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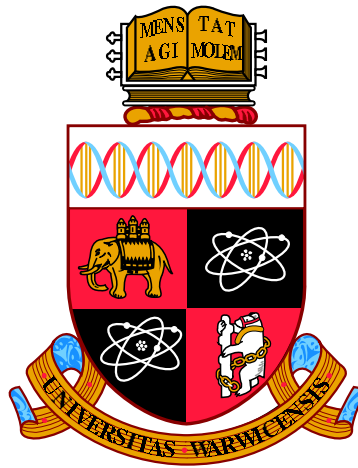
Dynamics of Degree Two Quasiregular Mappings of the Plane of Constant Dilatation

by

Robert Neil Fryer

Thesis

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Mathematics

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To the rest of my friends, thanks for entertaining me. Finally thank you to Lauren for having made me a better person.

Declarations

I declare that this thesis has not been submitted for a degree at another university.

Chapters 2 and 3 are introductory and survey some material on complex and quasiregular dynamics, based on various sources including [1, 4, 5, 10, 20, 29]. The remaining work is original and includes material from the paper [17] and the preprint [18] coauthored with Alastair Fletcher and presented here in more detail, with some extra diagrams.

Figures 3.1-3.5 are due to Dan Goodman and are computed using the Python programming language and the Numerical Python extension package. Figures 2.1-2.4 are copyright free and available widely. All other figures were created by the author using the programme Xfig.

Abstract

Let $h : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear map. In this thesis, we explore the dynamics of the quasiregular mapping $h(z)^2 + c$.

It is well-known that a polynomial can be conjugated by a holomorphic map ϕ to $w \mapsto w^d$ in a neighbourhood of infinity. This map ϕ is called a Böttcher coordinate for f near infinity. We construct a Böttcher type coordinate for compositions of h and polynomials, a class of mappings first studied in [19]. As an application, we prove that if h is affine and $c \in \mathbb{C}$, then $h(z)^2 + c$ is not uniformly quasiregular. Via the Böttcher type coordinate, we are able to obtain results for any degree two mapping of the plane with constant complex dilatation.

We show that any such mapping has either one, two or three fixed external rays, that all cases can occur, and exhibit how the dynamics changes in each case. We use results from complex dynamics to prove that these mappings are nowhere uniformly quasiregular in a neighbourhood of infinity. Finally, we show that in most cases, two such mappings are not quasiconformally conjugate on a neighbourhood of infinity.

Karma police arrest this man,

he talks in maths,

he buzzes like a fridge,

he's like a detuned radio.

<div align="right">

— Thom Yorke/Jonny Greenwood/Ed O'Brien/

Colin Greenwood/Phil Selway, 1997

</div>

Chapter 1

Introduction

The field of complex dynamics was first popularised nearly a century ago by Fatou [15, 16] and Julia [27]. Earlier work by Böttcher [8] and others had focused on the linearisability of analytic functions in neighbourhoods of fixed or periodic points. In these neighbourhoods the dynamics is well understood and there is some sense of stability. Julia and Fatou were concerned with the iteration of rational functions of the plane. Independently, they studied the boundary of the sets where linearisation was possible; informally this was the set of points where the iterates were *badly behaved*, these sets are now known as Julia sets. Fatou was concerned with the set of points where the iterates were not a normal family. Julia studied the closure of the set of repelling periodic points. Later it was proved that these two sets were equivalent.

They showed these maps had rich chaotic behaviour on the Julia set. Further Julia knew that quadratic polynomials, when iterated, had Julia sets that were either connected or totally disconnected and that the orbit of the only critical point determined which case occurred. However Julia never studied the parameter that caused this. Research into the area of complex dynamics largely ground to a halt, due to the fact that a complete classification of the stable domains (which became known as the Fatou set) eluded proof. Although notably Baker [34] did much work on the iteration of entire transcendental mappings in this time.

However, interest into complex dynamics was renewed in the 1980s when Sullivan [35], Douady and Hubbard [12], and others introduced powerful new techniques to the subject, including quasiconformal mappings. Further, computer generated images of Julia sets and the Mandelbrot set created wider interest in the subject outside of the field itself. They showed the wonderful intricacies at play for functions that could be stated very simply.

More recently there has been interest into whether the ideas of complex dynamics can be

extended to more general functions. In particular we are concerned with *quasiregular maps* of the plane. Recall that a holomorphic function sends infinitesimal circles to infinitesimal circles; informally a quasiregular mapping sends infinitesimal circles to infinitesimal ellipses and the greater the eccentricity of the ellipses, the greater the distortion of the mappings. Quasiregular mappings can be defined in any dimension, see Rickman's monograph [33] for more details.

The first quasiregular mappings to be iterated were *uniformly quasiregular mappings*, these are mappings with a uniform bound on the distortion of the iterates, for example holomorphic functions. These are special cases; in particular, due to Hinkkanen [23], every uniformly quasiregular mapping of the plane is quasiconformally conjugate to a holomorphic function. For more on uniformly quasiregular dynamics see for example [24, 26].

For general quasiregular mappings it is difficult to define the Fatou set, as we may not have a common bound on the distortion of the iterates and so may not have normality. We do not have an analogue of Montel's Theorem for general quasiregular mappings, which is a key ingredient in proofs of complex dynamics, hence we cannot just adapt existing proofs for their quasiregular analogues. It is however always possible to define the escaping set $I(f)$ of a quasiregular mapping f, this is the set of points z such that $f^n(z) \to \infty$ as $n \to \infty$. It is well known that for an analytic function, the boundary of $I(f)$ coincides with the Julia set of f. Therefore it is natural to consider $\partial I(f)$ as a substitute for the Julia set of quasiregular mappings. However it is much harder to prove analogous results with the holomorphic case due to the fact we can no longer use any results that use normality. Fletcher and Goodman [19], and Fletcher and Nicks [21] showed that for certain quasiregular mappings we can obtain analogous results for the sets $\partial I(f)$ compared to Julia sets of holomorphic functions.

Further, Fletcher and Goodman [19] studied the quasiregular mappings

$$f_{K,\theta,c}(z) := h_{K,\theta}(z)^2 + c,$$

where $h_{K,\theta}$ is an affine stretch of magnitude K in direction θ and $c \in \mathbb{C}$. These mappings are quasiregular analogues of the quadratic polynomials $f_c(z) = z^2 + c$ that were studied by Douady and Hubbard. They showed many similar properties to the iteration of quadratic polynomials and they introduced quasiregular versions of the Mandelbrot set, that depend on the parameters K and θ.

In this thesis we will continue the study of the dynamics of these quasiregular mappings $f_{K,\theta,c}$. Any mapping of degree 2 of constant dilatation is linearly conjugate to a mapping of

the form $f_{K,\theta,c}$, for some $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. We will construct *Böttcher coordinates* that conjugate $f_{K,\theta,c}$ to $f_{K,\theta,0} := H_{K,\theta}$ on some neighbourhood of infinity. We will see that we have rays that are fixed under the mappings $H_{K,\theta}$. These will play a key role in the dynamics of $H_{K,\theta}$. In particular we can calculate the complex dilatation of iterates of $H_{K,\theta}$ for points on the fixed rays, by seeing they are equal to iterating a Möbius mapping that is defined on each fixed ray. We then use these results to show that $H_{K,\theta}$, and so $f_{K,\theta,c}$ by the *Böttcher coordinate* result, is *nowhere uniformly quasiregular*. Finally we use more results from the iteration of Möbius maps to obtain certain conditions on maps not being quasiconformally conjugate on any neighbourhood of infinity.

1.1   Outline of thesis and key results

In Chapter 2 we survey some complex dynamics and include some results that will be needed later on logarithmic coordinates and hyperbolic Möbius maps of the disk $\mathbb{D}$. Chapter 3 introduces quasiregular mappings and some of their properties, we then go on to mention some known results including those relevant to the direction we will explore. In Chapter 4 we define the affine stretch $h_{K,\theta}$ and the maps $H = H_{K,\theta} = (h_{K,\theta})^2$ and $f = f_{K,\theta,c} = (h_{K,\theta})^2 + c$, which will be the main objects that we will study. We show that any degree two quasiregular map of polynomial type of constant complex dilatation is linearly conjugate to this special form.

Proposition 4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be quasiregular of degree two and let f have constant complex dilatation that is not identically 0. Then f is linearly conjugate to a unique mapping of the form $f_{K,\theta,c}(z) := h_{K,\theta}(z)^2 + c$ for some $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$.*

In Chapter 5 we prove the following theorem, a quasiregular version of Böttcher coordinates.

Theorem 5.1. *Let $h : \mathbb{C} \to \mathbb{C}$ be an affine mapping and $c \in \mathbb{C}$. Then there exists a neighbourhood $U = U(h,c)$ of infinity and a quasiconformal map $\psi = \psi(h,c)$ such that*

$$h(\psi(z))^2 = \psi(f(z)), \tag{1.1}$$

for $z \in U$, where $f(z) = h(z)^2 + c$. Further, ψ is asymptotically conformal as $|z| \to \infty$.

A *ray* is a semi-infinite line $R_\phi = \{te^{i\phi} : t \geq 0\}$. In Chapter 6 we consider fixed rays of $H_{K,\theta}$ and show that there exist one, two or three fixed rays and study the dynamics of these rays. We obtain the following result.

3

Theorem 6.1. *Let $\theta \in (-\pi/2, \pi/2) \setminus \{0\}$, $K > 1$ and let $H(z) = h_{K,\theta}(z)^2$. Then there exists $K_\theta > 1$ such that:*

- *for $K < K_\theta$, there is one fixed ray that is locally repelling;*

- *for $K = K_\theta$, there are two fixed rays, one of which is locally repelling and one that is neutral. Further, the neutral fixed ray is repelling on one side and attracting on the other;*

- *for $K > K_\theta$, there are three fixed rays, one of which is locally attracting and two that are locally repelling.*

When $\theta = 0$ the first and third statements above hold, but when $K = K_\theta$ there is just one neutral fixed ray which is locally attracting on both sides. When $\theta = \pi/2$ there is only one fixed ray for all $K > 1$ and it is always locally repelling.

We then go on to study the preimages of their fixed rays and *basins of attraction*. These are the sets $\Lambda \subset \mathbb{C}$ such that $\arg[H_{K,\theta}^n(z)] \to \phi$ for $z \in \Lambda$, where ϕ is the angle of the attracting fixed ray, R_ϕ, of $H_{K,\theta}$. In particular we prove the following key result.

Theorem 6.2. *If H has one fixed ray R_ϕ then $\{H^{-k}(R_\phi)\}_{k=0}^\infty$ is dense in $\mathbb{C}$. If H has two or three fixed rays, then Λ is dense in $\mathbb{C}$.*

We use these results to show that $\mathbb{C}$ decomposes nicely into different dynamical sets.

Corollary 6.3. *Let $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $H(z) = h_{K,\theta}(z)^2$. Then $\mathbb{C} = I(H) \cup \partial I(H) \cup \mathcal{A}(0)$, where $\mathcal{A}(0)$ is the basin of attraction of the fixed point 0.*

In Chapter 7 we show that our mappings $H_{K,\theta}$, and so $f_{K,\theta,c}$ by the Böttcher coordinate result, are *nowhere uniformly quasiregular*. We will define a *nowhere uniformly quasiregular* mapping later, but informally it is a mapping f that for all points $z \in \mathbb{C}$ and every neighbourhood $U \ni z$ there exists $w \in U$ such that f is not *uniformly quasiregular* at w, that is the distortion of the iterates of f is not bounded at w.

Theorem 7.3. *Let $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. Then the mapping $h_{K,\theta}(z)^2 + c$ is nowhere uniformly quasiregular.*

Finally in Chapter 8 we use a Möbius map, that is derived from the dilatation on fixed rays, to give the following conditions on our maps $H_{K,\theta}$, and again $f_{K,\theta,c}$, not being quasiconformally conjugate on any neighbourhood of infinity. Denote the fixed rays of $H_{K_1,\theta_1} := H_1$ by R_{ϕ_i} and the fixed rays of $H_{K_2,\theta_2} := H_2$ by R_{ψ_j}, the corresponding Möbius transformations of each

fixed ray R_{ϕ_i} by $A_i(z)$ and the corresponding Möbius transformations of each fixed ray R_{ψ_j} by $B_j(z)$, where

$$A_i(z) = \frac{\mu + e^{-i\phi_i} z}{1 + e^{-\phi_i}\overline{\mu} z},$$

where $\mu = e^{2i\theta_1}(K_1 - 1)/(K_1 + 1) \in \mathbb{D}$ and

$$B_j(z) = \frac{\nu + e^{-i\phi_j} z}{1 + e^{-\phi_j}\overline{\nu} z},$$

where $\nu = e^{2i\theta_2}(K_2 - 1)/(K_2 + 1) \in \mathbb{D}$. Then we prove the following theorem.

Theorem 8.1. *With the notation above, there is no quasiconformal conjugacy between H_1 and H_2 in any neighbourhood of infinity if any of the following conditions hold:*

(i) *the mappings H_1, H_2 have different numbers of fixed rays;*

(ii) *H_1 and H_2 both have one fixed ray, R_{ϕ_1} and R_{ψ_1} respectively, and $\mathrm{Tr}(A_1)^2 \neq \mathrm{Tr}(B_1)^2$;*

(iii) *if H_1 and H_2 both have two fixed rays R_{ϕ_i} and R_{ψ_i} for $i = 1, 2$, where $\phi_1 > \phi_2$ and $\psi_1 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i;*

(iv) *if H_1 and H_2 both have three fixed rays R_{ϕ_i} and R_{ψ_j}, $i, j \in \{0, 1, 2\}$ respectively, where $\phi_1 > \phi_0 > \phi_2$ and $\psi_1 > \psi_0 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i.*

Then we reduce the possibility of a quasiconformal equivalence existing on a neighbourhood of infinity, when we fix one of K or θ to the possible cases given in the following theorem.

Theorem 8.2.   • *If $K > 1$ is fixed and $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ then H_{K,θ_1} and H_{K,θ_2} are not quasiconformally conjugate on any neighbourhood of infinity, except if $\theta_1 = \theta_2$ or possibly one case where H_{K,θ_1} and H_{K,θ_2} both have one fixed ray and*

$$\theta_1 = \phi - \tan^{-1}\left(\frac{K}{\tan(\phi - \theta_2)}\right),$$

where ϕ is the fixed point of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$.

• *If $\theta \in (-\pi/2, \pi/2)$ is fixed and $K_1 \neq K_2 > 1$ then $H_{K_1,\theta}$ and $H_{K_2,\theta}$ are not quasiconformally conjugate on any neighbourhood of infinity.*

Chapter 2

Complex Dynamics

Complex dynamics was originally concerned with the behaviour of rational functions under iteration. Many results have been extended to more general mappings, such as for example to entire functions. We are interested in seeing whether the concepts and ideas of complex analysis can be extended to quasiregular maps. We begin with an overview of some relevant complex dynamics.

2.1 Rational functions

We say $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function if it can be expressed as $f(z) = P(z)/Q(z)$ where $P, Q : \mathbb{C} \to \mathbb{C}$ are polynomials. Here $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. Recall that a family of functions is called normal if there is *nice behaviour*, we define this more precisely now.

Definition 2.1. A family, $\mathcal{F}$, of meromorphic functions on a domain $\mathcal{D} \subset \overline{\mathbb{C}}$ is a normal family if every sequence $\{f_k\}$ in $\mathcal{F}$ contains a subsequence that converges uniformly in the spherical metric, on compact subsets of $\mathcal{D}$, to a meromorphic function f.

The basic objects studied in the iteration of rational functions, introduced by Fatou [15, 16] and Julia [27], are defined as follows.

Definition 2.2. The Fatou set of the rational function f is defined as:

$$F(f) := \left\{ z \in \overline{\mathbb{C}} \mid \{f^k\}_{k \in \mathbb{N}} \text{ is normal in some neighbourhood of } z \right\}.$$

The Julia set of f is defined as:

$$J(f) := \left\{ z \in \overline{\mathbb{C}} \mid \{f^k\}_{k \in \mathbb{N}} \text{ is not normal in some neighbourhood of } z \right\} = \overline{\mathbb{C}} \setminus F(f).$$

Also recall Montel's Theorem, which is invaluable in holomorphic dynamics.

Theorem 2.3 (Montel's Theorem, [10] Theorem 3.2)**.** *A family of meromorphic functions on* $\mathcal{D}$ *omitting three fixed values is normal.*

Let's fix some notation. For $z \in \mathbb{C}$ let

$$O^+(z) := \{f^n(z) \mid n \geq 0\}$$

be the forward orbit of z, let

$$O^-(z) := \bigcup_{n \geq 0} f^{-n}(z) = \bigcup_{n \geq 0} \left\{w \in \overline{\mathbb{C}} \mid f^n(w) = z\right\}$$

be the backward orbit of z, and let

$$O(z) := O^+(z) \cup O^-(z)$$

be the orbit of z. For $\mathcal{A} \subset \overline{\mathbb{C}}$ we let $O^{\pm}(\mathcal{A}) = \cup_{z \in \mathcal{A}} O^{\pm}(z)$ and we say $\mathcal{A}$ is completely invariant if $O(\mathcal{A}) = \mathcal{A}$. If ξ is an attracting periodic point of periop p, then

$$\Lambda(\xi) := \left\{w \in \overline{\mathbb{C}} \,\middle|\, \lim_{n \to \infty} f^{pn}(w) = \xi\right\}$$

is the basin of attraction of ξ. The exceptional set $E(f)$ is defined as the set of all points whose backwards orbit is finite. Given this notation we list some results noted in the review of Bergweiler [5]. This first theorem summarises some of the results shown by Fatou and Julia.

Theorem 2.4 ([5] Theorem 2.1)**.** *Let f be a rational function of degree at least 2. Then:*

   (i)   *$F(f)$ is open and $J(f)$ closed.*

  (ii)   *$F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$.*

 (iii)   *$F(f)$ and $J(f)$ are completely invariant.*

 (iv)   *$J(f)$ is perfect.*

  (v)   *If $X \subset \overline{\mathbb{C}}$ is closed and completely invariant and if $|X| \geq 3$, then $X \supset J(f)$.*

 (vi)   *If ξ is an attracting periodic point, then $\Lambda(\xi) \subset F(f)$ and $\partial\Lambda(\xi) = J(f)$.*

(vii)   *If $U$is open and $U \cap J(f) \neq \emptyset$, then $O^+(U) \supset \overline{\mathbb{C}} \setminus E(f)$.*

(viii) $|E(f)| \leq 2$ *and* $E(f) \cap J(f) = \emptyset$.

(ix) *If* $z \in J(f)$ *then* $J(f) = \overline{O^-(z)}$.

Another basic result of the dynamics of rational functions is the following.

Theorem 2.5 ([4] Theorem 4.2.7)**.** *The Julia set of a rational function is the closure of the set of repelling periodic points.*

2.1.1 Polynomials

A special set of rational functions is the set of polynomials. All polynomials fix the point at infinity; further infinity is always an attracting fixed point if we require the degree to be greater than one. We define the escaping set of a mapping f to be

$$I(f) := \{z \in \overline{\mathbb{C}} \mid f^n(z) \to \infty \text{ as } n \to \infty\}.$$

We can also define the non-escaping set, the set of points that remain bounded under iterations of f, as

$$N(f) := \mathbb{C} \setminus I(f).$$

Note that here we use $N(f)$ instead of the usual $K(f)$, so as not to confuse the non-escaping set with the distortion of f. If f is a polynomial of degree $d \geq 2$, then we always have ∞ as an attracting fixed point and there always exists some neighbourhood U of infinity such that $U \subset I(f)$. Also when f is a polynomial we can obtain an estimate on large values of z.

Lemma 2.6. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a polynomial* $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ *where* $n \geq 2$, $a_i \in \mathbb{C}$ *and* $a_n \neq 0$. *Then there exists some* $R > 0$ *such that if* $|z| > R$ *then* $|f(z)| \geq 2|z|$.

Remark 2.7. *Notice that this lemma implies that if* $|z| \geq R$ *then* $z \in I(f)$. *In fact we can do even better and note; if* $|f^m(z)| \geq R$ *for some* $m \in \mathbb{N}$, *then* $z \in I(f)$.

Proof. We can choose R large enough so that if $|z| \geq R$ then

$$\frac{|a_n||z|^n}{2} \geq 2|z|,$$

and

$$\frac{|a_n||z|^n}{2} \geq |a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|.$$

Then if $|z| \geq R$,

$$|f(z)| \geq |a_n||z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|) \geq \frac{1}{2}|a_n||z|^n \geq 2|z|.$$

$\square$

The motivation behind introducing the escaping set is the following result linking it to Julia sets.

Proposition 2.8. *If f is a polynomial then $J(f) = \partial I(f)$.*

Remark 2.9. *The proof of this is immediate from part (vi) of Theorem 2.4, but we include a proof to show the methods at play.*

Proof. First we show $I(f)$ is open. If $z \in I(f)$ then $|f^m(z)| > R$ for some $m \in \mathbb{N}$, for any $R > 0$. By continuity there exists some $\varepsilon > 0$ such that $|f^m(w)| > R$ for all $w \in B_\varepsilon(z)$. By Lemma 2.6 and the remark afterwards, $w \in I(f)$ showing $I(f)$ is open.

Pick $z \in \partial I(f)$. Then every neighbourhood $U \ni z$ contains points $w \in U$ such that $f^n(w) \to \infty$ as $n \to \infty$, but $f^n(z)$ remains bounded. Hence no subsequence of $\{f^n(z)\}$ is uniformly convergent on U; hence $\{f^n\}$ is not normal at z, so $z \in J(f)$ and

$$\partial I(f) \subset J(f). \tag{2.1}$$

Now suppose $z \notin \partial I(f)$. Then either $z \in I(f) \setminus \partial I(f)$ or $z \in \overline{\mathbb{C}} \setminus (I(f) \cup \partial I(f))$. If $z \in I(f)$ then, as it is not on the boundary, there exists a neighbourhood $V \ni z$ such that $V \subset I(f)$, then $f^n(w) \to \infty$ as $n \to \infty$ for all $w \in V$, hence f^n converges uniformly to infinity on V and $\{f^n\}$ is normal at z. If $z \in \overline{\mathbb{C}} \setminus (I(f) \cup \partial I(f))$ then there exists a neighbourhood $V \ni z$ and some $C > 0$ such that $|f^n(w)| < C$ for all $w \in V$. Applying Theorem 2.3 we see that $\{f^n\}$ is normal at z. Hence $z \notin J(f)$ and $(\partial I(f))^c = (J(f))^c$, using this and (2.1) we see $J(f) = \partial I(f)$. $\square$

2.1.2 Quadratic polynomials

We will be investigating a quasiregular version of quadratic polynomials and so mention some more results focusing on this.

Proposition 2.10. *Any quadratic polynomial of the form $P(z) = \alpha z^2 + \beta z + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha \neq 0$, is linearly conjugate to the form $f_c(z) = z^2 + c$ for some $c \in \mathbb{C}$.*

Proof. Let $\phi(z) := \eta z + \tau$, where $\eta, \tau \in \mathbb{C}$ and $\eta \neq 0$; then $\phi^{-1}(z) = z/\eta - \tau/\eta$. Let us consider

$$
\begin{aligned}
\phi \circ P \circ \phi^{-1}(z) &= \phi \circ P \left(\frac{z}{\eta} - \frac{\tau}{\eta} \right) \\
&= \phi \left(\alpha \left(\frac{z}{\eta} - \frac{\tau}{\eta} \right)^2 + \beta \left(\frac{z}{\eta} - \frac{\tau}{\eta} \right) + \gamma \right) \\
&= \eta \alpha \left(\frac{1}{\eta^2} z^2 - 2 \frac{\tau}{\eta^2} z + \frac{\tau^2}{\eta} \right) + \beta z - \beta \tau + \eta \gamma + \tau \\
&= \left(\frac{\alpha}{\eta} \right) z^2 + \left(\beta - 2 \frac{\alpha \tau}{\eta} \right) z + \frac{\alpha \tau^2}{\eta} - \beta \tau + \eta \gamma + \tau. \quad (2.2)
\end{aligned}
$$

We are trying to show $f_c = \phi \circ P \circ \phi^{-1}$ for some c. Hence by (2.2) we require $\alpha = \eta$ and $\beta = 2\alpha\tau/\eta$, this implies $\beta = 2\tau$. Hence $c = (3\beta^2 + 2\beta)/4 + \alpha\gamma$. $\qquad\square$

The advantage of conjugating every quadratic polynomial to some f_c is that 0 is now the only *branch point*. Recall the definition of the set of branch points of a mapping.

Definition 2.11. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous mapping. Then $B(f) := \{ z \in \mathbb{C} \mid f$ is not locally injective at $z \}$, denotes the branch set of f.

Let $J_c := J(f_c)$, then we have the following Theorem proved in [10].

Theorem 2.12 ([10] VIII. Theorem 1.1). *If $f_c^n(0) \to \infty$ as $n \to \infty$ then the Julia set J_c is totally disconnected. Otherwise $f_c^n(0)$ is bounded and J_c is connected.*

We now illustrate these concepts by considering some examples of $N(f_c)$ for different values of $c \in \mathbb{C}$, figures are shown on the next two pages.

Figure 2.1 depicts $N(f_{-1+0.1i})$; in this case $0 \notin I(f_{-1+0.1i})$ and so the Julia set, $J_{-1+0.1i} = \partial I(f_{-1+0.1i}) = \partial N(f_{-1+0.1i})$, is connected. Also notice that the interior of $N(f_{-1+0.1i})$ is non-empty.

Figure 2.2 depicts $N(f_i)$; in this case $0 \notin I(f_i)$ and so the Julia set J_i is connected. Also notice that the interior of $N(f_i)$ is empty, so $N(f_i) = J_i$.

Figure 2.3 depicts $N(f_{0.285})$; in this case $0 \notin I(f_{0.285})$ and so the Julia set $J_{0.285}$ is not connected and is again equal to $N(f_{0.285})$, in fact it is totally disconnected. The reason some regions look connected is due to the fact that nearby points escape very slowly and so more iterations would be needed for a more defined picture. Also by Theorem 2.4 we know that $J_{0.285}$ is perfect, this means given $z \in J_{0.285}$ every neighbourhood $U \ni z$ has the property that $U \cap J_{0.285} \neq \emptyset$.
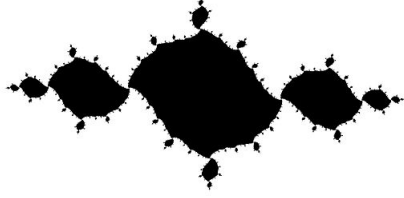
Figure 2.1: The black region denotes $N(f_{-1+0.1i})$.
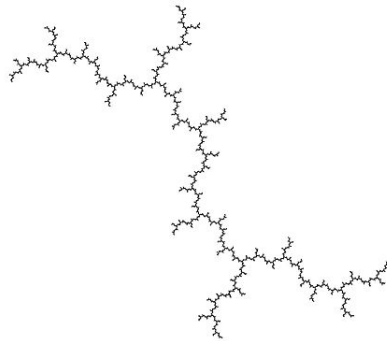
Figure 2.2: The black region denotes $N(f_i)$.

The points $c \in \mathbb{C}$ where J_c is connected is known as the Mandelbrot set, denoted by $\mathcal{M}$. In our examples $c = -1 + 0.1i$ and $c = i$ are points of $\mathcal{M}$ but $c = 0.285$ is not contained in $\mathcal{M}$. By Theorem 2.12 the Mandelbrot set is defined as follows.
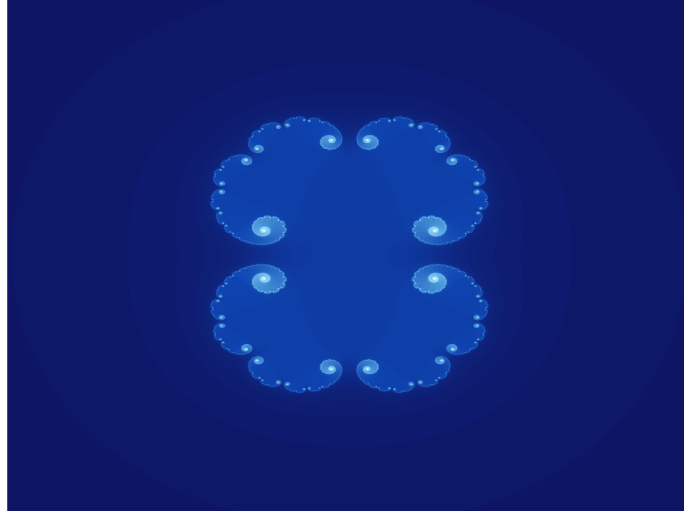
Figure 2.3: The white denotes $N(f_{0.285})$, the blue regions are points in $I(f_{0.285})$.

Definition 2.13. The Mandelbrot set is defined as

$$\mathcal{M} := \{c \in \mathbb{C} \mid f_c^n(0) \text{ is bounded}\}.$$
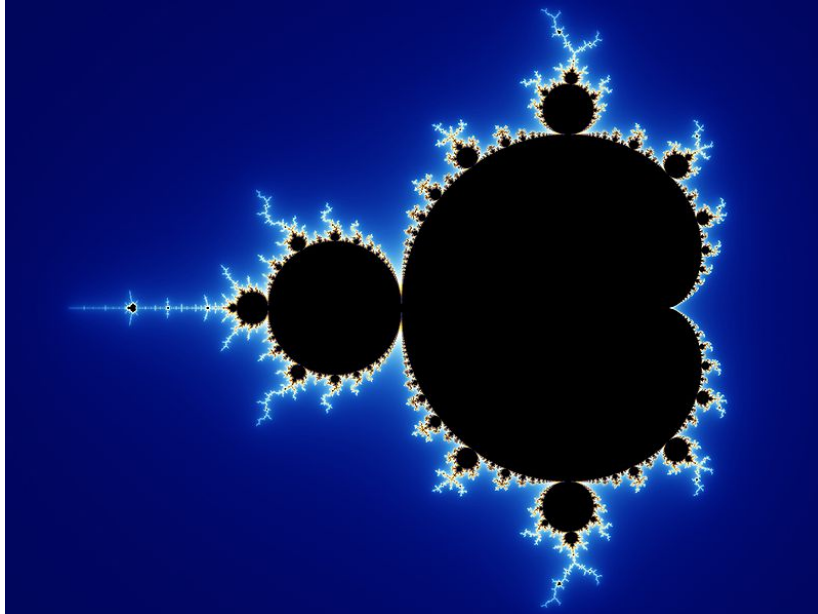
Figure 2.4: The black region denotes the Mandelbrot set, $\mathcal{M}$.

Many results have been proved about the Mandelbrot set, we will mention some of these briefly.

Theorem 2.14 ([10] VIII Theorem 1.1). $\mathcal{M}$ *is a closed simply connected subset of the disk*

$\mathbb{D}_2 := \{|c| < 2\} \subset \mathbb{C}$, *which meets the real line in the interval* $[-2, 1/4]$. *Further* $c \in \mathcal{M}$ *if and only if* $f_c^n(0) \leq 2$ *for all* $n \in \mathbb{N}$.

By definition if $c \in \mathcal{M}$ then J_c is connected however, as can be seen in Figures 2.1 and 2.2, we have cases where $N(f)$ has non-empty and empty interior respectively. If c is in the interior of $\mathcal{M}$ then $N(f_c)$ has non-empty interior, however this may still be the case for $c \in \partial\mathcal{M}$. Points $c \in \partial\mathcal{M}$ where $N(f_c)$ has empty interior are called Misiurewicz points. There is much literature about iterations of quadratic polynomials and the Mandelbrot set, see for instance [4, 10, 11, 14].

2.2   Logarithmic coordinates

To prove Theorem 5.1, we will need to use the logarithmic transform which we briefly outline here.

Let f be a function defined in a neighbourhood $U = \{|z| > R\}$ of infinity and which grows like a polynomial. That is, there exist constants A, B, n such that

$$A \leq \frac{|f(z)|}{|z|^n} \leq B.$$

Then f lifts to a function

$$\widetilde{f}(X) = \log f(e^X)$$

for $\operatorname{Re} X > \log R$.

Definition 2.15. The function $\widetilde{f}$ is called the logarithmic transform of f, and is unique up to addition of an integer multiple of $2\pi i$.

Lemma 2.16. *Suppose* f, g *are two functions whose logarithmic transforms exist. Then* $\widetilde{f \circ g} = \widetilde{f} \circ \widetilde{g}$ *in a suitable neighbourhood of infinity.*

Proof. We know that f is defined on a neighbourhood of infinity $\widetilde{U}$. Choose $R > 0$ large enough so that g is defined on $U = \{|z| > R\}$ and $g(U) \subset \widetilde{U}$, then f must be defined on the

neighbourhood of infinity $g(U)$. We have,

$$
\begin{aligned}
\widetilde{f \circ g}(X) &= \log[f(g(\exp X))] \\
&= \log[f \exp(\log[(g(\exp X))])] \\
&= \log[f(\exp[\widetilde{g}(X)])] \\
&= \widetilde{f} \circ \widetilde{g}(X).
\end{aligned}
$$

$\square$

Lemma 2.17. *Let* $g(z) = z^2 + c$. *Then* $\widetilde{g}(X) = 2X + \rho(X)$, *where* $\rho(X) = O(e^{-2\operatorname{Re}(X)})$ *as* $\operatorname{Re}(X) \to +\infty$.

Proof. We have

$$
\begin{aligned}
\widetilde{g}(X) &= \log(e^{2X} + c) \\
&= \log(e^{2X}(1 + ce^{-2X})) \\
&= 2X + \log(1 + ce^{-2X}),
\end{aligned}
$$

which proves the lemma. $\square$

2.3 Böttcher coordinates

Böttcher showed the following theorem, which we will prove a quasiregular version of in the next chapter.

Theorem 2.18 ([8]). *Let f be holomorphic in a neighbourhood U of infinity, and let infinity be a superattracting fixed point of f, that is, there exists $n \geq 2$ such that*

$$
f(z) = a_n z^n (1 + o(1)),
$$

for $z \in U$, where $a_n \in \mathbb{C} \setminus \{0\}$. Then there exists a holomorphic change of coordinate $w = \psi(z)$, with $\psi(\infty) = \infty$, which conjugates f to $w \mapsto w^n$ in some neighbourhood of infinity. Further, ψ is unique up to multiplication by an $(n-1)$-th root of unity.

The map ψ is called a *Böttcher coordinate* for f near infinity. In Chapter 5 we will find an analogous Böttcher coordinate for mappings of the form $f = g \circ h$, where g is a polynomial

of degree $n \geq 2$, and h is an affine mapping of the plane to itself. As we will follow a similar method later, we now prove this following the proof of Milnor ([29] Theorem 6.7).

Proof. Suppose our map has the Laurent series expansion

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + a_{-1} z^{-1} + \cdots$$

where $n \geq 2$, which is convergent for $|z| > r$. First notice that the linearly conjugate map $z \mapsto \alpha f(z/\alpha)$, where $\alpha^{n-1} = a_n$, has leading coefficient 1. So we may assume $a_n = 1$. Hence

$$f(z) = z^n(1 + o(1)),$$

for large $|z|$. Now we utilise logarithmic coordinates, introduced in the previous section. If we choose the correct lift we obtain

$$\widetilde{f}(X) = nX + O(e^{-\operatorname{Re}(X)}), \tag{2.3}$$

for large enough $\operatorname{Re}(X)$. This implies

$$|\widetilde{f}(X) - nX| < 1, \tag{2.4}$$

for large enough $\operatorname{Re}(X)$. Choose $\sigma > 1$ large enough so that (2.4) is satisfied for all X in the half plane $\mathbb{H}_\sigma$ defined as the points $X \in \mathbb{C}$ such that $\operatorname{Re}(X) > \sigma$. By construction $\widetilde{f}$ maps this half plane into itself. Also as $\widetilde{f}(X + 2\pi i) - \widetilde{f}(X)$ is a multiple of $2\pi i$ this implies $\widetilde{f}(X + 2\pi i) - \widetilde{f}(X) = 2\pi i n$ by (2.3) and because $\widetilde{f}(X + 2\pi i) - \widetilde{f}(X)$ and $n(X + 2\pi i) - nX$ differ by at most 2 by (2.4).

Suppose $X_0 \mapsto X_1 \mapsto X_2 \cdots$ is an orbit under $\widetilde{f}$ in $\mathbb{H}_\sigma$, then we know $|X_{k+1} - nX_k| < 1$. Setting $W_k := X_k/n^k$ we see

$$|W_{k+1} - W_k| < 1/n^{k+1}.$$

Hence the sequence of holomorphic functions $W_k = W_k(X_0)$ converges uniformly and geometrically as $k \to \infty$ to a holomorphic limit

$$\Psi(X_0) = \lim_{k \to \infty} W_k(X_0).$$

This mapping satisfies the identity

$$\Psi(\widetilde{f}(X)) = n\Psi(X).$$

Also $\Psi(X + 2\pi i) = \Psi(X) + 2\pi i$, so the mapping $\psi(z) = e^{\Psi(\log(z))}$ is well defined near infinity and satisfies

$$\psi(f(z)) = \psi(z)^n,$$

as required.

All that is left is to prove uniqueness. It is enough to study mappings $\zeta \mapsto \eta(\zeta)$ near infinity that satisfy $\eta(\zeta^n) = \eta(\zeta)^n$. Setting

$$\eta(\zeta) = c_1\zeta + c_0 + c_{-1}\zeta^{-1} + \cdots,$$

this implies

$$c_1\zeta^n + c_0 + c_{-1}\zeta^{-n} + \cdots = (c_1\zeta + c_0 + c_{-1}\zeta^{-1}\cdots)^n = c_1^n\zeta^n + nc_1^{n-1}c_0\zeta^{n-1} + \cdots.$$

This implies $c_1 = c_1^n$. Since $c_1 \neq 0$, we have that c_1 must be an $(n-1)$-th root of unity. Comparing the remaining coefficients we see $c_i = 0$ for $i \neq 1$. $\qquad\square$

Remark 2.19. *In particular, if f is a quadratic polynomial then by Proposition 2.10 it is linearly conjugate to $f_c(z) = z^2 + c$ for some $c \in \mathbb{C}$. Then each f_c is conformally conjugate to z^2 by Theorem 2.18 on some neighbourhood of infinity.*

2.4   Möbius maps and Blaschke products

To prove theorems in Chapters 7 and 8 we will need some results on hyperbolic Möbius maps of $\mathbb{D}$. We briefly recall some standard definitions and results from hyperbolic geometry; for more background and detail see [1].

Definition 2.20. Let $A : \mathbb{D} \to \mathbb{D}$ be a Möbius map, where $A(z) = (az + b)/(cz + d)$ for some $a, b, c, d \in \mathbb{R}$ and let $\lambda = 1/\sqrt{ad - bc}$. Then

$$\widehat{A} := \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{\widehat{a}z + \widehat{b}}{\widehat{c}z + \widehat{d}},$$

is the normalised form and

$$\text{Tr}\, A = \widehat{a} + \widehat{d}.$$

Definition 2.21. A Möbius map $A : \mathbb{D} \to \mathbb{D}$ is called hyperbolic if $\text{Tr}(A)^2 > 4$ and parabolic if $\text{Tr}(A)^2 = 4$.

When A is hyperbolic more is known. There exists a unique geodesic that is preserved set wise under A and we denote this by $\text{Ax}(A)$. Further if A is a hyperbolic Möbius map of $\mathbb{D}$ then there exist $\alpha, \beta \in \partial\mathbb{D}$ such that $A^n(z) \to \alpha$ and $A^{-n}(z) \to \beta$ as $n \to \infty$ for all $z \in \mathbb{D}$. Also $\text{Ax}(A)$ is the geodesic joining α and β. If A is parabolic then there exists one fixed point $\alpha \in \mathbb{D}$ and $A^n(z) \to \alpha$ as $n \to \infty$ for all $z \in \mathbb{D}$. We will require the following lemma which gives the standard form for a hyperbolic Möbius map of the upper half plane $\mathbb{H}$.

Lemma 2.22. *Let $A : \mathbb{D} \to \mathbb{D}$ be a hyperbolic Möbius map. Then A is conjugate to a Möbius map $\widetilde{A} : \mathbb{H} \to \mathbb{H}$ given by*

$$\widetilde{A}(z) = kz$$

where

$$k = (T - 2 - (T^2 - 4T)^{\frac{1}{2}})/2 < 1, \tag{2.5}$$

and $T := \text{Tr}^2(A)$.

Proof. We know that $\text{Tr}^2 A > 4$ and so by standard hyperbolic geometry we can lift to the upper half plane to obtain $\overline{A} : \mathbb{H} \to \mathbb{H}$. We see $\overline{A}$ has the same trace as A and is of hyperbolic type so must be conjugate to $\widetilde{A}(z) = kz$ for some $k > 0$. Conjugation preserves trace hence

$$k + 1/k + 2 = \text{Tr}^2(\widetilde{A}) = \text{Tr}^2(A) = T.$$

Solving this for k and taking the negative square root gives equation (2.5) and $k < 1$. Taking the positive square root would give the reciprocal. $\qquad\square$

The following theorem on sequences of hyperbolic Möbius transformations is a combination of results from [22] and [28].

Theorem 2.23 ([22, 28]). *Let A, A_j be hyperbolic Möbius maps of $\mathbb{D}$ such that $A^n(z) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$ and $A_j \to A$ locally uniformly as $j \to \infty$. Suppose we have sequences t_n, s_n of*

hyperbolic Möbius maps of $\mathbb{D}$ defined by

$$t_n(z) = A_1 \circ A_2 \circ \ldots \circ A_n(z),$$

$$s_n(z) = A_n \circ A_{n-1} \circ \ldots \circ A_1(z).$$

Then both $t_n(z) \to \alpha$ and $s_n(z) \to \alpha$ as $n \to \infty$ for all $z \in \mathbb{D}$.

We use this result to prove the following theorem, which is a new result.

Theorem 2.24. *Let $A, A_j : \mathbb{D} \to \mathbb{D}$ be hyperbolic Möbius maps such that $A^n(z) \to \alpha \in \partial\mathbb{D}$ as $n \to \infty$ and $A_j \to A$ locally uniformly as $j \to \infty$. Let*

$$t_n(z) = A_1 \circ A_2 \circ \ldots \circ A_n(z).$$

Then

$$d_h(0, t_n(z)) = \log\left[\frac{1}{\prod_{j=1}^n k_j}\right] + O(1),$$

for large n, where d_h denotes the hyperbolic metric on $\mathbb{D}$, $k_j < 1$ for all j and $k_j \to k$, where k_j, k are the quantities defined in Lemma 2.22.

Remark 2.25. *In particular, if $A_j = A$ for every $j \in \mathbb{N}$, then*

$$d_h(0, A^n(z)) = \log\left[1/k^n\right] + O(1) \text{ as } n \to \infty.$$

Proof. First if $A^n(z) \to \alpha$ for $z \in \mathbb{D}$ then $t_n(z) \to \alpha$ by Theorem 2.23. Now let $B = A^{-1}$ and $B_j = A_j^{-1}$. Then if $\alpha, \beta \in \partial\mathbb{D}$ are the attracting and repelling fixed points of A respectively, then β is the attracting fixed point and α is the repelling fixed point of B. Similarly if $\alpha_j, \beta_j \in \partial\mathbb{D}$ are the attracting and repelling fixed points of A_j respectively, then β_j is the attracting fixed point and α_j is the repelling fixed point of B_j. Further, we have $B_j \to B$ and so $\alpha_j \to \alpha$ and $\beta_j \to \beta$ as $j \to \infty$.

We write $\widetilde{B}$ for the lift of B to $\mathbb{H}$ via $\gamma : \mathbb{D} \to \mathbb{H}$ so that $\widetilde{B} = \gamma \circ B \circ \gamma^{-1}$. We choose γ so that $\gamma(\alpha) = \infty$ and $\gamma(0) = i$. This then means that $\gamma(\beta) = X \in \mathbb{R}$ and $\gamma(\beta_j) = X_j \in \mathbb{R}$, where $X_j \to X$ as $j \to \infty$.

We can also conjugate by the maps $\Phi, \Phi_i : \mathbb{H} \to \mathbb{H}$ where $\Phi(z) = z - X$ and $\Phi_i(z) = z - X_i$. This means that

$$\widetilde{B}(z) = \Phi^{-1} \circ \widehat{B} \circ \Phi(z) \text{ and } \widetilde{B}_j(z) = \Phi_j^{-1} \circ \widehat{B}_j \circ \Phi_j(z),$$

where $\widehat{B}(z) = kz$ and $\widehat{B}_j(z) = k_j z$. The factors k and k_j are determined as in Lemma 2.22 so that $k, k_j < 1$ and $k_j \to k$ as $j \to \infty$.
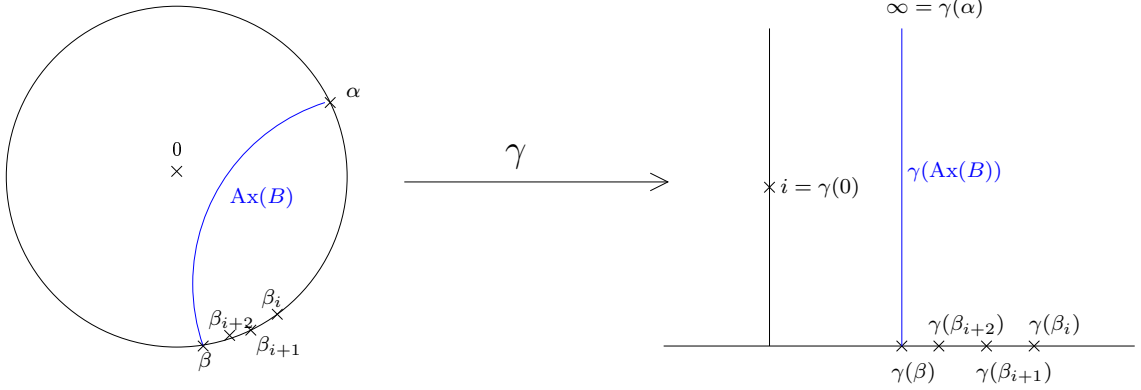
Figure 2.5: How the maps lift to $\mathbb{H}$, with points $\gamma(\beta_i)$ tending to $\gamma(\beta)$.

Let

$$s_n := t_n^{-1} = B_n \circ \ldots \circ B_1.$$

Writing $\rho_{\mathbb{H}}$ for the hyperbolic metric on $\mathbb{H}$, by conformal invariance we have

$$d_h(0, t_n(z)) = \rho_{\mathbb{H}}(i, \gamma(t_n(z)))$$
$$= \rho_{\mathbb{H}}(i, \widetilde{t_n}(\gamma(z))$$
$$= \rho_{\mathbb{H}}(\widetilde{s_n}(i), \gamma(z)).$$

We can rewrite $\widetilde{s}_n(i)$ as

$$\widetilde{s}_n(i) = \Phi_n^{-1} \circ \widehat{B}_n \circ \Phi_n \circ \Phi_{n-1}^{-1} \circ \widehat{B}_{n-1} \circ \Phi_{n-1} \circ \ldots \circ \Phi_1^{-1} \circ \widehat{B}_1 \circ \Phi_1(i).$$

It is not hard to see that

$$\widetilde{s}_n(i) = \sum_{j=1}^{n} \left((X_{j-1} - X_j) \left(\prod_{i=j}^{n} k_i \right) \right) + X_n + i \prod_{i=1}^{n} k_i,$$

where we use the convention $X_0 = 0$. Writing

$$P_n = \prod_{i=1}^{n} k_i, \quad R_n = \sum_{j=1}^{n} \left((X_{j-1} - X_j) \left(\prod_{i=j}^{n} k_i \right) \right)$$

and $\gamma(z) = x + iy \in \mathbb{H}$, we see that since $X_n \to X$ and $\widetilde{s}_n(i) \to X$ by Theorem 2.23, we have

$R_n \to 0$ as $n \to \infty$. By the formula for the hyperbolic metric in $\mathbb{H}$ [1, see §3.4],

$$
\begin{aligned}
\rho_{\mathbb{H}}(\widetilde{s}_n(i), \gamma(z)) &= \cosh^{-1}\left(\frac{(R_n + X_n - x)^2 + (P_n - y)^2}{2P_n y}\right) \\
&= \cosh^{-1}\left(\frac{P_n}{2y} + \frac{R_n^2 + X_n^2 + x^2 - 2xR_n - 2xX_n + 2X_nR_n - y}{2P_n}\right).
\end{aligned}
\tag{2.6}
$$

Since $R_n \to 0$, $X_n \to X$ and x, y are fixed,

$$
(R_n^2 + X_n^2 + x^2 - 2xR_n - 2xX_n + 2X_nR_n - y) \longrightarrow (X^2 + x^2 - 2xX - y),
\tag{2.7}
$$

as $n \to \infty$. This expression is bounded. We also have

$$
\frac{P_n}{2y} \to 0 \text{ as } n \to \infty.
\tag{2.8}
$$

Hence, from (2.6),(2.7), (2.8) and using the identity $\cosh^{-1}(z) = \log(z + \sqrt{z^2 + 1})$, we can write

$$
\begin{aligned}
\rho_{\mathbb{H}}(\widetilde{s}_n(i), \gamma(z)) &= \cosh^{-1}\left[O\left(\frac{1}{P_n}\right)\right] \\
&= \log\left[O\left(\frac{1}{P_n}\right)\right] \tag{2.9} \\
&= \log\left(\frac{1}{P_n}\right) + O(1), \tag{2.10}
\end{aligned}
$$

which proves the lemma. $\qquad\square$

We will need to use the following result on Blaschke products, see for example [4, 10]. A Blaschke product B is given by

$$
B(z) := \zeta \prod_{i=1}^{n} \left(\frac{z - a_i}{1 - \overline{a_i}z}\right)^{m_i},
$$

where $\zeta \in \partial\mathbb{D}$ and $|a_i| < 1$.

We are only concerned with Blaschke products of degree two and in this case we have the following standard result.

Proposition 2.26. *Let B be a Blaschke product of degree 2. Then the Julia set $J(B)$ is contained in S^1 and we have the following cases:*

- *If B has one fixed point in S^1, one fixed point in $\mathbb{D}$ and one fixed point in $\mathbb{C} \setminus \overline{\mathbb{D}}$, then $J(B) = S^1$.*

- *If B has one fixed point in S^1 of multiplicity three, and no other fixed points, then $J(B) = S^1$.*

- *If B has one repelling and one neutral fixed point in S^1, then $J(B)$ is a Cantor subset of S^1.*

- *If B has three fixed points in S^1, then $J(B)$ is a Cantor subset of S^1.*

This proposition is shown in [10, p58]. However, let's discuss the cases separately. Up to multiplicity B must have three fixed points. If z_0 is a fixed point then $1/\overline{z_0}$ must be a fixed point also, so there must always be at least one fixed point on S^1.

Suppose $z_0 \in \mathbb{D}$ is an attracting fixed point then the Denjoy-Wolff Theorem [10, §IV Theorem 3.1] tells us $B^n(z) \to z_0$ for all $z \in \mathbb{D}$ and also that the fixed point on S^1 must be repelling. Using the inversion $g(z) = 1/\overline{z}$ we see that $1/\overline{z_0}$ is an attracting fixed point also, such that $B^n(z) \to 1/\overline{z_0}$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. As $J(B)$ must be the boundary of the attracting basins of the fixed points, this implies $J(B) = S^1$. Similarly if $z_0 \in S^1$ is a fixed point of multiplicity three, then by the Denjoy-Wolff Theorem $B^n(z) \to z_0$ for $z \in \mathbb{D}$ and using the inversion g again we have $B^n(z) \to z_0$ for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, hence $J(B) = S^1$.

If there are three distinct fixed points on S^1, then the Denjoy-Wolff Theorem tells us that precisely one of them must be attracting and the other two are repelling. $J \neq S^1$ as the attracting fixed point is not in $J(B)$ and so $J(B)$ is a Cantor set in S^1. If there is fixed point of multiplicity two, then it must have one attracting direction of points on S^1 and $J(B)$ is a Cantor set.

Chapter 3

Quasiregular maps and dynamics

3.1 Quasiregular maps

A comprehensive study into quasiregular mappings is given by Rickman in his monograph [33], for our purposes the following is more than sufficient. Let $d > 1$ and let $\mathcal{D} \subset \mathbb{R}^d$ be a domain.

Definition 3.1. Let $ACL(\mathcal{D})$ be the set of all continuous maps $f = (f_1, \ldots, f_d) : \mathcal{D} \to \mathbb{R}^d$ which are absolutely continuous on almost all lines parallel to the coordinate axes. For a map $f \in ACL(\mathcal{D})$ the partial derivatives $\partial_k f_j$ exist almost everywhere. For $p \geq 1$ we let $ACL^p(\mathcal{D})$ denote the set of all $f \in ACL(\mathcal{D})$ for which all partial derivatives are locally L^p-integrable.

If $f : \mathcal{D} \to \mathbb{R}^d$ is a continuous map, then $f \in ACL^p(\mathcal{D})$ if and only if f belongs to the Sobolev space $W^1_{p,loc}(\mathcal{D})$.

Definition 3.2.

$$W^1_{p,loc} = \{f : \mathcal{D} \to \mathbb{R}^d \mid \text{ each } \partial_k f_j \text{ exists and is locally in } L^p\}.$$

Denote the Euclidean norm of $x \in \mathbb{R}^n$ by $|x|$.

Definition 3.3. A map $f \in ACL^d(\mathcal{D})$ is called quasiregular if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^d \leq K_O J_f(x) \quad \text{a.e.;} \tag{3.1}$$

where $Df(x)$ denotes the derivative,

$$|Df(x)| := \sup_{|h|=1} |Df(x)(h)|,$$

denotes its norm, and $J_f(x)$ denotes the Jacobian determinant. Let

$$\ell(Df(x)) := \inf_{|h|=1} |Df(x)(h)|.$$

The condition that (3.1) holds for some $K_O \geq 1$ is equivalent to the condition that

$$J_f(x) \leq K_I \ell(Df(x)) \quad \text{a.e.,} \tag{3.2}$$

for some $K_I \geq 1$. The smallest constants $K_O = K_O(f)$ and $K_I = K_I(f)$ for which (3.1) and (3.2) hold are called the outer and inner dilation of f. Further $K := \max\{K_I(f), K_O(f)\}$ is called the dilation of f. We say that f is K-quasiregular if $K(f) \leq K$.

Note that an injective K-quasiregular map is K-quasiconformal. Further it is clear from the equations (3.1) and (3.2) that the composition of two, and so inductively a finite number, of quasiregular maps is itself quasiregular.

Lemma 3.4. *If f and g are quasiregular then $f \circ g$ is quasiregular, assuming that the composition is well defined. Further;*

$$K(f \circ g) \leq K(f)K(g).$$

We denote the one point compactification of $\mathbb{R}^d$ by $\overline{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$ in the usual way (see for instance [29]). We say $p \in \mathbb{R}^d$ is a pole of a quasiregular mapping $f : \mathbb{R}^d \to \overline{\mathbb{R}}^d$ if $f(z_n) \to \infty$ as $n \to \infty$ for every sequence of points z_n that tend to p and we write $f(p) = \infty$.

Many properties of holomorphic maps hold for quasiregular maps as well. For example, non-constant quasiregular maps are open and discrete (Chapter I, Theorem 4.1 [33]). Also a modified version of Picard's Theorem is true, shown by Rickman.

Theorem 3.5 ([32] Theorem 1.1). *Let $d \in \mathbb{N}, d \geq 2$, and $K \geq 1$. There exists a constant $q = q(d, K)$ with the following property: if $a_1, \ldots, a_q \in \mathbb{R}^d$ are distinct points and if $f : \mathbb{R}^d \to \overline{\mathbb{R}}^d \setminus \{a_1, \ldots, a_q\}$ is K-quasiregular, then f is constant.*

Equivalently the theorem tells us that a non-constant K-quasiregular map $f : \mathbb{R}^d \to \overline{\mathbb{R}}^d$ omits at most q values. Note that Picard's theorem is a special case of this where $q(2, 1) = 2$. Also we have an analogue of Montel's Theorem, shown by Miniowitz.

Theorem 3.6 ([30] Theorem 4). *Let $d \geq 2$ and $K \geq 1$. Let $a_1, \ldots, a_q \in \overline{\mathbb{R}}^d$ be distinct points, where $q = (d, K)$ is as in Theorem 3.5. Let $\Omega \subset \mathbb{R}^d$ be a domain. Then the family of all K-quasiregular maps $f : \Omega \to \overline{\mathbb{R}}^d \setminus \{a_1, \ldots, a_q\}$ is normal.*

Equivalently this tells us, if $a_1, \ldots, a_q \in \overline{\mathbb{R}}^d$ are distinct, then the family of all K-quasiregular maps $f : \mathbb{R}^d \to \overline{\mathbb{R}}^d \setminus \{a_1, \ldots, a_q\}$ is normal.

3.1.1 Quasiregular maps of the plane

We will be concerned with quasiregular maps of $\mathbb{C}$, that is we have $d = 2$, where a lot more is known. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider quasiregular maps $f : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subset \mathbb{C}$ is a domain. For a detailed study see, for instance, [2] or [31]. We have

$$|Df(z)| = |f_z(z)| + |f_{\bar{z}}(z)|,$$

$$\ell(Df(z)) = |f_z(z)| - |f_{\bar{z}}(z)|,$$

and

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2$$

whenever the partial derivatives of f exist. It follows that

$$K(f) = K_O(f) = K_I(f) = \frac{1+k}{1-k}$$

where

$$k := \operatorname{ess\,sup}_{z \in \mathcal{D}} \left| \frac{f_{\bar{z}}(z)}{f_z(z)} \right|.$$

Note that the 1-quasiregular maps are precisely the holomorphic, or meromorphic, functions. We also have the following standard definition as with quasiconformal maps, see for example [20].

Definition 3.7. If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z then the complex dilatation of f at z is defined as,

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z}.$$

The distortion at z is defined as,

$$K(f)(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

We will also require the following useful result about composed mappings.

Lemma 3.8 ([20] p.6). *Suppose $f, g : \mathbb{C} \to \mathbb{C}$ are quasiregular maps with complex dilatation μ_f*

and μ_g. Then

$$\mu_{g \circ f} = \frac{\mu_f + r_f(\mu_g \circ f)}{1 + r_f \overline{\mu_f}(\mu_g \circ f)},$$

where $r_f = \overline{f_z}/f_z$.

We now state more some results given in the survey [5].

Theorem 3.9 ([5] Theorem 3.3). *Let $\mu : \overline{\mathbb{C}} \to \mathbb{C}$ be a measurable function with $k := ||\mu||_\infty < 1$. Then there exists a K-quasiconformal homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $K := (1 + k)/(1 - k)$ such that*

$$\frac{f_{\overline{z}}(z)}{f_z(z)} = \mu(z) \quad a.e. \tag{3.3}$$

The map f may be chosen to fix $0, 1$ and ∞; with this normalisation it is unique.

Equation (3.3) is called the Beltrami equation. A consequence of this theorem is the following.

Theorem 3.10 ([5] Theorem 3.4). *Let $U, V \subset \mathbb{C}$ be simply connected domains with $U, V \neq \mathbb{C}$. Let $\mu : U \to \mathbb{C}$ be measurable with $k := ||\mu||_\infty < 1$ and put $K := (1 + k)/(1 - k)$. Then there exists a K-quasiconformal homeomorphism $f : U \to V$ such that $f_{\overline{z}}(z)/f_z(z) = \mu(z)$ a.e.*

Notice that the case $\mu(z) \equiv 0$ is the Riemann mapping theorem. Therefore Theorems 3.9 and 3.10 are also called the measurable Riemann mapping theorem. In the plane, every quasiregular mapping has a useful decomposition which we will be using extensively.

Theorem 3.11 (The Stoilow factorisation, see for example [26] p.254). *Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiregular mapping. Then there exists an analytic function g and a quasiconformal mapping h such that $f = g \circ h$.*

A direct consequence of this and Montel's Theorem is that $q(2, K) = 2$. The Stoilow factorisation tells us what the branch set of quasiregular maps of the plane can be. Recall the definition of the set of branch points $B(f)$ from Definition 2.11. A quasiconformal map is a homeomorphism by definition, so has no branch points. A polynomial must have finitely many branch points, as must a mapping of *polynomial type* which we define precisely now.

Definition 3.12. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be of polynomial type if $|f(x)| \to \infty$ as $|x| \to \infty$.

Using this we have the following corollary of the Stoilow factorisation.

Corollary 3.13. *Let $f : \mathbb{C} \to \mathbb{C}$ be quasiregular. Then $B(f)$ is a discrete set of points. If f is quasiregular of polynomial type, then $B(f)$ is a finite set of points.*

3.2 Quasiregular dynamics

3.2.1 Uniformly quasiregular dynamics

We call a quasiregular mapping $f : \mathcal{D} \to \mathbb{C}$ *uniformly quasiregular* if there exists $K \geq 1$ such that $K_{f^n}(z) \leq K$ for all $n \in \mathbb{N}$ and for all $z \in \mathcal{D}$.

If f is uniformly quasiregular then direct analogues of Fatou and Julia sets can be defined. However for non-uniformly quasiregular functions, $f : \mathbb{C} \to \mathbb{C}$, we have no common bound on the distortion of the family of functions $\{f^k\}_{k \in \mathbb{N}}$, so cannot define the Fatou set (and so the Julia set also) easily. It is however still possible to define the escaping set $I(f)$. By Proposition 2.8 we know $J(f) = \partial I(f)$ when f is a polynomial. In fact this is still true when f is just a transcendental entire function, shown by Eremenko [13]. It is therefore natural to consider $\partial I(f)$ for quasiregular mappings and see to what extent it can be considered an analogue of $J(f)$. We will be considering quasiregular mappings of polynomial type, so that infinity is an attracting fixed point. We also have the definition of the degree of a quasiregular mapping, which as expected is defined as the maximal cardinality of the preimage of a point of $\mathbb{C}$.

Definition 3.14. The degree of a quasiregular mapping $f : \mathbb{C} \to \mathbb{C}$ is given by

$$\deg(f) = \sup_{z \in \mathbb{C}} |\{f^{-1}(z)\}|.$$

We have the following results on quasiregular mappings of polynomial type from the paper by Fletcher and Nicks [21].

Theorem 3.15 ([21] Theorem 1.1). *Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiregular of polynomial type. If the degree of f is greater than K_I, then $I(f)$ is a non-empty open set and $\partial I(f)$ is perfect.*

Notice how these properties are the same as for a polynomial f. Further, compare the following theorem to the earlier Theorem 2.4 for rational functions to see the similar properties.

Theorem 3.16 ([21] Theorem 1.2). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiregular of polynomial type and suppose that the degree of f is greater than K_I. Then:*

(i) *for any $k \geq 2$ we have $I(f^k) = I(f)$,*

(ii) *the family of iterates $\{f^k \mid k \in \mathbb{N}\}$ is equicontinuous on $I(f)$ and not equicontinuous on any point of $\partial I(f)$, with respect to the spherical metric on $\overline{\mathbb{R}}^n$,*

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(*iii*) $\partial I(f)$ *is infinite,*

(*iv*) $I(f), \partial I(f)$ *and* $\mathbb{R}^n \setminus \overline{I(f)}$ *are completely invariant,*

(*v*) $I(f)$ *is connected.*

To see that the condition that the degree of f is greater than K_I is necessary, consider the following example.

Example 3.17 ([19] Example 4.1). *Consider the winding map* $f : (r, \theta) \mapsto (r, 2\theta)$ *in polar coordinates. This map decomposes as* $f = g \circ h$, *where* $g(z) = z^2$ *and* $h(r, \theta) = (r^{\frac{1}{2}}, \theta)$. *We have the following equalities regarding partial derivatives of* h,

$$h_z = \frac{1}{2}(h_x - ih_y) \tag{3.4}$$

$$h_{\overline{z}} = \frac{1}{2}(h_x + ih_y) \tag{3.5}$$

$$rh_r = xh_x + yh_y \tag{3.6}$$

$$h_\theta = xh_y - yh_x. \tag{3.7}$$

Combining (3.6) *and* (3.7) *we obtain,*

$$h_y = \frac{yrh_r + xh_\theta}{x^2 + y^2} \tag{3.8}$$

and

$$h_x = \frac{-xrh_r + yh_\theta}{x^2 + y^2} \tag{3.9}$$

Using (3.4),(3.5),(3.8) *and* (3.9) *we can obtain an expression for the complex dilatation of* h.

$$\mu_h = \frac{h_z}{h_{\overline{z}}} = \frac{h_x - ih_y}{h_x + ih_y} = \frac{yh_\theta - xrh_r - iyrh_r + ixh_\theta}{yh_\theta - xrh_r + iyrh_r + ixh_\theta}. \tag{3.10}$$

Grouping partial derivatives and multiplying the numerator and denominator of (3.10) *by -1 we see,*

$$\mu_h = \frac{(x + iy)[rh_r + ih_\theta]}{(x - iy)[rh_r - ih_\theta]}. \tag{3.11}$$

Recalling $z = x + iy = re^{i\theta}$, $\overline{z} = x - iy = re^{-i\theta}$ *and dividing through by* r *we are left with the expression,*

$$\mu_h = e^{2i\theta} \frac{h_r + \frac{i}{r}h_\theta}{h_r - \frac{i}{r}h_\theta}. \tag{3.12}$$

$$h_r = \frac{e^{i\theta}}{2r^{\frac{1}{2}}}, h_\theta = ir^{\frac{1}{2}}e^{i\theta}.$$

Hence, by (3.12), the complex dilatation is

$$\mu_h = \frac{-e^{2i\theta}}{3}.$$

So $\|\mu_h\| = 1/3$ and the distortion of f is 2 and the degree is 2. However $I(f)$ is empty since $|f(z)| = |z|$ for all $z \in \mathbb{C}$.

When f is uniformly quasiregular we have a bound on the distortion of the iterates. We can define the Julia set and have the following result, which is analogous to the case where f is a rational function.

Theorem 3.18 ([21] Theorem 1.3). *Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a uniformly K-quasiregular mapping which is not injective. Then $\partial I(f) = J(f)$ and is an infinite, perfect set.*

See Hinkkanen, Martin and Mayer [25] for more examples of uniformly quasiregular dynamics.

3.2.2 Quasiregular dynamics in the plane

When we are in the complex plane things are much simpler. We have the very useful Stoilow decomposition of quasiregular functions and also the following theorem due to Hinkkanen.

Theorem 3.19 ([23] Theorem 1). *Every uniformly quasiregular map $f : \mathbb{C} \to \mathbb{C}$ is quasiconformally conjugate to a holomorphic map.*

So if we are studying quasiregular dynamics of the plane, then if the map is uniformly quasiregular we can apply results from analytic functions. Hence our study is only of independent interest if we consider non-uniformly quasiregular maps of the plane.

We will see in the next section that there is a quasiregular analogue of quadratic polynomials, it is these mappings that we will be studying. Informally they consist of an affine stretch of magnitude K in direction θ, given by $h_{K,\theta}$, then composition with a quadratic polynomial. We will see in Proposition 4.1 that this composition is linearly conjugate to a special form

$$f_{K,\theta,c} := (h_{K,\theta})^2 + c, \tag{3.13}$$

for $K > 1, \theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$. The following results are due to Fletcher and Goodman.

Proposition 3.20 ([19] Corollary 4.4). *Let $f_{K,\theta,c}$ be defined as in (3.13). Then $I(f_{K,\theta,c})$ is a non-empty open set and $\partial I(f_{K,\theta,c})$ is a perfect set.*

Theorem 3.21 ([19] Theorem 4.5). *Let $f = f_{K,\theta,c}$ be defined as in (3.13). Then for any $k > 2, I(f^k) = I(f)$. The family of iterates $\{f^k \mid k \in \mathbb{N}\}$ is equicontinuous on $I(f)$ and not equicontinuous at any point of $\partial I(f)$. The set $\partial I(f)$ is infinite. The sets $I(f), \partial I(f)$ and $\overline{I(f)}^c$ are all completely invariant. The escaping set is a connected neighbourhood of infinity.*

Recall that the non-escaping set $N(f_{K,\theta,c}) = I(f_{K,\theta,c})^c$, $N(f_{K,\theta,c})$ is completely invariant by Theorem 3.21. This set $N(f_{K,\theta,c})$ is the analogue of the filled in Julia set for a polynomial. Fletcher and Goodman showed they share similar properties.

Theorem 3.22 ([19] Theorem 5.2). *$N(f_{K,\theta,c})$ is connected if and only if $I(f_{K,\theta,c}) \cap B(f_{K,\theta,c}) = \emptyset$.*

We now consider some examples to visualise these concepts.
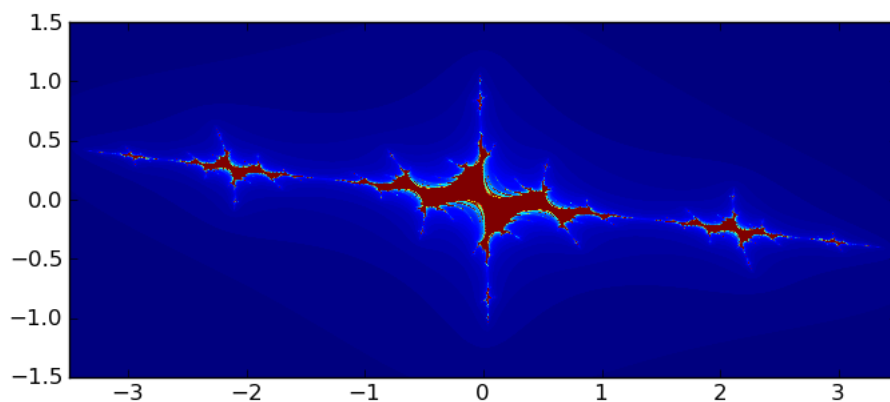
Figure 3.1: $N(f_{K,\theta,c})$ for $K = 1.2, \theta = 0.7\pi$ and $c = 2.297 - 0.295i$.
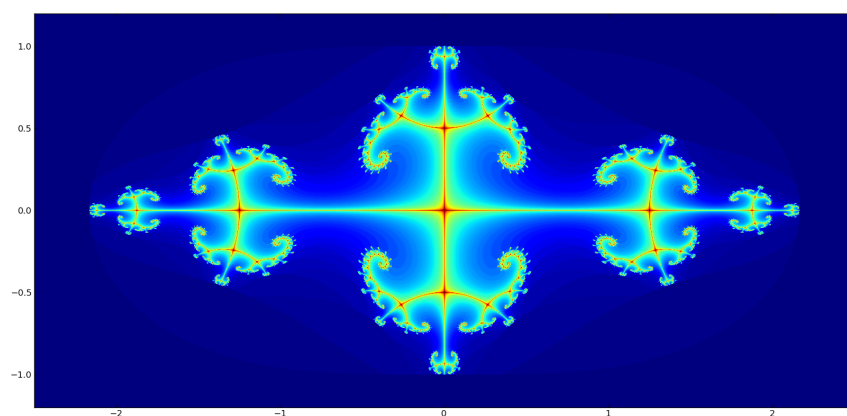
Figure 3.2: $N(f_{K,\theta,c})$ for $K = 0.8, \theta = 0$ and $c = -1.1$.
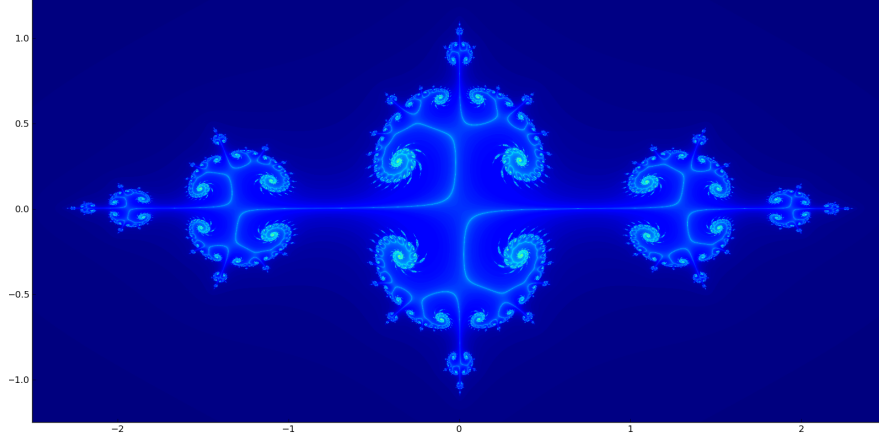
Figure 3.3: $N(f_{K,\theta,c})$ for $K = 0.8, \theta = 0$ and $c = -1.1 + 0.003i$.

Figure 3.1 shows $N(f_{1.2,0.7,2.297-0.295i})$; it has non-empty interior and $\partial I(f_{1.2,0.7,2.297-0.295i})$ is connected. Figure 3.2 shows $N(f_{0.8,0,-1.1})$; it has empty interior and $\partial I(f_{0.8,0,-1.1})$ is connected. Figure 3.3 shows $N(f_{0.8,0,-1.1+0.003i})$; here $\partial I(f_{0.8,0,-1.1+0.003i})$ is totally disconnected.

For any choice of $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$ we have that 0 is the only branch point of $f_{K,\theta,c}$. Hence in the previous theorem we are only interested in whether 0 escapes to know whether $N_{f_{K,\theta,c}}$ is connected or not. As with the traditional Mandelbrot set, we define the K,θ-*Mandelbrot set* to be

$$\mathcal{M}_{K,\theta} := \{c \in \mathbb{C} \mid f_{K,\theta,c}^n(0) \text{ is bounded}\}.$$

Note that $\mathcal{M}_{1,0} = \mathcal{M}$. By Theorem 3.22 we have the equivalent form,

$$\mathcal{M}_{K,\theta} = \{c \in \mathbb{C} \mid \partial I(f_{K,\theta,c}) \text{ is connected}\}.$$

Fletcher and Goodman also showed the following results that show similarities to the traditional Mandelbrot set (compare with Theorem 2.14).

Theorem 3.23 ([19] Theorem 6.3). *Let* $K \geq 1, \theta \in (\pi/2, \pi/2]$. *Then*

$$\mathcal{M}_{K,\theta} \subset \{c \in \mathbb{C} \mid |c| \leq 2K^{-2}\},$$

Further $\mathcal{M}_{K,\theta}$ *is compact and can be characterised as the set of* $c \in \mathbb{C}$ *for which* $f_{K,\theta,c}^n(0) \leq 2K^{-2}$ *for all* $n \in \mathbb{N}$.

Theorem 3.24 ([19] Theorem 6.4). *There exists $\phi_0 \in [0, 2\pi)$ and a real number η such that the line segment*

$$te^{i\phi_0} \subset \mathcal{M}_{K,\theta},$$

for

$$t \in \left[-\frac{2}{\eta}, \frac{1}{4\eta^2} \right].$$

Remark 3.25. *Further, it is shown that the angle ϕ_0 is the angle of a fixed ray R_{ϕ_0} of the mapping $h_{K,\theta}^2$. We will be studying these in more detail in Chapter 6.*

It is conjectured that $\mathcal{M}_{K,\theta}$ will share more properties with the Mandelbrot set, for instance that it is connected.
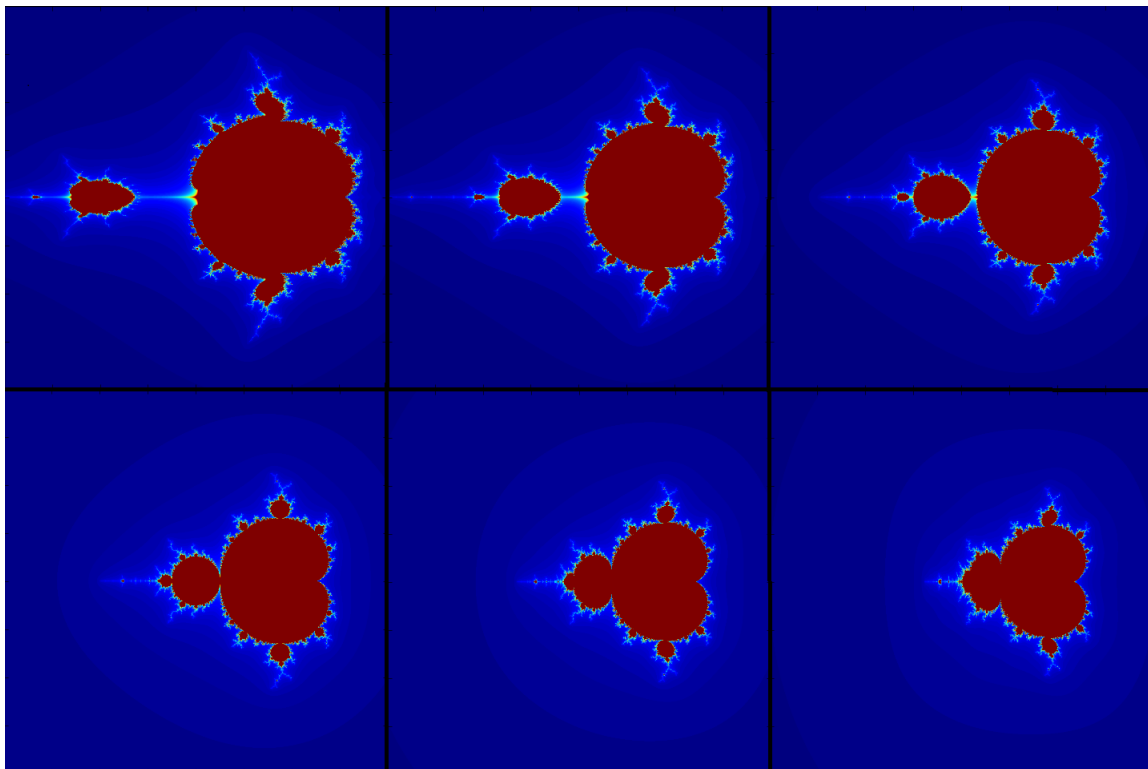
Figure 3.4: $\mathcal{M}_{K,0}$ for, starting top left and moving clockwise, $K = 0.7, 0.8, 0.9, 1.2, 1.1$ and 1.

Figure 3.4 shows how $\mathcal{M}_{K,0}$ varies for $K > 1$ and $K < 1$. Note that the bottom left set is the traditional Mandelbrot set.
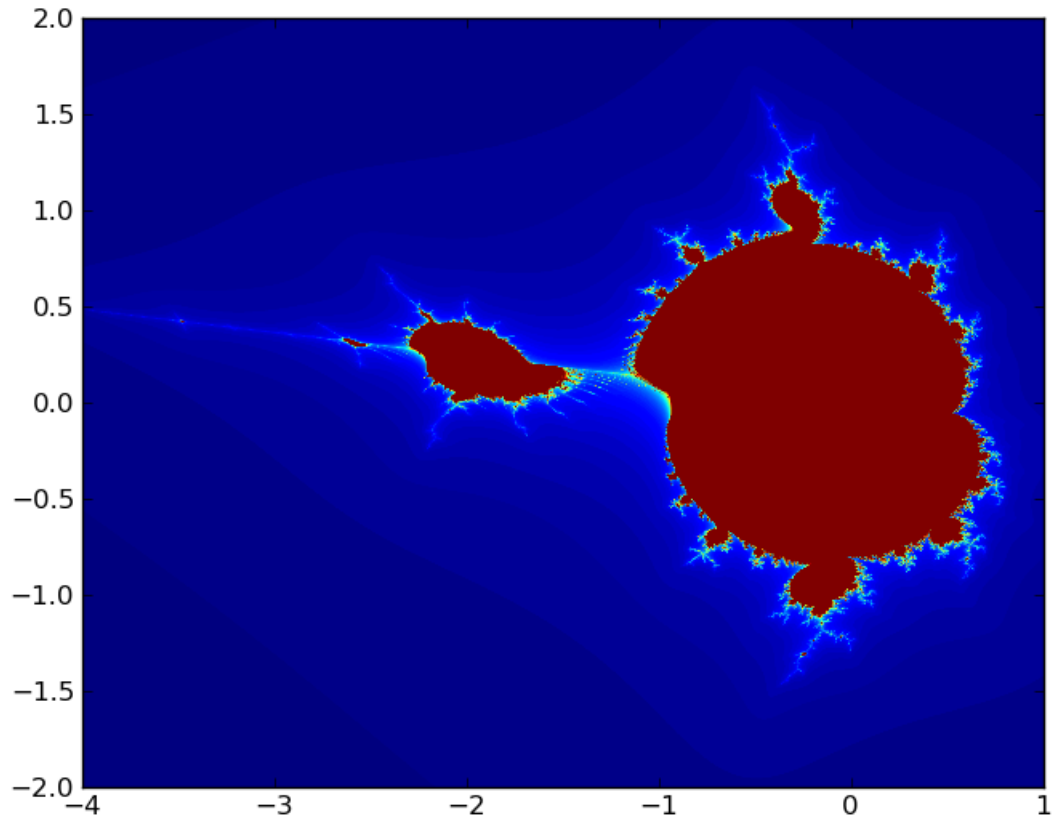
Figure 3.5: $\mathcal{M}_{0.7,\pi/12}$

Figure 3.5 depicts an example where $\theta \neq 0$; notice how there is still an interval contained in $\mathcal{M}_{0.7,\pi/12}$ but it is no longer contained in the real axis.

Chapter 4

Quasiregular maps of the plane of constant dilatation

We will now begin our main object of study. We consider the simplest non-trivial quasiregular examples, where we have degree two quasiregular mappings of the plane with constant complex dilatation not equal to zero.

4.1 Maps of constant dilatation

We first define an affine stretch, which will form part of a canonical example which our maps of constant complex dilatation will be conjugate to.

4.1.1 The affine stretch $h_{K,\theta}$

Consider an affine mapping $h := h_{K,\theta} : \mathbb{C} \to \mathbb{C}$ which stretches by a factor $K > 0$ in the direction $e^{i\theta}$. If $\theta = 0$, then

$$h_{K,0}(x + iy) = Kx + iy.$$

For general θ, pre-compose $h_{K,0}$ by a rotation of $-\theta$ and post-compose by a rotation of θ to give the expression

$$h_{K,\theta}(x + iy) = x(K \cos^2 \theta + \sin^2 \theta) + y(K - 1) \sin \theta \cos \theta$$

$$+i \left[x(K - 1) \cos \theta \sin \theta + y(K \sin^2 \theta + \cos^2 \theta) \right] \tag{4.1}$$

or

$$h_{K,\theta}(z) = \left(\frac{K + 1}{2} \right) z + e^{2i\theta} \left(\frac{K - 1}{2} \right) \overline{z}. \tag{4.2}$$

Using the formula for complex dilatation given in Definition 3.7, we see that

$$\mu_{h_{K,\theta}} = e^{2i\theta}\frac{K-1}{K+1}, \tag{4.3}$$

and so $||\mu_{h_{K,\theta}}||_\infty < 1$ which means that $h_{K,\theta}$ is quasiconformal with constant complex dilatation.

If $K = 1$, then this mapping is the identity and does not depend on θ. In this thesis we continue the study of the dynamics of the quasiregular mappings $h(z)^2 + c$ initiated in [19], where $h = h_{K,\theta}$ for $K > 1$ and $\theta \in (\pi/2, \pi/2]$, and $c \in \mathbb{C}$. If the mapping h is fixed, we will write $H(z) = h(z)^2$.

4.1.2 The canonical form $h_{K,\theta}^2 + c$

The justification for studying these mappings in the class of degree two quasiregular mappings of the plane with constant complex dilatation is given by the following proposition, first shown in a similar form by Fletcher and Goodman [19]. We include the proof here so that we can compare the extra complication given by quasiregular mappings, with the corresponding holomorphic version given in Proposition 2.10.

Proposition 4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be quasiregular of degree two and let f have constant complex dilatation that is not identically 0. Then f is linearly conjugate to a unique mapping of the form $f_{K,\theta,c} := h_{K,\theta}(z)^2 + c$ for some $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $c \in \mathbb{C}$.*

Note that this proof is a slight modification of the proof of [19, Proposition 3.1], with a different normalisation.

Proof. Let f satisfy the hypotheses of the proposition and let $\mu_f \equiv \mu$. By Theorem 3.11, we can write $f = \widetilde{g} \circ \widetilde{h}$ for some quadratic polynomial $\widetilde{g}$ and quasiconformal map $\widetilde{h}$ with constant complex dilatation. We may assume that $\widetilde{h}$ fixes 0.

Let $h_{K,\theta}$ be defined as in (4.2), where K, θ are chosen such that $\left(\frac{K-1}{K+1}\right)e^{2i\theta} = \mu$. Then by the formula for the complex dilatation of a composition, see Lemma 3.8, we have

$$\mu_{\widetilde{h} \circ h_{M,\phi}^{-1}} \equiv 0.$$

Therefore, there exists a conformal map $\Upsilon : \mathbb{C} \to \mathbb{C}$ such that $\widetilde{h} = \Upsilon \circ h_{M,\phi}$. We can therefore write $f = g \circ h_{M,\phi}$, where $g = \widetilde{g} \circ \Upsilon$ is a quadratic polynomial.

Let $g(z) = \alpha z^2 + \beta z + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha \neq 0$. Let $h = h_{M,\phi}$, for $M > 0$ and $\phi \in [-\pi, \pi]$, and write $f(z) = g(h(z))$. We need to see how h behaves under pre-composition by

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translations and dilations. Let $\upsilon(z) = Az$ for some $A \in \mathbb{C} \setminus \{0\}$. Then using (4.2)

$$
\begin{aligned}
h(\upsilon(z)) = h(Az) &= \left(\frac{M+1}{2}\right) Az + e^{2i\theta} \left(\frac{M-1}{2}\right) \overline{A}\overline{z} \\
&= A \left(\left(\frac{M+1}{2}\right) z + e^{2i(\theta - \arg(A))} \left(\frac{M-1}{2}\right) \overline{z} \right) \\
&= A h_{M, \theta - \arg(A)}(z).
\end{aligned}
\tag{4.4}
$$

Let $\tau(z) = z + B$ for some $B \in \mathbb{C}$. Again using (4.2) and noting that h is $\mathbb{R}$-linear,

$$
h(\tau(z)) = h(z) + h(B).
\tag{4.5}
$$

Using (4.4) with $A = 1/a$ we see,

$$
\begin{aligned}
\upsilon^{-1} \circ f \circ \upsilon(z) &= \alpha(\alpha(h_{M,\phi}(z/\alpha))^2 + \beta h_{M,\phi}(z/\alpha) + \gamma) \\
&= (h_{M,\phi+\arg(\alpha)}(z))^2 + \beta h_{M,\phi+\arg(\alpha)}(z) + \alpha\gamma \\
&= \left(h_{M,\phi+\arg(\alpha)}(z) + \frac{\beta}{2} \right)^2 + \alpha\gamma - \frac{\beta^2}{4}.
\end{aligned}
$$

Applying (4.5) with $B = h_{M,\phi+\arg(\alpha)}^{-1}(-\beta/2)$, we see

$$
\tau^{-1} \circ \upsilon^{-1} \circ f \circ \upsilon \circ \tau(z) = (h_{M,\phi+\arg(\alpha)}(z))^2 + \alpha\gamma - \frac{\beta^2}{4} - h_{M,\phi+\arg(\alpha)}^{-1}\left(-\frac{\beta}{2}\right).
$$

Hence f is linearly conjugate to $f_{K,\theta,c}$ with $K = M, \theta = \phi + \arg(\alpha)$ and $c = \alpha\gamma - \beta^2/4 - h_{K,\theta}^{-1}(-\beta/2)$.

For the uniqueness, we note that the choice of $K > 0$ and $\theta \in [-\pi, \pi]$ for a given complex dilatation μ is not unique. However there are the symmetries $(\theta \mapsto \theta + \pi)$ and $(K \mapsto 1/K, \theta \mapsto \theta + \pi/2)$. The first is obvious as $h_{K,\theta} = h_{K,\theta+\pi}$ and the second symmetry corresponds to the equality $h_{K,\theta} = K h_{1/K,\theta+\pi/2}$. There are no other symmetries.

We see that $f_{K,\theta,C}$ is linearly conjugate to $f_{1/K,\theta+\pi/2,CK^2}$ via the conjugation $L(z) = z/K^2$, so if $M < 1$ we can apply L so that $1/M > 1$, hence we are conjugate to $f_{K,\theta,C}$ for some $K > 1$. Also if $\phi + \arg(a) \notin (\pi/2, \pi/2]$ we take $-\phi - \arg(a)$ instead, so $\theta \in (-\pi/2, \pi/2]$.

Finally noting that all stretches with $K = 1$ correspond to the identity, so are equivalent, and noting that they have complex dilatation 0 and so not considered completes the proof. $\square$

In this thesis we mostly study the case where $c = 0$ and we suppress the subscripts K

and θ where there will be no confusion. We can restrict ourselves to studying only the $c = 0$ case because of the following theorem. This theorem gives an analogue of Böttcher coordinates for these mappings, see Theorem 2.18 for the analytic case. This theorem will be proved in the next chapter.

Theorem 5.1. *Let $h : \mathbb{C} \to \mathbb{C}$ be an affine mapping and $c \in \mathbb{C}$. Then there exists a neighbourhood $U = U(h, c)$ of infinity and a quasiconformal map $\psi = \psi(h, c)$ such that*

$$h(\psi(z))^2 = \psi(f(z)), \tag{4.6}$$

for $z \in U$, where $f(z) = h(z)^2 + c$. Further, ψ is asymptotically conformal as $|z| \to \infty$.

By Proposition 4.1 we know that any degree two mapping of constant complex dilatation is linearly conjugate to a mapping $f_{K,\theta,c}$ for some K, θ, c. Then Theorem 5.1 tells us that $f_{K,\theta,c}$ is quasiconformally conjugate to $H_{K,\theta} = h_{K,\theta}^2$ in a neighbourhood of infinity. Therefore we may restrict our attention to the study of dynamics of the mappings $H_{K,\theta}$. We also note that for a fixed K the maps $H_{K,\theta}$ and $H_{K,-\theta}$ are related.

Lemma 4.2. *We have $H_{K,-\theta}(z) = \overline{H_{K,\theta}(\overline{z})}$.*

Proof.

$$\overline{H_{K,\theta}(\overline{z})} = \left(\frac{K+1}{2} \right) z + e^{-2i\theta} \left(\frac{K-1}{2} \right) \overline{z}$$
$$= H_{K,-\theta}(z).$$

$\square$

Lemma 4.2 means that we can just study the range $\theta \in [0, \pi/2]$ then transfer the results using complex conjugation to extend to $\theta \in (-\pi/2, 0)$.

4.2   Polar form of H

Fix $K > 1, \theta \in (-\pi/2, \pi/2]$ and $H = h_{K,\theta}^2$. We now formulate the polar form of H.

4.2.1 Calculation of argument and magnitude

Lemma 4.3. *Let $z = re^{i\varphi}$, then the following equation holds.*

$$H(re^{i\varphi}) = r^2(1 + (K^2 - 1)\cos^2(\varphi - \theta))\exp\left[2i\left(\tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right) + \theta\right)\right]. \qquad (4.7)$$

Where $\tan^{-1}$ *takes values in* $(-\pi/2, \pi/2)$.

Proof. First we show $\arg[H(re^{i\varphi})] = 2\left(\theta + \tan^{-1}\left(\tan(\varphi - \theta)/K\right)\right)$. Recall that $h_{K,0}(x + iy) = Kx + iy$ hence $h_{K,0}(re^{i\varphi}) = Kr\cos\varphi + ir\sin\varphi$, so

$$\arg[h_{K,0}(re^{i\varphi})] = \tan^{-1}(r\sin\varphi/Kr\cos\varphi) = \tan^{-1}(\tan\varphi/K).$$

It was noted in (4.1) that $h_{K,\theta}$ is given by pre-composing $h_{K,0}$ by the rotation $-\theta$ and post-composing by the rotation θ. Hence,

$$\arg[h_{K,\theta}(re^{i\varphi})] = \tan^{-1}\left(\tan(\varphi - \theta)/K\right) + \theta.$$

As $H = h_{K,\theta}^2$ implies $\arg[H(z)] = 2\arg[h_{k,\theta}(z)]$, we have

$$\arg[h_{K,\theta}(re^{i\varphi})] = 2(\tan^{-1}\left(\tan(\varphi - \theta)/K\right) + \theta). \qquad (4.8)$$

We are left to show that

$$|H(re^{i\varphi})| = r^2(1 + (K^2 - 1)\cos^2(\varphi - \theta)). \qquad (4.9)$$

Notice that $|H(z)| = |h_{K,\theta}(z)|^2$, so we need to calculate $|h_{K,\theta}|^2$. Substitute $x = r\cos\varphi$ and $y = r\sin\varphi$ into (4.1) to obtain;

$$\begin{aligned}
h_{K,\theta}(re^{i\varphi}) =&\, r\cos\varphi(K\cos^2\theta + \sin^2\theta) + r\sin\varphi(K - 1)\sin\theta\cos\theta + \\
&\, i[r\cos\varphi(K - 1)\cos\theta\sin\theta + r\sin\theta(K\sin^2\theta + \cos^2\theta)].
\end{aligned}$$

We can calculate;

$$
\begin{aligned}
|h_{K,\theta}(re^{i\varphi})|^2 &= (r\cos\varphi(K\cos^2\theta + \sin^2\theta) + r\sin\varphi(K-1)\sin\theta\cos\theta)^2 + \\
&\quad (r\cos\varphi(K-1)\cos\theta\sin\theta + r\sin\varphi(K\sin^2\theta + \cos^2\theta))^2 \\
&= r^2\cos^2\varphi(K\cos^2\theta + \sin^2\theta)^2 + 2r^2\cos\varphi\sin\varphi\sin\theta\cos\theta(K-1)(K\cos^2\theta + \sin^2\theta) + \\
&\quad r^2\sin^2\varphi(K-1)^2\sin^2\theta\cos^2\theta + r^2\cos^2\varphi(K-1)^2\cos^2\theta\sin^2\theta + \\
&\quad 2r^2\cos\varphi\sin\varphi\cos\theta\sin\theta(K-1)(K\sin^2\theta + \cos^2\theta) + r^2\sin^2\varphi(K\sin^2\theta + \cos^2\theta)^2 \\
&= r^2(K-1)^2\cos^2\theta\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \\
&\quad 2r^2\cos\varphi\sin\varphi\cos\theta\sin\theta(K-1)((K+1)(\cos^2\theta + \sin^2\theta)) + \\
&\quad r^2(\cos^2\varphi(K\cos^2\theta + \sin^2\theta)^2 + \sin^2\varphi(K\sin^2\theta + \cos^2\theta)^2) \\
&= r^2[(K-1)^2\cos^2\theta\sin^2\theta + 2(K-1)(K+1)\cos\varphi\sin\varphi\cos\theta\sin\theta + \\
&\quad \cos^2\varphi(K\cos^2\theta + \sin^2\theta)^2 + \sin^2\varphi(K\sin^2\theta + \cos^2\theta)^2] \\
&= r^2[(K^2 - 2K + 1)\cos^2\theta\sin^2\theta + 2(K^2 - 1)\cos\varphi\sin\varphi\cos\theta\sin\theta + \\
&\quad K^2\cos^2\varphi\cos^4\theta + 2K\cos^2\varphi\cos^2\theta\sin^2\theta + \cos^2\varphi\sin^4\theta + \\
&\quad K^2\sin^2\varphi\sin^4\theta + 2K\sin^2\varphi\sin^2\theta\cos^2\theta + \sin^2\varphi\cos^4\theta] \\
&= r^2[K^2(\cos^2\theta\sin^2\theta + 2\cos\varphi\sin\varphi\cos\theta\sin\theta + \cos^2\varphi\cos^4\theta + \sin^2\varphi\sin^4\theta) - \\
&\quad 2K(\cos^2\theta\sin^2\theta - \cos^2\varphi\cos^2\theta\sin^2\theta - \sin^2\varphi\sin^2\theta\cos^2\theta) + \\
&\quad \cos^2\theta\sin^2\theta - 2\cos\varphi\sin\varphi\cos\theta\sin\theta + \cos^2\varphi\sin^4\theta + \sin^2\varphi\cos^4\theta].
\end{aligned}
$$

Now let's consider the different K^n coefficients separately. First the K coefficient which is;

$$
2(\cos^2\theta\sin^2\theta - \cos^2\varphi\cos^2\theta\sin^2\theta - \sin^2\varphi\sin^2\theta\cos^2\theta)
$$

$$
= 2(\cos^2\theta\sin^2\theta - \cos^2\theta\sin^2\theta(\cos^2\varphi + \sin^2\varphi)) = 0. \tag{4.10}
$$

Next let's consider the K^2 coefficient, this is given above as;

$$
\cos^2\theta\sin^2\theta + 2\cos\varphi\sin\varphi\cos\theta\sin\theta + \cos^2\varphi\cos^4\theta + \sin^2\sin^4\theta \tag{4.11}
$$

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To simplify this equation we will need to utilise several trigonometric identities, namely:

$$\cos^2\psi = \frac{1+\cos 2\psi}{2}, \tag{4.12}$$

$$\sin^2\psi = \frac{1-\cos 2\psi}{2}, \tag{4.13}$$

$$\cos^4\psi = \frac{3+4\cos 2\psi+\cos 4\psi}{8}, \tag{4.14}$$

$$\sin^4\psi = \frac{3-4\cos 2\psi+\cos 4\psi}{8}, \tag{4.15}$$

$$\cos(\psi-\phi) = \cos\psi\cos\phi+\sin\psi\sin\phi, \tag{4.16}$$

$$\sin 2\psi = 2\sin\psi\cos\psi. \tag{4.17}$$

First using (4.17) we note that;

$$2\sin\varphi\cos\varphi\sin\theta\cos\theta = \frac{1}{2}\sin 2\varphi\sin 2\theta. \tag{4.18}$$

Next we use equations (4.12)-(4.15) to simplify:

$$\begin{aligned}
&\cos^2\theta\sin^2\theta+\cos^2\varphi\cos^4\theta+\sin^2\varphi\sin^4\theta\\
={}&\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)+\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{3+4\cos 2\psi+\cos 4\psi}{8}\right)\\
&+\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{3-4\cos 2\psi+\cos 4\psi}{8}\right)\\
={}&\frac{1}{16}(4-4\cos^2 2\theta+3+4\cos 2\theta+\cos 4\theta+3\cos 2\varphi+4\cos 2\varphi\cos 2\theta+\cos 4\theta\cos 2\varphi\\
&+3-4\cos 2\theta+\cos 4\theta-3\cos 2\varphi+4\cos 2\varphi\cos 2\theta-\cos 2\varphi\cos 4\theta)\\
={}&\frac{1}{8}(4-(2\cos^2 2\theta-1)+\cos 4\theta+4\cos 2\varphi\cos 2\theta).
\end{aligned}$$

Using (4.12) we see;

$$\begin{aligned}
\frac{1}{8}(4-(2\cos^2 2\theta-1)+\cos 4\theta+4\cos 2\varphi\cos 2\theta) &= \frac{1}{8}(4-\cos 4\theta+\cos 4\theta+4\cos 2\theta\cos 2\theta)\\
&= \frac{1}{2}(1+\cos 2\varphi\cos 2\theta). \tag{4.19}
\end{aligned}$$

Combining (4.18), (4.19) and equation (4.11) we see,;

$$\cos^2\theta\sin^2\theta+2\cos\varphi\sin\varphi\cos\theta\sin\theta+\cos^2\varphi\cos^4\theta+\sin^2\sin^4\theta$$

$$= \frac{1}{2}(1 + \cos 2\varphi \cos 2\theta + \sin 2\varphi \sin 2\theta). \tag{4.20}$$

Using (4.16) and (4.20) we see;

$$\frac{1}{2}(1 + \cos 2\varphi \cos 2\theta + \sin 2\varphi \sin 2\theta) = \frac{1}{2}(1 + \cos(2(\varphi - \theta))). \tag{4.21}$$

Next we apply (4.12) to (4.21) to obtain.

$$\cos^2 \theta \sin^2 \theta + 2 \cos \varphi \sin \varphi \cos \theta \sin \theta + \cos^2 \varphi \cos^4 \theta + \sin^2 \varphi \sin^4 \theta$$

$$= \cos^2(\varphi - \theta). \tag{4.22}$$

Finally we are left to consider the K^0 coefficient:

$$\cos^2 \theta \sin^2 \theta - 2 \cos \varphi \sin \varphi \cos \theta \sin \theta + \cos^2 \varphi \sin^4 \theta + \sin^2 \varphi \cos^4 \theta. \tag{4.23}$$

We can rearrange (4.23) and use $\cos^2 \psi + \sin^2 \psi = 1$ to obtain:

$$\cos^2 \theta \sin^2 \theta - 2 \cos \varphi \sin \varphi \cos \theta \sin \theta - (\sin^2 \varphi \sin^4 \theta + \cos^2 \varphi \cos^4 \theta$$

$$- \cos^4 \theta - \sin^4 \theta). \tag{4.24}$$

Writing $\cos^4 \theta + \sin^4 \theta = (\cos^2 \theta + \sin^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta$, using the equations $\cos^2 \psi + \sin^2 \psi = 1$ and (4.22), (4.24) becomes;

$$1 - \cos^2(\varphi - \theta). \tag{4.25}$$

Combining (4.25), (4.10) and (4.22) we have;

$$|h_{K,\theta}(re^{i\varphi})|^2 = r^2(1 + (K^2 - 1)\cos^2(\varphi - \theta)). \tag{4.26}$$

By (4.26) and (4.8) we have proved the lemma. $\qquad\square$

4.3   Fixed rays of H exist

We define the ray of angle $\varphi \in [0, 2\pi)$ to be $R_\varphi := \{te^{i\varphi} \mid t \in \mathbb{R}\}$. As the argument of H does not depend on r we have that H maps rays to rays. First observed in [19] is the fact that H has

at least one fixed ray and that fixed rays correspond to roots of the cubic polynomial,

$$P(t) := Kt^3 + (2-K)\tan(\theta/2)t^2 + (2-K)t + K\tan(\theta/2), \tag{4.27}$$

where $t = \tan[(\varphi - \theta)/2]$. It is shown that $P(t)$ always has a root $t_0 \in (-1, 1)$ which corresponds to a fixed ray with angle in $(-\pi/2, \pi/2)$, so there is always at least one fixed ray. The fact that $P(t)$ is a cubic suggests there could be one, two or three fixed rays and we will show in Chapter 6 that all cases are possible; we will however use a different method to achieve this.

Chapter 5

Böttcher coordinates

In this chapter we aim to prove a quasiregular version of Böttcher coordinates (Theorem 2.18), namely the following.

Theorem 5.1. *Let $h : \mathbb{C} \to \mathbb{C}$ be an affine mapping and $c \in \mathbb{C}$. Then there exists a neighbourhood $U = U(h, c)$ of infinity and a quasiconformal map $\psi = \psi(h, c)$ such that*

$$h(\psi(z))^2 = \psi(f(z)), \tag{5.1}$$

for $z \in U$, where $f(z) = h(z)^2 + c$. Further, ψ is asymptotically conformal as $|z| \to \infty$.

Remark 5.2. *Theorem 5.1 also holds for $p(h(z))$, where p is any polynomial of degree $d \geq 2$ and h is affine. For simplicity, we restrict to the case p is a quadratic and recall from Proposition 4.1 that any composition of a quadratic and an affine mapping is linearly conjugate to a composition of a quadratic of the form $z^2 + c$ and an affine mapping.*

Recall the escaping set $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$. The quasiconformal map ψ constructed in Theorem 5.1 is initially defined in a neighbourhood of infinity, but we may extend its domain of definition. Again we write $H(z) = h(z)^2$.

Theorem 5.3.   (i) *If $0 \notin I(H + c)$, then ψ can be continued injectively to a locally quasiconformal map $I(H + c) \to I(H)$.*

(ii) *If $0 \in I(H + c)$, then ψ cannot be extended to the whole of $I(H + c)$, but may be extended injectively to a domain containing c.*

Remark 5.4. *In case (i) of Theorem 5.3, we can only assert local quasiconformality. The map ψ is extended by pulling back under (5.1), and each time we pull back the distortion will increase.*

Therefore the distortion will be unbounded as one approaches $\partial I(H+c)$.

5.1 Proof of Theorem 5.1

5.1.1 Outline

Let $g(z) = z^2 + c$ and $h = h_{K,\theta}$ for $K > 1$ and $\theta \in (-\pi/2, \pi/2]$ and consider $f = g \circ h$ in a neighbourhood of infinity, say $U = \{|z| > R\}$. To prove Theorem 5.1, we will do the following.

- Writing $H = h^2$, define a branch ψ_1 of $H^{-1} \circ f$ in U.

- Show $\psi_1(z) = z + o(1)$ near infinity and ψ_1 is asymptotically conformal.

- Inductively define a branch ψ_{k+1} of $H^{-(k+1)} \circ f^{k+1}$ in U by considering $H^{-1} \circ \psi_k \circ f$.

- Show $\psi_k(z) = z + o(1)$ near infinity and ψ_k is asymptotically conformal.

- Show the sequence ψ_k converges locally uniformly to the required Böttcher coordinate.

5.1.2 The sequence ψ_k

Firstly, define an analytic branch p_1 of $\log(1 + c/z^2)$ in U, shrinking U if necessary, so that $\lim_{z \to \infty} p_1(z) = 0$. Then $g(z) = z^2 \exp p_1(z)$ in U and we can choose an analytic square root q_1 given by

$$q_1(z) = z \exp p_1(z)/2.$$

such that $q_1^2 = g$ in U. We can also assume that q_1 is injective in U, since if $q_1(z) = q_1(w)$, then $g(z) = g(w)$ and so $z = \pm w$, but $q_1(w) \neq q_1(-w)$ since expanding the expression for q_1 gives $q_1(z) = z + o(1)$ near infinity. Then we define

$$\psi_1(z) = h^{-1}(q_1(h(z))).$$

We can write $\psi_1(z) = z + R_1(z)$, and assume for now that $R_1(z) = o(1)$ for large $|z|$.

We continue defining the functions $\psi_k(z) = z + R_k(z)$ by induction. For $k \geq 1$, define a continuous branch p_k of

$$\log\left(1 + \frac{c + R_{k-1}(z^2 + c)}{z^2}\right)$$

in U so that $\lim_{z \to \infty} p_k(z) = 0$, assuming $R_{k-1}(z) = o(1)$. Then $\psi_{k-1}(g(z)) = z^2 \exp p_k(z)$ in U and we can choose a continuous square root $q_k = z \exp(p_k/2)$ such that $q_k^2 = \psi_{k-1} \circ g$ in

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U. We also observe that q_k is injective near infinity, since if $q_k(z) = q_k(w)$, then $\psi_{k-1}(g(z)) = \psi_{k-1}(g(w))$ and so $z = \pm w$ since ψ_{k-1} is injective, but $q_k(z) \neq q_k(-w)$ since expanding the expression for q_k gives $q_k(z) = z + o(1)$ in U. This means that $\psi_k = h^{-1} \circ q_k \circ h$ is injective in a neighbourhood of infinity.

To prove Theorem 5.1, we will need to prove the following propositions.

Proposition 5.5. *The functions ψ_k can be written as*

$$\psi_k(z) = z + R_k(z),$$

in U, where $R_k(z) = o(1)$. Moreover, the ψ_k converge uniformly to a function ψ in U and

$$\psi(z) = z + R(z),$$

where $R(z) = o(1)$.

Proposition 5.6. *The function ψ is quasiconformal in U and, further, is asymptotically conformal.*

We will postpone the proof of these two propositions until the next section. It seems difficult to prove these propositions directly, and so the proofs make use of the logarithmic transforms of the functions ψ_k.

With these results in hand, by the construction,

$$h(z)^2 = \psi_{k-1}(f(\psi_k^{-1}(z)))$$

for all $k \geq 1$. Taking the limit as $k \to \infty$, we have $\psi(f(\psi^{-1}(z))) = h(z)^2$ for $z \in U$. That is, the following diagram commutes.

$$
\begin{array}{ccc}
U & \xrightarrow{\psi} & \psi(U) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h^2} \\
f(U) & \xrightarrow{\psi} & h^2(\psi(U))
\end{array}
$$

This proves the theorem.

5.2 Logarithmic transforms of ψ_k

In this section, we will take the logarithmic transforms (recall §2.2) of the ψ_k and use them to prove Propositions 5.5 and 5.6. Let L be the half-plane $\mathrm{Re}(X) > \sigma$, where σ is large, and so L corresponds to a neighbourhood U of infinity in the z-plane. In L, for $k \geq 0$, define $F_0(X) = X$ and

$$F_{k+1} = \widetilde{h^{-1}} \circ \widetilde{S} \circ F_k \circ \widetilde{g} \circ \widetilde{h}, \tag{5.2}$$

where $\widetilde{S}(X) = X/2$, and write

$$F_k(X) = X + T_k(X).$$

Here, T_k measures how far away F_k is from the identity in L. Then the logarithmic transform of our sequence ψ_k is $\widetilde{\psi_k}(X) = F_k(X)$ by Lemma 2.16.

5.2.1 Preliminary observations

We first fix $\alpha \in (1, 2)$. The role that α plays will be seen in Lemmas 5.15 and 5.16. We will work with $X \in L = \{Z : \mathrm{Re}\, Z > \sigma\}$ where σ may be larger than $\log R$, and will depend on K, θ, c, α. The constants C_j which appear will all depend on at least K, θ, c, and may have other dependencies, which will be stated.

Lemma 5.7. *Let $h = h_{K,\theta}$ be given by (4.2). Then*

$$\widetilde{h}(X) = X + \log\left(\frac{K+1}{2} + e^{2i\theta}\left(\frac{K-1}{2}\right)e^{-2i\,\mathrm{Im}\,X}\right).$$

and

$$\widetilde{h^{-1}}(X) = X + \log\left(\frac{K+1}{2K} - e^{2i\theta}\left(\frac{K-1}{2K}\right)e^{-2i\,\mathrm{Im}\,X}\right).$$

Proof. This is immediate from the definition of h. $\qquad\square$

Definition 5.8. We define the functions

$$\varphi(X) = \widetilde{h}(X) - X,$$

and

$$\xi(X) = \widetilde{h^{-1}}(X) - X.$$

It is clear from the definition that $|\varphi|, |\xi|$ are both bounded above and below.

Recalling that $f = g \circ h$ and that the logarithmic transform of f is well defined by Lemma 2.16, it follows that - using the notation above - the logarithmic transform of f is

$$\widetilde{f}(X) = 2X + 2\varphi(X) + \rho(X + \varphi(X)). \tag{5.3}$$

To see that $\widetilde{f}$ is well-defined, note that

$$\widetilde{f}(X + 2\pi i) = 2X + 4\pi i + 2\varphi(X + 2\pi i) + \rho(X + 2\pi i + \varphi(X + 2\pi i)).$$

It is easy to see that $\varphi(X + 2\pi i) = \varphi(X)$, and so

$$\widetilde{f}(X + 2\pi i) - \widetilde{f}(X) = 4\pi i + \rho(X + 2\pi i + \varphi(X + 2\pi i)) - \rho(X + \varphi(X)).$$

The left hand side of this equation is a multiple of $2\pi i$, whereas the right hand side differs from a multiple of $2\pi i$ by something small for large $\operatorname{Re} X$, and hence by 0.

Lemma 5.9. *There exists a constant $C_1 > 0$ such that $|\varphi(X)| < C_1$ and $|\xi(X)| < C_1$ for all $X \in L$. Further, we have*

$$\varphi(X) + \xi(X + \varphi(X)) = 0$$

and

$$\xi(X) + \varphi(X + \xi(X)) = 0.$$

Proof. The first part follows from the definition of φ and ξ since $e^{2i\theta}(K-1)/(K+1) \in \mathbb{D}$. The second part is just translating the fact that h and h^{-1} are mutual inverses to the logarithmic coordinate setting. $\square$

The following corollary follows by differentiating the identities of Lemma 5.9.

Corollary 5.10. *The partial derivatives of φ and ξ satisfy*

$$\varphi_X(X) + \xi_X(X + \varphi(X))(1 + \varphi_X(X)) + \xi_{\overline{X}}(X + \varphi(X))\overline{\varphi_{\overline{X}}(X)} = 0$$

and

$$\varphi_{\overline{X}}(X) + \xi_X(X + \varphi(X))\varphi_{\overline{X}}(X) + \xi_{\overline{X}}(X + \varphi(X))(\overline{1 + \varphi_X(X)}) = 0.$$

Next, we consider small variations of φ and ξ.

Lemma 5.11. *Given $\delta > 0$, there exists $C_2 > 0$ depending on δ such that for $|Y| < \delta$, we have*

$$|\varphi(X + Y) - \varphi(X)| < C_2|Y|$$

for any $X \in L$, and the same holds for ξ.

Proof. Write

$$\nu = e^{2i\theta}\left(\frac{K-1}{K+1}\right),$$

with $K \geq 1$, so that $\nu \in \mathbb{D}$. Then, expanding $e^{-2i\operatorname{Im} Y}$ shows that

$$
\begin{aligned}
|\varphi(X + Y) - \varphi(X)| &= \left| \log \frac{1 + \nu e^{-2i\operatorname{Im}(X+Y)}}{1 + \nu e^{-2i\operatorname{Im}(X)}} \right| \\
&= \left| \log\left(1 - \left(\frac{2i\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}}\right)\operatorname{Im}(Y) + O((\operatorname{Im} Y)^2) \right) \right| \\
&\leq \left| \frac{2i\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}} \right| |\operatorname{Im} Y| + o(|\operatorname{Im} Y|).
\end{aligned}
$$

Since $|\operatorname{Im} Y| \leq |Y|$ and the coefficient of $|\operatorname{Im} Y|$ in the latter expression is uniformly bounded because $\nu \in \mathbb{D}$, we have the required conclusion. Analogous calculations hold for ξ. $\qquad\square$

The following lemma is the analogue of Lemma 5.11 for the partial derivatives.

Lemma 5.12. *Given $\delta > 0$, there exists $C_3 > 0$ depending on δ such that for all $|Y| < \delta$, we have*

$$|\varphi_X(X + Y) - \varphi_X(X)| < C_3|Y|$$

for any $X \in L$, and the same holds for $\varphi_{\overline{X}}, \xi_X$ and $\xi_{\overline{X}}$. Further, there exists $C_4 > 0$ such that the modulus of each of the partial derivatives is uniformly bounded above, i.e. $|\varphi_X(X)| < C_4$ for $X \in L$ etc.

Proof. We note that the partial derivatives of φ and ξ are

$$\varphi_X(X) = -\frac{\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}}, \quad \varphi_{\overline{X}}(X) = \frac{\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}}$$

and

$$\xi_X(X) = \frac{\nu e^{-2i\operatorname{Im}(X)}}{1 - \nu e^{-2i\operatorname{Im}(X)}}, \quad \xi_{\overline{X}}(X) = -\frac{\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}}.$$

Then, we have

$$
\begin{aligned}
|\varphi_X(X+Y) - \varphi_X(X)| &= \left| -\frac{\nu e^{-2i\operatorname{Im}(X+Y)}}{1 + \nu e^{-2i\operatorname{Im}(X+Y)}} + \frac{\nu e^{-2i\operatorname{Im}(X)}}{1 + \nu e^{-2i\operatorname{Im}(X)}} \right| \\
&= \left| \frac{\nu e^{-2i\operatorname{Im}(X)}(1 - e^{-2i\operatorname{Im}(Y)})}{(1 + \nu e^{-2i\operatorname{Im}(X)})(1 + \nu e^{-2i\operatorname{Im}(X+Y)})} \right| \\
&\leq \left| \frac{2i\nu e^{-2i\operatorname{Im}(X)}}{(1 + \nu e^{-2i\operatorname{Im}(X)})(1 + \nu e^{-2i\operatorname{Im}(X+Y)})} \right| |\operatorname{Im} Y| + o(|\operatorname{Im} Y|).
\end{aligned}
$$

The denominator in the coefficient of $|\operatorname{Im} Y|$ is uniformly bounded since $\nu \in \mathbb{D}$, and since $|\operatorname{Im} Y| \leq |Y|$, we have the desired conclusion. The calculations for the other partial derivatives run analogously. The final part of the lemma follows since $\nu \in \mathbb{D}$. $\qquad \square$

We may assume that σ is chosen so large that there exists $C_5 > 0$ such that

$$
|\rho(X + \varphi(X))| < C_5 e^{-2\operatorname{Re} X} \tag{5.4}
$$

for all $X \in L$. Next, consider the behaviour of $\widetilde{f}$ for $X \in L$, recalling (5.3).

Lemma 5.13. *There exists a constant $C_6 > 0$ such that*

$$
|\operatorname{Re} \widetilde{f}(X) - 2\operatorname{Re} X| < C_6,
$$

for $X \in L$.

Proof. Recall the definition of $\widetilde{f}$ from (5.3). Then

$$
|\operatorname{Re} \widetilde{f}(X) - 2\operatorname{Re} X| \leq 2|\varphi(X)| + |\rho(X + \varphi(X))|.
$$

By Lemma 5.9 and (5.4), this gives

$$
|\operatorname{Re} \widetilde{f}(X) - 2\operatorname{Re} X| < 2C_1 + C_5 e^{-2\operatorname{Re} X},
$$

which proves the lemma. $\qquad \square$

We note that in applications of Lemma 5.13, we will usually use the inequality

$$
\operatorname{Re} \widetilde{f}(X) > 2\operatorname{Re} X - C_6,
$$

for $X \in L$.

5.2.2 Growth of F_k

In this section, we will estimate how $|F_k|$ grows for large $\operatorname{Re} X$, and also show that the difference between successive terms in the sequence gets smaller as k increases.

First, recall that $F_1 = \widetilde{h^{-1}} \circ \widetilde{S} \circ \widetilde{g} \circ \widetilde{h}$. Writing this out in full gives

$$F_1(X) = X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \xi\left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2}\right). \qquad (5.5)$$

Recall also that $T_k(X) = F_k(X) - X$ is the function that shows how far F_k deviates from the identity.

Lemma 5.14. *There exists a constant $C_7 > 0$ such that*

$$|T_1(X)| \le C_7 e^{-2\operatorname{Re} X},$$

for $X \in L$.

Proof. Applying Lemma 5.11 with $Y = \frac{\rho(X + \varphi(X))}{2}$ shows that

$$\left| \xi\left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2}\right) - \xi\left(X + \varphi(X)\right) \right| < C_2 \left| \frac{\rho(X + \varphi(X))}{2} \right|.$$

Recall from Lemma 5.9 that $\varphi(X) + \xi(X + \varphi(X)) = 0$. Then from (5.5) we obtain that

$$|T_1(X)| < (1 + C_2) \left| \frac{\rho(X + \varphi(X))}{2} \right|.$$

Finally, using (5.4) implies the lemma. $\qquad\qquad\square$

Recall that $\alpha \in (1, 2)$. The reason α is introduced is the following lemma. Namely, the fact that α is less than 2 allows us to give an estimate on the growth of the T_k which is valid for all k.

Lemma 5.15. *There exists a constant $C_8 > 0$ depending on α such that for all $k \ge 1$, we have*

$$|T_k(X)| < C_8 e^{-\alpha \operatorname{Re} X},$$

for $X \in L$.

Proof. We will proceed by induction. By Lemma 5.14, the result is true for $k = 1$ if $C_8 > C_7 e^{(\alpha-2)\sigma}$, recalling that $\operatorname{Re} X > \sigma$. Let us assume then that

$$|T_k(X)| < C_8 e^{-\alpha \operatorname{Re} X}. \tag{5.6}$$

We may assume that σ is large enough that (5.4) is satisfied and we may apply Lemma 5.11 with $Y = \rho(X + \varphi(X))/2 + T_k(\widetilde{f}(X))/2$, so that

$$\left| \xi\left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2} \right) - \xi(X + \varphi(X)) \right|$$
$$< C_2 \left| \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2} \right|, \tag{5.7}$$

for $X \in L$. Using (5.2), we can write F_{k+1} as

$$F_{k+1}(X) = X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2}$$
$$+ \xi\left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2} \right). \tag{5.8}$$

Recalling from Lemma 5.9 that $\varphi(X) + \xi(X + \varphi(X)) = 0$, then (5.7) and (5.8) imply that

$$|T_{k+1}(X)| < \left(\frac{1 + C_2}{2} \right) \left| \rho(X + \varphi(X)) + T_k(\widetilde{f}(X)) \right|.$$

By the inductive hypothesis and Lemma 5.13,

$$|T_k(\widetilde{f}(X))| < C_8 e^{-\alpha \operatorname{Re} \widetilde{f}(X)}$$
$$< C_8 e^{\alpha C_6} e^{-2\alpha \operatorname{Re} X}.$$

Using this and (5.4), we obtain

$$|T_{k+1}(X)| < \left(\frac{1 + C_2}{2} \right) \left(C_5 e^{-2 \operatorname{Re} X} + C_8 e^{\alpha C_6} e^{-2\alpha \operatorname{Re} X} \right)$$
$$= e^{-\alpha \operatorname{Re} X} \left(\frac{(1 + C_2)C_5}{2} e^{(\alpha-2) \operatorname{Re} X} + \frac{(1 + C_2)C_8}{2} e^{\alpha C_6} e^{-\alpha \operatorname{Re} X} \right)$$
$$< e^{-\alpha \operatorname{Re} X} \left(\frac{(1 + C_2)C_5}{2} e^{(\alpha-2)\sigma} + \frac{(1 + C_2)C_8}{2} e^{\alpha C_6} e^{-\alpha \sigma} \right).$$

We may assume that σ was chosen so large that

$$(1 + C_2) \, e^{\alpha C_6} e^{-\alpha \sigma} < 1,$$

and also C_8 is large enough that

$$(1 + C_2) C_5 \, e^{(\alpha - 2)\sigma} < C_8,$$

from which it follows that

$$|T_{k+1}(X)| < C_8 e^{-\alpha \operatorname{Re} X},$$

which proves the lemma. $\qquad\square$

Lemma 5.16. *For all $k \geq 1$, there exists a constant C_9 depending on α such that*

$$|F_{k+1}(X) - F_k(X)| < C_9 e^{-\alpha^k \operatorname{Re} X},$$

for $X \in L$.

Proof. Recalling that $F_0(X) = X$, the lemma holds for $k = 0$ by Lemma 5.14. We proceed by induction, and assume that for some $k \geq 1$, we have

$$|F_k(X) - F_{k-1}(X)| < C_9 e^{-\alpha^{k-1} \operatorname{Re} X},$$

noting that this is equivalent to

$$|T_k(X) - T_{k-1}(X)| < C_9 e^{-\alpha^{k-1} \operatorname{Re} X}. \tag{5.9}$$

Using (5.8) applied to F_{k+1} and F_k, we have that

$$
\begin{aligned}
F_{k+1}(X) - F_k(X) = {} & \xi \left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2} \right) \\
& - \xi \left(X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_{k-1}(\widetilde{f}(X))}{2} \right) \\
& + \frac{T_k(\widetilde{f}(X))}{2} - \frac{T_{k-1}(\widetilde{f}(X))}{2}.
\end{aligned}
$$

Using Lemma 5.11 applied to ξ with

$$Y = \frac{T_k(\widetilde{f}(X))}{2} - \frac{T_{k-1}(\widetilde{f}(X))}{2},$$

we see that

$$|F_{k+1}(X) - F_k(X)| \leq \left| (1 + C_2) \left(\frac{T_k(\widetilde{f}(X))}{2} - \frac{T_{k-1}(\widetilde{f}(X))}{2} \right) \right|. \tag{5.10}$$

The inductive hypothesis and Lemma 5.13 imply that

$$|T_k(\widetilde{f}(X)) - T_{k-1}(\widetilde{f}(X))| < C_9 e^{-\alpha^{k-1} \operatorname{Re} \widetilde{f}(X)}$$

$$< C_9 e^{C_6 \alpha^{k-1}} e^{-2\alpha^{k-1} \operatorname{Re} X}$$

$$< C_9 e^{\alpha^{k-1}(C_6 - (2-\alpha)\sigma)} e^{-\alpha^k \operatorname{Re} X},$$

for $X \in L$. Hence if σ is chosen large enough that $e^{\alpha^{k-1}(C_6 - (2-\alpha)\sigma)} < 2(1 + C_2)^{-1}$ for $k \geq 1$, then we obtain from (5.10) that

$$|F_{k+1}(X) - F_k(X)| < C_9 e^{-\alpha^k \operatorname{Re} X},$$

which proves the lemma. □

5.2.3 Complex dilatation of F_k

In this section, we will estimate the growth of the complex dilatation of F_k for large $\operatorname{Re} X$. We will use the following formula for the complex derivatives of a composition repeatedly, see for example [20].

Lemma 5.17. *The complex derivatives of compositions are*

$$(g \circ f)_z = (g_z \circ f)f_z + (g_{\bar{z}} \circ f)\overline{f_{\bar{z}}},$$

and

$$(g \circ f)_{\bar{z}} = (g_z \circ f)f_{\bar{z}} + (g_{\bar{z}} \circ f)\overline{f_z}.$$

As a first application of this, we consider the complex derivatives of $\rho(X + \varphi(X))$.

Lemma 5.18. *Let $\rho_1(X) = \rho(X + \varphi(X))$. Then there exists a constant $C_{10} > 0$ such that*

$$|(\rho_1)_X(X)| \leq C_{10}e^{-2X} \text{ and } |(\rho_1)_{\overline{X}}(X)| \leq C_{10}e^{-2X},$$

for $X \in L$.

Proof. Recall from Lemma 2.17 that $\rho(X) = \log(1 + ce^{-2X})$. Since ρ is analytic, it follows that $\rho_{\overline{X}} \equiv 0$, and also

$$\rho_X(X) = \frac{-2ce^{-2X}}{1 + ce^{-2X}}.$$

Then using Lemma 5.17 and Lemma 5.12, we have

$$|(\rho_1)_X(X)| \leq |\rho_X(X + \varphi(X)) \cdot (1 + \varphi_X(X)) + \rho_{\overline{X}}(X + \varphi(X)) \cdot \overline{\varphi_{\overline{X}}(X)}|$$
$$\leq (1 + C_4)|\rho_X(X + \varphi(X))|,$$

which gives the desired conclusion for $(\rho_1)_X$, since $|\varphi|$ is bounded above by Lemma 5.9. Similar calculations give the growth for $(\rho_1)_{\overline{X}}$. $\qquad\square$

We now want to estimate the complex dilatations μ_k of F_k.

Proposition 5.19. *There exist constants $C_{11}, C_{12} > 0$ such that for all $k \geq 1$,*

$$|(F_k)_X(X)| \geq 1 - C_{11}e^{-\alpha \operatorname{Re} X}$$

and

$$|(F_k)_{\overline{X}}(X)| \leq C_{12}e^{-\alpha \operatorname{Re} X}$$

for all $X \in L$.

The proof of this proposition will proceed by induction. Since $F_0(X) = X$, it is clear that the proposition holds for $k = 0$. Hence assume the result is true for k. Recalling that $F_k(X) = X + T_k(X)$, this means that

$$|(T_k)_X(X)| \leq C_{11}e^{-\alpha \operatorname{Re} X}, \quad |(T_k)_{\overline{X}}(X)| \leq C_{12}e^{-\alpha \operatorname{Re} X}. \tag{5.11}$$

Lemma 5.20. *There exists constants $C_{13}, C_{14} > 0$ such that*

$$\left| \left[T_k(\widetilde{f}(X)) \right]_X \right| < C_{13}e^{-2 \operatorname{Re} X}$$

and

$$\left| \left[T_k(\widetilde{f}(X)) \right]_{\overline{X}} \right| < C_{14} e^{-2\operatorname{Re}X},$$

for $X \in L$.

Proof. By the inductive hypothesis (5.11), we have

$$|(T_k)_X(\widetilde{f}(X))| \le C_{11} e^{-\alpha \operatorname{Re}\widetilde{f}(X)}.$$

Recalling the growth of $\widetilde{f}$ from Lemma 5.13, this gives

$$|(T_k)_X(\widetilde{f}(X))| < C_{11} e^{\alpha C_6} e^{-2\alpha \operatorname{Re}X}$$
$$< C_{11} e^{C_6\alpha + 2(1-\alpha)\sigma} e^{-2\operatorname{Re}X},$$

for $X \in L$, which gives the result for $\left[T_k(\widetilde{f}(X)) \right]_X$. The result for $\left[T_k(\widetilde{f}(X)) \right]_{\overline{X}}$ follows analogously. $\square$

Recalling the definition of F_{k+1} from (5.2), we have

$$F_{k+1}(X) = \frac{F_k(\widetilde{f}(X))}{2} + \xi\left(\frac{F_k(\widetilde{f}(X))}{2} \right).$$

For convenience let us write

$$P(X) = \frac{F_k(\widetilde{f}(X))}{2} = X + \varphi(X) + \frac{\rho(X + \varphi(X))}{2} + \frac{T_k(\widetilde{f}(X))}{2}, \qquad (5.12)$$

so that

$$F_{k+1}(X) = P(X) + \xi(P(X)).$$

The complex derivatives of P are

$$P_X(X) = 1 + \varphi_X(X) + \left[\frac{\rho(X + \varphi(X))}{2} \right]_X + \left[\frac{T_k(\widetilde{f}(X))}{2} \right]_X, \qquad (5.13)$$

and

$$P_{\overline{X}}(X) = \varphi_{\overline{X}}(X) + \left[\frac{\rho(X + \varphi(X))}{2} \right]_{\overline{X}} + \left[\frac{T_k(\widetilde{f}(X))}{2} \right]_{\overline{X}}. \qquad (5.14)$$

We are now in a position to prove Proposition 5.19.

Proof of Proposition 5.19. The complex derivative of F_{k+1} with respect to X is

$$(F_{k+1})_X(X) = P_X(X) + P_X(X)\xi_X(P(X)) + \overline{P_{\overline{X}}(X)}\xi_{\overline{X}}(P(X)).$$

Using the identity from Corollary 5.10, we can write

$$\begin{aligned}
(F_{k+1})_X(X) - 1 &= (P_X(X) - 1 - \varphi_X(X)) \\
&\quad + (P_X(X)\xi_X(P(X)) - (1 + \varphi_X(X))\xi_X(X + \varphi(X))) \\
&\quad + \left(\overline{P_{\overline{X}}(X)}\xi_{\overline{X}}(P(X)) - \overline{\varphi_{\overline{X}}(X)}\xi_{\overline{X}}(X + \varphi(X))\right) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, by (5.13) we have

$$\begin{aligned}
|I_1| = |P_X(X) - 1 - \varphi_X(X)| &= \left| \left[\frac{\rho(X + \varphi(X))}{2}\right]_X + \left[\frac{T_k(\widetilde{f}(X))}{2}\right]_X \right| \\
&< \frac{C_{10}}{2}e^{-2\operatorname{Re}X} + \frac{C_{13}}{2}e^{-2\operatorname{Re}X} \\
&= \frac{(C_{10} + C_{13})}{2}e^{-2\operatorname{Re}X}
\end{aligned}$$

by Lemmas 5.18 and 5.20.

For I_2, first observe that by (5.4) and Lemma 5.15, we may assume that σ is large enough that $|P(X) - X - \varphi(X)| < \delta$ for $X \in L$, and so Lemma 5.12 shows that

$$|\xi_X(P(X)) - \xi_X(X + \varphi(X))| < C_3|P(X) - (X + \varphi(X))|,$$

for $X \in L$. By the definition of P, (5.4) and the proof of Lemma 5.15, this implies that there exists $C_{15} > 0$ such that

$$|\xi_X(P(X)) - \xi_X(X + \varphi(X))| < C_3\left(\frac{C_5}{2}e^{-2\operatorname{Re}X} + \frac{C_8}{2}e^{-\alpha\operatorname{Re}\widetilde{f}(X)}\right) \quad (5.15)$$

$$< C_{15}e^{-2\operatorname{Re}X},$$

for $X \in L$. Next, by (5.13), Lemma 5.12 and the calculation for I_1, we have

$$|P_X(X)| < 1 + C_4 + \left(\frac{C_{10} + C_{13}}{2}\right)e^{-2\operatorname{Re}X} < C_{16}, \quad (5.16)$$

55

for $X \in L$. Then (5.15), (5.16), Lemma 5.12 for $|\xi_X|$ and the calculation for I_1 give

$$
\begin{aligned}
|I_2| &= |P_X(X)\xi_X(P(X)) - (1 + \varphi_X(X))\xi_X(X + \varphi(X))| \\
&= |P_X(X)[\xi_X(P(X)) - \xi_X(X + \varphi(X))] + \xi_X(X + \varphi(X))[P_X(X) - (1 + \varphi_X(X))]| \\
&< C_{16}C_{15}e^{-2\operatorname{Re}X} + \frac{C_4(C_{10} + C_{13})}{2}e^{-2\operatorname{Re}X}
\end{aligned}
$$

For I_3, observe first that since we may assume σ is large enough that $|P(X) - X - \varphi(X)| < \delta$ for $X \in L$, Lemma 5.12 implies that

$$
|\xi_{\overline{X}}(P(X)) - \xi_{\overline{X}}(X + \varphi(X))| < C_3|P(X) - (X + \varphi(X))|.
$$

As in the calculation for I_2, this implies that there exists $C_{17} > 0$ such that

$$
|\xi_{\overline{X}}(P(X)) - \xi_{\overline{X}}(X + \varphi(X))| < C_{17}e^{-2\operatorname{Re}X}, \tag{5.17}
$$

for $X \in L$. Also observe that by (5.14), Lemma 5.12 and the calculation for I_1 there exists $C_{18} > 0$ such that

$$
|P_{\overline{X}}(X)| < C_4 + \left(\frac{C_{10} + C_{13}}{2}\right)e^{-2\operatorname{Re}X} < C_{18}, \tag{5.18}
$$

for $X \in L$. Further, (5.14) and calculations analogous to those for I_1 show that there exists $C_{19} > 0$ such that

$$
|P_{\overline{X}}(X) - \varphi_{\overline{X}}(X)| < C_{19}e^{-2\operatorname{Re}X}. \tag{5.19}
$$

Then (5.17), (5.18), (5.19) and Lemma 5.12 for $|\xi_{\overline{X}}|$ give

$$
\begin{aligned}
|I_3| &= |\overline{P_{\overline{X}}(X)}\xi_{\overline{X}}(P(X)) - \overline{\varphi_{\overline{X}}(X)}\xi_{\overline{X}}(X + \varphi(X))| \\
&= |\overline{P_{\overline{X}}(X)}[\xi_{\overline{X}}(P(X)) - \xi_{\overline{X}}(X + \varphi(X))] + \xi_{\overline{X}}(X + \varphi(X))[\overline{P_{\overline{X}}(X)} - \overline{\varphi_{\overline{X}}(X)}]| \\
&< C_{18}C_{17}e^{-2\operatorname{Re}X} + C_4 C_{19}e^{-2\operatorname{Re}X},
\end{aligned}
$$

for $X \in L$. The estimates for I_1, I_2, I_3 show that there exists $C_8' > 0$ such that

$$
|(F_{k+1})_X(X) - 1| < C_8' e^{-2\operatorname{Re}X},
$$

for $X \in L$ and hence if σ is chosen large enough so that $C_8' e^{(\alpha-2)\sigma} < C_8$, then

$$|(F_{k+1})_X(X) - 1| < C_8 e^{-\alpha \operatorname{Re} X},$$

Therefore

$$|(F_{k+1})_X(X)| > 1 - C_8 e^{-\alpha \operatorname{Re} X}$$

for $X \in L$ as required.

We next move on to estimate $|(F_{k+1})_{\overline{X}}(X)|$. The calculations are very similar to those above, but are included for the reader's convenience. From the definition of F_{k+1} and Lemma 5.17, we have

$$(F_{k+1})_{\overline{X}}(X) = P_{\overline{X}}(X) + \xi_X(P(X)) P_{\overline{X}}(X) + \xi_{\overline{X}}(P(X)) \overline{P_X(X)}.$$

Using the second identity from Corollary 5.10, we can write this as

$$
\begin{aligned}
(F_{k+1})_{\overline{X}}(X) = {}& \big(P_{\overline{X}}(X) - \varphi_{\overline{X}}(X) \big) \\
& + \big(P_{\overline{X}}(X) \xi_X(P(X)) - \varphi_{\overline{X}}(X) \xi_X(X + \varphi(X)) \big) \\
& + \big(\overline{P_X(X)} \xi_{\overline{X}}(P(X)) - \overline{1 + \varphi_X(X)} \xi_{\overline{X}}(X + \varphi(X)) \big) \\
= {}& J_1 + J_2 + J_3.
\end{aligned}
$$

By (5.19), we have

$$
\begin{aligned}
|J_1| &= |P_{\overline{X}}(X) - \varphi_{\overline{X}}(X)| \\
&< C_{19} e^{-2 \operatorname{Re} X},
\end{aligned}
$$

for $X \in L$. Taking advantage of estimates already calculated, by (5.15), (5.18), (5.19) and Lemma 5.12,

$$
\begin{aligned}
|J_2| &= | \big(P_{\overline{X}}(X) \xi_X(P(X)) - \varphi_{\overline{X}}(X) \xi_X(X + \varphi(X)) \big) | \\
&= |P_{\overline{X}}(X)[\xi_X(P(X)) - \xi_X(X + \varphi(X))] + \xi_X(X + \varphi(X))[P_{\overline{X}}(X) - \varphi_{\overline{X}}(X)]| \\
&< C_{18} C_{15} e^{-2 \operatorname{Re} X} + C_4 C_{19} e^{-2 \operatorname{Re} X},
\end{aligned}
$$

for $X \in L$. Also, by (5.16), (5.17), the calculation for I_1 and Lemma 5.12, we have

$$
\begin{aligned}
|J_3| &= | \left(\overline{P_X(X)} \xi_{\overline{X}}(P(X)) - \overline{(1 + \varphi_X(X))} \xi_{\overline{X}}(X + \varphi(X)) \right) | \\
&= | \overline{P_X(X)} [\xi_{\overline{X}}(P(X)) - \xi_{\overline{X}}(X + \varphi(X))] + \xi_{\overline{X}}(X + \varphi(X)) [\overline{P_X(X)} - \overline{(1 + \varphi_X(X))}] | \\
&< C_{16} C_{17} e^{-2 \operatorname{Re} X} + \left(\frac{C_4(C_{10} + C_{13})}{2} \right) e^{-2 \operatorname{Re} X},
\end{aligned}
$$

for $X \in L$. The estimates for J_1, J_2 and J_3 show that

$$
|(F_{k+1})_{\overline{X}}(X)| < C_9' e^{-2 \operatorname{Re} X}
$$

for $X \in L$. Hence if σ is chosen large enough so that $C_9' e^{(\alpha - 2)\sigma} < C_9$, then

$$
|(F_{k+1})_{\overline{X}}(X)| < C_9 e^{-\alpha \operatorname{Re} X},
$$

for $X \in L$. This completes the proof of the proposition. $\qquad \square$

Corollary 5.21. *There exists a constant $C_{20} > 0$ such that the complex dilatation μ_k of F_k satisfies, for all $k \geq 1$,*
$$
|\mu_k(X)| \leq C_{20} e^{-\alpha \operatorname{Re} X}
$$

for all $X \in L$.

Proof. This is an immediate corollary of Proposition 5.19. $\qquad \square$

5.2.4   Proof of Proposition 5.5

Choose $\sigma > 0$ large enough so that the results of the previous sections hold in the half-plane $L = \{\operatorname{Re} X > \sigma\}$. Recall the definition of the functions ψ_k and assume that they are defined in a neighbourhood of infinity $U = \{|z| > R\}$ where $R > e^\sigma$. Recall that under a logarithmic change of variable, we have $\widetilde{\psi_k} = F_k$.

Write
$$
\psi_k(z) = \prod_{j=1}^k \frac{\psi_j(z)}{\psi_{j-1}(z)},
$$

where $\psi_0(z) \equiv 1$. To show that ψ_k converges uniformly on U, it is enough to show that $\log \psi_k(z)$ converges uniformly on U, where the principal branch of the logarithm is chosen. Then, writing

$z = e^X$, Lemma 5.16 implies that

$$|\log \psi_k(z)| = \left| \sum_{j=1}^{k} (\log \psi_j(z) - \log \psi_{j-1}(z)) \right|$$

$$= \left| \sum_{j=1}^{k} F_j(X) - F_{j-1}(X) \right|$$

$$\leq \sum_{j=1}^{k} |F_j(X) - F_{j-1}(X)|$$

$$< C_9 \sum_{j=1}^{k} \exp\{-\alpha^j \operatorname{Re}(X)\}$$

$$= C_9 \sum_{j=1}^{k} |z|^{-\alpha^j},$$

for some constant $C_9 > 0$ and $\alpha \in (1,2)$. As $k \to \infty$, this clearly converges on $U = \{|z| > R\}$. Hence ψ_k converges uniformly on U to ψ, and we may write $\psi(z) = z + R(z)$.

For the second part of the proposition, we need to show that $R(z) = o(1)$. We know that T_k converges uniformly to T for $\operatorname{Re} X > \sigma$ (this is just the content of the first part of the proof). By this fact and by Lemma 5.15, we have

$$|T(X)| < C_8 e^{-\alpha \operatorname{Re} X},$$

for $\operatorname{Re} X > \sigma$. Now, $\widetilde{\psi}(X) = X + T(X)$ and so, using the fact that $z = e^X$, we have that

$$|R(z)| = |\exp(\log z + T(\log z)) - z|$$

$$= |z (\exp T(\log z) - 1)|$$

$$\leq |z| (|T(\log z)| + o(|T(\log z)|))$$

$$\leq |z| \left(C_8 e^{-\alpha \log |z|} + o(|T(\log z)|) \right)$$

$$= C_8 |z|^{1-\alpha} + o(|z|^{1-\alpha}).$$

Since $\alpha \in (1,2)$, we have that $R(z) = o(1)$ for large $|z|$. In fact, although the constants C_j may change, we actually have that $R(z) = O(|z|^{1-\alpha})$ for any $\alpha \in (1,2)$, completing the proof of Proposition 5.5.

5.2.5 Proof of Proposition 5.6

As indicated in the construction of ψ_k in the introductory section, each ψ_k is injective on some neighbourhood U of infinity. Further, Corollary 5.21 shows that the complex dilatation μ_k of $\widetilde{\psi_k}$, which is ψ_k in logarithmic coordinates, satisfies

$$|\mu_k(X)| \leq C_{20} e^{-\alpha \operatorname{Re} X}, \tag{5.20}$$

for $\alpha \in (1,2)$ and all $X \in L$. Since $\widetilde{\psi_k}(X) = \log \psi_k(e^X)$, where $z = e^X$, and $\log, \exp$ are both holomorphic, it follows that

$$|\mu_k(X)| = |\mu_{\psi_k}(z)|.$$

Since $\operatorname{Re} X > \sigma$ corresponds to $|z| > e^{\sigma}$, it follows that ψ_k is quasiconformal in a neighbourhood of infinity. Moreover, (5.20) shows that $\mu_{\psi_k}(z) \to 0$ as $|z| \to \infty$, which means that ψ_k is asymptotically conformal.

By Proposition 5.5, ψ_k converges uniformly on U to a function ψ. Since we may assume each ψ_k is K-quasiconformal on U for some $K > 1$ then by Theorem 3.6, the quasiregular Montel's theorem, it follows that the limit ψ is also K-quasiconformal; moreover that ψ is asymptotically conformal, completing the proof of Proposition 5.6.

5.3 Proof of Theorem 5.3

Recall that $H = h^2$ and the statement of the theorem.

Theorem 5.3. *(i) If $0 \notin I(H + c)$, then ψ can be continued injectively to a locally quasiconformal map $I(H + c) \to I(H)$.*

(ii) If $0 \in I(H + c)$, then ψ cannot be extended to the whole of $I(H + c)$, but may be extended injectively to a domain containing c.

Assume that K, θ are fixed and the quasiconformal map ψ conjugates $f = H + c$ to H in a neighbourhood U of infinity. Without loss of generality, we can assume that $U = -U$ where $-U = \{z \in \mathbb{C} : -z \in U\}$. To prove the theorem, we need to show that the domain of definition of ψ may be extended. To this end we prove the following lemma; the proof of which contains standard ideas, see for example (§17 [29]).

Lemma 5.22. *Let $V \subset I(f)$ be a connected neighbourhood of infinity with connected complement, satisfying $V = -V$ and such that $f : f^{-1}(V) \to V$ is a two-to-one covering map. If*

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ψ is defined on V, then ψ can be extended to a quasiconformal map defined on $f^{-1}(V)$ which conjugates f to h^2.

Remark 5.23. *If $V = -V$, then since $h(-z) = -h(z)$ and $g(z) = g(-z)$, it is clear that $f^{-1}(V) = -f^{-1}(V)$.*

Proof. Let V satisfy the hypotheses of the lemma. Let $w \in V$ and γ be a curve connecting w to infinity in V. Since f is a two-to-one covering map from $f^{-1}(V)$ onto V, then given $z \in f^{-1}(w)$, γ lifts to a curve γ' connecting z and infinity in $f^{-1}(V)$. We note that since $V \cup \{\infty\}$ is simply connected and $f : f^{-1}(V) \to V$ is a covering map, $f^{-1}(V) \cup \{\infty\}$ is also simply connected.

Figure 5.1: How ψ extends to $f^{-1}(V)$

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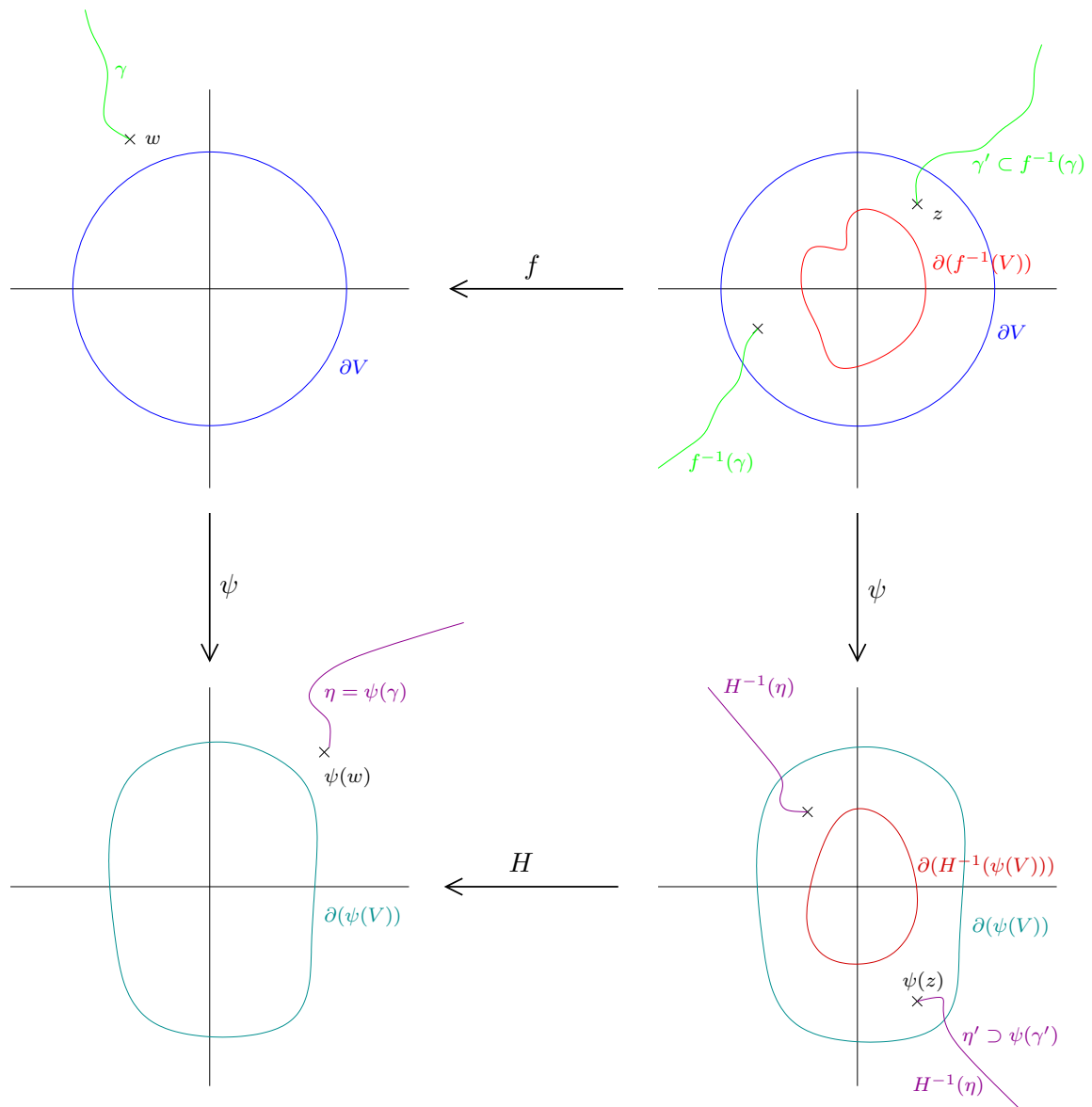
Now, $\eta = \psi(\gamma)$ is a curve in $\psi(V)$ connecting $\psi(w)$ and infinity in $\psi(V) \subset I(H)$. Since $H : H^{-1}(\psi(V)) \to \psi(V)$ is a two-to-one covering, η lifts to two curves in $H^{-1}(\psi(V))$, each terminating at one of the two points of $H^{-1}(\psi(w))$. Since ψ is defined in a neighbourhood of infinity, there is only one of these two curves, say η', which is the image of γ' under ψ near infinity. We then define $\psi(z)$ to be the end-point of η'. Note that the other lift of η corresponds to the other pre-image of w under f.

In this way, ψ extends to a map $f^{-1}(V) \to H^{-1}(\psi(V))$, with $\psi(z) \in H^{-1}(\psi(f(z)))$. Since f is continuous, ψ is continuous on V and H is a local homeomorphism away from 0, the extension of ψ is continuous. By construction, ψ still satisfies the conjugacy $H \circ \psi = \psi \circ f$ on its enlarged domain and hence is still locally quasiconformal. To finish the proof of the lemma, we have to show that ψ is injective.

Suppose this was not the case, and $\psi(z_1) = \psi(z_2)$ for $z_1, z_2 \in f^{-1}(V)$ (and at least one of z_1, z_2 must be in $f^{-1}(V) \setminus V$ since ψ is injective in V). Then

$$\psi(f(z_1)) = H(\psi(z_1)) = H(\psi(z_2)) = \psi(f(z_2)),$$

and since $f(z_1), f(z_2) \in V$ and ψ is injective there, we must have $f(z_1) = f(z_2)$. Thus $z_1 = -z_2$ and $\psi(z_1) = \psi(-z_1)$. Since $V = -V$, we obtain a contradiction: choose curves $\pm\gamma$ from $\pm z_1$ to infinity, and then by continuity we have $\psi(-z) = -\psi(z)$ on γ. $\qquad\square$

To prove part (i) of Theorem 5.3, observe that if $c \notin I(f)$, then $f : f^{-n}(U) \to f^{1-n}(U)$ is a two-to-one covering map for any $n \in \mathbb{N}$. Applying Lemma 5.22 repeatedly to $f^{-n}(U)$ for $n \in \mathbb{N}$ and noting that

$$I(f) = \bigcup_{n \geq 1} f^{-n}(U)$$

shows that ψ can be extended to all of $I(f)$. The extension of ψ to $f^{-n}(U)$ is a quasiconformal map, but the distortion may increase as n increases. Hence we can only conclude that $\psi : I(f) \to I(h^2)$ is an injective locally quasiconformal map.

For part (ii) of Theorem 5.3, the same reasoning applies as in part (i), but here we can only apply Lemma 5.22 finitely many times, since $c \in I(f)$. That is, once $c \in f^{-n}(U)$, then $f : f^{-(n+1)}(U) \to f^{-n}(U)$ is no longer a two-to-one covering map and we cannot apply Lemma 5.22. However, ψ can be extended to a neighbourhood of infinity which contains c, which completes the proof of the theorem.

Chapter 6

Behaviour of rays under H

We know that a fixed ray of $H = h_{K,\theta}^2$ must be a root of the cubic $P(t)$ given in (4.27). This would suggest that we will have one, two or three fixed rays, corresponding to the roots of the cubic $P(t)$. So if we fix θ and vary K continuously one might expect that we will move from one fixed ray, which is repelling, to two fixed rays of which one is repelling and one neutral; then as we continue to vary K the neutral fixed ray will split to leave two repelling and one attracting fixed rays. Or the one repelling fixed ray could split into three fixed rays of which one is attracting and two are repelling. All of this would be determined by how the cubic varies as we vary K. The first case is shown in Figure 6.1 and the second in Figure 6.2. We will see that this is exactly what occurs, but we will not use the cubic P to show this.
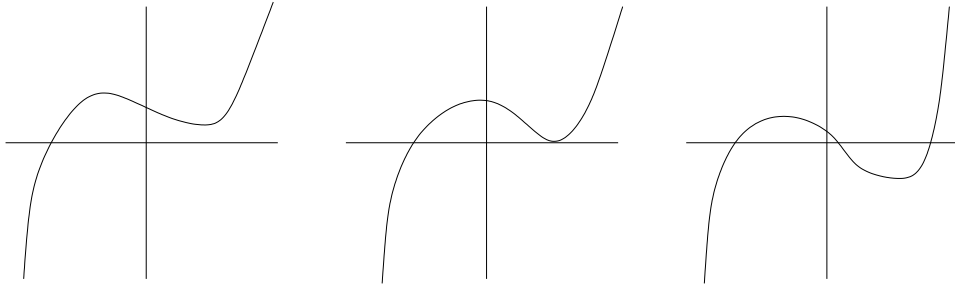
Figure 6.1: How the cubic P may vary with K to give, 1, 2 or 3 fixed rays.

6.1 Statement of chapter's results

Recall a *ray* is a semi-infinite line $R_\phi = \{te^{i\phi} : t \geq 0\}$. Recalling the polar form (4.7) it is obvious that $h_{K,\theta}$ maps rays to rays, and so $H = h_{K,\theta}^2$ also maps rays to rays. This means that H induces an increasing mapping $\widetilde{H} : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic, and is given by $\widetilde{H}(\varphi) = \arg[H(re^{i\varphi})]$, for
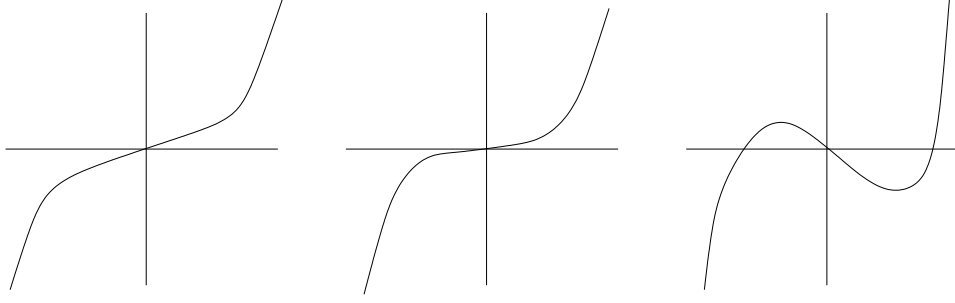
Figure 6.2: How the cubic P may vary with K to give, 1 or 3 fixed rays.

any $r > 0$ (note that this φ is a variable and is different from the function φ used in §5). We say that a ray R_ϕ which is fixed by H is *locally repelling*, *locally expanding* or *neutral* if the induced mapping satisfies $\widetilde{H}'(\phi) < 1$, $\widetilde{H}'(\phi) > 1$ or $\widetilde{H}'(\phi) = 1$ respectively.

Theorem 6.1. *Let $\theta \in (-\pi/2, \pi/2) \setminus \{0\}$, $K > 1$ and let $H(z) = h_{K,\theta}(z)^2$. Then there exists $K_\theta > 1$ such that:*

- *for $K < K_\theta$, there is one fixed ray that is locally repelling;*

- *for $K = K_\theta$, there are two fixed rays, one of which is locally repelling and one that is neutral. Further, the neutral fixed ray is repelling on one side and attracting on the other;*

- *for $K > K_\theta$, there are three fixed rays, one of which is locally attracting and two that are locally repelling.*

When $\theta = 0$ the first and third statements above hold, but when $K = K_\theta$ there is just one neutral fixed ray which is locally attracting on both sides. When $\theta = \pi/2$ there is only one fixed ray for all $K > 1$ and it is always locally repelling.

We next investigate the pre-images of these fixed rays. If H has two or three fixed rays, then denote by Λ the basin of attraction of the fixed ray that is not locally repelling.

Theorem 6.2. *If H has one fixed ray R_ϕ then $\{H^{-k}(R_\phi)\}_{k=0}^\infty$ is dense in $\mathbb{C}$. If H has two or three fixed rays, then Λ is dense in $\mathbb{C}$.*

We can use Theorem 6.2 to give a complete decomposition of the plane into dynamically important sets for H. For a quasiregular mapping of polynomial type whose degree is larger than the distortion, it was proved in [21] that the escaping set is a connected neighbourhood of infinity. However, such mappings can have dynamically undesired behaviour (such as sensitivity to initial conditions) outside the closure of the escaping set, for example in [5] a mapping is constructed

which locally behaves like a winding mapping. We show that this does not happen for H. Recalling that the escaping set of H is given by $I(H) = \{z \in \mathbb{C} \mid |H^n(z)| \to \infty \text{ as } n \to \infty\}$, we have the following corollary which will be proved later.

Corollary 6.3. *Let $K > 1$, $\theta \in (-\pi/2, \pi/2]$ and $H(z) = h_{K,\theta}(z)^2$. Then $\mathbb{C} = I(H) \cup \partial I(H) \cup \mathcal{A}(0)$, where $\mathcal{A}(0)$ is the basin of attraction of the fixed point 0.*

Via the Böttcher coordinate constructed in Theorem 5.1, Theorems 6.1 and 6.2 have analogues for mappings of the form $h(z)^2 + c$ for $c \in \mathbb{C}$. We first make the following definition in analogy with complex dynamics.

Definition 6.4. *Let $f(z) = h(z)^2 + c$. Then the external ray E_φ of f with angle $\varphi \in [0, 2\pi)$ is given by the image of the ray R_φ under the quasiconformal Böttcher coordinate $\psi = \psi(K, \theta, c)$ which conjugates f to H. The external ray E_φ is only defined in the range of ψ, that is, a neighbourhood of infinity.*

We remark that each E_φ is an asymptotically conformal arc of a quasi-circle, since the Böttcher coordinate is asymptotically conformal as $|z| \to \infty$. The collection $\{E_\varphi : \varphi \in [0, 2\pi)\}$ foliate a neighbourhood of infinity. We define an external ray E_φ which is fixed by f to be attracting, repelling or neutral if the corresponding fixed ray R_φ of H is attracting, repelling or neutral respectively. The following corollary is an immediate application of Theorem 6.1.

Corollary 6.5. *Let $\theta \in (-\pi/2, \pi/2) \setminus \{0\}$, $K > 1$, $c \in \mathbb{C}$ and let $f(z) = h_{K,\theta}(z)^2 + c$. Then, with K_θ as in Theorem 6.1,*

- *for $K < K_\theta$, there is one fixed external ray of f that is locally repelling;*

- *for $K = K_\theta$, there are two fixed rays, one of which is locally repelling and one that is neutral. Further, the neutral fixed ray is repelling on one side and attracting on the other;*

- *for $K > K_\theta$, there are three fixed rays, one of which is locally attracting and two that are locally repelling.*

When $\theta = 0$ the first and third statements above hold, but when $K = K_\theta$ there is just one neutral fixed external ray which is locally attracting on both sides. When $\theta = \pi/2$ there is only one fixed external ray for all $K > 1$ and it is always locally repelling.

In particular, the value of c plays no role in how many fixed external rays f has. Theorem 6.2 also has the following immediate corollary.

Corollary 6.6. *With the notation as above, if f has one fixed external ray E_ϕ then $\{f^{-k}(E_\phi)\}_{k=0}^{\infty}$ is dense in a neighbourhood of infinity. If f has two or three fixed external rays, then the basin of attraction of the non-repelling fixed external ray is dense in $\mathbb{C}$.*

6.2 Fixed rays of H

6.2.1 Outline of proof of Theorem 6.1

To prove Theorem 6.1 we use the following strategy.

- Given K, θ show that the argument of $H = H_{K,\theta}$ induces a map $\widetilde{H} : S^1 \to S^1$.

- Determine the possible locations of fixed points of $\widetilde{H}$.

- When $\theta = 0$, show that if $K \leq 2$ then $\widetilde{H}$ has one repelling fixed point, or neutral in the case $K = 2$, and if $K > 2$ then $\widetilde{H}$ has three fixed points, two repelling and one attracting.

- When $\theta = \pi/2$, show that $\widetilde{H}$ only ever has one repelling fixed point.

- For $\theta \in (0, \pi/2)$, show that there exists $K_\theta > 2$ such that $\widetilde{H}$ has two fixed points, one repelling and one neutral. If $K < K_\theta$, then $\widetilde{H}$ has one repelling fixed point. If $K > K_\theta$, then $\widetilde{H}$ has three fixed points, one attracting and two repelling.

6.2.2 Locations of fixed rays of H

First, we will narrow down the sectors where any possible fixed rays can be.

Lemma 6.7. *If $\theta > 0$ then any fixed ray R_ϕ of H lies in the sectors,*

$$\mathbb{F}_\theta^+ = \{R_\varphi \mid 2\theta < \varphi < \theta + \pi/2\},$$

or

$$\mathbb{F}_\theta^- = \{R_\varphi \mid \theta - \pi/2 < \varphi < 0\}.$$

If $\theta = 0$ any fixed ray is in $\mathbb{F}_\theta^\pm \cup \{R_0\}$. If $\theta = \pi/2$ then R_0 is the only fixed ray.

Proof. Recall that our normalisation for θ requires $\theta \in (-\pi/2, \pi/2]$ and that by Lemma 4.2 we
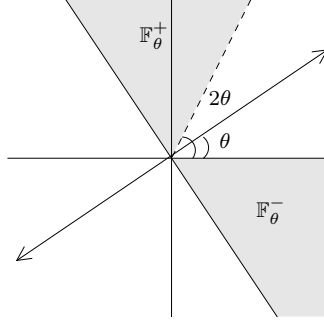
Figure 6.3: Diagram showing the regions $\mathbb{F}_\theta^\pm$.

need only consider $\theta \geq 0$. Define the following quadrants of $\mathbb{C}$:

$$Q_1 = \{R_\varphi \mid 0 < \varphi - \theta < \pi/2\},$$
$$Q_2 = \{R_\varphi \mid -\pi/2 < \varphi - \theta < 0\},$$
$$Q_3 = \{R_\varphi \mid -\pi/2 < \varphi - \theta < -\pi\},$$
$$Q_4 = \{R_\varphi \mid \pi/2 < \varphi - \theta < \pi\}.$$

Recall the definition of $h = h_{K,\theta}$ given in (4.1). Notice that the rays that bound the quadrants Q_i are fixed under h, hence the argument of the bounding rays is doubled under H. Let $0 < \theta < \pi/2$. First we consider Q_4. Under H, the image of Q_4 is

$$H(Q_4) = \{R_\varphi \mid -\pi < \varphi - 2\theta < 0\}.$$

We notice that $Q_4 \cap H(Q_4) = \emptyset$ and so there can be no fixed ray in the sector Q_4. Next we consider the rays in Q_3. For simplicity we will consider rays to have angle between -2π and 0. Now

$$H(Q_3) = \{R_\varphi \mid -2\pi < \varphi - 2\theta < \pi\}.$$

Recalling that $0 < \theta < \pi/2$, we have $H(Q_3) \cap Q_3 \neq \emptyset$, so it is possible that there is a fixed ray in Q_3. However note that $h(Q_3) = Q_3$ and that for $R_\varphi \in Q_3$ if $R_\psi = h(R_\varphi)$ then $-\pi < \psi < \varphi < 0$. Squaring doubles the angle so if $R_\tau = H(R_\varphi)$ the angles must satisfy

$$-2\pi < \tau < \psi < \varphi < 0.$$

This holds for all $R_\varphi \in Q_3$ and so there can be no fixed ray in Q_3.

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Next consider the quadrant Q_1. Note that $H(R_\theta) = R_{2\theta}$, so $H(Q_1) \cap Q_1 = \mathbb{F}_\theta^+$. Hence any fixed ray of Q_1 must lie in $\mathbb{F}_\theta^+$. Finally, if $R_\varphi \in Q_2$ and $\varphi > 0$ then, similar to the case of rays in Q_3, if $R_\psi = h(R_\varphi)$ then $0 < \varphi < \psi < \theta$. Squaring further increases the angle so if $R_\tau = H(R_\varphi)$ then the angles must satisfy

$$0 < \varphi < \psi < \tau < 2\theta.$$

Hence any fixed ray of Q_2 must lie in $\mathbb{F}_\theta^-$ as claimed.

When $\theta = 0$ the above holds with the addition that the ray R_0 is always fixed. It is easy to see that R_0 is fixed when $\theta = \pi/2$, and similar arguments to the above show that this is the only possible fixed ray. $\qquad\square$

6.2.3 The induced map $\widetilde{H}$ of S^1

Given K, θ the map $H = H_{K,\theta}$ induces a map $S^1 \to S^1$ as follows.

Definition 6.8. Define $\widetilde{H} : S^1 \to S^1$ by:

$$\widetilde{H}(\varphi) = \psi \text{ where } \arg[H(re^{i\varphi})] = \psi$$

for any $r > 0$.
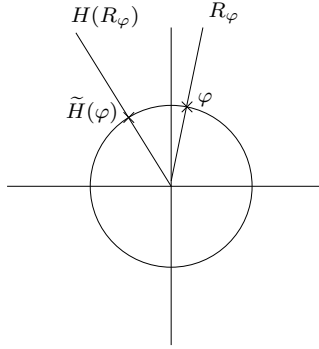
Figure 6.4: Diagram showing how $\widetilde{H}$ is induced from the action of H on the ray R_φ.

By lifting $\widetilde{H}$ to $\mathbb{R}$, we obtain a 2π-periodic mapping $\mathbb{R} \to \mathbb{R}$. We will often use $\widetilde{H}$ and its lift to $\mathbb{R}$ interchangeably, but the usage of $\widetilde{H}$ should be clear from the context. We also remark

that by the definition of H, $\widetilde{H}$ is actually a π-periodic mapping. We have that

$$\widetilde{H}(\varphi) = 2\theta + 2\tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right), \tag{6.1}$$

when $\widetilde{H}$ is viewed as the mapping lifted to $\mathbb{R}$.

Remark 6.9. *We can also view H as a map $H : \Omega \to \mathbb{C}$ where $\Omega := (1, \infty) \times (\pi/2, \pi/2] \times \mathbb{C}$. By the formulation of (6.1) we see that H is not just continuous in φ but in θ and K also. Further it is differentiable in all variables too.*

Viewed as a mapping on S^1, $\widetilde{H}$ is two-to-one. Points in S^1 correspond to rays in $\mathbb{C}$ and so fixed points of $\widetilde{H}$ correspond to fixed rays of H. In this way, we reduce our study of fixed rays of H to fixed points of the circle endomorphism $\widetilde{H}$. Given a sector $S \subset \mathbb{C}$, we will denote by $\widetilde{S}$ the corresponding subset of S^1 or interval in $\mathbb{R}/2\pi\mathbb{Z}$.

We also define the homeomorphism $\widetilde{h}$ on S^1 induced by $h = h_{K,\theta}$.

Definition 6.10. Define the map $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ by:

$$\widetilde{h}(\varphi) = \psi \text{ where } \arg[h(re^{i\varphi})] = \psi$$

for any $r > 0$.

To try to simplify matters we will use a tilde to denote the induced maps or sets of S^1 (and so $\mathbb{R}/2\pi\mathbb{Z}$ also) from $\mathbb{C}$. For example, the sector of rays $\mathbb{F}_\theta^+ \subset \mathbb{C}$ will induce an interval $\widetilde{\mathbb{F}}_\theta^+ \subset \mathbb{R}/2\pi\mathbb{Z}$. However a ray R_φ will correspond to the point $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, a tilde here would be superfluous. We want to study the dynamics of this map. What happens to nearby points when they are iterated under $\widetilde{H}$? Are points locally attracted, repelled or both? Is the point fixed? Initially we are only concerned with points in the semicircle containing $\widetilde{Q}_1$ and $\widetilde{Q}_2$, as this is where any fixed points are located from Lemma 6.7. This is also convenient because when we consider $\widetilde{Q}_i \subset (-\pi, \pi]$ for $i = 1, 2$ we have

$$\widetilde{H}(\widetilde{Q}_i) \subset (-\pi, \pi].$$

This means that fixed points of $\widetilde{H} : S^1 \to S^1$ just correspond to fixed points of $\widetilde{H} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$.

Recall from Lemma 4.2 that

$$H_{K,-\theta}(z) = \overline{H_{K,\theta}(\bar{z})}.$$

Hence we only need to consider the case $\theta \in [0, \pi/2]$. The ray R_φ is a fixed ray of $H_{K,\theta}$ if and only if $R_{-\varphi}$ is a fixed ray of $H_{K,-\theta}$ and they have the same behaviour. For the rest of this section we assume $\theta \in [0, \pi/2]$.

6.2.4 Local expansion and contraction

In this subsection we will study $\widetilde{H}'(\varphi)$, as this determines whether a small neighbourhood of φ is contracted or expanded under $\widetilde{H}$. Since $\widetilde{H}$ is sense-preserving, $\widetilde{H}' > 0$. Let us make this more precise.

Definition 6.11. An interval $I \subset \mathbb{R}/2\pi\mathbb{Z}$ is *expanded* by $\widetilde{H}$ if

$$|\widetilde{H}(I)| > |I|,$$

and is *contracted* by $\widetilde{H}$ if

$$|\widetilde{H}(I)| < |I|.$$

It is easy to see that if $\widetilde{H}'(\varphi) < 1$ or $\widetilde{H}'(\varphi) > 1$ then there exists some neighbourhood V of φ that is contracted or expanded respectively by $\widetilde{H}$. Further if there is some closed interval I such that $\widetilde{H}'(\varphi) < 1$ or $\widetilde{H}'(\varphi) > 1$ for all $\varphi \in I$ then it follows that I is contracted or expanded respectively by $\widetilde{H}$.

Lemma 6.12. *For $K < 2$ and any θ, $\widetilde{H}'(\varphi) > 1$ for all $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. When $K > 2$ there is a single interval $J \subset (\theta - \pi/2, \theta + \pi/2)$ where we have $\widetilde{H}'(\varphi) < 1$ and further, θ is the midpoint of J. When $K = 2$, $H'(\theta) = 1$ and $H'(\varphi) > 1$ for all $\varphi \in (\theta - \pi/2, \theta + \pi/2) \setminus \{\theta\}$.*

Proof. By differentiating the expression for $\widetilde{H}$ we obtain,

$$\widetilde{H}'(\varphi) = \frac{2K}{1 + (K^2 - 1)\cos^2(\varphi - \theta)}. \tag{6.2}$$

Note that $\widetilde{H}'(\varphi)$ is continuous and that

$$\widetilde{H}'(\varphi) \leq 1 \iff 2K \leq 1 + (K^2 - 1)\cos^2(\varphi - \theta). \tag{6.3}$$

It is easy to see that

$$1 + (K^2 - 1)\cos(\varphi - \theta) \leq K^2,$$

and hence if $K < 2$

$$2K > K^2 \geq 1 + (K^2 - 1)\cos^2(\varphi - \theta). \tag{6.4}$$

Then (6.3) and (6.4) imply $\widetilde{H}'(\varphi) > 1$ when $K < 2$ proving the first part of the lemma. Considering $\varphi = \theta$, we see

$$\widetilde{H}'(\theta) = \frac{2K}{1 + (K^2 - 1)\cos^2(0)} = \frac{2}{K}. \tag{6.5}$$

If $K = 2$, then it follows from (6.5) that $\widetilde{H}'(\theta) = 1$ and, from (6.3) that, if $\varphi \neq \theta$, then $\widetilde{H}'(\varphi) > 1$. For $K > 2$ we have $\widetilde{H}'(\theta) < 1$ by (6.5), this shows that $J \neq \emptyset$ when $K > 2$. As $\widetilde{H}'$ is continuous we must have some interval J containing θ such that $\widetilde{H}'(\varphi) < 1$ for $\varphi \in J$. We want to show this is the only interval of $(\theta - \pi/2, \theta + \pi/2)$ with this property. Note that

$$\widetilde{H}'(\varphi) \to 2K \text{ as } \varphi - \theta \to \pm\pi/2,$$

so $J \neq (\theta - \pi/2, \theta + \pi/2)$. We have to show there is no other region where $\widetilde{H}' < 1$. To do this we differentiate again to obtain

$$\widetilde{H}''(\varphi) = \frac{(2K^3 - 2K)\tan(\varphi - \theta)}{\cos^2(\varphi - \theta)(K^2 + \tan^2(\varphi - \theta))^2}.$$

It is easy to see that when $\widetilde{H}''(\varphi) = 0$, we are at a local minimum or maximum of $\widetilde{H}'$. Now $\widetilde{H}''(\varphi) = 0$ implies

$$(2K^3 - 2K)\tan(\varphi - \theta) = 0$$

which, as $K > 2$ and $\varphi - \theta \in (-\pi/2, \pi/2)$, implies that $\varphi = \theta$.

We know that $\widetilde{H}'(\varphi) > 1$ for φ near $\pm\pi/2$ and that there is only one critical point of $\widetilde{H}$ at $\varphi = \theta$. Hence J is the only interval such that $\varphi \in (\theta - \pi/2, \theta + \pi/2)$ implies $\widetilde{H}'(\varphi) < 1$. The final statement that θ is the midpoint of J follows from the fact that $\cos^2(\varphi - \theta)$, and so $\widetilde{H}'(\varphi)$ too, is symmetric about θ. $\qquad\square$

Definition 6.13. Given $K > 2$ and θ, denote by $J = J_K$ the interval $(\theta - \eta, \theta + \eta)$, for $\eta = \eta_K$, where $\widetilde{H}' < 1$. Note that η does not depend on θ.

We remark that as $\widetilde{H}$ is π-periodic, the translate of J by π is a second interval where $\widetilde{H}'(\varphi) < 1$. However, there can be no fixed points here from Lemma 6.7 and so we are not concerned with this other interval in this section.

6.2.5   Special cases

We will now investigate the fixed points of $\widetilde{H}$. The cases where $\theta = 0$ and $\theta = \pi/2$ are special cases and we deal with these now. We first show that if $\theta = 0$, then $\widetilde{H}$ can never have a neutral fixed point unequal to 0.

Lemma 6.14. *Let $\theta = 0$. Suppose $\varphi \neq 0$ and $\widetilde{H}'(\varphi) = 1$, then φ cannot be fixed.*

Proof. As $\widetilde{H}'(\varphi) = 1$ then (6.2) implies

$$\varphi = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right],$$

where we take the positive square root since $|\varphi| < \pi/2$. Suppose that φ is fixed so that $\widetilde{H}(\varphi) = \varphi$, then (6.1) implies

$$\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right] = 2\tan^{-1}\left[\frac{\tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)}{K}\right]. \tag{6.6}$$

Applying cos to both sides of (6.6) and using the double angle formula for cos, we obtain

$$\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}} = 2\cos^2\left(\tan^{-1}\left[\frac{\tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)}{K}\right]\right) - 1.$$

Applying the identity $\cos^2\tan^{-1}x = (1+x^2)^{-1}$, we obtain

$$\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}} = \frac{2}{1+\tan^2\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)/K^2} - 1$$

$$= \frac{1-\tan^2\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)/K^2}{1+\tan^2\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)/K^2}$$

$$= \frac{K^2-\tan^2\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)}{K^2+\tan^2\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)}. \tag{6.7}$$

Using the formula $\tan^2(\cos^{-1} X) = (1 - X^2)/X^2$ and rearranging, (6.7) becomes

$$(2K-1)^{\frac{1}{2}}(K^2(2K-1) + (K^2-1) - (2K-1)) = (K^2-1)^{\frac{1}{2}}(K^2(2K-1) - (K^2-1) + (2K-1)).$$

Rearranging and squaring both sides we see

$$4K^2(2K-1)(K^2-1)^2 = 4K^2(K^2-1)(K^2-K+1)^2. \tag{6.8}$$

Hence $K = 0$ and $K = 1$ are solutions to (6.6). Factoring these solutions out of (6.8) and expanding, we obtain

$$K^2(K-2)^2 = 0 \tag{6.9}$$

$K = 0$ and $K = 2$ are solutions of (6.9). Hence all possible solutions to (6.6) are $K = 0, 1, 2$. Since $K = 0$ and $K = 1$ are not permissible values, the only valid solution for us is $K = 2$. This implies $\varphi = \cos^{-1}(0)$. We have assumed that $\varphi \neq 0$ and any other fixed point must be in $(-\pi/2, \pi/2)$ by Lemma 6.7, so there are no more possible solutions. This completes the proof. $\square$

Lemma 6.15. *If $\theta = 0$ then $\widetilde{H}$ has one repelling fixed point $\phi_0 = 0$ when $K < 2$, has one neutral fixed point when $K = 2$ and has three fixed points $\pi/2 < \phi_2 < \phi_0 = 0 < \phi_1 < \pi/2$ when $K > 2$, where ϕ_1 and ϕ_2 are repelling and ϕ_0 is attracting. Further $\phi_2 = -\phi_1$.*
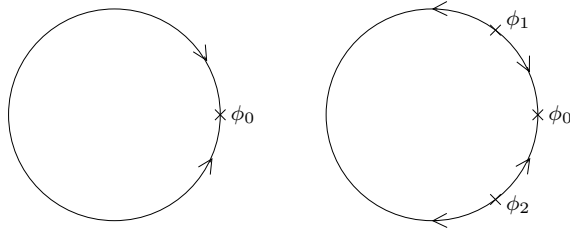

Figure 6.5: Diagram showing the local dynamics of ϕ_i in the two cases.

Proof. First substitute $\theta = 0$ into (6.1) to obtain

$$\widetilde{H}(\varphi) = 2\tan^{-1}[(\tan\varphi)/K].$$

Then any fixed point ϕ must satisfy the equation

$$K\tan(\phi/2) = \tan\phi. \tag{6.10}$$

Since $\phi_0 = 0$ satisfies (6.10), it is always a fixed point when $\theta = 0$. Lemma 6.12 implies that for $K < 2$, $\widetilde{H}'(\phi_0) > 1$, so ϕ_0 is repelling. When $K > 2$ we see $\widetilde{H}'(0) < 1$ and $\phi_0 = 0$ is attracting. Let $K < 2$ and suppose we had some other fixed point ϕ. Without loss of generality, assume $\phi > \phi_0$ then the interval $[\phi_0, \phi]$ is fixed under $\widetilde{H}$, since the image of $[0, \pi]$ under $\widetilde{H}$ is $[0, 2\pi]$. However this interval must also be expanded, as all $\varphi \in [\phi_0, \phi]$ satisfy $\widetilde{H}'(\varphi) > 1$, a contradiction.

Now let $K > 2$. Recall Definition 6.13 and the interval J. Write $J = (-\eta, \eta)$ and recall that Lemma 6.14 implies that neither $\pm\eta$ can be fixed for $K > 2$, and hence

$$|\widetilde{H}(J)| < |J|.$$

Since $\widetilde{H}(\pi/2) > \pi/2$ and $\widetilde{H}(\eta) < \eta$, by continuity there exists a fixed point $\phi_1 \in (\eta, \pi/2)$. Similarly, there is a fixed point $\phi_2 \in (-\pi/2, -\eta)$. Further, $\widetilde{H}'(\phi_i) > 1$ for $i = 1, 2$, so they are repelling. As ϕ_1 and ϕ_2 must satisfy (6.10) and since tan is odd we must have $\phi_2 = -\phi_1$. Since $\widetilde{H}$ can have at most three fixed points in S^1, these account for them all.

Finally we deal with the case when $K = 2$. Here we have from Lemma 6.12 that $\widetilde{H}'(0) = 1$ and so ϕ_0 is a neutral fixed point. However $\widetilde{H}'(\varphi) > 1$ for $\varphi \in (-\pi/2, \pi/2) \setminus \{0\}$, so any interval with one end-point 0 is expanded by $\widetilde{H}$. This implies there are no other fixed points. $\qquad\square$

Lemma 6.16. *If $\theta = \pi/2$ then $\phi_0 = 0$ is the only fixed point of $\widetilde{H}$ for all $K > 1$ and it is always repelling.*

Proof. By Lemma 6.7 we know $\phi_0 = 0$ is the only fixed point of $\widetilde{H}$. Substituting $\varphi = 0$ and $\theta = \pi/2$ into (6.2) we see

$$\widetilde{H}'(\phi_0) = 2K.$$

As $K > 1$ we have that ϕ_0 is repelling. $\qquad\square$

6.2.6 The general case $\theta \in (0, \pi/2)$

For the rest of this section assume $\theta \in (0, \pi/2)$. Recall the sectors $F_\theta^\pm \subset \mathbb{C}$ from Lemma 6.7, and the corresponding intervals $\widetilde{F}_\theta^\pm \subset S^1$.

Lemma 6.17. *There is exactly one fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}$ for all $K > 1$. Further, it is a repelling fixed point.*

Proof. Recalling the notation of Lemma 6.7, we know any fixed point in $\widetilde{Q}_2$ must lie in $\widetilde{\mathbb{F}}_\theta^-$. We also have that $\widetilde{Q}_2 \subset \widetilde{H}(\widetilde{Q}_2)$. Recall that $\widetilde{H}$ is orientation preserving, injective when restricted

to $\widetilde{Q}_2$, and continuous. Hence there must be a fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$. We will see that ϕ is the only fixed point in $\widetilde{\mathbb{F}}_\theta^-$.

From Lemma 6.12 and Definition 6.13, $J \subset Q_1 \cup Q_2$ where $\widetilde{H}'(\varphi) < 1$ for $\varphi \in J$ and also $\theta \in J$. If $K \leq 2$ then $J = \emptyset$ and every interval with end-point ϕ is expanded by $\widetilde{H}$. Therefore $\widetilde{H}$ has no other fixed points.

Finally suppose $K > 2$ and $J \neq \emptyset$. Then since $\phi < 0$ is fixed and $\theta > 0$, the interval $[\phi, \theta]$ is expanded by $\widetilde{H}$ and so $\phi \notin J$. Suppose there is some other fixed point $\phi' \in \widetilde{\mathbb{F}}_\theta^-$. Without loss of generality, assume $\phi < \phi'$. Then the interval $I = [\phi, \phi']$ satisfies $\widetilde{H}(I) = I$ and $I \cap J = \emptyset$. Therefore I is expanded by $\widetilde{H}$ which is a contradiction. Therefore there can only be one fixed point in $\widetilde{\mathbb{F}}_\theta^-$. $\qquad\square$

In the next step, we will see that given $\theta \in (0, \pi/2)$, we can choose K so that there are exactly two fixed points of $\widetilde{H}$. First we show a uniqueness lemma for neutral fixed points by similar calculations to Lemma 6.14.

Lemma 6.18. *Let $\theta \in (0, \pi/2)$. There exists one $K_\theta > 2$ such that $\widetilde{H}_{K_\theta, \theta}$ has a neutral fixed point ϕ_{K_θ}.*

Proof. Note that (6.2) implies that if ϕ is fixed and $\widetilde{H}'_{K,\theta}(\phi) = 1$ then,

$$\phi = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right] + \theta.$$

From (6.1) and the assumption ϕ is fixed we must satisfy

$$\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right] + \theta = 2\tan^{-1}\left[\tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)\Big/ K\right] + 2\theta. \qquad (6.11)$$

Rearranging and applying tan to both sides of (6.11), we obtain

$$\tan\left[\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right] - \theta\right)\Big/ 2\right] = \tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)\Big/ K. \qquad (6.12)$$

Using the formula $\tan(\cos^{-1} X) = (1 - X^2)^{\frac{1}{2}}/X$, (6.12) becomes

$$\tan\left[\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]-\theta\right)/2\right]=\left(1-\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\Big/\left(K\left(\frac{2K-1}{K^2-1}\right)\right)^{\frac{1}{2}}$$

$$=\frac{1}{K}\left(\frac{K(K-2)}{K^2-1}\right)^{\frac{1}{2}}\left(\frac{K^2-1}{2K-1}\right)^{\frac{1}{2}}$$

$$=\left(\frac{K-2}{K(2K-1)}\right)^{\frac{1}{2}}. \tag{6.13}$$

Squaring both sides and applying the identity $\tan^2 x/2=(1-\cos x)/(1+\cos x)$, we see (6.13) is equivalent to

$$\frac{1-\cos[\cos^{-1}([(2K-1)/(K^2-1)]^{\frac{1}{2}})-\theta]}{1+\cos[\cos^{-1}([(2K-1)/(K^2-1)]^{\frac{1}{2}})-\theta]}=\frac{K-2}{K(2K-1)}.$$

Applying the addition formula for cos and the formula $\sin(\cos^{-1} x)=(1-x^2)^{\frac{1}{2}}$ and clearing denominators, we obtain

$$\frac{(K^2-1)^{\frac{1}{2}}-(2K-1)^{\frac{1}{2}}\cos\theta-(K(K-2))^{\frac{1}{2}}\sin\theta}{(K^2-1)^{\frac{1}{2}}+(2K-1)^{\frac{1}{2}}\cos\theta+(K(K-2))^{\frac{1}{2}}\sin\theta}=\frac{K-2}{K(2K-1)} \tag{6.14}$$

Rearranging (6.14), by grouping the $\cos\theta$ and $\sin\theta$ terms together, we see

$$(K(2K-1)-(K-2))(K^2-1)^{\frac{1}{2}}=((K-2)+K(2K-1))[(2K-1)^{\frac{1}{2}}\cos\theta+(K(K-2))^{\frac{1}{2}}\sin\theta].$$

Expanding and cancelling we obtain

$$(K^2-K+1)=(K^2-1)^{\frac{1}{2}}((2K-1)^{\frac{1}{2}}\cos\theta+K^{\frac{1}{2}}(K-2)^{\frac{1}{2}}\sin\theta)$$

We can use $\sin X=(1-\cos^2 X)^{\frac{1}{2}}$, rearrange and square; to see that

$$((K^2-K+1)-(K^2-1)^{\frac{1}{2}}(2K-1)^{\frac{1}{2}}\cos\theta)^2=K(K-2)(K^2-1)(1-\cos^2\theta).$$

Expanding and solving the quadratic in $\cos\theta$, we obtain

$$\theta=\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{3}{2}}(K-1)\right], \tag{6.15}$$

since $\theta>0$. Writing (6.15) as $\theta=\cos^{-1}[f(K)]$, and viewing f as a function $(2,\infty)\to\mathbb{R}$, one

can calculate that

$$f'(K) = -(2K-1)^{1/2}(K-1)^{-3/2}(K+1)^{-3/2}(K+4) < 0$$

for $K > 2$. Hence f is a decreasing function which converges to 0 as $K \to \infty$. Hence $\cos^{-1} \circ f :$ $(2, \infty) \to \mathbb{R}$ is an increasing function which is therefore injective. Further $f(2) = 1$ and $f(K) > 0$, hence

$$\cos^{-1} \circ f : (2, \infty) \to (0, \pi/2) \tag{6.16}$$

is bijective. By observing that given $\theta \in (0, \pi/2)$ we can find exactly one $K := K_\theta > 2$ satisfying (6.15), this completes the proof. $\qquad\square$

We next show that for this value K_θ, the corresponding $\widetilde{H}$ has only the two fixed points constructed thus far.

Lemma 6.19. *Let $\theta \in (0, \pi/2)$ and let $K = K_\theta$. Then $\widetilde{H}$ has two fixed points, one of which is the neutral fixed point of Lemma 6.18, $\phi_{K_\theta} \in \widetilde{\mathbb{F}}_\theta^+$, and one of which is the repelling fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$.*
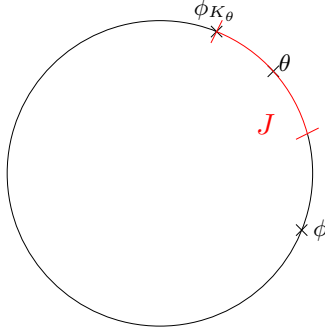
Figure 6.6: Example of when $\widetilde{H}$ has two fixed points.

Proof. Fix $\theta \in (0, \pi/2)$. First, by Lemma 6.17, there is always exactly one repelling fixed point of $\widetilde{H}$ in $\widetilde{\mathbb{F}}_\theta^-$, and so any remaining fixed points will lie in $\widetilde{\mathbb{F}}_\theta^+$. We know from Lemma 6.18 that for a neutral fixed point we require $K > 2$ for any $\theta \in (0, \pi/2)$. Recall the interval $J = J_K = (\theta - \eta_K, \theta + \eta_K)$ from Definition 6.13, which is non-empty for $K > 2$. Consider the subinterval

$$J_K^+ = (\theta, \theta + \eta_K).$$

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Writing $\varphi_K^+ = \theta + \eta_K$, we note that $\widetilde{H}_K'(\varphi_K^+) = 1$. We want to show that some value $K = K_\theta$ will give us the neutral fixed point $\varphi_{K_\theta}^+$.

First, by (6.2), we have

$$\varphi_K^+ = \cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right] + \theta. \tag{6.17}$$

Next, (6.1) implies that

$$\widetilde{H}_K(\varphi_K^+) = 2\tan^{-1}\left[\left(\tan\left(\cos^{-1}\left[\left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right]\right)\right)/K\right] + 2\theta.$$

Using the formula $\tan\cos^{-1} x = (1-x^2)^{1/2}/x$ and recalling (6.17), we see that

$$\widetilde{H}_K(\varphi_K^+) = 2\tan^{-1}\left[\left(\left(1 - \left(\frac{2K-1}{K^2-1}\right)\right)^{\frac{1}{2}}\Big/\left(K\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}\right)\right] + 2\theta$$

$$= 2\tan^{-1}\left[\frac{1}{K}\left(\frac{K(K-2)}{K^2-1}\right)^{\frac{1}{2}}\left(\frac{K^2-1}{2K-1}\right)^{\frac{1}{2}}\right] + 2\theta$$

$$= 2\tan^{-1}\left[\left(\frac{K-2}{K(2K-1)}\right)^{\frac{1}{2}}\right] + 2\theta.$$

For K just above 2, we see that $\widetilde{H}_K(\varphi_K^+) \approx 2\theta > \varphi_K^+$. Now as $K \to \infty$, we have both $\varphi_K^+ \to \pi/2 + \theta$ and $\widetilde{H}_K(\varphi_K^+) \to 2\theta$. Since $\theta < \pi/2$, we have $\widetilde{H}_K(\varphi_K^+) < \varphi_K^+$ for all large enough K.

By continuity, there exists some K_θ such that $\widetilde{H}_{K_\theta}(\varphi_{K_\theta}^+) = \varphi_{K_\theta}^+$. By construction, $\widetilde{H}_{K_\theta}'(\phi_{K_\theta}^+) = 1$ and hence it is a neutral fixed point. By Lemma 6.18 we know it is the only neutral fixed point for our given θ.

To see this is the only fixed point in $\widetilde{\mathbb{F}}_\theta^+$, consider any interval contained in $\widetilde{\mathbb{F}}_\theta^+$ with one endpoint at $\phi_{K_\theta}^+$. Then the interior of the interval is either contained in J and the interval is contracted, or it is contained in the complement of J and the interval is expanded. In either case, the other endpoint of the interval cannot be a fixed point. $\qquad\square$

For $K < K_\theta$, the next lemma shows that we only have one fixed point.

Lemma 6.20. *Let $\theta \in (0, \pi/2)$. For $K < K_\theta$ there exists only one fixed point ϕ of $\widetilde{H}$. Further $\phi \in \widetilde{\mathbb{F}}_\theta^-$ and ϕ is repelling.*

Figure 6.7: Example of when $\widetilde{H}$ has one fixed point.

Proof. From Lemma 6.17 we know there must be exactly one repelling fixed point in $\widetilde{\mathbb{F}}_\theta^-$. We are left to show there are no fixed points in $\widetilde{\mathbb{F}}_\theta^+$ by Lemma 6.7.

Using the notation of the previous lemma, when $K < K_\theta$ we know that $\widetilde{H}_K(\varphi_K^+) > \varphi_K^+$. Suppose we have a fixed point $\xi > 0$. Then if $\xi < \varphi_K^+$, the interval $I = (\xi, \varphi_K^+) \subset J$ and $|\widetilde{H}(I)| < |I|$. However, $\widetilde{H}_K(\varphi_K^+) > \varphi_K^+$ which gives a contradiction. On the other hand, suppose that $\xi > \varphi_K^+$. Then the interval $I' = (\varphi_K^+, \xi)$ similarly satisfies $|\widetilde{H}(I')| > |I'|$. Again the fact that $\widetilde{H}_K(\varphi_K^+) > \varphi_K^+$ gives a contradiction. $\qquad\square$

For $K > K_\theta$ we have three fixed points.

Lemma 6.21. *Let $\theta \in (0, \pi/2)$. For $K > K_\theta$ there exists three fixed points of $\widetilde{H}$. There are fixed points ϕ_0, ϕ_1 and ϕ_2 such that $\phi_2 < \phi_0 < \phi_1$, ϕ_1 and ϕ_2 are repelling and ϕ_0 is attracting. Further we have $\phi_1, \phi_0 \in \widetilde{\mathbb{F}}_\theta^+$ and $\phi_2 \in \widetilde{\mathbb{F}}_\theta^-$.*
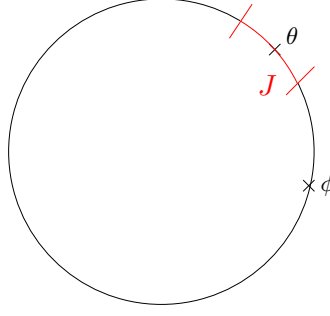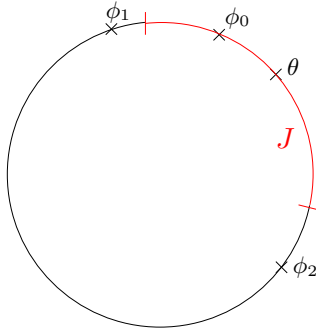

Figure 6.8: Example of when $\widetilde{H}$ has three fixed points.

Proof. From Lemma 6.17 we know there must be exactly one repelling fixed point $\phi_2 \in \widetilde{\mathbb{F}}_\theta^-$. We are left to show there are two fixed points in $\widetilde{\mathbb{F}}_\theta^+$ by Lemma 6.7.

By the methods of Lemma 6.19, we see that for $K > K_\theta$ we have $\widetilde{H}(\phi_{K_\theta}) < \phi_{K_\theta}$. Since $\widetilde{H}(\theta) = 2\theta > \theta$, by continuity there exists some $\phi_0 \in (\theta, \phi_{K_\theta})$ which is fixed by $\widetilde{H} = \widetilde{H}_K$. Similarly as $\widetilde{H}(\pi/2 + \theta) = \pi + 2\theta > \pi/2 + \theta$, there exists $\phi_1 \in (\phi_{K_\theta}, \pi/2 + \theta)$ that is fixed by $\widetilde{H}$. Hence for $K > K_\theta$ we have three fixed points. Note that we can have at most three fixed points since the fixed points of $\widetilde{H}$ correspond to roots of the cubic P given in (4.27).

Finally we have that $\phi_0 \in J$ and $\phi_1, \phi_2 \notin J$ by construction, and so ϕ_0 is attracting and ϕ_1, ϕ_2 are repelling. $\qquad\square$

The preceding lemmas prove Theorem 6.1.

6.3 How fixed rays of $H_{K,\theta}$ vary with K and θ

Recall from Lemmas 6.18 and 6.19 that for a fixed $\theta \in (0, \pi/2)$ there exists a unique $K_\theta > 2$ such that $\widetilde{H}_{K_\theta,\theta}$ has two fixed points. Similarly, because (6.16) is a bijection, for a fixed $K > 2$ there exists a unique $\theta_K \in (0, \pi/2)$ such that $\widetilde{H}_{K,\theta_K}$ has two fixed points. In either case we denote the neutral fixed point by ϕ_{K_θ}. Also recall, from Definition 6.13, the sets

$$J_K = (\theta - \eta_K, \theta + \eta_K) = \{\varphi \in (\theta - \pi/2, \theta + \pi/2) \mid \widetilde{H}'_{K,\theta}(\varphi) < 1\}.$$

6.3.1 Fixing θ and varying $K > 1$

First we will see how the fixed points of $\widetilde{H}_{K,\theta}$ behave if we fix $\theta \in (0, \pi/2)$ and vary $K > 1$.

Proposition 6.22. *Fix $\theta \in (0, \pi/2)$. As $K > 1$ increases the fixed point $\phi_K \in \widetilde{\mathbb{F}}_\theta^-$ decreases and tends to $\theta - \pi/2$, as $n \to \infty$. When $K = K_\theta$ we have a second fixed point $\phi_{K_\theta} \in \widetilde{\mathbb{F}}_\theta^+$. As $K > K_\theta$ increases the neutral fixed point ϕ_{K_θ} becomes two fixed points $\phi_K^\pm$, such that*

$$\theta + \pi/2 > \phi_K^+ > \phi_{K_\theta} > \phi_K^- > 2\theta. \tag{6.18}$$

Further $\phi_K^+ \to \theta + \pi/2$ and $\phi_K^- \to 2\theta$ as $K \to \infty$.

Proof. Recall from Lemma 6.17 that there always exists a fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}$, that is $0 > \phi > \theta - \pi/2$. Let $K_1, K_2 > 1$, $\widetilde{h}_1 := \widetilde{h}_{K_1,\theta}$, $\widetilde{h}_2 := \widetilde{h}_{K_2,\theta}$ and $0 > \varphi > \theta - \pi/2$; then

$$\widetilde{h}_1(\varphi) - \widetilde{h}_2(\varphi) = \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K_1}\right) + \theta - \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K_2}\right) - \theta. \tag{6.19}$$
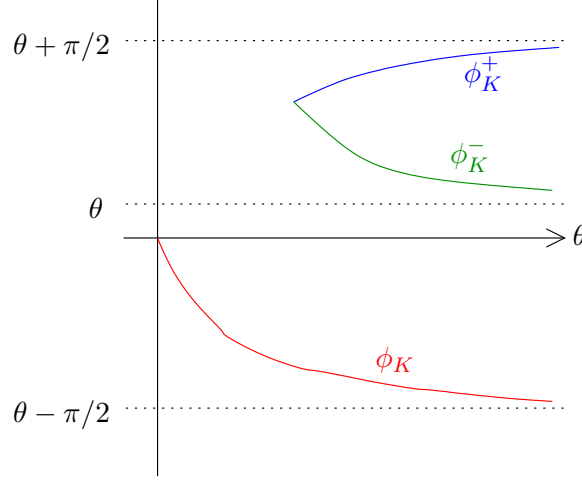
Figure 6.9: How the fixed points of $\widetilde{H}_{K,\theta}$ may vary as we vary K for a fixed $\theta \neq 0$.

Using the addition formula for $\tan^{-1}$, multiplying by $K_1 K_2 / K_1 K_2$ and simplifying (6.19) becomes

$$\widetilde{h}_1(\varphi) - \widetilde{h}_2(\varphi) = \tan^{-1}\left(\frac{(K_2 - K_1)\tan(\varphi - \theta)}{K_1 K_2 + \tan^2(\varphi - \theta)}\right). \qquad (6.20)$$

Here $\tan(\varphi - \theta) < 0$ is fixed, hence if $K_1 > K_2$ then (6.20) implies $\widetilde{h}_1(\varphi) > \widetilde{h}_2(\varphi)$.

Let ϕ_{K_i} be the fixed points of $\widetilde{h}_{K_i,\theta}$ in $\widetilde{\mathbb{F}}_\theta^-$. By Lemma 6.20 we know $\widetilde{h}_1(\phi_{K_1}) = \phi_{K_1}/2$ and $\widetilde{h}_1(\varphi) < \widetilde{h}_1(\varphi)/2$ for $0 < \varphi < \phi_{K_1}$. By (6.20) if $K_2 > K_1$ then $\widetilde{h}_2(\phi_{K_1}) < \phi_{K_1}/2$ and $\widetilde{h}_2(\varphi) < \widetilde{h}_2(\varphi)/2$ for $0 < \varphi < \phi_{K_1}$. Hence it must be the case that $\phi_{K_2} < \phi_{K_1}$. This shows that ϕ_K decreases as K increases.

Now consider the possible remaining fixed points in $\widetilde{\mathbb{F}}_\theta^+$. By Lemma 6.19 when $K = K_\theta$ we have a neutral fixed point $\phi_{K_\theta} \in \widetilde{\mathbb{F}}_\theta^+$. Then, by Lemma 6.21, when $K > K_\theta$ there are two fixed points $\phi_K^+ > \phi_K^-$ in $\widetilde{\mathbb{F}}_\theta^+$. Further, by Lemma 6.21, $\phi_K^+ \notin J_K$ and $J_{K_\theta} \subset J_K$. When $K > K_\theta$ this shows $\phi_K^+ > \phi_{K_\theta}$. Also noted in the proof of Lemma 6.12 was that $J_K \to (\theta - \pi/2, \theta + \pi/2)$ as $K \to \infty$; hence as $\phi_K^+ \notin J_K$ this implies $\phi_K^+ \to \theta + \pi/2$ as $K \to \infty$. We are only left to consider the fixed point ϕ_K^-, this was shown in Lemma 6.19 but we include a more detailed argument here. Contained in the proof of Lemma 6.21 is the fact $2\theta < \phi_K^- < \phi_{K_\theta}$, hence we have shown (6.18). Finally we want to show $\phi_K^- \to 2\theta$ as $K \to \infty$. Fixing $\varphi \in (2\theta, \theta + \pi/2)$ we see

$$\widetilde{H}_{K,\theta}(\varphi) = 2\theta + 2\tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right).$$

We can choose K large enough so that $\tan(\varphi - \theta)/K$ is as close to 0 as we want, hence $\tan^{-1}(\tan(\varphi - \theta)/K)$ is as close to 0 as we want also. As $\varphi > 2\theta$ we can pick K large enough

so that $\widetilde{H}_K(\varphi) < \varphi$. Hence by continuity we must have $\varphi > \phi_K^- > 2\theta$. We are free to pick φ as close to 2θ as we like, proving $\phi_K^- \to 2\theta$ as $K \to \infty$. $\qquad\square$

Recall from Lemma 6.16 that $\phi = 0$ is the only fixed point of $\widetilde{H}_{K,\pi/2}$. So we are only left to explain the behaviour of the fixed points when $\theta = 0$.

Lemma 6.23. *Fix* $\theta = 0$. *When* $1 < K \le 2$, $\phi = 0$ *is the only fixed point of* $\widetilde{H}_{K,0}$. *For* $K > 2$ *there exist fixed points* $\phi_0 = 0, \phi_\theta^- \in (-\pi/2, 0)$ *and* $\phi_\theta^+ \in (0, \pi/2)$; *further* $\phi_\theta^- = -\phi_\theta^+$. *As* $K > 2$ *increases,* ϕ_θ^+ *increases and* ϕ_θ^- *decreases. Also* $\phi_\theta^\pm \to \pm\pi/2$ *as* $K \to \infty$.

Proof. Lemma 6.15 implies the first two sentences of the lemma. Assume $K_2 > K_1 > 2$, then (6.20) also holds for $\theta = 0$, that is $\widetilde{h}_{K_2,0}(\phi_{K_1}^-) < \phi_{K_1}^-/2$ and $\widetilde{h}_{K_2,0}(\varphi) < \widetilde{h}_{K_2,\theta}(\varphi)/2$ for $0 < \varphi < \phi_{K_1}^-$. Hence it must be the case that $\phi_{K_2}^- < \phi_{K_1}^-$. This shows that ϕ_K^- decreases as K increases. The fact $\phi_\theta^- = -\phi_\theta^+$ shows ϕ_K^+ increases as K increases.

Noted in the proof of Lemma 6.12 was that $J_K \to (-\pi/2, \pi/2)$ as $K \to \infty$; hence as $\phi_K^\pm \notin J_K$ this implies $\phi_K^\pm \to \pm\pi/2$ as $K \to \infty$. $\qquad\square$

6.3.2 Fixing $K > 1$ and varying $0 \le \theta \le \pi/2$

It is much less natural to fix K and vary θ. This is because if $\theta \ge 0$, we have the equivalence $H_{K,\theta} = KH_{1/K,\pi/2-\theta}$. However we can still gain some knowledge. We first prove a useful lemma.

Lemma 6.24. *Fix* $K > 1$. *If* ϕ *is a fixed point of* $\widetilde{H}_{K,\theta_1}$ *and* $\widetilde{H}_{K,\theta_2}$, *then either* $\theta_1 = \theta_2$ *or* $\theta_2 = \phi - \tan^{-1}(K/\tan(\phi - \theta_1))$.

Proof. Assume ϕ is a fixed point of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$. Recalling the definition of $\widetilde{H}_{K,\theta_i}$ and using the fact ϕ is a fixed point if and only if $\widetilde{H}_1(\phi) = \widetilde{H}_2(\phi)$. We must have,

$$\theta_1 + \tan^{-1}\left(\frac{\tan(\phi - \theta_1)}{K}\right) = \theta_2 + \tan^{-1}\left(\frac{\tan(\phi - \theta_2)}{K}\right).$$

Rearranging we obtain,

$$\theta_1 - \theta_2 = \tan^{-1}\left(\frac{\tan(\phi - \theta_2)}{K}\right) - \tan^{-1}\left(\frac{\tan(\phi - \theta_1)}{K}\right).$$

Using the addition formula for $\tan^{-1}$ and simplifying we see,

$$\theta_1 - \theta_2 = \tan^{-1}\left(\frac{K(\tan(\phi - \theta_2)) - \tan(\phi - \theta_1)}{K^2 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)}\right).$$

Adding and subtracting ϕ from the left hand side and applying tan to both sides we are left with,

$$\tan((\phi - \theta_2) - (\phi - \theta_1)) = \frac{K(\tan(\phi - \theta_2) - \tan(\phi - \theta_1))}{K^2 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)}.$$

Applying the addition formula for tan we obtain,

$$\frac{\tan(\phi - \theta_2) - \tan(\phi - \theta_1)}{1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)} = \frac{K(\tan(\phi - \theta_2) - \tan(\phi - \theta_1))}{K^2 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)}.$$

Cancelling the denominators and grouping the different K coefficients together we obtain the quadratic equation,

$$K^2(\tan(\phi - \theta_2) - \tan(\phi - \theta_1))$$
$$- K(\tan(\phi - \theta_2) - \tan(\phi - \theta_1))(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2))$$
$$+ (\tan(\phi - \theta_2) - \tan(\phi - \theta_1))\tan(\phi - \theta_1)\tan(\phi - \theta_2) = 0.$$

This obviously has the solution $\tan(\phi - \theta_1) = \tan(\phi - \theta_2)$, which implies $\theta_1 = \theta_2$. Factoring out this solution we are left with,

$$K^2 - K(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)) + \tan(\phi - \theta_1)\tan(\phi - \theta_2) = 0.$$

Applying the quadratic formula for K we see

$$K = \frac{1}{2}(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)$$
$$\pm \sqrt{(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2))^2 - 4\tan(\phi - \theta_1)\tan(\phi - \theta_2)})$$

$$= \frac{1}{2}(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2)$$
$$\pm \sqrt{1 - 2\tan(\phi - \theta_1)\tan(\phi - \theta_2) + (\tan(\phi - \theta_1)\tan(\phi - \theta_2))^2})$$
$$= \frac{1}{2}(1 + \tan(\phi - \theta_1)\tan(\phi - \theta_2) \pm (1 - \tan(\phi - \theta_1)\tan(\phi - \theta_2)).$$

This leaves the solutions $K = 1$, which is not permissible, and

$$K = \tan(\phi - \theta_1)\tan(\phi - \theta_2). \tag{6.21}$$

Hence ϕ is a fixed point of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$ if and only if $\theta_1 = \theta_2$ or if

$$\theta_2 = \phi - \tan^{-1}(K/\tan(\phi - \theta_1))$$

.
$\square$

Remark 6.25. *If $K > 1$ is fixed then ϕ is a fixed point of $\widetilde{H}_{K,\theta_i}$ for at most two values of θ_i.*

We can now apply Lemma 6.24 to prove two propositions.

Proposition 6.26. *Fix $1 < K \leq 2$. There is exactly one fixed point ϕ_θ of $\widetilde{H}_{K,\theta}$ and*

- $\phi_0 = \phi_{\pi/2} = 0$,

- *for $\theta \in (0, \pi/2)$, $\phi_\theta \in (\theta - \pi/2, 0)$,*

- *there exists $\theta_0 \in (0, \pi/2)$ such that ϕ_θ decreases with θ for $\theta \in (0, \theta_0)$ and ϕ_θ increases with θ for $\theta \in (\theta_0, \pi/2)$.*
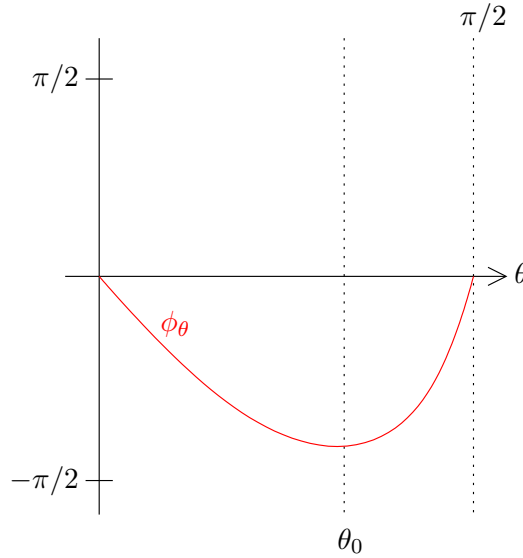
Figure 6.10: How the fixed points of $\widetilde{H}_{K,\theta}$ may vary as we vary θ for a fixed $K \leq 2$.

Proof. By Lemma 6.20 we know there is only one fixed point $\phi_\theta \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}_{K,\theta}$ when $K \leq 2$. When $\theta = 0$ or $\pi/2$ we have that $\phi_\theta = 0$ is fixed by Lemma 6.7. When $\theta \in (0, \pi/2)$ the fixed point satisfies $\phi_\theta \in (\theta - \pi/2, 0)$. By Lemma 6.24, $\phi_{\theta_1} = \phi_{\theta_2}$ if and only if $\theta_1 = \theta_2$ or if $\theta_2 = \phi_{\theta_1} - \tan^{-1}(K/\tan(\phi_{\theta_1} - \theta_1))$. That is, when $K > 1$ is fixed, there are at most two values of θ for which a point $\phi \in (-\pi/2, 0]$ is a fixed point of $\widetilde{H}_{K,\theta}$.

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We have noted that $\phi_0 = 0 = \phi_{\pi/2}$ and that $\phi_\theta \in (-\pi/2, 0)$ for $\theta \in (0, \pi/2)$. Hence as θ increases from 0 it must be the case that first ϕ_θ decreases; then as $\theta \to \pi/2$ it must be the case that ϕ_θ increases, so that $\phi_{\pi/2} = 0$. The fact that $\widetilde{H}_{K,\theta}$ varies continuously in θ and that there are at most two values of θ such that a given ϕ is a fixed point implies that there must be some value θ_0 where ϕ_θ stops decreasing away from 0 and starts to increase all the way back to 0, thus satisfying the hypothesis of the proposition. $\qquad \square$

Remark 6.27. *Further, by (6.21), the point where ϕ_θ is at a minimum must occur when*

$$\phi_{\theta_0} = \theta_0 - \tan^{-1}(\sqrt{K}).$$

Proposition 6.28. *Fix $K > 2$. Let $\phi_\theta, \phi_\theta^-, \phi_\theta^+$ be the the three possible fixed points of $\widetilde{H}_{K,\theta}$, so that when they exist they satisfy the inequality*

$$\theta - \pi/2 < \phi_\theta \leq 0 \leq \phi_\theta^- < \phi_\theta^+ < \theta + \pi/2. \tag{6.22}$$

These fixed points vary as follows.

- *As $\theta > 0$ increases, ϕ_θ^- increases and $\phi_\theta^- \to \phi_{K_\theta}$ as $\theta \to \theta_K$.*

- *There exists some $\theta_+ \in [0, \theta_K]$ such that as $\theta \in (0, \theta_+)$ increases ϕ_θ^+, increases and as $\theta > \theta_+$ increases, ϕ_θ^+ decreases. Further, $\phi_\theta^+ \to \phi_{K_\theta}$ as $\theta \to \theta_K$.*

- *There exists some $\theta_0 \in [0, \pi/2)$ such that as $\theta \in (\theta_0, \pi/2)$ increases, ϕ_θ decreases and as $\theta_0 < \theta < \pi/2$ increases, ϕ_θ increases. Further, $\phi_\theta \to 0$ as $\theta \to \pi/2$.*

Proof. Firstly Lemma 6.21 implies (6.22). The statements about the behaviour of the fixed rays will again follow from Lemma 6.24 and Remark 6.25, that $\phi_{\theta_1} = \phi_{\theta_2}$ for at most two values $\theta_1, \theta_2 \in [0, \pi/2]$. First let $\theta = 0$ and consider the fixed point $\phi_\theta \in \widetilde{\mathbb{F}}_\theta^-$. Then $\phi_0 \neq 0$, but we know that $\phi_{\pi/2} = 0$. If ϕ_θ begins increasing then it cannot start decreasing for larger θ as, by continuity in θ, this would imply there are at least three solutions for $\phi_{\theta_i} = \phi_{\theta_j}$, contradicting Lemma 6.24. Hence there must exist some $\theta_0 \in [0, \pi/2)$ such that $\phi_\theta > \phi_{\theta_0}$ increases as $\theta > \theta_0$ increases.

Next consider $\phi_\theta^- \in \widetilde{\mathbb{F}}_\theta^+$. When $\theta = 0$ we know $\phi_\theta^- = 0$ by Lemma 6.15. Also by Lemma 6.21 when $\theta > 0$ we have $\phi_\theta^- > 2\theta$, when it exists. Hence as $\theta > 0$ increases, ϕ_θ^- must begin to increase also. Further we know that $\phi_\theta^- \to \phi_{\theta_K}$ as $\theta \to \theta_K$. Hence ϕ_θ^- must continue

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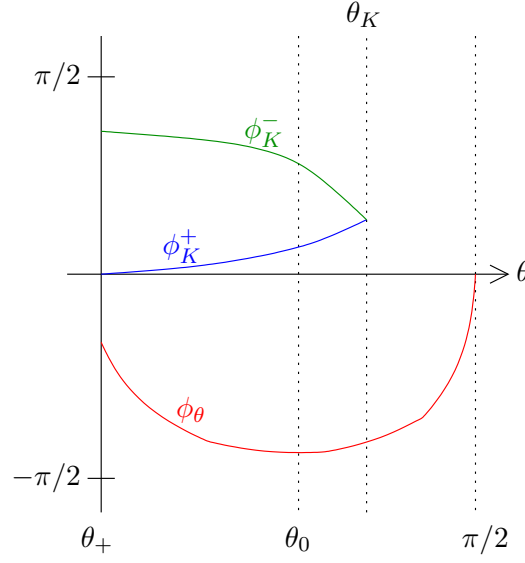
Figure 6.11: How the fixed points of $\widetilde{H}_{K,\theta}$ may vary as we vary θ for a fixed $K > 2$.
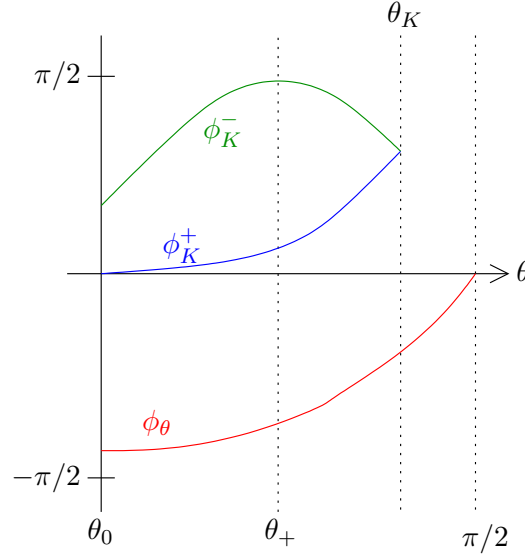
Figure 6.12: A further example of how the fixed points of $\widetilde{H}_{K,\theta}$ may vary as we vary θ for a fixed $K > 2$.

to increase, as if not it would require $\phi^-_{\theta_i} = \phi^-_{\theta_j}$ to have three solutions by continuity in θ, a contradiction. Also $\phi^-_\theta < \phi_{\theta_K}$, when it exists.

Next consider $\phi^+_\theta \in \widetilde{\mathbb{F}}^+_\theta$. Suppose, for a contradiction, that ϕ^+_θ begins to decrease as $\theta > 0$ increases, reaches a minimum and then starts to increase. As with ϕ^-_θ we know by continuity that $\phi^+_\theta \to \phi_{\theta_K}$ as $\theta \to \theta_K$. Hence there must exist $\theta_1 \neq \theta_2$ such that $\phi^+_{\theta_1} = \phi^+_{\theta_2} < \phi_K$. However we already know that ϕ^-_θ takes every value in $[0, \theta_K)$, so there exists some θ_3 such that $\phi^-_{\theta_3} = \phi^+_{\theta_1}$.

This is a contradiction, as Lemma 6.24 implies there can be only two values of θ that satisfy this. So if ϕ_θ^+ begins to decrease as θ increases then it must continue to decrease until $\theta = \theta_K$. It may be the case that ϕ_θ^+ begins to increase as $\theta > 0$ increases. Hence there exists $\theta_+ \in [0, \theta_K]$ such that if $\theta_+ \neq \theta_K$ then $\phi_\theta^+ < \phi_{\theta_+}^+$ for all $\theta > \theta_+$, or if $\theta_+ = \theta_K$ then ϕ_θ^+ increases as $\theta \in (0, \theta_K)$ increases. $\qquad\square$

6.4 Pre-images of fixed rays and basins of attraction

In this section we will study the preimages of fixed rays and their basins of attraction, where they exist. The basin of attraction of a non-repelling fixed point is defined as follows.

Definition 6.29. The *basin of attraction* $\widetilde{\Lambda}$ of a non-repelling fixed point ϕ of $\widetilde{H}$ is given by

$$\widetilde{\Lambda} := \{\varphi \in S^1 \mid \widetilde{H}^n(\varphi) \to \phi \text{ as } n \to \infty\}.$$

The *immediate basin of attraction* $\widetilde{\Lambda}^*$ is the component of $\widetilde{\Lambda}$ containing ϕ.

Recall that we use a tilde to denote sets in S^1 and that the basin of attraction of the non-repelling fixed ray R_ϕ of H in $\mathbb{C}$ will be $\Lambda = \{R_\varphi \mid \varphi \in \widetilde{\Lambda}\}$. The fixed points form a Cantor set and have the following properties.

Theorem 6.2. *If H has one fixed ray R_ϕ then $\{H^{-k}(R_\phi)\}_{k=0}^\infty$ is dense in $\mathbb{C}$. If H has two or three fixed rays, then Λ is dense in $\mathbb{C}$.*

We prove Theorem 6.2 by studying the backward orbits of the fixed points of $\widetilde{H}$ and any basins of attraction. We will see that $\widetilde{H}$ restricted to S^1 is actually a Blaschke product. We take advantage of this fact, and use properties of Julia sets and Fatou sets of rational functions. In view of Lemma 4.2, throughout this section we assume that $\theta \in [0, \pi/2]$.

6.4.1 Basins of attraction

In the case where $\widetilde{H}$ has two or three fixed points on S^1, we will see that the non-repelling fixed point has a basin of attraction. When $\widetilde{H}$ has three fixed points, the basin is formed by a union of open intervals, whereas when $\widetilde{H}$ has two fixed points, the basin is formed by a union of half-open intervals.

Lemma 6.30. *Recalling the notation of Lemma 6.18, suppose $\widetilde{H}$ has two fixed points. Then the neutral fixed point ϕ_{K_θ} has an immediate basin of attraction $\widetilde{\Lambda}^*$ that is the interval bounded by ϕ_{K_θ} and the repelling fixed point $\phi \in \widetilde{\mathbb{F}}_\theta^-$.*

Proof. From Lemma 6.19 we know that intervals of the form $[\phi_{K_\theta}, \varphi]$ are expanded and so

$$\widetilde{\Lambda}^* = (\psi, \phi_{K_\theta}],$$

for some ψ. Since the repelling fixed point $\phi \notin \widetilde{\Lambda}^*$, we have $\psi \geq \phi$. However, from Lemma 6.19 we also know that all intervals $[\varphi, \phi_{K_\theta}]$ such that

$$\phi < \varphi < \phi_{K_\theta} \tag{6.23}$$

are contracted under $\widetilde{H}$. Hence if φ satisfies (6.23) then $\varphi \in \widetilde{\Lambda}^*$. Therefore we have

$$\widetilde{\Lambda}^* = (\phi, \phi_{K_\theta}].$$

$\square$

Lemma 6.31. *When $\widetilde{H}$ has three fixed points $\phi_2 < \phi_0 < \phi_1$ as in Lemma 6.21, the attracting fixed point ϕ_0 has an immediate basin of attraction*

$$\widetilde{\Lambda}^* = (\phi_2, \phi_1).$$

Proof. As ϕ_0 is an attracting fixed point

$$\widetilde{\Lambda}^* = (\varphi_2, \varphi_1),$$

for some $\varphi_2 < \phi_0 < \varphi_1$. By Lemma 6.21 we know that all intervals of the form $[\phi_0, \varphi_1]$ and $[\varphi_2, \phi_0]$, where

$$\phi_2 < \varphi_2 < \phi_0 < \varphi_1 < \phi_1,$$

are contracted under $\widetilde{H}$ and so $\varphi_1, \varphi_2 \in \widetilde{\Lambda}^*$. Further as $\phi_1, \phi_2 \notin \widetilde{\Lambda}^*$ this implies that

$$\widetilde{\Lambda}^* = (\phi_2, \phi_1).$$

$\square$
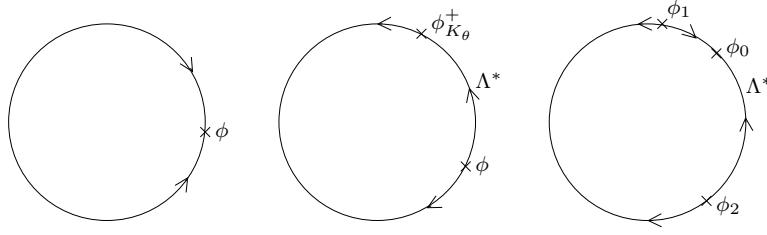
Figure 6.13: Diagram showing the local dynamics of one, two and three fixed points.

6.4.2 Writing $\widetilde{H}$ as a Blaschke product

We will consider the case $\theta = 0$, since $h_{K,\theta}$ can be obtained from $h_{K,0}$ by pre-composing and post-composing by the corresponding rotations. Let $h = h_{K,0}$. The induced map $\widetilde{h}$ on S^1 cannot be a Möbius map, however it is π periodic and so we can renormalise $\widetilde{h} : (-\pi/2, \pi/2] \to (-\pi/2, \pi/2]$ to a map $\hat{h} : (-\pi, \pi] \to (-\pi, \pi]$ by defining

$$\hat{h}(\varphi) = 2\widetilde{h}(\varphi/2). \tag{6.24}$$

This map $\hat{h}$ has an attracting fixed point at $\varphi = 0$ and a repelling fixed point at $\varphi = \pi$.

Lemma 6.32. *The map $\hat{h}$ agrees with the Möbius map*

$$A_K(z) = \frac{z + \alpha}{1 + \overline{\alpha} z},$$

where $\alpha = (K - 1)/(K + 1)$ on S^1.

Remark 6.33. *Here the complex conjugation is superfluous as α is real, however it will be necessary when we generalise in the next lemma.*

Proof. Recall that the induced map is given by $\widetilde{h}(\varphi) = \tan^{-1}(\tan(\varphi)/K)$ and so

$$\begin{aligned}
\tan(\hat{h}(\varphi)) &= \tan(2\tan^{-1}(\tan(\varphi/2)/K)) \\
&= \frac{2\tan(\varphi/2)/K}{1 - \tan^2(\varphi/2)/K^2} \\
&= \frac{(2K\sin\varphi)/(1 + \cos\varphi)}{K^2 - (\sin^2\varphi)/(1 + \cos(\varphi))^2} \\
&= \frac{2K\sin\varphi}{(K^2 - 1) + \cos\varphi(K^2 + 1)}.
\end{aligned}$$

Define A_K to be the Möbius map

$$A_K(z) = \frac{z + \alpha}{1 + \overline{\alpha} z}, \tag{6.25}$$

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where $\alpha = (K-1)/(K+1)$. By construction $A_K(S^1) = S^1$. Further by writing $z = x + iy$ we see

$$A_K(x + iy) = \frac{(x + \alpha + iy)(1 + \alpha x - i\alpha y)}{(1 + \alpha x + i\alpha y)(1 + \alpha x - i\alpha y)}$$
$$= \frac{x(1 + \alpha^2) + \alpha(1 + x^2 + y^2) + iy(1 - \alpha^2)}{(1 + \alpha x + i\alpha y)(1 + \alpha x - i\alpha y)}.$$

Noticing that $x, y \in S^1$ and so $x^2 + y^2 = 1$ and the fact that

$$\arg(A_K(z)) = \tan^{-1}[\mathrm{Im}(A_{K,0}(z))/\mathrm{Re}(A_{K,0}(z))]$$

we have

$$\tan[\arg(A_K(z))] = \frac{y(1 - \alpha^2)}{2\alpha + x(1 + \alpha^2)}.$$

It is easy to see that if φ denotes the argument of the point $z \in S^1$ and $z = x + iy$ then $x = \cos\varphi$ and $y = \sin\varphi$, hence using $\widetilde{A}_K$ to denote the map A_K induces on the argument of z we see

$$\widetilde{A}_K(\varphi) = \frac{2K\sin\varphi}{(K^2 - 1) + \cos\varphi(K^2 + 1)} = \hat{h}(\varphi).$$

This shows that $\hat{h}$ is a Möbius map of S^1. $\qquad\square$
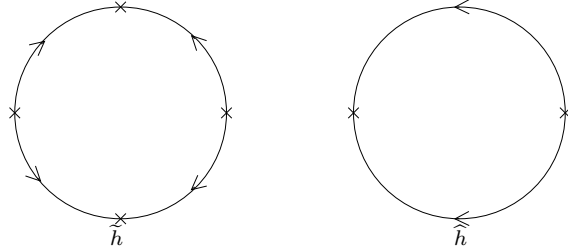
Figure 6.14: Diagram for $\theta = 0$ showing how we obtain a Möbius map with two fixed points.

Lemma 6.34. *Let* $H = H_{K,\theta}$. *Then* $\widetilde{H} : S^1 \to S^1$ *agrees with a Blaschke product B on S^1 given by*

$$B(z) = \frac{z^2 + \mu}{1 + \overline{\mu}z^2} = \left(\frac{z - a}{1 - \overline{a}z}\right)\left(\frac{z + a}{1 + \overline{a}z}\right),$$

where $\mu = e^{2i\theta}\left(\frac{K-1}{K+1}\right)$ *is the complex dilatation of H and* $a = e^{i(\theta - \pi/2)}\left(\frac{K-1}{K+1}\right)^{1/2}$.

Proof. For $\varphi \in (-\pi/2 + \theta, \pi/2 + \theta]$, we have $\widetilde{h}_{K,\theta}(\varphi) = \widetilde{h}_{K,0}(\varphi - \theta) + \theta$. Using Lemma 6.32, we

see that

$$\widetilde{h}_{K,\theta}(\varphi) = \frac{\hat{h}(2\varphi - 2\theta)}{2} + \theta$$
$$= \frac{A_K(2\varphi - 2\theta)}{2} + \theta$$

As $\widetilde{H}_{K,\theta} = 2\widetilde{h}_{K,\theta}$ we obtain

$$\widetilde{H}_{K,\theta}(\varphi) = 2\theta + A_K(2\varphi - 2\theta)$$

for $\varphi \in (-\pi/2 + \theta, \pi/2 + \theta]$, and by π-periodicity, for the remaining values of φ, we have

$$\widetilde{H}_{K,\theta}(\varphi) = 2\theta + 2\pi + A_K(2\varphi + 2\pi - 2\theta)$$
$$= 2\theta + A_K(2\varphi - 2\theta).$$

Letting $z = e^{i\varphi}$, recalling (6.25) we see

$$\widetilde{H}_{K,\theta}(e^{i\varphi}) = e^{2i\theta} \frac{e^{i(2\varphi - 2\theta)} + \alpha}{1 + \alpha e^{i(2\varphi - 2\theta)}}$$
$$= \frac{e^{2i\varphi} + \alpha e^{2i\theta}}{1 + \alpha e^{-2i\theta} e^{2i\varphi}}$$
$$= \frac{z^2 + \mu}{1 + \overline{\mu} z^2}, \tag{6.26}$$

where $\mu = e^{2i\theta}(K-1)/(K+1)$ is the complex dilatation of H.

$\square$

6.4.3   Proof of Theorem 6.2

We can now use the standard results on the iteration theory of Blaschke products as given in Proposition 2.26. Our Blaschke product B has three fixed points, counting multiplicity. Write $J(B)$ and $F(B)$ for the Julia and Fatou sets of B, respectively. Note that Theorem 6.1 tells us how many fixed points B has on S^1.

Suppose that $\widetilde{H}$ has one fixed point $\phi \in S^1$. We know from Lemma 6.34 and Proposition 2.26 that the Julia set of B is S^1 and so $\phi \in J$. Since $J(B) = \overline{O^-(z)}$ for any $z \in J(B)$, we immediately have that $\widetilde{\mathcal{P}} = \{\widetilde{H}^{-k}(\phi)\}_{k=0}^{\infty}$ is dense in S^1.

Suppose that $\widetilde{H}$ has more than one fixed point. Let ϕ be the non-repelling fixed point. By Lemma 2.26 we know that the Julia set $J(B)$ of B is a Cantor subset of S^1. This implies that $E = F(B) \cap S^1$ is a dense subset of S^1. Consider a point $z \in E$, then any neighbourhood

$U \subset F(B)$ of z contains points in $U \cap \mathbb{D}$. By the Denjoy-Wolff Theorem [10, §IV Theorem 3.1], we have $B^n(w) \to \phi$ as $n \to \infty$ for every $w \in U \cap \mathbb{D}$. As U is contained in the Fatou set, the iterates $\{B^n\}$ are a normal family on U and so $B^n(z) \to \phi$ as $n \to \infty$. This implies $z \in \widetilde{\Lambda}$, and hence $\widetilde{\Lambda}$ is dense in S^1.

This completes the proof of Theorem 6.2.

6.5   Decomposition of $\mathbb{C}$

We now show how we can decompose $\mathbb{C}$ into three sets. A basin of attraction of the fixed point 0, another basin of attraction of the fixed point ∞ and a set that is our analogue of the closure of the Julia set.

Corollary 6.3. *Let* $K > 1$, $\theta \in (-\pi/2, \pi/2]$ *and* $H(z) = h_{K,\theta}(z)^2$. *Then* $\mathbb{C} = I(H) \cup \partial I(H) \cup \mathcal{A}(0)$, *where* $\mathcal{A}(0)$ *is the basin of attraction of the fixed point* 0.

Proof. Fix $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. By Theorem 3.21, the escaping set $I(H)$ is a connected, completely invariant, open neighbourhood of infinity and $\partial I(H)$ is a completely invariant closed set. The point 0 is clearly fixed by H and since

$$|z|^2 \leq |H(z)| \leq K^2 |z|^2,$$

there is a neighbourhood of 0 contained in the basin of attraction $\mathcal{A}(0)$. It is therefore clear that $\mathcal{A}(0)$ is completely invariant and open.

Let R_ϕ be a fixed ray of H. Then on R_ϕ, we have

$$H(re^{i\phi}) = \alpha r^2 e^{i\phi},$$

where $\alpha = (1 + (K^2 - 1)\cos^2(\phi - \theta))$ by the polar form (4.7) of H. For $r = 1/\alpha$, this point is fixed, for $r > 1/\alpha$ the point is in $I(H)$ and for $r < 1/\alpha$, the point is in $\mathcal{A}(0)$. By complete invariance and the fact $I(H)$ and $\mathcal{A}(0)$ are open, any pre-image of R_ϕ breaks up into $\mathcal{A}(0), I(H)$ and $\partial I(H)$ in the same way.

Assume that $K < K_\theta$, then by Theorem 6.1 H has one fixed ray R_ϕ. By Theorem 6.2 the set $\{H^{-k}(R_\phi) : k \geq 0\}$ is dense in $\mathbb{C}$. Since $I(H)$ and $\mathcal{A}(0)$ are open, this proves the result in this case.

On the other hand, if $K \geq K_\theta$, then by Theorem 6.1 write Λ for the basin of attraction of the non-repelling fixed ray R_ϕ. By Theorem 6.2, Λ is dense in $\mathbb{C}$. Suppose that $R_\varphi \in \Lambda$. Then

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$H^n(R_\varphi) \to R_\phi$. Since $\mathcal{A}(0)$ and $I(H)$ are open, it is not hard to see that R_φ breaks up in the same way that R_ϕ does. Since Λ is dense in $\mathbb{C}$, the openness of $\mathcal{A}(0)$ again implies the result in this case, which proves the corollary. $\qquad\square$

Chapter 7

Failure of uniform quasiregularity

7.1 Statement of chapter's results

We can use Theorem 5.1 to prove the following result on the mapping $h(z)^2 + c$.

Theorem 7.1. *Let h be affine and $c \in \mathbb{C}$. Then the mapping $f(z) = h(z)^2 + c$ is not uniformly quasiregular.*

The significance of Theorem 7.1 is as follows. By Theorem 3.19, every uniformly quasiregular mapping in the plane is a quasiconformal conjugate of an analytic mapping. As mentioned earlier, this is a generalisation of results of Sullivan [35] and Tukia [36] for uniformly quasiconformal mappings. The upshot of this is that the study of uniformly quasiregular mappings in the plane reduces to the standard theory of complex dynamics. Therefore, for the study of the dynamics of mappings of the form $h(z)^2 + c$ to be of independent interest, we need to know that they are not uniformly quasiregular.

In view of Theorem 5.1, the proof of Theorem 7.1 reduces to the following result.

Theorem 7.2. *Let h be an affine mapping. Then $H = h^2$ is not uniformly quasiregular.*

This theorem will be proved by showing that the complex dilatation of the iterates of H on a ray fixed by H has a particularly nice form. Using this, and some basic iteration theory of Möbius transformations, we show that the modulus of the complex dilatation of the iterates converges to 1 on this fixed ray, which is equivalent to the maximal distortion of the iterates being unbounded. Assuming this result for the moment, the proof of Theorem 7.1 runs as follows.

Proof of Theorem 7.1. Write $H(z) = h(z)^2$ and $f(z) = h(z)^2 + c$. By Theorem 7.2, H is not uniformly quasiregular in any neighbourhood of infinity. By Theorem 5.1, $H = \psi \circ f \circ \psi^{-1}$ in a neighbourhood of infinity U. Therefore

$$K(H^n) = K(\psi \circ f^n \circ \psi^{-1}) \leq K(\psi)^2 K(f^n),$$

where $K(g)$ denotes the maximal dilatation of g. Since $K(H^n) \to \infty$ in U, we have $K(f^n) \to \infty$ in U. $\qquad\square$

In fact we can show more about our maps. We prove Theorem 7.1 by studying the complex dilatation on a fixed ray. By Theorem 6.2 we know that any neighbourhood of a point either intersects the preimage of a fixed ray or basin of attraction - if it exists. Hence we can show that in any neighbourhood of a point there exist points where the distortion is unbounded and we call such a function *nowhere uniformly quasiregular*.

Theorem 7.3. *Let $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. Then the mapping $h_{K,\theta}(z)^2 + c$ is nowhere uniformly quasiregular.*

7.2   Proof of Theorem 7.2

7.2.1   Fixed rays of h^2

Let the ray R_ϕ be a fixed ray of H. Let μ_{H^n} be the complex dilatation of H^n. Then by Lemma 3.8,
$$\mu_{H^n}(z) = \frac{\mu_H(z) + r_H(z)\mu_{H^{n-1}}(H(z))}{1 + r_H(z)\overline{\mu_H(z)}\mu_{H^{n-1}}(H(z))},$$
where $r_H(z) = \overline{H_z(z)}/H_z(z)$. As noted at the beginning of Chapter 4, $\mu_H = e^{2i\theta}(K-1)/(K+1)$ is constant in $\mathbb{C}$. The next lemma shows that μ_{H^n} is a constant on the fixed ray R_ϕ.

Lemma 7.4. *Let $z \in R_\phi$. Then for $n \geq 1$*

$$\mu_{H^n}(z) \equiv \frac{\mu_H + e^{-i\phi}\mu_{H^{n-1}}(z)}{1 + e^{-i\phi}\overline{\mu_H}\mu_{H^{n-1}}(z)}.$$

Proof. To find r_H, we observe that

$$H_z(z) = \left[h(z)^2\right]_z = 2(h_z(z))h(z) = (K+1)h(z).$$

Since $z \in R_\phi$, we have $z = re^{i\phi}$ for some $r > 0$. By the fact that R_ϕ is a fixed ray of H, it follows that $h(z) = r'e^{i\phi/2}$ for some $r' > 0$. Therefore

$$r_H(z) = e^{-i\phi}$$

for $z \in R_\phi$. Since $\mu_H \equiv e^{2i\theta}(K-1)/(K+1)$, by induction we see that μ_{H^n} is a constant on R_ϕ and takes the claimed form by the formula for the complex dilatation of a composition. $\square$

We will also need the following corollary, which is just a reformulation of Lemma 6.7.

Corollary 7.5. *Any fixed ray R_ϕ of H lies in the half plane*

$$\mathbb{H}_\theta = \{R_\varphi \mid -\pi/2 < \varphi - \theta < \pi/2\},$$

or if $\theta = \pi/2$ then R_0 is the only fixed ray.

7.2.2   Möbius transformations

Define

$$A(z) = \frac{\mu_H + e^{-i\phi}z}{1 + e^{-i\phi}\overline{\mu_H}z}$$

so that $\mu_{H^n} = A^{n-1}(\mu_H)$ on the fixed ray R_ϕ. Note that A depends only on K, θ. We can rewrite A as

$$A(z) = e^{-i\phi}\left(\frac{z + e^{i\phi}\mu_H}{1 + \overline{e^{i\phi}\mu_H}z}\right). \tag{7.1}$$

Now A is a Möbius map of the disk $\mathbb{D}$, and the behaviour of the iterates is determined by the trace. By standard theory (see §2.4) if $\operatorname{Tr}^2 A \geq 4$, then A has all of its fixed points on $\partial\mathbb{D}$ and $|A^n(z)| \to 1$ for all $z \in \mathbb{D}$. In particular, we would have $|A^n(\mu_H)| \to 1$ and so $|\mu_{H^n}| \to 1$. Therefore to prove Theorem 7.2, we need to prove the following proposition.

Proposition 7.6. *Given the Möbius transformation A as in (7.1), we have $\operatorname{Tr}^2 A \geq 4$.*

7.2.3   Proof of Proposition 7.6

The rest of this section is devoted to proving the proposition. We first calculate an expression for $\operatorname{Tr}^2 A$.

Lemma 7.7. *The trace of A satisfies*

$$\operatorname{Tr}^2 A = \frac{(K+1)^2(1 + \cos\phi)}{2K}.$$

Proof. To compute the trace of a Möbius transformation $(az + b)/(cz + d)$, we first need to ensure that $ad - bc = 1$, and then calculate $a + d$. Putting A into this normalised form yields

$$A(z) = \frac{e^{-i\phi/2}\left(\frac{K+1}{2K^{1/2}}\right)z + \mu_H e^{i\phi/2}\left(\frac{K+1}{2K^{1/2}}\right)}{e^{-i\phi/2}\left(\frac{K+1}{2K^{1/2}}\right)\overline{\mu_H}z + e^{i\phi/2}\left(\frac{K+1}{2K^{1/2}}\right)}.$$

From this we can calculate that

$$\mathrm{Tr}^2 A = \frac{(K+1)^2(e^{i\phi/2} + e^{-i\phi/2})^2}{4K} = \frac{(K+1)^2(1 + \cos\phi)}{2K},$$

which proves the lemma. $\qquad\square$

To prove Proposition 7.6 by using Lemma 7.7 we need to obtain a lower bound on $\cos\phi$, where ϕ is the angle of a fixed ray of H corresponding to K, θ. Recall from Corollary 7.5 that $\phi \in \widetilde{\mathbb{H}}_\theta$, so we need only consider rays R_φ where $\varphi - \theta \in (-\pi/2, \pi/2)$. To find a lower bound, first consider the function

$$G(\varphi) = \varphi - \theta - \tan^{-1}\left(\frac{\tan(\varphi - \theta)}{K}\right).$$

Recalling the polar form of h given in (4.7), and since h maps rays to rays, the function G describes the change in angle undergone by a ray of angle φ under h. Clearly $G(\theta) = 0$ since h stretches in the direction $e^{i\theta}$. Further, for the fixed ray of H with angle ϕ, $G(\phi) = \phi/2$.

We want to know how large G can be, that is, how much of an angle can h move a ray through. This maximum occurs when the derivative $\frac{\partial G}{\partial \varphi} = 0$. Calculating the derivative gives

$$\frac{\partial G}{\partial \varphi} = 1 - \frac{K}{(K^2 - 1)\cos^2(\varphi - \theta) + 1}.$$

Hence the maximum value of G occurs when

$$\cos^2(\varphi - \theta) = \frac{1}{K + 1}.$$

Since $\varphi - \theta \in (-\pi/2, \pi/2)$, then the maxima of G are attained at

$$\varphi_\pm = \theta \pm \cos^{-1}[(K + 1)^{-1/2}],$$

and the values of G attained there are

$$G_\pm := G\left(\varphi_\pm\right) = \pm\left(\cos^{-1}[(K+1)^{-1/2}] - \tan^{-1}\left(\frac{\tan(\cos^{-1}[(K+1)^{-1/2}])}{K}\right)\right).$$

Using these local maxima, if $0 < \varphi - \theta < \pi/2$, then

$$0 \le G(\varphi) \le G_+ \le \pi/2,$$

and in particular if the fixed ray of angle ϕ satisfies $0 < \phi - \theta < \pi/2$ we have

$$1 \ge \cos\phi \ge \cos 2G_- \ge 0$$

recalling that $G(\phi) = \phi/2$. On the other hand, if $0 < \varphi - \theta < -\pi/2$, then

$$0 \ge G(\varphi) \ge G_+ \ge -\pi/2$$

and in particular if $0 < \phi - \theta < -\pi/2$

$$1 \ge \cos\phi \ge \cos 2G_+ \ge 0.$$

In either case, we have

$$\cos\phi \ge \cos 2\left(\cos^{-1}[(K+1)^{-1/2}] - \tan^{-1}\left(\frac{\tan(\cos^{-1}[(K+1)^{-1/2}])}{K}\right)\right) \ge 0. \qquad (7.2)$$

We can simplify this expression by using standard trigonometric formula and the expressions

$$\cos(\tan^{-1} x) = (1 + x^2)^{-1/2}, \qquad (7.3)$$

$$\sin(\tan^{-1} x) = x(1 + x^2)^{-1/2}, \qquad (7.4)$$

$$\tan(\cos^{-1} x) = (1 - x^2)^{1/2}/x, \qquad (7.5)$$

$$\sin(\cos^{-1} x) = (1 - x^2)^{1/2}. \qquad (7.6)$$

First, using (7.5) and the addition formula for cos, the right hand side of (7.2) is

$$\cos\left[2\cos^{-1}[(K+1)^{-1/2}] - 2\tan^{-1}\left(\left(\frac{(1-\frac{1}{K+1})^{1/2}}{(K-1)^{-1/2}}\right)/K\right)\right]$$

$$= \cos\left[2\cos^{-1}[(K+1)^{-1/2}]) - 2\tan^{-1}(K^{-1/2})\right]$$

$$= \cos(2\cos^{-1}[(K+1)^{-1/2}])\cos(2\tan^{-1}(K^{-1/2})) + \sin(2\cos^{-1}[(K+1)^{-1/2}])\sin(2\tan^{-1}(K^{-1/2})).$$

Using the double angle formula and (7.3),(7.4) and (7.6), one can calculate that

$$\cos(2\cos^{-1}[(K+1)^{-1/2}]) = \frac{1-K}{1+K},$$
$$\cos(2\tan^{-1}(K^{-1/2})) = \frac{K-1}{K+1},$$
$$\sin(2\cos^{-1}[(K+1)^{-1/2}]) = \frac{2K^{1/2}}{K+1},$$
$$\sin(2\tan^{-1}(K^{-1/2})) = \frac{2K^{1/2}}{K+1}.$$

Therefore, the right hand side of (7.2) is equal to

$$-\frac{(K-1)^2}{(K+1)^2} + \frac{4K}{(K+1)^2} = \frac{-K^2+6K-1}{(K+1)^2}.$$

In conclusion, we have

$$\cos\phi \geq \frac{-K^2+6K-1}{(K+1)^2}. \tag{7.7}$$

From Lemma 7.7 and (7.7) we have that:

$$\begin{aligned}
\mathrm{Tr}^2 A &\geq \frac{(K+1)^2}{2K} + \frac{(K+1)^2(-K^2+6K-1)}{2K(K+1)^2} \\
&= \frac{K^2+2K+1-K^2+6K-1}{2K} \\
&= \frac{8K}{2K} \\
&= 4,
\end{aligned}$$

which completes the proof of Proposition 7.6 and hence of Theorem 7.1 also.

7.3 Nowhere Uniformly Quasiregular Mappings

The next result is a refinement of Theorem 7.2, that is we show our maps are *nowhere uniformly quasiregular*. This true because any neighbourhood of every point either intersects a fixed ray

or the basin of attraction.

7.3.1 Definitions

We aim to prove the following theorem.

Theorem 7.3. *Let $K > 1$ and $\theta \in (-\pi/2, \pi/2]$. Then the mapping $h_{K,\theta}(z)^2 + c$ is nowhere uniformly quasiregular.*

This theorem implies Theorem 7.2, however we use Theorem 7.2 to prove it so the original theorem is not superfluous. In the previous section we showed H was not uniformly quasiregular by considering points on a fixed ray. Here, we will use density of the pre-images of the fixed ray in the one fixed ray case, and the density of the basin of attraction in the remaining cases. Recall that due to Theorem 5.1 it suffices to show only the $c = 0$ case of Theorem 7.3. Let us first define what we mean by nowhere uniformly quasiregular. Recall the definition of distortion given in Definition 3.3.

Definition 7.8. We define the distortion of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $z \in \mathbb{C}$ as:

$$K_z(f) := \limsup_{\operatorname{diam}(U) \to 0} K(f|U),$$

where U is any neighbourhood of z and the $\limsup$ is taken as the diameters of these neighbourhoods tend to 0.

Definition 7.9. A function $f : \mathbb{C} \to \mathbb{C}$ is *nowhere uniformly quasiregular* if

$$K_z(f^n) \to \infty \text{ as } n \to \infty \text{ for all } z \in \mathbb{C}.$$

Note the difference between $K_z(f)$ and $K(f)(z)$ given in Definition 3.7.

7.4   Proof of Theorem 7.3

We will prove Theorem 7.3 using a couple of lemmas. Recall that if R_ϕ is the repelling fixed ray of H, we have the notation

$$\mathcal{P} = \{H^{-k}(R_\phi)\}_{k=0}^{\infty}.$$

Lemma 7.10. *If H has one fixed ray R_ϕ then H is nowhere uniformly quasiregular.*

Proof. Fix $z \in \mathbb{C}$. Theorem 6.2 tells us that $\mathcal{P}$ is dense. If z lies on a ray $R_\varphi \in \mathcal{P}$ then there must exist some m such that $H^m(R_\varphi) = R_\phi$. That is, $H^m(z)$ lies on the ray R_ϕ. We can apply Lemma 3.8, the formula for the complex dilatation of the composition of functions, to obtain:

$$\mu_{H^n \circ H^m}(z) = \frac{\mu_{H^m}(z) + r_{H^m}(z)\mu_{H^n}(H^m(z))}{1 + r_{H^m}(z)\overline{\mu_{H^m}(z)}\mu_{H^n}(H^m(z))}, \tag{7.8}$$

where $r_{H^m}(z) = \overline{(H^m)_{\bar{z}}(z)}/(H^m)_z(z)$. Notice that $|r_{H^m}(z)| = 1$ and that we can define

$$B(w) := r_{H^m}(z)\left(\frac{w + \mu_{H^m}(z)\overline{r_{H^m}(z)}}{1 + \overline{[r_{H^m}(z)\mu_{H^m}(z)]}w} \right). \tag{7.9}$$

We see that B is a Möbius map of the disk. Further we see that

$$B[\mu_{H^n}(H^m(z))] = \mu_{H^n \circ H^m}(z),$$

for $n \geq 1$. Using the fact that $H^{n+m}(z) \in R_\phi$ for $n \geq 0$, (7.9) and the fact that $\mu_{H^n} = A^{n-1}(\mu_H)$, we see that (7.8) becomes

$$\mu_{H^n \circ H^m}(z) = B(A^{n-1}(\mu_{H^m}(z))), \tag{7.10}$$

for $n \geq 1$. We know $|A^n(w)| \to 1$ as $n \to \infty$ for any $w \in \mathbb{D}$, $B(\partial\mathbb{D}) = \partial\mathbb{D}$ and so we have

$$|\mu_{H^\ell(z)}(z)| \to 1 \text{ as } \ell \to \infty.$$

Any neighbourhood $U \ni z$ trivially contains z and so $K_z(H^\ell)$ is unbounded as $\ell \to \infty$.

Next suppose z lies on a ray not in $\mathcal{P}$. As $\mathcal{P}$ is dense, any neighbourhood $U \ni z$ must intersect a ray $R_\varphi \in \mathcal{P}$. Picking one such ray there must exist m (depending on the neighbourhood U) such that $H^m(R_\varphi) = R_\phi$. We can now apply the same argument above to conclude $K_z(H^\ell)$ is unbounded as $\ell \to \infty$ for any $z \in \mathbb{C}$. $\qquad\square$

When we have more than one fixed ray we no longer have that any neighbourhood of a point contains the pre-image of a fixed ray; however we do know for $z \in \mathbb{C}$ that if we take n large enough then $H^n(z)$ will either end up on a fixed ray, or the argument of $H^n(z)$ tends to the argument of the non-repelling fixed ray. We take advantage of this to prove the remainder of Theorem 7.3, we will also require Lemma 2.23.

Lemma 7.11. *If H has more than one fixed ray then H is nowhere uniformly quasiregular.*

Proof. Fix $z \in \mathbb{C}$. From Theorem 6.2 we know that either z lies on the preimage of a fixed ray,

or $z \in \Lambda$. In the first case the result follows from the methods of the previous lemma. In the second case we know that the argument of $H^n(z)$ tends to the argument of the non-repelling fixed ray ϕ as $n \to \infty$.

We define the sequence of points $\phi_n \in S^1$ by $H^n(z) \in R_{\phi_n}$. Then $\phi_n \to \phi$ as $n \to \infty$, where ϕ is the non-repelling fixed point of $\widetilde{H}$. Again we use the formula for the complex dilatation of composition of functions, from Lemma 3.8, reformulated slightly differently than in (7.8) to see,

$$\mu_{H^n}(z) = \mu_{H^{n-1} \circ H}(z) = \frac{\mu_H(z) + r_H(z)\mu_{H^{n-1}}(H(z))}{1 + r_H(z)\overline{\mu_H(z)}\mu_{H^{n-1}}(H(z))}.$$

Recalling that μ_H is constant, we can write

$$\mu_{H^n}(z) = A_1(\mu_{H^{n-1}}(H(z)),$$

where A_1 is the Möbius map

$$A_1(w) = \frac{\mu_H + r_H(z)w}{1 + r_H(z)\overline{\mu_H}w}.$$

Using the same method, we may write

$$\mu_{H^{n-1}}(H(z)) = A_2(\mu_{H^{n-2}}(H^2(z)),$$

where A_2 is the Möbius map

$$A_2(w) = \frac{\mu_H + r_H(H(z))w}{1 + r_H(H(z))\overline{\mu_H}w}.$$

By induction, we may write

$$\mu_{H^n}(z) = A_1 \circ A_2 \circ \ldots \circ A_{n-1}(\mu_H(H^{n-1}(z))),$$

where each A_j is a Möbius map given by

$$A_j(w) = \frac{\mu_H + r_H(H^{j-1}(z))w}{1 + r_H(H^{j-1}(z))\overline{\mu_H}w}.$$

Now it is not hard to see that $H_z(z) = (K+1)h(z)$, and so

$$r_H(H^{j-1}(z)) = \exp(-2i \arg[h(H^{j-1}(z))]).$$

As $j \to \infty$, we have $\arg[h(H^{j-1}(z))] \to \arg[h(re^{i\phi})]$ for any $r > 0$. Since ϕ is a fixed ray of H, $\arg[h(re^{i\phi})] = \phi/2$. In particular, we have that the Möbius maps A_j converge to the Möbius map

$$A(w) = \frac{\mu_H + e^{-i\phi}w}{1 + e^{-i\phi}\overline{\mu_H}w}.$$

By Proposition 7.6, A is a hyperbolic Möbius map with fixed point $\alpha \in \partial\mathbb{D}$. Recalling that $\mu_H(z) = e^{2i\theta}(K-1)/(K+1)$ for all $z \in \mathbb{C}$, we can write

$$\mu_{H^n}(z) = A_1 \circ A_2 \circ \ldots \circ A_{n-1}(\mu_H) =: t_{n-1}(\mu_H). \tag{7.11}$$

Then by Theorem 2.23, $\mu_{H^n}(z) \to \alpha$ and in particular $|\mu_{H^n}(z)| \to 1$. This proves the lemma. $\qquad\square$

Note that we fixed z at the beginning of the proof and that different choices of z will give rise to different Möbius maps A_i. Together Lemmas 7.10 and 7.11 prove Theorem 7.3.

Chapter 8

Failure of quasiconformal equivalence on any neighbourhood of infinity

Let R_ϕ be a fixed ray of H. We recall from (7.1) that the complex dilatation of H^n at $z \in R_\phi$ is given by $\mu_{H^n}(z) = A^{n-1}(\mu_H)$ where A is the Möbius transformation

$$A(z) = \frac{\mu_H + e^{-i\phi}z}{1 + e^{-\phi}\overline{\mu_H}z}$$

and $\mu_H = e^{2i\theta}(K-1)/(K+1) \in \mathbb{D}$. Recall the trace of A satisfies

$$\text{Tr}^2 A = \frac{(K+1)^2(1 + \cos\phi)}{2K},$$

by Lemma 7.7.

8.1 Statement of results

Given $K_1, K_2 > 1$ and $\theta_1, \theta_2 \in (-\pi/2, \pi/2]$, let $H_1 := h^2_{K_1,\theta_1}$ and $H_2 := h^2_{K_2,\theta_2}$. Denote the fixed rays of H_1 by R_{ϕ_i} and the fixed rays of H_2 by R_{ψ_j}, the corresponding Möbius transformations of each fixed ray R_{ϕ_i} by $A_i(z)$ and the corresponding Möbius transformations of each fixed ray R_{ψ_j} by $B_j(z)$. We prove the following.

Theorem 8.1. *With the notation above, there is no quasiconformal conjugacy between H_1 and H_2 in any neighbourhood of infinity if any of the following conditions hold:*

(i) the mappings H_1, H_2 have different numbers of fixed rays;

(ii) H_1 and H_2 both have one fixed ray, R_{ϕ_1} and R_{ψ_1} respectively, and $\text{Tr}(A_1)^2 \neq \text{Tr}(B_1)^2$;

(iii) H_1 and H_2 both have two fixed rays R_{ϕ_i} and R_{ψ_i} for $i = 1, 2$, where $\phi_1 > \phi_2$ and $\psi_1 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i;

(iv) H_1 and H_2 both have three fixed rays R_{ϕ_i} and R_{ψ_j}, $i, j \in \{0, 1, 2\}$ respectively, where $\phi_1 > \phi_0 > \phi_2$ and $\psi_1 > \psi_0 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i.

Using this theorem we then rule out more equivalences.

Theorem 8.2. • *If $K > 1$ is fixed and $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ then H_{K,θ_1} and H_{K,θ_2} are not quasiconformally conjugate on any neighbourhood of infinity, except if $\theta_1 = \theta_2$ or possibly one case where H_{K,θ_1} and H_{K,θ_2} both have one fixed ray and*

$$\theta_1 = \phi - \tan^{-1}\left(\frac{K}{\tan(\phi - \theta_2)}\right),$$

where ϕ is the fixed point of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$.

• *If $\theta \in (-\pi/2, \pi/2)$ is fixed and $K_1 \neq K_2 > 1$ then $H_{K_1,\theta}$ and $H_{K_2,\theta}$ are not quasiconformally conjugate on any neighbourhood of infinity.*

8.1.1 Outline

The outline of our strategy is as follows.

- Each fixed ray R_ϕ of H has a corresponding hyperbolic Möbius automorphism of $\mathbb{D}$ which encodes how the complex dilatation of the iterates H^n behaves on R_ϕ.

- If there is a quasiconformal equivalence Ψ between H_1 and H_2 such that $\Psi \circ H_1 = H_2 \circ \Psi$ on a neighbourhood of infinity U, then $1/C \leq K_{H_1^n}(z)/K_{H_2^n}(\Psi(z)) \leq C$ for some constant $C > 0$, all $n \in \mathbb{N}$ and all $z \in U$.

- We show that in the various cases of different numbers of fixed rays, if there is a quasiconformal equivalence Ψ, then the image of a fixed ray of H_1 under Ψ will either be a fixed ray of H_2, intersect a fixed ray of H_2 or converge to a fixed ray of H_2.

- In each case, by comparing the behaviour of the corresponding Möbius maps for the respective fixed rays, we show that if the corresponding traces are different, then there can be no quasiconformal equivalence.

8.2 Consequences of a quasiconformal equivalence

Throughout this section, we will consider two maps H_1, H_2 associated with K_i, θ_i for $i = 1, 2$. Recall that two maps $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ are quasiconformally equivalent on a neighbourhood U of infinity if there exists a quasiconformal map $\Psi : U \to \Psi(U)$ such that

$$\Psi^{-1} \circ f_2 \circ \Psi(z) = f_1(z),$$

for all $z \in U$.

From Theorem 5.1 we know that $H(z)$ and $H(z) + c$ are quasiconformally equivalent on a neighbourhood of infinity. Therefore, if we are interested in knowing when $H_i + c_i$ can be quasiconformally equivalent for $i = 1, 2$, we can reduce to the situation where $c_i = 0$.

If H_1, H_2 are quasiconformally equivalent on a neighbourhood U of infinity, then

$$(H_2)^n(\Psi(z)) = \Psi((H_1)^n(z)), \tag{8.1}$$

for all $n \in \mathbb{N}$ and $z \in U$.

Lemma 8.3. *If H_1 and H_2 are quasiconformally equivalent on a neighbourhood U of infinity, then there exists $C > 0$ such that*

$$\frac{1}{C} \leq \frac{K_{H_1^n}(z)}{K_{H_2^n}(\Psi(z))} \leq C, \tag{8.2}$$

for all $n \in \mathbb{N}$ and $z \in U$.

Proof. This follows immediately from (8.1) and the fact that distortion is sub-multiplicative with respect to composition, see for example [20]. We may even take $C = (K_\Psi)^2$. $\quad\square$

Recall we use A_i to denote the Möbius map corresponding to the fixed ray R_{ϕ_i} of H_1 and B_i for the Möbius map corresponding to the fixed ray R_{ψ_j} of H_2.

Lemma 8.4. *Let H_1 and H_2 be quasiconformally equivalent on a neighbourhood U of infinity. Then if R_{ϕ_i} and R_{ψ_j}, are fixed rays of H_1 and H_2 respectively and*

$$\arg[H_2^n(\Psi(z))] \to \psi_j$$

for some $z \in R_{\phi_i}$, then

$$\mathrm{Tr}(A_i)^2 = \mathrm{Tr}(B_j)^2.$$

Note that this lemma takes care of the cases where $\Psi(R_{\phi_i})$ is either a fixed ray, intersects a fixed ray in one point (which means it must intersect in infinitely many) or is a curve which converges to a fixed ray of H_2. This lemma is our key tool in this section.

Proof. Suppose for a contradiction that $\arg[H_2^n(\Psi(z))] \to \psi_j$ for some $z \in R_{\phi_i}$, but $\text{Tr}(A_i)^2 \neq \text{Tr}(B_j)^2$. By Theorem 2.24

$$d_h(0, A_i^n(\mu_{H_1})) = \log\left[O(1/k^n)\right], \tag{8.3}$$

where

$$k = (\text{Tr}(A_i)^2 - 2 - (\text{Tr}(A_i)^4 - 4\,\text{Tr}(A_i)^2)^{\frac{1}{2}})/2.$$

Then as $\mu_{H^n} = A^{n-1}(\mu(H))$ and $d_h(0, w) = \exp[(1 + |w|)/(1 - |w|)]$, we see that

$$
\begin{aligned}
K_{H_1^n}(z) &= \frac{1 + |\mu_{H_1^n}(z)|}{1 - |\mu_{H_1^n}(z)|} \\
&= \exp\left(d_h(0, A_i^{n-1}(\mu_{H_1}))\right) \\
&= O(e^{\log[(1/k^{n-1})]}) \\
&= O(1/k^{n-1}).
\end{aligned}
$$

By hypothesis, $H_2^n(\Psi(z)) \in R_{\gamma_n}$ for some sequence of rays R_{γ_n} where $\gamma_n \to \psi_j$. As in Lemma 7.11, we may write

$$\mu_{H_2^n}(\Psi(z)) = B_1 \circ \ldots \circ B_{n-1}[\mu_{H_2}(H_2^{n-1}(\Psi(z)))],$$

where μ_{H_2} is a constant and each B_m is a Möbius map given by

$$B_m(w) = \frac{\mu_{H_2} + r_{H_2}(H_2^{m-1}(\Psi(z)))w}{1 + r_{H_2}(H_2^{m-1}(\Psi(z)))\overline{\mu_{H_2}}w},$$

and we have $r_{H_2}(H_2^{m-1}(\Psi(z))) \to e^{-i\psi_j}$. Hence $B_m \to B_j$ as $m \to \infty$. Let $t_n = B_1 \circ \ldots \circ B_{n-1}$. Then by Theorem 2.24

$$d_h(0, t_n(\mu_{H_2}(\Psi(z)))) = \log\left[\frac{1}{\prod_{j=1}^n \ell^j}\right] + O(1),$$

where $\ell_m \to \ell$ as $m \to \infty$. Here, ℓ is the quantity from Lemma 2.22 involving the trace squared of the Möbius map B_j corresponding to the fixed ray R_{ψ_j}. By our hypothesis, $k \neq \ell$. By

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Theorem 2.24, we have

$$K_{H_2^n}(\Psi(z)) = \frac{1 + |\mu_{H_2^n}(\Psi(z))|}{1 - |\mu_{H_2^n}(\Psi(z))|}$$

$$= \exp\left(d_h(0, t_{n-1}(\mu_{H_2}(\Psi(z))))\right)$$

$$= O\left(e^{\log\left[\left(\frac{1}{\prod_{j=1}^{n-1} \ell_j}\right)\right]}\right)$$

$$= O\left(\frac{1}{\prod_{j=1}^{n-1} \ell_j}\right). \tag{8.4}$$

As $\ell_j \to \ell \neq k$ then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $k < \ell$ then $\ell_j/k \geq \alpha > 1$ for all $j > N$ and if $k > \ell$ then $\ell_j/k \leq \beta < 1$ for all $j > N$. So first if $k > \ell$

$$\frac{1}{k^n} \bigg/ \frac{1}{\prod_{i=1}^{n} \ell_j} \leq \left(\frac{\prod_{i=1}^{N} \ell_j}{k^N}\right) \beta^{n-N} \to 0 \text{ as } n \to \infty$$

and if $k < \ell$ then,

$$\frac{1}{k^n} \bigg/ \frac{1}{\prod_{i=1}^{n} \ell_j} \geq \left(\frac{\prod_{i=1}^{N} \ell_j}{k^N}\right) \alpha^{n-N} \to \infty \text{ as } n \to \infty.$$

In either case, we contradict Lemma 8.3. $\qquad\square$

8.2.1 The one fixed ray case

We next show that if one of our mappings has one fixed ray, then a quasiconformal equivalence implies the other mapping must have one fixed ray.

Lemma 8.5. *Suppose H_2 has one fixed ray, and H_1 has more than one fixed ray. Then H_1 and H_2 are not quasiconformally equivalent.*

Proof. Suppose H_2 has one fixed ray R_ψ, H_1 has two or three fixed rays R_{ϕ_0}, R_{ϕ_1} and possibly R_{ϕ_2} and there is a quasiconformal equivalence Ψ between them. If $\Psi(R_{\phi_i})$ is a ray then (8.1) implies that it must be fixed by H_2, but as there is only one fixed ray R_ψ of H_2 this implies $\Psi(R_{\phi_i}) = R_\psi$ and so $\Psi(R_{\phi_j})$ cannot be a ray for $j \neq i$. Since $\Psi(R_{\phi_j})$ is not a ray then by Theorem 6.2 it must intersect $\{H_2^{-k}(R_\psi)\}_{k=0}^{\infty}$ and so by (8.1) it must intersect R_ψ contradicting Ψ being injective.

Therefore $\Psi(R_{\phi_i})$ is not a ray for any i and hence again by Theorem 6.2, there exists $z_i \in R_{\phi_i}$ such that $\Psi(z_i) \in R_\psi$. We can apply Lemma 8.4 to see that the corresponding traces

squared of the Möbius maps A_i of the fixed rays R_{ϕ_i} must equal the trace squared of the fixed ray R_ψ. Therefore $\mathrm{Tr}(A_i)^2 = \mathrm{Tr}(A_j)^2$ for each i, j. Recall from Lemma 7.7 that

$$\mathrm{Tr}(A_i)^2 = \frac{(K_1 + 1)^2(1 + \cos\phi_i)}{2K_1}.$$

As K_1 is fixed, this just depends on $\cos\phi$. To finish the proof, we need to show that $\cos\phi_i \neq \cos\phi_j$ for some pair of fixed rays of H_1, to give us a contradiction. Equivalently, we need to show that $\phi_i \neq -\phi_j$ for some pair of fixed rays of H_1. We will do this in Lemmas 8.6, 8.7 and 8.8 below. $\qquad\square$

We have already seen in Lemma 6.15 that when $\theta = 0$ the repelling fixed points satisfy $\phi_1 = -\phi_2$ and the attracting fixed point is always $\phi_0 = 0$. We now consider the remaining cases when $\theta \neq 0$. The following result tells us that if a point of S^1 is moved a given amount by the map $\widetilde{h}$ induced by h, then there are only two possibilities.

Lemma 8.6. *Define G by $G(\varphi) := \varphi - \widetilde{h}(\varphi)$ for $\varphi - \theta \in (0, \pi/2)$. Then $G(\varphi_1) = G(\varphi_2)$ implies that either*

$$\varphi_1 = \varphi_2 \text{ or } \varphi_1 = \tan^{-1}(K/\tan(\varphi_2 - \theta)) + \theta.$$

Proof. Suppose $G(\varphi_1) = G(\varphi_2)$ then this implies

$$\varphi_1 - \tan^{-1}\left(\frac{\tan(\varphi_1 - \theta)}{K}\right) + \theta = \varphi_2 - \tan^{-1}\left(\frac{\tan(\varphi_2 - \theta)}{K}\right) + \theta,$$

which we rearrange and use the addition formula for $\tan^{-1}$ to yield,

$$\varphi_1 - \varphi_2 = \tan^{-1}\left(\frac{K(\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta))}{K^2 + \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta)}\right). \tag{8.5}$$

We can add and subtract θ to the left hand side then apply tan to both sides and use the addition formula for tan and the fact tan is an odd function to see that (8.5) is equal to

$$\frac{\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta)}{1 + \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta)} = \frac{K(\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta))}{K^2 + \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta)}. \tag{8.6}$$

We can then rearrange (8.6) to get

$$K^2[\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta)] + \tan^2(\varphi_1 - \theta)\tan(\varphi_2 - \theta) - \tan(\varphi_1 - \theta)\tan^2(\varphi_2 - \theta) \tag{8.7}$$

$$= K[\tan^2(\varphi_1 - \theta)\tan(\varphi_2 - \theta) - \tan(\varphi_1 - \theta)\tan^2(\varphi_2 - \theta) + \tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta)].$$

Rearranging and factorising (8.7) we obtain

$$(\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta))[K^2 - K + \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta)(1 - K)] = 0. \qquad (8.8)$$

This shows that $\tan(\varphi_1 - \theta) = \tan(\varphi_2 - \theta)$ is a solution, which for our range of possible values implies $\varphi_1 = \varphi_2$. Dividing by $\tan(\varphi_1 - \theta) - \tan(\varphi_2 - \theta)$ and rearranging (8.8) we see

$$\frac{K^2 - K}{K - 1} = \tan(\varphi_1 - \theta)\tan(\varphi_2 - \theta). \qquad (8.9)$$

We know $K \neq 1$ hence we can cancel $K - 1$ on the left hand side of (8.9) and rearrange to obtain

$$\varphi_1 = \tan^{-1}(K/\tan(\varphi_2 - \theta)) + \theta. \qquad (8.10)$$

$\square$

We show that if H has two fixed rays, then they cannot be symmetric about the real axis.

Lemma 8.7. *Let $\theta \neq 0$ and $K > 1$. If the corresponding map H has two fixed rays R_{ϕ_1} and R_{ϕ_2} then*

$$\phi_1 \neq -\phi_2.$$

Proof. Assume to the contrary that $\phi_1 > 0$ and $\phi_2 = -\phi_1$. As ϕ_1 is a neutral fixed point from Lemma 6.19, we have that $\widetilde{H}'(\phi_1) = 1$. From (6.2) this implies

$$\phi_1 = \cos^{-1}\left[\left(\frac{2K - 1}{K^2 - 1}\right)^{\frac{1}{2}}\right] + \theta. \qquad (8.11)$$

Further we know that the ϕ_i are fixed under $\widetilde{H}$ and also that they are moved the same magnitude under $\widetilde{h}$. These imply

$$\widetilde{h}(\phi_1) = -\widetilde{h}(\phi_2) \qquad (8.12)$$

and

$$G(\phi_1) = -G(\phi_2). \qquad (8.13)$$

By reflecting in the θ axis we see that for $\varphi - \theta \in (-\pi/2, 0)$ we have

$$G(\varphi) = -G(-\varphi + \theta).$$

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Hence (8.13) implies

$$G(\phi_1) = G(\phi_1 + \theta) \tag{8.14}$$

We can apply Lemma 8.6 with $\varphi_1 = \phi_1 + \theta$ and $\varphi_2 = \phi_1$ to see

$$\phi_1 + \theta = \tan^{-1}\left(K/\tan(\phi_1 - \theta)\right) + \theta. \tag{8.15}$$

Substituting (8.11) into (8.15), writing $X = \left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}$ and rearranging we see

$$\tan[\cos^{-1} X]\tan[\cos^{-1}(X+\theta)] = K. \tag{8.16}$$

Next apply the addition formula to $\tan[\cos^{-1}(X+\theta)]$ to see (8.16) becomes

$$\tan[\cos^{-1} X]\frac{\tan[\cos^{-1} X] + \tan\theta}{1 - \tan[\cos^{-1} X]\tan\theta} = K. \tag{8.17}$$

Let $Y = \tan[\cos^{-1} X]$ and rearrange (8.17) to obtain

$$\tan\theta = \frac{K - Y^2}{Y(K+1)}. \tag{8.18}$$

Next we substitute (8.11) into (8.12), and again write $X = \left(\frac{2K-1}{K^2-1}\right)^{\frac{1}{2}}$ to see

$$\tan^{-1}\left(\frac{\tan[\cos^{-1} X]}{K}\right) + \theta = -\tan^{-1}\left(\frac{\tan[-\cos^{-1} X - 2\theta]}{K}\right) - \theta. \tag{8.19}$$

Rearranging (8.19) and using the fact $\tan$ and $\tan^{-1}$ are odd functions, we obtain

$$2\theta = \tan^{-1}\left(\frac{\tan[\cos^{-1} X + 2\theta]}{K}\right) - \tan^{-1}\left(\frac{\tan[\cos^{-1} X]}{K}\right). \tag{8.20}$$

Next we apply the addition formula for $\tan^{-1}$ to (8.20) and then apply $\tan$ to both sides to obtain

$$\tan 2\theta = \frac{K(\tan[\cos^{-1} X + 2\theta] - \tan[\cos^{-1} X])}{K^2 + \tan[\cos^{-1} X + 2\theta]\tan[\cos^{-1} X]}. \tag{8.21}$$

Rearranging (8.21), applying the addition formula to $\tan[\cos^{-1} X + 2\theta]$ and then writing $Y = \tan[\cos^{-1} X]$ we see

$$\tan 2\theta \left(K^2 + Y\frac{Y + \tan 2\theta}{1 - Y\tan 2\theta}\right) = K\left(\frac{Y + \tan 2\theta}{1 - Y\tan 2\theta} - Y\right). \tag{8.22}$$

Cancelling the denominators $1 - Y \tan 2\theta$ in (8.21) we see

$$\tan 2\theta (K^2(1 - Y \tan 2\theta) + Y(Y + \tan 2\theta)) = K(Y + \tan 2\theta - Y(1 - Y \tan 2\theta)). \qquad (8.23)$$

Expanding and cancelling (8.23) then rearranging we obtain

$$\tan 2\theta = \frac{Y^2 + K^2 - Y^2 K - K}{Y(K^2 - 1)} = \frac{K - Y^2}{Y(K + 1)}. \qquad (8.24)$$

Together (8.18) and (8.24) imply $\tan \theta = \tan 2\theta$. Letting $\tan \theta = T = (K - Y^2)/(Y(K+1))$ and using the double angle formula we must have

$$T = 2T/(1 - T^2). \qquad (8.25)$$

This only has solutions $T = 0, i$ and $-i$. As K and so Y are real this implies T is real also, hence the only possible solution left is $T = 0$. Substituting (8.18) into (8.25) for $T = 0$ we see we must have

$$\frac{K - Y^2}{Y(K + 1)} = 0, \qquad (8.26)$$

which implies $K = Y^2$. We can express Y^2 in terms of K as follows.

$$Y^2 = \tan^2[\cos^{-1} X] = (1 - X^2)/X^2 = X^{-2} - 1 = K(K - 2)/(2K - 1). \qquad (8.27)$$

Substituting (8.27) into (8.26) and rearranging we see

$$K(K + 1) = 0,$$

which implies $K = 0$ or $K = -1$. However $K = 0$ and $K = -1$ are not valid values of K; hence (8.11),(8.12) and (8.13) are never satisfied simultaneously, contradicting $\phi_1 = -\phi_2$. $\qquad \square$

We have to deal with the case where H has three fixed rays. It is clear that it is not possible for $\cos \phi_i$ to be the same for all three fixed rays, but we find a condition under which they are all different.

Lemma 8.8. *Let $\theta \neq 0$ and $K > 1$. If H has three fixed rays R_{ϕ_i} satisfying $\phi_2 < \phi_0 < \phi_1$, as in Lemma 6.21, then*

$$\phi_1 \neq -\phi_0.$$

Further if

$$\theta \geq \pi/6 \text{ then } \phi_i \neq -\phi_j \text{ for all } i \neq j.$$

However if $\theta < \pi/6$ then there exists some K such that

$$\phi_i = -\phi_2.$$

for $i = 0$ or $i = 1$.

Proof. As $\phi_1, \phi_0 > 0$ we must have $\phi_1 \neq -\phi_0$. Suppose $\phi_2 = -\phi_0$. Recall from Lemma 6.7 that for this to be possible

$$\widetilde{\mathbb{F}}_\theta^+ \cap -\widetilde{\mathbb{F}}_\theta^- \neq \emptyset.$$

This implies that ϕ_1 must satisfy the two inequalities

$$2\theta < \phi_1 < \pi/2 + \theta \text{ and } 0 < \phi_1 < \pi/2 - \theta,$$

which implies

$$0 < 2\theta < \pi/2 - \theta \Rightarrow 0 < \theta < \pi/6.$$

If $\theta < \pi/6$ then, by Lemma 8.7, when $K = K_\theta$ (recall Lemma 6.19) we know $\phi_i \neq \phi_2$, for $i = 0, 1$. For $K > K_\theta$, the neutral fixed point splits into two fixed points ϕ_0 and ϕ_1. Further $\phi_1 \to \pi/2 + \theta$ and $\phi_0 \to 2\theta$ as $K \to \infty$; also $\phi_2 \to -\pi/2 + \theta$ as $K \to \infty$. Hence by continuity there must exist some $K > K_\theta$ such that $\phi_2 = -\phi_i$ for $i = 0$ or $i = 1$. $\square$

The previous lemmas show that if H_2 has one fixed ray, then if H_1 has two or three fixed rays, H_1 and H_2 cannot be quasiconformally equivalent on a neighbourhood of infinity.

8.2.2 The two fixed ray case

We move on to the case where both H_1 and H_2 have more than one fixed ray. To start, we will show that if there is a quasiconformal equivalence between H_1 and H_2, it must map the immediate basin of attraction of the non-repelling fixed ray of H_1 into the immediate basin of attraction of the non-repelling fixed ray of H_2. Recall that the immediate basins of attraction are sectors of $\mathbb{C}$ bounded by two of the fixed rays of H_i.

Lemma 8.9. *If H_1 and H_2 have immediate basins of attraction Λ_1^* and Λ_2^* respectively for the non-repelling fixed rays, and are quasiconformally equivalent in a neighbourhood U of infinity via*

the map Ψ, *then* $\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap \Psi(U)$.

Proof. Since $\overline{\Lambda_1^*}$ is fixed by H_1, we have

$$H_1(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_1^*} \cap H_1(U).$$

Since Ψ is injective,

$$\Psi(H_1(\overline{\Lambda_1^*} \cap U)) = \Psi(\overline{\Lambda_1^*}) \cap \Psi(H_1(U)).$$

Using the quasiconformal equivalence,

$$H_2(\Psi(\overline{\Lambda_1^*} \cap U)) = \Psi(\overline{\Lambda_1^*}) \cap H_2(\Psi(U)),$$

but we also have

$$H_2(\Psi(\overline{\Lambda_1^*} \cap U)) \subset H_2(\Psi(\overline{\Lambda_1^*})) \cap H_2(\Psi(U)).$$

Therefore, in a neighbourhood U' of infinity, we have

$$\Psi(\overline{\Lambda_1^*}) \cap U' \subset H_2(\Psi(\overline{\Lambda_1^*})) \cap U'.$$

This argument also shows that in a neighbourhood U'_n of infinity, we have

$$\Psi(\overline{\Lambda_1^*}) \cap U'_n \subset H_2^n(\Psi(\overline{\Lambda_1^*})) \cap U'_n \tag{8.28}$$

for any $n \in \mathbb{N}$.

Now $\Psi(\overline{\Lambda_1^*})$ cannot spiral, as in that case, it would intersect all fixed rays of H_2. So we can apply Lemma 8.4 to see the corresponding traces squared of the Möbius maps A_i of the fixed ray R_{ϕ_i} and B_j of the fixed rays R_{ψ_j} must be equal. However Lemmas 8.7 and 8.8 show that this cannot be the case. Therefore $\Psi(\overline{\Lambda_1^*})$ must be contained in some sector. By (8.28) $\Psi(\Lambda_1^*)$ must be contained in each iterate of itself under H_2 and Theorem 6.2 tells us the pre-images of Λ_2^* under H_2 are dense; hence we must have that

$$\Psi(\overline{\Lambda_1^*} \cap U) \subset \overline{\Lambda_2^*} \cap \Psi(U)$$

for some neighbourhood U of infinity. The reverse argument shows that

$$\Psi^{-1}(\overline{\Lambda_2^*} \cap \Psi(U)) \subset \overline{\Lambda_1^*} \cap U,$$

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and the lemma is proved. □

We will next show that if H_2 has two fixed rays and H_1 has three fixed rays, then there cannot be a quasiconformal equivalence between them.

Lemma 8.10. *Let H_1, H_2 have three and two fixed rays respectively. Then there cannot be a quasiconformal equivalence between them in any neighbourhood of infinity.*

Proof. Suppose that H_2 has fixed rays R_{ψ_1}, R_{ψ_2} with $\psi_1 > \psi_2$ and H_1 has fixed rays R_{ϕ_i}, $i = 1, 2, 3$ with $\phi_2 < 0 < \phi_0 < \phi_1$. Suppose for a contradiction that there is a quasiconformal equivalence Ψ between them. Then by Lemma 8.9, we have $\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap \Psi(U)$, where $\overline{\Lambda_1^*}$ is the closed sector bounded by the rays R_{ϕ_2} and R_{ϕ_1}, and $\overline{\Lambda_2^*}$ is the closed sector bounded by the rays R_{ψ_1} and R_{ψ_2}.

We can lift these sectors to the strip

$$\mathbb{S} := \{z \in \mathbb{C} \mid -\pi/2 \leq \operatorname{Im}(z) \leq \pi/2\}$$

via the quasiconformal maps $F_i : \overline{\Lambda_i^*} \to \mathbb{S}$ given by

$$F_1(re^{is}) = \begin{cases} \log r + i\left(\frac{\pi(s-\phi_0)}{2(\phi_1-\phi_0)}\right) & \text{if } \phi_0 \leq s \leq \phi_1 \text{ or;} \\ \log r - i\left(\frac{\pi(s-\phi_0)}{2(\phi_2-\phi_0)}\right) & \text{if } \phi_2 \leq s < \phi_0. \end{cases},$$

and

$$F_2(re^{is}) = \log r + i\left(\frac{\pi(2s - (\psi_1 + \psi_2))}{2(\psi_1 - \psi_2)}\right).$$

Then $\Psi : \overline{\Lambda_1^*} \cap U \to \overline{\Lambda_2^*} \cap \Psi(U)$ lifts to a quasiconformal map $P : \Omega_1 \to \Omega_2$, where $\Omega_i = F_i(\overline{\Lambda_i^*}) \subset \mathbb{S}$ for $i = 1, 2$ and satisfies $P \circ F_1 = F_2 \circ \Psi$. Note that Ω_i is a connected subset of $\mathbb{S}$ whose boundary consists of two semi-infinite lines contained in the boundary of $\mathbb{S}$, and a curve γ_i in $\mathbb{S}$ connecting them. See Figure 8.1.

We want to extend P to a quasiconformal map from $\mathbb{S}$ to itself. There are many ways to do this, and we outline one here. Let T_i be the triangle $\mathbb{S} \setminus \Omega_i$ with vertices at the endpoints of γ_i and at $-\infty$. Define $q : \partial T_1 \to \partial T_2$ by translation on the respective horizontal semi-infinite lines, and agreeing with P on γ_1.

Let $g_i : T_i \to \mathbb{D}$ be conformal maps of the triangles onto the disk, sending the respective vertices to $-1, i, 1$ respectively. Then $\widetilde{q} = g_2 \circ q \circ g_1^{-1}$ from S^1 to itself is a quasisymmetric map by construction. Extend to a quasiconformal map $\widetilde{q} : \mathbb{D} \to \mathbb{D}$ via, for example, the Douady-Earle

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extension (see for example [20]). Then we may extend P on the strip $\mathbb{S}$ by setting $P = g_2^{-1} \circ \widetilde{q} \circ g_1$ on T_1. This extension of P is a quasiconformal map by construction.

Now, consider the attracting fixed ray R_{ϕ_0} of H_1 which is contained in the interior of the region $\Lambda_1^* \cap U$. The image of R_{ϕ_0} under Ψ must be contained in $\overline{\Lambda_2^*}$ by Lemma 8.9. Then by (8.1)

$$\arg[H_2^n(\Psi(z))] \to \psi_1 \tag{8.29}$$

as $n \to \infty$, for $z \in R_{\phi_0}$, since all points in Λ_2^* converge to the neutral fixed ray R_{ψ_1} of H_2. In particular, by lifting to the strip, $F_1(R_{\phi_0})$ is contained in the real line, but $P(F_1(R_{\phi_0})) = F_2(\Psi(R_{\phi_0}))$ is a curve which converges to the upper boundary component $\{\operatorname{Im} z = \pi/2\}$ of $\mathbb{S}$.

This contradicts the lemma below applied to P, completing the proof. $\qquad\square$

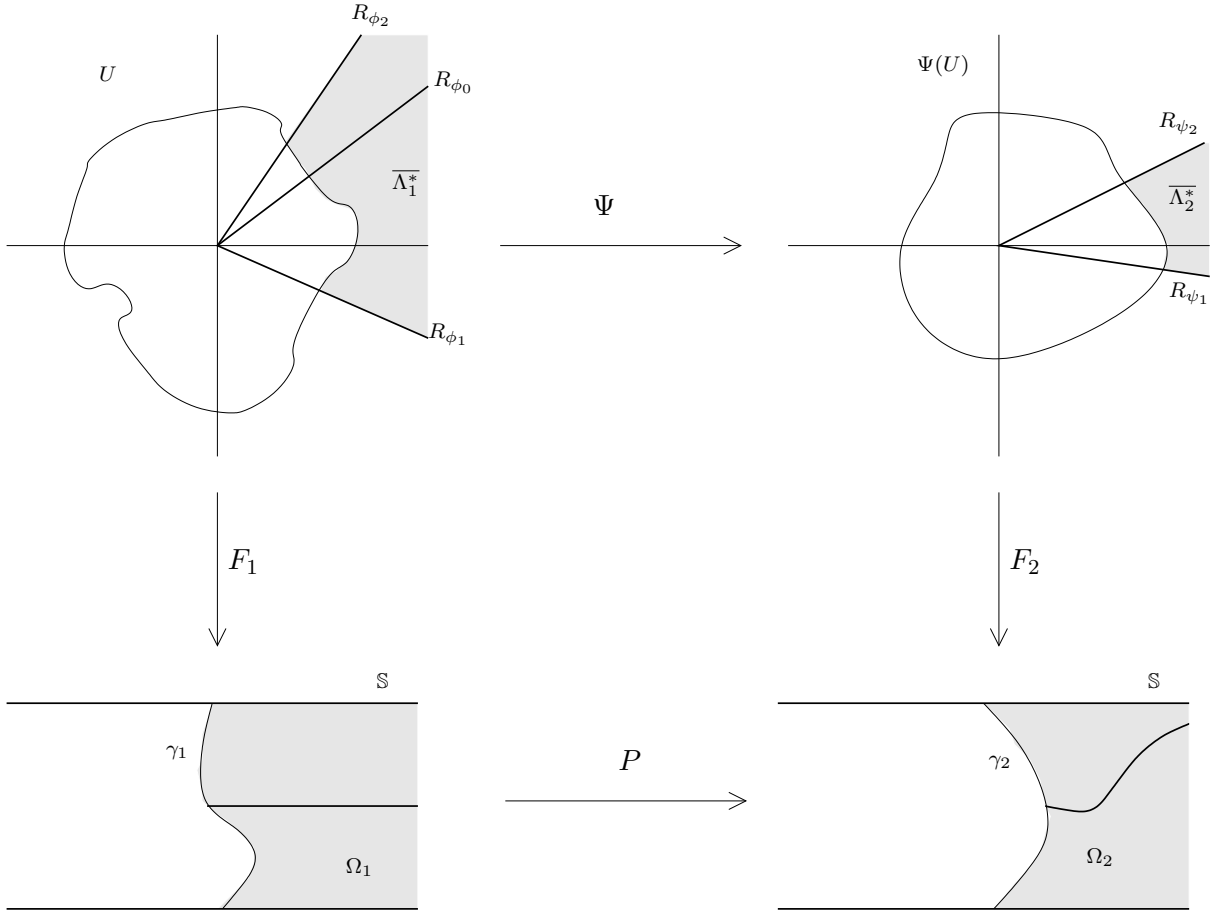Figure 8.1: Diagram showing how P is induced from the action of Ψ on the sector $\overline{\Lambda_1^*}$.

Lemma 8.11. *Let $f : \mathbb{S} \to \mathbb{S}$ be a K-quasiconformal map which sends $\pm\infty$ to $\pm\infty$ respectively. Then there exists $\delta < \pi/2$ such that $f(\mathbb{R})$ is contained in the sub-strip $\{z : |\operatorname{Im} z| < \delta\}$ of $\mathbb{S}$.*

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Proof. This is a strip version of a well-known result in the disk, and using the fact that $\mathbb{R}$ is a geodesic in $\mathbb{S}$. More specifically, by Theorem 4.3.2 of [20], if $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal, there exists some a, depending on K, such that f is a (K, a)-quasi-isometry. Then by Lemma 4.3.1 of [20], given a geodesic $\gamma \subset \mathbb{D}$, there exists $C > 0$ depending on K such that $f(\gamma)$ is contained in a C-neighbourhood of some geodesic γ'. Lifting to the strip, $\gamma = \mathbb{R}$ and the corresponding γ' is also $\mathbb{R}$. This proves the lemma. $\qquad\square$

8.3   Proof of Theorem 8.1

By Lemmas 8.5 and 8.10, we know that if H_1 and H_2 are quasiconformally equivalent in a neighbourhood of infinity, then they must have the same number of fixed rays. In the next two lemmas, we show that under a quasiconformal equivalence, the image of a fixed ray of H_1 must either intersect or approach a fixed ray of H_2.

Lemma 8.12. *Suppose H_1 and H_2 are quasiconformally equivalent on a neighbourhood U of infinity and both have one fixed ray R_ϕ and R_ψ respectively. Then there exists $z \in R_\phi \cap U$ such that $\Psi(z) \in R_\psi$.*

Proof. If $\Psi(R_\phi)$ is a ray then the result follows from (8.1), using the same argument as in the proof of Lemma 8.5. Suppose $\Psi(R_\phi)$ is not a ray, then it must intersect a sector Δ. By Theorem 6.2

$$\Delta \cap \{H_2^{-k}(R_\psi)\} \neq \emptyset,$$

hence there exists some ray $R \subset \Delta$ and $n \in \mathbb{N}$ such that $(H_2)^n(R) = R_\psi$. We can then choose $w \in R_\phi$ such that $\Psi(w) \in R$. From (8.1) we know

$$(H_2)^n(\Psi(w)) = \Psi((H_1)^n(w)).$$

Choosing $z = (H_1)^n(w)$ completes the proof. $\qquad\square$

Lemma 8.13. *Suppose H_1 and H_2 are quasiconformally equivalent on a neighbourhood of infinity and have three fixed rays. Let R_{ϕ_0} and R_{ψ_0} be the attracting fixed rays of H_1 and H_2 respectively. Then for $z \in R_{\phi_0}$*

$$\arg[H_2^n(\Psi(z))] \to \psi_0 \ as \ n \to \infty.$$

The remaining fixed rays R_{ϕ_i} for $i = 1, 2$ of H_1 and R_{ψ_i} for $i = 1, 2$ of H_2 such that $\phi_2 < \phi_0 < \phi_1$ and $\psi_2 < \psi_0 < \psi_1$ must satisfy $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i = 1, 2$.

Proof. We know from Lemma 8.9 that

$$\Psi(\overline{\Lambda_1^*} \cap U) = \overline{\Lambda_2^*} \cap U.$$

By Lemma 6.31 we know $\widetilde{\Lambda_1^*} = (\phi_1, \phi_2)$ and $\widetilde{\Lambda_2^*} = (\psi_1, \psi_2)$, proving the final part of the lemma.

If $\Psi(R_{\phi_0})$ is a ray then $\Psi(R_{\phi_0}) = R_{\psi_0}$ from (8.1), by using the same argument as in Lemma 8.5 and the fact that we already know $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i, j = 1, 2$. Assume $\Psi(R_{\phi_0})$ is not a ray, then by Theorem 6.2

$$\Psi(R_{\phi_0}) \cap \Lambda_2 \neq \emptyset.$$

Choosing $w \in \Psi(R_{\phi_0}) \cap \Lambda_2$ implies $\arg[H_2^n(w)] \to \psi_0$; choosing $z = \Psi^{-1}(w)$ proves the lemma. $\qquad\square$

Now finally we piece everything together to prove

Theorem 8.1. *With the previous notation, there is no quasiconformal conjugacy between H_1 and H_2 in any neighbourhood of infinity if any of the following conditions hold:*

(i) *the mappings H_1, H_2 have different numbers of fixed rays;*

(ii) *H_1 and H_2 both have one fixed ray, R_{ϕ_1} and R_{ψ_1} respectively, and $\mathrm{Tr}(A_1)^2 \neq \mathrm{Tr}(B_1)^2$;*

(iii) *if H_1 and H_2 both have two fixed rays R_{ϕ_i} and R_{ψ_i} for $i = 1, 2$, where $\phi_1 > \phi_2$ and $\psi_1 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i;*

(iv) *if H_1 and H_2 both have three fixed rays R_{ϕ_i} and R_{ψ_j}, $i, j \in \{0, 1, 2\}$ respectively, where $\phi_1 > \phi_0 > \phi_2$ and $\psi_1 > \psi_0 > \psi_2$, and $\mathrm{Tr}(A_i)^2 \neq \mathrm{Tr}(B_i)^2$ for some i.*

Proof. Suppose that there is a quasiconformal equivalence Ψ between H_1 and H_2 on some neighbourhood U of infinity. First notice that any neighbourhood of infinity intersects every ray. In particular, it intersects fixed rays of H_1 and H_2. By Lemma 8.5 and 8.10, H_1 and H_2 must have the same number of fixed rays. Each fixed ray ϕ_i and ψ_j of H_1 and H_2 respectively has a corresponding Möbius map A_i, B_j respectively.

Suppose H_1 and H_2 have one fixed ray. Then Lemma 8.12 tells us that we contradict Lemma 8.4 unless $\mathrm{Tr}(A_1)^2 = \mathrm{Tr}(B_1)^2$.

Suppose H_1 and H_2 have two fixed rays R_{ϕ_i} and R_{ψ_j} respectively, where $\phi_2 < \phi_1$ and $\psi_2 < \psi_1$. Lemma 8.9 implies $\Psi(R_{\phi_i}) = R_{\psi_i}$ for $i = 1, 2$, and so we contradict Lemma 8.4 unless $\mathrm{Tr}(A_i)^2 = \mathrm{Tr}(B_i)^2$ for both $i = 1, 2$.

Finally suppose H_1 and H_2 have three fixed rays. Again, by Lemma 8.13 we contradict Lemma 8.4 unless $\text{Tr}(A_i) = \text{Tr}(B_i)$ for $i = 1, 2, 3$. $\qquad \square$

8.3.1 Proof of Theorem 8.2

We can do better than this and rule out more cases, as stated earlier and restated and proved here.

Theorem 8.2.
- *If $K > 1$ is fixed and $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ then H_{K,θ_1} and H_{K,θ_2} are not quasiconformally conjugate on any neighbourhood of infinity, except if $\theta_1 = \theta_2$ or possibly one case where H_{K,θ_1} and H_{K,θ_2} both have one fixed ray and*

$$\theta_1 = \phi - \tan^{-1}\left(\frac{K}{\tan(\phi - \theta_2)}\right),$$

where ϕ is the fixed point of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$.

- *If $\theta \in (-\pi/2, \pi/2)$ is fixed and $K_1 \neq K_2 > 1$ then $H_{K_1,\theta}$ and $H_{K_2,\theta}$ are not quasiconformally conjugate on any neighbourhood of infinity.*

Proof. First recall from Lemma 4.2 that we have the identity $H_{K,-\theta}(z) = \overline{H_{K,\theta}(\bar{z})}$. So if we assume the theorem to be true for $H_{K,\theta}$ where $\theta \in [0, \pi/2)$, then we know that $H_{K,-\theta}$ is quasiconformally conjugate to $H_{K,\theta}$ with the opposite orientation. As we have assumed the theorem holds for $\theta \in [0, \pi/2)$ this implies that it now holds for any $\theta \in (-\pi/2, 0]$ also as in this range we have the opposite orientation. For the rest of this proof we assume $\theta \in [0, \pi/2)$.

Suppose $K > 1$ is fixed. If H_{K,θ_1} is quasiconformally conjugate to H_{K,θ_2} then by Theorem 8.1 they must have the same number of fixed rays and the corresponding traces, $\text{Tr}(A_i)$ and $\text{Tr}(B_i)^2$ for the fixed points ϕ_i and ψ_i of $\widetilde{H}_{K,\theta_1}$ and $\widetilde{H}_{K,\theta_2}$ respectively, squared must be equal. By Lemma 7.7 this implies

$$\frac{(K+1)^2}{2K}(1 + \cos\phi_i) = \frac{(K+1)^2}{2K}(1 + \cos\psi_i),$$

which in turn implies $\cos\phi_i = \cos\psi_i$. As $\theta \geq 0$ we know that if $\phi_i \geq 0$ then $\psi_i \geq 0$ also. Similarly if $\phi_i < 0$ then $\psi_i < 0$. Hence $\phi_i = \psi_i$, for each i. By Lemma 6.24 we know that this occurs only if $\theta_1 = \theta_2$ or

$$\theta_1 = \phi_i - \tan^{-1}\left(\frac{K}{\tan(\phi_i - \theta_2)}\right).$$

Further, by Proposition 6.28 we know that if we have three fixed points then the attracting fixed

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point ϕ_0 continues to increases until there are two fixed points, so $\phi_0 = \psi_0$ when they exist implies $\theta_1 = \theta_2$.

Now suppose $\theta \in (0, \pi/2)$ is fixed. Then by Proposition 6.22 the fixed point $\phi_K \in \widetilde{\mathbb{F}}_\theta^-$ decreases as K increases. Suppose that $H_{K_1,\theta}$ is quasiconformally conjugate to $H_{K_2,\theta}$. Then by Theorem 8.1 they must have the same number of fixed rays and the corresponding traces $\text{Tr}^2(A_i) = \text{Tr}^2(B_i)$, for the fixed points $\phi_{K_1}, \phi_{K_2} \in \widetilde{\mathbb{F}}_\theta^-$ of $\widetilde{H}_{K_1,\theta}$ and $\widetilde{H}_{K_2,\theta}$ respectively, must be equal. By Lemma 7.7 this implies

$$\frac{(K_1+1)^2}{2K_1}(1 + \cos\phi_{K_1}) = \frac{(K_2+1)^2}{2K_2}(1 + \cos\phi_{K_2}).$$

Rearranging we obtain

$$\frac{K_2(K_1+1)^2}{K_1(K_2+1)^2} = \frac{1 + \cos\phi_{K_1}}{1 + \cos\phi_{K_2}}. \tag{8.30}$$

Suppose $K_1 < K_2$. Then, as we know $\phi_{K_1} > \phi_{K_2}$, this implies the right hand side of (8.30) is greater than 1; however

$$K_2(K_1+1)^2 - K_1(K_2+1)^2 = (K_2 - K_1)(1 - K_1 K_2) < 0.$$

Hence the left hand side of (8.30) is less than 1, a contradiction.

If $\theta = 0$ then $\phi_0 = 0$ is always a fixed point of $\widetilde{H}_{K_1,0}$ and $\widetilde{H}_{K_2,0}$ for any $K_1, K_2 > 1$. Hence the right hand side of (8.30) is always equal to one, but the left hand side is only equal to one if $K_1 = K_2$. $\qquad\qquad\square$

Remark 8.14. *We have that H_{K,θ_1} and H_{K,θ_2} are not quasiconformally equivalent on a neighbourhood of infinity for $\theta_1 \neq \theta_2$ when both maps have three fixed rays. However we cannot rule out a quasiconformal equivalence when both have one fixed ray using our methods. If $\theta_0 < \theta_K$ then we have no quasiconformal equivalence for all θ_i. Unfortunately there is no obvious way to ascertain when this occurs. If $K \leq 2$ we know that we only have one fixed ray for all θ. So for all $\theta_i \neq \theta_0$ there will exist $\theta_j \neq \theta_i$ such that $\phi_{\theta_i} = \phi_{\theta_j}$; hence we cannot rule out a quasiconformal equivalence here using our methods.*

8.4   Concluding remarks

I conjecture that if $(K_1, \theta_1) \neq (K_2, \theta_2)$ for $K_i > 1$ and $\theta_i \in (-\pi/2, \pi/2]$, then H_{K_1,θ_1} is not quasiconformally conjugate to H_{K_2,θ_2} on any neighbourhood of infinity. However the methods

we have used will not show this when both maps have one fixed ray and their corresponding traces are equal. This is because we just compare the magnitudes of the complex dilatation of H_{K_i,θ_i} on the fixed rays, but have no grasp on their direction. If it were possible to obtain an expression for the direction given (K_i, θ_i) then we could possibly progress. However I have not achieved this yet. If it were possible to prove the conjecture it would complete this study nicely.

Do me a favour, and break my nose.

Do me a favour, and tell me to go away.

Oh do me a favour, and stop asking questions.

— Alex Turner, 2007

Bibliography

[1] J.W.Anderson, *Hyperbolic geometry*, Springer, 2005.

[2] K.Astala, T.Iwaniec, G.Martin, *Elliptic partial differential equations and quasi-conformal mappings in the plane.* Princeton Math., 2009.

[3] A.Beurling, L.V.Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, **96**, 125-142, 1956.

[4] A.F.Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, **132**, Springer-Verlag, New York, 1991.

[5] W.Bergweiler Iteration of quasiregular mappings, *Comput. Methods Funct. Theory*, **10**, 455-481, 2010.

[6] W.Bergweiler, A.Eremenko, Dynamics of a higher dimensional analog of the trigonometric functions, *Ann. Acad. Sci. Fenn. Math.*, **36**, 165-175, 2011.

[7] W.Bergweiler, A.Fletcher, J.K.Langley, J.Meyer, The escaping set of a quasiregular mapping, *Proc. Amer. Math. Soc.*, **137**, 641-651, 2009.

[8] L.E.Böttcher, The principal laws of convergence of iterates and their application to analysis (Russian), *Izv. Kazan. Fiz.-Mat. Obshch.*, **14**, 155-234, 1904.

[9] X.Buff, A.Epstein, S.Koch, Böttcher coordinates, arXiv:1104.2981.

[10] L.Carleson, T.Gamelin, *Complex dynamics*, Springer-Verlag, 1993.

[11] R.L.Devaney, *An Introduction to Chaotic Dynamical Systems, 2nd edition*, Westview Press, 2003.

[12] A.Douady, J.H.Hubbard, Étude dynamique des polynômes complexes I & II, *Publications Mathématiques d'Orsay*, **84-85**, 1984-85.

[13] A.E.Eremenko, On the iteration of entire functions, *Dynamical systems and ergodic theory*, Banach Center Publications, **23**, 339-345, 1989.

[14] K.Falconer, *Fractal Geometry: Mathematical Foundations and Applications, 2nd edition*, Wiley, 2003.

[15] P.Fatou, Sur les équations fonctionelles I. *Bull. Soc. Maths.*, **47**, 1919.

[16] P.Fatou, Sur les équations fonctionelles II. *Bull. Soc. Maths.*, **48**, 1920.

[17] A.Fletcher, R.Fryer, On Böttcher coordinates and quasiregular maps, to appear in *Contemp. Math.*, volume title: "Quasiconformal Mappings, Riemann Surfaces, and Teichmüller Spaces" (in honor of Clifford Earle's 75th birthday).

[18] A.Fletcher, R.Fryer, Dynamics of Mappings of Constant Dilatation, arXiv:1205.4044.

[19] A.Fletcher, D.Goodman, Quasiregular mappings of polynomial type in $\mathbb{R}^2$, *Conform. Geom. Dyn.*, **14**, 322-336, 2010.

[20] A.Fletcher, V.Markovic, *Quasiconformal mappings and Teichmüller spaces*, OUP, 2007.

[21] A.Fletcher, D.A.Nicks, Quasiregular dynamics on the n-sphere, *Erg. Th. and Dyn. Sys.*, **31**, 23-31, 2011.

[22] J.Gill, Outer compositions of hyperbolic/loxodromic linear fractional transformations, *Internat. J. Math. Math. Sci.*, **15**, no. 4, 819-822, 1992.

[23] A.Hinkkanen, Uniformly quasiregular semigroups in two dimensions, *Ann. Acad. Sci. Fenn.*, **21**, no.1, 205-222, 1996.

[24] A.Hinkkanen, G.J.Martin, Attractors in quasiregular semigroups, *XVIth Rolf Nevanlinna Colloquium (Joensuu, 1995)*, 135-141, de Gruyter, Berlin, 1996.

[25] A. Hinkkanen, G. Martin, V. Mayer, Local dynamics of uniformly quasiregular mappings, *Math. Scand.*, **95**, no. 1, 80-100, 2004.

[26] T.Iwaniec, G.Martin, Quasiregular semigroups, *Ann. Acad. Sci. Fenn.*, **21**, no. 2, 241-254, 1996.

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[27] G.Julia, Sur l'itération des fonctions rationnelles, *J. Math. Pures Appl.*, **8**, 47-245, 1918.

[28] M.Mandell, A.Magnus, On convergence of sequences of linear fractional transformations, *Math.Z.*, **115**, 11-17, 1970.

[29] J.Milnor, *Dynamics in one complex variable*, Third edition, Annals of Mathematics Studies, **160**, 2006.

[30] R. Miniowitz, Normal families of quasimeromorphic mappings, *Proc. Amer. Math. Soc.*, **84**, no.1, 35-43, 1982.

[31] O.Lehto, K.I.Virtanen, *Quasiconformal Mappings in the Plane*, Springer, 1973.

[32] S.Rickman, On the number of omitted values of entire quasiregular mappings, *Journal D'Analyse Mathématique*, **37**, 1980.

[33] S. Rickman, Quasiregular mappings, Ergebnisse der Mathematik und ihrer Grenzgebiete 26, Springer, 1993.

[34] P.Rippon, G.Stallard, Transcendental dynamics and complex analysis: a tribute to Noel Baker, Cambridge University Press, 2008.

[35] D.Sullivan, The ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, *Riemann Surfaces and Related Topics, Proceedings of the 1978 Stony Brook Conference. Ann. of Math. Stud.*, **97**, 465-496, 1981.

[36] P.Tukia, On two-dimensional quasiconformal groups, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **5**, 73-78, 1980.

And, in the end, the love you take is equal to the love you make.

— Lennon/McCartney, 1969