's δK^3 argument for general flips

The following discussion applies to any flip, not just the toric ones.

Theorem 14 (Reid, unpublished)
Let $X^- \to X \leftarrow X^+$ be any flip. Then $\delta K^3 > 0$.
Moreover, if $R = \operatorname{lcm}\{\operatorname{index}(P) \mid P$ is any singularity on X^- or $X^+\}$ then $\delta K^3 \geq 1/R^3$.

Proof. Let $g^\pm : Y \to X^\pm$ be a simultaneous resolution of X^+ and X^-. In other words, Y is a smooth 3-fold and I have the following commuting diagram.

$$\begin{tikzcd} & Y \arrow[dl, "g^-"'] \arrow[dr, "g^+"] & \\ X^- \arrow[dr, "f^-"'] & & X^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

Figure 2.7: The Shokurov picture

Certainly, always omitting $(g^\pm)^*$, on Y I have $K_Y = K_\pm + E_\pm$ so $K_- = K_+ + D$ where $D = E_+ - E_-$. Since $g^\pm_* D$ is one dimensional, $K^2_\pm D = 0$ so

$$K_-^3 = K_-^2(K_+ + D) = K_-(K_+ + D)K_+ = (K_+ + D)K_+^2 + K_-DK_+$$
$$= K_+^3 + K_-DK_+.$$

In other words,

$$\delta K^3 = -K_-DK_+.$$

Since this is a terminal flip, D is effective, as in [26], proof of (2.13). Let E be any exceptional divisor in Y. If E is contracted to a point in X^+ or X^- then $K_+EK_- = 0$. On the other hand, if E dominates both C^+ and C^- then

$$-K_+EK_- = (\text{coefficient of } E \text{ in } D)(K_+C^+)(-K_-C^-) > 0.$$

To complete the proof, I just need a component that dominates both C^+ and C^-. By definition of the product, the morphisms $g^\pm$ factor through $X^+ \times_X X^-$

and the complement of D maps onto (at least) $X^+ \times_X X^- \setminus C^+ \times_P C^-$. The image of Y is closed in the product so must be the whole product; now any divisor mapping onto $C^+ \times_P C^-$ will do.

The last statement holds because both RK_- and RK_+ are Cartier so RD is also Cartier and

$$\delta K^3 = \frac{1}{R^3}(-RK_-)(RD)(RK_+) \geq \frac{1}{R^3}.$$

Q.E.D.

Remark The same argument (with $D \geq 0$) also shows that for a terminal (or canonical) flop $\delta K^3 = 0$.

Indeed, in any dimension n, and for any *toric* flip diagram (that is, in the convex polyhedron $\langle e_1, \ldots, e_{n+1}\rangle$ replace the single internal wall $\langle e_1, \ldots, e_r\rangle$ with $\langle e_{r+1}, \ldots, e_{n+1}\rangle$)

$$\delta K^n = 0 \text{ iff the 'flip' was really a flop}$$

where 'flop' means that $KC = 0$ for all contracted curves on either side. This is simply a rephrasing of [22], Proposition (4.3)(ii) when compared with the calculation in the next section. In the smooth case it's easy to calculate δK^n for all toric flips. In the general theory of Thaddeus (and others; Brion–Procesi, Dolgachev–Hu) many real-life 'flips' associated to moduli spaces reduce to exactly this smooth case, see [28].

The calculation for toric flips

Now I fix attention back on toric flips and calculate δK^3 for the general one. I use the notation $t = a + 1 - b - c$ and $d = abc$ in the flip $(a, 1, -b, -c)$. I calculate δK^3 in 2 steps; first calculate $K_\pm$ on the exceptional curves and then pull back to a common resolution to compare.

In the notation of Figure 2.8, Proposition (2.7) of [22] says that, for each $i = 1, 2, 3$,

$$D_i C^- = -e_i^*(e_4) D_4 C^-$$

where $D_i := D_{e_i}$, $C^- = l_\omega$ and, for safety, I should tensor with $\mathbb{Q}$ before talking about $e_i^* \in M_\mathbb{Q}$. I will give the proof of [22]; I must do this in any case to show why it doesn't matter that my fan isn't complete.

Choose any embedding $\Sigma^- \subset \widetilde{\Sigma}$ of Σ^- in a complete fan $\widetilde{\Sigma}$. Let $\widetilde{X} = X_{\widetilde{\Sigma}}$. Choose any $m \in M$ such that $m(e_1) > 0$, $m(e_2) = m(e_3) = 0$; m is some multiple of $e_1^* \in M_\mathbb{Q}$. Let $D = \mathrm{Div}(m)$. Then

$$DC^- = \sum_{v \in \mathcal{V}(\widetilde{\Sigma})} m(v) D_v C^-$$

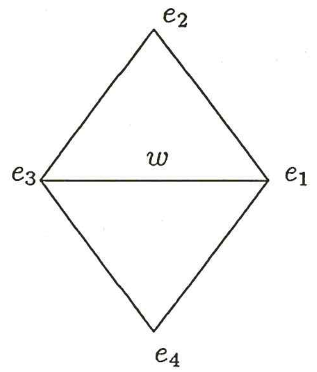


Figure 2.8: Intersection numbers

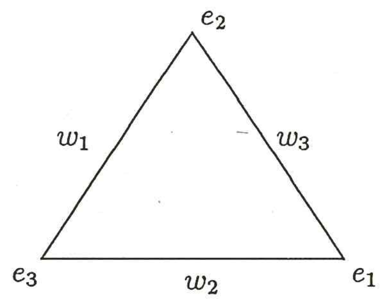
$$= \sum_{i=1}^{4} m(e_i) D_i C^-$$

because all other D_v are disjoint from C^-. But $D = 0$ so $DC^- = 0$ and the other side of the equation is $m(e_1)D_1C^- + m(e_4)D_4C^-$ by choice of m. So

$$D_1C^- = -\frac{m(e_4)}{m(e_1)} D_4C^- = -e_1^*(e_4)D_4C^-,$$

as claimed.

Lemma 15 *Let $\sigma \in \Sigma$ be the simplicial cone*



in $\mathbb{Z}^3$. If $0 \in X_\sigma \subset X_\Sigma$ is an isolated singularity or a smooth point then

$$D_i l_i = \Delta^{-1}$$

for each $i = 1, 2, 3$, where $l_i = l_{w_i}$ and Δ is the index in $\mathbb{Z}^3$ of the sublattice $\langle e_1, e_2, e_3 \rangle_{\mathbb{Z}}$.

Proof. I do not have to move anything in $A^*(X_\Sigma)$ to calculate $D_i l_i$ so Σ can have any other cones I like. By the isolatedness assumption I can extend σ

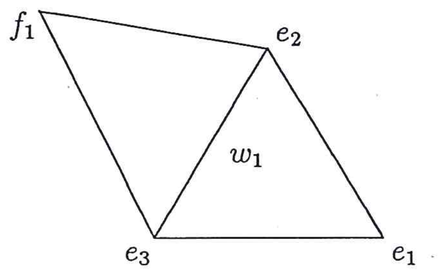


Figure 2.9: Bounding a singularity

as in Figure 2.9. where $\langle f_1, e_2, e_3 \rangle_{\mathbb{Z}} = \mathbb{Z}^3$. Using the matrix $(e_2, -f_1, e_3)$ I can suppose that

$$e_2 = (1,0,0), e_3 = (0,0,1), f_1 = (0,-1,0), e_1 = (l,m,n), m > 0.$$

In this case $\Delta = m$. By the calculation of [22] given above $D_1 l_1 = -e_1^*(f_1) D_{f_1} l_1$ and $e_1^* = (0, 1/m, 0)$ so

$$D_1 l_1 = 1/m = \Delta^{-1}.$$

Q.E.D.

Now for $K_{\widetilde{X}}$. As I recalled at the beginning of this chapter, or see [5] 6.6,

$$K_{\widetilde{X}} = -\sum_{v \in \mathcal{V}(\widetilde{\Sigma})} D_v$$

so, ignoring zeroes as above,

$$K_{\widetilde{X}} C^- = -\sum_{i=1}^{4} D_i C^-.$$

So clearly the choice of completion, $\widetilde{\Sigma}$, is irrelevant; in fact, I do not need a complete fan at all and happily conclude that

$$K_- C^- = -\sum_{i=1}^{4} D_i C^-.$$

Exactly the same thing holds for $K_+ C^+$.

Now I just choose dual vectors and calculate the contributions $D_i C^-$.

Lemma 16 *In the toric flip* $(a, 1, -b, -c)$

$$K_- C^- = -t/a,$$

$$K_+ C^+ = t/bc,$$

where $t = a + 1 - b - c$.

Proof. Using e_1, e_2, e_3 as a basis of $N_{\mathbb{Q}}$, the dual basis of $M_{\mathbb{Q}}$ is

$$e_1^* = (1,0,0), e_2^* = (0,0,1), e_3^* = (0,-1,0).$$

Now $a_1 = -e_1^*(e_4) = -b$ so

$$D_1C^- = -bD_4C^-$$

and similarly,

$$D_2C^- = -cD_4C^-$$

and

$$D_3C^- = aD_4C^-.$$

But $D_3C^- = 1$ because it is a coordinate line meeting a coordinate plane in $\mathbb{C}^3$. Altogether I get

$$\begin{aligned} K_-C^- &= -\sum D_iC^- \\ &= \frac{b}{a} + \frac{c}{a} - 1 - \frac{1}{a} \\ &= -t/a. \end{aligned}$$

The calculation on X^+ is the same using the basis e_2, e_3, e_4 and Lemma 15 to add that

$$D_1C^+ = 1/c.$$

Q.E.D.

Theorem 17 *In the toric flip* $(a, 1, -b, -c)$,

$$\delta K^3 = t^3/d.$$

where $t = a + 1 - b - c$ *and* $d = abc$.

Proof. In [6], Danilov gives an explicit common resolution of X^+ and X^- in terms of the fans; this is in Figure 2.2, where 'junk' means any decomposition of the cone into regular cones which matches the decomposition shown on the boundary.

The only contribution to δK^3 is from the divisor $E = \mathbb{P}^1 \times \mathbb{P}^1$ at the centre of the diagram because that is the only exceptional divisor dominating both C^+ and C^-. In other words, in the notation of the previous section, $D = (a_+ - a_-)E$ where $K_Y = K_\pm + a_\pm E +$(other exceptional contributions). So I must calculate the two discrepancies a_- and a_+. The central point is (b,0,c) so by Proposition 12

$$a_- = g_-(b,0,c) - 1, a_+ = g_+(b,0,c) - 1.$$

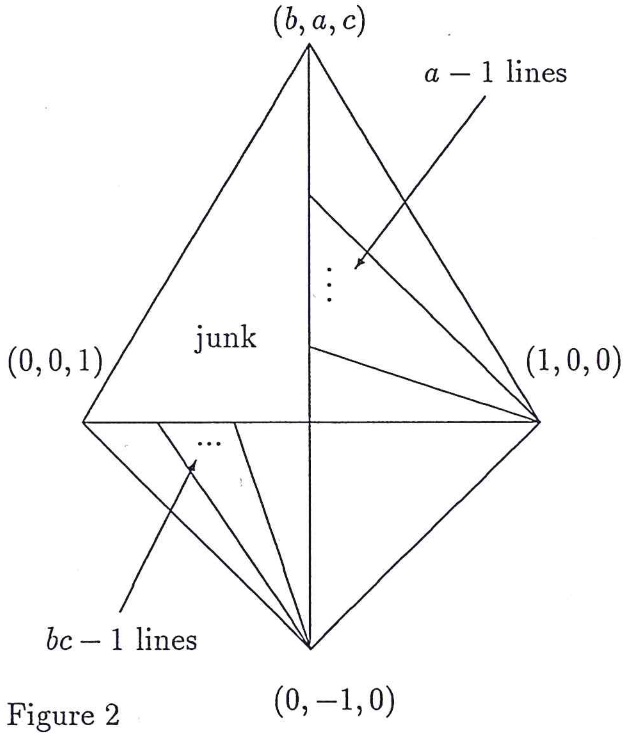


Figure 2.10: Toric resolution

I can take $g_- = x + z$ and $g_+ = \frac{a+1}{b+c}(x+z) - y$ giving $a_- = b + c - 1$ and $a_+ = a$ so that

$$a_+ - a_- = a - (b + c - 1) = t.$$

Recall from Lemma 16 that

$$(K_+C^+)(-K_-C^-) = t^2/d.$$

Now the result is clear. Q.E.D.

Remark (1) The proof didn't really use detailed properties of Danilov's resolution, it just used the existence of a resolution with the obvious fact that the central point must be part of it and I can choose it to be the only part dominating both C^+ and C^-. It also didn't rely on terminal singularities; the form of the fan is enough to guarantee that K is $\mathbb{Q}$-Cartier and that the same calculation works. I only used the fact that the singularities were isolated. It is more convenient to give the general answer later since the notation here is rigged up for the terminal case. Of course, the general answer will also work for varieties with canonical singularities since blowing up the 1 dimensional singular locus is a crepant process.

(2) The number δK^3 can be arbitrarily small. Indeed, for the flip $(n,1,1-n,-1)$

$$\delta K^3 = \frac{1}{n(n-1)}.$$

Maybe this is no surprise — you could think of these flips as being close to a (canonical) flop for large n because $K_-C^- = 1/n$.

The number δK^3 can also be arbitrarily large; $(n,1,-1,-1)$ has $\delta K^3 = (n-1)^3/n$.

2.3 The calculation of the pluricanonical difference

Suppose that $X^- \to X \leftarrow X^+$ is a flip of projective varieties. Then the *pluricanonical difference* is the function of $m \in \mathbb{Z}$ defined by

$$\delta\chi(mK) := \chi(X^+, mK_+) - \chi(X^-, mK_-).$$

I already know that $\delta\chi(\mathcal{O}) = 0$ (use normality when comparing $X^\pm$ with resolutions and then Hodge theory as usual). By Serre duality (terminal singularities *are* Cohen–Macaulay!)

$$\delta\chi(mK) = -\delta\chi((1-m)K)$$

so I also know that $\delta\chi(K) = 0$ and that I only need to consider $m \geq 2$.

A cohomological approach to general flips

The following lemma that reduces the calculation to that of just one cohomology group.

Lemma 18 *For any flip of projective varieties, not necessarily toric,*

(1)

$$\delta\chi(mK) = R^1 f_*^-(mK_-) - R^1 f_*^+(mK_+).$$

(2) If $m \geq 2$ then $R^1 f_*^+(mK_+) = 0$; if $m \leq -1$ then $R^1 f_*^-(mK_-) = 0$.

Of course, by $R^1 f_*(\mathcal{E})$ I mean $\dim R^1 f_*(\mathcal{E})_P$ where $P \in X$ is the flipping point.

Proof. (1) Firstly, $f_*^-(mK_-) = f_*^+(mK_+)$ because they are both, by definition, divisorial sheaves and they agree in codimension 1. Now you can write out the Leray spectral sequence for both sides of the flip and read the result straight from it; of course, $R^i f_*^\pm = 0$ for $i \geq 2$ because the fibres are, at most, one dimensional.

(2) is standard by vanishing since $-K_-$ and K_+ are relatively ample; you can use Leray again to interpret absolute cohomology vanishing in the relative case.

Here is the proof for X^+ in detail. I have taken the outline from Fletcher's thesis, [10]. I am given the morphism $f^+ \colon X^+ \to X$. Let $g \colon Y \to X^+$ be any resolution of the singularities of X^+. Now fix some $m \geq 2$. Let A be any ample divisor on X and let

$$D_1 = (m-1)K_+ + (f^+)^*A.$$

Since X^- is $\mathbb{Q}$-Gorenstein I can define

$$D = g^* D_1.$$

I can also assume that D is a simple normal crossing divisor: just blowup Y some more if necessary. Now the proof is in four steps.

Step 1 $H^i(Y, K_Y + \lceil D \rceil) = 0$, for all $i > 0$

By Kleiman's criterion D_1 is ample so D is nef and big on Y. Now the vanishing theorem of Kawamata and Viehweg gives the result.

Step 2 $H^i(X^+, g_*(K_Y + \lceil D \rceil)) = H^i(Y, K_Y + \lceil D \rceil)$

Let B be any ample divisor on X^+ and suppose, again without loss of generality, that $K_Y + \lceil D \rceil + g^*B$ is a simple normal crossing divisor. Then, as in Step 1, I have that, for all $i > 0$,

$$H^i(Y, K_Y + \lceil D \rceil + g^*B) = 0$$

because $\lceil D + g^*B \rceil = \lceil D \rceil + g^*B$. Now I write out the Leray spectral sequence for the map $g \colon Y \to X^+$ and the sheaf $\mathcal{F}_1 = \mathcal{F} \otimes \mathcal{O}_Y(g^*B)$ on Y where $\mathcal{F} = \mathcal{O}_Y(K_Y + \lceil D \rceil)$. Set $\mathcal{G}_i = R^i g_* \mathcal{F} \otimes \mathcal{O}_{X^+}(B)$.

$$\mathrm{I}_2^{p,q} = H^p(X^+, \mathcal{G}_q) \Longrightarrow H^*(Y, \mathcal{F}_1).$$

Since all the sheaves in sight are coherent, I can choose B ample enough to kill all cohomology groups in the spectral sequence except H^0s while maintaining the triviality of the abuttment, $H^*(Y, \mathcal{F}_1) = H^0(Y, \mathcal{F}_1)$, to conclude that

$$H^0(X^+, R^i g_* \mathcal{F} \otimes B) = H^i(Y, \mathcal{F}_1) = 0$$

for all $i > 0$. But B is locally trivial so I must have that for all $i > 0$, $R^i g_* \mathcal{F} = 0$. Now the Leray spectral sequence

$$\mathrm{II}_2^{p,q} = H^p(X^+, R^q g_* \mathcal{F}) \Longrightarrow H^*(Y, \mathcal{F})$$

for the same map and the sheaf $\mathcal{F}$ gives that for all $i > 0$,

$$H^i(Y, \mathcal{F}) = H^i(X^+, g_* \mathcal{F}).$$

Step 3 $H^i(X^+, mK_+ + (f^+)^*A) = H^i(X^+, g_*(K_Y + \lceil D \rceil))$

I want to say that the two reflexive sheaves

$$\mathcal{E}_1 = g_*(K_Y + \lceil D \rceil)$$

and

$$\mathcal{E}_2 = mK_+ + (f^+)^*A$$

are equal in codimension 1 so, since X^+ is normal, they are equal.

Let

$$\Delta = \lceil D \rceil - D = \lceil g^*((m-1)K_+) \rceil - g^*((m-1)K_+);$$

this is a divisor on Y supported only on the exceptional locus, $\cup E_j$, of g. Let $U \subset X^+ \setminus \mathrm{Sing} X^+$ be an open set. Then, remembering that $\lceil D \rceil = \Delta + D$,

$$\begin{aligned}
\Gamma(U, \mathcal{E}_1) &= \Gamma(g^{-1}U, K_Y + \lceil D \rceil) \\
&= \Gamma(g^{-1}U, mg^*K_+ + \sum_j a_j E_j + g^* f^+ * A + \Delta) \\
&= \Gamma(g^{-1}U, mg^*K_+ + g^* f^{+*} A) \\
&= \Gamma(U, g_* g^*(mK_+ + f^{+*}A)) \\
&= \Gamma(U, mK_+ + f^{+*}A) \\
&= \Gamma(U, \mathcal{E}_2).
\end{aligned}$$

So $\mathcal{E}_1 \equiv \mathcal{E}_2$ in codimension 1 since the singularities of X^+ are certainly in codimension 2.

Step 4 Conclusion

By Steps 1–3 I know that for any ample A on X

$$H^i(X^+, mK_+ + (f^+)^*A) = 0.$$

As in Step 2, if A is sufficiently ample,

$$0 = H^i(X^+, mK_+ + (f^+)^*A) = H^0(R^i f^+_*(mK_+) \otimes \mathcal{O}(A)),$$

and, if A is possibly even more ample,

$$R^i f^+_*(mK_+) = 0,$$

as required.

The case for X^- is exactly the same.

Q.E.D.

Corollary 19 *For any flip of projective varieties, not necessarily toric,*

$$\delta\chi(mK) \geq 0 \textit{ if } m \geq 2,$$
$$\delta\chi(mK) = 0 \textit{ if } m = 0, 1,$$
$$\delta\chi(mK) \leq 0 \textit{ if } m \leq -1.$$

Remark (1) Thinking of $\chi(X, mK_X)$ as a function $\mathbb{Z} \longrightarrow \mathbb{Z}$ there is no reason why it should be monotonic; indeed, using Fletcher's notation from [9], the canonical variety $X_{10,12} \subset \mathbb{P}(2^2, 3, 4, 5^2)$ has $\chi(K_X) = 1$, $\chi(2K_X) = 2$ and $\chi(3K_X) = 1$. Another example is the canonical variety $X_{6,6,6} \subset \mathbb{P}(2^4, 3^3)$.

In contrast, it is tempting to believe that the function $\delta\chi(mK)\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ *is* nondecreasing. Although I can't prove this in general, I give the proof in special cases in section 4.2. The result for toric flips is also a corollary of the calculation that follows.

(2) It is clear that eventually $\delta\chi(mK)$ will be nonzero: in the spirit of this section,

$$R^1 f_*^-(mK_-) \text{ is dual to } R^0 f_*^-((1-m)K_-) \overset{\sim RR}{>} (1-m)K_- C^-$$

where the relative duality works by the Theorem on Formal Functions. I have just noticed this gap and have no intention of working out any details, not least because it is much easier to do less technically using the Plurigenus Formula.

(3) For any (not necessarily toric) flip $\delta\chi(mK)$ is determined by X^- and f^-. This is no surprise since X^+ itself is determined by these two. It is interesting to notice, however, that this already gives some information about X^+. In [9], Fletcher shows how to calculate the 'record of pluridata' of a variety Y from the function $\chi(Y, mK_Y)$. This pluridata includes K_Y^3 and the basket of singularities of Y in the sense of the plurigenus formula. I will use this in Chapter 5.

(4) Now I can define the pluricanonical difference for any flip neighbourhood, not necessarily just for projective flips, to be

$$\delta\chi(mK) = -\delta R^1 f_*(mK).$$

This number is dependent only on an analytic neighbourhood of the flip and will coincide with the genuine $\delta\chi(mK)$ on any projective flip by the Lemma.

The toric result

In toric geometry it is possible to 'see' the plurigenera change. The prototype is the following Čech calculation of $h^1(\mathbb{P}^1, \mathcal{O}(n))$. Consider $\mathbb{P}^1 = U \cup V$ where $U = \{x \neq 0\}$ and $V = \{y \neq 0\}$. The Čech complex of this cover is

$$0 \to H^0(U, \mathcal{O}(n)) \times H^0(V, \mathcal{O}(n)) \to H^0(U \cup V, \mathcal{O}(n)) \to 0.$$

Writing bases of monomials for these vector spaces gives

$$H^0(V, \mathcal{O}(n)) = \left\langle x^i y^{n-i}, i \geq 0 \right\rangle$$

and so on. The coboundary map, δ, obviously maps basis elements to basis elements so $h^1(\mathbb{P}^1, \mathcal{O}(n)) = \dim \operatorname{coker} \delta$ is just the number

$$\dim \operatorname{coker} \delta = \#\{i \in \mathbb{Z} \mid 0 > i > n\}$$

of monomials in $H^0(U \cup V, \mathcal{O}(n))$ not hit by δ. You can see this happening in Figure 2.11; watch the cohomology change as the left hand marker moves with n (where the labelling is by $i \in \mathbb{Z}$, the power of x).

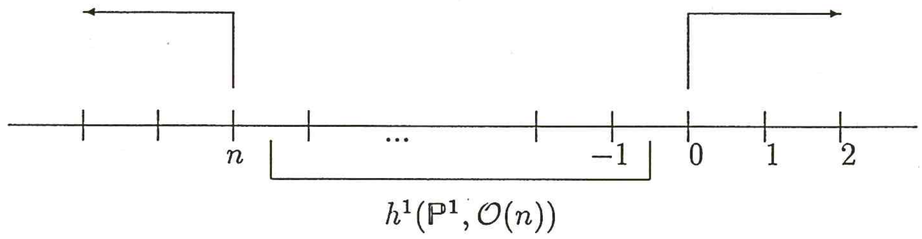


Figure 2.11: The cohomology of projective space

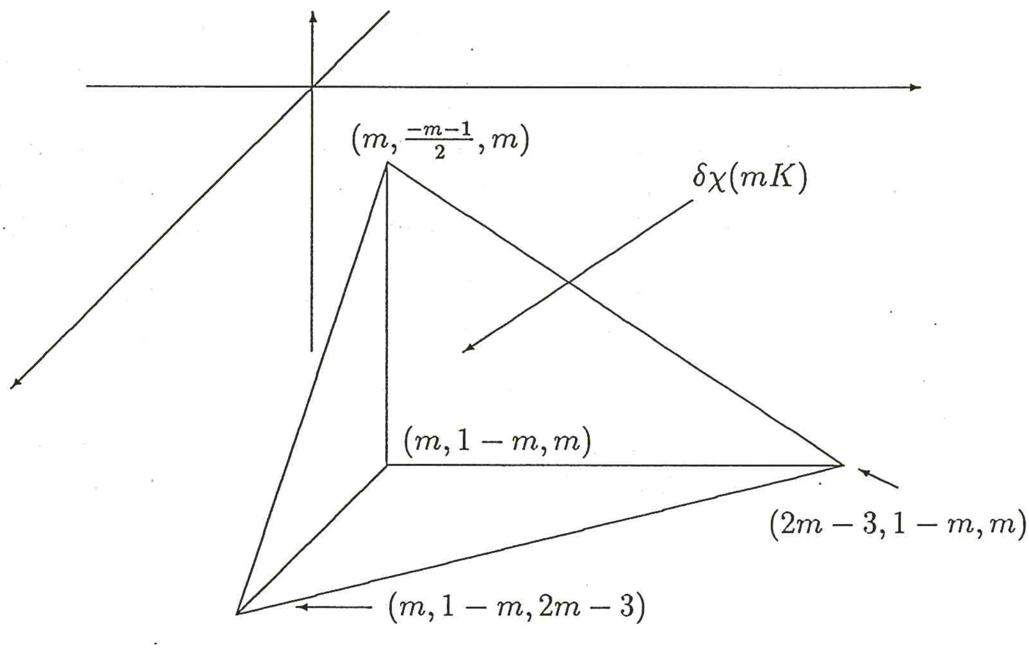


Figure 2.12: The relative cohomology of a flip

As usual I will start with the example of the Francia flip, $(2, 1, -1, -1)$. I want to present the answer as Figure 2.12. In the picture the number of lattice points in the tetrahedron, including the boundary, is $\delta\chi(mK)$ for $m > 0$. If $m < 1$ the picture is a similar tetrahedron on the other side of the origin.

To justify this description I'll look at $h^1(X^-, \mathcal{E}_m)$ where $\mathcal{E}_m = \mathcal{O}(mK_-)$. By the Lemma this is sufficient, since I want to calculate the dimension of $R^1 f^{-*}(mK_-)$ and X is affine. To write the Čech complex for $\mathcal{E}_m$ I need to know its sections over all intersections of opens in the covering. I showed how to do this this in the section 2.1.

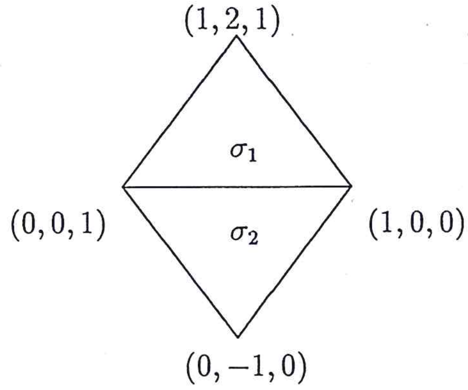


Figure 2.13: The left hand side of the Francia flip

The fan for X^- is in Figure 2.13. Let $X_i := X_{\sigma_i}$ and $X_{12} := X_1 \cap X_2$. I want to define $\mathcal{E}_1 = K_-$ first. The walls of σ_2 are defined by the dual elements (required to be positive on σ_2) $(1,0,0)$, $(0,-1,0)$ and $(0,0,1)$ so

$$k_2 = (1,-1,1),$$

since σ_2 is a regular cone. The walls of σ_1 are defined by $(2,-1,0)$, $(0,-1,2)$ and $(0,1,0)$. Adding these three together gives the linear function $(2,-1,2)$ However, σ_1 is not a regular cone — $0 \in X_1$ is the quotient singularity $\frac{1}{2}(1,1,1)$. So scale this linear function to get

$$k_1 = (1,-1/2,1),$$

where k_1 and k_2 now agree on $\sigma_1 \cap \sigma_2$ and define the line bundle $\mathcal{E}_1$.

Now I have the $\{m_\sigma\}$ for any of the sheaves $\mathcal{E}_m = mK_-$ by multiplication; explicitly, mk_1 amd mk_2 represent $\mathcal{E}_m$

At last I have a grip on some cohomology;

$$\begin{aligned} H^0(X_2, mK_-) &= \{(\alpha,\beta,\gamma) \in M \mid (\alpha,\beta,\gamma) - mk_2 \geq 0 \text{ on } \sigma_2\} \\ &= \{(\alpha,\beta,\gamma) \in M \mid (\alpha - m, \beta + m, \gamma - m) \geq 0 \text{ on } \sigma_2\}. \end{aligned}$$

It is sufficient to check the condition on the vertices of σ_2 so

$$\begin{aligned} H^0(X_2, mK_-) &= \{(\alpha,\beta,\gamma) \in M \mid \alpha - m \geq 0, \gamma - m \geq 0, -(\beta + m) \geq 0\} \\ &= \{(\alpha,\beta,\gamma) \in M \mid \alpha,\gamma \geq m, \beta \leq -m\}. \end{aligned}$$

Similarly,

$$H^0(X_1, mK_-) = \{(\alpha,\beta,\gamma) \in M \mid \alpha,\gamma \geq m, \alpha + 2\beta + \gamma \geq m\}.$$

The conditions for a section to be in $H^0(X_{12}, mK_-)$ are in codimension 1 so I just take the common conditions to be in $H^0(X_1, mK_-)$ and $H^0(X_2, mK_-)$,

$$H^0(X_{12}, mK_-) = \{(\alpha,\beta,\gamma) \in M \mid \alpha,\gamma \geq m\}.$$

So, noticing what coker δ is,

$$H^1(X^-, mK_-) = \{(\alpha, \beta, \gamma) \in M \mid \alpha, \gamma \geq m, \beta > -m, \alpha + 2\beta + \gamma < m\}.$$

which is the tetrahedron I drew at the beginning.

It is easy to calculate the first few terms of the sequence of plurigenera changes.

m	1	2	3	4	5	6	7	8
$\delta\chi(mK)$	0	0	1	3	7	13	22	34

Figure 2.14: Pluricanonical change data

The same calculation as above gives the general result. Recall from [22] that the *shed*, **Shed**(X), of a fan of a $\mathbb{Q}$-factorial toric variety, $X = X_\Sigma$, is the union of the primitive parts of its cones; that is, cone by cone, the shed is the closed polyhedron whose vertices are the origin and the primitive lattice points on all the one dimensional faces.

Theorem 20 *Let $a > b > 0$ be coprime integers and let $c = 1$ or $a - b$. For the toric flip $(a, 1, -b, -c)$ and for $m \geq 1$*

$$\delta\chi(mK) = h^1(X^-, mK_-)$$

where, equivalently,

(I) $H^1(X^-, mK_-) = \{(\alpha, \beta, \gamma) \mid \alpha, \gamma \geq m, \beta > -m, b\alpha + a\beta + c\gamma < m\}$;

(II) let Δ_m be the tetrahedron in $\mathbb{R}^3$ whose vertices are

$$(0,0,0), (mt/b, 0, 0), (0, mt/a, 0), (0, 0, mt/c),$$

where $t = a + 1 - b - c$ as before; $h^1(X^-, mK_-)$ is the number of integer lattice points of Δ_m which do not lie on either the XZ-plane or the front (sloping) face;

(III) in the standard model of the cones of a flip, the one in Figure 2.2 with coordinates chosen as in Danilov's Theorem 9, the shed of X^+ is contained in the shed of X^-. In this model $h^1(X^-, mK_-)$ is the number of lattice points (x, y, z) divisible by (b, a, c) in the difference of the interiors of multiples of the two sheds. Precisely, $H^1(X^-, mK_-)$ is

$$\{(x, y, z) \in N \mid (x, y, z) \in m\left(\mathbf{Shed}(X^-)^\circ \setminus \mathbf{Shed}(X^+)^\circ\right), b|x, a|y, c|z\}.$$

Figure 2.15 shows the tetrahedron that sits in the difference of the two sheds.

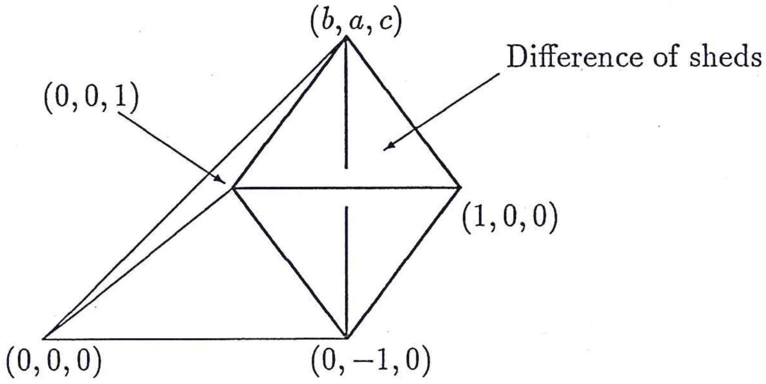


Figure 2.15: The difference of the sheds

Proof. (I) is identical to the previous calculation. The thing you need to see, calculating as in the example, is that in the general cone σ_1,

$$k_1 = \left(1, \frac{1-b-c}{a}, 1\right)$$

so that $(\alpha, \beta, \gamma) - mk_1 \geq 0$ when evaluated at the vertex (b, a, c) gives the condition

$$b\alpha + a\beta + c\gamma \geq m(b + (1-b-c) + c) = m.$$

(II) and (III) are affine restatements of (I). Q.E.D.

Corollary 21 *(a) In the flip* $(a, 1, -b, b-a)$

$$\delta\chi(mK) = 0 \textit{ iff } m \leq a.$$

In the flip $(a, 1, -b, -1)$

$$\delta\chi(mK) = 0 \textit{ iff } m \leq \frac{a}{a-b}.$$

(b) For all toric flips,

$$\delta\chi((a+1)K) \neq 0,$$

where a is the index of the singularity on X^-.

Proof. By the description (II) of the theorem

$$\delta\chi(mK) \neq 0 \text{ iff } (0,1,0) \in \Delta_m.$$

This condition easily translates as $mt > a$. Q.E.D.

Remark (1) Using this it is easy to see that the sequence of flips $(n, 1, 1-n, -1)$ gives examples where $\delta\chi(mK) = 0$ for arbitrarily large m. Also, the sequence of flips $(n, 1, -1, -1)$ gives examples where $\delta\chi(2K)$ is arbitrarily large.

(2) These descriptions show very clearly why $\delta\chi(mK)$ is nondecreasing in the toric case.

2.4 Other changes in the Minimal Model Program

As an exercise it is not hard to repeat everything here for toric divisorial contractions to a point. The result is very similar; $\delta\chi(mK) = h^2(X^-, mK_-)$ is the number of suitably divisible lattice points in the difference of the interiors of multiples of the two sheds. The ordinary blowup of the origin of $\mathbb{C}^3$ in toric terms is the typical calculation. I give some examples of the calculation of δK^3.

The case of a toric blowup of a point

The two types of toric extremal divisorial contractions to a point are shown in Figure 2.16; $\mathrm{hcf}(a, b) = 1$. It is easier to show this later; see the remark after Theorem 31, although, of course, it is all in [6]. A blowup of type (A) I denote by $\mathrm{Bl}^A_{(a,b)}$ and one of type (B) I denote by $\mathrm{Bl}^B_{(a,b)}$.

Let $Y \longrightarrow X$ be the blowup.
For the blowup $\mathrm{Bl}^A_{(a,b)}$ the relation

$$v = ae_1 + be_2 + e_3$$

holds among the vertices and the internal point of the cone and I define $t = a+b$, $d = -ab$; for the blowup $\mathrm{Bl}^B_{(a,b)}$ the relation

$$(a+b)v = ae_1 + be_2 + e_3$$

holds among the vertices and the internal point of the cone and I define $t = 1$, $d = -ab(a+b)$. (Compare section 3.2.)
I claim that in either case

$$\delta K^3 = K_X^3 - K_Y^3 = t^3/d$$

as before.

For the blowup of type (A), the exceptional divisor is $E = \mathbb{P}(a, b, 1)$ and it has discrepancy $(x+y+z)_{|(a,b,1)} - 1 = a+b$. So $K_Y = K_X + (a+b)E$ and

$$\delta K^3 = -(a+b)^3 E^3.$$

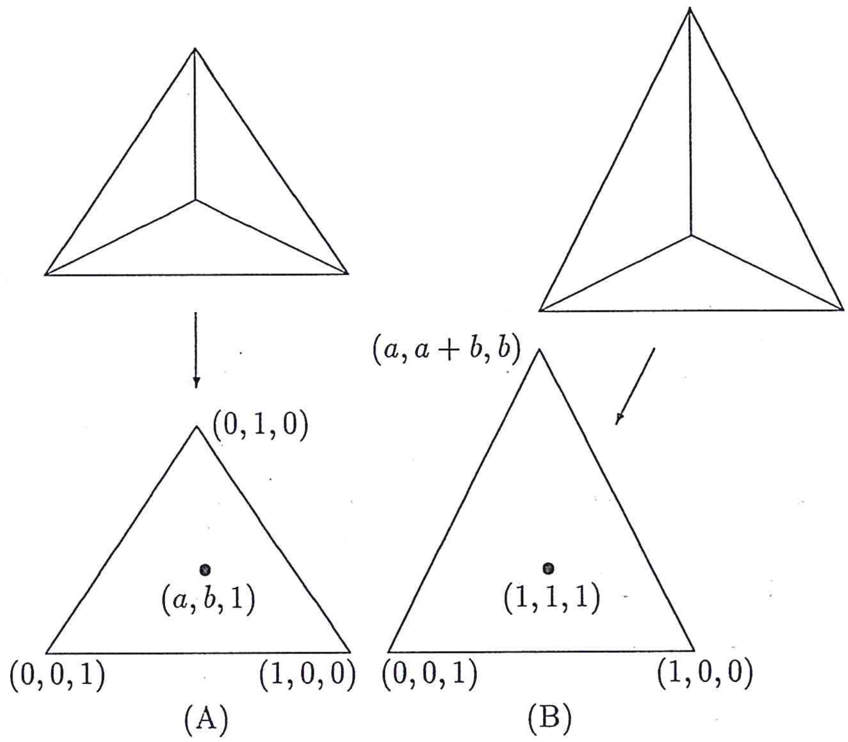


Figure 2.16: Divisorial contractions to a point

As before, since $K_{Y|E} = (a+b)E_{|E}$,

$$\mathcal{O}(-(a+b+1)) = K_E = (K_Y + E)_{|E} = (a+b+1)E_{|E}$$

so $E_{|E} = \mathcal{O}(-1)$ and $E^3 = \left(E_{|E}\right)^2 = 1/ab$ giving

$$\delta K^3 = -(a+b)^3 E^3 = -(a+b)^3/ab$$

as required.

I have to worry about my use of the adjunction formula in calculating $E_{|E}$ since Y is not smooth. However, if I remove the 2 singular 0-strata from Y, then the formula holds since both Y and E are smooth outside them. But I have only removed something in codimension 2 on E so the formula also holds on E since all sheaves in sight are divisorial.

For the blowup of type (B) I convert the calculation into a flipping calculation. You can see this in terms of fans in Figure 2.17.

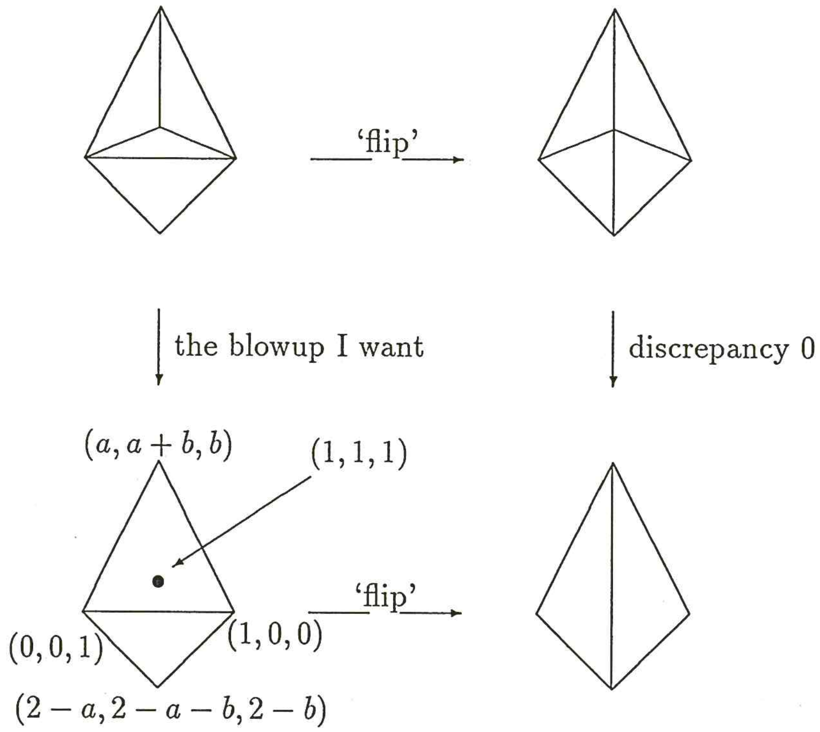


Figure 2.17: Factorising a blowup

The discrepancy of the exceptional divisor in the blowup of the line on the right is 0 — that was the whole point of choosing such a complicated lower vertex — so $\delta K^3 = 0$ there. The lower arrow is a (nonterminal) flip with $t = 2$, $d = 4ab(a+b)(a+b-2)$; calculate these numbers by writing down the relation holding among the vertices and the internal point. So $\delta K^3 = 2/ab(a+b)(a+b-2)$ there. Finally, the upper arrow is a (nonterminal) flip with $t = 1$, $d = ab(a+b-2)$ so $\delta K^3 = 1/ab(a+b-2)$ there. Altogether, for the blowup of type (B), I end up with

$$\delta K^3 = 2/ab(a+b)(a+b-2) - 1/ab(a+b-2) = -1/ab(a+b)$$

as required.

The problem with this calculation is that I haven't checked that the formula t^3/d holds for flips as general as the ones I use to factorise the blowup — the lower flip will certainly have nonisolated singularities on it, for instance.

A corollary of this is that $\delta K^3 = 0$ in the blowup of a line given by the decomposition of fans in Figure 2.18. This was the calculation I did in the case of the Francia flip at the beginning of the chapter. To see it, draw the same diagram as Figure 2.17 but with lower vertex $(0, -1, 0)$. In this case, the upper 'flip' is the Atiyah flop and the lower 'flip' is the genuine flip $(a+b, 1, -a, -b)$

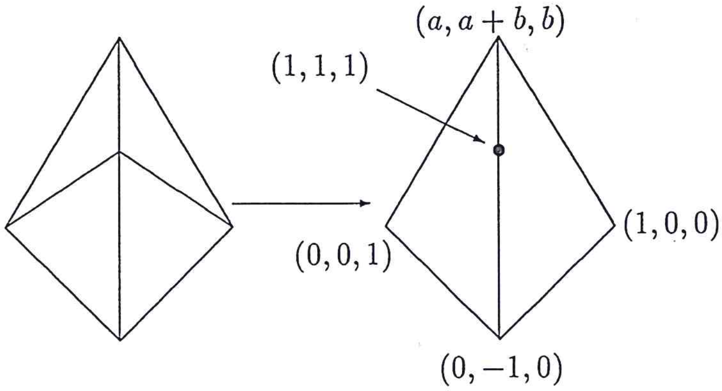


Figure 2.18: A weighted blowup of a line

which has $\delta K^3 = 1/ab(a+b)$ cancelling the contribution from the blowup on the lefthand side.

Remark I had hoped to get more out of the factorisation idea, but in the end the calculations for flips seemed to be easier than those for divisorial contractions.

Chapter 3

Flips and Geometric Invariant Theory

The aim of this chapter is to begin Miles Reid's programme for the classification of flips as described in [25]. This proceeds as follows. Take a normal affine Gorenstein 4-fold A with a $\mathbb{C}^*$ action fixing a unique point $0 \in A$. Taking quotients of A in different ways results in a 3-fold flip diagram. In some cases, this flip diagram will satisfy the conditions of a flip. I would like to make a list of all such cases.

When $A = \mathbb{C}^4$ I recover the toric flips of the last chapter. The main case in this chapter is when $A \subset \mathbb{C}^5$ is a hypersurface.

The issue of why all flips can be represented in this way is discussed in [25]; I resist discussing it further, but see Appendix B for a small part of the theory.

3.1 The three quotients

I am taking this description directly from Miles Reid's preprint [25], although there are certainly other authors I could mention. My aim is to get quickly into the explicit calculations of the following 2 sections so I miss out much of the detail.

Let $\mathbb{C}^*$ act linearly on $\mathbb{C}^N$, that is, on eigencoordinates $x_1, \ldots, x_N$ of $\mathbb{C}^N$, $\mathbb{C}^*$ acts by

$$\varepsilon\colon x_i \mapsto \varepsilon^{c_i} x_i,$$

where $c_i \in \mathbb{Z}$. I say that x_i *has weight* c_i and denote this by $\mathrm{wt} x_i = c_i$. The weights are also called the *characters* of the action. Often I will know the signs of the c_i. In that case I will say that $\mathbb{C}^*$ acts on eigencoordinates $x_1, \ldots, x_r$, $y_1, \ldots, y_s$, $z_1, \ldots, z_t$ of $\mathbb{C}^N$ by,

$$\varepsilon\colon x_i \mapsto \varepsilon^{a_i} x_i, \quad \varepsilon\colon y_j \mapsto \varepsilon^{-b_j} y_j, \quad \varepsilon\colon z_k \mapsto z_k,$$

where the a_i and the b_j are strictly positive integers. I abbreviate this by $(a_1, \ldots, a_r, -b_1, \ldots, -b_s, 0, \ldots, 0)$ or even more briefly by any of $(a^r, -b^s, 0^t)$, $(a, b, 0)$ or $(+^r, -^s, 0^t)$.

It is convenient to name certain subspaces of $\mathbb{C}^N$:

$$\widetilde{B^-} : (x_1 = \ldots = x_r = 0) \subset \mathbb{C}^N;$$

$$\widetilde{B^+} : (y_1 = \ldots = y_s = 0) \subset \mathbb{C}^N;$$

$$\widetilde{B^0} : (x_1 = \ldots = x_r = y_1 = \ldots = y_s = 0) \subset \mathbb{C}^N.$$

The sets are important since $\left(\widetilde{B^-} \cup \widetilde{B^+}\right) \setminus \widetilde{B^0}$ comprises the nonclosed $\mathbb{C}^*$ orbits in $\mathbb{C}^N$; the points of $\widetilde{B^0}$ lie in the closure of these orbits. This explanation is clarified by the following picture where the central point is $\widetilde{B^0} = \text{closure}\widetilde{B^-} \cap \text{closure}\widetilde{B^-}$.

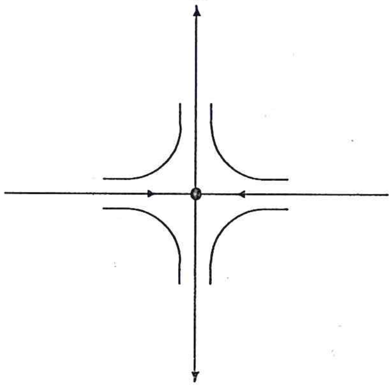


Figure 3.1: The bad loci of an action

Making the flip

Suppose that $\mathbb{C}^*$ acts on $\mathbb{C}^N$ by $(a^r, b^s, 0^t)$. Let $A \subset \mathbb{C}^N$ be an affine variety which is invariant under the $\mathbb{C}^*$ action. In other words, A is cut out by polynomials which are semi-invariants for the $\mathbb{C}^*$ action or, equally, $I_A \subset k[\mathbb{C}^N]$ is a homogeneous ideal with respect to the $\mathbb{Z}$-grading given by $\{a_i, -b_j, 0\}$. In particular, $k[A] = k[\mathbb{C}^N]/I_A$ inherits a $\mathbb{C}^*$ action. If $g_1, \ldots, g_m$ generate I_A and have homogeneous weights $e_1, \ldots, e_m$, I denote this action by

$$(a_1, \ldots, a_r, -b_1, \ldots, -b_s, 0, \ldots, 0; e_1, \ldots, e_m).$$

To construct the flip I start with such a $\mathbb{C}^*$ invariant subspace. The construction is in two steps.

Step 1 Let $B^- = \widetilde{B^-} \cap A \subset \mathbb{C}^N$ and $B^+ = \widetilde{B^+} \cap A \subset \mathbb{C}^N$. Clearly $\mathbb{C}^*$ acts on $A^- = A \setminus B^-$ and $A^+ = A \setminus B^+$.

Step 2 Let $X = A/\!\!/\mathbb{C}^*$, $X^- = A^-/\!\!/\mathbb{C}^*$ and $X^+ = A^+/\!\!/\mathbb{C}^*$.

This is all very reminiscent of constructing projective space as a $\mathbb{C}^*$ quotient. Because A is affine,

$$X = \mathbf{Spec}\left(k[A]^{\mathbb{C}^*}\right).$$

On the other hand, when $r \geq 2$, A^- is not affine but it is the union of r affine open subsets as follows: let $B_i^- := (x_i = 0) \cap A$ and $A_i^- := A \setminus B_i^- \subset A^-$, so,

$$A^- = \bigcup_{i=1}^{r} A_i^- \subset A.$$

In this case, as for projective space itself, X^- is covered by r affine patches,

$$X_i^- := A_i^-/\!\!/\mathbb{C}^* = \mathbf{Spec}\left(k[A][x_i^{-1}]^{\mathbb{C}^*}\right).$$

Exactly the same description works for X^+.

Proposition 22 *(a) There exist morphisms $f^-\colon X^- \longrightarrow X$ and $f^+\colon X^+ \longrightarrow X$ so the three quotients lie in a diagram*

$$X^- \to X \leftarrow X^+.$$

(b) If the morphisms f^- and f^+ are not isomorphisms, that is, their exceptional loci do not contain any divisors, then $r \geq 2$ and $s \geq 2$. In this case, and indeed whenever $rs \neq 0$, the dimensions are related by $\dim X = \dim A - 1$. $\square$

(a) is the statement that functions that are invariant on A are also invariant on open subvarieties of A.

I leave the proof of (b); the following example is the key.

Example The GIT description is more general than flips. Take $A = \mathbb{C}^3$ and $\mathbb{C}^*$ action $(1, 1, -1)$ on eigencoordinates x_1, x_2, y. Then

$$\begin{aligned} X &= \mathbf{Spec}\left(k[x_1, x_2, y]^{\mathbb{C}^*}\right) \\ &= \mathbf{Spec}\left(k[x_1 y, x_2 y]\right) \\ &= \mathbb{C}^2, \\ X_1^- &= \mathbf{Spec}\left(k[x_1, x_2, y, x_1^{-1}]^{\mathbb{C}^*}\right) \\ &= \mathbf{Spec}\left(k[x_1 y, x_2 y, x_2/x_1]\right), \end{aligned}$$

and

$$\begin{aligned} X_2^- &= \mathbf{Spec}\left(k[x_1, x_2, y, x_2^{-1}]^{\mathbb{C}^*}\right) \\ &= \mathbf{Spec}\left(k[x_1 y, x_2 y, x_1/x_2]\right). \end{aligned}$$

You recognise X^- as the ordinary blowup of the origin in $\mathbb{C}^2$ with coordinates $z_1 = x_1 y$, $z_2 = x_2 y$. In this example $X^+ = X$; doing the calculation shows that you could have predicted this since there is only one negative character.

Let $C^- \subset X^-$ and $C^+ \subset X^+$ be the exceptional loci of f^- and f^+ respectively. A $\mathbb{C}^*$ action is called *flipping* if $\dim C^- > 0$ and $\dim C^+ > 0$.

I am after 3-fold flips so I want to choose A to be a 4-fold and the action to be flipping. From now on I restrict myself to the cases $A = \mathbb{C}^4$ and $A \subset \mathbb{C}^5$ is a hypersurface. To fix notation, in the hypersurface case, I will always denote the equation by g and its weight by e.

I also make the simplifying assumption that in each case the $\mathbb{C}^*$ action has no quasireflections. By definition, this means that no element of $\mathbb{C}^*$ contains a divisor of A in its fixed locus. In practice, this means the following: in the action $(n_1, \dots, n_4)$ no $h > 1$ divides 3 of the n_i; in the action $(n_1, \dots, n_5; e)$ no $h > 1$ divides 4 of the n_i and if it divides exactly 3 of them it must divide e. I call quasireflections of this form *global quasireflections*.

Flip condition II

From Proposition 22 I immediately get

Corollary 23 *If $A = \mathbb{C}^4$ then a flipping action must be of the form $(++--)$.*

Let g be a homogeneous polynomial on $\mathbb{C}^5$. I say that $\mathrm{x} \in g$ if g contains a monomial, called x, purely in the positively weighted x coordinates.

Proposition 24 *If $A \subset \mathbb{C}^5$ and for some flipping action $r = 3$, $s = 2$ then $\mathrm{x} \in f$. Other cases are similar. In particular, when $A \subset \mathbb{C}^5$, any flipping action must be of the form*

$$(+++--;+) \text{ or } (++--0;0) \text{ or } (++---;-).$$

Proof. With respect to the action $(a_1, a_2, a_3, -b_1, -b_2; e)$,

$$C^- = \mathbf{Proj}\left(\frac{k[x_1, x_2, x_3]}{g(x_1, x_2, x_3, 0, 0)}\right),$$

$$C^+ = \mathbf{Proj}(k[y_1, y_2]).$$

Since I want both C^- and C^+ to be one dimensional I require at least two positive and two negative coordinates, which I have, and that $g(x_1, x_2, x_3, 0, 0) \neq 0$; in other words there should be at least one monomial $\mathrm{x} \in g$. Since g is equivariant it must now have $e > 0$. The other cases are similar. Q.E.D.

Flip condition I

The proof of the following proposition is elementary; ask your tutor for a hint.

Proposition 25 *Suppose $A \subset \mathbb{C}^5$ as above. When $P_1 \in X^-$ there is a local isomorphism of pairs of affine varieties*

$$\left(P_1 \in X^-\right) \cong \left(0 \in \frac{1}{a_1}(a_2,\dots;e)\right)$$

where the equation on the righthand side is $\pi_1 g = g(1, x_2, \dots)$.

When $A = \mathbb{C}^4$, just forget g to get the analogous result:

$$\left(P_1 \in X^-\right) \cong \left(0 \in \frac{1}{a_1}(a_1, -b_1, -b_2)\right).$$

Using this Proposition, the flip condition (I) is easy to check in particular cases; I simply compare the description in the Proposition with the classification of terminal singularities in Theorem 5 of the introduction. One point of the classification of terminal singularities is that the result is a list of the analytic types of the singularities, so I must allow analytic changes of coordinates when trying to identify the singularities of the flip.

I must worry a bit about local quasireflections. The following lemma cures almost all my problems.

Lemma 26 *(a) When $A = \mathbb{C}^4$, the description of $P_1 \in X^-$ in the last Proposition has no local quasireflections.*

(b) When $A \subset \mathbb{C}^5$ and $P_1 \in X^-$ is a hyperquotient singularity, the description of $P_1 \in X^-$ in the last Proposition has no local quasireflections.

Proof. (a) Suppose $P_1 = \frac{1}{a_1}(a_2, -b_1, -b_2)$ has local quasireflections; in other words, there is an $h > 1$ such that $h \mid a_1$, $h \mid a_2$ and $h \mid b_1$. But stop right there — I explicitly excluded these global quasireflections when setting up the notation.

(b) Suppose $P_1 = \frac{1}{a_1}(a_2, a_3, -b_1, -b_2; e)$ has local quasireflections; in other words,

either there is an $h > 1$ such that $h \mid a_1$, $h \mid a_2$, $h \mid a_3$ and $h \mid b_1$,

or there is an $h > 1$ such that $h \mid a_1$, $h \mid a_2$ and $h \mid a_3$ but $h \nmid e$.

But stop right there — I explicitly excluded these global quasireflections when setting up the notation.

Q.E.D.

Flip condition III

If $A = \mathbb{C}^4$ let $\tau = a_1 + a_2 - b_1 - b_2$; if $A\colon (f = 0) \subset \mathbb{C}^5$ let $\tau = \sum a_i - \sum b_j - e$. I call τ the *index* of the flip.

The key to the canonical class is the following lemma of Dolgachev, see [7], Theorem 3.3.4.

Lemma 27 *Suppose that the $\mathbb{C}^*$ action on A has no quasireflections. Then, as $k[X]$-modules,*

$$\Gamma(X, \mathcal{O}_X(-nK_X)) \cong \Gamma(X, \mathcal{O}_X(n\tau)) \cong k[A]_{n\tau}$$

where M_m denotes the mth graded part of a graded module M. The grading here is by **wt** *and the action of $k[X]$ on $k[A]$ is induced by the natural inclusion. The twisted module $\mathcal{O}_X(\nu)$ is defined in the same way as for ordinary projective space.*

□

Corollary 28

$$-K_- \textit{ is } f^-\textit{-ample} \iff K_+ \textit{ is } f^+\textit{-ample} \iff \tau > 0.$$

Proof. Suppose $\tau > 0$. So for suitable $m > 0$ and n_i,

$$x_i^{n_i} \in k[A]_{m\tau} = \Gamma(-mK) = \Gamma(-mK_-).$$

The last equality is by the definition of the canonical class; I'm free to ignore chosen codimension 2 subsets of X and X^-.

Let $C = \mathbb{P}^1 \subset C^-$ be any f^- exceptional curve. Then

$$-mK_-C = \deg_{\mathbb{P}^1}(-mK_-)_{|C} > 0$$

because the x_i form coordinates on C. So $-K_-$ is f^--ample. Similarly, $y_i^{n_j} \in \Gamma(mK_+)$ so $mK_+C^+ > 0$.

Repeat the argument for $\tau < 0$ to conclude. Notice that in the case $\tau = 0$, K has degree 0 on C^- and C^+ so $KC = 0$. Q.E.D.

Examples

(1) An example in codimension 0.

Let $A = \mathbb{C}^4$ and let $\mathbb{C}^*$ act on eigencoordinates by $(2, 1, -1, -1)$. The three quotients of A will produce a flip diagram by Corollary 23. Moreover, Corollary 28 ensures that it will be directed by the canonical class since $\tau = 2+1-1-1 > 0$. The only thing that I need to check to confirm that this does indeed give a terminal flip is that X^- and X^+ both have only terminal singularities.

I first calculate that, omitting **Spec**,

$$\begin{aligned} X &= k[x_1y_1^2, x_1y_1y_2, x_1y_2^2, x_2y_1, x_2y_1] \\ &= k[u_0, u_1, u_2, v_0, v_1]/(\mathrm{rk}M_0 \leq 1) \end{aligned}$$

where

$$M_0 = \begin{pmatrix} u_0 & u_1 & v_0 \\ u_1 & u_2 & v_1 \end{pmatrix}.$$

You recognise X as the cone on the rational ruled surface, $\mathbb{F}_1$. As described above, I construct X^- by introducing the weighted ratios of the x_i coordinates. In this case, X^- is covered by 2 patches, X_1^- and X_2^-. I calculate them as follows:

$$X_1^- = k[u_0, u_1, u_2, v_0, v_1, t]/(\mathrm{rk} M_1 \leq 1)$$

where $t = x_2^2/x_1$ and

$$M_1 = \begin{pmatrix} u_0 & u_1 & v_0 \\ u_1 & u_2 & v_1 \\ v_0 & v_1 & t \end{pmatrix};$$

These are also the equations of the singularity $\frac{1}{2}(1,1,1)$ so this calculation agrees with Proposition 25;

$$\begin{aligned} X_2^- &= k[u_0, u_1, u_2, v_0, v_1, t^{-1}]/(u_0 = t^{-1}v_0^2, u_1 = t^{-1}v_0v_1, u_2 = t^{-1}v_1^2) \\ &= k[v_0, v_1, t^{-1}] \end{aligned}$$

which is simply $\mathbb{C}^3$.

Similarly, you can calculate that X^+ has no singularities so this really is an example of a flip. In fact, you must recognise this flip by now: compare the calculations with the Francia flip. For flips with *terminal* singularities, this correspondence between codimension 0 and toric geometry holds; I comment briefly later.

(2) An example in codimension 1.

Let $A\colon (g = 0) \subset \mathbb{C}^5$ where $g = x_1y_1 + x_2x_3$ and let $\mathbb{C}^*$ act on eigencoordinates by $(3, 1, 1, -1, -1; 2)$. By Lemma 25, $P_1 = \frac{1}{3}(1, -1, -1)$ and all other points are Gorenstein. Indeed, the only other singularity on the flip is the point Q_2 on X^+ which is the hypersurface singularity $(g = 0) \subset \mathbb{C}^4$. This singularity is compound DuVal and isolated so it is terminal. Again, this is a genuine flip.

In both these examples the blowup $F^+\colon X^+ \longrightarrow X$ can be understood easily in terms of fibrings over $\mathbb{P}^1$.

In example (1), $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$ is the usual map given by the ratio $v_0 : v_1$. This extends to a rational map $p\colon X \dashrightarrow \mathbb{P}^1$ which is not defined at the vertex of the cone. Now X^+ is the closure of the graph of p in $X \times \mathbb{P}^1$ and the exceptional locus of f^+ projects isomorphically to the $\mathbb{P}^1$ factor.

In example (2), let $p\colon W_0 = \mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)) \longrightarrow \mathbb{P}^1$ be the natural projection. This extends to a rational map from W, the cone on W_0, to $\mathbb{P}^1$ which is not defined at the vertex of the cone. Let W^+ be the closure of the graph of p in $W \times \mathbb{P}^1$. Again, $W^+ \longrightarrow W$ is the contraction of a smooth rational curve. Now g is not weighted bihomogeneous so it does not determine a subvariety of W_0, but $X\colon (g = 0) \subset W$ is welldefined. Since every monomial in g contains some x_i, $X^+\colon (g = 0) \subset W^+$ contains the exceptional $\mathbb{P}^1$.

3.2 Codimension zero: toric geometry

These results originate in [22] and are completely listed in [14], §5.2. The calculations I do here are the prototype for the much longer codimension 1 case, so I take my time.

By Corollary 23, the action is $(a_1, a_2, -b_1, -b_2)$. I only need to consider singularities that lie on C^-. In this case, by Lemma 25,

$$P_1 = \frac{1}{a_1}(a_2, -b_1, -b_2)$$

and

$$P_2 = \frac{1}{a_2}(a_1, -b_1, -b_2).$$

Moreover, by Lemma 26, P_1 and P_2 do not suffer from local quasireflections.

Theorem 29 *The following are the only actions on $A = \mathbb{C}^4$ that give rise to flips:*

$$(a, 1, -b, -1)$$
$$(a, 1, -b, b-a)$$

where $a > b > 0$ are coprime integers.

Proof. The action is still $(a_1, a_2, -b_1, -b_2)$ by Corollary 23. I have 2 conditions to satisfy:

- P_1 and P_2 are terminal quotient singularities, or possibly smooth; moreover they cannot have local quasireflections by Lemma 26. In practice, this condition means that P_1 and P_2 are smooth or satisfy the condition (T) of Theorem 5 when written, as above, in the form of Proposition 25;

- by Corollary 28,
$$a_1 + a_2 > b_1 + b_2. \tag{3.1}$$

I can suppose that $a_1 \geq a_2$ and study P_1 first.

If $a_1 > 1$ then, without loss of generality, condition (T) offers me a choice of two situations:

- $b_1 + b_2 \equiv 0 \bmod a_1$;

- $a_2 \equiv b_2 \bmod a_1$.

In the former case, $b_1 + b_2 = ka_1$ for some $k \geq 1$. Substituting this condition into equation (3.1) gives the condition

$$a_1 + a_2 > ka_1$$

whence $k = 1$. In the latter case, $a_2 = b_2 + ha_1$ where $h \leq 0$ since $a_1 \geq a_2$. Substituting this condition into equation (3.1) gives the condition

$$(h+1)a_1 > b_1$$

whence $h = 0$.

The conclusion so far is that either $b_1 + b_2 = a_1$ or $a_2 = b_2$. In the former case I have

$$P_2 = \frac{1}{a_2}(a_1, -b_1, b_1 - a_1).$$

This would be fine, except that as soon as a_2 divides the sum of any two of the characters, as required by condition (T), it divides the third so that, by Lemma 26, the only possibility is $a_2 = 1$. In the latter case, looking at P_2 shows immediately that $a_2 = b_2 = 1$.

The case $a_1 = 1$, that is, when P_1 is a smooth point, fails inequality (3.1) immediately as, by assumption, I must have $a_1 = a_2 = 1$. Q.E.D.

Compare this result with Danilov's classification of toric flips in Theorem 9. In a moment, I make the formal connection between toric flips and codimension 0 flips.

As an aside, it is easy to calculate $K_{\pm}C$ on either side of the flip in this context to confirm the toric calculation of Chapter 2. The contracted curve in X^- is

$$C^- = \mathbf{Proj}\ k[x_1, x_2] = \mathbb{P}(a_1, a_2).$$

If $h \in k[\mathbb{P}(a_1, a_2)]$ is of weight d I have the following Bézout style formula from [9];

$$\#\{(x:y) \mid h(x:y) = 0\} = \left\lfloor \frac{d \cdot (a_1, a_2)}{a_1 a_2} \right\rfloor$$

where the zeros are counted with multiplicity and for $q \in \mathbb{Q}$, $\lfloor q \rfloor$ means 'integral part of q' in the usual sense.

Suppose the action is $(a, 1, -b, b-a)$. Then $-aK$ is Cartier on X^- and by Lemma 27 is defined by a function of weight $a \cdot (a + 1 - b + b - a) = a$, so

$$-aKC^- = \left\lfloor \frac{a}{a} \right\rfloor = 1$$

and

$$KC^- = -\left(\frac{1}{a}\right).$$

Similarly, in this case,

$$KC^+ = \frac{1}{b(a-b)}.$$

When the action is $(a, 1, -b, -1)$ I have the following intersections;

$$KC^- = -\left(\frac{a-b}{a}\right),$$

$$KC^+ = \frac{a-b}{b}.$$

It is possible to say this in a slightly different way. For example, in the Francia flip, $(2,1,-1,-1)$, x_2 is a section of $\mathcal{O}(1/2)$ in the sense that after eliminating quasireflections on $\mathbb{P}^1 = C^-$, $\mathbb{P}^1 = \mathbf{Proj}\ k[x_1, x_2^2]$ and $H^0(\mathcal{O}(1)) = k[x_1, x_2^2]$. But $-K_-$ is defined locally by $x_2 = 0$, so $-K_-C^- = \deg_{C^-} \mathcal{O}(1/2) = 1/2$.

Comparison with toric geometry

Proposition 30 *(a) Any codimension* 0 *flip, that is, one of those in Theorem 29, is isomorphic to a toric flip.*

(b) All toric flips can be represented as codimension 0 *flips.*

Proof. The reason is that there is a recipe, although certainly not a welldefined bijection, for passing between toric geometry and the $\mathbb{C}^*$ description.

(a) **Actions into cones.** Given a $\mathbb{C}^*$ action on $A = \mathbb{C}^4$ of type $(a, 1, -b_1, -b_2)$ we construct the base cone by projecting the first quadrant, τ, of $\mathbb{R}^4$ away from $(a, 1, -b_1, -b_2) \in \mathbb{R}^4$. There are lots of ways of doing this. To get the standard flipping cone, use

$$e_1 = (1,0,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1), e = (a,1,-b_1,-b_2)$$

as a basis of $\mathbb{R}^4$ in that order and forget the fourth coordinate, e. The generators e_1, e_2, e_3, e_4 of the semigroup $\tau \cap \mathbb{Z}^4$ project to

$$(1,0,0), (-a, b_1, b_2), (0,1,0), (0,0,1)$$

which is an easy change of basis in $\mathbb{Z}^3$ away from the standard model of a toric flip. The important point is that the map of lattices $\mathbb{Z}^4 \longrightarrow \mathbb{Z}^3$ is surjective. This follows from the last two paragraphs of [6] which say that three of the vertices of the cone of *any* toric flip generate $\mathbb{Z}^3$.

It is easy to check that the flip given by the decompositions of the cone is the same as that given by the $\mathbb{C}^*$ quotients of $\mathbb{C}^4$. I see that $k[\mathbb{C}^4/\!\!/\mathbb{C}^*] = k[\check{\sigma} \cap M]$ by describing an isomorphism as follows:

$$(\alpha, \beta, \gamma) \in \check{\sigma} \cap M \text{ iff } \alpha, \beta, \gamma \geq 0 \text{ and } -a\alpha + b\beta + c\gamma \geq 0.$$

But in that case, $x_1^\alpha x_2^\delta y_1^\beta y_2^\gamma$ is a $\mathbb{C}^*$ invariant monomial on $\mathbb{C}^4$, where $\delta = -a\alpha + b\beta + c\gamma$. Conversely, the powers of any invariant monomial determine an element of $k[\check{\sigma} \cap M]$. To write the four patches that cover the two sides of the flip means dropping one condition in in each desription. In the toric case, you lose a duality inequality; in the $\mathbb{C}^*$ case you allow one coordinate into the denominator of invariant monomials. But these present the same condition under the isomorphism I just described so all the patches are isomorphic too.

(b) **Cones into action.** Given a cone on four vertices v_1, v_2, v_3, v_4 satisfying a convex relation

$$a_1v_1 + a_2v_2 = a_3v_3 + a_4v_4,$$

$a_1, \ldots, a_4 > 0$, write down the $\mathbb{C}^*$ action $(a_1, a_2, -a_3, -a_4)$ on $\mathbb{C}^4$. If one of the a_i is 1 then the previous argument shows that the action I've written down does give the same flip as the decompositions of the cone determined by the convex relation. But Danilov's Theorem 9 says that this always happens.

Q.E.D.

Cheap extras in codimension 0

Examples aid even idle thoughts and codimension 0 quotients regularly bring mine back to reality. I illustrate briefly.

(1) I spend most of my time working with flips here mainly because that is the GIT problem I set up at the beginning of the chapter. Dealing with other MMP phenomena in codimension 0 is just as easy: see [25], §3, for the GIT setup. Proofs are the same as in the flipping case and I omit them.

Theorem 31 *The only codimension 0 terminal flop is,*

$$(1, 1, -1, -1).$$

The only codimension 0 terminal divisorial contractions to a point are,

$$(a_1, a_2, 1, -1),\ (a_1, a_2, 1, -a_1 - a_2),$$

where $a_1 > a_2$ and $(a_1, a_2) = 1$.

The flop is, of course, Atiyah's flop. The same argument as used for flips shows that codimension 0 divisorial contractions are the same as toric divisorial contractions, so this is the classification of toric divisorial contractions that I stated in section 2.4.

I remark that the toric versus codimension 0 correspondence no longer holds for blowups with canonical but nonterminal singularities.

(2) Returning to flips, the next lemma is just what you would have guessed: flipping from singularities that are nearly terminal will result in singularities that *are* terminal. Recall from [19] that an isolated canonical quotient singularity is either a terminal point, a Gorenstein quotient or one of the 2 exceptional singularities, $\frac{1}{14}(1, 9, 11)$ and $\frac{1}{9}(1, 4, 7)$. Also, from [21], index 1 canonical singularities are the total spaces of one parameter deformations of DuVal singularities; these are either smoothing directions or trivial deformations.

Proposition 32 *(a) If $X^- \to X \leftarrow X^+$ is a codimension 0 flip and X^- has isolated canonical singularities then X^+ has terminal singularities.*

(b) If $X^- \to X \leftarrow X^+$ is a codimension 0 flip and X^- has index 1 canonical singularities then X^+ has terminal singularities.

Examples include $(9,1,-5,-2)$ and $(14,5,-3,-11)$. The latter is unusual in that it has 2 singularities on X^-; with terminal singularities there was only ever 1.

Not surprisingly, neither of these statements holds for flops. The example $(3,1,-2,-2)$ shows this one way or the other. Indeed, compare this with the situation for terminal flops where the singularities on the two sides of the flop are analytically isomorphic; see, for example, [15], (4.11).

(3) In higher dimensions things are very different. It is easy to write down terminal flips with X^- smooth and X^+ singular, the smallest example being the 5-fold flip $(1,1,1,1,-2,-1)$. (In contrast, I believe that whenever $C^- = \mathbb{P}^1$ then X^- must have at least one singularity along C^- by Mori's deformation argument, although I haven't checked this. In codimension 0 it is clear: in $(a_1, a_2, -b_1, \ldots, -b_n)$ I require $a_1 + a_2 > b_1 + \ldots + b_n \geq 2$.)

3.3 Codimension one: toric hypersurfaces

The setup is the usual one:

$$A\colon (g = 0) \subset \mathbb{C}^5$$

has a singularity at $0 \in A$ and $\mathbb{C}^*$ acts on some eigencoordinates, by Proposition 24, with some weights $a_i > 0$, $-b_j < 0$ and, possibly, with one 0 weight. Moreover, this action has no quasireflections and I have the local conclusions of this from Lemma 26(b). I assume throughout that a_1 is the largest, although possibly only equal largest, of the a_i. Any other ordering on the a_i will be local to the paragraph.

Both the statement and the proof of this classification are similar to the toric case in Theorem 29. Literally the only difference *is* the equation.

Theorem 33 *(I) Any terminal flip given by some $A\colon (g = 0) \subset \mathbb{C}^5$ and $\mathbb{C}^*$ action is one of the following:*

	monomials in g	$\mathbb{C}^*$*action*	
(1)	$x_1y_1 + g'(x_2, x_3)$	$(a_1, a_2, 1, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, b_1$
(2)	$x_1y_1 + x_3^n$	$(a_1, a_2, a_3, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, a_3, b_1$
(3)	$x_2^2 + x_1y_2^2 + x_1^ny_1^{2n-1}$	$(4, 1, 1, -2, -1; 2)$	
(4)	$x_2y_1 + z^n$	$(a, 1, -1, -b, 0; 0)$	$a > b$
(5)	$z^2 + x_1y_2^3$	$(3, 1, -2, -1, 0; 0)$	
(6)	$x_2y_1 + y_3^2 + x_1^ny_2^{2n+1}$	$(4, 1, -3, -2, -1; -2)$	

In both cases (1) and (2), a_2 divides $a_1 - b_1$ and all the characters are coprime except that possibly $\mathrm{hcf}(a_1, b_1) > 1$. *If $a_2 > 1$ then $g \ni x_2^r y_2^s$ where, by the equivariance of g, $(r-s)a_2 = e = a_1 - b_1$. In case (4), a and b are coprime. In case (5), g must also contain one of $x_2^2 y_1$, $x_1 x_2 y_1^2$, $x_1^2 y_1^3$.*

(II) The flips given by the classes (1)–(6) are terminal iff the 0-strata P_i that lie on X^- are terminal, iff all hyperquotient singularities at the 0-strata are isolated singularities.

An explanation: I am not writing the full expression for g. There may be other monomials, indeed normally there must be. For example, in (3),

$$P_1 = 0 \in \frac{(\pi_1 g = 0)}{\mu_4(1,1,2,3;2)}.$$

For this to be terminal I need $0 \in (\pi_1 g = 0) \subset \mathbb{C}^4$ to be an isolated singularity. But it is clearly not isolated if I don't add some more monomials to g. There is a choice of things that will do, for example, adding x_3^2, so I don't list them all. In case (1), I write $g'(x_2, x_3)$ to mean one or more monomials purely in x_2 and x_3 as in the proof of Proposition 24.

I add the remark that if m is a monomial, when I say $m \in g$ I am assuming that there is only one occurrence of m in g. Clearly I lose no generality in making this assumption, while gaining the confidence that I will not come across $-m \in g$ later on.

Remark (1) Compare the classification of Theorem 33 with the list of terminal singularities in Theorem 5 or in [23], Theorem (6.1). From the point of view of the General Elephant in the next chapter, one might expect there to be a link since both terminal singularities and codimension 1 flips are closely related to 1 parameter deformations of DuVal surface singularities.

(2) In the proof of Theorem 33 it would be legitimate to use the fact the X^+ has terminal singularities. However, it is not necessary so I don't mention X^+ at all preferring instead to use this condition as a first check on the calculations. You can see in Appendix A that this 'parity check' is successful.

(3) I should also say that the proof is really only a guide to a reasonable way of proceeding with a large calculation. I have omitted many steps in it. If you don't want to do the whole calculation yourself, I guess you have to rely on my integrity.

The following well-known lemma lists allowable assumptions using equivariant analytic changes of coordinates.

Lemma 34 *(a) Suppose that $\xi_1, \ldots, \xi_5$ have $\mathbb{Z}$ weights, denoted* wt, *with respect to which $g \in \mathcal{M}^2$ is homogeneous, where $\mathcal{M}$ is the maximal ideal $(\xi_1, \ldots, \xi_5)$.*

If $g(\xi_1, \ldots, \xi_5)$ contains the monomial $\xi_1\xi_2$ I can assume that

$$g(\xi_1, \ldots, \xi_5) = \xi_1\xi_2 + g'(\xi_3, \xi_4, \xi_5).$$

If $g(\xi_1, \dots, \xi_5)$ contains the monomial ξ_1^2 and $\mathrm{wt}\xi_1 \neq 0$*, I can assume that*

$$g(\xi_1, \dots, \xi_5) = \xi_1^2 + g'(\xi_2, \xi_3, \xi_4, \xi_5).$$

(b) If $0 \in (g(\xi_1, \dots, \xi_4) = 0) \subset \mathbb{C}^4$ is an isolated singularity then, for each $i = 1, \dots, 4$, $g \ni \xi_i^{n_i}\xi_j$, for some n_i and j, possibly $j = i$.

Proof. (a) The first statement follows immediately from the Inverse Function Theorem. The second will follow by Weierstrass Preparation followed by completing the square. The resulting g is the Weierstrass polynomial of the initial one with respect to ξ_1. There are two points to check. First of all, g does not contain a higher pure power of ξ_1 because it is homogeneous of nonzero weight. Secondly, I need to know that the Weierstrass polynomial is itself homogeneous with respect to the same system of weights. But this is clear since Weierstrass Preparation gives the equation $g = uw$ where w is the Weierstrass polynomial and u is a unit. If these are not already homogeneous then the product of highest (or lowest) weight monomials in w and u will live in high (or low) degree and will not be cancelled by other terms in the product. But g is homogeneous so it cannot contain these terms so w is homogeneous as required.

(b) This is the statement that $(g = 0)$ is nonsingular along the coordinate axes away from the origin. Q.E.D.

The proof of Theorem 33

Statement (II) compares with the similar converse statement in the classification of terminal hyperquotient singularities, as in [23] Theorem (6.1). I have to check all the flip conditions. The dimension and polarisation conditions are all immediately clear. All that is left is to check that X^- has only terminal singularities. Given this at the 0-strata I must check the higher dimensional strata, for example $P_1P_2 \cap X^-$. This is easy using the description of the higher strata in [9] and terminal singularities in Theorem 5. You will see that the higher strata don't impose any new conditions on the flip. I won't carry out the calculation here because there is a very similar one in case 1 of the proof of Lemma 35.

Statement (I) is the real content of this chapter. Just as in the toric case the proof is essentially a tree search: list the possibilities for P_1, for each one check the possibilities for P_2, and so on. The proof is rather large; I've broken it up a bit to try to emphasise the major parts of the result. The result comes from Theorems 36–39.

You can look at Theorems 36–39 for details of the conditions in (II) you need for each case.

By my assumption that a_1 is the largest of the a_i, the point P_1 is potentially the singularity of highest index on X^-. But *a priori* I don't know that it lies on X^-. The next lemma is the first step in the tree search — it gives me a singularity to get working on.

Lemma 35 $P_1 \in X^-$.

Proof. Suppose on the contrary that $P_1 \notin X^-$. So $g \ni x_1^{n_1}$ for some $n_1 > 1$; if $n_1 = 1$ the point $0 \in A$ wouldn't be singular. In particular, $e = \mathrm{wt}\, g > 0$ and I must be in the case $(+++--;+)$ of Proposition 24. By Corollary 28 I know that

$$a_2 + a_3 > b_1 + b_2 + (n_1 - 1)a_1.$$

Since a_1 was chosen to be the biggest of the a_i, $2a_1 \geq a_2 + a_3$ so the only possibility is that $n_1 = 2$ and $g \ni x_1^2$. Corollary 28 now reads

$$a_2 + a_3 > b_1 + b_2 + a_1 \tag{3.2}$$

and by Lemma 34 I can assume that

$$g = x_1^2 + g'(x_2, x_3, x_4, x_5).$$

Suppose $a_2 \geq a_3$ so by inequality (3.2) $a_2 > a_1/2$. I now show that all the possibilities for the point P_2 lead to a contradiction.

Case 1 $P_2 \notin X^-$.

In this case $g \ni x_1^2 + x_2^{n_2}$ and, since $a_2 > a_1/2$, $n_2 = 2$ or 3; correspondingly, $a_2 = a_1$ or $2a_1/3$. So the 1-stratum P_1P_2 is not contained in X^- and, as in [9], Lemma I.6.4, I can count the points of $X^- \cap P_1P_2$; indeed, since $a_2 \mid 2h_{12}$

$$\left|X^- \cap P_1P_2\right| = \left\lfloor \frac{2a_1h_{12}}{a_1a_2} \right\rfloor \geq 1,$$

where for $q \in \mathbb{Q}$, $\lfloor q \rfloor$ means 'the integral part of q' in the usual sense. Take any one of these points, P. By the Implicit Function Theorem I can move X^- locally so it intersects P_1P_2 transversely. Each such point looks like

$$\frac{1}{h_{12}}(a_3, -b_1, -b_2).$$

Now check the cases $n_2 = 2, 3$ separately. For example, the case $n_2 = 2$ has $h_{12} = a_1$ and (T) says that either $b_1 + b_2 \equiv 0$ or $a_3 - b_1 \equiv 0 \bmod a_1$; since $a_1 > a_3$ either of these cases contradict the inequality (3.2).

Case 2 $P_2 \in X^-$ is a quotient singularity.

All of these cases fail condition (T) of Theorem 5. For example, if $g \ni x_2^{n_2}x_3$ then

$$P_2 = \frac{1}{a_2}(a_1, -b_1, -b_2).$$

Now (T) requires one of the following:

- $a_1 - b_1 \equiv 0 \bmod a_2$
- $b_1 + b_2 \equiv 0 \bmod a_2$

In the first case $b_1 = a_1 + ka_2$ for some $k \in \mathbb{Z}$. Substituting in inequality (3.2) for b_1 gives

$$a_2 + a_3 > a_1 + ka_2 + b_2 + a_1$$

so $k \leq -1$. On the other hand, substituting for a_1 in the same way gives $k \geq 0$. this is a contradiction so the first case cannot happen. The same calculation shows that all the possibilities also lead to a contradiction.

Case 3 $P_2 \in X^-$ is a hyperquotient singularity.

All these cases fail condition (M) of Theorem 5. In more detail:

$$P_2 = \frac{1}{a_2}(a_1, a_3, -b_1, -b_2; 2a_1).$$

For (M1) to hold I must have $2a_1 \equiv 0 \bmod a_2$. If $a_1 \equiv 0 \bmod a_2$ then the inequality (3.2) demands that $a_1 = a_2$ and $a_3 > b_1 + b_2$. But now I cannot satisfy the other requirement of (M1). On the other hand, if a_1 is not congruent to 0 mod a_2 then $(a_1, a_2) = 1$ by the conditions of (M1). But $2a_1 \equiv 0 \bmod a_2$ so $a_2 = 2$. Now inequality (3.2) reads $4 > b_1 + b_2 + a_1$ which is impossible.

For (M2), $a_2 = 4$, $a_1 \geq 5$ and $e = 2a_1$ so inequality (3.2) reads

$$7 \geq 4 + a_3 > b_1 + b_2 + a_1 \geq 7$$

which is a contradiction.

Finally, (M3) fails immediately as inequality (3.2) reads

$$2 > b_1 + b_2 + a_1$$

which is clearly impossible. This contradiction proves Lemma 35. Q.E.D.

Now I take the three cases $e > 0$, $e = 0$, $e < 0$ separately and in that order; I will split the case $e > 0$ as two theorems since I already know the answer. The first one gives cases (1) and (2) of Theorem 33, the second gives case (3). I emphasise that there is no order on a_2 and a_3 throughout the following two theorems and their proofs.

Theorem 36 (Main case)

Let $g = x_1y_1 + g'(x_2, x_3, y_2)$ and $\mathbb{C}^*$ act on $A\colon (g = 0) \subset \mathbb{C}^5$ by $(a_1, a_2, a_3, -b_1, -a_2; a_1 - b_1)$ where $a_1 > a_2, a_3, b_1$ and $\mathrm{hcf}(a_1, a_2a_3) = 1$. Suppose the resulting quotients give a terminal flip. Then the four integers a_1, a_2, a_3, b_1 are pairwise coprime except that possibly $\mathrm{hcf}(a_1, b_1) > 1$. If $a_2 > 1$ then also $g \ni x_2^r y_2^s$ where $(r - s)a_2 = a_1 - b_1$. In any case, one of the following conditions on g' and the a_i holds.

	monomials in g		a_i conditions
(i)	$x_2^{n_2} + x_3^{n_3}$		
(ii)	$x_2^{n_2} + x_3^{n_3}\xi$	$\xi = x_2$ or y_2	$a_3 = 1$
(iii)	$x_2^{n_2}$		$a_3 = 1$
(iv)	$x_2^{n_2}\eta + x_3^{n_3}$	$\eta = x_3$ or y_2	$a_2 = 1$
(v)	$x_2^{n_2}\eta + x_3^{n_3}\xi$	$\xi = x_2$ or y_2, $\eta = x_3$ or y_2	$a_2 = a_3 = 1,$
(vi)	$x_2^{n_2}x_3$		$a_2 = a_3 = 1$
(vii)	$x_3^{n_3}$		$a_2 \mid a_1 - b_1$
$(viii)$	$x_3^{n_3}\xi$	$\xi = x_2$ or y_2	$a_3 = 1,\ a_2 \mid a_1 - b_1$
(ix)	$x_2^{n_2}x_3^{n_3}$	$n_2,\ n_3 \geq 2$	$a_3 = 1,\ a_2 \mid a_1 - b_1$

Proof. First of all,

$$P_1 = \frac{1}{a_1}(a_2, a_3, -a_2)$$

is a terminal singularity so that is OK. Also notice that the inequality of Corollary 28 is already satisfied so I don't have to think about that at all this time. I write $e = a_1 - b_1$.

The proof breaks up into 3 cases depending on the 3 possibilities for P_2: either $P_2 \notin X^-$, or it is a (possibly smooth) quotient singularity, or it is a (possibly smooth) hyperquotient singularity. In the first case, $g \ni x_2^{n_2}$. In the other 2 cases, if X^- contains the x_2y_2 stratum then it has the (possibly smooth) quotient singularity $\frac{1}{a_2}(a_1, a_3, -b_2)$ along it. If $a_2 > 1$ then this quotient singularity really is singular and P_2 is no longer an isolated singularity; in particular, it is certainly not terminal. The conclusion is that, whenever $a_2 > 1$, I must insist that $g \ni x_2^r y_2^s$ so that X^- does not contain this singular locus.

Case 1 $P_2 \notin X^-$.
So $g \ni x_2^{n_2}$ and $a_2 \mid e$.

Subcase 1.1 $P_3 \notin X^-$.
So $a_3 \mid e$. I claim that $h_{23} = 1$ and $\mathrm{hcf}(b_1, a_2a_3) = 1$. The second claim is trivial since both $a_2 \mid e$ and $a_3 \mid e$. The first holds because $g \ni x_2^{n_2} + x_3^{n_3}$ so $P_2P_3 \cap X^-$ is a finite positive number of points (still given by [9], Lemma I.6.4, if you want to know the exact number) each of which looks like

$$\frac{1}{h_{23}}(a_1, -b_1, -a_2)$$

by the Implicit Function Theorem. If $h_{23} > 1$ these can only be terminal if h_{23} has a common factor with at least one, and hence both, of a_1 and b_1; since already $h_{23} \mid a_2$ and $h_{23} \mid a_3$ this gives global quasireflections which I assumed had all been removed. So $h_{23} = 1$.

Subcase 1.2 $P_3 \in X^-$ is a quotient singularity.
So $\pi_3 g$ must contain a linear term and the only things left are x_2 and y_2. If

$\pi_3 g \ni x_2$ then $g \ni x_2 x_3^{n_3}$ so

$$P_3 = \frac{1}{a_3}(a_1, -b_1, -a_2).$$

and $a_1 - b_1 = e = a_2 + n_3 a_3$ which I rewrite as

$$n_3 a_3 = a_1 - b_1 - a_2.$$

If $a_3 > 1$ then the conditions (T) give me global quasireflections again; for example, if

$$a_1 - a_2 \equiv 0 \bmod a_3$$

then $b_1 \equiv 0 \bmod a_3$ so the singularity can only be terminal if I have local quasireflections. But the last equation shows that local quasireflections will result in global quasireflections, as required.

If $\pi_3 g \ni y_2$ exactly the same arguments hold. So in this case $a_3 = 1$.

Subcase 1.3 $P_3 \in X^-$ is a hyperquotient singularity.

So

$$P_3 = \frac{1}{a_3}(a_1, a_2, -b_1, -a_2; a_1 - b_1).$$

If (M1) holds then $a_1 - b_1 \equiv 0 \bmod a_3$. But now it is impossible to get just one zero in the action.

If (M2) holds then $a_3 = 4$ and exactly one of the characters must be even. It cannot be a_2 because that occurs twice in the action and it cannot be a_1 or b_1 since, in particular, $a_1 - b_1$ is required to be even.

So in this case, again, $a_3 = 1$.

Case 2 $P_2 \in X^-$ is a quotient singularity.

Suppose that $g \ni x_2^{n_2} x_3$ so

$$P_2 = \frac{1}{a_2}(a_1, -b_1, 0).$$

This is only terminal if $a_2 \mid a_1$ or $a_2 \mid b_1$. But in either case the $\mathbb{C}^*$ action on A has quasireflection. So throughout this case $a_2 = 1$.

Subcase 2.1 $P_3 \notin X^-$.

In this case $a_3 \mid e$ so $(a_3, b_1) = 1$ and that's all I can say.

Subcase 2.2 $P_3 \in X^-$ is a quotient singularity.

Identical to subcase 1.2; that argument only used the assumptions of the theorem and was independent of P_2.

Subcase 2.3 $P_3 \in X^-$ is a hyperquotient singularity.

$$P_3 = \frac{1}{a_3}(a_1, a_2, -b_1, -a_2; a_1 - b_1).$$

As before, (M1) and (M2) fail leaving $a_3 = 1$.

The other possibility in case 2 is that $g \ni x_2^{n_2} y_2$; the arguments and conclusions don't change.

Case 3 $P_2 \in X^-$ is a hyperquotient singularity.

$$P_2 = \frac{1}{a_2}(a_1, a_3, -b_1, 0; a_1 - b_1)$$

so clearly (M2) fails immediately. If (M1) holds then $(a_2, a_3 b_1) = 1$ and $a_2 \mid e$.

Subcase 3.1 $P_3 \notin X^-$.
In this case again all I can say is that $(a_3, b_1) = 1$.

Subcase 3.2 $P_3 \in X^-$ is a quotient singularity.
This is the same as subcase 2.2.

Subcase 3.3 $P_3 \in X^-$ is a hyperquotient singularity.
This is the same as subcase 2.3. Q.E.D.

Theorem 37 *The only terminal flips given by some* $A\colon (g = 0) \subset \mathbb{C}^5$ *where* $\mathbb{C}^*$ *acts by* $(a_1, a_2, a_3, -b_1, -b_2; e)$ *with* $e > 0$ *are*
(i)–(ix) of Theorem 36,
(x) $g = x_2^2 + x_1 y_2^2$ *with* $\mathbb{C}^*$ *action* $(4, 1, 1, -2, -1; 2)$.
Conversely, cases (i)–(x) always give terminal flips if I add the additional assumptions to certain cases

- *in (iii) and (vi),* $0 \in (\pi_3 g = 0)$ *is an isolated singularity,*
- *in (vii) and (viii),* $0 \in (\pi_2 g = 0)$ *is an isolated singularity,*
- *in (ix),* $0 \in (\pi_i g = 0)$ *is an isolated singularity for both* $i = 1, 2$,
- *in (x),* $0 \in (\pi_1 g = 0)$ *is an isolated singularity.*

Remember that as in Proposition 24, in this case I need $\mathbf{x} \in g$. Also, by Lemma 35 I know that $P_1 \in X^-$.

Proof. **Case 1** P_1 is a quotient singularity.
Recall that at this stage I do *not* rule out the possibility that $a_2 < a_3$. Since P_1 is a quotient singularity I must have that $g \ni x_1^{n_1} w$ where $w \neq x_1$.
If $w = x_2$ then by Corollary 28, $g \ni x_1 x_2$ and $a_3 > b_1 + b_2$. So I fail immediately by checking (T): $P_1 = \frac{1}{a_1}(a_3, -b_1, -b_2)$ and all (T) violate the inequality.
So, without loss of generality, $g \ni x_1^{n_1} y_1$ and

$$P_1 = \frac{1}{a_1}(a_2, a_3, -b_2).$$

By Corollary 28 I also have

$$a_2 + a_3 > b_2 + (n_1 - 1)a_1. \tag{3.3}$$

Now I check (T); there are two possibilities. One is that $a_2 + a_3 = ka_1$, $k > 0$. In fact, $k = 1$ since a_1 is the largest of the a_i and $\mathrm{hcf}(a_1, a_2a_3) = 1$. Comparing with inequality (3.3) above it is clear that $n_1 = 1$ and $a_1 > b_2$. I can suppose that $a_2 > a_1/2$ (equality here, with coprimeness in the description of P_1, leads to $e = 1$ which prohibits $\mathrm{x} \in g$). So, since $e < a_1$, the only possibility for $\mathrm{x} \in g$ is $x_3^{n_3}$ and P_2 must lie on X^-. Now repeating these arguments for P_2 results (easily, but not immediately) in $b_2 = a_2$ which completes the hypotheses of Theorem 36.

The other possibility is that $a_2 - b_2 = ka_1$. Arguing as before, $k = 0$ and $n_1 = 1$; again, these are the hypotheses of Theorem 36.

It is still possible that P_1 is a smooth point. The case $a_1 = 1$ fails simply because $e = a_1 - b_1 = 1 - b_1 \leq 0$, contradicting the case assumption.

There are two possibilities for local quasireflections. The first one is that $a_1 = a_2 = a_3$ and is easily dispatched since $e < a_1$ so there is no possibility of a term $\mathrm{x} \in g$. The second possibility is that, using inequality (3.3) above, $a_1 = a_2 = b_2$ and $n_1 = 1$. So $g = x_1y_1 + g'(x_2, x_3, y_2)$. Since $e < a_1$, P_2 must be in X^- and must be a hyperquotient singularity. But then is looks like

$$P_2 = \frac{1}{a_1}(0, a_3, -b_1, 0; -b_1)$$

which is illegal.

Case 2 P_1 is a hyperquotient singularity.

Indeed,

$$P_1 = \frac{1}{a_1}(a_2, a_3, -b_1, -b_2; e).$$

Theorem 5 and Corollary 28 say that g contains one of the following monomials;

$$x_2x_3, x_1x_2^2, x_2y_1, x_1y_1y_2, x_1x_2y_1, x_1^2y_1y_2, x_1^{n_1}y_1^2, x_2^2.$$

The first four fail or reduce to the others by checking (M). I'll do the next three cases in a bit more detail.

(2a) $g \ni x_1x_2y_1$. The only problem is (M1). In that case I require $e \equiv 0 \bmod a_1$ so $b_1 = a_2$ (because $a_2 \leq a_1$ and $e > 0$) and $e = a_1$. I also require a zero in the action. The only possibility by Corollary 28 is $a_3 = a_1$. But now the term $\mathrm{x} \in g$ can only be $x_2^{n_2}$. So a_2 divides $e = a_1$ and so by the absence of quasireflections $a_2 = b_1 = 1$.

Now check P_3: since $e = a_1$ and $b_2 < a_1$ this must be a hyperquotient point; but, contrary to the condition in Lemma 34(b), I cannot get a power of x_1 into the jacobian ideal of $\pi_3 g$ by equivariance so $0 \in (\pi_3 g = 0)$ is not an isolated singularity and so P_3 is not terminal.

(2b) $g \ni x_1^2 y_1 y_2$. Again, the only problem is (M1). The same argument works: $e \equiv 0 \bmod a_1$ leads to $e = a_1$; a zero in the action leads to $a_2 = a_1$; checking P_2 reveals it to be a hyperquotient singularity that cannot be terminal because $0 \in (\pi_2 g = 0)$ cannot be isolated.

(2c) $g \ni x_1^{n_1} y_1^2$. In fact $1 \leq n_1 \leq 5$ as follows:

Corollary 28 reads $b_1 + a_1 + a_2 + a_3 > b_2 + n_1 a_1$ and since $e > 0$ I know that $n_1 a_1 > 2b_1$; substituting for $n_1 a_1$ certainly gives me $a_1 + a_2 + a_3 > b_1$; substituting for b_1 back in the inequality of Corollary 28 gives the result. Now check each case individually; they all fail (M).

Finally, suppose $x_2^2 \in g$. So $e = 2a_2$. Obviously $a_1 > 1$ to satisfy the inequality of Corollary 28.

If (M1) holds then $2a_2 \equiv 0 \bmod a_1$. Since $(a_1, a_2) = 1$ this implies that $a_1 = 2$ and $a_2 = 1$. Now the inequality of Corollary 28 can only be satisfied by choosing $a_3 = 2$ and $b_1 = b_2 = 1$. But then $0 \in (\pi_1 g = 0)$ is not isolated; try getting $x_1^{n_1} x_3^{n_3} w$ into g.

Now the only possibility is (M2). Checking all five possibilities for the action with $a_2 = 3$ shows that this would contradict Corollary 28. So $a_2 = 1$. Now try getting $0 \in (\pi_1 g = 0)$ isolated, in particular, getting a power of x_3 into the jacobian ideal as in Lemma 34(b).

As a final note, Theorem 5 gives the other monomial I've listed in (x). Q.E.D.

Theorem 38 *The only terminal flips given by some $A\colon (g = 0) \subset \mathbb{C}^5$ where $\mathbb{C}^*$ acts by $(a_1, a_2, -b_1, -b_2, 0; 0)$ are*

(xi) $g = x_2 y_1 + z^n$ with action $(a, 1, -1, -b, 0; 0)$ where $a > b$ and $(a, b) = 1$,

(xii) $g = z^2 + x_2^2 y_1$ with action $(3, 1, -2, -1, 0; 0)$.

Conversely, if the singularity $0 \in (\pi_1 g = 0)$ is isolated then cases (xi) and (xii) always gives terminal flips.

Proof. Throughout this case I know that $g \ni z^n$ for some $n \geq 2$ by the proof of Proposition 24. Also, by Corollary 28, $a_1 > 1$ since I chose $a_1 \geq a_2$ at the beginning.

Case 1 P_1 is a quotient singularity.

So $g \ni x_1^{n_1} y_1$ and Corollary 28 reads

$$a_2 > b_2 + (n_1 - 1)a_1.$$

So $n_1 = 1$ and since $e = 0$, $b_1 = a_1$. Now check (T):

$$P_1 = \frac{1}{a_1}(a_2, -b_2, 0).$$

Since $a_1 > 1$ and $a_1 \geq a_2$, the only possibility is that $a_1 \mid a_2$; but $a_1 = b_1$ already so I have 4 characters divisible by a_1 which is a contradiction.

Case 2 P_1 is a hyperquotient singularity.

So

$$P_1 = \frac{1}{a_1}(a_2, -b_1, -b_2, 0; 0).$$

The only possibility is (M1). Corollary 28 is simply

$$a_1 + a_2 > b_1 + b_2. \tag{3.4}$$

Now $0 \in (\pi_1 g = 0)$ must be an isolated singularity. In particular, since also $e = 0$, $g \ni x_1^{n_1} x_2^{n_2} y_1$ for some $n_1 \geq 0$, $n_2 > 0$. Inequality (3.4) demands that $n_1 = 0$. Checking (T) for $P_2 = \frac{1}{a_2}(a_1, -b_2, 0)$ shows that $a_2 = 1$.

If $n_2 = 1$ then $a_2 = b_1$ and I get case (xi). So suppose that $n_2 \geq 2$. Then Theorem 5 insists on some quadratic part in $\pi_1 g$. A case by case check of all monomials of the form $x_1^{n_1} w_1 w_2$ shows that the only possibilities are $x_2 y_2$, $x_1 y_1 y_2$ and z^2. The first one gives case (xi) again after changing variables. So I only check the other two:

(2a) if $g \ni x_1 y_1 y_2 + x_2^{n_2} y_1$ then $a_1 = b_1 + b_2$ and the action is $(a_1, 1, -n_2, n_2 - a_1, 0; 0)$. In $\pi_1 g$ the coordinate x_2 can only appear next to some y_i. But $\pi_1 g \ni y_1 y_2$ so I can make a change of coordinates so that no other y_1 appear. This change of coordinates shows that $0 \in (\pi_1 g = 0)$ is not an isolated singularity so I don't get a case here.

(2b) suppose $g \ni z^2 + x_2^{n_2} y_1$. If the rank of the quadratic part of $\pi_1 g$ is at least 2 then I must be in case (2a). So I can assume that this rank is just 1 and the get case (xii) straight from the list in (M1) of Theorem 5. Q.E.D.

Theorem 39 *The only terminal flip given by some $A\colon (g = 0) \subset \mathbb{C}^5$ where $\mathbb{C}^*$ acts by $(a_1, a_2, -b_1, -b_2, -b_3; e)$ with $e < 0$ is*

(xiii) $g = x_2 y_1 + y_3^2 + (x_1 y_2^2)^n y_2$ with action $(4, 1, -3, -2, -1; -2)$.

Conversely, if the singularity $0 \in (\pi_1 g = 0)$ is isolated then case (xiii) always gives a terminal flip.

Proof. Obviously $P_1 \in X^-$. Checking (T) and Corollary 28 immediately rules out P_1 being a quotient singularity. So P_1 is a hyperquotient singularity and I have to check (M).

I gain an extra condition by noting that $P_2 \in X^-$ must be a quotient singularity; isolatedness in the hyperquotient case leads to $g \ni x_1 y_1$ contradicting P_1 being hyperquotient. So $g \ni x_2^{n_2} y_1$ and Corollary 28 with this monomial giving e reads

$$a_1 > b_2 + b_3 + (n_2 - 1) a_2. \tag{3.5}$$

With this extra information I now check the conditions (M) for P_1 in detail; for reference

$$P_1 = \frac{1}{a_1}(a_2, -b_1, -b_2, -b_3; e).$$

For (M1) first note that $\pi_1 g$ must have some nonzero quadratic part and a zero in the action. Comparing a quadratic monomial $q \in \pi_1 g$ with Corollary 28 gives another inequality,

$$a_1 + a_2 > b_1 + b_2 + b_3 + n_1 a_1 + \mathrm{wt}(q), \tag{3.6}$$

where $x_1^{n_1} q \in g$; don't forget that $n_1 a_1 + \mathrm{wt}(q) = e < 0$ and $n_1 \geq 0$. Using inequality (3.5) shows that the zero in the action cannot be b_2 or b_3 and since $g \ni x_2^{n_2} y_1$ it cannot be a_2 either (because $e \equiv 0 \bmod a_1$ so I would then have $b_1 \equiv 0$ aswell). So $b_1 \equiv 0 \bmod a_1$. Now I do a case by case check. By inequality (3.6), one of the following monomials occurs in $\pi_1 g$:

$$y_1 x_2, y_1 y_2, y_2 y_3, y_2^2, y_2 x_2, y_1^2.$$

These fail as follows:

- $y_1 x_2$ and $y_1 y_2$ put two zeros in the action;
- $y_2 y_3$ fails inequality (3.5);
- y_2^2 implies (by coprimeness) that $a_1 = 2$ so inequality (3.5) fails;
- $y_2 x_2$ implies, by inequality (3.5), that $a_2 = b_2$ so e fails to be negative;
- $g \ni x_1^{n_1} y_1^2 + x_2^{n_2} y_1$ and $b_1 = k a_1$ so $(n_1 - k) a_1 > 0$ (since $\mathrm{wt}(x_1^{n_1} y_1) = \mathrm{wt}(x_2^{n_2}) > 0$) but this contradicts inequality (3.6).

For (M2), $a_1 = 4$ and since e is even, a_2 and b_1 are both odd. So, by the inequality (3.5), $b_2 = 2$, $b_3 = 1$ and $n_2 = 1$. If $a_2 = 3$ then $-b_1 \equiv 3 \bmod 4$ to get $e \equiv 2 \bmod 4$. But $-b_3 = -1 \equiv 3 \bmod 4$ already which contradicts the conditions of (M2). So $a_2 = 1$. By (M2), $\pi_1 g \ni y_2^2$ so $g \ni x_1^{n_1} y_2^2$. Since $e < 0$, $n_1 = 0$ and $g \ni y_2^2$. Now $e = -2$ so $b_1 = -3$ and I'm finished. Q.E.D.

Chapter 4

Flips and their elephants

My main aim here is to construct examples of flips in large codimension.

Let $X^- \to X \leftarrow X^+$ be an affine neighbourhood of a flip. Let S be any element of the linear system $|-K_X|$ and define $S^\pm$ to be the birational transforms of S on $X^\pm$ respectively. These three varieties fit into a flip picture of surfaces, $S^- \to S \leftarrow S^+$. I call this the *flip of elephants.* The maps between the elephants I denote by f_S^- and f_S^+.

The General Elephant conjecture asserts that, if I choose S sufficiently generally, S has only DuVal singularities and S^- and S^+ are partial resolutions of it. This has been proved by Kollár and Mori in [16], Theorem 2.2, under the additional assumption that the flip be analytically extremal. In section 4.2 and in the Appendix I give explicit calculations of elephants in codimensions 0 and 1.

To build examples, I want to work the other way round. I can think of the whole flip as being the total space of a deformation of the flip of elephants so the trick here, due to Miles Reid as far as I know, is to write down a DuVal singularity with two partial resolutions and deform this picture to produce a flip diagram. Indeed, just as with flips, I can construct a $\mathbb{C}^*$ cover of the 3 elephants. With that done I can deform the elephants by deforming the $\mathbb{C}^*$ cover. This has to be done in an equivariant way but then, if I am careful about the dimensions, the result is a canonically directed flip diagram. All I have left to check is the singularity conditions.

4.1 Higher codimension flips from elephants

I use the general method outlined above for constructing flips under the assumption that the $\mathbb{C}^*$ cover is an affine toric variety. I will describe a toric variety, B, by its cone of monomials, $\square \subset M$, or by the dual cone, $\diamondsuit \subset N$. I show how to proceed both in M and in N. I want B to be Gorenstein; it is, after all, going to be some $\mathbb{C}^*$ cover of a flip in the end so I expect this to be a good condition.

In toric terms, a $\mathbb{C}^*$ action on B can be described in two ways; either as a lattice map from the dual cone, $\diamondsuit \subset N$, onto a cone in a lattice of rank 1 less or, dually, as a linear map from M to $\mathbb{Z}$.

Now I want to deform B by adding a new variable, x_0. Mori has criteria for this to work in an equivariant way, but as I am only really interested in constructing special examples I am happy to use trial and error. In fact, all the conditions are automatic for this setup, except possibly the conditions on the singularities, but these are easy to check in particular cases.

Explicit example in N

One benefit of calculating in N is that I can nominate the behaviour that I want at the level of elephants in a straightforward way. Having said that, I find the calculations difficult to control; this paragraph is only here because I like the way you can *see* the flip of elephants appear as projections of different faces — it was Miles Reid who first showed me the picture. Please excuse this aside; I will be brief.

Let $\Diamond \subset N$ be an n dimensional cone which has a coplanar set of generators; Corollary 11 implies that $B = X_\Diamond$ is Gorenstein. Let $p\colon N \longrightarrow N'$ be a surjective linear lattice map to an rank $n-1$ lattice and $\sigma \subset N'$ be the saturation in N' of the image of $\Diamond$. Define Σ to be the fan consisting of σ and its faces. Let κ be a primitive vector in $\ker p$. This determines weights on the dual cone, $\Box \subset M$, and I can calculate the monomials of weight 0. In other words, the projection determines a $\mathbb{C}^*$ action on $X_\Diamond$ as in section 3.1. I want to realise the 3 quotients in a toric way, that is, I want to see their fans.

Lemma 40 *Let $p\colon N \longrightarrow N'$ and all other notation be as above. Let $S^- \to S \leftarrow S^+$ be the diagram constructed from the 3 quotients of the corresponding $\mathbb{C}^*$ action on $X_\Diamond$. Define 2 fan decompositions of Σ, Σ_1 and Σ_2, by projecting first the top faces of $\Diamond$ and then the bottom faces of $\Diamond$ as in Figure 4.1. Then*

(a)

$$S = X_\sigma \cong X_\Diamond /\!\!/ \mathbb{C}^*;$$

(b)

$$S^\pm \cong X_{\Sigma_1} \text{ and } S^\mp \cong X_{\Sigma_2}.$$

The choice of $\pm$ in (b) depends on the choice of κ or $-\kappa$.

Proof. If A is a subset of M, denote the invariant monomials in A, those m satisfying $m\kappa = 0$, by A^{inv}.

Let $M' = M^{\mathrm{inv}} \subset M$. There is a duality between M' and N': for $m \in M'$, $v \in N'$ let $mv = mw$ for any $w \in p^{-1}(v)$ which is nonempty because p is surjective; this is a welldefined perfect pairing since $m(w_1 - w_2) = 0$ iff $w_1 - w_2 = \lambda\kappa$.

First note that for $v \in N'$, $v \in \sigma$ iff $p^{-1}(v)_\mathbb{Q} \cap \Diamond_\mathbb{Q}$ is nonempty in $N_\mathbb{Q} = N \otimes \mathbb{Q}$.

$$\begin{aligned} v \in \sigma \quad &\text{iff} \quad mw \geq 0 \quad \text{for some } w \in p^{-1}(v)_\mathbb{Q} \cap \Diamond_\mathbb{Q}, \quad \text{for all } m \in \Box \\ &\text{iff} \quad mw \geq 0 \quad \text{for all } m \in \Box^{\mathrm{inv}}, \quad \text{for all } w \in p^{-1}(v) \end{aligned}$$

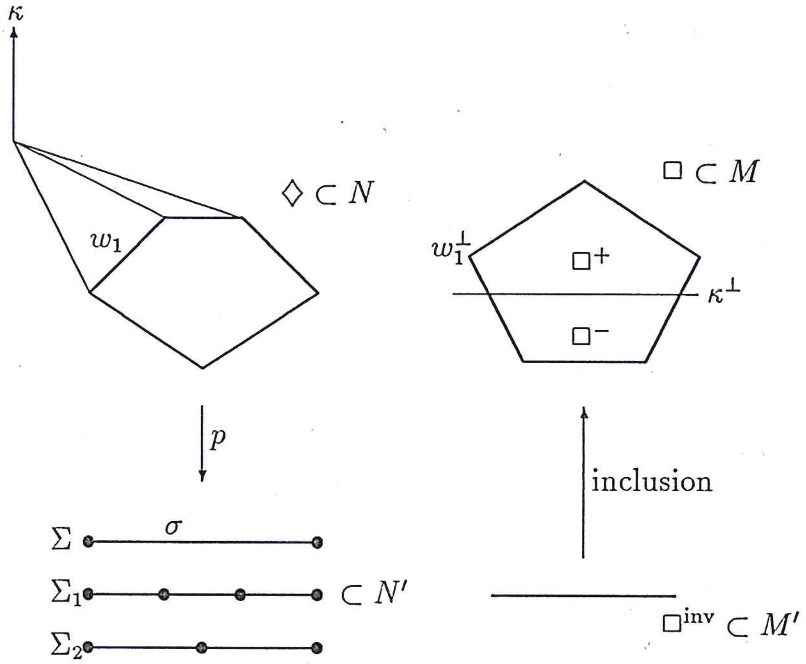


Figure 4.1: Quotients of toric varieties

$$\begin{aligned} &\text{iff} \quad mv \geq 0 \quad \text{for all } m \in \square^{\text{inv}} \\ &\text{iff} \quad v \in \left(\square^{\text{inv}}\right)^\vee \end{aligned}$$

The second equivalence holds because $\square^{\text{inv}}$ splits $\square$ into 2 convex chambers, $\square^+$ and $\square^-$ (possibly empty), disinguished by the sign of $m\lambda\kappa$ for $\lambda \gg 0$; as w varies in $p^{-1}(v)$, $\square^{\text{inv}}$ stays in the halfplane $w \geq 0$ while the hyperplane $w^\perp$ swings round from $\square^+$ to $\square^-$; everything is convex, so at some intermediate rational point, $w \in p^{-1}(v)_{\mathbb{Q}}$, I have $\square \subset (w \geq 0)$.

But now $k[S] = k[\square^{\text{inv}}]$ and so $S = X_\sigma$. This proves (a).

(b) holds by a similar calculation; I carry out a typical case. Let w_1 be the leftmost upper face, as in the figure, and let $w_1^\perp$ be the dual corner of $\square$. Now set $\square_{w_1} = \langle \square, -w_1 \rangle$. As before,

$$\begin{aligned} v \in \sigma_{w_1} \quad &\text{iff} \quad w \in w_1 \text{ for some } w \in p^{-1}(v)_{\mathbb{Q}} \\ &\text{iff} \quad w \in \square_{w_1}^\vee \text{ for some } w \in p^{-1}(v)_{\mathbb{Q}} \\ &\text{iff} \quad mw \geq 0 \text{ for all } w \in \left(\square_{w_1}^{\text{inv}}\right)^\vee \text{ and for all } w \in p^{-1}(v) \\ &\text{iff} \quad v \in \left(\square_{w_1}^{\text{inv}}\right)^\vee. \end{aligned}$$

Q.E.D.

Example Let $\triangle$ be the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Suppose that p is given by the matrix

$$M_p = \begin{pmatrix} a & b & 0 \\ 0 & a-b & a \end{pmatrix}.$$

The new cone, $\triangle' = M_p(\triangle)$, is the interval $I = [(0,a),(a,0)]$, the cone on which is the fan of the DuVal singularity A_{a-1}. Look how the faces of $\triangle$ decompose I. The bottom face, e, of $\triangle$ maps surjectively to I whereas the top two faces, f_1 and f_2, decompose I into

$$[(0,a),(b,a-b)] \cup [(b,a-b),(a,0)].$$

By the lemma above, this is describing the following partial resolutions of A_{a-1} where $\bullet r$ denotes the DuVal singularity A_{r-1}:

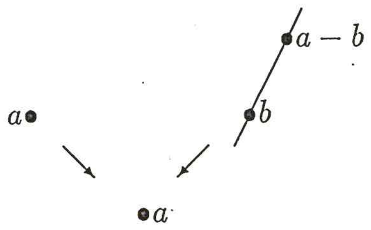


Figure 4.2: The elephants in a triangle flip

To calculate the deformations, I work in M;

$$\triangle^\vee = \langle (1,0,0),(0,1,0),(0,0,1)\rangle .$$

Call these coordinates y_1, x, y_2 respectively. The $\mathbb{C}^*$ action is given by a generator of the kernel of M_p, $(-b, a, b-a)$. To deform B I add a new variable x_0 with $\mathrm{wt} x_0 = \delta > 0$ which gives the $\mathbb{C}^*$ action $(a, \delta, -b, b-a)$ on $A = \mathrm{Spec} k[\triangle^\vee, x_0]$. But now I can check the terminal flip conditions for this flip and conclude that $\delta = 1$. This is one of the families of toric flips again.

Remark Using this approach, I am in a position to use the results of [16] on the classification of extremal neighbourhoods and their elephants: choose a different P-resolution of the flipping elephant and then attempt to hit the whole flip of elephants with a toric projection; now deform to find flips. But since I'm really interested in constructing some examples, I don't need to be so explicit in determining the cones and the projection. By working in M and using Lemma 41 and Corollary 42 below on the structure of cones in lattices, I only need to nominate the geometry of the cone $\square \subset M$ and its generators.

Explicit calculations in M

This is where I find it easier to come up with examples. I need to be able to calculate the generators of a cone. The following lemma does this. I say that $\square \subset M$ is a *Gorenstein* cone if $X_\diamond$ is Gorenstein. I am interested in Gorenstein cones with a minimal set of generators containing 1 internal point. Corollary 11 then implies that this internal point is the generator of the ideal of internal points.

Lemma 41 *Let $\square \subset M$ be a cone. The following two statements are equivalent.*

(a) $\square$ is a Gorenstein cone with interior ideal $u \cdot \square$.

(b) Let $p_1, \ldots, p_n \in \partial\square$ be the boundary elements of a minimal set of generators of $\square$in an order with p_1 adjacent to p_2, and so on. For some element u of the interior of $\square$, and taking subscripts mod n *throughout, for all i,*

$$\langle u, p_i, p_{i+1} \rangle$$

is a basic cone.

Proof. Suppose that (a) holds. Let q be a lattice point which lies in the positive rational span of u, p_i, p_{i+1}. If q lies on $\partial\square \cap \langle u, p_i, p_{i+1} \rangle$ then it is in the semigroup span of p_i and p_{i+1}. So suppose that q lies in the interior of $\square$. But then qu^{-1} lies in the simplicial cone and I continue with that, eventually reaching a point on the boundary, $qu^{-t} = p_i^{t_i} p_{i+1}^{t_{i+1}}$. This gives me a relation $q = u^t p_i^{t_i} p_{i+1}^{t_{i+1}}$ as required.

The converse is clear using Corollary 11(a). Q.E.D.

Corollary 42 *Let $\square \subset M$ be a Gorenstein cone and let u and p_i be as in the Lemma. Then, taking subscripts* mod n *throughout,*

(a) if p_i is a vertex of $\square$ then there is a relation

$$p_{i-1} p_{i+1} = p_i^{n_i} u^{m_i}$$

for some n_i, $m_i \geq 1$ except possibly for 2 adjacent vertices where the relation may be of the form

$$p_{i-1} p_{i+1}^{k_i} = p_{j_i}^{n_i} u^{m_i} \text{ or } p_{i-1}^{k_i} p_{i+1} = p_{j_i}^{n_i} u^{m_i}$$

for some j_i and for some n_i, $m_i \geq 1$.

if p_i is not a vertex then there is a relation

$$p_{i-1} p_{i+1} = p_i^{n_i}$$

for some $n_i \geq 2$;

(b) the relations in (a) generate all the relations in $\square$.

Proof. (a) Suppose first that u does not lie in the interior of the simplicial subcone with vertices p_{i-1}, p_i, p_{i+1} — this holds for all p_i except possibly for 2 adjacent vertices. Without loss of generality, $p_{i-1}p_{i+1}$ lies in the cone on vertices u, p_i, p_{i+1} so

$$p_{i-1}p_{i+1} = p_i^{n_i} p_{i+1}^{l_i} u^{m_i}.$$

But p_{i-1} is a generator so $l_i = 0$ as required. This proof holds for the second statement about edge relations aswell.

In the other case, $p_{i-1}p_{i+1}$ lies, without loss of generality, in the basic cone with vertices u, p_{i-1}, p_i^{-1}. Now use the good neighbouring relation to substitute for p_i^{-1}.

(b) Suppose $j_i = i$ for $i = 1, \ldots, n-2$. Let $\rho = u^r \prod_{i=1}^n p_i^{r_i} = 1$ be a relation in $\square$ where r, $r_i \in \mathbb{Z}$. From (a) I have the relations $\rho_i = p_{i-1}^{-1} p_{i+1}^{-1} p_i^{n_i} u^{m_i}$. Now

$$\rho = \rho p_3^{r_1} / p_3^{r_1} = \rho_2^{r_1} \rho'$$

where $\rho' = \left(u^r \prod_{i=2}^n p_i^{r_i}\right) \left(p_2^{n_2} u^{m_2}\right)^{r_1} p_3^{r_1}$ is a relation with no p_1 term in it. Iterating this results in

$$\rho = \left(\prod_{i=1}^{n-2} \rho_{i+1}^{r_i}\right) \rho^{(n-2)}$$

where $\rho^{(n-2)}$ is a relation among u, p_{n-1}, p_n; but this can only be 1. Q.E.D.

Remark In fact, an unpublished remark of Reid's on Mori's work shows that the internal point of a square Gorenstein cone always lies on the boundary of one of the outer simplicial cones. This means that there can only be at most 1 bad vertex, p_i, occurring in (b). Now the proof of (b) says that both forms of the relation hold at p_i so the relation must be

$$p_{i-1}p_{i+1} = p_{j_i}^{n_i} u^{m_i}.$$

You can see this drop out of the calculations in the codimension 3 case below.

Triangle flips

In fact, to make examples, there is no reason why I should use the elephant of a flip. For example, take the Gorenstein triangle in Figure 4.3.

By the lemma, it has the relation $x_1 y_1 = z^\alpha$. To deform, I add a new coordinate x_0 with weight $\delta > 0$. The flip I get is $A\colon (x_1 y_1 = z^\alpha + x_0^{b_2} y_2^\delta) \subset \mathbb{C}^5$ with $\mathbb{C}^*$ action $(\delta, a_1, -a_1, -b_2, 0; 0)$. On checking whether the singularities on the flip are terminal or not you come up with the condition $a_1 = 1$ so this flip is one of class (3) in Theorem 33.

In search of more ambitious examples I will use more complicated polygons.

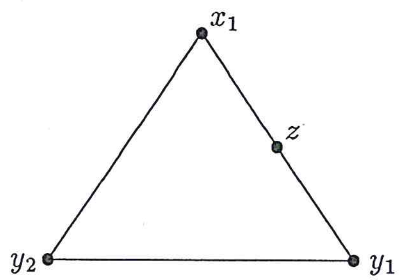


Figure 4.3: A codimension 1 cone in M

A codimension 2 quadrilateral flip after Mori

I have adapted this example of Mori from the appendix of [25]. The answer is given there with more sophistication.

Take the cone in M drawn in Figure 4.4. (The coincidence that x_2, y_2 and z are coplanar is not really a hypothesis; it can be shown that this is necessary for a codimension 2 flip cone with a single internal generator.)

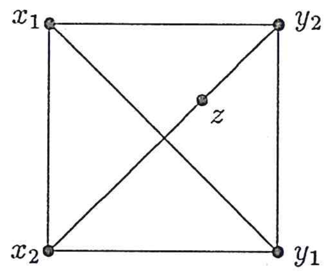


Figure 4.4: A codimension 2 cone in M

By the previous lemma, the relations in this cone are

$$\begin{aligned} x_1y_1 &= x_2^\gamma z^\beta \\ x_2y_2 &= z^\alpha. \end{aligned}$$

I choose a system of weights on $\square \subset M$ so that the x coordinates have positive weight, the ys negative weight and $\mathbf{wt}z = 0$. Let $\mathbf{wt}x_1 = a_1$ and $\mathbf{wt}x_2 = a_2$ with $\mathrm{hcf}(a_1, a_2) = 1$. Then, by the equivariance of the equations, $\mathbf{wt}y_2 = -a_2$ and $\mathbf{wt}y_1 = \gamma a_2 - a_1$; I must have $a_1 > \gamma a_2$ for the weight of y_1 to be negative.

Now I deform B by adding the new coordinate x_0 with weight $\mathbf{wt} x_0 = \delta > 0$. I choose δ so that $\mathrm{hcf}(a_1 a_2, \delta) = 1$. A deformation of the equations is

$$\begin{aligned} x_1 y_1 &= x_2^{\gamma} z^{\beta} + x_0^{\gamma a_2/\delta} \\ x_2 y_2 &= z^{\alpha} + x_0^{\tau} y_1^{\tau(a_1 - \gamma a_2)/\delta}. \end{aligned}$$

Since $\mathrm{hcf}(a_1 a_2, \delta) = 1$, I must have $\delta \mid \gamma$ and $\delta \mid \tau$.

Now it is easy to check as before that the singularities are terminal and so, indeed, these equations define $A \subset \mathbb{C}^6$ with a $\mathbb{C}^*$ action given by $(\delta, a_1, a_2, \gamma a_2 - a_1, -a_2, 0; \gamma a_2, 0)$ whose quotients give a flip.

A codimension 3 quadrilateral flip

Take the cone in M drawn in Figure 4.5. (The coincidence that x_3, y_1 and z are coplanar is not really a hypothesis; it can be shown that this is necessary for a codimension 3 flip cone with a single internal generator.)

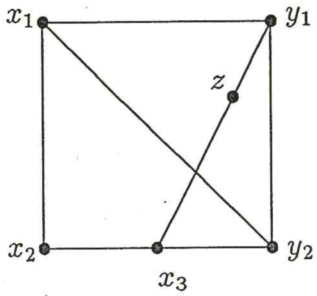


Figure 4.5: A codimension 3 cone in M

By the previous lemma, the relations in this cone are

$$\begin{aligned} x_1 x_3 &= x_2^a z^b \\ x_2 y_2 &= x_3^c \\ x_3 y_1 &= z^d \\ x_1 y_2^k &= x_3^e z^f \\ y_1 x_2 &= x_1^g z^h. \end{aligned}$$

Working out from the middle term each time I get syzygy conditions:

$$x_2^{a-1} x_1^g z^{b+h} = y_1 x_2^a z^b = x_1 x_3 y_1 = x_1 z^d$$

so $a = g = 1$ and $b + h = d$;

$$x_3^e z^{h+f} = y_2^k x_1 z^h = x_2 y_1 y_2^k = y_1 y_2^{k-1} x_3^c = x_3^{c-1} z^d y_2^{k-1}$$

so $k = 1$, $h + f = d$ and $e = c - 1$.

Setting $\alpha = b$, $\beta = c - 1$ and $\gamma = d$ gives

$$\begin{aligned} x_1 x_3 &= x_2 z^\alpha \\ x_2 y_2 &= x_3^{\beta+1} \\ x_3 y_1 &= z^\gamma \\ x_1 y_2 &= x_3^\beta z^\alpha \\ x_2 y_1 &= x_1 z^{\gamma-\alpha} \end{aligned}$$

where α, $\beta \geq 1$, $\gamma \geq 2$ and $\gamma > \alpha$. These can be rewritten as the 5 maximal Pfaffians of the 5×5 antisymmetric matrix

$$m_B = \begin{pmatrix} 0 & x_1 & x_2 & 0 & x_3^\beta \\ & 0 & 0 & z^\alpha & y_1 \\ & & 0 & x_3 & z^{\gamma-\alpha} \\ & & & 0 & y_2 \\ & & & & 0 \end{pmatrix}$$

where the bottom left hand triangle is the antisymmetrical thing.

The great thing about the matrix representation is that (at least, a component of the space of) deformations of B can be written down by deforming the entries of the matrix since obstructions coming from the syzygies are automatically taken care of. So I introduce a new coordinate x_0 and add a polynomial with a factor of x_0 to each entry of the matrix in a symmetric way. But I have to do this in an equivariant way so I need weights on the coordinates in such a way so that the equations are homogeneous.

By choice $\mathbf{wt}z = 0$. The intention of the notation was that, as before, an x has positive weight and a y has negative weight. If I can choose consistent weights satisfying this then the resulting $\mathbb{C}^*$ quotients will satisfy the dimension condition of flips: I get one equation of weight 0 and the rest all have positive weight. Say that $\mathbf{wt}x_1 = a_1 > 0$ and $\mathbf{wt}x_2 = a_2 > 0$. Now the weights of the other variables follow from the homogeneity of the equations. I can choose any positive number δ as the weight of x_0, say $\mathbf{wt}x_0 = \delta = 1$.

Now finding a solution to the deformation problem is easy: I just add a positive power of x_0 to the two 0 entries in the matrix to get the deformed matrix

$$m_B(x_0) = \begin{pmatrix} 0 & x_1 & x_2 & x_0^\nu & x_3^\beta \\ & 0 & x_0^\mu & z^\alpha & y_1 \\ & & 0 & x_3 & z^{\gamma-\alpha} \\ & & & 0 & y_2 \\ & & & & 0 \end{pmatrix}$$

The whole flip is described by the $\mathbb{C}^*$ quotients of

$$A = \mathbf{Spec}\left(\frac{k[x_0, x_1, x_2, x_3, z, y_1, y_2]}{\mathrm{Pf}(m_B(x_0))}\right)$$

where $\mathbb{C}^*$ acts by

$$(1, a_1, a_2, a_2 - a_1, 0, a_1 - a_2, \beta a_2 - (\beta + 1)a_1).$$

The sign requirement is satified by choosing a_1 and a_2 such that $a_2 > a_1$ and $\beta a_2 - (\beta + 1)a_1 < 0$.

The singularities are easy to see: on X^-

- $P_1 = \frac{1}{a_1}(1, a_2, 0, -a_2; 0)$ with equation $x_2 y_1 = z^{\gamma-\alpha} + x_0^{\mu}\left(x_2 z^{\alpha} + x_0^{\mu+\nu}\right)^{\beta}$
- $P_2 = \frac{1}{a_2}(1, a_1, -a_1, 0; 0)$ with equation $x_1 x_3 = z^{\alpha} + x_0^{\mu+\nu}$

and on X^+

- $Q_1 = \frac{1}{a_2 - a_1}(1, a_1, 0, -a_1; 0)$ with equation $x_1 y_2 = z^{\beta\gamma+\alpha} + y_2^{\beta} x_0^{\mu\beta} z^{\alpha} + x_0^{\nu}$
- $Q_2 = \frac{1}{b_2}(1, a_2 - a_1, 0, a_1 - a_2; 0)$ with equation $x_3 y_1 = z^{\gamma} + x_0^{\mu}$

where $b_2 = (\beta+1)a_1 - \beta a_2$. It is easy to check that these are isolated singularities so they really are terminal.

Remark The experience of toric hypersurfaces was that the problem of constructing flips with many negative characters was in forcing the singularities to be isolated. In this context I see it more as a problem of there not being enough deformations of the elephant. As an example let me give the flip generated by the following $\square \subset M$.

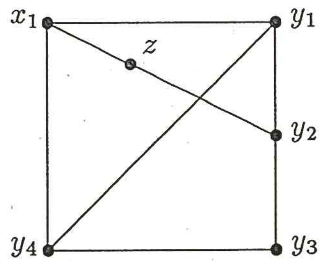


Figure 4.6: A bad codimension 3 cone in M

Calculating the weights shows that this generates 4 equations of negative weight and 1 of weight 0. They lie in Pfaffian form as before with two zero entries in the upper half of the matrix. The crunch is that only one of these is of 'positive weight' so I can only add a power of x_0 to one matrix entry (actually, there is one other entry of positive weight, but adding x_0 to it doesn't change anything). It is easy to see that most of the singularities on the flip are not isolated exactly because of this missing entry.

4.2 The pluricanonical difference via the general elephant

Let $X^- \to X \leftarrow X^+$ be an affine neighbourhood of a flip and $S^- \to S \leftarrow S^+$ an elephant contained in it. I assume that at the flipping point on the elephant $O \in S$ has a DuVal singularity.

Lemma 43 *If $f_S^- : S^- \longrightarrow S$ is an isomorphism then $\delta\chi(mK)$ is a nondecreasing function.*

Proof. Suppose that $m \geq 2$, so that $\delta\chi(mK) = R^1 f_*^-(mK_-)$ by Lemma 18. Applying f_*^- to the sequence

$$0 \to \mathcal{O}((m+1)K_-) \to \mathcal{O}(mK_-) \to \mathcal{O}_{S^-}(mK_-) \to 0$$

gives

$$R^1 f_*^-((m+1)K_-) \to R^1 f_*^-(mK_-) \to R^1 f_*^-(\mathcal{O}_{S^-}(mK_-)).$$

The sheaf $\mathcal{O}_{S^-}(mK_-)$ is supported on S^- so the last term of this sequence is equal to $R^1 \left(f_S^-\right)_*$ of the sheaf. But now, the hypothesis is that the fibres of f_S^- are 0 dimensional so this last term vanishes by dimension as required. Q.E.D.

Corollary 44 *Let $O \in S$ be a DuVal singularity as above. If $f_S^- : S^- \longrightarrow S$ is a crepant morphism and S^- contains a singularity isomorphic to $0 \in S$ then f^- is an isomorphism. In particular, if f^- is a map of elephants in a flip, then $\delta\chi(mK)$ is a nondecreasing function for that flip.*

Proof. Since S is normal, it is sufficient to check that f^- is an isomorphism in codimension 1. Let $\Gamma \subset S^-$ be an f_S^- exceptional curve. Since S^- is Gorenstein, adjunction still holds so

$$2g(\Gamma) - 2 = K_{S^-}\Gamma + \Gamma^2.$$

But $K_{S^-} = 0$ so $\Gamma^2 = -2$ and $g(\Gamma) = 0$. Going to the minimal resolution of S^- you can see that this is a contradiction. So S^- contains no exceptional curves and the result follows from the lemma. Q.E.D.

Remark Theorem (2.2) in [16] lists all possibilities for $S^- \longrightarrow S$ in analytically extremal flips. You see there that 'most' flips pull out some curve in S^- so I don't expect too much of this as it stands.

The elephants in toric flips

In the case of toric flips, I can easily give explicit partial resolutions of the general elephant.

For the toric flips $(a, 1, -b, b-a)$, Lemma 27 shows that $-K = k[\mathbb{C}^4]_1$ so a (possibly nongeneral) elephant is (omitting **Spec**)

$$\begin{aligned} S &= (x_2 = 0) /\!\!/ \mathbb{C}^* \\ &= k\left[x_1, y_1, y_2\right]^{\mathbb{C}^*(a,-b,b-a)} \\ &= k\left[x_1^b y_1^a, x_1^{a-b} y_2^a, x_1 y_1 y_2\right] \\ &= k\left[u, v, w\right] / (uv - w^a) \end{aligned}$$

which is an A_{a-1} singularity. This is partially resolved in X^+ by,

$$\begin{aligned} S_1^+ &= k\left[x_1, y_1, y_2, 1/y_1\right]^{\mathbb{C}^*} \\ &= k\left[u, v, w, t\right] / (uv - w^a, v - tw^{a-b}, u^{a-b}t^a - v^b);\ t = y_2^b / y_1^{a-b} \\ &= k\left[u, w, t\right] / (ut - w^b) \end{aligned}$$

which is an A_{b-1} singularity, and,

$$\begin{aligned} S_2^+ &= k\left[x_1, y_1, y_2, 1/y_2\right]^{\mathbb{C}^*} \\ &= k\left[v, w, s\right] / (sv - w^{a-b});\ s = y_1^{a-b} / y_2^b \end{aligned}$$

which is an A_{a-b-1} singularity.

The negative elephant, S^-, is isomorphic to S since after removing x_2 there is only one positive variable left.

Thinking of A_n as its dual graph, a chain of n points, you can see in Figure 4.7 how f^+ has broken the chain in two by extracting one of the central curves in the resolution; $\bullet r$ denotes the DuVal singularity A_{r-1}.

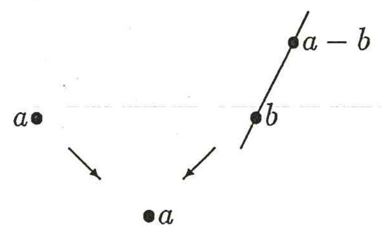


Figure 4.7: The elephants in a toric flip

The result for flips of type $(a, 1, -b, -1)$ is exactly the same. This time

$$\begin{aligned} S &= (x_1 y_1 - x_2^{a-b} = 0) /\!\!/ \mathbb{C}^* \\ &= \left(\frac{k\left[x_1, x_2 y_1, y_2\right]}{x_1 y_1 - x_2^{a-b}} \right)^{\mathbb{C}^*(a,1,-b,-1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{k\left[x_1^b y_1^a, x_1 y_2^a, x_2^b y_1, x_2 y_2\right]}{x_1 y_1 - x_2^{a-b}} \\
&= k[u, v, w]/(uv - w^a)
\end{aligned}
$$

since the monomial $x_1^b y_1^a$ is killed by the denominator.

It is already obvious from the description of $\delta\chi(mK)$ in Chapter 2, but nevertheless, by the corollary above, I see that

Corollary 45 *For toric flips, $\delta\chi(mK)$ is a nondecreasing function.*

Elephants in hypersurface flips

I have written these down in Appendix A. In most cases, whichever elephant I use, curves are pulled out on both sides of the flip so I cannot conclude directly from Lemma 43 that $\delta\chi(mK)$ is nondecreasing. Moving the elephant alone is not enough. Since I only want to calculate a *number* related to the variety A I can deform the variety as well: in other words, choose a general *flip* as well as a general elephant. This is just like finding an elephant for terminal hyperquotient singularities as in [23], 6.2.

The only possible problems are the flips of type $(+++--)$. However, in each case, there is a variable whose character divides the index of the flip. In families (1) and (2)

$$g = x_1 y_1 + g'(x_2, x_3)$$

and $na_2 = a_1 - b_1 = e$. I can deform g by $g \rightsquigarrow g_\lambda = g + \lambda x_2^n$. But now the elephant is

$$
\begin{aligned}
S &= \left(\frac{k[x_1, x_2, y_1, y_2]}{x_1 y_1 - x_2^n}\right)^{\mathbb{C}^*(a_1, a_2, -b_1, -a_2)} \\
&= \frac{k\left[x_1^{a_2} y_2^{a_1}, x_2^{b_1} y_1^{a_2}, x_2 y_2\right]}{x_1 y_1 - x_2^n} \\
&= k[u, v, w]/uv - w_1^a
\end{aligned}
$$

which is isomorphic to $P_1 \in S^-$. So $\delta\chi(mK)$ is nondecreasing by Corollary 44.

I have written down the other elephants in Appendix A.

Proposition 46 *For the hypersurface flips of Theorem 33, $\delta\chi(mK)$ is a nondecreasing function.*

Chapter 5

Baskets of flips

I want to be able to decompose flips into a union of simple flips with respect to some given property. The best result would be if I could make a small deformation of any flip into a bunch of toric flips in a predictable way. In general, I cannot do this but there are weaker results I can aim for. (However, see [16], Theorem (1.2) which shows how to flip in families and may very well include — although I haven't understood it sufficiently well — this deformation statement under slightly different hypotheses.)

Recall that for a projective flip $X^- \to X \leftarrow X^+$,

$$\delta\chi(mK) = \chi(X^+, mK_+) - \chi(X^-, mK_-)$$

and that by Lemma 18 this expression is local to a neighbourhood of the flip and makes sense on any flip neighbourhood.

The case here is this: suppose I have a flip given by some A with a $\mathbb{C}^*$ action. Then I look for an expression of the type

$$\delta\chi(mK, A) = \sum_i \delta\chi(mK, A_i)$$

where the $A_i = \mathbb{C}^4$ are toric flips $(a, 1, -b, -1)$ or $(a, 1, -b, b-a)$ as in Theorem 9; the collection $\{A_i\}_i$ is called the *basket* by analogy with the Plurigenus Formula which I recall in a moment. The Plurigenus Formula involves baskets of *singularities* on projective varieties. I always denote them by a $\mathcal{B}$ in contrast to baskets of *flips* which will be denoted by an $\mathcal{A}$.

You could think of this as being a study of the $\mathbb{Z}$-module $\mathcal{V}$ generated by all the $\delta\chi(mK)$. If I let $\widetilde{\mathcal{V}}$ be the $\mathbb{Z}$-module of all functions from $\mathbb{N}_+$ to $\mathbb{Z}$ and $\mathcal{A}$ be the set of all flips then I have a function

$$\delta\chi\colon \mathcal{A} \longrightarrow \widetilde{\mathcal{V}}$$

defined in the obvious way. Now $\mathcal{V}$ is the span of the set $\delta\chi(\mathcal{A})$. The first question is to determine the span of $\delta\chi(\mathcal{A}_{\mathrm{toric}})$, where $\mathcal{A}_{\mathrm{toric}}$ is the set of toric flips. The immediate result, Theorem 53, of overspanning by toric flips is easy to come by.

For functions $\zeta\colon \mathbb{N}_+ \to \mathbb{Z}$, I use the shorthand $\zeta = [\zeta(1), \zeta(2), \zeta(3), \ldots]$. A standard combinatorical technique is *differencing*, that is, given ζ define a new function $D\zeta$ by

$$D\zeta(n) = \zeta(n+1) - \zeta(n).$$

Before I start weaving baskets, I mention the Plurigenus Formula and its relevance to earlier results.

The Plurigenus Formula

As an example of why I want to use a RR formula, suppose that X^+ is a smooth surface and that $X^- \longrightarrow X^+$ is the blowup of a point. Then RR gives

$$\begin{aligned}\delta\chi(mK) &= \frac{m(m-1)}{2}(K_+^2 - K_-^2)\\ &= \frac{m(m-1)}{2},\end{aligned}$$

which is a formula of the type I'm always chasing.

For 3-folds I can do exactly the same thing but I will need to use a RR formula in the presence of canonical singularities of index greater than 1. The Plurigenus Formula is convenient; see [23], §10. I define

$$k(m) = \frac{(2m-1)m(m-1)}{12}.$$

Theorem 47 (Barlow–Fletcher–Reid)
If X is a projective 3-fold with canonical singularities then

$$\chi(X, mK_X) = k(m)K_X^3 + (1-2m)\chi(\mathcal{O}_X) + \ell(m)$$

where $\ell(m)$ is a correction term dependent on a 'basket' $\mathcal{B}$ of quotient singularities.

□

The *basket* referred to in the Theorem is a collection of terminal quotient singularities which contribute in a calculable way exactly the same amount to the formula as the actual singularities of X do. The fact that the singularities contribute to the formula rather than destroy it becomes clear when you attempt to calculate χ in a resolution of singularities; this point is made most effectively in the proof of Theorem (9.1)(I) in [23].

The calculation of the basket and of the contributions is as follows. When the singularities of X are all terminal quotient singularities, the basket $\mathcal{B}$ is actually collection of the singularities of X, and the contribution is explicit: $\ell(m) = \sum_{Q\in X} \ell(Q,m)$ is the sum of contributions $\ell(Q,m)$, one for each singularity of type $Q = \frac{1}{r}(a, -a, 1)$,

$$\ell(Q,m) = \frac{r^2-1}{12r}(m - \overline{m}) + \sum_{j=1}^{\overline{m}-1} \frac{\overline{bj}(r - \overline{bj})}{2r},$$

where $\overline{m}$ denotes the smallest residue of m mod r and $ab \equiv 1$ mod r. If X has worse singularities, the basket consists of the quotient singularities that result from making a crepant resolution of the nonisolated singular locus and then deforming the isolated singularities; see the proof of the Plurigenus Formula in [23]. The contribution is then calculated as though the singularities of X were those of the basket. I have written down the first few quotient singularities and the corresponding contributions in Appendix A.

If $X^- \to X \leftarrow X^+$ is a flip I define $\delta\chi(m) = \delta\chi(mK)$ as in section 2.3.

Corollary 48 *For any flip*

$$\delta\chi(m) = k(m)\delta K^3 + \ell_+(m) - \ell_-(m),$$

where $\ell_-(m)$ and $\ell_+(m)$ are the correction terms in the Plurigenus Formula for X^- with mK_- and X^+ with mK_+ respectively.

All the terms on the righthand side are local to the flip so this formula makes sense on any flipping neighbourhood. I will write $\delta\ell(m) = \ell_+(m) - \ell_-(m)$ from now on.

If I know the singularities that occur on the flip, the only other piece of information I need is δK^3. But this is the leading coefficient in the essentially cubic polynomial function $\delta\chi(m)$ so it is not surprising that given three values of this function I can retrieve the pluricanonical changes without explicit reference to δK^3. Two values are $\delta\chi(0) = \delta\chi(1) = 0$ so just one more will do.

Corollary 49 *Since $k(2)\delta K^3 = \delta\chi(2) - \delta\ell(2)$,*

$$\delta\chi(m) = 2k(m)(\delta\chi(2) - \delta\ell(2)) + \delta\ell(m).$$

Example Take the toric flip $(n, 1, -(n-1), -1)$ for some fixed $n \geq 2$. The singularities are one point of type $\frac{1}{n}(1, 1, -1)$ on X^- and one of type $\frac{1}{n-1}(1, 1, -1)$ on X^+. Choosing $m = n$ gives

$$\ell_-(n) = \frac{n^2-1}{12n} \cdot n = \frac{n^2-1}{12}$$

$$\ell_+(n) = \frac{(n-1)^2-1}{12(n-1)} \cdot (n-1) = \frac{(n-1)^2-1}{12}$$

so $\delta\chi(n) = 0$ since $\delta K^3 = 1/n(n-1)$. Using the result that $\delta\chi(m)$ is a nondecreasing function of m (for toric flips) from section 4.2, this example shows again that $\chi(X, mK_X)$ may be constant across a flip for arbitrarily many consecutive values of m.

5.1 General decompositions into a basket

I view $A \in \mathcal{A}$ both as the variety $A \subset \mathbb{C}^N$ with a $\mathbb{C}^*$ action, and as the flip $(X^-(A) \to X(A) \leftarrow X^+(A)) = (X^- \to X \leftarrow X^+)$; I refer to either of these as 'the flip A'.

Example Take the hypersurface flip $(a, 1, -b, -1, 0)$ with equation $f = x_2y_2 + z^k + x_1^b y_1^a$ from Theorem 33. The flipping curves are

$$C^- = \mathbf{Proj}\frac{\mathbb{C}[x_1, x_2, z]}{z^k}$$

and

$$C^+ = \mathbf{Proj}\frac{\mathbb{C}[y_1, y_2, z]}{z^k},$$

where C^- is the exceptional curve of the flipping contraction $f^-: X^- \longrightarrow X$ and C^+ is the flipped curve.

Let $f_\lambda = f + \lambda z$. As λ varies, $f_\lambda = 0$ defines a 1-parameter family of flips. Of course, the point is that z and f lie in the same eigenspace of the $\mathbb{C}^*$ action so λ has eigenvalue 0 and the projection $\lambda : (f_\lambda = 0) \subset \mathbb{C}^5 \to \mathbb{C}$ descends to the different quotients. (See [12], §2, for exactly the same argument in a different context; indeed in the philosophy of [25], the context is the same too!) At $\lambda = 0$ there is the flip I started with. Away from $\lambda = 0$ the fixed point of the action splits into k fixed points at each of which $(f_\lambda = 0)$ is smooth (because I can eliminate z), and the quotients are the toric flips $(a, 1, -b, -1)$. The euler characteristics are constant in this flat family so for the original flip I have

$$\delta\chi(mK) = \sum_{i=1}^{k} \delta\chi(mK, A_i),$$

where each A_i is the toric flip $(a, 1, -b, -1)$.

Looking at the list of hypersurface flips in Theorem 33 you see that simple examples like this one, where there is an eigencoordinate of the same weight as the equation, are rare so this stunt won't work very often.

Setting up the problem

I'm aiming at a weaker statement than the deformation example. I think of it as follows. On the set of all flips $\mathcal{A}$, define an equivalence relation $\sim$ by $A_1 \sim A_2$ iff the two flips are related by a deformation; that is, iff there exists a base space, $T \ni t_1, t_2$, and three varieties, $\mathcal{X}^-$, $\mathcal{X}$ and $\mathcal{X}^+$, with flat maps $F^{(\pm)}: \mathcal{X}^{(\pm)} \longrightarrow T$ and T-homomorphisms $\mathcal{X}^\pm \longrightarrow \mathcal{X}$ such that

$$\left(X^-(A_i) \to X(A_i) \leftarrow X^+(A_i)\right) = \left(\mathcal{X}^-_{t_i} \to \mathcal{X}_{t_i} \leftarrow \mathcal{X}^+_{t_i}\right).$$

Now let $\Lambda = \mathrm{FrAb}(\mathcal{A}/\sim)$ and define the group homomorphism

$$\delta\chi: \Lambda \longrightarrow \widetilde{\mathcal{V}}.$$

in the obvious way. This is well-defined by flatness as in the example. Let $\mathcal{V} = \delta\chi(\Lambda)$. As a corollary of Chapter 2, $\mathcal{A}_{\text{toric}} \hookrightarrow \Lambda$; let Λ_{toric} be its span. Any deformation of a flip into a sum of other flips determines an element of $\ker \delta\chi$. Rather than trying to find deformations, the easier problem I try is to find elements of this kernel. This still gives me results of the kind I want; the next proposition follows immediately from the list of toric flips and is an indication of the use of toric baskets.

Proposition 50 *If $A \in \mathcal{A}$ and $A_i \in \mathcal{A}_{\text{toric}}$ satisfy $A - \sum A_i \in \ker \delta\chi$ then*
(a) $\delta\chi(A)$ is a nondecreasing function;
(b) the highest index singularity, of index r_- say, in the flip lies on X^-;
(c) $\delta\chi(A, r_- + 1) > 0$.

Remark The elements of $\delta\chi(\mathcal{A}_{\text{toric}})$ are not linearly independent in $\mathcal{V}$. For example, the two baskets of toric flips

$$\mathcal{A}_1 = \{(5, 1, -3, -2), (3, 1, -1, -1)\}$$

and

$$\mathcal{A}_2 = \{(5, 1, -2, -1)\}$$

have the same basket of singularities, $\left\{\frac{1}{5}(1, 2, 3), -\frac{1}{2}(1, 1, 1)\right\}$, and the same $\delta K^3 = 27/10$ so give the same function $\delta\chi(mK)$.

This is a negative statement — the choice of toric basket is not unique — but its positive side is that the kernel of $\delta\chi$ is going to be big which is what I really need for the decomposition to work at all.

A virtual decomposition

Start with any flip $A \in \mathcal{A}$. I will write $\delta K^3(A)$ for δK^3 in A and $\delta\ell(A)$ for the function $\ell_+ - \ell_-$ in A.

Lemma 51 *If*

$$\delta\chi(mK, A) = \sum_{A_i \in \mathcal{A}^+} \delta\chi(mK, A_i) - \sum_{B_j \in \mathcal{A}^-} \delta\chi(mK, B_j),$$

where $\mathcal{A}^+$ and $\mathcal{A}^-$ are any baskets of flips, then

$$\delta K^3(A) = \sum_i \delta K^3(A_i) - \sum_j \delta K^3(B_j)$$

and

$$\delta\ell(A) = \sum_i \delta\ell(A_i) - \sum_j \delta\ell(B_j).$$

Proof. Collecting the polynomial terms together, which includes part of the correction term of the form $m(r^2-1)/12r$, you can see that the general form of the plurigenus formula is

$$\chi(X, mK_X) = (\text{cubic polynomial in } m) + (\text{periodic term in } m).$$

For

$$\delta\chi(mK, A) = \sum_i \delta\chi(mK, A_i) - \sum_{B_j \in \mathcal{A}^-} \delta\chi(mK, B_j)$$

to hold, it is certainly necessary that the coefficients of the cubic terms are the same, that is,

$$\delta K^3(A) = \sum_i \delta K^3(A_i) - \sum_j \delta K^3(B_j),$$

since the periodic terms are all zero for the same m infinitely often. Putting this back in the Plurigenus Formula shows that

$$\delta\ell(A) = \sum_i \delta\ell(A_i) - \sum_j \delta\ell(B_j)$$

as functions of m. Q.E.D.

But getting the correction term $\delta\ell(A)$ right is easy. It is dependent only on the type of the singularities so I just have to rig up the same singularities in the basket of toric flips as appear in the basket of the flip of A. If this can be done I say that I can *hit the singularities* of A with the toric basket. If I can hit the singularities so that the δK^3 equality holds then I'm done.

Lemma 52 *If A is a hypersurface in $\mathbb{C}^5$ then I can hit its singularities with a toric basket.*

In this case it is easy to construct toric baskets by hand for each flip in the list of Theorem 33. I will do this for the flip $(3,1,1,-1,-1)$ on $f=0$ where $f = x_1y_1 + h(x_2, x_3, y_2)$. The singularities are all of order 1, so don't contribute to the basket, with the exception of $P_1 \in X^-$ which is of type $\frac{1}{3}(1,1,2)$. Now there are two ways of getting the singularities in the basket right.

(1) The basket just contains $(3,1,-1,-1)$. This basket has $\delta K^3 = 2\frac{2}{3}$ and $\delta\chi(2K) = 1$.

(2) The basket is $\{(3,1,-2,-1),(2,1,-1,-1)\}$; the two singularities of type $\frac{1}{2}(1,1,1)$ 'cancel'. This has $\delta K^3(A_1) + \delta K^3(A_2) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$ and $\delta\chi(2K) = 0$.

I refer to Appendix A for the complete list of baskets for hypersurface flips.

Remark Let $A \in \mathcal{A}$. If I can hit the singularities of A with a toric basket then

$$\sum_{P \in \mathcal{B}^+(A)} \left(\frac{r(P)^2-1}{r(P)}\right) - \sum_{Q \in \mathcal{B}^-(A)} \left(\frac{r(Q)^2-1}{r(Q)}\right) < 0$$

where $\mathcal{B}^+(A)$ is the basket of singularities on $X^+(A)$, $\mathcal{B}^-(A)$ is the basket of singularities on $X^-(A)$ and $r(P)$ is the index of the terminal singularity P. This follows from the list of toric flips and the fact that the inequality depends only on the baskets of singularities that occur in the flip.

All the flips I know satisfy this inequality so finding baskets seems hopeful. As an example, the codimension 3 flip of section 4.1 has the following baskets of singularities: in the notation of section 4.1, $\mathcal{B}^-$ consists of $\gamma - \alpha$ singularities of index a_1 and α of index a_2 while $\mathcal{B}^+$ is $\beta\gamma + \alpha$ singularities of index $a_2 - a_1$ and γ of index $(\beta+1)a_1 - \beta a_2$.

Incidentally, the bad flip given at the end of section 4.1 doesn't satisfy this condition. I don't know how to take this. I haven't made up my mind whether this flip is very bad — it's far too singular on X^+ to be a terminal flip — or only just bad. After all, its singularities certainly are canonical. Of course, the inequality doesn't hold for canonical flips; take $(3,3,-2,-1)$ for a simpler example.

Theorem 53 *For any flip $A \in \mathcal{A}$, there exist two baskets of toric flips $\mathcal{A}^+$ and $\mathcal{A}^-$, so that*

$$\delta\chi(mK, A) = \sum_{A_i \in \mathcal{A}^+} \delta\chi(mK, A_i) - \sum_{B_j \in \mathcal{A}^-} \delta\chi(mK, B_j).$$

In other words, the elements of $\delta\chi(\mathcal{A}_{\text{toric}})$ span $\mathcal{V}$.

Proof. In 2 steps.

Step 1 Hitting the singularities.

The flip $(a,1,-b,-1)$ has baskets of singularities $\left\{\frac{1}{a}(1,-b,-1)\right\}$ on X^- and $\left\{\frac{1}{b}(a,1,-1)\right\}$ on X^+. I denote this by one basket as

$$\left\{-\frac{1}{a}(1,-b,-1), \frac{1}{b}(a,1,-1)\right\},$$

where singularities occurring with a minus sign are those of X^-; the basket of singularities of a flip is a formal *signed* collection of flips. Given this, I can account for singularities of highest index on $X^\pm$ at the expense of introducing singularities of lower index on $X^\mp$. Now a descending induction on the index of the singularities in the plurigenus basket of A will give me two baskets of flips of the form $(a,1,-b,-1)$, say $\mathcal{A}_1^+$ and $\mathcal{A}_1^-$, which between them get the singularities right.

Step 2 Getting δK^3 right.

For some $\alpha \in \mathbb{Q}$,

$$\delta\chi(mK, A) - \sum_{A_i \in \mathcal{A}_1^+} \delta\chi(mK, A_i) + \sum_{B_j \in \mathcal{A}_1^-} \delta\chi(mK, B_j) = \frac{m(m-1)(2m-1)}{12}\alpha.$$

But the left hand side is a numerical polynomial, so setting $m = 2$ I must have

$$\frac{\alpha}{2} = \frac{2(2-1)(4-1)}{12}\alpha \in \mathbb{Z}.$$

So $\alpha \in 2\mathbb{Z}$. Of course, if $\alpha = 0$ I've already finished; take $\mathcal{A}^{\pm} = \mathcal{A}_1^{\pm}$.

Suppose $\alpha > 0$. The baskets

$$\mathcal{A}_2^- = \{(3,1,-2,-1),(2,1,-1,-1)\}$$

and

$$\mathcal{A}_2^+ = \{(3,1,-1,-1)\}$$

do not contribute to the singularities at all but together have

$$\delta K^3 = \frac{8}{3} - \frac{1}{6} - \frac{1}{2} = 2.$$

So

$$\mathcal{A}^{\pm} = \mathcal{A}_1^{\pm} \cup \bigcup_{k=1}^{\alpha/2} \mathcal{A}_2^{\pm}$$

are the required baskets.

If $\alpha < 0$ use the baskets $-\mathcal{A}_2^+$ and $-\mathcal{A}_2^-$. Q.E.D.

As suggested by this proof, I define a basket of flips and antiflips $\mathcal{A}_2 = \mathcal{A}_2^+ - \mathcal{A}_2^-$. A corollary of the proof is

Corollary 54 *I can calculate* $\delta K^3 \bmod 2$ *for any flip* $X^- \to X \leftarrow X^+$ *from the knowledge of its baskets of singularities alone.*

Remark (1) Of course I would really like the sum of toric flips to be strictly positive as it would be if the flip actually deformed to toric flips; in particular, I want some division of the flip into baskets with $\mathcal{A}^- = \emptyset$. As a weak first statement, I can say that certainly $\mathcal{A}^+$ is not empty in any decomposition into baskets; for if it was, I would calculate $\delta K^3 \leq 0$ in the original flip.

(2) It is not hard to see that in addition I can always hit the singularities of a flip A with a basket of toric flips and antiflips, say $\mathcal{A}_0$, all of which have $\delta\chi(2K) = 0$. Let $k = \sum \delta K^3(A_i)$, where the sum is taken over $\mathcal{A}_0$. Since $\delta\chi(2K) \geq 0$ and $\delta\chi(2K) = 1$ for the basket $\mathcal{A}_2$, I get a genuine basket by adding a non-negative number of copies of $\mathcal{A}_2$ to $\mathcal{A}_0$. But δK^3 for $\mathcal{A}_2$ is 2 so

$$\delta K^3(A) \geq k.$$

For example, if the plurigenus baskets of A are $\mathcal{B}^- = \left\{5 \cdot \frac{1}{2}(1,1,1)\right\}$, $\mathcal{B}^+ = \emptyset$ then $\delta K^3(A) \equiv 1/2 \bmod 2$ but $\delta K^3(A) \neq 1/2$.

(3) If X^+ is smooth, or for that matter, Gorenstein, it is easy to hit the singularities of the whole flip using only the positive basket of flips. This, together with the Corollary above gives the bound

$$\delta K^3 \geq \frac{1}{(r_-)!}$$

where r_- is the index of the highest index singularity on X^- and ! is factorial. When r_- is small, this is a better bound than the one given in Corollary 14.

(4) Exactly the same lemma holds for divisorial contractions.

5.2 What does X^+ look like?

For the most part of this thesis, flips have been an *a posteriori* phenomenon. But, as I mentioned in the introduction, this is not how they occur in Mori theory. In that setup I am given a small extremal contraction

$$f^-: (C^- \subset X^-) \to (P \in X)$$

and the task is to construct the flipped variety, X^+. I can read $\mathcal{B}^-$, the basket of singularities of X^-, straight from X^- if it is described explicitly enough. Then, Lemma 18 says that, in principle, I can calculate what $\delta\chi(mK)$ would be *if the flip, $f^+: X^+ \to X$ did in fact exist.*

Attempting to prove the existence of flips by construction using this is too ambitious; nonetheless, it does give *a priori* information about X^+. The following proposition was pointed out to me by Miles Reid.

Proposition 55 *For a flipping contraction $f^-: X^- \to X$, if you know the dimensions of the vector spaces $R^1 f^-_*(mK_-)$ for all $m \in \mathbb{N}$ then (under the assumption that the flip exists)*

(1) knowing $\mathcal{B}^-$ makes $\mathcal{B}^+$ calculable

(2) knowing K^3_- makes K^3_+ calculable.

Proof. This is straight from the Plurigenus Formula and Theorem II.7.1 of [10]. Q.E.D.

Remark As I said in the Introduction, the question of the existence of flips has already been settled. In [18], Mori proves the existence of the flip as follows: first, in all cases, find a good element of $|-K_-|$ or $|-2K_-|$; then use a technique of Kawamata's from [13] to construct the flip using double covers and flops — see [4], §4, for a brief account of this. The hardest bit is finding the good member of the antibi- or anti-canonical linear system. Mori does this by a detailed analysis of singularity germs that could appear on an extremal neighbourhood and how they can combine globally on the extremal neighbourhood. Then he finds an explicit member in each case.

Example Suppose $f^-\colon X^- \to X$ is a small extremal contraction with

$$\mathcal{B}^- = \left\{\frac{1}{2}(1,1,1)\right\}.$$

If X^+ exists, then,

$$\delta\chi(mK) = k(m)\delta K^3 + \delta\ell(m).$$

From the contraction f^- I can calculate the dimensions of $R^1 f^-_*(mK)$. For this example, suppose it is

$$\delta\chi = [0,0,1,3,7,13,\ldots].$$

From $\mathcal{B}^-$ I can calculate that

$$\ell_- = \frac{1}{4}[0,1,1,2,2,3,\ldots].$$

Let $\zeta = \delta\chi + \ell_-$ and difference ζ;

$$\zeta = \frac{1}{4}[0,1,5,14,30,55,\ldots]$$

$$D\zeta = \frac{1}{4}[1,4,9,16,25,\ldots]$$

$$D^2\zeta = \frac{1}{4}[3,5,7,9,\ldots]$$

$$D^3\zeta = \frac{1}{4}[2,2,2,\ldots]$$

So the period of ζ is 1. But by the plurigenus formula, ζ and ℓ_+ have the same period. So the period of ℓ_+ is 1 and I conclude that $\ell_+ \equiv 0$ giving the result that $\mathcal{B}^+ = \emptyset$.

In this case I can see in addition that

$$\frac{1}{2}\delta K^3 = \zeta(2) = \frac{1}{4};$$

In other words, $\delta K^3 = 1/2$. The data I used came from the Francia flip and the conclusions happily match it.

One point I skipped in the example was that I concluded that $D^3\zeta$ had period 1 from only its first 3 terms. That's clearly fraudulent, but I get away with it by this lemma.

Lemma 56 *Let $X^- \to X \leftarrow X^+$ be a flip and let r_- be the index of the highest index singularity on X^-. If $Q \in X^+$ is any point on X^+ then the index of Q is strictly smaller than r_-.*

This is proved by Kawamata in the appendix to Shokurov's paper [27]. Let $P \in X$ be a terminal 3-fold singularity of index r and let $Y \longrightarrow X$ be a resolution. The minimum discrepancy that any exceptional divisor can have is $1/r$ by definition of the index. Kawamata proves that this minimum is attained by some exceptional divisor. Now the lemma follows from Shokurov's assertion that discrepancies in a common resolution of a flip must strictly increase; this is the proof of Theorem 4 in the introduction.

Corollary 57 *In the notation of the lemma, if R_+ is the global index of X^+ along C^+ then*

$$R_+ \leq \prod_{i=1}^{r_- - 1} i.$$

In particular, given $3 + \prod_{i=1}^{r_- -1} i$ consecutive terms of $\zeta(m) = \delta\chi(m) + \ell_-(m)$, R_+ is the minimum period of the resulting terms of $D^3\zeta$.

Either of these statements immediately justifies the conclusion in the example.

Effective bounds

In some situations there is a more effective way of extracting $\mathcal{B}^+$ from the Plurigenus Formula. Again rearranging the formula so that the information I hope to have is on the left gives

$$\delta\chi(m) + \ell_-(m) = k(m)\delta K^3 + \ell_+(m). \tag{5.1}$$

All the terms in this expression are nonnegative.

Recall from Fletcher's thesis, [10],

Lemma 58 *Let $P = \frac{1}{r}(1, -1, b)$. Then,*
(a) $\ell(P, m)$ is monotonic increasing with m;
(b) $\ell(P, m) \geq 1/4$ for all $m \geq 2$, with equality iff $r = 2$ and $m = 2$ or 3.

Proof. (a) is Lemma II.2.7 from [10]; (b) holds since if $r > 2$,

$$\ell(P, 2) = \frac{b(r-b)}{2r} = \frac{r-1}{2r} > 1/4;$$

now calculate the first few terms of $\ell(P, m)$ with $r = 2$ as in Appendix A. Q.E.D.

Corollary 59 *Let $N_+ = \#\mathcal{B}^+$ be the number of singularities in the basket $\mathcal{B}^+$ of singularities of X^+. For any flip*

$$N_+ < 4(\delta\chi(2) + \ell_-(2))$$

and

$$\delta K^3 \leq 2(\delta\chi(2) + \ell_-(2)).$$

So knowing $\mathcal{B}^-$ *and* $\delta\chi(2) = \dim R^1 f_*^-(2K_-)$ *leaves only a finite choice for* $\mathcal{B}^+$ *and* δK^3.

Proof. By part (b) of the Lemma, $N_+ \leq 4\ell_+(2)$. Now both statements follow by equation (5.1) with $m = 2$. Q.E.D.

I find this bound on N_+ a bit strange. I'm really thinking that the change in $\chi(mK)$ is giving a hint about the change in features of the variety. So, in particular, a large change in $\chi(mK)$ should be forcing fewer singularities on X^+, not sanctioning more. With that in mind, I conclude with the following effective result. I think of it as saying that when $\delta\chi(A, 2)$ is as large as possible I get control over the singularities on X^+.

Proposition 60 *If* $A \in \mathcal{A}$ *admits a decomposition into toric flips,* $\{A_i\}$, *such that* $\delta\chi(A) = \sum_i \delta\chi(A_i)$ *as functions on* $\mathbb{N}$ *then*

$$\delta\chi(A, 2) \leq \sum_{P \in \mathcal{B}^-(A)} \binom{r(P) - 1}{2}$$

and, equivalently,

$$\delta K^3 \leq \sum_{P \in \mathcal{B}^-(A)} \frac{(r(P) - 1)^3}{r(P)}$$

where $r(P)$ *is the index of the point* $P \in X^-(A)$.

Moreover, if equality holds in these relations, then $\mathcal{B}^+(A) \subset \mathcal{B}^-(A)$.

Proof. Suppose that $A \sim \sum_{i \in I} A_i$ where $I = \{1, \ldots, n\}$ and A_i is the toric fllp $(a_i, 1, -b_i, -c_i)$. Name the singular points of A_i as

$$P(i) = \frac{1}{a_i}(1, -b_i, -c_i),$$

$$Q_1(i) = \frac{1}{b_i}(a_i, 1, -c_i),\ Q_2(i) = \frac{1}{c_i}(a_i, 1, -b_i).$$

There are two partially defined orders on I:

$$i \leftarrow_1 j \text{ iff } P(i) = Q_1(j);$$

$$i \leftarrow_2 j \text{ iff } P(i) = Q_2(j).$$

A *chain* σ in I is a set of distinct elements $\left\{i^{\sigma}_{\alpha,\beta}\right\}$ of I satisfying a collection of relations of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & i^{\sigma}_{1,2} \arrow[dl, "1"'] & \\
 & i^{\sigma}_{1,1} \arrow[dl, "1"'] & & \\
i^{\sigma}_{1,0} & & {} \arrow[ul, "2"'] \arrow[dl, "1"'] & \cdots \\
 & i^{\sigma}_{2,1} \arrow[ul, "2"'] & & \\
 & & i^{\sigma}_{4,2} \arrow[ul, "2"'] &
\end{tikzcd}$$

which may terminate anywhere along any branch; call these leaves $i^{\sigma}_{1,\infty}, \dots, i^{\sigma}_{\gamma,\infty}$.

By Lemma 51 and Fletcher's result that the functions $\ell(Q,m)$ are linearly independent, [10], Theorem II.7.1, I know that

$$\begin{aligned}
\mathcal{B}^{-}(A) - \mathcal{B}^{+}(A) &= \sum_{i\in I} \left(\mathcal{B}^{-}(A_i) - \mathcal{B}^{+}(A_i)\right) \\
&= \sum_{\nu} \left(P(i^{\sigma_\nu}_{1,0}) - \sum_{\mu} \left(Q_1(i^{\sigma_\nu}_{\mu,\infty}) + Q_2(i^{\sigma_\nu}_{\mu,\infty})\right)\right)
\end{aligned}$$

for any collection of chains $\{\sigma_\nu\}$ that partitions I.

I want to choose a collection of chains, Σ, which partition I and such that

$$\mathcal{B}^{-}(A) = \mathcal{P}_\Sigma := \left\{ P_\nu := P(i^{\sigma_\nu}_{1,0}) \mid \nu \in \Sigma \right\};$$

that is, each singularity of $X^{-}(A)$ is the intial singularity of some chain. Take any partition Σ. If $P_\nu \notin \mathcal{B}^{-}(A)$ than it must be cancelled by some $Q(i^{\mu}_{\infty})$ so glue σ_ν to σ_μ at i^{μ}_{∞}. Eventually $\mathcal{P}_\Sigma \subset \mathcal{B}^{-}(A)$. If $P \in \mathcal{B}^{-}(A)$ then $P = P^i$ for some $i \in I$ which doesn't start some chain. If $P \notin \mathcal{P}_\Sigma$ just break the chain containing i at i.

For A_1, $A_2 \in \mathcal{A}$, I write $A_1 >_K A_2$ to mean $\delta K^3(A_1) > \delta K^3(A_2)$ and similarly for $\geq_K$ and $=_K$. The following inequalities are easy:

$$(a,1,-1,-1) >_K (a,1,-b,-1) + (b,1,-1,-1),$$

$$(a,1,-1,-1) >_K (a,1,-b,b-a) + (b,1,-1,-1) + (a-b,1,-1,-1),$$

where $b > 1$ in both cases. Now, for the choice of partition above, and using the inequalities inductively,

$$\begin{aligned}
\sum_{P\in\mathcal{B}^{-}(A)} (r(P),1,-1,-1) &\geq_K \sum_{\sigma\in\Sigma} \left(\sum_{i^\sigma \in \sigma} A_{i^\sigma} \right) \\
&=_K \sum_i A_i \\
&=_K A
\end{aligned}$$

with equality iff each chain is from a flip of the form $(a, 1, -1, -1)$.

This is the δK^3 inequality; halving and adding $\delta\ell(2)$ to each side gives the $\delta\chi$ inequality.

Q.E.D.

Appendix A

Tables of singularities and baskets

Toric hypersurface flips

Recall Theorem 33: any terminal flip given by some $A\colon (g = 0) \subset \mathbb{C}^5$ and $\mathbb{C}^*$ action containing no quasireflections is one of the following:

	monomials in g	$\mathbb{C}^*$ action	
(1)	$x_1y_1 + g'(x_2, x_3)$	$(a_1, a_2, 1, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, b_1$
(2)	$x_1y_1 + x_3^n$	$(a_1, a_2, a_3, -b_1, -a_2; a_1 - b_1)$	$a_1 > a_2, a_3, b_1$
(3)	$x_2^2 + x_1y_2^2 + x_1^n y_1^{2n-1}$	$(4, 1, 1, -2, -1; 2)$	
(4)	$x_2y_1 + z^n$	$(a, 1, -1, -b, 0; 0)$	$a > b, (a, b) = 1$
(5)	$z^2 + x_1y_2^3$	$(3, 1, -2, -1, 0; 0)$	
(6)	$x_2y_1 + y_3^2 + x_1^n y_2^{2n+1}$	$(4, 1, -3, -2, -1; -2)$	

where in both (1) and (2), a_2 divides $a_1 - b_1$ and all the characters are coprime except that possibly $\operatorname{hcf}(a_1, b_1) > 1$. If $a_2 > 1$ then $g \ni x_2^r y_2^s$ where, by the equivariance of g, $(r - s)a_2 = e = a_1 - b_1$. In case (4), a and b are coprime. In case (5), g must also contain one of $x_2^2y_1$, $x_1x_2y_1^2$, $x_1^2y_1^3$.

Pluricanonical series of some singularities

I list the first r terms of the correction function

$$\ell(Q, m) = \frac{r^2 - 1}{12r}(m - \overline{m}) + \sum_{j=1}^{\overline{m}-1} \frac{\overline{bj}(r - \overline{bj})}{2r}$$

associated to the singularity $Q = \frac{1}{r}(1, -1, b)$. The calculation of the other terms is now easy: for $i = 1, \ldots, r - 1$

$$\ell(Q, nr + i) = n\ell(Q, r) + \ell(Q, i).$$

Q vs. m	1	2	3	4	5	6	7	8	9
$\frac{1}{2}(1,-1,1)$	0	1/4							
$\frac{1}{3}(1,-1,1)$	0	1/3	2/3						
$\frac{1}{4}(1,-1,1)$	0	3/8	7/8	5/4					
$\frac{1}{5}(1,-1,1)$	0	2/5	1	8/5	2				
$\frac{1}{5}(1,-1,2)$	0	3/5	1	7/5	2				
$\frac{1}{6}(1,-1,1)$	0	5/12	13/12	22/12	30/12	35/12			
$\frac{1}{7}(1,-1,1)$	0	3/7	10/7	16/7	22/7	27/7	4		
$\frac{1}{7}(1,-1,2)$	0	5/7	11/7	14/7	17/7	23/7	4		
$\frac{1}{7}(1,-1,3)$	0	6/7	9/7	15/7	21/7	24/7	4		
$\frac{1}{8}(1,-1,1)$	0	7/16	19/16	34/16	50/16	65/16	77/16	21/4	
$\frac{1}{9}(1,-1,1)$	0	4/9	11/9	20/9	30/9	40/9	49/9	56/9	20/3
$\frac{1}{9}(1,-1,2)$	0	7/9	17/9	26/9	30/9	34/9	43/9	53/9	20/3
$\frac{1}{9}(1,-1,4)$	0	10/9	14/9	23/9	30/9	37/9	46/9	50/9	20/3

Singularity baskets of the hypersurface flips

The *basket of singularities of a flip* is the formal signed collection of singularities consisting of the difference of the pluricanonical baskets of X^+ and X^-. These baskets are determined by deformations so I can deform the equation g if it makes the calculations easier. When a singularity P deforms to the sum of quotient singularities $Q_1 + \ldots + Q_q$ I write $P \rightsquigarrow \sum_i Q_i$.

(1)&(2) $\quad g = x_1y_1 + x_2^r y_2^s$ where $(r-s)a_2 = a_1 - b_1$.
The singularities are

$$\begin{aligned} P_1 &= \tfrac{1}{a_1}(a_2, 1, -a_2) \\ P_2 &= \tfrac{1}{a_2}(a_1, 1, -a_1, 0; 0) \rightsquigarrow s \cdot \tfrac{1}{a_2}(a_1, 1, -a_1) \\ Q_1 &= \tfrac{1}{b_1}(a_2, 1, -a_2) \\ Q_2 &= \tfrac{1}{a_2}(a_1, 0, 1, -a_1; 0) \rightsquigarrow r \cdot \tfrac{1}{a_2}(a_1, 1, -a_1) \end{aligned}$$

so the basket of singularities is

$$\mathcal{B} = \left\{ \frac{1}{a_1}(a_2, 1, -a_2), -\frac{1}{b_1}(a_2, 1, -a_2), -(r-s) \cdot \frac{1}{a_2}(a_1, 1, -a_1) \right\}.$$

(3)

This includes $P_1 = \frac{1}{4}(1,1,2,3;2) \rightsquigarrow \frac{1}{4}(1,1,3) + n \cdot \frac{1}{2}(1,1,1)$. The whole basket is

$$\mathcal{B} = \left\{ \frac{1}{4}(1,1,3), -\frac{1}{2}(1,1,1) \right\}.$$

(4) $$\mathcal{B} = \left\{ n \cdot \tfrac{1}{a}(1,-1,-b), -n \cdot \tfrac{1}{b}(a,1,-1) \right\}.$$

(5) $$\mathcal{B} = \left\{ 2 \cdot \tfrac{1}{3}(1,1,2), -2 \cdot \tfrac{1}{2}(1,1,1) \right\}.$$

(6) $$\mathcal{B} = \left\{\tfrac{1}{4}(1,1,3), -\tfrac{1}{3}(1,1,2)\right\}.$$

Effective toric flip baskets for the hypersurface flips

The following baskets of toric flips hit the singularities of the corresponding hypersurface flips. Recall from Corollary 54 that this gives enough information to calculate δK^3 mod 2. Cases (4)–(6) are constructed by degeneration so they give δK^3 exactly.

(1)&(2) $$\mathcal{A} = \{(a_1, 1, -a_2, a_2 - a_1), (a_1 - a_2, 1, -a_2, 2a_2 - a_1), \ldots, (b_1 + a_2, 1, -a_2, -b_1)\}.$$
$$\delta K^3 \equiv 1/t \bmod 2$$
$$\text{where } t = a_1 a_2^n b_1 (a_1 - a_2)^2 (a_1 - 2a_2)^2 \ldots (a_1 - (n-1)a_2)^2.$$

(3) $$\mathcal{A} = \{(4, 1, -3, -1), (3, 1, -2, -1)\}.$$
$$\delta K^3 \equiv 1/4 \bmod 2.$$

(4) $$\mathcal{A} = \{n \cdot (a, 1, -b, -1)\}.$$
$$\delta K^3 = n(a-b)^3/ab.$$

(5) $$\mathcal{A} = \{2 \cdot (3, 1, -2, -1)\}.$$
$$\delta K^3 = 1/3.$$

(6) $$\mathcal{A} = \{(4, 1, -3, -1)\}.$$
$$\delta K^3 = 1/12.$$

Elephants of the hypersurface flips

Recall that the *index* of a hypersurface flip is the number $\tau = \sum a_i - \sum b_j - e$.

(1)&(2) When $a_2 > 1$, $g = x_1 y_1 + x_2^r y_2^s$ where $(r - s)a_2 = a_1 - b_1$. The index of this flip is a_3 so an elephant is given by $x_3 = 0$.

$$\begin{aligned} S &= \left(\frac{k[x_1, x_2, y_1, y_2]}{x_1 y_1 + x_2^r y_2^s}\right)^{\mathbb{C}^*(a_1, a_2, -b_1, -a_2)} \\ &= \frac{k\left[x_1^{a_2} y_2^{a_1}, x_2^{b_1} y_1^{a_2}, x_2 y_2\right]}{x_1 y_1 + x_2^r y_2^s} \\ &= k[u, v, w] / uv + w^{a_1 + a_2 s} \end{aligned}$$

This is partially resolved in X^- by

$$\begin{aligned} S_1^- &= k[u, v, w, t] / uv + w^{a_1 + a_2 s}, ut + w^{a_1} \\ &= k[u, w, t] / ut + w^{a_1} \end{aligned}$$

where $t = x_2^{a_1} / x_1^{a_2}$ and

$$S_2^- = k\left[v, w, t^{-1}\right] / vt^{-1} + w^{a_2 s}$$

A similar thing happens in X^+.

I draw the whole elephant in terms of its dual graph; $\bullet$ represents a -2 curve in the minimal resolution of $0 \in S$, $\circ$ represents one of these curves which has been extracted in S^- or S^+. For this elephant, the picture is

$$S^- = \bullet - \overset{a_1-1}{\cdots} - \bullet - \circ - \bullet - \overset{a_2s-1}{\cdots} - \bullet$$

$$S = \bullet - \overset{a_1+a_2s-1}{\cdots} - \bullet$$

$$S^+ = \bullet - \overset{a_1r-1}{\cdots} - \bullet - \circ - \bullet - \overset{b_1-1}{\cdots} - \bullet$$

When $a_2 = 1$, I get a similar picture using the elephant $x_2 - x_3 = 0$

$$S^- = S = \bullet - \overset{a_1-1}{\cdots} - \bullet \qquad S^+ = \bullet - \overset{a_1-b_1-1}{\cdots} - \bullet - \circ - \bullet - \overset{b_1-1}{\cdots} - \bullet$$

(3) $\qquad g = x_2^2 + x_1y_2^2 + x_1^ny_1^{2n-1}$.

The index of this flip is 1 so an elephant is $x_2+x_3 = 0$. I use this to eliminate x_3.

$$\begin{aligned}
S &= \left(\frac{k\,[x_1, x_2, y_1, y_2]}{x_2^2 + x_1y_2^2 + x_1^ny_1^{2n+1}}\right)^{\mathbb{C}^*(4,1,-2,-1)} \\
&= \frac{k[x_1y_1^2, x_1y_1y_2^2, x_1y_2^4, x_2y_2]}{x_2^2 + x_1y_2^2 + x_1^ny_1^{2n+1}} \\
&= \frac{k[u_0, u_1, u_2, v]}{u_0u_2 - u_1^2, v^2 + u_2 + u_1u_0^{n-1}} \\
&= \frac{k[u_0, u_1, v]}{u_1^2 + u_0v^2 + u_1u_0^n} \\
&= k[u, u_0, v]/u^2 + u_0v^2 + u_0^{2n}
\end{aligned}$$

where $u = u_1 + u_0^n$. This is the DuVal singularity D_{2n+1}.

$$S^- = S = \begin{array}{l} \phantom{\bullet - {}}\bullet \\ \phantom{\bullet - {}}| \\ \bullet - \bullet - \bullet - \overset{2n-2}{\cdots} - \bullet \end{array} \qquad S^+ = \begin{array}{l} \phantom{\bullet - {}}\bullet \\ \phantom{\bullet - {}}| \\ \bullet - \bullet - \bullet - \overset{2n-3}{\cdots} - \bullet - \circ \end{array}$$

The other calculations are similar.

(4) $\qquad$ The elephant given by $x_1y_2 + x_2^{a-b} = 0$ is

$$S^- = S = \bullet - \overset{an-1}{\cdots} - \bullet \qquad S^+ = \bullet - \overset{(a-b)n-1}{\cdots} - \bullet - \circ - \bullet - \overset{bn-1}{\cdots} - \bullet$$

(5) $\qquad$ For any choice of equation g, the elephant given by $x_2 + x_1y_1 = 0$ is

$$S^- = S = \begin{array}{l} \phantom{\bullet - {}}\bullet \\ \phantom{\bullet - {}}| \\ \bullet - \bullet - \bullet - \bullet \end{array} \qquad S^+ = \begin{array}{l} \phantom{\bullet - {}}\bullet \\ \phantom{\bullet - {}}| \\ \bullet - \bullet - \bullet - \circ \end{array}$$

(6) $\qquad g = x_2y_1 + y_3^2 + x_1^n y_2^{2n+1}$

The index of this flip is 1 so an elephant is given by $x_2 = x_1y_1$.

$$\begin{aligned}
S &= \left(\frac{k\,[x_1, y_1, y_2, y_3]}{x_1y_1^2 + y_3^2 + x_1^n y_2^{2n+1}}\right)^{\mathbb{C}^*(4,-3,-2,-1)} \\
&= \frac{k[x_1^3y_1^4, x_1^2y_1^2y_2, x_1y_1y_3, x_1y_2^2]}{x_1y_1^2 + y_3^2 + x_1^n y_2^{2n+1}} \\
&= \frac{k[u_0, u_1, u_2, v]}{u_0v - u_1^2, u_0 + u_2^2 + u_1v^n} \\
&= \frac{k[u_1, u_2, v]}{u_1^2 + vu_2^2 + u_1v^{n+1}} \\
&= k[u, u_2, v]/u^2 - vu_2^2 + v^{2n}
\end{aligned}$$

where $u = u_1 + v^{n+1}$. This is the DuVal singularity D_{2n+3}.

$S^- = S =$ 

In this case, S^+ does not have DuVal singularities. In fact, the singularity on S_1^+ is a triple point.

Appendix B

Subcanonical covers of flips are quasi-Gorenstein

In [25], Reid shows that any flip diagram has a $\mathbb{C}^*$ cover A. The flip itself can be retrieved by taking the three different quotients as described in section 3.1. He is also granted the revelation that A is a Gorenstein variety. The aim of this appendix is prove half of this assertion, namely that the canonical class K_A of A is locally free. This condition is known as *quasi-Gorenstein*. In fact, I work with an affine neighbourhood of a flip so I prove that K_A is a free $\mathcal{O}_A$ module.

Covers of varieties

Very quickly, I recall the construction of the $\mathbb{C}^*$ cover of a variety. Let X be a normal variety and D a Weil divisor on it. I assume that D is nontorsion in $\mathrm{Cl}X$ and so $\langle D\rangle = \mathbb{Z}$ is the subgroup it generates. Define the $\mathcal{O}_X$ algebra

$$\mathcal{R}(D) = \oplus_{n\in\mathbb{Z}}\mathcal{O}(nD)$$

where the multiplication $\mathcal{O}(nD) \times \mathcal{O}(mD) \longrightarrow \mathcal{O}((n+m)D)$ is determined by multiplication of functions. Now I have the variety

$$A(D) = \mathbf{Spec}_{\mathcal{O}_X}\mathcal{R}(D)$$

with its structure map $A(D) \longrightarrow X$. The dual group of $\langle D\rangle$ is defined as

$$\langle D\rangle^d = \mathrm{Hom}\,(\langle D\rangle\,, \mathbb{C}^*)$$

which is isomorphic to $\mathbb{C}^*$ and it acts on $\mathcal{R}(D)$ by

$$\begin{array}{ccc} \langle D\rangle^d \times \mathcal{R}(D) & \longrightarrow & \mathcal{R}(D) \\ \varepsilon \times \oplus_n f_n & \mapsto & \oplus_n \varepsilon(D)^n f_n. \end{array}$$

Clearly the invariant part of the algebra $\mathcal{R}(D)$ is $\mathcal{O}_X$ so

$$X = A(D)/\!\!/\mathbb{C}^*.$$

Example (Reid and others, [21] Corollary (1.9))

If X is affine with singular locus $P \in X$ a terminal singularity of index r, then $\mathrm{Cl}(X)$ has no torsion part, but nonetheless I can make the same construction. In this case I take $D = K_X$, using the trivialisation of rK_X to define the multiplication on

$$\mathcal{R}(G) = \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \ldots \oplus \mathcal{O}_X((r-1)K_X)$$

where $G = \langle K_X \rangle \cong \mu_r$. Taking $\mathbf{Spec}_{\mathcal{O}_X}$ gives the index 1 cover

$$Y = A(G) \longrightarrow X.$$

Temporarily, I work with a *smooth* variety $X = M$. The vector bundle associated to the total space of a locally free sheaf $\mathcal{E}$ on M is denoted $\mathsf{V}_{\mathcal{E}}$:

$$\mathsf{V}_{\mathcal{E}} = \left(\mathbf{Spec}_{\mathcal{O}_M} \mathcal{S}(\mathcal{E})\right)^{\vee},$$

where

$$\mathcal{S}(\mathcal{E}) = \oplus_{n \in \mathbb{N}} \mathcal{S}^n(\mathcal{E}).$$

Lemma 61 *If $X = M$ is a smooth variety then $A(D)$ is a principal $\mathbb{C}^*$ bundle over M. Indeed $A(D) \cong \mathsf{V}_{-D} \setminus M_0$ where M_0 is the zero section.*

Proof. The point is just that since $\mathcal{E} = \mathcal{O}(-D)$ is rank 1 there are no symmetry conditions in $\mathcal{S}^n(\mathcal{E})$ so $\oplus_{n \in \mathbb{N}} \mathcal{S}^n(\mathcal{E})$ is just the positive part of the full tensor algebra. In other words, locally over M,

$$\mathcal{R}(-D) = k[U][t, 1/t]$$

and

$$\mathcal{S}(-D) = k[U][t].$$

But this is locally the inclusion $U \times \mathbb{G}_m \hookrightarrow U \times \mathbb{G}_a$ as required. Q.E.D.

Remark Repeating the lemma for V_{+D} shows that $A(D)$ is the natural $\mathbb{C}^*$ bundle $\mathsf{V}_D \cap \mathsf{V}_{-D}$. Gluing over $A(D)$ you can see that $\mathsf{V}_D \cup \mathsf{V}_{-D}$ forms a $\mathbb{P}^1$ bundle over M.

Corollary 62 *$A(D)$ is a manifold of dimension* $\dim M + 1$.

Proposition 63 *The manifold $A(K_M)$ has globally trivial canonical bundle.*

Proof. Let $p: V = \mathbb{V}_{K_M} \longrightarrow M$. By Lemma 61 it is sufficient to prove that either K_V or $-K_V$ is the trivial bundle; in fact, $-K_V$ is the more convenient. In other words, I prove that given any two trivialising neighbourhoods for $-K_V$, say $U_1, V_1 \subset V$, I can trivialise $-K_V$ over the union $U_1 \cup V_1$. The main calculation is done over particular trivialising neighbourhoods.

On M:

Suppose that U and V are trivialising neighbourhoods for K_M with coordinates u_j and v_i respectively. The crossover is given by $\phi: U \to V$; $v_i = \phi_i(u)$.

On V:

Let $\widetilde{U} = U \times \mathbb{C}$ and $\widetilde{V} = V \times \mathbb{C}$ be patches on V with coordinates u_j, s and v_i, t respectively. Then the crossover is given by $\widetilde{\phi}: \widetilde{U} \to \widetilde{V}$; $(v,t) = (\phi(u), \chi(u) \cdot s)$ where $\chi: U \cap V \to \mathbb{C}^*$ is given by

$$\chi(u) = (\det J\phi(u))^{-1} = \det\left(\frac{\partial \phi_i}{\partial u_j}(u)\right)^{-1}.$$

Now *assume* that $\widetilde{U}$ and $\widetilde{V}$ are trivialising neighbourhoods for $-K_V$.

On $-K_V$:

The crossover is given by $\widetilde{\phi}: \widetilde{U} \times \mathbb{C} \to \widetilde{V} \times \mathbb{C}$; $(v,t,\eta) = (\widetilde{\phi}(u,s), \widetilde{\chi}(u,s) \cdot \xi)$ where $\widetilde{\chi}: \widetilde{U} \cap \widetilde{V} \to \mathbb{C}^*$ is given by

$$\widetilde{\chi}(u,s) = \left(\det J\widetilde{\phi}\right)(u,s).$$

But now I calculate that

$$\begin{aligned}
\widetilde{\chi}(u,s) &= \left(\det J\widetilde{\phi}\right)(u,s) \\
&= \det\begin{pmatrix} \frac{\partial \phi_i}{\partial u_j} & \frac{\partial(\chi \cdot s)}{\partial u_j} \\ \frac{\partial \phi_i}{\partial s} & \frac{\partial(\chi \cdot s)}{\partial s} \end{pmatrix}(u,s) \\
&= \det\begin{pmatrix} \frac{\partial \phi_i}{\partial u_j}(u) & * \\ 0 & \chi(u) \end{pmatrix} \\
&= (\det J\phi(u))(\det J\phi(u))^{-1} \\
&= 1.
\end{aligned}$$

This is now sufficient using the identity in M to trivialise $-K_V$ across the whole of $\widetilde{U} := p^{-1}p(U_1)$. Q.E.D.

Corollary 64 *For a 3-fold X whose singularities lie in codimension 2 and whose canonical divisor is not torsion in* $\mathrm{Cl}X$, *$A(D)$ has quasi-Gorenstein singularities for any divisor D such that $K_X = nD$.*

Proof. The last lemma says that $A(K_X)$ has only quasi-Gorenstein singularities. I claim that $A(D) = A(K_X) /\!\!/ \mu_n$ where $\mu_n = \{\varepsilon \mid \varepsilon^n = 1\} \subset \mathbb{C}^*$. The only thing that I need to check now is that if a finite abelian group, G, acts on an algebraic variety, U, then

$$U /\!\!/ G \text{ is quasi-Gorenstein} \Rightarrow U \text{ is quasi-Gorenstein.}$$

In fact I only need the conclusion in the case of a finite map, étale outside a point. But this is trivial by lifting a section of a trivial bundle.

The claim is clear since $\mathcal{R}(D)^{\mu_n} = \mathcal{R}(nD)$: this is the Veronese map, if you prefer. Q.E.D.

If $D \in \mathrm{Cl}X$ and $K_X = \delta D$ then I call $A(D)$ a *subcanonical cover* of X. I call $A(K_X)$ the *canonical cover* of X.

Covers of flip diagrams

Let X be an affine variety with an isolated singular point $P \in X$. I write Cl_P in place of $\mathrm{Cl}X$ since I like to think of covers as being a property of the germ $P \in X$ rather than of X and these two groups are the same in this context. Suppose $D \in \mathrm{Cl}_P$ is nontorsion. Then I can construct a flip diagram by following Reid and defining

$$X^{\pm}(D) = \mathbf{Proj}_{\mathcal{O}_X} \mathcal{R}^{\pm}(D) \text{ where } \mathcal{R}^{\pm}(D) = \bigoplus_{n \geq 0} \mathcal{O}_X(\pm nD).$$

These two varieties have morphisms to X and so fit into a flip diagram

$$X^{-}(D) \longrightarrow X \longleftarrow X^{+}(D).$$

Lemma 65 (Kawamata, [13], Lemma 3.1)
With the same notation as above, the following conditions are equivalent.
(a) $\mathcal{R}^{\pm}(D)$ is a fg $\mathcal{O}_X$ algebra.
(b) There exists a small projective morphism, $f^{\pm}: Y^{\pm} \longrightarrow X$ such that $D^{\pm}$ is an $f^{\pm}$-ample $\mathbb{Q}$-Cartier divisor where $D^{\pm}$ is the birational transform of D on $Y^{\pm}$.
(c) $f^{\pm}: X^{\pm}(D) \longrightarrow X$ is a small projective morphism such that $D^{\pm}$ is an $f^{\pm}$-ample $\mathbb{Q}$-Cartier divisor where $D^{\pm}$ is the birational transform of D on $X^{\pm}(D)$.
□

I say that $D \in \mathrm{Cl}_P$ *polarises the flip diagram*, $X^{-} \to X \leftarrow X^{+}$, if D^{+} is an f^{+}-ample $\mathbb{Q}$-Cartier divisor and D^{-} is an f^{-}-ample $\mathbb{Q}$-Cartier divisor where $D^{\pm}$ are the proper transforms of D.

Corollary 66 *Let $X^{-} \to X \leftarrow X^{+}$ be a flip diagram polarised by D. Then $\mathcal{R}^{(\pm)}(D)$ is an fg $\mathcal{O}_X$ algebra and $X^{\pm}(D) \cong X^{\pm}$ over X.*

In this situation, D is nontorsion so $\langle D \rangle = \mathbb{Z} \subset \mathrm{Cl}_P$. I let $A = A(D)$. Since X is affine, I have a $\mathbb{C}^*$ action on an affine variety, A, and I can start playing. Recall the notation $X^{(\pm)}(A)$ for the three different quotients of $A(D)$ by $\mathbb{C}^*$.

Corollary 67 *Let $X^- \to X \leftarrow X^+$ be a flip diagram polarised by D. Then*

$$X^{\pm}(A) \cong X^{\pm}$$

over $X(A) = X$ where $A = A(D)$.

Proof. This comes straight from the definition of **Proj**; just calculate affine patches on each side to see that $X^{(\pm)}(A) \cong X^{(\pm)}(D)$ and use the previous corollary. All that you need to check is that $\mathcal{I}_{B^{\pm}} = \mathcal{R}^{\pm}(D)$ as in section 3.1. Q.E.D.

Theorem 68 *Any flip can be expressed locally as the three different $\mathbb{C}^*$ quotients of a $\mathbb{C}^*$ action on some quasi-Gorenstein affine variety.*

If a flip diagram is polarised by K_X then $A(K_X)$ is called the *canonical cover* of the flip diagram and any $A(D)$ with $K_X = rD \in \mathrm{Cl}_P(X)$ is called a *subcanonical cover* of the flip diagram.

Bibliography

[1] BEAUVILLE, A., *Complex Surfaces*, LMS Lecture Note Series **68**

[2] BROWN, G.D., *Pluricanonical Euler Characteristics in Toric Flips*, preprint, 19pp, April 1994

[3] BROWN, G.D., *Some 3-fold Flips as Toric Hypersurfaces*, preprint, 23pp, June 1994

[4] CLEMENS, H., KOLLÁR, J., MORI, S., *Higher Dimensional Complex Geometry*, Astérisque **166** (1988)

[5] DANILOV, V.I., *The Geometry of Toric Varieties*, Russ. Math. Surveys **33**:2 (1978), 97–154

[6] DANILOV, V.I., *Birational Geometry of Toric Threefolds*, Math. USSR Izvestiya **21** (1983) No. 2, 269–80

[7] DOLGACHEV, I., *Weighted Projective Varieties*, in Group actions and vector fields, Proc. Vancouver 1981, SLN **956**, 34–71

[8] LECTURES ON VANISHING THEOREMS, Birkhäuser DMV Series **20** (1992)

[9] FLETCHER, A.R., *Working with Weighted Complete Intersections*, MPI preprint, Bonn, MPI/89–35

[10] FLETCHER, A.R., *Plurigenera of 3-folds and Weighted Hypersurfaces*, Warwick Thesis, 1988

[11] FRANCIA, P., *Some Remarks on Minimal Models I*, Comp. Math. **40** (1980), 301–13

[12] KAWAMATA, Y., *On the Plurigenera of Minimal Algebraic 3-folds with $K \equiv 0$*, Math. Ann. 275 (1986), 539–46

[13] KAWAMATA, Y., *Crepant Blowups of 3-dimensional Canonical Singularities and Applications to Deformations of Surfaces*, Ann. Math. **127** (1988), 93–163

[14] KAWAMATA, Y., MATSUDA K., MATSUKI K. *Introduction to the Minimal Model Program*, in Alg. Geom. Sendai, 1985, ed. T. Oda, Adv. Studies. in Pure Math. **10** (1987), 283–360

[15] KOLLÁR, J., *Flops*, Nagoya Math. J. **113** (1989), 14–36

[16] KOLLÁR, J., MORI, S., *Classification of Three Dimensional Flips*, J. Am. Math. Soc. **5**, 533–703

[17] MORI, S., *Threefolds whose Canonical Bundles are not Numerically Effective*, Ann. of Math. **110** (1979), 593–606

[18] MORI, S., *Flip Theorem and the Existence of Minimal Models for 3-folds*, J. Am. Math. Soc., vol.1, No.1 (1988), 117–253

[19] MORRISON, D. R., *Canonical Quotient Singularities in Dimension Three*, Proc. Am. Math. Soc. **93**(3) (1985), 393–6

[20] MUMFORD, D., FOGARTY, J., *Geometric Invariant Theory*, 2nd Edition, Springer Verlag, Erg. Math. **34** (1982)

[21] REID, M., *Canonical 3-folds*, in Journées de Géométrie Algébrique d'Angers, ed. A. Beauville, Sijthoff and Noordhoff, (1980), 273–310

[22] REID, M., *Decomposition of Toric Morphisms*, in Arithmetic & Geometry, II, Progress in Math. **36** (1983), 395–418

[23] REID, M., *Young Person's Guide to Canonical Singularities*, Proc. of Symp. of Pure Math. (Bowdoin) Part 1 **46** (1987), 345–414

[24] REID, M., *Tendencious Survey of 3-folds*, Proc. of Symp. Pure Math. (Bowdoin) Part 1 **46** (1987), 333–344

[25] REID, M., *What is a Flip?* Notes from Colloquium talk, University of Utah, December 1992

[26] SHOKUROV, V.V., *The Non-vanishing Theorem*, Math. USSR Izvestiya **26** (1986), 591–604

[27] SHOKUROV, V.V., *3-fold Log Flips*, Izv. Math. **40** (1993) No. 1, 95–202

[28] THADDEUS, M., *Geometric Invariant Theory and Flips*, alg-geom/9405004 available by email from `alg-geom@publications.maths.duke.edu`

[29] WILSON, P.M.H., *Towards a Birational Classification of Algebraic Varieties*, Bull. LMS **19** (1987), 1–48

[30] ZARISKI, O., *Normal Varieties and Birational Correspondences*, Bull. Am. math. Soc., vol.48 (1942), 402–13