Quantifying the Efficiency of Price-Only Contracts in Push Supply Chains over Demand Distributions of Known Supports*

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Abstract

In this paper, we quantify the efficiency of price-only contracts in supply chains with demand distributions by imposing prior knowledge only on the support, namely, those distributions with support $[a, b]$ for $0 < a \leq b < +\infty$. By characterizing the *price of anarchy* (PoA) under various push supply chain configurations, we enrich the application scope of the PoA concept in supply chain contracts along with complementary managerial insights. One of our major findings is that our quantitative analysis can identify scenarios where the price-only contract actually maintains its efficiency, namely, when the demand uncertainty, measured by the *relative range* b/a, is relatively low, entailing the price-only contract to be more attractive in this regard.

Keywords: supply chain management, price of anarchy, Stackelberg game, Nash equilibrium

1 Introduction

Price of anarchy (PoA), a quantifier measuring the inefficiency of a multi-agent system due to selfish behavior of its agents, has been an extremely popular concept in computer science and operations research communities in the last decade (Nisan et al. (2007)). Perakis and Roels (2007) pioneered its application in supply chain contracts and obtained the PoA for price-only contract for several configurations of the underlying supply chain when the demand distributions possess the (weakly) *increasing generalized failure rate* (IGFR) property. A nonnegative random variable X with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ is of IGFR property if $xf(x)/\bar{F}(x)$ is nondecreasing for all x such that $\bar{F}(x) > 0$, where $\bar{F}(x) = 1 - F(x)$. We will use $\mathcal{F}_{\text{IGFR}}$ to denote the class of all distributions with IGFR property.

One of the most important managerial insights observed from their analysis is that the worst PoA under $\mathcal{F}_{\text{IGFR}}$ is at least 1.71 (a 71% loss of efficiency) even for the simple two-stage chain, and consequently price-only contract may not be a viable practical contract with certain demand distributions due to this large loss of efficiency. Nevertheless, the price-only contract has been widely adopted in many real-life practices. This popularity has been constantly attributed to its low administrative cost (cf. Cachon (2003)).

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Therefore, the following important questions arise naturally: Is it possible that the assumption of IGFR on demand distributions leads to the overwhelmingly negative image on the price-only contract? Can we identify and justify those situations where price-only contract is attractive not just because of its low administrative cost?

These questions of significant practical consequences serve the main motivation of this work, which investigates another class $\mathcal{F}[a,b]$ $(0 < a \leq b < +\infty)$ of all distributions with support of the form $[a,b]$. The adoption of this class is a distributionally robust (or distribution-free, semi-parameter or min-max) approach similar to those by Scarf (1965) (see also, e.g., Gallego (1992), Gallego and Moon (1993), Gallego (1998), Gallego (2001), Godfrey, and Powell (2001), Jiang et al. (2011), Yue et al. (2006), Gallego (2007), Perakis and Roels (2008) etc.), when only distribution parameters, such as support, mean or variance, rather than the full distribution itself, are assumed to be known.

The main contribution of this work is to derive PoA bounds over all distributions in the class $\mathcal{F}[a,b]$ under various supply chain configurations, as compared with the work of Perakis and Roels (2007) for distribution class $\mathcal{F}_{\text{IGFR}}$. These two classes of distributions are overlapping but not inclusive: there are IGFR distributions with support $[0,\infty)$ and there are distributions of support $[a,b]$ that are not IGFR. The bounds derived by Perakis and Roels (2007) depend on the number of supply chain partners n and the profit margin, whereas the bounds derived here depend on the number of supply chain partners n and the relative range b/a. Hence, different and complementary managerial insights are obtained, especially with regard to the degree of uncertainty of demand, measured by the parameter b/a.

We only present the results on the *push* mode (Cachon, 2004), where the downstream partner(s) hold(s) the supply chain inventory. Interested readers are referred to our working paper (Du et al. 2011) for results concerning the other mode, pull mode. Moreover, throughout this paper, we will only consider pure equilibria for all the (sub-)games involved, as mixed strategies are not well-accepted in supply chain management (Cachon and Netessine, 2004). Finally, we only focus on the nontrivial cases where the upstream partner in the game is the leader and where full efficiency cannot be achieved, that is, PoA > 1.

The paper is organized as follows. After this introduction, we first provide some preliminary results in Sec. 2, and then consider the serial supply chain system, the assembly system, and two distribution systems depending on two different customer behaviors in Sec. 3, Sec. 4, and Sec. 5, respectively. We conclude the paper by some important observations in Sec. 6.

All technical proofs can be found in the Appendices.

2 Preliminaries

2.1 The centralized setting

For any given supply chain system, we imagine a centralized system facing a standard newsvendor problem where a single decision maker operates the entire supply chain. Without loss of generality, we assume the uncertain demand X follows a continuous distribution with probability density function f, cumulative probability function F and complementary cumulative probability function $\bar{F} = 1 - F(x)$ defined on support $[a,b]$ $(0 < a \leq b)$. Let r be the unit inbound cost and w.l.o.g. $p = 1$ be the normalized out-bound cost. Therefore, the imaginary decision-maker seeks to decide an inventory level x to maximize the profit $\Pi(x)$ of the entire supply chain:

$$\max_{a \leq x \leq b} \Pi(x) \equiv -rx + \mathbb{E}[\min\{x, X\}] = \max_{a \leq x \leq b} \left(-rx + \int_0^x \bar{F}(t)dt \right)$$

with optimal order quantity x^c equal to the upper support b (when $\bar{F}(x) > r$ for all $x \in [a, b]$, namely the objective function increases within $[a, b]$) or uniquely determined by:

$$\bar{F}(x^c) = r, \text{ for some } x^c \in [a, b].$$

2.2 The decentralized setting

The game-theoretic settings to be considered can be described by the following simple scenario with two players. There is a manufacturer producing a product with cost r per unit whose goal is to set the price of the product to some value w so that he maximizes his profit. There is also a retailer who buys the product from the manufacturer at price w per unit and sells it with price 1 per unit. The demand X for the product follows a probability distribution. Hence, the goal of the retailer is, given the price w of the product, to determine the inventory x that optimizes his expected profit, i.e., to determine x so that $-wx+\mathbb{E}[\min x, X]$ is maximized. Now, given the decision of the retailer for x, the manufacturer's optimal price is a value for w so that her profit $(w-r)x$ is maximized. When both the manufacturer and the retailer are profit maximizers, the total profit at equilibrium will be in general suboptimal. The PoA captures the profit loss due to the selfishness of the manufacturer and the retailer.

More complicated settings that generalize the one above will be considered in the sections to follow. They include those with more than two parties (Sec. 3) and with a tree-like structure (Sec. 4, and Sec. 5).

2.3 The formal definition of PoA

Before we formally define PoA, we note an essential difference between the classes $\mathcal{F}[a, b]$ and $\mathcal{F}_{\text{IGFR}}$, which poses some technical challenges in our analysis later. The price-only contract can be formulated as a multi-level mathematical program, where multiple optimal solutions (equilibria) may exist in the lower level problem for some parameter values, leading to ambiguity in the definition of the problem. To avoid this ambiguity, this work adopts the well-accepted *optimistic* approach in the multi-level programming literature (e.g., Dempe (2002)) with the economic interpretation that the follower is willing to support the leader, namely the follower will select, among all solutions optimal to himself, one that is best for the leader. Note that this is not a concern for the two-stage price-only contract under $\mathcal{F}[a, b]$ and two- or three-stage problem under $\mathcal{F}_{\text{IGFR}}$.

Throughout the rest of this paper, let x^c denote the optimal inventory level of the centralized system and x^d any inventory level of the decentralized system at equilibrium.

To capture the essence of the issues, we assume, where applicable, that $F(x)$ is smooth enough to ensure differentiability almost everywhere. For convenience we denote $\rho = b/a$ and

$$\alpha_F(x, y) := \int_x^y \bar{F}(t)dt, \ \forall\, 0 \le x \le y,$$

dropping off the subscript F whenever no confusion is caused. Let us formally define the price of anarchy (PoA) for a given price-only contract as follows:

Definition 1

$$\begin{aligned}\text{PoA} &= \sup_{F\in\mathcal{F}[a,b]} \frac{\Pi(x^c)}{\min_{x^d}\Pi(x^d)} = \sup_{F\in\mathcal{F}[a,b]} \frac{-rx^c + \alpha_F(0, x^c)}{\min_{x^d}\{-rx^d + \alpha_F(0, x^d)\}} \\ &= 1 + \sup_{F\in\mathcal{F}[a,b]} \min_{x^d} \frac{-r(x^c - x^d) + \alpha_F(x^d, x^c)}{-rx^d + \alpha_F(0, x^d)}.\end{aligned}$$

3 Serial supply chain

The organization of this section is as follows: we first describe the problem in Sec. 3.1, then present the exact PoA in Sec. 3.2, and finally utilize the uniform distribution to show detailed different behaviors between the classes $\mathcal{F}[a,b]$ and $\mathcal{F}_{\text{IGFR}}$ in Sec. 3.3.

3.1 Problem description

Let us label the stages of the decentralized supply chain in an increasing order from downstream to upstream: $1,\dots,n$. Each upstream stage i ($i=n,\dots,2$) as a leader offers a wholesale price w_{i-1} to his next downstream stage $i-1$ as a follower, who accepts his offer as long as his expected profit is non-negative. The price-only contract under this supply chain system can be formulated as an n-level optimization problem (refer to Fig. 1):

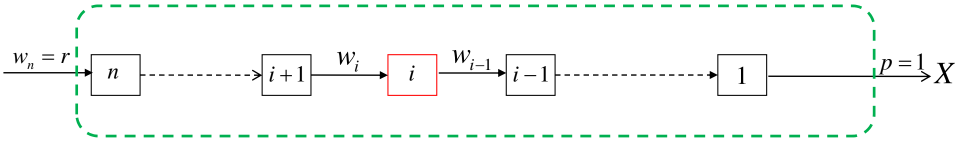

Figure 1: Decentralized multistage supply chain with the upstream stages as leaders

Level 1. Stage 1 as the retailer with given transferring price w_1 offered by Stage 2, faces the random customer demand X and chooses his order quantity x as inventory in such a way that his profit is to be maximized after selling the products to customers at a unit price of $p=1$:

$$\max_{a\le x\le b}\left(-w_1x+\int_0^x \bar{F}(t)dt\right).$$

Level i $(2\le i\le n-1)$. Stage i with a given transferring price w_i from his upstream stage, offers a transferring price w_{i-1} to his downstream stage $i-1$ in such a way that will maximize his profit, anticipating the order quantity x ($a\le x\le b$) from the downstream stage:

$$\max_{w_{i-1}:w_{i-1}\ge w_i}(w_{i-1}-w_i)x.$$

Level n. Stage n with the unit production cost r, offers a whole price w_{n-1} to his downstream stage $n-1$ to maximize his profit, anticipating the order quantity x ($a\le x\le b$) from the downstream stage:

$$\max_{w_{n-1}\ge r}(w_{n-1}-r)x.$$

Note that we adopt the *optimistic* approach in the multi-level programming literature (e.g., Dempe (2002)) to guarantee that this multi-level program is well-defined. Note also that the existence of a Stackelberg equilibrium is evident as all the optimization problems across the n levels are feasible with compact domain and continuous objectives. But multiple local optima and hence multiple equilibria may exist.

3.2 PoA

Denote $\ln^k\rho=(\ln\rho)^k$, where $\rho=b/a\ge 1$. The next result offers the exact PoA for this contract system along with graph illustration in Fig. 2.

Theorem 1 *The price of anarchy is given by*

$$\mathrm{PoA} = 1 + \sum_{k=1}^{n-1} \frac{1}{k!} \ln^k \rho.$$

The bound is achieved by the following worst distribution:

$$\bar{F}(x) = \begin{cases} 1, & 0 \le x \le a, \\ r + \frac{(1-r)a}{x} \sum_{k=0}^{n-2} \frac{1}{k!} \ln^k \left(\frac{x}{a}\right), & a < x \le b, \\ 0, & x > b. \end{cases}$$

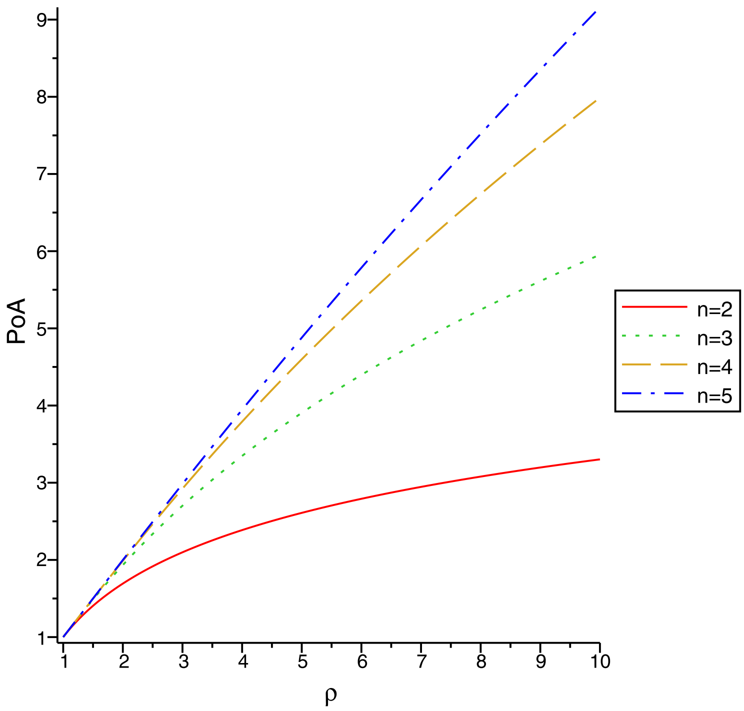

Figure 2: PoA in Theorem 1

3.3 Uniform distribution $U[a, b]$

We use the two-stage supply chain under the uniform distribution U$[a, b]$ to further illustrate the results obtained here. Moreover it also serves the purpose of showing the limitations of any worst-case analysis, namely, more accurate insights should be expected when more information about the demand distributions is available.

Under U$[a, b]$, we have $x^c = (1 - r)b + ra$ and

$$x^d = \begin{cases} a, & \text{if } (1-r)(\rho-1) \le 1, \\ \frac{(1-r)b+ra}{2}, & \text{if } (1-r)(\rho-1) > 1. \end{cases}$$

Recall that, for $\mathrm{U}[a,b]$,

$$\bar{F}(x) = \begin{cases} 1, & \text{if } x < a, \\ \frac{b-x}{b-a}, & \text{if } a \le x < b, \\ 0, & \text{if } x \ge b, \end{cases}$$

which implies that, for any $x \in [a,b)$,

$$\alpha(0,x) = a + \alpha(a,x) = a + \int_a^x \bar{F}(t)dt = a + \frac{(x-a)(2b-a-x)}{2(b-a)}.$$

Therefore, we can obtain the PoA as follows:

$$\mathrm{PoA}_{\mathrm{U}[a,b]} = \begin{cases} 1 + \frac{1}{2}(1-r)(\rho-1), & \text{if } (1-r)(\rho-1) \le 1, \\ 1 + \frac{1}{3-4/(1+(1-r)(\rho-1))^2}, & \text{if } (1-r)(\rho-1) > 1. \end{cases}$$

A few observations from this characterization (see Figure 3) are as follows:

Impacts of profit margin and relative range. Note that $\mathrm{PoA}_{\mathrm{U}[a,b]} \in [1, 3/2]$. The PoA depends on the profit margin $1-r$ and relative range ρ in a similar way: with increased values of $1-r$ or ρ, it increases initially and then decreases. The highest efficiency is attained at the lowest profit margin and relative range.

Closeness to $\mathbf{PoA}_{\mathcal{F}[a,b]}$. Figure 3 below illustrates the gap between the uniform bounds and the general PoA bound $\mathrm{PoA}_{\mathcal{F}[a,b]}$ when ρ varies.

Impact of the coefficient of variation (CV). Note that the CV of $\mathrm{U}[a,b)$ is given by

$$\mathrm{CV} = \frac{\sigma}{\mu} = \frac{\sqrt{1/12}(b-a)}{1/2(a+b)} = \frac{1}{\sqrt{3}}\frac{b-a}{b+a},$$

implying that

$$\rho - 1 = \frac{b}{a} - 1 = \frac{2\sqrt{3}}{1/\mathrm{CV} - \sqrt{3}}.$$

Therefore, ρ increases with CV, which implies that dependence on CV of the PoA for the uniform distribution is similar to that on ρ, i.e., with increase of CV, the PoA first increases and then decreases after CV goes beyond certain threshold.

4 Assembly supply chain

In such a system, the manufacturer produces a unit product at the cost of c_0 by assembling n components supplied by n upstream competitive suppliers with unit production cost c_i for supplier i ($i = 1, \ldots, n$), and sells the end product at unit price $p = 1$ (refer to Fig. 4).

The organization of this section is as follows: we first describe the problem in Sec. 4.1, then raise and address the issue of possible non-existence of an equilibrium under class $\mathcal{F}[a,b]$ in Sec. 4.2, and finally present the exact PoA in Sec. 4.3.

4.1 Problem description

Each supplier i ($1 \le i \le n$) as a leader offers a unit wholesale price w_i for his component to the manufacturer as the follower, who accepts the offer to produce for his inventory as long as his profit is nonnegative. The price-only contract under the system can be formulated as an bi-level optimization problem:

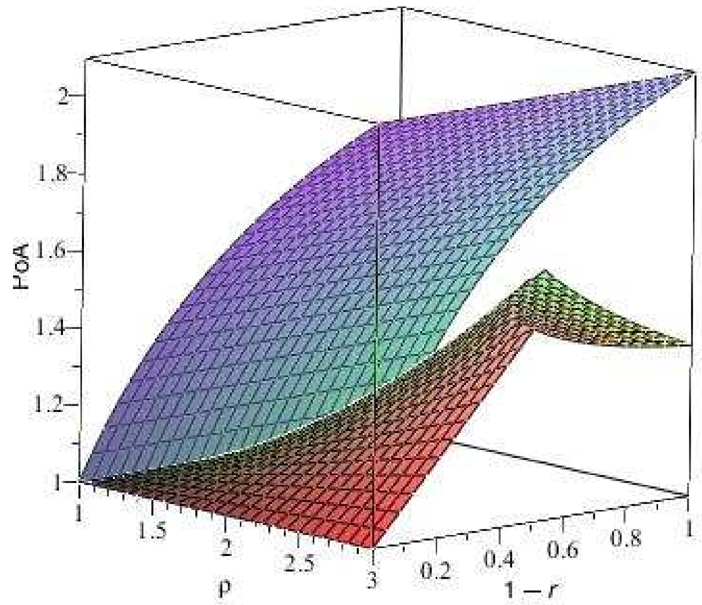

Figure 3: Comparison: the top surface $\text{PoA}_{\mathcal{F}[a,b]}$ vs. bottom surface $\text{PoA}_{\text{U}[a,b]}$

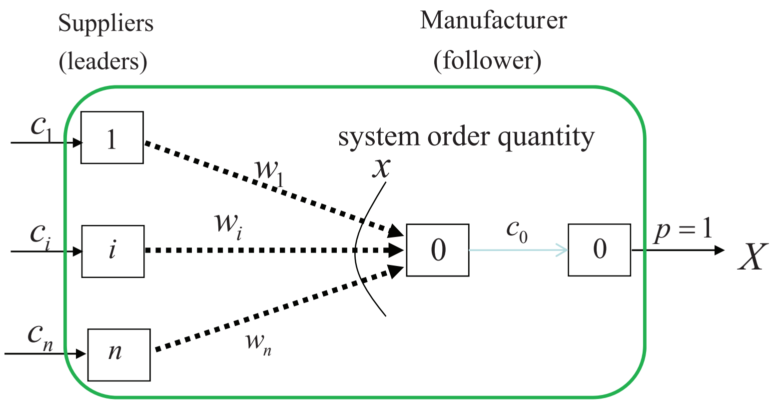

Figure 4: Decentralized assembly system with the suppliers (left) as leaders

Level 1. The manufacturer orders x units of each component (to be assembled to x units of end product) before observing the demand X to maximize his profit

$$\max_x \left(\mathbb{E}[\min\{x, X\}] - \left(\sum_{k=1}^{n} w_i + c_0 \right) x \right),$$

with optimal order quantity:

$$x = \bar{F}^{-1}\left(c_0 + \sum_{k=1}^{n} w_k\right).$$

Level 2. The suppliers choose their Nash equilibrium wholesale prices $w_1, \ldots, w_n$ to maximize their profits, anticipating the order quantity x from the manufacturer:

$$\begin{array}{ll} \max_{w_i} & \Pi_i(w_1, \ldots, w_n) \equiv (w_i - c_i)x, \;\; i = 1, \ldots, n \\ \text{s.t.} & x = \bar{F}^{-1}\left(c_0 + \sum_{k=1}^{n} w_k\right). \end{array}$$

4.2 Possible non-existence of an equilibrium

Unfortunately, unlike the other systems we consider, in the push assembly system here, the uniqueness of Nash equilibrium in the sub-games involved and, more seriously, the existence of Stackelberg equilibria in the decentralized supply chain *may not* be guaranteed for general class $\mathcal{F}[a,b]$ of demand distributions, as indicated in Appendix B, where a whole sub-class of distributions allows only those equilibria at which the assembler has zero profit in the supply chain, entailing that the assembler cannot remain in business! Gerchak and Wang (2004) require that, at an equilibrium, every partner in the supply chain system has a positive profit to make, so that all partners can remain in business and the supply chain system is sustainable. While we technically allow such an equilibrium in *the worst-case* (as we have seen in the previous sections), it is not clear at all whether an equilibrium will exist for *any* given distribution in class $\mathcal{F}[a,b]$. Therefore, in this section we are interested only in those distributions that guarantee the existence of an equilibrium.

4.3 PoA

The following result offers the exact PoA for the push assembly system along with the graph illustration in Fig. 5.

Theorem 2 *Let $r = \sum_{i=0}^{n} c_i$. The price of anarchy for all the demand distributions in class $\mathcal{F}[a,b]$ that permit existence of an equilibrium is given by*

$$PoA = \frac{1 + (n-1)\rho + \ln \rho}{n}.$$

The worst distribution is achieved by

$$\bar{F}(x) = \begin{cases} 1, & 0 \le x \le a, \\ 1 + \frac{1-r}{n}\left(\frac{a}{x} - 1\right), & a < x \le b, \\ 0, & x > b. \end{cases}$$

4.4 Uniform distribution $U[a,b]$

Analogously to the analysis in Sec. 3.3, under $\mathrm{U}[a,b]$ we obtain the PoA as follows:

$$\mathrm{PoA}_{\mathrm{U}[a,b]} = \begin{cases} 1 + \frac{1}{2}(1-r)(\rho-1), & \text{if } (1-r)(\rho-1) \le n, \\ 1 + \frac{n^2}{(2n+1) - \left(\frac{n+1}{1+(1-r)(\rho-1)}\right)^2}, & \text{if } (1-r)(\rho-1) > n. \end{cases}$$

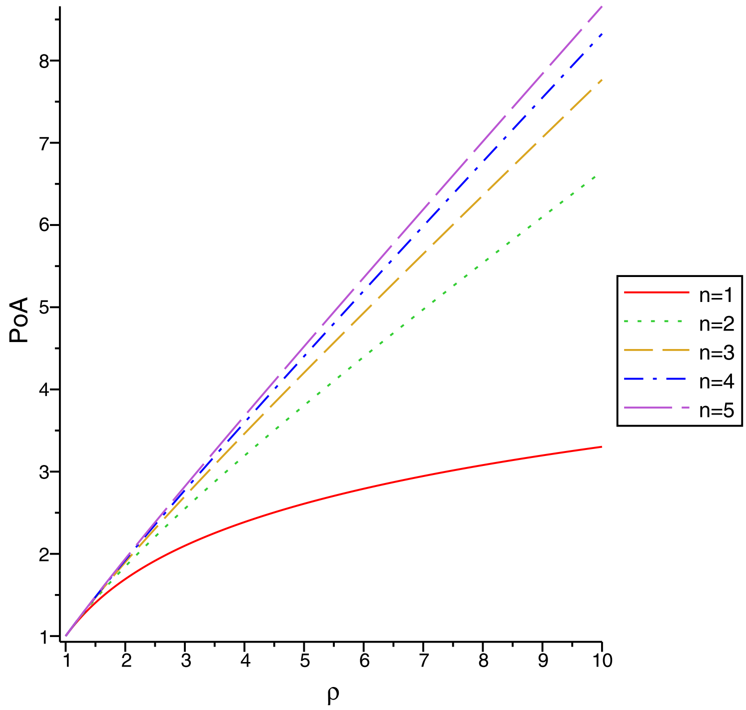

Figure 5: PoA in Theorem 2: PoA is almost linear in ρ for larger n

5 Competitive distribution system

In such a system, the manufacturer produces certain product at a unit cost of r, and n identical retailers, each with unit selling price $p = 1$, compete for the same aggregate demand X, which is allocated to the retailers according to some rule as specified in Lippman and McCardle (1997) or Cachon (2003) (refer to Fig. 6).

We distinguish between the herd-behaved customers and the proportionally-split customers respectively in Sec. 5.1 and 5.2.

5.1 Herd behavior of customers

5.1.1 Problem description

The manufacturer as the leader offers wholesale prices w_i $(i = 1, \ldots, n)$ to retailers as followers, who accept the offers for their inventories as long as their individual profits are nonnegative. Assume that aggregate demand X is allocated to the retailers according to the herd behavior of the customers (Lippman and McCardle, 1997), namely, these customers, randomly choosing an order $\pi = (\pi_1, \ldots, \pi_n)$ among retailers with equal probability among all permutations, visit the retailer one at a time until the total demand is met. The price-only contract under this supply chain systems can be formulated as a bi-level optimization problem:

Level 1. Each retailer i $(1 \leq i \leq n)$, facing the random demand allocated to him based on the herd behavior rule, decides his order quantity x_i at given wholesale price w_i to

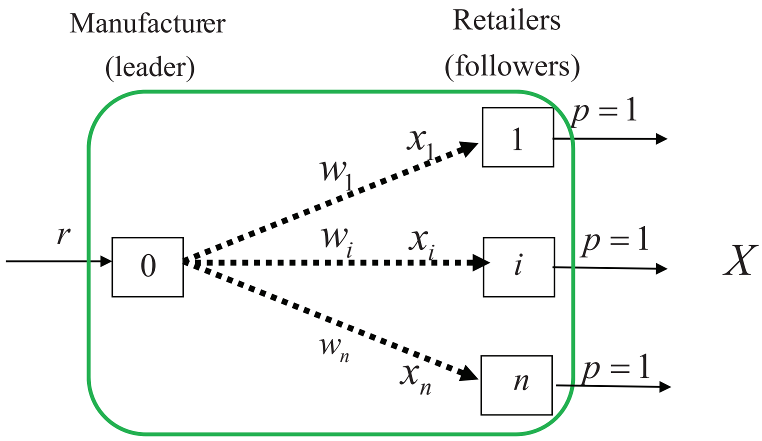

Figure 6: Decentralized push supply chain with the manufacturer (left) as the leader

maximize his profit:

$$\max_{x_i} \sum_{k=1}^{n} \sum_{\pi: i=\pi_k} \frac{1}{n!} \mathbb{E}\left[\min\left\{x_i, \left(X - \sum_{j>i} x_{\pi_j}\right)^+\right\}\right] - w_i x_i, \; i = 1, \ldots, n.$$

This subgame has a unique Nash equilibrium and it is symmetric (Lippman and McCardle, 1997). Due to symmetry because of the equi-probability of all permutations, the manufacturer offers the same wholesale price w to all retailers, each of whom orders the same quantity $y := x_i$ ($i = 1, \ldots, n$) at any Nash equilibrium. Therefore, each retailer faces the same optimization problem:

$$\max_{y} \sum_{k=1}^{n} \frac{1}{n} \mathbb{E}[\min\{y, (X - ky)^+\}] - wy$$

with optimal order quantity y for each retailer satisfying

$$\frac{1}{n} \sum_{k=1}^{n} \bar{F}(ky) = w.$$

Level 2. The manufacturer chooses the wholesale price w to maximize his profit, anticipating each retailer's order quantity y:

$$\begin{aligned} \max_{w} \quad & \sum_{k=1}^{n} (w - r) y \\ \text{s.t.} \quad & \frac{1}{n} \sum_{k=1}^{n} \bar{F}(ky) = w, \end{aligned}$$

or equivalently, with $x = ny$ being the system order quantity,

$$\max_{a \le x \le b} \left(\frac{1}{n} \sum_{k=1}^{n} \bar{F}\left(\frac{kx}{n}\right) - r \right) x.$$

5.1.2 PoA

Now we establish bounds on the PoA along with the graph illustration in Fig. 7.

Theorem 3 *The price of anarchy for the class $\mathcal{F}[a, b]$ has the following upper and lower bounds:*

$$(1 + \ln \rho)/(1 + \ln n) \leq PoA \leq 1 + \ln \rho.$$

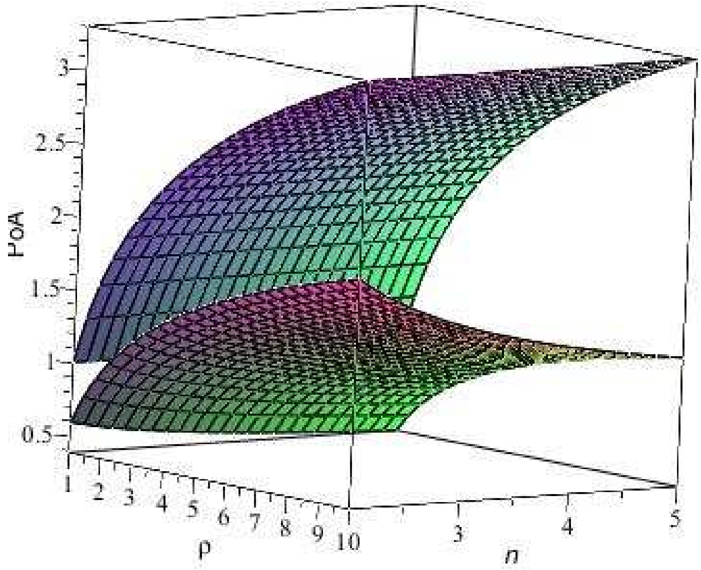

Figure 7: PoA bounds in Theorem 3: top surface is the upper bound and the bottom surface is the lower bound

5.1.3 Uniform distribution $U[a, b]$

Analogously to the analysis in Sec. 3.3, under U$[a, b]$ we obtain the PoA as follows:

$$\mathrm{PoA}_{\mathrm{U}[a,b]} = \begin{cases} 1 + \frac{1}{2}(1-r)\,(\rho-1)\,, & \text{if } (1-r)\,(\rho-1) \leq \frac{1}{n}, \\ 1 + \frac{1}{n(n+2) - \left(\frac{n+1}{1+(1-r)(\rho-1)}\right)^2}, & \text{if } (1-r)\,(\rho-1) > \frac{1}{n}. \end{cases}$$

Note that when $a = 0$ and hence $\rho \to \infty$, the second case above leads to PoA $= (n+1)^2/(n(n+2))$ under U$[0, b)$. We can derive similar insights as those in Sec. 3.3.

5.2 Splitting customers

5.2.1 Problem description

Assume that the aggregate demand X is allocated to the retailers in proportion to their inventory levels and there is no reallocation of the unmet demand (Cachon, 2003). The manufacturer as the leader offers wholesale price w to all retailers as followers, who accept the offer as long as his profit is nonnegative. The price-only contract in this supply chain systems can be formulated as a bi-level optimization problem (refer to Fig. 6):

Level 1. Each retailer i ($1 \le i \le n$), facing the random demand proportionally allocated to him, decides his order quantity x_i at wholesale price w to maximize his profit:

$$\max_{x_i} \left(\mathbb{E}\left[\min\left\{ x_i, \frac{x_i}{x} X \right\} \right] - w x_i \right) = \frac{x_i}{x} \int_0^x \bar{F}(t) dt - w x_i, \ i = 1, \dots, n,$$

where $x = \sum_{k=1}^n x_k$. This subgame has a unique Nash equilibrium, which is symmetric (Cachon, 2003).

Level 2. The manufacturer chooses the wholesale price w to maximize his profit, anticipating retailers' order quantity $x_1, \dots, x_n$:

$$\begin{aligned} \max_{w} \quad & \sum_{i=1}^n (w - r) x_i \\ \text{s.t.} \quad & x_i = \arg\max_{x_i} \mathbb{E}\left[\min\left\{ x_i, \frac{x_i}{x} X \right\} \right] - w x_i, \ i = 1, \dots, n. \end{aligned}$$

5.2.2 PoA

The following result offers the exact PoA for the system along with the graph illustration in Fig. 8.

Theorem 4 *The price of anarchy is given by*

$$PoA = 1 + \frac{1 - \rho^{1-n}}{n - 1}.$$

The worst distribution is achieved by

$$\bar{F}(t) = \begin{cases} 1, & 0 \le t < a, \\ r - \frac{1-r}{n-1} \left(\frac{a}{t} \right)^n, & a \le t < b, \\ 0, & t \ge b. \end{cases}$$

5.2.3 Uniform distribution

The result is exactly the same as that in Sec. 5.1.3.

6 Concluding remarks

We have extended the application of the PoA analysis in supply chain management. Our results have revealed some new performance behavior of the price-only contract in various supply chain systems and hence deepened our understanding of it. The following observations follow readily from our analysis:

1. The bounds derived in this paper are independent of costs, prices and the boundaries a and b of the demand distribution support. In particular,
   (a) The bounds in the present work do not depend on upstream supply costs. This property is an attractive feature in environments of fluctuating commodity prices.

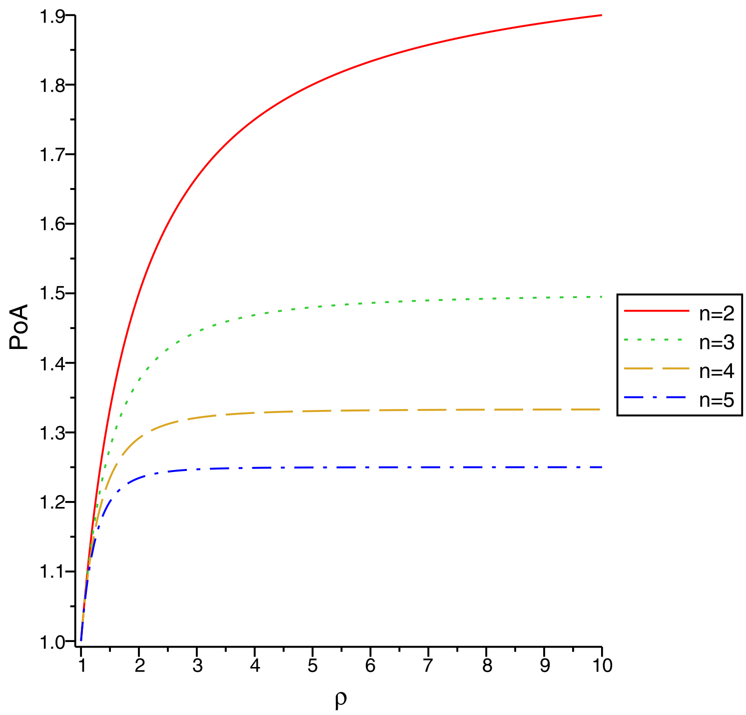

Figure 8: PoA in Theorem 4

(b) It is also significant that the bounds are independent of retail prices, individual values of a and b. Instead, the bounds only depend on their ratio $\rho = b/a$.

(c) Moreover, in the assembly setting where multiple equilibria can exist in the absence of the IGFR property, our PoA bounds serve the purpose to necessitate the need of coordinating contracts in settings where ambiguity surrounds the demand distribution, cost/price parameters, and/or the particular equilibrium reached.

2. One of the major contributions of this work is the identification of the *relative range* $\rho = b/a$, a measure of uncertainty, as a pivotal parameter in that the PoA obtained usually improves with reduced fluctuation ratio, although the exact analytical formula for PoA is highly nontrivial. Intuitively, on the one hand, when there is demand certainty, namely $\rho = 1$, it should be clear that both the decentralized and centralized solution will be to order the exact demand, leading to perfect coordination with PoA $= 1$. On the other hand, when there is uncertainty in the demand, namely, $\rho > 1$, the decentralized solution can be forced to order a while the centralized solution is to order b in the worst case.

3. Our analysis under $\mathcal{F}[a, b]$ in this work shows that the price-only contract actually maintains its efficiency when the demand uncertainty, measured by ratio $\rho = b/a$, is relatively low, entailing the price-only contract to be more attractive in this regard than those administratively more expensive contracts as those considered, e.g., by Chen et al. (2012), Jörnsten et al. (2013) and Palsule-Desai (2013), and hence justifying the efforts in demand forecasting to reduce uncertainties. Moreover, it actually offers a deeper reason on the wide acceptance and popularity of the price-only contract in

many real-life practices, besides its low administrative cost.

This insight of efficiency improvement with decrease of ρ is *empirically* and *qualitatively* intuitive, given that the main source of double marginalization is demand uncertainty (Lariviere and Porteus (2001)). However, to the best of our knowledge, our work here is the first one to *theoretically quantify* this effect, achieved by the introduction of an uncertainty measure, ρ.

4. Our analysis also shows that worst-case PoAs under $\mathcal{F}[a, b]$ and $\mathcal{F}_{\mathrm{IGFR}}$ are complementary in the following sense: the former is in general independent of the profit margin $1 - r$, while the latter is in general increasing in the profit margin.

Note that the aforementioned insights are obtained through worst-case analysis and for a given demand distribution other than the worse-case distribution. The PoA may not perform as described above (see Sec. 3.3, Sec. 4.4, Sec. 5.1.3, and Sec. 5.2.3 for the exact PoA under the uniform distribution, for which the PoA behaves differently from the worst-case situation). Therefore, one should exercise caution and care when applying the observations based on worst-case analysis to a given demand distribution other than the worst-case distributions.

An apparent open problem is to find a tight bound for the distribution system with herd behavior (Theorem 3). Moreover, due to the practical importance of the price-only contract, an investigation of other demand classes would be important for identifying the situations where the price-only contract is relatively efficient.

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Appendix A: Technical proofs

A.1 Proof for Theorem 1

We need two lemmas first. Define the following functions iteratively:

$$\begin{cases} m_1(x) = \bar{F}(x), \\ m_i(x) = (x m_{i-1}(x))', i = 2, \ldots, n-1. \end{cases}$$

Here the prime operator $'$ is the standard derivative in calculus.

Lemma 1 *If the decentralized equilibrium inventory level x^d is an interior solution of the n-level optimization problem with w_i being the corresponding equilibrium transfer price at stage i ($i = 1, \ldots, n-1$), then*

$$w_i = m_i(x^d), \ i = 1, \ldots, n-1, \tag{1}$$

implying that

$$m_1(x^d) \geq \cdots \geq m_{n-1}(x^d). \tag{2}$$

Proof of Lemma 1. We prove equations (1) by induction. For $i = 1$, the optimization problem at stage 1 is:

$$\max_{a \leq \xi \leq b} \left(-w_1 \xi + \int_0^{\xi} \bar{F}(t) dt \right).$$

So, if the optimal solution x is interior, then it satisfies the first-order condition $w_1 = \bar{F}(x) = m_1(x)$.

Assume that the equation in (1) is correct for $i = \ell - 1$ ($\ell \geq 2$), namely, the optimal interior solution x of the optimization problem at stage $\ell - 1$ satisfies $w_{\ell-1} = m_{\ell-1}(x)$.

Consider the optimization problem at stage ℓ:

$$\max_{w_{\ell-1}:\, w_{\ell-1} \geq w_\ell} (w_{\ell-1} - w_\ell)x.$$

If optimal $w_{\ell-1}$ results in optimal interior x, then according to the induction hypothesis, the above optimization problem is equivalent to:

$$\max_{a<\xi<b} (m_{\ell-1}(\xi) - w_\ell)\xi.$$

Therefore, the optimal solution x satisfies the first-order condition:

$$w_\ell = m_{\ell-1}(x) + x m'_{\ell-1}(x) = m_\ell(x),$$

which completes our induction. $\square$

Denote the following quantities:

$$\beta_i = x^d(m_i(x^d) - r),\ i = 1, \ldots, n-1.$$

Then from (2) in Lemma 1, we have

$$\beta_1 \geq \cdots \geq \beta_{n-1}. \tag{3}$$

Lemma 2 *Assume that the decentralized equilibrium inventory level x^d is an interior solution to the n-level optimization problem. Then, for any $x \geq x^d$, we have*

$$m_{n-i}(x) \leq r + \frac{1}{x}\sum_{k=0}^{i-1} \beta_{n-i+k} \frac{1}{k!} \ln^k\left(\frac{x}{x^d}\right),\ i = 1, \ldots, n-1,$$

implying with (3) that

$$\begin{aligned}\int_{x^d}^{x} m_{n-i}(\xi)d\xi &\leq r(x - x^d) + \sum_{k=1}^{i} \beta_{n-i-1+k} \frac{1}{k!} \ln^k\left(\frac{x}{x^d}\right) \\ &\leq r(x - x^d) + \beta_1 \sum_{k=1}^{i} \frac{1}{k!} \ln^k\left(\frac{x}{x^d}\right),\ i = 1, \ldots, n-1.\end{aligned}$$

Proof of Lemma 2. We prove the lemma by induction on i. For $i = 1$, the optimization problem at stage n is

$$\max_{w_{n-1} \leq 1} (w_{n-1} - r)x.$$

Since optimal w_{n-1} (i.e., w^*_{n-1}) results in an interior x^d, according to equations (1), the above problem is equivalent to

$$\max_{a<\xi<b} (m_{n-1}(\xi) - r)\xi,$$

implying that

$$\beta_{n-1} \geq (m_{n-1}(\xi) - r)\xi,\ \forall a < \xi < b,$$

or equivalently

$$m_{n-1}(\xi) \leq r + \beta_{n-1}\frac{1}{\xi},\ \forall a < \xi < b.$$

Therefore,

$$\int_{x^d}^{x} m_{n-1}(\xi)d\xi \le r(x-x^d) + \beta_{n-1}\ln\left(\frac{x}{x^d}\right),$$

which implies the basis step in the induction. Assume that the desired result is correct for $i=\ell-1$, that is, we have the inductive hypothesis,

$$\int_{x^d}^{x} m_{n-\ell+1}(\xi)d\xi \le r(x-x^d) + \sum_{k=1}^{\ell-1}\beta_{n-\ell+k}\frac{1}{k!}\ln^k\left(\frac{x}{x^d}\right).$$

Note that the LHS of the above inequality is equal to

$$\int_{x^d}^{x} m_{n-\ell+1}(\xi)d\xi = \int_{x^d}^{x}(\xi m_{n-\ell}(\xi))'d\xi = xm_{n-\ell}(x) - x^d m_{n-\ell}(x^d).$$

Therefore,

$$xm_{n-\ell}(x) - x^d m_{n-\ell}(x^d) \le r(x-x^d) + \sum_{k=1}^{\ell-1}\beta_{n-\ell+k}\frac{1}{k!}\ln^k\left(\frac{x}{x^d}\right),$$

or equivalently

$$m_{n-\ell}(x) \le r + \frac{1}{x}\left(\beta_{n-\ell} + \sum_{k=1}^{\ell-1}\beta_{n-\ell+k}\frac{1}{k!}\ln^k\left(\frac{x}{x^d}\right)\right),$$

which, by integration, implies

$$\int_{x^d}^{x} m_{n-\ell}(\xi)d\xi \le r(x-x^d) + \sum_{k=1}^{\ell}\beta_{n-\ell-1+k}\frac{1}{k!}\ln^k\left(\frac{x}{x^d}\right).$$

This completes the inductive step. □

Taking $i=n-1$ and $x=x^c$ in Lemma 2, we obtain

$$\begin{aligned}\alpha(x^d,x^c) &\le r(x^c-x^d) + \beta_1\sum_{k=1}^{n-1}\frac{1}{k!}\ln^k\left(\frac{x^c}{x^d}\right)\\ &= r(x^c-x^d) + x^d\left(\bar{F}(x^d)-r\right)\sum_{k=1}^{n-1}\frac{1}{k!}\ln^k\left(\frac{x^c}{x^d}\right),\end{aligned} \tag{4}$$

where x^d is interior: $a < x^d < b$.

Proof of Theorem 1

We consider three cases depending on whether the decentralized solution x^d of the n-level optimization problem is achieved at the upper support, or an interior point, or the lower support. If $x^d = b$, we have PoA = 1, since $x^d \le x^c \le b$ implying that $x^d = x^c = b$.

Now assume $x^d = a$. Recall that any optimal solution x for the optimization problem at stage 1 satisfies the first order condition:

$$w_1 = \bar{F}(x).$$

Consider the optimization problem at stage 2:

$$\max_{w_1}\left((w_1-w_2)x(w_1) : w_1 = \bar{F}(x) \ge w_2, a \le x \le b\right),$$

or equivalently

$$\max_x \left((\bar{F}(x) - w_2)x : \bar{F}(x) \geq w_2, a \leq x \leq b\right).$$

That $x^d = a$ being the equilibrium solution for the entire system implies that there exists a $\bar{w}_2 \in [r, 1]$ such that the optimal objective value at stage 2 is $(1 - \bar{w}_2)a \leq (1 - r)a$. Fix $w_2 = r$ in the above optimization problem at stage 2. Then the optimal value should be no more than $(1 - r)a$, implying that

$$(\bar{F}(x) - r)x \leq (1 - r)a, \ \forall x \geq a,$$

or equivalently

$$\bar{F}(x) \leq r + \frac{(1 - r)a}{x}, \ \forall x \geq a,$$

which in turn implies that

$$\alpha(a, x^c) \leq r(x^c - a) + (1 - r)a \ln\left(\frac{x^c}{a}\right),$$

making inequality (4) valid also for $x^d = a$.

Therefore, we assume $a \leq x^d < b$ with satisfaction of inequality (4), from which and the following inequality directly implied by the monotonicity of function $\bar{F}$,

$$\inf_{F \in \mathcal{F}[a,b]} \alpha_F(0, x^d) \geq x^d \bar{F}(x^d), \forall \ x^d \geq 0, \tag{5}$$

we obtain

$$\begin{aligned}
\frac{\Pi(x^c)}{\Pi(x^d)} &\leq 1 + \frac{-r(x^c - x^d) + \alpha(x^d, x^c)}{-rx^d + \alpha(0, x^d)} \\
&\leq 1 + \frac{x^d \left(\bar{F}(x^d) - r\right) \sum_{k=1}^{n-1} \frac{1}{k!} \ln^k \left(\frac{x^c}{x^d}\right)}{x^d(\bar{F}(x^d) - r)} = 1 + \sum_{k=1}^{n-1} \frac{1}{k!} \ln^k \left(\frac{x^c}{x^d}\right),
\end{aligned}$$

implying that

$$\text{PoA} \leq 1 + \sum_{k=1}^{n-1} \frac{1}{k!} \ln^k \rho.$$

Now we show the bound is tight. Under the worst distribution, we have that

$$\alpha(a, x) = \int_a^x m_1(t)dt = r(x - a) + (1 - r)a \sum_{k=1}^{n-1} \frac{x^k}{k!},$$

which implies that $x^c = b$ as the objective of the centralized system

$$\Pi(x) = -rx + a + \int_a^x m_1(\xi)d\xi = (1 - r)a \sum_{k=0}^{n-1} \frac{x^k}{k!},$$

is an increasing function.

On the other hand, x^d can be any value within $[a, b)$ because the objective at stage n in the decentralized system:

$$(m_{n-1}(t) - r)t = (1 - r)a$$

is a constant throughout. So for this worst distribution:

$$\sup_{x^d} \frac{\Pi(x^c)}{\Pi(x^d)} = \frac{(1 - r)a \sum\limits_{k=0}^{n-1} \frac{x^i}{k!}}{(1 - r)a} = 1 + \sum_{k=1}^{n-1} \frac{x^k}{k!}.$$

A.2 Proof for Theorem 2

Lemma 3 *Let* $r = \sum_{i=0}^{n} c_i$*. Then*

$$\alpha(x^d, x^c) \le (x^c - x^d)\left(r + \frac{n-1}{n}(\bar{F}(x^d) - r)\right) + (\bar{F}(x^d) - r)\frac{x^d}{n}\ln\left(\frac{x^c}{x^d}\right).$$

Proof of Lemma 3. At Nash Equilibrium, the utility $\Pi_i(w_1, \dots, w_n)$ of each supplier i $(i = 1, \dots, n)$ is maximized with respect to w_i. Assume that the Nash Equilibrium is $(w_1^*, \dots, w_n^*)$ with the corresponding order quantity x^d. Hence, for any $x \in [a, b]$, we have

$$\left(\bar{F}(x^d) - r - \sum_{k \ne i}(w_k^* - c_k)\right) x^d \ge \left(\bar{F}(x) - r - \sum_{k \ne i}(w_k^* - c_k)\right) x.$$

Summation for $i = 1, \dots, n$ leads to

$$\left(\bar{F}(x^d) - r\right) x^d \ge \left(n(\bar{F}(x) - r) - (n-1)(\bar{F}(x^d) - r)\right) x,$$

or equivalently:

$$\bar{F}(x) \le r + \frac{\bar{F}(x^d) - r}{n}\left(n - 1 + \frac{x^d}{x}\right),$$

which, after integration, gives

$$\alpha(x^d, x^c) \le (x^c - x^d)\left(r + \frac{n-1}{n}(\bar{F}(x^d) - r)\right) + (\bar{F}(x^d) - r)\frac{x^d}{n}\ln\left(\frac{x^c}{x^d}\right).$$

□

Proof of Theorem 2

From (5) and Lemma 3, we obtain

$$\begin{aligned}\frac{\Pi(x^c)}{\Pi(x^d)} - 1 \quad &\le \quad \frac{(x^c - x^d)\frac{n-1}{n}(\bar{F}(x^d) - r) + (\bar{F}(x^d) - r)\frac{x^d}{n}\ln\left(\frac{x^c}{x^d}\right)}{x^d(\bar{F}(x^d) - r)} \\ &= \quad \frac{n-1}{n}\left(\frac{x^c}{x^d} - 1\right) + \frac{1}{n}\ln\left(\frac{x^c}{x^d}\right),\end{aligned}$$

implying that

$$\text{PoA} \le 1 + \frac{n-1}{n}(\rho - 1) + \frac{1}{n}\ln\rho = \frac{1 + (n-1)\rho + \ln\rho}{n}.$$

Under the worst distribution, we have that

$$\alpha(a, x) = (x - a)\left(r + \frac{n-1}{n}(1 - r)\right) + (1 - r)\frac{a}{n}\ln\frac{x}{a},$$

implying that $x^c = b$, because the objective of the centralized system

$$\Pi(x) = -rx + a + \alpha(a, x) = (1 - r)a + \frac{(1 - r)a}{n}\left(1 + (n-1)\frac{x}{a} + \ln\frac{x}{a}\right)$$

is an increasing function.

It can be further shown that $w_i^d = c_i + \frac{1-r}{n}$ $(i = 1, \ldots, n)$ form an Nash equilibrium, and x^d can be any value within $[a, b]$, because the objective for each supplier i at w_i^d

$$\begin{aligned}(w_i - c_i)x &= (\bar{F}(x) - r - \sum_{k \neq i}(w_k^d - c_k))x \\ &= \left((1-r)\left(1 + \frac{1}{n}\left(\frac{a}{x} - 1\right)\right) - \frac{n-1}{n}(1-r)\right)x = \frac{(1-r)a}{n}\end{aligned}$$

is a constant, implying that $x^d = a$ is an Nash equilibrium. So for this worst distribution:

$$\sup_{x^d} \frac{\Pi(x^c)}{\Pi(x^d)} = \frac{1 + (n-1)\rho + \ln \rho}{n}.$$

A.3 Proof for Theorem 3

We first prove the following two lemmas.

Lemma 4 *Fix $\bar{F}(x) = \phi$ $(a \leq x \leq b)$. Then*

$$\inf_{F \in \mathcal{F}[a,b]} \int_0^x \bar{F}(t)dt = a + (x - a)\phi.$$

Hence,

$$PoA = 1 + \sup_{\substack{F \in \mathcal{F}[a,b] \\ \bar{F}(x^d) = \phi}} \frac{\alpha(x^d, x^c) - r(x^c - x^d)}{(\phi - r)x^d + (1 - \phi)a}. \tag{6}$$

The following parameterized distribution is a minimizing distribution with $\epsilon \to 0$:

$$\bar{F}_\epsilon(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq a + (x-a)\phi^{1/\epsilon}, \\ \phi(x-a)^\epsilon (t-a)^{-\epsilon}, & \text{if } a + (x-a)\phi^{1/\epsilon} < t \leq x. \end{cases}$$

Proof of Lemma 4. The integration concerned is equal to

$$a - \frac{\epsilon}{1-\epsilon}(x-a)\phi^{1/\epsilon} + \frac{1}{1-\epsilon}\phi(x-a),$$

which approaches $a + (x-a)\phi$ as $\epsilon \to 0$. The equation for PoA above follows from Definition 1. □

This lemma basically says that restricting on the minimizing distributions above will not preclude any distribution that maximizes the PoA containing parameters x^d, x^c and ϕ.

Lemma 5 *For any given $\delta > 0$, let $F(\cdot) = F_\epsilon(\cdot)$ be a worst distribution as given in Lemma 4 for some $\epsilon > 0$ sufficiently small. Denote $\phi = \bar{F}(x^d)$ and the manufacturer's utility function*

$$\Pi_1(Q) = \left(\frac{1}{n}\sum_{k=1}^{n} \bar{F}(kQ/n) - r\right)Q.$$

Then $\Pi_1(x^d) \leq (\phi - r)x^d + (1 - \phi)a + \delta$.

Proof of Lemma 5. If $x^d = a$, then $\phi = 1$ and $\Pi_1(x^d) = (1-r)x^d$, implying that the claimed inequality holds. If $na < x^d$ and $\epsilon > 0$ is sufficiently small, then $\Pi_1(x^d) = (\phi - r)x^d$ according to the definition of $F(\cdot)$. Therefore, without loss of generality, we assume $a_{\ell-1} < x^d \leq a_\ell$ for some ℓ with $1 \leq \ell \leq n-1$, where

$$a_i = \frac{n}{n-i}a, \ i = 0, 1, \ldots, n-1. \tag{7}$$

Consequently, when $\epsilon > 0$ is sufficiently small, we have

$$\begin{aligned}\Pi_1(x^d) &= \left(\frac{(n-\ell)+\ell\phi}{n} - r\right)x^d = \left((\phi - r) + \frac{n-\ell}{n}(1-\phi)\right)x^d \\ &= (\phi - r)x^d + (1-\phi)ax^d/a_\ell,\end{aligned}$$

which leads to our desired result due to $x^d \le a_\ell$. $\square$

Proof of Theorem 3

According to the definition,

$$\Pi_1(x^d) \ge \Pi_1(x) = \left(\frac{1}{n}\sum_{k=1}^{n} \bar{F}(kx/n) - r\right)x, \text{ for any } x \in [a, b].$$

Therefore, since $kx/n \le x$ for all $k = 1, \dots, n$ and at least one of the inequalities is strict, with strict monotonicity of $\bar{F}(\cdot)$ we have

$$\Pi_1(x^d) > (\bar{F}(x) - r)x, \text{ or } \bar{F}(x) < r + \frac{\Pi_1(x^d)}{x} \text{ for any } x \in (a, b].$$

Taking integration for the above from x^d to x^c, we obtain

$$\int_{x^d}^{x^c} \bar{F}(x)dx < r(x^c - x^d) + \Pi_1(x^d)\ln\left(\frac{x^c}{x^d}\right),$$

which (with strict inequality) together with the fact that $x^c/x^d \le \rho$ implies that, for some $\delta > 0$,

$$\int_{x^d}^{x^c} \bar{F}(x)dx \le r(x^c - x^d) + (\Pi_1(x^d) - \delta)\ln\rho.$$

Now, using Equation (6) we obtain

$$\text{PoA} \le 1 + \frac{(\Pi_1(x^d) - \delta)\ln\rho}{(\phi - r)x^d + (1-\phi)a},$$

which implies the upper bound in our theorem with Lemma 5.

To show the lower bound, we consider the following distribution with the assumption that $\rho > n$:

$$\bar{F}(x) = \begin{cases} 1, & 0 \le x < a, \\ r + \frac{(1-r)a}{x}, & a \le x < b, \\ 0, & x \ge b. \end{cases}$$

The global utility function is

$$\Pi(x) = -rx + \int_0^x \bar{F}(t)dt = a(1-r)(1 + \ln(x/a)), \tag{8}$$

which is maximized at $x^c = b$. On the other hand, if $\rho > n$, i.e., $b > na$, then with notation $a_n = b$ in addition to (7), the utility function of the manufacturer is: for any $a_{\ell-1} < x \le a_\ell$ $(1 \le \ell \le n)$,

$$\begin{aligned}\Pi_1(x) &= \left(\frac{n-\ell}{n} + \frac{1}{n}\sum_{k=n-\ell+1}^{n} \bar{F}(kx/n) - r\right)x \\ &= (1-r)\left(\frac{n-\ell}{n} + \frac{a}{x}\sum_{k=n-\ell+1}^{n} 1/k\right)x \\ &= (1-r)\left(\frac{n-\ell}{n}x + a(H_n - H_{n-\ell})\right),\end{aligned}$$

where $H_n = \sum_{k=1}^n 1/k$ denotes the nth harmonic number. For any $\ell = 1, \ldots, n-1$, the above is maximized uniquely at a_ℓ to $a(1-r)(1+H_n - H_{n-\ell}) \leq (1-r)aH_n$, the constant that $\Pi_1(x)$ is always equal to while $\ell = n$. Therefore, if $b > na$, we can have $x^d = na$, which together with (8) implies that

$$\text{PoA} \geq \frac{\Pi(b)}{\Pi(na)} = \frac{1+\ln\rho}{1+\ln n}.$$

A.4 Proof for Theorem 4

Lemma 6 *For any $t \geq 0$,*

$$\alpha(0,t) \leq rt + (1-r)a\left(\frac{a}{t}\right)^{n-1} + \left(1 - \left(\frac{a}{t}\right)^{n-1}\right)\left(\alpha(0,x^d) + \frac{x^d}{n-1}(\bar{F}(x^d) - nr)\right).$$

Proof of Lemma 6. For the subgame played by the n retailers, the profit function of retailer i is given by:

$$\Pi_i(x_1, \ldots, x_n) = -wx_i + \frac{x_i}{x}\int_0^x \bar{F}(y)dy, \text{ where } x = \sum_{i=1}^n x_i.$$

Any Nash equilibrium satisfies the first-order condition:

$$\frac{\partial \Pi_i(x)}{\partial x_i} = -w + \frac{1}{x}\int_0^x \bar{F}(y)dy + x_i \frac{x\bar{F}(x) - \int_0^x \bar{F}(y)dy}{x^2} = 0,\ i = 1, \ldots, n.$$

Due to symmetry: $x = nx_i$ $(i = 1, \ldots, n)$, the above is reduced to a simple equation:

$$w = \frac{1}{x}\int_0^x \bar{F}(y)dy + \frac{x}{n}\frac{x\bar{F}(x) - \int_0^x \bar{F}(y)dy}{x^2} = \frac{1}{x}\int_0^x \bar{F}(y)dy + \frac{1}{n}\frac{x\bar{F}(x) - \int_0^x \bar{F}(y)dy}{x}.$$

Therefore, the objective of the manufacturer's problem becomes:

$$\begin{aligned} \Pi_0(x) &\equiv (w-r)x = \left(\frac{1}{x}\int_0^x \bar{F}(y)dy + \frac{1}{n}\frac{x\bar{F}(x) - \int_0^x \bar{F}(y)dy}{x} - r\right)x \\ &= \left(1 - \frac{1}{n}\right)\alpha(0,x) + \frac{1}{n}x\alpha'(0,x) - rx, \end{aligned}$$

which implies

$$\Pi_0(x^d) \geq \left(1 - \frac{1}{n}\right)\alpha(0,x) + \frac{1}{n}x\alpha'(x) - rx,\ \forall a \leq x < x^c,$$

or equivalently

$$-\left(\alpha(0,x) - rx - \frac{n\Pi_0(x^d)}{n-1}\right) \geq \frac{x}{n-1}(\alpha'(0,x) - r) > 0,\ \forall a \leq x < x^c.$$

Therefore,

$$\frac{-(\alpha'(0,x) - r)}{-\left(\alpha(0,x) - rx - \frac{n\Pi_0(x^d)}{n-1}\right)} \geq \frac{1-n}{x},\ \forall x,$$

or equivalently

$$\left(\ln\left[-\left(\alpha(0,x) - rx - \frac{n\Pi_0(x^d)}{n-1}\right)\right]\right)' = \frac{-\left(\alpha(0,x) - rx - \frac{n\Pi_0(x^d)}{n-1}\right)'}{-\left(\alpha(0,x) - rx - \frac{n\Pi_0(x^d)}{n-1}\right)} \leq \frac{1-n}{x},\ \forall x.$$

Integrating on both sides from a to t and substituting $\gamma(x^d)$ back leads to the desired result.
□

Proof of Theorem 4

Denote $w = \left(\frac{a}{x^c}\right)^{n-1}$.

$$\frac{\Pi(x^c)}{\Pi(x^d)} \le 1 + \frac{-r(x^c - x^d) + \alpha(0, x^c) - \alpha(0, x^d)}{-rx^d + \alpha(0, x^d)}.$$

Note that the numerator above can be bounded above as follows:

$$\begin{aligned}
-r(x^c - x^d) \quad &- \quad \alpha(0, x^d) + \alpha(0, x^c) \\
&\le \quad rx^d - \alpha(0, x^d) + (1-r)aw + (1-w)\left(\alpha(0, x^d) + \frac{x^d}{n-1}(\bar{F}(x^d) - nr)\right) \\
&= \quad rx^d + (1-r)aw + (1-w)\left(\frac{x^d}{n-1}(\bar{F}(x^d) - nr)\right) - w\alpha(0, x^d) \\
&\le \quad rx^d + (1-r)aw + (1-w)\left(\frac{x^d}{n-1}(\bar{F}(x^d) - nr)\right) - wx^d\bar{F}(x^d) \\
&= \quad (1-r)aw + \frac{1-nw}{n-1}x^d(\bar{F}(x^d) - r) \\
&\le \quad x^d(\bar{F}(x^d) - r)w + \frac{1-nw}{n-1}x^d(\bar{F}(x^d) - r) \\
&= \quad \frac{1-w}{n-1}x^d(\bar{F}(x^d) - r),
\end{aligned}$$

where the first inequality follows from Lemma 6, the second inequality follows from (5), and the third inequality follows from the optimality of x^d. Moreover the denominator above is bounded from below as follows from (5):

$$-rx^d + \alpha(0, x^d) \ge x^d(\bar{F}(x^d) - r).$$

Therefore,

$$\frac{\Pi(x^c)}{\Pi(x^d)} - 1 \le \frac{1-w}{n-1} = \frac{1 - \left(\frac{a}{x^c}\right)^{n-1}}{n-1},$$

implying that

$$\frac{\Pi(x^c)}{\Pi(x^d)} \le 1 + \frac{1 - \left(\frac{a}{b}\right)^{n-1}}{n-1}.$$

Under the worst distribution, we have that

$$\begin{aligned}
\alpha(0, x) \quad &= \quad rx + \frac{n(1-r)a}{n-1} + \left((1-r)a - \frac{n(1-r)a}{n-1}\right)\left(\frac{a}{x}\right)^{n-1} \\
&= \quad rx + \frac{(1-r)a}{n-1}\left(n - \left(\frac{a}{x}\right)^{n-1}\right),
\end{aligned}$$

implying that $x^c = b$, because the objective of the centralized system

$$\Pi(x) = -rx + \alpha(0, x) = \frac{(1-r)a}{n-1}\left(n - \left(\frac{a}{x}\right)^{n-1}\right)$$

is an increasing function.

Secondly, note that x^d can be any value within $[a, b]$ because the objective in the decentralized system

$$\Pi_0(x) = (1-r)a,$$

a constant, implying that $x^d = a$ is a Nash Equilibrium. So for this worst distribution:

$$\sup_{x^d} \frac{\Pi(x^c)}{\Pi(x^d)} = \frac{n}{n-1} - \frac{1}{n-1}\rho^{1-n} = 1 + \frac{1 - \rho^{1-n}}{n-1}.$$

Appendix B: Possible non-existence of equilibria

Let us consider a class $\widetilde{\mathcal{F}}[a,b]$ of distributions in $\mathcal{F}[a,b]$, with each of which there are infinite number of Nash equilibria for the sub-game concerned at a stage of the supply chain and any such Nash equilibrium makes the supply chain partner at another stage no profit to make, which is in direct conflict with our obvious requirement (see Sec. 2.3) that everyone involved in the supply chain system should have a positive profit in order for them to remain in business.

For simplicity, let us assume without loss of generality that, in the push assembly system, there are only $n = 2$ suppliers with the same component cost $c_1 = c_2 = 1/4$ and the assembly cost is $c_0 = 0$.

A family of distribution functions

Take any member distribution $F_0(x)$ from class $\mathcal{F}[c,b]$, where c is between a and b: $a < c < b$ and $c \le 3a$. Define class $\widetilde{\mathcal{F}}[a,b]$ in such a way that its typical member distribution $F(x)$ is as follows:

$$F(x) = \begin{cases} (x-a)/d, & a \le x \le c, \\ \frac{1}{2}(1 + F_0(x)), & c < x \le b, \end{cases}$$

where $d = 2(c-a) \le 4a$. It is straightforward to see that, if $\frac{1}{2} \le w \le 1$, then $F^{-1}(w) = a + d(1-w)$. Let

$$\Pi_1(w;\lambda) = (w-\lambda)F^{-1}(w), \ \frac{1}{2} \le \lambda \le w \le 1.$$

Then the set $W(\lambda) \subseteq \mathbb{R}_+$ of all optimal solutions to the problem of

$$\max_{w:\ \lambda \le w \le 1} \Pi_1(w;\lambda)$$

is as follows:

$$W(\lambda) = \begin{cases} \{\frac{1}{2}(\lambda+1) + a/(2d)\}, & \frac{1}{2} \le \lambda < \lambda^*, \\ \{1\}, & \lambda^* \le \lambda \le 1, \end{cases} \tag{9}$$

where $\lambda^* = 1 - a/d$ if $d \ge 2a$ (i.e., $c \ge 2a$) and $\lambda^* = \frac{1}{2}$ otherwise (note: $\frac{1}{2} \le \lambda^* \le \frac{3}{4}$).

Push assembly system

Suppose the demand distribution is $F(x)$ as given above. In the sub-game of suppliers, the utility function of player 1 is

$$\Pi_1(w_1;w_2) = (w_1 - \frac{1}{4})F^{-1}(w_1+w_2) = (w-\lambda)F^{-1}(w),$$

where $w = w_1 + w_2$ and $\lambda = w_2 + \frac{1}{4}$, while the utility function of player 2, $\Pi_2(w_2;w_1)$, is symmetric with w_1 and w_2 above swapped. According to (9), we see that the *best-response correspondence* $\mathrm{BEST}_1(w_2) \subseteq \{w_1 : \frac{1}{4} \le w_1 \le \frac{3}{4}\}$ for supplier 1 is as follows:

$$\mathrm{BEST}_1(w_2) = \begin{cases} \{-\frac{1}{2}w_2 + \frac{7}{8} + a/(2d)\}, & \frac{1}{4} \le w_2 < \lambda^* - \frac{1}{4}, \\ \{1 - w_2\}, & \lambda^* - \frac{1}{4} \le w_2 \le \frac{3}{4}. \end{cases}$$

With a symmetric best-response correspondence $\mathrm{BEST}_2(w_1)$ for supplier 2, we conclude from the fact $\lambda^* \le \frac{3}{4}$ that any Nash equilibrium $w^* = (w_1^*, w_2^*)$, i.e., $w_1^* \in \mathrm{BEST}_1(w_2^*)$ and $w_2^* \in \mathrm{BEST}_2(w_1^*)$, satisfies that $w_1^* + w_2^* = 1$, which implies that the assembler has zero profit in the supply chain!