Some problems in irregular ordinary differential equations

by

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Contents

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Declarations

I declare that the work in this thesis was carried out in accordance with the Regulations of the University of Warwick. The work is original except where indicated by special reference in the text and no part of the thesis has been submitted for any other degree.

Abstract

We study the non-autonomous ordinary differential equation $\dot{x} = f(t,x)$ in the situation when the vector field f is of limited regularity, typically belonging to a space $L^p(0,T;L^q(\mathbb{R}^n))$. Such equations arise naturally when switching from an Eulerian to a Lagrangian viewpoint for the solutions of partial differential equations. We discuss some measurability issues in the foundations of the theory of regular Lagrangian flow solutions. Further, we examine the sensitivity of the choice of representative vector field f on solutions of the ordinary differential equation and, in particular, we demonstrate that every vector field can be altered on a set of measure zero to introduce non-uniqueness of solutions.

We develop some geometric tools to quantify the behaviour of solutions, notably a non-autonomous version of subset avoidance and the r-codimension print that encodes the dimension of a subset $S \subset \mathbb{R}^n \times [0,T]$ while distinguishing between the spatial and temporal detail of S. We relate this notion of dimension to the more familiar box-counting dimensions, for which we prove some new inequalities.

Finally, motivated by the issues with measurability that can arise with irregular vector fields we prove some fundamental results in the theory of Bochner integration in order to be able to manipulate the representatives of the equivalence classes in $L^p(0,T;L^q(\mathbb{R}^n))$.

Chapter 1

Introduction

This thesis is concerned with the study of the non-autonomous ordinary differential equation

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = f(\xi(t), t) \tag{1.1}$$

when the vector field $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is not necessarily continuous but is measurable and sufficiently integrable to belong to the space $L^p(0,T;L^q(\mathbb{R}^n))$ for some $1 \le p, q \le \infty$. Such problems arise naturally when switching from an Eulerian to a Lagrangian viewpoint for the solutions of partial differential equations which can have only very limited regularity, such as the Navier-Stokes equations.

In Chapter 2 we discuss the foundations of the theory of irregular ordinary differential equations. We recall some elements of the classical theory, where typically the vector field f is continuous and the objects of study are classical flow solutions of (1.1). In this familiar setting we introduce the concepts of solution concatenation and the avoidance of sets by classical flow solutions, which we develop for irregular ordinary differential equations in Chapters 3 and 4. In Section 2.2 we follow the seminal paper of DiPerna and Lions [1989] and the refinements in Ambrosio [2004] and De Lellis [2008] to motivate and discuss the appropriate notion of solution of (1.1) when the vector field is merely integrable. In particular we note that a classical flow solution is too strong to be of use for two reasons: firstly, a classical solution ξ requires the vector field f to be continuous on the trajectory $f(\xi(t), t)$, which is a strong restriction for vector fields that are only assumed to be integrable. Secondly, we recall that the elements of the space $L^p(0,T;L^q(\mathbb{R}^n))$ are equivalence classes of maps that are equal almost everywhere. Consequently, in order to be able to use the tools of functional analysis to find solutions of (1.1) we require that solutions are invariant under a choice of representative of the equivalence class of f.

An appropriate notion of solution of (1.1) in the irregular setting, identified in DiPerna and Lions [1989] and developed in Lions [1998], Ambrosio [2004], Hauray et al. [2007], De Lellis [2008] and others, is that of a regular Lagrangian flow solution, which we detail in Definition 2.32. Roughly, a regular Lagrangian flow solution is an integrable map $X\colon\ [0,T]\times\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ satisfying

$$\int_0^T X\left(t,x,s\right)\frac{\mathrm{d}\phi}{\mathrm{d}t}\left(t\right)\mathrm{d}t+\int_0^T f\left(X\left(t,x,s\right),t\right)\phi\left(t\right)\mathrm{d}t=0$$

for all test maps $\phi\in C_c^\infty\left(\left(0,T\right)\right)$, for almost every $\left(x,s\right)\in\mathbb{R}^n\times[0,T]$ with the additional Lusin property that for every Borel set $A\subset\mathbb{R}^n$ of zero measure the set $\left\{x\in\mathbb{R}^n|X\left(t,x,s\right)\in A\right\}$ has zero measure for almost every $t\in[0,T]$ and almost every $s\in[0,T]$. Essentially, this property guarantees that spatial sets of positive measure are not transported under the action of the flow into sets of zero measure. In the example of Section 2.1.5 we describe a classical flow solution that does not have the Lusin property, illustrating that regular Lagrangian flows are not strictly a generalisation of classical flows.

In DiPerna and Lions [1989] the authors demonstrate that a regular Lagrangian flow solution X necessarily has some Sobolev regularity with respect to time. Further, as one dimensional maps with Sobolev regularity have absolutely continuous representatives the authors conclude that there is a representative of X that is absolutely continuous in time, simplifying much of the theory. In Section 2.4 we describe this argument and examine a potential obstruction: we claim that if X is a regular Lagrangian flow solution of (1.1) then there is a map $\tilde{X}\colon\ [0,T]\times\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ such that

- for almost every $\left(x,s\right)\in\mathbb{R}^n\times[0,T]$ the map $t\mapsto\tilde{X}\left(t,x,s\right)$ is absolutely continuous, and
- $\tilde{X}\left(t,x,s\right)=X\left(t,x,s\right)$ for almost every $t\in[0,T]$ for almost every $\left(x,s\right)\in\mathbb{R}^n\times[0,T]$

however, we argue that it does not follow that the map $\tilde{X}$ is equal to X almost everywhere on $[0,T]\times\mathbb{R}^n\times[0,T]$. In particular, the map $\tilde{X}$ may not be measurable, in which case we are unable to consider the measure of the inverse images $\tilde{X}^{-1}\left(t,\cdot,s\right)A$ which are significant to the theory of regular Lagrangian flows. Fortunately, in Chapter 8 which was completetd after the submission of this thesis, we are able to demonstrate that the map $\tilde{X}$ is measurable. We end the chapter by discussing a similar problem in a classical result from the theory of Sobolev maps, and how adapting the proof of this result may remove the obstruction to the theory of regular Lagrangian flows. We discuss the sensitivity of choosing representatives of equivalence classes in a more general setting and at greater length in Chapter 7.

In the remainder we use the result of Chapter 8 that guarantees there is a measurable absolutely continuous representative of each regular Lagrangian flow and we restrict ourselves to this representative. With the additional regularity of this representative a map X is a regular Lagrangian flow solution of (1.1) if it sastisfies the Lusin condition and for almost every $(x, s) \in \mathbb{R}^n \times [0, T]$

$$X(t, x, s) = x + \int_s^t f(X(\tau, x, s), \tau)\, d\tau \quad \forall t \in [0, T],$$

which is the definition used in Lions [1998], Hauray et al. [2007], Crippa and De Lellis [2008], and others. With this formulation we can regard the map X as an aggregate of absolutely continuous solutions, with one solution for almost all initial data.

In Chapter 3 we discuss the uniqueness of absolutely continuous solutions of (1.1). We revisit the technique of solution concatenation and demonstrate in Theorem 3.1 that the concatenation of any absolutely continuous solutions is itself a solution. As a consequence, we see that examples of vector fields with non-unique solutions are easy to construct: we give two examples of vector fields with arbitrary Sobolev regularity (that is in $W^{k,\infty}(\mathbb{R}^n; \mathbb{R}^n)$ for all $k \in \mathbb{N}$) such that there are non-unique solutions for all initial data.

In Theorem 3.4, which is the main result of Chapter 3, we illustrate that the uniqueness of solutions is sensitive to the choice of representative vector field f. We prove that if there exists a regular Lagrangian flow solution of (1.1) then there is a vector field equivalent to g such that the ordinary differential equation $\dot{\xi} = g(\xi, t)$ has non-unique solutions on a set of initial data of positive measure. We end the chapter by demonstrating that if (1.1) has a regular Lagrangian flow solution X and unique solutions for almost all initial data, and if the set

$$N := \{(x, t) \in \mathbb{R}^n \times [0, T] \,|\, f(x, t) \neq g(x, t)\}$$

is sufficiently small that the flow X 'avoids' N, then the solutions of $\dot{\xi} = g(\xi, t)$ are unique for almost all initial data. An article containing the discussion and results of Chapter 3, co-authored with James Robinson, is currently in preparation.

We discuss the avoidance of subsets at length in Chapter 4 where, in Theorem 4.8, we adapt a result of Aizenman [1978b] to the non-autonomous case to give a sufficient condition for a regular Lagrangian flow to avoid a subset $S \subset \mathbb{R}^n \times [0,T]$. This condition is written in terms of both the spatial and the temporal regularity of the vector field f and an integral quantity dependent on the set S. In Chapter 5 we use this integral quantity to define a two-parameter 'r-codimension print', similar to the Hausdorff dimension print of Rogers [1988], which encodes the 'dimension' of the set in such a way that the temporal detail is distinguished from the spatial detail. In Theorem 5.13 we partially describe the r-codimension print of S in terms of the box-counting dimensions of the projections of S onto the coordinate axes. These results on non-autonomous avoidance and the r-codimension print are described in an article, co-authored with James Robinson, that is currently under review for publication in the *Journal of Differential Equations.*

Theorem 5.13 gives a partial description of the r-codimension print of a subset S, in terms of the upper and lower box-counting dimensions of its projections. In order to obtain the sharpest results for this Theorem, we need the best possible control of the upper and lower box counting dimensions of the set S and of its projections. In Chapter 6 we recall the definition of the box-counting dimension in a general metric space and in Theorem 6.8 we prove that for compact subsets $F, G \in \mathbb{R}^n$ the upper and lower box-counting dimensions of the product set $F \times G$ satisfy

$$\begin{aligned}
\dim_{LB}(F) + \dim_{LB}(G) &\leq \dim_{LB}(F \times G) \\
&\leq \min\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\
&\leq \max\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\
&\leq \dim_B(F \times G) \\
&\leq \dim_B(F) + \dim_B(G).
\end{aligned}$$

As far as we are aware the second and fourth inequalities are new. In the second half of Chapter 6, we provide a method of constructing 'compatible generalised Cantor sets' $F, G \subset \mathbb{R}$ such that the upper and lower box-counting dimensions of F, G and the product set $F \times G$ take arbitrary values subject to the above inequalites. The results in this chapter are described in an article, co-authored with James Robinson, that has been accepted for publication in *Real Analysis Exchange.*

Throughout this thesis we avoid the abuse of notation in which we would write $f \in L^1(\mathbb{R}^n)$ for a map f as properly the elements of this space are equivalence classes of maps. We write $f \in \mathcal{L}^1(\mathbb{R}^n)$ if the map $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable and the integral $\int_{\mathbb{R}^n} |f(x)|\, dx$ is finite. We observe that the space $\mathcal{L}^1(\mathbb{R}^n)$ is only equipped with a semi-norm, which we denote by $\|\cdot\|_{\mathcal{L}^1(\mathbb{R}^n)}$. In Chapter 7 we discuss Bochner integration with a view to defining and manipulating elements of the spaces $L^p(0,T;L^q(\mathbb{R}^n))$, which feature prominently in the theory of irregular non-autonomous ordinary differential equations. An important result of Chapter 7 is that the space $L^1\left(0,T;L^1(\mathbb{R}^n)\right)$, which consists of equivalence classes of equivalence-class-valued maps, is isometrically isomorphic to a space $\mathcal{L}^1\left(0,T;\mathcal{L}^1(\mathbb{R}^n)\right)$ of real valued maps modulo the equivalence relation defined by $f \sim g$ iff $f(t,x) = g(t,x)$ for almost every $x \in \mathbb{R}^n$, for almost every $t \in [0,T]$ is a Banach space. With this characterisation we can regard a real valued map $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ as being a representative of an equivalence class of $L^1\left(0,T;L^1(\mathbb{R}^n)\right)$.

In Section 7.2.2 we illustrate an important consequence of dealing with maps rather than equivalence classes: even though the spaces $L^1((0,T)\times\mathbb{R}^n)$ and $L^1\left(0,T;L^1(\mathbb{R}^n)\right)$ are isometrically isomorphic the inclusion

$$\mathcal{L}^1((0,T)\times\mathbb{R}^n) \subset \mathcal{L}^1\left(0,T;\mathcal{L}^1(\mathbb{R}^n)\right)$$

is strict. Indeed, in Corollary 7.13 we demonstrate that the indicator map of a set first described in Sierpiński [1920] is in $\mathcal{L}^1\left(0,T;\mathcal{L}^1(\mathbb{R}^n)\right)$ but is not measurable as a map from $[0,T]\times\mathbb{R}^n \to \mathbb{R}$. In particular, if $f(x,t) = g(x,t)$ for almost every $x \in \mathbb{R}^n$, for almost every $t \in [0,T]$ then it is not necessarily the case that $f(x,t) = g(x,t)$ almost everywhere on $\mathbb{R}^n \times [0,T]$.

In general, manipulating the 'almost everywhere' quantifier of measure theory requires caution, as we discuss in Appendix A. In this appendix we demonstrate that the only implication between statements with almost everywhere quantifiers is

$$P(x,y) \quad \forall\, x, \quad \text{a.e. } y \qquad \Rightarrow \qquad P(x,y) \quad \text{a.e. } y, \quad \forall\, x,$$

and in fact the validity of the implication

$$P(x,y) \quad \text{a.e. } x, \quad \text{a.e. } y \qquad \Leftrightarrow \qquad P(x,y) \quad \text{a.e. } y, \quad \text{a.e. } x,$$

depends upon our choice of axioms. To avoid these difficulties whenever we manipulate these quantifiers we will do so explicitly.

The spaces $L^p(0,T;L^p(\mathbb{R}^n))$ for $1 \le p < \infty$ similarly are isometrically isomorphic to $L^p((0,T)\times\mathbb{R}^n)$ but this is not true of $L^\infty(0,T;L^\infty(\mathbb{R}^n))$. In Section 7.4 we describe a simple example of a map f, due to Juan Arias de Reyna (University of Seville), that is in $\mathcal{L}^\infty([0,1]\times[0,1])$ but not in $\mathcal{L}^\infty(0,1;\mathcal{L}^\infty(0,1))$. Further, we show that no map that is equal to f almost everywhere on $[0,1]\times[0,1]$ is in both spaces. We use this example to show that there is an isomorphism between $L^\infty(0,T;L^\infty(\mathbb{R}^n))$ and a proper subspace of $\mathcal{L}^\infty([0,T]\times\mathbb{R}^n)$.

An article containing the discussion and results of Chapter 7 co-authored with James Robinson and José Real (University of Seville) is currently in preparation. Both James and I were saddened to learn of the death of José Real on the 27^{th} of January 2012.

Chapter 2

Ordinary Differential Equations

Our interest is in the non-autonomous ordinary differential equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = f\left(\xi\left(t\right), t\right) \tag{ODE}$$

when the vector field $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is of limited regularity, typically belonging to the space $L^p\left(0,T; L^q\left(\mathbb{R}^n\right)\right)$ for some $p, q \in [1, \infty]$. Intimately related to (ODE) are two partial differential equations, the transport equation

$$\frac{\partial u}{\partial t} + f \cdot \nabla_x u = 0 \quad \text{on} \quad \mathbb{R}^n \times (0,T) \tag{TE}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(f\rho\right) = 0 \quad \text{on} \quad \mathbb{R}^n \times (0,T)\,. \tag{CE}$$

There are two notions of solution of (ODE): trajectories, which describe a single continuous curve through the phase space $\mathbb{R}^n$ that satisfy (ODE) in some sense; and flow solutions, which describe an aggregate of trajectories satisfying some additional properties.

First, we recall the definitions of classical trajectories and flow solutions, where we require the vector field f to be continuous, before defining appropriately weakened analogues for vector fields of limited regularity.

2.1 Classical solutions

In the classical case we require a solution to the equation (ODE) to hold in the pointwise sense; for each point $t \in [0,T]$ we require the derivative to exist and to be equal to the vector field evaluated at this point.

Definition 2.1. *For each* $(x, s) \in \mathbb{R}^n \times [0, T]$ *a map* $\xi\colon\ [0, T] \to \mathbb{R}^n$ *is a solution of* (ODE) *with initial data* (x, s) *if*

- ξ *is continuously differentiable on* $[0, T]$
- *for each* $t \in [0, T]$ *the pointwise derivative* $\frac{\mathrm{d}\xi}{\mathrm{d}t} = f(\xi(t), t)$, *and*
- $\xi(s) = x$.

For such ξ *the map* $t \mapsto (\xi(t), t)$ *is called a* trajectory *of* (ODE) *with initial data* (x, s).

Note that we require the map to be defined and satisfy the equation (ODE) over the entire temporal domain. We can consider local solutions, where the map is defined on just a small neighbourhood of the initial time s but this is beyond the scope of this thesis.

There is a useful equivalent formulation in terms of integral equations:

Lemma 2.2. *The map* $\xi\colon\ [0, T] \to \mathbb{R}^n$ *is a solution of* (ODE) *with initial data* $(x, s) \in \mathbb{R}^n \times [0, T]$ *if and only if* ξ *is continuously differentiable and satisfies*

$$\xi(t) = x + \int_s^t f(\xi(\tau), \tau)\,\mathrm{d}\tau \qquad \forall\, t \in [0, T].$$

A classical flow solution of (ODE) is an aggregation of a solution of (ODE) for each initial data with an additional group property:

Definition 2.3. *A map* $X\colon\ [0, T]_t \times \mathbb{R}^n_x \times [0, T]_s \to \mathbb{R}^n$ *is a* classical flow solution *of* (ODE) *if for all initial data* (x, s)

- $X(\cdot, x, s)$ *is continuously differentiable on* $[0, T]$, *and*
- $X(t, x, s) = x + \int_s^t f(X(\tau, x, s), \tau)\,\mathrm{d}\tau \qquad \forall\, t \in [0, T]$,

and further the map X *satisfies the group property*

$$X(t, X(\tau, x, s), \tau) = X(t, x, s) \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, t, \tau, s \in [0, T]. \tag{GP}$$

The group property requires that distinct trajectories in the flow do not intersect; that is if two trajectories in the flow intersect then they are equal. It also guarantees that the flow is invertible, as $X(t, X(s, x, t), s) = x$ for all $x \in \mathbb{R}^n$ and

$t, s \in [0,T]$. Consequently for each fixed $t, s \in [0,T]$ the map $X(t,\cdot,s) : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection.

The existence of a solution for all initial data is not sufficient to guarantee the existence of a classical flow. However if there is a unique solution for all initial data then the existence of a unique classical flow is guaranteed, which is the content of the following lemma:

Lemma 2.4. *If there exists a unique solution of* (ODE) *for all initial data* $(x,s) \in \mathbb{R}^n \times [0,T]$ *then there exists a unique classical flow solution of* (ODE).

Proof. For each $(x,s) \in \mathbb{R}^n \times [0,T]$ let $\xi_{(x,s)}$ be the unique solution of (ODE) with initial data $(x,s) \in \mathbb{R}^n \times [0,T]$. Clearly the map $X(t,x,s) := \xi_{(x,s)}(t)$ is the unique aggregate of solutions. Assume for a contradiction that (GP) does not hold then there exist $x \in \mathbb{R}^n$ and $t, s, \tau \in [0,T]$ such that

$$X(t, X(\tau,x,s), \tau) \neq X(t,x,s)$$

Consequently, the solutions $\xi_{(X(\tau,x,s),\tau)}$ and $\xi_{(x,s)}$ are distinct, yet

$$\xi_{(X(\tau,x,s),\tau)}(\tau) = \xi_{(x,s)}(\tau) = X(\tau,x,s)$$

so there are two distinct solutions to (ODE) with the initial data $(X(\tau,x,s),\tau)$, which contradicts the uniqueness of solutions. □

2.1.1 Dependence on initial conditions, the transport equation and the continuity equations

It is well known that a classical flow solution of (ODE) inherits the regularity of the vector field f in the sense that if $f \in C^k(\mathbb{R}^n \times [0,T]; \mathbb{R}^n)$ and there exists a classical flow solution X of (ODE) then X is C^k with respect to x and s and C^{k+1} with respect to t. Essentially, for $k = 1$ we consider the system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = f(\xi,t) & \xi(s) = x \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \nabla_x f(\xi,t)\,\eta & \eta(s) = I \end{cases}$$

where $\eta(t,x,s)$ is an $n \times n$ matrix and by assumption the matrix $\nabla_x f$ exists and is continuous. As the latter ordinary differential equation is linear it has a unique solution and by approximating by difference quotients it can be shown that this solution is $\nabla_x X$ (see Chapter V Theorem 3.1 of Hartman [1964] for details). For $k > 1$ the existence of higher derivatives is demonstrated by extending the above system of ordinary differential equations by formally differentiating the right hand sides in the above system of ordinary differential equations repeatedly.

If the vector field is continuously differentiable then the additional regularity of the classical flow solution allows us to find solutions of the transport and continuity equations. For the remainder of this section we assume that the vector field f is at least in $C^1(\mathbb{R}^n \times [0,T]; \mathbb{R}^n)$.

Proposition 2.5. *If the vector field $f \in C^1(\mathbb{R}^n \times [0,T]; \mathbb{R}^n)$ and X is a classical flow solution of* (ODE) *then for all $s \in [0,T]$ and all $u_s \in C^\infty(\mathbb{R}^n)$ the function $u(x,t) := u_s(X(s,x,t))$ is the unique solution of the transport equation* (TE) *with the initial condition $u|_{t=s} = u_s$.*

Proof. Clearly $u(x,s) = u_s(x)$. Further

$$\frac{\mathrm{d}}{\mathrm{d}t} u(X(t,x,s),t) = \frac{\mathrm{d}}{\mathrm{d}t} u_s(x) = 0$$

so u is constant on the trajectories of X. Consequently,

$$\begin{aligned}
&\left.\frac{\partial u}{\partial t}\right|_{(X(t,x,s),t)} + f(X(t,x,s),t) \cdot \nabla_x u|_{(X(t,x,s),t)} \\
=&\left.\frac{\partial u}{\partial t}\right|_{(X(t,x,s),t)} + \left.\frac{\partial X}{\partial t}\right|_{(t,x,s)} \cdot \nabla_x u|_{(X(t,x,s),t)} \\
=&\frac{\mathrm{d}}{\mathrm{d}t} u(X(t,x,s),t) = 0
\end{aligned}$$

for all $t,s \in [0,T]$ and $x \in \mathbb{R}^n$, and as $X(t,\cdot,s)$ is bijective we conclude that the pointwise equality (TE) holds everywhere on $\mathbb{R}^n \times (0,T)$. $\square$

Proposition 2.6. *If the vector field $f \in C^1(\mathbb{R}^n \times [0,T]; \mathbb{R}^n)$ and X is a classical flow solution of* (ODE) *then*

$$\det \nabla_x X(t,x,s) = \mathrm{e}^{\int_s^t \operatorname{div} f(X(\tau,x,s),\tau)\mathrm{d}\tau} \quad \forall x \in \mathbb{R}^n \quad \forall t,s \in [0,T] \tag{2.1}$$

Proof. Applying the Liouville Theorem (see Chapter IV Theorem 1.2 of Hartman [1964]) to the linear differential equation

$$\frac{\mathrm{d}\nabla_x X}{\mathrm{d}t} = \nabla_x f(X(t,x,s),t)\nabla_x X$$

yields

$$\det \nabla_x X(t,x,s) = \mathrm{e}^{\int_s^t \operatorname{tr} \nabla_x f(X(\tau,x,s),\tau)\mathrm{d}\tau}$$

where

$$\operatorname{tr} \nabla_x f\left(X\left(\tau, x, s\right), \tau\right) := \sum_{i=1}^{n} \frac{\partial f_i}{\partial i}\left(X\left(\tau, x, s\right), \tau\right)$$
$$= \operatorname{div} f\left(X\left(\tau, x, s\right), \tau\right)$$

giving the result. □

We remark that for each $t, s \in [0, T]$ the familiar quantity $\det \nabla_x X\left(t, x, s\right)$ is the Jacobian of the map $x \mapsto X\left(t, x, s\right)$ and from (2.1) this Jacobian is strictly positive. Further as the map $x \mapsto X\left(t, x, s\right)$ has inverse $x \mapsto X\left(s, x, t\right)$ the Jacobians of these maps are related by

$$\left[\det \nabla_x X\left(s, x, t\right)\right]^{-1} = \left[\det \nabla_x X\right]\left(t, X\left(s, x, t\right), s\right). \tag{2.2}$$

In fact, this identity allows us to show that the Jacobian of the map $x \mapsto X\left(s, x, t\right)$ solves the continuity equation (CE) provided that the vector field f is sufficiently regular.

Proposition 2.7. *If the vector field $f \in C^2\left(\mathbb{R}^n \times [0, T]; \mathbb{R}^n\right)$ and X is a classical flow solution of* (ODE) *then for all $s \in [0, T]$ and all $\rho_s \in C^\infty\left(\mathbb{R}^n\right)$ the function*

$$\rho\left(x, t\right) := \left[\det \nabla_x X\left(s, x, t\right)\right] \rho_s\left(X\left(s, x, t\right)\right)$$

is the unique solution of the continuity equation (CE) *with the initial condition $\rho|_{t=s} = \rho_s$.*

Proof. Observe that $\rho\left(x, s\right) = \left[\det \nabla_x X\left(s, x, s\right)\right] \rho_s\left(x\right) = \left[\det \nabla_x x\right] \rho_s\left(x\right) = \rho_s\left(x\right)$ so the initial condition is satisfied. Using the identity (2.2) we write

$$\rho\left(X\left(t, x, s\right), t\right) = \left[\det \nabla_x X\left(t, x, s\right)\right]^{-1} \rho_s\left(x\right)$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho\left(X\left(t, x, s\right), t\right) = -\left[\det \nabla_x X\left(t, x, s\right)\right]^{-2} \frac{\mathrm{d}\left[\det \nabla_x X\right]}{\mathrm{d}t}\left(X\left(t, x, s\right), t\right) \rho_s\left(x\right)$$

which, from the equality (2.1),

$$= -\left[\det \nabla_x X\left(t, x, s\right)\right]^{-1} \operatorname{div} f|_{\left(X\left(t, x, s\right), t\right)} \rho_s\left(x\right)$$
$$= -\rho\left(X\left(t, x, s\right), t\right) \operatorname{div} f|_{\left(X\left(t, x, s\right), t\right)}.$$

Further, from the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(X\left(t,x,s\right),t\right) &= \left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{(X(t,x,s),t)} + \left.\frac{\mathrm{d}X}{\mathrm{d}t}\right|_{(t,x,s)} \nabla_x\rho|_{(X(t,x,s),t)} \\ &= \left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{(X(t,x,s),t)} + f\left(X\left(t,x,s\right),t\right)\nabla_x\rho|_{(X(t,x,s),t)}\end{aligned}$$

as X solves (ODE). Consequently,

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{(X(t,x,s),t)} + f\left(X\left(t,x,s\right),t\right)\nabla_x\rho|_{(X(t,x,s),t)} \\ + \rho\left(X\left(t,x,s\right),t\right)\operatorname{div} f|_{(X(t,x,s),t)} = 0.$$

Finally, as $X\left(t,\cdot,s\right)$ is bijective this implies

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + f\nabla_x\rho + \rho\operatorname{div} f = 0 \qquad \text{which is precisely} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} + \operatorname{div}\left(f\rho\right) = 0$$

so ρ satisfies the continuity equation (CE) in the pointwise sense. □

2.1.2 Existence and uniqueness of flows: Lipschitz vector fields

To demonstrate the existence of a unique flow solution of (ODE), in light of the aggregation Lemma 2.4, it is sufficient to demonstrate that for all initial data (x,s) there exists a unique solution of (ODE) with this initial data. However, the requirement in Definition 2.1 that solutions are defined on the entire temporal domain $[0,T]$ is remarkably strong. We illustrate in Example 2.17 below that smooth vector fields may not have such solutions: in this case any map satisfying (ODE) 'blows up' which is to say that it tends to infinity in finite time, so is not defined on the entire temporal domain. A sufficient condition to prevent this 'blow up', and guarantee the existence of solutions on $[0,T]$, is for the growth of the vector field to be controlled in that the vector field satisfies the globally Lipschitz condition, which we define below. The global Lipschitz condition, and in fact the weaker local Lipschitz condition, also defined below, is sufficient to guarantee that solutions of (ODE) are unique. This uniqueness theorem is classical, and is the first of the myriad uniqueness theorems in Agarwal and Lakshmikantham [1993].

In the remainder, we recall the 'local' approach to (ODE), in which we consider 'local' solutions that satisfy (ODE) on some subinterval of the temporal domain $[0,T]$. Further, using the terminology of Sobolevskii [1998] and Sobolevskii [1999] we consider local non-uniqueness points, defined below, which are roughly the points of the domain $\mathbb{R}^n \times [0,T]$ at which distinct solutions of (ODE) arise. Ultimately, by finding the necessarily unique local solutions that do not take values in the local non-uniqueness points, we can identify all the solutions of (ODE) as the concatenation (also defined below) of these local solutions. This analysis is epitomised in Example 2.19, below. Finally, the local viewpoint is advantageous as local solutions of (ODE) can be found with much milder assumptions on the vector field than the global Lipschitz condition that is generally required to demonstrate the existence of solutions defined on the entire temporal domain. We begin by defining local solutions:

Definition 2.8. *We say that the map $\xi\colon I \to \mathbb{R}^n$ is a* local solution *of* (ODE) *on the interval I with initial data $(x,s) \in \mathbb{R}^n \times [0,T]$ if the interval $I \subset [0,T]$ contains s, $\xi(s) = x$ and*

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}(t) = f(\xi(t), t) \quad \forall\, t \in I.$$

Further, we say that the local solution $\tilde{\xi}\colon J \to \mathbb{R}^n$ is an extension *of the local solution $\xi\colon I \to \mathbb{R}^n$ if $I \subsetneq J$ and*

$$\tilde{\xi}(t) = \xi(t) \quad \forall\, t \in I.$$

Finally, we say that a local solution ξ is maximal *if there does not exist an extension of ξ.*

The continuity of the vector field is sufficient for the existence of local solution of (ODE), which is the content of the following classical theorem:

Theorem 2.9 (Peano Existence Theorem). *If $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a continuous vector field then for all $(x,s) \in \mathbb{R}^n \times [0,T]$ there exists a local solution of* (ODE) *with initial data (x,s) defined on some interval $I \subset [0,T]$.*

Proof. See, for example, Theorem 2.1 in Chapter 2 of Hartman [1964]. □

Further, each local solution is either maximal or admits a maximal extension (see, for example, Theorem 3.1 in Chapter 2 of Hartman [1964]), so we may restrict our attention to maximal local solutions. We remark that this local existence theorem does not imply that the local solution is unique, nor that the local solution can be extended onto the entire temporal domain.

We first examine the uniqueness of solutions by defining the local non-uniqueness points of (ODE) which are those points on which every temporal neighbourhood admits multiple local solutions:

Definition 2.10 (Sobolevskii). *A point $(x,s) \in \mathbb{R}^n \times [0,T]$ is a* local non-uniqueness point *of* (ODE) *if for every open interval $I \subset [0,T]$ containing s there exist two local* *solutions* $\xi_1, \xi_2 \colon I \to \mathbb{R}^n$ *of* (ODE) *with initial data* (x, s) *such that* $\xi_1(t) \neq \xi_2(t)$ *for some* $t \in I$.

Definition 2.11. *We say that a vector field* $f \colon \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ *is*

- uniformly Lipschitz *on the domain* $U \subset \mathbb{R}^n \times [0, T]$ *with* Lipschitz constant $L_U > 0$ *if*

$$|f(x,t) - f(y,t)| \leq L_U |x - y| \quad \forall\ (x,t), (y,t) \in U, \tag{2.3}$$

- locally Lipschitz *if for each point* $(x, s) \in \mathbb{R}^n \times [0, T]$ *there exists a neighbourhood* U *of* (x, s) *such that* f *is uniformly Lipschitz on* U, *and*

- globally Lipschitz *if* f *is uniformly Lipschitz on* $U = \mathbb{R}^n \times [0, T]$, *that is if there exists a constant* $L > 0$ *such that*

$$|f(x,t) - f(y,t)| \leq L |x - y| \quad \forall\, x, y \in \mathbb{R}^n \quad \forall\, t \in [0, T]. \tag{2.4}$$

Clearly a globally Lipschitz vector field is locally Lipschitz. Further, a continuously differentiable vector field is locally Lipschitz as for each convex compact neighbourhood $K \subset \mathbb{R}^n \times [0, T]$ the constant

$$L_K = \sup_{(x,t) \in K} \sup_{|u|=1} |\nabla_x f(x,t) \cdot u| < \infty$$

satisfies (2.3) for all $U \subset K$. In fact, a continuously differentiable vector field f is globally Lipschitz if and only if this spatial derivative of f is bounded on $\mathbb{R}^n \times [0, T]$, i.e.

$$\|\nabla_x f\|_\infty := \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^n} \sup_{|u|=1} |\nabla_x f(x,t) \cdot u| < \infty$$

in which case the above supremum is the smallest constant L such that (2.4) holds. The significance of Lipschitz vector fields in the study of ordinary differential equations is evident from the following classical theorems:

Theorem 2.12. *If the continuous vector field* $f \colon \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ *is uniformly Lipschitz on the domain* $U \subset \mathbb{R}^n \times [0, T]$ *then no point of* U *is a local non-uniqueness point of* (ODE).

Proof. Follows from Theorem 1.2.4 of Agarwal and Lakshmikantham [1993] or Theorem 1.1 in Chapter 2 of Hartman [1964]. □

Corollary 2.13. *If the continuous vector field $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is locally Lipschitz then every local solution (and hence every solution) of* (ODE) *is unique.*

Proof. Suppose $\xi_1, \xi_2\colon I \to \mathbb{R}^n$ are distinct solutions of (ODE) on the interval $I \subset [0,T]$ with initial data (x,s). Assume that $\xi_1(\tau) \neq \xi_2(\tau)$ for some $\tau \in I$ with $\tau > s$ and define $t^* := \sup\{t > s | \xi_1(t) = \xi_2(t)\}$ so that certainly $t^* \leq \tau$. We show that $(\xi_1(t^*), t^*) \in \mathbb{R}^n \times [0,T]$ is a local non-uniqueness point contradicting the result of the above theorem: by continuity $\xi_1(t^*) = \xi_2(t^*)$ so both ξ_1 and ξ_2 are local solutions of (ODE) with initial data $(\xi_1(t^*), t^*) \in \mathbb{R}^n \times [0,T]$, however $\xi_1(t^* + \varepsilon) \neq \xi_2(t^* + \varepsilon)$ for all $\varepsilon > 0$. □

Theorem 2.14 (Cauchy-Lipschitz Theorem). *If the continuous vector field $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is globally Lipschitz then there exists a unique solution of* (ODE) *for all initial data $(x,s) \in \mathbb{R}^n \times [0,T]$.*

Proof. See, for example, Theorem 1.1 in Chapter 2 of Hartman [1964]. □

Corollary 2.15. *If the continuous vector field $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is globally Lipschitz then there exists a unique classical flow solution of* (ODE).

Proof. Follows from Theorem 2.14 and the aggregation Lemma 2.4. □

We remark that if f is globally Lipschitz then the Lipschitz bound guarantees some regularity of the classical flow solution:

Proposition 2.16. *Let f be a globally Lipschitz vector field with Lipschitz constant $L > 0$. The classical flow solution X of* (ODE) *satisfies*

$$|X(t,x,s) - X(t,y,s)| \leq \mathrm{e}^{LT} |x - y| \quad \forall\, x,y \in \mathbb{R}^n \quad \forall\, t,s \in [0,T], \tag{2.5}$$

and for all compact $K \subset \mathbb{R}^n$ there exists a constant $C > 0$ dependent on K, L and T such that

$$|X(t_1,x_1,s_1) - X(t_2,x_2,s_2)| \leq C\,|(t_1,x_1,s_1) - (t_2,x_2,s_2)| \tag{2.6}$$

for all $t_i, s_i \in [0,T]$ and $x_i \in K$.

Proof. Follows from a straightforward application of Gronwall's inequality. □

The classical zoo of pathologies include the following:

Example 2.17. *Let $f\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ be defined by $f(x,t)=x^2$. For all initial data $(x,s)\in\mathbb{R}\times[0,T]$ with $x>0$ the map*

$$\xi_{x,s}\colon\ [0,s+x^{-1})\cap[0,T]\to\mathbb{R}$$
$$\xi_{x,s}(t):=-\left(t-x^{-1}-s\right)^{-1}$$

is the unique maximal local solution of (ODE) *with initial data (x,s).*

The vector field f is locally Lipschitz but not globally so as $\left|x^2-y^2\right|=|x-y|\,|x+y|$ and $|x+y|$ is unbounded for $x,y\in\mathbb{R}$, so from Corollary 2.13 local solution $\xi_{x,s}$ is unique. Further, this local solution is maximal as $\xi_{x,s}(t)\to\infty$ as $t\to s+x^{-1}$ so for initial data (x,s) such that $s+x^{-1}<T$ the local solution $\xi_{x,s}$ cannot be extend onto the entire temporal domain $[0,T]$. Consequently, there does not exist a solution in the sense of Definition 2.1 for all initial data, so there is no classical flow solution.

In the following example we use the technique of *solution concatenation*, defined below and discussed further in Chapter 3 to extend local solutions and to construct multiple solutions for given initial data.

Definition 2.18. *Let $\xi_1\colon I\to\mathbb{R}^n$ and $\xi_2\colon J\to\mathbb{R}^n$ be local solutions of* (ODE) *such that $I\cap J\neq\emptyset$ such that there exists a point $\tau\in I\cap J$ with $\xi_1(\tau)=\xi_2(\tau)$. We define the* concatenation *of ξ_1 to ξ_2 at time τ by*

$$\vee_\tau(\xi_1,\xi_2):\ (I\cap(-\infty,\tau))\cup(J\cap[\tau,\infty))\to\mathbb{R}^n$$
$$\vee_\tau(\xi_1,\xi_2)(t):=\begin{cases}\xi_1(t) & t\in I,\ t<\tau\\ \xi_2(t) & t\in J,\ \tau\le t.\end{cases}$$

We remark, however, that the concatenation of two local solutions is not necessarily a local solution as the 'join' may not be differentiable.

Example 2.19. *Let $f\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=|x|^{\frac{1}{2}}$. Observe that for all $c,d\in\mathbb{R}$ each of the maps $\xi^0(t):=0\quad\forall\, t\in\mathbb{R}$,*

$$\xi_c^+(t):=\frac{1}{4}(t-c)^2 \qquad t\in[c,\infty),\quad \textit{and}$$
$$\xi_d^-(t):=-\frac{1}{4}(t-d)^2 \qquad t\in(-\infty,d]$$

is a local solution of (ODE). *Further, for each point $(x,s)\in\mathbb{R}\times\mathbb{R}$ there is a local solution with this initial data:*

- *if $x>0$ the local solution $\xi^+_{s-2\sqrt{x}}$ suffices,*

- *if* $x < 0$ *the local solution* $\xi^{-}_{s+2\sqrt{-x}}$ *suffices, and*

- *if* $x = 0$ *any of the local solutions* ξ^{0}, ξ^{+}_{s} *or* ξ^{-}_{s} *suffice.*

Further, for each $c \in \mathbb{R}$ *the concatenation of* ξ^{0} *to* ξ^{+}_{c} *at time* c *is a solution of* (ODE) *as the 'join' is sufficiently smooth:*

$$\lim_{h\to 0+} \frac{1}{h}\left(\vee_c\left(\xi^0,\xi^+_c\right)(c+h) - \vee_c\left(\xi^0,\xi^+_c\right)(c)\right) = \lim_{h\to 0+} \frac{1}{h}\left(\frac{1}{4}h^2 - \frac{1}{4}0\right) = 0$$
$$\lim_{h\to 0-} \frac{1}{h}\left(\vee_c\left(\xi^0,\xi^+_c\right)(c+h) - \vee_c\left(\xi^0,\xi^+_c\right)(c)\right) = \lim_{h\to 0-} \frac{1}{h}\left(0 - \frac{1}{4}0\right) = 0$$

so the derivative is defined at $t = c$ *and satisfies* (ODE) *at this point since*

$$\frac{\mathrm{d}}{\mathrm{d}t}\vee_c\left(\xi^0,\xi^+_c\right)(c) = 0 = f(0) = f\left(\vee_c\left(\xi^0,\xi^+_c\right)(c)\right).$$

Consequently, the concatenation satisfies (ODE) *for all* $t \in \mathbb{R}$*, and so is a solution of* (ODE) *in the sense of Definition 2.1. Similarly, for all* $c, d \in \mathbb{R}$ *with* $d < c$*,*

- *the concatenation of* ξ^{-}_{d} *to* ξ^{0} *at time* d*,*

- *the concatenation of* ξ^{-}_{d} *to* ξ^{+}_{d} *at time* d*, and*

- *the concatenation of* ξ^{-}_{d} *to* $\vee_c\left(\xi^0,\xi^+_c\right)$ *at time* d

are solutions of (ODE) *(see Figure 2.1). We see, therefore, that each of the local solutions* ξ^{0}, ξ^{+}_{c} *and* ξ^{-}_{d} *admit at least a one parameter family of distinct extensions onto the entire temporal domain. Consequently, for each point* $(x, s) \in \mathbb{R}\times\mathbb{R}$ *there is a local solution of* (ODE) *with this initial data and this local solution admits multiple distinct extensions, so there are multiple solutions in the sense of Definition 2.1 of* (ODE) *with initial data* (x, s)*.*

We remark that each point in $\{0\} \times [0, T]$ is a local non-uniqueness point of (ODE), and as the vector field f is locally Lipschitz away from this set by Theorem 2.12 there are no other local non-uniqueness points. Despite the non-uniqueness of solutions for all initial data there is a unique classical flow solution of $\frac{\mathrm{d}\xi}{\mathrm{d}t} = |\xi|^2$, which is composed of the solutions that are concatenations of ξ^{+} and ξ^{-}, and not of ξ^{0}, which are exactly those solutions that do not 'loiter' at the origin: we define

$$X(t, x, s) := \begin{cases} \vee_{s-2\sqrt{x}}\left(\xi^{-}_{s-2\sqrt{x}}, \xi^{+}_{s-2\sqrt{x}}\right) & x \geq 0 \\ \vee_{s+2\sqrt{-x}}\left(\xi^{-}_{s+2\sqrt{-x}}, \xi^{+}_{s+2\sqrt{-x}}\right) & x < 0 \end{cases}$$

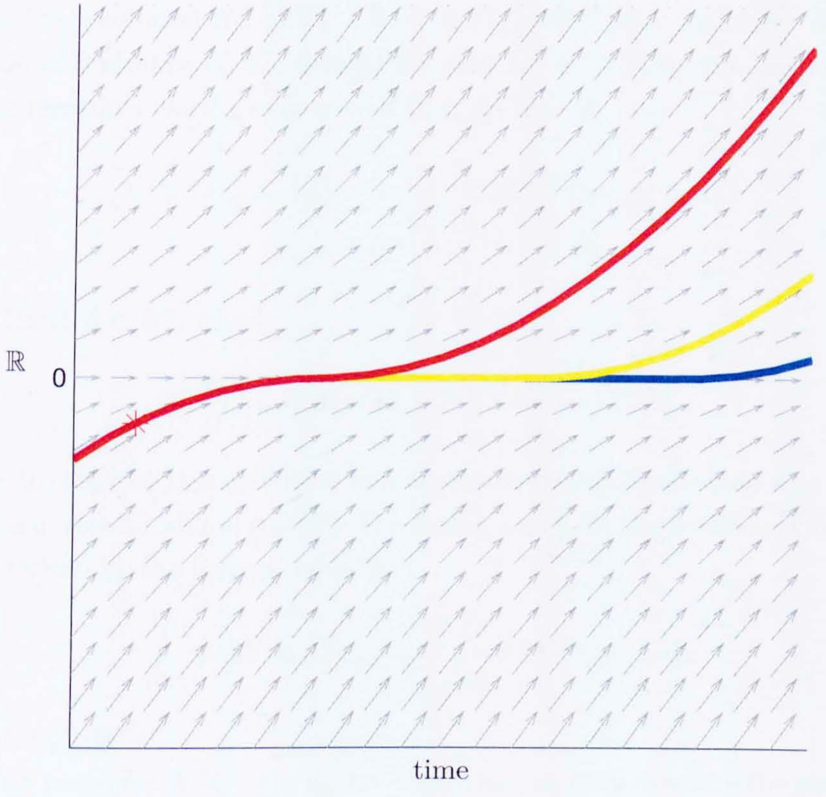

Figure 2.1: The trajectories of three distinct solutions of $\dot{\xi} = |\xi|^{\frac{1}{2}}$ with the same initial data (illustrated by the star). In the figure we plot the vector field with a unit time component, i.e. $\left(|x|^{\frac{1}{2}}, 1\right)$ as this is the derivative of the trajectories $(\xi(t), t)$.

and remark that X satisfies the group property. Further, any other aggregate of solutions contains a solution that 'loiters' at the origin for some time interval, which, as is evident from Figure 2.1, ensures that there are distinct trajectories that intersect violating the group property (GP).

The essential features of the above example are that the vector field f has a fixed point at the origin, all solutions of (ODE) reach the origin in finite time and the solutions can be concatenated in a sufficiently smooth way. Together, these features ensure that we can construct solutions that 'loiter' at the origin for an arbitrary time interval, which yields non-uniqueness of trajectories. We revisit solution concatenation in Chapter 3 where in the weaker setting we will see that every concatenation of solutions of (ODE) is itself a solution of (ODE).

2.1.3 Compressibility

A classical flow solution $X\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ induces a family of measures on the Borel σ-algebra of $\mathbb{R}^n$, defined for each $t,s \in [0,T]$ by the push forward of Lebesgue measures via the map $x \mapsto X(t,x,s)$, that is

$$\begin{aligned} X(t,\cdot,s)_{\#}\,\mu_n(A) &:= \mu_n(\{x \in \mathbb{R}^n | X(t,x,s) \in A\}) \\ &= \mu_n\left(X^{-1}(t,\cdot,s)\,A\right) \end{aligned}$$

for all Borel $A \subset \mathbb{R}^n$, where

$$X^{-1}(t,\cdot,s)\,A := \{x \in \mathbb{R}^n | X(t,x,s) \in A\}\,.$$

Of course in order for this definition to make sense we require the map $x \mapsto X(t,x,s)$ to be measurable for all $t,s \in [0,T]$. For each $t,s \in [0,T]$ the measure $X(t,\cdot,s)_{\#}\,\mu_n$ is characterised by the integral equality

$$\int_{\mathbb{R}^n} \phi \,\mathrm{d}X(t,\cdot,s)_{\#}\,\mu_n = \int_{\mathbb{R}^n} \phi(X(t,x,s))\,\mathrm{d}x \tag{2.7}$$

for all $\phi \in C_c^{\infty}(\mathbb{R}^n)$.

The measures $X(t,\cdot,s)_{\#}\,\mu_n$ are significant as they describe the evolution of the measure of spatial sets under the action of the flow.

Definition 2.20. *We say that a map* $X\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$

(i) *is* incompressible *if* $X(t,\cdot,s)_{\#}\,\mu_n(A) = \mu_n(A)$ *for each Borel subset* $A \subset \mathbb{R}^n$ *and all* $t,s \in [0,T]$*,*

(ii) *is* nearly incompressible *if there exists a constant* $C > 0$ *such that*

$$\frac{1}{C}\mu_n(A) \le X(t,\cdot,s)_{\#}\,\mu_n(A) \le C\mu_n(A)$$

for each Borel subset $A \subset \mathbb{R}^n$ *and all* $t,s \in [0,T]$*, and*

(iii) satisfies the Lusin condition *if* $X(t,\cdot,s)_{\#}\,\mu_n(A) = 0$ *for all* $t,s \in [0,T]$ *for each Borel set* $A \subset \mathbb{R}^n$ *such that* $\mu_n(A) = 0$.

We remark that these properties are related by the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

It is well known that if the vector field f is sufficiently regular then the flow is nearly incompressible, which is the content of the following proposition:

Proposition 2.21. *Let the vector field f be globally Lipschitz and continuously differentiable on $[0,T] \times \mathbb{R}^n$ and X be the classical flow solution X of* (ODE). *For all $t, s \in [0,T]$ the push forward measure $X(t,\cdot,s)_{\#}\,\mu_n$ satisfies*

$$e^{-T\|\operatorname{div} f\|_\infty}\mu_n(A) \leq X(t,\cdot,s)_{\#}\,\mu_n(A) \leq e^{T\|\operatorname{div} f\|_\infty}\mu_n(A) \tag{2.8}$$

for all Borel subsets $A \subset \mathbb{R}^n$, where

$$\|\operatorname{div} f\|_\infty = \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^n} |\operatorname{div} f(x,t)| < \infty.$$

In particular the flow is nearly incompressible, and further, if $\operatorname{div} f = 0$ the flow is incompressible.

Proof. As f is continuously differentiable the spatial divergence of f exists and is bounded uniformly in t as

$$\begin{aligned}|\operatorname{div} f(x,t)| = \left|\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}(x,t)\right| &\leq \sum_{j=1}^{n}\left|\frac{\partial f}{\partial x_j}(x,t)\right| \\ &\leq \sum_{j=1}^{n} |\nabla_x f(x,t)\cdot e_j| \leq n\,\|\nabla_x f\|_\infty .\end{aligned}$$

Next, for each $t, s \in [0,T]$ the Jacobian of the map $x \mapsto X(s,x,t)$ is defined and from Proposition 2.6

$$\det \nabla_x X(s,x,t) = e^{\int_t^s \operatorname{div} f(X(\tau,x,t),\tau)\mathrm{d}\tau}$$

so for each $\phi \in C_c^\infty(\mathbb{R}^n)$ by the change of variables formula the integral

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(X(t,x,s))\,\mathrm{d}x &= \int_{\mathbb{R}^n} \phi(X(t,X(s,x,t),s))\,|\det \nabla_x X(s,x,t)|\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \phi(x)\; e^{\int_t^s \operatorname{div} f(X(\tau,x,t),\tau)\mathrm{d}\tau}\mathrm{d}x.\end{aligned}$$

Further, as the divergence is bounded,

$$e^{-|t-s|\|\operatorname{div} f\|_\infty}\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^n}\phi(X(t,x,s))\,\mathrm{d}x \leq e^{|t-s|\|\operatorname{div} f\|_\infty}\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x$$

which extends to hold for all Borel maps $\phi \in \mathcal{L}^1(\mathbb{R}^n)$. Consequently, (2.8) holds for all Borel $A \subset \mathbb{R}^n$. □

2.1.4 Fractal geometry

In DiPerna and Lions [1989] the authors remark that there is a more direct proof of the claim in Proposition 2.21 that classical flow solutions of (ODE) are nearly incompressible if the vector field f is globally Lipschitz. Indeed, from the bound (2.4) and the group property (GP) it is easy to derive the estimates

$$\mathrm{e}^{-L|t-s|}\,|x-y| \leq |X(t,x,s) - X(t,y,s)| \leq \mathrm{e}^{L|t-s|}\,|x-y| \tag{2.9}$$

for $x, y \in \mathbb{R}^n$ and for all $t, s \in [0, T]$. As the image measure of Lipschitz maps is controlled by the Lipschitz constant (see, for example, Proposition 2.2 in Falconer [2003]) we conclude that

$$\mathrm{e}^{-nL|t-s|}\mu_n(A) \leq \mu_n(X(t,A,s)) \leq \mathrm{e}^{nL|t-s|}\mu_n(A)$$

for all Borel subsets $A \subset \mathbb{R}^n$ and so

$$\mathrm{e}^{-nLT}\mu_n(A) \leq X(s,\cdot,t)_{\#}\,\mu_n(A) \leq \mathrm{e}^{nLT}\mu_n(A)\,.$$

DiPerna and Lions further remark that this is the "wrong" explanation as the approach in Proposition 2.21 is sharper and more easily generalised. However, in addition to controlling the n-dimensional Lebesgue measure the Lipschitz bounds (2.9) show that the geometry of the phase space is sufficiently preserved under the action of the flow to maintain some structure of fractal sets. In order to make this more precise we now recall the definitions of the Hausdorff measure, Hausdorff dimension, and upper and lower box-counting dimensions:

Definition 2.22. *Let F be a non-empty subset of $\mathbb{R}^n$. For all $d \geq 0$ the d-* dimensional Hausdorff measure *$\mathcal{H}^d(F)$ of F is defined by*

$$\mathcal{H}^d(F) := \lim_{\delta\to 0}\mathcal{H}^d_\delta(F) \qquad \textit{where}$$

$$\mathcal{H}^d_\delta(F) := \inf\left\{\sum_{i=1}^{\infty}\mathrm{diam}(U_i)^d \,\middle|\, F \subset \bigcup_{i=1}^{\infty} U_i, \quad \mathrm{diam}(U_i) < \delta \quad \forall\, i\right\}$$

and the Hausdorff dimension *of F is* $\dim_H(F) := \sup\left\{d | \mathcal{H}^d(F) = 0\right\}$.
If F is bounded then the upper and lower box-counting dimensions of F are defined

by

$$\dim_B(F) := \limsup_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta} \qquad \textit{and}$$
$$\dim_{LB}(F) := \liminf_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta}$$

respectively, where $N(F,\delta)$ is the smallest number of sets of diameter δ whose union contains F.

Further, we recall that the Hausdorff measure of the image of a set under a Lipschitz map is controlled by the Lipschitz constant, and that $\dim_H, \dim_B$ and $\dim_{LB}$ are non-increasing under Lipschitz maps:

Lemma 2.23. *Let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with Lipschitz constant $c > 0$. For all subsets $F \subset \mathbb{R}^n$ the d-dimensional Hausdorff measure of the set $g(F)$ satisfies*

$$\mathcal{H}^d(g(F)) \le c^d \mathcal{H}^d(F) \quad \forall\, d \ge 0$$

and consequently, $\dim_H(g(F)) \le \dim_H(F)$.

If in addition F is bounded then $\dim_B(g(F)) \le \dim_B(F)$ *and* $\dim_{LB}(g(F)) \le \dim_{LB}(F)$.

Proof. See Proposition 2.2 in Falconer [2003] for the proof of the Hausdorff measure inequality, which immediately implies the Hausdorff dimension inequality. For the box-counting dimensions see §3.2 (iv) of Falconer [2003]. □

With this well known result we can characterise the geometry of spatial subsets under the action of a classical flow solution of a sufficiently regular vector field, which is the content of the following lemma:

Lemma 2.24. *If X is a classical flow solution of* (ODE) *and the vector field f is globally Lipschitz then for $t, s \in [0,T]$ and all $d \ge 0$ the d-dimensional Hausdorff measure $\mathcal{H}^d$ satisfies*

$$\mathrm{e}^{-dL|t-s|}\mathcal{H}^d(F) \le \mathcal{H}^d\left(X^{-1}(t,\cdot,s)\,F\right) \le \mathrm{e}^{dL|t-s|}\mathcal{H}^d(F)$$

for all subsets $F \subset \mathbb{R}^n$. In particular for all $t, s \in [0,T]$

$$\dim_H(F) = \dim_H\left(X^{-1}(t,\cdot,s)\,F\right)$$

and, in fact, the stronger property

$$\mathcal{H}^d(F) = 0 \quad \textit{if and only if} \quad \mathcal{H}^d\left(X^{-1}(t,\cdot,s)\,F\right) = 0.$$

holds. Further, if F is bounded,

$$\begin{aligned} \dim_B(F) &= \dim_B\left(X^{-1}(t,\cdot,s)\,F\right) \quad \textit{and} \\ \dim_{LB}(F) &= \dim_{LB}\left(X^{-1}(t,\cdot,s)\,F\right). \end{aligned}$$

Proof. As f is globally Lipschitz the flow satisfies (2.9). Consequently, for all $t, s \in [0,T]$ the map $x \mapsto X(t,x,s)$ is bi-Lipschitz so the results follow from Lemma 2.23. □

For a subset $S \subset \mathbb{R}^n \times [0,T]$ we can consider the set $P_X(S) \subset \mathbb{R}^n$ of initial data at time $s = 0$ whose trajectories intersect S, that is

$$P_X(S) := \{x \in \mathbb{R}^n | (X(t,x,0), t) \in S \quad \text{for some } t \in [0,T]\},$$

which can be thought of as the 'projection' of the set $S \subset \mathbb{R}^n \times [0,T]$ onto $\mathbb{R}^n \times \{0\}$ along the trajectories of X. As $x \mapsto X(t,x,s)$ is invertible we can write this explicitly as

$$P_X(S) = \{X(0,x,t) \,|\, (x,t) \in S\}.$$

Recall that the dimensions do not increase under the canonical projections in Euclidean space, which follows immediately from Lemma 2.23 and the fact that projections are Lipschitz. For sufficiently regular flows the projection along trajectories has the same property:

Lemma 2.25. *If X is a classical flow solution of* (ODE) *and the vector field f is globally Lipschitz then for all bounded subsets $S \subset \mathbb{R}^n \times [0,T]$ the projected set $P_X(S) \subset \mathbb{R}^n$ satisfies,*

$$\dim_H(P_X(S)) \leq \dim_H(S),$$

and the stronger statement that

$$\mathcal{H}^d(P_X(S)) = 0 \quad \textit{if} \quad \mathcal{H}^d(S) = 0.$$

Further,

$$\dim_B\left(P_X\left(S\right)\right) \leq \dim_B\left(S\right), \quad \textit{and}$$
$$\dim_{LB}\left(P_X\left(S\right)\right) \leq \dim_{LB}\left(S\right).$$

Proof. If f is globally Lipschitz then from (2.6) the flow X is locally Lipschitz on $[0,T] \times \mathbb{R}^n \times [0,T]$. As S is bounded there exists a compact $K \subset \mathbb{R}^n$ such that $S \subset K \times [0,T]$ so the projection

$$(x,t) \mapsto X\left(0,x,t\right) \quad x \in K \quad t \in [0,T]$$

is Lipschitz. Consequently, the set

$$P_X\left(S\right) = \left\{X\left(0,x,t\right) \mid (x,t) \in S \subset K \times [0,T]\right\}$$

is the image of S under a Lipschitz map and the result follows from Lemma 2.23. $\square$

In a typical application the vector field f is degenerate in some sense on a set $S \subset \mathbb{R}^n \times [0,T]$ and we wish to demonstrate that only a 'small' number of trajectories of a flow solution X intersect S. While the above lemma illustrate that we can have quite precise control of the 'size' of $P_X\left(S\right)$ we restrict our attention to the n-dimensional Lebesgue measure and give the following definition adapted from Aizenman [1978b]:

Definition 2.26. *Let X be a classical flow solution and $S \subset \mathbb{R}^n \times [0,T]$ be a compact subset. We say that X* avoids *the set S if $\mu_n\left(P_X\left(S\right)\right) = 0$.*

We discuss avoidance at length in Chapter 4 where we give some new sufficient conditions for a flow to avoid a set in terms of the geometry of S and the regularity of the vector field f generating the flow. The avoidance property has useful applications including the following: if S is the set of points at which f is not locally Lipschitz and a flow solution X avoids S then there is a unique solution of (ODE) for almost every initial data $(x,0)$. This argument is epitomised in Robinson and Sadowski [2009] where the authors use this approach to demonstrate the almost everywhere uniqueness of trajectories where the vector field f is a suitable weak solution of the $3D$ Navier-Stokes equations.

We end this section with an example of a discontinuous vector field for which there exists a flow solution of (ODE) that does not preserve the geometry of subsets in the sense of Lemmas 2.24 and 2.25. As far as we are aware, there is no such example in the literature.

2.1.5 A flow into a Cantor set

In this section we construct the map illustrated in Figure 2.2. The figure indicates that under X the unit interval $[0, 1)$ is mapped at time $t = 2 - 2^{1-j}$ into the j^{th} stage in the construction of the Cantor middle third set. After rigorously defining X we demonstrate in Lemma 2.28 that for each $x \in [0, 1)$ as $t \to T$ the trajectory $X(t, x)$ converges to a point in the Cantor set. Consequently, X does not preserve the Hausdorff or box-counting dimensions of the unit interval.

Further, in Lemma 2.29 we demonstrate that X is sufficiently regular to give rise to a classical flow solution and in Lemma 2.30 we demonstrate that the trajectories of X do not intersect.

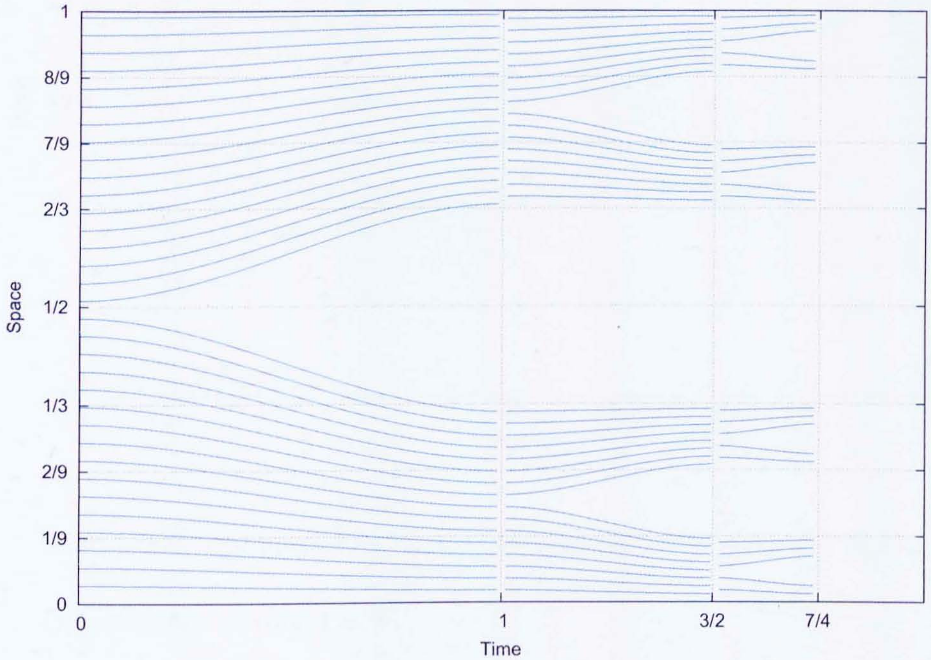

Figure 2.2: The maps $t \mapsto X(t, x)$ plotted up to time $t = \frac{7}{4}$ for various initial data $x \in [0, 1)$.

First, we recall that the j^{th} stage of the Cantor middle third set is the set of intervals $C_j := \cup_{p \in P_j} \left[p, p + 3^{-j+1}\right]$ where the set of left end points of these intervals P_j is defined inductively by $P_1 := \{0\}$ and $P_{j+1} := \left(2 \cdot 3^j P_j\right) \cup P_j$. We write $Q_j := P_j + 3^{-j+1}$ for the set of right end point of the intervals C_j. The Cantor middle third set C is defined as the intersection of the sets C_j.

Define the map $Y_1 \colon [0,1] \times [0,1) \to [0,1)$ by

$$Y_1(t,x) := \begin{cases} \left(\frac{2}{3}t^3 - t^2 + 1\right) x & 0 \le x < \frac{1}{2} \\ \left(\frac{2}{3}t^3 - t^2 + 1\right)(x-1) + 1 & \frac{1}{2} \le x < 1 \end{cases} \tag{2.10}$$

illustrated in Figure 2.3.

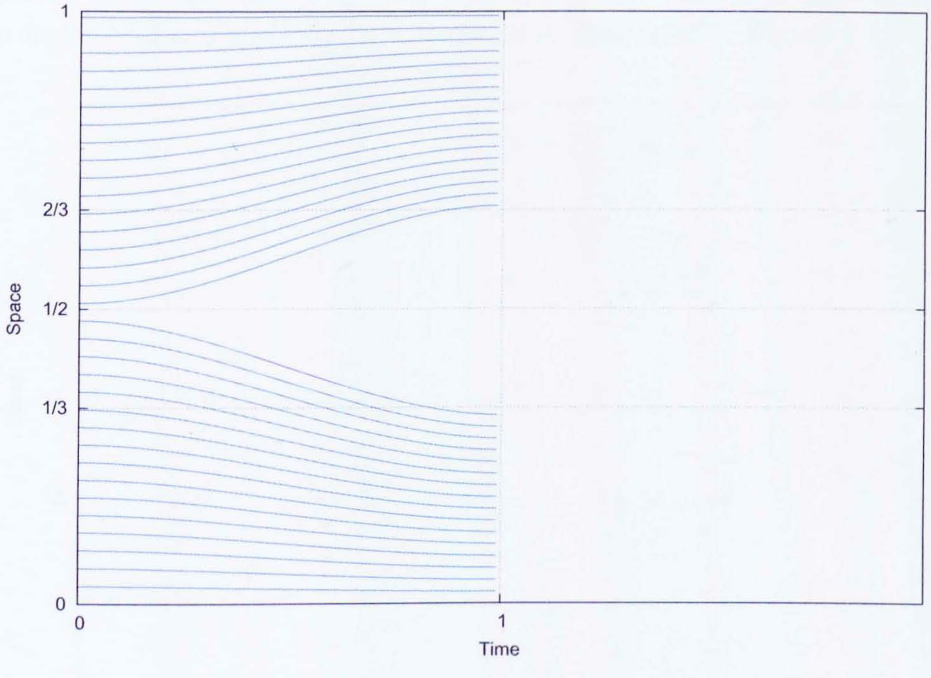

Figure 2.3: The maps $t \mapsto Y_1(t,x)$ for various initial data $x \in [0,1)$.

Observe that at time $t = 1$

$$Y_1(1,x) = \begin{cases} \frac{2}{3}x & 0 \le x < \frac{1}{2} \\ \frac{2}{3}x + \frac{1}{3} & \frac{1}{2} \le x < 1 \end{cases}$$

and in particular the images of the intervals $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$ under the map Y at time $t = 1$ are $Y_1\left(1, \left[0, \frac{1}{2}\right)\right) = \left[0, \frac{1}{3}\right)$ and $Y_1\left(1, \left[\frac{1}{2}, 1\right)\right) = \left[\frac{2}{3}, 1\right)$ respectively. More concisely, we write

$$Y_1(1, [0,1)) = C_2 \setminus Q_2, \tag{2.11}$$

so that the range of Y_1 at time $t = 1$ is contained in the second stage of construction of the Cantor middle third set C_2. By rescaling, duplicating and translating the map Y_1 we can iterate this map so that the range at subsequent times is contained in later stages C_j:

define the map $Y_2\colon \left[1, 1+\frac{1}{2}\right] \times C_2 \setminus Q_2 \to [0,1)$ by

$$Y_2(t,x) := \begin{cases} Y_1(2t-2, 3x)/3 & 0 \le x < \frac{1}{3} \\ Y_1\left(2t-2, 3\left(x-\frac{2}{3}\right)\right)/3 + \frac{2}{3} & \frac{2}{3} \le x < 1, \end{cases}$$

which can be thought of as two translated copies of the map Y_1 rescaled temporally by a factor of 2 and spatially by a factor of 3, illustrated in Figure 2.4. Observe

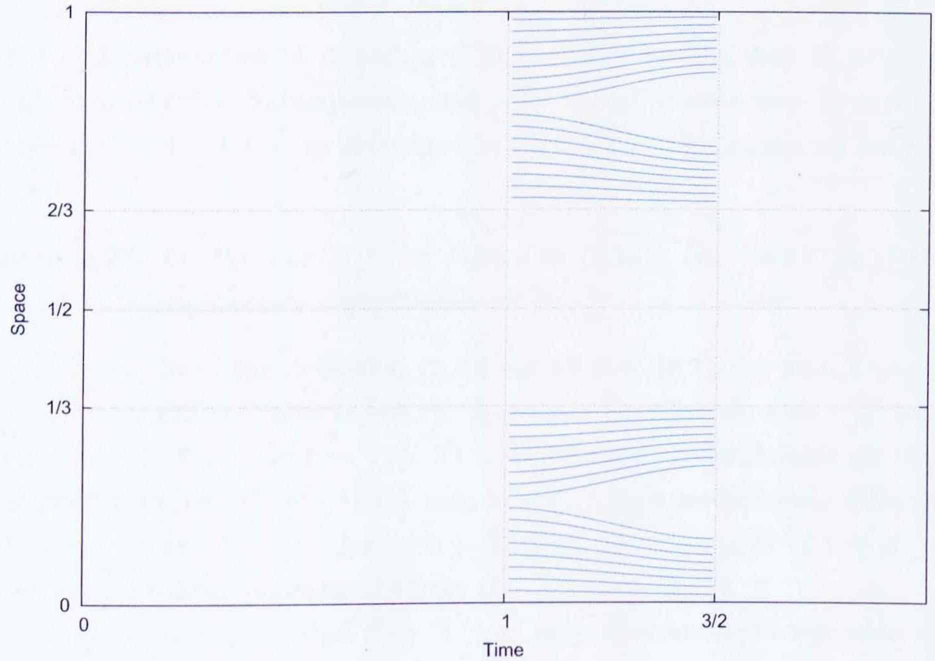

Figure 2.4: The maps $t \mapsto Y_2(t,x)$ for various initial data $x \in [0,1)$.

that

$$Y_2\left(1+\frac{1}{2}, Y_1(1,[0,1))\right) = Y_2\left(1+\frac{1}{2}, C_2 \setminus Q_2\right) = C_3 \setminus Q_3.$$

For all $j \in \mathbb{N}$ with $j \ge 2$ we define $t_j := 2-2^{-(j-1)}$ and the map $Y_j\colon [t_{j-1}, t_j] \times C_j \setminus Q_j \to [0,1)$ by

$$Y_j(t,x) := \begin{cases} Y_{j-1}(2t-2, 3x)/3 & 0 \le x < \frac{1}{3} \\ Y_{j-1}\left(2t-2, 3\left(x-\frac{2}{3}\right)\right)/3 + \frac{2}{3} & \frac{2}{3} \le x < 1, \end{cases} \tag{2.12}$$

which can be thought of as two translated and rescaled copes of the map Y_{j-1}.

Observe that for all $j \in \mathbb{N}$

$$Y_j\left(t_j, Y_{j-1}\left(t_{j-1}, C_{j-1} \setminus Q_{j-1}\right)\right) = Y_j\left(t_j, C_j \setminus Q_j\right) = C_{j+1} \setminus Q_{j+1}.$$

Next, we let $T := \lim_{j\to\infty} t_j = 2$ and assemble the maps Y_j in the following way: let the map $X\colon [0,T) \times [0,1) \to [0,1)$ be defined by

$$X(t,x) := \begin{cases} Y_1(t,x) & 0 \le t \le 1 \\ Y_j\left(t, X\left(t_{j-1},x\right)\right) & t_{j-1} < t \le t_j \end{cases} \tag{2.13}$$

so that under the action of X each $x \in [0,1)$ is sent via the map Y_1 to a point in $C_2 \setminus Q_2$ at time $t = 1$. Subsequently, this point is sent via the map Y_2 to a point in $C_3 \setminus Q_3$ at time $t = 1 + \frac{1}{2}$, as illustrated in Figure 2.2. This continues for all $j \in \mathbb{N}$ as $t \to T$.

Lemma 2.27. *Let the map X be as defined in* (2.13)*. For all $x \in [0,1)$ the map $t \mapsto X(t,x)$ is continuously differentiable on $[0,T)$.*

Proof. Observe from the definition (2.10) for all $x \in [0,1)$ the map $t \mapsto Y_1(t,x)$ is continuously differentiable on $(0,1)$. It follows that for all with $j \ge 2$ and for each $x \in C_j \setminus Q_j$ the map $t \mapsto Y_j(t,x)$ is continuously differentiable on (t_{j-1}, t_j). Consequently, for each $x \in [0,1)$ the map $t \mapsto X(t,x)$ is continuously differentiable on the open interval (t_{j-1}, t_j) for each j. Further, the continuity of $t \mapsto X(t,x)$ at t_{j-1} for each j follows immediately from the definition (2.13).

It remains to show that $t \mapsto X(t,x)$ is continuously differentiable at each t_{j-1}. From (2.10) it is apparent that

$$\frac{\partial Y_1}{\partial t}(0,x) = \frac{\partial Y_1}{\partial t}(1,x) = 0 \quad \forall\, x \in [0,1)$$

and so, as the Y_j are rescaled Y_1, for each $j \ge 2$

$$\frac{\partial Y_j}{\partial t}(t_{j-1},x) = \frac{\partial Y_j}{\partial t}(t_j,x) = 0 \quad \forall\, x \in C_j \setminus Q_j.$$

Consequently, the derivatives of $t \mapsto X(t,x)$ from above and below at t_{j-1} are equal to zero, so $\frac{\partial X}{\partial t}$ exists and is continuous at t_{j-1}. $\square$

Next, we extend X to the closed time interval $[0,T]$.

Lemma 2.28. *Let the map X be as defined in* (2.13)*. For all $x \in [0,1)$ the limit $\lim_{t\to T} X(t,x)$ exists and take values in the Cantor middle third set C.*

Proof. It is immediate from the definition of Y_1 that

$$Y_1(t,x) \in [0,1) \quad \forall\, t \in [0,1) \quad \forall; x \in [0,1)\,.$$

Consequently, as the Y_j consists of rescaled copies of the map Y_1, for all $j \geq 2$

$$Y_j(t,x) \in C_j \setminus Q_j \quad \forall\, t \in [t_{j-1}, t_j] \quad \forall\, x \in C_j \setminus Q_j.$$

Consequently, for each fixed $x \in [0,1)$

$$X(t,x) \in C_j \quad \forall t \in [t_{j-1}, t_j]\,.$$

As $t \mapsto X(t,x)$ is continuous on $[0,T)$ as $t \to T$ the map $X(t,x)$ takes values in nested intervals whose length tends to zero, so $X(t,x)$ converges as $t \to T$. □

We define the map $\tilde{X}\colon [0,T] \times [0,1) \to [0,1)$ by

$$\tilde{X}(t,x) := \begin{cases} X(t,x) & t \in [0,T) \\ \lim_{t \to T} X(t,x) & t = T. \end{cases} \tag{2.14}$$

Lemma 2.29. *Let the map $\tilde{X}$ be as defined in* (2.14)*. For each $x \in [0,1)$ the map $t \mapsto \tilde{X}(t,x)$ is continuously differentiable on the closed interval $[0,T]$.*

Proof. In light of Lemma 2.27 it is sufficient to demonstrate that X is continuously differentiable at $t = T$. Clearly, from (2.14), $t \mapsto X(t,x)$ is continuous at $t = T$. Further, from the definition (2.10), the derivative

$$\frac{\partial Y_1}{\partial t} = \begin{cases} 2\left(t^2 - t\right)x & 0 \leq x < \frac{1}{2} \\ 2\left(t^2 - t\right)(x-1) & \frac{1}{2} \leq x < 1 \end{cases}$$

and it is straightforward to show that

$$\left|\frac{\partial Y_1}{\partial t}(t,x)\right| \leq \frac{1}{4} \quad \forall\, t \in [0,1]\,, \quad \forall\, x \in [0,1)\,. \tag{2.15}$$

Further, from the definition (2.12), for all $t \in [t_{j-1}, t_j]$ and all $x \in C_j \setminus Q_j$ the derivative

$$\left|\frac{\partial Y_j}{\partial t}\right|(t,x) = \begin{cases} 2\frac{\partial Y_{j-1}}{\partial t}\left(2t-2, 3x\right)/3 & 0 \leq x < \frac{1}{3} \\ 2\frac{\partial Y_{j-1}}{\partial t}\left(2t-2, 3\left(x - \frac{2}{3}\right)\right)/3 + \frac{2}{3} & \frac{2}{3} \leq x < 1, \end{cases}$$

so, inductively from (2.15),

$$\left|\frac{\partial Y_j}{\partial t}(t,x)\right| \le \left(\frac{2}{3}\right)^{j-1}\frac{1}{4} \quad \forall\, t \in [t_{j-1}, t_j] \quad \forall\, x \in C_j \setminus Q_j. \tag{2.16}$$

Consequently, from the definition (2.13), for $t \in (t_{j-1}, t_j]$

$$\left|\frac{\partial X}{\partial t}(t,x)\right| \le \left(\frac{2}{3}\right)^{j-1}\frac{1}{4}$$

and so $\frac{\partial X}{\partial t}(t,x) \to 0$ as $t \to T$. We conclude that for all $x \in [0,1)$ the derivative $\frac{\partial \tilde{X}}{\partial t}(t,x)$ exists and is continuous at $t = T$. □

Next, we demonstrate that the trajectories of $\tilde{X}$ do not intersect:

Lemma 2.30. *Let the map $\tilde{X}$ be as defined in* (2.14). *For all $t \in [0,T]$ the map $x \mapsto \tilde{X}(t,x)$ is injective.*

Proof. Clearly for all $t \in [0,1]$ the map $x \mapsto \tilde{Y}_1(t,x)$ is injective. Consequently, this injectivity is inherited by each Y_j so from the definition (2.13) the map $x \mapsto X(t,x)$ is injective for all $t \in [0,T)$. It remains to show that the map $X(T,\cdot) : [0,1) \to C$ is injective. However, it is straightforward to show that if x is written as non-terminating binary expansion $x = 0.x_1x_2x_3\ldots_2$ where $x_i \in \{0,1\}$ then $\tilde{X}(T,x)$ has non-terminating ternary expansion $\tilde{X}(T,x) = 0.(2x_1)(2x_2)(2x_3)\ldots_3$ (see, for instance, Examples 1, 4 and 14 in Gelbaum and Olmsted [2003]). Recall that $w, z \in [0,1)$ are equal if and only if their ternary expansions $w = 0.w_1w_2\ldots_3$ and $z = 0.z_1z_2\ldots_3$ satisfy either

- $w_i = z_i$ for all $i \in \mathbb{N}$, or
- for some $k \in \mathbb{N}$

$$\begin{cases} w_i = z_i & i < k \\ w_i = 0, z_i = 1 & i = k \\ w_i = 2, z_i = 0 & i > k, \end{cases} \qquad \text{or} \qquad \begin{cases} w_i = z_i & i < k \\ w_i = 1, z_i = 0 & i = k \\ w_i = 0, z_i = 2 & i > k. \end{cases}$$

Consequently, if $x = 0.x_1x_2x_3\ldots_2$ and $y = 0.y_1y_2y_3\ldots_2$ then $X(T,x) = X(T,y)$ if and only if

$$0.(2x_1)(2x_2)(2x_3)\ldots_3 = 0.(2y_1)(2y_2)(2y_3)\ldots_3.$$

However, as no digit in these expansions is equal to 1 this equality holds if and only if $x_i = y_i$ for all i, in which case $x = y$. We conclude that $x \mapsto \tilde{X}(T,x)$ is injective. □

From the above Lemma we see that for each $t \in [0,T]$ the inverse $\tilde{X}^{-1}(t,x)$ is defined for all $x \in \tilde{X}(t,[0,1))$. Consequently, the function defined by

$$g(x,t) := \frac{\partial \tilde{X}}{\partial t}\left(\tilde{X}^{-1}(t,x)\right) \quad \forall x \in \tilde{X}(t,[0,1)) \quad \forall t \in [0,T]$$

trivially satisfies

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} = g\left(\tilde{X}(t,x),t\right) \quad \forall x \in \tilde{X}(t,[0,1)) \quad \forall t \in [0,T].$$

To summarise, we have defined an aggregate of non-intersecting trajectories of the ordinary differential equation $\dot{X} = g(X,t)$, defined on the image of $\tilde{X}$. It is straightforward to extend g and $\tilde{X}$ onto the entire space so that $\dot{X} = g(X,t)$ is satisfied everywhere (for example, set $g = 0$ outside the image of $\tilde{X}$ so that every point is fixed). Finally, as the trajectories do not intersect it is trivial to further extend $\tilde{X}$ to provide trajectories of the ordinary differential equation for initial times s other than $s = 0$. The resulting map is a classical flow solution of $\dot{X} = g(X,t)$.

This example can be adapted to generalised Cantor sets (as discussed in Chapter 6) to produce flows such that the image of the unit interval has arbitrary fractal dimension less than 1 in finite time.

2.2 Generalised solutions

There are two considerations in identifying a suitably weakened notion of solution of (ODE) for irregular vector fields $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n \times [0,T];\mathbb{R}^n)$: first, through the equality (ODE) any regularity requirements on the solution become regularity constraints on the class of vector fields we hope to define solutions for. Indeed, the existence of a classical solution, having a continuous derivative, has the strong requirement that the vector field f is continuous along trajectories, which precludes a large class of vector fields from having solutions to (ODE). The standard approach to relaxing this strong requirement is to require (ODE) to hold distributionally and accordingly we give the following definition, which does not require the solution to be differentiable in t.

Definition 2.31. *The map X is a* weak flow solution *of* (ODE) *if*

(i) *the maps X and $f \circ (X, \mathrm{id})$ are in $\mathcal{L}^1_{\text{loc}}((0,T)_t \times \mathbb{R}^n_x \times (0,T)_s;\mathbb{R}^n)$, and*

(ii) *the integral equalities*

$$\int_0^s X(t,x,s)\,\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\,\mathrm{d}t + \int_0^s f(X(t,x,s),t)\,\phi(t)\,\mathrm{d}t - x\;\phi(s) = 0 \quad (2.17)$$

and

$$\int_s^T X(t,x,s)\,\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\,\mathrm{d}t + \int_s^T f(X(t,x,s),t)\,\phi(t)\,\mathrm{d}t + x\;\phi(s) = 0 \quad (2.18)$$

hold for all test maps $\phi \in C_c^\infty((0,T))$, for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$.

Note that the condition (i) ensures that the integrals in equation (2.17) and (2.18) are defined and finite for almost every $(x,s) \in \mathbb{R}^n \times (0,T)$. We remark that we can extend the condition (ii) to account for initial data at $s=0$, which is significant in many applications, by requiring that

$$\int_0^T X(t,x,s)\,\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\,\mathrm{d}t + \int_0^T f(X(t,x,s),t)\,\phi(t)\,\mathrm{d}t + x\phi(0) = 0$$

for all test maps $\phi \in C_c^\infty([0,T))$.

The second consideration in the search for a suitably weakened notion of solution is a desire to use the tools of functional analysis to derive the existence, uniqueness and other properties of solutions: to demonstrate existence the usual approach (see for example, DiPerna and Lions [1989], Ambrosio [2004], and Crippa and De Lellis [2008]) is to approximate f in some sense with smooth globally Lipschitz vector fields f_ε, each of which give rise to a unique generalised flow solution X_ε and to demonstrate that the flows X_ε converge in some sense to a map X satisfying (i) and (ii) above.

To be able to apply such approximation arguments we must operate on elements of the Banach space $L^1(\mathbb{R}^n \times [0,T])$, which are equivalence classes of vector fields that are equal almost everywhere. Consequently, an appropriate notion of solution must be independent of the representative vector field of the equivalence class, which is to say that if X is a solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X,t)$ then for any vector field g equal to f almost everywhere we require that X is a solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = g(X,t)$. The appropriate restriction is to require the map X to satisfy one of the properties of Definition 2.20 relaxed to hold for almost every $t,s \in [0,T]$. In some treatments (for example in Hauray et al. [2007]) incompressible maps are studied for convenience, although much of the theory was developed for nearly incompressible maps (for example, in DiPerna and Lions [1989], and the extension to BV vector fields in Ambrosio [2004]). However, as noted in De Lellis [2008] a relaxed version of the

Lusin condition (iii) of Definition 2.20 is sufficient for the invariance of solutions under a change of vector field.

Definition 2.32. *The map* $X\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ *is a* regular Lagrangian flow solution *of* (ODE) *if*

(i) X *is a weak flow solution of* (ODE), *and*

(ii) *for each Borel set* $A \subset \mathbb{R}^n$ *such that* $\mu_n(A) = 0$ *the pushforward measure of* A, $X(t, \cdot, s)_{\#}\, \mu_n(A) = 0$ *for almost every* $t \in [0,T]$, *for almost every* $s \in [0,T]$.

Observe that in the example of Section 2.1.5 the map X is a weak flow solution but is not a regular Lagrangian flow solution as the inverse image of the Cantor set (which has zero measure and is a Borel set as it is the intersection of a countable number of half open intervals) has positive measure.

In the following lemma we demonstrate that a regular Lagrangian flow solution is invariant under equivalent vector fields. The proof follows a sketch in De Lellis [2008]. We first define the following notation:

Definition 2.33. *Let* $S \subset \mathbb{R}^n \times [0,T]$. *For each* $t \in [0,T]$ *we write*

$$S^t := \{x \in \mathbb{R}^n |\, (x,t) \in S\}.$$

and call such sets the sections of S.

Lemma 2.34. *Let* $X\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ *be a regular Lagrangian flow solution of* (ODE). *For every vector field* g *such that*

$$g(x,t) = f(x,t) \quad \text{for almost every } (x,t) \in \mathbb{R}^n \times [0,T]$$

the map X *is a regular Lagrangian flow solution of* $\frac{\mathrm{d}X}{\mathrm{d}t} = g(X,t)$.

Proof. Let $f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ be a vector field, X a regular Lagrangian flow solution of (ODE) and g a vector field equal to f almost everywhere. From Definition 2.31 we wish to demonstrate that $g \circ (X, \mathrm{id}_t)$ is in $\mathcal{L}^1_{\mathrm{loc}}\left((0,T)_t \times \mathbb{R}^n_x \times (0,T)_s ; \mathbb{R}^n\right)$ and that (2.17) and (2.18) hold with g in place of f. Consider the push-forward of the $(n+2)$-dimensional Lebesgue measure via the map $(t,x,s) \mapsto (t, X(t,x,s), s)$ which we denote

$$(\mathrm{id}_t, X, \mathrm{id}_s)_{\#}\, \mu_{n+2}(A) := \mu_{n+2}\left(\{(t,x,s) \,|\, (t, X(t,x,s), s) \in A\}\right)$$

for $A \subset [0,T] \times \mathbb{R}^n \times [0,T]$. As the map $(t,x,s) \mapsto X(t,x,s)$ is measurable this push-forward is a measure on the Borel σ-algebra of $[0,T] \times \mathbb{R}^n \times [0,T]$. Now, let $N := \{(t,x) \,|\, f(x,t) \neq g(x,t)\}$ so by assumption $\mu_{n+1}(N) = 0$ and consequently there exists a Borel set U containing N with $\mu_{n+1}(U) = 0$. By Fubini's theorem the push-forward measure of the product set $U \times [0,T]$ is given by

$$
\begin{aligned}
(\mathrm{id}_t, X, \mathrm{id}_s)_{\#} \mu_{n+2}(U \times [0,T]) &= \int_0^T \int_0^T \mu_n(\{x \,|\, (t, X(t,x,s), s) \in U \times [0,T]\}) \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_0^T \int_0^T \mu_n(\{x \,|\, (t, X(t,x,s)) \in U\}) \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_0^T \int_0^T X(t,\cdot,s)_{\#} \mu_n(U^t) \, \mathrm{d}s \, \mathrm{d}t.
\end{aligned}
$$

where $U^t := \{x \in \mathbb{R}^n \,|\, (t,x) \in U\}$. However, again by Fubini's theorem, $\mu_n(U^t) = 0$ for almost every $t \in [0,T]$ so, as the map X satisfies the Lusin condition (ii) of Definition 2.32, this final integral is equal to zero. Consequently the set of points (t,x,s) for which $f(X(t,x,s),t) \neq g(X(t,x,s),t)$

$$
\{(t,x,s) \,|\, (t, X(t,x,s), s) \in N \times [0,T]\} \subset \{(t,x,s) \,|\, (t, X(t,x,s), s) \in U \times [0,T]\},
$$

has μ_{n+2} measure zero. We conclude that

$$
f \circ (X, \mathrm{id}_t) = g \circ (X, \mathrm{id}_t) \quad \text{for almost every } (t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]
$$

so $g \circ (X, \mathrm{id}_t) \in \mathcal{L}^1_{loc}((0,T)_t \times \mathbb{R}^n \times (0,T)_s)$. Applying Fubini's theorem once more, we see that $f(X(t,x,s),t) = g(X(t,x,s),t)$ for almost every $t \in [0,T]$ and almost every $(x,s) \in \mathbb{R}^n \times [0,T]$ so certainly (2.17) and (2.18) hold with g in place of f. □

Essentially, we can interchange vector fields that are equal almost everywhere as we only require (ODE) to be satisfied at almost every point. Similarly, any map equal to a a regular Lagrangian flow solution almost everywhere is also a regular Lagrangian flow solution. This is mentioned in passing in the literature (see, for example, De Lellis [2008]) but here we provide the following proof:

Lemma 2.35. *If X is a regular Lagrangian flow solution of* (ODE) *and $X(t,x,s) = Y(t,x,s)$ for almost every $(t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]$ then Y is also a regular Lagrangian flow solution of* (ODE).

Proof. Immediately,

$$
f(X(t,x,s),t) = f(Y(t,x,s),t) \quad \text{for almost every } (t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]
$$

so Y is a weak solution of (ODE). Further, let $N := \{(t,x,s) \,|X(t,x,s) \neq Y(t,x,s)\}$ and note that $N_{t,s} := \{x| \,(t,x,s) \in N\}$ has measure zero for almost every $t \in [0,T]$, for almost every $s \in [0,T]$. Consequently, for all sets $A \subset \mathbb{R}^n$

$$\begin{aligned}\{x|Y(t,x,s) \in A\} &= (\{x|Y(t,x,s) \in A\} \setminus N_{t,s}) \cup (\{x|Y(t,x,s) \in A\} \cap N_{t,s}) \\ &= (\{x|X(t,x,s) \in A\} \setminus N_{t,s}) \cup (\{x|Y(t,x,s) \in A\} \cap N_{t,s})\end{aligned}$$

so for Borel sets $A \subset \mathbb{R}^n$

$$Y(t,\cdot,s)_{\#}\,\mu_n(A) = X(t,\cdot,s)_{\#}\,\mu_n(A) + \mu_n(N_{t,s}).$$

Consequently, for almost every $s \in [0,T]$, for almost every $t \in [0,T]$

$$Y(t,\cdot,s)_{\#}\,\mu_n(A) = X(t,\cdot,s)_{\#}\,\mu_n(A).$$

so Y satisfies the almost everywhere Lusin condition (ii) of Definition 2.32. □

Accordingly, we say that a regular Lagrangian flow solution X of (ODE) is unique if every regular Lagrangian flow solution of (ODE) is equal to X almost everywhere on $[0,T] \times \mathbb{R}^n \times [0,T]$.

2.3 Existence of regular Lagrangian flows

To demonstrate the existence of regular Lagrangian flow solutions the seminal paper of DiPerna and Lions [1989] exploits the relationship between the equations (ODE), (TE) and (CE) described above for regular vector fields. First, the authors demonstrate that it is comparatively straightforward to establish the existence of weak solutions of (TE) and (CE) under mild assumptions on the vector field f. Further, the authors demonstrate that if a vector field f has the following renormalization property, then these solutions are unique.

Definition 2.36. *A vector field $f \in L^1_{\text{loc}}(\mathbb{R}^n \times [0,T])$ has the* **renormalization property** *if for every $\beta \in C^1(\mathbb{R})$ whenever the map $u\colon\ [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a weak solution of* (TE), *respectively* (CE), *with initial data $u_s \in \mathcal{L}^\infty(\mathbb{R}^n)$, then the map $\beta(u)$ is a weak solution of* (TE), *respectively* (CE), *with initial data $\beta(u_s)$.*

Essentially, the argument is that for any solution u of (TE) with initial data equal to zero by the renormalization property the non-negative map u^2 is also a solution of (TE) with initial data equal to zero. It can be shown (see for example, Theorem 1.7 in De Lellis [2008]) that this non-negative map satisfies a distributional form of Gronwall's inequality from which we conclude that $u = 0$. Once the uniqueness of (TE) and (CE) are established the existence and uniqueness of regular Lagrangian flow solutions follow by an approximation argument: mollified vector fields f^ε give rise to smooth flow solutions X^ε of (ODE), which from Propositions 2.5 and 2.7 yield solutions u_ε to (TE) and ρ_ε to (CE). Consequently, u_ε and ρ_ε converge to the unique solutions of (TE) and (CE) respectively, and the unique flow solution of (ODE) can be recovered from these solutions by a 'reverse' theory of characteristics.

It remains to highlight which vector fields f have the renormalization property, and so by the above argument have a unique regular Lagrangian flow solution of (ODE). The two main results are contained in DiPerna and Lions [1989], in which the authors demonstrate that vector fields belonging to

$$L^1\left(0,T;W^{1,1}_{\text{loc}}\left(\mathbb{R}^n;\mathbb{R}^n\right)\right)$$

have the renormalization property, and in Ambrosio [2004], in which the authors demonstrate that this property extends to vector fields belonging to

$$L^1\left(0,T;BV\left(\mathbb{R}^n;\mathbb{R}^n\right)\right).$$

2.4 From regular Lagrangian flows to absolutely continuous flows

In the first result of this section we demonstrate that in order for a map X to satisfy (ODE) distributionally it is necessary for the map X to have some Sobolev regularity with respect to t. This was observed in DiPerna and Lions [1989] and the authors subsequently concluded that X is absolutely continuous in t using the well known embedding result of Lemma B.9. Absolute continuity of the flow with respect to time is desirable as it allows us to interpret (ODE) in the classical sense, albeit with the equality holding almost everywhere, which greatly simplifies the treatment of regular Lagrangian flows.

Initially, we recall that the above claim of DiPerna & Lions is that there exists a map $\tilde{X}$ equivalent to X in some sense to be made precise such that $\tilde{X}$ is absolutely continuous in t. After we make this equivalence precise we use a result from Section 7.2.2 to demonstrate that the map $\tilde{X}$ may not be measurable from

$[0,T]_t \times \mathbb{R}^n \times [0,T]_s$ to $\mathbb{R}^n$. Consequently, $\tilde{X}$ may not be a regular Lagrangian flow solution of (ODE) and in particular the pushforward measures $\tilde{X}(t,\cdot,s)_{\#}\mu_n$ used throughout the theory may not be defined so this potential loss of measurability is not trivial.

We continue this discussion by highlighting possible approaches to solving this measurability issue, in particular an approach inspired by a standard result in the theory of Sobolev maps. Fortunately since the preliminary submission of this thesis we have been able to address these issues, which we delay until Chapter 8 as the results are heavily dependent on the material of Chapter 7.

We begin by demonstrating that weak solutions of (ODE) have some Sobolev regularity, which is the content of the following lemma:

Lemma 2.37. *If the map $X\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ is a weak flow solution of* (ODE) *then the map*

$$t \mapsto X(t,x,s) \ \textit{is in} \ \mathcal{W}^{1,1}([0,T]) \quad \textit{for almost every}\,(x,s) \in \mathbb{R}^n \times [0,T]$$

where $\mathcal{W}^{1,1}([0,T])$ is the space of Sobolev maps defined in Definition B.5 of Appendix B.

Proof. Immediately from Definition 2.31 we see that the integral quantity

$$\int_0^T X(t,x,s)\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^T f(X(t,x,s),t)\,\phi(t)\,\mathrm{d}t \quad \forall \phi \in C_c^\infty((0,T))$$

for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$, which is precisely the statement that the map

$$t \mapsto X(t,x,s)$$

has weak derivative given by the map

$$t \mapsto f(X(t,x,s),t)$$

for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$. Further, as the maps X and $f(X,\mathrm{id})$ are in $\mathcal{L}^1([0,T]_t \times \mathbb{R}^n_x \times [0,T]_s)$ then, by Fubini's theorem, the map $t \mapsto X(t,x,s)$ and its weak derivative $t \mapsto f(X(t,x,s),t)$ are in $\mathcal{L}^1([0,T])$ for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$, so we conclude that the map $t \mapsto X(t,x,s)$ is in $\mathcal{W}^{1,1}([0,T])$ for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$. $\square$

Next, we see from Lemma B.9 that for each $(x,s) \in \mathbb{R}^n \times [0,T]$ such that the map $t \mapsto X(t,x,s)$ is in $\mathcal{W}^{1,1}([0,T])$ there exists an absolutely continuous map

$\tilde{X}_{x,s}\colon [0,T]_t \to \mathbb{R}^n$ equal to $X(t,x,s)$ for almost every $t \in [0,T]_t$. Consequently, we have the following corollary of the above lemma:

Corollary 2.38. *If the map X satisfies the hypothesis of Lemma 2.37 then there exists a map $\tilde{X}\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ such that*

$$t \mapsto \tilde{X}(t,x,s) \text{ is absolutely continuous for almost every } (x,s) \in \mathbb{R}^n \times [0,T]$$

and

$$\Big[X(t,x,s) = \tilde{X}(t,x,s) \quad \text{for almost every } t \in [0,T]\Big] \\ \text{for almost every } (x,s) \in \mathbb{R}^n \times [0,T]. \qquad (2.19)$$

In fact, in light of Lemma B.9 we can explicitly write the map $\tilde{X}$ (up to equality for almost every (x,s)) as

$$\tilde{X}(t,x,s) := \begin{cases} \lim_{\varepsilon\to 0} \fint_{t-\varepsilon}^{t+\varepsilon} X(\tau,x,s)\,\mathrm{d}\tau & \text{if this limits exists, and} \\ 0 & \text{otherwise.} \end{cases}$$

However, as we demonstrate in Section 7.2.2 the equality between X and $\tilde{X}$ given by (2.19) is significantly weaker than equality for μ_{n+2}-almost every $(t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]$. In particular, as we illustrate in Lemma 7.16, the equality (2.19) is not sufficient to ensure that $\tilde{X}$ inherits the measurability of X, which is to say that the map

$$\tilde{X}\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$$

may not be measurable.

This measurability issue is resolved in Chapter 8 where in Corollary 8.5 we demonstrate that the map $\tilde{X}$ given above is measurable. Consequently, in the remainder we exclusively consider regular Lagrangian flows that are absolutely continuous in t for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$. In Theorem 8.6 we demonstrate that by restricting our attention to the absolutely continuous representative of a regular Lagrangian flow we can use the following equivalent definition used in Lions [1998], Hauray et al. [2007] and Crippa and De Lellis [2008]:

Definition 2.39. *A map $\xi\colon [0,T] \to \mathbb{R}^n$ is an* **absolutely continuous solution** *of* (ODE) *with initial data $(x,s) \in \mathbb{R}^n \times [0,T]$ if*

- *ξ is absolutely continuous, and*
- *$\xi(t) = x + \int_s^t f(\xi(\tau),\tau)\ \mathrm{d}\tau \quad \forall; t \in [0,T]$.*

Equivalently, ξ is an absolutely continuous solution of (ODE) with initial data (x, s) if

- ξ is absolutely continuous,
- $\xi(s) = x$, and
- $\frac{\mathrm{d}\xi}{\mathrm{d}t} = f(\xi(t), t)$ for almost every $t \in [0, T]$.

Clearly absolutely continuous solutions are weaker than classical solutions. Henceforth by 'solution' of (ODE) we mean an absolutely continuous solution.

Definition 2.40. *A map $X\colon [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ is a* **regular Lagrangian flow solution** *of* (ODE) *if*

- $X \in \mathcal{L}^1_{\mathrm{loc}}\left([0,T]_t \times \mathbb{R}^n_x \times [0,T]_s ; \mathbb{R}^n\right)$
- *for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$*

$$X(t,x,s) = x + \int_s^t f(X(\tau,x,s),\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T], \quad \textit{and}$$

- *for each Borel set $A \subset \mathbb{R}^n$ with $\mu_n(A) = 0$ the pushforward*

$$\mu_n(\{x \in \mathbb{R}^n | X(t,x,s) \in A\}) = 0$$

  for all $t \in [0,T]$, for almost every $s \in [0,T]$.

Consequently, we see that a regular Lagrangian flow solution is an aggregate of absolutely continuous solutions of (ODE) that satisfy the Lusin condition.

2.4.1 A related result in the theory of Sobolev maps

In this section we outline an approach to considering the measurability of the absolutely continuous map $\tilde{X}$. Ultimately, the result is proved in Chapter 8 and does not require any of the material in this section. This section is included as it appears in the original thesis before we developed the results of Chapter 8 and may be skipped by the reader.

We consider the following problems:

(OP 1) Can the equality (2.19) be strengthened to equality for almost every $(t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]$?

(OP 2) Conversely, is there a vector field f such that for every regular Lagrangian flow solution X of (ODE) and every map $\tilde{X}$ satisfying the result of Corollary 2.38 the map $\tilde{X}$ is not measurable from $[0,T]_t \times \mathbb{R}^n_x \times [0,T]_s$ into $\mathbb{R}^n$?

We suggest an avenue that may lead to a positive answer for (OP 1) which takes inspiration from the following classical result in the theory of Sobolev maps:

Theorem 2.41. *If $g \in \mathcal{W}^{1,1}(\mathbb{R}^n)$ then there exists a map h, equal to g almost everywhere on $\mathbb{R}^n$ such that h is absolutely continuous on almost every line parallel to a coordinate axis, which is to say that for all $i = 1 \dots n$ the map*

$$t \mapsto h(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$$

is absolutely continuous in t for almost every $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$.

Proof. See §4.9.2 of Evans and Gariepy [1992]. □

Naïvely we may try and prove the above theorem using a similar approach to Corollary 2.38: for a fixed i (we take $i = n$ for brevity) and $\tilde{x} := (x_1, \dots, x_{n-1}) \in \mathbb{R}^n$ we consider the map

$$\begin{aligned} g_{n,\tilde{x}} \colon \mathbb{R} &\to \mathbb{R} \\ t &\mapsto g(\tilde{x}, t) \end{aligned}$$

which by Fubini's theorem, is in $\mathcal{W}^{1,1}(\mathbb{R})$ for almost every $\tilde{x} \in \mathbb{R}^{n-1}$. Consequently, by Lemma B.9 there exists an absolutely continuous map $h_{n,\tilde{x}} \colon \mathbb{R} \to \mathbb{R}$ that is equal to $g_{n,\tilde{x}}$ almost everywhere on $\mathbb{R}$. However, we have the same problem as Corollary 2.38: the aggregate of these maps $h_n(x) := h_{n,\tilde{x}}(x_n)$ is absolutely continuous in x_n for almost every $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ however we only have the equality

$$[g(x) = h_n(x) \quad \text{for a.e. } x_n \in \mathbb{R}] \quad \text{for a.e. } (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$$

so, in light of the discussion in Section 7.2.2, the map h_n is not necessarily equal to g almost everywhere.

We sketch the proof of the above theorem, which is surprisingly involved and relies on the content of Appendix B. We recall from Proposition B.7 that the mollifiers g^ε of g satisfy

- $g^\varepsilon \to g$ in $\mathcal{W}^{1,1}(\mathbb{R}^n)$,
- $g^\varepsilon(x) \to g(x)$ for every point $x \in \mathbb{R}^n$ that is a Lebesgue point of g.

Further, we recall from Theorem B.8 that for the precise representative g^* of g there is a set $N \subset \mathbb{R}^n$ with $\mathcal{L}^{n-1}(N) = 0$ such that every point $x \in \mathbb{R}^n \setminus N$ is a Lebesgue point of g^*. Next, as the projection of this set

$$P(N) = \left\{ \tilde{x} \in \mathbb{R}^{n-1} \middle| (\tilde{x}, x_n) \in N \quad \text{for some } x_n \in \mathbb{R} \right\}$$

also has $\mu_{n-1}(P(N)) = 0$, we conclude that for almost every $\tilde{x} \in \mathbb{R}^{n-1}$ the point $(\tilde{x}, x_n)$ is a Lebesgue point of g^* for all $x_n \in \mathbb{R}$. Consequently, the mollifier

$$g^{\varepsilon}(\tilde{x}, x_n) \to g^*(\tilde{x}, x_n) \quad \text{for all } x_n \in \mathbb{R}, \quad \text{for almost every } \tilde{x} \in \mathbb{R}^{n-1}. \tag{2.20}$$

Next, a straightforward application of Fubini's theorem to the convergence $g^{\varepsilon} \to g^*$ in $\mathcal{W}^{1,1}(\mathbb{R}^n)$ implies the convergence

$$g^{\varepsilon}(\tilde{x}, \cdot) \to g^*(\tilde{x}, \cdot) \quad \text{in } \mathcal{W}^{1,1}(\mathbb{R}) \quad \text{for almost every } \tilde{x} \in \mathbb{R}^{n-1}. \tag{2.21}$$

Finally, we know from Lemma B.9 that the convergence (2.21) implies that there is a continuous representative $g_{\tilde{x}}^{\text{cont.}}$ of $g^*(\tilde{x}, \cdot)$, and further from the properties of mollifiers in Proposition B.7, the values $g^{\varepsilon}(\tilde{x}, x_n)$ converge to $g_{\tilde{x}}^{\text{cont.}}(x_n)$ for all $x_n \in \mathbb{R}$, so from (2.20) $g_{\tilde{x}}^{\text{cont.}}(x_n) = g^*(\tilde{x}, x_n)$ for all $x_n \in \mathbb{R}$.

We conclude that the precise representative g^* is absolutely continuous on almost every line parallel to the x_n axis. Similarly, by considering the other projections we can demonstrate that g^* is absolutely continuous on almost every line parallel to a coordinate axis. Finally, it is trivial that the precise representative retains the measurability properties of g.

In order to adapt this method to the regular Lagrangian flow solution and obtain a positive answer to (OP 1) we require that the precise representative X^* of X has Lebesgue points outside a set $N \subset [0,T] \times \mathbb{R}^n \times [0,T]$ where $\mu_{n+1}(N) = 0$. However, as we only wish to demonstrate continuity with respect to t the above proof is sufficient to answer (OP 1) under the weaker hypothesis that the point (t, x, s) is a Lebesgue point of X^* for all $t \in [0,T]$, for almost every $(x, s) \in \mathbb{R}^n \times [0,T]$, which we leave as an open problem.

Chapter 3

Uniqueness

3.1 Trajectory non-uniqueness

In Example 2.19 we demonstrated that there are multiple solutions for each initial condition by finding trajectories that intersect and proving that the concatenation of these trajectories at the point of intersection is differentiable and so defines an additional solution. However, generally there is no guarantee that a concatenation of solutions is differentiable so in the general case we do not necessarily define additional solutions through this process. However, in the weaker setting for vector fields in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ where we consider regular Lagrangian flow solutions we only require solutions to be absolutely continuous and satisfy the pointwise equality (ODE) almost everywhere. Consequently, as the concatenation procedure introduces at most one extra point of non-differentiability, in this weaker setting the concatenation of arbitrary intersecting solutions is always a solution, which is the content of the following theorem. For clarity we restrict our attention to autonomous ODEs throughout this chapter and remark that the main results are easily adapted to the non-autonomous case. For the autonomous case the solutions do not depend upon an initial time. Consequently for each regular Lagrangian flow solution X of (ODE) we suppress the third argument and define

$$X(t,x) := X(t,x,0).$$

Theorem 3.1. *If $\xi_1, \xi_2 \colon [0,T] \to \mathbb{R}^n$ are solutions of* (ODE) *and $\xi_1(\tau) = \xi_2(\tau)$ for some $\tau \in [0,T]$ then the concatenation of ξ_1 to ξ_2 at time τ*

$$\vee_\tau(\xi_1,\xi_2)(t) := \begin{cases} \xi_1(t) & t < \tau \\ \xi_2(t) & \tau \le t \end{cases}$$

is a solution of (ODE).

Proof. For notational clarity let $\xi \equiv \vee_\tau(\xi_1, \xi_2)$. From Lemma B.11, the map ξ is absolutely continuous and so is differentiable almost everywhere. Further, whenever the derivative exists, for $t \in [0,\tau) \cup (\tau, T]$

$$\begin{aligned}\dot{\xi}(t) &= \begin{cases}\dot{\xi}_1(t) & t < \tau\\ \dot{\xi}_2(t) & \tau < t\end{cases}\\ &= \begin{cases}f(\xi_1(t)) & t < \tau\\ f(\xi_2(t)) & \tau < t\end{cases}\\ &= f(\xi(t)).\end{aligned}$$

Consequently, the derivative $\xi'(t)$ is equal to $f(\xi(t))$ for almost every $t \in \mathbb{R}$. $\square$

3.1.1 A scalar example

In the following we extract the desirable properties of Example 2.19: as we are considering generalised flows we only require the trajectories to be differentiable almost everywhere, so in particular the trajectories do not have to be differentiable when they arrive at or leave the origin. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the autonomous function

$$f(x) = \begin{cases}1 & x \neq 0\\ 0 & x = 0.\end{cases}$$

Immediately, we see that $f = 1$ almost everywhere, so by Lemma 2.35 X is a regular Lagrangian flow solution to (ODE) if and only if X is a regular Lagrangian flow solution to

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = 1.$$

This ordinary differential equation trivially has the unique regular Lagrangian flow solution $X(t,x) = x + t$ which is clearly measure preserving.

Alternatively, we can naïvely plug the function f into the results of DiPerna and Lions [1989]: clearly $f \in \mathcal{L}^\infty(\mathbb{R})$ so f is arbitrarily locally integrable. As the function f is equal to one except at $x = 0$ the derivative f' exists and is equal to zero for all $x \in \mathbb{R} \setminus \{0\}$. Consequently, for each $\phi \in C_c^\infty(\mathbb{R})$

$$0 = \int_{\mathbb{R}} \phi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\phi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} f'(x)\,\phi(x)\,\mathrm{d}x$$

so f has weak derivative and weak divergence equal to zero. Consequently, from the results of DiPerna and Lions [1989] discussed in Section 2.3 there exists a unique regular Lagrangian flow solution to (ODE).

The absolutely continuous maps $\phi(t) := 0$, $\xi(t) := t$ and the temporal translate $\xi_s(t) := t - s$ are solutions of (ODE) and as their trajectories intersect we can use Theorem 3.1 to define new trajectories. Essentially, as $f = 1$ away from the fixed point at the origin, every solution intersects the origin where they can wait for arbitrary time, which resembles the qualitative features of the classic Example 2.19.

For each $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$ the map

$$\xi_{a,b}(t) = \begin{cases} (t-a) & t < a \\ 0 & a \leq t \leq b \\ (t-b) & b < t \end{cases}$$

is a solution of (ODE). Further,

- for each inital data $x < 0$ the family of trajectories $\xi_{-x,b}$ parameterised by $b \in [-x, \infty]$ satisfy the initial condition $\xi(0) = x$
- for each initial data $x > 0$ the family of trajectories $\xi_{a,-x}$ paramterised by $a \in [-\infty, -x]$ satisfy the initial condition $\xi(0) = x$
- the family of trajectories $\xi_{a,b}$ parameterised by $a, b \in [-\infty, \infty]$ such that $a \leq b$ satisfy the initial condition $\xi(0) = 0$.

This example exhibits non-uniqueness on a large scale: for each initial data $x \in \mathbb{R}$ there are uncountably many distinct trajectories ξ of (ODE) with initial condition $\xi(0) = x$. However, there is a unique regular Lagrangian flow solution to (ODE).

It is possible to demonstrate that this flow is unique without invoking any of the DiPerna-Lions theory, nor relying on the results of Lemma 2.35. To do this we first demonstrate that all solutions of (ODE) have the form $\xi_{a,b}$, which follows as the function f is locally Lipschitz away from the origin. Next, we let X be a regular Lagrangian flow solution of (ODE), which has the form

$$X(t, x) := \xi_{a_x, b_x}(t)$$

for some a_x, b_x such that the initial condition $X(0, x) = x$ is satisfied. Now, for $a, b \in \mathbb{R}$ with $a < b$ we define the set of initial data $A_{a,b} := \{x \in \mathbb{R} | X(t, x) = 0 \quad \text{for } t \in (a, b)\}$. As X is a regular Lagrangian flow, $\mu_1(A_{a,b}) = 0$ for all $a < b$ as otherwise a set of positive measure is compressed to a point under the action of the flow at all times

$t \in (a, b)$. Consequently, the set

$$\{x \in \mathbb{R} | a_x < b_x\} \subset \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} A_{p,q}$$

has measure zero, so we conclude that for almost every $x \in \mathbb{R}$ the parameters a_x and b_x are equal, so the regular Lagrangian flow $X(t, x) = \xi_{-x,-x}(t)$ for all $t \in \mathbb{R}$ and almost every $x \in \mathbb{R}$.

3.1.2 A planar example

In the following we give an example of a vector field $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ for which every initial data $(x, y) \in \mathbb{R}^2$ gives rise to multiple distinct trajectories of (ODE) yet there is a unique regular Lagrangian flow solution. Further, each component of f is non-zero at every point.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the autonomous vector field

$$f(x, y) := \begin{cases} (1, 1) & x = y \\ (2, 1) & x \neq y \end{cases} \tag{3.1}$$

illustrated in Figure 3.1.

Again, the vector field is equal almost everywhere to the vector field $g \equiv (2, 1)$ which clearly has a unique measure preserving flow solution. Alternatively, we can reason as in the previous example by constructing the flow out of individual solutions: as f is locally Lipschitz away from $\{x = y\}$ solutions are unique outside a neighbourhood of $\{x = y\}$. However, we will see that every solution intersects the set $\{x = y\}$ in finite time where it may remain for an arbitrary time interval. In the same manner as Example 2.19 and the scalar example above, this arbitrary wait yields non-uniqueness of solutions for every initial data in $\mathbb{R}^2$. Once again we will see that the unique regular Lagrangian flow solution consists of those individual solutions that do not 'loiter' on the line $\{x = y\}$.

- for each initial data (x, y) with $x < y$ and each $b \in [y - x, \infty]$ the map

$$\xi^+_{(x,y),b}(t) = \begin{cases} (x + 2t, y + t) & t < y - x \\ (y + t, y + t) & y - x \leq t < b \\ (y + b + 2(t - b), y + t) & b \leq t \end{cases}$$

  is a solution of (ODE) with initial condition $\xi^+_{(x,y),b}(0) = (x, y)$,

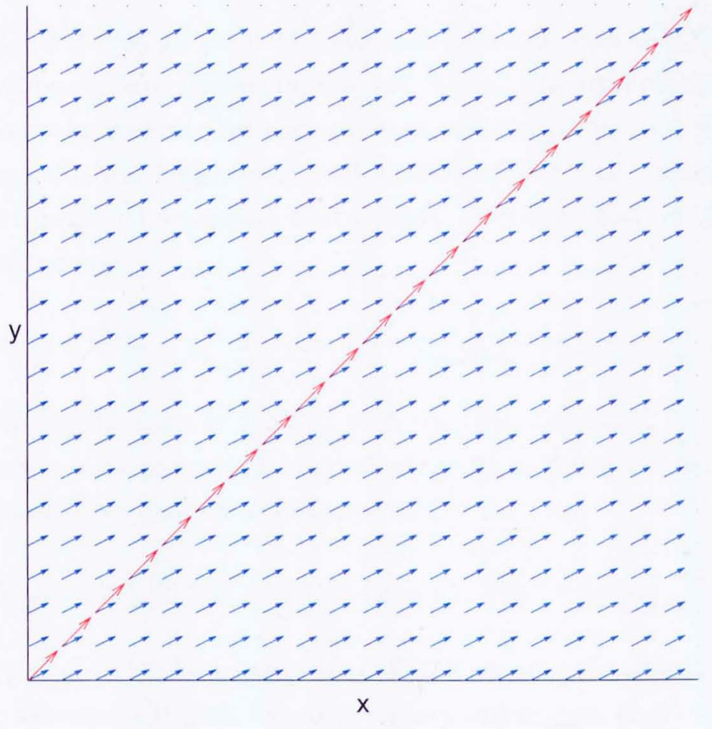

Figure 3.1: A plot of the vector field f. Clearly every trajectory of (ODE) intersects the line $x = y$, and as the line $x = y$ is invariant, solutions may remain on the line for arbitrary time. Consequently, every initial data of (ODE) has non-unique solutions.

- for each initial data (x, y) with $x > y$ and each $a \in [-\infty, y - x]$ the map

$$\xi^{-}_{(x,y),a}(t) = \begin{cases} (x + a + 2(t - a), y + t) & t < a \\ (y + t, y + t) & a \leq t < y - x \\ (x + 2t, y + t) & y - x \leq t \end{cases}$$

  is a solution of (ODE) with initial condition $\xi^{-}_{(x,y),a}(0) = (x, y)$, and

- for all initial data (x, y) with $x = y$, each $a \in [-\infty, 0]$ and $b \in [0, \infty]$ the map

$$\xi^{0}_{(x,y),a,b}(t) = \begin{cases} (x + a + 2(t - a), y + t) & t < a \\ (x + t, y + t) & a \leq t < b \\ (x + b + 2(t - b), y + t) & b \leq t \end{cases}$$

is a solution of (ODE) with initial condition $\xi^0_{(x,y),a,b}(0) = 0$.

From Theorem 3.1 it is easy to see that these maps are solutions of (ODE) as they are the concatenation of three trivial trajectories. Again, this example demonstrates non-uniqueness on a large scale: for each piece of initial data $(x, y) \in \mathbb{R}^2$ there are uncountably many distinct trajectories ξ of (ODE) with initial condition $\xi(0) = (x, y)$. Let X be a regular Lagrangian flow solution of (ODE) and let A be the set of initial data defined by

$$A := \left\{(x, y) \in \mathbb{R}^2 | x < y \quad X(t, (x, y)) = \xi^+_{(x,y),b}(t) \quad \text{for } b > y - x\right\}$$

so A consists of the initial data in $\{x < y\}$ such that the solution $X(t, (x, y))$ takes values in the line $y = x$ for some time interval. Arguing as in the previous example, the set A is contained in a countable union of sets of the form

$$A_{p,q} := \{x \in \mathbb{R} | X(t, (x, y)) \in \{x = y\} \quad \text{for } t \in (p, q)\}$$

for $p, q \in \mathbb{Q}$ with $p < q$, which have measure zero as X satisfies the almost everywhere Lusin condition. We conclude that for almost every initial data (x, y) with $x < y$ a regular Lagrangian flow solution X must take the values

$$X(t, (x, t)) = \xi^+_{(x,y),y-x}(t).$$

The result for initial data (x, y) with $x > y$ follows similarly.

Note that this autonomous planar ordinary differential equation can easily be written as a non-autonomous scalar ordinary differential equation as the vector field f has constant unit y-component.

3.2 A general non-uniqueness result

In this section we demonstrate that for each vector field $f \neq 0$ for which there exists a regular Lagrangian flow solution of (ODE) we can find a vector field g equal to f almost everywhere such that the ordinary differential equation $\dot{\xi} = g(\xi)$ has non-unique solutions for a set of initial data of positive measure.

Essentially the proof is a generalisation of the above examples: for a subset $N \subset \mathbb{R}^n$ of the phase space with $\mu_n(N) = 0$ we redefine f on N so that N is an invariant manifold of (ODE). While particular choices of sets N and values of the vector field $f|_N$ will yield invariant lines such as that in the example of Section 3.1.2, or other interesting invariant manifolds, we simplify proceedings by setting

$f|_N = 0$. With this alteration any solution that enters the set N can remain there for arbitrary time before 'rejoining' a solution that leaves N. Consequently, if a solution ξ of (ODE) with initial data $x \in \mathbb{R}^n$ intersects N then there are uncountably many solutions of (ODE) with initial data $x \in \mathbb{R}^n$.

In order to guarantee non-uniqueness of solutions for all initial data in a set $D \subset \mathbb{R}^n$ of positive measure, we need to find a set $N \subset \mathbb{R}^n$ of measure zero such that for each $x \in D$ the solution $X(\cdot, x)$ intersects N. This is the content of Lemma 3.3. First, we remark that if f is a smooth vector field then finding such sets is straightforward due to the continuity of the flow with respect to the initial data. We examine the smooth case in the following lemma to motivate the proof of Lemma 3.3.

Lemma 3.2. *Let $f \in C^\infty(\mathbb{R}^n)$ be a non-zero vector field and let X be a classical flow solution of* (ODE). *For each $x \in \mathbb{R}^n$ with $f(x) \neq 0$ there exists a $\delta > 0$ such that for all $y \in B_\delta(x)$ the solution $X(t, y) \in \partial B_\delta(x)$ for some $t \in \mathbb{R}$.*

Proof. Take $x \in \mathbb{R}^n$ such that $f(x) \neq 0$ and observe that there exists a $\tau > 0$ such that $X(\tau, x) \neq x$. Let $\varepsilon = |X(\tau, x) - x|/2$, so that $B_\varepsilon(x) \cap B_\varepsilon(X(\tau, x)) = \emptyset$. Now, as X is continuous with respect to initial data there exists a δ with $0 < \delta < \varepsilon$ such that

$$y \in B_\delta(x) \Rightarrow X(\tau, y) \in B_\varepsilon(X(\tau, x)).$$

Consequently, for each $y \in B_\delta(x)$ the solution takes the value $X(\tau, y) \in B_\varepsilon(X(\tau, x))$ at time τ. Further, as $B_\delta(x) \subset B_\varepsilon(x)$ is disjoint from $B_\varepsilon(X(\tau, x))$ we conclude that consequently $X(t, y) \in \partial B_\delta(x)$ for some $t \in (0, \tau]$ as the solutions are continuous in time. □

Essentially, away from the zeroes of f, if a ball of initial data is sufficiently small then every solution with this initial data crosses the boundary of the ball. Consequently, it is trivial to find sets of initial data of positive measure such that every solution intersects a set of null measure by taking these respective sets to be the ball and the boundary of the ball in the Lemma 3.2. With a regular Lagrangian flow solution we typically do not have continuous dependence with respect to initial data. Consequently, rather than find a particular ball of initial data whose solutions move outside the ball, we abandon the 'local' approach of Lemma 3.2 and consider for each $\varepsilon > 0$ the set of all initial data whose solutions 'move' more than ε from their initial data.

Lemma 3.3. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a vector field that is not equivalent to the zero vector field and let $X: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a regular Lagrangian flow solution of* (ODE).

There exist sets $N, D \subset \mathbb{R}^n$ with $\mu_n(N) = 0$ and $\mu_n(D) > 0$ such that for almost every $x \in D$ the map $X(\cdot, x)$ intersects N in finite time.

Proof. For a fixed $\varepsilon > 0$ examine the set of initial data whose solutions intersect a ball of radius ε centred on their initial value:

$$D_\varepsilon := \left\{ x \in \mathbb{R}^n \,\middle|\, \sup_{t \in \mathbb{R}} |X(t, x) - x| > \varepsilon \right\}.$$

We show that $\mu_n(D_\varepsilon) > 0$ for some $\varepsilon > 0$. Assume for a contradiction that $\mu_n(D_\varepsilon) = 0$ for all $\varepsilon > 0$. In this case, as the trajectories are continuous for almost every $x \in \mathbb{R}^n$, the set

$$\{x \in \mathbb{R}^n | X(\cdot, x) \in C(\mathbb{R}) \quad X(t, x) \neq x \quad \text{for some } t \in \mathbb{R}\} \subset \bigcup_{i=1}^{\infty} D_{1/i}$$

has measure zero, so $X(t, x) \equiv x$ for almost every $x \in \mathbb{R}^n$. Consequently,

$$0 = \frac{\mathrm{d}X}{\mathrm{d}t} = f(X(t, x)) = f(x) \quad \text{for almost every } t \in \mathbb{R}, \quad \text{for almost every } x \in \mathbb{R}^n$$

so $f(x) = 0$ for almost every $x \in \mathbb{R}^n$, which contradicts our assumption on the vector field.

Next, fix $\varepsilon > 0$ such that $\mu_n(D_\varepsilon) > 0$ and examine the mesh

$$N := \left\{ x \in \mathbb{R}^n \,\middle|\, x_i = z \frac{\varepsilon}{2\sqrt{n}} \quad \text{for some } z \in \mathbb{Z} \quad \text{for some } i = 1 \ldots n \right\}$$

which are the boundaries of n-dimensional cubes of side-length $\varepsilon/2\sqrt{n}$, which have diameter $\varepsilon/2$. Note that any ε-separated points lie in distinct cubes and, as N is a countable collection of $(n-1)$-dimensional planes, $\mu_n(N) = 0$. Consequently, for each $x \in D_\varepsilon$ the solution $X(\cdot, x)$ intersects N in finite time. □

Theorem 3.4. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a vector field that is not equivalent to the zero vector field and let $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a regular Lagrangian flow solution of* (ODE). *There exists a vector field g equal to f almost everywhere on $\mathbb{R}^n$ such that the set of initial data for which there are non-unique solutions of $\dot{\xi} = g(\xi)$ has positive measure.*

Proof. Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a vector field that is not equivalent to the zero vector field and let $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a regular Lagrangian flow solution of (ODE). From Lemma 3.3 there exist sets $D, N \subset \mathbb{R}^n$ with $\mu_n(N) = 0$ and $\mu_n(D) > 0$ such that for almost every $x \in D$ the trajectory $X(\cdot, x)$ intersects N in finite time. Define the vector field g by

$$g(x) = \begin{cases} f(x) & x \notin N \\ 0 & x \in N \end{cases}$$

and note that $f = g$ almost everywhere, so from Lemma 2.35 X is also a regular Lagrangian flow solution of $\dot{\xi} = g(\xi)$. Consequently, for almost every $x \in \mathbb{R}^n$ the map $X(\cdot, x)$ is a solution of $\dot{\xi} = g(\xi)$.

Next, observe that every point $x \in N$ is a fixed point of $\dot{\xi} = g(\xi)$ which is to say that the ordinary differential equation has a solution given by the trivial map $\phi_x(t) = x$ for all $t \in \mathbb{R}$.

Finally, for almost every $x \in D$ there exists a time $t_x \in \mathbb{R}$ such that $X(t_x, x) \in N$. Consequently, if $t_x \geq 0$ then for each $s \in \mathbb{R}$ with $s \geq t_x$ the map

$$\xi^+_{x,s}(t) := \begin{cases} X(t, x) & t < t_x \\ X(t_x, x) & t_x \leq t < s \\ X(t + t_x - s, h) & s \leq t \end{cases}$$

is the concatenation of the intersecting solutions $X(\cdot, x)$ and $\phi_{X(t_x,x)}(\cdot)$ and so, from Theorem 3.1 is a solution of $\dot{\xi} = g(\xi)$ with initial condition $\xi^+_{x,s}(0) = x$. Similarly if $t_x < 0$ then for each $s \in \mathbb{R}$ with $s \leq t_x$ the map

$$\xi^-_{x,s}(t) := \begin{cases} X(t + t_x - s, x) & t < s \\ X(t_x, x) & s \leq t < t_x \\ X(t, h) & t_x \leq t \end{cases}$$

is a solution of $\dot{\xi} = g(\xi)$ with initial condition $\xi^-_{x,s}(0) = x$. Consequently, as the solutions $\xi^\pm_{x,s}$ are distinct for each $s \in \mathbb{R}$, almost every $x \in D$ has a one parameter family of distinct solutions of $\dot{\xi} = g(\xi)$. □

In the above theorem we alter the vector field f on a set of zero measure, which from Lemma 2.34 does not affect the existence or uniqueness of regular Lagrangian flow solutions to (ODE). Consequently, even if we use the above technique to introduce non-uniqueness of trajectories for almost all initial data we will a priori be unable to aggregate these solutions into a distinct regular Lagrangian flow. We can see this directly by observing that in the above construction the additional solutions 'loiter' in a set of zero measure for an interval of time. Consequently, if there are too many 'loitering' solutions in an aggregate then the Lusin condition will not be satisfied. We make this rigorous in the following lemma:

Lemma 3.5. *Let $D, N \subset \mathbb{R}^n$ be sets with $\mu_n(D) > 0$ and $\mu_n(N) = 0$. Suppose that for each $x \in D$ there exist times $a_x, b_x \in \mathbb{R}$ with $a_x \leq b_x$ and a solution $\xi_{a_x,b_x,x} \colon \mathbb{R} \to \mathbb{R}^n$ of* (ODE) *with initial data x such that*

$$\xi_{a_x,b_x,x}(t) \in N \quad \forall t \in [a_x, b_x].$$

If X is an aggregate of solutions of (ODE) *such that $X(t,x) = \xi_{a_x,b_x,x}(t)$ for each $x \in D$ then X is a regular Lagrangian flow only if $b_x = a_x$ for almost every $x \in D$.*

Proof. We assume that the aggregate X is a regular Lagrangian flow solution of (ODE) and argue as in the examples of Sections 3.1.1 and 3.1.2: for $a < b$ define $D_{a,b} := \{x \in D | a_x < b_x\}$ and observe that $D_{a,b} \subset X^{-1}(t, N)$ for $t \in [a,b]$. Consequently, as $\mu_n(N) = 0$ and X satisfies the Lusin condition, $\mu_n(D_{a,b}) = 0$. Consequently, the set

$$\{x \in D | a_x < b_x\} \subset \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} D_{p,q}$$

has measure zero as it is contained in a countable union of sets of measure zero, and we conclude that $X(t,x) = \xi_{a_x,a_x,x}(t)$ for almost every $x \in D$. □

We remark that the vector fields in the scalar and planar examples of Section 3.1.1 and 3.1.2, and the redefined vector fields of Theorem 3.4 are somehow artificial examples, but they illustrate that the qualitative property of 'almost everywhere uniqueness of solutions' is highly sensitive to the choice of representative of equivalence classes in $L^p(\mathbb{R}^n)$. The result of Theorem 3.4 illustrates that, regardless of the regularity of f, if there is a regular Lagrangian flow solution of (ODE) then there is a representative vector field with non-unique solutions for a set of initial data of positive measure.

It is reasonable to search for a canonical representative that mitigates against this pathological behaviour. One such candidate is the precise representative f^* of the vector field f (defined in Appendix B (B.1)), which is determined by the average value in the neighbourhood of each point. Intuitively, the precise representative f^* seems incapable of containing an invariant manifold such that locally to the manifold the vector field takes values both heading towards and heading away from the manifold, and such that there is a unique regular Lagrangian flow solution to (ODE). Roughly, the thought is that if the vector field f has values heading towards and away from the manifold then the average value of the component of f transverse to the manifold is non-zero. If f has non-zero component transverse to the manifold then the manifold is not invariant, so solutions cannot 'loiter' on them arbitrarily.

This leads us to the following open problems:

(OP 1) If f is a vector field such that there exists a unique nearly incompressible regular Lagrangian flow solution of (ODE) then does the precise representative of f have unique solutions for almost every initial data?

(OP 2) Conversely, is there a vector field f equal to its precise representative, such that there exists a unique nearly incompressible regular Lagrangian flow solution of (ODE) and such that the set of initial data with non-unique solutions has positive measure?

(OP 3) Is there a vector field f such that there exists a unique nearly incompressible regular Lagrangian flow solution of (ODE) and such that for every vector field equivalent to f the set of initial data with non-unique solutions has positive measure?

We remark that a positive answer to (OP 3) implies a positive answer to (OP 2).

3.2.1 Vector field redefinition and avoidance

The examples in the previous section can be viewed as the 'well behaved' vector fields

$$f(x) = 1 \qquad \text{and} \qquad f(x,y) = (2,1)$$

respectively, which have unique flow solutions and unique solutions for all initial data, that are subsequently altered on a set of zero measure to introduce non-uniqueness of trajectories. In the first example altering the scalar field $f = 1$ at a single point is sufficient to introduce non-uniqueness, whereas in the second example the planar field is altered on a line. It seems intuitively reasonable that altering the vector field in the second example on a set much 'smaller' than a line can not introduce multiple solutions for a 'large' set of initial data. The following theorem formalises this observation in terms of the avoidance property of Definition 2.26. This property is easily extended to regular Lagrangian flow solutions and is discussed more fully in the following chapter.

Theorem 3.6. *Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ be a vector field and X a regular Lagrangian flow solution of* (ODE) *such that for almost every initial data $x \in \mathbb{R}^n$, the solution $X(t,x)$ is the unique solution of* (ODE) *with initial data $x \in \mathbb{R}^n$. Let g be a vector field equal to f outside a compact set $N \subset \mathbb{R}^n$. If the flow X avoids N then* *for almost every initial data $x \in \mathbb{R}^n$ the solution $X(t,x)$ is the unique solution of $\dot{\xi} = g(\xi)$ with initial data $x \in \mathbb{R}^n$.*

Proof. We recall that X avoids N if the set

$$P_X(N) := \{x \in \mathbb{R}^n | X(t,x) \in N \quad \text{for some } t \in \mathbb{R}\}$$

has measure zero. For almost every $x \in \mathbb{R}^n \setminus P_X(N)$ the unique solution $t \mapsto X(t,x)$ of (ODE) does not intersect the set N. Further, as both N and the trace of $t \mapsto X(t,x)$ are closed the solution $t \mapsto X(t,x)$ takes values outside a neighbourhood of N. Consequently, for all $t \in \mathbb{R}$ the point $X(t,x)$ is in a neighbourhood B on which f is identical to g so $t \mapsto X(t,x)$ is the unique solution of $\dot{\xi} = g(\xi)$ with initial data $x \in \mathbb{R}^n$. □

In the following chapter we give a sufficient condition for avoidance, which is dependent on the regularity of f and the box-counting dimension of N.

We remark that the above theory can be extended for non-autonomous vector fields with the caveat that the vector field cannot be redefined arbitrarily: to introduce a fixed point $x \in \mathbb{R}^n$ to a non-autonomous vector field f we have to set $f(x,t) = 0$ for all $t \in [0,T]$, which means redefining f on a line in $\mathbb{R}^n \times [0,T]$. As a consequence it is easier to consider redefining f on sets of the form $N \times [0,T]$ with $N \subset \mathbb{R}^n$.

Chapter 4

Avoidance

4.1 Avoidance

In Theorem 3.4 of the previous chapter we altered the vector field f on set N of zero measure to introduce non-uniqueness of solutions. We recall that the set N was constructed so that each solution from a set of initial data of positive measure would intersect the set N at which point the pathological behaviour is introduced. In terms of the avoidance property we see that the N was constructed so that the flow solution X failed to avoid N. Recall that in Theorem 3.6 we demonstrated that if we alter a vector field on a set that the flow avoids, then we can not introduce non-uniqueness of solutions for a significant amount of initial data.

In this section we develop a non-autonomous theory of avoidance that extends the autonomous theory discussed in Aizenman [1978b]. The main result of this chapter (Theorem 4.8) states that a nearly incompressible regular Lagrangian flow solution of (ODE) avoids a subset $S \subset \mathbb{R}^n \times [0, T]$ provided that the vector field f is sufficiently regular and the set S has a sufficiently small dimension. As a consequence of this theorem, if we alter a vector field on a set N then the dimension of N has to be sufficiently large in order to affect the qualitative properties of the flow solution in a significant way.

Definition 4.1. *For a compact subset $S \subset \mathbb{R}^n \times [0, T]$ we say that a regular Lagrangian flow X avoids S if almost every (by the Lebesgue measure of the initial conditions) trajectory does not intersect a point of S, that is if the set*

$$\{x \in \mathbb{R}^n | \left(X\left(t, x, 0\right), t\right) \in S \quad \textit{for some } t \in [0, T]\} \tag{4.1}$$

has zero n-dimensional Lebesgue measure.

In the case that $S = A \times [0,T]$ with $A \subset \mathbb{R}^n$ we can regard the subset S as a set of spatial points which are to be avoided at all times. As $(X(t,x,0),t) \in S$ if and only if $X(t,x,0) \in A$ the above definition of avoidance reduces to that used in the literature (see, for example, Aizenman [1978b], Cipriano and Cruzeiro [2005] and Robinson and Sadowski [2009]) which only considers avoidance of sets of this form. Throughout we assume S is compact.

In this section we consider nearly incompressible regular Lagrangian flow solutions X, that is a regular Lagrangian flow solution such that there exists a constant $C > 0$ satisfying

$$\frac{1}{C}\mu_n(B) \leq \mu_n(\{x \in \mathbb{R}^n | X(t,x,s) \in B\}) \leq C\mu_n(B) \tag{4.2}$$

for all Borel subsets $B \subset \mathbb{R}^n$, for all $t, s \in [0,T]$.

First, we give some trivial conditions for avoidance and non-avoidance for the non-autonomous case in the following lemmas.

Lemma 4.2. *If $S \subset \mathbb{R}^n \times [0,T]$ is compact and consists of a countable union of temporal sections of zero n-dimensional measure then every regular Lagrangian flow solution avoids S.*

Proof. Suppose $S = \bigcup_{i=1}^{\infty} S^{\tau_i} \times \{\tau_i\}$ with $\tau_i \in [0,T]$ and $\mu_n(S^{\tau_i}) = 0$ for all $i \in \mathbb{N}$ and let X be a regular Lagrangian flow solution. As S is compact each S^{τ_i} is compact, and so Borel. First, by the Lusin condition,

$$\mu_n(\{x \in \mathbb{R}^n | X(\tau_i,x,0) \in S^{\tau_i}\}) = 0 \quad \text{for all } i \in \mathbb{N}. \tag{4.3}$$

as $\mu_n(S^{\tau_i}) = 0$ for all $i \in \mathbb{N}$. Next, the set of initial conditions at time 0 that give rise to trajectories that intersect a point of S

$$\begin{aligned} &\{x \in \mathbb{R}^n | (X(t,x,0),t) \in S \quad \text{for some } t \in [0,T]\} \\ &= \{x \in \mathbb{R}^n | (X(\tau_i,x,0),\tau_i) \in S \quad \text{for some } i \in \mathbb{N}\} \\ &= \bigcup_{i=1}^{\infty} \{x \in \mathbb{R}^n | X(\tau_i,x,0) \in S^{\tau_i}\} \end{aligned}$$

which has zero measure as from (4.3) it is the countable union of sets of zero measure. Consequently, the set S is avoided by the regular Lagrangian flow X. □

Lemma 4.3. *If $S \subset \mathbb{R}^n \times [0,T]$ has a temporal section S^τ of positive n-dimensional measure for some $\tau \in [0,T]$ then no nearly incompressible regular Lagrangian flow solution X avoids S.*

Proof. Suppose $\mu_n(S^\tau) > 0$ and let X be a nearly incompressible regular Lagrangian flow. First, from (4.2)

$$\mu_n(\{x \in \mathbb{R}^n | X(\tau, x, 0) \in S^\tau\}) \geq \frac{1}{C}\mu_n(S^\tau) > 0. \tag{4.4}$$

Next, the set

$$\{x \in \mathbb{R}^n | (X(t, x, 0), t) \in S \quad \text{for some } t \in [0, T]\}$$

contains

$$\begin{aligned} \{x \in \mathbb{R}^n | (X(\tau, x, 0), \tau) \in S\} &= \{x \in \mathbb{R}^n | X(\tau, x, 0) \in S^\tau\} \\ &= X(\tau, \cdot, 0)^{-1} S^\tau. \end{aligned}$$

which, from (4.4) has positive n-dimensional Lebesgue measure. □

4.2 Avoidance criteria

In Aizenman [1978b] the author gives the following avoidance criterion for autonomous ordinary differential equations:

Theorem 4.4 (Aizenman). *Let the vector field $f \in \mathcal{L}^q(\mathbb{R}^n)$ for some $1 \leq 1 \leq \infty$ and $A \subset \mathbb{R}^n$ be a compact subset. A nearly incompressible regular Lagrangian flow solution of* (ODE) *avoids A if the integral*

$$\int_{\{x \in \mathbb{R}^n | r_A(x) < r_0\}} r_A(x)^{-q^*} \, \mathrm{d}x \tag{4.5}$$

is finite, where $r_A(x) := \operatorname{dist}(x, A)$, $r_0 > 0$ is arbitrary, and q^* is the Hölder conjugate of q, which is to say that

$$\frac{1}{q} + \frac{1}{q^*} = 1.$$

Proof. See Aizenman [1978b]. This Theorem is also a consequence of Theorem 4.8, below. □

Aizenman also proves a sufficient condition for the integral (4.5) to be finite in terms of the upper box-counting dimension of A. We supplement this with a necessary condition in terms of the lower box-counting dimension of A in the following lemma:

Lemma 4.5. *Let $A \subset \mathbb{R}^n$ be a compact set.*

- *If* $q^* < n - \dim_B(A)$ *then the integral* (4.5) *is finite, and*
- *if* $q^* > n - \dim_{LB}(A)$ *then the integral* (4.5) *is infinite.*

Proof. See Aizenman [1978b] or Lemma 5.7 for the former. The latter is the content of Lemma 5.9. □

Aizenman summarises Theorem 4.4 and Lemma 4.5 in the following sufficient condition for avoidance in the autonomous case:

Corollary 4.6. *Let the vector field* $f \in \mathcal{L}^q(\mathbb{R}^n)$ *and* $A \subset \mathbb{R}^n$ *be a compact subset. A nearly incompressible regular Lagrangian flow solution of* (ODE) *avoids* A *if*

$$\frac{1}{q} + \frac{1}{n - \dim_B(A)} < 1. \tag{4.6}$$

It may seem unnecessary to detail the intermediary results leading to Corollary 4.6, however in the remainder we generalise the above Theorem 4.4 to non-autonomous vector fields $f \in \mathcal{L}^p(0,T;\mathcal{L}^q(\mathbb{R}^n))$ and compact sets $S \subset [0,T] \times \mathbb{R}^n$. This generalisation is the content of Theorem 4.8, which is the main result of this section and again gives a sufficient condition for avoidance in terms of the finiteness of some integral quantities. Unfortunately, in the non-autonomous case it is not straightforward to derive a relationship between the appropriate integral quantities and the geometry of S analogous to Lemma 4.5, so we delay such a discussion until Chapter 5.

In Cipriano and Cruzeiro [2005] the authors provide a limited extension of Aizenman's avoidance results to the non-autonomous case: the same criterion (4.6) guarantees avoidance of sets of the form $S = A \times [0,T]$ for vector fields $f \in \mathcal{L}^1(0,T;\mathcal{L}^q(\mathbb{R}^n))$. For a general compact subsets $S \subset \mathbb{R}^n \times [0,T]$ the set $S \subset P_x(S) \times [0,T]$ where $P_x(x,t) = x$ is the canonical projection onto the spatial component. Consequently, as avoidance of the superset implies avoidance of the subset the result in Cipriano and Cruzeiro [2005] implies that X avoids S if

$$\frac{1}{q} + \frac{1}{n - \dim_B(P_x(S))} < 1 \tag{4.7}$$

This approach ignores the temporal regularity of f and the temporal detail of S in that it does not distinguish between the subsets $A \times [0,T]$ and $A \times \{0\}$ of the phase space despite the fact that the latter set is smaller and intuitively feels more 'avoidable'. It is useful, therefore, to find an appropriate way of encoding the spatial and temporal detail of the set $S \subset \mathbb{R}^n \times [0,T]$. To this end we adapt the integral (4.5).

We encode the 'dimension' of a compact subset $S \subset \mathbb{R}^n \times [0,T]$ in the set of $\alpha, \beta \in (0,\infty)$ for which the integral

$$I_{\alpha,\beta}(S) := \left(\int_0^T \left(\int_{\{x \in \mathbb{R}^n | r_S(x,t) < r_0\}} r_S(x,t)^{-\alpha} \, \mathrm{d}x \right)^{\frac{\beta}{\alpha}} \mathrm{d}t \right)^{\frac{1}{\beta}},$$

is finite, where $r_S(x,t) := \operatorname{dist}((x,t), S)$. This encoding is the subject of Chapter 5, in which we discuss the properties of this encoding, compute the integrals for some simple sets and, in a main result analogous to Lemma 4.5 above, we provide some sufficient and some necessary conditions for the integral $I_{\alpha,\beta}(S)$ to be finite in terms of more established notions of dimensions. We remark that we can extend the quantity $I_{\alpha,\beta}$ to take parameters $\alpha, \beta = \infty$ by interpreting the integrals as essential suprema where appropriate. Before turning to the main result of this section we will need the following technical result which we prove in Chapter 5.

Lemma 4.7. *Let $S \subset \mathbb{R}^n \times [0,T]$. If $q^* \geq 1$ and $I_{q^*,p^*}(S) < \infty$ then each temporal section S^τ has Lebesgue measure zero.*

Proof. Follows from Lemmas 5.5 and 5.12 of the following Chapter, in which we discuss the properties of the integral $I_{\alpha,\beta}(S)$. □

Theorem 4.8. *Let $f \in \mathcal{L}^p(0,T;\mathcal{L}^q(\mathbb{R}^n))$ for $1 \leq p,q \leq \infty$, let X be a nearly incompressible regular Lagrangian flow solution of* (ODE) *and let S be a compact subset of $\mathbb{R}^n \times [0,T]$. If the integral $I_{q^*,p^*}(S) < \infty$, where p^* and q^* are the Hölder conjugates of p and q, which is to say that*

$$\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1,$$

then the flow X avoids the subset S.

Proof. We assume that $I_{q^*,p^*}(S) < \infty$. As the Hölder conjugate $q^* \geq 1$ then, from Lemma 4.7, the temporal section S^0 has zero Lebesgue measure.

Following Aizenman [1978b], for $\delta > 0$ we define

$$\tau_\delta(x) := \begin{cases} \sup\{u | r_S(X(t,x,0),t) \geq \delta \quad \forall t \in [0,u]\} & r_S(x,0) > \delta \\ 0 & r_S(x,0) \leq \delta \end{cases}$$

the latest time for which the trajectory with initial data $(x,0)$ stays at least δ away from S. As almost every $x \in \mathbb{R}^n$ gives rise to a continuous trajectory we restrict our attention to such points and note that from this continuity

$$r_S\left(X\left(\tau_\delta\left(x\right),x,0\right),\tau_\delta\left(x\right)\right)=\delta. \tag{4.8}$$

Let $CP_X\left(S\right)$ be the points of $P_X\left(S\right)$ that give rise to continuous trajectories, so from the definition of the flow $\mu_n\left(P_X\left(S\right)\setminus CP_X\left(S\right)\right)=0$. If $x\in CP_X\left(S\right)$ then the (continuous) trajectory with initial data x approaches S arbitrarily closely, so $\tau_\delta\left(x\right)<T$ for all $\delta>0$. Finally, as $r_S\left(x,0\right)=0$ only if $\left(x,0\right)\in S$, that is if $x\in S^0$ which has zero n-dimensional Lebesgue measure, we only consider those $x\in CP_X\left(S\right)$ for which $r_S\left(x,0\right)>0$. Consequently, for all $\delta>0$ the set $CP_X\left(S\right)\setminus S^0$ is equal to a countable union of sets of the form

$$\Omega_{m,\delta}:=\left\{x\middle|r_S\left(x,0\right)\geq\frac{1}{m},\quad\tau_\delta\left(x\right)<T,\quad X\left(\cdot,x,0\right)\in C\left(\left[0,T\right]\right)\right\}. \tag{4.9}$$

Fix $r_0>0$ and $0<\delta<r_0$ and let

$$F\left(\delta\right):=\left\{x|r_S\left(x,0\right)\geq r_0,\tau_\delta\left(x\right)<T,X\left(\cdot,x,0\right)\in C\left(\left[0,T\right]\right)\right\}.$$

We now show that $\mu_n\left(F\left(\delta\right)\right)\to 0$ as $\delta\to 0$: first, introduce the Lipschitz function

$$g\left(y\right)=\begin{cases}\log\left(\frac{r_0}{y}\right) & \delta\leq y\leq r_0\\ 0 & r_0<y\end{cases}$$

chosen so that $g\left(r_S\left(x,0\right)\right)=0$ for $x\in F\left(\delta\right)$, and from (4.8),

$$g\left(r_S\left(X\left(\tau_\delta\left(x\right),x,0\right),\tau_\delta\left(x\right)\right)\right)=g\left(\delta\right)$$

so that

$$\mu_n\left(F\left(\delta\right)\right)\left|g\left(\delta\right)\right| = \int\limits_{F(\delta)}\left|g\left(r_S\left(X\left(\tau_\delta\left(x\right),x,0\right),\tau_\delta\left(x\right)\right)\right)-g\left(r_S\left(x,0\right)\right)\right|\mathrm{d}x. \tag{4.10}$$

Next, as

$$\left|r_S\left(X\left(t_1,x,0\right),t_1\right)-r_S\left(X\left(t_2,x,0\right),t_2\right)\right|\leq\left|X\left(t_1,x,0\right)-X\left(t_2,x,0\right)\right|+\left|t_1-t_2\right|$$

the map $t\mapsto r_S\left(X\left(t,x,0\right),t\right)$ is absolutely continuous and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}r_S\left(X\left(t,x,0\right),t\right)\right|\leq\left|\frac{\mathrm{d}}{\mathrm{d}t}X\left(t,x,0\right)\right|+1.$$

Consequently, the composition $g\left(r_S\left(X\left(t,x,0\right),t\right)\right)$ is absolutely continuous in t and so from the chain rule for almost everywhere differentiable functions Serrin and Varberg [1969] we have for almost every t

$$\frac{\mathrm{d}}{\mathrm{d}t}g\left(r_S\left(X\left(t,x,0\right),t\right)\right)=g'\left(r_S\left(X\left(t,x,0\right),t\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}r_S\left(X\left(t,x,0\right),t\right).$$

From (4.10) we write

$$\begin{aligned}\mu_n\left(F\left(\delta\right)\right)\left|g\left(\delta\right)\right| &= \int_{F(\delta)}\left|\int_0^{T_\delta}\frac{\mathrm{d}}{\mathrm{d}t}g\left(r_S\left(X\left(t,x,0\right),t\right)\right)\mathrm{d}t\right| \\ &\leq \int\limits_{F(\delta)}\int_0^{T_\delta}\left|g'\left(r_S\left(X\left(t,x,0\right),t\right)\right)\right|\left(\left|\frac{\mathrm{d}}{\mathrm{d}t}X\left(t,x,0\right)\right|+1\right)\mathrm{d}t\,\mathrm{d}x \\ &= \int\limits_{F(\delta)}\int_0^{T_\delta}\left|g'\left(r_S\left(X\left(t,x,0\right),t\right)\right)\right|\left(\left|f\left(X\left(t,x,0\right),t\right)\right|+1\right)\mathrm{d}t\,\mathrm{d}x \\ &\leq C\int\limits_{F(\delta)}\int_0^{T_\delta}\left|g'\left(r_S\left(x,t\right)\right)\right|\left|f\left(x,t\right)+1\right|\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

where we use the fact that X is a nearly incompressible regular Lagrangian flow solution with compressibility constant C. From Fubini's theorem we conclude

$$\mu_n\left(F\left(\delta\right)\right)\left|g\left(\delta\right)\right|\leq C\int_0^T\int\limits_{F(\delta)}\left|g'\left(r_S\left(x,t\right)\right)\right|\left|f\left(x,t\right)+1\right|\mathrm{d}x\,\mathrm{d}t. \tag{4.11}$$

Next, as $\left|g\left(\delta\right)\right|=\log\left(\frac{r_0}{\delta}\right)$ and the derivative

$$g'\left(y\right)=\begin{cases}-\frac{1}{y} & \text{for almost every } y\in\left(\delta,r_0\right] \\ 0 & \text{for almost every } y>r_0,\end{cases}$$

the inequality (4.11) is

$$\mu_n\left(F\left(\delta\right)\right)\leq\log\left(\frac{r_0}{\delta}\right)^{-1}\int_0^T\int\limits_{\{x|r_S(x,t)<r_0\}}r_S\left(x,t\right)^{-1}\left|f\left(x,t\right)+1\right|\mathrm{d}x\,\mathrm{d}t$$

which, after applying Hölder's inequality, gives

$$\mu_n\left(F\left(\delta\right)\right)\leq\log\left(\frac{r_0}{\delta}\right)^{-1}I_{q^*,p^*}\left(S\right)\left\|f+1\right\|_{L^p;L^q\left(S_{r_0}\right)}.$$

This is finite as $I_{q^*,p^*}\left(S\right)<\infty$, and $f+1\in\mathcal{L}^p\left(0,T;\mathcal{L}^q_{loc}\left(\mathbb{R}^n\right)\right)$ implies $\left\|f+1\right\|_{L^p;L^q\left(S_{r_0}\right)}$

is finite. As $\delta > 0$ was arbitrary we let $\delta \to 0$ whence $\log\left(\frac{r_0}{\delta}\right)^{-1} \to 0$ giving the desired result. $\square$

We remark that for subsets of the form $S = A \times [0, T]$ as $r_S(t, x) = r_A(x) = \operatorname{dist}(x, A)$ the integral $I_{\alpha,\beta}(S)$ reduces to

$$\begin{aligned} I_{\alpha,\beta}(S) &= \left(\int_0^T \left(\int_{\{x \in \mathbb{R}^n | r_A(x) < r_0\}} r_A(x)^{-\alpha} \, \mathrm{d}x \right)^{\frac{\beta}{\alpha}} \mathrm{d}t \right)^{\frac{1}{\beta}} \\ &= T^{\frac{1}{\beta}} \left(\int_{\{x \in \mathbb{R}^n | r_A(x) < r_0\}} r_A(x)^{-\alpha} \, \mathrm{d}x \right)^{\frac{1}{\alpha}} \end{aligned}$$

consequently, $I_{\alpha,\beta}(S) < \infty$ if and only if

$$\int_{\{x \in \mathbb{R}^n | r_A(x) < r_0\}} r_A(x)^{-\alpha} \, \mathrm{d}x < \infty.$$

In the autonomous treatment of avoidance in Aizenman [1978b] the author demonstrates that this integral is finite if $0 \leq \alpha < n - \dim_B(A)$, which we repeat in Lemma 5.7 of Chapter 5. Combined with Theorem 4.8 (which is presented in autonomous terms in Aizenman [1978b]) this condition on the finiteness of the integral gives the avoidance criterion (4.6).

For regular Lagrangian flow solutions of (ODE) Theorem 4.8 allows us to determine if the flow avoids a specified subset S of the phase space knowing nothing more than the regularity of f and the anisotropic detail of S encoded in its r-codimension print. This criterion is sufficient but not necessary for avoidance: indeed, if S consists of the rational coordinates of a compact rectangle $R \subset \mathbb{R}^n \times [0, T]$ then we will see in Example 5.16 that $I_{\alpha,\beta}(S) = \infty$ for all $\alpha, \beta \in (0, \infty]$, so no regular Lagrangian flow solution satisfies the hypothesis of the theorem. However, as S has only a countable number of non-empty temporal sections and each temporal section has zero n-dimensional Lebesgue measure, the elementary avoidance result of Lemma 4.2 guarantees that every regular Lagrangian flow avoids the subset S.

For the autonomous case Aizenman [1978a] highlights an example by Nelson [1962] of a generalised flow solution to $\dot{x} = f(x)$ with the property that almost every trajectory intersects the origin and $f \in \mathcal{L}^q(\mathbb{R}^n)$ for all $q < n/(n-1)$. In particular, the flow fails to avoid a set of zero box-counting dimension and

$$\frac{1}{q}+\frac{1}{n-\dim_B(\{0\})}>\frac{n-1}{n}+\frac{1}{n}=1$$

which gives a borderline example for the avoidance criterion (4.6). In Aizenman [1978a] the author constructs a similar example where a non-null set of trajectories intersects a set $A\subset\mathbb{R}^3$ of upper and lower box-counting dimension $\log k/\log m$ (for $k,m\in\mathbb{N}$ such that $k\le m^2$) and the vector field is in $\mathcal{L}^q(\mathbb{R}^3)$ for all $q<\log(m^3/k)/\log(m^2/k)$. Consequently,

$$\frac{1}{q}+\frac{1}{3-\dim_B(A)}>\frac{2\log m-\log k}{3\log m-\log k}+\frac{1}{3-\log k/\log m}=1$$

which again gives a borderline example for the avoidance criterion (4.6) but with a non-trivial subset A. Note that neither these examples nor the criterion (4.6) cover the case

$$\frac{1}{q}+\frac{1}{n-\dim_B(A)}=1$$

and it remains of interest to determine whether a non-avoiding example can be constructed which satisfies this equality.

It is of interest to produce examples similar to those of Aizenman [1978a] for the non-autonomous case in order to understand the sharpness of the avoidance criterion of Theorem 4.8. Trivially, the Aizenman examples can be adapted to the non-autonomous setting by writing $\tilde{f}=(f,1)$ and $S=A\times[0,T]$ (for sufficiently large T) in which case

$$\tilde{f}\in\mathcal{L}^\infty\left(0,T;\mathcal{L}^q\left(\mathbb{R}^3\right)\right)\quad\text{for all } q<\log\left(m^3/k\right)/\log\left(m^2/k\right)$$

so that q has Hölder conjugate $q^*>3-\log(k/m)$. Using the result of Theorem 5.13, we see that $I_{q^*-\varepsilon,1}(S)<\infty$ for all $\varepsilon>0$ yet $I_{q^*,1}(S)=\infty$ so there is a sense in which the hypothesis of Theorem 4.8 is 'not quite' satisfied.

This approach demonstrates the sharpness of Theorem 4.8 for vector fields $f\in\mathcal{L}^1(0,T;\mathcal{L}^q(\mathbb{R}^n))$ and product sets $S=A\times[0,T]$, but it is of interest to find similar borderline cases for vector fields with arbitrary temporal regularity and subsets S that are not products of a spatial set with the entire temporal domain.

In the next chapter we examine the relationship between the geometry of a subset $S\subset\mathbb{R}^n\times[0,T]$ and the set

$$\{(\alpha,\beta)\in(0,\infty]\,|I_{\alpha,\beta}(S)<\infty\}$$

We will see that this set encodes some anisotropic detail of S and we will explicitly find some points of this set in terms of the box-counting dimensions of S and its projections.

Chapter 5

Dimension prints

In the previous chapter we introduced the two parameter family of integrals $I_{\alpha,\beta}(S)$ for subsets $S \subset \mathbb{R}^n \times [0,T]$ and demonstrated that if $I_{q^*,p^*}(S) < \infty$ then the set S is sufficiently 'small' to be avoided by flow solutions with vector fields in $\mathcal{L}^p(0,T;L^q(\mathbb{R}^n))$. In this chapter we demonstrate that the set

$$\operatorname{print}_r(S) := \left\{(\alpha,\beta) \in (0,\infty]^2 \,|I_{\alpha,\beta}(S) < \infty\right\},$$

which we call the r-codimension print, carries significant geometric information about the subset S. In Section 5.1.1 we demonstrate that the prints are well behaved with respect to unions and inclusions and are invariant under some transformations of S. Further, we describe some of the structure of the prints by demonstrating that the print consists of a union of open rectangles $(0,\alpha)\times(0,\beta)$. In Section 5.1.2 we give the main results of this Chapter in which we partially describe the print $\operatorname{print}_r(S)$ in terms of the upper and lower box-counting dimensions of the projections of S. In particular from Corollary 5.14 we see that the point $(\gamma,\beta) \in \operatorname{print}_r(S)$ if either

- $\gamma < n - \dim_B(P_x(S))$,
- $\beta < 1 - \dim_B(P_t(S))$, or
- $\gamma\beta < \gamma(1 - \dim_B(P_t(S))) + \beta(n - \dim_B(P_x(S)))$,

where $P_x(S)$ and $P_t(S)$ are the canonical projections of S onto $\mathbb{R}^n$ and $[0,T]$ respectively. These results indicate that $\operatorname{print}_r(S)$ distinguishes between the spatial and the temporal detail of S in a manner similar to the Hausdorff dimension print of Rogers [1988], which we recall below.

The box-counting dimension encodes the degree to which the points of a set are 'spread out' but fails to capture some significant geometry of this detail: if C is the Cantor 'middle half' set, which has Hausdorff and box-counting dimension equal to $\frac{1}{2}$, then the product set $C \times C \subset \mathbb{R}^2$ has Hausdorff and box-counting dimension equal to 1 (the result for the box-counting dimension is a trivial consequences of Theorem 6.17. Alternatively, see example 7.6 in Falconer [2003]). Consequently, $C \times C$ has the same Hausdorff and box-counting dimension as a line segment in $\mathbb{R}^2$ yet the sets have different anisotropic (i.e. directionally dependent) detail in the sense that the product set has detail in two directions whilst the line segment has detail in only one.

In Rogers [1988] the author captures the anisotropic properties of subsets of $\mathbb{R}^n$ defining an n-parameter family of measures $\mathcal{H}^\alpha$ similar to the Hausdorff measures. The dimension print of a set S is the set of points α for which $\mathcal{H}^\alpha(S)$ is non-zero.

Definition 5.1 (Rogers [1988]). *For a subset $S \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$ with $\alpha_j \geq 0$ for all j we define for all $\delta > 0$ the quantity*

$$\mathcal{H}^\alpha_\delta(S) := \inf\left\{\sum_{i=1}^{\infty} l_1(B_i)^{\alpha_1} \dots l_n(B_i)^{\alpha_n} \middle| B_i \in \mathcal{B},\ \operatorname{diam} B_i \leq \delta,\ \cup_{i=1}^{\infty} B_i \supset S\right\}$$

where $\mathcal{B}$ is the set of open rectangular parallelepipeds (henceforth 'boxes') in $\mathbb{R}^n$, $l_1(B_i), l_2(B_i), \dots, l_n(B_i)$ are the side lengths of the box B_i taken in a non-increasing order and $l_j(B_i)^0 = 1$ for all i, j.

We say that α is in the Hausdorff dimension print of S and write $\alpha \in \operatorname{print}_H(S)$, if and only if the Hausdorff-type measure

$$\mathcal{H}^\alpha(S) := \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(S)$$

is positive.

As each measure weights the side lengths of the boxes differently it is possible to distinguish between sets that are easily covered by long thin boxes, such as a line, and sets which are not, such as the product set $C \times C$. Note that the measure $\mathcal{H}^{(d,0,\dots,0)}$ is equal to the usual d-dimensional Hausdorff measure multiplied by a constant depending only on n, so it is possible to read the Hausdorff dimension of a set directly from the Hausdorff dimension print. Also note that we do not require the boxes B_i to have sides parallel to the coordinate axes so that the Hausdorff dimension print captures the degree to which a set has directionally dependent detail but not the direction in which this detail lies. In particular the dimension print is invariant under Euclidean transformations of a set as we can simply apply the same transformation to each of the covering boxes B_i. Whilst this is generally regarded as a desirable property for any notion of 'dimension' we ultimately wish to distinguish between spatial detail and temporal detail when we consider subsets of the space $\mathbb{R}^n \times [0,T]$ for applications to non-autonomous ordinary differential equations.

At the expense of this Euclidean invariance we can use dimension prints to capture the direction in which the detail lies by instead restricting the class of boxes $\mathcal{B}$ in Definition 5.1 to be those with sides parallel to the coordinate axes and each $l^j(B_i)$ to be the length of the side of the box B_i which is parallel to the j^{th} coordinate axis.

In Lee and Baek [1995] a box-counting dimension print is defined in a similar way from the premeasure

$$\mu^\alpha(S) = \liminf_{\delta \to 0} \{N_l(S)\, l_1^{\alpha_1} \ldots l_n^{\alpha_n} | 0 < l_1 \le l_2 \le \ldots \le l_n \le \delta\}$$

where, after dividing $\mathbb{R}^n$ into mesh boxes with dimensions $l_1 \times l_2 \times \ldots \times l_n$, the quantity $N_l(S)$ is the number of mesh boxes which intersect the set S. In the next section we define a similar print, $\text{print}_r(S)$, by integrating the distance to S over a neighbourhood of S. We demonstrate that some of the structure of $\text{print}_r(S)$ can be determined from the box-counting dimension of S and of its projections.

5.1 r-codimension print

We introduce the r-codimension print, $\text{print}_r(S)$, as another way of encoding the anisotropic fractal detail of a subset $S \subset \mathbb{R}^n \times [0,T]$, so called as it is defined in terms of the function $\text{dist}(\cdot, S) : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ which Aizenman [1978b] denotes by r. Throughout we assume that S is compact. We demonstrate a relationship between this print and the box-counting dimension of S before generalising this result for product sets $S = A \times \mathcal{T}$ for $A \subset \mathbb{R}^n$ and $\mathcal{T} \subset [0,T]$.

Definition 5.2. *The r-codimension print of a subset $S \subset \mathbb{R}^n \times [0,T]$, denoted* $\text{print}_r(S)$, *is the set of points $(\gamma, \beta) \in (0,\infty]^2$ such that the integral*

$$\int_0^T \left(\int_{S_{r_0}} r_S(x,t)^{-\gamma}\, \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t$$

is finite, where $r_S(x,t) :=$ $\text{dist}((x,t), S)$ and S_{r_0} is the r_0-neighbourhood of S for some positive constant r_0.

As r_S^{-1} is bounded outside each neighbourhood of S the choice of positive constant r_0 is arbitrary. Equivalently, $(\gamma,\beta) \in \mathrm{print}_r(S)$ if and only if

$$\frac{1}{r_S}\mathbf{1}_{S_{r_0}} \in \mathcal{L}^{\beta}\left(0,T;\mathcal{L}^{\gamma}\left(\mathbb{R}^n\right)\right)$$

for some $r_0 > 0$, where $\mathbf{1}_{S_{r_0}}$ is the characteristic function for the set S_{r_0}. For brevity of notation we write

$$\|g\|_{\mathcal{L}^{\beta};\mathcal{L}^{\gamma}\left(S_{r_0}\right)} := \left\|g \cdot \mathbf{1}_{S_{r_0}}\right\|_{\mathcal{L}^{\beta}(0,T;\mathcal{L}^{\gamma}(\mathbb{R}^n))}$$

however this is not a seminorm on $\mathcal{L}^{\beta}\left(0,T;\mathcal{L}^{\gamma}\left(\mathbb{R}^n\right)\right)$ as $\|g\|_{\mathcal{L}^{\beta};\mathcal{L}^{\gamma}\left(S_{r_0}\right)} = 0$ for any maps g equal to zero almost everywhere on S_{r_0}

As r_S^{-1} is unbounded on S_{r_0} for fixed γ,β, the quantity $\left\|r_S^{-1}\right\|_{\mathcal{L}^{\beta};\mathcal{L}^{\gamma}\left(S_{r_0}\right)}$ is finite if r_S^{-1} is not too singular. By allowing γ,β to vary we can capture how singular r_S^{-1}, and therefore how 'spread out' the set S is. Further, by allowing γ,β to vary independently we can weight the norms so that contribution from the spatial component is more or less significant than the contribution from the temporal component. Consequently, the r-codimension print encodes how 'spread out' the points of S are and the extent to which this spread is temporal rather than spatial.

This definition is easily generalised to consider the anisotropic detail of a subset $S \subset \mathbb{R}^{n+1}$ with respect to each of the $(n+1)$ coordinates: in this case the r-codimension print of S is the set of points $\alpha \in (0,\infty]^{n+1}$ such that

$$\frac{1}{r_S(x,t)}\mathbf{1}_{S_{r_0}} \in \mathcal{L}^{\alpha_{n+1}}\left(\mathbb{R}\left(\mathcal{L}^{\alpha_n}\left(\mathbb{R};\mathcal{L}^{\alpha_{n-1}}\left(\mathbb{R};\ldots;\mathcal{L}^{\alpha_1}\left(\mathbb{R}\right)\right)\right)\right)\right). \tag{5.1}$$

This broader definition more closely mimics the dimension prints of Rogers [1988] and Lee and Baek [1995] however, in the application to non-autonomous ODEs in Chapter 4 we only distinguish between the spatial and temporal detail of a subset. The 'spatio-temporal' r-codimension print of Definition 5.2 is simply the restriction of the more general codimension print to points $\alpha \in (0,\infty]^{n+1}$ such that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ so that the spatial contributions to the norm are all weighted equally.

Note that the definition of the r-codimension print presupposes an ordering of the coordinate axes in the order of integration of (5.1). It is not immediately clear how the print varies under reordering of axes. However, for our application we use the canonical spatio-temporal order presupposed in our choice of vector fields: the norm of the vector field $f \in \mathcal{L}^{p}\left(0,T;\mathcal{L}^{q}\left(\mathbb{R}^n\right)\right)$ is defined by first integrating with respect to the spatial variables, then with respect to time. Henceforth, we consider $S \subset \mathbb{R}^n \times [0,T]$ and use Definition 5.2.

5.1.1 Properties of the r-codimension print

The r-codimension print reverses inclusions, which is a property shared by the so called codimensions $n - \dim(A)$, justifying the use of the term 'codimension'.

Lemma 5.3. *If* $A \subset B \subset \mathbb{R}^n \times [0,T]$ *then* $\operatorname{print}_r(B) \subset \operatorname{print}_r(A)$

Proof. Clearly $\operatorname{dist}((x,t), A) \geq \operatorname{dist}((x,t), B)$ so $0 < r_A(x,t)^{-1} \leq r_B(x,t)^{-1}$ and $A_{r_0} \subset B_{r_0}$. Consequently, for all $(\gamma, \beta) \in (0,\infty]^2$,

$$\|r_A^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(A_{r_0})} \leq \|r_B^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(B_{r_0})}$$

so $(\gamma, \beta) \in \operatorname{print}_r(B)$ implies $(\gamma, \beta) \in \operatorname{print}_r(A)$. □

Applying this lemma to a countable collection of subsets $A_i \subset \mathbb{R}^n \times [0,T]$ yields the inclusion

$$\operatorname{print}_r\left(\bigcup_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} \operatorname{print}_r(A_i). \tag{5.2}$$

The r-codimension print is invariant under translation and scaling of sets, which we demonstrate in the following lemma. However, the print is not invariant under more general similarities, such as rotations.

Lemma 5.4. *Let* $S \subset \mathbb{R}^n \times [0,T]$. *If* $y \in \mathbb{R}^n$, $t \in [-T,T]$ *and* $\lambda > 0$ *are such that the sets*

$$S + (y,t) := \{(x+y, s+t) \mid (x,s) \in S\}$$
$$\lambda S := \{(\lambda x, \lambda s) \mid (x,s) \in S\}$$

are subsets of $\mathbb{R}^n \times [0,T]$ *then*

$$\operatorname{print}_r(S) = \operatorname{print}_r(S + (y,t)) = \operatorname{print}_r(\lambda S).$$

Proof. It is clear that

$$r_{\lambda S}(\lambda x, \lambda s) = \lambda r_S(x,s) \qquad \text{and} \qquad r_{S+(y,t)}(x+y, s+t) = r_S(x,s)$$

so

$$r_{\lambda S}(x,s)^{-1} = \frac{1}{\lambda} r_S(x,s)^{-1} \qquad \text{and} \qquad r_{S+(y,t)}(x+y, s+t)^{-1} = r_S(x,s)^{-1}.$$

Consequently, for each $(\gamma, \beta) \in (0, \infty]^2$,

$$\left\| r_S^{-1} \right\|_{\mathcal{L}^\gamma; \mathcal{L}^\beta(S_{r_0})} = \left\| r_{S+(y,t)}^{-1} \right\|_{\mathcal{L}^\gamma; \mathcal{L}^\beta\left((S+(y,t))_{r_0}\right)} = \lambda \left\| r_{\lambda S}^{-1} \right\|_{\mathcal{L}^\gamma; \mathcal{L}^\beta\left((\lambda S)_{r_0}\right)}$$

so $\operatorname{print}_r(S) = \operatorname{print}_r(S + (y, t)) = \operatorname{print}_r(\lambda S)$. □

A straightforward application of Hölder's inequality gives some of the structure of the r-codimension print:

Lemma 5.5. *For each point in the r-codimension print $(\gamma, \beta) \in \operatorname{print}_r(S)$ the rectangle $(0, \gamma] \times (0, \beta]$ is a subset of the print.*

Proof. Follows immediately from the inclusion

$$\mathcal{L}^{\beta^*}\left(0, T; \mathcal{L}^{\gamma^*}(\mathbb{R}^n)\right) \subset \mathcal{L}^{\beta}\left(0, T; \mathcal{L}^{\gamma}(\mathbb{R}^n)\right)$$

for $0 < \beta^* \leq \beta, \quad 0 < \gamma^* \leq \gamma$. □

In Example 5.15 we explicitly find the r-codimension print of the singleton $\{0\} \subset \mathbb{R}^n \times [0, T]$, which, from Lemma 5.4, has the same r-codimension print as any singleton subset of $R^n \times [0, T]$. Consequently, in light of the reversal of inclusion in Lemma 5.3, the r-codimension print of any non-empty subset $S \subset \mathbb{R}^n \times [0, T]$ is a subset of the union

$$\begin{aligned} \operatorname{print}_r(\{0\}) = \quad & \{(\gamma, \beta) \,|\gamma\beta < \gamma + \beta n \quad 0 < \gamma, \beta < \infty\} \\ & \cup \{(\gamma, \infty), 0 < \gamma < n\} \\ & \cup \{(\infty, \beta), 0 < \beta < 1\}, \end{aligned}$$

illustrated in Figure 5.1.

5.1.2 Relationship with box-counting dimension

In this section we initially present two lemmas that provide conditions for the integral

$$\int_{S_{r_0}} r_S(x)^{-\gamma} \, \mathrm{d}x$$

to converge or diverge respectively. This allows us to obtain some points in the r-codimension print of S and to exclude others from the print. The first lemma yields an inclusion criterion in terms of the upper box-counting dimension of S and is due to Aizenman [1978b]. We include the proof as a similar argument will be useful later.

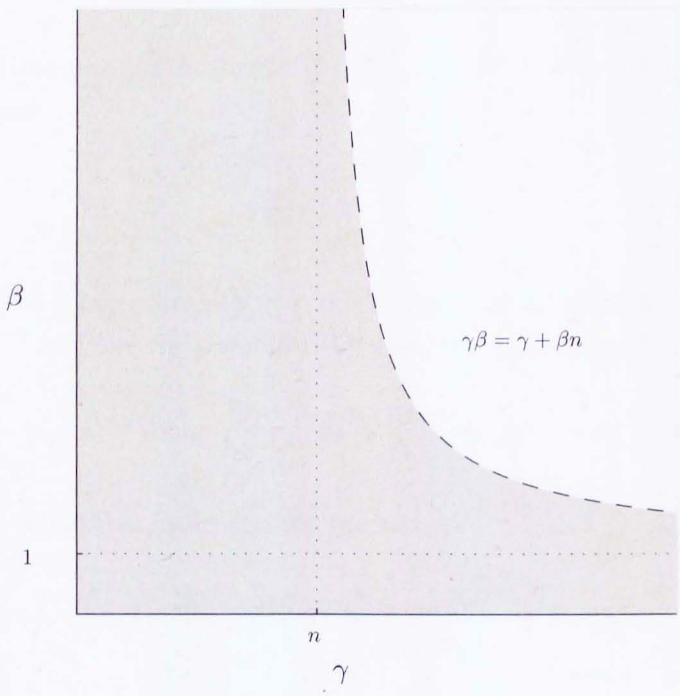

Figure 5.1: The r-codimension print of the singleton $\{0\} \subset \mathbb{R}^n \times [0, T]$.

We discuss the box-counting dimensions at length in Chapter 6. However, here we recall the following alternative 'Minkowski sausage' formulation of the upper and lower box-counting dimensions (see, for example, Proposition 3.2 of Falconer [2003]):

Lemma 5.6. *For a bounded non-empty $F \subset \mathbb{R}^n$*

$$\dim_B(F) = n - \liminf_{\delta \to 0} \frac{\log(\mu_n(F_\delta))}{\log \delta} \tag{5.3}$$

$$\dim_{LB}(F) = n - \limsup_{\delta \to 0} \frac{\log(\mu_n(F_\delta))}{\log \delta} \tag{5.4}$$

where $F_\delta = \{x \in \mathbb{R}^n | \operatorname{dist}(x, F) < \delta\}$ is the δ-neighbourhood of F. Further, for each α and β such that $\alpha > n - \dim_{LB}(F)$ and $\beta < n - \dim_B(F)$ and each $\delta^* > 0$ there exists a constant $C > 0$ such that

$$\frac{1}{C}\delta^\alpha \leq \mu_n(F_\delta) \leq C\delta^\beta \qquad \forall \delta \in [0, \delta^*] \tag{5.5}$$

Proof. See Lemma 6.5. □

We this formulation of the box-counting dimension, we can prove the following:

Lemma 5.7 (Aizenman [1978b]). *For $S \subset \mathbb{R}^n$, any $0 \leq \gamma < n - \dim_B(S)$ and any $r_0 > 0$ the integral*

$$\int_{S_{r_0}} r_S(x)^{-\gamma} \, \mathrm{d}x$$

is finite.

Proof. Choose $\varepsilon \in (0, n - \dim_B(S) - \gamma)$. We split the integral into the minimum value for $r_S(x)^{-\gamma}$ and the the difference between the minimum and actual value

$$\int_{S_{r_0}} r_S(x)^{-\gamma} \, \mathrm{d}x = \int_{S_{r_0}} r_0^{-\gamma} \mathrm{d}x + \int_{S_{r_0}} \left[r_S(x)^{-\gamma} - r_0^{-\gamma} \right] \mathrm{d}x$$

Clearly the first integral is finite and for the second

$$\int_{S_{r_0}} \left[r_S(x)^{-\gamma} - r_0^{-\gamma} \right] \mathrm{d}x = \int_{S_{r_0}} \int_{r_0^{-\gamma}}^{r_S(x)^{-\gamma}} 1 \, \mathrm{d}u \, \mathrm{d}x$$

which, from Fubini's theorem, is equal to

$$\int_{r_0^{-\gamma}}^{\infty} \int_{\left\{x \mid r_S(x) < u^{-\frac{1}{\gamma}}\right\}} 1 \, \mathrm{d}x \, \mathrm{d}u = \int_{r_0^{-\gamma}}^{\infty} \mu_n \left(S_{u^{-\frac{1}{\gamma}}} \right) \mathrm{d}u$$

As $\gamma + \varepsilon < n - \dim_B(S)$ we have from Lemma 6.5 that there exists a constant C such that $\mu_n \left(S_{u^{-\frac{1}{\gamma}}} \right) \leq C \left(u^{-\frac{1}{\gamma}} \right)^{(\gamma+\varepsilon)}$ for all $u^{-\frac{1}{\gamma}} < r_0$. Consequently the above integral is bounded above by

$$\int_{r_0^{-\gamma}}^{\infty} C \left(u^{-\frac{1}{\gamma}} \right)^{\gamma+\varepsilon} \mathrm{d}u = C \int_{r_0^{-\gamma}}^{\infty} u^{-\left(1+\frac{\varepsilon}{\gamma}\right)} \mathrm{d}u$$

which is convergent as $1 + \frac{\varepsilon}{\gamma} > 1$. □

Corollary 5.8. *For $S \subset \mathbb{R}^n \times [0, T]$ and $0 < \gamma < n + 1 - \dim_B(S)$ the point (γ, γ) is in the r-codimension print of S.*

Proof. From the above lemma the integral

$$\begin{aligned} \int \int_{S_{r_0}} r_S(x,t)^{-\gamma} \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_{\mathbb{R}^n} r_S(x,t)^{-\gamma} \, \mathbf{1}_{S_{r_0}} \mathrm{d}x \, \mathrm{d}t \\ &= \left(\| r_S^{-1} \|_{\mathcal{L}^\gamma; \mathcal{L}^\gamma(S_{r_0})} \right)^\gamma \end{aligned}$$

is finite, so $(\gamma,\gamma) \in \text{print}_r(S)$. □

The second lemma yields an exclusion criterion in terms of the lower box-counting dimension of S:

Lemma 5.9. *For $S \subset \mathbb{R}^n$, any $\gamma > n - \dim_{LB}(S)$ and any $r_0 > 0$ the integral*

$$\int_{S_{r_0}} r_S(x)^{-\gamma}\,\mathrm{d}x$$

diverges.

Proof. As in the proof of Lemma 5.7 we write the above integral

$$\int_{S_{r_0}} r_S(x)^{-\gamma}\,\mathrm{d}x = \int_{S_{r_0}} r_0^{-\gamma}\mathrm{d}x + \int_{r_0^{-\gamma}}^{\infty} \mu_n\left(S_{u^{-\frac{1}{\gamma}}}\right)\mathrm{d}u.$$

We ignore the first term and note that from Lemma 6.5 as $\gamma > n - \dim_{LB}(S)$ there exists a constant C such that $\frac{1}{C}\left(u^{-\frac{1}{\gamma}}\right)^{\gamma} \leq \mu_n\left(S_{u^{-\frac{1}{\gamma}}}\right)$ for all $u^{-\frac{1}{\gamma}} < r_0$. Consequently the above integral is bounded below by

$$\int_{r_0^{-\gamma}}^{\infty} \frac{1}{C}u^{-1}\mathrm{d}u$$

which diverges. □

Corollary 5.10. *For $S \subset \mathbb{R}^n \times [0,T]$ and $\gamma > n + 1 - \dim_{LB}(S)$ the point (γ,γ) is not in the r-codimension print of S.*

Proof. From the above lemma the integral

$$\begin{aligned}\int\int_{S_{r_0}} r_S(x,t)^{-\gamma}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^n} r_S(x,t)^{-\gamma}\,\mathbf{1}_{S_{r_0}}\mathrm{d}x\,\mathrm{d}t \\ &= \left(\left\|r_S^{-1}\right\|_{\mathcal{L}^{\gamma};\mathcal{L}^{\gamma}(S_{r_0})}\right)^{\gamma}\end{aligned}$$

diverges, so $(\gamma,\gamma) \notin \text{print}_r(S)$. □

Corollary 5.11. *For a subset $S \subset \mathbb{R}^n \times [0,T]$ every point of the open square $(0, n+1-\dim_B(S))^2$ is in r-codimension print of S. Further, every point of the square $(n+1-\dim_{LB}(S), \infty]^2$ is not in the r-codimension print of S. These points are illustrated in Figure 5.2.*

Proof. Follows from the previous two corollaries and the Lemma 5.5. □

There is a gap between the inclusion criterion of Lemma 5.7 and the exclusion criterion of Lemma 5.9. Indeed, the hypothesis for either lemma is not satisfied for γ in the range $n - \dim_B(S) \leq \gamma \leq n - \dim_{LB}(S)$ and, as we demonstrate in Chapter 6 that there are sets for which $\dim_{LB}(S) = 0$ and $\dim_B(S) = n$, this gap can be large.

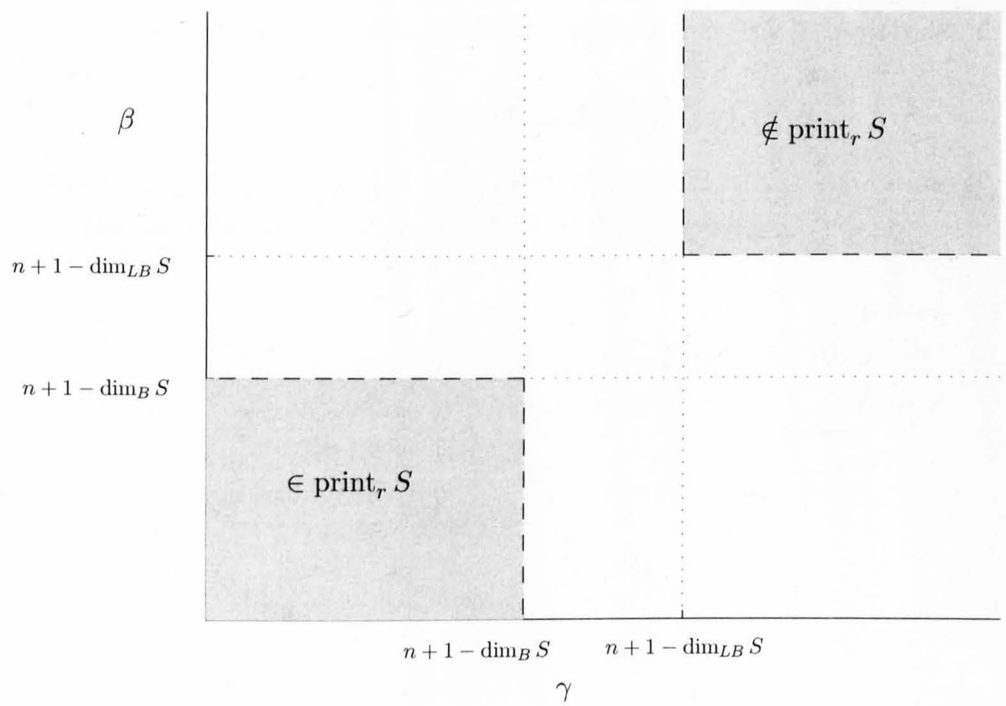

Figure 5.2: A subset of points (γ, β) that are in $\text{print}_r(S)$ and a subset of points (γ, β) that are not in $\text{print}_r(S)$.

The following lemma relates the Lebesgue measure of a set to its codimension print. We consider the Lebesgue $(n+1)$-dimensional measure of the set S and the Lebesgue n-dimensional measure of the temporal sections $S^\tau := \{x \in \mathbb{R}^n | (x, \tau) \in S\}$.

Lemma 5.12. *For* $S \subset \mathbb{R}^n \times [0, T]$

- *If* $\mu_{n+1}(S) > 0$ *then* $\text{print}_r(S) = \emptyset$.
- *If S has a temporal section with positive n-dimensional Lebesgue measure, that is $\mu_n(S^\tau) > 0$ for some $\tau \in [0, T]$, then the r-codimension print of S does not contain any point (γ, β) with $\beta \geq 1$.*

Proof. If $\mu_{n+1}(S) > 0$ then the Hausdorff dimension of S, $\dim_H(S)$, is at least $n+1$. Consequently, from the relationship between the Hausdorff and box-counting dimensions (see, for example, pp.46 of Falconer [2003] or §3.2 of Robinson [2011]), $n+1 \leq \dim_H(S) \leq \dim_{LB}(S)$ and so, from the exclusion result of Corollary 5.11, no point of $(0,\infty]^2$ is in $\mathrm{print}_r(S)$ so the print is empty. More directly, if $\mu_{n+1}(S) > 0$ then the function $r_S(x,t)^{-\gamma}$ is unbounded on a set of positive $(n+1)$ measure so $\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})}$ is infinite for all $\gamma, \beta > 0$.

Next, suppose that $\mu_n(S^\tau) > 0$ for some $\tau \in [0,T]$. Clearly $S^\tau \times \{\tau\} \subset S$ and

$$S^\tau \times [\tau - r_0, \tau + r_0] \subset (S^\tau \times \{\tau\})_{r_0} \subset S_{r_0}.$$

Further, $r_{S^\tau \times \{\tau\}}(x,t) = |t-\tau|$ for all $(x,t) \in S^\tau \times [\tau - r_0, \tau + r_0]$ so, as $S^\tau \times \{\tau\} \subset S$, the distance function $r_S(x,t) \leq |\tau - t|$. Consequently,

$$\begin{aligned}
\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} &= \int_0^T \left(\int_{\{x | r_S(x,t) < r_0\}} r_S(x,t)^{-\gamma}\, \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \\
&\geq \int_{\tau - r_0}^{\tau + r_0} \left(\int_{S^\tau} |t-\tau|^{-\gamma}\, \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \\
&= \mu_n(S^\tau)^{\frac{\beta}{\gamma}} \int_{\tau - r_0}^{\tau + r_0} |t-\tau|^{-\beta}\, \mathrm{d}t
\end{aligned}$$

which, as $\mu_n(S^\tau) > 0$, diverges for $\beta \geq 1$. □

5.1.3 Product sets

We now consider sets of the form $S := A \times \mathcal{T}$ where $A \subset \mathbb{R}^n$ and $\mathcal{T} \subset [0,T]$. With this product structure we can write the distance $r(x,t)$ in terms of the distance from x to A and the distance from t to $\mathcal{T}$: we introduce the notation $r_S(x,t)$, $r_A(x)$ and $r_{\mathcal{T}}(t)$ for these respective distances and note that

$$r_S(x,t)^2 = r_A(x)^2 + r_{\mathcal{T}}(t)^2. \tag{5.6}$$

The following theorem provides conditions for points to be in the r-codimension print of product sets. Conditions (i) and (ii) are consequences of Lemma 5.7; our interest is in conditions (iii) and (iv).

Theorem 5.13. *Suppose $S \subset \mathbb{R}^n \times [0,T]$ is such that $S = A \times \mathcal{T}$ for some $A \subset \mathbb{R}^n$ and $\mathcal{T} \subset [0,T]$. The point (γ,β) is in $\mathrm{print}_r(S)$ if one of the following conditions* *holds:*

(i) $\gamma < n - \dim_B(A)$

(ii) $\beta < 1 - \dim_B(\mathcal{T})$

(iii) $\gamma\beta < \gamma(1 - \dim_B(\mathcal{T})) + \beta(n - \dim_B(A))$.

Further, the point (γ, β) *is not in* $\mathrm{print}_r(S)$ *if the following condition holds*

(iv) $\gamma\beta > \gamma(1 - \dim_{LB}(\mathcal{T})) + \beta(n - \dim_{LB}(A))$.

These points are represented in Figure 5.3.

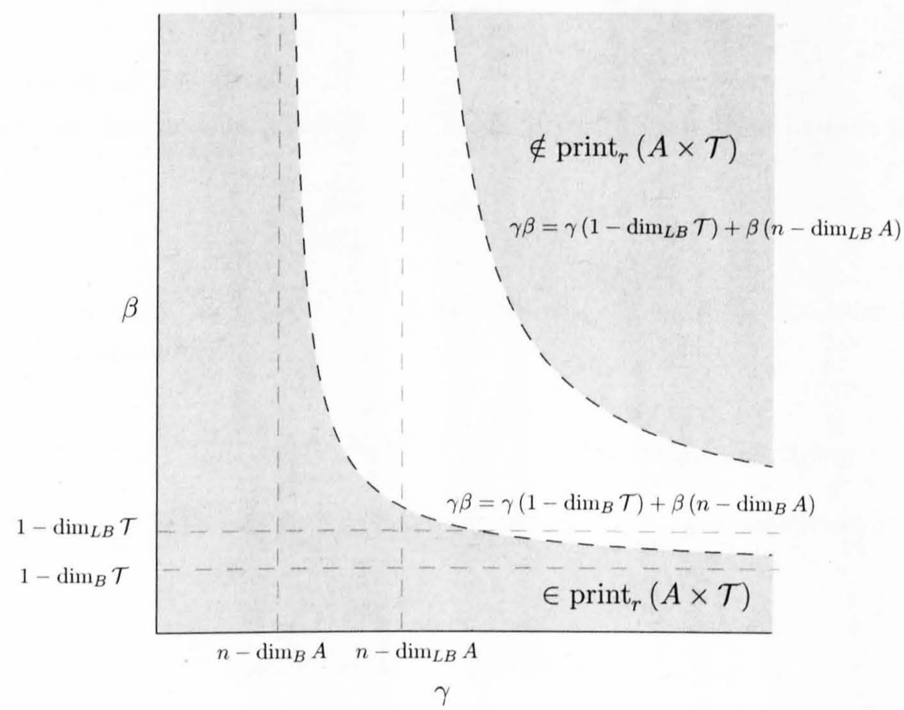

Figure 5.3: The result of Theorem 5.13: the region below the lower hyperbola consists of points $(\gamma, \beta) \in \mathrm{print}_r(A \times \mathcal{T})$; the region above the upper hyperbola consists of points $(\gamma, \beta) \notin \mathrm{print}_r(A \times \mathcal{T})$. The theorem provides no information about points on the hyperbolas themselves or in the region between them.

Proof. Note that in light of the equality (5.6)

$$\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} = \left\|\left(r_A^2 + r_\mathcal{T}^2\right)^{-\frac{1}{2}}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})}. \tag{5.7}$$

First, we assume that condition (i) holds in which case, from Lemma 5.7, the integral

$$\int_{A_{r0}} r_A(x)^{-\gamma}\,\mathrm{d}x$$

is finite. Consequently, as $\left(r_A^2 + r_T^2\right)^{-\frac{1}{2}} \le r_A^{-1}$ and $S_{r0} \subset A_{r0} \times [0,T]$, the quantity (5.7) is bounded above by

$$\begin{aligned}
\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r0})} &\le \|r_A^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(X_{r0}\times[0,T])} \\
&= \left\| r_A^{-1}\mathbf{1}_{(A_{r0}\times[0,T])} \right\|_{\mathcal{L}^\beta(0,T;\mathcal{L}^\gamma(\mathbb{R}^n))} \\
&= \left\| \left(\int_{A_{r0}} r_A(x)^{-\gamma}\,\mathrm{d}x \right)^{\frac{1}{\gamma}} \right\|_{\mathcal{L}^\beta([0,T])}
\end{aligned}$$

which is finite for all $\beta \in (0,\infty]$.

Next, we assume that condition (ii) holds in which case, from Lemma 5.7, the integral

$$\int_{\mathcal{T}_{r0}} r_T(t)^{-\beta}\,\mathrm{d}t$$

is finite. Consequently, as $\left(r_A^2 + r_T^2\right)^{-\frac{1}{2}} \le r_T^{-1}$ and $S_{r0} \subset A_{r0} \times \mathcal{T}_{r0}$ the quantity (5.7) is bounded above by

$$\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r0})} \le \|r_T^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(A_{r0}\times\mathcal{T}_{r0})} = \left\| r_T^{-1}\mathbf{1}_{(A_{r0}\times\mathcal{T}_{r0})} \right\|_{\mathcal{L}^\beta(0,T;\mathcal{L}^\gamma(\mathbb{R}^n))}$$

which, as $\mathbf{1}_{(A_{r0}\times\mathcal{T}_{r0})}(x,t) = \mathbf{1}_{(A_{r0})}(x)\,\mathbf{1}_{(\mathcal{T}_{r0})}(t)$ and $r_T^{-1}\mathbf{1}_{(\mathcal{T}_{r0})}$ is independent of x, is equal to

$$\begin{aligned}
&\left\|\mathbf{1}_{(A_{r0})}\right\|_{\mathcal{L}^\gamma(\mathbb{R}^n)} \left\| r_T^{-1}\mathbf{1}_{(\mathcal{T}_{r0})} \right\|_{\mathcal{L}^\beta([0,T])} \\
&= \left\|\mathbf{1}_{(A_{r0})}\right\|_{\mathcal{L}^\gamma(\mathbb{R}^n)} \left(\int_{\mathcal{T}_{r0}} r_{\mathcal{T}_{r0}}(t)^{-\beta}\,\mathrm{d}t \right)^{\frac{1}{\beta}}
\end{aligned}$$

which is finite for all $\gamma \in (0,\infty]$.

For conditions (iii) and (iv) both γ and β are finite so

$$\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r0})} = \left[\int_0^T \left(\int_{\mathbb{R}^n} r_S(x,t)^{-\gamma}\,\mathbf{1}_{(S_{r0})}(x,t)\,\mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}. \tag{5.8}$$

From (5.6)

$$S_{r_0} = \{(x.t)\,|r_S(x,t) < r_0\} = \left\{(x,t)\,|r_A(x) < \sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}, r_{\mathcal{T}}(t) < r_0\right\}$$

which with (5.8) yields

$$\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} = \left[\int_{\mathcal{T}_{r_0}} \left(\int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} r_S(x,t)^{-\gamma}\,\mathrm{d}x\right)^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}}. \tag{5.9}$$

We now write the middle integral of (5.9) in a more useful form: Denote

$$\begin{aligned} J(t) :&= \int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} r_S(x,t)^{-\gamma}\,\mathrm{d}x \\ &= \int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x \end{aligned}$$

from (5.6). Fix $t \in \mathcal{T}_{r_0}$ and, proceeding in a similar fashion to the proof of Lemma 5.7, we write $J(t)$ as the sum

$$J(t) = \int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} r_0^{-\gamma}\mathrm{d}x \tag{5.10}$$

$$+ \int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} - r_0^{-\gamma}\mathrm{d}x. \tag{5.11}$$

The second integral (5.11) is equal to

$$\int_{A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}} \int_{r_0^{-\gamma}}^{\left(r_A(x)^2+r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}}} 1\mathrm{d}u\,\mathrm{d}x = \int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \int_{A_{\sqrt{u^{-\frac{2}{\gamma}}-r_{\mathcal{T}}(t)^2}}} 1\mathrm{d}x\,\mathrm{d}u$$

from Fubini's theorem. Consequently,

$$J(t) = r_0^{-\gamma}\mu_n\left(A_{\sqrt{r_0^2-r_{\mathcal{T}}(t)^2}}\right) + \int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \mu_n\left(A_{\sqrt{u^{-\frac{2}{\gamma}}-r_{\mathcal{T}}(t)^2}}\right)\mathrm{d}u.$$

so from (5.9)

$$\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} = \left[\int\limits_{\mathcal{T}_{r_0}} \left[r_0^{-\gamma}\mu_n\left(A_{\sqrt{r_0^2-r_\mathcal{T}(t)^2}}\right)\right.\right.$$

$$\left.\left. + \int\limits_{r_0^{-\gamma}}^{r_\mathcal{T}(t)^{-\gamma}} \mu_n\left(A_{\sqrt{u^{-\frac{2}{\gamma}}-r_\mathcal{T}(t)^2}}\right)\mathrm{d}u\right]^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}. \tag{5.12}$$

Next, assume that (iii) holds. In light of the previous two cases, we assume additionally that $\gamma \geq n - \dim_B(A)$ and that $n - \dim_B(A) > 0$ as condition (iii) reduces to (ii) if $\dim_B(A) = n$. With these assumptions there exists an η such that $0 \leq \eta < n - \dim_B(A)$ and

$$\gamma\beta < \gamma(1 - \dim_B(\mathcal{T})) + \beta\eta. \tag{5.13}$$

Consequently, from Lemma 6.5 there exists a constant C such that $\mu_n(A_\delta) \leq C\delta^\eta$ for all $0 < \delta \leq r_0$. From (5.12),

$$\|r_S^{-1}\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} \leq \left[\int_0^T \left[r_0^{-\gamma}C\left(r_0^2 - r_\mathcal{T}(t)^2\right)^{\frac{\eta}{2}} + \int\limits_{r_0^{-\gamma}}^{r_\mathcal{T}(t)^{-\gamma}} C\left(u^{-\frac{2}{\gamma}} - r_\mathcal{T}(t)^2\right)^{\frac{\eta}{2}}\mathrm{d}u\right]^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}$$

$$\leq C^{\frac{1}{\gamma}}\left[\int_0^T \left(r_0^{\eta-\gamma} + \int\limits_{r_0^{-\gamma}}^{r_\mathcal{T}(t)^{-\gamma}} u^{-\frac{\eta}{\gamma}}\mathrm{d}u\right)^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}$$

and, as $\gamma > \eta$,

$$\leq C^{\frac{1}{\gamma}}\left[\int_0^T \left(r_0^{\eta-\gamma} + \frac{1}{1-\frac{\eta}{\gamma}}\left(r_\mathcal{T}(t)^{\eta-\gamma} - r_0^{\eta-\gamma}\right)\right)^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}$$

$$\leq \left(\frac{C}{1-\frac{\eta}{\gamma}}\right)^{\frac{1}{\gamma}}\left[\int_0^T \left(r_\mathcal{T}(t)^{\eta-\gamma}\right)^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}$$

$$= \left(\frac{C}{1-\frac{\eta}{\gamma}}\right)^{\frac{1}{\gamma}}\left[\int_0^T r_\mathcal{T}(t)^{\frac{\beta(\eta-\gamma)}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}$$

which, from Lemma 5.7, is finite as $0 \leq (\beta\gamma - \beta\eta)/\gamma < 1 - \dim_B(\mathcal{T})$ from (5.13).

Next, assume that (iv) holds so there exists an η such that

$$\begin{aligned} & \eta > n - \dim_{LB}(A) \\ \text{and} \quad & \gamma\beta > \gamma(1 - \dim_{LB}(\mathcal{T})) + \beta\eta. \end{aligned} \tag{5.14}$$

From Lemma 6.5 there exists a constant C such that $\mu_n(A_\delta) \geq C^{-1}\delta^\eta$ for all $0 < \delta \leq r_0$ and consequently, from (5.12),

$$\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} \geq C^{-\frac{1}{\gamma}} \left[\int_{\mathcal{T}_{r_0}} \left(\int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \left(u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2\right)^{\frac{\eta}{2}} \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}.$$

By restricting the domain of the first integral to $\mathcal{T}_{r_0/\sqrt{2}}$ and the domain of the second to u such that $r_0^{-\gamma} \leq u \leq \left(\sqrt{2}r_{\mathcal{T}}(t)\right)^{-\gamma}$, we write

$$\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} \geq C^{-\frac{1}{\gamma}} \left[\int_{\mathcal{T}_{r_0/\sqrt{2}}} \left(\int_{r_0^{-\gamma}}^{(\sqrt{2}r_{\mathcal{T}}(t))^{-\gamma}} \left(u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2\right)^{\frac{\eta}{2}} \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}$$

and for u in this range, $u^{-\frac{2}{\gamma}} \geq 2r_{\mathcal{T}}(t)^2$ so that

$$\begin{aligned} \left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} &\geq C^{-\frac{1}{\gamma}} \left[\int_{\mathcal{T}_{r_0/\sqrt{2}}} \left(\int_{r_0^{-\gamma}}^{(\sqrt{2}r_{\mathcal{T}}(t))^{-\gamma}} r_{\mathcal{T}}(t)^\eta \, \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\ &= C^{-\frac{1}{\gamma}} \left[\int_{\mathcal{T}_{r_0/\sqrt{2}}} \left(2^{-\frac{\gamma}{2}} r_{\mathcal{T}}(t)^{\eta-\gamma} - r_0^{-\gamma} r_{\mathcal{T}}(t)^\eta\right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\ &\geq C^{-\frac{1}{\gamma}} 2^{-\frac{1}{\gamma}} \left[\int_{\mathcal{T}_{r_0/\sqrt{2}}} r_{\mathcal{T}}(t)^{\frac{\beta(\eta-\gamma)}{\gamma}} \, \mathrm{d}t \right]^{\frac{1}{\beta}} \end{aligned}$$

which, from Lemma 5.9, diverges as $(\beta\gamma - \beta\eta)/\gamma > 1 - \dim_{LB}(\mathcal{T})$ from (5.14). $\square$

Note that the conditions are related by the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) for finite γ, β, so the condition (iii) is sufficient for finite γ, β.

Corollary 5.14. *For a general set $S \subset \mathbb{R}^n \times [0,T]$ the point $(\gamma,\beta) \in \operatorname{print}_r(S)$ if either*

- $\gamma < n - \dim_B(P_x(S))$, *or*
- $\beta < 1 - \dim_B(P_t(S))$, *or*
- $\gamma\beta < \gamma(1 - \dim_B(P_t(S))) + \beta(n - \dim_B(P_x(S)))$,

where $P_x(S)$ and $P_t(S)$ are the canonical projections of S onto $\mathbb{R}^n$ and $[0,T]$ respectively.

Proof. Follows from the trivial inclusion $S \subset P_x(S) \times P_t(S)$, the reversal of inclusion Lemma 5.3, and the Theorem 5.13. □

5.1.4 Examples

We compute the r-codimension print for some subsets of $\mathbb{R}^n \times [0,T]$. Whilst the calculations are straightforward, we find that computing the r-codimension print of even the most elementary subsets is quite involved. Fortunately, the result of Theorem 5.13 greatly simplifies this calculation.

Example 5.15. *The singleton $S = \{0\} \subset \mathbb{R}^n \times [0,T]$ has r-codimension print the union*

$$\begin{aligned}\operatorname{print}_r(S) = \{(\gamma,\beta) \,|\gamma\beta < \gamma + \beta n \quad 0 < \gamma,\beta < \infty\} \\ \cup \{(\gamma,\infty), 0 < \gamma < n\} \cup \{(\infty,\beta), 0 < \beta < 1\},\end{aligned}$$

illustrated in Figure 5.1. As S can be written as the product set $\{0\} \times \{0\}$ and $\dim_B(\{0\}) = 0$ *conditions* (i), (ii) *and* (iii) *of Theorem 5.13 guarantee that the print contains this union. Further, as* $\dim_{LB}(\{0\}) = 0$, *condition* (iv) *guarantees that no point of $\{(\gamma,\beta) \,|\gamma\beta > \gamma + \beta n \quad 0 < \gamma,\beta < \infty\}$ is in the print.*

In this case, Theorem 5.13 yields the majority of the structure of $\operatorname{print}_r(S)$ *as only the borderline cases remain: we now show that points on the hyperbola $\gamma\beta = \gamma + \beta n$, the points (γ,∞) for $\gamma \geq n$ and the points (∞,β) for $\beta \geq 1$ are not in* $\operatorname{print}_r(S)$*:*

For simplicity we assume that $T \geq \sqrt{2}$. The distance function is given by $r_S(x,t) = \sqrt{|x|^2 + |t|^2}$ and by taking $r_0 = \sqrt{2}$ the rectangular set $[-1,1]^n \times [0,1] \subset S_{r_0}$. Consequently, by reducing the domain of integration, for $0 < \gamma,\beta < \infty$ such *that $\gamma\beta = \gamma + \beta n$*

$$\begin{aligned}
\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})} &\geq \left[\int_0^1 \left(\int_{\{x||x|<|t|\}} \left(|x|^2+|t|^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x\right)^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}} \\
&\geq \left[\int_0^1 \left(\int_{\{x||x|<|t|\}} \left(2\,|t|^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x\right)^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}} \\
&\geq \left[\int_0^1 2^{-\frac{\beta}{2}}\,|t|^{-\beta}\,\mu_n\{x|\,|x|<|t|\}^{\frac{\beta}{\gamma}}\mathrm{d}t\right]^{\frac{1}{\beta}}
\end{aligned}$$

and as $\mu_n\{x|\,|x|<|t|\} = \omega_n\,|t|^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, the quantity $\left\|r_S^{-1}\right\|_{\mathcal{L}^\beta;\mathcal{L}^\gamma(S_{r_0})}$ is bounded below by

$$\left[\omega_n^{\frac{\beta}{\gamma}} 2^{-\frac{\beta}{2}} \int_0^1 |t|^{n\frac{\beta}{\gamma}-\beta}\,\mathrm{d}t\right]^{\frac{1}{\beta}} = \left[\omega_n^{\frac{\beta}{\gamma}} 2^{-\frac{\beta}{2}} \int_0^1 |t|^{-1}\,\mathrm{d}t\right]^{\frac{1}{\beta}}$$

which diverges.

Next, as $\sup_{\{x|r_S(x,t)<r_0\}} \left(|x|^2+|t|^2\right)^{-\frac{1}{2}} = |t|^{-1}$,

$$\begin{aligned}
\left\|r_S^{-1}\right\|_{\mathcal{L}^1;\mathcal{L}^\infty(S_{r_0})} &= \int_0^{\sqrt{2}} \operatorname*{ess\,sup}_{\{x|r_S(x,t)<r_0\}} \left(|x|^2+|t|^2\right)^{-\frac{1}{2}} \mathrm{d}t \\
&= \int_0^{\sqrt{2}} |t|^{-1}\,\mathrm{d}t
\end{aligned}$$

which diverges, so $(\infty,1) \notin \mathrm{print}_r(S)$. Consequently, from Lemma 5.5, $(\infty,\beta) \notin \mathrm{print}_r(S)$ for all $\beta \geq 1$.

Finally, the domain $\left\{x\middle|\,|x|^2+|t|^2<r_0^2\right\}$ and the integrand $\left(|x|^2+|t|^2\right)^{-\frac{n}{2}}$ are both largest at $t=0$ so we clearly have

$$\begin{aligned}
\left\|r_S^{-1}\right\|_{\mathcal{L}^\infty;\mathcal{L}^1(S_{r_0})} &= \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\left\{x\middle|\,|x|^2+|t|^2<r_0^2\right\}} \left(|x|^2+|t|^2\right)^{-\frac{n}{2}} \mathrm{d}x \\
&= \int_{\{x||x|<r_0\}} |x|^{-n}\,\mathrm{d}x
\end{aligned}$$

which diverges, so $(n,\infty) \notin \mathrm{print}_r(S)$. Again, Lemma 5.5 yields $(\gamma,\infty) \notin \mathrm{print}_r(S)$ for all $\gamma \geq n$.

In the following example the set is merely countable, but is sufficiently 'large' to have an empty r-codimension print:

Example 5.16. *Let S consist of the rational coordinates of a compact rectangle $R \subset \mathbb{R}^n \times [0,T]$. As the upper and lower box-counting dimensions are invariant under closure (Falconer [2003], proposition 3.4) and $\bar{S} = R$, $\dim_{LB}(S) = \dim_{LB}(R) = n+1$. Consequently, by Corollary 5.11, $\mathrm{print}_r(S) = \emptyset$.*

Unfortunately, Theorem 5.13 does not necessarily capture the entire r-codimension print, even for product sets, as the following example demonstrates:

Example 5.17. *Let $A \subset \mathbb{R}^n$ and $\mathcal{T} \subset [0,T]$ be such that $\dim_B(A \times \mathcal{T}) < \dim_B(A) + \dim_B(\mathcal{T})$, for example if A and $\mathcal{T}$ are the generalised compatible Cantor sets of Example 6.18. From this inequality there exists γ for which*

$$n + 1 - \dim_B(A) - \dim_B(\mathcal{T}) < \gamma < n + 1 - \dim_B(A \times \mathcal{T})$$

Consequently, the point (γ, γ) is in the print of S from Corollary 5.11. However this point is not captured by Theorem 5.13 as $\gamma^2 \geq \gamma(n - \dim_B(A)) + \gamma(1 - \dim_B(\mathcal{T}))$ and so does not satisfy condition (iii).

In light of this example it is of interest to determine exactly how badly Theorem 5.13 can fail to capture the r-codimension print. We can first attempt to optimise this example so that the inequality $\dim_B(X \times Y) < \dim_B(X) + \dim_B(Y)$ is as wide as possible. Indeed, if $1 = \dim_B(X \times Y) < \dim_B(X) + \dim_B(Y) = 2$ (which, as we see in the next chapter, is the widest inequality of this type and is given by the sets in Example 6.19) then Theorem 5.13 does not provide any points of $\mathrm{print}_r(X \times Y)$. However, we see from Corollary 5.11, that the open square $(0,1) \times (0,1) \subset \mathrm{print}_r(X \times Y)$.

Chapter 6

Box-counting dimension

6.1 Box-counting dimension

In the previous chapter we saw that as a result of Corollary 5.11 and Theorem 5.13 we can find points in and exclude points from the r-codimension print of a set $S \subset \mathbb{R}^n \times [0,T]$ if we know the upper and lower box-counting dimensions of the sets S and its projections $P_x(S)$ and $P_t(S)$. Further, we demonstrated that there is a discrepancy between these results and that the size of this discrepancy depends upon the size of $\dim_B(P_x(S)) + \dim_B(P_t(S)) - \dim_B(S)$. In this chapter we derive and discuss the box-counting dimension product formula in order to determine the sharpness of Theorem 5.13.

First, we define and prove some properties of the box-counting dimension of a set $F \subset X$ in an abstract setting where X is a general metric space. We will see that some proofs are simplified in the Euclidean case $X = \mathbb{R}^n$. We begin by defining the sets for which we can make sense of the box-counting dimension:

Definition 6.1. *A set $F \subset X$ is* **totally bounded** *if for any length $\delta > 0$ there exists a finite collection of sets of diameter δ that form a cover of F.*

In the following lemma we demonstrate that in the Euclidean case the totally bounded sets are precisely those that are bounded: that is those sets contained in a ball about the origin of some finite radius.

Lemma 6.2. *A set $F \subset \mathbb{R}^n$ is totally bounded if and only if it is bounded.*

Proof. Suppose F is totally bounded and fix $\delta > 0$. Let $\{U_i\}_{i \in I}$ be a finite collection of sets of radius $\delta > 0$ that forms a cover of F and for each $i \in I$ select a point $u_i \in U_i$. Clearly, for each $i \in I$ the set U_i is contained in the ball of radius δ centred on u_i, so no point of the cover, hence no point of F, is further than distance

$\max_{i\in I}\{|u_i|\} + \delta$ from the origin.

Conversely, suppose F is bounded then so that the closure of F, denoted by $\overline{F}$, is compact. For a fixed $\delta > 0$ consider the cover of $\overline{F}$ given by $\left\{B_{\delta/2}(x)\right\}_{x\in\overline{F}}$. As each element of this cover is open and $\overline{F}$ is compact there exists a finite subcover of $\overline{F}$ (and hence of F) of balls of diameter δ. As $\delta > 0$ was arbitrary we conclude that F is totally bounded. □

For a general metric space X boundedness is not sufficient to ensure total boundedness, as illustrated by the space $X = \mathbb{R}$ endowed with the discrete metric

$$\rho(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Note that the metric space $(\mathbb{R}, \rho)$ is contained within the unit ball centred on the origin, however the interval $[0,1]$ requires infinitely many balls of diameter $1/2$ to cover as no two points of $[0,1]$ are in the same ball.

Definition 6.3. *The* **upper** *and* **lower box-counting dimensions** *of a totally bounded set $F \subset X$ are defined by*

$$\dim_B(F) := \limsup_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} \qquad \textit{and} \tag{6.1}$$

$$\dim_{LB}(F) := \liminf_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} \tag{6.2}$$

respectively, where $N(F,\delta)$ is the smallest number of sets of diameter δ that form a cover of F.

It is sufficient to only consider $\delta < 1$, which avoids the singularity at $\delta = 1$. Essentially, if $N(F,\delta)$ scales like $\delta^{-\varepsilon}$ as $\delta \to 0$ then the box-counting dimensions capture ε which gives an indication of how 'spread out' the set F is at small length-scales.

Definition 6.3 immediately yields the inequality $\dim_{LB}(F) \leq \dim_B(F)$ for all sets $F \subset X$. This inequality can be strict, even in the Euclidean case, as illustrated in exercise 3.8 of Falconer [2003] or the example in §3.1 of Robinson [2011]. The existence of such sets is also a consequence of Theorem 6.17 albeit with a less direct construction than that of Falconer [2003] or Robinson [2011]. If $\dim_{LB}(F) = \dim_B(F)$ then we say that the box-counting dimension of the set $F \subset X$ exists and is equal to this common value. However, as the literature does not use the term 'box-counting dimension' consistently we will exclusively refer to the upper and lower box-counting dimensions.

6.1.1 Equivalent definitions

The limits in Definition 6.3 are unchanged if we replace $N(F, \delta)$ with one of many similar geometric quantities. These equivalences are discussed at length in Falconer [2003] §3.1 and the proofs proceed by finding relationships between the quantities and demonstrating that the difference vanishes at the limit. For completeness we derive these relationships as we will use them to prove the box-counting dimension product formulas in the next section. The geometric quantities of interest are

- the largest number of disjoint balls of diameter δ with centres in F, which we denote $N'(F, \delta)$, and
- in the Euclidean case $X = \mathbb{R}^n$, the number of δ-mesh boxes, that is sets of the form

$$[m_1\delta, (m_1+1)\delta] \times \ldots \times [m_n\delta, (m_n+1)\delta]$$

for integers $m_1, \ldots, m_n$, that intersect F, which we denote by $M(F, \delta)$.

We take the following from §3.1 of Falconer [2003]:

Lemma 6.4. *For each set $F \subset X$ and length $\delta > 0$ the geometric quantities N and N' are related by*

$$N(F, 2\delta) \leq N'(F, \delta), \tag{6.3}$$

$$N'(F, 2\delta) \leq N(F, \delta), \tag{6.4}$$

and, if $X = \mathbb{R}^n$, the geometric quantities N and M are related by

$$N\left(F, \delta\sqrt{n}\right) \leq M(F, \delta) \quad \textit{and} \tag{6.5}$$

$$M(F, \delta) \leq 3^n N(F, \delta) \tag{6.6}$$

Proof. Let $x_i \in F$ for $i = 1 \ldots N'(F, \delta)$ be the centres of disjoint balls of diameter 2δ. For $j = 1 \ldots m$ let each U_j be a set with diameter δ. If the U_j cover F then they certainly cover the centres x_i, but if $x_i \in U_j$ then U_j is contained in the ball of diameter 2δ centred on x_i. As these balls are disjoint, there must be at least one U_j for each x_i, yielding (6.4).
Next, let $x_i \in F$ for $i = 1 \ldots N'(F, \delta)$ be the centres of disjoint balls of diameter δ. Consider the balls $B_{2\delta}(x_i)$ of diameter 2δ with centres x_i. Suppose that these balls do not form a cover of F then there exists a $y \in F$ such that $|x_i - y| > \delta$. Consequently, there exists an additional disjoint ball of diameter δ with centre $y \in F$,

which is a contradiction, so the $B_{2\delta}(x_i)$ for $i = 1 \dots N'(F,\delta)$ are a cover of F, yielding (6.3).

Next, if $X = \mathbb{R}^n$ and F intersects $M(F,\delta)$ mesh boxes of side length δ, then these boxes have diameter $\delta\sqrt{n}$ and form a cover of F yielding (6.5).

Finally, any set of diameter δ is contained in at most 3^n boxes of side length δ, yielding (6.6). $\square$

In the Euclidean case $X = \mathbb{R}^n$ there is an alternative 'Minkowski sausage' formulation (see Falconer [2003] for proof of equivalence). For a compact $F \subset \mathbb{R}^n$

$$\dim_B(F) = n - \liminf_{\delta \to 0} \frac{\log(\mu_n(F_\delta))}{\log \delta} \tag{6.7}$$

$$\dim_{LB}(F) = n - \limsup_{\delta \to 0} \frac{\log(\mu_n(F_\delta))}{\log \delta} \tag{6.8}$$

where $F_\delta = \{x \in \mathbb{R}^n | \operatorname{dist}(x,F) < \delta\}$ is the δ-neighbourhood of F. Essentially, if $\mu_n(F_\delta)$ scales like $\delta^{n-\varepsilon}$ as $\delta \to 0$ then the box-counting dimensions capture ε giving an indication of the growth of the δ-neighbourhood of F. In fact we have the following bounds on $\mu_n(F_\delta)$:

Lemma 6.5. *Let F be a compact, non-empty subset of $\mathbb{R}^n$. For each α and β such that $\alpha > n - \dim_{LB}(F)$ and $\beta < n - \dim_B(F)$ and each $\delta^* > 0$ there exists a constant $C > 0$ such that*

$$\frac{1}{C}\delta^\alpha \le \mu_n(F_\delta) \le C\delta^\beta \qquad \forall \delta \in [0,\delta^*] \tag{6.9}$$

Proof. Immediately from (5.3) and (5.4) there is an η with $0 < \eta < \delta^*$ such that

$$\frac{\log(\mu_n(F_\delta))}{\log \delta} \le \alpha \quad \text{and} \quad \frac{\log(\mu_n(F_\delta))}{\log \delta} \ge \beta \qquad \text{for all } \delta \text{ in the range } 0 < \delta \le \eta$$

Consequently,

$$\delta^\alpha \le \mu_n(F_\delta) \le \delta^\beta \qquad \text{for } 0 < \delta \le \eta. \tag{6.10}$$

For δ in the range $\eta < \delta \le \delta^*$ observe that F_δ is contained in a ball B^+ of radius $\operatorname{diam}(F)/2 + \delta^*$ and, as F is non-empty, F_δ contains a ball B^- of radius η. Consequently,

$$\mu_n(B^-) \le \mu_n(F_\delta) \le \mu_n(B^+) \qquad \text{for } \eta < \delta \le \delta^*.$$

and so

$$\mu_n\left(B^-\right)\frac{\delta^\alpha}{\delta^{*\alpha}} \leq \mu_n\left(F_\delta\right) \leq \mu_n\left(B^+\right)\frac{\delta^\beta}{\eta^\beta} \qquad \text{for } \eta < \delta \leq \delta^*. \tag{6.11}$$

From, (6.10) and (6.11) it suffices to choose C sufficiently large that

$$C \geq \max\left(1, \mu_n\left(B^+\right)\eta^{-\beta}\right) \qquad \text{and} \qquad \frac{1}{C} \leq \min\left(1, \mu_n\left(B^-\right)\delta^{*-\alpha}\right).$$

The constant

$$\begin{aligned} C &= \max\left(1, \mu_n\left(B^+\right)\eta^{-\beta}, \mu_n\left(B^-\right)^{-1}\delta^{*\alpha}\right) \\ &= \max\left(1, \omega_n\left(\operatorname{diam}\left(F\right)/2+\delta^*\right)^n\eta^{-\beta}, \omega_n^{-1}\eta^{-n}\delta^{*\alpha}\right), \end{aligned}$$

where ω_n is the n-dimensional measure of the unit ball in $\mathbb{R}^n$, is sufficient.

Finally, we extend this bound to $\delta = 0$ by noting that $F_0 = F$ and that if $\mu_n(F) = 0$ then the bounds (5.5) are trivially satisfied as $\alpha > 0$. If $\mu_n(F) > 0$ then the lower bound is trivially satisfied and $n \leq \dim_H(F) \leq \dim_B(F)$. In this case $\beta < 0$ and the upper bound is vacuous. □

The growth of $\mu_n(F_\delta)$ at small length scales reflect how 'spread out' the set is at these length scales: rapid growth as δ increases indicates that the δ neighbourhoods around a significant number of individual points of F do not intersect by a large amount, which is that these individual points are 'spread out'.

6.1.2 Product sets

For the remainder of this section, let (X, d_X) and (Y, d_Y) be metric spaces and endow the product space $X \times Y$ with the metric

$$\mathrm{d}_{X\times Y,p} = \begin{cases} \left(\mathrm{d}_X^p + \mathrm{d}_Y^p\right)^{\frac{1}{p}} & p \in [1,\infty) \\ \max\left(\mathrm{d}_X, \mathrm{d}_Y\right) & p = \infty \end{cases} \tag{6.12}$$

for some $p \in [1, \infty]$. It is well known that if $F \subset X$ and $G \subset Y$ are compact subsets then the lower and upper box-counting dimensions of the Cartesian product $F \times G$ satisfy

$$\dim_{LB}\left(F\times G\right) \geq \dim_{LB}\left(F\right) + \dim_{LB}\left(G\right) \qquad \text{and} \tag{6.13}$$

$$\dim_B\left(F\times G\right) \leq \dim_B\left(F\right) + \dim_B\left(G\right) \tag{6.14}$$

respectively. These inequalities follow from the good behaviour of the above geometric quantities on taking products: for each of these geometric quantities we can derive a relationship between the values taken for the sets F, G and the set $F \times G$, which is the content of the next lemma. Further, we require these geometric relationships to establish the useful equivalent definitions for the lower and upper box-counting dimensions of products sets given in Lemma 6.7.

Lemma 6.6. *If $F \subset X$ and $G \subset Y$ are compact subsets then for all $\delta > 0$*

$$N\left(F \times G, 2^{\frac{1}{p}}\delta\right) \leq N(F,\delta)\, N(G,\delta) \tag{6.15}$$

where we define $2^{\frac{1}{p}} = 1$ at $p = \infty$ and

$$N'(F \times G, \delta) \geq N'(F,\delta)\, N'(G,\delta)\,. \tag{6.16}$$

Further, if $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $X \times Y = \mathbb{R}^{n+m}$ then for all $\delta > 0$

$$M(F \times G, \delta) = M(F,\delta)\, M(G,\delta)\,. \tag{6.17}$$

The proof of (6.15) is standard (see, for example, Falconer [2003] or Robinson [2011]) and relies on the fact that the Cartesian product of a cover of F with a cover of G is a cover of $F \times G$, possibly with a larger diameter. The inequality (6.16) is less familiar (see Robinson [2011]), but follows from the fact that the Cartesian product of a set of disjoint balls with centres in F and a set of disjoint balls with centres in G is a set of disjoint balls with centres in $F \times G$. We now prove the equality (6.17):

Proof. Let $\{U_i\}_{i=1}^{M(F,\delta)}$ be the set of δ-mesh boxes of $\mathbb{R}^n$ which intersect F and let $\{V_j\}_{j=1}^{M(G,\delta)}$ be the set of δ-mesh boxes of $\mathbb{R}^m$ which intersect G. For each $i = 1 \ldots M(F,\delta)$ and $j = 1 \ldots M(G,\delta)$ the set $U_i \times V_j$ is a δ-mesh box in $\mathbb{R}^{n+m}$, and as there exists $f_i \in U_i \bigcap F$ and $g_j \in V_j \bigcap G$ the point $f_i \times g_j$ is in $(U_i \times V_j) \bigcap (F \times G)$ so the product box $U_i \times V_j$ intersects $F \times G$.

Next, an arbitrary point (f,g) of $F \times G$ has $f \in U_i$ for some $i = 1 \ldots M(F,\delta)$ and $g \in V_j$ for some $j = 1 \ldots M(G,\delta)$, so that $(f,g) \in U_i \times V_j$ for some $i = 1 \ldots M(F,\delta)$ and $j = 1 \ldots M(G,\delta)$. Consequently, the set of δ-mesh boxes of $\mathbb{R}^{n+m}$ which intersect $F \times G$ is precisely the set

$$\{U_i \times V_j | i = 1 \ldots M(F,\delta)\,,\ j = 1 \ldots M(G,\delta)\}$$

which has exactly $M(F,\delta)\, M(G,\delta)$ members. $\square$

The proof that different geometric quantities provide equivalent definitions of the box-counting dimensions is relatively easy (see Falconer [2003]) and proceeds by finding relationships between the quantities and demonstrating that the difference vanishes at the limit. Using a similar technique we derive the following equivalent definitions for the box-counting dimensions of a product set:

Lemma 6.7. *Let* $(X, \mathrm{d}_X), (Y, \mathrm{d}_Y)$ *be metric spaces and endow the product space* $X \times Y$ *with the metric* (6.12) *for some* $p \in [1, \infty]$. *For compact subsets* $F \subset X$ *and* $G \subset Y$

$$\dim_{LB}(F \times G) = \liminf_{\delta \to 0+} \left(\frac{\log N(F, \delta)}{-\log \delta} + \frac{\log N(G, \delta)}{-\log \delta} \right) \tag{6.18}$$

and

$$\dim_{B}(F \times G) = \limsup_{\delta \to 0+} \left(\frac{\log N(F, \delta)}{-\log \delta} + \frac{\log N(G, \delta)}{-\log \delta} \right). \tag{6.19}$$

Proof. From the geometric inequality (6.15)

$$\begin{aligned}\liminf_{\delta \to 0+} \left(\frac{\log N(F, \delta)}{-\log \delta} + \frac{\log N(G, \delta)}{-\log \delta} \right) &= \liminf_{\delta \to 0+} \frac{\log (N(F, \delta) N(G, \delta))}{-\log \delta} \\ &\geq \liminf_{\delta \to 0+} \frac{\log N(F \times G, \delta)}{-\log \delta}.\end{aligned}$$

which is $\dim_{LB}(F \times G)$. From the equivalence of the definitions of the box-counting dimension

$$= \liminf_{\delta \to 0+} \frac{\log N'(F \times G, \delta)}{-\log \delta} \geq \liminf_{\delta \to 0+} \frac{\log (N'(F, \delta) N'(G, \delta))}{-\log \delta}$$

using the inequality (6.16). Finally, from the inequality (6.3)

$$\geq \liminf_{\delta \to 0+} \frac{\log (N(F, 2\delta) N(G, 2\delta))}{-\log \delta} = \liminf_{\delta \to 0+} \frac{\log (N(F, \delta) N(G, \delta))}{-\log \delta}$$

so there is equality throughout, yielding (6.18). The upper box-counting equivalence (6.19) follows similarly.

Note that in the Euclidean case we can immediately write

$$\dim_{LB}(F \times G) = \liminf_{\delta \to 0+} \left(\frac{\log M(F, \delta)}{-\log \delta} + \frac{\log M(G, \delta)}{-\log \delta} \right)$$

and

$$\dim_{B}(F \times G) = \limsup_{\delta \to 0+} \left(\frac{\log M(F, \delta)}{-\log \delta} + \frac{\log M(G, \delta)}{-\log \delta} \right).$$

from the equality (6.17). □

This observation simplifies the proof of the main theorem and the calculation of the box-counting dimensions in the subsequent examples.

Theorem 6.8. *For compact subsets $F \subset X$ and $G \subset Y$ the upper and lower box-counting dimensions of the product set $F \times G$ satisfy*

$$\begin{aligned}\dim_{LB}(F) + \dim_{LB}(G) \le \dim_{LB}(F \times G) & \\ &\le \min\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\ &\le \max\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\ &\le \dim_B(F \times G) \\ &\le \dim_B(F) + \dim_B(G).\end{aligned}$$

Proof. The result follows immediately from the equivalent definitions (6.18) and (6.19) together with the elementary analytic inequalities

$$\liminf A + \liminf B \le \liminf (A + B), \tag{6.20}$$

$$\liminf (A + B) \le \liminf A + \limsup B, \tag{6.21}$$

$$\liminf A + \limsup B \le \limsup (A + B) \qquad \text{and} \tag{6.22}$$

$$\limsup (A + B) \le \limsup A + \limsup B. \tag{6.23}$$

The inequalities (6.20) and (6.21) yield

$$\begin{aligned}\dim_{LB}(F) + \dim_{LB}(G) &= \liminf_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log\delta} + \frac{\log N(G,\delta)}{-\log\delta} \\ &\le \liminf_{\delta \to 0+} \left(\frac{\log N(F,\delta)}{-\log\delta} + \frac{\log N(G,\delta)}{-\log\delta}\right) \qquad (6.24) \\ &\le \liminf_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log\delta} + \limsup_{\delta \to 0+} \frac{\log N(G,\delta)}{-\log\delta} \\ &= \dim_{LB}(F) + \dim_B(G)\end{aligned}$$

and the result follows as (6.24) is equal to $\dim_{LB}(F \times G)$ from (6.18). The remaining inequalities are proved similarly. □

It is possible to derive similar product formulas for the product of m compact sets $F_1, \ldots, F_m$ by introducing the bounds

$$\dim_{LB}(F_1 \times \ldots \times F_m) \leq \min_{i=1\ldots m} \left(\dim_{LB}(F_i) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \dim_B(F_j) \right) \tag{6.25}$$

and
$$\dim_B(F_1 \times \ldots \times F_m) \geq \max_{i=1\ldots m} \left(\dim_B(F_i) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \dim_{LB}(F_j) \right) \tag{6.26}$$

which follow from the analytic inequalities $\liminf(A_1 + \ldots + A_m) \leq \liminf A_1 + \sum_{j=2}^{m} \limsup A_j$ and $\limsup(A_1 + \ldots + A_m) \geq \limsup A_1 + \sum_{j=2}^{m} \liminf A_j$. Note that the right hand side of (6.25) can be greater than the right hand side of (6.26) as illustrated by the Cartesian product $F \times F \times F$ if $\dim_{LB}(F) = 0$ and $\dim_B(F) = 1$. Consequently, we cannot write the inequalities (6.25) and (6.26) as a chain of inequalities as in the statement of Theorem 6.8.

If each of the sets F and G have equal upper and lower box-counting dimension then the box-counting dimension of their product is also well behaved:

Corollary 6.9. *If* $\dim_B(F) = \dim_{LB}(F)$ *and* $\dim_B(G) = \dim_{LB}(G)$ *then*

$$\dim_B(F \times G) = \dim_{LB}(F \times G) = \dim_B(F) + \dim_B(G).$$

Proof. As $\dim_{LB}(F) + \dim_{LB}(G) = \dim_B(F) + \dim_B(G)$ we clearly have equality throughout the statement of Theorem 6.8. □

6.2 Compatible generalised Cantor sets

A generalised Cantor set (see §4.11 in Mattila [1995]) is a variation of the well known Cantor middle-third set that permits the proportion removed from each interval to vary throughout the iterative process. Formally, for $b > 1$ we define the application of the generator gen_b to a set of disjoint intervals $\mathcal{I}$ as the procedure in which the middle $1 - 2^{1-b}$ proportion of each interval in $\mathcal{I}$ is removed.

With generators of this form, we can produce sets F of arbitrary box-counting dimension in the range $(0, 1)$ through the repeated application of a single generator.

Lemma 6.10. *Fix* $b > 1$. *Starting from the initial set* $F_0 = [0, 1]$ *let* $F_j = \text{gen}_b(F_{j-1})$ *for all* $j \in \mathbb{N}$. *The resulting set* $F = \bigcap F_j$ *has upper and lower box-counting dimension equal to* $\frac{1}{b}$.

Proof. See §4.10 in Mattila [1995]. This is also a consequence of Theorem 6.17. □

In the following we detail a method to construct a generalised Cantor sets F from an arbitrary sequence of generators gen_{b_i}. Roughly, the $(j-1)^{th}$ stage of the construction F_{j-1} will consist of a finite number of disjoint intervals and we define the j^{th} stage of the construction F_j by iteratively applying the generator gen_{b_j} to the set F_{j-1}. As we are applying the generator gen_{b_j} to a finite number of disjoint intervals we are effectively creating $\#(F_{j-1})$ disjoint copies of the set from Lemma 6.10 with box-counting dimension equal to $\frac{1}{b_j}$, so by applying the generator gen_{b_j} a sufficient number of times the 'box-counting function'

$$\frac{\log N(F,\delta)}{-\log \delta}$$

approaches $\frac{1}{b_j}$ for length scales δ of approximately the length of the intervals of F_j. While it is relatively straightforward to calculate the required number of iterations, the length scales δ and the value of $N(F,\delta)$ it is prohibitively difficult to calculate these quantities for the set $F \times G$ where G is another arbitrary generalised Cantor set.

We rectify this incompatibility by constructing the generalised Cantor sets F and G in parallel from two arbitrary sequences of generators gen_{b_i} and gen_{c_i}. At the j^{th} stage of the construction, as above, we iteratively apply the generators gen_{b_j} and gen_{c_j} respectively to the intermediary sets F_{j-1} and G_{j-1} a sufficient number of times for each set to ensure that

- the intermediary sets F_j and G_j consist of intervals of the same length, which is the content of Lemma 6.11,

- for δ equal to the common length of the intervals of F_j and G_j the 'box-counting functions'

$$\frac{\log N(F,\delta)}{-\log \delta} \qquad \text{and} \qquad \frac{\log N(G,\delta)}{-\log \delta} \tag{6.27}$$

  approach $\frac{1}{b_j}$ and $\frac{1}{c_j}$ respectively, which is the content of Corollary 6.14, and that

- for all δ the 'box-counting functions' (6.27) are tightly controlled, which is the content of Lemma 6.15.

As a consequence, for a given length scale δ we have good bounds on the values of the box-counting functions (6.27) which from Lemma 6.7 yields good bounds on the 'box counting function' of the set $F \times G$.

We proceed by defining the construction of these compatible generalised Cantor sets before describing their geometry in the next section in enough detail to find minimal covers by sets of diameter δ.

We construct the set F from a sequence of generators gen_{b_i} and the set G from a sequence of generators gen_{c_i}. For simplicity, we assume that b_i, c_i are rational numbers greater than 1 for all i. Let $K_0 = 0$, $b_0, c_0 = 1$ and take a set of increasing positive integers $\{K_i\}_{i=1}^{\infty}$ such that

$$\frac{K_j}{b_j}, \frac{K_{j-1}}{b_j}, \frac{K_j}{c_j}, \frac{K_{j-1}}{c_j} \in \mathbb{N} \tag{6.28}$$

for all $j \in \mathbb{N}$ and

$$\left(\sum_{i=1}^{j-1} (K_i - K_{i-1}) / b_i\right) / K_j \to 0 \qquad \text{as} \quad j \to \infty \tag{6.29}$$

$$\left(\sum_{i=1}^{j-1} (K_i - K_{i-1}) / c_i\right) / K_j \to 0 \qquad \text{as} \quad j \to \infty. \tag{6.30}$$

The sequence $K_j := 2^{2^j} \prod_{i=1}^{j+1} \text{num}\,(b_i)\, \text{num}\,(c_i)$, where $\text{num}\, b_i$ is is the numerator of the rational number b_i, is sufficient.

We define $F := \bigcap_{j \in \mathbb{N}} F_j$ where $F_0 = [0,1]$ and the set at the j^{th} stage of the construction, F_j, is formed by applying the generator gen_{b_j} a total of $(K_j - K_{j-1}) / b_j$ times to the set of disjoint intervals F_{j-1}. This is well defined as, from (6.28), $(K_j - K_{j-1}) / b_j$ is a positive integer. Similarly, we define $G := \bigcap_{j \in \mathbb{N}} G_j$ where $G_0 = [0,1]$ and G_j is formed by applying the generator gen_{c_j} a total of $(K_j - K_{j-1}) / c_j$ times to the set G_{j-1}.

In order to calculate the box-counting dimensions of the sets F and G we need to find reasonable bounds for one of the geometric quantities from the previous section. For the remainder of this section we use the geometric quantity $N(F, \delta)$ which, we recall, is the minimum number of sets of diameter δ that form a cover of F.

6.2.1 Geometry of generalised Cantor sets

For $n = 1 \ldots (K_j - K_{j-1}) / b_j$ define $F_{j-1,n}$ to be the result of n successive applications of the generator gen_{b_j} to the set F_{j-1}. The sets $F_{j-1,n}$ are the sets which make up the 'substages' in the construction of F, with one such substage for every application of a generator. Note that $F_j = F_{j-1,(K_j - K_{j-1})/b_j}$ and as the sets are monotonically decreasing in n,

$$F_j = \bigcap_{n=1}^{(K_j - K_{j-1})/b_j} F_{j-1,n}. \tag{6.31}$$

We write $\#(F_{j,n})$ for the number and $l(F_{j,n})$ for the length of the intervals in $F_{j,n}$.

Lemma 6.11. *For all $j \in \mathbb{N}$ and $n = 1 \ldots (K_j - K_{j-1})/b_j$ the number of intervals in the set $F_{j-1,n}$ is*

$$\#(F_{j-1,n}) = 2^{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n}$$

and these intervals are of length

$$l(F_{j-1,n}) = 2^{-K_{j-1} - b_j n}.$$

Proof. As F_{j-1} is formed by applying $\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i$ generators, the set $F_{j-1,n}$ is formed by applying a total of $\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n$ generators. Further, as each generator splits all intervals into two, and our initial set F_0 is a single interval, $F_{j-1,n}$ consists of

$$\#(F_{j-1,n}) = 2^{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n}$$

intervals.

Next, applying the generator gen_b to a set $\mathcal{I}$ of disjoint intervals of length λ removes the middle $1 - 2^{1-b}$ of each interval and so the intervals in $\mathrm{gen}_b(\mathcal{I})$ are of length $\lambda 2^{-b}$. Consequently, as the set F_{j-1} is formed from the unit interval by applying the generator gen_{b_i} a total of $(K_i - K_{i-1})/b_i$ times for each $i = 1 \ldots j-1$, the set F_{j-1} consists of intervals of length

$$l(F_{j-1}) = 2^{\sum_{i=1}^{j-1} -b_i(K_i - K_{i-1})/b_i} = 2^{-K_{j-1}}.$$

Finally, as the set $F_{j-1,n}$ is the result of a further n applications of the generator gen_{b_j}

$$l(F_{j-1,n}) = l(F_{j-1})\, 2^{-b_j n} = 2^{-K_{j-1} - b_j n}.$$

□

Replacing the b_i with c_i throughout the above lemma gives the corresponding result for the intermediary sets $G_{j-1,n}$ used in the construction of G. Note that the intervals in F_{j-1} are the same length as the intervals in G_{j-1} despite the arbitrarily chosen sequences $\{b_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$. These common lengths greatly simplify the calculation of the box-counting dimensions of the product set $F \times G$, and in this sense the sets F and G are 'compatible'.

We will need to find explicitly some points of the generalised Cantor set F. Recall that F is defined by $F := \bigcap_{j \in \mathbb{N}} F_j$ which, in light of (6.31), is the intersection of every intermediary substage $F_{j-1,n}$ and can be written

$$F := \bigcap_{j \in \mathbb{N}} \bigcap_{n=1}^{K_j - K_{j-1}/b_j} K_{j-1,n}.$$

When applying a generator gen_b to an intermediary set of disjoint intervals $\mathcal{I}$, a proportion is removed from the middle of each interval. Consequently the endpoints of the intervals $\mathcal{I}$ are in the set $\text{gen}_b(\mathcal{I})$ and remain endpoints of intervals. Inductively we see that the endpoints of each intermediary set $F_{j-1,n}$ are in the final set F.

As each intermediary set $F_{j-1,n}$ is a cover of F and the length of the intervals in $F_{j-1,n}$ approach zero as $j \to \infty$ it is natural to use the minimal cover formulation of the box-counting dimension for these sets: we immediately have that if $l(F_{j-1,n}) \leq \delta$ the set $F_{j-1,n}$ is a suitable cover of F. Unfortunately this cover is not always a minimal cover at this length-scale. However, a reasonable lower bound on $N(F, \delta)$ is easy to find and if we restrict the choice of generators so that the intervals are suitably separated then the sets $F_{j-1,n}$ are minimal covers at the approriate length-scale.

Lemma 6.12. *For δ in the range $l(F_{j-1,n}) \leq \delta < l(F_{j-1,n-1})$ the minimum number of sets of diameter δ which cover F satisfies*

$$\#(F_{j-1,n-1}) \leq N(F, \delta) \leq \#(F_{j-1,n}). \tag{6.32}$$

Further, if the choice of generators is restricted so that $b_i \geq \log(3)/\log(2)$ for all i then

$$N(F, \delta) = \#(F_{j-1,n}).$$

Proof. The upper bound follows immediately from the fact that $F_{j-1,n}$ is a cover of F consisting of $\#(F_{j-1,n})$ sets of diameter less than δ. For the lower bound consider the following points in F: let E consist of all the left endpoints of the intervals in $F_{j-1,n-1}$ so that E consists of $\#(F_{j-1,n-1})$ points. Now, any two points of E are separated by one of the intervals of $F_{j-1,n-1}$ so no set of diameter $\delta < l\left(F_{j-1,n-1}\right)$ can intersect two points of E (see Figure 6.1). Consequently, at least $\#\left(F_{j-1,n-1}\right)$ sets of diameter δ are required to cover E therefore at least this many are required to cover F, yielding

$$\#\left(F_{j-1,n-1}\right) \leq N\left(F,\delta\right) \leq \#\left(F_{j-1,n}\right).$$

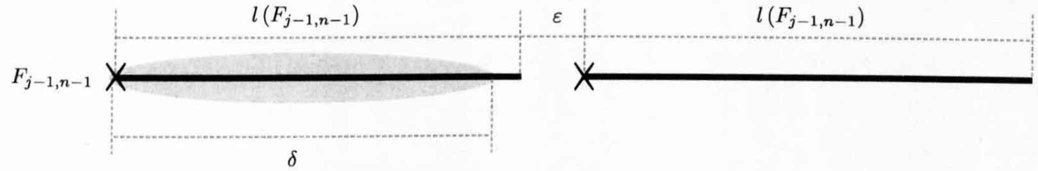

Figure 6.1: Two intervals in the set $F_{j-1,n-1}$ (black lines) to illustrate that sets with diameter $\delta < l\left(F_{j-1,n-1}\right)$ (grey ellipses) can not intersect two left endpoints (black crosses). Consequently we require at least one set of diameter δ for each interval in $F_{j-1,n-1}$ to cover all the left endpoints. Generally this cannot be improved as the distance between intervals ε can be arbitrarily small.

If we restrict the generators to those gen_b with $b \geq \log(3)/\log(2)$ then with every application of a generator at least the middle third of each interval is removed. Consequently, the intervals in $F_{j-1,n-1}$ are separated by at least the length $l\left(F_{j-1,n-1}\right)$ so that if E is the set of all (both left and right) endpoints of the intervals in $F_{j-1,n-1}$ then no set of diameter $\delta < l\left(F_{j-1,n-1}\right)$ can intersect two points of E (see Figure 6.2). As E consists of $2\#\left(F_{j-1,n-1}\right) = \#\left(F_{j-1,n}\right)$ points at least this many sets of diameter δ are required to cover E and hence required to cover F, yielding

$$\#\left(F_{j-1,n}\right) \leq N\left(F,\delta\right) \leq \#\left(F_{j-1,n}\right).$$

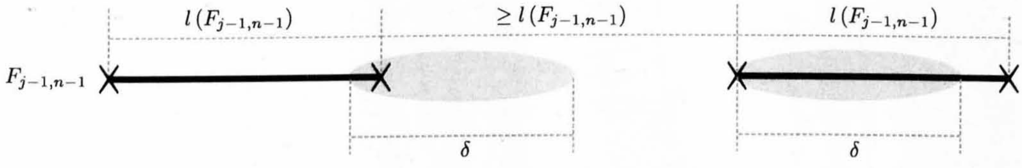

Figure 6.2: Two intervals in the set $F_{j-1,n-1}$ (black lines) constructed from generators gen_b with $b \geq \log(3)/\log(2)$. As the distance between intervals is at least the length of the interval a set of diameter $\delta < l\left(F_{j-1,n-1}\right)$ (grey ellipses) can not intersect two endpoints (black crosses). Consequently we require at least two sets of diameter δ for each interval in $F_{j-1,n-1}$ to cover all the endpoints.

□

6.2.2 Box-counting dimension

We can now calculate the box-counting dimensions of compatible generalised Cantor sets.

Lemma 6.13. *For all $j \in \mathbb{N}$ and $n = 1 \ldots (K_j - K_{j-1})/b_j$ if δ is in the range $2^{-K_{j-1}-b_j n} \leq \delta < 2^{-K_{j-1}-b_j(n-1)}$ then the 'box-counting function' of F satisfies*

$$\frac{\sum_{i=1}^{j-1} (K_i - K_{i-1})/b_i + n - 1}{K_{j-1} + b_j n} \leq \frac{\log N(F, \delta)}{-\log \delta} < \frac{\sum_{i=1}^{j-1} (K_i - K_{i-1})/b_i + n}{K_{j-1} + b_j (n-1)}. \quad (6.33)$$

Proof. Immediate from Lemmas 6.11 and 6.12. □

Replacing b_i with c_i throughout the above lemma gives the corresponding result for the set G.

Corollary 6.14. *With F and G constructed as above*

$$\dim_{LB}(F) \leq \liminf_{j \to \infty} \frac{1}{b_j} \qquad \limsup_{j \to \infty} \frac{1}{b_j} \leq \dim_B(F)$$

$$\dim_{LB}(G) \leq \liminf_{j \to \infty} \frac{1}{c_j} \qquad \limsup_{j \to \infty} \frac{1}{c_j} \leq \dim_B(G)$$

$$\dim_{LB}(F \times G) \leq \liminf_{j \to \infty} \frac{1}{b_j} + \frac{1}{c_j} \qquad \limsup_{j \to \infty} \frac{1}{b_j} + \frac{1}{c_j} \leq \dim_B(F \times G)$$

Proof. Consider the sequence $\delta_j := 2^{-K_j}$ and apply Lemma 6.13 with $n = (K_j - K_{j-1})/b_j$ to yield

$$\frac{\sum_{i=1}^{j} (K_i - K_{i-1})/b_i - 1}{K_j} \leq \frac{\log N(F, \delta_j)}{-\log \delta_j} < \frac{\sum_{i=1}^{j} (K_i - K_{i-1})/b_i}{K_j - 1}.$$

Consequently,

$$\frac{\log N(F, \delta_j)}{-\log \delta_j} \geq \frac{1}{b_j} + \frac{-K_{j-1}/b_j + \sum_{i=1}^{j-1} (K_i - K_{i-1})/b_i - 1}{K_j} \quad (6.34)$$

and

$$\frac{\log N(F, \delta_j)}{-\log \delta_j} < \frac{1}{b_j} + \frac{1/b_j - K_{j-1}/b_j + \sum_{i=1}^{j-1} (K_i - K_{i-1})/b_i}{K_j - 1} \quad (6.35)$$

and from (6.29) the second terms tend to zero as $j \to \infty$. Consequently,

$$\liminf_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} \le \liminf_{j\to\infty} \frac{\log N(F,\delta_j)}{-\log\delta_j} = \liminf_{j\to\infty} \frac{1}{b_j}$$
$$\text{and} \qquad \limsup_{\delta\to 0+} \frac{\log N(F,\delta)}{-\log\delta} \ge \limsup_{j\to\infty} \frac{\log N(F,\delta_j)}{-\log\delta_j} = \limsup_{j\to\infty} \frac{1}{b_j}.$$

The result for the set G follows similarly using the same sequence δ_j. Next, we sum each of (6.34) and (6.35) with their equivalent inequalities for the set G so that at the limit

$$\liminf_{j\to\infty} \left(\frac{\log N(F,\delta_j)}{-\log\delta_j} + \frac{\log N(G,\delta_j)}{-\log\delta_j} \right) = \liminf_{j\to\infty} \frac{1}{b_j} + \frac{1}{c_j}$$
$$\text{and} \qquad \limsup_{j\to\infty} \left(\frac{\log N(F,\delta_j)}{-\log\delta_j} + \frac{\log N(G,\delta_j)}{-\log\delta_j} \right) = \limsup_{j\to\infty} \frac{1}{b_j} + \frac{1}{c_j}$$

and the result for the product set $F \times G$ follows from the equivalent definitions (6.18) and (6.19). □

Finally, we find some bounds on the 'box-counting' function for all δ.

Lemma 6.15. *For δ in the range*

$$2^{-K_j} \le \delta < 2^{-K_{j-1}} \tag{6.36}$$

$$\min\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) - \varepsilon_j \le \frac{\log N(F,\delta)}{-\log\delta} < \max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \varepsilon_j \tag{6.37}$$

$$\min\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) - \varepsilon_j \le \frac{\log N(G,\delta)}{-\log\delta} < \max\left(\frac{1}{c_j}, \frac{1}{c_{j-1}}\right) + \varepsilon_j \tag{6.38}$$

where $\varepsilon_j \to 0$ as $j \to \infty$.

Proof. For δ in the range (6.36) there exists $n \in \mathbb{N}$ such that

$$2^{-K_{j-1}-b_j n} \le \delta < 2^{-K_{j-1}-b_j(n-1)}. \tag{6.39}$$

Lemma 6.13 yields the lower bound

$$\begin{aligned}\frac{\log N(F,\delta)}{-\log\delta} &\ge \frac{\sum_{i=1}^{j-1}(K_i - K_{i-1})/b_i + n - 1}{K_{j-1} + b_j n} \\ &= \frac{1}{b_{j-1}} \frac{K_{j-1} + b_{j-1} n}{K_{j-1} + b_j n} - \frac{K_{j-2}/b_{j-1} - \sum_{i=1}^{j-2}(K_i - K_{i-1})/b_i + 1}{K_{j-1} + b_j n}\end{aligned}$$

and writing the second term as ε_j^- we consider the separate cases

$$\geq \begin{cases} \frac{1}{b_{j-1}} \frac{K_{j-1}+b_j n}{K_{j-1}+b_j n} + \varepsilon_j^- & b_{j-1} \geq b_j \\ \frac{1}{b_{j-1}} \frac{b_{j-1}}{b_j} \frac{K_{j-1}+b_j n}{K_{j-1}+b_j n} + \varepsilon_j^- & b_{j-1} < b_j \end{cases}$$
$$\geq \min\left(\frac{1}{b_i}, \frac{1}{b_{i-1}}\right) + \varepsilon_j^-.$$

The term ε_j^- tends to zero as $j \to \infty$ as

$$\left|\varepsilon_j^-\right| \leq \frac{K_{j-2}/b_{j-1} + \sum_{i=1}^{j-2} (K_i - K_{i-1})/b_i + 1}{K_{j-1}} \to 0$$

as $j \to \infty$ from (6.29).

Similarly, for δ in the range (6.39), we have the upper bound

$$\begin{aligned} \frac{\log N(F,\delta)}{-\log \delta} &< \frac{\sum_{i=1}^{j-1} (K_i - K_{i-1})/b_i + n}{K_{j-1} + b_j(n-1)} \\ &= \frac{1}{b_{j-1}} \frac{K_{j-1} + b_{j-1}(n-1)}{K_{j-1} + b_j(n-1)} \\ &\quad - \frac{(K_{j-2} - 1)/b_{j-1} - \sum_{i=1}^{j-2} (K_i - K_{i-1})/b_i}{K_{j-1} + b_j(n-1)}. \end{aligned}$$

Again, writing the second term as ε_j^+ we consider the separate cases

$$\leq \begin{cases} \frac{1}{b_{j-1}} \frac{K_{j-1}+b_j(n-1)}{K_{j-1}+b_j(n-1)} + \varepsilon_j^+ & b_j \geq b_{j-1} \\ \frac{1}{b_{j-1}} \frac{b_{j-1}}{b_j} \frac{K_{j-1}+b_j(n-1)}{K_{j-1}+b_j(n-1)} + \varepsilon_j^+ & b_j < b_{j-1} \end{cases}$$
$$\leq \max\left(\frac{1}{b_j}, \frac{1}{b_{j-1}}\right) + \varepsilon_j^+.$$

The term ε_j^+ tends to zero as $j \to \infty$ as

$$\left|\varepsilon_j^+\right| \leq \frac{(K_{j-2} - 1)/b_{j-1} - \sum_{i=1}^{j-2} (K_i - K_{i-1})/b_i}{K_{j-1}} \to 0$$

as $j \to \infty$ from (6.29).

Similarly, for δ in the range (6.36) there exists an $m \in \mathbb{N}$ such that

$$2^{-K_{j-1} - c_j m} \leq \delta < 2^{-K_{j-1} - c_j(m-1)} \tag{6.40}$$

and similarly we can find η_j^- and η_j^+ such that

$$\min\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)-\eta_j^- \leq \frac{\log N(G,\delta)}{-\log\delta} < \max\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)+\eta_j^+$$

for δ in the range (6.36) where $\eta_j^-,\eta_j^+ \to 0$ as $j\to\infty$.
Taking $\varepsilon_j = \max\left(\left|\varepsilon_j^-\right|,\left|\varepsilon_j^+\right|,\left|\eta_j^-\right|,\left|\eta_j^+\right|\right)$ completes the proof. □

Corollary 6.16. *With F and G constructed as above*

$$\liminf_{j\to\infty}\frac{1}{b_j} \leq \dim_{LB}(F) \qquad \dim_B(F) \leq \limsup_{j\to\infty}\frac{1}{b_j}$$

$$\liminf_{j\to\infty}\frac{1}{c_j} \leq \dim_{LB}(G) \qquad \dim_B(G) \leq \limsup_{j\to\infty}\frac{1}{c_j}$$

and

$$\liminf_{j\to\infty}\left(\frac{1}{b_j}+\frac{1}{c_j}\right) \leq \dim_{LB}(F\times G) \quad \dim_B(F\times G) \leq \limsup_{j\to\infty}\left(\frac{1}{b_j}+\frac{1}{c_j}\right)$$

Proof. The results for the sets F and G follow immediately from (6.37) and (6.38) after taking limits. Next, if δ is in the range (6.36) then

$$\begin{aligned}\min\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\min\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)-2\varepsilon_j &\leq \frac{\log N(F,\delta)}{-\log\delta}+\frac{\log N(G,\delta)}{-\log\delta}\\ &< \max\left(\frac{1}{b_j},\frac{1}{b_{j-1}}\right)+\max\left(\frac{1}{c_j},\frac{1}{c_{j-1}}\right)+2\varepsilon_j\end{aligned}$$

from (6.37) and (6.38) and the result for the product set $F\times G$ follows after taking limits from the equivalent definitions (6.18) and (6.19). □

Using these bounds we can explicitly write the box-counting dimensions of the sets F, G and their Cartesian product $F\times G$:

Theorem 6.17. *With F and G constructed as above,*

$$\dim_{LB}(F) = \liminf_{i\to\infty}\frac{1}{b_i} \qquad \dim_B(F) = \limsup_{i\to\infty}\frac{1}{b_i}$$

$$\dim_{LB}(G) = \liminf_{i\to\infty}\frac{1}{c_i} \qquad \dim_B(G) = \limsup_{i\to\infty}\frac{1}{c_i}$$

$$\dim_{LB}(F\times G) = \liminf_{i\to\infty}\frac{1}{b_i}+\frac{1}{c_i} \qquad \dim_B(F\times G) = \limsup_{i\to\infty}\frac{1}{b_i}+\frac{1}{c_i}$$

Proof. Immediate from Corollaries 6.14 and 6.16. □

Example 6.18. *If F and G are constructed as above from the sequences defined by*

$$b_i = \begin{cases} 2 & i = 6n-5 \\ 6 & i = 6n-4 \\ 4 & \text{otherwise} \end{cases} \quad \text{and} \quad c_i = \begin{cases} 3 & i = 6n-2 \\ 5 & i = 6n-1 \\ 4 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$ then

$$\begin{aligned} \dim_{LB}(F) &= \frac{1}{6} & \dim_B(F) &= \frac{1}{2} \\ \dim_{LB}(G) &= \frac{1}{5} & \dim_B(G) &= \frac{1}{3} \\ \dim_{LB}(F \times G) &= \frac{5}{12} & \dim_B(F \times G) &= \frac{3}{4} \end{aligned}$$

and in particular,

$$\begin{aligned} \dim_{LB}(F) + \dim_{LB}(G) &< \dim_{LB}(F \times G) \\ &< \min\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\ &< \max\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\ &< \dim_B(F \times G) \\ &< \dim_B(F) + \dim_B(G). \end{aligned}$$

Example 6.19. *If F and G are constructed as above from the sequences defined by*

$$b_i = \begin{cases} 2 & i = 1 \\ i/(i-1) & i \text{ odd}, i > 1 \\ i & i \text{ even} \end{cases} \quad \text{and} \quad c_i = \begin{cases} 2 & i = 1 \\ i & i \text{ odd}, i > 1 \\ i/(i-1) & i \text{ even} \end{cases}$$

for $n \in \mathbb{N}$ then

$$\begin{aligned} & \dim_{LB}(F) = \dim_{LB}(G) = 0, \qquad \dim_{LB}(F \times G) = 1 \\ \text{and} \quad & \dim_B(F) = \dim_B(G) = \dim_B(F \times G) = 1. \end{aligned}$$

The Example 6.18 illustrates that there are sets F and G for which the chain of inequalities in Theorem 6.8 are all strict. Further Example 6.19 demonstrates some counter-intuitive behaviour as two sets with zero lower box-counting dimension have a product with positive lower box-counting dimension, and for the same sets the upper box-counting dimension does not increase upon taking the product. It is clear from Theorem 6.17 that we are able to construct sets $F, G \subset \mathbb{R}$ such that the upper and lower box-counting dimensions of the set F, G and $F \times G$ can take any arbitary values subject to the chain of inequalities in Theorem 6.8.

These results are relevant to the r-codimension prints of the previous chapter, as we see in Example 5.17 that the main theorem to determine the content of the r-codimension print fails to capture the entire print whenever the sets have strict inequality in the box-counting dimension product formula.

Chapter 7

Function spaces and measurability

In Section 2.4 we described a measurability issue in the foundations of the theory of regular Lagrangian flows. This issue arose as a map was considered to be a member of two functions spaces on which there are two different equivalence relations. Consequently, when a particular representative of this map was selected with respect to one equivalence relation there was no guarantee that the resulting map was also equivalent with respect to the other equivalence relation. In this chapter we discuss the measurability issues that arise as a consequence of considering maps belonging to both spaces

$$L^1\left(0,T;L^1\left(\mathbb{R}^n\right)\right) \qquad \text{and} \qquad L^1\left((0,T)\times\mathbb{R}^n\right).$$

Further, we examine the analogous L^∞ spaces and in Lemma 7.20 we demonstrate that the map

$$f(t,x) := \begin{cases} 1 & x \leq t \\ 0 & x > t \end{cases}$$

belongs to $L^\infty\left((0,1)\times(0,1)\right)$ but does not belong to $L^\infty\left((0,1);L^\infty\left((0,1)\right)\right)$. Ultimately we demonstrate that the latter space is isometically isomorphic to a proper subset of the former space.

It is well known that the spaces $L^1\left(\mathcal{I}\times\Omega\right)$ and $L^1\left(\mathcal{I};L^1\left(\Omega\right)\right)$ are isometrically isomorphic where the former space consists of equivalence classes of maps $f\colon \mathcal{I}\times\Omega\to\mathbb{R}$ and the latter space consists of equivalence classes of maps $f\colon \mathcal{I}\to$

$L^1(\Omega)$. However, with abused notation we often see statements like

$$f \in L^1(\mathcal{I} \times \Omega) \Rightarrow f \in L^1\left(\mathcal{I}; L^1(\Omega)\right) \tag{7.1}$$

for a map $f: \mathcal{I} \times \Omega \to \mathbb{R}$. We consider the sense in which a map $f: \mathcal{I} \times \Omega \to \mathbb{R}$ can 'belong' to both of these spaces. This is our first main result (Theorem 7.10) in which we demonstrate that the space $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ can be written in a natural way as a space of maps $f: \mathcal{I} \times \Omega \to \mathbb{R}$ modulo an equivalence relation. However, it is erroneous to conclude that the equivalence classes of $L^1(\mathcal{I} \times \Omega)$ and those of $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ contain the same maps even though these spaces are isometrically isomorphic, as they have different equivalence relations:

- in the equivalence classes of $L^1(\mathcal{I} \times \Omega)$ two vector fields f and g are equivalent, and we write $f \approx g$, if and only if

$$f(t,x) = g(t,x) \quad \text{for almost every } (t,x) \in \mathcal{I} \times \Omega, \quad \text{and} \tag{7.2}$$

- in the equivalence classes of $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ two vector fields f and g are equivalent, and we write $f \sim g$, if and only if

$$[f(t,x) = g(t,x) \quad \text{for almost every } x \in \Omega] \quad \text{for almost every } t \in \mathcal{I}. \tag{7.3}$$

We demonstrate that every map in an equivalence class of $L^1(\mathcal{I} \times \Omega)$ is in an equivalence class of $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ but that the converse does not hold: in Corollary 7.13 we give a map which 'belongs' to $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ but is not measurable as a map $\mathcal{I} \times \Omega \to \mathbb{R}$.

The situation is recovered in Lemma 7.17 in which we demonstrate that if a map f is in an equivalence class of $L^1\left(\mathcal{I}; L^1(\Omega)\right)$ then there is a map $g \sim f$ such that g is in an equivalence class of $L^1(\mathcal{I} \times \Omega)$. Consequently, there always exists a 'nice' representative map for which the implication (7.1) and its converse hold. In many applications we are free to choose this well behaved map as a representative, but we must cautious when applying these inclusions to particular maps.

Initially, we establish the necessary theory to be able to integrate maps that take values in Banach spaces, the so called Bochner integration after Bochner [1933]. We draw upon the treatment of Bochner integration in Lang [1993] Chapter 6, which has the advantages of brevity as it does have the theory of integration of real valued functions as a prerequisite. Here, we develop the theory with greater generality in order to examine the integration of maps that take values in spaces which are not necessarily complete and only have a seminorm. This generalisation is motivated by the desire to integrate maps whose values are integrable maps, which will allow us to write the space $L^p(\mathcal{I}; L^q(\Omega))$ in a more accessible way.

7.1 Definitions of the L^p spaces

Let X be a (semi)normed space and $A \subset \mathbb{R}^n$.

7.1.1 Measurability and strong measurability

The fundamental maps in measure theory are those that take finitely many values and such that the inverse image of any value is a measurable set of finite measure. Such maps can be written as a finite sum as follows:

Definition 7.1. *A map $f\colon A \to X$ is a* **step map** *if for some $m \in \mathbb{N}$*

$$f(t) = \sum_{i=1}^{m} x_i \mathbf{1}_{A_i}(t) \tag{7.4}$$

where $x_i \in X$, the sets A_i are Lebesgue measurable subsets of A with $\mu_n(A_i) < \infty$ and $\mathbf{1}_{A_i}$ is the characteristic function of the set A_i.

The following definitions of measurability and integrability only require a map to take values in X for almost every point in the domain. We use the notation

$$f\colon A \rightarrowtail X$$

if there is a set $\mathcal{N} \subset A$ of zero measure such that $f(t) \in X$ for all $t \in A \setminus \mathcal{N}$. In particular the map f can take values outside X, or even be undefined, on a set of zero measure.

Definition 7.2. *A map $f\colon A \rightarrowtail X$*

- *is* **measurable** *if for every open subset $O \subset X$ the inverse image $f^{-1}(O)$ is a Lebesgue measurable subset of A,*
- *is* **strongly measurable**, *and we write $f \in \mathcal{L}^0(A; X)$, if there is a sequence of step maps f_k such that*

$$f(t) = \lim_{k\to\infty} f_k(t) \quad \textit{for almost every } t \in A, \quad \textit{and}$$

- *is* **almost separably valued** *if there exists a null set $\mathcal{N} \subset A$ such that the image $f(A \setminus \mathcal{N}) \subset X$ contains a countable set that is dense in $f(A \setminus \mathcal{N})$.*

Immediately we see that step maps are strongly measurable and also, as the inverse image of an open set under a step map is a finite union of Lebesgue measurable sets, step maps are measurable.

It is known (see, for example, Lang [1993] pp.124) that a map $f: A \to X$ is strongly measurable if and only if it is measurable and almost separably valued. Consequently, if X is separable then measurability and strong measurability are equivalent. Also, it can be shown (see Lemma 451Q Fremlin [2000] vol.4 pp.425) that if $A \subset \mathbb{R}^n$ is of finite measure and f is measurable then f is almost separably valued and so is strongly measurable.

Strongly measurable maps are closed under pointwise limits almost everywhere, which is the content of the following lemma:

Lemma 7.3. *If the maps $f_k \in \mathcal{L}^0(A;X)$ for all $k \in \mathbb{N}$ and $f: A \rightarrowtail X$ is a map such that*

$$f(t) = \lim_{k\to\infty} f_k(t) \quad \textit{for almost every} \quad t \in A$$

then $f \in \mathcal{L}^0(A;X)$.

Proof. As a map is strongly measurable if and only if it is almost separably valued and measurable it is sufficient to demonstrate that the set of almost separably valued maps and the set of measurable maps are closed under pointwise limits almost everywhere. The fact that the set of measurable maps is closed is standard (see, for example, Corollary 2.2.2 of Cohn [1994] pp.59). Further, as a countable union of null sets is null and a countable union of separable sets is separable it follows that the set of almost separably valued maps is closed. For further details see, for example, Lang [1993] pp.125. □

We recall the definitions of the L^p spaces below, where we follow Lang [1993].

7.1.2 Spaces of maps $\mathcal{L}^p(A;X)$

Case $1 \leq p < \infty$:

Recall that the $\mathcal{L}^p$ semi-norm of the step map (7.4) is the real number

$$\|f\|_{\mathcal{L}^p(A;X)} := \left(\sum_{i=1}^{m} \|x_i\|_X^p \mu_n(A_i)\right)^{\frac{1}{p}}$$

and the integral of the step map (7.4) is the element of X given by

$$\int_A f \, \mathrm{d}\mu_n := \sum_{i=1}^{m} x_i \mu_n \left(A_i\right).$$

Definition 7.4. *For $p < \infty$ the map $f \colon A \rightarrowtail X$ is in $\mathcal{L}^p\left(A; X\right)$ if and only if there exists a sequence of step maps $f_k \colon A \to X$ such that*

- *$f_k\left(t\right) \to f\left(t\right)$ for almost every $t \in A$, and*
- *the maps f_k are Cauchy with respect to the $\|\cdot\|_{\mathcal{L}^p(A;X)}$ seminorm.*

Note that a fortiori $\mathcal{L}^p\left(A; X\right) \subset \mathcal{L}^0\left(A; X\right)$. We think of the space $\mathcal{L}^p\left(A; X\right)$ as a 'quasicompletion' of the step maps with respect to the $\mathcal{L}^p$ seminorm and we extend the $\mathcal{L}^p$ seminorm to maps in $\mathcal{L}^p\left(A; X\right)$ by continuity. Consequently, a map $f \in \mathcal{L}^p\left(A; X\right)$ has finite $\mathcal{L}^p$ seminorm.

For a map $f \in \mathcal{L}^1\left(A; X\right)$ we say that an element $x \in X$ is an integral of f if there exists a sequence of step maps f_k that approximate f as per Definition 7.4 such that

$$\lim_{k \to \infty} \left\| x - \int_A f_k \, \mathrm{d}\mu_n \right\|_X = 0.$$

It is easy to show that if the space X is complete then for each map $f \in \mathcal{L}^1\left(A; X\right)$ there is an element of X that is an integral of f, and further if $x, y \in X$ are both integrals of f then $\|x - y\|_X = 0$. In particular, if X is a Banach space then there exists a unique integral of $f \in \mathcal{L}^1\left(A; X\right)$, which we denote $\int_A f \, \mathrm{d}\mu_n$.

Case $p = \infty$:

Definition 7.5. *The map $f \colon A \rightarrowtail X$ is in $\mathcal{L}^\infty\left(A; X\right)$ if and only if*

- *$f \in \mathcal{L}^0\left(A; X\right)$, and*
- *there exists a constant $c \geq 0$ such that $\|f\left(t\right)\|_X \leq c$ for almost every $t \in A$*

The $\mathcal{L}^\infty$ semi-norm is the infimum of such constants c, that is

$$\|f\|_{\mathcal{L}^\infty(A;X)} := \operatorname*{ess\,sup}_{t \in A} \|f\left(t\right)\|_X := \inf\left\{c \in \mathbb{R} \middle| \quad \mu_n\left(\left\{t \in A \middle| \, \|f\left(t\right)\|_X > c\right\}\right) = 0\right\}.$$

We remark that generally it is not necessary for f to be strongly measurable to define a vector space of maps with finite $\mathcal{L}^\infty$ semi-norm. However, the restriction to strongly measurable maps ensures that each $f \in \mathcal{L}^\infty\left(A; X\right)$ is locally integrable, that is for each compact $K \subset A$ the restriction of f to K, denoted $f|_K$, is an element of $\mathcal{L}^1\left(A; X\right)$.

7.1.3 Spaces of equivalence classes $L^p(A;X)$

The spaces of maps $\mathcal{L}^p(A;X)$ are only equipped with a seminorm so are not Banach spaces. Indeed, if two maps $f,g\colon A \rightarrowtail X$ are identical except on a subset of A with zero measure, then the seminorm $\|f-g\|_{\mathcal{L}^p(A;X)}$ is equal to zero. Recall that we say the maps $f,g\colon A \rightarrowtail X$ are equivalent if $f(t)=g(t)$ for almost every $t \in A$, and that this is an equivalence relation on $\mathcal{L}^p(A;X)$.

Definition 7.6.

- *The space $L^p(A;X)$ is the set of equivalence classes of maps in $\mathcal{L}^p(A;X)$,*
- *for each map $f \in \mathcal{L}^p(A;X)$ we denote by $[f]_A$ the equivalence class in $L^p(A;X)$ that contains f, and*
- *for each equivalence class $\mathfrak{F} \in L^p(A;X)$ we define the quantity*

$$\|\mathfrak{F}\|_{L^p(A;X)} := \|f\|_{\mathcal{L}^p(A;X)} \tag{7.5}$$

where f is any representative of the equivalence class $\mathfrak{F}$.

It is entirely standard to show that (7.5) is independent of the representative of the equivalence class $\mathfrak{F}$ and that (7.5) is a seminorm on the space $L^p(A;X)$. Further, if X is normed space then (7.5) is a norm and if X is a Banach space then $L^p(A;X)$ is a Banach space with this norm (see, for example, Lang [1993] chapter VI). Further, if $f \in \mathcal{L}^1(A;X)$ and X is a Banach space we define the integral $\int_A \mathfrak{F}\ \mathrm{d}\mu_n := \int_A f\ \mathrm{d}\mu_n \in X$ which again is independent of the choice of representative $f \in \mathfrak{F}$.

7.2 $L^p(\mathcal{I};L^p(\Omega))$ and $L^p(\mathcal{I}\times\Omega)$ as equivalence classes of real valued maps.

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{R}^n$ be an open subset. From Definition 7.6 we recall two familiar spaces:

- the space $L^p(\mathcal{I}\times\Omega;\mathbb{R})$, given by setting $A = \mathcal{I}\times\Omega$ and the Banach space $X=\mathbb{R}$, and
- the space $L^p(\mathcal{I};L^p(\Omega;\mathbb{R}))$, given by setting $A=\mathcal{I}$ and the Banach space $X = L^p(\Omega;\mathbb{R})$.

For brevity of notation we write $L^p(A)$ for $L^p(A;\mathbb{R})$ so that the spaces considered here are $L^p(\mathcal{I}\times\Omega)$ and $L^p(\mathcal{I};L^p(\Omega))$ respectively.

In the remainder we will occasionally have to write the measure of a subset $A \subset \mathcal{I} \times \Omega$ in terms of the measure of the sections $A^t := \{x \in \Omega | (t,x) \in A\}$. Recall from the definition of the Lebesgue measure on the product space $\mathbb{R}^{n+1}$ the following lemma:

Lemma 7.7. *If the set $A \subset \mathbb{R}^{n+1}$ is Lebesgue measurable then for almost every $t \in \mathbb{R}$ the section $A^t \subset \mathbb{R}$ defined by*

$$A^t := \{x \in \mathbb{R}^n | (t,x) \in A\}$$

is measurable and

$$\mu_{n+1}(A) = \int_{\mathcal{I}} \mu_n\left(A^t\right) \mathrm{d}t.$$

Proof. See, for example, Cohn [1994] Theorem 5.1.3. □

Corollary 7.8. *If $A \subset \mathbb{R}^n$ is non-measurable and $Z := S \times A$ for some set $S \subset \mathbb{R}^m$ of positive measure then Z is not measurable as a subset of $\mathbb{R}^{n+m}$.*

Proof. Consider the case when $m = 1$. Suppose for a contradiction that Z is measurable, then by Lemma 7.7 almost every section Z^t is measurable. As S has positive measure it follows that for some $t \in S$ the set $Z^t = A$ is measurable, which is a contradiction. The result for general $m \in \mathbb{N}$ follows by induction. □

7.2.1 A characterisation of $L^p(\mathcal{I}; L^p(\Omega))$ for $1 \leq p \leq \infty$.

From Definition 7.6 the space $L^p(\mathcal{I}; L^p(\Omega))$ consists of equivalence classes of equivalence class valued maps, which are neither easy to manipulate nor intuitively accessible. To unpack these nested equivalence relations we give an alternative formulation of the space $L^p(\mathcal{I}; L^p(\Omega))$.

To this end we additionally examine the following spaces:

- $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$, which consists of maps from $\mathcal{I}$ into the seminormed space $\mathcal{L}^p(\Omega)$, and
- $\mathcal{L}^p(\mathcal{I}; L^p(\Omega))$, which consists of maps from $\mathcal{I}$ into the Banach space $L^p(\Omega)$.

Lemma 7.9. *There exist (semi)norm preserving operators π_1 and π_2*

$$\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) \xrightarrow{\pi_1} \mathcal{L}^p(\mathcal{I}; L^p(\Omega)) \xrightarrow{\pi_2} L^p(\mathcal{I}; L^p(\Omega)).$$

such that π_2 is linear and surjective and π_1 has the following properties:

- *for all $f, g \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and $\alpha \in \mathbb{R}$*

$$\pi_1(\alpha f + g)(t) = \alpha \pi_1(f)(t) + \pi_1(g)(t) \quad \textit{for almost every } t \in \mathcal{I}, \quad \textit{and}$$

- *for every $\mathfrak{f} \in \mathcal{L}^p(\mathcal{I}; L^p(\Omega))$ there exists an $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ such that*

$$\pi_1(f)(t) = \mathfrak{f}(t) \quad \textit{for almost every } t \in \mathcal{I},$$

which we call 'almost linearity' and 'almost surjectivity' respectively.

Proof. Let π_2 be the operator that for each $\mathfrak{f} \in \mathcal{L}^p(\mathcal{I}; L^p(\Omega))$ assigns the equivalence class of $L^p(\mathcal{I}; L^p(\Omega))$ that contains $\mathfrak{f}$, which is to say that $\pi_2(\mathfrak{f}) := [\mathfrak{f}]_{\mathcal{I}}$. As this is precisely how the space $L^p(\mathcal{I}; L^p(\Omega))$ is defined it is immediate from Definiton 7.6 that π_2 is surjective, linear and (semi)norm preserving.

Let the operator π_1 be defined by

$$\pi_1(f)(t) = \begin{cases} [f(t)]_\Omega & f(t) \in \mathcal{L}^p(\Omega) \\ [0]_\Omega & f(t) \notin \mathcal{L}^p(\Omega) \end{cases} \tag{7.6}$$

for each $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$. This operator is well defined: from Definition 7.4 the map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ has values $f(t) \in \mathcal{L}^p(\Omega)$ for almost every $t \in \mathcal{I}$, so $[f(t)]_\Omega$ is defined for almost every $t \in \mathcal{I}$ so $\pi_1(f) : \mathcal{I} \rightarrowtail L^p(\Omega)$.

We now show π_1 is almost linear: let $f, g \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and $\alpha \in \mathbb{R}$ and let

$$\mathcal{N} = \{t \in \mathcal{I} | f(t) \notin \mathcal{L}^p(\Omega)\} \cup \{t \in \mathcal{I} | g(t) \notin \mathcal{L}^p(\Omega)\}$$

then for all $t \in \mathcal{I} \setminus \mathcal{N}$ the value $(\alpha f + g)(t) \in \mathcal{L}^p(\Omega)$ and

$$\begin{aligned} \pi_1(\alpha f + g)(t) &= [(\alpha f + g)(t)]_\Omega \\ &= \alpha [f(t)]_\Omega + [g(t)]_\Omega \\ &= \alpha \pi_1(f)(t) + \pi_1(g)(t). \end{aligned}$$

We conclude that π_1 is almost linear as the set $\mathcal{N}$, being the union of two sets of measure zero, has zero measure.

We now demonstrate that $\pi_1(f) \in \mathcal{L}^p(\mathcal{I}; L^p(\Omega))$: from Definition 7.4 there exists a sequence of step maps $f_k : \mathcal{I} \to \mathcal{L}^p(\Omega)$ such that $f_k(t) \to f(t)$ with respect to the $\mathcal{L}^p(\Omega)$ seminorm for almost every $t \in \mathcal{I}$. Observe that $\pi_1(f_k) : \mathcal{I} \to L^p(\Omega)$

is a step map and that for each $t \in \mathcal{I}$ with $f(t) \in \mathcal{L}^p(\Omega)$

$$\begin{aligned}\|\pi_1(f_k)(t) - \pi_1(f)(t)\|_{L^p(\Omega)} &= \|[f_k(t)]_\Omega - [f(t)]_\Omega\|_{L^p(\Omega)} \\ &= \|[f_k(t) - f(t)]_\Omega\|_{L^p(\Omega)} \\ &= \|f_k(t) - f(t)\|_{\mathcal{L}^p(\Omega)} \end{aligned} \tag{7.7}$$

which tends to zero as $k \to \infty$. Consequently, the step maps $\pi_1(f_k)$ converge almost everywhere to $\pi_1(f)$, so the map $\pi_1(f)$ is strongly measurable. Further, if $p < \infty$ we assume that the step maps f_k are Cauchy with respect to the $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ seminorm. Observe that π_1 preserves the semi-norm of step maps: if $g\colon \mathcal{I} \to \mathcal{L}^p(\Omega)$ is a step map defined by $g(t) = \sum_{i=1}^m g_i \mathbf{1}_{A_i}(t)$ for maps $g_i \in \mathcal{L}^p(\Omega)$ and measurable sets $A_i \subset \mathcal{I}$ then

$$\begin{aligned}\|\pi_1(g)\|_{\mathcal{L}^p(\mathcal{I};L^p(\Omega))} &= \left(\sum_{i=1}^m \|[g_i]_\Omega\|^p_{L^p(\Omega)} \mu_n(A_i)\right)^{\frac{1}{p}} \\ &= \left(\|g_i\|^p_{\mathcal{L}^p(\Omega)} \mu_n(A_i)\right)^{\frac{1}{p}} \\ &= \|g\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))}\end{aligned} \tag{7.8}$$

and this extends to maps $g \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ by continuity, so π_1 is seminorm preserving.

Consequently, from the linearity of π_1 and (7.8), as the difference of step maps is also a step map,

$$\|\pi_1(f_k) - \pi_1(f_j)\|_{\mathcal{L}^p(\mathcal{I};L^p(\Omega))} = \|f_k - f_j\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))} \tag{7.9}$$

so the step maps $\pi_1(f_k)$ are Cauchy with respect to $\|\cdot\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))}$, so the map $\pi_1(f)$ is in $\mathcal{L}^p(\mathcal{I}; L^p(\Omega))$. Finally, if $p = \infty$ the map $\pi_1(f)$ satisfies

$$\|\pi_1(f)(t)\|_{L^\infty(\Omega)} = \|f(t)\|_{L^\infty(\Omega)} \quad \text{for almost every } t \in \mathcal{I} \tag{7.10}$$

so $\|\pi_1(f)(t)\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\mathcal{I};L^\infty(\Omega))} < \infty$ for almost every $t \in \mathcal{I}$, so $\pi_1(f) \in \mathcal{L}^\infty(\mathcal{I}; L^\infty(\Omega))$. Finally, it follows from (7.10) that

$$\|\pi_1(f)\|_{\mathcal{L}^\infty(\mathcal{I};L^\infty(\Omega))} = \|f\|_{\mathcal{L}^\infty(\mathcal{I};L^\infty(\Omega))}. \tag{7.11}$$

Next, we show that π_1 has the 'almost surjectivity' property: let $\mathfrak{f} \in \mathcal{L}^p(\mathcal{I}; L^p(\Omega))$ and for each $t \in \mathcal{I}$ such that the equivalence class $\mathfrak{f}(t)$ is in $L^p(\Omega)$ choose a representative $f^t \in \mathfrak{f}(t)$. Consider the map $f\colon \mathcal{I} \rightarrowtail \mathcal{L}^p(\Omega)$ defined by $f(t) := f^t$ and note that by construction $\pi_1(f)(t) = \mathfrak{f}(t)$ for almost every $t \in \mathcal{I}$. It remains to show that the map f is in $\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))$: from Definition 7.4 there exist step maps $\mathfrak{f}_k\colon \mathcal{I} \to L^p(\Omega)$ such that $\mathfrak{f}_k(t) \to \mathfrak{f}(t)$ with respect to the $L^p(\Omega)$ norm for almost every $t \in \mathcal{I}$. For each step map

$$\mathfrak{f}_k(t) := \sum_{i=1}^{m_k} \mathfrak{g}_{k,i}\mathbf{1}_{A_{k,i}}(t),$$

where $\mathfrak{g}_{k,i} \in L^p(\Omega)$ and the sets $A_{k,i} \subset \mathcal{I}$ are measurable and of finite measure, we choose a representative map $g_{k,i} \in \mathcal{L}^p(\Omega)$ of each equivalence class $\mathfrak{g}_{k,i}$ and consider the maps f_k defined by

$$f_k(t) := \sum_{i=1}^{m_k} g_{k,i}\mathbf{1}_{A_{k,i}}(t).$$

Clearly, each map $f_k\colon \mathcal{I} \to \mathcal{L}^p(\Omega)$ is a step map and $\pi_1(f_k) = \mathfrak{f}_k$. Further, from (7.7), for almost every $t \in \mathcal{I}$

$$\|f_k(t) - f(t)\|_{\mathcal{L}^p(\Omega)} = \|\pi_1(f_k)(t) - \pi_1(f)(t)\|_{L^p(\Omega)} = \|\mathfrak{f}_k(t) - \mathfrak{f}(t)\|_{L^p(\Omega)}$$

which tends to zero as $k \to \infty$ for almost every $t \in \mathcal{I}$. Consequently, $f_k(t) \to f(t)$ for almost every $t \in \mathcal{I}$ so the map f is strongly measurable. Finally, if $p < \infty$ then (7.9) guarantees that the step maps f_k are Cauchy with respect to $\|\cdot\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))}$ and if $p = \infty$ then (7.11) guarantees that f is essentially bounded, so we conclude that $f \in \mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))$ and so the operator π_1 has the 'almost surjectivity' property. $\square$

In the following theorem we extend the result of the previous lemma to provide an isomorphism between the space $L^p(\mathcal{I};L^p(\Omega))$ and a quotient space of $\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))$.

Theorem 7.10. *There exists an isometric isomorphism*

$$\Phi\colon \mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))/\sim\ \to L^p(\mathcal{I};L^p(\Omega)),$$

where $\sim$ is the equivalence relation defined by

$$[f(t)(x) = g(t)(x) \quad \textit{for almost every}\, x \in \Omega] \quad \textit{for almost every}\, t \in \mathcal{I}, \qquad (7.12)$$

We denote by $\overset{\sim}{[f]}$ the equivalence class containing the map $f\colon \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$ under this equivalence relation.

This theorem allows us to regard the space $L^p(\mathcal{I};L^p(\Omega))$ as a space of maps

$f\colon \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$ modulo the equivalence relation defined by (7.12), which is perhaps more intuitively accessible. In Corollary 7.13 we demonstrate a distinction between the equivalence relation (7.12), that is equality for almost every $x \in \Omega$ for almost every $t \in \mathcal{I}$, and the equivalence relation given by equality for almost every $(t, x) \in \mathcal{I} \times \Omega$, which, from Definition 7.6, defines the space $L^p(\mathcal{I} \times \Omega)$. As the equivalence relations are distinct we will see that the spaces $L^p(\mathcal{I} \times \Omega)$ and $L^p(\mathcal{I}; L^p(\Omega))$ have no elements in common: in fact each equivalence class of $L^p(\mathcal{I} \times \Omega)$ is a proper subset of some equivalence class of $L^p(\mathcal{I}; L^p(\Omega))$. Nevertheless, there is an isomorphism between these space for $1 \leq p < \infty$, as we demonstrate in the next section.

To prove Theorem 7.10 we demonstrate that the composition

$$\pi_2 \circ \pi_1 \colon \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) \to L^p(\mathcal{I}; L^p(\Omega))$$

is a bounded surjective linear map with kernel $\ker(\pi_2 \circ \pi_1) = \{f \sim 0\}$ which, from the first isomorphism theorem for seminormed spaces (see, for example, Megginson [1998] Theorem 1.7.14), yields the isomorphism Φ. These maps are summarised in the following diagram

$$\begin{tikzcd} \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) \arrow[r, "\pi_1"] \arrow[d, "\pi_\sim"'] & \mathcal{L}^p(\mathcal{I}; L^p(\Omega)) \arrow[r, "\pi_2"] & L^p(\mathcal{I}; L^p(\Omega)) \\ \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) / \sim \arrow[urr, "\Phi"] & & \end{tikzcd} \tag{7.13}$$

where $\pi_\sim$ is the canonical projection from $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ to the quotient space $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) / \sim$.

Proof of Theorem 7.10. Let π_1, π_2 be the maps from Lemma 7.9. We immediately see that the composition $\pi_2 \circ \pi_1$ is a (semi)norm preserving map. Further, $\pi_2 \circ \pi_1$ is linear: if $f, g \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and $\alpha \in \mathbb{R}$ then, from the almost linearity of π_1, the maps $\pi_1(\alpha f + g)$ and $\alpha \pi_1(f) + \pi_1(g)$ are equal almost everywhere on $\mathcal{I}$ and so are in the same equivalence class $[\pi_1(\alpha f + g)]_{\mathcal{I}} \in L^p(\mathcal{I}; L^p(\Omega))$. Consequently,

$$\begin{aligned} \pi_2(\pi_1(\alpha f + g)) = [\pi_1(\alpha f + g)]_{\mathcal{I}} &= [\alpha \pi_1(f) + \pi_1(g)]_{\mathcal{I}} \\ &= \pi_2(\alpha \pi_1(f) + \pi_1(g)) \\ &= \alpha \pi_2 \circ \pi_1(f) + \pi_2 \circ \pi_1(g) \end{aligned}$$

as π_2 is linear.

Next, we show that $\pi_2 \circ \pi_1$ is surjective: let $\mathfrak{F} \in L^p(\mathcal{I}; L^p(\Omega))$. From Lemma

7.9 the map π_2 is surjective so there exists a map $\mathfrak{f} \in \mathcal{L}^p(\mathcal{I}; L^p(\Omega))$ such that $\pi_2(\mathfrak{f}) = \mathfrak{F}$. Further, from Lemma 7.9, the map π_1 is almost surjective so there exists a map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ such that

$$\pi_1(f)(t) = \mathfrak{f}(t) \quad \text{for almost every } t \in \mathcal{I}$$

so the maps $\pi_1(f)$ and $\mathfrak{f}$ are in the same equivalence class $\mathfrak{F}$. Consequently, $\pi_2(\pi_1(f)) = \mathfrak{F}$ so $\pi_2 \circ \pi_1$ is surjective.

As $\pi_2 \circ \pi_1$ is surjective, linear and bounded the first isomorphism theorem guarantees the existence of a unique isomorphism

$$\Phi \colon \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) / \ker(\pi_2 \circ \pi_1) \to L^p(\mathcal{I}; L^p(\Omega))$$

such that the diagram (7.13) commutes.

Next, recall that the zero element of $L^p(\Omega)$, which we write $0_{L^p(\Omega)}$, is the equivalence class containing the constant map $x \mapsto 0 \in \mathbb{R}$. Also recall that the zero element of $L^p(\mathcal{I}; L^p(\Omega))$, which we write $0_{L^p(\mathcal{I}; L^p(\Omega))}$, is the equivalence class containing the constant map $t \mapsto 0_{L^p(\Omega)}$. Consequently,

$$\begin{aligned} \pi_2 \circ \pi_1(f) = 0_{L^p(\mathcal{I}; L^p(\Omega))} &\Leftrightarrow \pi_1(f) \in \left[0_{L^p(\Omega)}\right]_{\mathcal{I}} \\ &\Leftrightarrow \pi_1(f)(t) = 0_{L^p(\Omega)} \quad \text{for a.e. } t \in \mathcal{I} \end{aligned}$$

and, recalling the definition of the operator π_1 in (7.6),

$$\begin{aligned} &\Leftrightarrow [f(t)]_\Omega = 0_{L^p(\Omega)} \quad \text{for a.e. } t \in \mathcal{I} \\ &\Leftrightarrow [f(t)(x) = 0 \quad \text{for a.e. } x \in \Omega] \text{ for a.e. } t \in \mathcal{I} \end{aligned}$$

so $\ker(\pi_2 \circ \pi_1) = \{f \sim 0\}$.

Finally, we recall that the quotient space $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) / \sim$ inherits the quotient seminorm, so by definition the projection $\pi_\sim$ is seminorm preserving. Consequently, as the operators π_1, π_2 and $\pi_\sim$ are (semi)norm preserving and for all $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$

$$\pi_2 \circ \pi_1(f) = \Phi \circ \pi_\sim(f)$$

we conclude that Φ is also (semi)norm preserving and so is an isometric isomorphism. $\square$

In the remainder we identify $L^p(\mathcal{I}; L^p(\Omega))$ with $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega)) / \sim$ through the isomorphism Φ. Further, we identify the map-valued maps $f \colon \mathcal{I} \rightarrowtail \mathcal{L}^p(\Omega)$ with the real valued map $f\colon \mathcal{I}\times\Omega \rightarrowtail \mathbb{R}$ by setting $f(t,x) := f(t)(x)$. Consequently, the elements of $L^p(\mathcal{I};L^p(\Omega))$ are equivalence classes of maps $f\colon \mathcal{I}\times\Omega \rightarrowtail \mathbb{R}$ under the equivalence relation (7.12).

7.2.2 Equivalence classes of $L^p(\mathcal{I}\times\Omega)$ and $L^p(\mathcal{I};L^p(\Omega))$.

Recall from Definition 7.6 that the elements of $L^p(\mathcal{I}\times\Omega)$ are equivalence classes of maps $f\colon \mathcal{I}\times\Omega \rightarrowtail \mathbb{R}$ that satisfy the equivalence relation defined by $f \approx g$ if and only if

$$f(t,x) = g(t,x) \quad \text{for almost every } (t,x) \in \mathcal{I}\times\Omega, \tag{7.14}$$

and we denote by $\overset{\approx}{[f]}$ the equivalence class containing the map $f\colon \mathcal{I}\times\Omega \to \mathbb{R}$ under this equivalence relation.

There is a link between the equivalence relations (7.14), which is equality for almost every (t,x), and (7.12), which is equality for almost every x for almost every t. However, they are not identical as a consequence of the existence of the following pathological sets due to Sierpiński [1920]:

Lemma 7.11. *There exists a non-measurable set $E \subset \mathbb{R}^2$ such that the intersection of E with any line consists of at most two points. In particular, for all $t \in \mathbb{R}$ the section $E^t \subset \mathbb{R}$ is measurable and of measure zero.*

Proof. See Sierpiński [1920] or Gelbaum and Olmsted [2003] pp.142. □

Corollary 7.12. *For any interval $\mathcal{I} \subset \mathbb{R}$ and open subset $\Omega \subset \mathbb{R}^n$ there exists a non-measurable set $Z \subset \mathcal{I}\times\Omega$ such that for all $t \in \mathcal{I}$ the section Z^t is measurable and of measure zero.*

Essentially, we take the product of the Sierpiński set with $\mathbb{R}^{n-2}$ and demonstrate that this is also non-measurable and, further, contains a bounded non-measurable subset. This bounded non-measurable subset can then be rescaled so that it lies in $\mathcal{I}\times\Omega$.

Proof of Corollary 7.12. Let $E \subset \mathbb{R}^2$ be the set from Lemma 7.11. Suppose for a contradiction that the intersection $[-r,r]^2 \cap E$ is measurable for all $r > 0$ then the countable union

$$\bigcup_{r\in\mathbb{N}} \left([-r,r]^2 \cap E\right) = \left(\bigcup_{r\in\mathbb{N}} [-r,r]^2\right) \cap E = \mathbb{R}^2 \cap E = E$$

is also measurable, which is a contradiction as E is non-measurable. Consequently, there exists an $r > 0$ such that the intersection $E_r := [-r, r]^2 \cap E$ is non-measurable. Further, for all $t \in \mathbb{R}$ the section $E_r^t \subset E^t$ and so E_r^t consists of at most two points.

Next, without loss of generality we assume that $0 \in \mathcal{I} \times \Omega$ and let $\delta > 0$ be sufficiently small that the cube $[0, 2\delta]^{n+1}$ is contained in $\mathcal{I} \times \Omega$. We translate and scale the set E_r so that it is contained in the square $[0, 2\delta]^2$: let $g \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map $g(t, x) = \left(\delta r^{-1}(t + r), \delta r^{-1}(x + r)\right)$ and observe that

- $g(E_r) \subset [0, 2\delta]^2$,
- $g(E_r)$ is non-measurable as measurability is preserved under the biLipschitz map g^{-1} (see, for example, Corollary 262E of Fremlin [2000]), and
- for all $t \in \mathbb{R}$ the section $g(E_r)^t$ consists of at most two points as

  $$x \in g(E_r)^t \Leftrightarrow \frac{r}{\delta}x - r \in E_r^{\frac{r}{\delta}t - r}$$

  and the section $E_r^{\frac{r}{\delta}t-r}$ consists of at most two points.

In particular, for all $t \in \mathbb{R}$ the section $g(E_r)^t$ is measurable and of measure zero.

Next, let $Z := g(E_r) \times [0, 2\delta]^{n-1}$ and observe that $Z \subset [0, 2\delta]^{n+1} \subset \mathcal{I} \times \Omega$. Further, as Z is the product of a non-measurable set from Corollary 7.8 it follows that Z is non-measurable. Finally, for all $t \in \mathcal{I}$ the section

$$\begin{aligned}
Z^t &= \{(x_1, x_2, \ldots, x_n) \,|\, (t, x_1, x_2, \ldots, x_n) \in Z\} \\
&= \left\{(x_1, x_2, \ldots, x_n) \,|\, (t, x_1, x_2, \ldots, x_n) \in g(E_r) \times [0, 2\delta]^{n-1}\right\} \\
&= \{x_1 | \, (t, x_1) \in g(E_r)\} \times [0, 2\delta]^{n-1} \\
&= g(E_r)^t \times [0, 2\delta]^{n-1}
\end{aligned}$$

which has $(n+1)$-dimensional Lebesgue measure zero as the section $g(E_r)^t$ has 1-dimensional Lebesgue measure zero. □

We remark that it is possible to adapt Sierpinski's argument directly to the set $\mathcal{I} \times \Omega$ to produce a suitable pathological set, however the proof is involved.

Corollary 7.13. *Let $f, g \colon \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$. If $f \approx g$ then $f \sim g$. Conversely, there exists a map f such that $f \sim 0$ but $f \not\approx 0$.*

Proof. Suppose that $f \approx g$ and let $\mathcal{N} := \{(t, x) \in \mathcal{I} \times \Omega | f(t, x) \neq g(t, x)\}$. From Lemma 7.7

$$\mu_{n+1}(\mathcal{N}) = \int_{\mathcal{I}} \mu_n(\mathcal{N}^t) \, \mathrm{d}t$$

and from the definition (7.14) $\mu_{n+1}(\mathcal{N}) = 0$ so we conclude $\mu_n(\mathcal{N}^t)$ for almost every $t \in \mathcal{I}$. Consequently, $f(t,x) = g(t,x)$ for almost every $x \in \Omega$ for almost every $t \in \mathcal{I}$.

Next, from Corollary 7.12 there exists a non-measurable subset $Z \subset \mathcal{I} \times \Omega$ such that for all $t \in \mathcal{I}$ the section Z^t has measure zero and examine the characteristic function $\mathbf{1}_Z : \mathcal{I} \times \Omega \to \mathbb{R}$. As every section Z^t has measure zero clearly $\mathbf{1}_Z(t,x) = 0$ for almost every $x \in \Omega$ for almost every $t \in \mathcal{I}$. However, as Z is not measurable it is not the case that $\mathbf{1}_Z(t,x) = 0$ for almost every $(t,x) \in \mathcal{I} \times \Omega$. Consequently $\mathbf{1}_Z \sim 0$ but $\mathbf{1}_Z \not\approx 0$. □

7.3 Inclusions between $L^p(\mathcal{I} \times \Omega)$ and $L^p(\mathcal{I}; L^p(\Omega))$.

In the previous sections we have defined the L^p spaces and demonstrated that their elements are equivalence classes of maps $f : \mathcal{I} \times \Omega \to \mathbb{R}$ with different equivalence relations that are related but not identical. For $1 \leq p < \infty$ we recall the Fubini theorem and its partial converse which relate the maps in the equivalence classes of $L^p(\mathcal{I} \times \Omega)$ and $L^p(\mathcal{I}; L^p(\Omega))$. Further in Lemma 7.17 we demonstrate that the partial converse is almost a full converse in the sense that every map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ can be modified on a 'negligible' set so that the modified map is in $\mathcal{L}^p(\mathcal{I} \times \Omega)$.

Theorem 7.14 (Fubini's Theorem). *For $1 \leq p < \infty$ if the map $f \in \mathcal{L}^p(\mathcal{I} \times \Omega)$ then $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and there is equality in the seminorms*

$$\|f\|_{\mathcal{L}^p(\mathcal{I}\times\Omega)} = \|f\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))} .$$

Proof. See, for example, Lang [1993] Theorem 8.4. □

Theorem 7.15 (Partial converse to Fubini's Theorem). *For $1 \leq p < \infty$ if the map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and f is strongly measurable as a map $f : \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$ then $f \in \mathcal{L}^p(\mathcal{I} \times \Omega)$ and there is equality in the seminorms*

$$\|f\|_{\mathcal{L}^p(\mathcal{I}\times\Omega)} = \|f\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))} .$$

Proof. See, for example, Lang [1993] Theorem 8.7. □

Theorem 7.15 is not a complete converse to the Fubini Theorem 7.14 as it has the additional requirement that $f \in \mathcal{L}^0(\mathcal{I} \times \Omega)$, which, as we see in the following lemma, is not guaranteed by $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$.

Lemma 7.16. *For $1 \leq p \leq \infty$ and each map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ there exists a map $g \sim f$ such that $g \notin \mathcal{L}^0(\mathcal{I} \times \Omega)$ hence $g \notin \mathcal{L}^p(\mathcal{I} \times \Omega)$.*

Proof. Let $Z \subset \mathcal{I} \times \Omega$ be the pathological set from Corollary 7.13 and $\mathbf{1}_Z : \mathcal{I} \times \Omega \to \mathbb{R}$ be the characteristic function of this set. Recall that $\mathbf{1}_Z \sim 0$ but $\mathbf{1}_Z \notin \mathcal{L}^0(\mathcal{I} \times \Omega)$. Now, let $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$. If $f \notin \mathcal{L}^0(\mathcal{I} \times \Omega)$ then let $g = f$. Otherwise, if $f \in \mathcal{L}^0(\mathcal{I} \times \Omega)$ then consider the map $f + \mathbf{1}_Z$. As $\mathbf{1}_Z \sim 0$ and $\sim$ is an equivalence relation on the vector space $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ we conclude that $f + \mathbf{1}_Z \sim f$. We suppose for a contradiction that $f + \mathbf{1}_Z \in \mathcal{L}^0(\mathcal{I} \times \Omega)$, but as $\mathcal{L}^0(\mathcal{I} \times \Omega)$ is a vector space we conclude that

$$\mathbf{1}_Z = f + \mathbf{1}_Z - f \in \mathcal{L}^0(\mathcal{I} \times \Omega)$$

which contradicts the result of Corollary 7.13. □

The previous lemma demonstrates that each map in $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ has an equivalent map which is not in $\mathcal{L}^0(\mathcal{I} \times \Omega)$. We now show that there is also an equivalent map that is in $\mathcal{L}^0(\mathcal{I} \times \Omega)$. We remark that this seems unlikely to be a new result, yet we have been unable to find a reference in the literature.

Lemma 7.17. *For $1 \leq p \leq \infty$ if $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ then there exists a map $g \sim f$ such that $g \in \mathcal{L}^0(\mathcal{I} \times \Omega)$.*

The proof is notationally demanding but reasonably straightforward; the most elementary step-map-valued step map

$$\begin{aligned} h\colon \mathcal{I} &\to \mathcal{L}^p(\Omega) \\ t &\mapsto \begin{cases} \mathbf{1}_B(x)\, z & t \in A \\ 0 & t \notin A \end{cases} \end{aligned}$$

where $A \subset \mathcal{I}$, $B \subset \Omega$ and $z \in \mathbb{R}$ can trivially be written as the step map

$$\begin{aligned} h\colon \mathcal{I} \times \Omega &\to \mathbb{R} \\ (t, x) &\mapsto \mathbf{1}_{A \times B} z. \end{aligned}$$

Consequently, any step-map-valued step map, being a finite sum of such elementary maps, can be written as a step map with the product domain $\mathcal{I} \times \Omega$. Next, a map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$, from the definition of the space, is the limit almost everywhere of $\mathcal{L}^p(\Omega)$-valued step maps f_k, each value of which is the limit almost everywhere of real valued step maps $g_{k,i,j}$. This gives rise to a sequence of step-map-valued step maps, which from the above observation can be written as step maps with the product domain.

The difficulty lies in ensuring that this sequence of step maps from $\mathcal{I} \times \Omega$ converges to f for almost every $(t, x) \in \mathcal{I} \times \Omega$; in Corollary 7.13 we demonstrated that this does not hold generally as a union of null sections can form a non-measurable set. However, by aggregating the troublesome spatial points for which the step maps $g_{k,i,j}$ do not converge and redefining all the step maps on this set, we can ensure that the modified step maps converge to f outside a measurable subset of $\mathcal{I} \times \Omega$. Further, as the troublesome spatial points are a countable union of null sets, the step maps converge to f almost everywhere on $\mathcal{I} \times \Omega$.

Proof. Let the map $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$. As $f\colon \mathcal{I} \rightarrowtail \mathcal{L}^p(\Omega)$ is strongly measurable there exists a sequence of step maps $f_k\colon \mathcal{I} \to \mathcal{L}^p(\Omega)$ such that

$$f_k(t) \to f(t) \quad \text{with respect to the seminorm } \|\cdot\|_{\mathcal{L}^p(\Omega)} \quad \text{for almost every } t \in \mathcal{I}$$

as $k \to \infty$. Let the set $\mathcal{N} := \{t \in \mathcal{I} | f_k(t) \nrightarrow f(t)\}$ and observe that $\mathcal{N}$ has zero measure. Define

$$f' := f \cdot \mathbf{1}_{\mathcal{I} \setminus \mathcal{N}} \qquad\qquad f'_k := f_k \cdot \mathbf{1}_{\mathcal{I} \setminus \mathcal{N}} \tag{7.15}$$

and observe that $f'(t) = f(t)$ for almost every $t \in \mathcal{I}$, the maps $f'_k\colon \mathcal{I} \to \mathcal{L}^p(\Omega)$ are step maps, and

$$f'_k(t) \to f'(t) \quad \text{with respect to the seminorm } \|\cdot\|_{\mathcal{L}^p(\Omega)} \quad \forall t \in \mathcal{I}. \tag{7.16}$$

For each $k \in \mathbb{N}$ the step map f'_k has the form

$$f'_k(t) := \sum_{i=1}^{m_k} g_{k,i} \mathbf{1}_{A_{k,i}}(t) \tag{7.17}$$

for some $m_k \in \mathbb{N}$, measurable sets $A_{k,i} \subset \mathcal{I}$ of finite measure and maps $g_{k,i} \in \mathcal{L}^p(\Omega)$. Now, as each $g_{k,i}\colon \Omega \rightarrowtail \mathbb{R}$ is strongly measurable there exists a sequence of step maps $g_{k,i,j}\colon \Omega \to \mathbb{R}$ such that

$$g_{k,i,j}(x) \to g_{k,i}(x) \quad \text{for almost every } x \in \Omega$$

as $j \to \infty$. Now, for each $k, i \in \mathbb{N}$ let the set $\omega_{k,i} := \{x \in \Omega | g_{k,i,j}(x) \nrightarrow g_{k,i}(x)\}$ and observe that $\omega_{k,i}$ has zero measure. Consequently, the countable union

$$\omega := \bigcup_{k,i \in \mathbb{N}} \omega_{k,i}$$

has zero measure. As before, define

$$g'_{k,i} = g_{k,i} \cdot \mathbf{1}_{\Omega \setminus \omega} \qquad\qquad g'_{k,i,j} = g_{k,i,j} \cdot \mathbf{1}_{\Omega \setminus \omega} \tag{7.18}$$

and observe that $g'_{k,i}(x) = g_{k,i}(x)$ for almost every $x \in \Omega$, that the $g'_{k,i,j}$ are step maps and that

$$g'_{k,i,j}(x) \to g'_{k,i}(x) \quad \forall x \in \Omega. \tag{7.19}$$

Define $f''_{k,j} : \mathcal{I} \times \Omega \to \mathbb{R}$ by

$$f''_{k,j}(t,x) := \sum_{i=1}^{m_k} g'_{k,i,j}(x) \, \mathbf{1}_{A_{k,i}}(t) \tag{7.20}$$

and consider for each $k \in \mathbb{N}$ the map $f''_k : \mathcal{I} \times \Omega \to \mathbb{R}$ defined by $f''_k(t,x) \coloneqq \lim_{j\to\infty} f''_{k,j}(t,x)$. From (7.20)

$$f''_k(t,x) = \sum_{k=1}^{m_k} \lim_{j\to\infty} g'_{k,i,j}(x) \, \mathbf{1}_{A_{k,i}}(t) \quad \forall (t,x) \in \mathcal{I} \times \Omega$$

which, from the convergence (7.19),

$$= \sum_{k=1}^{m_k} g'_{k,i}(x) \, \mathbf{1}_{A_{k,i}}(t) \quad \forall (t,x) \in \mathcal{I} \times \Omega$$

further, from the definition of $g'_{k,i}$ in (7.18),

$$= \begin{cases} \sum_{k=1}^{m_k} g_{k,i}(x) \, \mathbf{1}_{A_{k,i}}(t) & \forall\, x \in \Omega \setminus \omega \quad \forall\, t \in \mathcal{I} \\ 0 & \forall\, x \in \omega \quad \forall\, t \in \mathcal{I} \end{cases}$$

which, from (7.17),

$$= \begin{cases} f'_k(t)(x) & \forall\, x \in \Omega \setminus \omega \quad \forall\, t \in \mathcal{I} \\ 0 & \forall\, x \in \omega \quad \forall\, t \in \mathcal{I}. \end{cases}$$

Now recall from (7.16) that

$$\left\| \lim_{k\to\infty} f'_k(t) - f'(t) \right\|_{\mathcal{L}^p(\Omega)} = 0 \quad \forall t \in \mathcal{I}$$

so $\lim_{k\to\infty} f'_k(t)(x) = f'(t)(x)$ for almost every $x \in \Omega$ for all $t \in \mathcal{I}$, so for each $t \in \mathcal{I}$ the set

$$\omega_t := \left\{ x \in \Omega | f'_k(t,x) \nrightarrow f'(t,x) \right\}$$

has zero measure.

Consequently, the pointwise limit

$$f''(t,x) := \lim_{k\to\infty} f''_k(t,x) = \begin{cases} f'(t)(x) & \forall\, x \in \Omega \setminus \omega,\ x \notin \omega_t \quad \forall\, t \in \mathcal{I} \\ \lim_{k\to\infty} f'_k(t)(x) & \forall\, x \in \Omega \setminus \omega,\ x \in \omega_t \quad \forall\, t \in \mathcal{I} \\ 0 & \forall\, x \in \omega \quad \forall\, t \in \mathcal{I} \end{cases}$$

where the limit $\lim_{k\to\infty} f'_k(t)(x)$ does not necessarily exist. Finally, from the definition of f' in (7.15),

$$f''(t,x) = \begin{cases} f(t)(x) & \forall\, x \in \Omega \setminus \omega,\ x \notin \omega_t \quad \forall\, t \in \mathcal{I} \setminus \mathcal{N} \\ 0 & \forall\, x \in \Omega \setminus \omega,\ x \notin \omega_t \quad \forall\, t \in \mathcal{N} \\ \lim_{k\to\infty} f'_k(t)(x) & \forall\, x \in \Omega \setminus \omega,\ x \in \omega_t \quad \forall\, t \in \mathcal{I} \\ 0 & \forall\, x \in \omega \quad \forall\, t \in \mathcal{I}. \end{cases} \tag{7.21}$$

We now show that $f'' \sim f$, indeed if $t \in \mathcal{I} \setminus \mathcal{N}$ then, from the equality (7.21), $f''(t,x) \neq f(t)(x)$ if and only if $x \in \omega_t \cup \omega$, which has measure zero as it is the union of two sets of measure zero. Consequently, as $\mathcal{N}$ is of measure zero, $f''(t,x) = f(t)(x)$ for almost every $x \in \Omega$ for almost every $t \in \mathcal{I}$.

Finally, we show that $f'' \colon \mathcal{I} \times \Omega \to \mathbb{R}$ is strongly measurable. It is straightforward to see that $f''_{k,j} \colon \mathcal{I} \times \Omega \to \mathbb{R}$ is a step map: indeed, if we write each step map $g'_{k,i,j}$ in (7.20) as

$$g'_{k,i,j}(x) = \sum_{l=1}^{m_{k,i,j}} x_{k,i,j,l} \mathbf{1}_{B_{k,i,j,l}}(x)$$

then, after ordering the $M_{k,j} := \sum_{i=1}^{m_k} m_{k,i,j}$ summands, (7.20) can be written

$$\begin{aligned} f''_{k,j}(t,x) &= \sum_{i=1}^{M_{k,j}} \mathbf{1}_{A_{k,i}}(t)\,\mathbf{1}_{B_{k,i,j}}(x)\,x_{k,i,j} \\ &= \sum_{i=1}^{M_{k,j}} \mathbf{1}_{A_{k,i}\times B_{k,i,j}}(t,x)\,x_{k,i,j}. \end{aligned}$$

Consequently, $f''_k : \mathcal{I} \times \Omega \to \mathbb{R}$ is strongly measurable by definition as it is the pointwise limit of step maps $f''_{k,j}$. Finally, $f'' : \mathcal{I} \times \Omega \to \mathbb{R}$ is the pointwise limit of strongly measurable maps and so by Lemma 7.3 is strongly measurable. $\square$

The Fubini Theorem 7.14 together with Lemma 7.16 demonstrate that for $1 \leq p < \infty$ each equivalence class $\mathfrak{F} \in L^p(\mathcal{I} \times \Omega)$ is a proper subset of some equivalence class $\mathfrak{G}$ in $L^p(\mathcal{I}; L^p(\Omega))$. Indeed, the Fubini theorem guarantees that each element $f \in \mathfrak{F}$ is an element of $\mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and Corollary 7.13 guarantees that each map $f \in \mathfrak{F}$ is in the same equivalence class $\mathfrak{G} \in L^p(\mathcal{I}; L^p(\Omega))$, so certainly $\mathfrak{F} \subset \mathfrak{G}$. Further, Lemma 7.16 demonstrates that there are elements of $\mathfrak{G}$ which are not in $\mathfrak{F}$ so this is a strict inclusion. Finally, Lemma 7.17 ensures that for $1 \leq p < \infty$ each equivalence class $\mathfrak{G} \in L^p(\mathcal{I}; L^p(\Omega))$ contains an element (and consequently, by Corollary 7.13, an entire equivalence class of elements) of $\mathcal{L}^p(\mathcal{I} \times \Omega)$, so each equivalence class $\mathfrak{G} \in L^p(\mathcal{I}; L^p(\Omega))$ is a superset of some equivalence class $\mathfrak{F} \in L^p(\mathcal{I} \times \Omega)$.

Due to this strict inclusion no element of $L^p(\mathcal{I} \times \Omega)$ is equal to an element of $L^p(\mathcal{I}; L^p(\Omega))$ so strictly these two spaces are disjoint. However, the spaces are isometrically isomorphic, which is the content of the following theorem.

Theorem 7.18. *For $1 \leq p < \infty$ the spaces $L^p(\mathcal{I} \times \Omega)$ and $L^p(\mathcal{I}; L^p(\Omega))$ are isometrically isomorphic.*

Proof. Consider the map $\Phi : L^p(\mathcal{I} \times \Omega) \to L^p(\mathcal{I}; L^p(\Omega))$ defined by $\Phi(\mathfrak{F}) := \overset{\sim}{[f]}$ where f is any representative of the equivalence class $\mathfrak{F}$. As every $f \in \mathcal{F}$ is a map $f \in \mathcal{L}^p(\mathcal{I} \times \Omega)$ Fubini's Theorem 7.14 yields $f \in \mathcal{L}^p(\mathcal{I}; \mathcal{L}^p(\Omega))$ and

$$\|\Phi(\mathfrak{F})\|_{L^p(\mathcal{I};L^p(\Omega))} = \|f\|_{\mathcal{L}^p(\mathcal{I};\mathcal{L}^p(\Omega))} = \|f\|_{\mathcal{L}^p(\mathcal{I}\times\Omega)} = \|\mathfrak{F}\|_{L^p(\mathcal{I}\times\Omega)}$$

so $\Phi(\mathfrak{F}) = \overset{\sim}{[f]} \in L^p(\mathcal{I}; L^p(\Omega))$ and Φ is norm preserving. Further, if f, g are two representatives of $\mathfrak{F}$ then $f \approx g$, which implies from Corollary 7.13 that $f \sim g$ so Φ is independent of the choice of representative of the equivalence class.

It is immediate that Φ is linear. Further, Φ is injective: indeed, $\Phi(\mathfrak{F}) = \widetilde{[0]}$ if and only if there exists $f \in \mathcal{F}$ such that $f \sim 0$. However, as $f\colon \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$ is strongly measurable it is measurable so the inverse image $\mathcal{N} := f^{-1}(\mathbb{R} \setminus \{0\})$ is a measurable subset of $\mathcal{I} \times \Omega$. As $\mathcal{N}$ is measurable, from Lemma 7.7

$$\mu_{n+1}(\mathcal{N}) = \int_{\mathcal{I}} \mu_n(\mathcal{N}^t)\,\mathrm{d}t,$$

which is equal to zero because

$$\mathcal{N}^t = \left\{x \in \Omega \,|\, (t,x) \in f^{-1}(\mathbb{R} \setminus \{0\})\right\} = \{x \in \Omega | f(t,x) \neq 0\}$$

has measure zero for almost every $t \in \mathcal{I}$ since $f(t,x) = 0$ for almost every $x \in \Omega$, for almost every $t \in \mathcal{I}$. As $\mu_{n+1}\left(f^{-1}(\mathbb{R} \setminus \{0\})\right) = 0$ we conclude that $f(t,x) = 0$ for almost every $(t,x) \in \mathcal{I} \times \Omega$ so $\mathfrak{F}$ is the equivalence class of $L^p(\mathcal{I} \times \Omega)$ that contains the zero map.

Finally, Φ is surjective as Lemma 7.17 guarantees that each equivalence class $\mathfrak{F} \in L^p(\mathcal{I}; L^p(\Omega))$ contains a map $f \in \mathfrak{F}$ that is strongly measurable from $\mathcal{I} \times \Omega$ into $\mathbb{R}$. Consequently by Theorem 7.15 $f \in \mathcal{L}^p(\mathcal{I} \times \Omega)$ and by the definition of Φ

$$\Phi\left(\overset{\approx}{[f]}\right) = \widetilde{[f]} = \mathfrak{F}.$$

□

7.4 Inclusions between $L^\infty(\mathcal{I} \times \Omega)$ and $L^\infty(\mathcal{I}; L^\infty(\Omega))$

When $p = \infty$ the situation is more complicated as the Fubini theorem does not hold, as we demonstrate in Lemma 7.20. However, the partial converse given in Theorem 7.15 does hold for $p = \infty$, which is the content of Theorem 7.19. As a consequence we will see that every equivalence class $\mathfrak{F} \in L^\infty(\mathcal{I}; L^\infty(\Omega))$ can be realised as a superset of some equivalence class $\mathfrak{G} \in L^\infty(\mathcal{I} \times \Omega)$ but there are equivalence classes in $\mathcal{L}^\infty(\mathcal{I} \times \Omega)$ that are disjoint from every equivalence class in $\mathcal{L}^\infty(\mathcal{I}; \mathcal{L}^\infty(\Omega))$. The discrepency is realised in Theorem 7.22 in which we demonstrate that $L^\infty(\mathcal{I}; L^\infty(\Omega))$ is isometrically isomorphic to a proper subspace of $L^\infty(\mathcal{I} \times \Omega)$.

Theorem 7.15, the partial converse to Fubini's theorem, also holds for $p = \infty$:

Theorem 7.19. *If the map $f \in \mathcal{L}^\infty(\mathcal{I}; \mathcal{L}^\infty(\Omega))$ and f is strongly measurable as a* *map $f : \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$ then $f \in \mathcal{L}^{\infty}(\mathcal{I} \times \Omega)$ and there is equality in the seminorms*

$$\|f\|_{\mathcal{L}^{\infty}(\mathcal{I};\mathcal{L}^{\infty}(\Omega))} = \|f\|_{\mathcal{L}^{\infty}(\mathcal{I}\times\Omega)}$$

Proof. Let $f \in \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$ and for each $c \in \mathbb{R}$ let

$$X_c := \{(t,x) \in \mathcal{I} \times \Omega |\, |f(t,x)| > c\}.$$

As $f : \mathcal{I} \times \Omega \to \mathbb{R}$ is strongly measurable the map $|f|$ is measurable, so for each $c \in \mathbb{R}$ the set X_c is measurable as it is the inverse image of the open interval (c, ∞) under the map $|f| : \mathcal{I} \times \Omega \to \mathbb{R}$. Consequently, by the definition of the product measure (see, for example, Cohn [1994] Theorem 5.1.3),

$$\mu_{n+1}(X_c) = \int_{\mathcal{I}} \mu_n(X_c^t)\ \mathrm{d}t \tag{7.22}$$

where $X_c^t := \{x \in \Omega |\, |f(t,x)| > c\}$, which is the inverse image of the open interval (c, ∞) under the map $|f(t)| : \Omega \to \mathbb{R}$. Now,

$$\begin{aligned}\|f\|_{\mathcal{L}^{\infty}(\mathcal{I};\mathcal{L}^{\infty}(\Omega))} \le c &\Leftrightarrow \|f(t)\|_{\mathcal{L}^{\infty}(\Omega)} \le c \quad \text{for almost every } t \in \mathcal{I}\\ &\Leftrightarrow \mu_n(X_c^t) = 0 \quad \text{for almost every } t \in \mathcal{I}\end{aligned}$$

which, from (7.22), holds

$$\Leftrightarrow \mu_{n+1}(X_c) = 0 \Leftrightarrow \|f\|_{\mathcal{L}^{\infty}(\mathcal{I}\times\Omega)} \le c$$

Consequently, the map f has finite $\|\cdot\|_{L^{\infty}(\mathcal{I}\times\Omega)}$ seminorm and there is equality in the seminorms

$$\|f\|_{\mathcal{L}^{\infty}(\mathcal{I}\times\Omega)} = \|f\|_{\mathcal{L}^{\infty}(\mathcal{I};\mathcal{L}^{\infty}(\Omega))}.$$

□

However, Fubini's theorem does not hold for $p = \infty$, as the following example demonstrates:

Lemma 7.20. *Let $\mathcal{I} = \Omega = (0, 1)$. There exists a map $f \in \mathcal{L}^{\infty}(\mathcal{I} \times \Omega)$ such that $f \notin \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$*

Proof. Let $\mathcal{I} = \Omega = (0,1)$ and consider the indicator map

$$f(t,x) := \mathbf{1}_{\{x \le t\}}(t,x) = \begin{cases} 1 & x \le t \\ 0 & x > t. \end{cases}$$

As the set $\{(t,x) \in \mathcal{I} \times \Omega | x \le t\}$ is measurable and $\mu_2(\{(t,x) \in \mathcal{I} \times \Omega | x \le t\}) = 1/2 < \infty$, the map $f \colon \mathcal{I} \times \Omega \to \mathbb{R}$ is a step map and so is a fortiori strongly measurable. Further,

$$\mu_2(\{(t,x) \in \mathcal{I} \times \Omega | \, |f(t,x)| > c\}) = \begin{cases} 0 & 1 \le c \\ 1/2 & 0 \le c < 1 \end{cases}$$

so $\|f\|_{\mathcal{L}^\infty(\mathcal{I}\times\Omega)} = 1$ and we conclude that $f \in \mathcal{L}^\infty(\mathcal{I} \times X)$.

Next, observe that

$$f(t,x) = \mathbf{1}_{(0,t]}(x) \quad \forall x \in \Omega, \; t \in \mathcal{I}$$

so for each $t \in (0,1)$ the map $f(t) : \Omega \to \mathbb{R}$ defined by

$$f(t)(x) = \mathbf{1}_{(0,t]}(x) = f(t,x)$$

is a step map, so is a fortiori strongly measurable. Further, $\left\|\mathbf{1}_{(0,t]}\right\|_{\mathcal{L}^\infty(\Omega)} = 1$ so $f(t) \in \mathcal{L}^\infty(\Omega)$ and the map $f \colon \mathcal{I} \to \mathcal{L}^\infty(\Omega)$ defined by

$$f \colon t \mapsto f(t) = \mathbf{1}_{(0,t]}$$

is a map from $\mathcal{I}$ into $\mathcal{L}^\infty(\Omega)$. We now demonstrate that the $\mathcal{L}^\infty(\Omega)$ valued map $f \colon t \mapsto f(t)$ is not strongly measurable and so $f \notin \mathcal{L}^\infty(\mathcal{I}; \mathcal{L}^\infty(\Omega))$:

Observe that for each $s, t \in \mathcal{I}$ with $s \le t$

$$\begin{aligned} \|f(t) - f(s)\|_{\mathcal{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_{x\in\Omega} |f(t)(x) - f(s)(x)| \\ &= \operatorname*{ess\,sup}_{x\in\Omega} \left|\mathbf{1}_{(0,t]}(x) - \mathbf{1}_{(0,s]}(x)\right| \\ &= \operatorname*{ess\,sup}_{x\in\Omega} \left|\mathbf{1}_{(s,t]}(x)\right| \\ &= \begin{cases} 1 & t \ne s \\ 0 & t = s. \end{cases} \end{aligned}$$

Consequently, for each $t \in \mathcal{I}$ the open ball in $\mathcal{L}^\infty(\Omega)$ of radius $1/2$ with centre the map $f(t)$, which we denote $B_{1/2}(f(t))$, contains no other map $f(s)$ for any $s \neq t$. Consequently,

$$f^{-1}\left(B_{1/2}(f(t))\right) := \left\{s \in \mathcal{I} | f(s) \in B_{1/2}(f(t))\right\} = \{t\}.$$

Now, if $V \subset \mathcal{I}$ is any non-measurable set then the union of open balls

$$Z := \bigcup_{t \in V} B_{1/2}(f(t)) \subset \mathcal{L}^{\infty}(\Omega)$$

is an open subset of $\mathcal{L}^{\infty}(\Omega)$ but $f^{-1}(Z) = V$, which is non-measurable. We conclude that $f: \mathcal{I} \to \mathcal{L}^{\infty}(\Omega)$ is a not a measurable map, so is a fortiori not strongly measurable. Consequently, $f \notin \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$. □

In fact, any map in the same equivalence class of the map in the above lemma is also not a member of $\mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$, which is the content of the following corollary:

Corollary 7.21. *There exists an equivalence class of maps $\mathfrak{F} \in L^{\infty}(\mathcal{I} \times \Omega)$ such that no representative map $g \in \mathfrak{F}$ is a member of $\mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$.*

Proof. Let f be the map from lemma 7.20 for which we demonstrated that $f \in \mathcal{L}^{\infty}(\mathcal{I} \times \Omega)$ and $f \notin \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$. Let $g \in \overset{\approx}{[f]}$ so that clearly $g \in \mathcal{L}^{\infty}(\mathcal{I} \times \Omega)$. Suppose for a contradiction that $g \in \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$: as $g \approx f$ it follows from corollary 7.13 that $g \sim f$, so we conclude that f is in the same equivalence class $\overset{\sim}{[g]}$ so $f \in \mathcal{L}^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$ which is the required contradiction. □

Consequently, there are equivalence classes in $L^{\infty}(\mathcal{I} \times \Omega)$ that have no elements in common with any equivalence class of $L^{\infty}(\mathcal{I}; L^{\infty}(\Omega))$. In this sense the space $L^{\infty}(\mathcal{I} \times \Omega)$ is 'bigger' than the space $L^{\infty}(\mathcal{I}; L^{\infty}(\Omega))$, which we formalise in the following theorem.

Theorem 7.22. *The space $L^{\infty}(\mathcal{I}; L^{\infty}(\Omega))$ is isomorphic to a proper subspace of $L^{\infty}(\mathcal{I} \times \Omega)$.*

Proof. From Lemma 7.17, in each equivalence class $\mathfrak{F} \in L^{\infty}(\mathcal{I}; \mathcal{L}^{\infty}(\Omega))$ there exists a representative map $f \in \mathfrak{F}$ that is also strongly measurable as a map $f: \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$. By Theorem 7.19 this representative map $f \in \mathfrak{F}$ is also an element of $\mathcal{L}^{\infty}(\mathcal{I} \times \Omega)$. Now, define the operator

$$\Phi: L^{p}(\mathcal{I}; L^{p}(\Omega)) \to L^{p}(\mathcal{I} \times \Omega)$$

by $\Phi(\mathfrak{F}) := \overset{\approx}{[f]}$ where $f \in \mathfrak{F}$ is any representative map for which $f \in \mathcal{L}^0(\mathcal{I} \times \Omega)$. Clearly, Φ is linear. Further, if $f, g \in \mathfrak{F}$ and $f, g \in \mathcal{L}^0(\mathcal{I} \times \Omega)$ then the set $\{(t,x) \in \mathcal{I} \times \Omega | f(t,x) \neq g(t,x)\}$ is measurable so from Lemma 7.7

$$\mu_{n+1}(\{(t,x) \in \mathcal{I} \times \Omega | f(t,x) \neq g(t,x)\}) = \int_{\mathcal{I}} \mu_n(\{x \in \Omega | f(t,x) \neq g(t,x)\})\,\mathrm{d}t,$$

which is equal to zero as $f \sim g$ implies that $\mu_n(\{x \in \Omega | f(t,x) \neq g(t,x)\}) = 0$ for almost every $t \in \mathcal{I}$. Consequently, $f \approx g$ so the map Φ is independent of the representative of the equivalence class $\mathfrak{F}$.

Further, by Theorem 7.19,

$$\|\Phi(\mathfrak{F})\|_{L^\infty(\mathcal{I} \times \Omega)} = \|f\|_{\mathcal{L}^\infty(\mathcal{I} \times \Omega)} = \|f\|_{\mathcal{L}^\infty(\mathcal{I}; \mathcal{L}^\infty(\Omega))} = \|\mathfrak{F}\|_{L^\infty(\mathcal{I}; L^\infty(\Omega))}$$

where the map $f \in \mathfrak{F} \cap \mathcal{L}^0(\mathcal{I} \times \Omega)$.

Next, we show that Φ is injective: if the equivalence class $\Phi(\mathfrak{F})$ is the zero element of $L^\infty(\mathcal{I} \times \Omega)$ then it contains the zero map. Consequently, any map $f \in \mathfrak{F} \cap \mathcal{L}^0(\mathcal{I} \times \Omega)$, which is in $\Phi(\mathfrak{F})$, is equivalent to zero map, that is $f \approx 0$. Consequently, from Corollary 7.13, $f \sim 0$ and so $\mathfrak{F}$ is the zero element of $L^\infty(\mathcal{I}; L^\infty(\Omega))$.

We conclude that Φ is an isometric isomorphism

$$\Phi \colon L^\infty(\mathcal{I}; L^\infty(\Omega)) \to \Phi(L^\infty(\mathcal{I}; L^\infty(\Omega)))$$

and that the range of Φ is a proper subspace as from Corollary 7.21 there are elements of $L^\infty(\mathcal{I} \times \Omega)$ that do not contain any map $f \in \mathcal{L}^\infty(\mathcal{I}; \mathcal{L}^\infty(\Omega))$. $\square$

Chapter 8

Addendum: Continuity of Sobolev Maps

In this chapter we resolve the measurability issue of Chapter 2: recall that each regular Lagrangian flow solution Y gives rise to a map $\tilde{Y}$ that is absolutely continuous with respect to the time variable t, but that it is not immediately apparent that $\tilde{Y}$ is measurable. Ultimately in Corollary 8.5 we prove that $\tilde{Y}$ is measurable and consequently, in Theorem 8.6 that with this continuity the regular Lagrangian flow can be defined in a more direct way. We begin with a general result relating the measurability of a map from a product space to the measurability and continuity of its sections.

For measurable Euclidean domains X and $\mathcal{T}$ the Fubini theorem guarantees that if $f\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ is a measurable map then

(i) the section $f^x\colon \mathcal{T} \rightarrowtail \mathbb{R}$ is measurable for almost every $x \in X$, and

(ii) the section $f_t\colon X \to \mathbb{R}$ is measurable for almost every $t \in \mathcal{T}$,

where here and throughout this chapter, $f^x(t) := f_t(x) := f(x,t)$.

The converse implication does not generally hold. Indeed, the characteristic function of the Sierpińksi set considered in Corollary 7.12 is non-measurable yet *every* section f^x and f_t is measurable. However, in Ursell [1939] the author demonstrates that f is measurable under the stronger hypothesis that the section f^x is continuous for all x and the section f_t is measurable for all t.

In Theorem 8.3 we generalise the result of Ursell to demonstrate that f is measurable if the section f^x is continuous for almost every x and the section f_t is measurable for almost every t.

Theorem 8.1 (Ursell). *If $f\colon X \times \mathcal{T} \to \mathbb{R}$ is a map such that*

(i) *the section* $f^x\colon \mathcal{T} \to \mathbb{R}$ *is continuous for every* $x \in X$, *and*

(ii) *the section* $f_t\colon X \to \mathbb{R}$ *is measurable for every* $t \in T$

then f *is measurable.*

Proof. First examine the one-dimensional case $\mathcal{T} \subset \mathbb{R}$. Suppose $f\colon X \times \mathcal{T} \to \mathbb{R}$ satisfies (i) and (ii). For each $n \in \mathbb{N}$ define the map $f_{[n]}\colon X \times \mathcal{T} \to \mathbb{R}$ by

$$f_{[n]}(x,t) = \begin{cases} f\left(x, \frac{m}{2^n}\right) & \frac{m}{2^n} \leq t < \frac{m+1}{2^n} \quad \text{and} \quad \frac{m}{2^n} \in \mathcal{T} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, from the continuity in (i), $f_{[n]}(x,t) \to f(x,t)$ as $n \to \infty$ for each fixed $x \in X$, so $f_{[n]} \to f$ pointwise on $X \times \mathcal{T}$.
Next, fix $n \in \mathbb{N}$. For each open $O \subset \mathbb{R}$

$$f_{[n]}^{-1}(O) = \bigcup_{t \in \mathcal{T}} f_{[n],t}^{-1}(O) \times \{t\},$$

where the section $f_{[n],t}\colon X \to \mathbb{R}$ is defined by $f_{[n],t}(x) := f_{[n]}(x,t)$. However, as $f_{[n]}$ is piecewise constant in t, for each $t \in \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right)$ the inverse image of the section $f_{[n],t}^{-1}(O) = f_{\frac{m}{2^n}}^{-1}(O)$. Consequently,

$$f_{[n]}^{-1}(O) = \bigcup_{m \in \mathbb{N}} f_{\frac{m}{2^n}}^{-1}(O) \times \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right)$$

and, as each $f_{\frac{m}{2^n}}^{-1}(O)$ is measurable from (ii), $f_{[n]}^{-1}(O)$ is a countable union of measurable sets, and so is measurable. We conclude that the map f is the pointwise limit of a sequence of measurable maps, and so is measurable.
The general case follows by induction on the dimension of $\mathcal{T}$. □

In the following theorem, we generalise the above by noting that for a given $f\colon X \times \mathcal{T} \to \mathbb{R}$ the approximation scheme induced by the set $\left\{\frac{m}{2^n} \middle| m, n \in \mathbb{N}\right\} \subset T$ can be replaced by a scheme induced by any dense subset $D \subset \mathcal{T}$ such that every section $f_t\colon X \to \mathbb{R}$ is measurable. First we give a condition for the existence of such a subset:

Lemma 8.2. *If* $D \subset \mathcal{T}$ *is such that* $\mathcal{T} \setminus D$ *has Lebesgue measure zero then* D *is dense in* $\mathcal{T}$.

Proof. Let $D \subset \mathcal{T}$ be such that $\mathcal{T} \setminus D$ has Lebesgue measure zero. We assume for a contradiction that D is not dense so there exists a $t \in \mathcal{T}$ and an $\varepsilon > 0$ such that

$B_\varepsilon(t) \cap D = \emptyset$. However, this implies that $B_\varepsilon(t) \subset \mathcal{T} \setminus D$ and so $\mathcal{T} \setminus D$ has positive measure, which is the desired contradiction. $\square$

Theorem 8.3. *If $f\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ is a map such that*

(i) *the section $f^x\colon \mathcal{T} \to \mathbb{R}$ is continuous for almost every $x \in X$, and*

(ii) *the section $f_t\colon X \rightarrowtail \mathbb{R}$ is measurable for almost every $t \in \mathcal{T}$*

then f is measurable.

Proof. First we examine the one-dimensional case $\mathcal{T} \subset \mathbb{R}$. Let $f\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ be a map satisfying (i) and (ii). Let the set

$$D := \{t \in \mathcal{T} | f_t\colon X \rightarrowtail \mathbb{R} \quad \text{is measurable}\} \tag{8.1}$$

so that, from (ii), $\mathcal{T} \setminus D$ has Lebesgue measure zero and so from Lemma 8.2 D is dense in $\mathcal{T}$. Next, fix $n \in \mathbb{N}$ and note that for each interval $\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ with non-empty intersection with $\mathcal{T}$ there exists a $c_{n,i} \in \left(\frac{i}{2^n}, \frac{i+1}{2^n}\right) \cap D$ as D is dense in $\mathcal{T}$. We use these points to define approximations to f in the following way: for each $n \in \mathbb{N}$ define the map $f_{[n]}\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ by

$$f_{[n]}(x,t) := \begin{cases} f(x, c_{n,i}) & c_{n,i} \leq t < c_{n,i+1} \\ 0 & t < c_{n,i} \quad \forall\, i. \end{cases}$$

Observe that if $x \in X$ is such that the section f^x is continuous then for each $t \in \mathcal{T}$ the point $f_{[n]}(x,t) \to f(x,t)$. Consequently, from (i),

$$f_{[n]}(x,t) \to f(x,t) \quad \forall\, t \in \mathcal{T}, \quad \text{for almost every } x \in X$$

so in particular $f_{[n]}(x,t) \to f(x,t)$ almost everywhere on $X \times \mathcal{T}$.
We now demonstrate that each map $f_{[n]}\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ is measurable. Fix $n \in \mathbb{N}$ and let $O \subset \mathbb{R}$ be an open subset. As in the proof of Theorem 8.1 we write

$$f_{[n]}^{-1}(O) = \bigcup_{t \in \mathcal{T}} f_{[n],t}^{-1}(O) \times \{t\}$$

and similarly, as $f_{[n]}$ is piecewise constant in t,

$$\begin{aligned} f_{[n]}^{-1}(O) &= \bigcup_i f_{[n],c_{n,i}}^{-1}(O) \times [c_{n,i}, c_{n,i+1}) \\ &= \bigcup_i f_{c_{n,i}}^{-1}(O) \times [c_{n,i}, c_{n,i+1})\,. \end{aligned}$$

Now, as $c_{n,i} \in D$ from (8.1) we see that each section $f_{c_{n,i}}$ is measurable. Consequently, $f_{[n]}^{-1}(O)$ is a union of at most a countable number of measurable sets, and so is measurable. We conclude that the map $f\colon X \times \mathcal{T} \rightarrowtail \mathbb{R}$ is the limit almost everywhere on $X \times \mathcal{T}$ of a sequence of measurable maps, and so from Lemma 7.3 is measurable.

The general case now follows from induction on the dimension of n. □

8.1 An application to nonautonomous Sobolev spaces

The following theorem characterises measurable maps that have Sobolev regularity in one component.

Theorem 8.4. *Let the map $f\colon X \times [0,T] \rightarrowtail \mathbb{R}$ be measurable. If the section*

$$f^x \in \mathcal{W}^{1,1}([0,T]) \quad \textit{for almost every } x \in X$$

where $\mathcal{W}^{1,1}([0,T])$ is the Sobolev space defined in Definition B.5 of Appendix B then the map

$$\tilde{f}(x,t) := \begin{cases} \lim_{\varepsilon\to 0} \fint_{t-\varepsilon}^{t+\varepsilon} f(x,\tau)\ \mathrm{d}\tau & \forall \tau \in [0,T] \quad \textit{if } f^x \in \mathcal{W}^{1,1}([0,T]) \\ 0 & \textit{otherwise} \end{cases} \tag{8.2}$$

satisfies

(i) *$\tilde{f}\colon X \times [0,T] \rightarrowtail \mathbb{R}$ is measurable,*

(ii) *$\tilde{f}(x,t) = f(x,t)$ for almost every $(x,t) \in X \times [0,T]$, and*

(iii) *the section $\tilde{f}^x\colon [0,T] \to \mathbb{R}$ is absolutely continuous for almost every $x \in X$.*

Proof. Define

$$\Omega := \left\{x \in X \,|\, f^x \in \mathcal{W}^{1,1}([0,T])\right\}$$

and note that $X \setminus \Omega$ has measure zero. For each $x \in \Omega$ we see from Lemma B.9 that $t \mapsto \tilde{f}(x,t)$ is the precise representative of the section f^x. Consequently, for each $x \in \Omega$ the map $t \mapsto \tilde{f}(x,t)$ is absolutely continuous in t, proving (iii), and $\tilde{f}(x,t) = f(x,t)$ for almost every $t \in [0,T]$. Consequently,

$$\left[\tilde{f}(x,t) = f(x,t) \quad \text{for almost every } t \in [0,T]\right] \quad \text{for almost every } x \in X. \tag{8.3}$$

Next, we demonstrate that $\tilde{f}$ is measurable: consider the map

$$\tilde{f}_{[n]}(x,t) := \begin{cases} ⨍_{t-\frac{1}{n}}^{t+\frac{1}{n}} f(x,\tau)\ \mathrm{d}\tau & \forall \tau \in [0,T] \quad x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

and note that from (8.2)

$$\tilde{f}_{[n]}(x,t) \to \tilde{f}(x,t) \quad \forall t \in [0,T], \quad \text{for almost every } x \in X.$$

as $n \to \infty$. Interchanging the quantifiers, from Corollary A.2 we see that

$$\tilde{f}_{[n]}(x,t) \to \tilde{f}(x,t) \text{ for almost every } x \in X, \quad \forall t \in [0,T]$$

as $n \to \infty$ and that in particular for all $t \in [0,T]$ the map $x \mapsto \tilde{f}(x,t)$ is the limit almost everywhere of the sequence of maps $x \mapsto \tilde{f}_{[n]}(x,t)$.

Further, as f is measurable, from Fubini's Theorem we see that for almost every $t \in [0,T]$ the map

$$x \mapsto ⨍_{t-\frac{1}{n}}^{t+\frac{1}{n}} f(x,\tau)\ \mathrm{d}\tau = \frac{2}{n} \int_{t-\frac{1}{n}}^{t+\frac{1}{n}} f(x,\tau)\ \mathrm{d}\tau$$

is measurable for each $n \in \mathbb{N}$. Consequently, for almost every $t \in [0,T]$ the map $x \mapsto \tilde{f}_{[n]}(x,t)$ is measurable for each $n \in \mathbb{N}$ and so the map $x \mapsto \tilde{f}(x,t)$ is the limit almost everywhere of a sequence of measurable maps which, from Lemma 7.3, is measurable.

Consequently, the map $\tilde{f}$ is such that

- the section $\tilde{f}^x\colon [0,T] \to \mathbb{R}$ is continuous for almost every $x \in X$, and
- the section $\tilde{f}_t\colon X \rightarrowtail \mathbb{R}$ is measurable for almost every $t \in [0,T]$

and so from Theorem 8.3, $\tilde{f}\colon X \times [0,T] \rightarrowtail \mathbb{R}$ is measurable, proving (i).

Finally, as $\tilde{f}$ and f are both measurable their difference $\tilde{f} - f$ is also measurable and so the set

$$N := \left\{(x,t) \in X \times [0,T] \mid \tilde{f}(x,t) \neq f(x,t)\right\}$$

is measurable. Consequently, from (8.3) and Fubini's Theorem we conclude that N has measure zero, proving (ii). □

This measurability result ensures that the absolutely continuous representative of a regular Lagrangian flow solution of (ODE) is itself measurable. Consequently we are able to interpret (ODE) in the classical sense with equality holding almost everywhere. We make this precise in the following:

Corollary 8.5. *Let $Y\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ be a regular Lagrangian flow solution of* (ODE) *in the sense of Definition 2.32. There exists a map $\tilde{Y}\colon\ [0,T]_t \times \mathbb{R}^n \times [0,T]_s \to \mathbb{R}^n$ such that*

- *$\tilde{Y}\colon\ [0,T]_t \times \mathbb{R}^n \times [0,T]_s \to \mathbb{R}^n$ is measurable,*
- *$\tilde{Y}(t,x,s) = Y(t,x,s)$ for almost every $(t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]$, and*
- *the map $t \mapsto \tilde{Y}(t,x,s)$ is absolutely continuous for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$,*

Proof. Let $X := \mathbb{R}^n_x \times [0,T]_s$ and $f((x,s),t) := Y(t,x,s)$. The result follows from the above Theorem. □

By identifying a regular Lagrangian flow with the absolutely continuous representative of Corollary 8.5 we give the following equivalent definition:

Theorem 8.6. *If $X\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ is a regular Lagrangian flow solution of* (ODE) *then there is a map $\tilde{X}\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ such that*

(i) $\tilde{X} \in \mathcal{L}^1_{\mathrm{loc}}([0,T]_t \times \mathbb{R}^n_x \times [0,T]_s ; \mathbb{R}^n)$,

(ii) *for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$*

$$\tilde{X}(t,x,s) = x + \int_s^t f\left(\tilde{X}(\tau,x,s),\tau\right) \,\mathrm{d}\tau \quad \forall t \in [0,T], \quad \textit{and}$$

(iii) *for each Borel set $A \subset \mathbb{R}^n$ with $\mu_n(A) = 0$ the pushforward*

$$\tilde{X}(t,\cdot,s)_{\#}\,\mu_n(A) = \mu_n\left(\left\{x \in \mathbb{R}^n | \tilde{X}(t,x,s) \in A\right\}\right) = 0$$

for all $t \in [0,T]$, for almost every $s \in [0,T]$.

Proof. Let $\tilde{X}\colon\ [0,T]_t \times \mathbb{R}^n_x \times [0,T]_s \to \mathbb{R}^n$ satisfy

- $t \mapsto \tilde{X}(t,x,s)$ is absolutely continuous for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$, and

- $X(t,x,s) = \tilde{X}(t,x,s)$ for almost every $(t,x,s) \in [0,T] \times \mathbb{R}^n \times [0,T]$.

The existence of such an $\tilde{X}$ is ensured by Corollary 8.5. We see that $\tilde{X}$ is a regular Lagrangian flow solution of (ODE) from Lemma 2.35 as $\tilde{X}$ is equal to X almost everywhere. Further, for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$ the map $t \mapsto \tilde{X}(t,x,s)$ is absolutely continuous with weak derivative given by the map $t \mapsto f\left(\tilde{X}(t,x,s), t\right)$. Consequently, from (B.3), for almost every $(x,s) \in \mathbb{R}^n \times [0,T]$

$$\tilde{X}(t,x,s) = x + \int_s^t f\left(\tilde{X}(\tau,x,s), \tau\right) \,\mathrm{d}\tau \quad \forall t \in [0,T].$$

□

This Theorem rigorously justifies the transition from the regular Lagrangian flows of Chapter 2 that are defined in terms of weak derivatives, to the regular Lagrangian flows of the subsequence Chapters that are defined in terms of almost everywhere equality of derivatives. This solves measurability obstruction in the theory of irregular ordinary differential equations as developed in DiPerna and Lions [1989], Lions [1998], Ambrosio [2004], Hauray et al. [2007], and De Lellis [2008].

Chapter 9

Conclusion

In Chapter 2 we identified that the temporal Sobolev regularity of a regular Lagrangian flow X implies that there exists a map $\tilde{X}$ such that

(i) for almost every $(x, s) \in \mathbb{R}^n \times [0, T]$ the map $t \mapsto \tilde{X}(t, x, s)$ is absolutely continuous, and

(ii) $\tilde{X}(t, x, s) = X(t, x, s)$ for almost every $t \in [0, T]$ for almost every $(x, s) \in \mathbb{R}^n \times [0, T]$.

Although the continuity with respect to time greatly simplifies the treatment of regular Lagrangian flows it is not immediately apparent that the map

$$\tilde{X}\colon [0, T] \times \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$$

is measurable. This is problematic as we are unable to define the pushforward measures $\tilde{X}(t, \cdot, s)_{\#}\,\mu_n$ that are fundamental to the theory of regular Lagrangian flows if the map $\tilde{X}$ is not measurable. However, in Chapter 8 we demonstrate that this absolutely continuous representative is measurable and so we can legitimately choose this representative in order to simplify the theory. In this manner we have addressed an issue in the theory of irregular ordinary differential equations that seems to have been overlooked in the literature.

In Chapter 3 we demonstrated through the examples in Sections 3.1.1 and 3.1.2 that there is no Sobolev regularity or integrability condition on the vector field f that is sufficient for (ODE) to have unique solutions for almost all initial data. Further, in Theorem 3.4, we demonstrated that the uniqueness of solutions for almost all initial data is sensitive to the choice of representative of the vector field f. In fact we demonstrated that every non-zero vector field f for which there is a regular Lagrangian flow solution there is a vector field g equal to f almost everywhere, such that the ordinary differential equation $\dot{\xi} = g(\xi, t)$ has non-unique solutions for a set of initial data of positive measure. These constructions can seem quite artificial as we deliberately introduce values of f that do not represent the local average in order to introduce qualitatively different behaviour. Consequently it is of interest to consider the uniqueness of solutions when we restrict the vector field f to one that represents the 'local average' such as the precise representative of f^*. We remark that the classical Example 2.19 has non-unique solutions for all initial data and the that vector field is equal to the precise representative, and further that if a vector field has sufficient Sobolev regularity then the precise representative of this vector field is Lipschitz continuous, and so have unique solutions. With these two examples established it is of interest to determine the extent to which the regularity of a vector field equal to its precise representative affects the non-uniqueness of solutions to (ODE).

In Chapter 4 we provide a sufficient condition for a nearly incompressible regular Lagrangian flow solution of (ODE) to avoid a subset $S \subset \mathbb{R}^n [0, T]$. This condition is written in terms of the r-codimension print of S, which encodes the spatio-temporal detail of S in a useful way, as we demonstrated in Chapter 5. In the exposition of avoidance in the autonomous case in Aizenman [1978b], the author gives examples illustrating the sharpness of his sufficient condition for avoidance. While these autonomous examples can trivially be embedded into the non-autonomous framework they fail to show that the the condition of Theorem 4.8 is sharp with respect to the temporal regularity of the vector field or the temporal detail of the subset. It would be interesting to explore the sharpness of Theorem 4.8, and generally to see if any other geometric information can be derived from the r-codimension print.

It may be fruitful to attempt to generalise the avoidance property for regular Lagrangian flows. We recall from Section 2.1.4 that for classical flow solutions we can project sets $S \subset \mathbb{R}^n \times [0, T]$ along the trajectories of the flow solution and consider the fractal properties of this projection rather than simply its n-dimensional Lebesgue measure. For flows with Lipschitz regularity, these projections are well understood as their box-counting and Hausdorff dimensions do not exceed that of the set S. By considering the fractal dimension of the set

$$\{x \in \mathbb{R}^n | (X(t, x, 0), t) \in S\}$$

we are effectively defining a 'finer' variant of the avoidance property, in which we can determine the size of the set of initial conditions whose trajectories intersect

S. Unfortunately, we cannot in general apply these finer variants of avoidance to regular Lagrangian flows as the flow is only defined for almost all initial data $x \in \mathbb{R}^n$.

However, there are regular Lagrangian flow solutions for which there is a trajectory defined for all initial data. The flow constructed in Foias et al. [1985] is one such example, where the authors demonstrate that for a suitable weak solution $f \in \mathcal{L}^1\left(0,T;\mathcal{L}^\infty\left(\mathbb{R}^3\right)\right)$ of the 3D Navier-Stokes equations there are absolutely continuous solutions of (ODE) for all initial data. Consequently, given a set $S \subset \mathbb{R}^3 \times [0,T]$ we can consider the Hausdorff measure or box-counting dimension of the set

$$\left\{x \in \mathbb{R}^3 \middle| \left(X\left(t,x,0\right),t\right) \in S\right\}$$

If we are able to adapt the tools of Chapter 4 to find these finer bounds on the fractal dimension of this set then we would be in a position to improve the result of Robinson and Sadowski [2009] in which the authors use the quasi-non-autonomous avoidance result of Cipriano and Cruzeiro [2005] to determine that a suitable weak solution f of the 3D Navier-Stokes equations avoids the set of singular points of S and so gives rise to unique solutions of (ODE) for almost all initial data. If we are able to develop a theory of finer avoidance then we would be able to use the best known bounds in Caffarelli et al. [1982] of the dimension of the singular set of S together with the best known regularity of f to determine a good bound on the dimension of the set $\left\{x \in \mathbb{R}^3 \middle| \left(X\left(t,x,0\right),t\right) \in S\right\}$ of initial data which gives rise to non-unique solutions.

In Chapter 6 we proved that the the upper and lower box-counting dimensions satisfy the chain of inequalities

$$\begin{aligned}
\dim_{LB}(F) + \dim_{LB}(G) &\leq \dim_{LB}(F \times G) \\
&\leq \min\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\
&\leq \max\left(\dim_{LB}(F) + \dim_B(G), \dim_B(F) + \dim_{LB}(G)\right) \\
&\leq \dim_B(F \times G) \\
&\leq \dim_B(F) + \dim_B(G).
\end{aligned}$$

As far as we are aware the second and fourth inequalities are new. In addition we detailed a method to construct generalised Cantor sets in such a way that their length-scales are compatible, making it easy to calculate the upper and lower box-counting dimensions of their product. We used this method to construct sets $F, G \subset \mathbb{R}$ such that the upper and lower box-counting dimensions of F, G and the product $F \times G$ take arbitrary values subject to the above chain of inequalities.

Finally, in Chapter 7 we illustrated the possible pathologies in measurability that may arise when specifying representative maps in function spaces. We saw that typical embeddings between function spaces are isometric isomorphisms of equivalence classes rather than a straightforward inclusion of maps. Further we illustrated that as these equivalence relation may differ, care is needed when manipulating maps in the intersection of two function spaces.

Appendix A

Measure theory

In this section we examine two planar sets that illuminate the difficulties in manipulating statements that hold almost everywhere. Critically, when making a statement dependent on multiple variables, the order of the quantifiers is important, and generally cannot be interchanged.

For a planar set $N \subset \mathbb{R}^2$ we define the sections

$$\begin{aligned} N^y &:= \{x \in \mathbb{R} |\, (x,y) \in N\} \quad \text{for all } y \in \mathbb{R}, \quad \text{and} \\ N_x &:= \{y \in \mathbb{R} |\, (x,y) \in N\} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Lemma A.1. *Let $N \subset \mathbb{R}^2$. The following implications hold:*

$$\mu_2(N) = 0 \qquad \Rightarrow \qquad \mu_1(N^y) = 0 \quad a.e.\ y \in \mathbb{R}, \tag{A.1}$$

$$\mu_2(N) = 0 \qquad \Rightarrow \qquad \mu_1(N_x) = 0 \quad a.e.\ x \in \mathbb{R}, \quad \textit{and} \tag{A.2}$$

$$N^y = \emptyset \quad a.e.\ y \in \mathbb{R} \qquad \Rightarrow \qquad \mu_1(N_x) = 0 \quad \forall x \in \mathbb{R}. \tag{A.3}$$

Generally, however,

$$\mu_1(N_x) = 0 \quad \forall x \in \mathbb{R} \qquad \not\Rightarrow \qquad N^y = \emptyset \quad a.e.\ y \in \mathbb{R}. \tag{A.4}$$

Proof. (A.1) and (A.2) are consequences of Fubini's theorem, as N is a measurable subset of $\mathbb{R}^2$. Next, observe that

$$N_x = \{y \in \mathbb{R} | x \in N^y\} \subset \{y \in \mathbb{R} | N^y \neq \emptyset\}$$

so if $N^y = \emptyset$ for almost every $y \in \mathbb{R}$ then $\mu_1(N_x) = 0$ for all x, yielding (A.3). Finally, observe that (A.4) follows from considering the example of the set $\{(x,y)\,|\,x = y\}$, for which every section is non-empty and of zero measure. $\square$

Corollary A.2. *If P is a statement dependent on two variables $x, y \in \mathbb{R}$ then*

$$\begin{array}{llcll} P(x,y) & a.e.\ (x,y) \in \mathbb{R}^2 & \Rightarrow & P(x,y) & a.e.\ x \in \mathbb{R},\ \textit{for a.e. } y \in \mathbb{R}, \quad \textit{and} \\ P(x,y) & \forall x \in \mathbb{R},\ a.e.\ y \in \mathbb{R} & \Rightarrow & P(x,y) & a.e.\ y \in \mathbb{R},\ \forall x \in \mathbb{R} \end{array}$$

but

$$P(x,y) \quad a.e.\ y \in \mathbb{R},\ \forall x \in \mathbb{R} \quad \not\Rightarrow \quad P(x,y) \quad \forall x \in \mathbb{R},\ a.e.\ y \in \mathbb{R}.$$

Proof. Let the set N consist of the points (x, y) such that $P(x, y)$ does not hold. The results follows from the above lemma. $\square$

It remains to establish whether $\mu_1(N_x) = 0$ for almost every $x \in \mathbb{R}$ implies $\mu_1(N^y) = 0$ for almost every $y \in \mathbb{R}$, and correspondingly, whether $P(x, y)$ holding for almost every $x \in \mathbb{R}$, for almost every $y \in \mathbb{R}$ implies that $P(x, y)$ holds for almost every $y \in \mathbb{R}$ for almost every $x \in \mathbb{R}$. Interestingly enough the answer depends on our choice of axioms. In Friedman [1980], and independently proved in Freiling [1986], the author takes the Zermelo-Fraenkel Axioms and the Axiom of Choice and constructs a model in which

$$\int_{\mathbb{R}} \mu_1(N_x)\, \mathrm{d}x = \int_{\mathbb{R}} \mu_1(N^y)\, \mathrm{d}y$$

for all sets $N \subset \mathbb{R}^2$ such that N_x is measurable for almost every $x \in \mathbb{R}$ and N^y is measurable for almost every $y \in \mathbb{R}$. Note that this is strictly weaker than requirement that N is a measurable subset of $\mathbb{R}^2$, as in the classic Fubini theorem.

Alternatively, if we assume the Zermelo-Fraenkel Axioms, the Axiom of Choice and the Continuum Hypothesis then we can demonstrate the existence of a set N such that

$$\begin{aligned} \mu_1(N_x) &= 0 \quad \forall\, x \in \mathbb{R} \quad \text{and} \\ \mu_1(N^y) &= 1 \quad \forall\, y \in \mathbb{R}. \end{aligned}$$

The existence of this set is classical (see for example Theorem 7.1.2 of Ciesielski [1997]) and hinges on the fact that with these axioms we can demonstrate the existence of a well-ordering on $\mathbb{R}$ such that each element has countably many predecessors.

Appendix B

Precise representatives and absolutely continuous maps

We list definitions and basic propositions concerning Sobolev maps, mollifiers, precise representatives and absolutely continuous maps taken from Evans [2010] and Evans and Gariepy [1992], and a lemma on the concatenation of absolutely continuous maps, due to the author.

B.1 Lebesgue points and precise representatives

For a map $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ we write the average of f over the ball $B_r(x) \subset \mathbb{R}^n$ as

$$\fint_{B_r(x)} f(x)\,\mathrm{d}x := \mu_n(B_r(x)) \int_{B_r(x)} f(x)\,\mathrm{d}x.$$

Definition B.1. *We say that the point $x \in \mathbb{R}^n$ is a* Lebesgue point *of the map $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ if*

$$\lim_{r \to 0} \fint_{B_r(x)} |f(x) - f(y)|\,\mathrm{d}y = 0.$$

Theorem B.2 (Lebesgue Differentiation Theorem). *If $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ then almost every $x \in \mathbb{R}^n$ is a Lebesgue point of f.*

Proof. See, for example, §1.7.1 of Evans and Gariepy [1992]. □

Definition B.3. *For a map $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ we define its* precise representative f^* *by*

$$f^*(x) := \begin{cases} \lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} f(y)\,\mathrm{d}y & \textit{if this limit exists, and} \\ 0 & \textit{otherwise.} \end{cases} \tag{B.1}$$

It is straightforward to demonstrate that if $x \in \mathbb{R}^n$ is a Lebesgue point of f then $f(x) = f^*(x)$ so in light of Theorem B.2 f^* is equal to f almost everywhere. However, not every point for which the limit in (B.1) is defined is a Lebesgue point.

B.2 Sobolev maps

Definition B.4. *For a map $g \in \mathcal{L}^1_{\text{loc}}(R^n)$ we say that the map $h \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ is the i^{th}* weak partial derivative *of g if*

$$\int_{\mathbb{R}^n} g(x) \frac{\mathrm{d}\phi}{\mathrm{d}x_i} \mathrm{d}x = -\int_{\mathbb{R}^n} h(x)\,\phi(x)\,\mathrm{d}x$$

for all test maps $\phi \in C_c^\infty(\mathbb{R}^n)$.

We denote the i^{th} weak partial derivative of g by $\frac{\mathrm{d}g}{\mathrm{d}x_i}$.

Definition B.5. *For $1 \le p \le \infty$ the* Sobolev space of maps *$\mathcal{W}^{1,p}(\mathbb{R}^n)$ consists of all maps $g \in \mathcal{L}^p(\mathbb{R}^n)$ such that for $i = 1 \dots n$ the i^{th} weak derivative $\frac{\mathrm{d}g}{\mathrm{d}x_i}$ exists and belongs to $\mathcal{L}^p(\mathbb{R}^n)$.*

The Sobolev space $\mathcal{W}^{1,p}(\mathbb{R}^n)$ is endowed with the seminorm

$$\|g\|_{\mathcal{W}^{1,p}(\mathbb{R}^n)} := \begin{cases} \|g\|_{\mathcal{L}^p(\mathbb{R}^n)} + \left(\sum_{i=1}^n \left\|\frac{\mathrm{d}g}{\mathrm{d}x_i}\right\|^p_{\mathcal{L}^p(\mathbb{R}^n)}\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \|g\|_{\mathcal{L}^\infty(\mathbb{R}^n)} + \sum_{i=1}^n \left\|\frac{\mathrm{d}g}{\mathrm{d}x_i}\right\|_{\mathcal{L}^\infty(\mathbb{R}^n)} & p = \infty. \end{cases}$$

Definition B.6. *The* Sobolev space *$W^{1,p}(\mathbb{R}^n)$ consists of equivalences classes of maps in $\mathcal{W}^{1,p}(\mathbb{R}^n)$ under the equivalence relation given by equality almost everywhere on $\mathbb{R}^n$.*

Under this equivalence relation the seminorm $\|\cdot\|_{\mathcal{W}^{1,p}(\mathbb{R}^n)}$ becomes a norm, which we denote $\|\cdot\|_{W^{1,p}(\mathbb{R}^n)}$, and the space $W^{1,p}(\mathbb{R}^n)$ is a Banach space.

B.3 Mollifiers and the precise representative of Sobolev maps

Next, we recall some fact about mollified maps: Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be positive have support in $B_1(0)$ and $\int_{\mathbb{R}^n} \eta\,\mathrm{d}x = 1$. For each $\varepsilon > 0$ define

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

For $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$ we define its mollification by

$$f^\varepsilon(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y)\,\mathrm{d}y.$$

Proposition B.7. *Let $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$.*

- $f^\varepsilon \in C^\infty(\mathbb{R}^n)$
- *if x is a Lebesgue point of f then $f^\varepsilon(x) \to f(x) = f^*(x)$*
- *if $f \in \mathcal{L}^p(\mathbb{R}^n)$ then $f^\varepsilon \to f$ in $L^p(\mathbb{R}^n)$*
- *if $f \in \mathcal{W}^{1,p}(\mathbb{R}^n)$ then $f^\varepsilon \to f$ in $W^{1,p}(\mathbb{R}^n)$.*

Proof. See, Theorem 1 §4.2 of Evans and Gariepy [1992]. □

The Lebesgue points therefore are the 'good' points of f, as the limit of the mollifiers is equal to f at these points. It is of interest to determine the 'size' of the set of points $N := \{x \in \mathbb{R}^n | x \text{ is not a Lebesgue point of } f\}$. However, as $f(x) = f^*(x)$ for all Lebesgue points the result $\mu_n(N) = 0$ provided by the Theorem B.2 any improvement of this result does not apply to every map g equal to f almost everywhere. Consequently, to hope to improve the result we must restrict ourselves to good representatives of f and the precise representative f^* is the obvious choice.

Theorem B.8. *If f is in $\mathcal{W}^{1,1}(\mathbb{R}^n)$ then the set*

$$N := \{x \in \mathbb{R}^n | x \textit{ is not a Lebesgue point of } f^*\}$$

has $\mathcal{H}^{n-1}(N) = 0$.

Proof. The proof proceeds via defining the capacity measure of a Sobolev map and is a consequence of Theorem 1 §4.8 and Theorem 3 §5.6.3 of Evans and Gariepy [1992]. □

The precise representative of a Sobolev map, then, is the limit of its mollification outside a set of zero $(n-1)$ dimensional Hausdorff measure. In one dimension the precise representative of a Sobolev map is absolutely continuous, which is the content of the following lemma:

Lemma B.9. *Let $g \in \mathcal{W}^{1,1}([0,T])$ and denote the weak derivative of g by $\frac{\mathrm{d}g}{\mathrm{d}t}$. The precise representative of g is given by*

$$g^*(t) = \lim_{\varepsilon \to 0} \fint_{B_\varepsilon(t)} g(\tau)\ \mathrm{d}\tau \quad \forall t \in [0,T]. \tag{B.2}$$

Further, g^* is absolutely continuous on $[0,T]$, and for each Lebesgue point t_0 of g

$$g^*(t) = g(t_0) + \int_{t_0}^{t} \frac{\mathrm{d}g}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T].$$

Finally, for all $t_0 \in [0,T]$

$$g^*(t) = g^*(t_0) + \int_{t_0}^{t} \frac{\mathrm{d}g^*}{\mathrm{d}\tau}(\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T]. \tag{B.3}$$

Proof. See Theorem 1, §4.9.1 of Evans and Gariepy [1992]. The equation (B.3) follows from the absolute continuity of g^*, but can also be derived from applying the first part of the lemma to the map g^* and noting that every point of $[0,T]$ is a Lebesgue point of g^* as the map is continuous. □

B.4 Absolutely continuous maps

Absolutely continuous maps satisfy the integration by parts formula:

Lemma B.10. *If $g\colon (0,T) \to \mathbb{R}$ is absolutely continuous and $\psi \in C_c^\infty((0,T))$ then*

$$\int_a^b g(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}(t)\,\mathrm{d}t = g(b)\psi(b) - g(a)\psi(a) - \int_a^b \frac{\mathrm{d}g}{\mathrm{d}t}(t)\psi(t)\,\mathrm{d}t \tag{B.4}$$

for all intervals $[a,b] \subset (0,T)$.

Proof. Observe that the product $g \cdot \psi$ is absolutely continuous on $(0,T)$ and is differentiable at t if and only if g is differentiable at t, in which case

$$\frac{\mathrm{d}}{\mathrm{d}t} g \cdot \psi(t) = \frac{\mathrm{d}g}{\mathrm{d}t}(t)\psi(t) + g(t)\frac{\mathrm{d}\psi}{\mathrm{d}t}(t)$$

and the formula (B.4) follows after integrating. □

Lemma B.11. *If $h\colon \mathbb{R} \to \mathbb{R}^n$ is continuous and piecewise absolutely continuous then h is absolutely continuous.*

Proof. It is sufficient to consider a continuous function in two absolutely continuous pieces as the general result will follow inductively. Let $h\colon \mathbb{R} \to \mathbb{R}^n$ be continuous and have the form

$$h(t) = \begin{cases} h_1(t) & t < \tau \\ h_2(t) & \tau \le t \end{cases} \tag{B.5}$$

for some $\tau \in \mathbb{R}$ and h_1, h_2 absolutely continuous. Let $\varepsilon > 0$. From the absolute continiuity of h_1 and h_2 there exists a $\delta_1 > 0$ such that

- for any finite sequence of disjoint intervals $[x_j, y_j] \subset (-\infty, \tau)$

$$\sum |y_j - x_j| < \delta \Rightarrow \sum |h_1(y_j) - h_1(x_j)| < \frac{\varepsilon}{3} \quad \text{and} \tag{B.6}$$

- for any finite sequence of disjoint intervals $[x_j, y_j] \subset [\tau, \infty)$

$$\sum |y_j - x_j| < \delta \Rightarrow \sum |h_2(y_j) - h_2(x_j)| < \frac{\varepsilon}{3}. \tag{B.7}$$

Further, as h is continuous at τ, there exists a $\delta_2 > 0$ such that $|x - \tau| < \delta_2$ implies $|h(x) - h(\tau)| < \varepsilon/6$. In particular if $[x, y]$ is an interval of length less than δ_2 containing τ then

$$|h(y) - h(x)| < \frac{\varepsilon}{3} \tag{B.8}$$

Let $\delta = \min(\delta_1, \delta_2)$ and let $\mathcal{I}$ index a finite sequence of disjoint intervals $[x_j, y_j] \subset \mathbb{R}$ such that

$$\sum_{j \in \mathcal{I}} |y_j - x_j| < \delta.$$

Write

$$\begin{aligned}
\mathcal{I}^- &= \{j \in \mathcal{I} | [x_j, y_j] \subset (-\infty, \tau)\}, \\
\mathcal{I}^+ &= \{j \in \mathcal{I} | [x_j, y_j] \subset [\tau, \infty)\}
\end{aligned}$$

so that

$$\mathcal{I} = \mathcal{I}^- \cup \mathcal{I}^+ \cup \{j \in \mathcal{I} | \tau \in (x_j, y_j]\}.$$

Consequently,

$$\begin{aligned}
\sum_{j \in \mathcal{I}} |h(y_j) - h(x_j)| = &\sum_{j \in \mathcal{I}^-} |h_1(y_j) - h_1(x_j)| + \sum_{j \in \mathcal{I}^+} |h_2(y_j) - h_2(x_j)| \\
&+ \sum_{\{j \in \mathcal{I} | \tau \in (x_j, y_j]\}} |h(y_j) - h(x_j)| \\
\leq &\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{\{j \in \mathcal{I} | \tau \in (x_j, y_j]\}} \frac{\varepsilon}{3}
\end{aligned}$$

from (B.6), (B.7) and (B.8) as the sum of the lengths of the intervals, and so the length of individual intervals, is less than δ. Next, as the intervals are disjoint only a single interval $[x_j, y_j]$ can contain τ so the final summation is over at most one term. Therefore,

$$\sum_{j \in \mathcal{I}} |h(y_j) - h(x_j)| < \varepsilon$$

so h is absolutely continuous. $\square$

List of Symbols

Geometry and Measure

For sets $A \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^n \times [0, T]$

For a set $F \subset \mathbb{R}$ consisting of disjoint intervals of equal length

Matrices and Functions

For a square matrix M

For a function $g\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and a vector field $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$

$\nabla_x g$	the spatial gradient of g	7
$\operatorname{div}(f)$	the spatial divergence of f	7
$\nabla_x f$	the spatial Hessian of f	9

For maps $\xi_1, \xi_2\colon [0,T] \to \mathbb{R}^n$

$\vee_\tau(\xi_1, \xi_2)$	the concatenation of the maps ξ_1 and ξ_2 at time τ	16

For $f\colon B \to C$ and $g\colon A \to B$

$f \circ g$	the composition of the maps f and g	31

For $f\colon \mathbb{R}^n \to \mathbb{R}$

$⨍_A f(x)\,\mathrm{d}x$	the average of f over the set $A \subset \mathbb{R}^n$	141

Flows

For $X\colon [0,T] \times \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$

$X^{-1}(t,\cdot,s)\,A$	the inverse image of $A \subset \mathbb{R}^n$ under the map $X(t,\cdot,s)$	19
$X(t,\cdot,s)_{\#}\,\mu_n$	the push-forward of the Lebesgue measure via the map $X(t,\cdot,s)$	19
$P_X(S)$	the projection of $S \subset \mathbb{R}^n \times [0,T]$ along the trajectories of X	23

Function Spaces and Measurability

$C^k((0,T))$	the space of k-times continuously differentiable maps $f\colon (0,T) \to \mathbb{R}$	9
$C_c^\infty((0,T))$	the space of maps $f \in C^\infty((0,T))$ with compact support	32

For a set $A \subset \mathbb{R}^n$, a seminormed space X, and $1 \le p < \infty$

$f\colon A \rightarrowtail X$	means that $f(t) \in X$ for almost every $t \in A$	105
$\mathcal{L}^0(A;X)$	the space of strongly measurable maps $f\colon A \rightarrowtail X$	106
$\mathcal{L}^p(A;X)$	the subspace of $\mathcal{L}^0(A;X)$ with finite $\mathcal{L}^p$ seminorm	107
$\mathcal{L}^1_{\mathrm{loc}}(A;B)$	the space of locally integrable maps $f\colon A \rightarrowtail X$	31
$\mathcal{L}^\infty(A;X)$	the seminormed space of essentially bounded maps $f\colon A \rightarrowtail X$	107
$L^p(A;X)$	the normed space of equivalence classes of maps in $\mathcal{L}^p(A;X)$	108
$L^\infty(A;X)$	the normed space of equivalence classes of maps in $\mathcal{L}^\infty(A;X)$	108
$\mathcal{W}^{1,1}_{\mathrm{loc}}(A;X)$	the subspace of $\mathcal{L}^1_{\mathrm{loc}}(A;X)$ whose elements have locally integrable weak derivatives	142
$W^{1,1}_{\mathrm{loc}}(A;X)$	the space of equivalence classes of $\mathcal{W}^{1,1}_{\mathrm{loc}}(A;X)$	36
$BV(A;X)$	the space of equivalence classes of maps $f\colon A \rightarrowtail X$ of bounded variation	36

For $f\colon A \rightarrowtail X$

$[f]_A$	the equivalence class of maps $g\colon A \rightarrowtail X$ such that $g(t) = f(t)$ for almost every $t \in A$ 108

For $f, g\colon \mathcal{I} \times \Omega \rightarrowtail \mathbb{R}$

$f \approx g$	means that $f(t,x) = g(t,x)$ for almost every $(t,x) \in \mathcal{I} \times \Omega$ 104
$\overset{\approx}{[f]}$	the equivalence class of maps $h \approx f$ 115
$f \sim g$	means that $f(t,x) = g(t,x)$ for almost every $x \in \Omega$, for almost every $t \in \mathcal{I}$ 104
$\overset{\sim}{[f]}$	the equivalence class of map $h \sim f$ 112

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