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TITLE: **Solitons of geometric flows and their applications**

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Solitons of geometric flows and their applications

by

Sebastian Thomas Wolfgang Helmensdorfer

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

08 2012

THE UNIVERSITY OF
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Acknowledgments

I would like to thank Peter Topping for his good supervision during my PhD and for sharing great ideas and insights with me. For enjoyable mathematical discussions, hints and corrections I am indebted to Gregor Giesen, Tom Ilmanen, John Lott, Mario Micallef, Vlad Moraru and Ben Sharp.

I would like to thank all current and former members of the geometric analysis group, especially Charles Baker, Simon Blatt, Michael Coffey, Gregor Giesen, Mario Micallef, Vlad Moraru, Reto Müller, Michael Munn, Huy Nguyen, Melanie Rupflin, Ben Sharp, Peter Topping and Miaomiao Zhu for many interesting conversations and nice evenings.

There are many good friends who made the time during my PhD a pleasant one and I could never come up with a complete list. Nevertheless I would like to thank Sergios Agapiou, Michael Allman, Leo Basel, Melk Bucher, Mark Dalling, Gregor Giesen, Fabian Haaßengier, Ines & Sebastian Hanke, Alexander Hullmann, Joachim Kaune, Michael Lauerer, Lorraine Lim, Thuy & Philip Milnes, Vlad Moraru, Georg Ostrovski, Amar Parmar, Heiko Schmidt, Felix Schmiedl, Tanja Seel, Ben Sharp, Eugenia Seah, Janina & Florian Sponsel, Jorge Vitória, Daniela Vogel, Miho Wako and Maximilian Wirth for having great times together and for helping me out whenever I needed it.

Most importantly I would like to thank my parents for all their support and good advice during my studies, without them I would never have made it this far. Also I would like to thank my brother Florian and his fiancé Julia Meyringer for supporting me many times.

Finally I would like to thank Stefanie Wunderlich for turning the last months

of my PhD into a wonderful time.

Declarations

The results in chapter 3 of this thesis are published in SIAM Journal on Mathematical Analysis (see Helmensdorfer [2012a]). Chapter 4 is joint work with Peter Topping. Chapter 5 has been submitted to the Arxiv (see Helmensdorfer [2012b]).

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

In this thesis we construct solitons of geometric flows with applications in three different settings.

The first setting is related to nonuniqueness for geometric heat flows. We show that certain double cones in Euclidean space have several self-expanding evolutions under mean curvature flow. The construction of the associated self-expanding solitons leads to an application in fluid dynamics. We present a new model for the behaviour of oppositely charged droplets of fluid, based on the mean curvature flow of double cones. If two oppositely charged droplets of fluid are close to each other, they start attracting each other and touch eventually. Surprisingly, experiments have shown, that if the strength of the charges is high enough, then the droplets are repelled from each other, after making short contact. The constructed self-expanders can be used to correctly predict the experimental results, using our theoretical model.

Secondly we employ space-time solitons of the mean curvature flow to give a geometric proof of Hamilton's Harnack estimate for the mean curvature flow. This proof is based on the observation that the associated Harnack quantity is the second fundamental form of a space-time self-expander. Moreover the self-expander is asymptotic to a cone over the convex initial hypersurface. Hence the self-expander can be seen as the mean curvature evolution of a convex cone, which we exploit to show that preservation of convexity directly implies the Harnack estimate.

In the last chapter we study solutions of the mean curvature flow in a Ricci flow backgound. We show that the space-time track of such a solution can be seen as a soliton. Moreover the second fundamental form of this soliton matches the evolution of a functional, which is the analogue of G. Perelman's $\mathcal{F}$-functional for the Ricci flow on a manifold with boundary and which also has relations to quantum gravity. Furthermore our construction provides a link between the Harnack estimate for the mean curvature flow and the Harnack estimate for the Ricci flow.

Chapter 1

Introduction

The field of geometric flows was started in 1964, when J. Eells and J.H. Sampson introduced the harmonic map flow (see Eells & Sampson [1964]). They realised that such a geometric flow can be used to obtain results in topology and differential geometry. In general a geometric flow deforms a geometric object (e.g. a map or a metric) in time via a partial differential equation in such a way that the initial object becomes more understandable or more suitable for some other purpose [1]. Prior to successfully applying a geometric flow one needs to gain good insight into the associated flow equation.

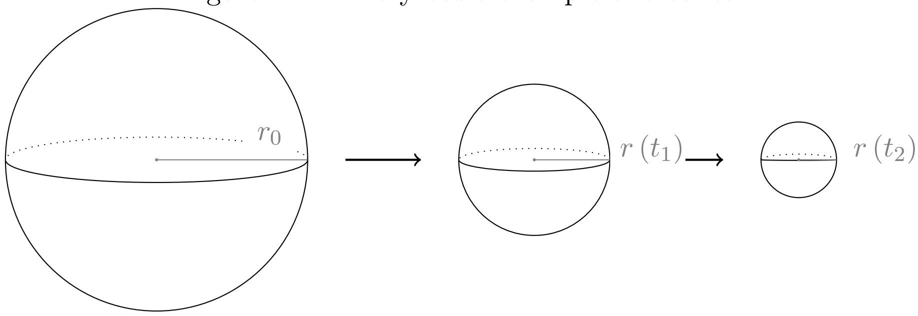
In order to understand partial differential equations it is often beneficial to consider appropriate types of special solutions. Such special solutions can often be stated explicitly or at least be classified in a suitable way and can then provide a great deal of insight into general solutions of the equation under consideration.

Solitons of geometric flows are a prime example of this concept. These are solutions that remain invariant in time to a certain degree under a particular flow. A very basic example would be a round sphere in Euclidean space, shrinking to a point in time at a specific rate, which provides a solution of the mean curvature flow (see figure 1.1). Solitons are a key element in understanding geometric flows.

One important reason for this is that they provide a model for singularities. At a singular time T_{sing} the solution of the flow equation ceases to exist in the classical sense and in many cases concentration of energy can be observed as $t \uparrow T_{\text{sing}}$. A well-known example is the curvature blow-up in a Ricci flow neck pinch on S^3, as shown in figure 1.2 (for more details, see Topping [2006]). The general idea is that in many cases magnifying the solution of a geometric flow close to a singularity reveals a soliton and therefore classification of solitons is an important step towards

[1] it is often crucial, that certain properties (e.g. the homotopy class) of the initial object are preserved

Figure 1.1: A very basic example of a soliton



Shrinking spheres $\partial B^{n+1}_{r(t)}$ provide a very basic example of a soliton for a geometric flow, in this case the mean curvature flow for $r(t) = \sqrt{r_0^2 - 2nt}$, $t < \frac{r_0^2}{2n}$.

understanding the singularities.

The Ricci flow (see Topping [2006] for an introduction) deforms a family of metrics $g(t)$ on a Riemannian manifold M^n according to

$$\frac{\partial}{\partial t} g = -2Ric. \tag{1.1}$$

It can bee seen as a heat equation for the metric and was introduced by R. Hamilton in 1982 with the aim of proving the famous Poincaré conjecture. Solitons of the Ricci flow are solutions which move according to

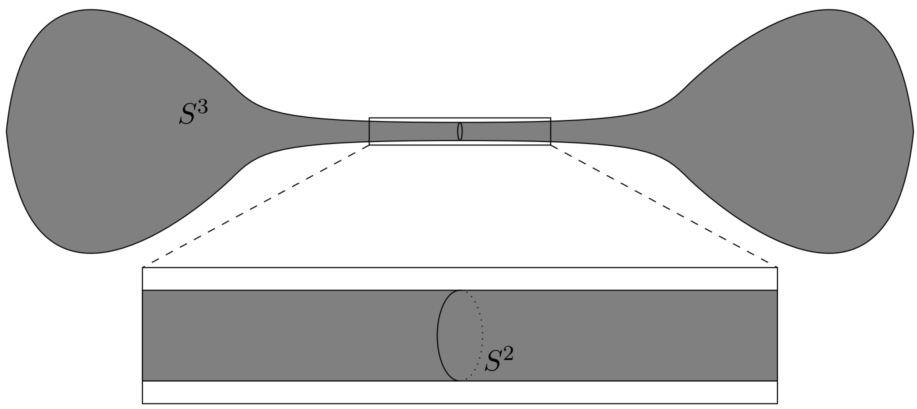
$$g(t) = (1 - 2\lambda t)\, \phi_t^* g_0 \tag{1.2}$$

where ϕ_t is family of diffeomporphisms of M^n and we call the solitons expanding for $\lambda < 0$, shrinking for $\lambda > 0$ and steady for $\lambda = 0$ (see also figure 1.2 for a shrinking soliton). If ϕ_t is generated by a gradient vector field ∇f, then we speak of a gradient Ricci soliton.

In 2003 G. Perelman managed to complete the so called Ricci flow programme, which yielded a proof of the Poincaré conjecture and also of W. Thurston's geometrisation conjecture (see Thurston [1982]), by making a huge leap in understanding the flow and especially its singularities respectively solitons (see Perelman [2002, 2003b,a]; Kleiner & Lott [2008]; Lott [2008]). Another remarkable application of the Ricci flow is S. Brendle's and R. Schoen's proof of the Differentiable Sphere Theorem (see Brendle & Schoen [2011]).

The mean curvature flow is a geometric flow, which has its origins in material science (see Ecker [2004] for an introduction). This flow deforms a family of smooth

Figure 1.2: Solitons are models for singularities of geometric flows



Zooming in on an S^3 neck pinch of the Ricci flow reveals a Ricci flow soliton, in this case part of a cylinder $S^2 \times \mathbb{R}$.

hypersurfaces $F_t : M^n \to \mathbb{R}^{n+1}$ [2] along their normal direction with the speed of the mean curvature

$$\frac{\partial}{\partial t} F_t = -H\nu. \tag{1.3}$$

Area decreases as fast as possible along the flow [3], it is the L^2-gradient flow of the area. Decreasing of area is one of the reasons why the mean curvature flow has proved to be very successful in a wide range of applications, including differential geometry, image processing, medicine and physics (see e.g. Sethian [1999]; Huisken & Ilmanen [2001]; Huisken & Sinestrari [2009]) [4].

Again it is crucial to understand the singularities of the flow and therefore the solitons. Solitons are in this case solutions that move by an isometry or homothety of the ambient space, for instance translations, rotations and scalings of Euclidean space (see figure 1.1 for a shrinking example). However direct classification of all relevant solitons has proved to be very hard. A new approach, started by T. Colding and W. Minicozzi, is to consider only singularities that can be generically observed [5]. This might provide a way to overcome the problem (see Colding & Minicozzi II [2009]; Colding *et al.* [2012]).

Understanding singularities of geometric flows is already a compelling moti-

[2] the ambient space can also be a manifold, possibly of higher codimension
[3] among all flows in the normal direction
[4] we include the inverse mean curvature flow here
[5] nongeneric singularities can be avoided by slightly perturbing the initial hypersurface

vation for studying solitons. But their role is by no means restricted to investigating singularities. In this work we provide three different instances of constructing and using solitons of the mean curvature flow and the Ricci flow for very different applications. Let us describe these three instances in more detail now.

1.1 Solitons, nonuniqess in geometric flows and charged fluid droplets

It is a natural question to ask if and how it is possible to continue a solution of a geometric flow past a singularity. One possibility is to consider smooth solutions of the flow with a singularity for $t \downarrow T_{\text{sing}}$, a setting also known as reverse bubbling (see Topping [2010]). Contrary to weak solutions, information about the singularity does not get lost with such an approach - a necessity for many applications.

But smooth continuation after a singularity poses other difficulties. First one needs to make precise in which sense the smooth solution approaches the singular object as $t \downarrow T_{\text{sing}}$. While this is usually not a big issue, one still has to deal with the question of uniqueness. For many of the important geometric flows, including harmonic map flow, mean curvature flow and Yang-Mills flow, examples of nonunique continuation after a singularity have been constructed (see Ilmanen [1995], Gastel [2003], Germain & Rupflin [2011]). This phenomenon and especially how to choose the correct solution in case of nonuniqueness is not only at the heart of investigating geometric flows, it also relates closely to other partial differential equations. A good example is given by nonuniqueness of solutions for the Euler equation in the theory of hydrodynamics (see e.g. Shnirelman [1997]; De Lellis & Székelyhidi [2009]).

Following work of S. Angenent, T. Ilmanen and D. Chopp (see Angenent *et al.* [1995]) we study a particular example of reverse bubbling for the mean curvature flow. The singular object under consideration is the linear double cone

$$D_\alpha = \left\{ x \in \mathbb{R}^{n+1} : \ |(x_2, \dots, x_{n+1})|^2 = \tan^2 \alpha \cdot x_1^2 \right\}$$

with angle α, which is supposed to be attained as an initial condition as $t \downarrow 0$ locally in the sense of Hausdorff distance. To investigate the evolution of the double cone we consider appropriate solitons, namely self-expanders [6]. We show that if the double cone angle is high enough [7], then there are at least three different self-expanding evolutions of the double cone. The existence of at least two such evolutions has

[6] these are solutions, which move by scaling with $\sqrt{t}$ in time, compare also with figure 1.1

[7] i.e. if the double cone is close enough to a plane

already been proved in Angenent *et al.* [1995], where also the existence of the third evolution has been stated first.

The question of selecting the right solution in this setting is closely related to a remarkable phenomenon from fluid dynamics. When two oppositely charged droplets of fluid are brought close together, they start attracting each other. In doing so the charges cause them to develop conical tips and consequently their local geometry is that of a double cone [8], when they touch eventually. Surprisingly experiments have shown that if the strength of the charges is higher than some critical value, the droplets are repelled from each other, after making only a short contact (see Ristenpart *et al.* [2009]). Otherwise the droplets merge, as one might expect. Moreover the strength of the charges corresponds to the angle of the double cone.

We follow an idea by P. Topping in trying to predict the critical angle, by applying the mean curvature flow to double cones. This seems reasonable, since it should be minimisation of energy which governs the motion of the touching fluid droplets. Using the constructed self-expanders, we show that this approach indeed correctly predicts the experimental results. Furthermore the physical application gives a clue about which of the self-expanding evolutions of the double cone one should use in case of nonuniqueness.

1.2 Solitons and Harnack estimates

The classical parabolic Harnack estimate of J. Moser from 1964 (see Moser [1964]) applies to solutions of the scalar heat equation. It says that for a nonnegative solution $u : [0, T] \to [0, \infty)$ of the heat equation

$$\frac{\partial}{\partial t} u = \triangle u \tag{1.4}$$

[9]on a complete Riemannian manifold (M^n, g) of bounded curvature one has the estimate

$$Cu\,(x_2, t_1) \geq u\,(x_1, t_1) \tag{1.5}$$

for $x_1, x_2 \in M^n$ and $0 < t_1 < t_2 \leq T$, where $C = C\,(x_1, t_1, x_2, t_2, T, M, g)$ is uniformly bounded when x_1, x_2 are varied over a compact set (with the other parameters fixed). The classical proofs of (1.5) do not give sharp bounds on the constant C.

The first sharp Harnack estimate for the heat equation is the differential

[8]smoothed out in some way at the singularity

[9] the Laplacian ist taken with respect to the metric g here

Harnack inequality, obtained by P. Li and S.T. Yau in Li & Yau [1986]. It says that if (M^n, g) is compact with nonnegative Ricci curvature and $u : [0,T] \to (0,\infty)$ is a positive solution of (1.4), then [10]

$$\frac{\partial}{\partial t}\log u - |\nabla \log u| + \frac{n}{2t} \geq 0 \tag{1.6}$$

[11] for $t \in (0,T]$. One can integrate such a differential Harnack estimate along geodesics in order to obtain a classical Harnack estimate:

$$u(x_2,t_2) \geq u(x_1,t_1)\left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-\frac{\Delta}{4}}$$

where

$$\Delta = \inf_{\gamma}\int_{t_1}^{t_2}\left|\frac{d}{dt}\gamma\right|^2 dt$$

and the infimum is taken over all C^1-paths γ with $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$.

By putting the heat kernel on $\mathbb{R}^n$ into (1.6), we see that the Li-Yau Harnack inequality is sharp. Moreover if equality holds in (1.6) for a $0 < t < T$, then (M^n, g) has to be isometric to $\mathbb{R}^n$ (see Ni [2004b,a]). A crucial observation in the proof of (1.6) is the log-convexity of the solution u, in the sense that

$$\mathrm{Hess}(\log u) + \frac{1}{2t} \geq 0. \tag{1.7}$$

R. Hamilton was the first to prove a differential Harnack inequality of this type for a geometric flow, namely the matrix Harnack estimate for the Ricci flow, in 1993 (see Hamilton [1993]). After that differential Harnack inequalities have been proved and applied for many different geometric flows, including the mean curvature flow (and more generally curvature flows of hypersurfaces), the (Kähler-) Ricci flow and the Yamabe flow (see e.g. Chow [1991, 1992]; Cao [1992]; Andrews [1994]; Hamilton [1995b]; Smoczyk [1997]; Perelman [2002]; Brendle [2009]; Cao & Hamilton [2009]; Cabezas-Rivas & Topping [2012a]).

In each case the crucial steps for proving a Harnack estimate are to find a suitable preserved curvature condition and then the equivalent of (1.6), a so called Harnack quantity, that has a sign along the flow, given the preserved curvature condition. Therefore it is clearly desirable to get a better insight into these Harnack quantities, in order to discover new estimates and also to find simplified proofs of existing estimates. R. Hamilton and N. Wallach first asked the question, whether Harnack quantities for geometric flows have a geometric interpretation (see Hamilton

[10] assuming u is bounded the estimate also holds for (M^n, g) complete with bounded curvature
[11] again ∇, $|\cdot|$ are taken with respect to g

[1995a]).

The answer to this question is affirmative and again solitons play a central part in it. The general idea is that Harnack quantities approximately correspond to the curvature of higher dimensional objects, which turn out to be approximate solitons in space-time. Harnack estimates can then be seen as nothing else but preserved curvature conditions one dimension up.

Based on pioneering works by B. Chow, S. Chu and D. Knopf (see Chow & Chu [1995, 2001]; Chow & Knopf [2002]) this concept has been established for the Ricci flow by E. Cabezas-Rivas and P. Topping (see Cabezas-Rivas & Topping [2012a]) and for curvature flows of hypersurfaces by B. Kotschwar (see Kotschwar [2009]). Using this geometric insight all known Harnack estimates for the Ricci flow were recovered respectively discovered in Cabezas-Rivas & Topping [2012a] and a new proof of Hamilton's Harnack estimate for the mean curvature flow was found in Kotschwar [2009].

We also want to mention, that there is an alternative Lie algebraic approach to understanding the Harnack quantities for the Ricci flow (see Wilking [2010]).

In this work we present a geometric proof of Hamilton's Harnack estimate for the mean curvature flow based on the constructions in Kotschwar [2009]. The preserved curvature condition, associated to the estimate, is in this case convexity. Our proof is based on the observation that the space-time soliton, of which the second fundamental form corresponds to the Harnack quantity, is a self-expanding [12] evolution of a cone over the initial hypersurface. This cone is itself convex and therefore preservation of convexity can be used directly to prove the Harnack estimate. One can compare this to the case of the scalar heat equation above, where convexity also implies the differential Harnack estimate (see (1.7)).

1.3 Space-time solitons of the mean curvature flow in a Ricci flow background

It is often beneficial to consider coupled geometric flows, for instance in order to model a physical problem or to overcome a mathematical one. Examples include the Ricci flow coupled with harmonic map flow, which shows interesting behaviour with respect to its singularities (see Müller [2012]), the Ricci flow coupled with the Yang-Mills flow (see Streets [2007]) and a coupled gradient flow, which can be used to show existence of branched minimal immersions in certain homotopy classes (see Ding *et al.* [2006]; Rupflin & Topping [2012]). Another particularly nice

[12] i.e. the solution moves by scaling with $\sqrt{t}$, $t > 0$

example is the relation between Ricci flow and optimal transportation, where the so called 2-Wasserstein distance of two measures evolving under the heat equation in a backwards Ricci flow background [13], decreases (see McCann & Topping [2010]). In case of a static background metric one needs to assume nonnegative Ricci curvature in order for the result to hold (see e.g. von Renesse & Sturm [2005]).

It seems therefore interesting to study the mean curvature flow in a (backwards) Ricci flow background as well. We look at a family of metrics $g^O(t)$ on an ambient manifold O^{n+1}, which satisfy (1.1) and a family of smooth immersions $F_t : M^n \to O^{n+1}$, satisfying (1.3). So far very little is known about this coupled flow, for instance it is not clear whether it is better to parametrise the ambient flow forwards or backwards in time. In the special case that the ambient Ricci flow is a gradient soliton, monotone quantities, analogous to G. Huisken's monotonicity formula for the mean curvature flow (see Huisken [1990]), have been found (see Magni *et al.* [2009]; Tsatis [2010]; Lott [2012]).

The Ricci flow on the ambient manifold O^{n+1} can be seen as the gradient flow of G. Perelman's $\mathcal{F}$-functional

$$\mathcal{F}\left(g^O, f\right) = \int_{O^{n+1}} \left(R\left(g^O\right) + \left| {}^{g^O}\nabla f \right|^2_{g^O} \right) e^{-f} d\mu\left(g^O\right) \tag{1.8}$$

[14], where

$$\frac{\partial}{\partial t} f = -R\left(g^O\right) - \triangle_{g^O} f.$$

The $\mathcal{F}$-functional also appears in string theory (see [Perelman, 2002, section 1] and Deligne *et al.* [1999]). In Lott [2012] J. Lott managed to construct such a functional for the modified Ricci flow on a manifold with boundary, by adding an appropriate boundary term to the original $\mathcal{F}$-functional. This new $\mathcal{F}$-functional also appears in quantum gravity (see York Jr [1972]; Gibbons & Hawking [1977]). The boundary term of its evolution matches Hamilton's Harnack quantity for the mean curvature flow, in case of a flat Ricci flow on $O^{n+1} = \mathbb{R}^{n+1}$.

In this setting we investigate a natural space-time construction. One can construct so called canonical solitons of the Ricci flow as metrics on $O^{n+1} \times (0, T]$, which have strong relations to optimal transportation and Harnack inequalities and are approximate gradient Ricci solitons (see Cabezas-Rivas & Topping [2012b,a]). The space-time track $\{(F_t(x), t) : x \in M^n, t \in (0, T]\}$ of the mean curvature flow can be immersed into each of the canonical Ricci solitons as a canonical mean curvature flow soliton. By this we mean that the mean curvature flow solution

[13] $\frac{\partial g}{\partial \tau} = 2Ric$

[14] we suppress the time dependence of $g^O(t)$ here

moves along the gradient vector field, which determines the motion of the ambient Ricci flow.

Moreover we point out that the second fundamental form of our space-time track canonical soliton matches the boundary term of the evolution of the modified $\mathcal{F}$-functional. In view of the relation between space-time solitons and Harnack estimates, described in the previous section 1.2, this also establishes a connection between the Harnack estimate for the Ricci flow and the Harnack estimate for the mean curvature flow.

1.4 Outline

This work consists of three chapters and an appendix. Each of the three chapters 3, 4 and 5 corresponds to one of the settings, which we describe in sections 1.1, 1.2 and 1.3. The chapters are set up in such a way, that each of them can be read independently from the others. In appendix A we give a brief overview over level-set flow techniques for the mean curvature flow, which are used throughout chapters 3 and 4.

Chapter 2

Notations and conventions

The following notations and conventions are used througout this thesis, unless stated otherwise.

Let $F : M^n \to O^{n+1}$ be a smooth immersion or embedding, where M^n respectively O^{n+1} denote smooth manifolds without boundary of dimensions n respectively $n+1$. We use coordinates $\left(p^i\right)_{i=1,\ldots,n}$ on M^n and $\left(x^j\right)_{j=1,\ldots,n+1}$ on the ambient space O^{n+1}.

A metric on the ambient space O^{n+1} is denoted by g^O and we also write $\langle\cdot,\cdot\rangle$ for g^O, in case $O^{n+1} = \mathbb{R}^{n+1}$ with the Euclidean metric, which is the case througout chapters 3 and 4. We then denote by ${}^{g^O}\nabla$, $\triangle_{g^O} = \operatorname{tr}_{12}\left({}^{g^O}\nabla^2\right)$ and $|\cdot|_{g^O}$ the Levi-Civita connection, the connection Laplacian and the norm associated to g^O. Here we use

$${}^{g^O}\nabla^2_{X,Y}A = {}^{g^O}\nabla_X {}^{g^O}\nabla_Y A - {}^{g^O}\nabla_{{}^{g^O}\nabla_X Y} A = \left({}^{g^O}\nabla A\right)(X, Y, \ldots)$$

for the second covariant derivative of any tensor field A, where $X, Y \in \Gamma\left(TO^{n+1}\right)$ are vector fields on O^{n+1} and tr_{12} denotes the trace over the first two entries. We also write ${}^{g^O}\nabla^2 f = \operatorname{Hess}_{g^O}(f)$ for a function $f \in C^\infty\left(O^{n+1}\right)$. We use the sign convention

$$\mathcal{R}\left(g^O\right)(X,Y) = {}^{g^O}\nabla_Y\, {}^{g^O}\nabla_X - {}^{g^O}\nabla_X\, {}^{g^O}\nabla_Y + {}^{g^O}\nabla_{[X,Y]} = {}^{g^O}\nabla^2_{Y,X} - {}^{g^O}\nabla^2_{X,Y}$$

for the curvature of g^O. By $\Gamma\left(g^O\right)$, $Rm\left(g^O\right)$, $Ric\left(g^O\right)$, $R\left(g^O\right)$ and $\mu\left(g^O\right)$ we denote the associated Christoffel symbols, the Riemann curvature tensor, the Ricci curvature, the scalar curvature and the volume form.

By ν we denote a choice of unit normal vector field associated to $F : M^n \to$

O^{n+1}. We then denote by g the induced metric

$$g_{ij} = g^O \left(F_* \frac{\partial}{\partial p^i}, F_* \frac{\partial}{\partial p^j} \right), \quad 1 \leq i, j \leq n$$

on TM^n [1], by h the second fundamental form

$$h_{ij} = -g^O \left(\nu, {}^{g^O}\nabla_{F_* \frac{\partial}{\partial p^i}} F_* \frac{\partial}{\partial p^j} \right) \quad 1 \leq i, j \leq n,$$

by ∇ the induced connection on TM^n and by H the mean curvature

$$H = g^{ij} h_{ij}.$$

In most cases we deal with families $(F_t)_{t \in I}$, I a real interval, of smooth immersions $F_t : M^n \to O^{n+1}$, which are solutions of the mean curvature flow. By this we mean - unless stated otherwise - that the family F_t classically solves the partial differential equation

$$\frac{\partial}{\partial t} F_t = -H\nu$$

and we write $F_t (M^n) = M_t$.

Similarly by a solution of the Ricci flow we mean a one-parameter family of metrics $g^O (t)$ on O^{n+1}, $t \in I$ for I some real interval, solving

$$\frac{\partial}{\partial t} g^O = -2Ric \left(g^O \right).$$

For more details on the geometry of submanifolds we refer to [Ecker, 2004, appendices A, B] and [Jost, 2011, section 4.7].

[1] $F_* \frac{\partial}{\partial p^i} = \frac{\partial F}{\partial p^i}$ denotes the push-forward of $\frac{\partial}{\partial p^i}$, we have $F_* \in \Gamma \left(T^* M^n \otimes F^{-1} TO^{n+1} \right)$, where $F^{-1} TO^{n+1}$ denotes the pull-back bundle

Chapter 3

Mean curvature flow of double cones and a model for the behaviour of oppositely charged fluid droplets

We consider solutions of the mean curvature flow, i.e. families $(M_t)_{t\in I}$ of smooth, immersed hypersurfaces $M_t = F_t\left(M^n\right) \subset \mathbb{R}^{n+1}$ for $n \geq 2$ satisfying

$$\frac{\partial}{\partial t} F_t = -H\nu. \tag{3.1}$$

Following work of S. Angenent, D. Chopp and T. Ilmanen (see Angenent *et al.* [1995]) we are interested in solutions of (3.1), which have the double cone with angle $0 \leq \alpha \leq \frac{\pi}{2}$

$$D_\alpha = \left\{x \in \mathbb{R}^{n+1} : \ \left|(x_2, \ldots, x_{n+1})\right|^2 = \tan^2 \alpha \cdot x_1^2\right\} \tag{3.2}$$

as an initial condition. The initial condition is here understood to be attained locally in the sense of Hausdorff distance.

We call a solution of (3.1) self-expanding, if we have $M_t = \sqrt{t} M_1$, where $I = (0, \infty)$. In the first part of this chapter we show that for α large enough, there are multiple self-expanding solutions of (3.1) with D_α as an initial condition. D_α can bee seen as the union of two Lipschitz graphs, each of which has a self-expanding evolution for any angle α (see Ecker & Huisken [1989]). On the other hand for α large D_α is close to a hyperplane of dimension n and drilling a hole significantly reduces area in this case, which means there should be an evolution of D_α with only one connected component (see also Ilmanen [1995]). We show existence of at least two such solutions. The existence of one solution was first shown in Angenent *et al.* [1995], where also the existence of a second solution was stated first.

Using the constructed self-expanders as barriers also enables us to study the pinching and the long-time behaviour of rotationally symmetric solutions of (3.1), which are asymptotic to D_α. In the second part of this chapter we show how to apply this theory to a problem from fluid dynamics. One can employ the theory of flowing double cones by mean curvature to create a theoretical model, which correctly predicts experimental behaviour of oppositely charged droplets of fluid.

3.1 Existence of self-expanders asymptotic to double cones

We want to study the evolution by mean curvature of the double cone D_α, following Angenent *et al.* [1995]. We are interested in solutions $(M_t)_{t\in(0,\infty)}$ of (3.1) that satisfy

$$M_t \text{ is rotationally symmetric and self-expanding.} \tag{3.3}$$

We write $x = \left(x^1, x^2, \ldots, x^{n+1}\right) = \left(x^1, \hat{x}\right)$ for $x \in \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. For a curve γ in $\mathbb{R} \times [0, \infty)$ we define the corresponding surface of rotation

$$M(\gamma) = \left\{\left(x^1, \hat{x}\right) \in \mathbb{R}^{n+1} : \left(x^1, |\hat{x}|\right) = (y, u) \in \gamma\right\}. \tag{3.4}$$

We say that a rotationally symmetric hypersurface $M(\gamma)$ is asymptotic to a double cone $D_\alpha = M(\gamma_\alpha)$, where $\gamma_\alpha = (y, \tan\alpha\,|y|)$, if the curve γ is asymptotic to the curve γ_α, in the sense that for $\varepsilon > 0$ we can make $|\gamma - \gamma_\alpha| < \varepsilon$ outside a large compact set $K = K(\varepsilon) \subset \mathbb{R} \times [0, \infty)$. $M(\gamma)$ is said to lie outside of $M(\delta)$, if $|u_\gamma| \geq |u_\delta|$ for any $(y, u_\gamma) \in \gamma$ and $(y, u_\delta) \in \delta$. If a solution of (3.3) is asymptotic to D_α it approaches the double cone as $t \to 0$ locally in the sense of Hausdorff distance and can therefore be considered an evolution of D_α. Our goal is to show existence of such self-expanders.

The main result of this section can be stated as follows (see figures 3.1 and 3.2 for illustrations):

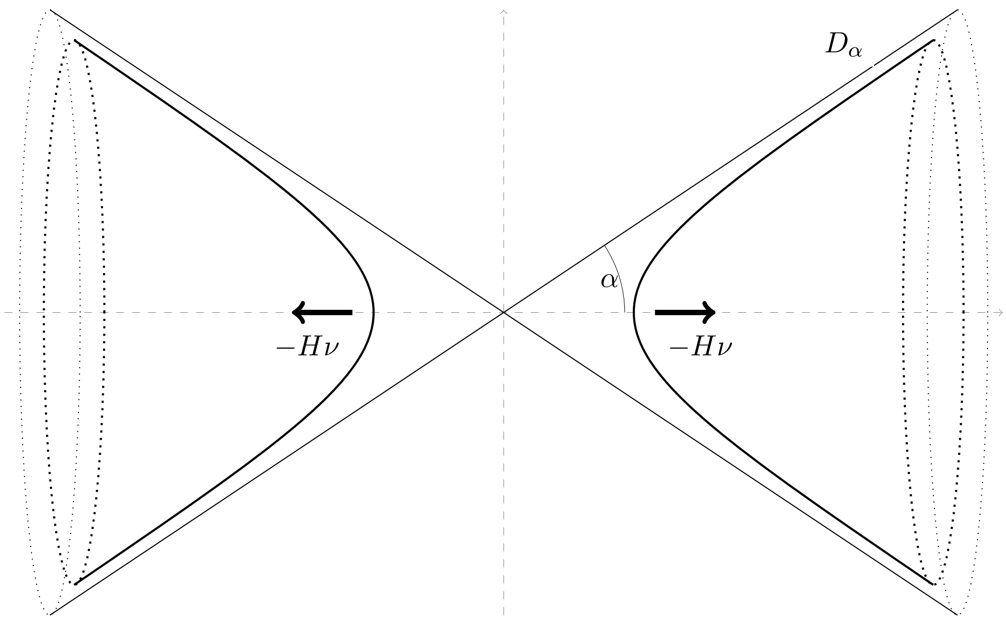
Theorem 3.1. *For $n \geq 2$ there exists a critical angle $\alpha^*_{crit}(n) \in \left(0, \frac{\pi}{2}\right)$ with the following properties.*

For any angle $\alpha > \alpha^*_{crit}$ there exist at least three distinct, smooth, rotationally symmetric evolutions of the double cone D_α, which are self-expanding. Two of these *evolutions have one connected component and one has two connected components.*

For $\alpha = \alpha^*_{crit}$ at least one one-component and one two-component self-expanding, smooth, rotationally symmetric evolution of D_α exist.

For $\alpha < \alpha^*_{crit}$ the two-component evolution of D_α is the unique evolution of D_α.

Figure 3.1: Self-expanders with two connected components asymptotic to double cones



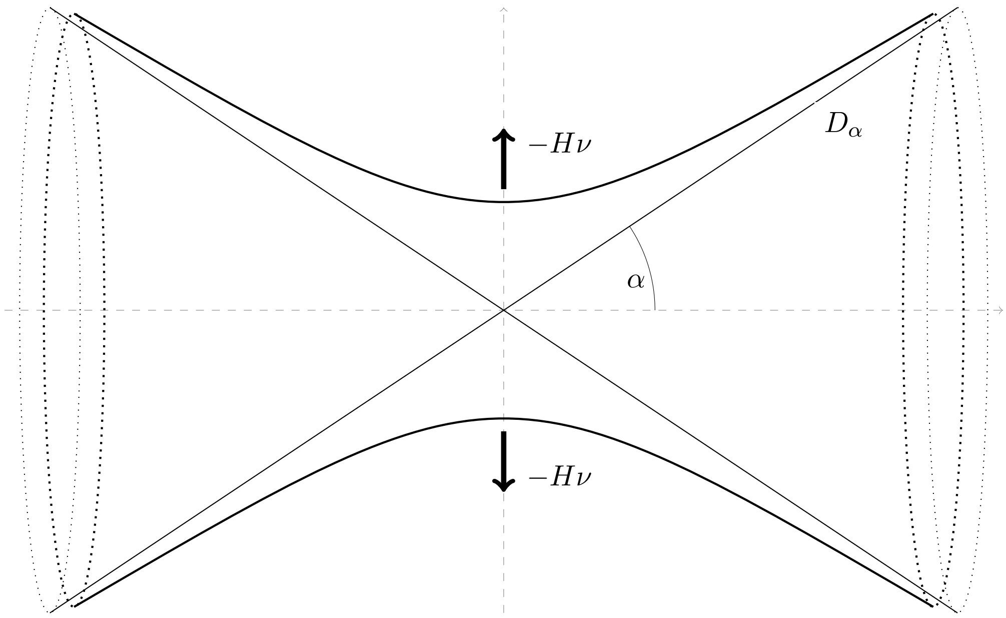
For any angle $0 < \alpha < \frac{\pi}{2}$ there are self-expanding evolutions of D_α, which have two connected components. The mean curvature vector $-H\nu$ indicates the direction of the flow.

The existence of one-sheeted self-expanders was first proved in Angenent *et al.* [1995], where also the uniqueness part for $\alpha < \alpha^*_{crit}$ is proved. Additionally our proof provides nonuniqueness among one-sheeted solutions, which was also first stated in Angenent *et al.* [1995]. The existence of the two-component evolution of the double cone comes from considering D_α as two entire Lipschitz graphs and then applying theory for graphical mean curvature flow (see Ecker & Huisken [1989]).

Before providing a proof of theorem 3.1 we want to discuss various aspects and implications of it.

There is a notion of weak solution for the mean curvature flow, called the level-set flow, which can be applied to D_α (more precisely we take the biggest flow as in definition A.7). Nonuniqueness here corresponds to fattening of the level-set

Figure 3.2: Self-expanders with one connected component asymptotic to double cones



For $\alpha \geq \alpha^*_{crit}$ there are self-expanding evolutions of D_α, which have one connected component. The mean curvature vector $-H\nu$ indicates the direction of the flow.

flow, i.e. the unique solution of the level-set flow $(\Gamma_t)_{t\in[0,\infty)}$ with initial condition D_α develops an interior for $t > 0$. For $\alpha < \alpha^*_{crit}$ the two-component evolution of D_α is unique among all evolutions of D_α and therefore we have non-fattening of the level-set flow.

Self-expanders are stationary for the functional

$$\mathbf{K}[M] = \int_M \exp\left(\frac{|x|^2}{4}\right) d\mathcal{H}^n(x).$$

The authors of Angenent *et al.* [1995] sketch a proof for the existence of one-sheeted self-expanders, asymptotic to D_α, which are minimizers of $\mathbf{K}$. Furthermore they indicate how one might prove a version of theorem 3.1 using this approach (see also Ilmanen [1995]). Ilmanen [1995] also provides further examples of nonuniqueness for the mean curvature flow in dimensions $3 \leq n \leq 6$.

It has already been conjectured in Angenent *et al.* [1995] that there is a a self-shrinking solution (i.e. $M_t = \sqrt{-t}M_1$, $t \in (-\infty, 0)$) of (3.1) which has high genus and approaches the double cone D_α as $t \to 0$ locally in the sense of Hausdorff distance. This is proved in Kleene & Møller [2010] and Kapouleas *et al.* [2011], using gluing techniques. Therefore, for $\alpha \geq \alpha^*_{crit}$, we have an example, where the mean curvature flow continues nonuniquely after the singlarity D_α and it is natural to ask the question, which evolution of D_α one should choose (see Ilmanen [1995]). In the next section we provide a partial answer to this question, based on an application from fluid dynamics.

In Bode [2007] we can find analogues of the classical estimates for graphical mean curvature flow (see Ecker & Huisken [1989]) for cylindrical graphs. Here the missing part in order to prove convergence for rotationally symmetric graphs is the existence of self-expanding barriers. More precisely theorem 3.1 complements [Bode, 2007, theorem 5.16]. The latter says that rotationally symmetric cylindrical graphs, which satisfy appropriate growth and straightness conditions and curvature bounds and are bounded self-similarly below, converge (after an appropriate rescaling) under mean curvature flow as $t \to \infty$ to a self-expander. Being bounded self-similarly below here means that the graphs lie outside of D_α for some $\alpha \geq \alpha^*_{crit}$, which implies together with an appropriate comparison principle ([Bode, 2007, theorem E.7]), that the self-expanders from theorem 3.1 can be employed as barriers in order to prevent pinching. This result is an analogue of [Ecker & Huisken, 1989, theorem 5.1]. Long-time existence for solutions, which are bounded self-similarly below, also follows from the work of M. Simon (see Simon [1990]).

3.1.1 Proof of theorem 3.1

First we derive the ordinary differential equations which describe rotationally symmetric self-expanders in our setting, arriving at (3.7). After that, before going through the proof, we give a short outline of the proof strategy.

For $-\pi \leq \alpha \leq \pi$ let σ_α be the closed ray $\{t(\cos\alpha, \sin\alpha) : \ t \geq 0\}$ in $\mathbb{R}^2$. The cone in $\mathbb{R}^{n+1}$ with angle α is defined as $C_\alpha = M\left(\sigma_\alpha \cup \sigma_{-\alpha}\right)$. We get $D_\alpha = C_\alpha \cup C_{\pi-\alpha}$.

We get self-expanding solutions of (3.1), if the hypersurface M_1 satisfies [1]

$$H + \frac{\langle x, \nu\rangle}{2} = 0 \tag{3.5}$$

for each $x \in M_1$. Under the condition of rotational symmetry (3.3) this equation becomes an ordinary differential equation.

Suppose $M_1 = M(\gamma)$ in (3.5). Parametrizing γ by arclength s we define $\theta \in [0, 2\pi)$ along γ by setting $\gamma_s = (\cos\theta, \sin\theta)$ for the tangent vector γ_s. The left-handed unit normal is given by $\bar{\nu} = (-\sin\theta, \cos\theta)$. Let $\mathbf{k}$ be the curvature vector

[1] this is up to tangential diffeomorphisms, for details see [Ecker, 2004, chapter 2]

of γ and $k = \mathbf{k} \cdot \bar{\nu}$. Equation (3.5) for $M(\gamma)$ becomes

$$k - \frac{n-1}{u}\cos\theta + \frac{y}{2}\sin\theta - \frac{u}{2}\cos\theta = 0, \quad (y,u) \in \gamma. \tag{3.6}$$

A solution of this equation creates a smooth surface $M(\gamma)$ and we have (this is from [Angenent *et al.* , 1995, (13)-(15)]):

a. Every critical point of $|u|$ restricted to a solution γ of (3.6) is a strict local minimum.
   Assume without loss of generality that we have $u > 0$ at a critical point $x \in \gamma$. Orient γ in the direction of increasing y which means $\nu = \frac{\partial}{\partial u}$ and $\theta = 0$ at x. Then (3.6) implies that $k > 0$ with respect to ν and therefore $|u|$ is a local minimum.

b. Every critical point of $|y|$ restricted to a solution γ of (3.6) is a strict local minimum, unless γ is the u-axis.
   Assume without loss of generality that $y > 0$ at the critical point x and locally orient γ so that u is increasing. Then $\nu = -\frac{\partial}{\partial y}$ and $\theta = \frac{\pi}{2}$ at x. Hence (3.6) shows that $k \leq 0$ and therefore $|y|$ must be a local minimum. We have $k < 0$ unless $y = 0$ at x, in which case γ must be the u-axis by uniqueness of the initial value problem.

The following lemma is from Angenent *et al.* [1995]). The first part is a consequence of results from Ecker and Huisken on graphical mean curvature flow (see Ecker & Huisken [1989]).

Lemma 3.2. *(i) (Two-component case)*

For $\alpha \in (0,\pi)$ there exists a unique, smooth, connected curve $\gamma(\alpha)$, solving (3.6) and asymptotic to C_α. Furthermore, unless γ is the u-axis, γ is the graph of a positive, convex (or negative, concave) even function $y = y(u)$.

(ii) (One-component case)

If γ is a smooth, connected curve solving (3.6) which meets $\{u > 0\}$, then γ lies in $\{u > 0\}$ and there exist α, β with $0 < \alpha < \beta < \pi$ such that γ is asymptotic to $\sigma_\alpha \cup \sigma_\beta$. Moreover, if γ meets the u-axis at a right angle, then $\beta = \pi - \alpha$ and γ is the graph of a positive, even function $u = u(y)$ which is monotone for $y \neq 0$.

We first show that for our purposes it is enough to consider surfaces $M(\gamma)$, where γ is a graph in $\mathbb{R}^2$.

We define

$$A = \left\{\alpha \in \left(0, \frac{\pi}{2}\right) : \exists \text{ connected } \gamma \text{ solving (3.6), asymptotic to } \sigma_\alpha \cup \sigma_{\pi-\alpha}\right\}$$

and

$$\alpha^*_{crit}(n) = \inf A.$$

Let γ be a connected solution of (3.6) which is asymptotic to $\sigma_\alpha \cup \sigma_{\pi-\alpha}$. There must be a point x on γ with $\theta \in \{0, \pi\}$. It follows now from a and b that there are at most two points on γ with $\theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Hence γ must be a graph $u(y)$ and we have shown

$$A = \left\{\alpha \in \left(0, \frac{\pi}{2}\right) : \exists \text{ a graph } \gamma \text{ solving (3.6), asymptotic to } \sigma_\alpha \cup \sigma_{\pi-\alpha}\right\}.$$

Hence we now want to focus on smooth graphs u solving (3.6). By symmetry it is enough to consider solutions with non-negative derivative in 0. After imposing initial conditions we get the following initial value problem for u

$$u_{yy} = \left(1 + (u_y)^2\right)\left(\frac{1}{2}(u - yu_y) + \frac{n-1}{u}\right), \quad u(0) = C > 0,\ u_y(0) = K \geq 0. \tag{3.7}$$

For symmetry reasons it is enough to consider $u|_{[0,\infty)}$ which we denote again by u. For $y > 0$ we denote by $\alpha(y) = \arctan\left(\frac{u}{y}\right)$ the signed angle, that u makes with the positive y-axis.

Our strategy for proving theorem 3.1 can now be outlined as follows. We show that for any fixed $K \geq 0$ the asymptotic angle $\alpha = \lim_{y\to\infty} \alpha(y)$ [2] of u goes to $\frac{\pi}{2}$ when C goes to 0 or ∞ in (3.7) (see figure 3.3 and lemmas 3.3, 3.4). Furthermore lemma 3.4 implies that α also goes to $\frac{\pi}{2}$ when K goes to ∞ in (3.7). Finally we show (see lemma 3.5) that the asymptotic angle α depends continuously on the parameters $C > 0$ and $K \geq 0$. We can then use a compactness argument to see that the asymptotic angle must achieve a minimum, namely $\alpha^*_{crit}(n)$ (see figure 3.4).

First we note the following observations regarding the solution u (for i, iii and iv see also Angenent *et al.* [1995]).

i. Every critical point of u is a strict local minimum.

   To see this compute $u_{yy} = \frac{u}{2} + \frac{n-1}{u} > 0$ from (3.7) whenever $u_y = 0$ (also follows from a).

ii. $u_y > 0$ for all $y > 0$ and every critical point of u_y is a strict local maximum.

   The first part is a direct consequence of i and the fact that $u_{yy}(0) = \frac{C}{2} + \frac{n-1}{C} > 0$. For the second part compute $u_y^{(3)} = -\left(1 + (u_y)^2\right)(n-1)\frac{u_y}{u^2} < 0$ from (3.7) using the first part whenever $u_{yy} = 0$.

[2] the limit always exists, see iii

iii. Every critical point of $\alpha(y)$ is a strict local minimum. Therefore u is asymptotic to a ray σ_α, $\alpha = \lim_{y\to\infty}\alpha(y)$.

Let $0 = \alpha_y(y) = \frac{1}{u^2+y^2}(yu_y - u)$, $y > 0$. Using (3.7) we get $0 = yu_y - u = \frac{2(n-1)}{u} - \frac{2u_{yy}}{1+(u_y)^2}$ and therefore $u_{yy} > 0$. But then $\alpha_{yy} = \frac{yu_{yy}}{u^2+y^2} > 0$.

iv. $0 < \alpha = \lim_{y\to\infty}\alpha(y) < \frac{\pi}{2}$.

This follows from lemma 3.2 (ii) which yields $0 < \alpha < \pi - \alpha < \pi$ for $\beta = \pi - \alpha$.

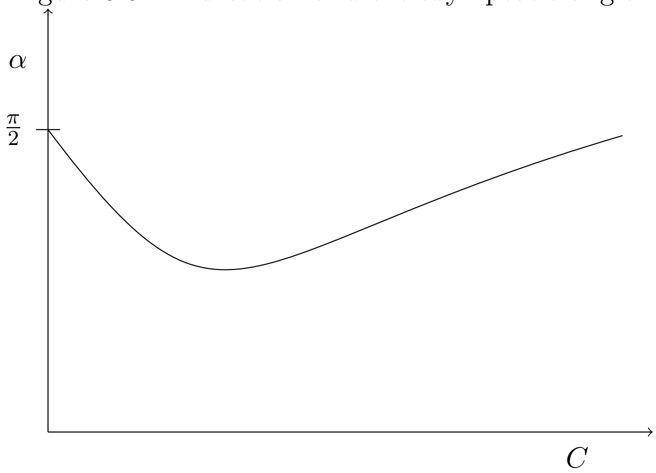
v. $u_y \to \tan\alpha$ as $y \to \infty$ and hence $|u_y|$ is bounded.

Since $u_{yy}(0) > 0$, ii shows that u_y is either strictly monotone increasing or has one strict local maximum. Therefore iii and iv yield the claim.

vi. u_{yy} has at most one zero and $u_{yy} \to 0$ as $y \to \infty$.

In v it was shown that u_{yy} can have at most one zero. By v and iii the right hand side of (3.7) and therefore u_{yy} goes to 0 as $y \to \infty$.

Figure 3.3: Bifurcation and the asymptotic angle



The profile of the asymptotic angle α for fixed $K \geq 0$. We believe there is exactly one minimum as in the picture, but our proof does not exclude the case that several minima occur.

As a first step we show that the first derivative of the solutions u blows up, when C goes to 0 or ∞ in (3.7). In view of v this is useful for investigating the behaviour of the corresponding asymptotic angles.

Lemma 3.3. *The solutions u of (3.7) satisfy $\sup_{[0,\infty)} u_y \to \infty$ if $C \to 0$ or $C \to \infty$.*

Proof. Let $y_0 > 0$ and $D = \sup_{(0,y_0)} u_y$. We can write (3.7) as

$$\frac{u_{yy}}{1+(u_y)^2} + \frac{yu_y}{2} = \frac{u}{2} + \frac{n-1}{u}. \tag{3.8}$$

Integrating from 0 to y_0 noting that $u_y \leq D$ and $y \leq y_0$ we can bound the left hand side of (3.8) by

$$\int_0^{y_0} \frac{u_{yy}}{1+(u_y)^2} + \frac{yu_y}{2} dy \leq \arctan\left(u_y\left(y_0\right)\right) + \frac{y_0 D}{2} \leq \frac{\pi}{2} + \frac{y_0 D}{2}. \tag{3.9}$$

By ii the integral of the right hand side of (3.8) is bounded below as follows

$$\int_0^{y_0} \frac{u}{2} + \frac{n-1}{u} dy \geq \int_0^{y_0} \frac{u}{2} dy \geq \frac{C y_0}{2}. \tag{3.10}$$

Inequalities (3.9) and (3.10) yield $\frac{Cy_0}{2} \leq \frac{\pi}{2} + \frac{y_0 D}{2}$ and therefore $D \to \infty$ as $C \to \infty$.

Since $u_y \leq D$ we have $u \leq C + Dy$ on $(0, y_0)$ and therefore

$$\int_0^{y_0} \frac{u}{2} + \frac{n-1}{u} dy \geq \int_0^{y_0} \frac{n-1}{C+Dy} dy = \frac{n-1}{D} \log\left(1 + \frac{Dy_0}{C}\right). \tag{3.11}$$

Combining inequalities (3.8), (3.9) and (3.11) and solving for C yields

$$C \geq \frac{Dy_0}{\exp\left(\frac{D}{n-1}\left(\frac{\pi}{2} + \frac{Dy_0}{2}\right)\right) - 1}.$$

This shows that $C \nrightarrow 0$ if D remains bounded. Therefore $D \to \infty$ as $C \to 0$. □

From the last lemma we know that u_y blows up somewhere on $(0, \infty)$ as $C \to 0$ or $C \to \infty$. The next lemma shows that the asymptotic angle α of solutions of (3.7) has to go to $\frac{\pi}{2}$ if the first derivative blows up somewhere on $[0, \infty)$. Together lemmas 3.3 and 3.4 show that the asymptotic angle has the desired behaviour for fixed $K \geq 0$, as shown in figure 3.3.

Lemma 3.4. *For any sequence of solutions u_k of (3.7) with $\sup_{[0,\infty)} (u_k)_y \to \infty$ as $k \to \infty$ we have $\alpha_k \to \frac{\pi}{2}$ as $k \to \infty$.*

Proof. At first we show that it is enough to prove the claim for a sequence of solutions of (3.7) with precisely one inflection point which we call $\hat{y}_k$. For such a sequence we can show that the quantity $\frac{u_k}{y} \to_{y\to\infty} \tan \alpha_k$ (see iii) is increasing on $[\hat{y}_k, \infty)$. Hence it remains to show $\sup_{[\hat{y}_k,\infty)} \frac{u_k}{y} \to \infty$ as $k \to \infty$. We achieve this by distinguishing the cases $C_k \geq C_0 > 0$ for $k \in \mathbb{N}$ and the case that the sequence $C_k > 0$ is not bounded below by a positive constant.

1. We may assume precisely one inflection point $\hat{y}_k$ for each u_k

We want to show that for any $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ for all $k \geq k_0$ holds for the asymptotic angles of the u_k. Let $\varepsilon > 0$. Since $\sup_{[0,\infty)} (u_k)_y \to \infty$ as $k \to \infty$, v implies that we can find $\tilde{k}_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ holds for all $k \geq \tilde{k}_0$ for which the solutions satisfy $(u_k)_{yy} > 0$ on $[0, \infty)$. By vi it is therefore enough to find $\overline{k}_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ for all $k \geq \overline{k}_0$ for which the solutions u_k have precisely one inflection point.

2. $\frac{u_k}{y}$ is increasing on $[\hat{y}_k, \infty)$

We have from (3.7) for $y > 0$

$$\left(\frac{u_k}{y}\right)_y = \frac{(u_k)_y}{y} - \frac{u_k}{y^2} = \frac{2}{y^2}\left(\frac{n-1}{u_k} - \frac{(u_k)_{yy}}{1 + \left((u_k)_y\right)^2}\right).$$

This means $\left(\frac{u_k}{y}\right)_y > 0$ on $[\hat{y}_k, \infty)$ for any $k \in \mathbb{N}$. By iii we have $\frac{u_k}{y} \to \tan \alpha_k$ as $y \to \infty$, so to prove the claim it is enough to show

$$\sup_{[\hat{y}_k,\infty)} \frac{u_k}{y} \to \infty \tag{3.12}$$

as $k \to \infty$.

3. $\sup_{[\hat{y}_k,\infty)} \frac{u_k}{y} \to_{k\to\infty} \infty$ in the case $(C_k)_{k\in\mathbb{N}}$ is bounded below by $C_0 > 0$

Suppose first that $C_k \geq C_0 > 0$ for all $k \in \mathbb{N}$ holds for the initial conditions. At $\hat{y}_k$ we can write (3.7) as

$$\frac{u_k(\hat{y}_k)}{\hat{y}_k} = (u_k)_y(\hat{y}_k) - \frac{2(n-1)}{u_k(\hat{y}_k)\,\hat{y}_k}.$$

So by ii

$$\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq (u_k)_y(\hat{y}_k) - \frac{2(n-1)}{C_k \hat{y}_k} \tag{3.13}$$

and $\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq \frac{C_k}{\hat{y}_k}$ which yields

$$\frac{1}{C_k \hat{y}_k} \leq \frac{u(\hat{y}_k)}{C_k^2 \hat{y}_k}. \tag{3.14}$$

Combining inequalities (3.13) and (3.14) yields

$$\left(1 + \frac{2(n-1)}{C_k^2}\right)\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq (u_k)_y(\hat{y}_k).$$

$\hat{y}_k$ is the only inflection point, so by assumption we have $(u_k)_y(\hat{y}_k) \to \infty$ as $k \to \infty$. Therefore $\frac{u_k(\hat{y}_k)}{\hat{y}_k} \to \infty$ as $k \to \infty$ which yields (3.12), as desired.

4. $\sup_{[\hat{y}_k,\infty)} \frac{u_k}{y} \to_{k\to\infty} \infty$ in the case $(C_k)_{k\in\mathbb{N}}$ is not bounded below by $C_0 > 0$

To show the claim for the case that the sequence C_k is not bounded below, it is without loss of generality enough to show the claim for a sequence of solutions u_k with $C_k \to 0$ as $k \to \infty$. Assume not, then there is a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ with $\frac{u_{k_i}}{y} \le D$ on $[\hat{y}_{k_i}, \infty)$ for all $i \in \mathbb{N}$ for a constant $D > 0$. (3.7) for $y > 0$ can be written as

$$\frac{(u_{k_i})_{yy}}{y\left(1+\left((u_{k_i})_y\right)^2\right)} + \frac{(u_{k_i})_y}{2} = \frac{(u_{k_i})}{2y} + \frac{n-1}{yu_{k_i}}. \tag{3.15}$$

Integrating (3.15) from $\hat{y}_{k_i}$ to $2\hat{y}_{k_i}$ and estimating each side using the bound on $\frac{u_{k_i}}{y}$ yields

$$\int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})_{yy}}{y\left(1+\left((u_{k_i})_y\right)^2\right)} + \frac{(u_{k_i})_y}{2} dy \le \int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})_y}{2} dy \le \frac{\hat{y}_{k_i} D}{2} \tag{3.16}$$

and

$$\int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})}{2y} + \frac{n-1}{yu_{k_i}} dy \ge \frac{n-1}{2u_{k_i}(2y_{k_i})} \ge \frac{n-1}{4\hat{y}_{k_i} D}. \tag{3.17}$$

Combining inequalities (3.16) and (3.17) yields $\hat{y}_{k_i}^2 \ge \frac{n-1}{2D}$ for $i \in \mathbb{N}$. To get a contradiction we show that $\hat{y}_{k_i} \to 0$ as $i \to \infty$.

Assume not, so there is a subsequence of $(k_i)_{i\in\mathbb{N}}$ which we call without loss of generality again $(k_i)_{i\in\mathbb{N}}$ and $\tilde{\varepsilon} > 0$ such that $\hat{y}_{k_i} \ge \tilde{\varepsilon}$ for all $i \in \mathbb{N}$. Evaluating (3.7) at $\hat{y}_{k_i}$ yields

$$\frac{u_{k_i}(\hat{y}_{k_i})}{\hat{y}_{k_i}} = (u_{k_i})_y(\hat{y}_{k_i}) - \frac{2(n-1)}{u_{k_i}(\hat{y}_{k_i})\hat{y}_{k_i}} \le D.$$

By assumption we have $(u_{k_i})_y(\hat{y}_{k_i}) \to \infty$ as $i \to \infty$ and therefore $u_{k_i}(\hat{y}_{k_i})\hat{y}_{k_i} \to 0$ as $i \to \infty$. Since $\hat{y}_{k_i} \ge \tilde{\varepsilon}$ and by ii we have therefore $u_{k_i} \to 0$ uniformly on $[0, \tilde{\varepsilon}] \subseteq [0, \hat{y}_{k_i}]$ as $i \to \infty$. Since $(u_{k_i})_{yy} > 0$ on $[0, \hat{y}_{k_i})$ we can estimate

$$u_{k_i}(\hat{y}_{k_i}) \ge \int_{\frac{\hat{y}_{k_i}}{2}}^{\hat{y}_{k_i}} (u_{k_i})_y dy \ge \frac{\hat{y}_{k_i}}{2}(\hat{u}_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right) \ge \frac{\tilde{\varepsilon}}{2}(\hat{u}_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right).$$

Hence $(u_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right) \to 0$ as $i \to \infty$. We get $(\hat{u}_{k_i})_y \to 0$ as $i \to \infty$ uniformly on $\left[0, \frac{\tilde{\varepsilon}}{2}\right] \subseteq \left[0, \frac{\hat{y}_{k_i}}{2}\right]$. Since $u_{k_i} \to 0$ and $(u_{k_i})_y \to 0$ uniformly on $\left[0, \frac{\tilde{\varepsilon}}{2}\right]$ as $i \to \infty$, (3.7) implies that $(u_{k_i})_{yy} \to \infty$ uniformly on $\left[0, \frac{\tilde{\varepsilon}}{2}\right]$ as $i \to \infty$. But

$$(u_{k_i})_y\left(\frac{\tilde{\varepsilon}}{2}\right) = (u_{k_i})_y(0) + \int_0^{\frac{\tilde{\varepsilon}}{2}} (u_{k_i})_{yy}\, dy \geq \frac{\tilde{\varepsilon}}{2} \min_{[0,\frac{\tilde{\varepsilon}}{2}]} (u_{k_i})_{yy}.$$

□

Finally we need a stability result for (3.7) in order to finish the proof.

Lemma 3.5. *The asymptotic angle of solutions of (3.7), $\alpha = \alpha(C, K) : (0, \infty) \times [0, \infty) \to \left(0, \frac{\pi}{2}\right)$, is a continuous function of the initial conditions $C > 0$ and $K \geq 0$.*

Proof. We show that for a sequence of solutions u_k of (3.7) with converging parameters C_k and K_k, we have uniform bounds on $(u_k)_y$ and $\left|(u_k)_{yy}\right|$. From these we can deduce an appropriate bound on $\left(\frac{u_k}{y}\right)_y$ which shows uniform (in $k \in \mathbb{N}$) convergence of $\frac{u_k}{y}$ towards $\tan \alpha_k$ as $y \to \infty$. This uniform convergence is then enough to make a contradiction argument work.

1. Uniform bounds on $(u_k)_y$ and $\left|(u_k)_{yy}\right|$

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions of (3.7) with $(C_k, K_k) \to (C_0, K_0)$ as $k \to \infty$, where C_0 and K_0 are associated to a solution u_0 of (3.7). We denote the asymptotic angles by α_k and α_0. In view of vi we assume that $(u_0)_{yy}$ has precisely one zero $\hat{y}_0$. The case $(u_0)_{yy} > 0$ on $[0, \infty)$ can be handled exactly in the same way. Since we have continuous dependence of u_k and $(u_k)_y$ on the initial conditions on compact intervals we may assume in view of the equation (3.7) that all (C_k, K_k) are close enough to (C_0, K_0), so that each $(u_k)_{yy}$ has precisely one zero $\hat{y}_k$ close to $\hat{y}_0$. So there is a compact interval J, containing all $\hat{y}_k$ and $\hat{y}_0$. Due to continuous dependence we may also assume that $(u_k)_y$ is bounded independently of $k \in \mathbb{N}$ on J. Since all $\hat{y}_k$ lie in J, v shows therefore that $(u_k)_y$ is uniformly bounded everywhere. So we have

$$0 < \sup_{y\geq 0} (u_k)_y \leq D \tag{3.18}$$

for all $k \in \mathbb{N}$, $D > 0$ a constant. In the following we will adjust the constant $D > 0$ as necessary without explicitly mentioning it.
The bound on $\left|(u_k)_y\right|$ and ii imply that the right hand side of (3.7) is bounded on J independently of $k \in \mathbb{N}$. Therefore $(u_k)_{yy}$ is bounded on J and since the zeros of $(u_k)_{yy}$ all lie in J, we get with v

$$\sup_{y\geq 0} (u_k)_{yy} \leq D$$

for all $k \in \mathbb{N}$. Differentiating (3.7) with respect to $y > 0$, setting $u_y^{(3)} = 0$ and employing ii yields

$$(u_k)_{yy} = \frac{-2(n-1)\,(u_k)_y}{y\,(u_k)^2} + \frac{4\,(u_k)_y\left((u_k)_{yy}\right)^2}{y\left(1+\left((u_k)_y\right)^2\right)^2} \geq \frac{-2(n-1)\,(u_k)_y}{y\,(u_k)^2}. \tag{3.19}$$

By ii and the fact that all C_k are close to $C_0 > 0$ we have

$$\frac{1}{D} \leq \inf_{y \geq 0} u_k \tag{3.20}$$

for all $k \in \mathbb{N}$. Since $\left|(u_k)_y\right|$ is bounded independently of $k \in \mathbb{N}$ and the zeros of $(u_k)_{yy}$ lie close to $\hat{y}_0 > 0$ and $(u_k)_{yy}(0) > 0$, (3.19) implies that $(u_k)_{yy}$ can not have arbitrarily small local extrema. Therefore $(u_k)_{yy}$ must be bounded uniformly below. Together with our upper bound on $(u_k)_{yy}$ we get

$$\sup_{y \geq 0} \left|(u_k)_{yy}\right| \leq D \tag{3.21}$$

for all $k \in \mathbb{N}$.

2. A bound on $\left(\frac{u_k}{y}\right)_y$

With (3.18), (3.20) and (3.21) we can estimate for $y > 0$ large enough, using (3.7)

$$\left(\frac{u_k}{y}\right)_y = \frac{1}{y}\left((u_k)_y - \frac{u_k}{y}\right) =$$
$$\frac{2}{y^2}\left(\frac{n-1}{u_k} - \frac{(u_k)_{yy}}{1+(u_y)^2}\right) \leq \frac{D}{y^2}.$$

So by iii for any $\delta > 0$ there exists $y_\delta > 0$ with

$$\left|\frac{u_k}{y} - \tan\alpha_k\right| < \delta, \quad y \geq y_\delta,\ k \in \mathbb{N}. \tag{3.22}$$

3. A contradiction argument

Assume not, i.e. without loss of generality there is $\tilde{\varepsilon} > 0$ such that

$$|\tan\alpha_k - \tan\alpha_0| \geq \tilde{\varepsilon} \tag{3.23}$$

holds for all $k \in \mathbb{N}$.

Using (3.22) we can choose y_δ large enough such that

$$\left|\frac{u_k}{y} - \tan\alpha_k\right| < \delta$$
$$\left|\frac{u_0}{y} - \tan\alpha_0\right| < \delta$$

for $y \geq y_\delta$ and $k \in \mathbb{N}$. The triangle inequality yields for small enough $\delta > 0$

$$\left|\frac{u_k}{y} - \frac{u_0}{y}\right| \geq \frac{\tilde{\varepsilon}}{2}, \quad y \geq y_\delta,\ k \in \mathbb{N}.$$

We can write the difference of the ordinary differential equations for u_0 and u_k at $y > 0$ as $\xi_1 = \xi_2 + \xi_3 + \xi_4$, where

$$\xi_1 = \frac{1}{y}\left(u_0 - u_k\right)$$
$$\xi_2 = \left(u_0\right)_y - \left(u_k\right)_y$$
$$\xi_3 = \frac{2}{y}\left(\frac{\left(u_0\right)_{yy}}{1 + \left(u_0\right)_y^2} - \frac{\left(u_k\right)_{yy}}{1 + \left(u_k\right)_y^2}\right)$$
$$\xi_4 = \frac{2}{y}\left(\frac{n-1}{u_k} - \frac{n-1}{u_0}\right).$$

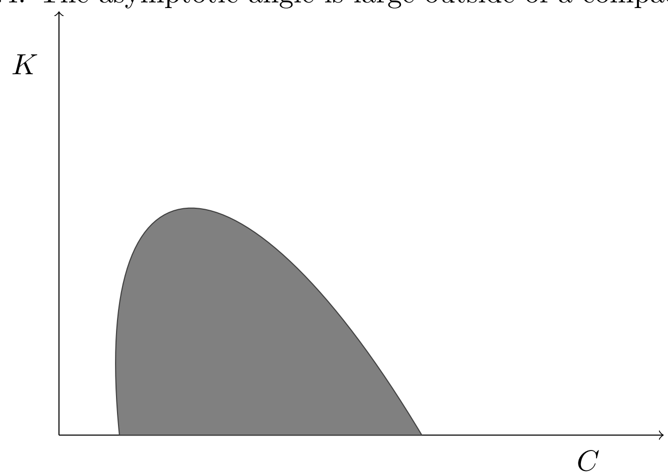
Using (3.18), (3.20) and (3.21) we can find $\tilde{y} > y_\delta$ such that $|\xi_3| < \frac{\tilde{\varepsilon}}{6}$ and $|\xi_4| < \frac{\tilde{\varepsilon}}{6}$ for any $k \in \mathbb{N}$. By continuous dependence we can find $\tilde{k} \in \mathbb{N}$ large enough such that $|\xi_2| < \frac{\tilde{\varepsilon}}{6}$ at $\tilde{y}$. Therefore we get $|\xi_2 + \xi_3 + \xi_4| < \frac{\tilde{\varepsilon}}{2}$ - a contradiction to our previous estimate for ξ_1.

□

Proof of theorem 3.1. The last three lemmas imply that the continuous function $\alpha = \alpha(C, K) : (0, \infty) \times [0, \infty) \to \left(0, \frac{\pi}{2}\right)$ attains its minimum at some $(C_{crit}, K_{crit}) \in (0, \infty) \times [0, \infty)$, since it must be large outside of a compact region (see figure 3.4). The solution u of (3.7) with initial conditions C_{crit}, K_{crit} must have asymptotic angle α^*_{crit}. Continuity and lemma 3.3 imply that for $\alpha > \alpha^*_{crit}$ there are at least two one-component, rotationally symmetric, self-expanding evolutions of D_α and that there is at least one such evolution of $D_{\alpha^*_{crit}}$. As stated in lemma 3.2 the existence of a two-sheeted, rotationally symmetric, self-expanding evolution of D_α (for any cone angle $\alpha \in \left(0, \frac{\pi}{2}\right)$) is asserted using results of K. Ecker and G. Huisken (see [Ecker & Huisken, 1989, theorem 5.1]), which ensure existence of self-expanding evolutions of the two Lipschitz graphs C_α and $C_{\pi-\alpha}$.

$\alpha^*_{crit} < \frac{\pi}{2}$ is a direct consequence of iv and $\alpha^*_{crit} > 0$ is proved in [Angenent et al. , 1995, lemma 3]. By our definition of α^*_{crit} we have uniqueness of rotationally symmetric evolutions of D_α for $\alpha < \alpha^*_{crit}$. Additionally one can show that the level-set flow (see appendix A and especially definition A.7) of D_α is rotationally symmetric with a smooth boundary, which implies uniqueness among all evolutions of D_α for $\alpha < \alpha^*_{crit}$ (see [Angenent et al. , 1995, theorem 4] for a proof, a similar argument occurs in the proof of lemma 3.17). This finishes the proof of theorem 3.1.

Figure 3.4: The asymptotic angle is large outside of a compact region



The asymptotic angle is large outside of a compact region (marked gray in the picture). Therefore the continuous angle function must have a global minimum.

□

Remark 3.6. *We can only prove a suitable comparison principle for solutions of (3.7) for* $C \geq \sqrt{2(n-1)}$. *But we believe that there are no more than two one-component, rotationally symmetric, self-expanding evolutions of* D_α *for* $\alpha > \alpha^*_{crit}$ *(both of these evolutions must then correspond to symmetric solutions of (3.7), i.e.* $K = 0$*). Additionally there might be non-rotationally symmetric evolutions of* D_α *(see Angenent* et al. *[1995]).*

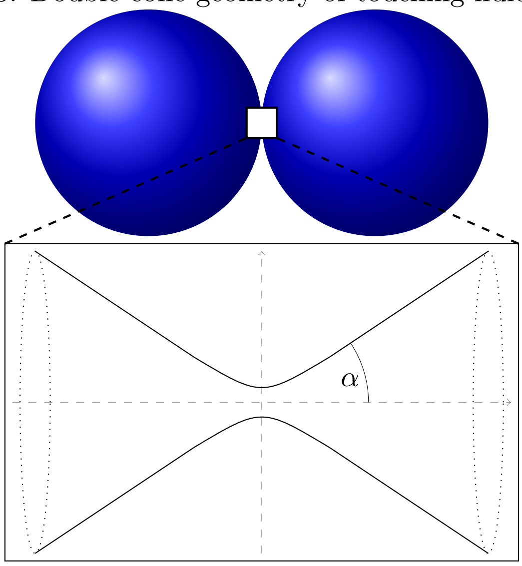
3.2 A model for the behaviour of oppositely charged fluid droplets

Droplet motion induced by electrical charges has been extensively studied at least since the 19th century (see e.g. Rayleigh [1878]). Such a process occurs in a wide variety of applications, including storm cloud formation, commercial ink-jet printing, petroleum and vegetable oil dehydration, electrospray ionization for use in mass spectrometry, electrowetting and lab-on-a-chip manipulations (see Trau *et al.* [1997]; Leunissen *et al.* [2005]; Baygents & Saville [1991]; Ochs & Czys [1987]; Eow *et al.* [2001]; Fenn *et al.* [1989]; Baret & Mugele [2006]; Link *et al.* [2006]).

Nevertheless the first experiments studying the behavior of two oppositely charged fluid droplets after touching are documented in Ristenpart *et al.* [2009] and Bird *et al.* [2009]. If two oppositely charged droplets of fluid are close enough, they start attracting each other. The electric field causes the droplets to develop conical tips, commonly revered to as Taylor cones (see Fernández De La Mora [2007]; Saville [1997]; Melcher & Taylor [1969]; Eggers *et al.* [1999]; Oddershede & Nagel [2000]; Collins *et al.* [2007]).

Eventually the droplets touch and one would expect them to merge into one big drop. Surprisingly though, if the strength of the charges is higher than some critical value, the droplets break apart after making short contact (repulsion). Otherwise the droplets indeed form one big drop (coalescence). Looking closely at the bridge between the touching fluid droplets reveals a double cone geometry (see 3.5). The angle α of the double cone corresponds to the field strength (a higher field strength corresponds to a lower angle α) and therefore determines the behaviour of the system. So according to experiments there is a critical angle α_{crit} such that we get coalescence if $\alpha > \alpha_{crit}$ and repulsion if $\alpha < \alpha_{crit}$.

Figure 3.5: Double cone geometry of touching fluid droplets



Zooming in on the touching fluid droplets reveals a double cone geometry.

Following an idea by P. Topping, we present a theoretical model for this phenomenon. It is assumed that after the two droplets have touched and exchanged their charges, the motion of the system is driven by minimisation of energy. To model this mathematically we use the mean curvature flow which is the gradient flow of the surface area (see e.g. Ecker [2004]). Our model yields a prediction for α_{crit}, which is in good agreement with experiments. Furthermore we point out several advantages of our theoretical model, compared to the ones in Ristenpart *et al.* [2009] and Bird *et al.* [2009].

3.2.1 Touching fluid droplets and mean curvature flow

We want to study the behaviour of touching fluid droplets. These are assumed to have locally conical, rotationally symmetric shape, i.e. the shape of a smoothing of D_α (see figure 3.5). The following definition makes this formulation precise. We use the notation from (3.4) for rotationally symmetric hypersurfaces and from now on we set the dimension to $n+1=3$.

Definition 3.7. *We call $M_\alpha = M(u_\alpha)$ a smoothing of the double cone D_α with angle $0 < \alpha < \frac{\pi}{2}$, $\gamma = \tan\alpha$, if for $a < 0 < b$*

$$u_\alpha(y) = \begin{cases} \gamma\,|y| & \text{if } y \le a \text{ or } b \le y \\ s(y) & \text{if } a \le y \le b \end{cases}$$

such that $s > 0$ and $u_\alpha \in C^{2,\beta}(\mathbb{R})$, $\beta > 0$. M_α is said to lie outside of the double cone D_α if $s(y) \ge \gamma\,|y|$ for $a < y < b$. u is the generating function of M_α.

Since the mean curvature vector inherits rotational symmetry, any smoothing of a double cone stays rotationally symmetric under mean curvature flow. One can compute (see Simon [1990]) that the rotationally symmetric mean curvature evolution M_t of M_α is satisfies

$$\frac{\partial}{\partial t}u = \frac{\frac{\partial^2}{\partial^2 y}u}{1+\left(\frac{\partial}{\partial y}u\right)^2} - \frac{1}{u} \tag{3.24}$$

where $u = u(\cdot,t),\ u(\cdot,0) = u_\alpha$ generates M_t.

For any smoothing M_α we have short-time existence of a solution M_t of (3.1) on a maximal time interval $[0,T),\ T > 0$. Furthermore the solution must be smooth for $t > 0$ and every finite time singularity must be due to pinching, i.e. $\inf_{\mathbb{R}} u(\cdot,t) \to 0$ as $t \to T$. This holds even without any growth assumption on the initial generating function (see Simon [1990] and Ladyženskaja *et al.* [1968]).

In fact a sphere comparison argument (see Ecker [2004]) shows that

$$\lim_{t\to T} \min_{\mathbb{R}} u(\cdot,t) \to 0 \text{ as } t \to T$$

must hold for finite time singularities.

This agrees with intuition about repulsion (pinching in finite time) and coalescence (long-time existence) of fluid droplets.

Using techniques by K. Ecker and G. Huisken (see Ecker & Huisken [1989]) one can derive global height estimates for rotationally symmetric cylindrical graphs evolving by mean curvature flow, which yield, using the results from Barles *et al.* [2003], the following comparison principle (see [Bode, 2007, theorem E.7]):

Lemma 3.8. *Let M_{α_1}, M_{α_2} be two smoothings of the double cone with $u_{\alpha_1} \le u_{\alpha_2}$ for the associated generating functions. Denoting the generating functions of the two evolutions with u^1 and u^2 we have $u^1 \le u^2$ as long as the solutions exist.*

Corollary 3.9. *Any smoothing of the double cone M_α, $0 < \alpha < \frac{\pi}{2}$ has a unique evolution by mean curvature.*

Using these results we can now define what coalescence and repulsion mean within our model.

Definition 3.10. *An angle $0 < \alpha < \frac{\pi}{2}$ is called a repulsion angle if there exists a smoothing of the double cone M_α which is outside of the double cone and such that the mean curvature flow evolution of M_α pinches in finite time.*

Lemma 3.11. *An angle $0 < \alpha < \frac{\pi}{2}$ is not a repulsion angle if and only if there is a smoothing of the cone M_α, for which the evolution under mean curvature flow exists for all $t > 0$.*

Proof. By definition any angle $0 < \alpha < \frac{\pi}{2}$, that is not a repulsion angle, must have a smoothing M_α for which the evolution under mean curvature flow M_t exists for all $t > 0$.

So suppose for $0 < \alpha < \frac{\pi}{2}$ there exists a smoothing M_α, such that the evolution M_t exists for all $t > 0$. Let $\hat{M}_\alpha$ be another smoothing, that is outside of the double cone. We denote its evolution by $\hat{M}_t,\ t \in [0, T)$.

We know that the mean curvature flow is invariant under parabolic rescaling

$$x \mapsto \lambda x,\ t \mapsto \lambda^2 t$$

for any scaling parameter $\lambda > 0$, $x \in M_t$ and $t \in I$. Let M_t^λ be the rescaling of M_t. Note here that any double cone D_α is invariant under the scaling $x \mapsto \lambda x$. Since

$\hat{M}_\alpha$ is outside of the double cone we can therefore choose $\lambda > 0$ sufficiently small in order to get initially

$$u_\alpha^\lambda \leq \hat{u}_\alpha$$

for the corresponding generating functions. By lemma 3.8 $\hat{M}_t$ must exist for all $t > 0$, therefore α is not a repulsion angle. □

In view of the last lemma we make the following definition:

Definition 3.12. *An angle $0 < \alpha < \frac{\pi}{2}$ is called a coalescence angle if it is not a repulsion angle in the sense of definition 3.10.*

Given a field strength (respectively a double cone angle) at which coalescence occurs, any lower field strength (respectively greater cone angle) should lead to coalescence as well. The same should hold for repulsion angles with higher field strength. The next two lemmas shows that this is true within our model.

Lemma 3.13. *Let α_0 be a coalescence angle. Then for any angle $\alpha \geq \alpha_0$ and for any smoothing M_α that is outside of the double cone the mean curvature flow with initial data M_α exists for all $t > 0$. Therefore any angle $\alpha > \alpha_0$ is a coalescence angle.*

Proof. As in the proof of lemma 3.11 we can scale appropriately and then use the comparison principle from lemma 3.8. □

Lemma 3.14. *Let α_0 be a repulsion angle. Then for $\alpha \leq \alpha_0$ any smoothing of the double cone M_α with angle α must pinch in finite time. This means any angle $\alpha < \alpha_0$ must be a repulsion angle.*

Proof. Again, as for lemma 3.11 we can scale appropriately and then use lemma 3.8. □

The observations from experiments in Ristenpart *et al.* [2009] suggest the existence of a critical angle for the behaviour of the system. As the next lemma shows, this is also true for our model.

Lemma 3.15. *There is a critical angle $0 \leq \alpha_{crit} \leq \frac{\pi}{2}$ such that any angle $\alpha < \alpha_{crit}$ is a repulsion angle and any angle $\alpha > \alpha_{crit}$ is a coalescence angle.*

Proof. Let

$$\overline{\alpha} = \sup\left\{0 < \alpha < \frac{\pi}{2} : \alpha \text{ is a repulsion angle}\right\}$$
$$\underline{\alpha} = \inf\left\{0 < \alpha < \frac{\pi}{2} : \alpha \text{ is a coalescence angle}\right\}.$$

According to lemma 3.14 any cone angle $\alpha < \overline{\alpha}$ is not a coalescence angle. This shows that $\overline{\alpha} \leq \underline{\alpha}$. Clearly any angle is either a repulsion or a coalescence angle, hence we have $\overline{\alpha} = \underline{\alpha} = \alpha_{crit}$. □

We can now determine α_{crit}. Remember the significance of α^*_{crit} from theorem 3.1: it is the critical angle for the existence of rotationally symmetric self-expanding evolutions of D_α with one connected component. We will use these self-expanders as barriers.

Determining α_{crit}

Lemma 3.16. *Any double cone smoothing with angle $\alpha > \alpha^*_{crit}(3)$, that lies outside of the double cone has an evolution which exists for all $t > 0$.*

Proof. Let M_α be a smoothing of D_α with $\alpha > \alpha^*_{crit}(3)$ and generating function u_α. Let s be the generating function of the self-expander asymptotic to $D_{\alpha^*_{crit}(3)}$ which exists by theorem 3.1. As in the proof of lemma 3.11 we can do a parabolic rescaling and therefore assume $s \leq u_\alpha$. Then the evolution of M_α must exist for all $t > 0$ by the comparison principle, lemma 3.8, since every finite time singularity must be due to pinching. □

Lemma 3.17. *Any double cone smoothing with angle $\alpha < \alpha^*_{crit}(3)$ must pinch in finite time.*

Proof. We follow a level-set flow argument from Angenent *et al.* [1995].

Assume not, i.e. there is a double cone smoothing M_α, $\alpha < \alpha^*_{crit}(3)$, such that its evolution by mean curvature M_t exists for all $t > 0$.

Let Γ_t , $t \geq 0$, be the level-set flow of D_α (see appendix A and especially definition A.7). Γ_t can be characterised as follows (see lemma A.11). $\mathbb{R}^3 \backslash \Gamma_t$ is the union of all level-set flows Δ_t such that Δ_0 is compact and lies in $\mathbb{R}^3 \backslash D_\alpha$.

First we show that $0 \in \Gamma_t$. So let Δ_t be a level-set flow with $\Delta_0 \subset \mathbb{R}^3 \backslash D_\alpha$ compact and connected. Δ_0 must either lie in the convex hull of one of C_α or $C_{\pi-\alpha}$ or outside of D_α. Using the maximum principle for one level-set flow and one smooth flow, we see that in the first case Δ_0 is pushed away from 0 by the graphical self-expanders of Ecker and Huisken (part (i) of lemma 3.2) and therefore $0 \notin \Delta_t$. In the second case we can parabolically rescale M_α as in the proof of lemma 3.11 and therefore assume $\Delta_0 \cap M_\alpha = \emptyset$. Then we can apply the maximum principle to see that $0 \notin \Delta_t$. With the same argument (possibly using a union of the graphical self-expanders and M_α) we can also show $0 \notin \Delta_t$, if Δ_0 has more than one connected component. Hence by the above characterization we must have $0 \in \Gamma_t$.

Γ_1 is rotationally symmetric, so $\Gamma_1 = M(X)$ for some closed set $X \subset \mathbb{R}^2$. In fact the boundary $\partial\Gamma_1$ is smooth and ∂X consists precisely of curves of type (i) and (ii) in lemma 3.2 (see [Angenent *et al.* , 1995, proof of theorem 4]).

Since $0 \in \Gamma_1$ there must be a curve of type (ii) in lemma 3.2 in ∂X, which is asymptotic to $\sigma_\alpha \cup \sigma_{\pi-\alpha}$. This yields a smooth, rotationally symmetric, one-sheeted self-expanding evolution of D_α - a contradiction. □

In view of the definition of α_{crit} the last two lemmas yield the following

Corollary 3.18. $\alpha_{crit} = \alpha^*_{crit}(3)$.

3.2.2 Conclusions

We can summarize our results, obtained by modelling the behaviour of oppositely charged droplets of fluid with the mean curvature flow, as follows:

Assume that two initially oppositely charged droplets of fluid after touching have the local shape of a smoothing of a double cone D_α in $\mathbb{R}^3$. Also assume that their motion is governed by minimisation of area, which we model using the mean curvature flow.

Then there is a critical angle α_{crit} with the following properties. If $\alpha < \alpha_{crit}$ the droplets are repelled from each other. If the associated smoothing lies outside of D_α and $\alpha > \alpha_{crit}$ the droplets coalesce and form one big drop.

Using appropriate barriers and a level-set flow argument we can conclude that α_{crit} is precisely the critical angle for the existence of one-component, self-expanding evolutions of $D_\alpha \subset \mathbb{R}^3$, which one can easily determine numerically as $\alpha_{crit} \approx 66°$ (see also Angenent et al. *[1995]).*

This coincides with observations from experiments (see Ristenpart *et al.* [2009]), which predict a critical angle of $60° - 70°$. This shows especially, that, contrary to prior belief (see Bird *et al.* [2009]), minimisation of energy can explain the phenomenon of coalescence and repulsion. One can also interpret this result as a way to choose from the self-expanding evolutions of D_α, in case $\alpha \geq \alpha_{crit}$: the solutions with one connected component correspond to the physical behaviour of the touching droplets.

Finally we want to compare our model with the one in Bird *et al.* [2009]. For that approach it is assumed that the bridge between the touching droplets minimizes area under a volume constraint. This corresponds to constant mean curvature surfaces of revolution (Delaunay surfaces) which are fitted to linear double cones, similarly as in definition 3.7 (unlike in definition 3.7 the associated generating function is only continuous). The associated capillary pressure p is assumed to determine the behaviour.

The critical shape has $p = 0$ which yields a rescaled catenoid. Suitable smoothings of the associated surfaces with $p > 0$ provide double cone smoothings in the sense of definition 3.7, which pinch in finite time. This is in agreement with Bird *et al.* [2009]. However lemma 3.17 shows that down to a certain $p_0 < 0$ the (smoothings of the) associated surfaces with $p \leq 0$ still pinch in finite time under mean curvature flow. Therefore our predicted critical angle is greater than the predicted critical angle from Bird *et al.* [2009], which is approximately 59°.

Our model has the advantage that we do not make assumptions on the precise shape of the bridge between the touching fluid droplets (apart from it being outside of the linear double cone). Also fitting the Delaunay surfaces to linear double cones in a continuous way can be done in several ways, which yield different predictions for α_{crit}. Therefore, unlike in our approach, the prediction here also depends on parameters within the theoretical model.

Chapter 4

The differential Harnack estimate for the mean curvature flow from a geometric viewpoint

We consider solutions of the mean curvature flow, i.e. families of smoothly immersed hypersurfaces $(M_t)_{t\in[0,T]} \subset \mathbb{R}^{n+1}$, $M_t = F_t(M^n)$, satisfying

$$\frac{\partial}{\partial t} F_t = -H\nu. \tag{4.1}$$

The differential Harnack estimate for the mean curvature flow was first proved by R. Hamilton (see Hamilton [1995b]). In the compact case it can be stated as follows:

Theorem 4.1. *Let $(M_t)_{t\in[0,T]}$ be a compact solution of (4.1) such that M_0 is convex. Then all M_t are convex and satisfy for $0 < t \leq T$*

$$Z(V,V) = \frac{\partial}{\partial t} H + 2\langle \nabla H, V\rangle + h(V,V) + \frac{H}{2t} \geq 0 \tag{4.2}$$

for any tangent vector V.

One can integrate this estimate along geodesics to obtain a classical Harnack inequality for the mean curvature along solutions of (4.1) (see also Hamilton [1995b]):

Corollary 4.2. *Under the assumptions of theorem 4.1 we have for $0 < t_1 < t_2 < T$, $x_1 \in M_{t_1}$, $x_2 \in M_{t_2}$*

$$H(x_2, t_2) \geq \sqrt{\frac{t_1}{t_2}} e^{-\frac{\Delta}{4}} H(x_1, t_1)$$

where the squared space-time distance $\Delta = \inf_\gamma \int_{t_1}^{t_2} \left|\frac{d}{dt}\gamma\right|_g^2 dt$ *is the infimum over all* C^1*-paths* $\gamma(t)$ *on* M_t *with* $\gamma(t_1) = x_1$ *and* $\gamma(t_2) = x_2$*.*

It was first proved by G. Huisken that convexity is preserved along the flow and that a convex compact solution shrinks to a point in finite time (see Huisken [1984]), in this case we denote the singular time by $T_{\max}$. The original proof of theorem 4.1 uses a tensor maximum principle type argument, which requires a great deal of computations. The original method for finding such a quantity is to look at expressions that vanish on self-expanding solutions of (4.1) and then to try to combine these in an appropriate way. While this method has proved to be effective, it does not give the desired geometric insight into the Harnack expression. Such an insight can be gained by considering appropriate space-time constructions for geometric flows.

The pioneering work in the direction of space-time constructions related to the Harnack quantity for the Ricci flow and for the mean curvature flow was done by B. Chow and S. Chu (see Chow & Chu [1995, 2001]). They managed to show that $Z(V,V) - \frac{H}{2t}$ - an expression which is constant along translating solitons [1] - approximately corresponds to the second fundamental form of an approximate translating space-time soliton and they obtained a similar result for the Ricci flow with a steady space-time soliton. The approximation here is with respect to a parameter $N > 0$ in the construction of the space-time solitons. One can take the limit as $N \to \infty$ of the inverse metric (the limit is degenerate) and of the second fundamental form, where the latter is exactly $Z(V,V) - \frac{H}{2t}$. B. Chow and D. Knopf developed this idea further for the Ricci flow and managed to obtain new Li-Yau-Hamilton inequalities in this way(see Chow & Knopf [2002]).

It was later on realised by E. Cabezas-Rivas and P. Topping in Cabezas-Rivas & Topping [2012a] that it is advantageous to consider an expanding space-time Ricci soliton in order to understand the Harnack quantity. It turned out that R. Hamilton's full matrix Harnack estimate corresponds [2] to this so called canonical soliton having nonnegative curvature operator. In fact with this approach all known Harnack quantities for the Ricci flow were recovered respectively discovered (see Hamilton [1993]; Brendle [2009]; Cabezas-Rivas & Topping [2012a]), they correspond to the curvature operator of the canonical self-expander being in appropriate cones.

B. Kotschwar has adapted the idea of using expanding respectively shrinking instead of translating approximate solitons to curvature flows (see Kotschwar [2009]). He was able to recover the Harnack quantities for a large class of curvature

[1] we have $M_t = M_0 + tX$, $t \in \mathbb{R}$ and $X \in \mathbb{R}^{n+1}$, for translating solitons of (4.1)

[2] again up to an approximation in the construction

flows, which are due to B. Andrews (see Andrews [1994]), as the limit as $N \to \infty$ of the second fundamental form of these space-time solitons, which we call canonical solitons in view of the Ricci flow analogues. Hence the Harnack estimates for curvature flows correspond to the canonical solitons being convex. In the case of the mean curvature flow the canonical soliton is a self-expander. B. Kotschwar also managed to give a proof of Hamilton's Harnack estimate for the mean curvature flow, using a generalised tensor maximum principle for the second fundamental form of the canonical self-expander.

In this chapter we provide a new method of showing that the canonical self-expander for the mean curvature flow is convex in the case of a compact convex solution. The canonical self-expander is asymptotic to a convex cone over M_0 in $\mathbb{R}^{n+2}$, which becomes steep in the approximation process as N becomes large. The canonical self-expander can therefore be considered an approximate mean curvature flow evolution of such a convex cone and should be convex itself.

To make this idea work, we first consider (in section 4.1) exact graphical self-expanding evolutions of the cone over M_0, which exist due to results from graphical mean curvature flow (see Ecker & Huisken [1989]). With the help of level-set flow theory we show that an appropriate limit of these self-expanding evolutions as $N \to \infty$ is given by an expanded space-time track of the solution. In section 4.2 we then review the link between the convexity of the canonical self-expanders, which are also constructed as expanded space-time tracks, and theorem 4.1. Finally in section 4.3 we put things together and show that preservation of convexity directly implies convexity of the canonical self-expander.

4.1 Mean curvature flow of space-time cones

Let $(M_t)_{t\in[0,T_{\max})}$ be a compact convex solution of (4.1). The cone C_N over M_0 for $N > 0$ with the tip (which we take without loss of generality to be at the origin) in the interior of M_0 can be written as an entire graph of a Lipschitz function f_N defined on the plane $\left\{y \in \mathbb{R}^{n+1} : \langle y, \omega\rangle = 0\right\}$, where $\omega = -\frac{\partial}{\partial t}$:

$$C_N = \{t\,(x, N) : x \in M_0, t \in [0, \infty)\} = \text{graph } f_N.$$

The cone C_N becomes more and more steep as N becomes large and the Lipschitz constant of f_N is bounded by CN, $C = C\,(M_0) > 0$.

C_N is an entire Lipschitz graph, so according to the work of K. Ecker and G. Huisken (see Ecker & Huisken [1989]) there is a graphical self-expanding mean curvature evolution of C_N (see figure 4.1). More precisely we have:

Lemma 4.3. *For any $N > 0$ there exists a smooth self-expanding graphical solution $\tilde{\Sigma}_N =$ graph $\tilde{v}_N$ of (4.1), which is asymptotic to C_N and satisfies* $\min_{y\in\mathbb{R}^{n+1}} \tilde{v}_N(y) \leq CN$ *and* $|D\tilde{v}_N(y)| \leq CN$ *for* $y \in \mathbb{R}^{n+1}$ *and* $C = C(M_0)$.

Proof. We apply a convergence result for graphs by K. Ecker and G. Huisken from Ecker & Huisken [1989]. [Ecker & Huisken, 1989, theorem 5.1] says that there is a smooth graphical solution of the mean curvature flow with C_N as an initial condition and moreover that we have (after rescaling with $\frac{1}{\sqrt{2t+1}}, t \geq 0$) convergence of this solution to our desired self-expander $\tilde{\Sigma}_N$. In order to apply the theorem we need to check two conditions:

- The gradient function of C_N satisfies $\upsilon^{C_N} = \frac{1}{\langle \nu^{C_N}, \omega\rangle} \leq CN$ for a constant $C(M_0) > 0$.

  Proof. The normal vector of C_N at a point $z = t(x, N) \neq 0$, $x \in M_0$, is

  $$\nu^{C_N} = \frac{\left(\nu^{M_0}, -\frac{\langle x, \nu^{M_0}\rangle}{N}\right)}{\sqrt{1 + N^{-2}\langle x, \nu^{M_0}\rangle^2}}.$$

  So we have (away from 0)

  $$\left\langle \nu^{C_N}, \omega\right\rangle = \frac{1}{\sqrt{1 + N^{-2}\langle x, v\rangle^2}N}\left\langle x, \nu^{M_0}\right\rangle \geq \frac{C(M_0)}{N}$$

  since M_0 is convex. □

- C_N is straight at infinity, i.e. $\left\langle z, \nu^{C_N}\right\rangle \leq \left(1 + |z|^2\right)$ for $z \in C_N$.

  Proof. We have for $z \in C_N$ (away from 0)

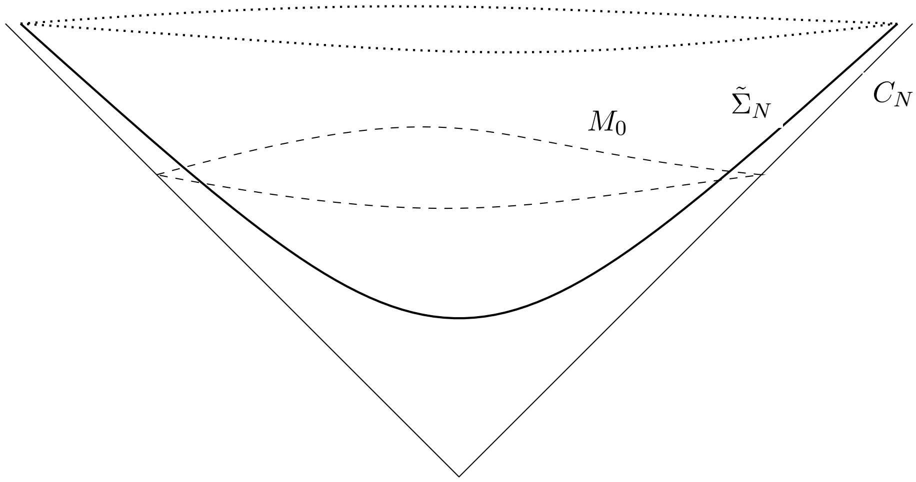
  $$\left\langle z, \nu^{C_N}\right\rangle = 0.$$

  □

Now [Ecker & Huisken, 1989, theorem 5.1] shows the existence of a graphical self-expander $\tilde{\Sigma}_N$, asymptotic to C_N, for each $N > 0$. The derivative bound follows from our bound on the gradient function for C_N and the fact that the gradient function along the flow remains bounded by its maximum at the initial time (see [Ecker & Huisken, 1989, corollary 3.2]). The C^0-bound on $\tilde{v}_N$ can be obtained by placing spheres of radius $\rho = \frac{h}{N} d(M_0)$, $d(M_0) > 0$, within C_N at a given height $h > 0$ [3] such that the spheres are disjoint from C_N. These spheres have extinction time $\frac{\rho^2}{2(n+2)}$. Hence they act as barriers, showing that the rescaled evolution of C_N can never go entirely beyond the height CN, $C = C(M_0)$.

$\square$

Figure 4.1: The graphical self-expander $\tilde{\Sigma}_N$



$\tilde{\Sigma}_N$ is an exact self-expanding solution of (4.1). The time slices (respectively level sets) are approximate solutions of (4.1).

Since $\tilde{\Sigma}_N =$ graph $\tilde{v}_N$ becomes degenerate as $N \to \infty$, we squash it down by defining

$$v_N = \frac{1}{N} \tilde{v}_N. \tag{4.3}$$

Now we can take the limit of the graphs v_N and level-set flow theory gives us a precise notion of what the limit is.

Proposition 4.4. *A subsequence $(v_{N_n})_{n \in \mathbb{N}}$ converges in C^0 to a continuous limit v_∞. The graph of v_∞ is the expanded space-time track*

$$\left\{ t^{-\frac{1}{2}} (x, 1) : \; x \in M_t, t \in (0, T_{\max}] \right\}. \tag{4.4}$$

Proof. Lemma 4.3 gives a suitable derivative bound for $\tilde{v}_N$ in order to have a uniform bound on $|Dv_N|$, independently of N. Together with the C^0-bound on $\tilde{v}_N$ from lemma 4.3, this implies that there is a subsequence $(v_{N_n})_{n \in \mathbb{N}}$ for which $N_n \to \infty$ as $n \to \infty$ and which converges in $C^0 \left(\mathbb{R}^{n+1} \right)$ to a continuous limit v_∞.

[3] points $\frac{h}{N}(x, N) \in C_N$ are meant to have height h

Since each $\tilde{v}_{N_n}$ corresponds up to tangential diffeomorphisms to a solution of the graphical mean curvature flow equation with respect to a new time variable $s \in (0, \infty)$ [4]

$$\frac{\partial}{\partial s}\tilde{v}_{N_n} = \sqrt{1 + |D\tilde{v}_{N_n}|^2}\ \mathrm{div}\ \left(\frac{D\tilde{v}_{N_n}}{\sqrt{1 + |D\tilde{v}_{N_n}|^2}}\right)$$

we see that v_{N_n} solves an approximation to the level-set flow equation

$$\frac{\partial}{\partial s}v_{N_n} = \sqrt{\frac{1}{N_n^2} + |Dv_{N_n}|^2}\ \mathrm{div}\ \left(\frac{Dv_{N_n}}{\sqrt{\frac{1}{N_n^2} + |Dv_{N_n}|^2}}\right).$$

We can now apply an approximation lemma (see lemma A.3) in order to show that the limit v_∞ [5] corresponds to a weak solution of the level-set flow equation (see Evans & Spruck [1991]; Chen *et al.* [1991])

$$\frac{\partial}{\partial s}v_\infty = \left(I - \frac{Dv_\infty \otimes Dv_\infty}{|Dv_\infty|^2}\right) : D^2 v_\infty.$$

v_∞ is a weak solution of the level-set flow equation with the cone C_1 as an initial condition. Therefore the smooth level sets of v_∞ at time $s = 1$ agree with the classical smooth evolution of the compact level sets of C_1 (see theorem A.6). A level set $t_0 M_{t_0}$ of C_N at height $t_0 > 0$ corresponds to the parabolically rescaled evolution $t_0 M_{t_0^{-2}}$ at time $t = 1$, which is also the level set at height t_0 of the expanded space-time track (4.4). Hence graph v_∞ must be the expanded space-time track.

□

4.2 Canonical self-expanders and the Harnack quantity

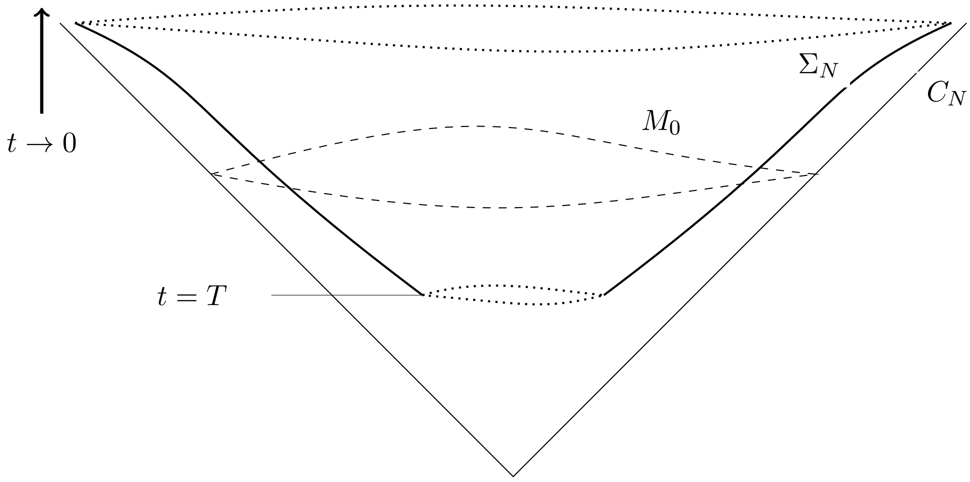
For a solution $(M_t)_{t\in[0,T]}$ of (4.1) there is an associated space-time construction by B. Kotschwar (see Kotschwar [2009]), which we call a canonical self-expander in analogy to the Ricci flow case (see Cabezas-Rivas & Topping [2012b]). The canonical self-expander Σ_N can be defined for a parameter $N > 0$ as (see figure 4.2)

$$\Sigma_N = \left\{t^{-\frac{1}{2}}(x, N) : \ x \in M_t, t \in (0, T]\right\}. \tag{4.5}$$

[4] so we have $\tilde{v}_{N_n} : \mathbb{R}^{n+1} \times (0,\infty) \to \mathbb{R}$, $\tilde{v}_{N_n} = \tilde{v}_{N_n}(y, s)$; abusing notation we also write $\tilde{v}_{N_n} = \tilde{v}_{N_n}(\cdot, 1)$

[5] again we abuse notation by writing $v_\infty = v_\infty(\cdot, 1)$, where appropriate

Figure 4.2: The canonical self-expander Σ_N



Σ_N is an approximate self-expanding solution of (4.1). The time slices (respectively level sets) are by construction exact solutions of (4.1).

Suppose now that the hypersurfaces $(M_t)_{t\in[0,T]}$ have uniformly bounded curvature. Then Σ_N is an approximate self-expanding solution of (4.1), i.e.

$$H^{\Sigma_N} + \frac{\langle z, \nu^{\Sigma_N} \rangle}{2} \approx 0 \tag{4.6}$$

for $z \in \Sigma_N$ (see [Kotschwar, 2009, 4.1]). By this we mean that $H^{\Sigma_N} + \frac{\langle z,\nu^{\Sigma_N} \rangle}{2} = E_N$, where $N\,|E_N|$ is locally uniformly bounded independently of N (we continue to use this notation).

Furthermore Σ_N is asymptotic to C_N and the second fundamental form of Σ_N, which we denote by h^{Σ_N}, satisfies

$$h^{\Sigma_N}\left(V + \frac{\partial}{\partial t}, V + \frac{\partial}{\partial t}\right) = \frac{Z\,(V,V)}{\sigma_N \sqrt{t}} \approx \frac{Z\,(V,V)}{\sqrt{t}} \tag{4.7}$$

for any tangent vector $V \in TM^n$, where $\sigma_N = \sqrt{1 + \frac{2t^{5/2}(t^{-1}\langle x,v\rangle + 2H)}{N^2}}$ (see [Kotschwar, 2009, theorem 7]).

4.3 A geometric proof of Hamilton's Harnack estimate

We can now deduce theorem 4.1 from preservation of convexity as follows:

Proof.

$$
\begin{aligned}
&M_0 \text{ convex} \Rightarrow \text{Cone } C_N \text{ over } M_0 \text{ convex}\\
\Rightarrow_{\text{preservation of convexity, see [Barles } et\ al.\text{ , 2002, theorem 10.2]}}\ &\tilde{\Sigma}_N = \text{ graph } \tilde{v}_N \text{ convex}\\
\Rightarrow\ &\text{graph } v_N = \text{ graph } \tfrac{1}{N}\tilde{v}_N \text{ convex}\\
\Rightarrow_{C^0\text{-limits of convex functions are convex}}\ &v_\infty = \lim_{N\to\infty} v_N \text{ convex}\\
\Rightarrow_{\text{see proposition 4.4}}\ &\text{Expanded space-time track convex}\\
\Rightarrow_{\text{see (4.5)}}\ &\text{Canonical self-expander } \Sigma_N \text{ convex}\\
\Rightarrow_{\text{see (4.7)}}\ &Z(V,V) \geq 0.
\end{aligned}
$$

□

Chapter 5

Space-time constructions for the mean curvature flow in a Ricci flow background

We want to study solutions of the mean curvature flow in a Ricci flow background. Consider families $F_t : M^n \to O^{n+1}$, $t \in I$ (I a real interval), of smooth immersions in a Riemannian manifold O^{n+1} equipped with a corresponding family of metrics $g^O(t)$ such that

$$\begin{aligned} \frac{\partial}{\partial t} F_t &= -H\nu \\ \frac{\partial}{\partial t} g^O &= S \end{aligned} \tag{5.1}$$

where

$$S = -2Ric\left(g^O\right) \text{ (Ricci flow background)}$$

or

$$S = 2Ric\left(g^O\right) \text{ (backwards Ricci flow background)} .$$

For $S = 2Ric$ we use τ as the reverse time parameter, in line with standard notation.

Canonical solitons for Ricci flow have been introduced by E. Cabezas-Rivas and P. Topping (see Cabezas-Rivas & Topping [2012b,a]), building up on results by B. Chow, S. Chu and D. Knopf (see Chow & Chu [1995, 2001]; Chow & Knopf [2002]). These are space-time constructions, which give geometric insight into optimal transportation in relation to Ricci flow (see Lott [2012]; McCann & Topping [2010]; Topping [2009]) and into Harnack estimates for the Ricci flow (see Hamilton [1993]; Brendle [2009]; Cabezas-Rivas & Topping [2012a]). The idea can be adapted to curvature flows in Euclidean space, which provides geometric insight into the corresponding Harnack estimates (see Kotschwar [2009] and chapter 3 of this thesis).

In this chapter we show that the construction of canonical solitons extends in a natural way to the case of mean curvature flow in a Ricci flow background. Given a solution of (5.1) one can construct the canonical shrinking, steady (in case $S = 2Ric$) and expanding (in case $S = -2Ric$) Ricci solitons, given by a metric on space-time $O^{n+1} \times I$. We show then that the space-time track $\{(F_t(x), t) : x \in M^n, t \in I\}$ of the associated mean curvature flow is itself a canonical soliton embedded within one of the canonical Ricci flow solitons.

This provides a relation between the Harnack quantity for the mean curvature flow and the Harnack quantity for the Ricci flow (see Hamilton [1993, 1995b]). Moreover J. Lott has introduced a version of G. Perelman's $\mathcal{F}$-functional (see Perelman [2002]) for the Ricci flow with boundary (see Lott [2012]), which is a weighted version of the Gibbons-Hawking-York functional from quantum gravity (see Gibbons & Hawking [1977]; York Jr [1972]). We show that the evolution of this $\mathcal{F}$-functional, corresponds to the second fundamental form of our canonical solitons.

5.1 Canonical solitons

To define the notion of a soliton in our setting, we follow Lott [2012].

Let $\left(O^{n+1}, g^O(t), f\right)$ for $t \in I$, $f : O^{n+1} \times I \to \mathbb{R}$ be a gradient Ricci flow soliton with potential f, i.e.

$$\begin{aligned} \frac{\partial}{\partial t} g^O &= -2Ric\left(g^O\right) \\ Ric\left(g^O\right) + Hess_{g^O} f + \frac{c}{2t} g^O &= 0 \\ \frac{\partial f}{\partial t} &= \left|{}^{g^O}\nabla f\right|^2 \end{aligned} \tag{5.2}$$

where $c = 0$ in the steady case ($I = \mathbb{R}$), $c = -1$ in the expanding case ($I = (0, \infty)$) and $c = 1$ in the shrinking case ($I = (-\infty, 0)$).

Definition 5.1. *At a given time t a hypersurface M_t in O^{n+1} is a soliton of the mean curvature flow (within the Ricci soliton) if*

$$H \pm {}^{g^O}\nabla_\nu f = 0. \tag{5.3}$$

Definition 5.1 can be understood as follows. Suppose $\left(O^{n+1}, g^O(t), f\right)$ is a gradient steady Ricci soliton. Let $(\psi_t)_{t \in I}$ be the one-parameter family of diffeomorphisms generated by the time-independent vector field ${}^{g^O}\nabla f$ and with $\psi_0 = Id$, such that $g^O(t) = \psi_t^* g^O(0)$, where ψ_t^* denotes the pull-back.

Then if a hypersurface M_0 satisfies (5.3) with a "+" sign, the hypersurfaces $M_t = \psi_t(M_0)$ provide a solution of (5.1) up to tangential diffeomorphisms. Changing the sign in (5.3) means that the solution M_t moves along the vector field $-{}^{g^O}\nabla f$.

For an expanding or shrinking Ricci soliton background we choose the one-parameter family of diffeomorphisms generated by $\pm\frac{1}{\tau(t)}{}^{g^O}\nabla f$ for $\tau(t) = ct + 1 > 0$ and we get an analogous statement as above.

Examples for solutions of (5.3) and more generally examples for solutions of (5.1) with a Ricci soliton background can be found in Tsatis [2010].

5.1.1 Canonical solitons within expanding Ricci flow solitons

We now consider solutions $\left(M^n, M_t, O^{n+1}, g^O(t)\right)$ of (5.1) for $S = -2Ric$ and we choose $I = (0, T]$.

The construction of canonical expanding solitons can be described as follows (see [Cabezas-Rivas & Topping, 2012a, theorem 2.1]). Suppose $\left(O^{n+1}, g^O(t)\right)$, $t \in (0, T]$, has uniformly bounded curvature. For a large enough parameter $N > 0$ we define the metric $\check{g}^O$ on $O^{n+1} \times (0, T]$ by

$$\check{g}^O_{ij} = \begin{cases} \frac{g^O_{ij}}{t}, \ i, j \geq 0 \\ 0, \ i \geq 1, j = 0 \\ \frac{N}{2t^3} + \frac{R\left(g^O\right)}{t} + \frac{n+1}{2t^2} \end{cases} \tag{5.4}$$

where 0 denotes the time direction on $O^{n+1} \times (0, T]$.

Then up to errors of order $\frac{1}{N}$ the metric $\check{g}$ is an approximate gradient expanding Ricci soliton:

$$E_N = Ric\left(\check{g}\right) + Hess_{\check{g}}\left(-\frac{N}{2t}\right) + \frac{1}{2}\check{g} \approx 0$$

by which we mean that the quantity $N\left|E_N\right|_{\check{g}}$ is bounded uniformly locally on $O^{n+1} \times (0, T]$ (independently of N).

We denote the potential of the canonical expanding Ricci soliton by $\check{f}(y, t) = -\frac{N}{2t}$. We can show that there is a natural corresponding construction for the coupled flow.

Theorem 5.2. *Suppose* $\left(M^n, M_t, O^{n+1}, g^O(t)\right)$ *is a solution of (5.1) such that* $g^O(t)$ *has uniformly bounded curvature and* M_t *has uniformly bounded second fundamental* *form.*

Define $\check{\Sigma} = \{(x,t) : x \in M_t, t \in (0,T]\}$ *to be the space-time track of* M_t *within the canonical Ricci expander* $\left(O^{n+1} \times (0,T], \check{g}\right)$. *Then* $\check{\Sigma}$ *is an approximate soliton, i.e.*

$$\check{E}_N = H^{\check{\Sigma}} - \nu^{\check{\Sigma}} \check{f} \approx 0$$

where $H^{\check{\Sigma}}$ *and* $\nu^{\check{\Sigma}}$ *denote the mean curvatue and the normal vector of* $\check{\Sigma}$. *By this we mean that* $N \left|\check{E}_N\right|_{\check{g}}$ *is bounded uniformly locally on* $\check{\Sigma}$ *(independently of* N*).*

The proof of theorem 5.2 consists of computing the relevant quantities (see section 5.4.1). We keep the terminology and call $\check{\Sigma}$ a canonical soliton of the mean curvature flow (within a canonical expanding Ricci soliton).

5.1.2 Canonical solitons within shrinking Ricci flow solitons

We consider solutions $\left(M^n, M_\tau, O^{n+1}, g^O(\tau)\right)$ of (5.1) for $S = 2Ric$ and we choose $I = (0,T]$.

Associated to the backwards Ricci flow $g^O(\tau)$ we have a canonical shrinking Ricci soliton which can be described as follows (see [Cabezas-Rivas & Topping, 2012b, theorem 1.1]). Suppose $\left(O^{n+1}, g^O(\tau)\right)$, $\tau \in (0,T]$, has uniformly bounded curvature. For a large enough parameter $N > 0$ we define the metric $\hat{g}^O$ on $O^{n+1} \times (0,T]$ by

$$\hat{g}_{ij} = \begin{cases} \frac{g^O_{ij}}{\tau}, \ i,j \geq 0 \\ 0, \ i \geq 1, j = 0 \\ \frac{N}{2\tau^3} + \frac{R\left(g^O\right)}{\tau} - \frac{n+1}{2\tau^2} \end{cases} \tag{5.5}$$

where 0 denotes the time direction on $O^{n+1} \times (0,T]$.

Then up to errors of order $\frac{1}{N}$ the metric $\hat{g}$ is an approximate gradient shrinking Ricci soliton:

$$E_N = Ric\left(\hat{g}\right) + Hess_{\hat{g}}\left(\frac{N}{2\tau}\right) - \frac{1}{2}\check{g} \approx 0$$

by which we mean that the quantity $N \left|E_N\right|_{\hat{g}}$ is bounded uniformly locally on $O^{n+1} \times (0,T]$ (independently of N). It has been shown in Cabezas-Rivas & Topping [2012a], that the metric $\hat{g}$ relates to $\mathcal{L}$-optimal transportation (for details see Cabezas-Rivas & Topping [2012a]; Topping [2009]).

We denote the potential of the canonical shrinking Ricci soliton by $\hat{f}\left(y,\tau\right) = \frac{N}{2\tau}$. As in the expanding case we can show that there is a natural corresponding construction for the coupled flow. The following result was first observed in a slightly different setting in [Cabezas-Rivas & Topping, 2012a, section 6].

Theorem 5.3. *Suppose* $\left(M^n, M_\tau, O^{n+1}, g^O(\tau)\right)$ *is a solution of (5.1) such that* $g^O(\tau)$ *has uniformly bounded curvature and* M_τ *has uniformly bounded second fundamental form.*

Define $\hat{\Sigma} = \{(x,\tau) : x \in M_t, \tau \in (0,T]\}$ *to be the space-time track of* M_τ *within the canonical Ricci shrinker* $\left(O^{n+1} \times (0,T], \hat{g}\right)$*. Then* $\hat{\Sigma}$ *is an approximate soliton, i.e.*

$$\check{E}_N = H^{\hat{\Sigma}} + \nu^{\hat{\Sigma}} \hat{f} \approx 0$$

where $H^{\hat{\Sigma}}$ *and* $\nu^{\hat{\Sigma}}$ *denote the mean curvatue and the normal vector of* $\hat{\Sigma}$*. By this we mean that* $N \left|\hat{E}_N\right|_{\hat{g}}$ *is bounded uniformly locally on* $\hat{\Sigma}$ *(independently of* N*).*

The proof of theorem 5.3 consists of computing the relevant quantities (see section 5.4.1). We keep the terminology and call $\hat{\Sigma}$ a canonical soliton of the mean curvature flow (within a canonical shrinking Ricci soliton).

5.1.3 Canonical solitons within steady Ricci flow solitons

We consider solutions $\left(M^n, M_\tau, O^{n+1}, g^O(\tau)\right)$ of (5.1) for $S = 2Ric$ and we choose $I = (0,T]$.

Associated to the backwards Ricci flow $g^O(\tau)$ we have a canonical steady Ricci soliton which can be described as follows (see [Cabezas-Rivas & Topping, 2012b, theorem 5.1]). Suppose $\left(O^{n+1}, g^O(\tau)\right)$, $\tau \in (0,T]$, has uniformly bounded curvature. For a large enough parameter $N > 0$ we define the metric $\bar{g}^O$ on $O^{n+1} \times (0,T]$ by

$$\bar{g}_{ij} = \begin{cases} g^O_{ij},\ i,j \geq 0 \\ 0,\ i \geq 1, j = 0 \\ N + R\left(g^O\right) \end{cases} \tag{5.6}$$

where 0 denotes the time direction on $O^{n+1} \times (0,T]$.

Then up to errors of order $\frac{1}{N}$ the metric $\bar{g}$ is an approximate gradient steady Ricci soliton:

$$E_N = Ric\left(\bar{g}\right) + Hess_{\bar{g}}\left(-N\tau\right) \approx 0$$

by which we mean that the quantity $N\left|E_N\right|_{\bar{g}}$ is bounded uniformly locally on $O^{n+1} \times (0,T]$ (independently of N).

We denote the potential of the canonical steady Ricci soliton by $\bar{f}(y,\tau) = -N\tau$. As for the expanding and shrinking cases we can show that there is a natural corresponding construction for the coupled flow.

Theorem 5.4. *Suppose $\left(M^n, M_\tau, O^{n+1}, g^O(\tau)\right)$ is a solution of (5.1) such that $g^O(\tau)$ has uniformly bounded curvature and M_τ has uniformly bounded second fundamental form.*

Define $\bar{\Sigma} = \{(x,\tau) : x \in M_\tau, \tau \in (0,T]\}$ to be the space-time track of M_τ within the canonical steady Ricci soliton $\left(O^{n+1} \times (0,T], \bar{g}\right)$. Then $\bar{\Sigma}$ is an approximate soliton, i.e.

$$\check{E}_N = H^{\bar{\Sigma}} + \nu^{\bar{\Sigma}} \bar{f} \approx 0$$

where $H^{\bar{\Sigma}}$ and $\nu^{\bar{\Sigma}}$ denote the mean curvatue and the normal vector of $\bar{\Sigma}$. By this we mean that $N \left|\bar{E}_N\right|_{\bar{g}}$ is bounded uniformly locally on $\bar{\Sigma}$ (independently of N).

The proof of theorem 5.4 consists of computing the relevant quantities (see section 5.4.1). We keep the terminology and call $\bar{\Sigma}$ a canonical soliton of the mean curvature flow (within a canonical steady Ricci soliton).

5.2 Canonical solitons and Harnack quantities

An application of canonical solitons for geometric flows is to generate Harnack quantities. The general idea is that a differential Harnack estimate corresponds to a preserved curvature condition on a canonical soliton.

E. Cabezas-Rivas and P. Topping have shown in Cabezas-Rivas & Topping [2012b] that this procedure works for the Ricci flow. The Ricci curvature of the canonical expanding Ricci soliton (see 5.4) converges as $N \to \infty$ to R. Hamilton's matrix Harnack quantity for the Ricci flow (see Hamilton [1993]). Moreover preserved curvature conditions on the canonical expanding Ricci soliton recover S. Brendle's Harnack quantity (see Brendle [2009]) and also provide new Harnack quantities. These correspond to the curvature operator of the canonical soliton lying in certain cones.

It has been shown in Kotschwar [2009] that the same approach works also for curvature flows in Euclidean space. The corresponding Harnack quantities which are due to R. Hamilton, B. Andrews and K. Smoczyk (see Hamilton [1995b], Andrews [1994] and Smoczyk [1997]) can be recovered as the second fundamental form of a canonical self-expander. In the case of mean curvature flow in Euclidean space it is possible to give a geometric proof of R. Hamilton's Harnack estimate in this way (see Kotschwar [2009] or chapter 3 of this thesis).

Our construction of canonical mean curvature flow solitons within expanding Ricci flow solitons from theorem 5.2 provides a link between the Harnack estimate for the mean curvature flow and the Harnack estimate for the Ricci flow. From (5.22) we get

Proposition 5.5. *The second fundamental form of the canonical soliton $\check{\Sigma}$ converges as $N \to \infty$ to a limit $h^{\check{\Sigma}}_{\infty}$ with*

$$h^{\check{\Sigma}}_{\infty}\left(V+\frac{\partial}{\partial t},V+\frac{\partial}{\partial t}\right)= \tag{5.7}$$
$$\frac{\partial}{\partial t}H+h\left(V,V\right)+\frac{H}{2t}+2\left\langle V,\nabla H\right\rangle+$$
$$2Ric\left(g^{O}\right)\left(V,\nu\right)-HRic\left(g^{O}\right)\left(\nu,\nu\right)+\frac{1}{2}\check{\nabla}_{\nu}R\left(g^{O}\right).$$

for any vector field V on M^n.

On the other hand we have (see [Cabezas-Rivas & Topping, 2012a, proposition 2.6])

Proposition 5.6. *The Ricci curvature of the canonical expanding Ricci soliton $\check{g}$ converges as $N \to \infty$ to $\left(Ric\left(\check{g}\right)\right)_{\infty}$ with*

$$\left(Ric\left(\check{g}\right)\right)_{\infty}\left(X+\frac{\partial}{\partial t},X+\frac{\partial}{\partial t}\right)= \tag{5.8}$$
$$Ric\left(g^{O}\right)\left(X,X\right)+g^{O}\left(X,{}^{g^{O}}\nabla R\left(g^{O}\right)\right)+\frac{1}{2}\left(\frac{\partial R\left(g^{O}\right)}{\partial t}+\frac{R\left(g^{O}\right)}{t}\right)$$

for any vector field X on O^{n+1}.

From proposition 5.6 we see that the Ricci curvature of the canonical expanding Ricci soliton converges to Hamilton's Harnack quantity for the Ricci flow (see Hamilton [1993])

$$Z\left(X,X\right)=Ric\left(g^{O}\right)\left(X,X\right)+g^{O}\left(X,{}^{g^{O}}\nabla R\left(g^{O}\right)\right)+\frac{1}{2}\left(\frac{\partial R\left(g^{O}\right)}{\partial t}+\frac{R\left(g^{O}\right)}{t}\right) \tag{5.9}$$

as $N \to \infty$. If $O^{n+1}=\mathbb{R}^{n+1}$ with the Euclidean metric, then we see from proposition 5.5 that the second fundamental form of the mean curvature flow soliton within the canonical expanding Ricci soliton converges as $N \to \infty$ to Hamilton's Harnack quantity for the mean curvature flow

$$\tilde{Z}\left(V,V\right)=\frac{\partial}{\partial t}H+h\left(V,V\right)+2\left\langle V,\nabla H\right\rangle+\frac{H}{2t} \tag{5.10}$$

(see Hamilton [1995b]), providing a link between the Harnack quantity for the Ricci flow and the Harnack quantity for the mean curvature flow.

This seems to suggest that the quantity (5.7) should have a similar meaning for the coupled flow. Unfortunately no suitable preserved curvature condition for proving a Harnack type estimate for the coupled flow is known.

5.3 Canonical solitons and the modified $\mathcal{F}$-functional

In view of the relation of canonical solitons to Harnack estimates (see last section) it seems natural that the curvature of canonical solitons also plays an important role for the coupled flow (5.2). Again we focus here on the case $S = -2Ric$, so we get canonical expanding Ricci solitons (see 5.4) and (within these) canonical mean curvature flow solitons (see 5.2).

Nonnegative second fundamental form is not preserved in general for the coupled flow, even for a Ricci soliton background. Hence we can not expect a direct analogue of the differential Harnack estimate for the mean curvature flow. However we want to point out a link between canonical solitons and a version of the $\mathcal{F}$-functional (see [Perelman, 2002, section 1.1]).

In Ecker [2007] K. Ecker found a remarkable connection between the $\mathcal{W}$-functional for the Ricci flow and the differential Harnack expression for the mean curvature flow in $\mathbb{R}^{n+1}$. Suppose that for a solution $(M_t)_{t\in[0,T)}$ of the mean curvature flow in $\mathbb{R}^{n+1}$ the hypersurfaces M_t are the boundary of open bounded sets Ω_t in R^{n+1}. One can consider the integral over Ω_t of G. Perelman's $\mathcal{W}$-integrand (see [Perelman, 2002, proposition 9.1])

$$\mathcal{W}(\Omega_t, f, \tau(t)) = \int_{\Omega_t} \Big(\tau\, |\nabla f|^2 + f - (n+1)\Big)\, u d\mu + 2\tau \int_{M_t} HudA$$

[1]for a reverse time parameter $\frac{d}{dt}\tau(t) = -1$, where the $\mathcal{W}$-integrand is itself defined using a positive solution of the backwards heat equation

$$u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$$
$$\frac{\partial}{\partial t} f + \triangle f = |\nabla f|^2 + \frac{n+1}{2\tau} \tag{5.11}$$

[1]here $d\mu$ denotes the volume density in $\mathbb{R}^{n+1}$ and dA the associated area density of a boundary

in Ω_t with von Neumann boundary condition

$$H = \langle \nabla f, \nu \rangle$$

on $M_t = \partial \Omega_t$.

The boundary term of the evolution of $\mathcal{W}$ then consists of Hamilton's Harnack quantity for the mean curvature flow (5.10):

$$\frac{d}{dt}\mathcal{W}\left(\Omega_t, f, \tau(t)\right) = 2\tau \int_{\Omega_t} \left| \nabla_i \nabla_j f - \frac{\delta_i^j}{2\tau} \right|^2 dV + 2\tau \int_{M_t} \tilde{Z}\left(-\nabla f, -\nabla f\right) u dA \quad (5.12)$$

(see [Ecker, 2007, proposition 3.4]).

Furthermore K. Ecker conjectured that (5.12) is nonnegative for any compact solution $M_t = \partial \Omega_t$ of the mean curvature flow in $\mathbb{R}^{n+1}$ with positive mean curvature. This conjecture is still open.

J. Lott showed in Lott [2012] that analogous $\mathcal{F}$-versions of these relations hold for solutions of (5.2). He defined a version I_∞ of G. Perelman's $\mathcal{F}$-functional for manifolds O^{n+1} with boundary by adding an appropriate (from a variational viewpoint) boundary term to the original integral. For $f \in C^\infty\left(O^{n+1}\right)$ we have

$$I_\infty\left(g^O, f\right) = \int_{O^{n+1}} R^\infty\left(g^O\right) e^{-f} d\mu\left(g^O\right) + 2 \int_{\partial O^{n+1}} H^\infty e^{-f} dA\left(g^O\right) \quad (5.13)$$

where

$$R^\infty\left(g^O\right) = R\left(g^O\right) + 2\triangle_{g^O} f - \left| {}^{g^O}\nabla f \right|_{g^O}^2$$

is the analogue of the scalar curvature on the smooth metric measure space $\mathcal{O} = \left(O^{n+1}, g^O, e^{-f}\mu\left(g^O\right)\right)$ and

$$H^\infty = H - \nu f$$

is the analogue of the mean curvature (ν denotes the outward-pointing unit normal on ∂O^{n+1} and $dA\left(g^O\right)$ the induced area on ∂O^{n+1}). I_∞ can be seen as a weighted [2] version of the Gibbons-Hawking-York functional (see Gibbons & Hawking [1977] and York Jr [1972])

[2] with respect to the measure $e^{-f}\mu\left(g^O\right)$

$$I_{GHY}\left(g^O\right) = \int_{O^{n+1}} R\left(g^O\right) d\mu\left(g^O\right) + 2\int_{\partial O^{n+1}} H dA\left(g^O\right)$$

which occurs in quantum gravity.
Now if $g^O(t)$ is a solution of the modified Ricci flow

$$\frac{\partial}{\partial t} g^O = -2\left(Ric\left(g^O\right) + Hess_{g^O} f\right)$$

for a solution $u = e^{-f}$ of the conjugate heat equation

$$\frac{\partial}{\partial t} u = \left(-\triangle_{g^O} + R\left(g^O\right)\right) u$$

with boundary condition

$$H - \nu f = 0$$

then we have (see [Lott, 2012, theorem 1])

$$\begin{aligned} \frac{d}{dt} I_\infty = 2\int_{O^{n+1}} \left|Ric\left(g^O\right) + Hess_{g^O} f\right|^2 e^{-f} d\mu\left(g^O\right) + \\ 2\int_{\partial O^{n+1}} \left(\frac{\partial}{\partial t} H - 2\langle \nabla f, \nabla H\rangle + h(\nabla f, \nabla f)\right. \\ \left. -2Ric\left(g^O\right)(\nu, \nabla f) + \frac{1}{2}\nu R\left(g^O\right) - HRic\left(g^O\right)(\nu,\nu)\right) e^{-f} dA\left(g^O\right) \end{aligned} \tag{5.14}$$

where ∇ denotes the derivative on ∂O^{n+1} and h, H denote the second fundamental form and the mean curvature of ∂O^{n+1}.

From proposition 5.5 we see that up to the $\frac{H}{2t}$ term the limit second fundamental form of our canonical soliton evaluated at $-\nabla f$ exactly matches the boundary integrand of (5.14). The former is 0 on mean curvature flow solitons in gradient expanding Ricci solitons, whereas the latter is 0 on mean curvature flow solitons in gradient steady Ricci solitons (see [Lott, 2012, proposition 7]).

In the case of Euclidean ambient space $\mathbb{R}^{n+1}$ (5.14) is the $\mathcal{F}$-version of K. Ecker's result (see [Ecker, 2007, proposition 3.2, 3.4]). In this case the limit second fundamental form of the canonical soliton from theorem 5.2 exactly matches Hamilton's Harnack quantity for the mean curvature flow (5.10), which in turn matches the boundary integrand in (5.14) up to the $\frac{H}{2t}$ term.

In the same fashion as proposition 5.5 the canonical solitons from theorem 5.3 and theorem 5.4 yield analogous quantities to the boundary integrand in (5.14) for a backward Ricci flow background.

5.4 Computations

We prove theorems 5.2, 5.3 and 5.4 by computing the relevant quantities on $\check{\Sigma}$, $\hat{\Sigma}$ and $\bar{\Sigma}$, which we denote by $\cdot^{\check{\Sigma}}$ (and analogously in the other cases). Points on $\check{\Sigma}$ are denoted by

$$z = F^{\check{\Sigma}}(p,t) = (F_t(p), t).$$

for $p \in M^n$ and $t \in I$. In index notation 0 always denotes the time direction and we also use $p^0 = t$ here. As before we use bare letters to denote quantities on the hypersurfaces M_t, e.g. g or h. By $A \approx B$ up to errors of order $\frac{1}{N}$ on $\check{\Sigma}$ we always mean $A = B + E_N$ and $N\,|E_N|_{\check{g}}$ is bounded uniformly locally on $\check{\Sigma}$ (independently of N).

5.4.1 Canonical mean curvature flow solitons within canonical expanding Ricci flow solitons

The computations of the Christoffel symbols and the curvature of $\check{g}$ can be found in Cabezas-Rivas & Topping [2012a]. To compute the second fundamental form of $\check{\Sigma}$ we need the Christoffel symbols of $\check{g}$, which are given by:

$$\check{\Gamma}^i_{jk} = \left(\Gamma\left(g^O\right)\right)^i_{jk}, \; \check{\Gamma}^i_{j0} = -\left(\left(Ric\left(g^O\right)\right)^i_j + \frac{\delta^i_j}{2t}\right), \; \check{\Gamma}^i_{00} = -\frac{1}{2}\left(g^O\right)_{ij}\frac{\partial R\left(g^O\right)}{\partial x^j} \tag{5.15}$$

$$\check{\Gamma}^0_{jk} = \check{g}_{00}^{-1}\left(\frac{\left(Ric\left(g^O\right)\right)_{jk}}{t} + \frac{g^O_{jk}}{2t^2}\right), \; \check{\Gamma}^0_{j0} = \frac{1}{2t}\check{g}_{00}^{-1}\frac{\partial R\left(g^O\right)}{\partial x^j}$$

$$\check{\Gamma}^0_{00} = -\frac{3}{2t} + \frac{\check{g}_{00}^{-1}}{2t}\left(\frac{R\left(g^O\right)}{t} + \frac{\partial R\left(g^O\right)}{\partial t} + \frac{n+1}{2t^2}\right).$$

A basis for $T_z\check{\Sigma}$ is given by

$$\left\{(F_t)_*\frac{\partial}{\partial p^1}, \ldots, (F_t)_*\frac{\partial}{\partial p^n}, -H\nu + \frac{\partial}{\partial t}\right\} = \left\{\frac{\partial}{\partial p^1}F_t, \ldots, \frac{\partial}{\partial p^n}F_t, -H\nu + \frac{\partial}{\partial t}\right\}$$

Therefore we get

$$\nu^{\check{\Sigma}} = \frac{1}{\sigma_N}\left(\nu + \frac{H}{\check{g}_{00}t}\frac{\partial}{\partial t}\right) \tag{5.16}$$

where $\sigma_N = \sqrt{t^{-1} + H^2 t^{-2} \check{g}_{00}^{-1}}$.
(5.16) yields

$$\nu^{\check{\Sigma}} \check{f} = \frac{HN}{2t^3 \check{g}_{00}\sigma_N}. \tag{5.17}$$

From

$$g_{ij}^{\check{\Sigma}} = \check{g}\left(\frac{\partial}{\partial p^i}F^{\check{\Sigma}}, \frac{\partial}{\partial p^j}F^{\check{\Sigma}}\right)$$

we obtain

$$g_{ij}^{\check{\Sigma}} = \begin{cases} \frac{1}{t}g_{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ \frac{H^2}{t} + \check{g}_{00},\ i = j = 0. \end{cases} \tag{5.18}$$

For the inverse metric we have then

$$\left(g^{\check{\Sigma}}\right)^{ij} = \begin{cases} tg^{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ \frac{t}{H^2 + t\check{g}_{00}},\ i = j = 0. \end{cases} \tag{5.19}$$

Hence the inverse metric $\left(g^{\check{\Sigma}}\right)^{-1}$ converges as $N \to \infty$ to the degenerate metric $\left(g_{\infty}^{\check{\Sigma}}\right)^{-1}$ given by

$$\left(g_{\infty}^{\check{\Sigma}}\right)^{ij} = \begin{cases} tg^{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ 0,\ i = j = 0. \end{cases} \tag{5.20}$$

Using the formula

$$h_{ij}^{\check{\Sigma}} = -\check{g}\left(\check{\nabla}_{F_*^{\check{\Sigma}}\frac{\partial}{\partial p^i}} F_*^{\check{\Sigma}}\frac{\partial}{\partial p^j}, \nu^{\check{\Sigma}}\right)$$

we can compute the second fundamental form of $\check{\Sigma}$ (see also (5.15))

$$h_{ij}^{\check{\Sigma}} = \frac{1}{t\sigma_N}\begin{cases} h_{ij} + \frac{H}{t\breve{g}_{00}}\left(Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_t, \frac{\partial}{\partial p^j}F_t\right) + \frac{1}{2t}g_{ij}\right),\ i,j \geq 1 \\ \frac{\partial}{\partial p^i}H + Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_t, \nu\right) - \frac{H}{2t\breve{g}_{00}}\frac{\partial}{\partial p^i}F_t\left(R\left(g^O\right)\right),\ i \geq 1, j = 0 \\ \frac{\partial}{\partial t}H + \frac{H}{2t} - HRic\left(g^O\right)(\nu,\nu) + \frac{1}{2}\check{\nabla}_\nu R\left(g^O\right) + \\ \frac{H}{t\breve{g}_{00}}\left(\frac{H^2}{2t} + H^2 Ric\left(g^O\right)(\nu,\nu) - H^2\check{\nabla}_\nu R\left(g^O\right) - \right. \\ \left. \frac{1}{t}R\left(g^O\right) - \frac{1}{2}\frac{\partial}{\partial t}R\left(g^O\right) + \frac{n}{4t^2}\right),\ i = j = 0. \end{cases} \tag{5.21}$$

Up to errors of order $\frac{1}{N}$ this means

$$h_{ij}^{\check{\Sigma}} \approx \frac{1}{t\sigma_N}\begin{cases} h_{ij},\ i,j \geq 1 \\ \frac{\partial}{\partial p^i}H + Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_t, \nu\right),\ i \geq 1, j = 0 \\ \frac{\partial}{\partial t}H + \frac{H}{2t} - HRic\left(g^O\right)(\nu,\nu) + \frac{1}{2}\check{\nabla}_\nu R\left(g^O\right),\ i = j = 0. \end{cases} \tag{5.22}$$

Hence (5.19) and (5.22) show that the mean curvature of $\check{\Sigma}$ is given by (up to errors of order $\frac{1}{N}$)

$$H^{\check{\Sigma}} \approx \frac{t}{\sigma_N}H \approx \sqrt{t}H. \tag{5.23}$$

Together (5.17) and (5.23) then yield

$$H^{\check{\Sigma}} \approx \nu^{\check{\Sigma}}\check{f} \tag{5.24}$$

from which we can deduce theorem 5.2.

5.4.2 Canonical mean curvature flow solitons within canonical shrinking Ricci flow solitons

The Christoffel symbols of $\hat{g}$ are given by (see Cabezas-Rivas & Topping [2012b]).

$$\hat{\Gamma}^i_{jk} = \left(\Gamma\left(g^O\right)\right)^i_{jk},\ \hat{\Gamma}^i_{j0} = \left(Ric\left(g_O\right)\right)^i_j - \frac{\delta^i_j}{2\tau},\ \hat{\Gamma}^i_{00} = -\frac{1}{2}g^O_{ij}\frac{\partial R\left(g^O\right)}{\partial x^j} \tag{5.25}$$

$$\hat{\Gamma}^0_{jk} = \hat{g}_{00}^{-1}\left(-\frac{\frac{g^O_{jk}}{2\tau^2} - \left(Ric\left(g^O\right)\right)_{jk}}{\tau}\right),\ \hat{\Gamma}^0_{j0} = \frac{1}{2\tau}\hat{g}_{00}^{-1}\frac{\partial R\left(g^O\right)}{\partial x^j}$$

$$\hat{\Gamma}^0_{00} = -\frac{3}{2\tau} + \frac{\hat{g}_{00}^{-1}}{2\tau}\left(\frac{R\left(g^O\right)}{\tau} + \frac{\partial R\left(g^O\right)}{\partial\tau} + \frac{n+1}{2\tau^2}\right).$$

We can proceed as in appendix 5.4.1 to compute the relevant quantities. For the normal vector we get

$$\nu^{\hat{\Sigma}} = \frac{1}{\sigma_N}\left(\nu + \frac{H}{\hat{g}_{00}\tau}\frac{\partial}{\partial\tau}\right) \tag{5.26}$$

where $\sigma_N = \sqrt{\tau^{-1} + H^2\tau^{-2}\hat{g}_{00}^{-1}}$.
(5.26) yields

$$\nu^{\hat{\Sigma}}\hat{f} = -\frac{HN}{2\tau^3\hat{g}_{00}\sigma_N}. \tag{5.27}$$

The induced metric on $\hat{\Sigma}$ is

$$g_{ij}^{\hat{\Sigma}} = \begin{cases} \frac{1}{\tau}g_{ij}, \ i,j \geq 0 \\ 0, \ i \geq 0, j = 0 \\ \frac{H^2}{\tau} + \hat{g}_{00}, \ i = j = 0. \end{cases} \tag{5.28}$$

For the inverse metric we have then

$$\left(g^{\hat{\Sigma}}\right)^{ij} = \begin{cases} \tau g^{ij}, \ i,j \geq 0 \\ 0, \ i \geq 0, j = 0 \\ \frac{\tau}{H^2+\tau\hat{g}_{00}}, \ i = j = 0. \end{cases} \tag{5.29}$$

Hence the inverse metric $\left(g^{\hat{\Sigma}}\right)^{-1}$ converges as $N \to \infty$ to the degenerate metric $\left(g_\infty^{\hat{\Sigma}}\right)^{-1}$ given by

$$\left(g_\infty^{\hat{\Sigma}}\right)^{ij} = \begin{cases} \tau g^{ij}, \ i,j \geq 0 \\ 0, \ i \geq 0, j = 0 \\ 0, \ i = j = 0. \end{cases} \tag{5.30}$$

We can compute the second fundamental form of $\hat{\Sigma}$ (see also (5.25))

$$h_{ij}^{\hat{\Sigma}} = \frac{1}{\tau\sigma_N} \begin{cases} h_{ij} + \frac{H}{\tau\hat{g}_{00}} \left(-Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \frac{\partial}{\partial p^j}F_\tau\right) + \frac{1}{2\tau}g_{ij}\right), \ i,j \geq 1 \\ \frac{\partial}{\partial p^i}H - Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \nu\right) - \frac{H}{2\tau\hat{g}_{00}}\frac{\partial}{\partial p^i}F_\tau\left(R\left(g^O\right)\right), \ i \geq 1, j = 0 \\ \frac{\partial}{\partial \tau}H + \frac{H}{2\tau} + HRic\left(g^O\right)(\nu,\nu) + \frac{1}{2}\hat{\nabla}_\nu R\left(g^O\right) + \\ \frac{H}{\tau\hat{g}_{00}}\left(-\frac{H^2}{2\tau} + H^2 Ric\left(g^O\right)(\nu,\nu) - H^2\hat{\nabla}_\nu R\left(g^O\right) - \right. \\ \left. \frac{1}{\tau}R\left(g^O\right) - \frac{1}{2}\frac{\partial}{\partial\tau}R\left(g^O\right) + \frac{n}{4\tau^2}\right), \ i = j = 0. \end{cases} \tag{5.31}$$

Up to errors of order $\frac{1}{N}$ this means

$$h_{ij}^{\hat{\Sigma}} \approx \frac{1}{\tau\sigma_N} \begin{cases} h_{ij}, \ i,j \geq 1 \\ \frac{\partial}{\partial p^i}H - Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \nu\right), \ i \geq 1, j = 0 \\ \frac{\partial}{\partial \tau}H + \frac{H}{2\tau} + HRic\left(g^O\right)(\nu,\nu) + \frac{1}{2}\hat{\nabla}_\nu R\left(g^O\right), \ i = j = 0. \end{cases} \tag{5.32}$$

Hence (5.29) and (5.32) show that the mean curvature of $\hat{\Sigma}$ is given by (up to errors of order $\frac{1}{N}$)

$$H^{\hat{\Sigma}} \approx \frac{\tau}{\sigma_N}H \approx \sqrt{\tau}H. \tag{5.33}$$

Together (5.27) and (5.33) then yield

$$H^{\hat{\Sigma}} \approx -\nu^{\hat{\Sigma}}\hat{f} \tag{5.34}$$

from which we can deduce theorem 5.3.

5.4.3 Canonical mean curvature flow solitons within canonical steady Ricci flow solitons

The Christoffel symbols of $\bar{g}$ are given by (see Cabezas-Rivas & Topping [2012b]).

$$\bar{\Gamma}^i_{jk} = \left(\Gamma\left(g^O\right)\right)^i_{jk}, \ \bar{\Gamma}^i_{j0} = \left(R\left(g^O\right)\right)^i_j, \ \bar{\Gamma}^i_{00} = -\frac{1}{2}g^O_{ij}\frac{\partial R\left(g^O\right)}{\partial x^j} \tag{5.35}$$

$$\bar{\Gamma}^0_{jk} = -\frac{1}{N + R\left(g^O\right)}\left(Ric\left(g^O\right)\right)_{jk}, \ \bar{\Gamma}^0_{j0} = \frac{1}{2}\frac{\partial R\left(g^O\right)}{\partial x^j}, \ \bar{\Gamma}^0_{00} = \frac{1}{2}\frac{\partial R\left(g^O\right)}{\partial \tau}.$$

We can proceed as in appendix 5.4.1 to compute the relevant quantities. For the normal vector we get

$$\nu^{\bar{\Sigma}} = \frac{1}{\sigma_N}\left(\nu + \frac{H}{\bar{g}_{00}}\frac{\partial}{\partial\tau}\right) \tag{5.36}$$

where $\sigma_N = \sqrt{1 + H^2\bar{g}_{00}^{-1}}$.
(5.36) yields

$$\nu^{\bar{\Sigma}}\bar{f} = -\frac{HN}{\bar{g}_{00}\sigma_N}. \tag{5.37}$$

The induced metric on $\bar{\Sigma}$ is

$$g_{ij}^{\bar{\Sigma}} = \begin{cases} g_{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ \frac{H^2}{+}\bar{g}_{00},\ i = j = 0. \end{cases} \tag{5.38}$$

For the inverse metric we have then

$$\left(g^{\bar{\Sigma}}\right)^{ij} = \begin{cases} g^{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ \frac{1}{H^2+\tau\bar{g}_{00}},\ i = j = 0. \end{cases} \tag{5.39}$$

Hence the inverse metric $\left(g^{\bar{\Sigma}}\right)^{-1}$ converges as $N \to \infty$ to the degenerate metric $\left(g_\infty^{\bar{\Sigma}}\right)^{-1}$ given by

$$\left(g_\infty^{\bar{\Sigma}}\right)^{ij} = \begin{cases} g^{ij},\ i,j \geq 0 \\ 0,\ i \geq 0, j = 0 \\ 0,\ i = j = 0. \end{cases} \tag{5.40}$$

We can compute the second fundamental form of $\bar{\Sigma}$ (see also (5.35))

$$h_{ij}^{\bar{\Sigma}} = \frac{1}{\sigma_N}\begin{cases} h_{ij} + \frac{H}{N+R(g^O)}Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \frac{\partial}{\partial p^j}F_\tau\right),\ i,j \geq 1 \\ \frac{\partial}{\partial p^i}H - Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \nu\right) + \\ \frac{H}{2}\frac{\partial}{\partial p^i}\ln\left(N + R\left(g^O\right)\right) + \frac{H^2}{N+R(g^O)}Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i}F_\tau, \nu\right),\ i \geq 1, j = 0 \\ \frac{\partial}{\partial\tau}H + HRic\left(g^O\right)(\nu,\nu) + \\ \frac{1}{2}\bar{\nabla}_\nu R\left(g^O\right) + \frac{H^2}{2}\bar{\nabla}_\nu \ln\left(N + R\left(g^O\right)\right) - \\ \frac{H}{2}\frac{\partial}{\partial t}\ln\left(N + R\left(g^O\right)\right) - \frac{H^3}{N+R(g^O)}Ric\left(g^O\right)(\nu,\nu),\ i = j = 0. \end{cases} \tag{5.41}$$

Up to errors of order $\frac{1}{N}$ this means

$$h_{ij}^{\bar{\Sigma}} \approx \frac{1}{\tau\sigma_N} \begin{cases} h_{ij},\ i,j \geq 1 \\ \frac{\partial}{\partial p^i} H - Ric\left(g^O\right)\left(\frac{\partial}{\partial p^i} F_\tau, \nu\right),\ i \geq 1, j = 0 \\ \frac{\partial}{\partial \tau} H + HRic\left(g^O\right)(\nu,\nu) + \frac{1}{2}\bar{\nabla}_\nu R\left(g^O\right),\ i = j = 0. \end{cases} \tag{5.42}$$

Hence (5.39) and (5.42) show that the mean curvature of $\bar{\Sigma}$ is given by (up to errors of order $\frac{1}{N}$)

$$H^{\bar{\Sigma}} \approx \frac{H}{\sigma_N} \approx H. \tag{5.43}$$

Together (5.37) and (5.43) then yield

$$H^{\bar{\Sigma}} \approx -\nu^{\bar{\Sigma}} \bar{f} \tag{5.44}$$

from which we can deduce theorem 5.4.

Appendix A

The level-set flow

We want to give a short overview of the level-set flow approach to mean curvature flow. This is one possible notion of weak solution for the mean curvature flow. Another important notion, which we do not want to discuss here, is K. A. Brakke's varifold approach (see Brakke [1978]; Ilmanen [1994]). The level-set approach was first introduced by Barles (see Barles [1985]) and Osher and Sethian (see Osher & Sethian [1988]). The theory for the level-set flow was developed independently by L.C. Evans and J. Spruck (see Evans & Spruck [1991, 1992a,b, 1995]) and by Y.G. Chen, Y. Giga and S. Goto (see Chen *et al.* [1991]).

The central idea for the level-set flow is to consider evolving hypersurfaces in $\mathbb{R}^{n+1}$ as level sets of a function $u : \Omega \times [0, \infty) \to \mathbb{R}$ for $\Omega \subset \mathbb{R}^{n+1}$ open. Let us first derive a partial differential equation, which says that all level sets of u move by mean curvature as defined in (3.1) or (4.1). To this end we assume now that u is smooth and that $Du \neq 0$ in a space-time neighbourhood. We consider the level sets $\Gamma_t = u^{-1}(const)$, which move smoothly in the neighbourhood under consideration. Let $x(t)$ be a smooth path such that $x(t) \in \Gamma_t$ and $\frac{dx}{dt}$ is perpendicular to Γ_t. Then

$$u(x(t), t) = const$$

and

$$\frac{dx}{dt} x(t) = -H\nu = -H \frac{Du}{|Du|}.$$

Therefore we get, putting the two equations together

$$0 = \frac{dx}{dt} u(x(t), t) = \frac{\partial}{\partial t} u(x(t), t) + \left\langle Du, \frac{dx}{dt} x(t) \right\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^{n+1}$. Hence we get

$$\frac{\partial}{\partial t}u = H\,|Du|\,.$$

We can express the mean curvature as (see [Ecker, 2004, appendix A])

$$H = \text{ div }_{\Gamma_t}\nu = \text{ div }_{\mathbb{R}^{n+1}}\nu - \langle \nu, D_\nu \nu\rangle = \frac{\left(I - \frac{Du\otimes Du}{|Du|^2}\right) : D^2 u}{|Du|}.$$

Substituting this into the last equation yields the level-set flow equation

$$\begin{cases} \frac{\partial}{\partial t}u = \left(I - \frac{Du\otimes Du}{|Du|^2}\right) : D^2 u \text{ on } \Omega\times[0,T) \\ u = f \text{ on } \Omega\times 0 \end{cases} \tag{A.1}$$

where I denotes the identity matrix on $\mathbb{R}^{n+1}$.

Since the level-set flow equation is completely degenerate in the direction perpendicular to the level sets and not even well-defined where $Du = 0$, we need a suitable notion of weak solution. This can be achieved, even for merely continuous functions u, by considering differential inequalities for smooth test functions touching the graph of u.

For $V \subset \Omega\times(0,T)$ open, a point $(x_0, t_0) \in V$ and a smooth function $\varphi : V \to \mathbb{R}$ we say φ is tangent to u from above (below) at (x_0, t_0) if

$$\varphi \geq u \ \ (\varphi \leq u) \ \text{ and } \varphi(x_0, t_0) = u(x_0, t_0)\,.$$

Definition A.1. *Let $u : \Omega\times[0,\infty) \to \mathbb{R}$ be continuous.*

u is called a weak subsolution of (A.1), if for all $(x_0, t_0) \in \Omega\times(0,T)$ and all φ tangent to u from above at (x_0, t_0) we have

$$\begin{cases} \frac{\partial}{\partial t}\varphi \leq \left(I - \frac{D\varphi\otimes D\varphi}{|D\varphi|^2}\right) : D^2\varphi \ \textit{at}\ (x_0,t_0)\ \textit{if}\ D\varphi \neq 0 \\ \frac{\partial}{\partial t}\varphi \leq (I - \tau\otimes\tau) : D^2\varphi \ \textit{at}\ (x_0,t_0)\ \textit{for some}\ \tau\in\mathbb{R}^{n+1}, |\tau|\leq 1\ \textit{if}\ D\varphi = 0. \end{cases}$$

Similarly u is called a weak supersolution of (A.1), if for all $(x_0, t_0) \in \Omega\times(0,T)$ and all φ tangent to u from below at (x_0, t_0) we have

$$\begin{cases} \frac{\partial}{\partial t}\varphi \geq \left(I - \frac{D\varphi\otimes D\varphi}{|D\varphi|^2}\right) : D^2\varphi \ \textit{at}\ (x_0,t_0)\ \textit{if}\ D\varphi \neq 0 \\ \frac{\partial}{\partial t}\varphi \geq (I - \tau\otimes\tau) : D^2\varphi \ \textit{at}\ (x_0,t_0)\ \textit{for some}\ \tau\in\mathbb{R}^{n+1}, |\tau|\leq 1\ \textit{if}\ D\varphi = 0. \end{cases}$$

u is called a weak solution of (A.1), if it is a weak subsolution and a weak supersolution.

This notion of weak solution originally comes from hyperbolic conservation laws and for historic reasons it used to be called viscosity solution. The theory of viscosity solutions has been developed for Hamilton-Jacobi equations by M.G. Crandall, L.C. Evans, H. Ishii and P.L. Lions (see Crandall & Lions [1983]; Crandall *et al.* [1984, 1992]). Based on the level-set flow formulation various powerful techniques for numerically solving the mean curvature flow have been developed and are especially capable of handling non-smooth surfaces and changes in topology. This has opened the door to a great variety of applications of the mean curvature flow and other curvature flows, including geometry, fluid dynamics and image processing (see e.g. Sethian [1985]; Osher & Sethian [1988]; Sethian [1999]).

We want to state two important properties of weak solutions of (A.1). The first one is the uniform convergence property. For a proof see [Evans & Spruck, 1991, theorem 2.7] or [Ilmanen, 1992, 3.1].

Lemma A.2. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of weak solutions of (A.1) and suppose $u_k \to u$ locally uniformly as $k \to \infty$. Then u is a weak solution of (A.1).*

The second lemma we are interested in is a version of an approximation lemma, which was originally used to show existence of solutions of (A.1) (see [Evans & Spruck, 1991, proof of theorem 4.2] or [Ilmanen, 1992, 5.1]).

Lemma A.3. *Let $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ be a sequence of smooth functions $u_{\varepsilon_k} : \Omega \times [0,\infty) \to \mathbb{R}$, classicly solving an approximation to the level-set flow*

$$\frac{\partial}{\partial t} u_{\varepsilon_k} = \left(I - \frac{Du_{\varepsilon_k} \otimes Du_{\varepsilon_k}}{|Du_{\varepsilon_k}|^2 + \varepsilon_k^2} \right) : D^2 u_{\varepsilon_k} + R_k$$

with $R_k \to 0$ as $k \to \infty$. Suppose that $\varepsilon_k \to 0$ as $k \to \infty$ and that $u_{\varepsilon_k} \to u$ uniformly in C^0 on any given $\Omega \times [0,T]$ as $k \to \infty$. Then u is a weak solution of (A.1).

We now follow Evans & Spruck [1991] and Ilmanen [1992] in presenting existence and uniqueness results for level-set flow evolutions in $\mathbb{R}^{n+1}$.

Definition A.4. *A family $(\Gamma_t)_{t\in[0,T)}$ of subsets of $\mathbb{R}^{n+1}$ is said to move by the level-set flow, if $\Gamma_t = u(\cdot,t)^{-1}(0)$ for some solution u of (A.1).*

For the compact case we find in [Evans & Spruck, 1991, theorem 4.2] respectively [Ilmanen, 1992, theorem 6.4] the following existence and uniqueness

Theorem A.5. *For any compact initial set $\Gamma_0 \subset \mathbb{R}^{n+1}$ there is a unique solution $(\Gamma_t)_{t\in[0,\infty)}$ of the level-set flow for all time and each Γ_t is compact.*

Naturally we expect the weak evolution to agree with the classical motion by mean curvature, as long as the latter exists. This was shown by L.C. Evans and J.Spruck (see [Evans & Spruck, 1991, theorem 6.1] and Evans & Spruck [1992a]; Hamilton [1982]; Gage & Hamilton [1986]). By classical motion by mean curvature we mean smooth solutions of (3.1) respectively (4.1).

Theorem A.6. *Let $\Gamma_0 \subset R^{n+1}$ be a smooth hypersurface, which is the connected boundary of a bounded open set. Then there exists a time $T > 0$ such that for $0 \le t < T$ a classical mean curvature evolution M_t of Γ_0 exists. Moreover we have $\Gamma_t = M_t$ for $0 \le t < T$ for the level-set flow evolution $(\Gamma_t)_{t\in[0,\infty)}$ of Γ_0.*

In the noncompact case the level-set flow evolution can be nonunique, as [Ilmanen, 1992, example 7.3] shows. However one gets a unique evolution by considering the maximal evolution in the following sense.

Definition A.7. *We call a family of closed sets $(\Gamma_t)_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$ a biggest flow, if $\Gamma_t = u(\cdot,t)^{-1}(0)$, where u solves (A.1) on $\mathbb{R}^{n+1}\times[0,T)$ and $\lim_{(x,t)\to\infty} u(x,t) = 0$.*

This means that all level-sets of u, apart from at most one, are compact. It is shown in [Ilmanen, 1992, theorem 7.2], that this definition is justified.

Theorem A.8. *For any closed set $\Gamma_0 \subset \mathbb{R}^{n+1}$ the biggest flow $(\Gamma_t)_{t\in[0,\infty)}$ is unique, exists forever and contains any other level-set flow with the same initial condition.*

Additionally we want to provide T. Ilmanen's very nice characterisation of the biggest flow from Ilmanen [1993]. One can define set-theoretic mean curvature flow subsolutions and establish a maximum principle (see [Ilmanen, 1993, definition 4A] and [Ilmanen, 1993, lemma 4E]):

Definition A.9. *A set-theoretic subsolution of the mean curvature flow starting at a closed set $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a family $(\Gamma_t)_{t\in[0,\infty)}$ of closed sets such that $\Delta_t \cap \Gamma_t = \emptyset$ for $t \in [0,T]$ and any compact, smooth solution $(\Delta_t)_{t\in[0,T]}$ of the mean curvature flow with $\Delta_0 \cap \Gamma_0 = \emptyset$.*

Lemma A.10. *Let $(\Gamma_t)_{t\in[0,\infty)}$, $(\Delta_t)_{t\in[0,\infty)}$ be two set-theoretic subsolutions of the mean curvature flow such that Δ_0 is compact and $\Delta_0 \cap \Gamma_0 = \emptyset$. Then the Hausdorff distance $dist(\Delta_t, \Gamma_t)$ is nonincreasing.*

Now suppose $(\Gamma_t)_{t\in[0,\infty)}$ is a biggest flow as in definition A.7. It is clear from the definition of weak solution of (A.1), that $(\Gamma_t)_{t\in[0,\infty)}$ is a set-theoretic subsolution. Furthermore any set set-theoretic subsolution, which is contained within the biggest flow at $t = 0$, must remain within the biggest flow. Otherwise it would collide with one of the other (compact) level-sets of $u(\cdot, t)$, $\Gamma_t = u(\cdot, t)^{-1}(0)$, which would violate the maximum principle from lemma A.10. Therefore we have (see Ilmanen [1994]) the following

Lemma A.11. *The biggest flow is the maximal set-theoretic subsolution in the sense that it contains all other set-theoretic subsolutions with the same initial condition.*

Bibliography

Andrews, B. 1994. Harnack inequalities for evolving hypersurfaces. *Mathematische Zeitschrift*, **217**(1), 179–197.

Angenent, S., Ilmanen, T., & Chopp, D.L. 1995. A computed example of nonuniqueness of mean curvature flow in $\mathbb{R}^3$. *Communications in Partial Differential Equations*, **20**(11-12), 1937–1958.

Baret, J.C., & Mugele, F. 2006. Electrical discharge in capillary breakup: Controlling the charge of a droplet. *Physical Review Letters*, **96**(1), 16106.

Barles, G. 1985. Remarks on a flame propagation model. *INRIA*, **451**.

Barles, G., Biton, S., & Ley, O. 2002. A geometrical approach to the study of unbounded solutions of quasilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, **162**(4), 287–325.

Barles, G., Biton, S., Bourgoing, M., & Ley, O. 2003. Uniqueness results for quasilinear parabolic equations through viscosity solutions' methods. *Calculus of Variations and Partial Differential Equations*, **18**(2), 159–179.

Baygents, J.C., & Saville, D.A. 1991. Electrophoresis of drops and bubbles. *Journal of the Chemical Society, Faraday Transactions*, **87**(12), 1883–1898.

Bird, J.C., Ristenpart, W.D., Belmonte, A., & Stone, H.A. 2009. Critical angle for electrically driven coalescence of two conical droplets. *Physical Review Letters*, **103**(16), 164502.

Bode, J. 2007. *Mean Curvature Flow of Cylindrical Graphs*. Ph.D. thesis, Freie Universität Berlin, Universitätsbibliothek.

Brakke, K.A. 1978. *The motion of a surface by its mean curvature*. Princeton University Press Princeton, New Jersey.

Brendle, S. 2009. A generalization of Hamilton's differential Harnack inequality for the Ricci flow. *Journal of Differential Geometry*, **82**, 207–227.

Brendle, S., & Schoen, R. 2011. Curvature, sphere theorems, and the Ricci flow. *Bulletin of the American Mathematical Society*, **48**, 1–32.

Cabezas-Rivas, E., & Topping, P.M. 2012a. The canonical expanding soliton and Harnack inequalities for Ricci flow. *Transactions of the American Mathematical Society*, **364**, 3001–3021.

Cabezas-Rivas, E., & Topping, P.M. 2012b. The canonical shrinking soliton associated to a Ricci flow. *Calculus of Variations and Partial Differential Equations*, **43**(1), 173–184.

Cao, H.D. 1992. On Harnack's inequalities for the Kähler-Ricci flow. *Inventiones Mathematicae*, **109**(1), 247–263.

Cao, X., & Hamilton, R.S. 2009. Differential Harnack estimates for time-dependent heat equations with potentials. *Geometric and Functional Analysis*, **19**(4), 989–1000.

Chen, Y.G., Giga, Y., & Goto, S. 1991. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Journal of Differential Geometry*, **33**(3), 749–786.

Chow, B. 1991. On Harnack's inequality and entropy for the Gaussian curvature flow. *Communications on Pure and Applied Mathematics*, **44**(4), 469–483.

Chow, B. 1992. The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. *Communications on Pure and Applied Mathematics*, **45**(8), 1003–1014.

Chow, B., & Chu, S.C. 1995. A geometric interpretation of Hamilton's Harnack inequality for the Ricci flow. *Mathematical Research Letters*, **2**, 701–718.

Chow, B., & Chu, S.C. 2001. Space-time formulation of Harnack inequalities for curvature flows of hypersurfaces. *Journal of Geometric Analysis*, **11**(2), 219–231.

Chow, B., & Knopf, D. 2002. New Li-Yau-Hamilton inequalities for the Ricci flow via the space-time approach. *Journal of Differential Geometry*, **60**(1), 1–54.

Colding, T.H., & Minicozzi II, W. 2009. Generic mean curvature flow I; generic singularities. *Arxiv preprint arXiv:0908.3788.*

Colding, T.H., Ilmanen, T., Minicozzi II, W., & White, B. 2012. The round sphere minimizes entropy among closed self-shrinkers. *Arxiv preprint arXiv:1205.2043.*

Collins, R.T., Jones, J.J., Harris, M.T., & Basaran, O.A. 2007. Electrohydrodynamic tip streaming and emission of charged drops from liquid cones. *Nature*, **4**(2), 149–154.

Crandall, M.G., & Lions, P.L. 1983. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, **277**(1).

Crandall, M.G., Evans, L.C., & Lions, P.L. 1984. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, **282**(2).

Crandall, M.G., Ishii, H., & Lions, P.L. 1992. *User's guide to viscosity solutions of second order partial differential equations.* American Mathematical Society.

De Lellis, C., & Székelyhidi, L. 2009. The Euler equations as a differential inclusion. *Annals of Mathematics*, **170**, 1417–1436.

Deligne, P., Etingof, P., & Freed, D.S. 1999. *Quantum fields and strings: a course for mathematicians.* Vol. 2. American Mathematical Society.

Ding, W., Li, J., & Liu, Q. 2006. Evolution of minimal torus in Riemannian manifolds. *Inventiones Mathematicae*, **165**(2), 225–242.

Ecker, K. 2004. *Regularity theory for mean curvature flow.* Vol. 57. Birkhäuser.

Ecker, K. 2007. A formula relating entropy monotonicity to Harnack inequalities. *Communications in Analysis and Geometry*, **15**(5), 1025–1061.

Ecker, K., & Huisken, G. 1989. Mean curvature evolution of entire graphs. *Annals of Mathematics*, **130**(3), 453–471.

Eells, J., & Sampson, J.H. 1964. Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics*, **86**(1), 109–160.

Eggers, J., Lister, J.R., & Stone, H.A. 1999. Coalescence of liquid drops. *Journal of Fluid Mechanics*, **401**(1), 293–310.

Eow, J.S., Ghadiri, M., Sharif, A.O., & Williams, T.J. 2001. Electrostatic enhancement of coalescence of water droplets in oil: a review of the current understanding. *Chemical Engineering Journal*, **84**(3), 173–192.

Evans, L.C., & Spruck, J. 1991. Motion of level sets by mean curvature I. *Journal of Differential Geometry*, **33**(3), 635–681.

Evans, L.C., & Spruck, J. 1992a. Motion of level sets by mean curvature II. *Transactions of the American Mathematical Society*, **330**(1), 321–332.

Evans, L.C., & Spruck, J. 1992b. Motion of level sets by mean curvature III. *Journal of Geometric Analysis*, **2**(2), 121–150.

Evans, L.C., & Spruck, J. 1995. Motion of level sets by mean curvature IV. *Journal of Geometric Analysis*, **5**(1), 77–114.

Fenn, J.B., Mann, M., Meng, C.K., Wong, S.F., & Whitehouse, C.M. 1989. Electrospray ionization for mass spectrometry of large biomolecules. *Science*, **246**(4926), 64–71.

Fernández De La Mora, J. 2007. The fluid dynamics of Taylor cones. *Annual Review of Fluid Mechanics*, **39**, 217–243.

Gage, M., & Hamilton, R.S. 1986. The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, **23**(1), 69–96.

Gastel, A. 2003. Nonuniqueness for the Yang–Mills heat flow. *Journal of Differential Equations*, **187**(2), 391–411.

Germain, P., & Rupflin, M. 2011. Selfsimilar expanders of the harmonic map flow. *In: Annales de l'Institut Henri Poincare (C) Non Linear Analysis*. Elsevier.

Gibbons, G.W., & Hawking, S.W. 1977. Action integrals and partition functions in quantum gravity. *Physical Review D*, **15**(10), 2752.

Hamilton, R.S. 1982. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, **17**(2), 255–306.

Hamilton, R.S. 1993. The Harnack estimate for the Ricci flow. *Journal of Differential Geometry*, **37**(1), 225–243.

Hamilton, R.S. 1995a. The formation of singularities in the Ricci flow. *Surveys in Differential Geometry*, **2**, 7–136.

Hamilton, R.S. 1995b. Harnack estimate for the mean curvature flow. *Journal of Differential Geometry*, **41**(1), 215–226.

Helmensdorfer, S. 2012a. A model for the behavior of fluid droplets based on mean curvature flow. *SIAM Journal on Mathematical Analysis*, **44**, 1359–1371.

Helmensdorfer, S. 2012b. Space-time constructions for the mean curvature flow in a Ricci flow background. *Arxiv preprint arXiv:1206.2829v2*.

Huisken, G. 1984. Flow by mean curvature of convex surfaces into spheres. *Journal of Differential Geometry*, **20**(1), 237–266.

Huisken, G. 1990. Asymptotic behavior for singularities of the mean curvature flow. *Journal of Differential Geometry*, **31**(1), 285–299.

Huisken, G., & Ilmanen, T. 2001. The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry*, **59**(3), 353–437.

Huisken, G., & Sinestrari, C. 2009. Mean curvature flow with surgeries of two–convex hypersurfaces. *Inventiones Mathematicae*, **175**(1), 137–221.

Ilmanen, T. 1992. Generalized flow of sets by mean curvature on a manifold. *Indiana University Mathematics Journal*, **41**(3), 671–705.

Ilmanen, T. 1993. The level-set flow on a manifold. *Pages 193–204 of: Proceedings of Symposia in Pure Mathematics*, vol. 54.

Ilmanen, T. 1994. *Elliptic regularization and partial regularity for motion by mean curvature*. American Mathematical Society.

Ilmanen, T. 1995. *Lectures on mean curvature flow and related equations*. ICTP, Trieste.

Jost, J. 2011. *Riemannian geometry and geometric analysis*. Springer Verlag.

Kapouleas, N., Kleene, S.J., & Møller, N.M. 2011. Mean curvature self-shrinkers of high genus: non-compact examples. *Arxiv preprint arXiv:1106.5454*.

Kleene, S.J., & Møller, N.M. 2010. Self-shrinkers with a rotational symmetry. *Arxiv preprint arXiv:1008.1609*.

Kleiner, B., & Lott, J. 2008. Notes on Perelmans papers. *Geometry and Topology*, **12**(5), 2587–2855.

Kotschwar, B. 2009. Harnack inequalities for evolving convex hypersurfaces from the space-time perspective. *Preprint*.

Ladyženskaja, O.A., Solonnikov, V.A., & Uralceva, N.N. 1968. *Linear and quasi-linear equations of parabolic type*. American Mathematical Society Providence, Rhode Island.

Leunissen, M.E., Christova, C.G., Hynninen, A.P., Royall, C.P., Campbell, A.I., Imhof, A., Dijkstra, M., Van Roij, R., & Van Blaaderen, A. 2005. Ionic colloidal crystals of oppositely charged particles. *Nature*, **437**(7056), 235–240.

Li, P., & Yau, S.T. 1986. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, **156**(1), 153–201.

Link, D.R., Grasland-Mongrain, E., Duri, A., Sarrazin, F., Cheng, Z., Cristobal, G., Marquez, M., & Weitz, D.A. 2006. Electric control of droplets in microfluidic devices. *Angewandte Chemie International Edition*, **45**(16), 2556–2560.

Lott, J. 2008. *Notes and commentary on Perelman's Ricci flow papers*. http://math.berkeley.edu/~lott/ricciflow/perelman.html.

Lott, J. 2012. Mean curvature flow in a Ricci flow background. *Communications in Mathematical Physics*, **313**, 517–533.

Magni, A., Mantegazza, C., & Tsatis, E. 2009. Flow by Mean Curvature inside a Moving Ambient Space. *Arxiv preprint arXiv:0911.5130*.

McCann, R.J., & Topping, P.M. 2010. Ricci flow, entropy and optimal transportation. *American Journal of Mathematics*, **132**(3), 711–730.

Melcher, J.R., & Taylor, G.I. 1969. Electrohydrodynamics: a review of the role of interfacial shear stresses. *Annual Review of Fluid Mechanics*, **1**(1), 111–146.

Moser, J. 1964. A Harnack inequality for parabolic differential equations. *Communications on Pure and Applied Mathematics*, **17**(1), 101–134.

Müller, R. 2012. Ricci flow coupled with harmonic map flow. *Annales Scientifiques de l'École Normale Supérieure*, **45**, 101–142.

Ni, L. 2004a. Addenda to The entropy formula for linear heat equation. *Journal of Geometric Analysis*, **14**(2), 369–374.

Ni, L. 2004b. The entropy formula for linear heat equation. *Journal of Geometric Analysis*, **14**(1), 87–100.

Ochs, H.T., & Czys, R.R. 1987. Charge effects on the coalescence of water drops in free fall. *Nature*, **327**(1038), 606–608.

Oddershede, L., & Nagel, S.R. 2000. Singularity during the onset of an electrohydrodynamic spout. *Physical Review Letters*, **85**(6), 1234–1237.

Osher, S., & Sethian, J.A. 1988. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, **79**(1), 12–49.

Perelman, G. 2002. The entropy formula for the Ricci flow and its geometric applications. *Arxiv preprint math/0211159*.

Perelman, G. 2003a. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *Arxiv preprint math/0307245*.

Perelman, G. 2003b. Ricci flow with surgery on three-manifolds. *Arxiv preprint math/0303109*.

Rayleigh, L. 1878. The influence of electricity on colliding water drops. *Proceedings of the Royal Society of London*, **28**(190-195), 405–409.

Ristenpart, W.D., Bird, J.C., Belmonte, A., Dollar, F., & Stone, H.A. 2009. Non-coalescence of oppositely charged drops. *Nature*, **461**(7262), 377–380.

Rupflin, M., & Topping, P.M. 2012. Flowing maps to minimal surfaces. *Arxiv preprint arXiv:1205.6298*.

Saville, D.A. 1997. Electrohydrodynamics: the Taylor-Melcher leaky dielectric model. *Annual Review of Fluid Mechanics*, **29**(1), 27–64.

Sethian, J.A. 1985. Curvature and the evolution of fronts. *Communications in Mathematical Physics*, **101**(4), 487–499.

Sethian, J.A. 1999. *Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*. Cambridge University Press.

Shnirelman, A. 1997. On the nonuniqueness of weak solution of the Euler equation. *Communications on Pure and Applied Mathematics*, **50**(12), 1261–1286.

Simon, M. 1990. *Mean curvature flow of rotationally symmetric hypersurfaces, Honours thesis.*

Smoczyk, K. 1997. Harnack inequalities for curvature flows depending on mean curvature. *New York Journal of Mathematics*, **3**(103), 118.

Streets, J.D. 2007. *Ricci Yang-Mills flow.* Ph.D. thesis, Duke University.

Thurston, W.P. 1982. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bulletin of the American Mathematical Society*, **6**(3).

Topping, P.M. 2006. *Lectures on the Ricci flow.* Vol. 325. Cambridge University Press.

Topping, P.M. 2009. $\mathcal{L}$-optimal transportation for Ricci flow. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, **636**, 93–122.

Topping, P.M. 2010. Reverse bubbling in geometric flows. *Preprint.*

Trau, M., Yao, N., Kim, E., Xia, Y., Whitesides, G.M., & Aksay, I.A. 1997. Microscopic patterning of orientated mesoscopic silica through guided growth. *Nature*, **390**(6661), 674–676.

Tsatis, E. 2010. Mean curvature flow on Ricci solitons. *Journal of Physics A: Mathematical and Theoretical*, **43**, 045202.

von Renesse, M.K., & Sturm, K.T. 2005. Transport inequalities, gradient estimates, entropy and Ricci curvature. *Communications on Pure and Applied Mathematics*, **58**(7), 923–940.

Wilking, B. 2010. A Lie algebraic approach to Ricci flow invariant curvature conditions and Harnack inequalities. *Arxiv preprint arXiv:1011.3561.*

York Jr, J.W. 1972. Role of conformal three-geometry in the dynamics of gravitation. *Physical Review Letters*, **28**(16), 1082–1085.