A Construction of Semi-Infinite de Rham Cohomology

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Contents

Declaration

I declare that to the best of my knowledge and except where indicated otherwise in the text, the work contained in this thesis is my own.

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Abstract

The purpose of this thesis is to describe a construction of semi-infinite de Rham cohomology for infinite dimensional manifolds equipped with the extra structure of a polarisation. We describe the construction for finite dimensions and show how it extends to other cases; in particular the semi-infinite. We then define variations for Hilbert manifolds which allow us to calculate the semi-infinite cohomology of the projective space and the Grassmannians of a polarised Hilbert space. Finally, we consider some of the implications of these results for index theory, in particular for the Witten genus.

Chapter 1

Introduction

There are many different constructions of cohomology for smooth, finite dimensional manifolds; for example, the de Rham, singular, Čech, Morse and cellular constructions. For Lie groups there is the Lie algebra cohomology which gives the de Rham cohomology when the group is a closed manifold. The de Rham theory is a good construction for smooth manifolds because it just involves the smooth structure of the manifold and is a direct cohomology construction. It is also a starting point for other analytical constructions such as Hodge theory, Harmonic theory and Index theory.

The theory of semi-infinite manifolds starts with polarisations. A polarisation of an infinite dimensional complex vector space E is a decomposition $E = E_- \oplus E_+$ where E_- and E_+ are closed, infinite dimensional subspaces of E. This is usually defined up to a particular notion of equivalence. A polarisation on a smooth infinite dimensional manifold is a local choice of equivalence class of polarisation on the complexified tangent bundle of the manifold.

The standard examples of polarised complex Hilbert spaces are the spaces $L^2(S^1, \mathbb{C}^n)$ for $n \in \mathbb{N}$. These have bases given by the Laurent monomials: $\{z_l^k : k \in \mathbb{Z}, 1 \leq l \leq n\}$. The polarisation $H_- \oplus H_+$ is such that H_- is the closure of the span of $\{z_l^k : k < 0, 1 \leq l \leq n\}$ and H_+ is the closure of the span of $\{z_l^k : k \geq 0, 1 \leq l \leq n\}$. The main examples of polarised manifolds are the projective space of a polarised space, the Grassmannian of k-planes in a polarised space, and the based loop space of a finite dimensional manifold.

At present, the main semi-infinite cohomology theory is given by Floer theory (see, for example, Salamon [20]). This is the analogue of Morse theory and was developed to solve the Arnold conjecture for monotone symplectic manifolds. It is also closely related to quantum cohomology and the theory of J-holomorphic curves. There is also a semi-infinite Lie algebra cohomology (see, for example, Feigin and Frenkel [8]) which is known in the physics literature as BRST cohomology and is closely linked to string theory. However, until now no theory of semi-infinite de Rham cohomology has been put forward. With a de Rham theory one would hope to be able to extend finite dimensional analytical objects to the infinite dimensional case; especially those objects which depend upon a "middle dimension", such as the signature operator.

We also construct a finite codimension cohomology theory. There have already been some suggestions for finite codimension cohomology theories, the closest to our construction being that of Ramer [19]; another is explained in Mukherjea [16]. Both of these theories rely on a duality and construct the cohomology theory as a dual to a standard homology theory. Our construction differs from these in that it does not use duality.

1.1 The Grassmannian Construction of Cohomology

The construction of the cohomology theory is based on the isomorphism $\Lambda^k V^* \cong A^k(V)$ for a finite dimensional complex vector space V, where $A^k(V) = \Gamma_{\text{hol}}(\mathrm{Gr}_k(V), D^*)$ is the space of holomorphic sections of the dual of the determinant line bundle over the Grassmannian of k dimensional subspaces of V. In finite dimensions this isomorphism allows us to translate the standard de Rham construction into one using the determinant line bundle. The advantage of this construction is that the space $\Gamma_{\text{hol}}(\mathrm{Gr}_k(V), D^*)$ makes sense in infinite dimensions as well. Thus the cochain groups in the cohomology theory are certain sections of $\Gamma_{\text{hol}}(\mathrm{Gr}_k(T_{\mathbb{C}}M), D^*)$ over the manifold M.

In finite dimensions the differential d is defined locally using differentiation $D : C^\infty(U, A^k(T_{\mathbb{C}}U)) \to C^\infty(U, \mathfrak{L}(T_{\mathbb{C}}U, A^k(T_{\mathbb{C}}U)))$ followed by a contraction map $\wedge : \mathfrak{L}(T_{\mathbb{C}}U, A^k(T_{\mathbb{C}}U)) \to A^{k+1}(T_{\mathbb{C}}U)$; where for topological vector spaces X and Y, $\mathfrak{L}(X, Y)$ denotes the space of continuous linear maps from X to Y. In infinite dimensions the domain of the contraction map is the tensor product $T^*_{\mathbb{C}}U \otimes A^k(T_{\mathbb{C}}U)$ and does not extend to the whole of $\mathfrak{L}(T_{\mathbb{C}}U, A^k(T_{\mathbb{C}}U))$. To overcome this problem, we first extend the contraction map to a larger space and secondly we restrict the sections to a class for which d is definable.

One factor in the construction of the fibre bundle $A(T_{\mathbb{C}}M)$ over M and the global extension of the differential is that in finite dimensions the space $A(V)$ is a representation of the general linear group $\mathrm{Gl}(V)$. In infinite dimensions this is no longer the case as not all infinite dimensional operators have a determinant. In order to surmount this problem, the group which acts on the space must be altered to keep track of the determinant. This is done by taking a central extension which encodes the action on the determinant line bundle. This introduces an obstruction $b_2 \in H^3(M;\mathbb{Z})$; when this obstruction vanishes, we say that the manifold is semi-infinite. However, we need yet further conditions on the type of manifold to be able to construct the de Rham cohomology. We show that these conditions are automatically satisfied for the based loop space of an almost complex manifold and for the projective space and the Grassmannian of k planes of a polarised complex Hilbert space.

In chapter 2 we outline the construction for finite dimensional manifolds. In chapter 3 we set up the infinite dimensional apparatus needed extend this to infinite dimensional manifolds and then in chapter 4 we define the semi-infinite cohomology and remark how to adapt the definition for the finite dimension and finite codimension cohomology. One interesting aspect of the semi-infinite theory is that it is periodic. In the case of the based loop space of an almost complex manifold, the period agrees with that found in Floer theory.

1.2 Calculating Semi-Infinite de Rham Cohomology

In calculating cohomology, one important tool is the Thom isomorphism. Given an inclusion of smooth orientable closed manifolds $i : N \to M$ it is possible to find an open neighbourhood U of $i(N)$ in M which is diffeomorphic to a vector bundle over N. There is a sequence of maps in cohomology:

$$\begin{array}{ccc} & H^k(M, M\setminus U) & \longrightarrow & H^k(M) \\ & \cong\big\downarrow & & \\ H^{k-r}(N) \xrightarrow{\cong} & H^k(\overline{U}, \partial\overline{U}) & & \end{array}$$

where the vertical isomorphism is due to excision, the horizontal isomorphism is the Thom isomorphism and r is the codimension of N in M. This gives a map $H^{k-r}(N) \to H^k(M)$.

The importance of this map is the jump in dimension. It gives a way to extend a low dimension calculation, which may be relatively easy to do, to a high dimension calculation, which may be more difficult by other means. For semi-infinite and finite codimension cohomology, the Thom isomorphism is very important since with the ability to jump dimensions it is possible to extend finite dimensional calculations to the semi-infinite and finite codimension cases.

In section 4.4 we show that the finite codimension cohomology of a Hilbert manifold is trivial. This shows that in order to define a useful cohomology theory, some alteration to the definition is needed. In chapter 5 we consider two possible alterations to the cohomology theory in which there is a Thom isomorphism. The first uses the theory of Wiener integration and Wiener manifolds and the second uses the concept of filtrations of infinite dimensional manifolds by finite dimensional submanifolds.

These allow us to calculate the semi-infinite cohomology of the Grassmannian of k planes in a polarised Hilbert space H. The cohomology is:

$$H_{\text{si}}(\text{Gr}_k(H)) = \left[c_1, \ldots, c_k, {c_k}^{-1}\right]$$

In particular, the semi-infinite cohomology of the projective space is given by $H_{\text{si}}(\mathbb{P}H) = [u, u^{-1}]$ where $|u| = 2$.

1.3 The Truncated Witten Genus

One advantage of a de Rham construction of semi-infinite cohomology is the possibility of extending various finite dimensional constructions which depend on de Rham theory to infinite dimensions. The most obvious of these is the signature operator and, through that, index theory. In chapter 6 we define a semi-infinite signature operator for polarised Hilbert manifolds.

Although there is not a semi-infinite analogue of index theory as yet, some calculations have been done by blindly applying index theory to infinite dimensional manifolds, see Witten [22] for the main example of this. If there is a semi-infinite index theory then the connection in Floer theory between the semi-infinite cohomology of a manifold and the limit of the cohomology of a family of submanifolds suggests a similar connection between the semi-infinite and finite index theories. In chapter 6 we define the truncated Witten genus and show that the blind calculation agrees with the limit of the finite dimension calculations by proving the following two theorems.

Let M be an orientable manifold of even dimension $2d$ and V an orientable real vector bundle of even dimension $2r$ with $w_2(V) = w_2(TM)$. Using notation defined in section 6.1, we define the *Witten genus twisted by V and truncated at m* to be the power series in ξ defined by:

$$W_m(M,V)(\xi) = \left\langle \widehat{A}(TM,V)\,\mathrm{ch}\left(\bigotimes_{k=1}^{m} \xi^{dk} S_{\xi^k} T_{\mathbb{C}} M \bigotimes_{l=1}^{m} \xi^{-rl} \Lambda_{\xi^l} V_{\mathbb{C}}\right), [M] \right\rangle$$

Let $p \in \mathbb{N}$ and set $n = 2p+1$. Let M^n be the n-fold product of M and V^n the n-fold product of V; V^n is a vector bundle over M^n. Let C_n act on M^n and V^n by cyclic permutation of coordinates. We identify C_n with a subgroup of S^1 by choosing a primitive nth root of unity ξ. We denote the spinor bundle constructed from V^n by $\Delta(V^n)$. Although this bundle may not be well-defined over M^n, we can consider the Dirac operator on M^n twisted by $\Delta(V^n)$ because $w_2(V) = w_2(TM)$.

Theorem A. *Let D_V be the Dirac operator on M^n twisted by $\Delta(V^n)$. Then:*

$$\mathit{Index}_\xi D_V = (-1)^{(p(d+r)+d)} W_p(M,V)(\xi)$$

Define the *kth Witten characteristic class truncated at m* for a real vector bundle U of dimension $2d$ to be the power series:

$$W_{k,m}(U)(\xi) = \widehat{A}(\xi^k U)\,\mathrm{ch}\left(\bigotimes_{s=1}^{m} \xi^{sd} S_{\xi^s}(\xi^k U)_{\mathbb{C}}\right)$$

where the notation ζU denotes a vector bundle U with an action of ξ of rotation by ζ.

Let n be an odd positive integer. For $a, b \in \mathbb{Z}$ with $a \leq b$ define $Y_a^b := \mathbb{C}[z, z^{-1}]_a^b$ to be the space of Laurent polynomials in z whose terms have degree between and including a and b. Let $X_a^b = \mathbb{P}(\mathbb{C}^{n+1} \otimes Y_a^b)$. Define an action of S^1 on Y_a^b by $\xi \cdot z^r = \xi^r z^r$. This action projects down to X_a^b.

Theorem B. *Let $r \in \mathbb{Z}$ and $q \in \mathbb{N}$. There is a S^1-equivariant Dirac operator on X_{r-q}^{r+q} and for $\xi \in S^1$ of either infinite order or finite order greater than $2q$:*

$$Index_\xi D_{X_{r-q}^{r+q}} = \sum_{k=-q}^{q} - \left\langle W_{k,q}(T\mathbb{CP}^n + \mathbb{R}^2)(\xi), [\mathbb{CP}^n] \right\rangle$$

Chapter 2

The Construction in Finite Dimensions

In this chapter we describe the basics of the Grassmannian construction of de Rham cohomology for finite dimensional manifolds. The construction of de Rham cohomology is the basis for several other useful mathematical constructions nearly all of which have natural correspondents in the Grassmannian construction. We shall show how some of these de Rham based constructions for finite dimensional manifolds can be realised naturally in the Grassmannian model.

2.1 Holomorphic Sections and Exterior Powers

Let U be a complex vector space of finite dimension n. For $k \in \mathbb{N}$ with $0 \leq k \leq n$, the kth Stiefel manifold $V_k(U)$ of U is the space of ordered sets of

k linearly independent[1] vectors in U. It is an open submanifold of U^k. The kth Grassmannian manifold $\mathrm{Gr}_k(U)$ of U is the space of complex subspaces of U of dimension k. Both $V_k(U)$ and $\mathrm{Gr}_k(U)$ are complex manifolds. There is a holomorphic map $s : V_k(U) \to \mathrm{Gr}_k(U)$ which sends a linearly independent set to the subspace it spans.

The general linear group of U, $\mathrm{Gl}(U)$, acts transitively on both $V_k(U)$ and $\mathrm{Gr}_k(U)$ and each is a homogeneous space for $\mathrm{Gl}(U)$. The map $s : V_k(U) \to \mathrm{Gr}_k(U)$ is equivariant for these actions. Let $w \in V_k(U)$ and let $W \in \mathrm{Gr}_k(U)$ be the space spanned by the elements of w. Let E_k be the stabiliser of w under the action of $\mathrm{Gl}(U)$ on $V_k(U)$ and let F_k be the stabiliser of W under the action of $\mathrm{Gl}(U)$ on $\mathrm{Gr}_k(U)$. The group E_k is a normal subgroup of F_k with factor group isomorphic to $\mathrm{Gl}_k(\mathbb{C})$.

Given a character χ of E_k (resp. F_k), there is a holomorphic line bundle over $V_k(U)$ (resp. $\mathrm{Gr}_k(U)$) defined as the space $V_k(U) \times_\chi \mathbb{C}$ (resp. $\mathrm{Gr}_k(U) \times_\chi \mathbb{C}$). This is the quotient of the space $\mathrm{Gl}(U) \times \mathbb{C}$ by E_k (resp. F_k) acting on the first factor by composition and on the second factor by χ^{-1}, thus (AC, λ) is equivalent to $(C, \chi(A)\lambda)$. Since E_k is a subgroup of F_k, a character χ of F_k is also a character of E_k. There is thus a map $s_\chi : V_k(U) \times_\chi \mathbb{C} \to \mathrm{Gr}_k(U) \times_\chi \mathbb{C}$ which covers the map $s : V_k(U) \to \mathrm{Gr}_k(U)$ and is linear on fibres.

The determinant line bundle $D \to \mathrm{Gr}_k(U)$ is constructed in this manner using the character defined by $\chi(B) = \det(B|_W)$. This character is trivial on E_k and so the corresponding holomorphic line bundle on $V_k(U)$ is trivial. Thus an element of D can be represented by the equivalence class of an element of $V_k(U) \times \mathbb{C}$ under the action of $\mathrm{Gl}_k(\mathbb{C})$. We shall write such an

[1] We use the convention that the emptyset, $\emptyset$, is a linearly independent set of 0 vectors.

element as $[w, \lambda]$ where $w \in V_k(U)$ and $\lambda \in \mathbb{C}$. If $[w, \lambda] = [u, \mu]$ then there is a transformation $B \in \mathrm{Gl}_k(\mathbb{C})$ such that $Bw = u$ and $\lambda = \mu \det B$. The action of $\mathrm{Gl}(U)$ on $\mathrm{Gr}_k(U)$ extends to an action on D via $B\,[w, \lambda] = [Bw, \lambda]$.

Define $A^k(U)$ to be the space of holomorphic sections of D^* over $\mathrm{Gr}_k(U)$, where D^* is the dual bundle to D. Let $A(U) = \bigoplus_k A^k(U)$. $A^k(U)$ can also be thought of as the space of holomorphic maps $D \to \mathbb{C}$ which are linear on the fibres of D.

Since $\mathrm{Gl}(U)$ acts on D, it acts on $A^k(U)$ via $(Bf)([w, \lambda]) = f(B\,[w, \lambda])$. If U is the complexification $Y \otimes \mathbb{C}$ of some n dimensional real vector space Y then there is an action of $\mathrm{Gl}(Y)$ on $A^k(U)$ via the natural inclusion $\mathrm{Gl}(Y) \to \mathrm{Gl}(U)$. It is a well-known result that as $\mathrm{Gl}(U)$ representations, $A^k(U)$ and $\Lambda^k U^*$ are isomorphic. See Pressley and Segal [18], proposition 2.9.2, for a proof of this. The map $\Lambda^k U^* \to A^k(U)$ is given by:

$$(a_1 \wedge \cdots \wedge a_k)([w, \lambda]) = \lambda \det(a_i(w_j))$$

where $(a_i(w_j))$ is the $k \times k$ matrix with value $a_i(w_j)$ in the (i, j) entry.

2.2 The Grassmannian Operators

Let M be a smooth manifold of dimension n. Let Q be a principal bundle over M with fibre $\mathrm{Gl}_m(\mathbb{F})$ where $\mathbb{F}$ is one of $\mathbb{R}$ or $\mathbb{C}$. Let $E \to M$ be the complex vector bundle over M given by the natural complex representation of $\mathrm{Gl}_m(\mathbb{F})$. There are also vector bundles over M with fibre $A^k(\mathbb{C}^m)$ given by $Q \times_{\mathrm{Gl}_m(\mathbb{F})} A^k(\mathbb{C}^m)$. Let $A^k(E) = Q \times_{\mathrm{Gl}_m(\mathbb{F})} A^k(\mathbb{C}^m)$ and let $A(E) =$

$\bigoplus_k A^k(E)$. Since for a vector space U, $A^k(U)$ is isomorphic to $\Lambda^k U^*$, the space $C^\infty(M; A^k(T_{\mathbb{C}}M))$ is isomorphic to $\Omega^k(M; \mathbb{C})$.

There are various standard operators on the spaces $\Omega^k(M; \mathbb{C})$. We now translate the definitions of these into the context of the Grassmannian construction.

Firstly, we define two contraction operators, $\wedge, \iota$. These are defined respectively as linear maps $U^* \otimes A^k(U) \to A^{k+1}(U)$ and $U \otimes A^k(U) \to A^{k-1}(U)$ which are equivariant with respect to the $\mathrm{Gl}(U)$ action. Because of this equivariance, they extend to fibrewise maps over the manifold.

The operator $\wedge$ is defined by $a \otimes f \to a \wedge f$ where the value of $a \wedge f$ on an element $[w, \lambda]$ of $D \to \mathrm{Gr}_{k+1}(U)$ is given by the formula:

$$(a \wedge f)([w, \lambda]) = \begin{cases} 0 & \text{if } k = n \text{ or } a|_{\langle w \rangle} = 0 \\ a(\alpha) f([u, \lambda]) & \text{otherwise} \end{cases}$$

where u is a basis of $\langle w \rangle \cap \ker a$ and $\alpha \in \langle w \rangle$ is such that $[w, \lambda] = [\{\alpha\} \cup u, \lambda]$.

The operator ι is defined by $\alpha \otimes f \to \iota_\alpha f$ where the value of $\iota_\alpha f$ on an element $[w, \lambda]$ of $D \to \mathrm{Gr}_{k-1}(U)$ is given by the formula:

$$(\iota_\alpha f)([w, \lambda]) = \begin{cases} 0 & \text{if } k = 0 \text{ or } \alpha \in \langle w \rangle \\ f([\{\alpha\} \cup w, \lambda]) & \text{otherwise} \end{cases}$$

We now define the Hodge $*$ operator. This is a duality map $* : A^k(U) \to A^{n-k}(U^*)$ which is invariant under the action of $\mathrm{Sl}(U)$, the normal subgroup of $\mathrm{Gl}(U)$ of operators with determinant 1. Thus it can be defined over a manifold M if and only if M is orientable. It depends upon a choice of orientation and so $*$ is well-defined and canonical on *oriented* manifolds.

The map $*$ is defined firstly as a map from the determinant bundle over $\mathrm{Gr}_k(U)$ to the determinant bundle over $\mathrm{Gr}_{n-k}(U^*)$, where U^* is the dual space of U, this extends to $* : A^k(U) \to A^{n-k}(U^*)$ via the formula $*f(\alpha) = f(*\alpha)$.

Consider first the case when $k = n$. The space $\mathrm{Gr}_n(U)$ consists of the single point $\{U\}$ and therefore $D \to \mathrm{Gr}_n(U)$ is isomorphic to $\mathbb{C}$. Thus also $A^n(U)$ is isomorphic to $\mathbb{C}$. These isomorphisms are not canonical and depend upon a choice of non-zero element, either η in D or g in $A^n(U)$. Given a non-zero $g \in A^n(\mathbb{C})$ there is a unique $\eta \in D$ such that $g(\eta) = 1$. The map $*$ is defined by $*(\lambda\eta) = [\emptyset, \lambda]$.

For $k \neq n$ the map is defined as follows. Given an element $[w, \lambda]$ of $D \to \mathrm{Gr}_k(U)$ there is an extension of w to a basis $w \cup v$ for U such that $[w \cup v, 1] = \eta$. This has dual basis $w' \cup v'$ in W^*. The set v' spans the subspace W^0 of U^* which consists of those maps which are zero on W. The map $*$ is defined by $*[w, \lambda] = [v', \lambda]$.

This map depends upon the choice of g and thus is invariant under the action of the subgroup of $\mathrm{Gl}(U)$ which preserves g. As the action of $\mathrm{Gl}(U)$ on $A^n(U)$ is given by $Bf = \det Bf$ this subgroup is $\mathrm{Sl}(U)$. Thus $*$ can only be defined over a manifold where the structure group of $T_{\mathbb{C}}M$ can be reduced to $\mathrm{Sl}_n(\mathbb{C})$, which is equivalent to the line bundle $A^n(T_{\mathbb{C}}M)$ being trivial. A smooth never zero section of this bundle, equivalent to a choice of orientation, gives a choice of g on every fibre and thus defines $*$ fibrewise. Thus $*$ is canonically defined for oriented manifolds.

As $(U^*)^* = U$ there is a similarly defined map $* : A^k(U) \to A^{n-k}(U^*)$; the

$*$ maps satisfy $*^2 = (-1)^{k(n-k)}$. There is a connection between the operators $\wedge, \iota, *$ given by the formula $\iota_\alpha f = (-1)^{nk+k+1} * (\alpha \wedge *f)$.

We now define two differential operators:

$$d : C^\infty(M, A^k(T_{\mathbb{C}}M)) \to C^\infty(M, A^{k+1}(T_{\mathbb{C}}M))$$
$$d^* : C^\infty(M, A^k(T^*_{\mathbb{C}}M)) \to C^\infty(M, A^{k-1}(T^*_{\mathbb{C}}M))$$

Let P be the frame bundle of M. Let $V \subseteq M$ be the domain of a chart in M. The chart map $\phi : V \to W \subseteq \mathbb{R}^n$ defines a trivialisation of $P|_V$ and thus a trivialisation of each of the bundles $A^k(T_{\mathbb{C}}V)$. A smooth section of $A^k(T_{\mathbb{C}}V)$ corresponds to a smooth map $W \to A^k(\mathbb{C}^n)$. Differentiation of such maps is itself a map $D : C^\infty(W, A^k(\mathbb{C}^n)) \to C^\infty(W, \mathfrak{L}(\mathbb{C}^n, A^k(\mathbb{C}^n)))$. As we are in finite dimensions, $\mathfrak{L}(\mathbb{C}^n, A^k(\mathbb{C}^n)) \cong \mathbb{C}^{n*} \otimes A^k(\mathbb{C}^n)$ and thus we can use the operator $\wedge$ to construct a map $d = \wedge D : C^\infty(W, A^k(\mathbb{C}^n)) \to C^\infty(W, A^{k+1}(\mathbb{C}^n))$. This map extends globally and uniquely over the manifold and satisfies $d^2 = 0$.

This map can also be constructed using a connection on M. Since each $A^k(T_{\mathbb{C}}M)$ is defined using a representation of the frame bundle of M, a connection on M defines a covariant differential operator on each $A^k(T_{\mathbb{C}}M)$. This is a map:

$$\begin{aligned}\nabla : C^\infty(M, A^k(T_{\mathbb{C}}M)) \to C^\infty(M, \mathfrak{L}(T_{\mathbb{C}}M, A^k(T_{\mathbb{C}}M)))\\ = C^\infty(M, T^*_{\mathbb{C}}M \otimes A^k(T_{\mathbb{C}}M))\end{aligned}$$

and combining this with $\wedge$ defines a differential operator d^∇. If the connection on M is torsion free then $d^\nabla d^\nabla = 0$ and d^∇ is unique. The uniqueness of both d and d^∇ together with the fact that the differentiation map D is a local connection show that $d = d^\nabla$.

The operator d^* is constructed in the same way except that the bundles $A^k(T^*_{\mathbb{C}}M)$ and the contraction operator ι are used. When the manifold is orientable a Riemann structure on the manifold gives a map $T_{\mathbb{C}}M \to T^*_{\mathbb{C}}M$ which is conjugate linear on fibres. Under these circumstances d^* induces a differential operator $d^* : C^\infty(M, A^k(T_{\mathbb{C}}M)) \to C^\infty(M, A^{k-1}(T_{\mathbb{C}}M))$ and we can define the signature and Laplacian operators acting on $C^\infty(M, A(T_{\mathbb{C}}M))$ as $d + d^*$ and $dd^* + d^*d$ respectively.

These operators are all defined so that under the isomorphism $\Lambda U^* \cong A(U)$ they translate to the corresponding operator in the standard de Rham theory. Using this fact shows that the de Rham cohomology of the manifold M is the cohomology of the cochain complex $(C^\infty(M, A^k(T_{\mathbb{C}}M)), d)$.

Chapter 3

The Apparatus in Infinite Dimensions

The goal of this chapter and the next is to extend the Grassmannian construction of de Rham cohomology to infinite dimensional manifolds. There are three basic extensions, namely to finite dimension[1] cohomology, semi-infinite cohomology and finite codimension cohomology. There are further variations on these themes which will be explored in chapter 5. We shall concentrate on the semi-infinite theory, making mention where necessary of how the theory needs to be altered for the other two possibilities.

In this chapter we develop the apparatus necessary for the construction of semi-infinite de Rham cohomology. Our aim is to construct the vector spaces $A_{\mathrm{si}}(X)$ for a certain type of infinite dimensional vector space X and to define the contraction map $\wedge$ on $X^* \otimes A_{\mathrm{si}}(X)$. These definitions mirror those of chapter 2 but because we are now in infinite dimensions, we need to

[1]This refers to the finite dimension of the forms, not to the dimension of the manifold.

examine these definitions in depth.

3.1 Topological Vector Spaces

In this section we gather together the necessary tools that we shall need from functional analysis. In studying an infinite dimensional manifold, it is necessary to have an understanding of the underlying infinite dimensional vector space. Various objects that we might wish to construct on an infinite dimensional manifold are only possible when the model space has a particular structure. Milnor explains the basic theory of smooth manifolds based on locally convex topological vector spaces in [14].

We first examine topological vector spaces and the spaces of linear maps between such spaces. This leads into the theory of tensor products, finite rank operators and Fredholm operators. We end this section with a choice of type of vector space with which to work. The material in this section is expository and is covered in greater detail elsewhere. The original reference for tensor products is Grothendieck [10]. A good overview of the subject can be found in Schaefer [21].

3.1.1 Locally Convex Topological Vector Spaces

In this section we shall refer to the following properties of subsets of vector spaces:

Definition 3.1.1.1. *Let X be a vector space over a field $\mathbb{F}$. Let B be a subset of X.*

1. *B is* convex *if $tx + (1-t)y \in B$ whenever $x, y \in B$ and $t \in [0,1]$,*

2. *B is* balanced *if $\lambda B \subseteq B$ whenever $\lambda \in \mathbb{F}$ is such that $|\lambda| \leq 1$,*

3. *B is* absorbing *if for each $x \in X$ there is some $\lambda \in \mathbb{F}$ with $x \in \lambda B$.*

The types of vector space we shall be considering are locally convex:

Definition 3.1.1.2. *A* topological vector space *consists of a vector space X over a field $\mathbb{F}$ with a topology satisfying:*

1. *the scalar multiplication map $\mathbb{F} \times X \to X$ is continuous,*

2. *the vector addition map $X \times X \to X$ is continuous.*

A locally convex topological vector space, *or* LCTV-space, *is a topological vector space which has a topological base of locally convex sets.*

A CLCTV-space *is an LCTV-space which is complete for the given topology.*

The topology on an LCTV-space X is completely determined by the family of open sets $\mathcal{B}$ of all the open, convex, balanced, absorbing neighbourhoods of the origin. This family $\mathcal{B}$ is directed under inclusion and each set B in $\mathcal{B}$ determines a continuous semi-norm ρ_B on X defined by:

$$\rho_B(x) = \inf\{\lambda \in \mathbb{R}^+ : x \in \lambda B\}$$

Given a continuous semi-norm ρ on X let X_ρ be the Banach space completion of $X/\ker\rho$ with the norm induced from ρ, there is a natural map $X \to X_\rho$ which is continuous.

The topology on X is the coarsest topology such that the semi-norms $\{\rho_B : B \in \mathcal{B}\}$ are continuous; alternatively, the coarsest such that for each $B \in \mathcal{B}$ the map $X \to X_{\rho_B}$ is continuous. Any LCTV-space can thus be isomorphically embedded as a dense subspace of a projective limit of Banach spaces. The embedding is surjective if the original space is complete, i.e. is a CLCTV-space. Conversely, any subspace of a projective limit of Banach spaces is an LCTV-space.

All Fréchet spaces are CLCTV-spaces, in particular Banach and Hilbert spaces. Thus the spaces $C^\infty(S^1, \mathbb{R}^n)$ and $C^\infty(S^1, \mathbb{C}^n)$ for $n \in \mathbb{N}$ are CLCTV-spaces. Subspaces of LCTV-spaces and quotients of LCTV-spaces by closed subspaces are also LCTV-spaces.

When considering subspaces of an LCTV-space X over a field $\mathbb{F}$, we shall mainly be concerned with closed subspaces. Thus we define:

Definition 3.1.1.3. *The* span $\langle E \rangle$ *of a subset* $E \subseteq X$ *is the closure of the linear hull of* E, $\{\lambda_1 e_1 + \cdots + \lambda_n e_n : n \in \mathbb{N}, \lambda_i \in \mathbb{F}, e_i \in E\}$.

We shall be particularly interested in operators on LCTV-spaces:

Definition 3.1.1.4. *Let X and Y be LCTV-spaces. We define the following notation:*

1. *let $L(X, Y)$ be the space of all linear maps from X to Y,*

2. *let $\mathfrak{L}(X, Y)$ be the space of all continuous linear maps from X to Y,*

3. *let $X' = L(X, \mathbb{F})$ be the algebraic dual of X,*

4. *let $X^* = \mathfrak{L}(X, \mathbb{F})$ be the topological dual of X,*

5. *let* $\mathrm{Gl}(X)$ *be the space of invertible continuous linear operators on* X*; that is,* $T \in \mathrm{Gl}(X)$ *if* T *is a continuous linear bijection on* X *with continuous inverse,*

6. *let* $\mathfrak{I}_0(X, Y)$ *be the space of finite rank continuous linear operators from* X *to* Y.

Whenever we have a notation of the form $E(X, Y)$ *for LCTV-spaces* X *and* Y*, we use the shortened notation* $E(X)$ *for* $E(X, X)$.

In order to discuss topologies on the spaces of linear operators, we need to consider bounded subsets of LCTV-spaces:

Definition 3.1.1.5. *Let* Y *be an LCTV-space over* $\mathbb{F}$ *and let* $\mathcal{B}$ *be an open neighbourhood base of* 0 *in* Y*; the family* $\mathcal{B}$ *determines the topology on* Y*. A* bounded *subset* S *of* Y *is one such that for each* $B \in \mathcal{B}$ *there is some* $\lambda \in \mathbb{F}$ *such that* $S \subseteq \lambda B$.

We say that Y *is* bornological *if every balanced, convex subset that absorbs every bounded set in* Y *is a neighbourhood of* 0.

Continuous maps take bounded sets to bounded sets and for bornological spaces a map which takes bounded sets to bounded sets is continuous. All Banach and Fréchet spaces are bornological. With the concept of bounded sets, we can define a topology on the spaces $\mathfrak{L}(X, Y)$ and X^* for LCTV-spaces X and Y. Firstly we define a topology on a more general space:

Definition 3.1.1.6. *Let* Y *be an LCTV-space and let* T *be a set. Let* Y^T *be the vector space of maps from* T *to* Y*. Let* $\mathfrak{S}$ *be a directed family of subsets of* T*, ordered by inclusion. For* $S \in \mathfrak{S}$ *and* $V \in \mathcal{B}$*, let* $M(S, V) = \{f \in Y^T :$

$f(S) \subseteq V\}$. The family of sets $\{M(S,V)\}$ generates a vector space topology on Y^T called the $\mathfrak{S}$ – topology.

A subspace Z of Y^T is a topological vector space if and only if $f(S)$ is bounded in Y for each $f \in Z$ and $S \in \mathfrak{S}$. If, in addition, T is a topological space with $\bigcup \mathfrak{S}$ dense in T and Z is contained in the space of continuous maps from T to Y then Z is an LCTV-space. Thus Z is an LCTV-space if T is a topological vector space, $\mathfrak{S}$ is a family of bounded sets with $\bigcup \mathfrak{S}$ total in T (i.e. the linear hull of $\bigcup \mathfrak{S}$ is dense in T) and Z is a subspace of the continuous maps from T to Y. In the following, $\mathfrak{S}$ will be a family of bounded sets with $\bigcup \mathfrak{S}$ total in the relevant space.

In particular, the space of continuous linear operators from one LCTV-space X to another LCTV-space Y can be given a locally convex topology:

Definition 3.1.1.7. *We use the notation $\mathfrak{L}_{\mathfrak{S}}(X,Y)$ to denote $\mathfrak{L}(X,Y)$ with the $\mathfrak{S}$-topology arising from $\mathfrak{S}$. If $\mathfrak{S}$ is the set of all bounded subsets of X the topology on $\mathfrak{L}(X,Y)$ is called the* strong *topology and is denoted by $\mathfrak{L}_b(X,Y)$.*

We denote the space X^* with the $\mathfrak{S}$-topology by $X^*_{\mathfrak{S}}$ and denote the space X^* with the strong topology by X^*_b.

The use of the word *strong* here follows Grothendieck [10], Introduction, III (3) rather than Schaefer [21], Chapter IV (5) where the strong topology is defined in terms of weakly bounded sets on X (i.e. those sets S for which $f(S)$ is bounded for each $f \in X^*$).

When X and Y are normed vector spaces the strong topologies on X^* and $\mathfrak{L}(X,Y)$ coincide with the norm topologies. This is one reason for preferring these topologies. Another is that when X is bornological and Y is complete the spaces $\mathfrak{L}_b(X, Y)$ and X_b^* are both complete.

There is another useful topology on X^* called the inductive topology:

Definition 3.1.1.8. *Let $\{\rho_B : B \in \mathcal{B}\}$ be a directed family of semi-norms which determines the topology on X. For each $B \in \mathcal{B}$ let X_B be the Banach space completion of $X/\ker \rho_B$ with the norm induced from ρ_B. The inductive topology on X^* is the weakest such that the natural maps $X_B^* \to X^*$ are continuous. Let X_i^* denote X^* with the inductive topology.*

The identity map $X_i^* \to X_b^*$ is continuous with equality if X is a Banach space.

3.1.2 Tensor Products and Finite Rank Operators

Let X and Y be two vector spaces. Let $B(X, Y)$ be the vector space of bilinear forms on $X \times Y$. For $(x, y) \in X \times Y$ the map $f \to f(x, y)$ is linear in f and hence defines an element of $B(X, Y)'$. The map $\chi : X \times Y \to B(X, Y)'$ so defined is bilinear and the linear hull of $\chi(X \times Y)$ is a particularly important space:

Definition 3.1.2.1. *The linear hull of $\chi(X \times Y)$ in $B(X, Y)'$ is called the* algebraic tensor product *of X and Y and is denoted by $X \otimes Y$. The image of (x, y) under χ is written $x \otimes y$. For $u \in X \otimes Y$, the* rank *of u is defined to be the minimum number of summands in a representation of u as $\sum x_i \otimes y_i$.*

One of the key properties of tensor products is the following: let Z be a vector space; the map $f \to f \circ \chi$ defines an isomorphism of $B(X, Y; Z)$ onto $L(X \otimes Y, Z)$, where $B(X, Y; Z)$ is the space of Z-valued bilinear forms on $X \times Y$.

Another property of the tensor product is that the space $X^* \otimes Y$ is algebraically isomorphic to the space $\mathfrak{J}_0(X, Y)$ of finite rank continuous linear operators, the isomorphism being such that $(f \otimes y)(x) = f(x)y$. This induces a subspace topology on $X^* \otimes Y$ with respect to a given topology on $\mathfrak{L}(X, Y)$. There are natural actions of $\mathfrak{L}(X)$ and $\mathfrak{L}(Y)$ on $\mathfrak{L}(X, Y)$ and the space of finite rank operators is invariant under both actions. The rank of an element in the tensor product corresponds to its rank when considered as an operator.

If X, Y and Z are topological vector spaces a topology on $X \otimes Y$ defines a subspace $\mathfrak{L}(X \otimes Y, Z)$ of $L(X \otimes Y, Z)$. It is possible to define topologies on $X \otimes Y$ such that the image of $\mathfrak{L}(X \otimes Y, Z)$ in $B(X, Y; Z)$ has some particular structure. One important such topology is the inductive topology on $X \otimes Y$. To define this, we need the concept of a separately continuous bilinear form:

Definition 3.1.2.2. *A bilinear form $f : X \times Y \to Z$ is* separately continuous *if for each $x \in X$ the map $y \to f(x, y)$ is continuous on Y and similarly for each $y \in Y$.*

Definition 3.1.2.3. *The* inductive *topology on $X \otimes Y$ is the finest topology such that $\mathfrak{L}(X \otimes Y, Z)$ corresponds to the space of separately continuous Z-valued bilinear forms. Let $X \overline{\otimes} Y$ denote the completion of $X \otimes Y$ in the inductive topology.*

The inductive topology on $X^*_{\mathfrak{S}} \otimes X$ is closely linked to the trace operator:

Definition 3.1.2.4. *There is a canonical bilinear form on $X^*_{\mathfrak{S}} \times X$ given by $(f, x) \to f(x)$. This is separately continuous and so defines a continuous linear functional called the* trace *on $X^*_{\mathfrak{S}} \overline{\otimes} X$ written as $u \to \operatorname{Tr} u$.*

At this point we encounter the "approximation problem" (Grothendieck, [10], section 5). An LCTV-space with the *approximation property* is one for which the identity operator on E lies in the closure of the space of finite rank operators under the topology of uniform convergence on precompact subsets. The conjecture concerning the approximation problem was that every LCTV-space had the approximation property, a conjecture that has now been disproved, see Enflo [7].

Let X be an LCTV-space and let R be the directed family of continuous semi-norms on X; X has the approximation property if for each $\rho \in R$, X_ρ has the approximation property. Thus the question of the approximation property essentially reduces to one on Banach spaces.

Spaces with the approximation property include all Hilbert spaces, all Banach spaces such that the finite rank operators are dense in the compact operators (with the subspace topology), in particular $l^p(\mathbb{F})$, $L^p(\mathbb{F}, \mathbb{F}^n)$ and $C^p(\mathbb{F}, \mathbb{F}^n)$ for $1 \leq p < \infty$,[2] and the Fréchet spaces derived from such spaces, in particular $C^\infty(\mathbb{F}, \mathbb{F}^n)$.

By restricting to bornological spaces with the approximation property, we can identify $X_b^* \overline{\otimes} X$ with a subspace of $\mathfrak{L}_b(X)$.

Definition 3.1.2.5. *Let $\mathfrak{J}_1(X)$ denote the image in $\mathfrak{L}_b(X)$ of $X_b^* \overline{\otimes} X$. The topology on $\mathfrak{J}_1(X)$ induced from $X_b^* \overline{\otimes} X$ is said to be the* trace topology *and operators in $\mathfrak{J}_1(X)$ are said to be of* trace class.

This is an ideal in $\mathfrak{L}_b(X)$ contained in the ideal of compact operators. The linear functional $\mathrm{Tr} : \mathfrak{J}_1(X) \to \mathbb{F}$ is well-defined in the trace topology. Closely

[2]This also holds for $l^\infty(\mathbb{F})$ and $L^\infty(\mathbb{F}, \mathbb{F}^n)$ but these are not separable so we are not concerned with them.

related to the operators of trace class are the operators with a determinant:

Definition 3.1.2.6. *Let $\mathfrak{D}(X) = I + \mathfrak{J}_1(X)$ denote those continuous linear operators on X which differ from the identity by an operator of trace class. For an operator A in $\mathfrak{D}(X)$, the* determinant *of A is defined by* $\det(A) = \exp \operatorname{Tr} \log(A)$. *Let $\mathfrak{D}_\times(X)$ denote the space of invertible operators with a determinant.*

The space $\mathfrak{D}(X)$ and the determinant operator have the following properties:

1. $\mathfrak{D}(X)$ is a semi-group and $\det : \mathfrak{D}(X) \to \mathbb{F}$ is a continuous semi-group homomorphism,
2. $\mathfrak{D}_\times(X) = \det^{-1} \mathbb{F}^\times$, where $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$, and this is a group,
3. If A has a determinant and $B \in \mathrm{Gl}(X)$ then $B^{-1}AB$ has a determinant with $\det B^{-1}AB = \det A$.

Let $\mathfrak{D}_1(X) = \det^{-1}(1)$ and when $\mathbb{F} = \mathbb{C}$ let $\mathfrak{D}_{S^1}(X) = \det^{-1}(S^1)$.

The set of topologies on a space forms a complete lattice (technically, a topped intersection structure). Using this, we can define another topology on $X_b^* \otimes X$. Considering $X_b^* \otimes X$ as $\mathfrak{J}_0(X)$, the set of continuous finite rank operators on X, there is a bilinear map $\mathfrak{J}_0(X) \times \mathfrak{J}_0(X) \to \mathfrak{J}_0(X)$ given by composition of operators. We give the right hand side the trace topology and consider the family of topologies on $\mathfrak{J}_0(X)$ for which the composition map is continuous when $\mathfrak{J}_0(X) \times \mathfrak{J}_0(X)$ is given the product topology. We further refine this family to those topologies finer than the subspace topology on $\mathfrak{J}_0(X)$ induced from $\mathfrak{L}_b(X)$ and those for which the left and right actions of $\mathfrak{L}_b(X)$ are continuous. This family is non-empty because it contains the trace topology and is closed under intersections in the lattice of topologies. Therefore there is an infimum topology. The closure of $\mathfrak{J}_0(X)$ under this topology maps continuously into $\mathfrak{L}_b(X)$.

Definition 3.1.2.7. *Let $\mathfrak{J}_2(X)$ denote the image of this map.*

Composition of operators is a bilinear map $\mathfrak{J}_2(X) \times \mathfrak{J}_2(X) \to \mathfrak{J}_1(X)$ and $\mathfrak{J}_2(X)$ is an ideal in $\mathfrak{L}_b(X)$. In the case of a Hilbert space H, $\mathfrak{J}_2(H)$ is the space of Hilbert-Schmidt operators.

3.1.3 Fredholm Operators

Definition 3.1.3.1. *A* Fredholm operator $F : X \to Y$ *between LCTV-spaces X and Y is a continuous linear operator for which:*

1. $\ker F$ *is finite dimensional,*
2. $\operatorname{im} F$ *is closed in Y and finite codimensional.*

Let $\mathfrak{F}(X,Y)$ denote the space of Fredholm operators from X to Y. The index *of a Fredholm operator $F : X \to Y$ is the integer* $\operatorname{Index} F = \dim\ker F - \dim\operatorname{coker} F$. *This is a continuous map $\mathfrak{F}(X,Y) \to \mathbb{Z}$. Let $\mathfrak{F}_k(X,Y)$ be the space of Fredholm operators of index k.*

If X and Y are complete, metrisable topological vector spaces then Banach's homomorphism theorem (Schaefer [21], Chapter III (2)) applies and the induced map $\tilde{F} : X/\ker F \to \operatorname{im} F$ is an isomorphism. From this, it is straightforward to prove the following result:

Theorem 3.1.3.2. *Let $F : X \to Y$ be a Fredholm operator of index 0 between metrisable CLCTV-spaces X, Y. There is a finite rank operator $T : X \to Y$ such that $F + T$ is an isomorphism.*

Proof. Since $\ker F$ is finite dimensional, a corollary of the Hahn-Banach theorem says that there is a closed subspace W of X such that $X = \ker F \oplus W$. The projection $W \to X/\ker F$ is an isomorphism so $F|_W \colon W \to \operatorname{im} F$ is an isomorphism.

As $\operatorname{im} F$ is closed and finite codimensional in Y, there is a finite dimensional closed subspace V in Y such that $Y = V \oplus \operatorname{im} F$. Since $\operatorname{Index} F = 0$, $\dim V = \dim \ker F$ and so there is an isomorphism $T : \ker F \to V$. The map $T + F : \ker F \oplus W \to V \oplus \operatorname{im} F$ defines an isomorphism from X to Y. □

In fact, we can characterise Fredholm operators of index 0 using this. Any such Fredholm operator can be written in the form $A(I + T)$ where A is invertible and T is finite rank. We can relax the condition that T be finite rank to the condition that T be compact and thus if $F \in \mathfrak{F}(X)$ and $T \in \mathfrak{J}_1(X)$ then $F + T \in \mathfrak{F}(X)$ and $\operatorname{Index} F + T = \operatorname{Index} F$.

For a Hilbert space H, $\mathfrak{F}_k(H)$ is not empty. For a general LCTV-space X, $\mathfrak{F}_k(H)$ is not empty for all k if and only if $X \cong X \oplus \mathbb{F}^k$ for all k, i.e. X is stable. This is not true in general, but it is true for the main examples of Fréchet spaces such as the Banach spaces $L^p(S^1, \mathbb{F}^n)$, $C^p(S^1, \mathbb{F}^n)$ and the Fréchet spaces $C^\infty(S^1, \mathbb{F}^n)$.

3.1.4 Bases

In the construction of the determinant line bundle over the Grassmannian in finite dimensions we used the Stiefel manifold to get a convenient representation for an element of the determinant line bundle. In the infinite dimensional case, we do something similar using admissible bases for subspaces. These depend upon a choice of a particular basis for the vector space, up to a notion of equivalence. The fact that not all such bases are equivalent gives the first hint that the theory in infinite dimensions has some extra twists not apparent in finite dimensions. Thus in order to understand the definition of the determinant line bundle, we first need to look at bases in general topological vector spaces.

A basis for an LCTV-space X is a subset $\{x_\alpha : \alpha \in A\} \subseteq X$ which spans X and is linearly independent. In infinite dimensions a set is linearly independent if any finite subset is a linearly independent set. This definition is not strong enough for our purposes, for example in the Hilbert space $l^2(\mathbb{R})$ with standard orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ the set $\{x, e_1, e_2, \dots\}$ where $x = (\frac{1}{i})_{i\in\mathbb{N}}$ is a basis for $l^2(\mathbb{R})$ but there is clearly some redundancy.

The property we require is that the set be topologically free:

Definition 3.1.4.1. *A subset $\{x_\alpha : \alpha \in A\}$ of a topological vector space is* topologically free *if each x_α is not contained in the (closed) span of $\{x_\beta : \beta \neq \alpha\}$.*

It is obvious that the property of being topologically free is stronger than that of being linearly independent. Thus for our purposes a basis will be required to be topologically free.

It is easily seen that a Fréchet space has a maximal topologically free set and that such a set forms a basis. Hence every Fréchet space has a topologically free basis. In the following, the word "basis" when unadorned by other adjectives will be taken to mean "topologically free basis".

Given a basis $\{x_\alpha : \alpha \in A\}$ of X and a finite subset $B \subseteq A$ there is a decomposition of X into $X_B \oplus X^B$ where X_B is the span of $\{x_\beta : \beta \in B\}$ and X^B is the span of $\{x_\beta : \beta \notin B\}$. There are projection operators $p_B : X \to X_B$ and $p^B : X \to X^B$.

We define an equivalence relation on the set of bases in the following way. Since all bases will have the same cardinality, we assume that they are indexed by the same set.

Definition 3.1.4.2. *Two bases $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ for the same space X are* equivalent *if there is an invertible operator B on X with a determinant such that $Bx_\alpha = y_\alpha$.*

The larger group $\mathrm{Gl}(X)$ acts on the space of bases. However, as $\mathfrak{D}_\times(X) \neq \mathrm{Gl}(X)$ there are bases for X which are inequivalent. For example, if $\{x_\alpha\}$ is a basis for X then $\{2x_\alpha\}$ is another basis, but the operator $2I$ does not have a determinant so these bases are inequivalent. In fact, even $\mathrm{Gl}(X)$ does not act transitively on the set of bases. Suppose that X is a Banach space with a normalised basis $\{x_n\}$ indexed by $\mathbb{N}$ (i.e. $\|x_n\| = 1$ for all n). The set $\{nx_n\}$ is a basis but there is no continuous invertible operator which carries one set to the other as such an operator would be unbounded.

Let X be a Fréchet space with an operator $\sigma \in \mathfrak{F}_{-1}(X)$ such that $\ker \sigma = \{0\}$. Choose some $x_1 \in X$ such that $x_1 \notin \sigma(X)$. Let $x_{k+1} = \sigma(x_k) = \sigma^k(x_1)$

and consider the set $\{x_k : k \in \mathbb{N}\}$. Since $\sigma(X)$ is closed in X, this set is topologically free and therefore is a basis for its span. The span is X if and only if $\bigcap \sigma^k(X) = \{0\}$ and this is the case if X has no σ-invariant subspaces.

Definition 3.1.4.3. *A basis $\{x_k : k \in \mathbb{N}\}$ for X and a Fredholm operator σ of index -1 on X are* compatible *if $\sigma(x_k) = x_{k+1}$.*

3.1.5 $\mathcal{F}$-Spaces

Although it is possible to define the cohomology theories for a manifold modelled on a general bornological CLCTV-space with the approximation property, by assuming that such a space is metrisable the theory is somewhat simplified. In such circumstances we are able to use Banach's homomorphism theorem and, as a consequence, theorem 3.1.3.2 from above. It is a consequence of Urysohn's lemma that a metrisable CLCTV-space is a Fréchet space.

It is also possible to define the cohomology theories for a manifold modelled on a space X where $\mathfrak{F}(X)$ does not contain Fredholm operators of each index. There are no mathematical difficulties in considering such a space, but as there are considerable notational difficulties and as Fredholm operators of all indices exist for all the main examples, we shall not consider such spaces. Note that the existence of Fredholm operators of all indices is equivalent to the existence of a Fredholm operator of index -1. We shall also insist that this Fredholm operator has no invariant subspace so that we can find a compatible basis. Thus we define:

Definition 3.1.5.1. *A vector space X is an $\mathcal{F}$-space if X is a separable*

Fréchet space with the approximation property and such that $\mathfrak{F}(X)$ contains a Fredholm operator of index -1 (i.e. X is stable) with no invariant subspace.

The class of $\mathcal{F}$-spaces includes all Hilbert spaces, the Banach spaces $L^p(S^1, \mathbb{F}^n)$, $C^p(S^1, \mathbb{F}^n)$ and the Fréchet spaces $C^\infty(S^1, \mathbb{F}^n)$. A closed subspace of an $\mathcal{F}$-space is an $\mathcal{F}$-space and the complexification of a real $\mathcal{F}$-space is an $\mathcal{F}$-space.

We shall often encounter $\mathcal{F}$-spaces X and Y such that $\mathfrak{F}(X, Y)$ is not empty. For Fréchet spaces the relation $\sim_F$ defined by $X \sim_F Y$ if $\mathfrak{F}(X, Y) \neq \emptyset$ is an equivalence relation which is dominated by the isomorphism relation, i.e. if $X \cong Y$ then $X \sim_F Y$. A key property of $\mathcal{F}$-spaces is that these relations are equivalent:

Theorem 3.1.5.2. *If X and Y are $\mathcal{F}$-spaces such that $X \sim_F Y$ then X and Y are isomorphic.*

Proof. Let X and Y be $\mathcal{F}$-spaces and suppose that there is some $F \in \mathfrak{F}(X, Y)$. As $\mathfrak{F}(X)$ contains Fredholm operators of all indices, by composing with a Fredholm operator of suitable index we can assume that $\operatorname{Index} F = 0$. Then there is a finite rank operator $T \in \mathfrak{I}_0(X, Y)$ such that $F + T : X \to Y$ is invertible. $\square$

Given a completion $\mathfrak{I}(X)$ of the tensor product $X^* \otimes X$, if $\mathfrak{F}(X, Y)$ is non-empty, there is a natural completion $\mathfrak{I}(X, Y)$ of the tensor product $X^* \otimes Y$. In particular, there are completions $\mathfrak{I}_i(X, Y)$ for $i = 1, 2$ which satisfy the property that the map induced from composition of operators, $\mathfrak{I}_2(X, Y) \times \mathfrak{I}_2(Y, X) \to \mathfrak{I}_1(X)$, is defined and continuous.

3.2 Polarised Spaces

The basic object of a semi-infinite theory is a polarised space. We start with the definition of a polarised space and the associated structure group. Then we introduce the notion of an admissible basis which we shall need in defining the determinant line bundle. As above, two bases for the same space are considered equivalent if there is an operator with a determinant which carries one to the other. In finite dimensions all bases are equivalent but in infinite dimensions not all invertible operators have a determinant so not all bases are equivalent. Thus we need to alter our structure groups in order to take this into account. This is done by defining a particular central extension. Although these definitions and theorems are mainly to do with semi-infinite theory, there are some aspects which are used in finite codimension theory and so are considered in the more general context. The work in this section is mainly a generalisation to $\mathcal{F}$-spaces of the work of Pressley and Segal [18], chapters 6 and 7.

3.2.1 Polarisations

Definition 3.2.1.1. *A* polarisation *of a complex $\mathcal{F}$-space X is a decomposition $X = X_- \oplus X_+$ into two closed infinite dimensional subspaces which are $\mathcal{F}$-spaces. A polarisation is* symmetric *if X_- and X_+ are isomorphic.*

Although the theory can be developed for non-symmetric polarisations, all the main examples are symmetric and so to simplify the exposition we shall only consider symmetric polarisations.

Let $X_- \oplus X_+$ and $X'_- \oplus X'_+$ be two symmetric polarisations for the same space X. The identity map on X decomposes as a map $X'_- \oplus X'_+ \to X_- \oplus X_+$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If a and d are Fredholm then X_+, X'_+, X_- and X'_- are all isomorphic $\mathcal{F}$-spaces. Thus there are completions $\mathfrak{J}_2(X'_-, X_+)$ and $\mathfrak{J}_2(X'_+, X_-)$ of the tensor products $X'^*_- \otimes X_+$ and $X'^*_+ \otimes X_-$ respectively.

Definition 3.2.1.2. *The polarisations $X_- \oplus X_+$ and $X'_- \oplus X'_+$ are said to be* equivalent *if a and d are Fredholm and $b \in \mathfrak{J}_2(X'_-, X_+)$, $c \in \mathfrak{J}_2(X'_+, X_-)$, where a, b, c, d are defined as above.*

Corresponding to a polarisation are continuous projection operators $p_\pm : X \to X_\pm$ and continuous inclusion maps $i_\pm : X_\pm \to X$. There is also a polarising operator $J = i_+p_+ - i_-p_-$. This polarising operator has spectrum $\{-1, +1\}$ and the decomposition $X_- \oplus X_+$ is into the negative and positive eigenspaces of J. This gives an alternative way to define a polarisation as the positive and negative eigenspaces of an operator $J : X \to X$ with spectrum $\{-1, +1\}$ (we also need the condition that the eigenspaces are infinite dimensional $\mathcal{F}$-spaces).

If the circle acts on a polarised space X then for each $k \in \mathbb{Z}$ there is a (possibly trivial) closed subspace $X(k)$ of X on which the action is given by $(\xi, x) \to \xi^k x$.

Definition 3.2.1.3. *We say that the circle action is* compatible *with the polarisation if $\bigoplus_{k \geq 0} X(k)$ is dense in X_+ and $\bigoplus_{k<0} X(k)$ is dense in X_-.*

The main example of a space with a polarisation is the Hilbert space $L^2(S^1, \mathbb{C})$. This has an orthonormal basis $\{z^n : n \in \mathbb{Z}\}$. The polarisation is into the subspaces spanned by $\{z^n : n < 0\}$ and $\{z^n : n \geq 0\}$. Each of the spaces $L^p(S^1, \mathbb{C})$ and $C^p(S^1, \mathbb{C})$ for $1 \leq p < \infty$ and $C^\infty(S^1, \mathbb{C})$ has a similar polarisation. These polarisations are all symmetric and compatible with the obvious circle action. Given two $\mathcal{F}$-spaces X and Y, the space $Z = X \oplus Y$ has an obvious polarisation which is only symmetric if X and Y belong to the same equivalence class of $\mathcal{F}$-spaces.

To extend this section to the real case, we make the following definitions. Let X be a real $\mathcal{F}$-space.

Definition 3.2.1.4. *A* polarisation *of X is a polarisation of $X \otimes \mathbb{C}$.*

Given a complex polarised space $X = X_- \oplus X_+$, the complexification $Y = X \otimes_{\mathbb{R}} \mathbb{C}$ also carries a polarisation. We have a choice in the exact polarisation, the choices having positive space either $X_+ \oplus \overline{X_-}$ or $X_+ \oplus \overline{X_+}$, where $\overline{X_-}$ denotes the space X_- with the conjugate action of $\mathbb{C}$ (here we are using the isomorphism $X \otimes_{\mathbb{R}} \mathbb{C} = X \oplus \overline{X}$). If the polarisation on X is compatible with a circle action then the polarisation with positive space $X_+ \oplus \overline{X_-}$ is compatible with the induced circle action on Y. However, the polarising operator on X extends to one on Y which has positive space $X_+ \oplus \overline{X_+}$. Although many naturally occurring polarisations do arise from a circle action, it is the polarising operator which actually defines the polarisation. Thus we define:

Definition 3.2.1.5. *Let $X = X_- \oplus X_+$ and let $Y = X \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of X. The preferred polarisation on Y is given by $Y_- = X_- \otimes_{\mathbb{R}} \mathbb{C}$*

and $Y_+ = X_+ \otimes_{\mathbb{R}} \mathbb{C}$.

Corresponding to a polarisation is a particular subgroup of $\mathrm{Gl}(X)$. An operator $B \in \mathrm{Gl}(X)$ can be decomposed according to the polarisation as

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Definition 3.2.1.6. *The* restricted general linear group, $\mathrm{Gl}_{res}(X)$, *corresponding to the polarisation of* X *is the space of operators* B *such that* a *and* d *are Fredholm,* $b \in \mathfrak{J}_2(X_-, X_+)$ *and* $c \in \mathfrak{J}_2(X_+, X_-)$ *in the above decomposition.*

Strictly speaking, we ought to have a notation for $\mathrm{Gl}_{\mathrm{res}}(X)$ which includes the polarisation, but such a notation would be unwieldy and unnecessary as this is usually implicit in the space under consideration. Because B is invertible, a and d are such that $\operatorname{Index} a + \operatorname{Index} d = 0$.

Given $B, C \in \mathrm{Gl}_{\mathrm{res}}(X)$ with decompositions:

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad C = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

the product BC has decomposition:

$$BC = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

then $bg \in \mathfrak{J}_1(X_+)$ so $ae + bg \in \mathfrak{F}(X_+)$ and $\operatorname{Index} ae + bg = \operatorname{Index} ae =$

$\operatorname{Index} a + \operatorname{Index} e$, similarly for $cf + dh$. As $\mathfrak{J}_2(X_+, X_-)$ is an ideal, $af + bh \in \mathfrak{J}_2(X_+, X_-)$ and similarly for $ce + dg$. Thus $\mathrm{Gl}_{\mathrm{res}}(X)$ is a group.

For X a real $\mathcal{F}$-space, the restricted general linear group of X, $\mathrm{Gl}_{\mathrm{res}}(X)$, is defined to be the preimage of the group $\mathrm{Gl}_{\mathrm{res}}(X \otimes \mathbb{C})$ under the inclusion $\mathrm{Gl}(X) \to \mathrm{Gl}(X \otimes \mathbb{C})$.

The map $j : \mathrm{Gl}_{\mathrm{res}}(X) \to \mathbb{Z}$ given by $B \to \operatorname{Index} p_+ B i_+$ is a continuous group homomorphism. Let $\mathrm{Gl}_{\mathrm{res},l}(X)$ denote the inverse image of $l \in \mathbb{Z}$, so $\mathrm{Gl}_{\mathrm{res},0}(X)$ contains the identity component of $\mathrm{Gl}_{\mathrm{res}}(X)$ and each $\mathrm{Gl}_{\mathrm{res},l}(X)$ is diffeomorphic to $\mathrm{Gl}_{\mathrm{res},0}(X)$. $\mathrm{Gl}_{\mathrm{res}}(X)$ is the semi-direct product of $\mathrm{Gl}_{\mathrm{res},0}(X)$ with $\mathbb{Z}$. The semi-direct product is given by the action of $\mathbb{Z}$ on $\mathrm{Gl}_{\mathrm{res},0}(X)$ as $(n, C) = B^{-n} C B^{n}$ where $B = in\, \mathrm{Gl}_{\mathrm{res},1}(X)$. Thus the structure of $\mathrm{Gl}_{\mathrm{res}}(X)$ is given by the structure of $\mathrm{Gl}_{\mathrm{res},0}(X)$. When X is a Hilbert space $\mathrm{Gl}_{\mathrm{res}}(X)$ can be identified with $\mathbb{Z} \tilde{\times} BU$, see Pressley and Segal [18], Proposition 6.2.4.

3.2.2 Equivalent Polarisations

In definition 3.2.1.2 we defined the notation of equivalent polarisations. In this section we explore this further, considering the relationship between the group $\mathrm{Gl}_{\mathrm{res}}(X)$ and the set of equivalent polarisations. The result we wish to show is the following:

Theorem 3.2.2.1. *For a polarised space $X = X_- \oplus X_+$, the group $\mathrm{Gl}_{\mathrm{res}}(X)$ acts transitively on the set of polarisations equivalent to $X_- \oplus X_+$.*

Proof. Clearly if $B \in \mathrm{Gl}_{\mathrm{res}}(X)$ then $BX_- \oplus BX_+$ is a polarisation of X equivalent to $X_- \oplus X_+$ and thus $\mathrm{Gl}_{\mathrm{res}}(X)$ acts on the set of polarisations equivalent to $X_- \oplus X_+$. To show that this action is transitive takes a few steps to establish. Firstly, note that if $B \in \mathrm{Gl}(X)$ is such that $BX_- \oplus BX_+$ is equivalent to $X_- \oplus X_+$ then $B \in \mathrm{Gl}_{\mathrm{res}}(X)$.

Now suppose that $X'_- \oplus X'_+$ is an equivalent polarisation of X such that $p_+i'_+ : X'_+ \to X_+$ is of index 0. There is a finite rank operator $t_+ : X'_+ \to X_+$ such that $p_+i'_+ + t_+ : X'_+ \to X_+$ is invertible. Similarly, there is a finite rank operator $t_- : X'_- \to X_-$ such that $p_-i'_- + t_- : X'_- \to X_-$ is invertible. Let $B : X \to X$ be the map $i_+(p_+i'_+ + t_+)p'_+ + i_-(p_-i'_- + t_-)p'_-$. This is a bijective, continuous map, hence is invertible. It satisfies $BX'_\pm = X_\pm$ and so lies in $\mathrm{Gl}_{\mathrm{res},0}(X)$.

To go between equivalent polarisations where $p_+i'_+ : X_+ \to X_+$ is of non-zero index, it is sufficient to construct an element of $\mathrm{Gl}_{\mathrm{res},1}(X)$. Using this element, one of the polarisations can be altered to an equivalent polarisation such that the Fredholm operator between the positive spaces is of index 0. The above method shows that there is an operator which takes one polarisation to the other.

To construct an element of $\mathrm{Gl}_{\mathrm{res},1}(X)$, we do the following. Since $X_\pm$ are $\mathcal{F}$-spaces, there is a Fredholm operator a of index -1 in $\mathfrak{F}(X_+)$ and d of index 1 in $\mathfrak{F}(X_-)$. By adding suitable operators of finite rank if necessary, we can assume that $\ker a$ and $\operatorname{coker} d$ are trivial. Let $x \in X_+$ be a vector not in the image of a and let $y \in \ker d$. There is a map $b : X_- \to X_+$ of rank 1 with $by = x$. The map $B = a + b + d$ is in $\mathrm{Gl}_{\mathrm{res}}(X)$ with $BX_+ = \operatorname{im} a$ and $BX_- = X_- \oplus \langle x \rangle$. Note that this operator satisfies the conditions for the construction of the semi-direct product structure of $\mathrm{Gl}_{\mathrm{res}}(X)$ as $\mathbb{Z} \tilde{\times} \mathrm{Gl}_{\mathrm{res},0}(X)$. □

Finally, we show the following lemma:

Lemma 3.2.2.2. *Let $W \subseteq X$ be a closed subspace of X such that $p_+|_W$: $W \to X_+$ is Fredholm and $p_-|_W$: $W \to X_-$ lies in $\mathfrak{J}_2(W, X_-)$ then there is a polarisation of X equivalent to $X_- \oplus X_+$ with W as positive space.*

Proof. The subspace $W \cap X_-$ is finite dimensional because $\ker p_+ = X_-$. Thus there is a closed finite codimension subspace U of X_- such that $U \cap W$ is trivial. As $p_+(W)$ is closed and finite codimensional in X_+, there is a complementary finite dimensional subspace V of X_+. Then $Y = U \oplus V$ is a closed subspace of X complementary to W. Clearly $p_-|_Y$: $Y \to X_-$ is Fredholm and $p_+|_Y$: $Y \to X_+$ is finite rank and therefore $Y \oplus W$ is an equivalent polarisation to $X_- \oplus X_+$. □

3.2.3 Admissible Bases

Let X be a polarised complex $\mathcal{F}$-space. Since X_+ is an $\mathcal{F}$-space, we can choose a basis (y_k) indexed by $\mathbb{N}$ on X_+ compatible with a given Fredholm operator $a \in \mathfrak{F}_{-1}(X_+)$. Since X_+ and X_- are isomorphic, this gives a basis (z_k) for X_- compatible with a Fredholm operator $d \in \mathfrak{F}_{-1}(X_-)$. Define the $\mathbb{Z}$ indexed basis (x_k) on X by $x_k = y_{k+1}$ for $k \geq 0$ and $x_k = z_{-k}$ for $k < 0$. As $\ker d$ is trivial, there is a Fredholm operator $\tilde{d} \in \mathfrak{F}_1(X_-)$ which is a left-inverse to d and such that z_1 spans $\ker \tilde{d}$. Define $b : X_- \to X_+$ by $bz_1 = y_1$. The operator $\sigma := a + b + \tilde{d}$ is in $\mathrm{Gl}_{\mathrm{res},-1}(X)$ such that $\sigma(x_k) = x_{k+1}$. In particular, $\sigma(X_+) \subseteq X_+$.

Definition 3.2.3.1. *σ is the* shift operator *associated to the basis (x_k).*

For each $k \in \mathbb{Z}$ there is a decomposition $X = X^k \oplus X_k$ where X^k is the span of $\{x_l : l < k\}$ and X_k is the span of $\{x_l : l \geq k\}$. Alternatively,

$X^k = \sigma^k(X_-)$ and $X_k = \sigma^k(X_+)$. In particular, $X^0 = X_-$ and $X_0 = X_+$. Given such a decomposition, there are projection operators $p^l : X \to X^l$ and $p_l : X \to X_l$.

We can now define the notion of an admissible bases for a subspace of X.

Definition 3.2.3.2. *An* admissible basis *for a subspace W of X for which $p_l : W \to X_l$ is Fredholm of index 0 consists of a basis $\{w_k\} \subseteq W$ which satisfies the following conditions:*

1. *There is an isomorphism $w : X_l \to W$ such that $w(x_k) = w_k$,*

2. *The map $p_l w : X_l \to X_l$ is an operator with a determinant.*

All admissible bases are also bases when considering W as an $\mathcal{F}$-space in its own right. The following lemmas are some basic results on admissible bases which we shall need later on.

Lemma 3.2.3.3. *Let $W \subseteq X$ satisfy the conditions in definition 3.2.3.2 then W has an admissible basis.*

Proof. There is a finite rank operator $t : W \to X_l$ such that $p_l + t$ is invertible. Let $w = (p_l + t)^{-1}$ and let $w_k = w(x_k)$. w and $\{w_k\}$ satisfy the first condition.

The map $p_l w$ can be written as $(p_l + t)w - tw$ which is $I - tw$. As t is finite rank, tw is finite rank and thus $p_l w$ has a determinant. □

Lemma 3.2.3.4. *Two admissible bases for $W \subseteq X$ are equivalent bases for W.*

Proof. Let w and w' be admissible bases for W. The maps $w, w' : X_l \to W$ are isomorphisms, thus the map $t = w'w^{-1}$ is an invertible map $W \to W$

such that $t(w_k) = w'_k$. The map $p_l : W \to X_l$ is Fredholm of index 0 so there is some finite rank operator, $s : W \to X_l$, such that $p_l + s$ is invertible. The operator $r := (p_l + s)t(p_l + s)^{-1}$ is invertible from X_l to itself and has a determinant if and only if t does. Now $(p_l + s)w' = p_l w' + sw'$ which is the sum of an operator with a determinant and a finite rank operator, hence has a determinant. Similarly, $w^{-1}(p_l + s)^{-1}$ is the inverse of an operator with a determinant, hence has a determinant. Thus r and hence t have determinants. Hence w and w' are equivalent. □

Lemma 3.2.3.5. *If w be an admissible basis for $W \subseteq X$ then $\sigma(w)$ is an admissible basis for $\sigma(W)$.*

Proof. This is a simple consequence of the fact that $\sigma : X_l \to X_{l+1}$ is an isomorphism which takes the basis for X_l to that for X_{l+1}. □

Lemma 3.2.3.6. *Let w be an admissible basis for $W \subseteq X$ with respect to (x_k). If (y_k) is a basis for X equivalent to (x_k) then there is an admissible basis w' for W such that $w'(y_k) = w(x_k)$.*

Proof. As (y_k) is equivalent to (x_k), there is an operator $A \in \mathfrak{D}_\times(X)$ such that $Ay_k = x_k$. In particular $AY_l = X_l$. Define $w' : Y_l \to W$ by $w' = wA$. Clearly $w'(y_k) = w(x_k)$. The projection $p^y_l : X \to Y_l$ coincides with the map $A^{-1}p^x_l A$ and thus $p^y_l w' = A^{-1}p^x_l AwA$.

As A has a determinant, it is of the form $I + T$ for some $T \in \mathfrak{J}_1(X)$. Thus $p^x_l Aw = p^x_l w + p^x_l Tw$, which is an operator with a determinant plus a trace class operator, thus is an operator with a determinant. Thus $p^y_l w'$ has a determinant. □

Lemma 3.2.3.7. *Let w be an admissible basis for $W \subseteq X$. Let V be a finite dimensional subspace of X such that $V \cap W = \{0\}$. Let v be a basis for V, then $v \cup w$ is an admissible basis for $V \oplus W$.*

Proof. Let $\dim V = n$ and suppose that l is such that $p_l : W \to X_l$ is Fredholm of index 0, then $p_{n+l} : V \oplus W \to X_{n+l}$ is Fredholm of index 0. Let $t : X_{n+l} \to V$ be the map which sends the set $\{x_j : n+l \leq j < l\}$ to v and is zero on X_l. The map $w + t : X_{n+l} \to V \oplus W$ satisfies the first condition for an admissible basis and $p_{n+l}(w+t)$ differs from $p_l w$ by an operator of finite rank and so is an operator with a determinant. Hence $v \cup w$ is an admissible basis for $V \oplus W$. □

Finally in this section we shall define some particular closed subspaces of X and find natural admissible bases for them.

Let $S \subseteq \mathbb{Z}$ be such that $|S - \mathbb{N}|$ and $|\mathbb{N} - S|$ are finite[3] with $|S - \mathbb{N}| - |\mathbb{N} - S| = l$. There is some $N < l$ such that $\{k \geq N\} \subseteq S$ and the complement of this in S has size $l - N$. Let $\{s_i\}_{i \geq l}$ be the natural ordering of S (i.e. $s_i > s_j$ if and only if $i > j$) then for $i \geq N$, $s_i = i$. The set $\{x_{s_i} : i < N\}$ is a finite linearly independent set so spans a finite dimensional subspace in X. Let $X_S = \langle x_{s_i} : l \leq i < N \rangle \oplus X_N$. There is a basis for X_S given by $\{x_s\}_{s \in S}$.

Since both spaces are of the same dimension, there is an isomorphism $\langle x_i : l \leq i < N \rangle \to \langle x_{s_i} : l \leq i < N \rangle$ which takes x_i to x_{s_i}. Extending this by the identity on X_N defines an isomorphism $x_S : X_l \to X_S$ such that $x_S(x_i) = x_{s_i}$. This is clearly an admissible basis for X_S.

[3]For two sets S, T, the set $S - T$ is defined to be $S \setminus (S \cap T)$

In the following, when we write "$S \subseteq \mathbb{Z}$ such that $|S - \mathbb{N}| - |\mathbb{N} - S| = l$" we shall take this to include the condition that both $|S - \mathbb{N}|$ and $|\mathbb{N} - S|$ are finite.

3.2.4 Central Extensions

For a polarised complex $\mathcal{F}$-space X the group $\mathrm{Gl}_{\mathrm{res}}(X)$ has an important central extension by $\mathbb{C}^{\times}$, the non-zero complex numbers. This is detailed in [18], section 6.6 for a Hilbert space. The definition involves first defining a central extension of the identity component, $\mathrm{Gl}_{\mathrm{res},0}(X)$, and then extending it to all of $\mathrm{Gl}_{\mathrm{res}}(X)$ by using the semi-direct product structure. The central extension is denoted by $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$.

The basis of this extension consists of the following. Define the group $\mathcal{E} = \{(B, q) \in \mathrm{Gl}_{\mathrm{res},0}(X) \times \mathrm{Gl}(X_+) : q^{-1}p_+Bi_+ \in \mathfrak{D}(X_+)\}$. This is a fibration by $\mathfrak{D}_{\times}(X_+)$ with action $p(B,q) = (B,pq)$. The quotient $\mathcal{E}/\mathfrak{D}_1(X_+)$ is the central extension $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ by $\mathbb{C}^{\times}$.

Given some $C \in \mathrm{Gl}_{\mathrm{res}}(X)$, $CX_- \oplus CX_+$ is an equivalent polarisation of X which gives rise to the same group. Let $\mathcal{E}_C$ be the group constructed using CX_+, there is an isomorphism $\mathcal{E} \to \mathcal{E}_C$ given by $(B,q) \to (CBC^{-1}, CqC^{-1})$. The action of $\mathfrak{D}_{\times}(X_+)$ translates into the action of $\mathfrak{D}_{\times}(CX_+)$ and so the two central extensions so defined are isomorphic.

Let $C \in \mathrm{Gl}_{\mathrm{res},1}(X)$ be such that $CX_+ \subseteq X_+$. The group $\mathcal{E}_C$ can be constructed relative to this polarisation. Let V be complementary to CX_+ in X_+. There is a map $\mathcal{E} \to \mathcal{E}$ given by $(B,q) \to (CBC^{-1}, I_V + CqC^{-1})$. Although this is not an isomorphism, because $\det q = \det(I_V + CqC^{-1})$ it induces an isomorphism on $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$. Thus we define $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ as the semi-direct product of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ with $\mathbb{Z}$ where $k \in \mathbb{Z}$ acts as C^k on $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$. This is a central extension of $\mathrm{Gl}_{\mathrm{res}}(X)$ by $\mathbb{C}^{\times}$.

If $Y = X \otimes_{\mathbb{R}} \mathbb{C}$ with polarisation defined by definition 3.2.1.5 then there is a map $\mathrm{Gl}_{\mathrm{res}}(X) \to \mathrm{Gl}_{\mathrm{res}}(Y)$ which we write as $A \to (A, \overline{A})$. There is a corresponding map $i_{\mathbb{C}} : \mathcal{E}_{X_+} \to \mathcal{E}_{Y_+}$ given by $(A, q) \to ((A, q), (\overline{A}, \overline{q}))$. There is an action of $\mathfrak{D}_{S^1}(X_+)$ on $\mathcal{E}_{X_+}$ given by $p(A, q) = (A, pq)$. Under the map $i_{\mathbb{C}}$, the action of p on $\mathcal{E}_{X_+}$ corresponds to the action of $(p, \overline{p})$ on $\mathcal{E}_{Y_+}$. This lies in $\mathfrak{D}_1(Y_+)$ and thus there is a map $\mathcal{E}_{X_+}/\mathfrak{D}_{S^1}(X_+) \to \mathcal{E}_{Y_+}/\mathfrak{D}_1(Y_+) = \mathrm{Gl}^{\sim}_{\mathrm{res}}(Y)$. Thus $\mathcal{E}_{X_+}/\mathfrak{D}_{S^1}(X_+) \cong \mathrm{Gl}_{\mathrm{res}}(X) \times \mathbb{R}^+$, a trivial extension of $\mathrm{Gl}_{\mathrm{res}}(X)$ by the strictly positive real numbers. It should also be noted that if $A \in \mathrm{Gl}_{\mathrm{res},l}(X)$ then $(A, \overline{A}) \in \mathrm{Gl}_{\mathrm{res},2l}(Y)$.

A similar construction can be applied to $\mathrm{Gl}(X)$ where instead of X_+ we take a closed subspace of finite dimension. The resulting central extension $\mathrm{Gl}^{\sim}(X)$ is independent of the closed finite codimension subspace used to define it. Thus in the definition of the group $\mathcal{E}$ we use the subspace X of X. This group has a cross-section given by $B \to (B, B)$ and so $\mathrm{Gl}^{\sim}(X) \cong \mathrm{Gl}(X) \times \mathbb{C}^{\times}$.

3.3 The Space of Holomorphic Sections

The key object in the cohomology theories is $A(X) = \Gamma_{\mathrm{hol}}(\mathrm{Gr}(X), D^*)$, the space of holomorphic sections of the dual of the determinant line bundle over a Grassmannian manifold. The type of Grassmannian determines the type of cohomology theory and a grading on the Grassmannian defines a grading on the cohomology theory. In this section we describe the definition of the restricted Grassmannian manifold and the determinant line bundle over it. We then consider the space $A_{\mathrm{si}}(X)$ and prove some properties of it; in particular we define the contraction map $X^* \otimes A_{\mathrm{si}}(X) \to A_{\mathrm{si}}(X)$. We also describe how the central extension of the restricted general linear group acts on this space.

3.3.1 The Restricted Grassmannian

Let X be polarised with symmetric polarisation $X = X_- \oplus X_+$ and with basis (x_k). For a closed subspace W of X such that $p_+ : W \to X_+$ is Fredholm, there is a completion $\mathfrak{J}_2(W, X_-)$ of $W^* \otimes X_-$. We define:

Definition 3.3.1.1. *Let* $\mathrm{Gr}_{res}(X)$ *be the set of all closed subspaces* W *such that* $p_+ : W \to X_+$ *is Fredholm and* $p_- : W \to X_-$ *lies in* $\mathfrak{J}_2(W, X_-)$.

This space has connected components corresponding to $\operatorname{Index} p_+$.

For $W \in \mathrm{Gr}_{\mathrm{res}}(X)$, there is some $S \subseteq \mathbb{Z}$ such that $S - \mathbb{N}$ and $\mathbb{N} - S$ are finite and such that the projection $p^S : X \to X^S$ is an isomorphism when restricted to W. The map $p_S(p^S|_W)^{-1} : X^S \to X_S$ lies in $\mathfrak{J}_2(X^S, X_S)$. Conversely, given a map $F \in \mathfrak{J}_2(X^S, X_S)$ the space $W_F := \{(w, Fw) : w \in X^S\}$ lies in $\mathrm{Gr}_{\mathrm{res}}(X)$. This defines an atlas for $\mathrm{Gr}_{\mathrm{res}}(X)$ of sets $\{\mathfrak{J}_2(X^S, X_S)\}$ where the indexing set is $\mathcal{S} = \{S \subseteq \mathbb{Z} : |\mathbb{N} - S|, |S - \mathbb{N}| < \infty\}$. Each $\mathfrak{J}_2(X^S, X_S)$ is isomorphic to $\mathfrak{J}_2(X_+, X_-)$. Two sets in the atlas lie in the same component of $\mathrm{Gr}_{\mathrm{res}}(X)$ if and only if $|S - \mathbb{N}| - |\mathbb{N} - S| = |T - \mathbb{N}| - |\mathbb{N} - T|$.

Let W be in the image of $\mathfrak{J}_2(X^S, X_S)$ and $\mathfrak{J}_2(X^T, X_T)$ and let F_S, F_T be the corresponding maps. Let $S' = \mathbb{Z} \setminus S$ and $T' = \mathbb{Z} \setminus T$. The identity map

$I : X \to X$ can be expressed as:

$$\begin{pmatrix} p_{S\cap T} & p_{S'\cap T} \\ p_{S\cap T'} & p_{S'\cap T'} \end{pmatrix} : X^S \oplus X_S \to X^T \oplus X_T$$

there is some isomorphism $q : X^S \to X^T$ such that:

$$\begin{pmatrix} p_{S\cap T} & p_{S'\cap T} \\ p_{S\cap T'} & p_{S'\cap T'} \end{pmatrix} \begin{pmatrix} 1 \\ F_S \end{pmatrix} = \begin{pmatrix} 1 \\ F_T \end{pmatrix} q$$

Thus $F_T = (p_{S\cap T'} + p_{S'\cap T'}F_S)(p_{S\cap T} + p_{S'\cap T}F_S)^{-1}$. This is a holomorphic function of F_S in the open set $\{F_S : p_{S\cap T} + p_{S'\cap T}F_S \text{ is invertible}\}$.

The determinant bundle D over a Grassmannian is defined in terms of admissible bases and thus requires a choice of basis for the original vector space. An element of the determinant line can be represented as an admissible basis and a complex number, $[w, \lambda]$. If w' is another choice of admissible basis then the map $t : W \to W$ which takes $w' \to w$ is an operator with a determinant. Then $[w, \lambda]$ is identified with $[w', \lambda \det(t)]$.

For W in the image of $\mathfrak{J}_2(X^S, X_S)$ there is a natural admissible basis given by the preimage of x_S under p_S. This gives an identification of the part of D over $\mathfrak{J}_2(X^S, X_S)$ with $\mathfrak{J}_2(X^S, X_S) \times \mathbb{C}$. The transition maps are given by $(\lambda, F_S) \to (\lambda', F_T)$ where $\lambda' = \lambda \det(p_{S\cap T} + p_{S'\cap T}F_S)$. This is the determinant of the finite dimensional matrix formed from F_S by considering the induced map from $\langle x_k : k \in S \cap T' \rangle$ to $\langle x_k : k \in S' \cap T \rangle$ with the canonical bases, which is a holomorphic function of F_S. Thus D is a holomorphic line bundle over $\mathrm{Gr}_{\mathrm{res}}(X)$.

The central extension constructed in section 3.2.4 acts on the determinant line bundle. This action is detailed in Pressley and Segal [18], theorem 7.7.3. The action is defined in stages. Firstly there is an action of the group $\mathcal{E}$ on D above $\mathrm{Gr}_{\mathrm{res},0}(X)$, the component of $\mathrm{Gr}_{\mathrm{res}}(X)$ containing X_+. This action is defined by $(A,q)\,[w,\lambda] = [Awq^{-1},\lambda]$. The group $\mathcal{E}$ was defined precisely so that Awq^{-1} is an admissible basis. The subgroup of $\mathcal{E}$ consisting of those elements of the form $(1,q)$ where q has determinant 1 acts trivially on D so the action is one of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$.

We extend this action using the semi-direct structure of $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$. The shift operator σ acts on D over the whole of $\mathrm{Gr}_{\mathrm{res}}(X)$ via $\sigma\,[w,\lambda] = [\sigma w,\lambda]$. From this we can construct an action of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ on D over the whole of $\mathrm{Gr}_{\mathrm{res}}(X)$. The action of $[A,q]$ on the component of D above $\mathrm{Gr}_{\mathrm{res},k}(X)$ is given by $\sigma^{-k}\tilde{\sigma}^k([A,q])\sigma^k$, where $\tilde{\sigma}$ is the action of σ on $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$. Since we now have an action of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ and of σ on D we have an action of $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ on D.

We can similarly define the Grassmannians $\mathrm{Gr}_{\mathrm{fd}}(X)$ of finite dimensional subspaces and $\mathrm{Gr}_{\mathrm{fc}}(X)$ of closed finite codimensional subspaces. These are modelled on the spaces $X_l^* \otimes X^l$ for finite dimensions and $X^{l*} \otimes X_l$ for finite codimensions, where in this case X_l is a subspace of dimension l and X^l is a complementary subspace. The determinant line bundle D is defined in the same way. The group $\mathrm{Gl}(X)$ acts on D over $\mathrm{Gr}_{\mathrm{fd}}(X)$ and the central extension $\mathrm{Gl}^{\sim}(X)$ of $\mathrm{Gl}(X)$ acts on D over $\mathrm{Gr}_{\mathrm{fc}}(X)$. However, as $\mathrm{Gl}^{\sim}(X)$ is a trivial central extension, there is an action of $\mathrm{Gl}(X)$ on D over $\mathrm{Gr}_{\mathrm{fc}}(X)$.

3.3.2 Holomorphic Sections

Since D is a holomorphic line bundle over a complex manifold, we can consider the following spaces:

Definition 3.3.2.1. *Let X be a complex polarised $\mathcal{F}$-space. Let:*

$$A^k_{si}(X) = \Gamma_{hol}(\mathrm{Gr}_{res,k}(X), D^*) \qquad \textit{for } k \in \mathbb{Z}$$

$$A_{si}(X) = \prod_{k \in \mathbb{Z}} A^k_{si}(X)$$

This space is a closed subspace of the space of smooth maps $D \to \mathbb{C}$ and so is a complete, locally convex topological space under the smooth compact-open topology (i.e. uniform convergence of all derivatives on compact sets).

For the finite dimension theory we use the notation A_{fd} and for the finite codimension theory we use the notation A_{fc}. The grading on A_{fd} is given by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and on A_{fc} by $-\mathbb{N}_0$. When we wish to refer to all three theories, or to one unspecified theory, we use the notation A. If X is finite dimensional then $A_{\mathrm{fd}}(X)$ coincides with the spaces constructed in chapter 2. Occasionally we will want to discuss spaces which may be of infinite or finite dimension without specifying which. To make the notation easier in this case, for a complex space V of finite dimension n and for $0 \leq l \leq n$, we define $A^{-l}_{\mathrm{fc}}(V)$ to be $A^{n-l}(V)$.

In order to show that these spaces are not trivial, we consider maps between $A(X)$ and $A(Y)$ given a linear map $F : X \to Y$. What we are aiming for is a formula like $F^* f([w, \lambda]) = f([F(w), \lambda])$. The other case that we wish to consider is when X is a closed subspace of Y with basis x. We want a map

$i_X^* : A(Y) \to A(Y/X)$ with formula like $i_X^* f([w + X, \lambda]) = f([w \cup x, \lambda])$.

The key to showing that these formulae are well-defined is the following lemma which shows that a holomorphic section defined over a certain part of the Grassmannian manifold can be extended over the whole.

Lemma 3.3.2.2. *For each l there is some non-zero $f \in A^l(X)$.*

Proof. Let $B_l \subseteq \mathrm{Gr}_l(X)$ be the set of those subspaces V for which $p_l : V \to X_l$ is an isomorphism. This is also the set $\{V \in \mathrm{Gr}_l(X) : V \cap \ker p_l = \{0\}\}$.

Let $V \in B_l$ and let v be an admissible basis for V; $pv : X_l \to X_l$ is an operator with a determinant so $p(v)$ is a basis for X_l equivalent to the given one. Define the map $f : D_{B_l} \to \mathbb{C}$ by $f([v, \lambda]) = \lambda \det pv$. This is well-defined, holomorphic and linear on fibres.

Now B_l is open and dense in $\mathrm{Gr}_l(X)$. Let $V \in \mathrm{Gr}_l(X)$ be such that $V \cap \ker p_l$ is one dimensional. There is an admissible basis for V of the form $\{v\} \cup w$ where $v \in V \cap \ker p^l$. Let $u \in X_l$ and $\epsilon > 0$. Let V_ϵ be the subspace with admissible basis $\{v + \epsilon u\} \cup w$ then $V_\epsilon \in B_l$. We have $p_l(\{v + \epsilon u\} \cup w) = \{\epsilon u\} \cup p(w)$. Let U be the subspace with admissible basis $\{u\} \cup w$ then $p_l(\{u\} \cup w) = \{u\} \cup p(w)$. Let μ be the determinant of the map which takes the standard basis to $\{u\} \cup p(w)$. Then $f([\{v + \epsilon u\} \cup w, \lambda]) = \lambda \epsilon \mu$ so as $\epsilon \to 0$, $f_{V_\epsilon} \to 0$.

This method clearly also works for $V \cap \ker p_l$ of finite non-zero dimension, and hence for all of $\mathrm{Gr}_l(X)$. Thus f can be extended over all of $\mathrm{Gr}_l(X)$ by defining it to be zero on the complement to B_l and this extension is clearly holomorphic. □

Corollary 3.3.2.3. *For each* $[w, \lambda] \in D$ *with* $\lambda \neq 0$ *there is some* $f \in A(X)$ *such that* $f([w, \lambda]) \neq 0$.

Proof. Let W be the space spanned by w. Let l be such that $W \in \mathrm{Gr}_l(X)$. Let g be the element of $A(X)$ as constructed in lemma 3.3.2.2. There is some B in the relevant group such that $BW = X_l$. Thus $B^* g$ has the required properties. □

In effect, lemma 3.3.2.2 constructs an injective map $A^0_{\mathrm{fc}}(X^l) \to A^l(X)$. The same technique can be used to construct maps $A^{-k}_{\mathrm{fc}}(X^l) \to A^{l-k}(X)$ and thus, by the same method as corollary 3.3.2.3, $A^{-k}_{\mathrm{fc}}(W) \to A^{l-k}(X)$ where $W \in \mathrm{Gr}_l(X)$.

Let $F : X \to Y$ be a continuous linear map with closed range. The first thing we need to ensure that the formula $F^* g([w, \lambda]) = g([F(w), \lambda])$ makes sense is that there is some $W \in \mathrm{Gr}(X)$ such that $F(W) \in \mathrm{Gr}(Y)$ and $F : W \to F(W)$ is injective (hence an isomorphism as $F(W)$ is closed). We also need to ensure that if w is an admissible basis for W then $F(w)$ is an admissible basis for $F(W)$. As the Grassmannian for X can be constructed relative to W and w and the Grassmannian for Y can be constructed relative to $F(W)$ and $F(w)$ then these conditions are also sufficient to ensure that $F(U) \in \mathrm{Gr}(Y)$ for all $U \in \mathrm{Gr}(X)$ such that $F : U \to F(U)$ is injective and that if u is an admissible basis for U then $F(u)$ is an admissible basis for $F(U)$.

This shows that the map F^* is well-defined over the set $\{U : F : U \to F(U) \text{ is injective}\}$ in $\mathrm{Gr}(X)$. The technique of lemma 3.3.2.2 extends F^*

over the whole of $\mathrm{Gr}(X)$. Thus the full definition is:

$$F^*g([w,\lambda]) = \begin{cases} g([F(w),\lambda]) & \text{if } F \text{ is injective on } \langle w \rangle \\ 0 & \text{otherwise} \end{cases}$$

Given a subspace X of Y and a basis x for X, provided X and x are such that the following formula makes sense, there is a map $i_X^* : A(Y) \to A(Y/X)$ (where these may be over different Grassmannians) such that $i_X^*([w,\lambda]) = [w \cup x, \lambda]$.

There are also certain cases where we may relax the condition that F have closed range. One of these is the finite dimension theory, the other is the semi-infinite theory. The relaxation in the finite dimension theory is to all continuous linear maps. The relaxation in the semi-infinite theory is to continuous linear maps for which the restricted map $X_+ \to Y_+$ is closed. However, such maps only occur in the context of polarisations which are not symmetric. Thus as we are mainly interested in symmetric polarised spaces, we shall only consider continuous, closed maps.

Let $F : X \to Y$ be a closed, continuous map. Then F induces maps between the spaces of holomorphic sections according to the following, where, if necessary, we assume that F takes an admissible basis to an admissible basis:

1. In all cases, there is a map $F^* : A^l_{\mathrm{fd}}(Y) \to A^l_{\mathrm{fd}}(X)$.

2. If F is Fredholm of index k then there is a map $F^* : A^l_{\mathrm{fc}}(Y) \to A^{l-k}_{\mathrm{fc}}(X)$.

3. If $p_+ F i_+ : X_+ \to Y_+$ is Fredholm of index k and $p_- F i_+ : X_+ \to Y_-$ is

in $\mathfrak{J}_2(X_+, Y_-)$ then there is a map $F^* : A^l_{\mathrm{si}}(Y) \to A^{l-k}_{\mathrm{si}}(X)$.

4. If $p_+F : X \to Y_+$ is Fredholm of index k and $p_-F : X \to Y_-$ is in $\mathfrak{J}_2(X, Y_-)$ then there is a map $F^* : A^l_{\mathrm{si}}(Y) \to A^{l-k}_{\mathrm{fc}}(X)$.

With the above, we can prove some properties of $A(X)$. We have already mentioned above the result that $A^0_{\mathrm{fc}}(X) = \mathbb{C}$ for all X.

Lemma 3.3.2.4. *Let $f \in A^l(X)$, $x, y \in X$ and $W \in \mathrm{Gr}_{l-1}(X)$ such that $x, y, x+y \notin W$. Then for any admissible basis w of W:*

$$f([\{x+y\} \cup w, \lambda]) = f([\{x\} \cup w, \lambda]) + f([\{y\} \cup w, \lambda])$$

Proof. If there is some $\lambda \in \mathbb{C}^\times$ such that $x - \lambda y \in W$ then the spaces $W + \langle x+y \rangle$, $W + \langle x \rangle$ and $W + \langle y \rangle$ are all the same space. The formula in the statement follows from considering the transformation operators between the bases $\{x+y\} \cup w$, $\{x\} \cup w$ and $\{y\} \cup w$.

Otherwise, the formula in the statement concerns the value of f over the spaces $W + \langle x+y \rangle$, $W + \langle x \rangle$ and $W + \langle y \rangle$ which all contain W and are contained within $W + \langle x, y \rangle$. Thus we can restrict to the Grassmannian of the space $V = (W + \langle x, y \rangle)/W$. The restriction of f to $\mathrm{Gr}(V)$, say $\tilde{f}$, lies in $A^1(V)$ which is isomorphic to V^* as V is finite dimensional. Thus there is some $c \in V^*$ such that $\tilde{f}([z, \lambda]) = \lambda c(z)$ for $z \in V$. Hence $\tilde{f}$ satisfies the condition:

$$\tilde{f}([\{x+y\}, \lambda]) = \tilde{f}([\{x\}, \lambda]) + \tilde{f}([\{y\}, \lambda])$$

Let $\tilde{x} = x + W$ and $\tilde{y} = y + W$, then:

$$\begin{aligned} f([\{x+y\} \cup w, \lambda]) &= \tilde{f}([\{\tilde{x}+\tilde{y}\}, \lambda]) \\ &= \tilde{f}([\{\tilde{x}\}, \lambda]) + \tilde{f}([\{\tilde{y}\}, \lambda]) \\ &= f([\{x\} \cup w, \lambda]) + f([\{y\} \cup w, \lambda]) \end{aligned}$$

□

Lemma 3.3.2.5. *For any complex vector space X, $A^1_{fd}(X) = X^*$.*

Proof. By lemma 3.3.2.4, any element f in $A^1_{\mathrm{fd}}(X)$ satisfies the relationship:

$$f([\{w_1 + \eta w_2\}, 1]) = f([\{w_1\}, 1]) + \eta f([\{w_2\}, 1])$$

and therefore there is a map $A^1_{\mathrm{fd}}(X) \to X'$, the algebraic dual of X, given by $\phi(f)(w) = f([\{w\}, 1])$. Since the map $w \to [\{w\}, 1]$ is continuous from $X \setminus \{0\}$ to $D \setminus \mathrm{Gr}^1_{\mathrm{fd}}(X)$, the image of ϕ is contained in X^*, the topological dual of X.

Conversely, given $a \in X^*$ the map $[\{w\}, \lambda] \to \lambda a(w)$ defines a holomorphic map $D \to \mathbb{C}$ which is linear on fibres, and hence an element of $A^1_{\mathrm{fd}}(X)$.

Finally we note that the topology on X^* induced by this isomorphism is that of uniform convergence on compact sets. □

Corollary 3.3.2.6. $A^k(X + \mathbb{C}) \cong A^k(X) + A^{k-1}(X)$

Proof. Let $\alpha \in \mathbb{C}$ and let $a \in (X + \mathbb{C})^*$ be such that $a(\alpha) = 1$ and $\ker a = X$.

Let $W \in \mathrm{Gr}_k(X)$ and let $\{u\} \cup w$ be an admissible basis for W such that $w \subseteq X$ and $u = v + \eta\alpha$ where $v \in X$ and $\eta \in \mathbb{C}$. If $p : X + \mathbb{C} \to X$ is the projection then $p(\{u\} \cup w) = \{v\} \cup w$. Also note that $a(u) = \eta$.

There is an extension map $e_1 : A^k(X) \to A^k(X + \mathbb{C})$ and a restriction map $r_1 : A^k(X + \mathbb{C}) \to A^k(X)$. The choice of $\alpha \in \mathbb{C}$ gives an extension map $e_2 : A^{k-1}((X + \mathbb{C})/\mathbb{C}) \to A^k(X + \mathbb{C})$ and a restriction map $r_2 : A^k(X + \mathbb{C}) \to A^{k-1}((X + \mathbb{C})/\mathbb{C})$. These maps satisfy the relations $r_1 e_1 = 1 = r_2 e_2$, $r_1 e_2 = 0 = r_2 e_1$.

On an element in D of the form $[\{u\} \cup w, \lambda]$ as defined above, we have:

$$(e_1 r_1) f([\{u\} \cup w, \lambda]) = f([\{v\} \cup w, \lambda])$$
$$(e_2 r_2) g([\{u\} \cup w, \lambda]) = g([\{\eta\alpha\} \cup w, \lambda])$$

For $f \in A^k(X + \mathbb{C})$, lemma 3.3.2.4 gives:

$$f([\{u\} \cup w, \lambda]) = f([\{\eta\alpha\} \cup w, \lambda]) + f([\{v\} \cup w, \lambda])$$

thus from above $f = (e_1 r_1) f + (e_2 r_2) f \in e_1(A^k(X)) + e_2(A^{k-1}(X))$.

Each e_i is injective onto its image and the images intersect trivially. Thus the corollary is proved. $\square$

For any pair of closed subspaces $V \subseteq W$ of X, the set $\{U \in \mathrm{Gr}_k(X) : V \subseteq U \subseteq W\}$ is a (possibly empty) submanifold of $\mathrm{Gr}_{\mathrm{res},k}(X)$ diffeomorphic to a component of $\mathrm{Gr}(W/V)$. In particular, considering $X_m \subseteq X_l$ for $l < m$ we have a submanifold of $\mathrm{Gr}_{\mathrm{res},k}(X)$ diffeomorphic to a component of $\mathrm{Gr}(X_l/X_m)$. The union of these spaces is dense in $\mathrm{Gr}_{\mathrm{res}}(X)$ and thus using the restriction and extension maps we have:

$$\bigcup A^{n+k}(X_{-n}/X_n) \to A^k_{\mathrm{si}}(X) \to \varprojlim A^{n+k}(X_{-n}/X_n)$$

This gives a basis for $A^k_{\mathrm{si}}(X)$ as the set $\{f^S : S \subseteq \mathbb{Z} : |S - \mathbb{N}| - |\mathbb{N} - S| = k\}$.

In the finite dimension case the restriction maps are $A^l_{\mathrm{fd}}(X) \to A^l(X^n)$. The basis is indexed by the set $\{S \subseteq \mathbb{N} : |S| = l\}$. In the finite codimension case the restriction maps are $A^{-l}_{\mathrm{fc}}(X) \to A^{-l}_{\mathrm{fc}}(X/X^n)$. The basis is indexed by the set $\{S \subseteq \mathbb{N} : |\mathbb{N} \setminus S| = l\}$.

3.3.3 The Action of $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ on $A_{\mathrm{si}}(X)$

There are actions of $\mathrm{Gl}(X)$ on $A_{\mathrm{fd}}(X)$, of $\mathrm{Gl}^{\sim}(X)$ on $A^k_{\mathrm{fc}}(X)$ and of $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ on $A_{\mathrm{si}}(X)$. These actions are defined by $Bf([w,\lambda]) = f([Bw,\lambda])$.

We describe the action of $\mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ on $A_{\mathrm{si}}(X)$ by examining the action of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ and the action of the shift map σ. We are particularly interested in their actions on the basis elements $\{f^S\}$.

Recall that $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ is defined as a quotient of the space $\mathcal{E} = \{(B,q) \in \mathrm{Gl}_{\mathrm{res},0}(X) \times \mathrm{Gl}(X_+) : q^{-1}p_+Bi_+ \in \mathfrak{D}(X_+)\}$. The action of $\mathcal{E}$ on D over $\mathrm{Gr}_{\mathrm{res},0}(X)$ is $(B,q)([w,\lambda]) = [Bwq^{-1},\lambda]$.

Consider $(B,q)f^S$ at $[x_T,1]$:

$$\begin{aligned}
(B,q)f^S([x_T,1]) &= f^S([(B,q)x_T,1]) \\
&= f^S([Bx_Tq^{-1},1]) \\
&= f^S([p_SBx_Tq^{-1},1])
\end{aligned}$$

For this to be non-zero, $p_S B : X_T \to X_S$ must be an isomorphism so $x_S^{-1} p_S B x_T : X_+ \to X_+$ is an isomorphism of X_+. Let $B_T^S = x_S^{-1} p_S B x_T$.

As S, T differ from $\mathbb{N}$ by only a finite amount, there is some N such that for $k \geq N$ then $k \in S, T$ and $k \geq 0$. Let $i_N : X_N \to X$ be the inclusion and $p_N : X \to X_N$ be the projection, then $x_T i_N = i_N$ and $p_N x_S^{-1} p_S = p_N$. Thus $p_N B_T^S i_N = p_N B i_N$ so B and B_T^S agree on a subspace of finite codimension in X_+ and thus on X_+ differ by a finite rank operator. Thus (B, B_T^S) is a choice of element above B in $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$. The space $\{B : p_S B : X_T \to X_S\}$ is open in $\mathrm{Gl}_{\mathrm{res},0}(X)$ and $B \to (B, x_S^{-1} p_S B x_T)$ is a cross section of $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$ above this set. The matrix of B_T^S is the submatrix of B corresponding to the rows of S and the columns of T.

Thus $(B, B_T^S) f^S([x_T, 1]) = f^S([p_S B x_T (p^S B x_T)^{-1} x_S, 1]) = f^S([x_S, 1]) = 1$ and so for any $(B, q) \in \mathrm{Gl}^{\sim}_{\mathrm{res},0}(X)$, $(B, q) f^S([x_T, 1]) = \det B_T^S q^{-1}$.

Now (B, q) defines a new basis for A_{si} which satisfies the relationship $\tilde{f}^S([(B, q) x_T, 1]) = \delta_T^S$. Then:

$$
\begin{aligned}
\tilde{f}^S([(B, q) x_T, q]) &= \delta_T^S \\
&= f^S([x_T, 1]) \\
&= (B^{-1}, q^{-1}) f^S([(B, q) x_T, 1])
\end{aligned}
$$

so $\tilde{f}^S = (B^{-1}, q^{-1}) f^S$. Thus:

$$
\tilde{f}^S([x_T, 1]) = \det(B^{-1})_T^S q
$$

The action of σ is particularly simple. Consider the set $\mathcal{S}$ of all subsets

$S \subseteq \mathbb{Z}$ such that $|S - \mathbb{N}|$ and $|\mathbb{N} - S|$ are both finite. Let $\mathcal{S}_l$ be the subset of $\mathcal{S}$ consisting of those S for which $|S - \mathbb{N}| - |\mathbb{N} - S| = l$. Define $\sigma : \mathcal{S}_l \to \mathcal{S}_{l-1}$ by $\sigma(S) = \{k : k - 1 \in S\}$. This defines a new basis $\{\tilde{f}^S\}$ which satisfies the relation $\tilde{f}^S([\sigma x_T, 1]) = \delta_T^S$. We have $\tilde{f}^S = f^{\sigma^{-1}(S)}$.

3.3.4 The Contraction Map

The differential of the cochain complex is defined using a contraction map. This map is defined as a map $X^* \times A^l(X) \to A^{l+1}(X)$ for a complex, polarised $\mathcal{F}$-space X which extends to a map $X^* \otimes A^l(X) \to A^{l+1}(X)$ by linearity.

Definition 3.3.4.1. *The map $X^* \times A^l(X) \to A^{l+1}(X)$ is written as $(a, f) \to a \wedge f$ and is defined by:*

$$(a \wedge f)([w, \lambda]) = \begin{cases} 0 & \text{if } a|_{\langle w \rangle} = 0 \\ a(\alpha) f([u, \lambda]) & \text{otherwise} \end{cases}$$

where $[w, \lambda]$ lies in the determinant bundle over $\mathrm{Gr}_{\mathrm{res},l+1}(X)$. If $a|_{\langle w \rangle} \neq 0$ then u is an admissible basis for $\langle w \rangle \cap \ker a$ and $\alpha \in \langle w \rangle$ is such that $[w, \lambda] = [\{\alpha\} \cup u, \lambda]$.

For finite dimensional spaces U, V, $\mathfrak{L}(U, V) = U^* \otimes V$ so in that context, this map can be thought of as a map $\mathfrak{L}(X, A^l(X)) \to A^{l+1}(X)$. However, in infinite dimensions $X^* \otimes A^l(X)$ is a strict subspace of $\mathfrak{L}(X, A^l(X))$ and the map does not extend. It is, however, possible to write down a formal expression which extends the contraction map but outside a completion of

$X^* \otimes A^l(X)$ this expression will not necessarily converge. The expression is:

$$\wedge F([w,\lambda]) = \sum_k (-1)^{k-1} F(w_k)([w \setminus \{w_k\}, \lambda]). \tag{3.1}$$

Theorem 3.3.4.2. *The map $X^* \times A^k_{si}(X) \to A^{k+1}_{si}(X)$ given by $(a,f) \to a \wedge f$ is well-defined.*

Proof. There are two things to show here. To show that the resulting function is an element of $A^{k+1}_{\mathrm{si}}(X)$ and that the contraction map is independent of the choices made. Let w be an admissible basis for an element of $\mathrm{Gr}_{\mathrm{res},k+1}(X)$. The choices made are of α and u when a is non-zero on $\langle w \rangle$, thus there are no choices to be made when a is zero on $\langle w \rangle$.

Thus to show independence of choices, assume that a is non-zero on $\langle w \rangle$. Suppose that α' and u' also satisfy the conditions for α and u. Note that as $a|_{\langle w \rangle} \neq 0$ and $u, u' \subseteq \ker a$, it must be the case that $a(\alpha) \neq 0$ and $a(\alpha') \neq 0$.

As $\{\alpha\} \cup u$ and $\{\alpha'\} \cup u'$ are both equivalent to w, they are equivalent to each other. Thus there is an element $B \in G$ such that $B\alpha = \alpha'$ and $Bu = u'$.

As u and u' span the same space, there is an element A of G such that $Au = u'$. Also there are $\mu, \nu_j \in \mathbb{C}$ such that $\alpha' = \mu\alpha + \nu_1 u_1 + \ldots$. Thus B is:

$$\begin{pmatrix} \mu & \nu_1 \ldots \\ 0 & A \end{pmatrix}$$

and so $\mu \det A = \det B = 1$. Now $a(\alpha') = \mu a(\alpha)$ so $a(\alpha') f([u', \lambda]) = \mu \det A a(\alpha) f([u, \lambda]) = a(\alpha) f([u, \lambda])$. Hence the definition is independent of the choices of α and u.

To show independence from the choice of w, it is sufficient to note that if w and w' differ by a transformation of determinant μ and $\{\alpha\} \cup u$ is equivalent to w then $\{\mu\alpha\} \cup u$ is equivalent to w'. The linearity of f and a then gives the required result.

To show that the resulting map is holomorphic, it suffices to note that it is a linear multiple of the map obtained by applying the extension and restriction maps of section 3.3.2 in the sequence:

$$A^k(X) \to A^k(\ker a) \to A^k(X/\langle\alpha\rangle) \to A^{k+1}(X)$$

□

This extends to $X_i^* \otimes A_{\text{si}}(X)$ by linearity. This map is separately continuous and so extends to $\mathfrak{J}_1(X, A_{\text{si}}^k(X)) \to A_{\text{si}}^{k+1}(X)$.

As we do not have the isomorphism $A(X) \cong \Lambda X^*$, we need to prove the properties of $\wedge$ from first principles. The properties of $\wedge$ that we wish to prove are:

Theorem 3.3.4.3. *Let $a, b \in X^*$, $\eta \in \mathbb{C}$, $f \in A_{si}^k(X)$:*

1. $a \wedge (a \wedge f) = 0$,
2. $(\eta a) \wedge f = \eta(a \wedge f)$,
3. $(a + b) \wedge f = a \wedge f + b \wedge f$,
4. $a \wedge (b \wedge f) = -b \wedge (a \wedge f)$,

Proof. The first two follow straight from the definition and the fourth follows from the first and third. However, the proof of the third is more complicated.

Consider first the case when $k = 1$. From lemma 3.3.2.5, $A^1_{\mathrm{fd}}(X) \cong X^*$. Let $f \in A^1_{\mathrm{fd}}(X)$; there exists $c \in X^*$ such that $f([w,\lambda]) = \lambda c(w)$. Let $W \in \mathrm{Gr}_{\mathrm{fd},2}(X)$ and $[w,\lambda] \in D$ above W.

Let $a, b \in X^*$ be non-zero and not collinear on W (otherwise the proposition is trivial). We can pick $\alpha, \beta \in W$ such that $a(\alpha) = 1$, $b(\beta) = 1$, $a(\beta) = 0$, $b(\alpha) = 0$. Thus $W \cap \ker a = \langle\beta\rangle$ and $W \cap \ker b = \langle\alpha\rangle$.

Since α, β are linearly independent, we may assume (by adjusting λ if necessary) that $w = \{\alpha, \beta\}$. Now $\{\alpha+\beta, \alpha-\beta\}$ is another basis for W. The matrix of transformation is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which has determinant -2. Thus $[\{\alpha,\beta\},\lambda] = [\{\alpha+\beta,\alpha-\beta\}, -\frac{1}{2}\lambda]$ and so:

$$\begin{aligned}
(a+b) \wedge f([\{\alpha,\beta\},\lambda]) &= (a+b) \wedge f([\{\alpha+\beta,\alpha-\beta\}, -\frac{1}{2}\lambda]) \\
&= -\frac{1}{2}(a+b)(\alpha+\beta) f([\{\alpha-\beta\},\lambda]) \\
&= -\frac{1}{2} \cdot 2\lambda c(\alpha-\beta) \\
&= \lambda\,(c(\beta) - c(\alpha)) \\
a \wedge f([\{\alpha,\beta\},\lambda]) + b \wedge f([\{\alpha,\beta\},\lambda]) &= a(\alpha) f([\beta,\lambda]) + b(\beta) f([\{\alpha\},-\lambda]) \\
&= \lambda c(\beta) - \lambda c(\alpha)
\end{aligned}$$

Now consider the general case. Let $f \in \mathrm{Gr}_{\mathrm{res},k}(X)$ and $a, b \in X^*$. Let $W \in \mathrm{Gr}_{\mathrm{res},k+1}(X)$. Let $c = a + b$. If, say, $a|_W = 0$ then $c|_W = b|_W$ and their respective pairings in W will also be the same. Thus in this case the proposition is trivial. Similarly, if $a|_W$ and $b|_W$ are collinear then again the proposition is trivial.

So suppose that $a|_W$ and $b|_W$ are non-zero and not collinear. Let $U = \ker a \cap \ker b \cap W$. Since a and b are not collinear on W the codimension of U in W is 2. In calculating $(a \wedge f)$, $(b \wedge f)$ and $(a+b) \wedge f$ we use the value of f only on spaces which contain U and we also choose a preferred basis of U. Thus we can consider the submanifold $\mathrm{Gr}_{\mathrm{fd}}(X/U)$ of $\mathrm{Gr}_{\mathrm{res}}(X)$. This inclusion induces the restriction map from holomorphic sections of D^* over $\mathrm{Gr}_{\mathrm{res}}(X)$ to holomorphic sections of D^* over $\mathrm{Gr}_{\mathrm{fd}}(X/U)$. Under this map, the restriction of f lies in $A^1_{\mathrm{fd}}(X/U)$.

Thus if j is the inclusion map of $\mathrm{Gr}_{\mathrm{fd}}(X/U)$ into $\mathrm{Gr}_{\mathrm{res}}(X)$ and j^* is the corresponding restriction map, we have:

$$\begin{aligned}((a+b) \wedge f)_W(\zeta) &= j^*((a+b) \wedge f)_W(\zeta)\\ &= ((j^*a + j^*b) \wedge j^*f)_{j^*W}(j^*\zeta)\\ &= (j^*a \wedge j^*f)_{j^*W}(j^*\zeta) + (j^*b \wedge j^*f)_{j^*W}(j^*\zeta)\\ &= j^*(a \wedge f)_W(\zeta) + j^*(b \wedge f)_W(\zeta)\\ &= (a \wedge f)_W(\zeta) + (b \wedge f)_W(\zeta)\end{aligned}$$

$\square$

Using the contraction map, the isomorphism in corollary 3.3.2.6 can be expressed more invariantly as $A^k(X + \mathbb{C}) \cong A^k(X) + a \wedge A^{k-1}(X)$ where $a \in (X + \mathbb{C})^*$ is such that $X = \ker a$.

Chapter 4

Semi-Infinite de Rham Cohomology

In this chapter we construct the semi-infinite de Rham cohomology of a certain type of manifold using the apparatus developed in chapter 3. Minor variations of the construction also define the finite dimension and finite codimension de Rham cohomology theories. We start with a discussion of the type of manifold for which semi-infinite cohomology can be defined. We then give the definition of the cochain complex and the differential and show that it defines a cohomology theory.

4.1 Infinite Dimensional Manifolds

For a general discussion of types of infinite dimensional manifolds, see Milnor [14]. All the manifolds that we shall be considering will be paracompact smooth manifolds modelled on $\mathcal{F}$-spaces. We shall also assume that the model space X has enough smooth maps. By this we mean that for each open $U \subseteq X$ there is a smooth, non-zero map $\rho : X \to [0,1]$ such that the support of ρ is contained within U. This ensures that a manifold modelled on X admits a partition of unity subordinate to any given open cover.

The theories of finite dimension and finite codimension cohomology can be defined for any such manifold. Semi-infinite cohomology can only be defined for a certain class of manifolds.

A real or complex polarised bundle $E \to M$ is a vector bundle modelled on a polarised $\mathcal{F}$-space X such that the transition functions lie in the group $\mathrm{Gl}_{\mathrm{res}}(X)$. With this we can define the concept of a polarised manifold.

Definition 4.1.0.4. *A manifold M modelled on a real polarised $\mathcal{F}$-space is* polarised *if the tangent bundle TM is a polarised bundle. The polarisation is* integrable *if the transition functions in the definition of the polarised structure arise from the transition functions of an atlas for M.*

In all the known examples of polarised manifolds the polarisation is integrable. Let M be a polarised manifold. There is a bundle $P \to M$ modelled on (a subgroup of) $\mathrm{Gl}_{\mathrm{res}}(X)$ such that $TM = P \times_{\mathrm{Gl}_{\mathrm{res}}(X)} X$. P is called the *structure bundle* of M.

There is some $l \in \mathbb{N}_0$ such that the structure bundle P of M can be modelled on $\mathrm{Gl}_{\mathrm{res},l\mathbb{Z}}(X)$. The period of M is defined to be the minimum l for which this is true. If the period of M is 0 then M is said to be *non-periodic.*

The period of a polarised manifold M can be derived cohomologically. There is a short exact sequence:

$$\mathrm{Gl}_{\mathrm{res},l\mathbb{Z}}(X) \longrightarrow \mathrm{Gl}_{\mathrm{res}}(X) \longrightarrow \mathbb{Z}/l\mathbb{Z}$$

which gives rise to an exact sequence of pointed sets:

$$\begin{aligned}\mathrm{Prin}(M;\mathrm{Gl}_{\mathrm{res},l\mathbb{Z}}(X)) \to \mathrm{Prin}(M;\mathrm{Gl}_{\mathrm{res}}(X)) \to \mathrm{Prin}(M;\mathbb{Z}/l\mathbb{Z}(X))\\ = H^1(M;\mathbb{Z}/l\mathbb{Z})\end{aligned}$$

where $\mathrm{Prin}(M;G)$ is the space of principal G-bundles over M. Thus M is of period at most l if the image of P in $H^1(M;\mathbb{Z}/l\mathbb{Z})$ is zero. This can be factored through $H^1(M;\mathbb{Z})$ and thus there is a characteristic class $b_1 \in H^1(M;\mathbb{Z})$ such that the period of M is 0 if $b_1 = 0$ or is the minimum $l > 0$ such that $b_1 \equiv 0 \pmod{l}$.

Definition 4.1.0.5. *A manifold M is* semi-infinite *if it is polarised and there is a $\mathrm{Gl}^{\sim}_{res}(X)$ bundle $\tilde{P}$ over M whose quotient by the action of the centre $\mathbb{C}^{\times} \subseteq \mathrm{Gl}^{\sim}_{res}(X)$ is the principal bundle P.*

There is an obstruction to a polarised manifold being semi-infinite which is easiest to describe in terms of Čech cocycles. Let $\mathcal{U}$ be an open covering of M such that $P|_U$ is trivial for each $U \in \mathcal{U}$ and choose trivialisations $P|_U \to U \times G$. The transition maps are functions $g_{UV} : U \cap V \to \mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ which satisfy the cocycle relation: $g_{UV}g_{VW}g_{WU} = 1$. The bundle $\tilde{P}$ exists if and only if there are lifts of these functions $\tilde{g}_{UV} : U \cap V \to \mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ which still satisfy the cocycle relation: $\tilde{g}_{UV}\tilde{g}_{VW}\tilde{g}_{WU} = 1$.

If we assume that the covering $\mathcal{U}$ is chosen such that each intersection $U \cap V$ is contractible then we may choose continuous functions $\tilde{g}_{UV} : U \cap V \to \mathrm{Gl}^{\sim}_{\mathrm{res}}(X)$ which are lifts of the g_{UV}. For $U, V, W \in \mathcal{U}$ with $U \cap V \cap W \neq \emptyset$, let $h_{UVW} = \tilde{g}_{UV}\tilde{g}_{VW}\tilde{g}_{WU} : U \cap V \cap W \to \mathbb{C}^{\times}$. The $\{h_{UVW}\}$

define a continuous Čech 2-cocycle, $h \in \check{C}^2_{\text{cts}}(M; S^1)$ (using the isomorphism $\check{C}_{\text{cts}}(M; \mathbb{C}^\times) \cong \check{C}_{\text{cts}}(M; S^1)$). We can choose $\{\tilde{g}_{UV}\}$ satisfying the cocycle condition if and only if h is a coboundary. Thus we have an obstruction in $\check{H}^2_{\text{cts}}(M; S^1)$ to the existence of the principal bundle $\tilde{P}$. Under the standard isomorphism $\check{H}^2_{\text{cts}}(M; S^1) \cong H^3(M; \mathbb{Z})$ the obstruction is an element $b_2 \in H^3(M; \mathbb{Z})$.

In constructing the semi-infinite cohomology of a semi-infinite manifold, it is not sufficient to have such a lift. The lifted transition maps must have a particular property. In essence this property is the requirement that the lift does not depend upon the point in the manifold. I do not know whether it is possible to choose a lift with the required property in all cases but there are three types of semi-infinite manifold where this is possible. In each of these three cases the structure group of the manifold is a subgroup of $\mathrm{Gl}_{\text{res}}(X)$ for which the central extension induced by $\mathrm{Gl}^{\sim}_{\text{res}}(X)$ is trivial.

When considering finite codimension cohomology we do not encounter the same problems since the central extension $\mathrm{Gl}^{\sim}(X)$ of $\mathrm{Gl}(X)$ is trivial, the lift being given by $A \to [A, A]$.

Definition 4.1.0.6. *The three types of semi-infinite manifold are as follows:*

1. *A polarised manifold M is* globally polarised *if there is a global decomposition $T_{\mathbb{C}}M = T_- \oplus T_+$ which agrees with the polarisation.*

2. *A polarised manifold M is* nuclear *if its structure group is $\mathfrak{D}_\times(X)$, the group of invertible operators with a determinant.*

3. *A polarisation of an almost complex manifold M is compatible with the almost complex structure if the tangent space of M when considered as*

a complex vector bundle is itself a polarised bundle and the polarisation of M is that arising from the complexification of the polarisation of TM.

It should be noted that these cases are not mutually exclusive. It is possible to have a polarised manifold which is of all three types - the trivial example being $X \otimes_{\mathbb{R}} \mathbb{C}$ where X is a complex polarised $\mathcal{F}$-space. In each case the group which acts on $A_{\text{si}}(X)$ is a trivial central extension of the structure group and thus by choosing a cross-section $A_{\text{si}}(X)$ can be made into a representation of the structure group.

The structure group of a globally polarised manifold is $\mathrm{Gl}(X_-) \times \mathrm{Gl}(X_+)$ and the lift is given by $A = (A_-, A_+) \to [A, A_+]$. The main example of this case is when the manifold M is the total space of a fibration $\pi : M \to N$ where both N and the fibre are modelled on infinite dimensional $\mathcal{F}$-spaces. The bundle TM decomposes according to the directions in the base space and in the fibre so declaring one to be the positive space and the other negative defines a polarisation.

In the case of a nuclear polarised manifold the lift is given by $A \to [A, I_+]$ where I_+ is the identity on the positive space X_+. It is a standard theorem of the theory of Hilbert manifolds that all manifolds modelled on Hilbert spaces are nuclear (this is often called a Fredholm structure in the literature). We shall return to this case in chapter 5.

The final case is the most interesting to us because all the known examples of polarised manifolds are of this type. If M is an almost complex manifold modelled on X such that the tangent bundle TM with its complex structure carries a polarisation then $T_{\mathbb{C}}M$ carries a polarisation defined by definition 3.2.1.5. The group which acts on $T_{\mathbb{C}}M$ is $\mathrm{Gl}_{\mathrm{res}}(X)$ acting through the homomorphism $\mathrm{Gl}_{\mathrm{res}}(X) \to \mathrm{Gl}_{\mathrm{res}}(X \otimes_{\mathbb{R}} \mathbb{C})$. Thus the central extension required is the trivial extension $\mathrm{Gl}_{\mathrm{res}}(X) \times \mathbb{R}^+$ and thus such a polarised manifold is always of semi-infinite type. Also because the map $\mathrm{Gl}_{\mathrm{res}}(X) \to \mathrm{Gl}_{\mathrm{res}}(X \otimes_{\mathbb{R}} \mathbb{C})$ doubles the degree of an element, the period of $T_{\mathbb{C}}M$ is twice that of TM. Thus $b_2(T_{\mathbb{C}}M) = 0$ and $b_1(T_{\mathbb{C}}M) = 2b_1(TM)$.

4.1.1 Loop Spaces

A particularly interesting example of the last case is the following. Let M be a finite dimensional almost complex manifold and let ΩM be the space of smooth based loops in M. There is an evaluation map $e : S^1 \times \Omega M \to M$ given by $e(t,\gamma) \to \gamma(t)$. This induces a map in cohomology $e^* : H^k(M;\mathbb{Z}) \to H^k(S^1 \times \Omega M;\mathbb{Z})$. Evaluation on the volume form of S^1 defines a map $\omega : H^k(S^1 \times \Omega M;\mathbb{Z}) \to H^{k-1}(\Omega M;\mathbb{Z})$. The composition $\tau : H^k(M;\mathbb{Z}) \to H^{k-1}(\Omega M;\mathbb{Z})$ is called the *transgression map.*

Given a complex n dimensional vector bundle $E \to M$, there is a vector bundle $\mathcal{E} \to \Omega M$ with fibre $\mathcal{E}_\gamma = \Gamma(S^1, \gamma^* E)$. This bundle is polarised with polarisation defined by the twisted Dirac operator $j\frac{\partial}{\partial\theta}$ on the circle.

In terms of classification spaces, E is represented by an element of the space homotopy class of maps $M \to B\,\mathrm{Gl}_n(\mathbb{C})$, $[M, B\,\mathrm{Gl}_n(\mathbb{C})]$. This defines an element of $[\Omega M, \Omega B\,\mathrm{Gl}_n(\mathbb{C})]$. As $\Omega B\,\mathrm{Gl}_n(\mathbb{C}) = B\Omega\,\mathrm{Gl}_n(\mathbb{C})$ and $\Omega\,\mathrm{Gl}_n(\mathbb{C})$ is a subgroup of $\mathrm{Gl}_{\mathrm{res}}(C^\infty(S^1, \mathbb{C}^n))$ the element of $[M, B\,\mathrm{Gl}_n(\mathbb{C})]$ defines a complex polarised vector bundle over ΩM. This vector bundle is the bundle $\mathcal{E}$ defined above.

This can also be done using K-theory as follows. The bundle E represents an element η in $K^0(M)$. The evaluation map $e : S^1 \times \Omega M \to M$ induces a map in K-theory and so $e^*\eta \in K^0(M)$. We have a slant product $K^0(S^1 \times \Omega M) \to K^0(S\Omega M)$ where $S\Omega M$ is the first suspension of ΩM. By definition, $K^0(S\Omega M) = K^{-1}(\Omega M)$ so we have a map $\omega : K^0(S^1 \times \Omega M) \to K^{-1}(\Omega M)$. The classification space for $K^{-1}(\Omega M)$ is the space of skew-adjoint Fredholm operators and so an element of this space, together with the complex structure, defines a polarising operator. Thus $K^{-1}(\Omega M)$ represents polarised bundles over ΩM. Using the natural maps from K-theory to cohomology, we have:

$$\begin{array}{ccccc} K^0(M) & \xrightarrow{e^*} & K^0(S^1 \times \Omega M) & \xrightarrow{\omega} & K^{-1}(\Omega M) \\ \downarrow & & \downarrow & & \downarrow \\ H^k(M) & \xrightarrow{e^*} & H^k(S^1 \times \Omega M) & \xrightarrow{\omega} & H^{k-1}(\Omega M) \end{array}$$

and thus the bundle $\mathcal{E}$ has characteristic classes $b_j(\mathcal{E}) \in H^{2j-1}(\Omega M; \mathbb{Z})$ given by $b_j(\mathcal{E}) = \tau c_j(E)$. The classes $b_1(\mathcal{E})$ and $b_2(\mathcal{E})$ are as defined in section 4.1. In particular, $b_1(\mathcal{E}) = \tau c_1(E)$ and $b_2(\mathcal{E}) = \tau c_2(E)$.

If M is an almost complex manifold then applying this construction to TM gives the complex polarised bundle $T\Omega M$. As ΩM is an almost complex manifold such that $T\Omega M$ is polarised, $b_2(T_{\mathbb{C}}\Omega M) = 0$ and $b_1(T_{\mathbb{C}}\Omega M) = 2b_1(T\Omega M) = 2\tau c_1(TM)$. Thus for a simply connected almost complex finite dimensional manifold M, the based loop space ΩM is a semi-infinite manifold of periodicity $2c_1(M)$.

4.2 Semi-Infinite Forms

In this section we extend the Grassmannian construction of de Rham cohomology to infinite dimensions. Let M be a semi-infinite manifold modelled on an $\mathcal{F}$-space X and assume that M is one of the three types defined in definition 4.1.0.6. Let G be the structure group of M and let P be the principal bundle of M. Because M is of one of the types defined in definition 4.1.0.6 $A_{\mathrm{si}}(X_{\mathbb{C}})$ is a representation of G. Thus we can define the bundles $A^k_{\mathrm{si}}(T_{\mathbb{C}}M) := P \times_G A^k_{\mathrm{si}}(X_{\mathbb{C}})$.

We first define the space of semi-infinite forms and exterior derivative over an open subset U of X. Then we show how these transform under diffeomorphisms and thus how to define semi-infinite forms and the exterior derivative for M.

4.2.1 Locally Tame Maps

Let $U \subseteq X$ be an open set. There is a natural isomorphism $A^k_{\mathrm{si}}(T_{\mathbb{C}}U) \cong U \times A^k_{\mathrm{si}}(X_{\mathbb{C}})$. Thus a smooth section s of $A^k_{\mathrm{si}}(T_{\mathbb{C}}U)$ is a smooth map $U \to A^k_{\mathrm{si}}(X_{\mathbb{C}})$ which we can differentiate to get a smooth map $Ds : T_{\mathbb{C}}U \to A^k_{\mathrm{si}}(X_{\mathbb{C}})$. Since $T_{\mathbb{C}}U$ can be naturally identified with $U \times X_{\mathbb{C}}$, the adjoint of Ds gives us a map $Ds : U \to \mathfrak{L}(X_{\mathbb{C}}, A^k_{\mathrm{si}}(X_{\mathbb{C}}))$. We wish to compose this with the contraction map $\wedge$ of section 3.3.4. In order to do this, we need to restrict to smooth maps $s : U \to A_{\mathrm{si}}(X_{\mathbb{C}})$ for which Ds is a map $U \to X^*_{\mathbb{C}} \otimes A_{\mathrm{si}}(X_{\mathbb{C}})$.

This can be considered as a generalisation of the concept of a tame map. A map $f : Y \to Z$ between LCTV-spaces Y and Z is said to be *tame* if there is a finite rank projection $P : Y \to Y$ such that $f(x) = f(Px)$. Given a differentiable tame map f, $Df : Y \to \mathfrak{L}(Y, Z)$ factors through $\mathfrak{L}(PY, Z)$ which is isomorphic to $(PY)^* \otimes Z$ as PY is finite dimensional. We shall come back to this idea in section 5.4.

The space of smooth maps $s : U \to A_{si}(X_{\mathbb{C}})$ for which Ds is a map $U \to X_{\mathbb{C}}^* \otimes A_{si}(X_{\mathbb{C}})$ is not invariant under the diffeomorphisms of U and thus does not extend to manifolds modelled on X. To make this extension possible we define the concept of locally tame maps. A smooth map $s : U \to A_{si}(X_{\mathbb{C}})$ is *locally tame* at a point $p \in U$ if there is an open set W of U containing p, an open set V of X and a diffeomorphism $\phi : V \to W$ such that the induced map $\phi^* s : V \to A_{si}(X_{\mathbb{C}})$ has the property that Ds is a map $V \to X_{\mathbb{C}}^* \otimes A_{si}(X_{\mathbb{C}})$. The triple (W, V, ϕ) is called a *locally tame chart* for s at p; s is locally tame on U if it is a locally finite linear combination of maps which are locally tame at each point in U, i.e. $s = \sum_{\alpha \in A} s_\alpha$ where each s_α is locally tame and each point $p \in U$ has a neighbourhood W on which s_α is non-zero for only a finite subset of A. This definition is invariant under diffeomorphism and thus extends over a manifold modelled on X. Note that this definition makes sense for vector bundles other than $A_{si}(X_{\mathbb{C}})$.

Let $s : U \to A_{si}(X_{\mathbb{C}})$ be a locally tame smooth map and let $g : U \to \mathbb{C}$ be a smooth function. gs is a smooth map $U \to A_{si}(X_{\mathbb{C}})$ and $Dgs = Dg \otimes s + gDs$ so gs is locally tame. Thus a locally tame smooth map can be obtained by patching together locally tame maps using a partition of unity.

The differential ds of a locally tame smooth map s at a point p is defined by $\wedge Ds$ in a locally tame chart for s at p. To show that this is well-defined we need to check that if (V_1, W_1, ϕ_1) and (V_2, W_2, ϕ_2) are locally tame charts for s at p then we get the same answer for ds.

The other potential problem with the differential map d is that if s is locally tame then it is not necessarily the case that ds is locally tame. There are two ways around this. The first is to restrict to maps s such that both s and ds are locally tame. We can show that $d^2s = 0$ and thus if $t = ds$ then t and dt are locally tame so d preserves this space of sections. The second way is to take the linear span of the space of maps which are either locally tame or are in the image of the locally tame maps under d. We formally define d to be zero on the image of d. We just need to check that if s is locally tame and $s = dt$ then $ds = 0$ anyway. In this section we use the latter method, though we shall use the former in chapter 5. The cohomology groups obtained are the same under either choice. The cohomology groups from the second method clearly contain those from the first. A non-zero element in the cohomology group obtained by the second method can be represented by a locally tame element s. The element s is locally tame and is such that $ds = 0$ so s and ds are locally tame. Similarly, if s is locally tame and is such that $s = dt$ for some locally tame t then t and dt are locally tame.

4.2.2 The Extended Contraction Map

The way we actually implement the idea of locally tame maps is to extend the domain of definition of the contraction map to a space $\mathfrak{K}(X_{\mathbb{C}}, A_{\mathrm{si}}(X_{\mathbb{C}}))$. This space has the required property that smooth maps $s : U \to A_{\mathrm{si}}(X_{\mathbb{C}})$ with Ds a map from U to $\mathfrak{K}(X_{\mathbb{C}}, A_{\mathrm{si}}(X_{\mathbb{C}}))$ are invariant under diffeomorphism.

Now let X be a complex $\mathcal{F}$-space; $\wedge$ is a continuous map from the space $\mathfrak{J}_l(X, A^l_{\mathrm{si}}(X))$ to $A^{l+1}_{\mathrm{si}}(X)$. As X is an $\mathcal{F}$-space and so has the approximation property, $\mathfrak{J}_1(X, A^l_{\mathrm{si}}(X))$ can be regarded as an ideal in $\mathfrak{L}(X, A^l_{\mathrm{si}}(X))$. We wish to extend $\wedge$ to a yet larger subspace of $\mathfrak{L}(X, A^l(X))$ which we do using the following method. Let Y be a subspace of $\mathfrak{L}(X, A^l(X))$. If we can give a recipe which for each $F \in Y$ and $[w, \lambda] \in D$ gives a manipulation of equation 3.1 which sums to zero, then we can extend the map $\wedge$ over Y by defining it to be zero on Y. For $F \in Y \cap \mathfrak{J}_1(X, A^l(X))$ then $\wedge F$ converges and so the manipulation does not change the limit, whence $\wedge F = 0$. Thus the extension is well-defined over $\mathfrak{J}_1(X, A^l_{\mathrm{si}}(X)) + Y$. There are two spaces which over which we shall extend $\wedge$. They are introduced to ensure two essential properties of the cochain differential. The first is to ensure that $d^2 = 0$ and the second to ensure that d is invariant under changes of bases.

The space $\mathfrak{L}(X, \mathfrak{J}_1(X, A^l(X)))$ is a subspace of $\mathfrak{L}(X \times X, A^l(X))$. The map $\wedge$ defines a map $\mathfrak{L}(X, \mathfrak{J}_1(X, A^l(X))) \to \mathfrak{L}(X, A^{l+1}(X))$. The symmetric subspace of a subspace of $\mathfrak{L}(X, \times X, A^l(X)))$ consists of those F for which $F(\alpha)(\beta) = F(\beta)(\alpha)$. Let F be in the image of the symmetric subspace of $\mathfrak{L}(X, \mathfrak{J}_1(X, A^l(X)))$ in $\mathfrak{L}(X, A^{l+1}(X))$. Equation 3.1 becomes:

$$\begin{aligned}
\wedge(\wedge F)([w,\lambda]) &= \sum_k (-1)^{k-1}(\wedge F)(w_k)([w \smallsetminus w_k, \lambda]) \\
&= \sum_k (-1)^{k-1}\Big(\sum_{j<k} (-1)^{j-1} F(w_k)(w_j)([w \smallsetminus \{w_k, w_j\}, \lambda]) \\
&\qquad + \sum_{j>k} (-1)^{j} F(w_k)(w_j)([w \smallsetminus \{w_k, w_j\}, \lambda]) \Big) \\
&= \sum_{j,k:j<k} \Big((-1)^{k+j} F(w_k)(w_j)([w \smallsetminus \{w_k, w_j\}, \lambda]) \\
&\qquad + (-1)^{k+j-1} F(w_k)(w_j)([w \smallsetminus \{w_k, w_j\}, \lambda]) \Big) = 0
\end{aligned}$$

The second space we wish to consider is more complicated to describe. Let (x_j) be a basis for X. Let $B : X \to W$ be a continuous linear map onto an element of $\mathrm{Gr_{res}}(X)$. Let $(b_{k,l})$ be a subset of X such that $b_{k,l} = b_{l,k}$. Define $B_{k,l}$ to be the continuous linear operator which agrees with B on $\langle x_j : j \neq k \rangle$ and maps x_k to $b_{l,k}$. Let $B_{R,k,l}$ be the restriction of $B_{k,l}$ to the space $\langle x_R \rangle$.

Let $F \in \mathfrak{L}(X, A^l(X))$ be such that there is such a set $(B_{R,k,l})$ as defined above and an operator q such that:

$$F(x_l)([x_R, 1]) = \sum_{k \in R} \det B_{R,k,l} q$$

Let $T = \{t_j : j \in \mathbb{N}\}$. The formal sum for $\wedge F$ on $[x_T, 1]$ is:

$$\begin{aligned}(\wedge F)([x_T, 1]) &= \sum_l (-1)^{l-1} \sum_{k \neq l} \det B_{T \setminus t_l, t_k, t_l} q \\ &= \sum_{l,k : l \neq k} (-1)^{l-1} \det B_{T \setminus t_l, t_k, t_l} q\end{aligned}$$

By construction $B_{T \setminus t_l, t_k, t_l}$ is $B_{T \setminus t_k, t_l, t_k}$ but with the column corresponding to t_l in the column corresponding to t_k. Thus there is a permutation of sign $(-1)^{l-k-1}$ which takes one to the other. Thus:

$$\begin{aligned}(\wedge F)([x_T, 1]) &= \sum_{l<k} (-1)^{l-1} \det B_{T \setminus t_l, t_k, t_l} q + (-1)^{k-1+l-k-1} \det B_{T \setminus t_l, t_k, t_l} q \\ &= \sum_{l<k} (-1)^{l-1} \left(\det B_{T \setminus t_l, t_k, t_l} q - \det B_{T \setminus t_l, t_k, t_l} q\right) \\ &= \sum_{l<k} 0\end{aligned}$$

Thus we can extend $\wedge$ over the linear span of such operators F by defining it to be zero. Let $\mathfrak{K}(X, A^l_{\mathrm{si}}(X))$ be the total space over which $\wedge$ is now defined. Note that the image of $\wedge$ remains unchanged by these extensions and thus the image of the symmetric subspace of $\mathfrak{L}(X, \mathfrak{K}(X, A^l_{\mathrm{si}}(X)))$ in $\mathfrak{L}(X, A^{l+1}_{\mathrm{si}}(X))$ coincides with the image of the symmetric subspace of $\mathfrak{L}(X, \mathfrak{J}_1(X, A^l_{\mathrm{si}}(X)))$. Since the extension of $\wedge$ over $\mathfrak{K}(X, A^l_{\mathrm{si}}(X))$ is by zero, the properties described in theorem 3.3.4.3 still hold.

4.2.3 The Semi-Infinite Cochain Complex

We can now define the space of semi-infinite forms and the differential d. Let X be a real $\mathcal{F}$-space and let $U \subseteq X$ be an open set. We use the identification of smooth sections of $A_{\mathrm{si}}(T_{\mathbb{C}}U)$ with smooth maps $U \to A_{\mathrm{si}}(X_{\mathbb{C}})$.

Definition 4.2.3.1. *The space of semi-infinite forms on U is:*

$$\mathfrak{A}^k_{si}(U) = \{s \in C^\infty(U, A^k_{si}(X_{\mathbb{C}})) : Ds \in C^\infty(U, \mathfrak{K}(X_{\mathbb{C}}, A^k_{si}(X_{\mathbb{C}}))) \\ \text{and } ds \in C^\infty(U, A^{k+1}_{si}(X))\}$$

The differential is the composition of D with $\wedge$:

$$d = \wedge D : \mathfrak{A}^k_{si}(U) \to C^\infty(U, A^{k+1}_{si}(X_{\mathbb{C}}))$$

In the light of the remarks in section 4.2.1, we note that this space contains the locally tame sections of $A^k_{\mathrm{si}}(X_{\mathbb{C}})$ but may actually be a larger space. We now prove the essential properties of semi-infinite forms and of d.

Lemma 4.2.3.2. *Let $s \in \mathfrak{A}^k_{si}(U)$ and $g \in C^\infty(U, \mathbb{C})$ then $gs \in \mathfrak{A}^k_{si}(U)$ and $d(gs) = dg \wedge s + gds$.*

Proof. Both parts of this lemma follow from the fact that $Dgs = dg \otimes s + gDs$ which lies in $C^\infty(U, \mathfrak{K}(X_\mathbb{C}, A^k_{\mathrm{si}}(X_\mathbb{C})))$. □

Lemma 4.2.3.3. *Let $s \in \mathfrak{A}^k_{si}(U)$ then $ds \in \mathfrak{A}^k_{si}(U)$ and $d^2 s = 0$.*

Proof. The map $\wedge$ commutes with D and thus the following diagram is commutative, where the horizontal maps are D and the vertical maps $\wedge$:

$$\begin{tikzcd}
\mathfrak{A}^k_{\mathrm{si}}(U) \arrow[r, "D"] & C^\infty(U, \mathfrak{K}(X, A^k_{\mathrm{si}}(X))) \arrow[r, "D"] \arrow[d, "\wedge"'] & C^\infty(U, \mathfrak{L}(X, \mathfrak{K}(X, A^k_{\mathrm{si}}(X)))) \arrow[d, "\wedge"'] \\
& C^\infty(U, A^{k+1}_{\mathrm{si}}(X)) \arrow[r, "D"] & C^\infty(U, \mathfrak{L}(X, A^{k+1}_{\mathrm{si}}(X)))
\end{tikzcd}$$

and hence $Dd = \wedge D^2$. If $s \in \mathfrak{A}^k_{\mathrm{si}}(U)$ then $D^2 s$ is a section of the symmetric subspace of $\mathfrak{L}(X, \mathfrak{K}(X, A^k_{\mathrm{si}}(X)))$ and therefore $Dds = \wedge D^2 s \in \mathfrak{K}(X, A^{k+1}_{\mathrm{si}}(X))$ and so $ds \in \mathfrak{A}^k_{\mathrm{si}}(U)$. Moreover, $\wedge D^2 s$ lies in the kernel of $\wedge$ so $d^2 s = 0$. □

Lemma 4.2.3.4. *Let $\phi : U \to V$ be a diffeomorphism between open subsets U and V of X. The induced isomorphism $\phi^* : \mathfrak{A}_{si}(U) \to \mathfrak{A}_{si}(V)$ is such that $\phi^* d = d\phi^*$.*

Proof. To prove this we examine $\phi^* D - D\phi^*$. This is invariant under the action of $C^\infty(U, \mathbb{C})$ and so it is sufficient to consider the pointwise situation. Thus we consider this as a continuous linear map $A^k_{\mathrm{si}}(X_\mathbb{C}) \to \mathfrak{L}(X_\mathbb{C}, A^k_{\mathrm{si}}(X_\mathbb{C}))$. As it is continuous, it is sufficient to consider the action on a basis element. Since we are dealing with a pointwise situation, the operator we are considering is $D\phi^*$.

We have bases $(\frac{\partial}{\partial x^i})$ for TU and $(\frac{\partial}{\partial \tilde{x}^i})$ for TV. Corresponding to these are bases (f^S) for $\mathfrak{A}^k_{\mathrm{si}}(U)$ and $(\tilde{f}^S)$ for $\mathfrak{A}^k_{\mathrm{si}}(V)$. The matrix of $\phi^* : TU \to TV$ relative to these basis is given by $b^j_i = \frac{\partial x^i}{\partial \tilde{x}^j}$.

Let $[B, q(B)]$ be the action of ϕ^* on $A^k_{\mathrm{si}}(X_{\mathbb{C}})$ at a point $p \in U$. The change of basis formula is given by $\tilde{f}^S\left(\left[\frac{\partial}{\partial x^T}, 1\right]\right) = \det(B^{-1})^S_T q(B)$. We need to show that $D\tilde{f}^S$ lies in $\mathfrak{K}(X_{\mathbb{C}}, A^k_{\mathrm{si}}(X_{\mathbb{C}}))$ and moreover that it lies in $\ker \wedge$.

For a matrix of functions E with a determinant, the derivative of the determinant in the direction x is given by $\sum_k \det E_{k,x}$ where $E_{k,x}$ is the matrix obtained from E by replacing the kth column by its column of derivatives in the x direction. Applying this to the above change of variables formula gives the expression:

$$D\tilde{f}^S\left(\frac{\partial}{\partial x^l}\right)\left(\left[\frac{\partial}{\partial x^T}, 1\right]\right) = \sum_{k \in T} \det(B^{-1})^S_{T,k,x^l} q(B) + \sum_{k \in \mathbb{N}} \det(B^{-1})^S_T q(B)_{k,x^l}$$

However, because q depends solely on B and the matrix of B is of the form $\frac{\partial x^i}{\partial \tilde{x}^j}$, $q(B)_{k,x^l}$ is singular for each k, l. Thus we have:

$$D\tilde{f}^S\left(\frac{\partial}{\partial x^l}\right)\left(\left[\frac{\partial}{\partial x^T}, 1\right]\right) = \sum_{k \in T} \det(B^{-1})^S_{T,k,x^l} q(B)$$

For $l \notin T$ the column of $(B^{-1})^S_{T,k,x^l}$ corresponding to k is identical to the column of $(B^{-1})^S_{R,l,x^k}$ corresponding to l, where $R = T \cup l \setminus \{k\}$. This is precisely the situation dealt with at the end of section 3.3.4 and so $\phi^* D - D\phi^*(A^k_{\mathrm{si}}(X_{\mathbb{C}})) \subseteq \mathfrak{K}(X_{\mathbb{C}}, A^k_{\mathrm{si}}(X_{\mathbb{C}}))$. Thus $\mathfrak{A}_{\mathrm{si}}(U)$ is invariant under diffeomorphism. Moreover, because $\phi^* D - D\phi^* \in \ker \wedge$, we have $\phi^* d - d\phi^* = 0$. □

The requirement that ϕ be a diffeomorphism can be relaxed to ϕ being a smooth map with $d\phi : TU \to TV$ preserving the semi-infinite structure, i.e. $d\phi : X_+ \to X_+$ is Fredholm and $d\phi : X_+ \to X_-$ lies in $\mathfrak{J}_2(X_+, X_-)$.

4.3 Semi-Infinite de Rham Cohomology

We can now define the groups $\mathfrak{A}^k_{\mathrm{si}}(M)$ and the differential operator $d : \mathfrak{A}^k_{\mathrm{si}}(M) \to \mathfrak{A}^{k+1}_{\mathrm{si}}(M)$. We define $H^k_{\mathrm{si}}(M)$ to be the cohomology group obtained from this complex. This is a complex vector space. Given a smooth map $f : M \to N$ with $df : T_{\mathbb{C}}M \to T_{\mathbb{C}}N$ a map which preserves the semi-infinite structure we have an induced map $f^* : \mathfrak{A}^k_{\mathrm{si}}(N) \to \mathfrak{A}^k_{\mathrm{si}}(M)$ such that $df^* = f^*d$.

For a submanifold $N \subseteq M$ such that the inclusion $i : N \to M$ preserves the semi-infinite structure we define the relative groups $\mathfrak{A}^k_{\mathrm{si}}(M,N) = \mathfrak{A}^k_{\mathrm{si}}(M) \times \mathfrak{A}^{k-1}_{\mathrm{si}}(N)$. The differential is defined by $d(a,b) = (da, i^*a - db)$. This gives a short exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathfrak{A}^{k-1}_{\mathrm{si}}(N) & \longrightarrow & \mathfrak{A}^k_{\mathrm{si}}(M,N) & \longrightarrow & \mathfrak{A}^k_{\mathrm{si}}(M) & \longrightarrow 0 \\ 0 \longrightarrow 0, & b & \longrightarrow & (0,-b), \quad (a,b) & \longrightarrow & a, & 0 \longrightarrow 0 \end{array}$$

these maps are all chain maps. Thus we have a long exact sequence:

$$\cdots \to H^{k-1}_{\mathrm{si}}(N) \to H^k_{\mathrm{si}}(M,N) \to H^k_{\mathrm{si}}(M) \to \cdots$$

This sequence is clearly functorial in (M,N) and the boundary map can be easily seen to be i^*.

Lemma 4.3.0.5. *Let (M,N,U) be a triple of semi-infinite manifolds with $U \subseteq N \subseteq M$ and all the inclusions preserving the semi-infinite structure. Suppose that the closure of U is contained within the interior of N, then:*

$$H_{si}(M,N) \cong H_{si}(M \setminus U, N \setminus U)$$

Proof. As M is modelled on a space with smooth functions, there is a smooth function $\tau : M \to [0,1]$ such that $\tau(M \setminus N) = 0$ and $\tau(U) = 1$. The inclusion $j : (M \setminus U, N \setminus U) \to (M, N)$ gives a cochain map $j^* : \mathfrak{A}^k_{\mathrm{si}}(M, N) \to \mathfrak{A}^k_{\mathrm{si}}(M \setminus U, N \setminus U)$. We define a reverse by:

$$\rho(a, b) = ((1 - \tau)a - d\tau \wedge b, (1 - \tau)b)$$

We define the map $h : \mathfrak{A}^k_{\mathrm{si}}(M, N) \to \mathfrak{A}^{k-1}_{\mathrm{si}}(M, N)$ by $h(a, b) = (\tau b, 0)$. Then:

$$\begin{aligned} dh(a,b) &= d(\tau b, 0) = (d\tau \wedge b + \tau db, \tau b) \\ hd(a,b) &= h(da, i^* a - db) = (\tau a - \tau db, 0) \\ \text{so:} \quad (dh + hd)(a,b) &= (\tau a + d\tau \wedge b, \tau b) \\ &= (a,b) - ((1 - \tau)a + d\tau \wedge b, (1 - \tau)b) \\ &= (1 - \rho j^*)(a,b) \end{aligned}$$

Thus $\rho j^* \simeq \mathrm{id}$.

We define $h : \mathfrak{A}^k_{\mathrm{si}}(M \setminus U, A \setminus U) \to \mathfrak{A}^{k-1}_{\mathrm{si}}(M \setminus U, A \setminus U)$ in the same way: $h(a, b) = (\tau b, 0)$. This satisfies $dh + hd = 1 - j^* \rho$ and so $j^* \rho \simeq \mathrm{id}$. Thus j^* is an isomorphism in cohomology. Hence we have excision. □

Lemma 4.3.0.6. *The semi-infinite cohomology of $M \times \mathbb{R}$ is isomorphic to that of M.*

Proof. We have a projection $\pi : M \times \mathbb{R} \to M$. Given a map $s : M \to \mathbb{R}$, we have a map $s : M \to M \times \mathbb{R}$ by $s(q) = (q, s(q))$. There are maps

$s^* : H_{si}(M \times \mathbb{R}) \to H_{si}(M)$ and $\pi^* : H_{si}(M) \to H_{si}(M \times \mathbb{R})$. Since $\pi s = \mathrm{id}_M$, the map $s^*\pi^*$ on $\mathfrak{A}^k_{si}(M)$ is the identity map.

The complexified tangent space of $M \times \mathbb{R}$ is isomorphic to $\pi^* T_{\mathbb{C}} M \oplus \mathbb{C}$. By corollary 3.3.2.6, at a point in $M \times \mathbb{R}$ we have $A^k_{si}(\pi^* T_{\mathbb{C}} M \oplus \mathbb{C}) \cong A^k_{si}(\pi^* T_{\mathbb{C}} M) \oplus dt \wedge A^{k-1}_{si}(\pi^* T_{\mathbb{C}} M)$. Thus an element in $\mathfrak{A}^k_{si}(M \times \mathbb{R})$ can be decomposed as $a = b + dt \wedge c$. With respect to the above decomposition, $\pi^* s^* a = b(q, s(q))$.

Define $K : \mathfrak{A}^k_{si}(M \times \mathbb{R}) \to \mathfrak{A}^{k-1}_{si}(M \times \mathbb{R})$ by:

$$Ka = (-1)^{|c|} \int_{s(q)}^{t} c(q, r) dr$$

This is well-defined since $A_{si}(\pi^* T_{\mathbb{C}} M \oplus \mathbb{C})$ is a complete locally convex space. Then:

$$\begin{aligned}
(dK - Kd)b(q,t) &= -Kdb(q,t) \\
&= -K\left(dt \wedge \frac{\partial b}{\partial t}(q,t) + d_M b(q,t)\right) \\
&= -(-1)^{|b|} \int_{s(q)}^{t} \frac{\partial b}{\partial t}(q,t) dr \\
&= -(-1)^{|b|}(b(q,t) - b(q, s(q))) \\
&= (-1)^{|b|-1}(1 - \pi^* s^*) b(q,t)
\end{aligned}$$

$$\begin{aligned}
(dK - Kd)dt \wedge c(q,t) &= (-1)^{|c|} d \int_{s(q)}^{t} c(q,r)dr - K(dt \wedge d_M c(q,t)) \\
&= (-1)^{|c|} dt \wedge c(q,t) + (-1)^{|c|} \int_{s(q)}^{t} d_M c(q,r)dr \\
&\quad + (-1)^{|c|+1} \int_{s(q)}^{t} d_M c(q,r)dr \\
&= (-1)^{|c|} dt \wedge c(q,t) \\
&= (-1)^{|c|} (1 - \pi^* s^*) dt \wedge c(q,t)
\end{aligned}$$

Since $|c| = |a| - 1 = |b| - 1$ we have $dK - Kd = (-1)^{|a|-1}(1 - \pi^* s^*)$ and so $\pi^* s^*$ is chain homotopic to the identity map. Hence $H^k_{\mathrm{si}}(M \times \mathbb{R}) \cong H^k_{\mathrm{si}}(M)$. $\square$

Lemma 4.3.0.7. *Semi-infinite cohomology is invariant under homotopies of semi-infinite manifolds.*

Proof. Suppose that $f, g : M \to N$ are homotopic smooth maps through a homotopy which preserves the semi-infinite structure. There is some smooth map $F : M \times \mathbf{R} \to N$ such that $F(q,0) = f(q)$ and $F(q,1) = g(q)$. Let $s_0 : M \to \mathbf{R}$ be the map $s_0(q) = 0$ and s_1 defined similarly. Then $F s_0 = f$ and $F s_1 = g$. Hence $f^* = s_0^* F^*$ and $g^* = s_1^* F^*$ on cohomology. By the above, $s_0^*, s_1^* : H^*(M \times \mathbf{R}) \to H^*(M)$ are both inverses to the map $\pi^* : H^*(M) \to H^*(M \times \mathbf{R})$ hence are the same map. Thus $f^* = g^*$. $\square$

Thus H_{si} satisfies the generalised Eilenberg-Steenrod axioms of cohomology. The same analysis results in the cohomology theories H_{fc} and H_{fd} except that we have the additional result that H_{fd} satisfies the dimension axiom.

This can either be proved directly or by using the fact that in finite dimensions the complex which defines H_{fd} coincides with the standard de Rham complex.

4.4 The Cohomology of a Hilbert Manifold

As it currently stands, it is only possible to calculate the finite codimension cohomology for a certain type of manifold called a Hilbert manifold:

Definition 4.4.0.8. *A* Hilbert manifold *is a smooth manifold modelled on a complete Hilbert space.*

The transition functions on a Hilbert manifold have bounded derivatives.

The calculation of the cohomology of a Hilbert manifold only goes to show that a further refinement is needed in order to have a useful cohomology theory. This calculation uses a key theorem of Hilbert manifolds from the theory of Fredholm structures (see Eells and Elworthy [3] for a brief introduction). The theorem we wish to use is:

Theorem (Burghelea and Kuiper [2]). *Two Hilbert manifolds are diffeomorphic if and only if they have the same homotopy type.*

Since M is homotopic to $M \times \mathbb{R}$, this implies that M is diffeomorphic to $M \times \mathbb{R}$. We are thus able to prove:

Theorem 4.4.0.9. *The finite codimensional cohomology of a Hilbert manifold is trivial.*

Proof. The map onto the first factor $M \to M \times \mathbb{R}$ is a homotopy equivalence. As this map is Fredholm of index -1, it induces an isomorphism $H^l_{\mathrm{fc}}(M \times \mathbb{R}) \to H^{l+1}_{\mathrm{fc}}(M)$. Using the diffeomorphism $M \to M \times \mathbb{R}$, $H^l_{\mathrm{fc}}(M \times \mathbb{R}) \cong H^l_{\mathrm{fc}}(M)$. Thus $H^l_{\mathrm{fc}}(M) \cong H^{l+1}_{\mathrm{fc}}(M)$. However, $H^{-1}_{\mathrm{fc}}(M) = 0$ and so by induction $H^l_{\mathrm{fc}}(M) = 0$ for all l. □

If the diffeomorphism $M \to M \times \mathbb{R}$ also preserves the semi-infinite structure this method can be used to prove that $H^l_{\mathrm{si}}(M) = H^k_{\mathrm{si}}(M)$ for all $l, k \in \mathbb{Z}$. Thus whenever $f : N \to M$ is an admissible map from an infinite dimensional manifold to a semi-infinite manifold the map $H_{\mathrm{si}}(M) \to H_{\mathrm{fc}}(N)$ is defined and is zero. Thus the semi-infinite cohomology is zero on all submanifolds compatible with the polarisation. This is strong evidence for the conjecture that $H_{\mathrm{si}}(M) = 0$.

Chapter 5

Extending Cohomology

5.1 Calculating Cohomology

Theorem 4.4.0.9 shows that the current definition of de Rham cohomology needs some alteration in order to be useful. The alteration required is to a theory which contains a Thom isomorphism. This is an important tool in calculating cohomology which in finite dimensions can be expressed as follows: let $U \to M$ be an n dimensional vector bundle over a manifold M. Let DU be the closed disc bundle of U and SU the sphere bundle (so $SU = \partial DU$). The Thom map is an isomorphism $H^k(M) \to H^{k+n}(DU, SU)$.

The importance of this map is the jump in dimension. It gives a way to extend a low dimension calculation, which may be relatively easy to do, to a high dimension calculation, which may be more difficult by other means. For semi-infinite and finite codimension cohomology, the Thom isomorphism is very important since with the ability to jump an infinite number of dimensions it is possible to extend finite dimensional calculations to the semi-infinite and finite codimension cases.

To show how useful such a map is, consider the finite codimension cohomology of the sphere S in the Hilbert space $l^2(\mathbb{R})$. Let $p \in S$ be the point $(1, 0, \dots)$ and $q = -p$. Let C_p be the set of points with first coordinate positive or zero and C_q those with first coordinate negative or zero. We have the relative exact sequence:

$$\to H^k_{\mathrm{fc}}(S, C_p) \to H^k_{\mathrm{fc}}(S) \to H^k_{\mathrm{fc}}(C_p) \to$$

Excision of the interior of C_p gives an isomorphism $H^k_{\mathrm{fc}}(S, C_p) \cong H^k_{\mathrm{fc}}(C_q, \partial C_q)$. The right hand side of this is an infinite dimensional vector bundle over q so assuming the validity of the Thom isomorphism this is equal to $H^k_{\mathrm{fd}}(\{q\})$. The space C_p is homotopic to $\{p\}$ so the finite codimension cohomology of C_p is trivial. Thus we have $H^k_{\mathrm{fc}}(S) \cong H^k_{\mathrm{fd}}(\{q\})$ and so $H^0_{\mathrm{fc}}(S) = \mathbb{C}$ and $H^k_{\mathrm{fc}}(S) = 0$ for $k \neq 0$.

In this chapter, we consider two alterations to the definition of de Rham cohomology. In both cases we consider only Hilbert manifolds though each is extendible to a slightly larger class of manifolds. The first case is manifolds with an integration theory, so-called Wiener-Hilbert manifolds and the theory we develop is the infinite dimensional analogue of cohomology with compact support. In this situation, we prove a general version of the Thom isomorphism in a formal context and show how it applies. The second case consists of manifolds which can be approximated in some fashion by finite dimensional submanifolds. In this case the Thom isomorphism exists by construction. In both cases we rely heavily on the theory of Fredholm manifolds.

A Wiener structure on a manifold defines a Fredholm structure and a Fredholm structure is precisely what is required to approximate a manifold by finite dimensional submanifolds. Thus we start this chapter with a short introduction to Fredholm structures and say what we mean by a semi-infinite structure and a Fredholm structure being compatible.

5.2 Fredholm Structures

Let H be a real Hilbert space and let $\mathrm{Gl}_c(H) = \{I + u \in \mathrm{Gl}(H) : u \in \mathcal{C}(H)\}$ where $\mathcal{C}(H)$ is the ideal of compact operators on H. In [17], Palais showed that $\mathrm{Gl}_c(H)$ has the homotopy type of $\mathrm{Gl}(\mathbb{R}^\infty) = \varinjlim \mathrm{Gl}(\mathbb{R}^n)$.

Definition 5.2.0.10. *A* Fredholm structure *on a Hilbert manifold is an integrable reduction of the structure group to* $\mathrm{Gl}_c(H)$.

It is a theorem of Elworthy in [5] that all Hilbert manifolds admit a Fredholm structure.

Definition 5.2.0.11. *A* layer structure *on a Hilbert manifold M is a maximal atlas with transition maps of the form $I + u$ where u is locally finite dimensional (i.e. every point in its domain has a neighbourhood with image under u lying in a finite dimensional subspace of the model space H).*

Clearly a layer structure determines a Fredholm structure, and it is also true that all Hilbert manifolds admit a layer structure.

The structure group of a Hilbert manifold can thus be assumed to be the group $\{I + u \in \mathrm{Gl}(H) : u \in \mathcal{J}_0(H)\}$ where $\mathcal{J}_0(H)$ is the ideal of finite rank operators. If H is a polarised Hilbert space then this is a subgroup of $\mathrm{Gl}_{\mathrm{res},0}(H)$ and thus a layer structure on a Hilbert manifold determines a polarisation. Moreover, since the lift to $\mathrm{Gl}^{\sim}_{\mathrm{res},0}(H)$ is trivial over this subgroup, a layer structure determines a semi-infinite structure. Conversely, given a semi-infinite structure on a Hilbert manifold, any layer structure is compatible with it.

We can illustrate this point by considering the semi-infinite structure of the Grassmannian manifold of k planes in a polarised Hilbert space H.

5.2.1 The Semi-Infinite Structure of the Grassmannian

Let $H = L^2(S^1, \mathbb{C})$. There is an orthonormal basis of H given by the functions z^k for $k \in \mathbb{Z}$. The polarisation of H is into the spaces $H_- = \langle z^k : k < 0 \rangle$ and $H_+ = \langle z^k : k \geq 0 \rangle$. There is also a natural circle action on H which acts on the orthonormal basis via $(\zeta, z^k) \to \zeta^k z^k$. H_- is the closure of the space on which the circle acts negatively and H_+ is the closure of the space on which it acts positively. Since the circle action preserves the polarisation, there is a map $S^1 \to \mathrm{Gl}_{\mathrm{res}}(H)$. As the polarisation is preserved exactly, there is a canonical lift to $\mathrm{Gl}^{\sim}_{\mathrm{res}}(H)$ given by (ζ, ζ_+) where ζ_+ is the restriction of ζ to H_+. The action of this on the induced basis of $A_{\mathrm{si}}(H)$ is given by:

$$(\zeta, \zeta_+) f^S = \zeta^{-E_S} f^S$$

$$\text{where } E_S = \left(\sum_{s \in S - \mathbb{N}} s - \sum_{s \in \mathbb{N} - S} s \right) \text{ is the energy of } f^S$$

Of particular note is the fact that the circle always acts positively on $A_{\mathrm{si}}(H)$.

For $S \subseteq \mathbb{Z}$ with $|S| = k$, let $H_S = \langle z^{s_1}, \ldots, z^{s_k} \rangle$ and let H^S be the orthogonal complement. Let $p_S : H \to H$ be the orthogonal projection onto H_S. Let $\mathcal{S}$ be the set of $S \subseteq \mathbb{Z}$ with $|S| = k$.

The Grassmannian $\mathrm{Gr}_k(H)$ of k dimensional subspaces of H can be covered by open sets U_S where $S \in \mathcal{S}$. The set U_S consists of those $W \in \mathrm{Gr}_k(H)$ such that $p_S : W \to H_S$ is an isomorphism. It is modelled on $L(H_S, H^S)$, the map being given by $B \to G_B = \{(z, Bz) : z \in H_S\}$.

The natural polarisation of $L(H_S, H^S)$ is in terms of the circle action. As H_S is finite dimensional, the circle equivariant map $H_S^* \otimes H^S \to L(H_S, H^S)$ is an isomorphism. The polarisation has positive space spanned by $\{\overline{z}^s \otimes z^t : t - s \geq 0\}$.

If $W \in \mathrm{Gr}_k(H)$ is in $U_S \cap U_T$ then there are $B_S \in L(H_S, H^S)$ and $B_T \in L(H_T, H^T)$ such that $W = G_{B_S} = G_{B_T}$. Thus W is the image of H_S under the map $I_S + B_S$ and of H_T under the map $I_T + B_T$, where I_S, I_T are the identity maps on H_S, H_T respectively. These maps are isomorphisms onto W so there is an isomorphism $C : H_T \to H_S$ such that $(I_T + B_T)C = I_S + B_S$. The decomposition on the left hand side of this equation is with respect to the decomposition of H as $H_T \oplus H^T$ and on the right with respect to $H_S \oplus H^S$. Thus to compare them, we first need to rewrite them with respect to the same decomposition. Let:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : H_S \oplus H^S \to H_T \oplus H^T$$

be the decomposition of the identity on H with respect to the two decompositions of H. Then:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} I_S \\ B_S \end{pmatrix} = \begin{pmatrix} C \\ B_T C \end{pmatrix}$$

and so $B_T = (c + dB_S)(a + bB_S)^{-1}$. The transition map is:

$$\begin{aligned} D\psi_{TS}(A)(B) &= dB(a+bA)^{-1} - (c+dA)(a+bA)^{-1}bB(a+bA)^{-1} \\ &= (d - \psi_{TS}(A)b)B(a+bA)^{-1} \end{aligned}$$

Under the isomorphism $L(H_S, H^S) \cong H_S^* \otimes H^S$ this becomes:

$$D\psi_{TS}(A)(\overline{z}^i \otimes z^j) = \overline{z}^i(a+bA)^{-1} \otimes (d - \psi_{TS}(A)b)z^j$$

The map $b : H^S \to H_T$ is such that $bz^j = \delta_T^j z^j$ where $\delta_T^j = 1$ if and only if $j \in T$. Thus:

$$D\psi_{TS}(A)(\overline{z}^i \otimes z^j) = \begin{cases} -\overline{z}^i(a+bA)^{-1} \otimes \psi_{TS}(A)z^j & \text{if } j \in T \\ \overline{z}^i(a+bA)^{-1} \otimes z^j & \text{otherwise} \end{cases}$$

Since S and T are both finite, there is some $N \geq 0$ such that $S \cup T \subseteq \{-N, \dots, N\}$. For $|j| > N$ and $i \in S \cup T$, the sign of the circle action on $\overline{z}^i \otimes z^j$ is determined by the sign of j. Since $D\psi_{TS} : H_S^* \otimes z^j \to H_T^* \otimes z^j$, $D\psi_{TS}$ preserves the sign of these subspaces. Thus the off-diagonal terms of $D\psi_{TS}(A)$ are finite rank and hence $D\psi_{TS}(A) \in \mathrm{Gl}_{\mathrm{res}}$.

Although there is a shift in degree when moving between the natural polarisations of U_S and U_T, because $H^1(\mathrm{Gr}_k(H); \mathbb{Z}) = 0$ polarisations can be defined over each U_S equivalent to the natural ones such that the degree of the transition maps is zero. Because the polarisation arises from the complex structure of $\mathrm{Gr}_k(H)$ it defines a semi-infinite structure of zero periodicity on $\mathrm{Gr}_k(H)$.

To calculate the degree of $D\psi_{TS}(A)$ it is sufficient to restrict to the case when T differs from S by one element, i.e. there are some $s \in S$ and $t \in T$ such that $S \setminus \{s\} = T \setminus \{t\}$ and $s \neq t$. If $A : H_S \to H^S$ is the map $Az^s = z^t$ and $Az^{s'} = 0$ for $s' \neq s$ then the degree of $\psi_{TS}(A)$ is $2s - 2t = 2\sum S - 2\sum T$. Since $U_S \cap U_T$ is connected, this holds for all $D\psi_{TS}(A)$. For generic S, T, it is possible to find a finite sequence $S = S_0, \ldots, S_l = T$ such that S_j differs from S_{j+1} by one element. The degree of $D\psi_{ST}(A)$ is $2\sum_{i=0}^{l-1} S_i - 2\sum_{i=1}^{l} S_i = 2\sum S - 2\sum T$.

Let $S = \{-1, \ldots, -k\}$. The polarisation of TU_S due to the circle action is $H_S^* \otimes (H^S \cap H_-) \oplus H_S^* \otimes H_+$. The inclusion $\mathrm{Gr}_k(H_-) \to \mathrm{Gr}_k(H)$ maps the tangent space of $\mathrm{Gr}_k(H_-)$ onto the negative part of this polarisation. Thus we choose the zero degree of the global polarisation such that the natural polarisation on U_S is of zero degree. For any $T \in \mathcal{S}$, the polarisation on U_T is of degree $-k(k-1) - 2\sum T$. In particular, under the inclusion of $\mathrm{Gr}_k(H_+)$ the tangent space of $\mathrm{Gr}_k(H_+)$ maps onto the positive part of the natural polarisation in U_R where $R = \{0, \ldots, k-1\}$. This has degree $-k(k-1) - 2\sum R = -2k^2$.

5.3 Integrable Cohomology

In this section we develop the concept of integrable cohomology and prove that it has a Thom isomorphism. Our strategy is to first prove that under certain conditions, the Thom isomorphism exists. Following this, we define the integrable cohomology of a certain type of Hilbert manifold and show that this definition satisfies the conditions for the Thom isomorphism.

5.3.1 The Thom Isomorphism

In this section we give a proof of the Thom isomorphism in a formal context. In the next section we shall discuss how to realise this context using the theory of Wiener integration. The argument is based upon Atiyah's proof of Bott periodicity in [1].

We shall consider the case of a chain complex $(C^k(\cdot), \partial)$ over a commutative ring with a one, R. We shall assume that this chain complex is defined for a class of Fréchet manifolds and is functorial for maps within that class. We further assume that this chain complex has the following properties for V a Fréchet space of the appropriate class:

1. the map on $V \times V$ defined by $(u, v) \to (v, u)$ induces a chain map on $C^k(V \times V)$ written $x \to \theta(x)$,
2. the map on $V \times V$ defined by $(u, v) \to (v, -u)$ induces a chain map on $C^k(V \times V)$ which is chain homotopic to the identity,
3. the map on V defined by $u \to -u$ induces a chain map on $C^k(V)$.
4. there is a map $\mu : C^0(V) \to R$ which is a cocycle, i.e. $\mu(\partial x) = 0$,

5. there is some $U \in C^0(V)$ such that $\mu(U) = 1$,

6. there are two chain maps $C^l(V) \to C^l(V \times V)$ written $x \to xU$ and $x \to Ux$ such that $\theta(xU) = Ux$,

7. there is a cocycle $\mu_V : C^k(V \times V) \to C^k(V)$ such that $\mu_V(xU) = \mu(x)U$ and $\mu_V(Ux) = \mu(U)x$,

Theorem 5.3.1.1. $H(C^k(V), \partial) \cong R$

Proof. As the maps μ and μ_V are cocycles, they induce maps in homology which we shall also denote by μ and μ_V respectively. The map in 3 is self-inverse so induces an isomorphism on $H(C^k(V), \partial)$ which we write as $x \to \tilde{x}$.

Since the map of point 2 is chain homotopic to the identity, it induces the identity on $H(C^k(V \times V), \partial)$. For $x \in H(C^k(V), \partial)$, applying this map to xU is the same as applying θ to $\tilde{x}U$ and thus $xU = U\tilde{x}$.

Define $\eta : R \to H(C^0(V), \partial)$ by $\eta(r) = rU$. Clearly $\mu\eta = 1$ on R. Let $x \in H(C^k(V), \partial)$ and consider $\eta\mu(x)$. This lies in $H(C^0(V), \partial)$ and is equal to $\mu(x)U$. From point 7, $\mu(x)U = \mu_V(xU)$. Now $xU \in H(C^k(V \times V), \partial)$ and from the above, $xU = U\tilde{x}$ so $\mu_V(xU) = \mu_V(U\tilde{x})$. Thus from point 7 $\mu_V(U\tilde{x}) = \mu(U)\tilde{x}$. However, $\mu(U) = 1$ and so $\eta\mu(x) = \tilde{x}$. The map $x \to \tilde{x}$ is an isomorphism on $H(C^k(V), \partial)$ and so $\eta\mu$ is an isomorphism on $H(C^k(V), \partial)$. Thus since $\mu\eta = 1$ we must also have $\eta\mu = 1$ and thus $H(C^k(V), \partial) \cong R$. □

We use a similar method to prove the Thom isomorphism for a vector bundle E over a manifold X with fibre modelled on V. We need to examine the definition of the vector bundle $E \oplus E$ over X. The space $E \times E$ has the structure of a vector bundle over $X \times X$ modelled on $V \times V$. We then embed X in $X \times X$ with the diagonal map and pull back the vector bundle. Thus $E \oplus E = \{(e,f) \in E \times E : \pi(e) = \pi(f)\}$ and the projection map is $\pi(e,f) = \pi(e)$. This can also be regarded as a vector bundle over E modelled on V, the projection being $\pi_E(e,f) = e$. The conditions we require for the Thom isomorphism are:

1. the map on $V \times V$ defined by $(u,v) \to (v,u)$ induces a chain map on $C^k(E \oplus E)$ written $x \to \theta(x)$,
2. the map on $E \times E$ defined by $(u,v) \to (v,-u)$ induces a chain map on $C^k(E \oplus E)$ which is chain homotopic to the identity,
3. the map on V defined by $u \to -u$ induces a chain map on $C^k(E)$.
4. there is a map $\mu : C^k(E) \to C^k(X)$ which is a cocycle, i.e. $\mu(\partial x) = 0$,
5. there is a cocycle map $\eta : C^k(X) \to C^k(E)$ written $x \to xU$ such that $\mu(xU) = x$,
6. there are two chain maps $C^l(E) \to C^l(E \oplus E)$ written $x \to xU$ and $x \to Ux$ such that $\theta(xU) = Ux$,
7. there is a cocycle $\mu_E : C^k(E \oplus E) \to C^k(E)$ such that $\mu_E(xU) = \mu(x)U$ and $\mu_E(Ux) = \mu(U)x$,

Theorem 5.3.1.2. $H(C^k(E), \partial) \cong H(C^k(X), \partial)$

Proof. The proof is exactly as in theorem 5.3.1.1. □

5.3.2 Wiener-Hilbert Manifolds

We shall use the above in the theory of Wiener integration and Wiener manifolds. These are studied in detail in Eells and Elworthy [4], Kuo [12], and Gross [9]. Wiener integration is based on the theory of Gaussian integration which typically depends upon a variance parameter. In the following we shall suppress this dependence, thus assuming the parameter to be 1.

Definition 5.3.2.1. *A* Wiener-Hilbert space *is a triple* (H_1, H_2, i) *where* H_1 *and* H_2 *are real Hilbert spaces and* $i : H_1 \to H_2$ *is an injective Hilbert-Schmidt map with dense image.*

This induces another injective Hilbert-Schmidt map $i^* : H_2^* \to H_1^*$. Identifying H_1 with H_1^* gives an injective trace class map $i^*i : H_2^* \to H_2$.

Definition 5.3.2.2. *A* Wiener-Hilbert manifold M *is one modelled on* H_2 *with an atlas* $\mathcal{U} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ *for which the transition maps* $\phi_\alpha {\phi_\beta}^{-1}$ *preserve the subspace* $i(H_1)$.

The differentials of the transition maps, $D(\phi_\alpha {\phi_\beta}^{-1}) : U_\alpha \cap U_\beta \to \mathrm{Gl}(H_2)$, lie in the subgroup $(I + i^*i\mathfrak{L}(H_1)) \cap \mathrm{Gl}(H_2)$. The structure group is thus a subgroup of the group of invertible operators which differ from the identity by an operator of trace class, which is precisely the space of invertible operators with a determinant. As the transition maps factor through i^*i, they preserve the subspaces H_1 and H_2^*. Thus there are corresponding subbundles of the tangent space of M.

The model space is over $\mathbb{R}$ so the group of invertible operators with a determinant has two connected components given by $\det^{-1}(\mathbb{R}^\pm)$. This leads to the following definition:

Definition 5.3.2.3. *If the structure reduces to the identity component then the manifold is said to be* orientable.

As the group of operators with a determinant is homotopic to $O = \lim O_n$ (see Palais [17]), there is a characteristic class $w_1(M) \in H^1(M; \mathbb{Z}_2)$ corresponding to this condition.

Definition 5.3.2.4. *An orientable Wiener-Hilbert manifold M has a trivial line bundle $\mathcal{G} \to M$ called the* bundles of densities.

A section of this line bundle corresponds to a choice of density. As this bundle is trivial, any two choices differ by a function in $C^\infty(M, \mathbb{R}^+)$. In addition to the space of Wiener densities, we also need to choose Wiener data on M. This consists of an inner product G on the subspace corresponding to H_1 and a position field $z : M \to TM$. The properties that these need to satisfy are defined in Elworthy [6].

It is an interesting fact that through any point $p \in M$ there are submanifolds $M_1(p)$ and $M_2(p)$ modelled on H_1 and H_2^* respectively. For distinct $p, q \in M$, either $M_i(p) = M_i(q)$ or $M_i(p) \cap M_i(q) = \emptyset$. The metric G is said to be *complete* if all the submanifolds $M_1(p)$ are complete with the induced Riemannian metric. The subbundle modelled on H_2^* can be identified with T^*M. The Wiener data defines a density on each tangent space such that $i : T_pM_1(p) \to T_pM$ is the underlying Wiener structure.

This data specifies a particular density $\mu(G, z)$ on M. Also, given a finite codimension submanifold N of M there is an induced set of data on N. The following form of the divergence theorem is stated in Elworthy [6]:

Theorem 5.3.2.5. *Let M be an abstract Wiener manifold with boundary ∂M and with complete Wiener data (G, z). Let X be a vector field on M which factors through the subbundle T^*M and such that the map $p \to \|X(p)\|_p$ is integrable. Then:*

$$\int_M \operatorname{Div} X d\mu(G_M, z_M) = -\int_{\partial M} \langle n(p)|X(p)\rangle\, e^{-\frac{1}{2}\langle n(p)|z(p)\rangle_p^2} d\mu(G_{\partial M}, z_{\partial M})$$

where $n(p)$ is the uniquely defined internal normal to ∂M at p with $\|n(p)\|_p = 1$.

The extra factor in the integrand on the right hand side is due to the fact that the bundle on ∂M induced from the bundle of densities $\mathcal{G}$ does not necessarily agree with the natural bundle of densities on ∂M.

Corollary 5.3.2.6. *If $\partial M = \emptyset$ then $\int_M \operatorname{Div} X d\mu(G, z) = 0$.*

5.3.3 Hilbert Cohomology

We now consider how we can use the theory of Wiener manifolds to provide a context for section 5.3.1. Because we are only considering Hilbert manifolds, there are some simplifications that can be made to the general theory and we have certain extra information about the groups and spaces involved.

Let H be a complex, separable, polarised Hilbert space. The group $\mathrm{U}(H)$ is defined to be the subgroup of $\mathrm{Gl}(H)$ of operators which preserve the inner product. Similarly, the group $\mathrm{U}_{\mathrm{res}}(H)$ is defined to be the subgroup of $\mathrm{Gl}_{\mathrm{res}}(H)$ of operators which preserve the inner product. We have the following well-known theorems:

Theorem (Kuiper [11]). *The groups* $\mathrm{Gl}(H)$ *and* $\mathrm{U}(H)$ *are contractible.*

Theorem. *The groups* $\mathrm{U}(H)$ *and* $\mathrm{U}_{res}(H)$ *are deformation retracts of* $\mathrm{Gl}(H)$ *and* $\mathrm{Gl}_{res}(H)$ *respectively.*

Theorem (Pressley and Segal [18]). *The group* $\mathrm{Gl}_{res}(H)$ *is homotopic to the space* $\mathfrak{F}(H)$ *of Fredholm operators on a Hilbert space. This has the homotopy type of* $\mathbb{Z} \times BU$.

Theorem (Pressley and Segal [18]). $A_{si}(H)$ *contains a dense subspace* $\mathcal{H}_{si}(H)$ *which can be given the structure of a Hilbert space. This space is preserved by the action of* $\mathrm{U}_{res}^{\sim}$ *and the action is unitary. Moreover, given an orthonormal basis of* H*, the corresponding basis* $\{f^S\}$ *constructed as in section 3.3.3 is an orthonormal basis for* $\mathcal{H}_{si}(H)$.

The space $\mathcal{H}_{si}(H)$ inherits the grading from $A^*_{si}(H)$. We recall that the tensor product of two Hilbert spaces can be given a Hilbert space structure and define:

Definition 5.3.3.1. *Let* H_1 *and* H_2 *be Hilbert spaces over the same field. Write* $H_1 \otimes_{hs} H_2$ *for the Hilbert space completion of the tensor product* $H_1 \otimes H_2$.

Using this notation, we have the following result which is a corollary of the existence of $\mathcal{H}_{si}(H)$:

Corollary 5.3.3.2. *The map* $H^* \otimes A^k_{si}(H) \to A^{k+1}_{si}(H)$ *preserves the subspace* $\mathcal{H}_{si}(H)$*. The map* $H^* \otimes \mathcal{H}^k_{si}(H) \to \mathcal{H}^{k+1}_{si}(H)$ *extends to the natural Hilbert space completion of the tensor product, denoted* $H^* \otimes_{hs} \mathcal{H}^k_{si}(H)$.

Proof. This is a natural consequence of the fact that $\wedge$ preserves the orthonormal bases. □

This result shows that we can use the space $\mathcal{H}_{\mathrm{si}}(H)$ rather than the space $A^*_{\mathrm{si}}(H)$. This is preferable as it allows us to use the Hilbert space structure. The spaces $A_{\mathrm{fd}}(H)$ and $A_{\mathrm{fc}}(H)$ also have dense subspaces $\mathcal{H}_{\mathrm{fd}}(H)$ and $\mathcal{H}_{\mathrm{fc}}(H)$ which can be given the structure of Hilbert spaces.

The class of manifolds which we are considering are Wiener-Hilbert manifolds modelled on a real Hilbert space H and thus the structure group is a subgroup of $\mathfrak{D}_\times(H)$. The key consequence of this is that for U an open subset of H and a smooth map $s : U \to \mathcal{H}_{\mathrm{si}}(H_{\mathbb{C}})$, the condition that Ds be a map $U \to H^*_{\mathbb{C}} \otimes_{\mathrm{hs}} \mathcal{H}_{\mathrm{si}}(H_{\mathbb{C}})$ is invariant under structure-preserving diffeomorphisms.

For a Wiener-Hilbert manifold M we can thus define the space of sections of $\mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M)$ which satisfy the property that for a section s and point $p \in M$ there is a local coordinate chart U at p compatible with the Wiener-Hilbert structure of M and $Ds : U \to H^*_{\mathbb{C}} \otimes_{\mathrm{hs}} \mathcal{H}^k_{\mathrm{si}}(H_{\mathbb{C}})$. Thus we can define ds for such sections. We encounter the problem of whether ds is a section of the same type. To get round this we define:

$$\begin{aligned}\Gamma_1(M, \mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M)) = \{&s \in \Gamma(M, \mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M)) \\ &: Ds \in \Gamma(M, T^*_{\mathbb{C}}M \otimes_{\mathrm{hs}} \mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M))\}\end{aligned}$$

thus we can define $d : \Gamma_1(M, \mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M)) \to \Gamma(M, \mathcal{H}^{k+1}_{\mathrm{si}}(T_{\mathbb{C}}M))$ in local coordinates by $d = \wedge D$. We make the following definition:

Definition 5.3.3.3. *The space of semi-infinite Hilbert-Schmidt forms is defined to be:*

$$\mathfrak{A}^k_{si,hs}(M) = d^{-1}\Gamma_1(M, \mathcal{H}^{k+1}_{si}(T_{\mathbb{C}}M))$$

This is the space of sections s of $\mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M)$ for which s and ds both lie in $\Gamma_1(M, \mathcal{H}^k_{\mathrm{si}}(T_{\mathbb{C}}M))$. As $d^2 = 0$ we have:

$$d : \mathfrak{A}^k_{\mathrm{si,hs}}(M) \to \mathfrak{A}^{k+1}_{\mathrm{si,hs}}(M)$$

We can do the same with the finite dimension and finite codimension cohomology theories.

5.3.4 Bounded Hilbert Cohomology

Consider the Hilbert spaces $\mathcal{H}_{\mathrm{fc}}(H)$ and $\mathcal{H}_{\mathrm{fd}}(H)$ for a complex Hilbert space H. Given an orthonormal basis for H we have orthonormal bases $\{f^S : S \subseteq \mathbb{N}, |\mathbb{N} \setminus S| < \infty\}$ for $\mathcal{H}_{\mathrm{fc}}(H)$ and $\{f^S : S \subseteq \mathbb{N}, |S| < \infty\}$ for $\mathcal{H}_{\mathrm{fd}}(H)$. Using these we can define a unitary linear map $* : \mathcal{H}_{\mathrm{fd}}(H) \to \mathcal{H}_{\mathrm{fc}}(H)^*$ as follows:

$$f^S(f^T) = \epsilon_{ST}$$

where ϵ_{ST} is zero unless $S \cup T = \mathbb{N}$ as unordered sets and otherwise is the sign of the permutation which takes $S \cup T$ to $\mathbb{N}$ as ordered sets. The orders on S and T are those inherited from $\mathbb{N}$.

Let H, H_1 be real Hilbert spaces and let $i : H_1 \to H$ be a Wiener structure on H. There is a trace class map $i^*i : H^* \to H$. This induces a map $\mathcal{H}_{\mathrm{fd}}(H^*_{\mathbb{C}})^* \to \mathcal{H}_{\mathrm{fd}}(H_{\mathbb{C}})^*$. Using the $*$ map defined above, we can also define a map $\mathcal{H}_{\mathrm{fc}}(H^*_{\mathbb{C}}) \to \mathcal{H}_{\mathrm{fc}}(H_{\mathbb{C}})$ via the diagram:

$$\begin{tikzcd}
\mathcal{H}_{\mathrm{fc}}(H^*_{\mathbb{C}}) \arrow[d, "*"'] & \mathcal{H}_{\mathrm{fc}}(H_{\mathbb{C}}) \\
\mathcal{H}_{\mathrm{fd}}(H^*_{\mathbb{C}})^* \arrow[r, "i^*i"] & \mathcal{H}_{\mathrm{fd}}(H_{\mathbb{C}})^* \arrow[u, "*"]
\end{tikzcd}$$

Now $\mathcal{H}^1_{\mathrm{fd}}(H_{\mathbb{C}}) \cong H^*_{\mathbb{C}}$ so $\mathcal{H}^1_{\mathrm{fc}}(H_{\mathbb{C}}) \cong H_{\mathbb{C}}$ and thus the image of $\mathcal{H}^1_{\mathrm{fc}}(H^*_{\mathbb{C}})$ in $\mathcal{H}^1_{\mathrm{fc}}(H_{\mathbb{C}})$ coincides with $i^*i(H^*_{\mathbb{C}}) \subseteq H_{\mathbb{C}}$.

As $\mathcal{H}_{\mathrm{fc}}(H_{\mathbb{C}})$ is a Hilbert bundle over H, within $\mathfrak{A}_{\mathrm{fc,hs}}(H)$ lies the space $\mathfrak{A}_{\mathrm{fc,bs}}(H)$ defined as follows:

Definition 5.3.4.1. *The space $\mathfrak{A}_{fc,bs}(H)$ consists of those elements s of the space $\mathfrak{A}_{fc,hs}(H)$ for which s and ds are bounded, have bounded support and which factor through the subbundle $\mathcal{H}_{fc}(H^*_{\mathbb{C}})$.*

As in the definition of $\mathfrak{A}_{\mathrm{fc,hs}}(H)$, $(\mathfrak{A}_{\mathrm{fc,bs}}(H), d)$ is a cochain complex because $d^2 = 0$.

Now let M be an orientable Wiener-Hilbert manifold. Since the structure group preserves the inner product and the subbundle T^*M, we can define the chain complex $\mathfrak{A}_{\mathrm{fc,bs}}(M)$ of bounded forms with bounded support.

5.3.5 The Bounded Cohomology of a Hilbert Space

We now show that the complex of bounded forms with bounded support satisfies the conditions for the Thom isomorphism as described in section 5.3.1. We start with the case of a real Wiener-Hilbert space H with Wiener structure given by $i : H_1 \to H$.

Theorem 5.3.5.1. *The bounded cohomology of a real Wiener-Hilbert space H is given by:*

$$H^k_{fc,bs}(H) = \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We prove this theorem by showing that the bounded cohomology satisfies the conditions for the Thom isomorphism as laid down in section 5.3.1. We group this proof into two lemmas:

Lemma 5.3.5.2. *The bounded cohomology theory satisfies the following conditions:*

1. *the map on $H \times H$ defined by $(u, v) \to (v, u)$ induces a chain map on $\mathfrak{A}^k_{fc,bs}(H \times H)$ written $x \to \theta(x)$,*
2. *the map on $H \times H$ defined by $(u, v) \to (v, -u)$ induces a chain map on $\mathfrak{A}^k_{fc,bs}(H \times H)$ which is chain homotopic to the identity,*
3. *the map on H defined by $u \to -u$ induces a chain map on $\mathfrak{A}^k_{fc,bs}(H)$.*

Proof. The Wiener structure on the Hilbert space $H \times H$ is given by $i \times i : H_1 \times H_1 \to H \times H$. The chain complex is defined in terms of bounded sets and in terms of the measure so certainly unitary maps which preserve i and $i \times i$ will induce chain maps. In particular, the two maps on $H \times H$ defined by $(u, v) \to (v, u)$ and $(u, v) \to (v, -u)$ induce chain maps on $\mathfrak{A}_{\mathrm{fc,bs}}(H \times H)$. Similarly the map on H defined by $u \to -u$ induces a chain map on $\mathfrak{A}_{\mathrm{fc,bs}}(H)$.

The map on $H \times H$ defined by $(u, v) \to (v, -u)$ is homotopic to the identity via the homotopy:

$$F_t(u, v) = \left(u \cos\frac{\pi t}{2} + v \sin\frac{\pi t}{2}, -u \sin\frac{\pi t}{2} + v \cos\frac{\pi t}{2}\right)$$

thus to show that the map $(u, v) \to (v, -u)$ induces a map on $\mathfrak{A}_{\mathrm{fc},sb}(H \times H)$ chain homotopic to the identity, it is sufficient to show that the maps $F_t(u, v)$ for fixed t induce chain maps. However, this is true since they are unitary and preserve the Wiener structure on $H \times H$. Thus the induced map from $(u, v) \to (v, -u)$ is chain homotopic to the identity. $\square$

Lemma 5.3.5.3. *The bounded cohomology theory satisfies the following conditions:*

1. *there is a map $\mu : \mathfrak{A}^0_{fc,bs}(H) \to R$ which is a cocycle, i.e. $\mu(\partial x) = 0$,*
2. *there is some $U \in \mathfrak{A}^0_{fc,bs}(H)$ such that $\mu(U) = 1$,*
3. *there are two chain maps $\mathfrak{A}^l_{fc,bs}(H) \to \mathfrak{A}^l_{fc,bs}(H \times H)$ written $x \to xU$ and $x \to Ux$ such that $\theta(xU) = Ux$,*
4. *there is a cocycle $\mu_H : \mathfrak{A}^k_{fc,bs}(H \times H) \to \mathfrak{A}^k_{fc,bs}(H)$ such that $\mu_H(xU) = \mu(x)U$ and $\mu_H(Ux) = \mu(U)x$,*

Proof. The space $\mathcal{H}^0_{\mathrm{fc}}(H^*_{\mathbb{C}})$ is a complex line bundle over H. A trivialisation of this bundle is given by a choice of basis for H and two trivialisations are the same if they arise from equivalent bases. Given a trivialisation, a smooth section of $\mathcal{H}^0_{\mathrm{fc}}(H_{\mathbb{C}})$ is a smooth map $H \to \mathbb{C}$. Since bounded functions on H with bounded support are integrable, we have a map $\mu : \mathfrak{A}^0_{\mathrm{fc,bs}}(H) \to \mathbb{C}$.

Since $\mathcal{H}_{\mathrm{fc}}^{-1}(H_{\mathbb{C}}) \cong H_{\mathbb{C}}$, a section of this bundle can be identified with a vector field on H. Then $\mathcal{H}_{\mathrm{fc}}^{-1}(H_{\mathbb{C}}^*) \cong H_{\mathbb{C}}^*$ sits inside $H_{\mathbb{C}}$ as the cotangent bundle and thus an element s of $\mathfrak{A}_{\mathrm{fc,bs}}(H)$ is a vector field which factors through $T_{\mathbb{C}}^*H$ and such that the map $p \to \|s(p)\|_p$ is integrable. Hence by the divergence theorem (theorem 5.3.2.5) $\int_H ds = 0$ and so $\mu(ds) = 0$.

There is an element $U \in \mathfrak{A}^0_{\mathrm{fc,bs}}(H)$ represented by a bounded positive function $\rho : H \to \mathbb{R}$ with bounded support such that $\int_H \rho = 1$. This satisfies $\mu(U) = 1$.

The chain maps $\mathfrak{A}^l_{\mathrm{fc,bs}}(H) \to \mathfrak{A}^l_{\mathrm{fc,bs}}(H \times H)$ are constructed as follows. Let $\{x_j\}_{j\in\mathbb{N}}$ be the unitary basis chosen above for H. There is a unitary basis for $H \times H$ given by $\{y_j\}_{j\in\mathbb{N}}$ where:

$$y_{2j} = (x_j, 0) \qquad y_{2j+1} = (0, x_j)$$

Given $S \subseteq \mathbb{N}$ such that $|\mathbb{N}\setminus S| < \infty$ we can define two similar subsets $\tau_1(S)$ and $\tau_2(S)$ as follows:

$$\tau_1(S) = \mathbb{N}\setminus 2(\mathbb{N}\setminus S) \qquad \tau_2(S) = \mathbb{N}\setminus\{2(\mathbb{N}\setminus S) + 1\}$$

We can define two maps $\tau_i : \mathcal{H}^l_{\mathrm{fc}}(H_{\mathbb{C}}) \to \mathcal{H}^l_{\mathrm{fc}}(H_{\mathbb{C}} \times H_{\mathbb{C}})$ for $i = 1, 2$ by $\tau_i(f^S) = f^{\tau_i(S)}$. These map the subspaces $\mathcal{H}^l_{\mathrm{fc}}(H^*_{\mathbb{C}})$ into the subspaces $\mathcal{H}^l_{\mathrm{fc}}(H^*_{\mathbb{C}} \times H^*_{\mathbb{C}})$. The chain maps $\tau_i : \mathfrak{A}^l_{\mathrm{fc,bs}}(H) \to \mathfrak{A}^l_{\mathrm{fc,bs}}(H \times H)$ are defined by:

$$\tau_i(s)(x_1, x_2) = \rho(x_{2-i})\tau_i(s(x_i))$$

Since the maps τ_i are unitary, they preserve the boundedness of a section.

As $s(x_i)$ has bounded support in the x_i directions and $\rho(x_{2-i})$ in the x_{2-i} directions, $\tau_i(s)$ has bounded support. These are chain maps because:

$$d\tau_i(s)(x_1, x_2) = d\rho(x_{2-i})\tau_i(s(x_i)) + \rho(x_{2-i})\tau_i(ds(x_i))$$

and $d\rho(x_{2-i})$ has components only in the x_{2-i} directions which are saturated already in $\tau_i(s(x_i))$. We write $\tau_1(x) = xU$ and $\tau_2(x) = Ux$.

The map $\mu_H : \mathfrak{A}^k_{\mathrm{fc,bs}}(H \times H) \to \mathfrak{A}^k_{\mathrm{fc,bs}}(H)$ is defined in a similar fashion. We define first a map $\tau : \mathcal{H}^l_{\mathrm{fc}}(H_{\mathbb{C}} \times H_{\mathbb{C}}) \to \mathcal{H}^l_{\mathrm{fc}}(H_{\mathbb{C}})$ by:

$$\tau(f^S) = \begin{cases} f^T & \text{if } S = \tau_1(T) \\ 0 & \text{otherwise} \end{cases}$$

and define $\mu_H : \mathfrak{A}^l_{\mathrm{fc,bs}}(H \times H) \to \mathfrak{A}^l_{\mathrm{fc,bs}}(H)$ by:

$$\mu_H(s)(y) = \int_H \tau(a(x, y))$$

where the integration is over the first factor. This clearly satisfies $\mu_H(xU) = \mu(x)U$ and $\mu_H(Ux) = \mu(U)x$. □

Hence by theorem 5.3.1.1 the homology of the chain complex $\mathfrak{A}_{\mathrm{fc},sb}(H)$ is given by:

$$H^k_{\mathrm{fc},sb}(H) = \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

5.3.6 The Thom Isomorphism in Bounded Cohomology

We can do a similar thing with a polarised Hilbert space, $H = H_- \oplus H_+$. We choose a Wiener structure on H_+, $i : H_1 \to H_+$, this gives:

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{si}}(H_- \oplus H_+^*) & & \mathcal{H}_{\mathrm{si}}(H_- \oplus H_+) \\ *\downarrow & & *\uparrow \\ \mathcal{H}_{\mathrm{si}}(H_+ \oplus H_-^*)^* & \xrightarrow{i^*i\oplus 1} & \mathcal{H}_{\mathrm{si}}(H_+^* \oplus H_-^*)^* \end{array}$$

We consider the case of a vector bundle over a Hilbert manifold $\pi : E \to M$ with fibre modelled on the Wiener-Hilbert space $i : H_1 \to H_+$. The tangent bundle of the total space of E decomposes as $\pi^*TM \oplus \pi^*E$. If M is infinite dimensional this defines a global polarisation of $T_{\mathbb{C}}E$ and we consider sections of the bundle $\mathcal{H}_{\mathrm{si}}(T_{\mathbb{C}}E)$. If M is finite dimensional we consider sections of the bundle $\mathcal{H}_{\mathrm{fc}}(T_{\mathbb{C}}E)$, but grade it according to the decomposition $\pi^*T_{\mathbb{C}}M \oplus \pi^*E_{\mathbb{C}}$. For simplicity, we shall here only consider the case where M is infinite dimensional.

We have the bundle $\mathcal{H}_{\mathrm{si}}(T_{\mathbb{C}}M \oplus E_{\mathbb{C}}^*)$ as constructed above.

Definition 5.3.6.1. *Define $\mathfrak{A}_{si,bs}(E)$ to be the space of sections a of this bundle for which da is defined and a_x and da_x are bounded with bounded support on the fibre E_x at any point $x \in M$.*

In a similar manner, we can define $\mathfrak{A}_{\mathrm{si,bs}}(E \oplus E)$.

Theorem 5.3.6.2. *The semi-infinite bounded cohomology of E is given by:*

$$H_{si,bs}(E) \cong H_{fd}(X)$$

We prove this as for theorem 5.3.5.1 by showing that the bounded cohomology theory satisfies the conditions for theorem 5.3.1.2. The fibrewise maps $(u,v) \to (-v,u)$ and $(u,v) \to (v,u)$ on $E \oplus E$ and $u \to -u$ on E induce chain maps with the required properties by the same argument as in lemma 5.3.5.2 above. The proof that the theory satisfies the rest of the conditions for theorem 5.3.1.2 is similar to lemma 5.3.5.3:

Lemma 5.3.6.3. *The bounded cohomology theory satisfies the following conditions:*

1. *there is a map $\mu : \mathfrak{A}^0_{fc,bs}(H) \to R$ which is a cocycle, i.e. $\mu(\partial x) = 0$,*
2. *there is some $U \in \mathfrak{A}^0_{fc,bs}(H)$ such that $\mu(U) = 1$,*
3. *there are two chain maps $\mathfrak{A}^l_{fc,bs}(H) \to \mathfrak{A}^l_{fc,bs}(H \times H)$ written $x \to xU$ and $x \to Ux$ such that $\theta(xU) = Ux$,*
4. *there is a cocycle $\mu_H : \mathfrak{A}^k_{fc,bs}(H \times H) \to \mathfrak{A}^k_{fc,bs}(H)$ such that $\mu_H(xU) = \mu(x)U$ and $\mu_H(Ux) = \mu(U)x$,*

Proof. We define the map $\mathfrak{A}_{\mathrm{fd}}(X) \to \mathfrak{A}_{\mathrm{si,bs}}(E)$ and the two maps $\mathfrak{A}_{\mathrm{si,bs}}(E) \to \mathfrak{A}_{\mathrm{si,bs}}(E \oplus E)$ in a similar way to the maps in section 5.3.5. Firstly we define maps $\mathcal{H}_{\mathrm{fd}}(H_-) \to \mathcal{H}_{\mathrm{si}}(H)$ and $\mathcal{H}_{\mathrm{si}}(H_- \oplus H_+) \to \mathcal{H}_{\mathrm{si}}(H_- \oplus (H_+ \oplus H_+))$. These maps are defined in terms of bases so first we define maps on certain subsets of $\mathbb{Z}$:

$$\begin{aligned}
\sigma_0(S) &= S \cup \mathbb{N}_0 \\
\sigma_1(S) &= (S \cap -\mathbb{N}) \cup (\mathbb{N}_0 \setminus 2(\mathbb{N}_0 - S)) \\
\sigma_2(S) &= (S \cap -\mathbb{N}) \cup (\mathbb{N}_0 \setminus \{2(\mathbb{N}_0 - S) + 1\})
\end{aligned}$$

where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the domain of σ_0 is $S \subseteq -\mathbb{N}$ with $|S| < \infty$ and the domain of σ_i for $i = 1, 2$ is $S \subseteq \mathbb{Z}$ with $|\mathbb{N} - S|, |S - \mathbb{N}| < \infty$. The maps σ_1 and σ_2 are the same as applying the maps τ_1 and τ_2 of section 5.3.5 respectively to the positive part of S (with a slight adjustment for the use of $\mathbb{N}_0$ rather than $\mathbb{N}$). The maps on $\mathcal{H}$ are defined by $\sigma_i(f^S) = f^{\sigma_i(S)}$ for $i = 0, 1, 2$.

In order to define the chain maps we first choose a function $\rho : E \to \mathbb{R}$ which is positive, bounded and such that on each fibre E_x, $\rho|_{E_x}$ has bounded support and integrates to 1. In addition, we assume that $d\rho \in \pi^*E^*$ under the decomposition $T^*E = \pi^*T^*M \oplus \pi^*E^*$. Recall that $E \oplus E$ can be thought of as a Hilbert bundle over E in two ways. Let $\pi_i : E \oplus E \to E$ for $i = 1, 2$ be the two projection maps. We define:

$$\sigma_0(s)(x) = \rho(x)\sigma_0(s(\pi x))$$
$$\sigma_i(s)(x) = \rho(\pi_{2-i}x)\sigma_i(s(\pi_i x))$$

We write $\sigma_0(x) = xU$, $\sigma_1(x) = Ux$ and $\sigma_2(x) = xU$.

The reverse maps are defined in a similar fashion. We define partial inverses to σ_0 and σ_1 on $\mathcal{H}_{\mathrm{si}}$ as follows:

$$\sigma^i(f^S) = \begin{cases} f^T & \text{if } S = \sigma_i(T) \\ 0 & \text{otherwise} \end{cases}$$

and define $\mu : \mathfrak{A}^k_{\mathrm{si,bs}}(E) \to \mathfrak{A}^k_{\mathrm{fd}}(X)$ and $\mu_E : \mathfrak{A}^k_{\mathrm{si,bs}}(E \oplus E) \to \mathfrak{A}^k_{\mathrm{si,bs}}(E)$ by:

$$\mu(s)(x) = \int_{E_x} \sigma^0(s(p)) \qquad \text{where } p \in E_x = \pi^{-1}(x)$$
$$\mu_E(s)(x) = \int_{E_x} \sigma^1(s(p)) \qquad \text{where } p \in \pi_1^{-1}(x)$$

These satisfy $\mu(xU) = x$, $\mu_E(xU) = \mu(x)U$ and $\mu_E(Ux) = \mu(U)x$ and are cocycles by the divergence theorem. □

Hence by theorem 5.3.1.2, $H_{\mathrm{si,bs}}(E) \cong H_{\mathrm{fd}}(X)$. When M is of finite dimension n we have $H^{-k}_{\mathrm{fc,bs}}(E) \cong H^{-k}_{\mathrm{fc}}(M) = H^{n-k}(M)$.

5.3.7 Semi-Infinite Bounded Cohomology

To apply this theory for finite codimension cohomology merely requires noticing that if $i : H_1 \to H$ is a Wiener structure on a Hilbert space H then $I_n \oplus i : \mathbb{R}^n \oplus H_1 \to \mathbb{R}^n \oplus H$ is a Wiener structure on the Hilbert space $\mathbb{R}^n \oplus H$. For a Hilbert bundle over a closed finite dimensional manifold, $E \to M$, the subbundle $\pi^* T^* M$ maps isomorphically onto $\pi^* TM$ under the inclusion $TE^* \to TE$ and thus there is no restriction on the forms on M (recall that in bounded cohomology we need to restrict to sections of $A_{\mathrm{fd}}(H^*_{\mathbb{C}})$).

To apply the theory in the semi-infinite case requires more care. We need to ensure that the bundle $A_{\mathrm{si}}(T_{\mathbb{C}}M_- \oplus T^*_{\mathbb{C}}M_+)$ makes sense. To ensure that this is the case, we choose a Wiener structure on the whole of M compatible with the polarisation. Thus the model space for M is a Wiener space with a polarisation: $i_- \oplus i_+ : H_{1-} \oplus H_{1+} \to H_- \oplus H_+$ and we assume that the transition maps preserve the polarisation of H_1 as well as of H.

We can illustrate all of this theory by calculating the cohomology of projective space.

5.3.8 The Bounded Cohomology of Projective Space

Let H be a complex Hilbert space and let $\sigma : H \to H$ be an inclusion of complex codimension 1. There is an induced inclusion $\sigma : \mathbb{P}H \to \mathbb{P}H$ of complex codimension 1. The orthogonal complement of $\sigma(H)$ in H is a one dimensional line and so corresponds to a point $p \in \mathbb{P}H$. Let C be the set $\{q \in \mathbb{P}H : |\langle q|p\rangle| \geq |\langle q|\sigma(\mathbb{P}H)\rangle|\}$ of lines closer to $\sigma(\mathbb{P}H)$ than to p.

There is a long exact sequence in cohomology:

$$\to H^l_{\mathrm{fc,bs}}(\mathbb{P}H, C) \to H^l_{\mathrm{fc,bs}}(\mathbb{P}H) \to H^l_{\mathrm{fc,bs}}(C) \to$$

The space C is homotopic to $\sigma(\mathbb{P}H)$. The inclusion of $\sigma(\mathbb{P}H)$ in C is of complex codimension 1. Excision of the interior of C gives an isomorphism $H^l_{\mathrm{fc,bs}}(\mathbb{P}H, C) \cong H^l_{\mathrm{fc,bs}}(D, \partial D)$ where D is the set $\{q \in \mathbb{P}H : |\langle q|p\rangle| \leq |\langle q|\sigma(\mathbb{P}H)\rangle|\}$ of lines closer to p than to $\sigma(\mathbb{P}H)$. This is a disc over $\{p\}$ of infinite dimension and so the Thom isomorphism gives an isomorphism $H^l_{\mathrm{fc,bs}}(D, \partial D) \cong H^l(\{p\})$.

The long exact sequence becomes:

$$\to H^l(\{p\}) \to H^l_{\mathrm{fc,bs}}(\mathbb{P}H) \to H^{l+2}_{\mathrm{fc,bs}}(\mathbb{P}H) \to$$

For $l \neq 0$, $H^l(\{p\})$ is zero and so $H^l_{\mathrm{fc,bs}}(\mathbb{P}H) \cong H^{l+2}_{\mathrm{fc,bs}}(\mathbb{P}H)$. Since finite codimension cohomology is zero for strictly positive degree, this implies that the key dimensions are $l = 0, 1$ in which case we have:

$$0 \to H^0(\{p\}) \to H^0_{\mathrm{fc,bs}}(\mathbb{P}H) \to 0$$
$$0 \to H^1_{\mathrm{fc,bs}}(\mathbb{P}H) \to 0$$

and so $H^l_{\mathrm{fc,bs}}(\mathbb{P}H)$ is $\mathbb{C}$ for $l \leq 0$ even and zero elsewhere.

Now let $H = H_- \oplus H_+$ be a polarised Hilbert space and $\mathbb{P}H$ the projective space of H. Let $\mathbb{P}_+$ be the projective space of H_+ and $\mathbb{P}_-$ of H_-. Let C_+ be the set of lines at least as close to $\mathbb{P}_+$ as $\mathbb{P}_-$ and C_- the converse.

The space C_+ is a polarised manifold which has the structure of a closed disc bundle over $\mathbb{P}_+$. This fibration has the property that with respect to the polarisation, directions in the fibres are negative and directions in the base are positive. A corresponding description holds for C_- with the positive and negative parts interchanged.

There is a long exact sequence in cohomology:

$$\to H^l_{\mathrm{si,bs}}(\mathbb{P}H, C_+) \to H^l_{\mathrm{si,bs}}(\mathbb{P}H) \to H^l_{\mathrm{si,bs}}(C_+) \to$$

The space C_+ is homotopic to $\mathbb{P}_+$ through a homotopy which collapses the negative directions. Thus $H^l_{\mathrm{si,bs}}(C_+) \cong H^l_{\mathrm{fc,bs}}(\mathbb{P}_+)$.

Excision of the interior of C_+ from the pair $(\mathbb{P}H, C_+)$ results in the pair $(C_-, \partial C_-)$. The Thom map integrates along the fibres which are in the positive direction and thus gives an isomorphism $H_{\mathrm{si,bs}}(C_-, \partial C_-) \cong H_{\mathrm{fc}}(\mathbb{P}_-)$. However, the natural polarisation on C_- is to label all the fibre directions as positive which means that the inclusion $C_- \to \mathbb{P}H$ is actually a map of degree 2 and thus the isomorphism is $H^l_{\mathrm{si,bs}}(\mathbb{P}H, C_+) \cong H^{l-2}_{\mathrm{fd}}(\mathbb{P}_-)$.

Including this information gives:

$$
\begin{tikzcd}
{} \arrow[r] & H^l_{\mathrm{si,bs}}(\mathbb{P}H, C_+) \arrow[r] \arrow[d, "\cong"'] & H^l_{\mathrm{si,bs}}(\mathbb{P}H) \arrow[r] & H^l_{\mathrm{si,bs}}(C_+) \arrow[r] \arrow[d, "\cong"'] & {} \\
& H^{l-2}_{\mathrm{si,bs}}(C_-, \partial C_-) \arrow[d, "\cong"'] & & H^l_{\mathrm{fc,bs}}(\mathbb{P}_+) & \\
& H^{l-2}_{\mathrm{fd}}(\mathbb{P}_-) & & &
\end{tikzcd}
$$

For l odd we get:

$$0 \to H^l_{\mathrm{si,bs}}(\mathbb{P}H) \to 0$$

and for l even we get:

$$0 \to H^{l-2}_{\mathrm{fd}}(\mathbb{P}_-) \to H^l_{\mathrm{si,bs}}(\mathbb{P}H) \to H^l_{\mathrm{fc,bs}}(\mathbb{P}_+) \to 0$$

Thus $H^l_{\mathrm{si,bs}}(\mathbb{P}H)$ is zero for l odd and is $\mathbb{C}$ for l even.

5.4 Tame Cohomology

Another area in which we have a Thom isomorphism is that of tame cohomology. In this case, the Thom isomorphism is a consequence of the construction, although it does fit into the context of section 5.3.1 as well.

A tame function f on an infinite dimensional vector space X is one for which there is a projection $P : X \to X$ of finite rank such that $f(x) = f(Px)$ for all $x \in X$. Thus a tame function is one which only depends upon a finite number of variables. In the theory of Wiener integration tame functions are dense in the space of integrable functions. If f is a tame function such that $f|_{PX}$ is integrable then $\int_X f = \int_{PX} f|_{PX}$ where PX has the induced measure from X (here X must be a Wiener space). The Wiener measure has the property that when restricted to any one dimensional subspace, it is a Gaussian measure.

Let $U \to M$ be a finite dimensional vector bundle over a finite dimensional manifold. Let DU be the closed disc bundle of U and SU the sphere bundle of U. One way of describing the Thom isomorphism is to say that the cohomology of (DU, SU) only depends upon the submanifold M, where we consider $M \subseteq DU$ embedded as the zero section.

Both of these concepts give rise to the idea of tame cohomology. In this section we shall discuss how this applies to manifolds and so define the tame finite codimension and tame semi-infinite cohomology of Hilbert manifolds.

5.4.1 Tame Finite Codimension Cohomology

Let $p : U \to M$ be a Hilbert bundle over a finite n dimensional manifold. The tangent space of U is given by $p^*TM \oplus p^*U$ and therefore given a finite codimension subspace of $T_{\mathbb{C}}U$ at a point x, we can ask whether the projection onto $p^*U_{\mathbb{C}}$ is surjective. If so, its kernel is a subspace of $p^*T_{\mathbb{C}}M$. Given a choice of basis u for U, this map can be extended to a map p_* on the determinant bundle, thus $p_*[w, \lambda] = [v, \lambda]$ where v is such that $[v \cup u, \lambda] =$

$[w, \lambda]$. For a form $a \in \Omega(M)$, we define the form p^*a on U by:

$$p^*a(x)([w,\lambda]) = \begin{cases} e^{-\pi\|x\|^2} a(p(x))(p_* [w,\lambda]) & \text{if } \langle w \rangle \to T_{\mathbb{C}}U_x \text{ is surjective} \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

the map p^* commutes with d since the function $e^{-\pi\|x\|^2}$ depends upon the fibre directions only.

Suppose that there is a finite dimensional subbundle V of U. Then U decomposes orthogonally as $V \oplus V^\perp$ and U can be considered as a Hilbert bundle over V. Let $p_V : V \to M$ and $p_U : U \to V$ denote the projection maps, then $p = p_V p_U$ and $\|x\|_U^2 = \|x\|_{V^\perp}^2 + \|x\|_V^2$. Thus $p^*a = p_U^* p_V^* a$. However, the map p_V^* is the standard Thom isomorphism from cohomology on M to the cohomology on V which decays rapidly at infinity.

Because the function $e^{-\pi\|x\|^2}$ decays rapidly at infinity, if U is an open subset of a manifold N then the form p^*a can be extended to the whole of N by defining it to be zero outside U. We can thus define:

Definition 5.4.1.1. *The complex of* tame forms *on N is the linear span of those forms which arise from such open sets.*

This complex is a variant of the finite codimension cohomology of N. In finite dimensions all forms are tame and thus we recover the standard cohomology. In infinite dimensions by construction we have a theory which admits a Thom isomorphism.

Let $\mathcal{M}$ denote the family of finite dimensional, closed submanifolds of a Hilbert manifold N. The following theorem is a direct consequence of the construction of the tame cohomology:

Theorem 5.4.1.2. *If $\mathcal{M}$ is a directed family then:*

$$TH_{fc}^{-k}(N) = \varinjlim_{M \in \mathcal{M}} H_{fc}^{-k}(M)$$

where for a finite n dimensional manifold M, $H_{fc}^{-k}(M) = H^{n-k}(M)$.

We can pursue this line further. In [15], Mukherjea constructs a sequence of closed submanifolds $(M_n)_{n \geq n_0}$ for a Fredholm manifold M with the following properties:

1. $M_n \subseteq M_{n+1}$,
2. $\dim M_n = n$,
3. $\bigcup M_n$ is homotopy equivalent to M,
4. $M_n \to M$ and $M_n \to M_{n+1}$ have trivial normal bundles,
5. there is a sequence of open sets Z_n such that each Z_n is a tubular neighbourhood of the corresponding M_n and $\bigcup Z_n = M$.

Moreover, every Hilbert manifold admits a parallelisable Fredholm structure (see, for example, [5] and [3]) so the above applies to all Hilbert manifolds.

For our purposes, we can relax these conditions to the following:

Definition 5.4.1.3. *A* filtration *of a Hilbert manifold M is a directed family $\{M_\lambda\}_{\lambda \in \Lambda}$ of finite dimensional closed submanifolds of M such that:*

1. $\dim M_\lambda \to \infty$,

2. $\bigcup M_\lambda$ *is homotopy equivalent to* M,

3. *there is a family of open sets* $\{Z_\lambda\}_{\lambda\in\Lambda}$ *such that each* Z_λ *is a tubular neighbourhood of the corresponding* M_λ *and* $\bigcup Z_\lambda = M$.

Theorem 5.4.1.4. *Let* M *be a Hilbert manifold with a filtration* $\{M_\lambda\}_{\lambda\in\Lambda}$. *The tame cohomology of* M *is given by:*

$$\varinjlim H_{fc}^{-k}(M_\lambda) \to TH_{fc}^{-k}(M)$$

Proof. Within the context of tame cohomology there are maps $H_{\text{fc}}^{-k}(M_\lambda) \to TH_{\text{fc}}^{-k}(M)$ and $H_{\text{fc}}^{-k}(M_\lambda) \to H_{\text{fc}}^{-k}(M_\mu)$ for $\mu > \lambda$ and these maps commute. Thus we have a map $\varinjlim H_{\text{fc}}^{-k}(M_\lambda) \to TH_{\text{fc}}^{-k}(M)$ which we wish to show is an isomorphism. Because $\bigcup M_\lambda$ is homotopy equivalent to M, a submanifold N of M can be perturbed to a submanifold of some M_μ and thus the map $H_{\text{fc}}^{-k}(N) \to TH_{\text{fc}}^{-k}(M)$ factors through $H_{\text{fc}}^{-k}(M_\mu)$. □

A similar method was used in [15] by Mukherjea to define finite codimension cohomology. However, in that paper the groups $H^{\infty-k}(M)$ are defined to be the direct limit of the groups $H^{n-k}(M_n)$ with maps given by Thom isomorphisms.

5.4.2 Tame Semi-Infinite Cohomology

The extension of tame cohomology to the semi-infinite case is a simple one. The idea is to consider those forms which are tame in directions corresponding to the positive part of the polarisation. As above in section 5.4.1, the case of a Hilbert bundle over a manifold is the simplest case. The difference between this case and that in section 5.4.1 is that the base manifold is now infinite dimensional.

Let $p : U \to M$ be a Hilbert bundle over a Hilbert manifold. The tangent bundle of U splits orthogonally as $p^*TM \oplus p^*U$. This defines a global polarisation of U and thus U is a semi-infinite manifold.

Definition 5.4.2.1. *A* tame semi-infinite form *on U is one which arises from a finite dimension form on M exactly as in equation 5.1.*

As before, if U splits as $V^{\perp} \oplus V$ where V is finite dimensional then the form arising from M can be considered as one arising from V factoring through the Thom isomorphism from forms on M to forms on V which decay rapidly at infinity. Thus a tame semi-infinite form depends upon the polarisation only up to equivalence.

The semi-infinite analogue of the filtration is as follows:

Definition 5.4.2.2. *A* semi-infinite filtration $\{M_\lambda\}_{\lambda\in\Lambda}$ *for a semi-infinite manifold M is a directed set of closed submanifolds such that:*

1. *the inclusion $i_\lambda : M_\lambda \to M$ satisfies $i_\lambda : T_{\mathbb{C}}M_\lambda \to T_{\mathbb{C}}M_-$ is Fredholm of index n_λ and $i_\lambda : T_{\mathbb{C}}M_\lambda \to T_{\mathbb{C}}M_+$ is Hilbert-Schmidt,*

2. *n_λ defined by the above satisfies $n_\lambda \to -\infty$,*

3. *there is a family of open sets $\{Z_\lambda\}_{\lambda\in\Lambda}$ such that each Z_λ is a tubular neighbourhood of the corresponding M_λ and $\bigcup Z_\lambda = M$*

The construction of the filtration in [15] depends upon the construction of a Fredholm map $f : M \to H$, where H is a Hilbert space. The filtration is defined by $M_n = f^{-1}H_n$ where $H_1 \subseteq H_2 \subseteq \ldots$ is a flag in H. Provided that the map f preserves the polarisation, the same idea can be used to construct a semi-infinite filtration using the semi-infinite flag $\ldots H_{-1} \subseteq H_0 \subseteq H_1 \subseteq \ldots$ where $H_0 = H_-$.

It is not immediately obvious what the semi-infinite analogue of the statement that $\bigcup M_\lambda$ be homotopy equivalent to M should be. Clearly what we are looking for is a condition such that if N is a submanifold of M such that $T_{\mathbb{C}}N \to T_{\mathbb{C}}M_-$ is Fredholm and $T_{\mathbb{C}}N \to T_{\mathbb{C}}M_+$ is Hilbert-Schmidt then N can be perturbed to a submanifold of some M_λ and thus the map from the cohomology of N into the tame semi-infinite cohomology of M factors through M_λ.

We can answer this question by considering a further refinement to the notion of "tame". In the above, "tame" really means "half tame" as it refers to the positive directions only. Let N be a submanifold of M. From [15], N has the homotopy type of a union of finite dimensional closed manifolds, (N_n). If, in addition, the homology of N is finite dimensional in each degree then for each k there is some n_k such that $H^k_{\text{fd}}(N) \to H^k(N_n)$ is an isomorphism for $n > n_k$. Forms on M which arise from N can be considered as "fully tame" since they are determined by finite dimensional closed submanifolds. Given a closed finite dimensional submanifold P of M, it can be perturbed to a submanifold of one of the M_λ and thus the tame semi-infinite cohomology of M is determined by the filtration.

Thus we define a tame filtration:

Definition 5.4.2.3. *A* tame filtration *of a semi-infinite Hilbert manifold is a family of closed, finite dimensional submanifolds* $\{M_{\lambda\mu}\}$ *such that:*

1. *there is a semi-infinite filtration* $\{M_\lambda\}$ *of* M *with* $\{M_{\lambda\mu}\}_\mu$ *a filtration of* $\{M_\lambda\}$,

2. *the homology of each* $\{M_\lambda\}$ *is finite dimensional in each degree.*

The following theorem is immediate:

Theorem 5.4.2.4. *Let* M *be a semi-infinite manifold with tame filtration* $\{M_{\lambda\mu}\}$*; we have:*

$$TH_{si}(M) = \varinjlim{}_\lambda \varprojlim{}_\mu H(M_{\lambda\mu})$$

5.4.3 The Tame Cohomology of the Grassmannian

Let H be a polarised complex Hilbert space and let $\mathrm{Gr}_k(H)$ be the space of k dimensional subspaces of H. In section 5.2.1 we showed that this is a semi-infinite manifold. Let $\mathcal{P}$ be the directed family of orthogonal projections on H for which $PH \in \mathrm{Gr}_{\mathrm{res}}(H)$. The family $\{\mathrm{Gr}_k(PH) : P \in \mathcal{P}\}$ is a semi-infinite filtration for $\mathrm{Gr}_k(H)$. Given $P \in \mathcal{P}$ let $\mathcal{Q}_P$ be the directed family of finite rank orthogonal projections dominated by P (i.e. $QP = Q$). The finite dimensional manifolds $\{\mathrm{Gr}_k(QH) : Q \in \mathcal{Q}_P\}$ form a filtration for $\mathrm{Gr}_k(PH)$. Thus there is a tame filtration of $\mathrm{Gr}_k(H)$ given by $\{G_{PQ} : G_{PQ} = \mathrm{Gr}_k(QH), P \in \mathcal{P}, Q \in \mathcal{Q}_P\}$. From section 5.4.2 we have:

$$\begin{aligned} TH_{\mathrm{si}}(\mathrm{Gr}_k(H)) &= \varinjlim{}_P \varprojlim{}_Q H_{\mathrm{fd}}(G_{PQ}) \\ &= \varprojlim{}_P H_{\mathrm{fd}}(\mathrm{Gr}_k(PH)) \end{aligned}$$

To calculate this, we consider a specific subfamily of $\mathrm{Gr}_k(PH)$. For $l \in \mathbb{Z}$, let H_l be the span of $\{z^n : n < l\}$ and let $P_l : H \to H_l$ be the orthogonal projection, in particular $H_- = H_0$. The inclusion $H_l \to H$ induces an inclusion $i_l : \mathrm{Gr}_k(H_l) \to \mathrm{Gr}_k(H)$. With the above choice of zero, this inclusion induces a map $\phi_l : H^n_{\mathrm{fd}}(\mathrm{Gr}_k(H_l)) \to TH^{n-2kl}_{\mathrm{si}}(\mathrm{Gr}_k(H))$. There are also injective maps $\phi^m_l : H^n_{\mathrm{fd}}(\mathrm{Gr}_k(H_l)) \to H^{n+2(m-l)k}_{\mathrm{fd}}(\mathrm{Gr}_k(H_m))$ for $m > l$ which commute with the above map given by the Thom isomorphism.

For a general $P \in \mathcal{P}$, there is some $l \in \mathbb{Z}$ such that $\ker P$ and $\ker P_l$ are in the same component of $\mathrm{Gr}_{\mathrm{res}}(H)$. Then there is some $Q \in \mathcal{P}$ such that $Q \geq P, P_l$. The maps $H_{\mathrm{fd}}(\mathrm{Gr}_k(PH)) \to H_{\mathrm{fd}}(\mathrm{Gr}_k(QH))$ and $H_{\mathrm{fd}}(\mathrm{Gr}_k(P_lH)) \to H_{\mathrm{fd}}(\mathrm{Gr}_k(QH))$ induce the same shift in dimension and have the same image. Thus in calculating the spaces $\varinjlim H_{\mathrm{fd}}(\mathrm{Gr}_k(PH))$ it is sufficient to calculate $\varinjlim H_{\mathrm{fd}}(\mathrm{Gr}_k(P_lH))$.

Using the standard description of $H_{\mathrm{fd}}(\mathrm{Gr}(H_l))$ as $\mathbb{C}[c_1, \ldots, c_k]$, the map $H_{\mathrm{fd}}(\mathrm{Gr}_k(H_l)) \to H_{\mathrm{fd}}(\mathrm{Gr}_k(H_m))$ is given by:

$$\phi^m_l(c_1^{r_1} \ldots c_k^{r_k}) = c_1^{r_1} \ldots c_k^{r_k+(m-l)}$$

If we choose an alternative description of $H_{\mathrm{fd}}(\mathrm{Gr}(H_l))$ as the $\mathbb{C}[c_1, \ldots, c_k]$ module $\mathbb{C}\left[\tilde{c}_1, \ldots, \tilde{c}_k, \tilde{c}_k^{-1}\right] / \tilde{c}_k^{-l} = 0$ then this map is:

$$\phi^m_l(\tilde{c}_1^{r_1} \ldots \tilde{c}_k^{r_k}) = \tilde{c}_1^{r_1} \ldots \tilde{c}_k^{r_k}$$

In $H_{\mathrm{fd}}(\mathrm{Gr}_k(H_0))$, the $\tilde{c}_j$ coincide with the standard c_j. In $H_{\mathrm{fd}}(\mathrm{Gr}(H_l))$, the $\tilde{c}_j$ are the images of c_j under ϕ^l_0. Thus:

$$\lim H_{\mathrm{fd}}(\mathrm{Gr}_k(H_l)) = \mathbb{C}\left[\tilde{c}_1, \ldots, \tilde{c}_k, \tilde{c}_k^{-1}\right]$$

Hence the semi-infinite tame cohomology of $\mathrm{Gr}_k(H)$ is given by:

$$TH_{\mathrm{si}}(\mathrm{Gr}_k(H)) = \mathbb{C}\left[c_1, \ldots, c_k, {c_k}^{-1}\right]$$

In particular, the semi-infinite tame cohomology of $\mathbb{P}H$ is:

$$TH_{\mathrm{si}}(\mathbb{P}H) = \mathbb{C}\left[c_1, {c_1}^{-1}\right]$$

$$TH^k_{\mathrm{si}}(\mathbb{P}H) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \mathbb{C} & \text{if } k \text{ is even} \end{cases}$$

Chapter 6

The Truncated Witten Genus

In chapter 4 the differential was defined using a contraction $\wedge : X^* \otimes A^k_{\mathrm{si}}(X) \to A^{k+1}_{\mathrm{si}}(X)$. There is an alternative contraction with domain $X \otimes A^k_{\mathrm{si}}(X)$ and range $A^{k-1}_{\mathrm{si}}(X)$ defined by $\alpha \otimes f \to \iota_\alpha f$ where:

$$(\iota_\alpha f)([w,\lambda]) = \begin{cases} 0 & \text{if } \alpha \in \langle w \rangle \\ f([\{\alpha\} \cup w, \lambda]) & \text{otherwise} \end{cases}$$

On a Hilbert manifold there is a conjugate linear isomorphism between $T^*_{\mathbb{C}}M$ and $T_{\mathbb{C}}M$. Because the inner contraction is alternating, it has the same properties as the $\wedge$ map and can be extended over the same domain. Thus we can use this contraction to define an operator δ on $\mathfrak{A}_{\mathrm{si}}(M)$ with degree -1. We can therefore define the signature and Laplacian operators as, respectively, $d+\delta$ and $d\delta+\delta d$. These can also be defined on $\mathfrak{A}_{\mathrm{fc}}(M)$ and $\mathfrak{A}_{\mathrm{fd}}(M)$ (we note that if $f \in A^0_{\mathrm{fd}}(X)$ then $\iota_\alpha f = 0$ for all $\alpha \in X$).

The constructions of tame cohomology and of Floer cohomology indicate a link in semi-infinite theory between a given semi-infinite manifold and a system of finite dimensional submanifolds which approximate the semi-infinite structure of the manifold. Given such a system of submanifolds, calculations which use the cohomology should respect this limit. In particular, index calculations on a semi-infinite manifold should be expressible as limits of calculations on the manifolds in the semi-infinite system. In this chapter, we do two calculations which support this conjecture. Section 6.1 contains some preliminary identities in equivariant K-theory, section 6.2 contains a discussion of twisted Dirac operators and the Witten genus, then section 6.3 and section 6.4 contain the two calculations.

On a point of notation, we shall often be dealing with the situation where a particular element of the circle acts fibrewise on a complex (resp. real) vector bundle U by multiplication (resp. rotation) by ζ. We shall denote this by ζU. The Chern character ch extends to an equivariant Chern character by defining $\operatorname{ch} \zeta U = \zeta \operatorname{ch} U$. In the literature this is often denoted by ch_g to emphasize the dependence on the group action.

Let M be an orientable manifold of even dimension $2d$ and V an orientable real vector bundle of even dimension $2r$ with $w_2(V) = w_2(TM)$. Using notation defined in section 6.1, we define the *Witten genus twisted by V and truncated at m* to be the power series in ξ defined by:

$$W_m(M,V)(\xi) = \left\langle \widehat{\mathcal{A}}(TM,V)\operatorname{ch}\left(\bigotimes_{k=1}^{m} \xi^{dk} S_{\xi^k} T_{\mathbb{C}} M \bigotimes_{l=1}^{m} \xi^{-rl} \Lambda_{\xi^l} V_{\mathbb{C}}\right), [M]\right\rangle$$

Let $p \in \mathbb{N}$ and set $n = 2p+1$. Let M^n be the n-fold product of M and let V^n be the n-fold product of V; this is a vector bundle over M^n. Let C_n act on M^n and V^n by cyclic permutation of coordinates. We identify C_n with a subgroup of S^1 by choosing a primitive nth root of unity ξ. We denote the spinor bundle constructed from V^n by $\Delta(V^n)$. Although this bundle may not be globally defined, we can consider the Dirac operator on M^n twisted by $\Delta(V^n)$ because $w_2(V) = w_2(TM)$.

Theorem A. *Let D_V be the Dirac operator on M^n twisted by $\Delta(V^n)$. Then:*

$$Index_\xi D_V = (-1)^{(p(d+r)+d)} W_p(M, V)(\xi)$$

Define the *kth Witten characteristic class truncated at m* for a real vector bundle U of dimension $2d$ to be the power series:

$$W_{k,m}(U)(\xi) = \widehat{A}(\xi^k U)\,\mathrm{ch}\left(\bigotimes_{s=1}^{m} \xi^{sd} S_{\xi^s}(\xi^k U)_{\mathbb{C}}\right)$$

Let n be an odd positive integer. For $a, b \in \mathbb{Z}$ with $a \leq b$ define $Y_a^b := \mathbb{C}[z, z^{-1}]_a^b$ to be the space of Laurent polynomials in z whose terms have degree between and including a and b. Let $X_a^b = \mathbb{P}(\mathbb{C}^{n+1} \otimes Y_a^b)$. Define an action of S^1 on Y_a^b by $\xi \cdot z^r = \xi^r z^r$. This action projects down to X_a^b.

Theorem B. *Let $r \in \mathbb{Z}$ and $q \in \mathbb{N}$. There is a S^1-equivariant Dirac operator on X_{r-q}^{r+q} and for $\xi \in S^1$ of either infinite order or finite order greater than $2q$ then:*

$$Index_\xi D_{X_{r-q}^{r+q}} = \sum_{k=-q}^{q} -\left\langle W_{k,q}(T\mathbb{CP}^n + \mathbb{R}^2)(\xi), [\mathbb{CP}^n]\right\rangle$$

6.1 Preliminary Results from K-Theory

Let M be an oriented manifold. For V a spin bundle, let $\Delta(V)$ denote the associated spinor bundle. If V is even dimensional, let $\Delta^+(V)$ and $\Delta^-(V)$ denote the positive and negative spinor bundles respectively.

Let W, V be orientable real vector bundles with W even dimensional. If $w_2(W) = w_2(V)$ then there is a vector bundle over M which locally has the form $(\Delta^+ - \Delta^-)(W)\Delta(V)$. Although this bundle cannot be so expressed globally, we shall refer to it using this notation. For such bundles, define the characteristic class $\widehat{\mathcal{A}}(W, V)$ to be:

$$\widehat{\mathcal{A}}(W, V) = \frac{\mathrm{ch}((\Delta^+ - \Delta^-)(W)\Delta(V))\,\mathrm{td}(W_{\mathbb{C}})}{\mathrm{e}(W)}$$

If $V = 0$ then this coincides with the standard definition of the $\widehat{\mathcal{A}}$ characteristic class of W. This can be rearranged into the following form:

$$\widehat{\mathcal{A}}(W) = -e(W)\,\mathrm{ch}\left(\frac{(\Delta^+ - \Delta^-)(W)}{\Lambda_{-1}W_{\mathbb{C}}}\right) \tag{6.1}$$

For a real vector bundle with a ξ action, ζW, define the equivariant $\widehat{\mathcal{A}}$ characteristic class as:

$$\widehat{\mathcal{A}}(\zeta W) = -e(W)\,\mathrm{ch}\left(\frac{(\Delta^+ - \Delta^-)(\zeta W)}{\Lambda_{-1}(\zeta W)_{\mathbb{C}}}\right) \tag{6.2}$$

If ξ acts on U by multiplication by ζ then ξ acts on S^kU and Λ^kU by multiplication by ζ^k. Thus $S\zeta U = S_\zeta U$ and $\Lambda\zeta U = \Lambda_\zeta U$. There is an important relationship between the functors $\Lambda_{-\zeta}$ and S_ζ given by:

$$(\Lambda_{-\zeta}U)^{-1} = S_\zeta U \tag{6.3}$$

Let L be a complex line bundle and suppose that $L_{\mathbb{R}}$ is a spin bundle. There is a complex line bundle L' such that $L' \otimes L' = L$. If ξ acts on L by multiplication by ζ then ξ acts on L' by multiplication by $\zeta^{1/2}$, where we can choose the sign of the square root arbitrarily. Then:

$$\left(\Delta^+ - \Delta^-\right)(\zeta L)_{\mathbb{R}} = \zeta^{1/2}L' - \zeta^{-1/2}\overline{L'}$$

This map from $KO(M)$ to $K(M)$ converts sums to products, thus considering $\zeta L + \xi\,\overline{L}$, we have:

$$\begin{aligned}\left(\Delta^+ - \Delta^-\right)(\zeta L + \xi\,\overline{L})_{\mathbb{R}} &= (\zeta^{1/2}L' - \zeta^{-1/2}\,\overline{L'})(\xi^{1/2}\,\overline{L'} - \xi^{-1/2}L')\\ &= \zeta^{1/2}\xi^{1/2}\mathbb{C} - \zeta^{1/2}\xi^{-1/2}L - \zeta^{-1/2}\xi^{1/2}\overline{L} + \zeta^{-1/2}\xi^{-1/2}\mathbb{C}\\ &= \zeta^{-1/2}\xi^{-1/2}\Lambda_{-1}(\zeta L + \xi\,\overline{L})\end{aligned}$$

Since $(\zeta L + \xi\,\overline{L})_{\mathbb{R}} = (\zeta^{-1}\overline{L} + \xi^{-1}L)_{\mathbb{R}}$, we also have:

$$\begin{aligned}\left(\Delta^+ - \Delta^-\right)(\zeta L + \xi\,\overline{L})_{\mathbb{R}} &= \left(\Delta^+ - \Delta^-\right)\left((\zeta^{-2}\overline{L} + \xi^{-1}L)_{\mathbb{R}}\right)\\ &= \zeta^{1/2}\xi^{1/2}\Lambda_{-1}(\zeta^{-1}\overline{L} + \xi^{-1}L)\end{aligned}$$

Similar calculations give the identities:

$$\begin{aligned}\Delta(\zeta L + \xi\,\overline{L})_{\mathbb{R}} &= \zeta^{1/2}\xi^{1/2}\Lambda(\zeta L + \xi\,\overline{L}) \\ &= \zeta^{-1/2}\xi^{-1/2}\Lambda(\zeta^{-1}\overline{L} + \xi^{-1}L)\end{aligned}$$

Now $(\zeta L + \xi\,\overline{L})_{\mathbb{RC}} = (\zeta L + \xi\,\overline{L}) + (\zeta^{-1}\overline{L} + \xi^{-1}L)$ and so:

$$\Lambda_{-1}(\zeta L + \xi\,\overline{L})_{\mathbb{RC}} = \Lambda_{-1}(\zeta L + \xi\,\overline{L})\Lambda_{-1}(\zeta^{-1}\overline{L} + \xi^{-1}L)$$

and thus, using equation 6.3:

$$\begin{aligned}\frac{(\Delta^+ - \Delta^-)(\zeta L + \xi\,\overline{L})_{\mathbb{R}}}{\Lambda_{-1}(\zeta L + \xi\,\overline{L})_{\mathbb{RC}}} &= \frac{\zeta^{1/2}\xi^{1/2}\Lambda_{-1}(\zeta^{-1}\overline{L} + \xi^{-1}L)}{\Lambda_{-1}(\zeta L + \xi\,\overline{L})\Lambda_{-1}(\zeta^{-1}\overline{L} + \xi^{-1}L)} \\ &= \frac{\zeta^{1/2}\xi^{1/2}}{\Lambda_{-1}(\zeta L + \xi\,\overline{L})} \\ &= \zeta^{1/2}\xi^{1/2}S(\zeta L + \xi\,\overline{L})\end{aligned}$$

Using the splitting principle we extend these results in two ways.

Firstly, suppose that V is an orientable real bundle of even dimension $2l$ and let $U = V_{\mathbb{C}}$. Note that $\overline{U} = U$. Then $w_2(U_{\mathbb{R}}) = w_2(V) + w_2(V) = 0$ so $U_{\mathbb{R}}$ is a spin bundle. Suppose that ξ acts on U by multiplication by ζ. The above identities and the splitting principle give:

$$\frac{(\Delta^+ - \Delta^-)\,(\zeta U)_{\mathbb{R}}}{\Lambda_{-1}(\zeta U)_{\mathbb{RC}}} = \zeta^l S_\zeta U \tag{6.4}$$

$$\Delta(\zeta U)_{\mathbb{R}} = \zeta^{-l}\Lambda_\zeta U \tag{6.5}$$

Secondly, suppose that U is a complex vector bundle of even complex dimension $2r$. The above identities and the splitting principle give:

$$\frac{(\Delta^+ - \Delta^-)\,(\zeta U + \xi\,\overline{U})_{\mathbb{R}}}{\Lambda_{-1}(\zeta U + \xi\,\overline{U})_{\mathbb{RC}}} = \zeta^r \xi^r S(\zeta U + \xi\,\overline{U}) \tag{6.6}$$

6.2 Twisted Dirac Operators and the Witten Genus

A connected finite even dimensional orientable manifold M is a spin manifold if the characteristic class $w_2(M)$ vanishes. In this case, there is a Dirac operator defined over M. The index of this operator is given by the Atiyah-Singer Index theorem. Even if $w_2(M)$ does not vanish, for a real vector bundle V such that $w_2(V) = w_2(M)$, we may construct a twisted Dirac operator which acts on sections of bundles which have the local structure of $\Delta^{\pm}(TM)\Delta(V)$. The restriction $w_2(V) = w_2(M)$ ensures that this bundle is well-defined globally, even though neither factor is globally defined. In the case that $V = TM$ this condition always holds and the twisted Dirac operator coincides with the signature operator.

Now suppose that a group G acts on M. This action induces a fibration:

$$\begin{array}{ccc} M & \longrightarrow & M \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

where BG is the classifying space of G and EG is the total space of the canonical G bundle over BG. For any bundle V over M upon which G acts in a way that preserves the G action on M, we can define a bundle V_G over

$M \times_G EG$ in a natural way by doing the same construction with the total space of V. This bundle restricted to any fibre is V.

We can do this with TM to get a bundle T_G. For any V such that $w_2(V_G) = w_2(T_G)$, the construction of the twisted Dirac operator can be done such that it is equivariant with respect to the natural G action on the bundles. There is then an equivariant version of the index theorem.

To state this theorem, we need some notation. Fix $g \in G$ and let M^g be the fixed point set of g. M^g has even dimension, say $2d$. Let $j : M^g \to M$ be the inclusion of the fixed point set and let N be the normal bundle to this inclusion.

The bundles j^*TM and j^*V decompose according to the fibrewise action of g into bundles TM_θ and V_θ where $\theta \in [0, \pi]$ and g acts on TM_θ and V_θ by rotation by θ. For $\theta \neq 0, \pi$ the g action induces a complex structure on V_θ and TM_θ. Given choices of orientation for TM_0, TM_π, V_0 and V_π, we thus have orientations for j^*TM and j^*V. We introduce a factor ϵ to measure the difference between these orientations and the induced orientations from TM and V.

The index of the twisted Dirac operator is given by the expression:

$$\epsilon(-1)^d \left\langle \frac{\mathrm{ch}\left((\Delta^+ - \Delta^-)(j^*TM)\Delta(j^*V)\right)\mathrm{td}(T_{\mathbb{C}}M^g)}{\mathrm{e}(TM^g)\,\mathrm{ch}(\Lambda_{-1}N_{\mathbb{C}})}, [M^g] \right\rangle$$

This expression can be simplified by extracting the parts of j^*TM and j^*V upon which g acts trivially. In the above notation, these are TM_0 and V_0. Clearly $TM_0 = TM^g$ and thus the remaining part of j^*TM is the normal bundle N. Let V^g be the remaining part of j^*V, i.e $j^*V = V_0 \oplus V^g$.

Substituting this in gives:

$$\epsilon(-1)^d \left\langle \frac{\operatorname{ch}\left((\Delta^+ - \Delta^-)(TM^g)\Delta(V_0)\right)\operatorname{td}(T_{\mathbb{C}}M^g)}{e(TM^g)} \frac{\operatorname{ch}\left((\Delta^+ - \Delta^-)(N)\Delta(V^g)\right)}{\operatorname{ch}(\Lambda_{-1}N_{\mathbb{C}})}, [M^g] \right\rangle$$

The first part of this is $\widehat{\mathcal{A}}(TM^g, V_0)$. In the cases we shall be considering, N and V^g will be spin bundles and thus the bundles $(\Delta^+ - \Delta^-)(N)$ and $\Delta(V^g)$ are defined globally, thus we shall rewrite the second part in a form that will be particularly useful in calculations:

$$\operatorname{Index}_g D_v = \epsilon(-1)^d \left\langle \widehat{\mathcal{A}}(TM^g, V_0) \operatorname{ch}\left(\frac{(\Delta^+ - \Delta^-)(N)}{\Lambda_{-1}N_{\mathbb{C}}} \Delta(V^g) \right), [M^g] \right\rangle \tag{6.7}$$

If V is the zero bundle, this simplifies to the untwisted version:

$$\operatorname{Index}_g D_v = \epsilon(-1)^d \left\langle \widehat{\mathcal{A}}(TM^g) \operatorname{ch}\left(\frac{(\Delta^+ - \Delta^-)(N)}{\Lambda_{-1}N_{\mathbb{C}}} \right), [M^g] \right\rangle \tag{6.8}$$

In [22], Witten applies this calculation to the loop space of a manifold M of even dimension $2d$ with the natural circle action. In his calculation, he decomposes the tangent bundle of the loop space at the fixed point manifold as representations of the circle. This decomposition suggests that the calculation is of the index of an operator acting on a finite energy subspace rather than the full vector space.

Given a vector bundle V over M of even dimension $2n$, he defines the infinite dimensional vector bundle $\mathcal{V}$ over LM as follows: given a loop γ :

$S^1 \to M$, let $\mathcal{V}_\gamma = \Gamma(S^1, \gamma^* V)$. This construction applied to TM yields the tangent space of LM. The compatibility condition is now that $w_2(V) = w_2(TM)$ and $\frac{1}{2}p_1(V) = \frac{1}{2}p_1(TM)$.

Witten calculates the index of the Dirac operator on LM twisted by $\Delta(\mathcal{V})$ to be the power series in ξ given by:

$$W(M,V)(\xi) = \xi^{\frac{n-d}{12}} \left\langle \widehat{A}(TM,V)\,\mathrm{ch}\left(\bigotimes_{k=1}^{\infty} S_{\xi^k} T_{\mathbb{C}} M \bigotimes_{l=1}^{\infty} \Lambda_{\xi^l} V_{\mathbb{C}}\right), [M] \right\rangle$$

In this calculation, Witten uses a renormalisation of the infinite product $\prod_{k=1}^{\infty} \xi^k$ as $\xi^{-1/12}$ which comes from considering the Riemann zeta function at $z = -1$.

The case when V is trivial is called the *Witten genus* of M. It is given by the power series:

$$W(M)(\xi) = \xi^{-d/12} \left\langle \widehat{A}(TM)\,\mathrm{ch}\left(\bigotimes_{k=1}^{\infty} S_{\xi^k} T_{\mathbb{C}} M\right), [M] \right\rangle$$

6.3 The Dirac Operator on Product Space

Let M be a closed orientable manifold. Let $n \in \mathbb{N}$. We define M^n as the n-fold product of M. That is:

$$M^n := \overbrace{M \times M \times \cdots \times M}^{n}$$

Let Δ^r be the r-simplex considered as the space of $r+1$ ordered points on the circle starting with 1. There is an evaluation map $\Delta^{n-1} \times LM \to M^n$. This gives a map $LM \to \mathrm{Map}(\Delta^{n-1} \times M^n)$ and thus a map $LM \to$

$\varprojlim \mathrm{Map}(\Delta^{n-1} \times M^n)$. Let ξ be a primitive nth root of unity. The ordered set $(1, \xi, \ldots, \xi^{n-1})$ is an element of Δ^{n-1} and evaluation at this point gives a map $LM \to M^n$:

$$\phi_\xi(\gamma) = \big(\gamma(1), \gamma(\xi), \gamma(\xi^2), \ldots, \gamma(\xi^{n-1})\big)$$

The group C_n acts on LM as the subgroup of S^1 consisting of the elements $\{1, \xi, \xi^2, \ldots, \xi^{n-1}\}$. This action preserves the set of chosen points of S^1 and thus projects down to an action on M^n given by cyclic permutation of coordinates:

$$\xi(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$$

The fixed point set of the C_n action is a copy of M embedded in M^n as the diagonal. Let $j : M \to M^n$ be this embedding.

Assume that $n = 2p+1$ is odd and that M is of even dimension $2d$. Let E be an oriented real vector bundle of even dimension $2r$ over M. E^n is a vector bundle over M^n isomorphic to the bundle $\bigoplus_{k=1}^n \pi_k^* E$ where $\pi_k : M^n \to M$ is the projection onto the kth factor. The group action of C_n on M^n extends to cyclic permutations of the factors in this splitting. This construction applied to TM results in the tangent bundle TM^n.

Since C_n acts on $j^* E^n$ fibrewise there is a global splitting of $j^* E^n$ into real representations of C_n. Let $\theta \in (0, 2\pi)$ be such that $\xi = e^{i\theta}$. We have:

$$j^* E^n = E_0 + E_1 + \cdots + E_p$$

where E_k corresponds to the real representation where ξ acts as rotation through angle $k\theta$.

There is an injective map $\psi_0 : E \to E_0$ given by $\psi_0(e) = (e, \dots, e)$. This is clearly surjective as well. Thus E_0 is isomorphic to E.

For $k \neq 0$, we can define injective maps $\psi_k : E + E \to E_k$ as follows:

$$\begin{aligned}\psi_k(e_1, e_2) = ((e_1 + e_2, (\cos k\theta + \sin k\theta)e_1 + (\cos k\theta - \sin k\theta)e_2,\\ (\cos 2k\theta + \sin 2k\theta)e_1 + (\cos 2k\theta - \sin 2k\theta)e_2, \dots,\\ (\cos(n-1)k\theta + \sin(n-1)k\theta)e_1 + (\cos(n-1)k\theta - \sin(n-1)k\theta)e_2))\end{aligned}$$

Using a dimension count, we can see that each of these must be surjective. Thus $E_k = E + E$. It is a simple calculation to show that since E is even dimensional, the orientation on the bundle E^n given by this decomposition coincides with that given by the decomposition $\bigoplus_{k=1}^n \pi_k^* E$.

The action of ξ on E_k induces a natural complex structure with respect to which the ξ action becomes multiplication by ξ^k. On $E + E$, this complex structure is given by $J(e_1, e_2) = (-e_2, e_1)$. As a complex bundle, this is just $E_{\mathbb{C}}$. Thus $E_k = (\xi^k E_{\mathbb{C}})_{\mathbb{R}}^{r*}$ and so when considering orientations for inclusion into the Atiyah-Singer index theorem, each E_k contributes $(-1)^r$ to the value of ϵ and hence E contributes $(-1)^{rp}$ to the value of ϵ.

Now let V be an orientable real vector bundle over M. Suppose that both V and M are even dimensional. We wish to consider the C_n-equivariant Dirac operator on M^n twisted by $\Delta(V^n)$. From section 6.2, we know that this is can be constructed when $w_2(V^n_{C_n}) = w_2(T^n_{C_n})$ where these bundles arise from the fibration:

$$\begin{array}{ccc} M^n & \longrightarrow & M^n \times_{C_n} S^\infty \\ & & \downarrow \\ & & L_n \end{array}$$

where C_n acts on S^∞ as a subgroup of the circle and L_n is the Lens space S^∞/C_n.

This fibration induces an exact sequence in $\mathbb{Z}_2$ cohomology. The part relevant to our calculation is:

$$H^2(L_n;\mathbb{Z}_2) \xrightarrow{\alpha} H^2(M^n \times_{C_n} S^\infty;\mathbb{Z}_2) \xrightarrow{\beta} H^2(M^n;\mathbb{Z}_2) \xrightarrow{\gamma} H^3(L_n;\mathbb{Z}_2)$$

Since n is odd, the $\mathbb{Z}_2$-cohomology of the Lens space is $\mathbb{Z}_2$ in degree 0 and 0 elsewhere. Thus β is an isomorphism.

The bundles $V^n_{C_n}$ and $T^n_{C_n}$ have the property that when restricted to the fibre they are V^n and T^n respectively. Thus $\beta w_2(V^n_{C_n}) = w_2(V^n)$ and similarly for $T^n_{C_n}$. Hence $w_2(V^n_{C_n}) = w_2(T^n_{C_n})$ if and only if $w_2(V^n) = w_2(TM^n)$. As V is orientable, $w_2(V^n) = \sum \pi_k^* w_2(V)$. As TM is orientable, similarly $w_2(T^n) = \sum \pi_k^* w_2(TM)$. As each π_k^* is injective, $w_2(V^n) = w_2(TM^n)$ if and only if $w_2(V) = w_2(TM)$. Thus provided $w_2(V) = w_2(M)$, we can define the twisted Dirac operator on M^n and compute its index.

From the above, we have decompositions of N and V_ξ as:

$$N = (\xi T_{\mathbb{C}} M)_{\mathbb{R}} + \dots (\xi^p T_{\mathbb{C}} M)_{\mathbb{R}}$$

$$V_\xi = (\xi V_{\mathbb{C}})_{\mathbb{R}} + \dots (\xi^p V_{\mathbb{C}})_{\mathbb{R}}$$

Using equation 6.4 and equation 6.5, we have:

$$\frac{(\Delta^+ - \Delta^-)(N)}{\Lambda_{-1} N_{\mathbb{C}}} = \bigotimes_{k=1}^{p} \xi^{kd} S_{\xi^k} T_{\mathbb{C}} M$$
$$\Delta(V_\xi) = \bigotimes_{k=1}^{p} \xi^{-kr} \Lambda_{\xi^k} V_{\mathbb{C}}$$

Substituting these into equation 6.7 shows that the index of the twisted Dirac operator is given by the expression:

$$(-1)^{(r+d)p+d} \left\langle \widehat{\mathcal{A}}(TM, V) \operatorname{ch} \left(\bigotimes_{k=1}^{p} \xi^{kd} S_{\xi^k} T_{\mathbb{C}} M \bigotimes_{k=1}^{p} \xi^{-kr} \Lambda_{\xi^k} V_{\mathbb{C}} \right), [M] \right\rangle$$

and thus:

$$\operatorname{Index}_\xi D_V = (-1)^{(r+d)p+d} W_m(M, V)(\xi)$$

6.4 The Dirac Operator on Projective Space

For $a, b \in \mathbb{Z}$ with $a \leq b$, let $Y_a^b := \mathbb{C}[z, z^{-1}]_a^b$ be the space of Laurent polynomials in z whose terms have degree between and including a and b. Let $X_a^b = \mathbb{P}(\mathbb{C}^{n+1} \otimes Y_a^b)$.

On the space $L_{alg}(\mathbb{C}^{n+1} \smallsetminus 0)$ we define an action of the group $L_{alg}(\mathbb{C} \smallsetminus 0)$ by pointwise multiplication. The quotient space of this action is precisely $L_{alg}(\mathbb{CP}^n)$, the space of algebraic loops on $\mathbb{CP}^n$.

We can factor this action through the action of $\mathbb{C} \smallsetminus 0$ also acting by pointwise multiplication and thus the algebraic loop space $L_{alg}\mathbb{CP}^n$ is a quotient space of the projective space $\mathbb{P}L_{alg}(\mathbb{C}^{n+1} \smallsetminus 0)$.

This gives the following commutative diagram of smooth maps:

$$\begin{tikzcd} L_{alg}(\mathbb{C}^{n+1} \setminus 0) \arrow[r, "\mathbb{C}\setminus 0"] & \mathbb{P}L_{alg}(\mathbb{C}^{n+1} \setminus 0) \arrow[r, "L_{alg}(\mathbb{C}\setminus 0)"] & L_{alg}\mathbb{CP}^n \\ \mathbb{C}^{n+1} \otimes Y_a^b \arrow[u] \arrow[r, "\mathbb{C}\setminus 0"] & X_a^b \arrow[u] & \end{tikzcd}$$

where the upward vertical maps are the inclusion maps. Standard results on Fourier series show that the unions over all a and b of $\mathbb{C}^{n+1} \otimes Y_a^b$ and X_a^b embedded in their relevant spaces are dense in those spaces.

We have a circle action on the top row of the diagram defined by rotating loops. Under the vertical inclusions, this action preserves the spaces $\mathbb{C}^{n+1} \otimes Y_a^b$ and X_a^b and corresponds to the action $\xi \cdot z^r \to \xi^r z^r$ on Y_a^b.

From section 6.2, we know that we can construct a circle equivariant Dirac operator over X_a^b if and only if $w_2(T_{S^1})$ vanishes, where T_{S^1} is the tangent bundle to the fibres of the fibration:

$$\begin{tikzcd} X_a^b \arrow[r] & X_a^b \times_{S^1} S^\infty \arrow[d] \\ & \mathbb{CP}^\infty \end{tikzcd}$$

This fibration induces an exact sequence in $\mathbb{Z}_2$ cohomology. The part relevant to our calculation is:

$$\begin{aligned} H^1(X_a^b; \mathbb{Z}_2) \to H^2(\mathbb{CP}^\infty; \mathbb{Z}_2) \to H^2(X_a^b \times_{S^1} S^\infty; \mathbb{Z}_2) \\ \to H^2(X_a^b; \mathbb{Z}_2) \to H^3(\mathbb{CP}^\infty; \mathbb{Z}_2) \end{aligned}$$

Since $X_a^b = \mathbb{CP}^{(n+1)(b-a+1)-1}$, we have:

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\alpha} H^2(X_a^b \times_{S^1} S^\infty; \mathbb{Z}_2) \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0$$

and thus $H^2(X_a^b \times_{S^1} S^\infty; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We wish to know when $w_2(T_{S^1}) = 0$. Let $m \in \mathbb{Z}$ be such that $a \leq m \leq b$. Consider $\mathbb{CP}^n$ embedded in X_a^b as the space of homogeneous polynomials of degree m. The circle action on this copy of $\mathbb{CP}^n$ is trivial and thus we have an inclusion $\mathbb{CP}^n \times \mathbb{CP}^\infty \to X_a^b \times_{S^1} S^\infty$ which is the identity on $\mathbb{CP}^\infty$.

The fibration $\mathbb{CP}^n \to \mathbb{CP}^n \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a similar exact sequence to that above:

$$0 \to \mathbb{Z}_2 \to H^2(\mathbb{CP}^n \times \mathbb{CP}^\infty; \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0$$

and the inclusion $\mathbb{CP}^n \times \mathbb{CP}^\infty \to X_a^b \times_{S^1} S^\infty$ induces a map between these exact sequences. This map is clearly an isomorphism on all but the middle group and hence by the five lemma is also an isomorphism on that group.

Given an inclusion $j : \mathbb{CP}^r \to \mathbb{CP}^s$ with $r \leq s$, $j^*T\mathbb{CP}^s = T\mathbb{CP}^r + (r-s)\overline{\gamma}$ where γ is the canonical bundle over $\mathbb{CP}^r$. Thus T_{S^1} pulls back to $T\mathbb{CP}^n + (n+1)(b-a)\overline{\gamma}$ and so $w_2(T_{S^1})$ pulls back to $w_2(T\mathbb{CP}^n) + (n+1)(b-a)w_2(\overline{\gamma}) = (n+1)(b-a+1)w_2(\overline{\gamma})$. Since $w_2(\gamma)$ generates $H^2(\mathbb{CP}^n; \mathbb{Z}_2)$, this is zero if and only if $(n+1)(b-a+1)$ is even.

Thus we can construct the equivariant spin bundle over X_a^b if and only if $(n+1)(b-a+1)$ is even. This is the same as the condition that X_a^b be a spin manifold. This condition is satisfied for all a, b if n is odd and this is the condition that $\mathbb{CP}^n$ be a spin manifold. Thus we assume that n is odd.

The fixed point set of X_a^b under the action of S^1 is the set of homogeneous polynomials. That is, defining:

$$X_m = \{[c_0 z^m, c_2 z^m, \ldots, c_n z^m],\, c_j \in \mathbb{C} \text{ not all zero}\}$$

then the fixed point set is the disjoint union: $X^{S^1} := \bigcup_{m=a}^{b} X_m$. Clearly each $X_m \cong \mathbb{CP}^n$.

Let $\xi \in S^1$ be of either infinite order or finite order greater than $b - a$, so $X^\xi = X^{S^1}$. Let $m \in \mathbb{Z}$ be such that $a \leq m \leq b$. Let $j_m : \mathbb{CP}^n \to X_a^b$ be the inclusion map of $\mathbb{CP}^n$ with image X_m and let N be the normal bundle to this inclusion, so $j_m^* T X_a^b = T\mathbb{CP}^n + N$. Let γ be the canonical line bundle over $\mathbb{CP}^r$. Then $j^*\gamma = \gamma$ where γ on the left hand side is over X_a^b and on the right hand side over $\mathbb{CP}^n$. Since for a projective space $\mathbb{CP}^r$ we have $T\mathbb{CP}^r + \mathbb{C} = (r+1)\overline{\gamma}$, and thus:

$$\begin{aligned} j_m^*(TX_a^b + \mathbb{C}) &= T\mathbb{CP}^n + \mathbb{C} + N \\ j_m^*(n+1)(b-a+1)\overline{\gamma} &= (n+1)\overline{\gamma} + N \\ (n+1)(b-a+1)\overline{\gamma} &= (n+1)\overline{\gamma} + N \end{aligned}$$

Thus $N = (b-a)(n+1)\overline{\gamma}$. Let $\Gamma = (n+1)\overline{\gamma}$. The rth copy of Γ with its natural complex structure corresponds to curves of the form:

$$\alpha_{rs}(t) = [(c_0 z^m, \ldots, c_n z^m) + (0, \ldots, 0, tz^r, 0, \ldots, 0)]$$

where the tz^r occurs in the sth place. ξ acts via:

$$\begin{aligned}\xi\alpha_{rs}(t) &= [(c_0\xi^m z^m, \dots, c_n\xi^m z^m) + (0, \dots, 0, t\xi^r z^r, 0, \dots, 0)]\\ &= [(c_0 z^m, \dots, c_n z^m) + (0, \dots, 0, t\xi^{r-m} z^r, 0, \dots, 0)]\end{aligned}$$

so ξ acts on the rth copy of Γ as ξ^{r-m}. Thus:

$$N = \sum_{\substack{s=a\\ s\neq m}}^{s} \left(\xi^{s-m}\Gamma\right)_{\mathbb{R}}$$

Assume that $b-a$ is even and let $2q = b-a$. Let $r = b-q = q-a$ be the midpoint of $[a,b]$. Let $k = m-r$ then $a \le m \le b$ implies that $-q \le k \le q$. Recall that $(\xi^s\Gamma)_{\mathbb{R}} = (\xi^{-s}\overline{\Gamma})_{\mathbb{R}}$. Thus:

$$N + \Gamma_{\mathbb{R}} = \sum_{s=a}^{b} (\xi^{s-m}\Gamma)_{\mathbb{R}} = \sum_{s=1}^{q} \left(\xi^{k+s}\Gamma + \xi^{-k+s}\overline{\Gamma}\right)_{\mathbb{R}} + \left(\xi^k\Gamma\right)_{\mathbb{R}}$$

Thus by equation 6.6 and the fact that $(\xi^k\Gamma_{\mathbb{R}})_{\mathbb{C}} = \xi^k\Gamma + \xi^{-k}\overline{\Gamma}$:

$$\begin{aligned}\frac{(\Delta^+ - \Delta^-)(N + \Gamma_{\mathbb{R}})}{\Lambda_{-1}(N+\Gamma_{\mathbb{R}})_{\mathbb{C}}} &= \bigotimes_{s=1}^{q} \left(\xi^{(n+1)(k+s)/2}\xi^{(n+1)(-k+s)/2} S(\xi^{k+s}\Gamma + \xi^{-k+s}\,\overline{\Gamma})\right)\\ &\quad \otimes \frac{(\Delta^+ - \Delta^-)(\xi^k\Gamma)_{\mathbb{R}}}{\Lambda_{-1}(\xi^k\Gamma)_{\mathbb{R}\mathbb{C}}}\\ &= \bigotimes_{s=1}^{q} \left(\xi^{(n+1)s} S_{\xi^s}(\xi^k\Gamma_{\mathbb{R}})_{\mathbb{C}}\right) \otimes \frac{(\Delta^+ - \Delta^-)(\xi^k\Gamma)_{\mathbb{R}}}{\Lambda_{-1}(\xi^k\Gamma)_{\mathbb{R}\mathbb{C}}}\end{aligned}$$

Using equation 6.1 and equation 6.2:

$$\begin{aligned}
\widehat{\mathcal{A}}(\Gamma_{\mathbb{R}})\,\mathrm{ch}\left(\frac{(\Delta^+-\Delta^-)(N)}{\Lambda_{-1}N_{\mathbb{C}}}\right) &= -e(\Gamma_{\mathbb{R}})\,\mathrm{ch}\left(\frac{(\Delta^+-\Delta^-)(N+\Gamma_{\mathbb{R}})}{\Lambda_{-1}(N+\Gamma_{\mathbb{R}})_{\mathbb{C}}}\right)\\
&= -e(\Gamma_{\mathbb{R}})\,\mathrm{ch}\left(\frac{(\Delta^+-\Delta^-)(\xi^k\Gamma_{\mathbb{R}})}{\Lambda_{-1}(\xi^k\Gamma_{\mathbb{R}})_{\mathbb{C}}}\right)\\
&\qquad \mathrm{ch}\bigotimes_{s=1}^{q}\left(\xi^{(n+1)s}S_{\xi^s}(\xi^k\Gamma_{\mathbb{R}})_{\mathbb{C}}\right)\\
&= \widehat{\mathcal{A}}(\xi^k\Gamma_{\mathbb{R}})\,\mathrm{ch}\bigotimes_{s=1}^{q}\left(\xi^{(n+1)s}S_{\xi^s}(\xi^k\Gamma_{\mathbb{R}})_{\mathbb{C}}\right)\\
&= W_{k,q}(\Gamma_{\mathbb{R}})(\xi)
\end{aligned}$$

Since $\widehat{\mathcal{A}}$ is a stable characteristic class, $\widehat{\mathcal{A}}(T\mathbb{CP}^n) = \widehat{\mathcal{A}}(\Gamma_{\mathbb{R}})$ and hence:

$$\widehat{\mathcal{A}}(T\mathbb{CP}^n)\,\mathrm{ch}\left(\frac{(\Delta^+-\Delta^-)(N)}{\Lambda_{-1}N_{\mathbb{C}}}\right) = W_{k,q}(\Gamma_{\mathbb{R}})(\xi)$$

Substituting this into equation 6.8 and using the equality $\Gamma_{\mathbb{R}} = T\mathbb{CP}^n + \mathbb{R}^2$ gives the index of the Dirac operator as:

$$\mathrm{Index}_\xi D = -\sum_{k=-q}^{q}\left\langle W_{k,q}(T\mathbb{CP}^n+\mathbb{R}^2)(\xi), [\mathbb{CP}^n]\right\rangle$$

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