Stochastic Flows and Sticky Brownian motion

by

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Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institution of tertiary education. Information derived from the published or unpublished work of others has been acknowledged in the text and a list of references is given.

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Abstract

Sticky Brownian motion is a one-dimensional diffusion with the property that the amount of time the process spends at zero is of positive Lebesgue measure and yet the process does not stay at zero for any positive interval of time. Sticky Brownian motion can be considered as qualitatively between standard Brownian motion and Brownian motion absorbed at zero.

A system of coalescing Brownian motions is a collection of paths, where each path behaves as a Brownian motion independent of all other paths until the first time two paths meet, at which point the two paths that have just met behave is a single Brownian motion independent of all remaining paths. Thus the difference between any two paths of a system of coalescing Brownian motion behaves as a Brownian motion absorbed at zero. In this thesis we consider systems of Brownian paths, where the difference between any two paths behaves as a sticky Brownian motion rather than a coalescing Brownian motion.

We consider systems of sticky Brownian motions starting from points in continuous time and space. The evolution of systems of this type may be described by means of a stochastic flow of kernels. A stochastic flow of kernels is characterised by its N-point motions which form a consistent family of Brownian motions. We characterise such a consistent family such that the difference between any pair of coordinates behaves as a sticky Brownian motion.

The Brownian web is a way of describing a system of coalescing Brownian motions starting in any point in space and time. We describe a coupling of Brownian webs such that the difference between one path in each web behaves as a sticky Brownian motion. Then by conditioning one Brownian web on the other we can construct a stochastic flow of kernels.

Finally we discuss the concept of duality in relation to flows and we prove some minor results relating to these dualities.

Chapter 1

Introduction

Suppose that $(\xi_{k,n}; (k,n) \in L)$ is a family of independent random signs with $\mathbf{P}(\xi_{k,n} = 1) = \mathbf{P}(\xi_{k,n} = -1) = \frac{1}{2}$, indexed by the points of the lattice $L = \{(k,n) \in \mathbf{Z}^2 : k+n \text{ is even}\}$. Now at each point in L we place an arrow from (k,n) to $(k+\xi_{k,n}, n+1)$. Starting from arbitrary points $(k,n) \in L$ and following the arrows, we construct in this way an infinite family, $\mathcal{S}$, of coalescing simple random walk paths, see figure 1.1.

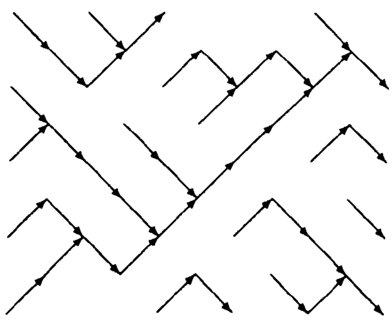

Figure 1.1: Coalescing random walks on the lattice L

Under a diffusive scaling the limit of this system of coalescing random walks is a system of coalescing Brownian motions. This limiting object was first investigated by Arratia, [Arr79], who was motivated by studying the scaling limit of coalescing random walks and voter models. Further work on systems of coalescing Brownian motions has been done by Tóth and Werner, [TW98], motivated by constructing continuum "self-repelling motions", and more recently Fontes, Isopi, Newman, and Ravishankar, [FINR04], motivated again by scaling limits of discrete coalescing systems. The underlying idea behind each of these is to construct a system of coalescing Brownian motions starting from every point in space ($\mathbb{R}$) and time ($\mathbb{R}$).

It is relatively straight forward to define a system of coalescing Brownian motions starting from a finite collection of points in $\mathbb{R} \times \mathbb{R}$ and then to extend this to a system of coalescing Brownian motions starting from a countable dense subset of $\mathbb{R} \times \mathbb{R}$, see Section 4.1.1. A question arises about what to do with the remaining starting points. If $C_{x,t_0}(t)$ is the position, at time t, of the Brownian motion started at (x, t_0). Then from [TW98] or [Arr79] the method would be, in a sense, to apply some right (or left) continuity condition to $x \mapsto C_{x,t_0}(t)$. Discussions of different regularity conditions can be found in [TW98].

The characterisation in [FINR04] attacks the problem from a different angle by defining a metric space of paths with starting points in $\mathbb{R}^2$. To construct the paths started from points outside some countable dense subset of $\mathbb{R}^2$, the closure is taken in this metric space. In doing so, a random object is created, called the Brownian web.

Effectively by taking the closure in this metric space of paths we are allowing limits to be taken from below and above a starting point. For any deterministic starting point this does not make a difference to the resulting path starting from that point but for some non-deterministic points the Brownian web construction leads to the possibility of two different paths starting from the same point.

The main advantage of the Brownian web construction is that it exists as a random point in a certain metric space, which allows the use of certain weak convergence results and will give us the ability to construct a Markov chain on the space itself. For more on the Brownian web see [FINR04], [FN06], [FN06], [NRS05] and [FFW05].

Returning to the system of coalescing random walks $\mathcal{S}$, we consider a generalisation of this system, in which we replace each arrow in the system with two weighted arrows, one up and one down. Weights of the arrows are chosen independently for each point in L, but with the sum of the weights of the two arrows emanating from a given point always being 1. Thus we have a family of i.i.d. $[0,1]$-valued random variables $(Q_{k,n};(k,n)\in L)$, and the weight of the arrow from (k,n) to $(k+1,n+1)$ is $Q_{k,n}$, whereas the weight of the arrow from (k,n) to $(k-1,n+1)$ is $1-Q_{k,n}$.

We can consider this new system as an evolution of mass. That is, if there is a mass $M_{k,n}$ at position $(k,n)\in L$ then a mass of $M_{k,n}Q_{k,n}$ moves to $(k+1,n+1)$ and the remaining $M_{k,n}(1-Q_{k,n})$ moves to $(k-1,n+1)$. See figure 1.2. Similarly $M_{k,n}=Q_{k-1,n-1}M_{k-1,n-1}+(1-Q_{k+1,n-1})M_{k+1,n-1}$. Alternatively we can consider the system as a random environment which governs the motion of a particle. That is, conditional on the environment given by $(Q_{k,n};(k,n)\in L)$ the probability of a particle currently at (k,n) moving to $(k+1,n+1)$ is given by $Q_{k,n}$, whereas the probability the particle moving to $(k-1,n+1)$ is given by $1-Q_{k,n}$.

In this case, letting $(M_{k,0};k\in 2\mathbb{Z};\sum M_{k,0}=1)$ be the initial distribution of a particle, $(M_{k,n},k\in 2\mathbb{Z}+n)$ gives the distribution of the particle at time n conditional on the environment given by $(Q_{k,n};(k,n)\in L)$. Particles are then sampled independently conditional on the environment. Consider N such particles. The N-dimensional process that results we will call the N-point motion of the system.

If we assume $\mathbf{E}[Q_{k,n}]=\frac{1}{2}$ and we observe only the path of a single particle

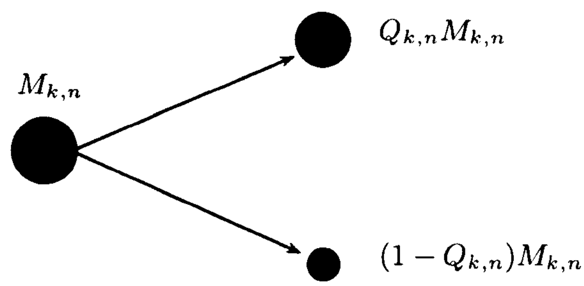

Figure 1.2: Splitting of mass

moving through the environment, we notice that this path behaves as a simple symmetric random walk. If we observe the paths of two particles they behave as independent simple symmetric random walks until the first moment that the two paths meet, at which point they move in the same direction with probability $\mathbf{E}[Q_{k,n}^2 + (1-Q_{k,n})^2]$ and they move in different directions with probability $\mathbf{E}[2Q_{k,n}(1-Q_{k,n})]$.

Taking a diffusive scaling of this system, scaling space by a multiplication factor of $\sqrt{\epsilon}$ and time by a factor of ϵ , produces an interesting limit if the distribution of $Q_{k,n}$ is also scaled, such that

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\mathbf{E}[2Q_{k,n}(1-Q_{k,n})] = \theta, \tag{1.1}$$

where θ is some positive constant.

In the limit the paths of single particles observed on their own behave as Brownian motion and the motions of pairs of particles behave as what we shall call a pair of θ-coupled Brownian motions. This is a pair of Brownian motions that move independently when apart with some interaction when they meet, such that the difference between the positions of the particles behaves as a

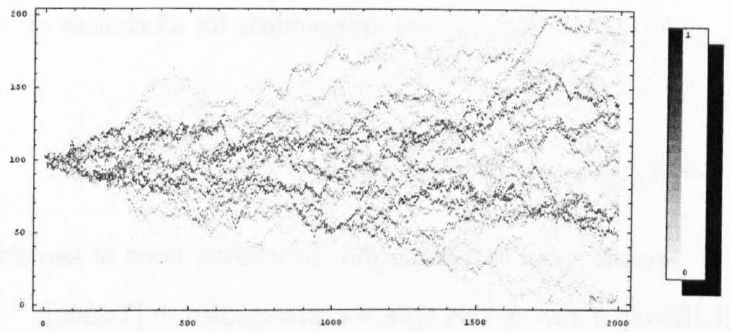

Figure 1.3: Realisation of a flow of mass

diffusion on $\mathbb{R}$ known as θ-sticky Brownian motion.

Le Jan and Raimond, [LJR04b] and Le Jan and Lemaire [LJL04], discuss the limits of such systems when

$$Q_{k,n} \sim \text{Beta}(\sqrt{\epsilon}\theta, \sqrt{\epsilon}\theta) \tag{1.2}$$

Sun and Swart, [SS06], discuss the limits of systems where

$$Q_{k,l} = \begin{cases} 1 & \text{with probability } \frac{1}{2} - \sqrt{\epsilon}\theta \\ \frac{1}{2} & \text{with probability } 2\sqrt{\epsilon}\theta \\ 0 & \text{with probability } \frac{1}{2} - \sqrt{\epsilon}\theta \end{cases} \tag{1.3}$$

One way of describing such random environments is via stochastic flows of kernels.

Definition 1. A stochastic flow of kernels on a measurable space $(E, \mathcal{E})$ is a double indexed family $(K_{s,t}; s \leq t)$ of random $E \times \mathcal{E}$ transition kernels satisfying the properties

1. $K_{s,u}(x, A) = \int_E K_{s,t}(x, dy) K_{t,u}(y, A) \qquad x \in E, A \in \mathcal{E}$ almost surely for each $s \leq t \leq u$.

2. $K_{t_1,t_2}, K_{t_2,t_3}, \ldots, K_{t_{n-1},t_n}$ are independent for all choices of $t_1 < t_2 \cdots < t_n$

3. $K_{s,t} \overset{d}{=} K_{s+h,t+h}$ for all $s \leq t$ and h.

Here '$\overset{d}{=}$' means equal in distribution. Stochastic flows of kernels are developed in [LJR04a]. Flows of this type are also studied in [Tsi04a].

We can define a stochastic flow of kernels K on the integers $\mathbb{Z}$ and indexed in $\mathbb{Z}$ from the set of weights $(Q_{k,l}; (k,l) \in L)$. For some fixed $(x,n) \in L$ let $M_{x,s} = 1$ and let $M_{k,n} = 0$ for all $k \neq x$. Then evolving $(M_{k,n}, (k,n) \in L)$, using the weights $(Q_{k,n}; (k,n) \in L)$ as described above we then define $K_{s,t}$ by

$$K_{s,t}(x,A) = \sum_{k \in A} M_{k,t} \qquad t > s, A \subset 2\mathbb{Z} + t.$$

We note that property 3 above does not hold due to the periodic nature of our current example. We need to replace it with

3'. $K_{s,t} \overset{d}{=} K_{s+h,t+h}$ for all $s \leq t$ and $h \in 2\mathbb{Z}$.

At the end of chapter 3 we discuss a flow similar to the above but in continuous time, which eliminates the problems of periodicity.

We can use the structure given by the stochastic flows of kernels to describe the random environment in the case where time and space are both continuous. Suppose we have a stochastic flow of kernels $K_{s,t}$ on $\mathbb{R}$ and indexed by $\mathbb{R}$. Then we can think of the stochastic flow $(K_{s,t}; s \leq t)$ as an evolution of mass in that $K_{s,t}(s,A)$ represents the proportion of mass which was located at x at time s, which is then located within set $A \in \mathcal{B}(\mathbb{R})$ at time t. Alternatively we can think of K as a random environment which governs the motion of a particle. Then $K_{s,t}(x,A)$ gives the conditional probability given the environment that a particle which is located at x and time s in located within the set A at time t.

We consider N-particles sampled from the flow conditionally independent of each other given the environment. We then have a Markov process in $\mathbb{R}^N$ whose transition probabilities are given by

$$P_t^N(x, A) = \mathbf{E}[K_{0,t}(x_1, A_1)K_{0,t}(x_2, A_2)\cdots K_{0,t}(x_N, A_N)]$$

for all $x = (x_1, x_2, \dots x_N) \in \mathbb{R}^N$ and $A = A_1 \times A_2 \times \cdots A_N \in \mathcal{B}(\mathbb{R}^N)$. We call the trajectories of these particles the N-point motion of the stochastic flow of kernels. In [LJR04a] they show how these N-point motions for all $N \geq 1$, give complete information about the environment. More precisely the family $((P_t^N; t \geq 0); N \geq 1)$ characterise the law of the flow of kernels K. We are led therefore to consider families of N-dimensional processes with the property that each coordinate process observed on its own behaves as a Brownian motion and each pair of coordinates behaves as a pair of θ-coupled Brownian motions.

In [LJR04a] they characterise the motion of N particles with the above properties via Dirichlet form methods. The motion they characterise corresponds to the limit of particles moving in a discrete system as described above with $Q_{k,n}$ as in (1.2).

Gawędzki and Horvai [GH04] discuss general systems with the property of pairs of particles behaving as θ-coupled Brownian motions, motivated by studying the compressible Kraichnan model of turbulent advection and taking limits in certain parameters. The following is a quote from the end of their paper.

> The main open problem, untouched by our analysis, is the construction of N-particle processes corresponding to the sticky behaviour of the two-particle dispersion. In particular it would be interesting to know whether the amount of two-particle glue is the only parameter that labels possible Lagrangian flows in the moderately

compressible phase of the Kraichnan model. The Dirichlet form approach used in [18] ([LJR04b]) in the 1-dimensional $\xi = 0$ case to tackle such questions is unavailable in the other instances, at least in its classical form, due to the lack of symmetry of the generators of the N-particle processes.

We investigate these general systems with pairs of particles behaving as θ-coupled Brownian motions and in doing so we answer the open problem above for the one dimensional case.

We begin by studying the one dimensional diffusion, sticky Brownian motion in chapter 2. We give constructions and characterisations of θ-sticky Brownian motion and the associated pair of θ-coupled Brownian motions. These characterisations will be fundamental to the rest of the thesis.

In chapter 3 we present a characterisation of any N-dimensional diffusion with the property that any single coordinate process is a Brownian motion and any pair of coordinates behave as a pair of θ-coupled Brownian motions. This diffusion corresponds to the N-point motions of possible limiting systems as above for any such i.i.d $(Q_{k,n}; (k,n) \in L)$ with the property $\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} E[2Q_{k,n}(1 - Q_{k,n})] = \theta$.

This characterisation will be given via a martingale problem. We show that the N-point motions are not specified by the two particle interactions alone. In fact there are many different N-dimensional diffusions with the property that each pair of coordinates behave as θ-coupled Brownian motions, and there is a family of parameters $(\theta(k:l); k,l \geq 1)$, where $\theta(k:l)$ in some way represents the rate that $k+l$ particles separate into k and l particles.

In chapter 4 we present a system based on taking perturbations of the Brownian web given in [FINR04]. Suppose that each sign $\xi_{k,n}$ in the construction of the system of coalescing random walks, S described above, is replaced with a stochastic process $(\xi_{k,n}(u); u \geq 0)$ such that for each $(k, n) \in L$ each process $(\xi_{k,n}(u); u \geq 0)$ is a stationary Markov chain on $\{-1, 1\}$ with unit rate of jumping between states. Assume that each process $(\xi_{k,n}(u); u \geq 0)$ is independent of all other processes $((\xi_{l,m}(u); u \geq 0); (l, m) \neq (k, n))$.

At any fixed time u, $(\xi_{k,n}(u); (k, n) \in L)$ is a system of random signs as described at the beginning of this chapter and hence we can construct a system of coalescing random walks $\mathcal{S}(u)$.

It is possible to consider $(\mathcal{S}(u); u \geq 0)$ as a Markov chain in some state space describing families of coalescing paths. For each fixed time u, $\mathcal{S}(u)$ is a system of coalescing random walks. However, if we consider two fixed times $u_1 \neq u_2$ and observe some fixed point (k, n) in both $\mathcal{S}(u_1)$ and $\mathcal{S}(u_2)$ then the probability that the arrow in $\mathcal{S}(u_1)$ at (k, n) and the arrow in $\mathcal{S}(u_2)$ at (k, n) are pointing in the same direction is $\frac{1}{2}\left(1 + e^{-2|u_2-u_1|}\right)$ whereas the probability they point in different directions is $\frac{1}{2}\left(1 - e^{-2|u_2-u_1|}\right)$. Considering a path in $\mathcal{S}(u_1)$ and a path in $\mathcal{S}(u_2)$ it is possible to see that individually they behave as simple symmetric random walks, and as a pair they behave independently when apart, and when the two paths meet they stay together for the next step with probability $\frac{1}{2}\left(1 + e^{-2|u_2-u_1|}\right)$.

Consider taking a diffusive scaling on both $\mathcal{S}(u_1)$ and $\mathcal{S}(u_2)$ simultaneously such that space is scaled by $\sqrt{\epsilon}$ and (discrete) time is scaled by ϵ. Letting $\mathcal{S}^\epsilon(u)$ be the scaled collection of paths, then $\mathcal{S}^\epsilon(u_1)$ and $\mathcal{S}^\epsilon(u_2)$ both converge to Brownian webs, as ϵ tends down to zero, in the sense of the metric space described in [FINR04], which will be discussed in chapter 4. If we consider a path in $\mathcal{S}^\epsilon(u_1)$ and a path in $\mathcal{S}^\epsilon(u_2)$, then individually the paths converge to Brownian motions and as a pair in the limit the two paths will become independent. If however we consider a path in $\mathcal{S}^\epsilon(\sqrt{\epsilon}u_1)$ and a path in $\mathcal{S}^\epsilon(\sqrt{\epsilon}u_2)$ and u_1 and u_2 are fixed such that $|u_1 - u_1| = \theta$ then $\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \frac{1}{2}\left(1 - e^{-2\sqrt{\epsilon}|u_2-u_1|}\right) = \theta$.

Comparing with (1.1) it is reasonable to suppose that the pair of paths will converge to a pair of θ-coupled Brownian motions.

More generally we construct a pair of objects $(\mathcal{W}, \mathcal{W}')$ that is the limit of the pair $(\mathcal{S}^\epsilon(\sqrt{\epsilon}u_1), \mathcal{S}^\epsilon(\sqrt{\epsilon}u_2))$, if such a limit exists. This pair has the property that $\mathcal{W}$ and $\mathcal{W}'$ are both Brownian webs and the joint distribution of a path from each web is that of a pair of θ-coupled Brownian motions. Essentially these properties characterise the pair of objects $(\mathcal{W}, \mathcal{W}')$ that we shall call a pair of θ-coupled Brownian webs.

Furthermore we go on to show that from this pair, conditioning one of the webs on the other, we derive a flow of kernels given by $K_{s,t}(x, A) = \mathbf{P}(\mathcal{W}'_{x,s}(t) \in A|\mathcal{W})$ for any $A \in \mathcal{B}(\mathbb{R})$, and where $W_{x,s}$ is the almost surely unique path in $\mathcal{W}$ started from (x, s). We will call this the erosion flow with parameter θ. We go on to find the N-point motion of the erosion flow and show that it solves the $\mathcal{A}^\theta_N$-martingale problem from chapter 3 with some particular family of parameters.

Reverting to the two system of arrows $\mathcal{S}(u_1)$ and $\mathcal{S}(u_2)$. At any particular point $(k, n) \in L$,

$$
\begin{aligned}
\mathbf{P}(\xi_{k,n}(u_2) = 1|\xi_{k,n}(u_1) = 1) = \mathbf{P}(\xi_{k,n}(u_2) &= -1|\xi_{k,n}(u_1) = -1) \\
&= \frac{1}{2}\left(1 + e^{-2|u_2-u_1|}\right)
\end{aligned}
$$

whereas

$$
\begin{aligned}
\mathbf{P}(\xi_{k,n}(u_2) = -1|\xi_{k,n}(u_1) = 1) = \mathbf{P}(\xi_{k,n}(u_2) &= 1|\xi_{k,n}(u_1) = -1) \\
&= \frac{1}{2}\left(1 - e^{-2|u_2-u_1|}\right).
\end{aligned}
$$

Each pair of arrows $(\xi_{k,n}(u_1), \xi_{k,n}(u_2))$ is mutually independent of every other pair and $\mathbf{P}(\xi_{k,n}(u_1) = 1) = \mathbf{P}(\xi_{k,n}(u_1) = -1) = \frac{1}{2}$. Thus the conditioned system of arrows $\mathcal{S}(u_2)|\mathcal{S}(u_1)$ can be seen to be equivalent to a system of weighted arrows with random weights given by

$$Q_{n,k} = \begin{cases} \frac{1}{2}\left(1 - e^{-2|u_2-u_1|}\right) & \text{with probability } \frac{1}{2} \\ \frac{1}{2}\left(1 + e^{-2|u_2-u_1|}\right) & \text{with probability } \frac{1}{2} \end{cases}$$

which, under the same diffusive scaling as above, is equivalent to

$$Q_{n,k} = \begin{cases} \theta\sqrt{\epsilon} & \text{with probability } \frac{1}{2} \\ 1 - \theta\sqrt{\epsilon} & \text{with probability } \frac{1}{2} \end{cases} \tag{1.4}$$

for small ϵ.

Staying with the discrete systems of arrows on $L = \{(k,n) \in \mathbb{Z}^2; k + n \text{ is even}\}$, it can be observed that they have very natural dual objects that can be achieved by placing arrows on $L' = \{(k,n) \in \mathbb{Z}^2; k + n \text{ is odd}\}$. That is if there is an arrow going from $(k,n) \in L$ to $(k \pm 1, n+1)$ then we place an arrow from $(k, n+1) \in L'$ to $(k \mp 1, n)$, see figure 1.4. Similarly, if we are using weighted arrows, then we let the arrow from $(k, n+1) \in L'$ to $(k \mp 1, n)$ have the same weight as the arrow from $(k,n) \in L$ to $(k \pm 1, n+1)$. Joining the arrows starting from points L' we have a system of paths, $\mathcal{S}'$, running backwards in time and it is easy to see that the distribution of $\mathcal{S}'$ reflected in the vertical (space) axis is equal to the distribution of $\mathcal{S}$. Arratia, [Arr79], Toth and Werner [TW98] and Fontes, Isopi, Newman, and Ravishankar, [FINR04], [FN06] have studied the limit of objects $\mathcal{S}$ together with their dual objects $\mathcal{S}'$. In this case paths in $\mathcal{S}$ and paths in $\mathcal{S}'$ do not cross. In the limit, observing one path in each object, the first path we observe behaves as a Brownian motion and the second path behaves as an independent Brownian motion which is reflected, in

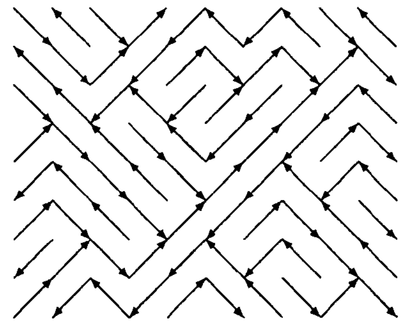

Figure 1.4: Coalescing random walks with dual

the sense of Skorokhod reflection, off the first path. See [STW00] and [War05] for results of this nature and other results concerning forwards and backwards dual flows.

The relationship between forward and backward paths in the flow is related to a duality relationship of one dimension processes. This duality is of the "H -dual" type found in [Lig85], where X and $\hat{X}$ are H-dual if and only if

$$\mathbf{E}_x[H(X(t),y)] = \mathbf{E}_y[H(x,\hat{X}(t))].$$

We are interested in cases where $H(x,y) = \mathbf{1}_{\{x<y\}}$. Examples of processes with this particular type of duality can be found in [WW04] and [Wat01]. In chapter 5 we show how sticky Brownian motion has this duality relationship with a new process which we call alternating Brownian motion, and we discuss how this relates to the forward and backward systems of the type with the weighted arrows.

We close this introduction by considering figure 1.5 below. The graphs are produced from discrete approximations using (1.2), (1.3) and (1.4). Then for each time $n \in \mathbb{Z}$ the cumulative mass function is calculated $C(k,n) = \sum_{j=-\infty}^{k} M_{j,n}$. We then plot the set $\{C(k,n) : k \in Z\}$ against n. In each of these plots we start at time $n = 0$ with a mass of 1 at zero, $M_{0,0} = 1$ and

$M_{k,0} = 0$ for all $k \neq 0$. Thus the total mass at each time $n \geq 0$ is always 1.

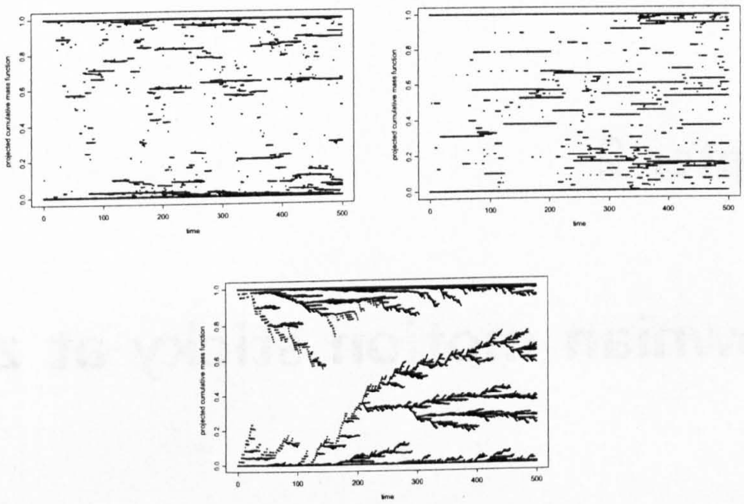

Figure 1.5: Projected cumulative mass against time with (1.2), (1.3) and (1.4) respectively.

At any time n the length of the gaps in the y-axis represent the size of the masses that exist at time n. We see in each figure at the start there is one mass of size one. In the second case we can see that masses can only split into 2 halves and in the final diagram the single mass at the start becomes smaller incrementally and new 'clumps' of mass emerge. We note that the 2-point motions are identical in each of the examples, thus the dramatically different structures evident in these figures illustrates how significant the full family of N-point motions are in determining the qualitative properties of the associated flow.

Chapter 2

Brownian motion sticky at zero

In this chapter we study the one dimensional diffusion known as sticky Brownian motion. Sticky Brownian motion is a real-valued continuous strong Markov process, $(X(t); t \geq 0)$, defined on some filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, that behaves as Brownian motion away from zero and has the property that the time spent at zero, $\int_0^t \mathbf{1}_{\{X(s)=0\}} ds$, has positive probability of being greater than 0. The expected amount of time spent at zero depends on a non-negative parameter θ, which in some sense gives the rate at which excursions leave zero.

Definition 2. For any choice of parameter $\theta > 0$ a sticky Brownian motion or a θ-sticky Brownian motion is a diffusion of natural scale and speed measure m, given by $m(A) = 2$ Leb$(A) + \frac{2}{\theta}\mathbf{1}_{\{0\in A\}}$.

The scale and speed of a diffusion in dimension 1 specify its law uniquely. To clarify the above definition, the diffusion being of natural scale implies that it is a (local) martingale. Then there is a well known result that tells us that any continuous martingale can be represented as a time change of a Brownian motion, see for example [RY99]. The speed measure gives this time change, via the following proposition.

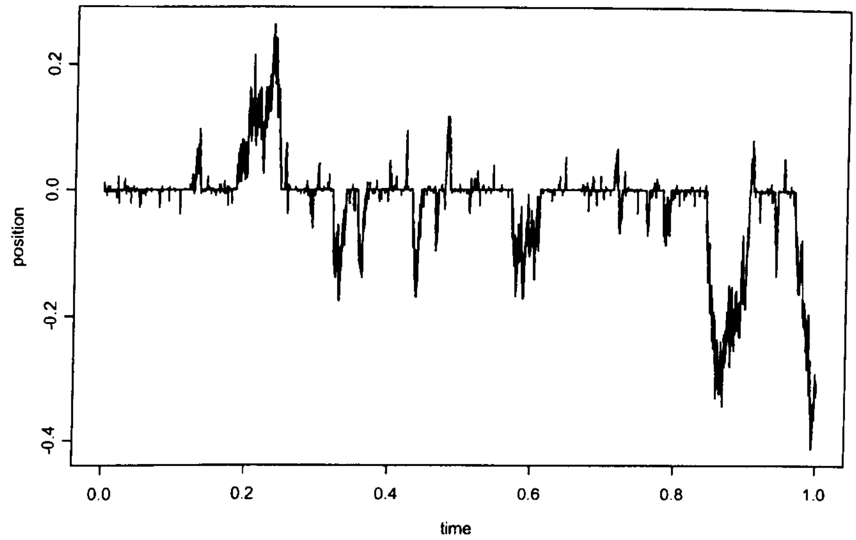

Figure 2.1: Sticky Brownian motion

Proposition 3. *For any Brownian motion B let $L_t(B)$ be the local time at zero of B, as given by Tanaka's formula. A random process $(X(t); t \geq 0)$ is a sticky Brownian motion if and only if there exists a Brownian motion B, such that $X(t) = B(A(t))$, where $A(t) = \inf\{u; u + \frac{1}{\theta}L_u(B) > t\}$.*

Note $A(t) < t$ for every $t \geq 0$, and so this time change can be interpreted as slowing the Brownian motion down when it is at zero. As θ tends to infinity the time change, A, becomes the identity and the process X leaves the origin instantaneously: it is simply a Brownian motion. On the other hand as θ tends down to zero the time change becomes $A(t) = t \wedge T_0$, where $T_0 = \inf\{t \geq 0; B(t) = 0\}$, therefore X becomes a Brownian motion absorbed at 0 and the rate X leaves the origin is 0.

2.1 Properties of sticky Brownian motion

Proposition 4. *Let X be a θ-sticky Brownian motion. Define the local time of X at zero, $L_t(X)$, via Tanaka's formula:*

$$L_t(X) = |X(t)| - |X(0)| + \int_0^t \operatorname{sgn}(X(s))dX(s)$$

where $\operatorname{sgn}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\leq 0\}}$. *Then*

$$L_t(X) = \theta \int_0^t \mathbf{1}_{\{X(s)=0\}}ds$$

and the quadratic variation of X is given by

$$\langle X\rangle_t = \int_0^t \mathbf{1}_{\{X(s)\neq 0\}}ds.$$

Proof. Let A be defined as in Proposition 3, thus $A_t = \inf\{u : u + \frac{1}{\theta}L_u(B) > t\}$ for some Brownian motion B and then $X(t) = B(A_t)$. Let $\alpha_t = t + \frac{1}{\theta}L_t(B) > t$ so that A is the right continuous inverse of α. A is strictly increasing and continuous and $A_\infty = \infty$, thus, by theory of continuous time changes, see [RY99], $X(t)$ is a continuous local martingale with respect to its natural filtration and $\langle X\rangle_t = \langle B\rangle_{A_t}$. The quadratic variation of Brownian motion is $\langle B\rangle_t = t = \int_0^t \mathbf{1}_{\{B(s)\neq 0\}}ds$, thus

$$\langle X\rangle_t = A_t = \int_0^{A_t} \mathbf{1}_{\{B(s)\neq 0\}}(ds + \frac{1}{\theta}dL_s(B)) = \int_0^t \mathbf{1}_{\{X(s)\neq 0\}}ds. \tag{2.1}$$

Tanaka's formula gives us that $|B(t)| = |B(0)| + \int_0^t \operatorname{sgn}B(s)dBs + L_t(B)$. It follows that $|X(t)| = |X(0)| + \int_0^{A_t} \operatorname{sgn}B(s)dB(s) + L_{A_t}(B)$. From [RY99] we have $\int_0^{A_t} \operatorname{sgn}B(s)dB(s) = \int_0^t \operatorname{sgn}B(A_s)dB(A_s) = \int_0^t \operatorname{sgn}X(s)dX(s)$. Finally $\alpha_t = t + \frac{1}{\theta}L_t(B)$ implies $t = A_t + \frac{1}{\theta}L_{A_t}(B)$, which gives us that

$L_{A_t}(B) = \theta(t - A_t) = \theta \int_0^t \mathbf{1}_{\{X(s)=0\}} ds.$ □

The following proposition is a result found in [Yor89]

Proposition 5. *The amount of time spent at zero,* $\int_0^t \mathbf{1}_{\{X(s)=0\}} ds$ *by a* θ*-sticky Brownian motion* X*, started at zero, is equal in distribution to*

$$\frac{|N|}{\theta}\sqrt{t + \frac{N^2}{4\theta^4}} - \frac{N^2}{2\theta^2}$$

where N *is a centred gaussian variable, with variance 1.*

Proof. It is well known that the local time of standard Brownian motion started at zero, $L_t(B)$, satisfies the property that for fixed t, $L_t(B) \stackrel{d}{=} |B(t)| \stackrel{d}{=} \sqrt{t}|N|$. Let $(\alpha_t; t \geq 0)$ be as in the proof of proposition 4. $\alpha_t = t + \frac{1}{\theta}L_t(B)$, thus $\alpha_t \stackrel{d}{=} t + \frac{\sqrt{t}}{\theta}|N|$. Then

$$\begin{aligned}(A_t < u) &= (\alpha_u > t)\\ &\stackrel{d}{=} \left(u + \frac{\sqrt{u}|N|}{\theta} > t\right)\\ &\stackrel{d}{=} \left(u > \left(\sqrt{t + \frac{N^2}{4\theta^2}} - \frac{|N|}{2\theta}\right)^2\right).\end{aligned}$$

Finally, by (2.1), we have $\int_0^t \mathbf{1}_{\{X(s)=0\}} ds = t - A_t$ and the desired result follows from this. □

2.2 Sticky Brownian motion as a solution to a martingale problem

A martingale problem in the most general terms is a set of functionals of some random process X that we require to be martingales relative to the natural filtration of X. We say that X is a solution to the given martingale problem if the specified functionals are all martingales, relative to the natural filtration of X, under the law of X. We say that the law of X is uniquely specified by the martingale problem if any solution has a law equal to that of X. If there exists a solution to a martingale problem and the law of the solution is uniquely specified, then the martingale problem is said to be well posed. Lévy's characterisation of Brownian motion is an example of such a martingale problem. See [SV79] for a thorough account of multidimensional martingale problems. We will use martingale problems within this thesis on a number of occasions and the following proposition will be our first example:

Proposition 6. *There exists a random process $(X(t); t \geq 0)$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, starting at $x \in \mathbb{R}$, such that X is a continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and so are the processes*

$$X(t)^2 - \int_0^t \mathbf{1}_{\{X(s)\neq 0\}} ds, \qquad t \geq 0 \tag{2.2}$$

$$|X(t)| - \theta \int_0^t \mathbf{1}_{\{X(s)=0\}} ds, \qquad t \geq 0. \tag{2.3}$$

Moreover the law of the process is uniquely specified and is equal to the law of a θ-sticky Brownian motion.

Proof. Existence follows from Proposition 4. To prove uniqueness, define a time change $(A_t; t \geq 0)$ by $A_t = \int_0^t \mathbf{1}_{\{X(s)\neq 0\}} ds$ and let $(\alpha_t; t \geq 0)$ be the right continuous inverse of A, that is $\alpha_t = \inf\{u; A_u > t\}$. Clearly $\langle X \rangle_t = A_t$. Thus by the Dambis,Dubins-Schwartz theorem, see [RY99], under the assumption that A is strictly increasing and $A_\infty = \infty$, $B(t) = (X(\alpha_t); t \geq 0)$ is a Brownian motion.

$$|X(\alpha_t)| - \theta \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=0\}} ds, \qquad t \geq 0$$

being a continuous local martingale implies that $L_t(B) = \theta \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=0\}} ds$.

Thus

$$\begin{aligned}\alpha_t &= \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=0\}} ds + \int_0^{\alpha_t} \mathbf{1}_{\{X(s)\neq 0\}} ds \\ &= \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=0\}} ds + t \\ &= t + \frac{1}{\theta} L_t(B).\end{aligned}$$

The assumption that $A_\infty = \infty$ almost surely can by verified by assuming that $\mathbf{P}(A) > 0$ where $A = \{\omega : A_\infty(\omega) < \infty\}$. As $\langle X \rangle_\infty = A_\infty$ the limit $X(\infty)(\omega)$ exists for all $\omega \in A$, see [RW00]. Also, as $\left\langle |X| - \theta \int_0^\cdot \mathbf{1}_{\{X(s)=0\}} ds \right\rangle_\infty = A_\infty$, the limit $X(\infty)(\omega) - \theta \int_0^\infty \mathbf{1}_{\{X(s)(\omega)=0\}} ds$ exists for all $\omega \in A$, but for such ω, $\int_0^\infty \mathbf{1}_{\{X(s)(\omega)=0\}} ds = \infty$, thus we have a contradiction. Similarly if $\langle X(\omega) \rangle_t - \langle X(\omega) \rangle_s = 0$ for some $s < t$ then $X(u)(\omega) = X(s)(\omega)$ for all $s \le u \le t$. Which in turn implies $L_t(X(\omega)) - L_s(X(\omega)) = 0$, which incurs a contradiction. Thus A is strictly increasing almost surely.

We have that the time change A can be written in terms of the Brownian motion B, since $A_t = \inf\{u : u + \frac{1}{\theta} L_u^0(B) > t\}$, so that $X(t) = B(A_t)$. By Proposition 3, X is a θ-sticky Brownian motion. Thus we have shown that any continuous local martingale satisfying (2.2) and (2.3) is a θ-sticky Brownian motion, which proves the uniqueness part of the proposition. □

2.3 Sticky Brownian motion as a solution to an S.D.E.

Closely related to the martingale characterisation of sticky Brownian motion is the idea of a process being a solution to a set of stochastic differential equations. A continuous local martingale can be represented as a time change of Brownian motion as discussed above. A continuous local martingale can also be represented as a stochastic integral with respect to a Brownian motion. If

X is sticky Brownian motion started at x, then X satisfies the following S.D.E.

$$X(t) = x + \int_0^t \mathbf{1}_{\{X(s)\neq 0\}} dB(s). \tag{2.4}$$

This S.D.E. is not enough to specify the law of the process, indeed there is no θ in the equation. On the other hand, consider the following S.D.E. for a non-negative process Y. For $\theta > 0$, $y \geq 0$ and some Brownian motion W,

$$Y(t) = y + \int_0^t \mathbf{1}_{\{Y(s)>0\}} dW(s) + \theta \int_0^t \mathbf{1}_{\{Y(s)=0\}} ds. \tag{2.5}$$

Proposition 7. *There exists a process Y which satisfies* (2.5) *and the law of Y is uniquely specified and equal to the law of $|X|$, where X is a θ-sticky Brownian motion started from x, with $y = |x|$.*

Proof. (2.5) implies that $\langle Y \rangle_t = \int_0^t \mathbf{1}_{\{Y(s)>0\}} ds$. For existence, Proposition 6 implies there exists a process Y such that $(Y(t) - \theta \int_0^t \mathbf{1}_{\{Y(s)=0\}} ds; t \geq 0)$ is a martingale. Then, by the martingale representation theorem, see proposition (3.8) of [RY99] we have

$$\begin{aligned} Y(t) - \theta \int_0^t \mathbf{1}_{\{Y(s)=0\}} ds &= Y(0) + \int_0^t (\mathbf{1}_{\{Y(s)>0\}})^{1/2} dW(s) \\ &= y + \int_0^t \mathbf{1}_{\{Y(s)>0\}} dW(s) \end{aligned}$$

for some Brownian motion W.

For uniqueness, let $A_t = \int_0^t \mathbf{1}_{\{Y(s)>0\}} ds$ and $\alpha_t = \inf\{u \geq 0 : A_u > t\}$. By similar techniques to those given in the proof of Proposition 6, we can show that A is strictly increasing and $A_\infty = \infty$. Applying the time change α to (2.5) it is possible to see that the law of $Y(\alpha_.)$ is that of a reflecting Brownian motion and that $L_t(Y(\alpha_.)) = \theta \int_0^{\alpha_t} \mathbf{1}_{\{Y(s)=0\}} ds$. Thus, by similar methods as in the proof of Proposition 6, $\alpha_t = t + \frac{1}{2\theta} L_t(Y(\alpha_.))$ and hence the law of Y is uniquely specified. □

Proposition 8. *There exists a local martingale X such that $|X|$ satisfies* (2.5). *Moreover the law of X is that of a θ-sticky Brownian motion.*

Proof. If X is the process as specified by Proposition 6 then X is a martingale and we have seen in the proof of the above proposition that $|X|$ satisfies (2.5). Then letting X be any martingale such that $|X|$ satisfies (2.5), $\langle X\rangle_t = \int_0^t \mathbf{1}_{\{X(s)\neq 0\}} ds$ and $L_t(X) = \theta \int_0^t \mathbf{1}_{\{X(s)=0\}} ds$. Thus as X itself is also a martingale, Proposition 6 implies that X is a θ-sticky Brownian motion. □

If X a θ-sticky Brownian motion we call the process $Y = |X|$ a one-sided sticky Brownian motion with parameter θ. Also known as a slowly reflecting Brownian motion the one sided process is also often referred to as simply a sticky Brownian motion. To avoid confusion we will use the term one-sided sticky Brownian motion and reserve sticky Brownian motion for the two-sided case.

In [War97], Warren studies one-sided sticky Brownian motion as a solution to the S.D.E. (2.5). The paper uses a duality relationship between Brownian paths and real trees. There is no pathwise solution to (2.5), which is shown in [Chi97], indeed the conditional distribution of $Y(t)$ given the path of the driving Brownian motion is found in [War97]. We quote the following theorem from this paper.

Theorem 9. *Suppose that Y is a (one-sided) sticky Brownian motion starting from zero, and that W is the driving Wiener process, in* (2.5), *also starting from zero, Letting $I(t) = -\inf_{s\leq t}(W(s))$, the conditional law of Y given W satisfies*

$$\mathbf{P}(Y(t) \leq y|\sigma(W)) = \exp(-2\theta(W(t) + I(t) - y))$$

almost surely for $y \in [0, W(t) + I(t)]$.

Corollary 10. *For a one sided sticky Brownian motion,* Y*, started at* $x \geq 0$*, with parameter* θ*,*

$$\begin{aligned}
&\mathbf{P}(Y(t) \leq y) \\
&\quad = \mathbf{E}\left[\exp(-2\theta(W(t) + I(t) - y))\mathbf{1}_{\{y \leq W(t)+I(t)\}}\mathbf{1}_{\{x \leq I(t)\}}\right] \\
&\qquad + \mathbf{E}\left[\mathbf{1}_{\{y > W(t)+I(t)\}}\mathbf{1}_{\{x \leq I(t)\}}\right] \\
&\qquad + \mathbf{E}\left[\mathbf{1}_{\{x+W(t) \leq y\}}\mathbf{1}_{\{x > I(t)\}}\right].
\end{aligned}$$

Proof. For $|Y|$ started at zero we have, from Theorem 9,

$$\begin{aligned}
\mathbf{P}(Y(t) \leq y) = \mathbf{E}&\left[\exp(-2\theta(W(t) + I(t) - y))\mathbf{1}_{\{y \leq W(t)+I(t)\}}\right] \\
&+ \mathbf{E}\left[\mathbf{1}_{\{y > W(t)+I(t)\}}\right].
\end{aligned} \tag{2.6}$$

Now consider $x > 0$. Y behaves as Brownian motion away from zero thus, letting $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$,

$$\mathbf{P}(Y(t) \leq y \;\&\; \tau_0 > t) = \mathbf{E}\left[\mathbf{1}_{\{x+W(t) \leq y\}}\mathbf{1}_{\{x > I(t)\}}\right]. \tag{2.7}$$

Let $Y(t) = x + W(t)$ for all $t \in [0, \tau_0]$. Then by the strong Markov property, combined with (2.6), and as $\inf_{\tau_0 \leq s \leq t} W(s) = \inf_{0 \leq s \leq t} W(s)$ for all $t \geq \tau_0$,we have

$$\begin{aligned}
&\mathbf{P}(Y(t) \leq y \;\&\; \tau_0 \leq t) \\
&\quad = \mathbf{E}\left[\exp(-2\theta(W(t) + I(t) - y))\mathbf{1}_{\{y \leq W(t)+I(t)\}}\mathbf{1}_{\{x \leq I(t)\}}\right] \\
&\qquad + \mathbf{E}\left[\mathbf{1}_{\{y > W(t)+I(t)\}}\mathbf{1}_{\{x \leq I(t)\}}\right].
\end{aligned} \tag{2.8}$$

(2.7) and (2.8) together give the desired result. $\square$

2.4 Resolvent and transition probabilities for sticky Brownian motion

In this section we calculate the transition kernels for sticky Brownian motion. If X is a time-homogenous Markov process, then $\mathbf{P}_x(A)$ denotes the probability of an event A when X starts from x. Let $p_t(x, dy)$ be the transition probability kernel for X, defined by $\int_A p_t(x, dy) = \mathbf{P}_x(X(t) \in A)$, for any $A \in \mathcal{B}(\mathbb{R})$. We calculate the transition probabilities for sticky Brownian motion by first finding the resolvent kernel $p_\lambda(x, dy)$, given by $\int_0^\infty e^{-\lambda t} p_t(x, dy) dt$. The resolvent kernel for the one-sided sticky Brownian is calculated in [War97] and in [Kni81], where the resolvent kernel is calculated for general processes that behave as Brownian motion on $(0, \infty)$ and have some boundary behaviour at 0. Here we give the calculation for two-sided sticky Brownian motion. The results can also be found in [BS02].

We note that in the following we use $\delta_0(dx)$ to represent the dirac measure, which has the property $\int_A \delta_0(dx) = \mathbf{1}_{\{0 \in A\}}$.

Proposition 11. *For all $x \in \mathbb{R}$, the resolvent kernel of two-sided sticky Brownian motion, $p_\lambda(x, dy)$, is given by*

$$p_\lambda(x, dy) = \frac{e^{-\gamma|x-y|}}{\gamma} dy - \frac{e^{-\gamma(|x|+|y|)}}{2\theta + \gamma} dy + \frac{e^{-\gamma|x|}}{\theta\gamma + \lambda} \delta_0(dy), \tag{2.9}$$

where $\gamma\sqrt{2\lambda}$.

Proof. Let $p_t(x, dy)$ be the transition probability kernel for sticky Brownian motion. We decompose, using the strong Markov property, by the first visit to zero:

$$p_t(x, dy) = \int_0^t f_x(s) p_{t-s}(0, dy) ds + \left\{ \frac{1}{\sqrt{2\pi t}} e^{\frac{(x-y)^2}{2t}} - \frac{1}{\sqrt{2\pi t}} e^{\frac{(|x|+|y|)^2}{2t}} \right\} dy,$$

where $f_x(ds)$ is the probability distribution of τ, the time of the first visit to zero of a Brownian motion starting from x. We note that the term in the curly brackets is the kernel for killed Brownian motion. It is equal to $\mathbf{P}_x(B(t) \in dy$ and $B(s) \neq 0$ for all $s \in [0,t])$ and is found via the reflection principle. The equivalent decomposition for the resolvent kernel follows from the above.

$$p_\lambda(x,dy) = \int_0^\infty e^{-\lambda t} p_t(x,dy)dt$$
$$= \int_0^\infty e^{-\lambda t} \left[\int_0^t f_x(s) p_{t-s}(0,dy)ds + \left\{ \frac{1}{\sqrt{2\pi t}} e^{\frac{(x-y)^2}{2t}} - \frac{1}{\sqrt{2\pi t}} e^{\frac{(|x|+|y|)^2}{2t}} \right\} dy \right] dt.$$

Letting $\gamma^2 = 2\lambda$ we have

$$p_\lambda(x,dy) = \gamma^{-1}(e^{-\gamma|x-y|} - e^{-\gamma(|x|+|y|)})dy + \int_0^\infty e^{-\lambda t} \int_0^t f_x(s) p_{t-s}(0,dy)ds\, dt. \tag{2.10}$$

Consider the last term on the right

$$\int_0^\infty e^{-\lambda t} \int_0^t f_x(s) p_{t-s}(0,dy)dsdt = \int_0^\infty f_x(s) \int_s^\infty e^{-\lambda t} p_{t-s}(0,dy)dt\, ds$$

then a change variables with $u = t - s$ gives us

$$\begin{aligned} \int_0^\infty f_x(s) \int_s^\infty e^{-\lambda t} p_{t-s}(0,dy)dtds &= \int_0^\infty f_x(s) \int_0^\infty e^{-\lambda(u+s)} p_u(0,dy)du\, ds \\ &= p_\lambda(0,dy) \int_0^\infty e^{-\lambda s} f_x(s)ds. \end{aligned} \tag{2.11}$$

Now we have

$$F_x(t) = \int_0^t f_x(s)ds = \mathbf{P}_x(\tau \le t) = 2\mathbf{P}_0(B(t) \ge |x|) = 2\int_{|x|}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{t}} dy,$$

and using integration by parts gives

$$\begin{aligned}\int_0^\infty e^{-\lambda s} f_x(s)ds &= 2\lambda \int_0^\infty \int_{|x|}^\infty \frac{1}{\sqrt{2\pi t}} e^{\frac{-y^2}{t}} dydt \\ &= 2\lambda\gamma^{-2} e^{-\gamma|x|} \\ &= e^{-\gamma|x|}.\end{aligned}$$

Putting this together with (2.10) and (2.11) gives

$$p_\lambda(x, dy) = \gamma^{-1}\left(e^{-\gamma|x-y|} - e^{-\gamma(|x|+|y|)}\right) + e^{-\gamma|x|} p_\lambda(0, dy). \tag{2.12}$$

We can find $p_\lambda(0, dy)$, by adapting a technique from [War97], which uses the fact that sticky Brownian motion started from zero is a time change of standard Brownian motion. We take $\{X(t); t \geq 0\}$ to be a two sided sticky Brownian motion starting at zero and we define

$$A_t^+ = \int_0^t \mathbf{1}_{\{X(s)\neq 0\}} ds; \qquad \alpha_t^+ = \inf\{u : A_u^+ > t\} \tag{2.13}$$

$$A_t^0 = \int_0^t \mathbf{1}_{\{X(s)=0\}} ds; \qquad \alpha_t^0 = \inf\{u : A_u^0 > t\}. \tag{2.14}$$

We take two independent exponential random variables, T_1 and T_2, both independent of X, and both with mean λ^{-1}. Let

$$T = \alpha_{T_1}^0 \wedge \alpha_{T_2}^+.$$

This is also exponentially distributed with mean λ^{-1}. Now $X_T = 0$ if and only if $\alpha_{T_1}^0 < \alpha_{T_2}^+$ or equivalently $T_1 < A^0_{\alpha_{T_2}^+}$. We have seen in Proposition 6 that, there exists a Brownian motion B, such that $B(t) = X(\alpha_t^+)$ and

$$L_t(B) = \theta \int_0^{\alpha_t^+} \mathbf{1}_{\{X(s)=0\}} ds = \theta A^0_{\alpha_t^+}.$$

Thus $\theta A^0_{\alpha^+_{T_2}} = L_{T_2}(B)$, which is exponentially distributed with mean γ^{-1}. T_1 is clearly independent of $\theta A^0_{\alpha^+_{T_2}}$ so the probability that $X_T = 0$ is equal to the probability that an exponential random variable of mean $(\theta\gamma)^{-1}$ is greater than an independent exponential random variable of mean λ^{-1}. Hence

$$\mathbf{P}(X_T = 0) = \int_0^\infty \lambda e^{-\lambda t} e^{-\theta\gamma t} dt = \frac{\lambda}{\lambda + \theta\gamma}.$$

Next we find the density $\mathbf{P}(X_T \in dy)$, in the case when $X_T \neq 0$. It can be seen (by factorising the transition probabilities, for example) that $B(T_2)$ and $L_{T_2}(B)$ are independent. Thus $X(\alpha^+_{T_2})$ is independent of the event $\left(T_1 > A^0_{\alpha^+_{T_2}}\right)$, and hence

$$\begin{aligned}
\mathbf{P}(X_T \in dy) &= \mathbf{P}(X_{\alpha^+_{T_2}} \in dy)\mathbf{P}(\alpha^0_{T_1} > \alpha^+_{T_2}) \\
&= \left(\lambda\gamma^{-1} e^{-\gamma|y|} dy\right)\left(\frac{\theta\gamma}{\lambda + \theta\gamma}\right) \\
&= \frac{\theta\lambda}{\lambda + \theta\gamma} e^{-\gamma|y|} dy.
\end{aligned}$$

It is easy to show that the resolvent kernel of X is equal to $\frac{1}{\lambda}\mathbf{P}(X_T \in dy)$, where T is an exponential random variable of mean λ^{-1} independent of X. Thus

$$p_\lambda(0, dy) = \frac{\theta}{\theta\gamma + \lambda} e^{-\gamma|y|} dy + \frac{1}{\theta\gamma + \lambda}\delta_0(dy).$$

This together with (2.12) gives

$$p_\lambda(x, y) = \frac{e^{-\gamma|x-y|}}{\gamma} dy - \frac{e^{-\gamma(|x|+|y|)}}{2\theta + \gamma} dy + \frac{e^{-\gamma|x|}}{\theta\gamma + \lambda}\delta_0(dy).$$

□

Using tables we can invert this Laplace transform to give the transition kernel

$$\begin{aligned} p_t(x,dy) =& \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}dy - \frac{1}{\sqrt{2\pi t}}e^{-\frac{(|x|+|y|)^2}{2t}}dy \\ &+ \theta e^{2\theta(|x|+|y|)}e^{2\theta^2 t}\text{erfc}\left(\theta\sqrt{2t}+\frac{|x|+|y|}{\sqrt{2t}}\right)dy \\ &+ \delta_0(dy)e^{2\theta(|x|)}e^{2\theta^2 t}\text{erfc}\left(\theta\sqrt{2t}+\frac{|x|}{\sqrt{2t}}\right). \end{aligned} \tag{2.15}$$

We note that we are using the convention of $\text{erfc}(x) = \int_x^\infty \frac{2}{\sqrt{\pi}}e^{-y^2}dy$.

Remark 12. Up to (2.12) we had not used the fact that we were looking at sticky Brownian motion. We had only used the fact that the process is moving as Brownian motion away from zero. Thus we can use (2.12) to find the resolvent kernels for some other processes.

Brownian motion on $[0,\infty)$ with reflection at 0:

$$p_\lambda(0,dy) = 2\gamma^{-1}e^{-\gamma y}dy \qquad y \geq 0$$

$$p_\lambda(x,dy) = \gamma^{-1}\left(e^{-\gamma|x-y|}+e^{-\gamma(x+y)}\right)dy \qquad x,y\geq 0.$$

One sided sticky Brownian motion:

$$p_\lambda(0,dy) = \frac{2\theta}{\theta\gamma+\lambda}e^{-\gamma y}dy + \frac{1}{\theta\gamma+\lambda}\delta_0(dy) \qquad y\geq 0$$

$$\begin{aligned} p_\lambda(x,dy) &= \gamma^{-1}\left(e^{-\gamma|x-y|}-e^{-\gamma(x+y)}\right)dy \\ &\quad + \frac{2\theta}{\theta\gamma+\lambda}e^{-\gamma(x+y)}dy + \frac{e^{-\gamma x}}{\theta\gamma+\lambda}\delta_0(dy) \\ &= \gamma^{-1}\left(e^{-\gamma|x-y|}+\frac{\theta\gamma-\lambda}{\theta\gamma+\lambda}e^{-\gamma(x+y)}\right)dy \\ &\quad + \frac{e^{-\gamma x}}{\theta\gamma+\lambda}\delta_0(dy) \qquad x,y\geq 0. \end{aligned}$$

Skew Brownian motion (with parameter $\alpha \in [0,1]$):

$$p_\lambda(0,dy) = \begin{cases} 2\alpha\gamma^{-1}e^{-\gamma|y|}dy & y \geq 0 \\ 2(1-\alpha)\gamma^{-1}e^{-\gamma|y|}dy & y < 0 \end{cases}$$

$$p_\lambda(x,dy) = \gamma^{-1}\left(e^{-\gamma|x-y|} + (2\alpha-1)\ \mathrm{sgn}(y)e^{-\gamma(|x|+|y|)}\right)dy.$$

Lemma 13. *An invariant measure, π, for θ-sticky Brownian motion is given by*

$$\pi(dx) = \theta dx + \delta_0(dx). \tag{2.16}$$

Proof. We have to show that (2.16) satisfies

$$\int_{-\infty}^{\infty} \pi(dx)p_t(x,dy) = \pi(dy),$$

alternatively it is equivalent to show

$$\int_{-\infty}^{\infty} \pi(dx)p_\lambda(x,dy) = \frac{1}{\lambda}\pi(dy).$$

$$\begin{aligned}&\int_{-\infty}^{\infty} \pi(dx)p_\lambda(x,dy) \\ &= \int_{-\infty}^{\infty}(\theta dx + \delta_0(dx))\left(\frac{e^{-\gamma|x-y|}}{\gamma}dy - \frac{e^{-\gamma(|x|+|y|)}}{2\theta+\gamma}dy + \frac{e^{-\gamma|x|}}{\theta\gamma+\lambda}\delta_0(dy)\right).\end{aligned}$$

This can be integrated using $\int_{-\infty}^{\infty} e^{-\gamma|x|}dx = 2\int_0^{\infty} e^{-\gamma x}dx = \frac{2}{\gamma}$ and a change of variables for the remaining terms. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \pi(dx)p_\lambda(x,dy) &= \frac{2\theta}{\gamma^2}dy - \frac{2\theta e^{-\gamma|y|}}{\gamma(2\theta+\gamma)}dy + \frac{2\theta}{\gamma(\theta\gamma+\lambda)}\delta_0(dy) \\ &\quad + \frac{e^{-\gamma|y|}}{\gamma}dy - \frac{e^{-\gamma|y|}}{2\theta+\gamma}dy + \frac{1}{\theta\gamma+\lambda}\delta_0(dy) \\ &= \frac{\theta}{\lambda}dy + \frac{1}{\lambda}\delta_0(dy).\end{aligned}$$

□

Lemma 14. *Two sided sticky Brownian motion is reversible.*

Proof. A random process is reversible if and only if its transition probability kernel, $p_t(x,dy)$, satisfies the equation of detailed balance:

$$\pi(dx)p_t(x,dy) = \pi(dy)p_t(y,dx). \tag{2.17}$$

This is satisfied if and only if

$$\pi(dx)p_\lambda(x,dy) = \pi(dy)p_\lambda(y,dx). \tag{2.18}$$

In fact it is known that any one dimensional diffusion is reversible, but it is good check of the formulas to show that (2.18) holds.

$$\begin{aligned}\pi(dx)p_\lambda(x,dy) = &\frac{\theta e^{-\gamma|x-y|}}{\gamma}\,dx\,dy - \frac{\theta e^{-\gamma(|x|+|y|)}}{2\theta+\gamma}\,dx\,dy + \frac{\theta e^{-\gamma|x|}}{\theta\gamma+\lambda}\,dx\,\delta_0(dy) \\ &+ \frac{e^{-\gamma|x-y|}}{\gamma}\,dy\,\delta_0(dx) - \frac{e^{-\gamma(|x|+|y|)}}{2\theta+\gamma}\,dy\,\delta_0(dx) \\ &+ \frac{e^{-\gamma|x|}}{\theta\gamma+\lambda}\,\delta_0(dy)\,\delta_0(dx)\end{aligned}$$

Any symmetric terms can be safely ignored. The measure $\delta_0(dx)$ is supported on the set $\{x = 0\}$, hence the term $\frac{e^{-\gamma|x|}}{\theta\gamma+\lambda}\delta_0(dx)\delta_0(dy)$ is equal to $\frac{1}{\theta\gamma+\lambda}\delta_0(dx)\delta_0(dy)$, which is also symmetric. This leaves us with the following:

$$f(dx, dy) = \frac{\theta e^{-\gamma|x|}}{\theta\gamma+\lambda} dx\, \delta_0(dy) + \frac{e^{-\gamma|y|}}{\gamma} dy\, \delta_0(dx) - \frac{e^{-\gamma|y|}}{2\theta+\gamma} dy\, \delta_0(dx).$$

Then (2.18) holds if and only $f(dx, dy) = f(dy, dx)$ which holds if and only if

$$\frac{1}{\gamma} - \frac{1}{2\theta+\gamma} = \frac{\theta}{\theta\gamma+\lambda},$$

which is indeed correct. □

2.5 θ-coupled Brownian motions

For a fixed parameter θ, a pair of Brownian motions X and X' defined on a common probability space are said to be θ-coupled if X and X' are both standard Brownian motions relative to the same filtration, and the difference between the coordinates $\frac{1}{\sqrt{2}}(X - X')$ is a $(\sqrt{2}\theta)$-sticky Brownian motion. Then, by virtue of Proposition 6 this last property can be specified via the two equalities

$$\langle X, X' \rangle_t = \int_0^t 1_{\{X(s)=X'(s)\}} ds \qquad t \geq 0 \tag{2.19}$$

$$L_t(X - X') = 2\theta \int_0^t 1_{\{X(s)=X'(s)\}} ds \qquad t \geq 0. \tag{2.20}$$

Proposition 15. *For each fixed $(x_1, x_2) \in \mathbb{R}^2$ and $\theta > 0$ there exists a pair of θ-coupled Brownian motions (X, X') started from (x_1, x_2) and its law is uniquely determined.*

Proof. The proof of the proposition would be in a similar style to Proposition 6, but we do not give the proof here as this result can be seen directly as a special case of Proposition 16 below. □

In the following we add the extra complexity of allowing both of the θ-coupled Brownian motions to have drift. For any continuous semi-martingale X let $\tilde{L}_t^a(X)$ denote the symmetric local time of X at a, that is

$$\tilde{L}_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{X(s) \in (a-\epsilon, a+\epsilon)\}} d\langle X \rangle_s .$$

We note that with this definition of local time the following version of the Itô-Tanaka formula holds for convex f

$$f(X(t)) = f(X(0)) + \int_0^t \frac{1}{2}(f'_+ + f'_-)(X(s))dX(s) + \frac{1}{2}\int \tilde{L}_t^a f''(da),$$

where f'_+ and f'_- are the right and left derivatives of f respectively. We denote $\tilde{L}_t^0$ simply as $\tilde{L}_t$. We use symmetric local time in order to be consistent with [HS81], and to use left or right continuous local times only adds extra complexity to the calculations. Up to this point discontinuity in local time has not been an issue. Indeed for any martingale, its local time is continuous in the space variable, see [RY99]. It is only because in the sequel we have processes with drift, together with points of singular nature, that the issue of discontinuities in local time needs to be considered.

Proposition 16. *Suppose that β_1, β_2 and θ are parameters satisfying $|\beta_1 - \beta_2| \leq 2\theta < \infty$ and $\theta > 0$. Then, for each starting point $(x_1, x_2) \in \mathbb{R}^2$, there exists a stochastic process $((X(t), X'(t)); t \geq 0)$ such that X is a Brownian motion with drift β_1 starting from x_1, and X' is a Brownian motion with drift β_2 starting from x_2 (relative to some common filtration), and*

$$\langle X, X' \rangle_t = \int_0^t \mathbf{1}_{\{X(s)=X'(s)\}} ds \quad t \geq 0, \tag{2.21}$$

$$\tilde{L}_t(X - X') = 2\theta \int_0^t \mathbf{1}_{\{X(s)=X'(s)\}} ds \quad t \geq 0. \tag{2.22}$$

Moreover the law of (X, X') is uniquely determined.

Proof. Consider the stochastic equation given by

$$Z(t) = \sqrt{2}B(t) + \frac{\beta_1 - \beta_2}{2\theta}\tilde{L}_t(Z) + (\beta_1 - \beta_2)t$$
$$Z(0) = x_1 - x_2 \tag{2.23}$$

where B is a standard Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We wish to show that there exists a solution to (2.23) and the law of such a solution is uniquely specified.

Let $M(t) = \exp\left[-\frac{\beta_1-\beta_2}{\sqrt{2}}B(t) - \frac{(\beta_1-\beta_2)^2}{4}t\right]$. Now M is a martingale and there exists a probability measure $\hat{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{F}_t$

$$\hat{\mathbf{P}}|_{\mathcal{F}_t}(A) = \mathbf{E}[M(t); A].$$

Then by Girsanov's theorem $\hat{B}$, given by $\hat{B}(t) = B(t) + \frac{\beta_1-\beta_2}{\sqrt{2}}t$ for all t, is a Brownian motion under $\hat{\mathbf{P}}$. Then we have

$$Z(t) = \sqrt{2}\hat{B}(t) + \frac{\beta_1 - \beta_2}{2\theta}\tilde{L}_t(Z),$$

so that under $\hat{\mathbf{P}}$ the process $(Z(t); t \geq 0)$ solves the stochastic equation

$$Z(t) = \sqrt{2}\hat{B}(t) + \frac{\beta_1 - \beta_2}{2\theta}\tilde{L}_t(Z)$$
$$Z(0) = x_1 - x_2 \tag{2.24}$$

where $\hat{B}$ is a standard Brownian motion. It is known that, see [HS81], (2.24) has a strong solution, for any β_1, β_2 with $|\beta_1 - \beta_2| \leq 2\theta$. This solution is Brownian motion (here there is a scalar multiple) skew at zero with the probability of positive excursions being $\frac{1}{2} + \frac{\beta_1-\beta_2}{4\theta}$. Therefore, if there exists a solution to

(2.23), its law is uniquely specified. To show that there exists a solution to (2.23) we can start with a solution to (2.24) and apply Girsanov's theorem in reverse.

To prove the existence part of the proposition let Z be a solution to (2.23) defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $(\alpha_t; t \geq 0)$ be a time change given by $\alpha_t = t + \frac{1}{2\theta}\tilde{L}_t(Z)$ and let $(A_t; t \geq 0)$ be inverse of α_t, that is $A_t = \inf\{u \geq 0 : \alpha_u > t\}$. Note that $2t - A_t$ is strictly increasing and continuous. We let $\gamma_t = \inf\{u \geq 0 : 2u - A_u > t\}$, and define the process $(Z'(t); t \geq 0)$ by

$$Z'(t) = \sqrt{2}B'(t) + (\beta_1 + \beta_2)\gamma_t + (x_2 + x_1), \tag{2.25}$$

where B' is an independent Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We now let $X(t) = \frac{1}{2}\left(Z'(2t - A_t) + Z(A_t)\right)$ and
$X'(t) = \frac{1}{2}\left(Z'(2t - A_t) - Z(A_t)\right)$. Then

$$X(t) = \frac{1}{\sqrt{2}}B'_{2t-A_t} + \frac{1}{\sqrt{2}}B_{A_t} + \beta_1 t + x_1 \tag{2.26}$$

and

$$X'(t) = \frac{1}{\sqrt{2}}B'_{2t-A_t} - \frac{1}{\sqrt{2}}B_{A_t} + \beta_2 t + x_2. \tag{2.27}$$

Now let $\mathcal{F}_t^B = \sigma(B(s); 0 \leq s \leq t)$ and $\mathcal{F}_t^{B'} = \sigma(B'(s); 0 \leq s \leq t)$ and then let $\mathcal{G}_t = \mathcal{F}_t^B \bigvee \mathcal{F}_\infty^{B'}$ and $\mathcal{H}_t = \mathcal{F}_t^{B'} \bigvee \mathcal{F}_\infty^B$.

For each t, A_t is an $\mathcal{F}_t^B$-stopping time. This implies that A_t is a $\mathcal{G}_t$ stopping time. Also as B' and B are independent, B is a $\mathcal{G}_t$-martingale. $\mathcal{F}_\infty^B$ contains all information about B, therefore $2t - A_t$ is an $\mathcal{H}_t$-stopping time and B' is a $\mathcal{H}_t$ martingale.

Let $\hat{\mathcal{G}}_t = \mathcal{G}_{A_t}$ and $\hat{\mathcal{H}}_t = \mathcal{H}_{2t-A_t}$. By the theory of continuous times-changes, see for example [RY99], B_{A_t} is a $\hat{\mathcal{G}}_t$-martingale and B'_{2t-A_t} is $\hat{\mathcal{H}}_t$-martingale.

The relationships (2.26) and (2.27) tell us that

$$\mathcal{F}_t^{X,X'} := \sigma(X(s), X'(s); 0 \leq s \leq t) = \sigma(B(A_s), B'(2s - A_s); 0 \leq s \leq t)$$

and then, as B_{A_t} and B'_{2t-A_t} are both measurable with respect to both $\hat{\mathcal{G}}_t$ and $\hat{\mathcal{H}}_t$, $\mathcal{F}_t^{X,X'} \subseteq \hat{\mathcal{G}}_t$ and $\mathcal{F}_t^{X,X'} \subseteq \hat{\mathcal{H}}_t$. By the tower property B_{A_t} and B'_{2t-A_t} are both $\mathcal{F}_t^{X,X'}$ martingales and hence $X(t) - \beta_1 t$ and $X'(t) - \beta_2 t$ are both $\mathcal{F}_t^{X,X'}$ - martingales. Also $\langle X \rangle_t = \langle X' \rangle_t = t$ and, $X(0) = \frac{1}{2}(Z(0) + Z'(0)) = x_1$ and $X'(0) = \frac{1}{2}(Z'(0) - Z(0)) = x_2$, hence X is a Brownian motion with drift β_1 started at x_1 and X' is a Brownian motion with drift β_2 started at x_2 with respect to a common filtration, $(\mathcal{F}_t^{X,X'})_{t \geq 0}$.

Next we observe that

$$X(t) - X'(t) = Z(A_t)$$

implies that $\langle X - X' \rangle_t = 2 \langle B_{A.} \rangle_t = 2A_t$. By Tanaka's formula it is easy to show that

$$\tilde{L}_t(X - X') = \tilde{L}_{A_t}(Z)$$

but as $\alpha_t = t + \frac{1}{2\theta}\tilde{L}_t(Z)$ this implies that $t = A_t + \frac{1}{2\theta}\tilde{L}_t(X - X')$. Using the occupation times formula and the fact that $\langle Z \rangle_t = 2t$, we have that $t = \int_0^t \mathbf{1}_{\{Z(s) \neq 0\}} ds$, from which it follows that $A_t = \int_0^t \mathbf{1}_{\{Z(A_s) \neq 0\}} dA_s = \int_0^t \mathbf{1}_{\{X(s) \neq X'(s)\}} ds$ and so

$$2\langle X, X' \rangle_t = \langle X \rangle_t + \langle X' \rangle_t - \langle X - X' \rangle_t = 2\int_0^t \mathbf{1}_{\{X(s) = X'(s)\}} ds$$

and

$$\tilde{L}_t(X - X') = 2\theta \int_0^t \mathbf{1}_{\{X(s) = X'(s)\}} ds.$$

This proves that we can construct a pair of Brownian motions (X, X') with drifts β_1 and β_2 respectively, which also satisfy (2.21) and (2.22).

Now assume that we have any pair of processes (X, X'), defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, that satisfy the properties of the proposition. We now define $(A_t; t \geq 0)$ in terms of X and X',

$$A_t = \int_0^t \mathbf{1}_{\{X(s)\neq X'(s)\}} ds$$

and we now let $\alpha_t = \inf\{u \geq 0 : A_u > t\}$ and $\gamma_t = \inf\{u \geq 0 : 2u - A_u > t\}$. Using similar arguments to those in Proposition 6 it is possible to show that $A_\infty = \infty$.

To prove the 'uniqueness in law' part of the proposition we must show that the joint laws of $(X(\alpha_t) - X'(\alpha_t); t \geq 0)$ and $(X(\gamma_t) + X'(\gamma_t); t \geq 0)$ are equal to the joint laws of Z and Z', where Z is the solution to stochastic equation (2.23) and Z' is given by (2.25), and also that $\alpha_t = t + \frac{1}{2\theta}\tilde{L}_t(X(\alpha_.) - X'(\alpha_.))$.

Let W and W' be Brownian motions given by

$$W(t) = X(t) - \beta_1 t - x_1 \text{ and } W'(t) = X(t) - \beta_2 t - x_2, \qquad t \geq 0.$$

Clearly $W - W'$ and $W + W'$ are both martingales, $\langle W - W'\rangle = \langle X - X'\rangle = 2A_t$ and $\langle W + W'\rangle = \langle X + X'\rangle = 4t - 2A_t$. It is also true that $\langle W + W', W - W'\rangle_t = \langle W\rangle - \langle W'\rangle = 0$. Thus it follows, from Knight's theorem, that $(W(\alpha_t) - W'(\alpha_t); t \geq 0)$ and $(W(\gamma_t) + W'(\gamma_t); t \geq 0)$ are independent and each equal in distribution to $(\sqrt{2}B(t); t \geq 0)$, where B is a standard Brownian motion.

Now observe that

$$X(\alpha_t) - X'(\alpha_t) = W(\alpha_t) - W'(\alpha_t) + (\beta_1 - \beta_2)\alpha_t + (x_1 - x_2). \qquad (2.28)$$

By Tanaka's formula

$$\tilde{L}_t(X(\alpha_\cdot) - X'(\alpha_\cdot)) = \tilde{L}_{\alpha_t}(X - X')$$

and it follows from (2.22) that

$$\tilde{L}_t(X(\alpha_\cdot) - X'(\alpha_\cdot)) = 2\theta \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=X'(s)\}} ds.$$

Hence

$$\alpha_t = \int_0^{\alpha_t} \mathbf{1}_{\{X(s)\neq X'(s)\}} ds + \int_0^{\alpha_t} \mathbf{1}_{\{X(s)=X'(s)\}} ds = t + \frac{1}{2\theta}\tilde{L}_t(X(\alpha_\cdot) - X'(\alpha_\cdot))$$

The above, together with (2.28), tell us that the process $(X(\alpha_t) - X'(\alpha_t); t \geq 0)$ solves the stochastic equation (2.23) and hence $(X(\alpha_t) - X'(\alpha_t); t \geq 0)$ is equal in distribution to Z. Then as $X(\gamma_t) + X'(\gamma_t) = W'(\gamma_t) + W'(\gamma_t) + (\beta_1 + \beta_2)\gamma_t + (x_1 + x_2)$. we have that the joint distribution of $(X(\alpha_t) - X'(\alpha_t); t \geq 0)$ and $(X(\gamma_t) + X'(\gamma_t); t \geq 0)$ is equal to the joint distribution of Z and Z', from which uniqueness in law follows.

□

2.6 Sticky Brownian motion as a scaling limit of sticky random walks

We consider a simple symmetric random walk $(S(t); t \in \mathbb{N})$, that is $S(t) = \sum_{i=0}^{t} \xi_i$, where $(\xi_i; i \geq 1)$ is a sequence of i.i.d random signs with $\mathbf{P}(\xi_i = 1) = \mathbf{P}(\xi_i = -1) = 1/2$. Let $S^{(n)}$ be derived from S via diffusive scaling and joining points $(t, S(t))$ to give continuous paths. Thus

$$S^{(n)}(t) = \frac{1}{\sqrt{n}} S([nt]+1)(nt-[nt]) + ([nt]+1-nt)S([nt]), \tag{2.29}$$

where $[x]$ gives the integer part of x. Donsker's theorem gives us that $S^{(n)}$ converges to a Brownian motion in the sense of weak convergence in the space of continuous paths $C([0,\infty),\mathbb{R})$ with the topology of locally uniform convergence.

We now construct a random walk $\hat{S}$, where each time the random walk reaches zero it is held there for $T \sim$ Geometric(p). More precisely, let $\nu(t)$ be the cardinality of the set $\{u \in \mathbb{Z}; u \le t, S(u) = 0\}$ and let t_i be the ith element. Let $(T_i; i \ge 1)$ be an independent sequence of random variables distributed as T. Define a time change C such that $C^{-1}(t) = t + \sum_{i=1}^{\nu(t)} T_i$ and $C(t) = \inf\{u \in \mathbb{N}; C^{-1}(u) \ge t\}$. Now define a sticky random walk $\hat{S}$ by

$$\hat{S}(t) = S(C(t)) \quad t \in \mathbb{N}.$$

We define $\hat{S}^{(n)}(t)$ as in (2.29) and then the process $(\hat{S}^{(n)}(t); t \ge 0)$ converges in distribution to a standard Brownian motion. In order to make the limit non-trivial we must vary the distribution of T with n. Let $T^{(n)} \sim$ Geometric($p(n)$) where $\lim_{n\to\infty} \sqrt{n}p(n) = \theta$ for some constant $\theta > 0$. Let $(T_i^{(n)}; i \ge 1)$ be an independent sequence of random variables distributed as $T^{(n)}$. We define the time change C and the sticky random walk $\hat{S}$ as before but with respect to $(T_i^{(n)}; i \ge 1)$.

$\hat{S}^{(n)}$ now converges in distribution to a θ-sticky Brownian motion. This result is obtained by Amir [Ami91]. In fact Amir uses an embedded random walk to get an almost sure convergence result. Results on sticky Brownian motion as a limit of sticky random walks are also found in [Tsi04b].

We note that $\mathbf{E}\left[T^{(n)}\right]$ is of order $\sqrt{n}$, so heuristically $\sum_{i=1}^{nt} \mathbf{1}_{\{\hat{S}(i)=0\}}$ is of the same order as $\nu(nt)\mathbf{E}[T^{(n)}]$, which is of order n. Consequently the amount of time the process $\hat{S}^n$ spends at 0, $\int_0^t \mathbf{1}_{\{\hat{S}^n(u)=0\}}du$, is of order 1. In fact it is a well known result, see [RW94] that $\lim_{n\to\infty} \frac{\nu(nt)}{\sqrt{n}} \stackrel{d}{=} \sqrt{t}|N|$ and as $\frac{E[T(n)]}{\sqrt{n}} \to \theta$ it is possible to see that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{C(nt)} \mathbf{1}_{\{\hat{S}(i)=0\}} = \frac{\sqrt{t}|N|}{\theta}$, which is comparable with the result from Proposition 5.

2.7 One-sided sticky Brownian motion as a limit of certain continuous processes

In this section we describe another way of studying sticky Brownian motion. This is as a limit of a sequence of processes, where, on each interval of length $1/n$, the nth process in the sequence behaves as either absorbing or reflecting Brownian motion according to a coin toss. For simplicity we consider processes over the time interval $[0,1]$, it then being an easy generalisation to processes on the half line. We describe a sequence of continuous processes, $(X^{(n)}(t); t \in [0,1])$. The time interval is partitioned into n equal length subintervals. Over each subinterval the process $X^{(n)}$ behaves as either absorbing Brownian motion or reflecting Brownian motion. The behaviour is determined by n independent Bernoulli trials $\{Y_k^{(n)} : k \in \{1,2,\ldots,n\}\}$, with $\mathbf{P}(Y_k^{(n)} = 1) = 1 - \mathbf{P}(Y_k^{(n)} = 0) = p$ for all k. Thus if a strip (subinterval) $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ is labelled with a 0, i.e. $Y_k^{(n)} = 0$, then $X^{(n)}$ behaves as reflecting Brownian motion for the duration of that strip. If a strip is labelled with a 1 then $X^{(n)}$ behaves as Brownian motion absorbed at zero while within that strip. If this probability depends on n such that $p = p(n)$ and $p(n)$ satisfies

$$\lim_{n\to\infty} \sqrt{\frac{2n}{\pi}}p(n) = \theta \in (0,\infty) \qquad (2.30)$$

then we will show that the process $X^{(n)}$ converges in distribution to one-sided sticky Brownian motion with parameter θ.

Consider a Brownian motion $\{B(t) : 0 \leq t \leq 1\}$. We can construct a new process $X^{(n)}$, with the properties above, from B and a sequence of n i.i.d Bernoulli trials $\{Y_k^{(n)} : k \in \{1, 2, \dots, n\}\}$ with $\mathbf{P}(Y_k^{(n)} = 1) = p(n)$ and $p(n)$ satisfying (2.30). Fix $x \geq 0$ and set $X^{(n)}(0) = x$. Then, for $t \in [\frac{k}{n}, \frac{k+1}{n}]$, let

$$X^{(n)}(t) = \begin{cases} B(t) + X^{(n)}(k/n) - B(k/n) & \text{if } B(s) > X^{(n)}(k/n) - B(k/n) \; \forall s \in [k/n, t) \\ (B(t) - \inf_{k/n \leq s \leq t}(B(s)))\mathbf{1}_{\{Y_k^{(n)}=0\}} & \text{otherwise .} \end{cases} \tag{2.31}$$

Proposition 17. *The law of $X^{(n)}$ converges weakly to a that of a one-sided sticky Brownian motion with parameter θ, in the space of continuous paths $C([0,1], \mathbb{R})$ with the uniform metric.*

Firstly we need the following proposition

Proposition 18. *Assuming $X^{(n)}(0) = 0$, if there exists a process X such that X is the weak limit of $X^{(n)}$, then X must have the property*

$$\begin{aligned} \mathbf{P}(X(1) \leq x) &= \lim_{n \to \infty} \mathbf{P}(X^{(n)}(1) \leq x) \\ &= \mathbf{E}\left[e^{-2\theta(L_1 - x)}\mathbf{1}_{\{0 \leq x \leq L_1\}} + \mathbf{1}_{\{x > L_1\}}\right], \end{aligned}$$

where $L_1 = B(1) - \inf_{0 \leq s \leq 1}(B(s))$.

Proof. Let $\tilde{B}(t) = B(1) - B(1-t)$ and let $\tilde{T}_x = \inf\{t \geq 0 : \tilde{B}(t) \geq x\} \wedge 1$ and $T_x = 1 - \tilde{T}_x$. First hitting times of a level x by $\tilde{B}$ are related to the last exit times B via $t \mapsto 1-t$. We shall call this set of exit times Z. Thus $Z = \{t \in [0,1] : B(t) = \inf_{t \leq s \leq 1} B(s)\}$.

We observe that the process $X^{(n)}$ never returns to zero after $\inf\{t \in Z : t \in [\frac{k}{n}, \frac{k+1}{n}) \text{ with } Y_k^{(n)} = 1\}$ and is at zero at this time. This leads to the relation.

$$\mathbf{P}(X^{(n)}(1) \leq x)$$
$$= \mathbf{P}\left(Y_k^{(n)} = 0, \forall k \in \mathbb{Z} \text{ s.t. } 0 \leq k/n < T_x \text{ and } \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z = \emptyset\right)$$

Let $M_x^{(n)} = \#\{k \in \mathbb{Z} : 0 \leq k/n < T_x, \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$, let $N^{(n)} = \#\{k \in \{0, 1, \ldots, n-1\} : \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$ and let $N_x^{(n)} = \#\{k \in \mathbb{Z} : T_x \leq k/n < 1, \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$.

We note that Z is the zero set of a Brownian motion B', where $|B'|$ is given by $|B'(t)| = \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{B}(t)$. The local time of B' at 0 is given by

$$L_t(B') = \sup_{0 \leq s \leq t} \tilde{B}(s) = B(1) - \inf_{(1-t) \leq s \leq 1} B(s) = L_1$$

Thus, we have from Proposition 27,

$$\frac{N^{(n)}}{\sqrt{n}} \xrightarrow{L^2} 2\sqrt{\frac{2}{\pi}} L_1 \qquad \text{as } n \to \infty.$$

Similarly, by Proposition 28,

$$\frac{N_x^{(n)}}{\sqrt{n}} \xrightarrow{L^2} 2\sqrt{\frac{2}{\pi}} L_{\tilde{T}_x}(B') = 2\sqrt{\frac{2}{\pi}}(x \wedge L_1) \qquad \text{as } n \to \infty.$$

We have $M_x^{(n)} = N^{(n)} - N_x^{(n)}$, therefore

$$\frac{M_x^{(n)}}{\sqrt{n}} \xrightarrow{L^2} 2\sqrt{\frac{2}{\pi}}(L_1 - x)\mathbf{1}_{\{x \leq L_1\}} \qquad \text{as } n \to \infty. \tag{2.32}$$

We now compute the probability, $\mathbf{P}(X^{(n)}(1) \leq x)$. Remember that $\mathbf{P}(Y_k^{(n)} = 0) = (1 - p(n))$, so

$$\mathbf{P}(X^{(n)}(1) \leq x) = \mathbf{E}\left[(1 - p(n))^{M_x^{(n)}}\right] = \mathbf{E}\left[\exp\left[M_x^{(n)} \ln(1 - p(n))\right]\right].$$

Now $-x - x^2 \leq \ln(1-x) \leq -x$ for all $x < 1/2$ so

$$\mathbf{E}\left[\exp\left[-\left(p(n)+p(n)^2\right)M_x^{(n)}\right]\right] \leq \mathbf{P}(X^{(n)}(1) \leq x) \leq \mathbf{E}\left[\exp\left[-p(n)M_x^{(n)}\right]\right]$$

By (2.30) and (2.32) $M_x^{(n)}p(n)$ converges in L^2 to $2\theta(L_1 - x)\mathbf{1}_{\{x \leq L_1\}}$, and $\frac{M_x^{(n)}}{n}$ converges to 0. Thus the above expectation converges, and

$$\lim_{n\to\infty} \mathbf{P}(X^{(n)}(1) \leq x) = \begin{cases} \mathbf{E}\left[e^{-2\theta(L_1-x)}\mathbf{1}_{\{x\leq L_1\}} + \mathbf{1}_{\{x>L_1\}}\right] & x \geq 0 \\ 0 & x < 0. \end{cases}$$

□

Let us now consider the value of $X^{(n)}(1)$ when $X^{(n)}(0) > 0$, using the same Brownian motion B and the same sequence of Bernoulli trials $\{Y_k^{(n)} : k \in \{1, \ldots, n\}\}$. Two possible situations can occur. Either
$X^{(n)}(0) > -\inf_{0\leq s\leq 1}(B(s))$ in which case $X^{(n)}(t) > 0 : \forall t \in [0,1]$ and $X^{(n)}(1) = X^{(n)}(0) + B(1)$, or $X^{(n)}(0) \leq -\inf_{0\leq s\leq 1}(B(s))$, in which case all the first exit points occur after the first time $X^{(n)}(t) = 0$. I.e. $[0, \inf\{t \in [0,1] : X^{(n)}(t) = 0\}] \cap Z = \emptyset$. Therefore the value of $X^{(n)}(1)$ is the same as the value of $X^{(n)}(1)$ when $X^{(n)}(0) = 0$. This argument generalises to considering the value of the process at any fixed time $t \in [0,1]$. Thus for a fixed time $t \in [0,1]$ the value of the $X_t^{(n)}(x) = X^{(n)}(t)$ as a function of the starting value $X_0^{(n)} = x$ is given by

$$X_t^{(n)}(x) = \begin{cases} X_t^{(n)}(0) & 0 \leq x \leq -\inf_{0\leq s\leq t}(B(s)) \\ B(t) + x & x > -\inf_{0\leq s\leq t}(B(s)). \end{cases}$$

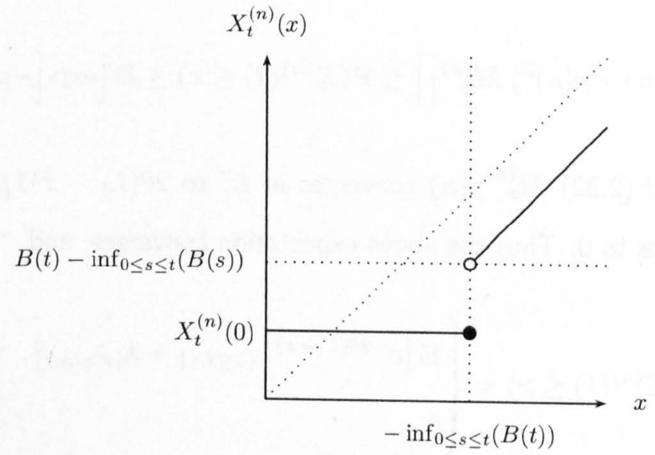

Figure 2.2: Final value versus starting value

Therefore the limit in distribution of $X_t^{(n)}(x)$ is given by

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}(X_t^{(n)}(x) \le y) \\
&\quad = \mathbf{E}\left[\left(e^{-2\theta(L_t-y)}\mathbf{1}_{\{y\le L_t\}} + \mathbf{1}_{\{y>L_t\}}\right)\mathbf{1}_{\{x\le -\inf_{0\le s\le t}(B(s))\}}\right] \\
&\quad\quad + \mathbf{E}\left[\mathbf{1}_{\{B(t)+x\le y\}}\mathbf{1}_{\{x>-\inf_{0\le s\le t}(B(s))\}}\right] \qquad (2.33)
\end{aligned}$$

where $L_t = B(t) - \inf_{0\le s\le t}(B(s))$.

Lemma 19. *For $t \in [0,1]$, let $P_t^{(n)}$ be given by $P_t^{(n)} f(x) = \mathbf{E}_x[X^{(n)}(t)]$, where under $\mathbf{P}_x$, $X^{(n)}(0) = x$. Suppose that f is bounded. Then $|P_t^{(n)} f(x) - P_t^{(n)} f(y)| \le C|x-y|$ for some constant C depending only on f and t.*

Proof. Let $(V_1^{(n)}, V_2^{(n)}, V_3^{(n)})$ be three Brownian motions, started at $0, \sqrt{2}x > 0$ and $\sqrt{2}y > 0$ respectively. Over each interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ with $Y_k^{(n)} = 0$, let $(V_1^{(n)}, V_2^{(n)}, V_3^{(n)})$ be coalescing Brownian motions. Over each interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ with $Y_k^{(n)} = 0$, let $(V_1^{(n)}, V_2^{(n)}, V_3^{(n)})$ be independent Brownian motions. We note that the pairs $(V_1^{(n)}, V_2^{(n)})$ and $(V_1^{(n)}, V_3^{(n)})$ are (p,n)-coupled Brownian motion, which we describe later. Let $\tau = \inf\{t \geq 0 : V_2^{(n)}(t) = V_3^{(n)}(t)\}$ and define a new Brownian motion $Z_3^{(n)}$ via

$$Z_3^{(n)}(t) = \begin{cases} V_3(t) & t \leq \tau \\ V_2(t) & t > \tau. \end{cases}$$

Let $X_1^{(n)} = \frac{1}{\sqrt{2}}|V_1^{(n)} - V_2^{(n)}|$ and $X_2^{(n)} = \frac{1}{\sqrt{2}}|V_1^{(n)} - Z_3^{(n)}|$. Thus $X_1^{(n)}$ and $X_2^{(n)}$ are both processes that behave as reflecting Brownian motions over the interval $[\frac{k}{n}, \frac{k+1}{n})$ if $Y_k^{(n)} = 0$, or absorbing Brownian motion if $Y_k^{(n)} = 1$. $X_1^{(n)}$ starts at x, and $X_2^{(n)}$ starts at y. Thus

$$\begin{aligned}
|P_t^{(n)}(f(x)) - P_t^{(n)}(f(y))| &= |\mathbf{E}[f(X_1^{(n)}(t)) - f(X_2^{(n)}(t))]| \\
&= |\mathbf{E}[(f(X_1^{(n)}(t)) - f(X_2^{(n)}(t)))\mathbf{1}_{\{t \leq \tau\}}]| \\
&\leq 2||f||_\infty \mathbf{P}(\tau \geq t) \\
&= 2||f||_\infty \int_0^{|x-y|} \frac{1}{\sqrt{\pi t}} \exp(-z^2/4t)dz \\
&\leq \frac{2||f||_\infty}{\sqrt{\pi t}}|x - y|.
\end{aligned}$$

The penultimate equality coming from the distribution of Brownian hitting times. $\square$

Proposition 20. *The finite dimensional distributions of $X^{(n)}$ converge to the finite dimensional distributions of a one-sided sticky Brownian motion with parameter θ.*

Proof. Let $f : \mathbb{R}^N \mapsto \mathbb{R}$ be some continuous bounded function. Let $(P_t; t \geq 0)$ be given by $P_t f(x) = \mathbf{E}_x[X(t)]$ where X is a one-sided sticky Brownian motion with parameter θ. Comparing (2.33) with corollary 10 it is easy to see that $P_t^{(n)} f(x)$ converges to $P_t f(x)$ pointwise. By Lemma 19 the family of functions

$(x \mapsto P_t^{(n)} f(x); n \geq 1)$ is uniformly equicontinuous. Thus, for any compact subset $K \subset \mathbb{R}$, we have that $P_t^{(n)} f(x)$ converges to $P_t f(x)$ uniformly for all $x \in K$.

$X^{(n)}$ isn't Markov! Consider the process at some time $t \in [k/n, (k+1)/n]$: Knowing that the process was reflected at some time between k/n and t would tell us how the process will behave for the rest of the interval. However $X^{(n)}$ is Markov at times k/n for $0 \leq k \leq n$ and, using the equicontinuity of $(X^{(n)}; n \geq 1)$ this will be all we need.

Let $f_1, \dots f_k$ be continuous, bounded functions, such that for each i, f_i is non-zero only on some compact set, K_i. We now use an induction argument. Clearly $X^{(n)}(t_1)$ converges in distribution to $X(t_1)$. Assume that $(X^{(n)}(t_1), \dots, X^{(n)}(t_{k-1}))$ converges in distribution to $(X(t_1), \dots, X(t_{k-1}))$. Let $t_{k-1}^n = [nt_{k-1}]/n$. The family of functions $(t \mapsto X^{(n)}(t); n \geq 1)$ is uniformly equicontinuous for $t \in [0,1]$, which comes from the way that each process in the sequence is constructed from the same Brownian motion. From this it follows that

$$
\begin{aligned}
&\mathbf{E}[f_1(X^{(n)}(t_1)) \cdots f_k(X^{(n)}(t_k))] \\
&= \lim_{n\to\infty} \mathbf{E}[f_1(X^{(n)}(t_1)) \cdots f_{k-1}(X^{(n)}(t_{k-1}^n)) P^n_{t_k - t_{k-1}^n} f_k(X^{(n)}(t_{k-1}^n))] \\
&= \lim_{n\to\infty} \mathbf{E}[f_1(X^{(n)}(t_1)) \cdots f_{k-1}(X^{(n)}(t_{k-1})) P^n_{t_k - t_{k-1}} f_k(X^{(n)}(t_{k-1}))].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbf{E}[f_1(X^{(n)}(t_1))\cdots f_{k-1}(X^{(n)}(t_{k-1}))P^n_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1}))] \\
&\quad= \mathbf{E}[f_1(X^{(n)}(t_1))\cdots f_{k-1}(X^{(n)}(t_{k-1}))P_{t_k-t_{k-1}}f_k(X^n(t_{k-1}))] \\
&\qquad+ \mathbf{E}\Big[f_1(X^{(n)}(t_1))\cdots f_{k-1}(X^{(n)}(t_{k-1})) \\
&\qquad\qquad (P^{(n)}_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1})) - P_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1})))\mathbf{1}_{\{X^{(n)}(t_{k-1})\in K_k\}}\Big] \\
&\qquad+ \mathbf{E}\Big[f_1(X^{(n)}(t_1))\cdots f_{k-1}(X^{(n)}(t_{k-1})) \\
&\qquad\qquad (P^{(n)}_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1})) - P_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1})))\mathbf{1}_{\{X^{(n)}(t_{k-1})\notin K_k\}}\Big]
\end{aligned}
$$

As for each t, $P^{(n)}_t f_k(x)$ converges to $P_t f_k(x)$ uniformly for all $x \in K_k$ as $n \to \infty$, the penultimate term above converges to zero. The final term above can also be seen to be trivial. The term can be seen to be bounded above by $C\mathbf{P}(X^{(n)}(t_{k-1}) \notin K_k)$ for some constant C. The compact set K_k is arbitrary. For any $\epsilon > 0$, tightness of the family $(X^{(n)}; n \geq 1)$ proven below shows means that we can choose a compact set $K_k(\epsilon)$ such that $\sup_n \mathbf{P}(X^{(n)}(t_{k-1}) \notin K_k) < \epsilon$.

As $x \mapsto P_{t_k-t_{k-1}}f_k(x)$ is bounded and continuous we have, as $n \to \infty$,

$$
\begin{aligned}
&\mathbf{E}[f_1(X^{(n)}(t_1))\cdots f_{k-1}(X^{(n)}(t_{k-1}))P_{t_k-t_{k-1}}f_k(X^{(n)}(t_{k-1}))] \\
&\quad\to \mathbf{E}[f_1(X(t_1))\cdots f_{k-1}(X(t_{k-1}))P_{t_k-t_{k-1}}f_k(X(t_{k-1}))] \\
&\qquad= \mathbf{E}[f_1(X(t_1))\cdots f_k(X(t_k))].
\end{aligned}
$$

□

Proposition 21. *The family of laws of $X^{(n)}$ for $n \geq 1$ is tight.*

Proof. Let $g : C([0,1],\mathbb{R}) \mapsto C([0,1],\mathbb{R})$ be the operator given by $g \circ f(t) = f(t) - \inf_{0\leq s\leq t} f(s)$. Fix $t \in [0,1]$, then for $\delta > 0$ let $t^\delta = [0 \vee (t-\delta), (t+\delta)\wedge 1]$.

It follows that

$$\begin{aligned}
&\sup_{s\in t^\delta}(g\circ f(s)) - g\circ f(t)\\
&\quad = \sup_{s\in t^\delta} f(s) - \inf_{0\le u\le (t-\delta)\vee 0} f(u) - f(t) + \inf_{0\le u\le t} f(u)\\
&\quad \le \sup_{s\in t^\delta} f(s) - f(t)
\end{aligned}$$

and also

$$\begin{aligned}
&g\circ f(t) - \inf_{s\in t^\delta}(g\circ f(s))\\
&\quad = f(t) - \inf_{0\le u\le t} f(u) - \inf_{s\in t^\delta} f(s) + \inf_{0\le u\le (t-\delta)\vee 0} f(u)\\
&\quad \le f(t) - \inf_{s\in t^\delta} f(s).
\end{aligned}$$

Then, as $\sup_{s\in t^\delta}|f(t)-f(s)| = \max(\sup_{s\in t^\delta} f(s) - f(t), f(t) - \inf_{s\in t^\delta} f(s))$, we have that $\sup_{s\in t^\delta}|g\circ f(s) - g\circ f(t)| \le \sup_{s\in t^\delta}|f(t)-f(s)|$ and from this it follows that for any $0<\delta\le 1$

$$\sup_{|s-t|\le\delta}|g\circ f(t) - g\circ f(s)| \le \sup_{|s-t|\le\delta}|f(t)-f(s)|.$$

Now take $h_k : C([0,1],\mathbb{R}) \mapsto C([0,1],\mathbb{R})$ to be the operator which makes f absorbed at zero between $\frac{k}{n}$ and $\frac{k+1}{n}$. Thus we let $\tau = \inf\{t\in[\frac{k}{n},\frac{k+1}{n}] : f(t)=0\}$, where $\tau=\infty$ if $f(t)\neq 0$ for all $t\in[\frac{k}{n},\frac{k+1}{n}]$ and then

$$h_k(f(t)) = \begin{cases} f(t) & \forall t\in[0,1] \quad \text{if } \tau=\infty\\ \begin{cases} f(t) & \forall t\in[0,\tau]\\ 0 & \forall t\in[\tau,\frac{k+1}{n}]\\ f(t) - \inf_{\frac{k+1}{n}\le s\le t} f(s) & \forall t\in[\frac{k+1}{n},1] \end{cases} & \text{if } \tau<\infty. \end{cases}$$

It possible to see by similar arguments to the above that

$$\sup_{|s-t|\leq\delta} |h_k \circ f(t) - h_k \circ f(s)| \leq \sup_{|s-t|\leq\delta} |f(t) - f(s)|.$$

Looking back to the construction at the beginning of Section 2.7, the sample paths of $X^{(n)}$ can be found from the sample paths of B using the operator g and then a finite number of operators of the form h_k according to the values of the random variables $(Y_k^{(n)} : k \in \{1, \ldots, n\})$. So if μ_n is the measure on path-space, $C([0,1],\mathbb{R})$, associated with the random variable $X^{(n)}$ and μ is the Wiener measure associated with B then

$$\mu_n(w \in C([0,1],\mathbb{R}) : \sup_{|s-t|\leq\delta} |w(t) - w(s)| \geq \epsilon)$$
$$\leq \mu(w \in C([0,1],\mathbb{R}) : \sup_{|s-t|\leq\delta} |w(t) - w(s)| \geq \epsilon).$$

As the space $C([0,1],\mathbb{R})$ is complete and separable any measure is tight, see [Bil99]. Thus Wiener measure on $C([0,1],\mathbb{R})$ is tight, which implies that $\mu(w \in C([0,1],\mathbb{R}) : \sup_{|s-t|\leq\delta} |w(t) - w(s)| \geq \epsilon) \to 0$ as $\delta \to 0$. It follows from the above arguments that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mu_n(w \in C([0,1],\mathbb{R}) : \sup_{|s-t|\leq\delta} |w(t) - w(s)| \geq \epsilon) = 0.$$

Then by Theorem 7.3 of [Bil99] the family of measures $(\mu_n; n \geq 1)$ is tight. □

Proof of Proposition 17. Tightness, from Proposition 21, and convergence of finite dimensional distributions on $C([0,1],\mathbb{R})$, from Proposition 20, gives the result. □

We now consider a sequence of pairs of Brownian motions whose difference behaves as the sequence of processes described in Proposition 17 above. We will need the following lemma.

Lemma 22. *Let $(\mu_1^n; n \geq 1)$ and $(\mu_2^n : n \geq 1)$ be two tight families of measures on metric spaces S and S' respectively. Let $(\mu^n; n \geq 1)$ be a family of measures on $S \times S'$ such that for each $n \geq 1, A \in \mathcal{B}(S), A' \in \mathcal{B}(S')$, $\mu_1^n(A) = \mu^n(A \times S')$ and $\mu_2^n(A') = \mu^n(S \times A')$. Then the family of measures $(\mu^n; n \geq 1)$ is tight.*

Proof. $(\mu_1^n; n \geq 1)$ and $(\mu_2^n : n \geq 1)$ being tight implies that for each $\epsilon > 0$ there exists compact subsets $K \subset S$ and $K' \subset S'$ such that

$$\inf_{n\geq 1} \mu_1^n(K) \geq 1 - \epsilon/2.$$

and

$$\inf_{n\geq 1} \mu_2^n(K') \geq 1 - \epsilon/2.$$

It is straight forward to show that $K \times K'$ is a compact subset of $S \times S'$. Then for each n

$$\mu^n(K \times K') \geq 1 - \mu_1^n(K) - \mu_2^n(K') \geq 1 - \epsilon.$$

Thus

$$\inf_{n\geq 1} \mu^n(K \times K') \geq 1 - \epsilon.$$

□

For $p \in (0,1)$, let $\{Y_k, k \geq 0\}$ be a sequence of independent Bernoulli trials with $\mathbf{P}(Y_k = 1) = 1 - \mathbf{P}(Y_k = 0) = p$. Then let X and $X^{(n)}$ both be Brownian motions started at x_1 and x_2 respectively such that over the interval $[k/n, (k+1)/n]$, X and $X^{(n)}$ behave as a pair of independent Brownian motion if $Y_k = 1$ and as a pair of coalescing Brownian motions if $Y_k = 0$. We call the pair $(X, X^{(n)})$ a pair of (p, n)-coupled Brownian motions. In the following convergence in distribution means weak convergence of probability measures on the path space $C([0,1],\mathbb{R}^2)$.

Proposition 23. *Let $p = p(n)$ be such that $p(n)$ satisfies*

$$\lim_{n\to\infty} 2\sqrt{\frac{n}{\pi}}p(n) = \theta \in (0,\infty).$$

Then a pair of (p,n)-coupled Brownian motions converges in distribution to a pair of θ-coupled Brownian motions.

Proof. If $(X, X^{(n)})$ are a pair of (p,n)-coupled Brownian motions the family of laws of $\{(X, X^{(n)}); n \geq 1\}$ is tight by Lemma 22 above, as the law of each marginal is the law of a Brownian motion for each n and this is tight because $C([0,1],\mathbb{R})$ is complete and separable, see [Bil99]. Assume $(n(k); k \geq 1)$ is some subsequence such that $(X, X^{(n(k))})$ converges in law to some pair (X, X'). The map $(X, X') \mapsto \frac{1}{\sqrt{2}}(X - X')$ is continuous in the spaces of continuous functions with the uniform metric. So by proposition 17, $\frac{1}{\sqrt{2}}(X - X')$ is equal in distribution to a $\sqrt{2}\theta$-sticky Brownian motion. Then as the law of a pair of θ-coupled Brownian motions, (X, X') is uniquely specified by the fact that $\frac{1}{\sqrt{2}}|X - X'|$ is a one sided $\sqrt{2}\theta$-sticky Brownian motion (X, X') must indeed be a pair of θ-coupled Brownian motions.

Every subsequence $((X, X^{(n(k))}); k \geq 1)$ that converges weakly at all, convergence to the law of a pair of θ-coupled Brownian motions. So the entire sequence must also converge to a pair of θ-coupled Brownian motions. □

We note that tightness in Proposition 17 could have been proved indirectly from the tightness of the family of (p,n)-coupled Brownian motions above, as $(X, X') \mapsto \frac{1}{\sqrt{2}}|X - X'|$ is a continuous mapping from $C([0,\infty),\mathbb{R}^2)$ to $C([0,\infty),\mathbb{R})$. It is with this method that we will show tightness of the sequence of processes in the following.

Returning to one dimensional continuous processes on $[0,1]$, again let the time interval be divided into n equal length strips (subintervals) and let $(Y_k; k \in \{1,\ldots,n\})$ be a sequence of independent Bernoulli(p) trials. Now let $(W^{(n)}(t), t \in [0,1])$ behave as a reflecting Brownian motion over the interval $[k/n,(k+1)/n]$ if $Y_k = 1$, and as a one-sided sticky Brownian motion with parameter θ if $Y_k = 0$.

Again we can construct a process with the above properties from a Brownian motion $(B(t) : 0 \leq t \leq 1)$. For $p \in (0,1)$ let $\{Y_k^{(n)} : k = 1,2,\ldots,n\}$ be a sequence of independent Bernoulli(p) trials. Let $(V^{(n,k)} : k = 1,2,\ldots,n)$ be independent of each other and of B, where each $V^{(n,k)}$ has an exponential distribution of rate $2\theta_1$. Fix $x \geq 0$ and let $W^{(n)} = x$. Then for $t \in [k/n,(k+1)/n)$, we have

$$W^{(n)}(t) = B(t) + W^{(n)}(k/n) - B(k/n) \qquad \begin{array}{c}\text{if } B(s) > B(k/n) - W^{(n)}(k/n) \\ \forall s \in [k/n,t)\end{array}.$$

If, however, $B(s) \leq B(k/n) - W^{(n)}(k/n)$ for some $s \in [k/n,(k+1)/n)$, so that $W^{(n)}(s) = 0$ for some $s \in [k/n,(k+1)n)$ then, from Proposition 18,

$$W^{(n)}((k+1)/n) = \left[B((k+1)/n) - \inf_{k/n \leq s \leq (k+1)/n}(B(s)) - V^{(n,k)} 1_{\{Y_k^{(n)}=1\}}\right]^+$$

gives the correct distribution at time $(k+1)/n$.

Proposition 24. *Let $p = p(n)$ satisfy*

$$\lim_{n\to\infty} \sqrt{\frac{2n}{\pi}}\, p(n) = \theta_2 \in (0,\infty).$$

Then $W^{(n)}$ converges weakly to a one-sided sticky Brownian motion with parameter $\theta_1 + \theta_2$.

Proof. The result follows from Proposition 25 below, and similar arguments to those given in the proof of proposition 17. Tightness follows from the tightness of the sequence of coupled processes in Proposition 26 below, together with the observation that $(W, W') \mapsto \frac{1}{\sqrt{2}}|W - W'|$ is a continuous map from $C([0,1], \mathbb{R}^2)$ to $C([0,1], \mathbb{R})$. □

Proposition 25. *Letting* $W^n(0) = 0$,

$$\lim_{n\to\infty} \mathbf{P}(W^{(n)}(1) \le x) = \mathbf{E}\left[e^{-2(\theta_1+\theta_2)(L_1-x)}\mathbf{1}_{\{x\le L_1\}} + \mathbf{1}_{\{x>L_1\}}\right],$$

where $L_1 = B(1) - \inf_{0\le s\le 1} B(s)$.

Proof. Let $\tilde{B}(t) = B(1) - B(1-t)$, $\tilde{T}_x = \inf\{t \ge 0 : \tilde{B}(t) \ge x\} \wedge 1$, $T_x = 1 - \tilde{T}_x$ and let $Z = \{t \in [0,1] : B(t) = \inf_{t\le s\le 1} B(s)\}$ as in the proof of Proposition 18.

We observe that the process $W^{(n)}$ will be at zero for some time after T_x if $Y_k^{(n)} = 0$ and $W^{(n)}((k+1)/n) < B((k+1)/n) - \inf_{(k+1)/n\le s\le 1} B(s)$ for all $k \in \mathbb{Z}$ with $0 \le k/n \le T_x$ such that $[\frac{k}{n}, \frac{k+1}{n}) \cap Z \ne \emptyset$. We observe that $W^{(n)}((k+1)/n) < B((k+1)/n) - \inf_{(k+1)/n\le s\le 1} B(s)$ is equivalent to $V^{(n,k)} > \inf_{(k+1)/n\le s\le 1} B(s) - \inf_{k/n\le s\le (k+1)/n} B(s)$. This leads to the following inequality

$$\begin{aligned}
&\mathbf{P}(W^{(n)}(1) \le x) \\
&\le \mathbf{P}\Bigg(Y_k^{(n)} = 0 \text{ and } V^{(n,k)} > \inf_{(k+1)/n\le s\le 1} B(s) - \inf_{k/n\le s\le (k+1)/n} B(s), \\
&\qquad \forall k \in \mathbb{Z} \text{ with } 0 \le k/n < T_x \text{ s.t. } \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \ne \emptyset\Bigg). \qquad (2.34)
\end{aligned}$$

The process $W^{(n)}$ never returns to zero after $\inf\{t \in Z : t \in [\frac{k}{n}, \frac{k+1}{n})$ with $Y_k^{(n)} = 1\}$ and $W^{(n)}$ also never returns to zero after $\inf\{t \in Z : t \in [\frac{k}{n}, \frac{k+1}{n})$ with $Y_k^{(n)} = 0$ and $V^{(n,k)} < \inf_{(k+1)/n\le s\le 1} B(s) - \inf_{k/n\le s\le (k+1)/n} B(s)$. This leads to the second inequality

$$\begin{aligned}
&\mathbf{P}(W^{(n)}(1) \leq x) \\
&\geq \mathbf{P}\left(Y_k^{(n)} = 0 \text{ and } V^{(n,k)} > \inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s),\right. \\
&\qquad \left.\forall k \in \mathbb{Z} \text{ with } 0 \leq k/n < (T_x - 1) \text{ s.t. } \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\right). \qquad (2.35)
\end{aligned}$$

If the expressions in (2.34) and (2.35) converge as $n \to \infty$ then they do so to the same value. Thus we set about finding the probability on the right hand side of (2.34).

We have for each k

$$\begin{aligned}
&\mathbf{P}\left(Y_k^{(n)} = 0 \text{ and } V^{(n,k)} > \inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s)\right) \\
&\quad = \mathbf{E}\left[(1 - p(n)) \exp\left(-2\theta_1 \left(\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s)\right)\right)\right].
\end{aligned}$$

Let $M_x^{(n)} = \#\{k \in \mathbb{Z} : 0 \leq k/n < T_x, \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$, let $N^{(n)} = \#\{k \in \{0, 1, \ldots, n-1\} : \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$ and let $N_x^{(n)} = N^{(n)} - M_x^{(n)}$ as before. We also define the set $\mathcal{S}$ by $\mathcal{S} = \{k \in \mathbb{Z} : 0 \leq k/n < T_x, \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\}$. Thus

$$\begin{aligned}
&\mathbf{P}\left(Y_k^{(n)} = 0 \text{ and } V^{(n,k)} > \inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s),\right. \\
&\qquad \left.\forall k \in \mathbb{Z} \text{ with } 0 \leq k/n \leq 1 \text{ s.t. } \left[\frac{k}{n}, \frac{k+1}{n}\right) \cap Z \neq \emptyset\right) \\
&= \mathbf{E}\left[\prod_{k \in \mathcal{S}} \left(1 - \frac{p}{\sqrt{n}}\right) \exp\left(-2\theta_1 \left(\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s)\right)\right)\right] \\
&= \mathbf{E}\left[\exp\left(\sum_{k \in \mathcal{S}} \ln\left(1 - \frac{p}{\sqrt{n}}\right)\right.\right. \\
&\qquad \left.\left. + \sum_{k \in \mathcal{S}} -2\theta_1 \left(\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s)\right)\right)\right].
\end{aligned}$$

Now, for each $k \in \mathcal{S}$, $[k/n, (k+1)/n)$ contains a point of last exit for B, therefore $\inf_{k/n \leq s \leq (k+1)/n} B(s) = \inf_{k/n \leq s \leq 1} B(s)$. Also, for all $k \in \mathbb{Z}$ such that $0 \leq k/n < T_x$ and $k \notin \mathcal{S}$ we have $\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq 1} B(s) = 0$. Thus

$$\begin{aligned}
&\sum_{k \in \mathcal{S}} -2\theta_1 \left(\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq (k+1)/n} B(s) \right) \\
&= \sum_{k=0}^{[nT_x]} -2\theta_1 \left(\inf_{(k+1)/n \leq s \leq 1} B(s) - \inf_{k/n \leq s \leq 1} B(s) \right) \\
&\to 2\theta_1 \left(\inf_{0 \leq s \leq 1} B(s) - \inf_{T_x \leq s \leq 1} B(s) \right) \qquad \text{as } n \to \infty \\
&= 2\theta_1 (L_1 - x \wedge L_1).
\end{aligned}$$

It follows therefore that

$$\lim_{n \to \infty} \mathbf{P}(W_1^{(n)} \leq x) = \lim_{n \to \infty} \mathbf{E} \left[\exp \left(M_x^{(n)} \ln (1 - p(n)) \right) e^{-2\theta_1 (L_1 - x \wedge L_1)} \right]$$

and we have seen in the proof of Proposition 18 that $\exp \left(M_x^{(n)} \ln (1 - p(n)) \right)$ converges in expectation to $e^{-2\theta_2 (L_1 - x \wedge L_1)}$. Thus because M_x^n converges in L^2 we have, for $x \geq 0$,

$$\lim_{n \to \infty} \mathbf{P}(W_1^{(n)} \leq x) = \mathbf{E} \left[e^{-2(\theta_1 + \theta_2)(L_1 - x \wedge L_1)} \right].$$

□

We now consider a sequence of pairs of Brownian motions whose difference behaves as the sequence of processes in proposition 24 above.

For $p \in (0, 1)$, let $\{Y_k, k \geq 0\}$ be a sequence of independent Bernoulli trials with $\mathbf{P}(Y_k = 1) = 1 - \mathbf{P}(Y_k = 0) = p$. Then let X and $X^{(n)}$ both be Brownian motions started at x_1 and x_2 respectively such that over the interval

$[k/n,(k+1)/n]$, X and $X^{(n)}$ behave as a pair of independent Brownian motions if $Y_k = 1$ and as a pair of θ-coupled Brownian motions if $Y_k = 0$. We call the pair $(X, X^{(n)})$ a pair of (p, θ, n)-coupled Brownian motions.

Proposition 26. *Let $p = p(n)$ be such that*

$$\lim_{n\to\infty} 2\sqrt{\frac{n}{\pi}}p(n) = \theta_2 \in (0,\infty).$$

Then a pair of (p, θ_1, n)-coupled Brownian motions converges in distribution to a pair of $\theta_1 + \theta_2$-coupled Brownian motions.

Proof. The proof follows a similar argument to the proof of proposition 23 except we use the result from proposition 24 instead of Proposition 17. □

2.7.1 Counting zeros and local time

We show some L^2 convergence results about the local time of Brownian motion, which have been used earlier in Section 2.7. Here $\underline{x} = \sup\{i \in \mathbb{Z} : i < x\}$.

Proposition 27. *Let B be a Brownian motion started from zero. For fixed $t \geq 0$ let*

$$N^{(n)} = \#\left\{k \in \{0,1,2,\ldots,\underline{nt}\} : \exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \text{ with } |B(t)| = 0\right\}$$

and let L_t be the local time at 0 of B by time t. Then $\frac{N^{(n)}}{\sqrt{n}}$ converges in L^2 to $2\sqrt{\frac{2}{\pi}}L_t$ as n tends to infinity.

Proposition 28. *Let τ be a bounded stopping time. Let B be a Brownian motion started from zero, let*

$$N^{(n)}_\tau = \#\left\{k \in \{0,1,2,\ldots,\underline{n\tau}\} : \exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \text{ with } |B(t)| = 0\right\}$$

and let L_t be the local time at 0 *of* B *by time* t*. Then* $\frac{N_\tau^{(n)}}{\sqrt{n}}$ *converges in* L^2 *to* $2\sqrt{\frac{2}{\pi}}L_\tau$ *as* n *tends to infinity.*

Proof. Assume Proposition 27. Without loss of generality assume τ is bounded by 1. Let $M_\tau^m = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{N_\tau^{(m)}}{\sqrt{m}}$, then

$$\begin{aligned}(M_\tau^m - L_\tau)^2 &\leq \sup_{0<s<1}(M_s^m - L_s)^2 \\ &= \sup_{k\in\{0,\dots,n-1\}}\ \sup_{\frac{k}{n}\leq s\leq\frac{k+1}{n}}(M_s^m - L_s)^2.\end{aligned}$$

As both M_t and L_t are increasing with t we have that

$$\begin{aligned}\sup_{\frac{k}{n}\leq s\leq\frac{k+1}{n}}(M_s^m - L_s)^2 &\leq \left(|L_{\frac{k+1}{n}} - M^m_{\frac{k+1}{n}}| + |L_{\frac{k}{n}} - M^m_{\frac{k}{n}}| + |L_{\frac{k+1}{n}} - L_{\frac{k}{n}}|\right)^2 \\ &\leq 3\left(\left(L_{\frac{k+1}{n}} - M^m_{\frac{k+1}{n}}\right)^2 + \left(L_{\frac{k}{n}} - M^m_{\frac{k}{n}}\right)^2 + \left(L_{\frac{k+1}{n}} - L_{\frac{k}{n}}\right)^2\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\mathbf{E}\left[(M_\tau^m - L_\tau)^2\right] \\ &\quad\leq 3\mathbf{E}\left[\sup_{k\in\{0,\dots,n-1\}}\left(M^m_{\frac{k}{n}} - L_{\frac{k}{n}}\right)^2\right] + 3\mathbf{E}\left[\sup_{k\in\{0,\dots,n-1\}}\left(M^m_{\frac{k+1}{n}} - L_{\frac{k+1}{n}}\right)^2\right] \\ &\quad\quad + 3\mathbf{E}\left[\sup_{k\in\{0,\dots,n-1\}}\left(L_{\frac{k+1}{n}} - L_{\frac{k}{n}}\right)^2\right]. \qquad (2.36)\end{aligned}$$

Firstly consider

$$\mathbf{E}\left[\sup_{k\in\{0,\dots,n-1\}}\left(M^m_{\frac{k}{n}} - L_{\frac{k}{n}}\right)^2\right] \leq \sum_{k=0}^{n-1}\mathbf{E}\left[\left(M^m_{\frac{k}{n}} - L_{\frac{k}{n}}\right)^2\right].$$

By Proposition 27 we have that M_t^m converges in L^2 to L_t, as $m\to\infty$, for any fixed time t. Therefore $\sum_{k=0}^{n-1}\mathbf{E}\left[\left(M^m_{\frac{k}{n}} - L_{\frac{k}{n}}\right)^2\right] \to 0$ as $m\to\infty$ for any fixed n. Similarly $\sum_{k=0}^{n-1}\mathbf{E}\left[\left(M^m_{\frac{k+1}{n}} - L_{\frac{k+1}{n}}\right)^2\right] \to 0$ as $m \to \infty$.This takes care of the first two terms of (2.36). Thus

$$\lim_{m\to\infty}\mathbf{E}\left[(M^m_\tau - L_\tau)^2\right] \leq 3\mathbf{E}\left[\sup_{k\in\{0,\ldots,n-1\}}\left(L_{\frac{k+1}{n}} - L_{\frac{k}{n}}\right)^2\right].$$

This holds for all $n \geq 1$, and the result follows, by letting $n \to \infty$, from the fact that the local time of Brownian motion is almost surely continuous. □

Proof of Proposition 27. Our strategy is to consider

$$\mathbf{E}\left[\left(\frac{N^{(n)}}{\sqrt{n}} - 2\sqrt{\frac{2}{\pi}}L_t\right)^2\right] = \mathbf{E}\left[\frac{(N^{(n)})^2}{n}\right] - 4\sqrt{\frac{2}{\pi}}\mathbf{E}\left[\frac{N^{(n)}}{\sqrt{n}}L_t\right] + \frac{8}{\pi}\mathbf{E}[L_t^2].$$

We then find the value of each term on the right as $n \to \infty$ and show the resulting summation is 0.

To make notation simpler we prove the theorem for $t = 1$, the case with a general fixed t being a straightforward generalisation. Firstly we use the fact that the local time at t, L_t, is equal in distribution to $|B(t)|$ so that

$$\mathbf{E}\left[L_1^2\right] = \mathbf{E}[B(1)^2] = 1.$$

Next we shall find the value of $\mathbf{E}\left[\frac{N^{(n)}}{\sqrt{n}}L_1\right]$. $N^{(n)}$ can be written as the sum of indicator functions,

$$N^{(n)} = \mathbf{1}_{A_0} + \mathbf{1}_{A_1} + \mathbf{1}_{A_2} + \cdots + \mathbf{1}_{A_{n-1}},$$

where A_k is the event $\exists t \in [\frac{k}{n}, \frac{k+1}{n})$ with $B(t) = 0$. Then we need to find $\mathbf{E}[L_1\mathbf{1}_{A_k}]$. Fix some $a \in [0,1]$ and consider A, the event $B_t = 0$ for some $t \in$

$[a, a+\epsilon)$. Then

$$\mathbf{E}[L_1 \mathbf{1}_A] = \mathbf{E}[L_a \mathbf{1}_A] + \mathbf{E}[L_{a,a+\epsilon} \mathbf{1}_A] + \mathbf{E}[L_{a+\epsilon,1} \mathbf{1}_A], \tag{2.37}$$

where $L_{s,t} = L_t - L_s$ is the local time (at zero) which has accrued between time s and t. Firstly calculate $\mathbf{E}[L_a \mathbf{1}_A]$. We have, using the reflection principle

$$\mathbf{P}(L_a \in dl, |B_a| \in dx) = \sqrt{\frac{2}{\pi a^3}}(x+l)e^{-\frac{(x+l)^2}{2a}} dl dx$$

and so for $x > 0$

$$\mathbf{E}[L_a \mathbf{1}_{\{|B_a| \in dx\}}] = \int_0^\infty \sqrt{\frac{2}{\pi a^3}}(x+l)le^{-\frac{(x+l)^2}{2a}} dl dx = \text{erfc}\left(\frac{x}{\sqrt{2a}}\right) dx. \tag{2.38}$$

Here we are using the convention that $\text{erfc}(x) = \int_x^\infty \frac{2}{\sqrt{\pi}} e^{-y^2} dy$. Given the value of $|B_a|$ we can find the probability of a zero in the interval $[a, a+\epsilon)$

$$P(A \| B_a| = x) = \mathbf{P}(|B_\epsilon| > x) = \int_x^\infty \sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{y^2}{2\epsilon}} dy = \text{erfc}\left(\frac{x}{\sqrt{2\epsilon}}\right). \tag{2.39}$$

Then combining (2.38) and (2.39) we have

$$\mathbf{E}[L_a \mathbf{1}_A] = \int_0^\infty \text{erfc}\left(\frac{x}{\sqrt{2a}}\right) \text{erfc}\left(\frac{x}{\sqrt{2\epsilon}}\right) dx. \tag{2.40}$$

For convenience we introduce some notation. Let $F_t(x) = \text{erfc}\left(\frac{x}{\sqrt{2t}}\right)$ and $f_t(x) = \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}}$. Therefore we have $\frac{d^2}{dx^2} F_t(x) = -\frac{d}{dx} f_t(x) = \frac{x}{t} f_t(x)$. Then integrating (2.40) by parts we have

$$\mathbf{E}[L_a \mathbf{1}_A] = \int_0^\infty x\left(-f_a(x) F_\epsilon(x) - f_\epsilon(x) F_a(x)\right) dx.$$

A further integration by parts gives

$$\begin{aligned}\mathbf{E}[L_a\mathbf{1}_A] &= \epsilon f_\epsilon(x)F_a(x)\Big|_0^\infty + af_a(x)F_\epsilon(x)\Big|_0^\infty \\ &\quad - \int_0^\infty \epsilon f_\epsilon(x)f_a(x) + af_a(x)f_\epsilon(x)dx \\ &= \sqrt{\frac{2}{\pi}}(\sqrt{a}+\sqrt{\epsilon}-\sqrt{a+\epsilon}). \qquad (2.41)\end{aligned}$$

Replacing a with k/n and ϵ with $1/n$ we have

$$\mathbf{E}[L_{k/n}\mathbf{1}_{A_k}] = \sqrt{\frac{2}{\pi n}}(1+\sqrt{k}-\sqrt{1+k}).$$

$\frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1} L_{k/n}\mathbf{1}_{A_k}\right]$ is then given by

$$\sqrt{\frac{2}{\pi}\frac{1}{n}}\sum_{k=0}^{n-1}(1+\sqrt{k}-\sqrt{1+k}) = \sqrt{\frac{2}{\pi}\frac{1}{n}}\int_0^n(1+\sqrt{\underline{x}}-\sqrt{1+\underline{x}})dx$$

where $\underline{x} = \sup\{i \in \mathbb{Z} : i < x\}$. Then we substitute $ny = x$ to give

$$\frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1} L_{k/n}\ \mathbf{1}_{A_k}\right] = \sqrt{\frac{2}{\pi}}\int_0^1(1+\sqrt{\underline{ny}}-\sqrt{1+\underline{ny}})dy$$

But $(1+\sqrt{\underline{ny}}-\sqrt{1+\underline{ny}})$ converges to 1 as $n \to \infty$ for all $y > 0$, and also $|1+\sqrt{\underline{ny}}-\sqrt{1+\underline{ny}}| < 1$ and so by the dominated convergence theorem,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1} L_{k/n}\mathbf{1}_{A_k}\right] = \sqrt{\frac{2}{\pi}}.$$

Next we look at the middle term in (2.37), namely $\mathbf{E}[L_{a,a+\epsilon}\mathbf{1}_A]$. If the event A does not occur, that is $|B(t)| > 0$ for all $t \in [a, a+\epsilon)$, then of course $L_{a,a+\epsilon} = 0$. Thus

$$\mathbf{E}[L_{a,a+\epsilon}\mathbf{1}_A] = \mathbf{E}[L_{a,a+\epsilon}] = \mathbf{E}[L_{a+\epsilon}] - \mathbf{E}[L_a] = \sqrt{\frac{2}{\pi}}(\sqrt{a+e}-\sqrt{a}).$$

Using the same method as before, it follows that

$$\frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1} L_{\frac{k}{n},\frac{k+1}{n}} \mathbf{1}_{A_k}\right] = \sqrt{\frac{2}{\pi}}\int_0^1 (\sqrt{1+\underline{ny}} - \sqrt{\underline{ny}})dy.$$

This time $(\sqrt{1+\underline{ny}} - \sqrt{\underline{ny}})$ converges to 0 as $n \to \infty$ for all $y > 0$, and $|(\sqrt{\underline{ny}} - \sqrt{1+\underline{ny}})| < 1$, so by the dominated convergence theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1} L_{\frac{k}{n},\frac{k+1}{n}} \mathbf{1}_{A_k}\right] = 0.$$

Now we move onto the last term in (2.37), $\mathbf{E}[L_{a+\epsilon,1}\mathbf{1}_A]$.

$y \mapsto \mathbf{E}_y[L_t]$ is decreasing (for $y \geq 0$), so by conditioning on $|B(a+\epsilon)|$ it follows that $\mathbf{E}_0[L_{a+\epsilon,1}\mathbf{1}_A] \leq \mathbf{E}_0[\mathbf{1}_A]\mathbf{E}_0[L_{0,1-a-\epsilon}]$. For B, a standard one dimensional Brownian motion starting from 0, the probability of being at zero for some time between a and $a+\epsilon$ is given by,

$$\mathbf{P}(\exists t \in [a, a+\epsilon) \text{ with } B(t) = 0) = \begin{cases} \frac{2}{\pi}\arctan\left(\sqrt{\frac{\epsilon}{a}}\right) & a > 0 \\ 1 & a = 0. \end{cases} \tag{2.42}$$

This is a well known result that can be found in many textbooks, see for example [Kni81]. For simplicity in later notation we will assume that $\frac{2}{\pi}\arctan(\infty) = 1$. This result gives us

$$\mathbf{E}[L_{a+\epsilon,1}\mathbf{1}_A] \leq \frac{2}{\pi}\arctan\left(\sqrt{\frac{\epsilon}{a}}\right)\sqrt{\frac{2}{\pi}}\sqrt{1-a-\epsilon}.$$

Now consider the value of $\mathbf{E}[L_{a,1}\mathbf{1}_A] = \mathbf{E}[L_{a,1}|A]\mathbf{P}(A)$. Given A has occurred there exists $t \in [a, a+\epsilon]$ with $B(t) = 0$. Let $\tau = \inf\{t \geq a : B(t) = 0\}$. Then let $\tilde{B}(t) = B_{t-\tau} - B_\tau$ which by the strong Markov property is a Brownian motion started at zero independent of $(B(t) : 0 \leq t \leq \tau)$. Consequently

$\mathbf{E}[L_{a,1}(B)|A] = \mathbf{E}[L_{\tau,1}(B)|A] \geq \mathbf{E}[L_{\tau,\tau+1-a-\epsilon}(B)|A] = \mathbf{E}[L_{1-a-\epsilon}(\tilde{B})] = \sqrt{\frac{2}{\pi}}\sqrt{1-a-\epsilon}$ and therefore

$$\mathbf{E}[L_{a,1}\mathbf{1}_A] \geq \frac{2}{\pi}\arctan\left(\sqrt{\frac{\epsilon}{a}}\right)\sqrt{\frac{2}{\pi}}\sqrt{1-a-\epsilon}.$$

This gives us an upper and lower bound on $\mathbf{E}[L_{a+\epsilon,1}\mathbf{1}_A]$

$$\begin{aligned}
&\left(\frac{2}{\pi}\right)^{3/2}\arctan\left(\sqrt{\frac{\epsilon}{a}}\right)\sqrt{1-a-\epsilon} - \mathbf{E}[L_{a,a+\epsilon}\mathbf{1}_A] \\
&\leq \mathbf{E}[L_{a,1}\mathbf{1}_A] - \mathbf{E}[L_{a,a+\epsilon}\mathbf{1}_A] = \mathbf{E}[L_{a+\epsilon,1}\mathbf{1}_A] \\
&\leq \left(\frac{2}{\pi}\right)^{3/2}\arctan\left(\sqrt{\frac{\epsilon}{a}}\right)\sqrt{1-a-\epsilon}.
\end{aligned}$$

Replacing a with k/n and ϵ with $1/n$ and taking the sum over k we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\left(\frac{2}{\pi}\right)^{3/2}\sum_{k=0}^{n-1}\arctan\left(\sqrt{\frac{1}{k}}\right)\sqrt{1-\frac{k}{n}-\frac{1}{n}} - \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\mathbf{E}[L_{\frac{k}{n},\frac{k+1}{n}}\mathbf{1}_A] \\
&\leq \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\mathbf{E}[L_{\frac{k+1}{n},1}\mathbf{1}_{A_k}] \\
&\leq \frac{1}{\sqrt{n}}\left(\frac{2}{\pi}\right)^{3/2}\sum_{k=0}^{n-1}\arctan\left(\sqrt{\frac{1}{k}}\right)\sqrt{1-\frac{k}{n}-\frac{1}{n}}.
\end{aligned}$$

We have already seen $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\mathbf{E}[L_{\frac{k}{n},\frac{k+1}{n}}\mathbf{1}_A] \to 0$ as $n \to \infty$ and so

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\mathbf{E}[L_{\frac{k+1}{n},1}\mathbf{1}_{A_k}] \\
&\quad = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{2}{\pi}\right)^{3/2}\sum_{k=0}^{n-1}\arctan\left(\sqrt{\frac{1}{k}}\right)\sqrt{1-\frac{k}{n}-\frac{1}{n}}.
\end{aligned}$$

Rewriting the summation on the right in terms of an integral gives us

$$
\begin{aligned}
&\frac{2}{\pi}\sqrt{\frac{2}{\pi n}}\sum_{k=0}^{n-1}\arctan\left(\frac{1}{\sqrt{k}}\right)\sqrt{1-\frac{k}{n}-\frac{1}{n}} \\
&=\frac{2}{\pi}\sqrt{\frac{2}{\pi n}}\int_0^n \arctan\left(\frac{1}{\sqrt{\underline{x}}}\right)\sqrt{1-\frac{\underline{x}}{n}-\frac{1}{n}}dx \\
&=\frac{2}{\pi}\sqrt{\frac{2n}{\pi}}\int_0^1 \arctan\left(\frac{1}{\sqrt{\underline{ny}}}\right)\sqrt{1-\frac{\underline{ny}}{n}-\frac{1}{n}}dy.
\end{aligned}
$$

Taking limits for the integrand we have

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{2}{\pi}\sqrt{\frac{2n}{\pi}}\arctan\left(\frac{1}{\sqrt{\underline{ny}}}\right)\sqrt{1-\frac{\underline{ny}}{n}-\frac{1}{n}} \\
&=\lim_{n\to\infty}\frac{2\sqrt{n}}{\pi}\arctan\left(\frac{1}{\sqrt{\underline{ny}}}\right)\sqrt{\frac{2}{\pi}}\sqrt{1-\frac{\underline{ny}}{n}} \\
&=\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\sqrt{\frac{1-y}{y}}.
\end{aligned}
$$

We also have the bounds $|\sqrt{n}\arctan(1/\sqrt{\underline{ny}})| \le 1/\sqrt{y}$ and $|\sqrt{1-\frac{\underline{ny}}{n}}| \le 1$for all $y \in [0,1]$ and for all $n \ge 1$, so by the dominated convergence theorem

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{\sqrt{n}}\mathbf{E}\left[\sum_{k=0}^{n-1}L_{\frac{k+1}{n},1}\mathbf{1}_{A_k}\right] &= \frac{2}{\pi}\sqrt{\frac{2}{\pi}}\int_0^1\sqrt{\frac{1-y}{y}}dy \\
&=\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\int_{\sin^{-1}(0)}^{\sin^{-1}(1)}\frac{\cos u}{\sin u}(2\sin u\cos u)du \\
&=\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{\pi/2}\frac{1}{2}(\cos(2u)+1)du \\
&=\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\frac{\pi}{2}=\sqrt{\frac{2}{\pi}}.
\end{aligned}
$$

Finally we put the three parts of (2.37) back together to get

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{E}\left[\frac{N^{(n)}}{\sqrt{n}}L_1\right] &= \lim_{n\to\infty} \mathbf{E}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} L_{k/n}\mathbf{1}_{A_k}\right] \\
&\quad + \lim_{n\to\infty} \mathbf{E}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} L_{\frac{k}{n},\frac{k+1}{n}}\mathbf{1}_{A_k}\right] \\
&\quad + \lim_{n\to\infty} \mathbf{E}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} L_{\frac{k+1}{n},1}\mathbf{1}_{A_k}\right] \\
&= 2\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

We now need to calculate the value of $\mathbf{E}\left[\frac{(N^{(n)})^2}{n}\right]$. Again we use the fact that $N^{(n)}$ can be written as the sum of indicator functions so that

$$\left(N^{(n)}\right)^2 = \sum_{k=0}^{n-1}\mathbf{1}_{A_k} + 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbf{1}_{A_i}\mathbf{1}_{A_j}. \tag{2.43}$$

Using the result (2.42)

$$\begin{aligned}
\mathbf{E}\left[\sum_{k=0}^{n-1}\mathbf{1}_{A_k}\right] = \sum_{k=0}^{n-1}\mathbf{P}(A_k) &= \sum_{k=0}^{n-1}\frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{k}}\right) \\
&= \int_0^n \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{x}}\right)dx.
\end{aligned}$$

So we have

$$\frac{1}{n}\mathbf{E}\left[\sum_{k=0}^{n-1}\mathbf{1}_{A_k}\right] = \int_0^1 \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{ny}}\right)dy$$

$\frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{ny}}\right)$ converges to zero as $n\to\infty$ for all $y>0$ and $|\arctan(x)|\le\frac{\pi}{2}$ so we use the dominated convergence theorem to give

$$\frac{1}{n}\mathbf{E}\left[\sum_{k=0}^{n-1}\mathbf{1}_{A_k}\right] \to 0 \text{ as } n\to\infty.$$

Now let us consider the second term in (2.43),

$$2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \mathbf{1}_{A_i}\mathbf{1}_{A_j} = 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \mathbf{P}(A_i \cap A_j).$$

Consider $\mathbf{P}(A_i \cap A_j)$ where $i < j$. $y \mapsto \mathbf{P}_y(A)$ is decreasing and consequently, by conditioning on $B((i+1)/n)$, we have

$$\begin{aligned}\mathbf{P}(A_i \cap A_j) &\leq \mathbf{P}(A_i)\mathbf{P}(A_{j-i-1}) \\ &= \frac{4}{\pi^2}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i-1}}\right).\end{aligned}$$

We also have $\mathbf{P}(A_i \cap A_j) = \mathbf{P}(A_i)\mathbf{P}(A_j|A_i)$ Given that A_i has occurred the Brownian motion must be at zero between $\frac{i}{n}$ and $\frac{i+1}{n}$. i.e. $\exists t \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$ with $|B(t)| = 0$. Let $\tau = \inf\{t \geq i/n : B(t) = 0\}$. Let $\tilde{B}(t) = B_{\tau+t}$ Then by the strong Markov property for Brownian motion $\tilde{B}$ is a Brownian motion started at zero independent of $(B(t); t \leq \tau)$. $\mathbf{P}(A_j|A_i) = \mathbf{P}(\exists t \in [j/n - \tau, (j+1)/n - \tau)$ with $\tilde{B}(t) = 0)$ Now as this probability increases as τ increases and $\tau \geq i/n$, so this probability is bounded below by $\mathbf{P}(\exists t \in [(j-i)/n, (j-i+1)/n)$ with $\tilde{B}(t) = 0) = \mathbf{P}(A_{j-i}) = \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{j-i}}\right)$. Therefore we have

$$\begin{aligned}&\frac{4}{\pi^2}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i}}\right) \\ &\qquad \leq \mathbf{P}(A_i \cap A_j) \leq \\ &\qquad \frac{4}{\pi^2}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i-1}}\right). \quad (2.44)\end{aligned}$$

First of all we show that the upper and lower bounds are equivalent when we take the sum over i and j and then take the limit. Consider the sum of the right hand side.

$$\frac{4}{\pi^2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i-1}}\right)$$
$$=\frac{4}{\pi^2}\sum_{i=0}^{n-1}\sum_{j=i}^{n-2}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i}}\right)$$

Now that when $i=j$ we have $\arctan\left(\frac{1}{\sqrt{j-i}}\right)=1$, and so this upper bound equals,

$$\frac{4}{\pi^2}\sum_{i=0}^{n-1}\arctan\left(\frac{1}{\sqrt{i}}\right)+\frac{4}{\pi^2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-2}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i}}\right).$$

The first term, we have already seen, will disappear when we divide by n and take the limit. The second term is less than

$$\frac{4}{\pi^2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i}}\right)$$

which is the same as the lower bound that would result from summing the left hand side (2.44) over values of i and j with $i < j$. Therefore we have

$$
\begin{aligned}
&\lim_{n\to\infty} 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} P(A_i \cap A_j) \\
&= \lim_{n\to\infty} \frac{8}{n\pi^2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j-i}}\right) \\
&= \lim_{n\to\infty} \frac{8}{n\pi^2}\sum_{i=0}^{n-1}\sum_{j=1}^{n-1-i} \arctan\left(\frac{1}{\sqrt{i}}\right)\arctan\left(\frac{1}{\sqrt{j}}\right) \\
&= \lim_{n\to\infty} \frac{8}{n\pi^2}\int_0^n\int_1^{n-\underline{x}} \arctan\left(\frac{1}{\sqrt{\underline{x}}}\right)\arctan\left(\frac{1}{\sqrt{\underline{y}}}\right) dydx \\
&= \lim_{n\to\infty} \frac{8n}{\pi^2}\int_0^1\int_0^{1-n\underline{x}/n} \arctan\left(\frac{1}{\sqrt{n\underline{x}}}\right)\arctan\left(\frac{1}{\sqrt{n\underline{y}}}\right) dydx
\end{aligned}
$$

Then taking the limit of the integrand

$$
\frac{8n}{\pi^2}\arctan\left(\frac{1}{\sqrt{n\underline{x}}}\right)\arctan\left(\frac{1}{\sqrt{n\underline{y}}}\right) \to \frac{8}{\pi^2}\frac{1}{\sqrt{x}}\frac{1}{\sqrt{y}} \qquad \text{as } n \to \infty.
$$

The above sequence approaches the limit from below, so again by DOM we have

$$
\begin{aligned}
\lim_{n\to\infty} \frac{8}{n\pi^2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} P(A_i \cap A_j) &= \frac{8}{\pi^2}\int_0^1\int_0^{1-x} \frac{1}{\sqrt{x}}\frac{1}{\sqrt{y}} dxdy \\
&= \frac{16}{\pi^2}\int_0^1 \sqrt{\frac{1-x}{x}} dx \\
&= \frac{16}{\pi^2}\frac{\pi}{2} = \frac{8}{\pi}
\end{aligned}
$$

Summarising the results so far we have $\mathbf{E}[L_1^2] = 1$, $\lim_{n\to\infty}\mathbf{E}\left[\frac{N^{(n)}}{\sqrt{n}}L_1\right] =$

$2\sqrt{\frac{2}{\pi}}$, and $\lim_{n\to\infty} \mathbf{E}\left[\frac{(N^{(n)})^2}{n}\right] = \frac{8}{\pi}$. This means we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\left[\left(\frac{N^{(n)}}{\sqrt{n}} - 2\sqrt{\frac{2}{\pi}}L_1\right)^2\right] \\
&\quad = \lim_{n\to\infty} \mathbf{E}\left[\frac{(N^{(n)})^2}{n}\right] - 4\sqrt{\frac{2}{\pi}} \lim_{n\to\infty} \mathbf{E}\left[\frac{N^{(n)}}{\sqrt{n}}L_1\right] + \frac{8}{\pi}\mathbf{E}[L_1^2] \\
&\quad = \frac{8}{\pi} - \frac{16}{\pi} + \frac{8}{\pi} = 0.
\end{aligned}$$

□

Chapter 3

A Martingale problem

In this chapter we aim to characterise an $\mathbb{R}^N$-valued diffusion $(X(t); t \geq 0)$ with the property that each pair of coordinates $(X_i(t), X_j(t); t \geq 0)$ is distributed as a pair of θ-coupled Brownian motions as given in Proposition 15. In fact we construct a family of diffusions with a natural consistency property, that is we have a diffusion for each $N \in \mathbb{N}$ such that if we observe any $M \leq N$ coordinates of the N dimensional diffusion in the family then these M coordinates are distributed as the M dimensional diffusion in the family.

We aim to characterise this family of diffusions via a certain martingale problem, which we call the $\mathcal{A}_N^\theta$-martingale problem. Here θ stands for a family of parameters $(\theta(k:l); k, l \geq 0)$ where $\theta(k:l)$ is some sense represents the rate at which $k+l$ coordinates, when taking the same value, separate into k and l coordinates with two distinct values. We will show that if the family θ has certain consistency properties then for each $N \geq 1$ and $x \in \mathbb{R}^N$, there exists a solution to the $\mathcal{A}_N^\theta$-martingale problem started at x and the law of any such solution is uniquely specified. Moreover the family of solutions for $N \geq 1$ are consistent in the sense described above. Material from this chapter appears in [HW06].

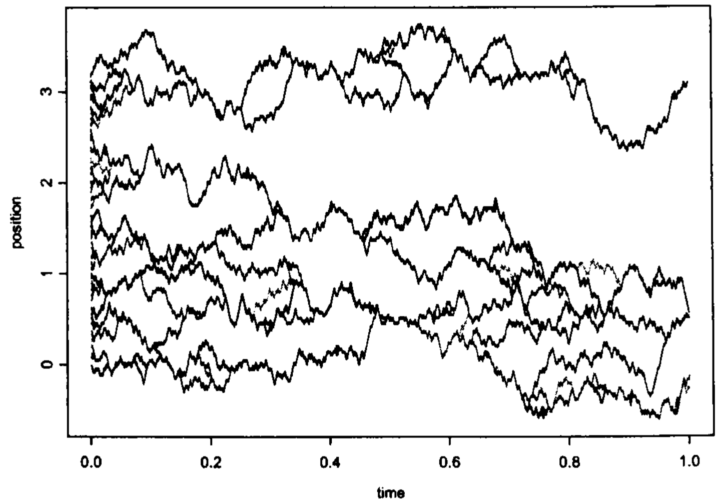

Figure 3.1: Solution to an $\mathcal{A}_N^\theta$-martingale problem

3.1 The martingale problem

First of all, we develop some notation. Suppose that I and J are disjoint subsets of $\{1,2,\ldots,N\}$. We define a vector $v = v_{IJ} \in \mathbb{R}^N$, that has components given by

$$v_i = \begin{cases} 0 & \text{if } i \notin I \cup J \\ +1 & \text{if } i \in I \\ -1 & \text{if } i \in J. \end{cases} \tag{3.1}$$

For each point $x \in \mathbb{R}^N$ we define a partition $\pi(x)$ of $\{1,2,\ldots,N\}$ such that i and j belong to the same component of $\pi(x)$ if and only if $x_i = x_j$. For example if $x \in \mathbb{R}^5$ is such that $x_5 < x_3 < x_2 = x_1 = x_4$ then $\pi(x) = \{\{1,2,4\},\{3\},\{5\}\}$. We define a set of vectors for each point x, which we denote by $\mathcal{V}(x)$. $\mathcal{V}(x)$ consists of every vector of the form $v = v_{IJ}$ where the disjoint union $I \cup J$ form one of the components of $\pi(x)$. Clearly given a partition π then the set of vectors $\mathcal{V}(x)$ is the same for all $x \in \mathbb{R}^N$ such that $\pi(x) = \pi$. In this way it is meaningful to write $\mathcal{V}(\pi)$ and $\mathcal{V}(x) = \mathcal{V}(\pi(x))$.

For each $x \in \mathbb{R}^N$ we define the cell containing x as

$$E(x) = \{y \in \mathbb{R}^N : y_i \leq y_j \text{ if and only if } x_i \leq x_j \text{ for all } 1 \leq i,j \leq N\}.$$

For example if $x \in \mathbb{R}^5$ is such that $x_5 < x_3 < x_2 = x_1 = x_4$ then $E(x)$ is the set of all possible $y \in \mathbb{R}^5$ such that $y_5 < y_3 < y_2 = y_1 = y_4$ holds. The collection of all possible cells in $\mathbb{R}^N$, which we call E_N, forms a partition of $\mathbb{R}^N$.

Note that $\pi(y) = \pi(x)$ for all $y \in E(x)$ hence it is meaningful to write $\pi(E)$, and therefore it is also meaningful to write $\mathcal{V}(E)$ for some cell E.

We note that the vectors in $\mathcal{V}(x)$ divide into two types. Firstly $v_{IJ} \in \mathcal{V}(x)$ such that either I or J are empty. In this case v_{IJ} points in a direction which remains in the cell $E(x)$ (for at least some small distance) . We write $\mathcal{V}_0(x)$ for the subset of $\mathcal{V}(x)$ containing all such v_{IJ}. For example if $x \in \mathbb{R}^5$ is such that $x_5 < x_3 < x_2 = x_1 = x_4$, then a possible $v_{IJ} \in \mathcal{V}_0(x)$ is $(-1,-1,0,-1,0)$ here I is empty and $J = \{1,2,4\}$. Clearly here if we move from x along this vector we remain in $E(x)$ at least for some small distance.

The remaining vectors in $\mathcal{V}(x)$ are vectors of the from v_{IJ} where both I and J are non-empty. In this case v_{IJ} points in a direction which immediately leaves the cell $E(x)$ into a new cell which we then call a neighbour of $E(x)$. We write $\mathcal{V}_+(x)$ for the subset of $\mathcal{V}(x)$ which contains all such v_{IJ}. For example if $x \in \mathbb{R}^5$ is such that $x_5 < x_3 < x_2 = x_1 = x_4$, then a possible $v_{IJ} \in \mathcal{V}_+(x)$ is $(1,1,0,-1,0)$ here $I = \{1,2\}$ and $J = \{4\}$. Clearly here if we move from x along this vector we immediately leave the cell $E(x)$ and enter the cell $\{y \in \mathbb{R}^5 : y_5 < y_3 < y_4 < y_2 = y_1\}$, which is then a neighbour of $E(x)$.

Let L_N be the space of real-valued functions defined on $\mathbb{R}^N$, that are continuous and whose restriction to each cell is linear.

Lemma 29. *L_N is a 2^N-1 dimensional vector space over $\mathbb{R}$, with one possible basis given by $f_1(x) = \min_i(x_i) + \max_i(x_i)$ and the 2^N-2 functions of the form*

$$f_v(x) = \inf_{i\in I, j\in J}(x_i - x_j)^+,$$

where I and J form a partition of $\{1,2,\dots,N\}$ and $v = v_{IJ}$.

Proof. It is straightforward to show that L_N is a vector space. It remains to show that the above functions span L_N and that they are linearly independent.

If D is the diagonal of $\mathbb{R}^N$, given by $D = \{x \in \mathbb{R}^N : x_1 = \cdots = x_N\}$, then $\mathcal{V}_+(D)$ is the set of vectors of the form $v = v_{IJ}$, where I and J form a partition of $\{1,\dots,N\}$. For each $u, v \in \mathcal{V}_+(D)$ with $u \neq v$, we have $f_v(v) = 2$, $f_u(v) = 0$ and $f_1(v) = 0$. Also for each $v \in \mathcal{V}_+(D)$, $f_1(\mathbf{1}) = 2$, $f_v(\mathbf{1}) = 0$, where $\mathbf{1} = (1,\dots,1)$. Thus the above functions are linearly independent.

Finally, for any $f \in L_N$

$$f(x) = \frac{1}{2}f(\mathbf{1})f_1(x) + \frac{1}{2}\sum_{v\in\mathcal{V}_+(D)} f(v)f_v(x). \tag{3.2}$$

We can verify this decomposition for the cell $\{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$. For any x in this cell $f(\mathbf{1})f_1(x) = f(\mathbf{1})(x_N + x_1) = f(x_1 + x_N, \dots, x_1 + x_N)$ and

$$\begin{aligned}
&\sum_{v\in\mathcal{V}_+(D)} f(v)f_v(x) \\
&\quad = f(-1,1,\dots,1)(x_2 - x_1) + f(-1,-1,\dots,1)(x_3 - x_2) + \cdots \\
&\qquad + f(-1,\dots,-1,1)(x_N - x_{N-1}) \\
&\quad = f(x_1 - x_N, 2x_2 - x_1 - x_N, \dots, 2x_{N-1} - x_1 - x_N, x_N - x_1).
\end{aligned}$$

Thus by the linearity of f within the cell $\{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$ and, by the continuity of f at the boundaries of the cell, the decomposition, (3.2), is verified for x in the cell $\{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$. Similarly, we can verify (3.2) for x in any cell such that coordinates are strictly ordered. Thus by the continuity of f we can verify (3.2) for all $x \in \mathbb{R}^N$. □

Let θ be a family of parameters $(\theta(k : l); k, l \geq 0)$. For some vector $v \in \mathcal{V}(x)$, let $\theta(v) = \theta(k : l)$ where $k = |I|$, and $l = |J|$, are the number of elements in I and J respectively, and are determined by $v = v_{IJ}$.

An operator $\mathcal{A}_N^\theta$, which acts on functions in L_N, is defined by

$$\mathcal{A}_N^\theta f(x) = \sum_{v \in \mathcal{V}(x)} \theta(v) \nabla_v f(x)$$

where $\nabla_v f(x)$ denotes the one sided gradient of f in the direction of v at point x, that is

$$\nabla_v f(x) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(f(x + v\epsilon) - f(x)).$$

We are now ready to define our martingale problem.

Definition 30. We say a continuous, $\mathbb{R}^N$-valued process $(X(t); t \geq 0)$, defined on a filtered probability space, solves the $\mathcal{A}_N^\theta$-martingale problem if for all $f \in L_N$,

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds \text{ is a martingale,}$$

and the bracket between components X_i and X_j is given by

$$\langle X_i, X_j \rangle_t = \int_0^t 1_{\{X_i(s) = X_j(s)\}} ds \text{ for } t \geq 0.$$

Note if X solves the $\mathcal{A}_N^\theta$-martingale problem then so does X under a permutation of coordinates. This fact, together with the proposition below, show that solutions to the $\mathcal{A}_N^\theta$-martingale problem, for appropriate choices of $(\theta(k:l); k, l \geq 0)$ form a consistent family of processes, in that if we take a subset of components of a solution to the martingale problem, then these components themselves solve the same martingale problem at a lower dimension.

Proposition 31. *Suppose that θ satisfies the consistency property:*

$$\theta(k:l) = \theta(k+1:l) + \theta(k:l+1) \text{ for all } k, l \geq 0 \tag{3.3}$$

Suppose that X is a solution to the $\mathcal{A}_N^\theta$-martingale problem, and let Y be the process consisting of the first $N-1$ components of X. Then Y is a solution to the $\mathcal{A}_{N-1}^\theta$-martingale problem.

Proof. Define $\rho : \mathbf{R}^N \to \mathbf{R}^{N-1}$ to be the projection onto the first $N-1$ components. Suppose that $g \in L_{N-1}$, and let $f = g \circ \rho$, which belongs to L_N. X being a solution to the $\mathcal{A}_N^\theta$-martingale problem implies that

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds \text{ is a martingale,}$$

and since $f(X(t)) = g(Y(t))$ we need to show that $\mathcal{A}_N^\theta f(X(s)) = \mathcal{A}_{N-1}^\theta g(Y(s))$. For this we verify that $\mathcal{A}_N^\theta f(x) = \mathcal{A}_{N-1}^\theta g(\rho(x))$ for all $x \in \mathbf{R}^N$.

Fix $x \in \mathbb{R}^N$ and let $y = \rho(x)$. We would like to show

$$\sum_{v \in \mathcal{V}(x)} \theta(v) \nabla_v f(x) = \sum_{u \in \mathcal{V}(y)} \theta(u) \nabla_u g(y) \tag{3.4}$$

We partition [1] $\mathcal{V}(x)$ into 3 sets $\mathcal{V}_1(x)$, $\mathcal{V}_2(x)$, and $\mathcal{V}_3(x)$, defined as

$$\mathcal{V}_1(x) = \{v_{IJ} \in \mathcal{V}(x); N \notin I \cup J\}$$
$$\mathcal{V}_2(x) = \{v_{IJ} \in \mathcal{V}(x); N \in I \cup J, I \cup J \neq \{N\}\}$$
$$\mathcal{V}_3(x) = \{v_{IJ} \in \mathcal{V}(x); I \cup J = \{N\}\}$$

and we partition [1] $\mathcal{V}(y)$ into two sets $\bar{\mathcal{V}}_1(y)$, $\bar{\mathcal{V}}_2(y)$, defined as

$$\bar{\mathcal{V}}_1(y) = \{\rho(v_{IJ}); v_{IJ} \in \mathcal{V}_1(x)\}$$
$$\bar{\mathcal{V}}_2(y) = \{\rho(v_{IJ}); v_{IJ} \in \mathcal{V}_2(x)\}$$

This is a partition of $\mathcal{V}(y)$ since for any $v_{IJ} \in \mathcal{V}_2(x)$, $(\rho(v), 0) \notin \mathcal{V}_1(x)$ and for any $v_{IJ} \in \mathcal{V}_1(x)$, $(\rho(v_{IJ}), \pm 1) \notin \mathcal{V}_2(x))$. We note that $\{\rho(v_{IJ}); v_{IJ} \in \mathcal{V}_3(x)\}$ contains only the vector of zeros and hence is not in $\mathcal{V}(y)$. $\mathcal{V}_3(x)$ is either empty or contains two vectors namely $v_1 = (0, \dots, 0, 1)$ and $v_2 = (0, \dots, 0, -1)$. As $f = g \circ \rho$ does not depend on x_N, we have $\nabla_{v_1} f(x) = \nabla_{v_2} f(x) = 0$ and so $\mathcal{V}_3(x)$ makes no contribution to the sum on the left of (3.4). The fact that $f = g \circ \rho$ does not depend on x_N means also that

$$\nabla_v f(x) = \nabla_{\rho(v)} g(y). \tag{3.5}$$

For all $v \in \mathcal{V}_1(x)$, there exists $u \in \bar{\mathcal{V}}_1(y)$ such that $u = \rho(v)$ and $(\rho(v), 0) = v$ hence $\theta(v) = \theta(u)$. This fact together with (3.5) gives us

$$\sum_{v \in \mathcal{V}_1(x)} \theta(v) \nabla_v f(x) = \sum_{u \in \bar{\mathcal{V}}_1(y)} \theta(u) \nabla_u g(y).$$

[1] By partition we mean exhaustive and mutually exclusive. Strictly speaking, we don't have a partition in the usual sense as the sets maybe empty, indeed one of $\mathcal{V}_2(x)$ and $\mathcal{V}_3(x)$ is always empty

We are left with showing

$$\sum_{v\in\mathcal{V}_2(x)} \theta(v)\nabla_v f(x) = \sum_{u\in\bar{\mathcal{V}}_2(y)} \theta(u)\nabla_u g(y). \tag{3.6}$$

There are twice as many vectors in $\mathcal{V}_2(x)$ as there are in $\bar{\mathcal{V}}_2(y)$. For each vector $u_{IJ} \in \bar{\mathcal{V}}_2(y)$ we have two different vectors $v_{I'J'}$ and $v_{I^*J^*}$ in $\mathcal{V}_2(x)$ such that $\rho(v_{I'J'}) = \rho(v_{I^*J^*}) = u_{IJ}$. One with $I' = I \cup \{N\}$ and $J' = J$ and the other with $I^* = I$ and $J^* = J \cup \{N\}$. If we assume that there are k elements in I and l elements in J then by the consistency property for θ,

$$\theta(u_{IJ}) = \theta(k:l) = \theta(k+1:l) + \theta(k:l+1) = \theta(v_{I'J'}) + \theta(v_{I^*J^*})$$

This together with (3.5) gives us

$$\theta(u_{IJ})\nabla_{u_{IJ}}g(y) = \theta(v_{I'J'})\nabla_{v_{I'J'}}f(x) + \theta(v_{I^*J^*})\nabla_{v_{I^*J^*}}f(x)$$

In this way we can match all the terms on the left and right of (3.6) and hence we have shown that (3.4) holds.

□

The parameter $\theta(k:l)$ with both k and l strictly positive may be loosely interpreted as the rate that $k+l$ components of X split into k and l components with k moving upwards and l moving downwards. For this reason we impose a further constraint on the family of parameters θ,

$$\theta(k:l) \geq 0 \text{ for all } k,l \geq 1. \tag{3.7}$$

The parameters $\theta(k:0)$ and $\theta(0:l)$ are not necessarily positive. Their rôle is probably best described as contributing correction terms to the generator $\mathcal{A}_N^\theta$

which ensure the consistency of the martingale problem as N varies. If we didn't have these correction terms in the generator $\mathcal{A}_N^\theta$ we would be forced to impose an symmetry condition $\theta(k : l) = \theta(l : k)$ which would greatly restrict the types of processes we could describe. Note that given any consistent family of parameters $(\theta(k : l); k, l \geq 1)$, we can find an extension to a consistent family of parameters $(\theta(k : l) : k, l \geq 0)$ via the relationships $\theta(k+1 : 0) = \theta(k : 0) - \theta(k : 1)$ and $\theta(0 : l+1) = \theta(0 : l) - \theta(1 : l)$. Given the parameters $\theta(1 : 0)$ and $\theta(0 : 1)$ this extension is unique.

The following is the main theorem of the chapter and the majority of the rest of the chapter is devoted to proving it.

Theorem 32. *Let θ be a family of parameters satisfying the consistency condition* (3.3) *and the positivity condition* (3.7)*. For each $N \geq 1$ and $x \in \mathbb{R}^N$ there exists a process solving the $\mathcal{A}_N^\theta$-martingale problem starting from x. Moreover the law of this process is uniquely determined.*

The following lemmas allow us to make some assumptions on the family of parameters θ.

Lemma 33. *Let θ be a family of parameters satisfying the consistency* (3.3) *and positivity* (3.7) *conditions and let $\tilde{\theta}$ be another family of parameters satisfying for some $\alpha \in \mathbb{R}$*

$$\tilde{\theta}(k : l) = \theta(k : l) + \alpha \mathbf{1}_{\{k=0\}} + \alpha \mathbf{1}_{\{l=0\}}$$

then $\tilde{\theta}$ satisfies the consistency and positivity conditions and $\mathcal{A}_N^{\tilde{\theta}} = \mathcal{A}_N^\theta$ hence there is no loss in generality in always assuming that $\theta(0 : 0) = 0$

Proof. Clearly $\tilde{\theta}$ satisfies the positivity condition and the consistency condition is satisfied as long as

$$\tilde{\theta}(k+1:0) = \tilde{\theta}(k:0) - \tilde{\theta}(k:1)$$

and

$$\tilde{\theta}(0:l+1) = \tilde{\theta}(0:l) - \tilde{\theta}(1:l)$$

for all $k, l \geq 0$ but the equivalent relations are satisfied for θ and to show the above we just need to add α to both sides of each equation.

The equivalence of the operators can be seen as follows

$$\begin{aligned}
\mathcal{A}_N^{\tilde{\theta}} f(x) &= \sum_{v \in \mathcal{V}(x)} \tilde{\theta}(v) \nabla_v f(x) \\
&= \sum_{v \in \mathcal{V}_+(x)} \tilde{\theta}(v) \nabla_v f(x) + \sum_{v \in \mathcal{V}_0(x)} (\theta(v) + \alpha) \nabla_v f(x) \\
&= \mathcal{A}_N^{\theta} f(x) + \alpha \sum_{v \in \mathcal{V}_0(x)} \nabla_v f(x),
\end{aligned}$$

but for all $v \in \mathcal{V}_0(x)$, v points in a direction which remains in the cell $E(x)$ at least for some small distance and f is linear within cells hence for all $v \in \mathcal{V}_0(x)$,

$$\nabla_v f(x) = \nabla_{-v} f(x)$$

and therefore

$$\sum_{v \in \mathcal{V}_0(x)} \nabla_v f(x) = 0.$$

□

We can now assume that $\theta(0:0) = 0$ and hence for some constant $\beta \in \mathbb{R}$ we have $\theta(1:0) = -\theta(0:1) = \beta$. The following lemma allows us to assume that $\beta = 0$ when proving existence and uniqueness of a solution to the $\mathcal{A}_N^{\theta}$

martingale problem.

Lemma 34. *Let X be a solution to the $\mathcal{A}_N^{\theta}$-martingale problem. For any $\beta in \mathbb{R}$ The process $\tilde{X}$ given by $\tilde{X}(t) = X(t) + 2\beta t\mathbf{1}$ solves the $\mathcal{A}_N^{\tilde{\theta}}$ problem. Where*

$$\tilde{\theta}(k:l) = \theta(k:l) + \beta\mathbf{1}_{\{k=0\}} - \beta\mathbf{1}_{\{l=0\}}$$

and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

Proof. For $x \in \mathbb{R}^N$, we consider the quantity $f(x - 2\beta t\mathbf{1})$. For any $f \in L_N$ we can write f as

$$f(x) = \sum_{E \in E_N} \sum_{i=1}^{N} a_i(E) x_i \mathbf{1}_{\{E(x)=E\}}.$$

Let E_N be the set of all possible cells in $\mathbb{R}^N$. For any $x \in \mathbb{R}^N$, adding multiples of $\mathbf{1}$ does not change the ordering of the coordinates of x, hence $E(x - 2\beta t\mathbf{1}) = E(x)$. It follows that

$$\begin{aligned} f(x - 2\beta t\mathbf{1}) &= \sum_{E \in E_N} \sum_{i=1}^{N} a_i(E)(x_i - 2\beta t)\mathbf{1}_{\{E(x)=E\}} \\ &= f(x) - 2\beta t \sum_{E \in E_N} \sum_{i=1}^{N} a_i(E)\mathbf{1}_{\{E(x)=E\}} \\ &= f(x) - 2\beta t f(\mathbf{1}). \end{aligned}$$

Note that $\nabla_v f(x - 2\beta t\mathbf{1}) = \nabla_v f(x)$ and from this it follows that

$$\mathcal{A}_N^{\theta} f(x - 2\beta t\mathbf{1}) = \mathcal{A}_N^{\theta} f(x).$$

We have

$$f(X(t)) - \int_0^t \mathcal{A}_N^{\theta} f(X(s))ds$$

is a martingale. Then as $X(t) = \tilde{X}(t) - 2\beta t$, the following is also a martingale

$$f(\tilde{X}(t) - 2\beta t\mathbf{1}) - \int_0^t \mathcal{A}_N^\theta f(\tilde{X}(s) - 2\beta s\mathbf{1})ds$$
$$= f(\tilde{X}(t)) - \int_0^t \mathcal{A}_N^\theta f(\tilde{X}(s))ds - 2\beta t f(\mathbf{1}).$$

We now just need to show that

$$\mathcal{A}_N^{\tilde{\theta}} f(x) = \mathcal{A}_N^\theta f(x) + 2\beta f(\mathbf{1}). \tag{3.8}$$

We consider the decomposition

$$\mathcal{A}_N^\theta f(x) = \sum_{v\in\mathcal{V}(x)} \theta(v)\nabla_v f(x) = \sum_{v\in\mathcal{V}_+(x)} \theta(v)\nabla_v f(x) + \sum_{v\in\mathcal{V}_0(x)} \theta(v)\nabla_v f(x) \tag{3.9}$$

From the hypothesis we have the following equality for the first term on the right

$$\sum_{v\in\mathcal{V}_+(x)} \theta(v)\nabla_v f(x) = \sum_{v\in\mathcal{V}_+(x)} \tilde{\theta}(v)\nabla_v f(x).$$

It is the final term of (3.9) we are interested in. $v \in \mathcal{V}_0(x)$ are vectors which point in a directions which remain within the cell $E(x)$ at least for some small distance. For such v, $\nabla_v f(x) = \sum_{i=1}^N v_i a_i(E)$ for all $x \in E$. Therefore the final term of (3.9) can be written as

$$\sum_{v\in\mathcal{V}_0(x)} \theta(v)\nabla_v f(x) = \sum_{E\in E_N} \sum_{i=1}^N a_i(E)[\theta(m_i(E):0) - \theta(0:m_i(E))]\mathbf{1}_{\{E(x)=E\}}.$$

Where $m_i(E)$ is the size of the element of $\pi(E)$ that contains i. From the hypothesis we have

$$\theta(m_i(E):0) - \theta(0:m_i(E)) = \tilde{\theta}(m_i(E):0) - \tilde{\theta}(0:m_i(E)) - 2\beta$$

for any i and E. Hence

$$\begin{aligned} \sum_{v\in\mathcal{V}_0(x)} \theta(v)\nabla_v f(x) &= \sum_{v\in\mathcal{V}_0(x)} \tilde{\theta}(v)\nabla_v f(x) - 2\beta \sum_{E\in E_N}\sum_{i=1}^{N} a_i(E)\mathbf{1}_{\{E(x)=E\}} \\ &= \sum_{v\in\mathcal{V}_0(x)} \tilde{\theta}(v)\nabla_v f(x) - 2\beta f(\mathbf{1}). \end{aligned}$$

which in turn gives us

$$\mathcal{A}_N^{\theta} f(x) = \mathcal{A}_N^{\tilde{\theta}} f(x) - 2\beta f(\mathbf{1}).$$

Thus we have the equality (3.8). $\square$

The previous lemma tells us that if we can show that there exists a process solving an $\mathcal{A}_N^{\theta}$-martingale problem with $\theta(1:0) = \theta(0:1) = 0$, then we can show that there exists a solution to any $\mathcal{A}_N^{\tilde{\theta}}$-martingale problem with $\tilde{\theta}(1:0) = -\tilde{\theta}(0:1) = \beta$ simply by adding on a drift of 2β to each component of a solution to the $\mathcal{A}_N^{\theta}$-martingale problem. Similarly if we can assume that the law of a $\mathcal{A}_N^{\theta}$-martingale problem, with $\theta(1:0) = \theta(0:1) = 0$, is uniquely specified, then the solution to any $\mathcal{A}_N^{\tilde{\theta}}$-martingale problem can be shown to be uniquely specified by removing a drift of 2β. So, for the purposes of proving existence and uniqueness, from now on we can assume not only $\theta(0:0) = 0$ but also $\theta(0:1) = \theta(1:0) = 0$. In this case each component X_i evolves as a Brownian motion with no drift.

3.2 Independent coupling of a Brownian motion and a Sticky Brownian Motion

In this section we study a two dimensional process that is the coupling of a standard Brownian motion with a θ-sticky Brownian motion. The main purpose of this is so we can apply the results to help prove Proposition 39 in Section 3.3.

First of all we have the following lemma, which gives us a couple results for planar Brownian motion, which will be used in the sequel.

Lemma 35. *For some $a > 0$, let (B_1, B_2) be a two dimensional standard Brownian motion started at some point $(x, y) \in (0, a) \times (0, \infty)$. Let $\tau_0 = \inf\{t \geq 0 : B_2(t) = 0\}$, $\tau_1 = \inf\{t \geq 0 : B_1(t) = 0\}$ and $\tau_2 = \inf\{t \geq 0 : B_1(t) = a\}$, then*

$$\mathbf{P}(\tau_1 < \tau_0) = \frac{2}{\pi} \tan^{-1}\left(\frac{y}{x}\right)$$
$$\mathbf{P}(\tau_2 < \tau_0) = \frac{2}{\pi} \tan^{-1}\left(\frac{y}{a-x}\right)$$

and also

$$\mathbf{P}(\tau_0 < \tau_1 \wedge \tau_2) \geq \sin\left(\frac{\pi}{a} x\right) e^{-\frac{\pi}{a} y}.$$

Proof. Let $h_1(x, y) = \frac{2}{\pi} \tan^{-1}\left(\frac{y}{x}\right)$, $h_2(x, y) = \frac{2}{\pi} \tan^{-1}\left(\frac{y}{a-x}\right)$ and $h_3(x, y) = \sin\left(\frac{\pi}{a} x\right) e^{-\frac{\pi}{a} y}$ it is possible to show that for each i, $\Delta h_i = 0$. Thus, using Itô's formula, we can see that for each i, $(h_i(B_1(t), B_2(t)) : t \geq 0)$ is a local martingale. Each function is also bounded, therefore by optional stopping and bounded convergence we have $h_1(x, y) = \mathbf{E}[h_1(B_1(\tau_0 \wedge \tau_1), B_2(\tau_0 \wedge \tau_1))]$ and similarly $h_2(x, y) = \mathbf{E}[h_1(B_1(\tau_0 \wedge \tau_2), B_2(\tau_0 \wedge \tau_2))]$. But $h_1(x, 0) = h_2(x, 0) = 0$ for all $x < 0 < a$ and $\lim_{x \to 0} h_1(x, y) = \lim_{x \to a} h_2(x, y) = 1$ for all $y > 0$. As $\mathbf{P}(B_1(\tau_0 \wedge \tau_1) = B_2(\tau_0 \wedge \tau_2) = 0) = \mathbf{P}(B_1(\tau_0 \wedge \tau_1) = a, B_2(\tau_0 \wedge \tau_2) = 0) = 0$

we have $h_1(x, y) = \mathbf{P}(\tau_1 > \tau_0)$ and $h_2(x, y) = \mathbf{P}(\tau_2 > \tau_0)$.

Again by optional stopping and bounded convergence we have $h_3(x, y) = \mathbf{E}[h_3(B_1(\tau_0 \wedge \tau_1 \wedge \tau_2), B_2(\tau_0 \wedge \tau_1 \wedge \tau_2))]$ and $h_3(0, y) = h_3(a, y) = 0$ for all $y \geq 0$. As $h_3(x, 0) \leq 1$ for all $x \in [0, a]$ we have $h_3(x, y) \leq \mathbf{P}(\tau_0 < \tau_1 \wedge \tau_2)$. $\square$

Consider the two dimensional process $((X(t), Y(t)); t \geq 0)$, defined as the independent coupling of $(X(t); t \geq 0)$, a standard Brownian motion, and $(Y(t); t \geq 0)$, a θ-sticky Brownian motion. Let $\tau_\epsilon = \inf\{t \geq 0 : |X(t) - \epsilon| = 2\epsilon\}$. Let $\mathbf{P}_{(x,y)}$ denote the probability measure governing the process $((X(t), Y(t)); t \geq 0)$ with $(X(0), Y(0)) = (x, y)$.

Lemma 36.

$$\mathbf{P}_{(x,0)}\left(Y(\tau_\epsilon) \neq 0\right) \leq 2\sqrt{\frac{2}{\pi}}\epsilon\theta \quad \forall x \in [0, 2\epsilon].$$

Proof. Let

$$f(t) = \mathbf{P}_{(x,0)}(Y(t) = 0) = \exp(2t\theta^2)\text{erfc}(\sqrt{2t}\theta), \tag{3.10}$$

where the second equality can easily be seen from (2.15). We note that here $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz$. Now we show that f is a convex with the aim of using Jensen's inequality.

$$\begin{aligned} \frac{d}{dt} f(t) &= 2\theta^2 \exp(2t\theta^2)\text{erfc}(\sqrt{2t}\theta) + \exp(2t\theta^2)\left(-\sqrt{\frac{2}{\pi t}}\theta \exp(-2t\theta^2)\right) \\ &= 2\theta^2 f(t) - \theta\sqrt{\frac{2}{\pi t}} \end{aligned} \tag{3.11}$$

and so

$$\frac{d^2}{dt^2} f(t) = 4\theta^4 f(t) - 2\theta^3\sqrt{\frac{2}{\pi t}} + \frac{1}{2}\theta\sqrt{\frac{2}{\pi t^3}}. \tag{3.12}$$

Now we note that $f(t) = \sqrt{\frac{2}{\pi}}R(2\theta\sqrt{t})$ where

$$R(x) = \frac{\int_x^\infty e^{-\frac{u^2}{2}}du}{e^{-\frac{x^2}{2}}}$$

which is usually called Mill's Ratio. There are many results on bounds for Mill's Ratio, see [Mit70]. The one we shall use, which can be seen by a simple integration by parts, is, for all $x \geq 0$,

$$R(x) \geq \frac{1}{x} - \frac{1}{x^3}$$

as this implies

$$f(t) \geq \sqrt{\frac{2}{\pi}}\left(\frac{1}{2\theta\sqrt{t}} - \frac{1}{(2\theta\sqrt{t})^3}\right).$$

From this it follows that

$$4\theta^4 f(t) \geq \sqrt{\frac{2}{\pi}}\left(\frac{2\theta^3}{\sqrt{t}} - \frac{\theta}{2\sqrt{t^3}}\right),$$

which, with (3.12), gives us that for all $t \geq 0$, $\frac{d^2}{dt^2}f(t) \geq 0$, and so f is convex. We now apply Jensen's inequality.

$$\mathbf{P}_{(x,0)}(Y(\tau_\epsilon) = 0) = \mathbf{E}_x[f(\tau_\epsilon)] \geq f(E_x[\tau_\epsilon]) = f((2\epsilon - x)x) \tag{3.13}$$

We also have by an integration by parts, $R(x) \leq \frac{1}{x}$ for all $x \geq 0$, which gives us

$$f(t) \leq \sqrt{\frac{2}{\pi}}\frac{1}{2\theta\sqrt{t}}.$$

Then it follows that

$$2\theta^2 f(t) \leq \theta\sqrt{\frac{2}{\pi t}}.$$

This, together with (3.11), gives that f is decreasing for all $t \geq 0$ as we would expect.

Clearly we have the inequality $(2\epsilon - x)x \leq \epsilon^2$ for all $x \in [0, 2\epsilon]$, therefore from (3.13) it follows that

$$\mathbf{P}_{(x,0)}(Y(\tau_\epsilon) = 0) \geq f(\epsilon^2) \quad \forall x.$$

Plugging this into the formula for f, (3.10), gives us

$$\mathbf{P}_{(x,0)}(Y(\tau_\epsilon) = 0) \geq (1 + 2\epsilon^2\theta^2)\left(1 - 2\sqrt{\frac{2}{\pi}}\epsilon\theta\right) \geq 1 - 2\sqrt{\frac{2}{\pi}}\epsilon\theta.$$

Thus $\mathbf{P}_{(x,0)}(Y(\tau_\epsilon) \neq 0) \leq 2\sqrt{\frac{2}{\pi}}\epsilon\theta$. □

Lemma 37.

$$\mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon) \neq 0) \leq \frac{\pi}{2}\frac{|y|}{\epsilon} + 2\sqrt{\frac{2}{\pi}}\theta\epsilon$$

Proof. Recall that $\tau_\epsilon = \inf\{t \geq 0 : |X(t) - \epsilon| = 2\epsilon\}$ and let $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$ then

$$\mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon) \neq 0) \leq \mathbf{P}_{(\epsilon,y)}(\tau_\epsilon < \tau_0) + \mathbf{P}_{(\epsilon,y)}(\{Y(\tau_\epsilon) \neq 0\} \cap \{\tau_0 < \tau_\epsilon\}) \quad (3.14)$$

The bound for the first probability comes from the second part of Lemma 35. This gives us

$$\mathbf{P}(\tau_\epsilon < \tau_0) \leq 1 - \sin\left(\frac{\pi}{2}\right)e^{-\frac{\pi}{2\epsilon}|y|}.$$

Thus

$$\mathbf{P}(\tau_\epsilon < \tau_0) \leq \frac{\pi|y|}{2\epsilon}.$$

For the second probability on the right of (3.14) we have by the strong

Markov property.

$$\mathbf{P}_{(\epsilon,y)}(\{Y(\tau_\epsilon) \neq 0\} \cap \{\tau_0 < \tau_\epsilon\}) \leq \mathbf{P}_{(X(\tau_0),0)}(Y(\tau_\epsilon) \neq 0)$$

which we have a bound for, given in Lemma 36. Thus

$$\mathbf{P}_{(\epsilon,y)}(\{Y(\tau_\epsilon) \neq 0\} \cap \{\tau_0 < \tau_\epsilon\}) \leq 2\sqrt{\frac{2}{\pi}}\epsilon\theta.$$

Putting these bounds in (3.14) gives the result. □

Now let $r : [0,2] \mapsto \mathbb{R}$ be a continuous non-negative function with

- $r(0) = r(2) = 0$
- $r(x) > 0$ for all $x \in (0,2)$.
- $\liminf_{x \downarrow 0} r'(x) > 0$ and $\limsup_{x \uparrow 2} r'(x) < 0$.

Let $r^\epsilon : [0, 2\epsilon] \mapsto \mathbb{R}$ be defined as $r^\epsilon(x) = \epsilon r(x/\epsilon)$ and let $D^\epsilon = \{(x,y) \in \mathbb{R}^2 : |y| = r^\epsilon(x), x \in [0,2\epsilon]\}$ be the union of the images of r^ϵ and $-r^\epsilon$ in $\mathbb{R}^2$.

Lemma 38. *Let $\tau_\epsilon^* = \inf\{t \geq 0 : (X(t), Y(t)) \in D^\epsilon\}$. Then, for any $\epsilon > 0$,*

$$\mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon^*) \neq 0) \leq C_1 \frac{|y|}{\epsilon} + C_2 \epsilon\theta$$

for some constants $0 < C_1, C_2 < \infty$, depending only on r.

Proof. Let

$$f(x,y) = \mathbf{P}_{(x,y)}(Y(\tau_\epsilon) \neq 0)$$

We have shown in Lemma 36 that $f(\epsilon, y) \leq \frac{2}{\pi}\frac{|y|}{\epsilon} + 2\sqrt{\frac{2}{\pi}}\epsilon\theta$. Now $f(X(t), Y(t))_{t \geq 0}$ is a local martingale and hence by the optional stopping theorem $f(X(\tau_\epsilon^* \wedge t), Y(\tau_\epsilon^* \wedge t))_{t \geq 0}$ is a local martingale and we note that it is also bounded. We

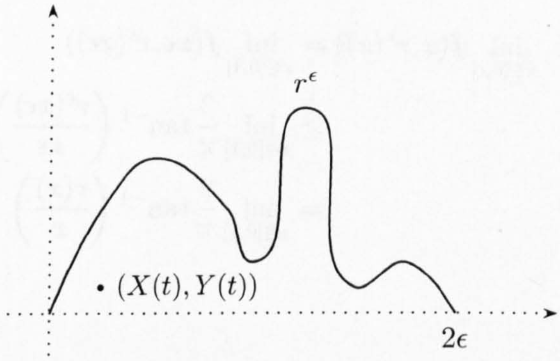

Figure 3.2: The curve r^ϵ

need to show there exists a constant $\delta > 0$, depending only on r, such that

$$\inf_{x\in[0,2\epsilon]} f(x, r^\epsilon(x)) \geq \delta. \tag{3.15}$$

Then we have

$$f(\epsilon, y) = \mathbf{E}_{(\epsilon,y)}\left[f\left(X(\tau_\epsilon^*), Y(\tau_\epsilon^*)\right)\right] \geq \delta \mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon^*) \neq 0) + 0\mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon^*) = 0)$$

which implies

$$\mathbf{P}_{(\epsilon,y)}(Y(\tau_\epsilon^*) \neq 0) \leq \frac{\pi}{2\delta}\frac{|y|}{\epsilon} + \frac{2}{\delta}\sqrt{\frac{2}{\pi}}\epsilon\theta$$

which leads to the result.

We need to show (3.15) holds. Let $\tau_0 = \inf\{t \geq 0 : Y(t) = 0\}$, $\tau_1 = \inf\{t \geq 0 : X(t) = 0\}$, $\tau_2 = \inf\{t \geq 0 : X(t) = 2\epsilon\}$ then for $x \in [0, \epsilon]$ $f(x, r^\epsilon(x)) \geq \mathbf{P}(\tau_1 < \tau_0)$. The first part of Lemma 35 gives us that

$$f(x, r^\epsilon(x)) \geq \frac{2}{\pi}\tan^{-1}\left(\frac{r^\epsilon(x)}{x}\right).$$

Then

$$\begin{aligned}
\inf_{x\in[0,\epsilon]} f(x,r^\epsilon(x)) &= \inf_{x\in[0,1]} f(x\epsilon,r^\epsilon(x\epsilon)) \\
&\geq \inf_{x\in[0,1]} \frac{2}{\pi}\tan^{-1}\left(\frac{r^\epsilon(x\epsilon)}{x\epsilon}\right) \\
&= \inf_{x\in[0,1]} \frac{2}{\pi}\tan^{-1}\left(\frac{r(x)}{x}\right)
\end{aligned}$$

As $r(x) > 0$ for all $x \in (0,1]$ and $\liminf_{x\downarrow 0} r'(x) > 0$ we must have $\frac{r(x)}{x} \geq \delta_1$, for some $\delta_1 > 0$, for all $x \in (0,1]$ hence $\inf_{x\in[0,\epsilon]} f(x,r^\epsilon(x)) \geq \frac{\delta_1}{2}$. Similarly for $x \in [\epsilon, 2\epsilon]$ we have $f(x,r^\epsilon(x)) \geq \mathbf{P}(\tau_2 < \tau_0)$ and Lemma 35 gives us that

$$f(x,r^\epsilon(x)) \geq \frac{2}{\pi}\tan^{-1}\left(\frac{r^\epsilon(x)}{2\epsilon - x}\right).$$

and

$$\inf_{x\in[\epsilon,2\epsilon]} f(x,r^\epsilon(x)) = \inf_{x\in[1,2]} \frac{2}{\pi}\tan^{-1}\left(\frac{r(x)}{2-x}\right)$$

Then as $r(x) > 0$ for all $x \in [1,2)$ and $\limsup_{x\uparrow 0} r'(x) < 0$ we must have $\frac{r(x)}{2-x} = -\frac{r(x)}{x-2} \geq \delta_2$, for some $\delta_2 > 0$, for all $x \in [1,2)$ hence $f(x,r(x)) \geq \frac{\delta_2}{2}$, for all $x \in [1,2]$. Take $\delta = \frac{1}{2}\min\{\delta_1,\delta_2\}$ and we are done. □

3.3 Leaving the Diagonal

We now begin our progress towards proving uniqueness in law of solutions to the $\mathcal{A}_N^\theta$-martingale problem. This will involve knowing certain results about how the process behaves near the diagonal $D = \{x \in \mathbb{R}^N; x_1 = \cdots = x_n\}$. In this section we prove some results about how the process leaves the diagonal. The following proposition (Proposition 39) tells us that if the process is stopped on leaving a small neighbourhood of D, the exit distribution is concentrated on the cells that are neighbours of D. Proposition 40 tells us that before leaving this neighbourhood of D, the process spends most of its time on the diagonal. We use these results to deduce Theorem 41 which states how the rate at which the process leaves the diagonal and the direction the process leaves in depend on the family parameters $(\theta(k:l);k,l\geq 0)$.

Let

$$T_\epsilon = \inf\{t \geq 0 : |X_i(t) - X_j(t)| \geq \epsilon \text{ for some } i,j \in \{1,2,\dots,N\}\}. \quad (3.16)$$

In the following let X be any solution to the $\mathcal{A}_N^\theta$-martingale problem starting from any point x on the diagonal D.

Proposition 39. *Let Λ_ϵ be the event that there are three or more distinct values taken by the components of $X(T_\epsilon)$. Then for some constant C,*

$$\mathbf{P}(\Lambda_\epsilon) \leq N(N-1)(N-2)C\theta\epsilon$$

The proof of this proposition uses results from Section 3.2 above. This takes some time and is be given at the end of this section.

Proposition 40.

$$\mathbf{E}\left[\int_0^{T_\epsilon} 1(X(s)\notin D)ds\right] \leq \frac{N(N-1)}{4}\epsilon^2.$$

Proof. Let X_i be the ith component of X then

$$X(s)\notin D \Rightarrow X_i(s)\neq X_j(s) \text{ for some } i\neq j$$

so

$$\int_0^{T_\epsilon} 1(X(s)\notin D)ds \leq \sum_{i<j}\int_0^{T_\epsilon} 1(X_i(s)\neq X_j(s))ds.$$

Let

$$T_\epsilon^{ij} = \inf\{t \geq 0 : |X_i(t) - X_j(t)| \geq \epsilon\}$$

so, for each $i \neq j$, $T_\epsilon = \inf_{k,l} T_\epsilon^{k,l} \leq T_\epsilon^{i,j}$ and therefore

$$\int_0^{T_\epsilon} \mathbf{1}(X_i(s) \neq X_j(s))ds \leq \int_0^{T_\epsilon^{i,j}} \mathbf{1}(X_i(s) \neq X_j(s))ds.$$

We know from Proposition 31 that (X_i, X_j) solves the $\mathcal{A}_2^\theta$-martingale problem so

$$(X_i(t) - X_j(t))^2 - 2\int_0^t \mathbf{1}(X_i(s) \neq X_j(s))ds$$

is a martingale. Therefore by the optional stopping theorem

$$\left(X_i(t \wedge T_\epsilon^{i,j}) - X_j(t \wedge T_\epsilon^{i,j})\right)^2 - 2\int_0^{t\wedge T_\epsilon^{i,j}} \mathbf{1}(X_i(s) \neq X_j(s))ds$$

is a martingale and so

$$\mathbf{E}\left[\left(X_i(t \wedge T_\epsilon^{i,j}) - X_j(t \wedge T_\epsilon^{i,j})\right)^2\right] = 2\mathbf{E}\left[\int_0^{t\wedge T_\epsilon^{i,j}} \mathbf{1}(X_i(s) \neq X_j(s))ds\right]$$

Now $(X_i(t \wedge T_\epsilon^{i,j}) - X_j(t \wedge T_\epsilon^{i,j}))^2$ is bounded above by ϵ^2 and $\int_0^{t\wedge T_\epsilon^{i,j}} \mathbf{1}(X_i(s) \neq X_j(s))ds$ is monotonic in t, so by the dominated convergence theorem and the monotone convergence theorem we take the limit as $t \to \infty$ to give

$$\mathbf{E}\left[(X_1(T_\epsilon^{i,j}) - X_2(T_\epsilon^{i,j}))^2\right] = 2\mathbf{E}\left[\int_0^{T_\epsilon^{i,j}} \mathbf{1}(X_1(s) \neq X_2(s))ds\right].$$

The left hand side of which is equal to ϵ^2 and so we have

$$\mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}(X_i(s) \neq X_j(s))ds\right] \leq \mathbf{E}\left[\int_0^{T_\epsilon^{i,j}} \mathbf{1}(X_i(s) \neq X_j(s))ds.\right] = \frac{\epsilon^2}{2}$$

and hence

$$\mathbf{E}\left[\int_0^{T_\epsilon} 1(X(s) \notin D)ds\right] \leq \sum_{i<j} \mathbf{E}\left[\int_0^{T_\epsilon} 1(X_i(s) \neq X_j(s))ds\right] \leq \frac{N(N-1)}{4}\epsilon^2$$

□

We can now use the above two propositions to prove the main theorem of this section.

In Section 3.1 we described how vectors $v_{IJ} \in V_+(x)$ point out of the cell $E(x)$ and into cell which we call a neighbour of $E(x)$. This neighbour is determined by the vector v. In particular $V_+(D)$ consists of vectors v_{IJ} such that I and J form a partition of $\{1, 2, \ldots, N\}$ and for each $v \in V_+(D)$ we have a neighbour of D, which is given by

$$E(v) = \{x \in \mathbb{R}^N : x_i = x_j \ \forall i, j \in I, \quad x_i = x_j \ \forall i, j \in J \\ \text{and } x_j < x_i \ \forall i \in I, j \in J\}.$$

Theorem 41. *The following limits exist and are determined by the family of parameters $(\theta(k:l), k, l \geq 0)$.*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\mathbf{E}[T_\epsilon] = \frac{1}{2\sum_{v \in \mathcal{V}_+(D)} \theta(v)}$$

and, for cells $E(v)$ such that $E(v)$ is a neighbour of D,

$$\lim_{\epsilon \to 0} \mathbf{P}\big(X(T_\epsilon) \in E(v)\big) = \frac{\theta(v)}{\sum_{u \in \mathcal{V}_+(D)} \theta(u)}.$$

Proof. For both of the results we apply the optional stopping theorem to some $f \in L_N$. For the first part we choose $f \in L_N$ to be $f(x) = \max_i(x_i) - \min_i(x_i)$ which gives us, for all $x \in D$, $\nabla_v f(x) = 2$ for all $v \in \mathcal{V}_+(x)$ and $\nabla_v f(x) = 0$

for all $v \in \mathcal{V}_0(x)$. Note also that 2 is an upper bound for the absolute value of the one sided derivative of the f in any direction and at any point. This means we have

$$f(X(t)) - \int_0^t g(X(s))\mathbf{1}_{\{X(s)\notin D\}}ds - \int_0^t \sum_{v\in\mathcal{V}_+(D)} 2\theta(v)\mathbf{1}_{\{X(s)\in D\}}ds$$

is a martingale, where g is some bounded function and so by the optional stopping theorem and by the dominated and monotone convergence theorems we have

$$\begin{aligned}\mathbf{E}\left[f(X(T_\epsilon))\right] = \mathbf{E}&\left[\int_0^{T_\epsilon} g(X(s))\mathbf{1}_{\{X(s)\notin D\}}ds\right] \\ &+ \mathbf{E}\left[\int_0^{T_\epsilon} \sum_{v\in\mathcal{V}_+(D)} 2\theta(v)\mathbf{1}_{\{X(s)\in D\}}ds\right]\end{aligned}$$

Now $f(X(T_\epsilon)) = \epsilon$ and g is bounded , so

$$\begin{aligned}1 = &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon} g(X(s))\mathbf{1}_{\{X(s)\notin D\}}ds\right] \\ &+ \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon} \sum_{v\in\mathcal{V}_+(D)} 2\theta(v)\mathbf{1}_{\{X(s)\in D\}}ds\right].\end{aligned}$$

The first term is zero by Proposition 40. Therefore we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}_{\{X(s)\in D\}}ds\right] = \frac{1}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}.$$

Finally

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[T_\epsilon\right] &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}_{\{X(s)\notin D\}}ds\right] + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}_{\{X(s)\in D\}}ds\right] \\ &= \frac{1}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)},\end{aligned}$$

since the first term on the right is zero, again by proposition 40.

For the second result of the proposition consider some neighbour of D, $E(v)$, where $v = v_{IJ}$ and I and J form some partition of $\{1, \ldots, N\}$. We define a function f_v by

$$f_v(x) = \inf_{i \in I, j \in J} (x_i - x_j)^+.$$

Then, for all $x \in D$, $\nabla_v f_v(x) = 2$ and $\nabla_u f_v(x) = 0$ for all $u \in \mathcal{V}(D), u \neq v$. We also have that for any $x \in \mathbb{R}^N$ and $u \in \mathcal{V}(x)$, $|\nabla_u f_v(x)| \leq 2$. Then we have, for some bounded function g that

$$f(X(t)) - \int_0^t g(X(s))\mathbf{1}_{\{X(s) \notin D\}} ds - \int_0^t 2\theta(v)\mathbf{1}_{\{X(s) \in D\}} ds$$

is a martingale and so by the optional stopping theorem and by the dominated and monotone convergence theorems we have

$$\mathbf{E}\left[f(X(T_\epsilon))\right] = \mathbf{E}\left[\int_0^{T_\epsilon} g(X(s))\mathbf{1}_{\{X(s) \notin D\}} ds\right] + \mathbf{E}\left[\int_0^{T_\epsilon} 2\theta(v)\mathbf{1}_{\{X(s) \in D\}} ds\right] \tag{3.17}$$

for some bounded g.

$$f(X(T_\epsilon)) = \begin{cases} \epsilon & \text{if } X(T_\epsilon) \in E(v) \\ 0 & \text{if } X(T_\epsilon) \in E \neq E(v) \text{ s.t. } E \text{ is a neighbour of } D. \end{cases}$$

The event $X(T_\epsilon) \notin \bigcup\{E; E \text{ is a neighbour of D}\}$ is equal to the event Λ_ϵ of proposition 39. This together with the fact $0 \leq f(X(T_\epsilon)) \leq \epsilon$ give us

$$\epsilon \mathbf{P}(X(T_\epsilon) \in E(v)) \leq \mathbf{E}\left[f(X(T_\epsilon))\right] \leq \epsilon \mathbf{P}(X(T_\epsilon) \in E(v)) + \epsilon \mathbf{P}(\Lambda_\epsilon).$$

Then by Proposition 39 we have

$$\lim_{\epsilon\to 0}\mathbf{P}(X(T_\epsilon)\in E(v))=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[f(X(T_\epsilon))\right].$$

As g is bounded, Proposition 40 applied to (3.17) gives us

$$\lim_{\epsilon\to 0}\mathbf{P}(X(T_\epsilon)\in E(v))=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon}2\theta(v)\mathbf{1}_{\{X(s)\in D\}}ds\right].$$

This is then equal to

$$2\theta(v)\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[T_\epsilon\right]-\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^{T_\epsilon}2\theta(v)\mathbf{1}_{\{X(s)\notin D\}}ds\right]$$

The second term is 0, again by Proposition 40, and by the first part of this theorem the first term above is given by

$$\frac{2\theta(v)}{2\sum_{u\in\mathcal{V}_+(D)}\theta(u)}$$

□

Corollary 42. *The rates of convergence of the above limits are given by*

$$\left|\mathbf{E}\left[T_\epsilon\right]-\frac{\epsilon}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}\right|\le C\epsilon^2$$

and

$$\left|\mathbf{P}\big(X(T_\epsilon)\in E(v)\big)-\frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\right|\le C\epsilon$$

for some constant C that depends only on N and θ.

Proof. Running through the proof of Theorem 41 we find

$$\left|\mathbf{E}\left[T_\epsilon\right]-\mathbf{E}\left[\int_0^{T_\epsilon}\mathbf{1}_{\{X(s)\in D\}}ds\right]\right|\le\frac{(N-1)N}{4}\epsilon^2$$

and

$$\left|\mathbf{E}\left[\int_0^{T_\epsilon} 1_{\{X(s)\in D\}} ds\right] - \frac{\epsilon}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}\right| \leq \frac{||g||_\infty}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}\frac{(N-1)N}{4}\epsilon^2.$$

The second part of the corollary follows in a similar way. □

For the rest of this section we will be proving proposition 39. We will prove a series of lemmas that will eventually lead to the proof of Proposition 39.

In the following we consider a solution to the $\mathcal{A}_3^\theta$-martingale problem projected onto the plane $x_1 + x_2 + x_3 = 0$. This projected process behaves as Brownian motion in the plane, which is sticky at the origin, and on six rays emanating from the origin. The origin being equivalent to the diagonal D in $\mathbb{R}^3$ and 6 rays being the equivalent to the neighbouring cells of D in $\mathbb{R}^3$.

Behaviour of similar types of processes in the plane are studied in [IW73]. Restricted to a certain wedge, this projected process behaves as a Brownian motion coupled with an independent sticky Brownian motion under a time change. This allows us to apply results from Section 3.2.

Let $X = (X_1, X_2, X_3)$ be a solution to the $\mathcal{A}_3^\theta$-martingale problem started $x = (x_1, x_2, x_3)$. Now let Y_1 be defined by

$$Y_1(t) = \frac{1}{\sqrt{2}}\inf_{i\neq j}|X_i(t) - X_j(t)| \tag{3.18}$$

$Y_1(t)$ measures distance $X(t)$ is from the diagonal $D = \{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\}$ and from the neighbouring cells of D, of which there are 6, given by $\{x_1 < x_2 = x_3\}$, $\{x_2 < x_1 = x_3\}$, $\{x_3 < x_1 = x_2\}$, $\{x_1 = x_2 < x_3\}$, $\{x_1 = x_3 < x_2\}$ and $\{x_2 = x_3 < x_1\}$. We also define a process Y_2 given by

$$Y_2(t) = \frac{1}{\sqrt{6}}\left(\sum_{i<j}|X_i(t) - X_j(t)| - \inf_{i\neq j}|X_i(t) - X_j(t)|\right). \tag{3.19}$$

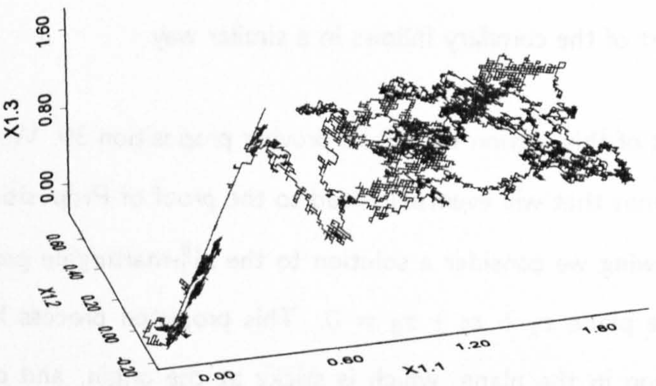

Figure 3.3: Approximation to a solution to the $\mathcal{A}_3^\theta$-martingale problem

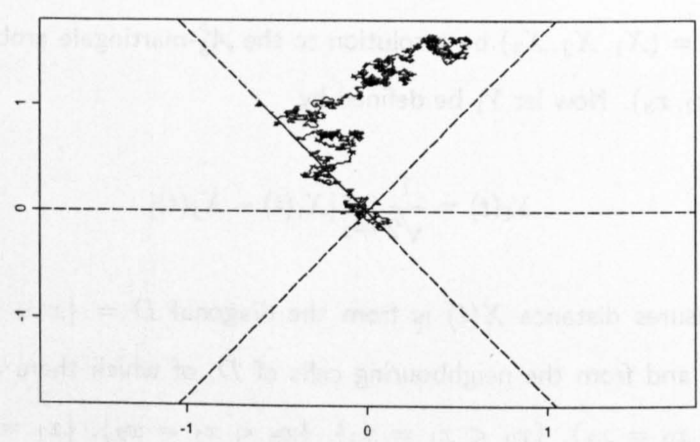

Figure 3.4: 3 dimensional problem projected on to the plane $x_1 + x_2 + x_3 = 0$

This gives the motion in a direction perpendicular to Y_1. We let $\tau_\epsilon = \inf\{t \geq 0 : Y_2(t) = \epsilon\}$ then we have the following lemma.

Lemma 43. *Suppose that X starts from the diagonal D, so that Y_1 and Y_2 both start from 0. Then*

$$E[Y_1(\tau_\epsilon)] \leq 6\sqrt{6}\theta\epsilon^2,$$

where $\theta = 2\theta(1:1)$.

Proof. Let

$$Z(t) = \prod_{i<j} |X_i(t) - X_j(t)|.$$

We proceed be showing $Y_1(\tau_\epsilon) < \frac{C}{\epsilon^2} Z(\tau_\epsilon)$ for some constant C and then applying Itô's formula to Z in order to find an appropriate bound for $E[Z(\tau_3)]$.

For $x \in \mathbb{R}^3$, let $S = \sup_{i \neq j} |x_i - x_j|$, $I = \inf_{i \neq j} |x_i - x_j|$ and $M = \sum_{i<j} |x_i - x_j| - S - I$ Consider values of (x_1, x_2, x_3) such that $\sum_{i<j} |x_i - x_j| - I = \epsilon$. We note that $\sum_{i<j} |x_i - x_j| = 2S$, hence $2S = \epsilon + I$ and $2M = \epsilon - I$ so that

$$\prod_{i<j} |x_i - x_j| = SIM = \frac{1}{4}(\epsilon^2 I - I^3).$$

$2S - I = S + M \geq 3I$, which implies that $\frac{1}{\sqrt{3}} Y_2(t) \geq Y_1(t)$ for all t. If $0 \leq I \leq \frac{1}{\sqrt{3}}\epsilon$ it is easy to see the following inequality holds

$$\frac{\epsilon^2}{6} I \leq \frac{1}{4}(\epsilon^2 I - I^3)$$

so as $Y_1(\tau_\epsilon) \leq \frac{1}{\sqrt{3}} Y_2(\tau_\epsilon) = \epsilon$, we must have.

$$\frac{\epsilon^2}{6} Y_1(\tau_\epsilon) \leq Z(\tau_\epsilon). \tag{3.20}$$

Now we apply Itô's formula to Z. If $f(u,v,w) = uvw$ then

$$\begin{aligned} f(U(t),V(t),W(t)) =& f(U_0,V_0,W_0) + \int_0^t VWdU(s) + \int_0^t UWdV(s) + \int_0^t UVdW(s) \\ &+ \int_0^t Wd\langle U,V\rangle_s + \int_0^t Vd\langle U,W\rangle_s + \int_0^t Ud\langle V,W\rangle_s \end{aligned} \tag{3.21}$$

We note that for some local martingale M

$$\begin{aligned} |X_i(t) - X_j(t)| &= \int_0^t \operatorname{sgn}(X_i(s) - X_j(s))d(X_i(s) - X_j(s)) + \int_0^t 2\theta \mathbf{1}_{\{X_i(s)=X_j(s)\}}ds \\ &= M(t) + \int_0^t 2\theta \mathbf{1}_{\{X_i(s)=X_j(s)\}}ds. \end{aligned}$$

We also note that

$$\begin{aligned} &\langle |X_1 - X_2|, |X_2 - X_3|\rangle_t = \\ &\quad \int_0^t \operatorname{sgn}(X_1(s) - X_2(s)) \operatorname{sgn}(X_2(s) - X_3(s))d\langle X_1 - X_2, X_2 - X_3\rangle_s \end{aligned}$$

then as $\langle X_1 - X_2, X_2 - X_3\rangle_t = \langle X_1, X_2\rangle_t + \langle X_2, X_3\rangle_t - \langle X_1, X_3\rangle_t - \langle X_2\rangle_t$ we can induce that

$$\langle X_1 - X_2, X_2 - X_3\rangle_t = -\int_0^t \mathbf{1}_{\{X_1(s)\neq X_2(s)\neq X_3(s)\}}ds - 2\int_0^t \mathbf{1}_{\{X_1(s)=X_3(s)\neq X_2(s)\}}ds.$$

We can find similar results for other combinations of coordinates. Putting these together with (3.21) we get for some local martingale M

$$\begin{aligned}
Z(t) = M(t) &+ 2\theta \int_0^t |X_1(s) - X_2(s)||X_2(s) - X_3(s)| \mathbf{1}_{\{X_1(s)=X_3(s)\}} ds \\
&+ 2\theta \int_0^t |X_2(s) - X_3(s)||X_3(s) - X_1(s)| \mathbf{1}_{\{X_1(s)=X_2(s)\}} ds \\
&+ 2\theta \int_0^t |X_3(s) - X_1(s)||X_1(s) - X_2(s)| \mathbf{1}_{\{X_2(s)=X_3(s)\}} ds \\
&- \int_0^t |X_1(s) - X_2(s)| \operatorname{sgn}(X_2(s) - X_3(s)) \operatorname{sgn}(X_3(s) - X_1(s)) \\
&\qquad (\mathbf{1}_{\{X_1(s)\neq X_2(s)\neq X_3(s)\}} ds + 2\mathbf{1}_{\{X_1(s)=X_2(s)\neq X_2(s)\}}) ds \\
&- \int_0^t |X_2(s) - X_3(s)| \operatorname{sgn}(X_3(s) - X_1(s)) \operatorname{sgn}(X_1(s) - X_2(s)) \\
&\qquad (\mathbf{1}_{\{X_1(s)\neq X_2(s)\neq X_3(s)\}} ds + 2\mathbf{1}_{\{X_2(s)=X_3(s)\neq X_2(s)\}}) ds \\
&- \int_0^t |X_3(s) - X_1(s)| \operatorname{sgn}(X_1(s) - X_2(s)) \operatorname{sgn}(X_2(s) - X_3(s)) \\
&\qquad (\mathbf{1}_{\{X_1(s)\neq X_2(s)\neq X_3(s)\}} ds + 2\mathbf{1}_{\{X_1(s)=X_3(s)\neq X_2(s)\}}) ds.
\end{aligned}$$

We can lose some of these terms. For example

$$\int_0^t |X_3(s) - X_1(s)| 2\mathbf{1}_{\{X_1(s)=X_3(s)\neq X_2(s)\}} ds = 0.$$

We note also that

$$\begin{aligned}
&|X_3(s) - X_1(s)| \operatorname{sgn}(X_1(s) - X_2(s)) \operatorname{sgn}(X_2(s) - X_3(s)) \\
&+ |X_2(s) - X_3(s)| \operatorname{sgn}(X_3(s) - X_1(s)) \operatorname{sgn}(X_1(s) - X_2(s)) \\
&+ |X_1(s) - X_2(s)| \operatorname{sgn}(X_2(s) - X_3(s)) \operatorname{sgn}(X_3(s) - X_1(s)) \\
&= 0.
\end{aligned}$$

as we have one positive term and two negative terms the positive term being the largest. This gives us $S - M - I$ which is equal to 0. We note also that for example $\mathbf{1}_{\{X_1(s)=X_2(s)\}} = \mathbf{1}_{\{X_1(s)=X_2(s)\neq X_3(s)\}} + \mathbf{1}_{\{X_1(s)=X_2(s)=X_3(s)\}}$ and

$$\int_0^t |X_2(s) - X_3(s)||X_3(s) - X_1(s)| \mathbf{1}_{\{X_1(s)=X_2(s)=X_3(s)\}} ds = 0.$$

We are left with

$$
\begin{aligned}
Z(t) = M(t) &+ 2\theta \int_0^t |X_1(s) - X_2(s)||X_2(s) - X_3(s)|\mathbf{1}_{\{X_1(s)=X_3(s)\neq X_2(s)\}}ds \\
&+ 2\theta \int_0^t |X_2(s) - X_3(s)||X_3(s) - X_1(s)|\mathbf{1}_{\{X_1(s)=X_2(s)\neq X_3(s)\}}ds \\
&+ 2\theta \int_0^t |X_3(s) - X_1(s)||X_1(s) - X_2(s)|\mathbf{1}_{\{X_2(s)=X_3(s)\neq X_1(s)\}}ds
\end{aligned}
$$

We use this formula with the optional stopping theorem to give our bound for $\mathbf{E}[Z(\tau_\epsilon)]$. For all $t < \tau_\epsilon$,we have $|X_i(t) - X_j(t)| < \epsilon \ \forall i, j$ so that

$$
\begin{aligned}
E[Z(\tau_\epsilon)] &\leq \epsilon^2 2\theta \mathbf{E}\bigg[\int_0^{\tau_\epsilon} \big(\mathbf{1}_{\{X_1(s)=X_2(s)\neq X_3(s)\}} + \mathbf{1}_{\{X_2(s)=X_3(s)\neq X_1(s)\}} \\
&\qquad\qquad + \mathbf{1}_{\{X_1(s)=X_3(s)\neq X_2(s)\}} ds\big)\bigg] \\
&\leq \mathbf{E}\left[\int_0^{\tau_\epsilon} \mathbf{1}_{\{Y_2(s)\notin D\}} ds\right].
\end{aligned}
$$

where D here is the diagonal $D = \{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\}$. As $2S - I \leq 2S$ it is possible to see that $\tau_\epsilon < T_{\left(\frac{\sqrt{2}}{\sqrt{3}}\epsilon\right)}$ hence by Proposition 40, we have.

$$
E[Z(\tau_\epsilon)] \leq \sqrt{6}\theta\epsilon^4
$$

Finally by the inequality (3.20) we have

$$
E[Y_1(\tau_\epsilon)] \leq 6\sqrt{6}\theta\epsilon^2
$$

□

Consider the two processes V and W given by $V(t) = \frac{1}{\sqrt{2}}(X_1(t) - X_2(t))$ and $W(t) = \frac{1}{\sqrt{6}}(X_1(t) + X_2(t) - 2X_3(t))$. Consider the region given by

$$
U = \left\{(v, w) \in \mathbb{R}^2 : w > 0, |v| < \frac{1}{\sqrt{3}} w, |v| < \sqrt{3}(2\epsilon - w)\right\}.
$$

Let $T(U) = \inf\{t \geq 0 : (V(t), W(t)) \notin U\}$, then we have the following lemma.

Lemma 44. *Let the starting values of W and V be given by $(w, v) \in U$. Let $(\beta_t : t \geq 0)$ be a time change given by*

$$\beta_t = \inf\{u \geq 0 : t + \frac{1}{3}\int_0^u \mathbf{1}_{\{V(s)=0\}} > t\}.$$

The stopped process

$$(W(T(U) \wedge \beta_t), V(T(U) \wedge \beta_t))_{t \geq 0}$$

is equal in distribution to $(\hat{W}(t \wedge T'(U)), \hat{V}(t \wedge T'(U)))_{t\geq 0}$ where $\hat{V}$ is a $(3/(2\sqrt{2})\theta)$-sticky Brownian motion started at $V(0) = v$ and $\hat{W}$ is a standard Brownian motion independent of $\hat{V}$ started at $W(0) = w$ and

$$T'(U) = \inf\{t \geq 0 : (\hat{W}(t), \hat{V}(t)) \notin U\}$$

Proof. V is a θ_0-sticky Brownian motion, where $\theta_0 = 2\sqrt{2}\theta(1:1) = \sqrt{2}\theta$. Let $A_t = \int_0^t \mathbf{1}_{\{V(s)\neq 0\}} ds$ and let $\alpha_t = \inf\{u \geq t : A_u > t\}$. It follows from results in chapter 2 that $(V(\alpha_t); t \geq 0)$ is distributed as a standard Brownian motion. Let $B(t) = V(\alpha_t)$, then, also from chapter 2, the local time at zero of B is given by

$$L_t(B) = \theta_0 \int_0^{\alpha_t} \mathbf{1}_{\{V(s)=0\}} ds \tag{3.22}$$

and the time change $(\alpha_t : t \geq 0)$ can be expressed in terms of B via

$$\alpha_t = t + \frac{1}{\theta_0} L_t(B). \tag{3.23}$$

Let

$$\alpha_t^{3\theta_0/4} = t + \frac{4}{3\theta_0} L_t(B)$$

and let $A_t^{3\theta_0/4} = \inf\{u \geq 0 : \alpha_u^{3\theta_0/4} > t\}$. It follows, from (3.22) and (3.23) that

$$\alpha_t + \frac{1}{3}\int_0^{\alpha_t} \mathbf{1}_{\{V(s)=0\}} ds = t + \frac{4}{3\theta_0} L_t(B)$$

and from this we find that

$$\beta_t = \alpha_{A_t^{3\theta_0/4}}.$$

Now consider the stopped process $W(t \wedge T(U)) = \frac{1}{\sqrt{6}}(X_1(t \wedge T(U)) + X_2(t \wedge T(U)) - 2X_3(t \wedge T(U)))$. The process $W(t \wedge T(U))$ is a martingale and also we have

$$\begin{aligned}\langle W\rangle_{t\wedge T(U)} &= t \wedge T(U) + \frac{1}{3}\int_0^{t\wedge T(U)} \mathbf{1}_{\{X_2(s)=X_1(s)\}} ds \\ &- \frac{2}{3}\int_0^{t\wedge T(U)} \mathbf{1}_{\{X_1(s)=X_3(s)\}} ds - \frac{2}{3}\int_0^{t\wedge T(U)} \mathbf{1}_{\{X_2(s)=X_3(s)\}} ds, \quad t \geq 0.\end{aligned}$$

The last two terms of the above are equal to zero as $T(U) \leq \inf\{t \geq 0 : X_1(t) = X_3(t)\}$ and $T(U) \leq \inf\{t \geq 0 : X_2(t) = X_3(t)\}$. We also note that $X_1(t) = X_2(t)$ if and only if $V(t) = 0$. Thus

$$\langle W\rangle_{t\wedge T(U)} = t \wedge T(U) + \frac{1}{3}\int_0^{t\wedge T(U)} \mathbf{1}_{\{V(s)=0\}} ds, \quad t \geq 0. \tag{3.24}$$

We note that

$$\langle V, W\rangle_t = \frac{1}{\sqrt{3}}\left(\int_0^t \mathbf{1}_{\{X_1(s)=X_3(s)\}} ds + \int_0^t \mathbf{1}_{\{X_2(s)=X_3(s)\}} ds\right) \tag{3.25}$$

and therefore $\langle V, W\rangle_{t\wedge T(U)} = 0$ for all t. Using (3.24) and (3.25) we apply Knight's theorem to (V, W). Let $T^* = \alpha_{A_{T(U)}^{3\theta_0/4}}$ then given a standard Brownian motion started at zero, B', which is independent of everything else, we have that

$$(V(\alpha_t), (W(\beta_{t\wedge T^*}) + B'(t - T^*)\mathbf{1}_{\{t \geq T^*\}},) : t \geq 0)$$

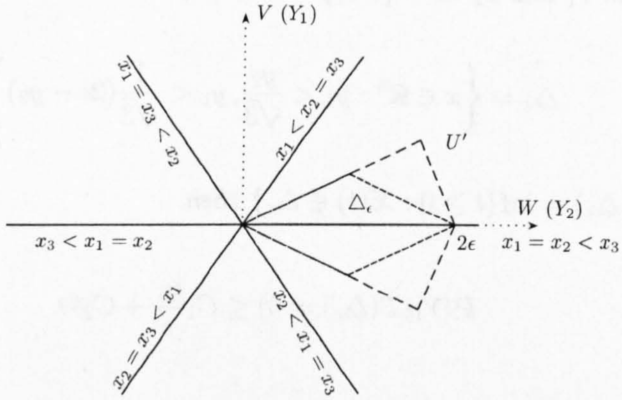

Figure 3.5: 3 dimensional process restricted to a wedge

is a two dimensional Brownian motion. Thus $(W(\beta_{t\wedge T^*}) + B'(t-T^*)\mathbf{1}_{\{t\geq T^*\}} : t \geq 0)$ is independent of $(V(\alpha_t) : t \geq 0)$. $V(\alpha_t) = B(t)$ for all t and the time change $(A_t^{3\theta_0/4} : t \geq 0)$ is a function of B and only B. Therefore $(B(A_t^{3\theta_0/4}); t \geq 0) = (V(\beta_t); t \geq 0)$ is independent of $(W(\beta_{t\wedge T^*}) + B'(t-T^*)\mathbf{1}_{\{t\geq T^*\}}; t \geq 0)$. Letting $\hat{V}(t) = V(\beta_t)$ and $\hat{W}(t) = W(\beta_{t\wedge T^*}) + B'(t-T^*)\mathbf{1}_{\{t\geq T^*\}}$ for all $t \geq 0$ it follows that $(\hat{V}, \hat{W})$ is an independent coupling of a $(3\theta_0/4)$-sticky Brownian motion and a standard Brownian motion. It remains to show that $T^* = \inf\{t \geq 0 : (W(\beta_t), V(\beta_t)) \notin U\}$. Note that when T is the exit time for some process (V, W) of some set C. i.e. $T = \inf\{t \geq 0 : W(t) \notin C\}$ Then if $(\gamma_t : t \geq 0)$ is some time change with $\gamma_t^{-1} = \inf\{u \geq 0 : \gamma_u > t\}$ then it clear that $\gamma^{-1}(T) = \inf\{t \geq 0 : W(\gamma_t) \notin C\}$. Thus we have the result. $\square$

Consider a general point $x \in \mathbb{R}^3$. Let $y_1 = \inf_{i<j} |x_i - x_j| = Y_1(0)$ and $y_2 = \sum_{i<j} |x_i - x_j| - \inf_{i<j} |x_i - x_j| = Y_2(0)$.

Lemma 45. *For any $\epsilon > 0$, let $X = (X_1, X_2, X_3)$ be a solution to the $\mathcal{A}_3^\theta$-martingale problem started at some point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that*

$y_2 = \epsilon$, *with* Y_1 *and* Y_2 *as in* (3.18) *and* (3.19). *Let*

$$\triangle_\epsilon = \left\{ x \in \mathbb{R}^3 : y_1 < \frac{y_2}{\sqrt{3}}, y_1 < \frac{1}{\sqrt{3}}(2\epsilon - y_2) \right\}$$

and let $T(\triangle_\epsilon) = \inf\{t \geq 0 : X(t) \notin \triangle_\epsilon\}$ *then*

$$\mathbf{P}(Y_1(T(\triangle_\epsilon) \neq 0) \leq C_1 \frac{y_1}{\epsilon} + C_2 \theta \epsilon$$

for some constants $0 < C_1, C_2 < \infty$.

Proof. Assume without loss of generality, because of the symmetry under permutations of coordinates, that our starting position $x = (x_1, x_2, x_3)$ satisfies the conditions $|x_1 - x_2| = \inf_{i \neq j} |x_i - x_j|$ and $x_1, x_2 \geq x_3$. We consider the region in $\mathbb{R}^3$ given by

$$U' = \{x \in \mathbb{R}^3 : |x_1 - x_2| < \min(x_1 - x_3, x_2 - x_3),$$
$$0 < \max(x_1, x_2, x_3) - \min(x_1, x_2, x_3) < \sqrt{6}\epsilon\}$$

and let $T(U') = \inf\{t \geq 0 : X(t) \notin U'\}$. While $X(t) \in U'$ we have $Y_1(t) = |V(t)|$ and $Y_2(t) = W(t)$. Indeed, under the given assumptions on the starting values, $(Y_2(t \wedge T(U')), Y_1(t \wedge T(U')); t \geq 0) = (W(t \wedge T(U)), |V(t \wedge T(U))|; t \geq 0)$.

Let $\triangle'_\epsilon = \triangle_\epsilon \cap U'$ so that under the given assumptions on the starting values, $T(\triangle'_\epsilon) = T(\triangle_\epsilon)$ by the continuity of X. Then as $T(\triangle'_\epsilon) \leq T(U')$ we can see that by Lemma 44 the process $(Y_2(\beta_t \wedge T(\triangle_\epsilon)), Y_1(\beta_t \wedge T(\triangle_\epsilon)))_{t \geq 0}$ behaves as the independent coupling of a sticky Brownian motion and a standard Brownian motion stopped at time τ^*_ϵ where $\beta_{\tau^*_\epsilon} = T(\triangle_\epsilon)$. Lemma, 44, means we can now apply some of the results from section, 3.2.

Note that continuous time changes do not effect exit distributions. Let

$(\gamma_t : t \geq 0)$ be some time change. Let $\tau = \inf\{t \geq 0 : X(t) \notin U\}$ and $\tau' = \inf\{t \geq 0 : X(\gamma_t) \notin U\}$ then clearly $\mathbf{P}(X(\tau) \in A) = \mathbf{P}(X(\gamma_{\tau'}) \in A)$. Thus

$$\mathbf{P}(Y_1(T_{\triangle_\epsilon}) \neq 0) = \mathbf{P}(Y_1(\beta_{\tau_\epsilon^*}) \neq 0)$$

and we can apply Lemma 38, with

$$r(x) = \min\left(\frac{x}{\sqrt{3}}, \frac{1}{\sqrt{3}}(2-x)\right),$$

which satisfies the properties required in the hypothesi of lemma 38.

$$r(0) = r(2) = 0,\ r(x) > 0 \text{ for all } x \in (0,2)$$

$$\lim_{x\downarrow 0} r'(x) = \frac{1}{\sqrt{3}} \text{ and } \lim_{x\uparrow 2} r'(x) = -\frac{1}{\sqrt{3}}.$$

This gives us

$$r^\epsilon(x) = \min\left(\frac{x}{\sqrt{3}}, \frac{1}{\sqrt{3}}(2\epsilon - x)\right).$$

and

$$\tau_\epsilon^* = \inf\{t \geq 0 : Y_1(\beta_t) = r^\epsilon(Y_2(\beta_t))\}$$

Lemma 38 can be applied to give $\mathbf{P}(Y_1(T_{\triangle_\epsilon}) \neq 0) \leq C_1 \frac{y_1}{\epsilon} + C_2\theta_0\epsilon$. □

Corollary 46. *By the symmetry of $\triangle_\epsilon$ about the line $y_2 = \epsilon$. It is possible to see that*

$$\mathbf{P}(Y_2(T_{\triangle_\epsilon}) = 0 | Y_1(T_{\triangle_\epsilon}) = 0) = \mathbf{P}(Y_2(T_{\triangle_\epsilon}) = 2\epsilon | Y_1(T_{\triangle_\epsilon}) = 0) = \frac{1}{2}.$$

Lemma 47. *Let $X = (X_1, X_2, X_3)$ be a solution to the $\mathcal{A}_3^\theta$-martingale problem started at $x = (x_1, x_2, x_3)$ and let Y_1 and Y_2 be defined as* (3.18) *and* (3.19) *respectively. Let x be on the diagonal D so that $Y_2(0) = Y_1(0) = 0$. Let*

$T_{\sqrt{6}\epsilon} = \inf\{t \geq 0 : \max(X_1(t), X_2(t), X_3(t)) - \min(X_1(t), X_2(t), X_3(t)) = \sqrt{6}\epsilon\}$ *then for some constant* C

$$\mathbf{P}(Y_1(T_{\sqrt{6}\epsilon}) \neq 0) \leq C\theta\epsilon.$$

Proof. We define a sequence of stopping times by $T_0 = 0$ then for $n \geq 0$

$$T_{2n+1} = \inf\{t \geq T_{2n} : Y_2(t) = \epsilon\}$$

$$T_{2n+2} = \inf\{t \geq T_{2n+1} : Y_2(t) = 0\}$$

Let $\mathcal{F}_t = \sigma(X(s) : s \leq t)$, then by properties of martingale problems given in [SV79] the conditional law of the process $(X(T_{2n} + t); t \geq 0)$ given $\mathcal{F}_{T_{2n}}$ is almost surely a solution to the $\mathcal{A}_3^\theta$-martingale problem started at from $X(T_{2n})$ hence by Lemma 43 $E[Y_1(T_{2n+1})|\mathcal{F}_{T_{2n}}] \leq 6\sqrt{6}\theta\epsilon^2$ almost surely. Similarly we can show from Lemma 45 that almost surely

$$\mathbf{P}(Y_1(T^n(\Delta_\epsilon)) \neq 0|\mathcal{F}_{T_{2n+1}}) \leq C_1 \frac{Y_1(T_{2n+1})}{\epsilon} + C_2\theta\epsilon$$

where $T^n(\Delta_\epsilon) = \inf\{t \geq T_{2n+1} : X(t) \notin \Delta_\epsilon\}$. Then combing these two estimates.

$$\begin{aligned}\mathbf{P}(Y_1(T^n(\Delta_\epsilon)) \neq 0|\mathcal{F}_{T_{2n}}) &\leq C_1 E\left[\frac{Y_1(T_{2n+1})}{\epsilon} + C_2\theta\epsilon \middle| \mathcal{F}_{T_{2n}}\right] \\ &\leq (6\sqrt{6}C_1 + C_2)\theta\epsilon\end{aligned}$$

almost surely. Letting $T^n_{\sqrt{6}\epsilon} = \inf\{t \geq T_{2n+1} : \max(X_1(t), X_2(t), X_3(t)) -$

$\min(X_1(t), X_2(t), X_3(t)) = \sqrt{6}\epsilon\}$ and using corollary 46 we have

$$\begin{aligned}
&\mathbf{P}(\{Y_1(T^n(\triangle_\epsilon)) = 0\} \cap \{T^n(\triangle_\epsilon) = T^n_{\sqrt{6}\epsilon}\}|\mathcal{F}_{T_{2n}}) \\
&= \mathbf{P}(\{Y_1(T^n(\triangle_\epsilon)) = 0\} \cap \{T^n(\triangle_\epsilon) = T_{2n+2}\}|\mathcal{F}_{T_{2n}}) \\
&\geq \frac{1}{2}(1 - (6C_1 + C_2)\theta\epsilon)
\end{aligned}$$

Let ϕ_n be the event given by

$$\phi_n = \{T^n(\triangle_\epsilon) = T^n_{\sqrt{6}\epsilon}\} \bigcap_{k=0}^{n-1} \{T^k(\triangle_\epsilon) = T_{2k+2}\} \bigcap_{k=0}^{n} \{Y_1(T^k(\triangle_\epsilon)) = 0\}.$$

By the estimates given above we have

$$\begin{aligned}
\mathbf{P}(\phi_n) &= \mathbf{E}[\mathbf{P}(\phi_n|\mathcal{F}_{T_{2n}})] \\
&= (\frac{1}{2}(1 - (6C_1 + C_2)\theta\epsilon)\mathbf{P}\left(\bigcap_{k=0}^{n-1} \{T^k(\triangle_\epsilon) = T_{2k+2}\} \cap \{Y_1(T^k(\triangle_\epsilon)) = 0\}\right) \\
&\geq \left(\frac{1}{2}(1 - (6C_1 + C_2)\theta\epsilon)\right)^{n+1}.
\end{aligned}$$

Finally we have that

$$\begin{aligned}
\mathbf{P}(Y_1(T_{\sqrt{6}\epsilon}) = 0) &\geq \mathbf{P}\left(\bigcup_{n=0}^{\infty} \phi_n\right) \\
&\geq \sum_{n=0}^{\infty} \left(\frac{1}{2}(1 - (6C_1 + C_2))\theta\epsilon\right)^{n+1} \\
&\geq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - (6C_1 + C_2))\theta\epsilon \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n \\
&= 1 - 2(6C_1 + C_2))\theta\epsilon
\end{aligned}$$

from which the result follows. □

We now use the above lemmas to complete the proof of proposition 39.

Proof of Proposition 39. In the proposition we have a solution, X, to the $\mathcal{A}_N^\theta$-martingale problem started from some point on the Diagonal,D and we have $T_\epsilon = \inf\{t \geq 0 : \max_{1\leq i\leq N}(X_i(t)) - \min_{1\leq i\leq N}(X_j(t)) \geq \epsilon\}$. Let

$$T_{i,j,k} = \inf\{t \geq 0 : \max(X_i(t), X_j(t), X_k(t)) - \min(X_i(t), X_j(t), X_k(t)) \geq \epsilon\}.$$

Now at the stopping time T_ϵ we must have $|X_i(T_\epsilon) - X_j(T_\epsilon)| = \epsilon$ for some i and j. Fix i and j such that this is true, then $T_{i,j,k} = T_\epsilon$ for all $k \notin \{i,j\}$. Recall that Λ_ϵ is the event that there are three or more distinct values taken by the components of $X(T_\epsilon)$. Thus,

$$\Lambda_\epsilon = \bigcup_{i\neq j} \bigcup_{k\notin\{i,j\}} \{X_k(T_{i,j,k}) = X_i(T_{i,j,k}) \text{ or } X_j(T_{i,j,k})\}$$

and so it follows that

$$\mathbf{P}(\Lambda_\epsilon) \leq \sum_{i\neq j} \sum_{k\notin\{i,j\}} \mathbf{P}(X_k(T_{i,j,k}) = X_i(T_{i,j,k}) \text{ or } X_j(T_{i,j,k})).$$

Note that

$$\begin{aligned}
&\mathbf{P}(X_k(T_{i,j,k}) = X_i(T_{i,j,k}) \text{ or } X_j(T_{i,j,k})) \\
&= \mathbf{P}(\inf\{|X_i(T_{i,j,k}) - X_j(T_{i,j,k})|, |X_j(T_{i,j,k}) - X_k(T_{i,j,k})|, \\
&\qquad |X_k(T_{i,j,k}) - X_i(T_{i,j,k})|\} = 0)
\end{aligned}$$

By Proposition 31 the process $((X_i(t), X_j(t), X_k(t)); t \geq 0)$ is a solution to the $\mathcal{A}_3^\theta$-martingale problem started from the diagonal of $\mathbb{R}^3$ and therefore from

Lemma 47

$$\mathbf{P}(\inf\{|X_i(T_{i,j,k}) - X_j(T_{i,j,k})|, |X_j(T_{i,j,k}) - X_k(T_{i,j,k})|,$$
$$|X_k(T_{i,j,k}) - X_i(T_{i,j,k})|\} = 0) \leq C\theta\epsilon$$

for some constant $0 < C < \infty$. Hence

$$\mathbf{P}(\Lambda_\epsilon) \leq N(N-1)(N-2)C\theta\epsilon$$

□

3.4 The process stopped on first hitting of the diagonal

For any solution to the $\mathcal{A}_N^\theta$-martingale problem started at some point $x \in \mathbb{R}^N$, let T_D be the first time the process reached the diagonal D, that is $T_D = \inf\{t \geq 0 : X_1(t) = \cdots = X_N(t)\}$. The aim of this section is to prove the following proposition:

Proposition 48. *Suppose that for every $n \leq N-1$ and $x \in \mathbb{R}^n$ the $\mathcal{A}_n^\theta$-martingale problem has a solution whose a law is uniquely determined. Then if X is any solution to the $\mathcal{A}_N^\theta$-martingale problem starting from a point x in $\mathbb{R}^N$, the law of the stopped process $(X(t \wedge T_D); t \geq 0)$ is uniquely determined.*

For each bi-partition of the set $\{1, 2, \ldots, N\}$ into two non-empty subsets S and S^c we define the projection $\rho_S : \mathbb{R}^N \mapsto \mathbb{R}^{|S|}$ by $\rho_S(x) = (x_i; i \in S)$ and similarly we define $\rho_{S^c} : \mathbb{R}^N \mapsto \mathbb{R}^{|S^c|}$ by $\rho_{S^c}(x) = (x_i; i \in S^c)$.

For a given S, $x \in \mathbb{R}^N$, and a family of parameters θ, we say that a $\mathbb{R}^N$-valued process X is a solution to the $\mathcal{A}_N^{\theta,S}$-martingale problem started at x

if $Y = (X_i : i \in S)$ is a solution to the $\mathcal{A}^\theta_{|S|}$-martingale problem started at $y = \rho_S(x)$, and $Z = (X_i : i \in S^c)$ is a solution to the $\mathcal{A}^\theta_{|S^c|}$ problem started at $z = \rho_{S^c}(x)$, and furthermore for any $i \in S$ and $j \in S^c$

$$\langle X_i, X_j \rangle \equiv 0.$$

Proposition 49. *Suppose that for every $n \leq N-1$ and $x \in \mathbb{R}^n$ the $\mathcal{A}^\theta_n$-martingale problem has a solution whose law is uniquely determined. Then if X is a solution to the $\mathcal{A}^{\theta,S}_N$-martingale problem starting from a point $x \in \mathbb{R}^N$, then the law of X is uniquely determined.*

Before we begin the proof of this proposition we need the following lemma.

Lemma 50. *Let X be a solution to the $\mathcal{A}^\theta_N$-martingale problem started at $x \in \mathbb{R}^N$. For each $f \in L_N$, the martingale M_f given by*

$$M_f(t) = f(X(t)) - \int_0^t \mathcal{A}^\theta_N(X(s))ds$$

can be represented as

$$M_f(t) = f(x) + \sum_{i=1}^N \int_0^t h_i(s)dX_i(s).$$

for some predictable processes $(h_i; 1 \leq i \leq N)$. Furthermore, for each $1 \leq i,j \leq N$

$$X_i^2(t) - t = x_i^2 + 2\int_0^t X_i(s)dX_i(s)$$

and

$$X_i(t)X_j(t) - \int_0^t 1_{\{X_i(s)=X_j(s)\}}ds = x_ix_j + \int_0^t X_i(s)dX_j(s) + \int_0^t X_j(s)dX_i(s).$$

Proof. For $f \in L_N$, f is a piecewise linear and continuous. It is possible to see that each function $f \in L_N$ can be constructed by starting from some linear functions g and h and iteratively applying functions of the form $\min(g, h)$ or $\max(g, h)$. Applying Tanaka's formula iteratively gives that, for some predictable $(h_i, 1 \leq i \leq N)$,

$$f(X_t) = \sum_{i=1}^{N} \int_0^t h_i(s) dX_i(s) + l(t)$$

where $l(t)$ is some combination of local times and is a process of finite variation. Thus the martingale part of $f(X_t)$ is $\sum_{i=1}^{N} \int_0^t h_i(s) dX_i(s)$. The remaining two representations in the lemma follow from Itô's formula. □

Proof of Proposition 49. Let X be a solution to the $\mathcal{A}_N^{\theta,S}$-martingale problem starting from $x \in \mathbb{R}^N$ and let Y and Z be as described above. By hypothesis, separately, the laws of Y and Z are both uniquely determined. Consider the set of martingales that determine the law of Y. These martingales are of the form

$$f(Y(t)) - \int_0^t \mathcal{A}_{|S|}^{\theta}(Y(s)) ds, \qquad t \geq 0$$

for $f \in L_N$, or of the form $(X_i(t)X_j(t) - \int_0^t 1_{\{X_i(s)=X_j(s)\}} ds; t \geq 0)$ for $i, j \in S$ or$(X_i(t)^2 - t; t \geq 0)$ for $i \in S$. If M is such a martingale then, from lemma 50, there exists predictable processes $(h_i; i \in S)$ such that $M(t) = M(0) + \sum_{i \in S} \int_0^t h_i(s) dX_i(s)$.

Consequently it is possible to show, see [Jac79], that any $F \in L^\infty(Y)$ can be written as

$$F = \mathbf{E}[F] + \sum_{i \in S} \int_0^\infty f_i(s) dX_i(s).$$

for some Y-predicable processes $(f_i; i \in S)$. Similarly any $G \in L^\infty(Z)$ can be written as

$$G = \mathbf{E}[G] + \sum_{i \in S^c} \int_0^\infty g_i(s) dX_i(s)$$

for some Z-predictable processes $(g_i; i \in S^c)$.

Then multiplying and taking expectations gives

$$\begin{aligned}\mathbf{E}[FG] &= \mathbf{E}[F]\mathbf{E}[G] \\ &+ \mathbf{E}[F]\mathbf{E}\left[\sum_{i \in S} \int_0^\infty f_i(s) dX_i(s)\right] + \mathbf{E}[G]\mathbf{E}\left[\sum_{i \in S^c} \int_0^\infty g_i(s) dX_i(s)\right] \\ &+ \mathbf{E}\left[\left(\sum_{i \in S} \int_0^\infty f_i(s) dX_i(s)\right)\left(\sum_{i \in S^c} \int_0^\infty g_i(s) dX_i(s)\right)\right].\end{aligned}$$

Note for $i \in S$ and $j \in S^c$ that X_i and X_j are orthogonal Brownian motions. It follows that $\int_0^t f_i(s) dX_i(s)$ and $\int_0^t g_j(s) dX_j(s)$ are orthogonal martingales. From this and the fact that $F \in L^\infty(Y)$ and $G \in L^\infty(Z)$ implies

$$\begin{aligned}\mathbf{E}\left[\int_0^\infty f_i(s) dX_i(s)\right] &= \mathbf{E}\left[\int_0^\infty g_j(s) dX_j(s)\right] \\ &= \mathbf{E}\left[\int_0^\infty f_i(s) dX_i(s) \int_0^\infty g_j(s) dX_j(s)\right] = 0.\end{aligned}$$

This leaves us with $\mathbf{E}[FG] = \mathbf{E}[F]\mathbf{E}[G]$ for any $F \in L^\infty(Y)$ and $G \in L^\infty(Z)$ which implies that Y and Z must be independent. This means that the joint law of $X = (Y, Z)$ is uniquely specified. □

Proof of Proposition 48. For a bi-partition $\{S, S^c\}$ we define the open set

$$U^S = \{x \in \mathbb{R}^N : x_i > x_j \text{ for all } i \in S, j \in S^c\}. \tag{3.26}$$

For a given S, let X be a solution to the $\mathcal{A}_N^\theta$-martingale problem started at $x \in U^S$ and let $T_S = \inf\{t \geq 0 : X(t) \notin U^S\}$. By Proposition 31, $\rho_S(X)$

is a solution to the $\mathcal{A}^{\theta}_{|S|}$-martingale problem started at $\rho_S(x)$ and $\rho_{S^c}(X)$ is a solution to the $\mathcal{A}^{\theta}_{|S^c|}$-martingale problem started at $\rho_{S^c}(x)$. Also it is clear that $\langle X_i, X_j\rangle_{t\wedge T_S} = 0$ for all $i \in S, j \in S^c$, and $t \geq 0$.

Conditional on X, let $\tilde{X}$ be a solution to the $\mathcal{A}^{\theta,S}_N$-martingale problem started at $X(T_S)$. Letting

$$\hat{X}(t) = \begin{cases} X(t) & t \leq T_S \\ \tilde{X}(t - T_S) & t > T_S, \end{cases}$$

it follows that $\hat{X}$ is a solution to the $\mathcal{A}^{\theta,S}_N$-martingale problem started at x. Thus, by the previous proposition, the law of $\hat{X}$ and hence the law of $(X(t\wedge T_S); t \geq 0)$ is uniquely determined. Thus if T is a stopping time, almost surely less than T_S, then the law of $(X(T \wedge t); t \geq 0)$ is uniquely specified.

Now let X be any solution to the $\mathcal{A}^{\theta}_N$-martingale problem started at some point $x \in \mathbb{R}^N$. We remove a small area of the state space around the diagonal D and we also specify extremities. For $\epsilon > 0$ we define

$$K^\epsilon = \{x \in \mathbb{R}^N : |x_i - x_j| > \epsilon \text{ for some } i \neq j\} \cap \{x \in \mathbb{R}^N : |x_i| < 1/\epsilon \text{ for all } i\}$$

and let $\tau_\epsilon = \inf\{t \geq 0 : X(t) \notin K^\epsilon\}$.

Looking back at the definition of U^S, (3.26), we can see that for each $x \in K^\epsilon$ there exists an $S(x) \subset \{1, \ldots, N\}$ such that $U^{S(x)}$ contains the ball $\{y \in \mathbb{R}^N : ||y - x|| < \epsilon/2N\}$. We fix $x \in K^\epsilon$ then we define the stopping times $T_0 = 0$ and

$$T_{i+1} = \inf\{t \geq T_i : X(t) \notin U^{S(i)} \cap K^\epsilon\},$$

where $S(i) = S(X(T_i))$.

As for any solution to the $\mathcal{A}^{\theta}_N$-martingale problem each coordinate is a

Brownian motion, we must have that τ_ϵ is almost surely finite. This together with the fact that $d(x, (U^{S(i)})^c) = \inf_{y \notin U^{S(i)}} ||X(T_i) - y|| \geq \epsilon/2N$, and the fact that paths of X are continuous means that $T_i = \tau_\epsilon$ for sufficiently large i with probability one.

Consider the conditional distribution of $(X(T_{i+1} \wedge (t + T_i)); t \geq 0)$ given $\mathcal{F}_{T_i} = \sigma(X(t \wedge T_i); t \geq 0)$. By standard theorems on martingale problems, see [SV79], this conditional law is almost surely a solution to the $\mathcal{A}_N^\theta$-martingale problem started at $X(T_i)$ and stopped upon exiting the set $U^{S(i)} \cap K^\epsilon$. By the arguments given above this conditional law is uniquely specified. As the conditional law $(X(T_{i+1} \wedge (t + T_i)); t \geq 0)$ given $\mathcal{F}_{T_i} = \sigma(X(t \wedge T_i); t \geq 0)$ is unique for every i it follows that the law of the process $(X(t \wedge T_i); t \geq 0)$ is uniquely specified. Then as, $T_i = \tau_\epsilon$ for sufficiently large i with probability one, it follows that the law of the process $(X(t \wedge \tau_\epsilon); t \geq 0)$ is uniquely specified. Finally, letting ϵ tend down to zero, gives the result of Proposition 48. We note that this type of localisation technique can be found in [SV79]. □

3.5 Uniqueness

We combine the results of the previous two sections to give uniqueness in law of a solution to the $\mathcal{A}_N^\theta$-martingale problem started at some fixed $x \in \mathbb{R}^N$. First of all we use the results of the previous sections to prove that the process projected on to the hyperplane $x_1 + x_2 + \cdots + x_N = 0$ has a uniquely determined law. This is a natural projection because the interesting interactions between the components of the process occur in this hyperplane. The second part of this section shows that we can construct the movement of the process in the remaining direction in a unique way.

3.5.1 Uniqueness of the projected process

We plan to use an induction argument on N. Thus throughout this subsection we will be assuming the following hypothesis.

If X is any solution to the $\mathcal{A}_N^\theta$-martingale problem starting from a point $x \in \mathbf{R}^N$, the law of the stopped process $\big(X(T_D \wedge t); t \geq 0\big)$ is uniquely determined. (3.27)

Suppose that $(\mathbf{P}_x; x \in \mathbb{R}^N)$ is some family of probability measures such that under $\mathbf{P}_x$, X solves the $\mathcal{A}_N^\theta$-martingale problem and $X(0) = x$. Then we define the function ψ_λ by

$$\psi_\lambda(x) = \mathbf{E}_x\left[\exp(-\lambda T_D)\right]$$

and, for any bounded test function f, we define $R_\lambda^0 f$ by

$$R_\lambda^0 f(x) = \mathbf{E}_x\left[\int_0^{T_D} e^{-\lambda s} f(X(s))\right] ds.$$

Note that, under the hypothesis (3.27), the two functions above are uniquely specified.

Let f be bounded and invariant under shifts parallel to the diagonal of $\mathbb{R}^N$, $D = \{x \in \mathbb{R}^N : x_1 = \cdots = x_N\}$. Our aim is to show that $R_\lambda f$ given by

$$R_\lambda f(x) = \mathbf{E}_x\left[\int_0^\infty e^{-\lambda s} f(X(s))ds\right]$$

which *a priori* depends on the possible choice of the family of measures $(\mathbf{P}_x; x \in \mathbb{R}^N)$, is in fact uniquely determined. We will show that

$$R_\lambda f(x) = R_\lambda^0 f(x) + R_\lambda f(0)\psi_\lambda(x) \tag{3.28}$$

and that the value of $R_\lambda f(0)$ is the same for any choice of measures $(\mathbf{P}_x; x \in \mathbb{R}^N)$. To achieve this we need results from Section 3.3, in particular Theorem 41.

The following lemma will be needed later.

Lemma 51. *For some constant C depending on λ,N and θ only*

$$1 - \psi_\lambda(x) \leq C\sqrt{dist(x, D)},$$

whenever $dist(x, D) \leq 1$.

Proof. First of all we bound ψ below by something which is easier to work with.

$$\begin{aligned}
\psi_\lambda(x) = \mathbf{E}_x[e^{-\lambda T_D}] &= \int_0^\infty e^{-\lambda s}\mathbf{P}_x(T_D \in ds) \\
&\geq \int_0^t e^{-\lambda s}\mathbf{P}_x(T_D \in ds) \geq e^{-\lambda t}\mathbf{P}_x(T_D \leq t) \geq e^{-\lambda t}\mathbf{P}_x(X(t) \in D) \\
&\geq e^{-\lambda t}(1 - \mathbf{P}_x(X(t) \notin D)) \\
&\geq e^{-\lambda t}\left(1 - \sum_{i \neq j}\mathbf{P}_x(X_i(t) \neq X_j(t))\right) \qquad (3.29)
\end{aligned}$$

This holds for all t, and in particular we can let $t = d$, where here $d = dist(x, D)$. We have that $dist(x, D) = \inf_{y \in \mathbb{R}} \sqrt{(x_i - y)^2 + (x_j - y)^2 + \cdots}$ and

$$(x_i - x_j)^2 \leq 2[(x_i - y)^2 + (x_j - y)^2] \ \ \forall y \in \mathbb{R}.$$

Thus we conclude that $|x_i - x_j| \leq \sqrt{2}d$. As $\frac{1}{\sqrt{2}}|X_i(t) - X_j(t)|$ is a $(\sqrt{2}\theta)$-sticky BM we have, see (2.15),

$$\begin{aligned}
\mathbf{P}_x(X_i(t) \neq X_j(t)) &\leq 1 - e^{2\sqrt{2}\theta|x_i - x_j|}e^{4\theta^2 t} \operatorname{erfc}\left(2\theta\sqrt{t} + \frac{|x_i - x_j|}{2\sqrt{t}}\right) \\
&\leq 1 - e^{4\sqrt{2}\theta d}e^{4\theta^2 t} \operatorname{erfc}\left(2\theta\sqrt{t} + \frac{d}{\sqrt{t}}\right).
\end{aligned}$$

Then letting $t = d$

$$\mathbf{P}_x(X_i(d) \neq X_j(d)) \leq 1 - e^{4\sqrt{2}\theta d} e^{4\theta^2 d} \operatorname{erfc}\left(2\theta\sqrt{d} + \sqrt{d}\right)$$

and so

$$\begin{aligned}\mathbf{P}_x(X_i(d) \neq X_j(d)) &\leq \\ 1 - \left[(1 + 4\sqrt{2}\theta d)(1 + 4\theta^2 d)\left(1 - \sqrt{d}\frac{(4\theta + 2)}{\sqrt{\pi}}\right)\right] &\leq \frac{(4\theta + 2)}{\sqrt{\pi}}\sqrt{d}.\end{aligned}$$

Combining this with (3.29) we have

$$\begin{aligned}1 - \psi_\lambda(x) &\leq 1 - e^{-\lambda d} + e^{-\lambda d} N(N-1)\frac{(4\theta + 2)}{\sqrt{\pi}}\sqrt{d} \\ &\leq \left(\lambda + N(N-1)\frac{(4\theta + 2)}{\sqrt{\pi}}\right)\sqrt{d}.\end{aligned}$$

□

For a solution to the $\mathcal{A}_N^\theta$-martingale problem X started from $x \in D$, recall that $T_\epsilon = \inf\{t \geq 0 : |X_i(t) - X_j(t)| \geq \epsilon, \text{ for some } i, j\}$.

Lemma 52. *For some constant C depending only on θ and N,*

$$\mathbf{E}\left[T_\epsilon^2\right] \leq \frac{\epsilon^2}{2\theta^2} + C\epsilon^3,$$

whenever $\epsilon \leq 1$.

Proof. Let $T_\epsilon^{i,j} = \inf\{t \geq 0 : |X_i(t) - X_j(t)| \geq \epsilon\}$. Then $T_\epsilon \leq T_\epsilon^{i,j}$ for all i, j. Thus, because of Proposition 31, we only have to prove the lemma for the two dimensional case and we are done. Let $Z(t) = |X_1(t) - X_2(t)|$ Then for some martingale M,

$$dZ(t) = dM(t) + 2\theta \mathbf{1}_{\{Z(t)=0\}} dt$$

and

$$d\langle Z\rangle_t = 2\mathbf{1}_{\{Z(t)\neq 0\}}dt.$$

Consider the function

$$f(z,t) = t\left(z^2 + \frac{z}{\theta}\right) - \frac{1}{12}\left(z^4 + \frac{2z^3}{\theta}\right).$$

Then

$$\frac{\partial f}{\partial z}(z,t) = t\left(2z + \frac{1}{\theta}\right) - \left(\frac{1}{3}z^3 + \frac{z^2}{2\theta}\right)$$
$$\frac{1}{2}\frac{\partial^2 f}{\partial z^2}(z,t) = t - \frac{1}{2}z^2 - \frac{z}{2\theta}$$
$$\frac{\partial f}{\partial t}(z,t) = \left(z^2 + \frac{z}{\theta}\right)$$

and so by Itô's formula

$$\begin{aligned} f(Z(t),t) =& f(0,0) + \int_0^t s\left(2Z(s) + \frac{1}{\theta}\right)dM(s) - \int_0^t \left(Z(s)^3 + \frac{Z(s)^2}{2\theta}\right)dM(s) \\ &+ 2\theta\int_0^t s\left(2Z(s) + \frac{1}{\theta}\right)\mathbf{1}_{\{Z(s)=0\}}ds - 2\theta\int_0^t \left(Z(s)^3 + \frac{Z(s)^2}{2\theta}\right)\mathbf{1}_{\{Z(s)=0\}}ds \\ &- \int_0^t 2s\mathbf{1}_{\{Z(s)\neq 0\}}ds - \int_0^t Z(s)^2 + \frac{Z(s)}{\theta}\mathbf{1}_{\{Z(s)\neq 0\}}ds \\ &+ \int_0^t Z(s)^2 + \frac{Z(s)}{\theta}ds \end{aligned}$$

So for some local martingale $\hat{M}$ with $\hat{M}(0) = 0$

$$f(Z(t),t) = \hat{M}(t) + \int_0^t 2sds = \hat{M} + t^2$$

hence by the OST

$$(t\wedge T_\epsilon)\left(Z(t\wedge T_\epsilon)^2 + \frac{Z_{t\wedge T_\epsilon}}{\theta}\right) - \frac{1}{12}\left(Z^4_{t\wedge T_\epsilon} + \frac{2Z(t\wedge T_\epsilon)^3}{\theta}\right) - (t\wedge T_\epsilon)^2$$

is a local martingale. Then because $Z(t \wedge T_\epsilon)$ is bounded above by ϵ and below by 0 it follows that

$$\mathbf{E}\left[(t \wedge T_\epsilon)^2\right] \leq \mathbf{E}\left[(t \wedge T_\epsilon)\right]\left(\epsilon^2 + \frac{\epsilon}{\theta}\right).$$

The monotone convergence theorem gives us

$$\mathbf{E}\left[T_\epsilon^2\right] \leq \mathbf{E}\left[T_\epsilon\right]\left(\epsilon^2 + \frac{\epsilon}{\theta}\right)$$

From corollary 42 we have that for some constant depending on θ and N only, $\mathbf{E}\left[T_\epsilon\right] \leq \frac{\epsilon}{4\theta} + C\epsilon^2$. Thus

$$\mathbf{E}\left[T_\epsilon^2\right] \leq \frac{\epsilon^2}{4\theta^2} + \frac{C}{\theta}\epsilon^3.$$

□

Lemma 53. *For any solution to the $\mathcal{A}_N^\theta$-martingale problem,X, started at some point on the diagonal D we have for some finite non-negative constant C, depending only on θ, N and λ.*

$$\left|\mathbf{E}\left[\int_0^{T_\epsilon} e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right] - \frac{\epsilon}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}\right| \leq C\epsilon^2. \tag{3.30}$$

Proof. Firstly note that from corollary 42 we have

$$\left|\mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}_{\{X_S\in D\}}ds\right] - \frac{\epsilon}{2\sum_{v\in\mathcal{V}_+(D)}\theta(v)}\right| \leq C\epsilon^2$$

Then for the upper bound of (3.30) we have

$$\mathbf{E}\left[\int_0^{T_\epsilon} e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right] \leq \mathbf{E}\left[\int_0^{T_\epsilon} \mathbf{1}_{\{X(s)\in D\}}ds\right]$$

and for the lower bound of (3.30) we have

$$\begin{aligned}\mathbf{E}\left[\int_0^{T_\epsilon} e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right] &\geq \mathbf{E}\left[\int_0^{T_\epsilon}(1-\lambda s)\mathbf{1}_{\{X(s)\in D\}}ds\right] \\ &\geq \mathbf{E}\left[\int_0^{T_\epsilon}\mathbf{1}_{\{X(s)\in D\}}ds\right] - \frac{\lambda}{2}\mathbf{E}\left[T_\epsilon^2\right]\end{aligned}$$

which, by Lemma 52, gives us (3.30) □

Lemma 54. *For X being any solution to the A_N^θ-martingale problem started at some point on the diagonal $x \in D$ we have, for some constant C depending only on θ, N and λ,*

$$\left|\mathbf{E}\left[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))\right] - \frac{1}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\left(\frac{\lambda\epsilon}{2} + \sum_{v\in\mathcal{V}_+(D)}(1-\psi_\lambda(\epsilon v))\theta(v)\right)\right| \leq C\epsilon^{3/2}. \quad (3.31)$$

Proof. In the following let $C_1, C_2, \ldots$ be constants depending on θ, N and λ only. Recall that Λ_ϵ is the event that there are three or more distinct values taken by the components of $X(T_\epsilon)$. We know from Proposition 39 that

$$\mathbf{P}(\Lambda_\epsilon) \leq C_1\epsilon. \quad (3.32)$$

For $E(v)$, a neighbour of D, from corollary 42 we have

$$\left|\mathbf{P}\big(X(T_\epsilon)\in E(v)\big) - \frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\right| \leq C_2\epsilon. \quad (3.33)$$

Using these approximations we can get bounds for $\mathbf{E}[1-\psi_\lambda(X(T_\epsilon))]$. We have

$$\begin{aligned}\mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] = \sum_{v\in\mathcal{V}_+(D)} &(1-\psi_\lambda(\epsilon v))\mathbf{P}(X(T_\epsilon)\in E(v)) \\ &+ \mathbf{E}\left[(1-\psi_\lambda(X(T_\epsilon)))\mathbf{1}_{\Lambda_\epsilon}\right]\end{aligned}$$

The second term is positive so for a lower bound we have

$$\begin{aligned}\mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] &\geq \sum_{v\in\mathcal{V}_+(D)} (1-\psi_\lambda(\epsilon v))\mathbf{P}(X(T_\epsilon)\in E(v)) \\ &\geq \sum_{v\in\mathcal{V}_+(D)} (1-\psi_\lambda(\epsilon v))\frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)} - C_2\epsilon \qquad (3.34)\end{aligned}$$

Lemma 51 tells us that, $1-\psi_\lambda(X(T_\epsilon)) \leq C_3\sqrt{\epsilon}$, and so for an upper bound we have

$$\begin{aligned}\mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] &\leq \sum_{v\in\mathcal{V}_+(D)} (1-\psi_\lambda(\epsilon v))\mathbf{P}(X(T_\epsilon)\in E(v)) \\ &\quad + C_3\sqrt{\epsilon}\mathbf{P}(\Lambda_\epsilon) \\ &\leq \sum_{v\in\mathcal{V}_+(D)} (1-\psi_\lambda(\epsilon v))\mathbf{P}(X(T_\epsilon)\in E(v)) \\ &\quad + C_1C_3\epsilon^{3/2} \\ &\leq \sum_{v\in\mathcal{V}_+(D)} (1-\psi_\lambda(\epsilon v))\frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)} + C_4\epsilon^{3/2}\end{aligned}$$

Now we can find the upper bound in (3.31), as

$$\begin{aligned}
&\mathbf{E}[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))] \\
&\quad \le \mathbf{E}[1-(1-\lambda T_\epsilon)\psi_\lambda(X(T_\epsilon))] \\
&\quad = \mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] + \lambda\mathbf{E}[T_\epsilon\psi_\lambda(X(T_\epsilon))] \\
&\quad \le \sum_{v\in\mathcal{V}_+(D)}(1-\psi_\lambda(\epsilon v))\frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)} + \lambda\mathbf{E}[T_\epsilon] + C_4\epsilon^{3/2}.
\end{aligned}$$

Thus using corollary 42 we have

$$\begin{aligned}
&\mathbf{E}[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))] \\
&\quad \le \frac{1}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\left(\frac{\lambda\epsilon}{2} + \sum_{v\in\mathcal{V}_+(D)}(1-\psi_\lambda(\epsilon v))\theta(v)\right) + C_5\epsilon^{3/2}.
\end{aligned}$$

For the lover bound of (3.31) we have

$$\begin{aligned}
&\mathbf{E}[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))] \\
&\quad \ge \mathbf{E}[1-(1-\lambda T_\epsilon+\lambda^2T_\epsilon^2)\psi_\lambda(X(T_\epsilon))] \\
&\quad = \mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] + \lambda\mathbf{E}[T_\epsilon\psi_\lambda(X(T_\epsilon))] - \lambda^2\mathbf{E}[T_\epsilon^2\psi_\lambda(X(T_\epsilon))] \\
&\quad \ge \mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] + \lambda\mathbf{E}[T_\epsilon(1-C_3\sqrt{\epsilon})] - \lambda^2\mathbf{E}[T_\epsilon^2]
\end{aligned}$$

Thus, by Lemma 52, and by corollary 42 we have

$$\mathbf{E}[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))] \ge \mathbf{E}[1-\psi_\lambda(X(T_\epsilon))] + \frac{\lambda\epsilon}{2\sum_{u\in\mathcal{V}_+(D)}\theta(u)} - C_6\epsilon^{3/2}.$$

Then by (3.34) we have

$$\mathbf{E}[1-e^{-\lambda T_\epsilon}\psi_\lambda(X(T_\epsilon))]$$
$$\geq \frac{1}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\left(\frac{\lambda\epsilon}{2}+\sum_{v\in\mathcal{V}_+(D)}(1-\psi_\lambda(\epsilon v))\theta(v)\right)-C_7\epsilon^{3/2}$$

□

Proposition 55. *The following limit exists*

$$\kappa_0=\lambda+\lim_{\epsilon\downarrow 0}\frac{2}{\epsilon}\sum_{v\in\mathcal{V}_+(D)}\theta(v)(1-\psi_\lambda(\epsilon v))$$

and the following equality is satisfied

$$\psi_\lambda(x)\kappa_0=\mathbf{E}_x\left[\int_0^\infty e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right]. \tag{3.35}$$

Thus, under hypothesis (3.27), $\mathbf{E}_x\left[\int_0^\infty e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right]$ *does not depend on possible choices of* $\mathbf{P}_x$.

Proof. Introduce stopping times $T_0^\epsilon=0$, $T_1^\epsilon=\inf\{t\geq 0: X(t)\in D\}$,

$$T_2^\epsilon=\inf\{t\geq T_1^\epsilon: |X_i(t)-X_j(t)|\geq\epsilon \text{ for some } i,j\in\{1,2,\dots N\}\},$$

and in general $T_{2k+1}^\epsilon=\inf\{t\geq T_{2k}^\epsilon: X(t)\in D\}$, and

$$T_{2k}^\epsilon=\inf\{t\geq T_{2k-1}^\epsilon: |X_i(t)-X_j(t)|\geq\epsilon \text{ for some } i,j\in\{1,2,\dots N\}\}.$$

For the LHS of (3.35) we have,

$$\begin{aligned}
\psi_\lambda(x) &= \lim_{n\to\infty} \mathbf{E}_x\left[\psi_\lambda(x) - e^{-\lambda T_n^\epsilon}\psi_\lambda(X(T_n^\epsilon))\right] \\
&= \lim_{n\to\infty} \mathbf{E}_x\left[\sum_{k=0}^{n} e^{-\lambda T_k^\epsilon}\psi_\lambda(X(T_k^\epsilon)) - e^{-\lambda T_{k+1}^\epsilon}\psi_\lambda(X(T_{k+1}^\epsilon)))\right] \\
&= \sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon}\psi_\lambda(X(T_k^\epsilon)) - e^{-\lambda T_{k+1}^\epsilon}\psi_\lambda(X(T_{k+1}^\epsilon)))\right] \\
&= \sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon}\tilde{\mathbf{E}}_k\left[1 - e^{-\lambda T^\epsilon}\psi_\lambda(X(T_\epsilon))\right]\right],
\end{aligned}$$

where $\tilde{\mathbf{E}}_k$ denotes expectation relative to the conditional distribution of $\left(X(T_k^\epsilon + u); u \geq 0\right)$ given $\left(X(u); u \leq T_k^\epsilon\right)$. We note that the even terms in the sum above are dropped because for k even $\tilde{\mathbf{E}}_k\left[\psi_\lambda(X(0)) - e^{-\lambda T_D}\right] = 0$.

Well known results , see [SV79] or [EK86], tell us that the conditional process also solves the martingale problem almost surely. Thus we can use Lemma 54 to give us, for some constant C depending only on N, θ and λ,

$$\left|\psi_\lambda(x) - \frac{\sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon}\right]}{\sum_{u\in\mathcal{V}_+(D)}\theta(u)}\left(\frac{\lambda\epsilon}{2} + \sum_{v\in\mathcal{V}_+(D)}(1-\psi_\lambda(\epsilon v))\theta(v)\right)\right| \leq C\epsilon^{3/2}. \tag{3.36}$$

For the right hand side of (3.35) we have

$$\begin{aligned}
\mathbf{E}_x\left[\int_0^\infty e^{-\lambda s}\mathbf{1}_{\{(X(s)\in D\}}ds\right] &= \sum_{k \text{ odd}} \mathbf{E}_x\left[\int_{T_k^\epsilon}^{T_{k+1}^\epsilon} e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right] \\
&= \sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon}\tilde{\mathbf{E}}_k\left[\int_0^{T^\epsilon} e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right]\right].
\end{aligned}$$

With the even terms in the sum dropped being dropped this time because

$X(s) \notin D$ for all $T_k^\epsilon \leq s < T_{k+1}$, when k is even. We use Lemma 53 to give us

$$\left| \mathbf{E}_x \left[\int_0^\infty e^{-\lambda s} \mathbf{1}_{\{X(s) \in D\}} ds \right] - \sum_{k \text{ odd}} \mathbf{E}_x \left[e^{-\lambda T_k^\epsilon} \right] \frac{\epsilon}{2 \sum_{v \in \mathcal{V}_+(D)} \theta(v)} \right| \leq C\epsilon^2. \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$\mathbf{E}_x \left[\int_0^\infty e^{-\lambda s} \mathbf{1}_{\{X(s) \in D\}} ds \right] = \lim_{\epsilon \to 0} \sum_{k \text{ odd}} \mathbf{E}_x \left[e^{-\lambda T_k^\epsilon} \right] \frac{\epsilon}{2 \sum_{v \in \mathcal{V}_+(D)} \theta(v)}. \tag{3.38}$$

and

$$\psi_\lambda(x) = \mathbf{E}_x \left[\int_0^\infty e^{-\lambda s} \mathbf{1}_{\{X(s) \in D\}} ds \right] \left(\frac{\lambda}{2} + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{v \in \mathcal{V}_+(D)} (1 - \psi_\lambda(\epsilon v)) \theta(v) \right).$$

Thus the limit defining κ_0 exists and (3.35) holds. □

Lemma 56. *For bounded f, and some constant C, depending only on λ, N, and θ.*

$$R_\lambda^0 f(x) \leq C ||f||_\infty (\sqrt{dist(x, D)})$$

whenever $dist(x, D) \leq 1$

Proof.

$$\begin{aligned} \left| \mathbf{E}_x \left[\int_0^{T_D} e^{-\lambda s} f(X(s)) ds \right] \right| &\leq \mathbf{E}_x \left[\int_0^{T_D} e^{-\lambda s} |f(X(s))| ds \right] \\ &\leq ||f||_\infty \mathbf{E}_x \left[\int_0^{T_D} e^{-\lambda s} ds \right] \\ &\leq \frac{||f||_\infty}{\lambda} \mathbf{E}_x \left[1 - e^{-\lambda T_D} \right] \end{aligned}$$

and so by Lemma 51 we have the result. □

Lemma 57. *Let X be a solution to the $\mathcal{A}_N^\theta$-martingale problem started at some point on the diagonal, D. Then for some constant C depending on N, θ and λ only,*

$$\left|\mathbf{E}[e^{-\lambda T_\epsilon} R_\lambda^0 f(X(T_\epsilon))] - \sum_{\mathcal{V}_+(D)} R_\lambda^0 f(v\epsilon) \frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)} \theta(u)}\right| \le C\epsilon^{3/2}.$$

Proof. In the following let $C_1, C_2, \ldots$ be constants depending on θ, N and λ only. First of all we find bounds for $\mathbf{E}_x[R_\lambda^0 f(X(T_\epsilon))]$. Let $A = \{x : \mathrm{dist}(x, D) = \epsilon$ and $x \notin E(v)$ for any $v \in \mathcal{V}_+(D)\}$ then

$$\mathbf{E}[R_\lambda^0 f(X(T_\epsilon))] = \sum_{v\in\mathcal{V}_+(D)} R_\lambda^0 f(v\epsilon)\mathbf{P}(X(T_\epsilon) \in E(v)) + \int_{x\in A} R_\lambda^0 f(x)\mathbf{P}(X(T_\epsilon) \in dx)$$

Lemma 56 applied to the second term gives us

$$\mathbf{E}[R_\lambda^0 f(X(T_\epsilon))] \le \sum_{v\in\mathcal{V}_+(D)} R_\lambda^0 f(v\epsilon)\mathbf{P}(X(T_\epsilon) \in E(v)) + C_1\sqrt{\epsilon}\mathbf{P}(\Lambda_\epsilon).$$

Proposition 39 and corollary 42 gives us

$$\left|\mathbf{E}[R_\lambda^0 f(X(T_\epsilon))] - \sum_{v\in\mathcal{V}_+(D)} R_\lambda^0 f(v\epsilon) \frac{\theta(v)}{\sum_{u\in\mathcal{V}_+(D)} \theta(u)}\right| \le C_2\epsilon^{3/2}. \tag{3.39}$$

We then have

$$\left|\mathbf{E}[e^{-\lambda T_\epsilon} R_\lambda^0 f(X(T_\epsilon))] - \mathbf{E}[R_\lambda^0 f(X(T_\epsilon))]\right| \le \mathbf{E}[\lambda T_\epsilon R_\lambda^0 f(X(T_\epsilon))] \tag{3.40}$$

and so by Lemma 56 and corollary 42

$$\mathbf{E}[\lambda T_\epsilon R_\lambda^0 f(X(T_\epsilon))] \le C_3 ||f||_\infty \sqrt{\epsilon}\mathbf{E}[\lambda T_\epsilon] = C_4\epsilon^{3/2}.$$

Hence (3.40) and (3.39) give us

$$\left| \mathbf{E}[e^{-\lambda T_\epsilon} R^0_\lambda f(X(T_\epsilon))] - \sum_{v \in \mathcal{V}_+(D)} R^0_\lambda f(v\epsilon) \frac{\theta(v)}{\sum_{u \in \mathcal{V}_+(D)} \theta(u)} \right| \le C_5 \epsilon^{3/2}.$$

□

Proposition 58. *Suppose that f is bounded, zero in a neighbourhood of D, and invariant under shifts along D: that is $f(x+y) = f(x)$ for all $y \in D$. The following limit exists*

$$\kappa_f = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \sum_{\mathcal{V}_+(D)} 2R^0_\lambda f(v\epsilon)\theta(v)$$

and the equality

$$\mathbf{E}_x \left[\int_0^\infty e^{-\lambda s} \mathbf{1}_{\{X(s) \in D\}} ds \right] \kappa_f = R_\lambda f(x) - R^0_\lambda f(x) \tag{3.41}$$

is satisfied. Consequently (3.28) *holds with $R_\lambda f(0) = \kappa_f / \kappa_0$.*

Proof. Using the same sequence of stopping times as in the proof of Proposition 55,

$$\begin{aligned}
R^0_\lambda f(x) &= \lim_{n \to \infty} \mathbf{E}_x \left[R^0_\lambda f(x) - e^{-\lambda T^\epsilon_n} R^0_\lambda f(X(T^\epsilon_n)) \right] \\
&= \lim_{n \to \infty} \mathbf{E}_x \left[\sum_{k=0}^n e^{-\lambda T^\epsilon_k} R^0_\lambda f(X(T^\epsilon_k)) - e^{-\lambda T^\epsilon_{k+1}} R^0_\lambda f(X(T^\epsilon_{k+1})) \right] \\
&= \sum_{k \text{ even}} \mathbf{E}_x \left[e^{-\lambda T^\epsilon_k} R^0_\lambda f(X(T^\epsilon_k)) - e^{-\lambda T^\epsilon_{k+1}} R^0_\lambda f(X(T^\epsilon_{k+1})) \right] \\
&\quad + \sum_{k \text{ odd}} \mathbf{E}_x \left[e^{-\lambda T^\epsilon_k} R^0_\lambda f(X(T^\epsilon_k)) - e^{-\lambda T^\epsilon_{k+1}} R^0_\lambda f(X(T^\epsilon_{k+1})) \right].
\end{aligned}$$

$X(T_k^\epsilon) \in D$ for k odd, hence $R_\lambda^0 f(X(T_k^\epsilon)) = 0$ for k odd. This means the first term above becomes

$$\sum_{k \text{ even}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon} R_\lambda^0 f(X(T_k^\epsilon))\right] = \sum_{k \text{ even}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon} \tilde{\mathbf{E}}_k\left[\int_0^{T_D} e^{-\lambda s} f(X(s))ds\right]\right]$$
$$= \sum_{k \text{ even}} \mathbf{E}_x\left[\int_{T_k^\epsilon}^{T_{k+1}^\epsilon} e^{-\lambda s} f(X(s))ds\right]$$

Then as we are assuming f is zero in a neighbourhood of D, and as $\text{dist}(X(s), D) \leq \epsilon$ for all $T_k \leq s \leq T_{k+1}$, when k is odd, we have

$$\int_{T_k^\epsilon}^{T_{k+1}^\epsilon} e^{-\lambda s} f(X(s))ds = 0$$

for small enough ϵ. Therefore, for ϵ small, we have

$$R_\lambda^0 f(x) = \mathbf{E}_x\left[\int_0^\infty e^{-\lambda s} f(X(s))ds\right]$$
$$+ \sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon} R_\lambda^0 f(X(T_k^\epsilon)) - e^{-\lambda T_{k+1}^\epsilon} R_\lambda^0 f(X(T_{k+1}^\epsilon))\right]$$

Then as $R_\lambda^0 f(X(T_k^\epsilon)) = 0$ for k odd, the second term becomes

$$-\sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_{k+1}^\epsilon} R_\lambda^0 f(X(T_{k+1}^\epsilon))\right] = -\sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon} \tilde{\mathbf{E}}_k\left[e^{-\lambda T_\epsilon} R_\lambda^0 f(X(T_\epsilon))\right]\right]$$

and so for small ϵ

$$R_\lambda^0 f(x) = R_\lambda f(x) - \sum_{k \text{ odd}} \mathbf{E}_x\left[e^{-\lambda T_k^\epsilon} \tilde{\mathbf{E}}_k\left[e^{-\lambda T_\epsilon} R_\lambda^0 f(X(T_\epsilon))\right]\right].$$

Applying Lemma 57 we get

$$\left| R_\lambda f(x) - R_\lambda^0 f(x) - \frac{\sum_{k \text{ odd}} \mathbf{E}_x[e^{-\lambda T_k^\epsilon}]}{\sum_{u \in \mathcal{V}_+(D)} \theta(u)} \sum_{v \in \mathcal{V}_+(D)} R_\lambda^0 f(v\epsilon)\theta(v) \right| \leq C\epsilon^{3/2}$$

which together with (3.38) gives us

$$R_\lambda f(x) - R^0_\lambda f(x) = \mathbf{E}_x\left[\int_0^\infty e^{-\lambda s}\mathbf{1}_{\{X(s)\in D\}}ds\right]\lim_{\epsilon\to 0}\frac{2}{\epsilon}\sum_{v\in\mathcal{V}_+(D)} R^0_\lambda f(v\epsilon)\theta(v).$$

Thus the limit use to define κ_f exists and (3.41) holds. □

Lemma 59. *Let X be a solution to the $\mathcal{A}^\theta_N$-martingale problem started at x. Let $\tilde{X}$ be defined by $\tilde{X}(t) = X(t) + \beta 1$ then $\tilde{X}$ solves the $\mathcal{A}^\theta_N$ martingale problem started at $x + \beta\mathbf{1}$. Hence by uniqueness of the stopped process the law of $\tilde{X}(t\wedge T_D)_{t\geq 0}$ under $\mathbf{P}_x$ is equal to the law of $X(t\wedge T_D)_{t\geq 0}$ under $\mathbf{P}_{x+\beta 1}$*

Proof. As in the proof of Lemma 34, it can be seen that for any $f\in L_N$, $f(x-\beta\mathbf{1}) = f(x) - \beta f(\mathbf{1})$. From this it follows that for all $f\in L_N$ and $v\in\mathcal{V}(x)$, $\nabla_v f(x-\beta\mathbf{1}) = \nabla_v f(x)$. Thus

$$\mathcal{A}^\theta_N f(x-\beta\mathbf{1}) = \mathcal{A}^\theta_N f(x).$$

We know that X solves the $\mathcal{A}^\theta_N$ martingale problem under $\mathbf{P}_x$ which tells us that for all $f\in L_N$

$$f(X(t)) - \int_0^t \mathcal{A}^\theta_N f(X(s))ds \text{ is a martingale.}$$

This in turn tells us that

$$f(\tilde{X}(t)-\beta\mathbf{1}) - \int_0^t \mathcal{A}^\theta_N f(\tilde{X}(s)-\beta\mathbf{1})ds = f(\tilde{X}(t)) - \int_0^t \mathcal{A}^\theta_N f(X(s)) - \beta f(\mathbf{1})$$

is a martingale under. Clearly for each $\tilde{X}_i$ is a Brownian motion and for each i,j $\langle\tilde{X}_i,\tilde{X}_j\rangle_t = \langle X_i,X_j\rangle_t = \int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}}ds = \int_0^t \mathbf{1}_{\{\tilde{X}_i(s)=\tilde{X}_j(s)ds\}}$. Thus $\tilde{X}$ solves the $\mathcal{A}^\theta_N$-martingale problem started at $x+\beta\mathbf{1}$. □

Lemma 60. *Let f be bounded, zero in a neighbourhood of D, and invariant under shifts parallel to the diagonal, D. Then the function $x \mapsto R_\lambda f(x)$ is also invariant under shifts parallel to the diagonal.*

Proof. Firstly $\psi_\lambda(x)$ is invariant under shifts parallel to D. As by the above lemma $\mathbf{E}_{x+\beta\mathbf{1}}[e^{-\lambda T_D}] = \mathbf{E}_x[e^{-\lambda T_D}]$.

$x \mapsto R^0_\lambda f(x)$ is also invariant under shifts parallel to D, when f is invariant under shifts parallel to D as by the above lemma

$$\mathbf{E}_{x+\beta\mathbf{1}}\left[\int_0^{T_D} e^{-\lambda s} f(X(s))ds\right] = \mathbf{E}_x\left[\int_0^{T_D} e^{-\lambda s} f(X(s)+\beta\mathbf{1})ds\right]$$

and as we are assuming f is invariant under shifts parallel to D the above is equal to

$$\mathbf{E}_x\left[\int_0^{T_D} e^{-\lambda s} f(X(s))ds\right]$$

From this it follows that $x \mapsto R_\lambda f(x)$ is invariant under shifts parallel to D, when f is zero in a neighbourhood of D, and invariant under shifts along D by virtue of the relationship (3.28). □

Proposition 61. *$R_\lambda f(x)$ is unique for all x and for all bounded f, invariant under shifts parallel to the diagonal. Also $x \mapsto R_\lambda f(x)$ is invariant under shifts parallel to the diagonal.*

Proof. Assume that the law of X is not unique. Say we have two families of laws $(\mathbf{P}_x; x \in \mathbb{R}^N)$ and $(\mathbf{P}'_x; x \in \mathbb{R}^N)$, such that under both $\mathbf{P}_x$ and $\mathbf{P}'_x$, X solves the $\mathcal{A}^\theta_N$-martingale problem started at x. Let $R_\lambda f(x)$ and $R'_\lambda f(x)$ be the corresponding resolvent operators. Consider the measures

$$\mu_x(A) = R_\lambda \mathbf{1}_A(x) \text{ and } \mu'_x(A) = R'_\lambda \mathbf{1}_A(x)$$

Let $D_\epsilon = \{x \in \mathbb{R}^N;\ \text{dist}(x, D) \le \epsilon\}$. By Proposition 58, $\mu_x(A) = \mu'_x(A)$ for any $A \in \mathbb{R}^N$ such that $A \cap D_\epsilon = \emptyset$ and A is invariant under shifts parallel to the diagonal. Now let $A \in \mathbb{R}^N$ be a set which is invariant under shifts parallel to the diagonal such that $A \cap D = \emptyset$. In this case $A = \bigcup_{\epsilon>0}(A \cap D_\epsilon^c)$ and hence by the monotone convergence theorem

$$\mu_x(A) = \lim_{\epsilon\downarrow 0} \mu_x(A \cap D_\epsilon^c) = \lim_{\epsilon\downarrow 0} \mu'_x(A \cap D_\epsilon^c) = \mu'_x(A).$$

We also have

$$\mu_{x+\beta\mathbf{1}}(A) = \lim_{\epsilon\downarrow 0} \mu_{x+\beta\mathbf{1}}(A \cap D_\epsilon^c) = \lim_{\epsilon\downarrow 0} \mu_x(A \cap D_\epsilon^c) = \mu_x(A)$$

Now consider A is any set which is invariant under shifts parallel to the diagonal then $A = D \cup (A \cap D^c)$. $\mu_x(\mathbb{R}^N) = \mu'_x(\mathbb{R}^N) = 1/\lambda$. From this it follows that $\mu_x(D) = \mu'_x(D)$. Thus

$$\mu_x(A) = \mu_x(D) + \mu_x(A \cap D^c) = \mu'_x(D) + \mu'_x(A \cap D^c) = \mu'_x(A)$$

and by Lemma 60 we have $\mu_{x+\beta\mathbf{1}}(D) = \mu_x(D)$ hence

$$\mu_{x+\beta\mathbf{1}}(A) = \mu_{x+\beta\mathbf{1}}(D) + \mu_{x+\beta\mathbf{1}}(A \cap D^c) = \mu_x(D) + \mu_x(A \cap D^c) = \mu_x(A)$$

We have shown that $R_\lambda \mathbf{1}_A(x)$ is uniquely specified by the martingale problem for any A which is invariant under shifts parallel to the diagonal. We have also shown that $x \mapsto R_\lambda \mathbf{1}_A(x)$ is also invariant under shifts parallel to the diagonal. It is straightforward to extend these properties to $R_\lambda f(x)$ for simple f invariant under shifts and finally for bounded f invariant under shifts parallel to the diagonal. □

Finally we have the proposition which has been the aim of this subsection

Proposition 62. *Let X be a solution to the $\mathcal{A}_N^\theta$-martingale problem started from $x \in \mathbb{R}^N$ and suppose that hypothesis* (3.27) *holds. Let $\hat{X}$ be the process X projected onto the hyperplane $\sum_{i=1}^n x_i = 0$ then the law of $\hat{X}$ is uniquely determined.*

Proof. Inverting the Laplace transform of the previous proposition tells us that $\mathbf{E}_x[f(X(t))]$ is uniquely specified for all x and t and for all bounded functions f that are invariant under shifts parallel to D. Letting the operators $(P_t; t \geq 0)$ be given by $P_t f(x) = \mathbf{E}_x[f(X(t))]$, the previous proposition also tells us that $x \mapsto P_t f(x)$ is invariant under shifts parallel to D.

Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, and for all i let $f_i : \mathbb{R}^N \mapsto \mathbb{R}$ be a function which is invariant under shifts parallel to D. Consider the expectation

$$\mathbf{E}_x\left[\prod_{i=1}^n f_i(X(t_i))\right]$$

Clearly this is uniquely determined for $n = 1$. Assume that

$$\mathbf{E}_x\left[\prod_{i=1}^{n-1} f_i(X(t_i))\right]$$

is uniquely specified then we have

$$\mathbf{E}_x\left[\prod_{i=1}^n f_i(X(t_i))\right] = \mathbf{E}_x\left[\prod_{i=1}^{n-1} f_i(X(t_i))\tilde{\mathbf{E}}\left[f_n(X(t_n))\right]\right].$$

where $\tilde{\mathbf{E}}$ denotes expectation relative to the conditional distribution of $\big(X(t_{n-1}+u); u \geq 0\big)$ given $\big(X(u); u \leq t_{n-1}\big)$. Well known results , see [SV79] or [EK86], tell us that the conditional process also solves the $\mathcal{A}_N^\theta$-martingale problem started from $X(t_{n-1})$ and so

$$\tilde{\mathbf{E}}\left[f_n(X(t_n))\right] = P_t f_n(X(t_{n-1})) \qquad a.s..$$

Then we have

$$\begin{aligned}\mathbf{E}_x\left[\prod_{i=1}^{n} f_i(X(t_i))\right] &= \mathbf{E}_x\left[\prod_{i=1}^{n-1} f_i(X(t_i))P_t f_n(X(t_{n-1}))\right] \\ &= \mathbf{E}_x\left[\prod_{i=1}^{n-2} f_i(X(t_i))f_{n-1}(X(t_{n-1}))P_t f_n(X(t_{n-1}))\right]\end{aligned}$$

but $x \mapsto f_{n-1}(x)P_t f_n(x)$ is invariant under shifts parallel to D hence by assumption

$$\mathbf{E}_x\left[\prod_{i=1}^{n-1} f_i(X(t_i))P_t f_n(X(t_{n-1}))\right]$$

is uniquely specified by the $\mathcal{A}_N^\theta$-martingale problem.

This gives uniqueness for the finite dimensional distributions of $\hat{X}$ and as we are working in the space of continuous paths we are done. □

3.5.2 Uniqueness of the whole process

Let X be a solution to the $\mathcal{A}_N^\theta$-martingale problem started at $x \in \mathbb{R}^N$. In this subsection we show the motion of X in the direction perpendicular to the hyperplane $\{x \in \mathbb{R}^N; \sum_{i=1}^N x_i = 0\}$ is uniquely specified by the martingale problem. Then the full process X is constructed from the projected process $\hat{X}$ and the perpendicular motion.

Let P_N be the set of partitions of $\{1, 2, ..., N\}$ and let $m_i(\pi)$ be the size of the component of a partition π which contains i and let $m(\pi)$ be the number of components of partition π . We note that

$$m(\pi) = \sum_{i=1}^{N} \frac{1}{m_i(\pi)}.$$

Let X be a solution to the $\mathcal{A}_N^\theta$-martingale problem started at $x \in R^N$ with the property $\theta(0:1) = \theta(1:0) = 0$. Thus each component X_i is a Brownian motion. We define the process M as follows,

$$M(t) = \int_0^t \sum_{\pi \in P_N} \left[\mathbf{1}_{\{\pi(X(s))=\pi\}} \sum_{i=1}^N \frac{1}{\sqrt{m(\pi)}} \frac{1}{m_i(\pi)} dX_i(s) \right].$$

Lemma 63. *M is a Brownian motion started at* 0.

Proof. As we are assuming $\theta(0:1) = \theta(1:0) = 0$, the N processes X_i, $i = 1, \ldots N$ are martingales hence M is a local martingale.

The bracket process of M is given by

$$\begin{aligned}
\langle M, M\rangle_t = & \int_0^t \sum_{\pi \in P_N} \sum_{i=1}^N \frac{1}{m(\pi)} \frac{1}{m_i(\pi)^2} \mathbf{1}_{\{\pi(X(s))=\pi\}} ds \\
& + \int_0^t \sum_{\pi \in P_N} \sum_{i \neq j} \mathbf{1}_{\{X_i(s)=X_j(s)\}} \frac{1}{m(\pi)} \frac{1}{m_i(\pi)^2} \mathbf{1}_{\{\pi(X(s))=\pi\}} ds \\
& = \int_0^t \sum_{\pi \in P_N} \sum_{i=1}^N \frac{1}{m(\pi)} \frac{1}{m_i(\pi)^2} \mathbf{1}_{\{\pi(X(s))=\pi\}} ds \\
& + \int_0^t \sum_{\pi \in P_N} \sum_{i=1}^N \sum_{j: x_i = x_j x \in \pi} \mathbf{1}_{\{X_i(s)=X_j(s)\}} \frac{1}{m(\pi)} \frac{1}{m_i(\pi)^2} \mathbf{1}_{\{\pi(X(s))=\pi\}} \\
& = \int_0^t \sum_{\pi \in P_N} \sum_{i=1}^N \frac{1}{m(\pi)} \frac{1}{m_i(\pi)} \mathbf{1}_{\{\pi(X(s))=\pi\}} ds
\end{aligned}$$

Then for any partition π we have that $\sum_{i=1}^N \frac{1}{m_i(\pi)} = m(\pi)$ hence

$$\langle M, M\rangle_t = \int_0^t \sum_{\pi \in P_N} \mathbf{1}_{\{\pi(X(s))=\pi\}} ds = t.$$

Thus M is a Brownian motion.

□

Fix some $b \in R$, and let Z be the exponential martingale given by

$$Z(t) = \exp\left(bM(t) - \frac{1}{2}b^2 t\right).$$

Lemma 64. *If X solves the $\mathcal{A}_N^\theta$-martingale problem under $\mathbf{P}$, then under the measure, $\tilde{\mathbf{P}}$ defined by*

$$\tilde{\mathbf{P}}(A) = \mathbf{E}[Z(t) : A], \qquad A \in \mathcal{F}_t$$

$X(t) - \int_0^t \beta(s)\mathbf{1}ds$ solves the $\mathcal{A}_N^\theta$ martingale problem. Where β is given by $\beta(t) = \sum_{\pi \in P_N} \frac{b}{\sqrt{m(\pi)}} \mathbf{1}_{\{\pi(X(t))=\pi\}}$.

Proof. Our aim is to use Girsanov's theorem. To the this end we must first find the covariation process $\langle M, f(X)\rangle$ for a general function $f \in L_N$.

As $\langle X_i, X_i\rangle_t = t$ and $\langle X_j, X_i\rangle_t = \int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}}ds$ for $i \neq j$ it is possible to see, letting $i \sim j$ mean that i and j belong to the same element of the partition π,

$$\begin{aligned}
\langle M, X_j\rangle_t &= \int_0^t \sum_{\pi \in P_N} \left[\mathbf{1}_{\{\pi(X(s))=\pi\}} \sum_{i:i\sim j} \frac{1}{\sqrt{m(\pi)}} \frac{1}{m_i(\pi)} ds \right] \\
&= \sum_{\pi \in P_N} \frac{1}{\sqrt{m(\pi)}} \int_0^t \mathbf{1}_{\{\pi(X(s))=\pi\}} ds
\end{aligned}$$

which we note does not depend on j. It follows from this fact and Lemma 50, that it is possible to calculate that $\langle M, f(X)\rangle_t = f(\mathbf{1}) \langle M, X_j\rangle_t$. Thus

$$\langle M, f(X)\rangle_t = \sum_{\pi \in P_N} \frac{f(\mathbf{1})}{\sqrt{m(\pi)}} \int_0^t \mathbf{1}_{\{\pi(X(s))=\pi\}} ds.$$

Under the measure $\mathbf{P}$

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s)) ds$$

is a martingale. Thus, Girsanov's theorem tells us that

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds - \sum_{\pi \in P_N} \frac{bf(\mathbf{1})}{\sqrt{m(\pi)}} \int_0^t 1_{\{\pi(X(s))=\pi\}} ds \tag{3.42}$$

is a martingale under $\tilde{\mathbf{P}}$.

Now consider any function in L_N. In a similar fashion to the proof of Lemma 34 we can show that $\pi\left(X(t) - \int_0^t \beta(s)\mathbf{1}ds\right) = \pi(X(t))$ and

$$f\left(X(t) - \int_0^t \beta(s)\mathbf{1}ds\right) = f(X(t)) - f(\mathbf{1})\int_0^t \beta(s)ds.$$

It then follows that for all $v = v_{IJ}$, $\nabla_v f\left(X(t) - \int_0^t \beta(s)\mathbf{1}ds\right) = \nabla_v f(X(t))$, hence

$$\mathcal{A}_N^\theta f\left(X(t) - \int_0^t \beta(s)\mathbf{1}ds\right) = \mathcal{A}_N^\theta f(X(t)).$$

Now $f(\mathbf{1})\int_0^t \beta(s) = \sum_{\pi\in P_N}\int_0^t \frac{bf(\mathbf{1})}{m(\pi(X(s)))} 1_{\{\pi(X(s))=\pi\}}ds$, so by (3.42) it follows that

$$f\left(X(t) - \int_0^t \beta(s)\mathbf{1}ds\right) - \int_0^t \mathcal{A}_N^\theta f\left(X(s) - \int_0^s \beta(u)\mathbf{1}du\right)ds$$

is a martingale under $\tilde{\mathbf{P}}$ for any $f \in L_N$. We can also see that for each i, $\left\langle X_i(.) - \int_0^{\cdot} \beta(s)ds\right\rangle_t = \langle X_i\rangle_t = t$ and for each $i \neq j$,

$$\left\langle X_i(.) - \int_0^{\cdot} \beta(s)ds, X_j(.) - \int_0^{\cdot} \beta(s)ds\right\rangle_t = \langle X_i, X_j\rangle_t = \int_0^t 1_{\{X_i(s)=X_j(s)\}}ds.$$

Thus under $\tilde{\mathbf{P}}$, $X(t) - \int_0^t \beta(s)\mathbf{1}ds$ solves the $\mathcal{A}_N^\theta$-martingale problem. □

Proposition 65. *Suppose hypothesis* (3.27) *holds, then a solution to the* $\mathcal{A}_N^\theta$*-martingale problem started at* $x \in \mathbb{R}^N$ *has a law which is uniquely specified.*

Proof. We will show, using Lemma 64, that under any solution to the $\mathcal{A}_N^\theta$-

martingale problem started at some point $x \in \mathbb{R}^N$, M is independent of the projected process $\hat{X}$. We know from Proposition 62, the law of $\hat{X}$ is uniquely specified and we know that the law of M is that of Brownian motion from Lemma 63. So if we can show that for X being any solution to the $\mathcal{A}_N^\theta$-martingale problem started at $x \in \mathbb{R}^N$ that $\hat{X}$ and M are independent then it is clear that the joint law of $(\hat{X}, M)$ is unique. Finally we would be left with showing that X can be constructed from $\hat{X}$ and M.

We first show that M is independent of $\hat{X}$. Let $\mathbf{P}$ and $\tilde{\mathbf{P}}$ be as described in Lemma 64 and let $\beta(t) = \sum_{\pi \in P_N} \frac{b}{\sqrt{m(\pi)}} \mathbf{1}_{\{\pi(X(t))=\pi\}}$ as in Lemma 64. Let $f : C([0,t], \mathbb{R}^N) \mapsto \mathbb{R}$ be a bounded continuous function then by the definition of $\tilde{\mathbf{P}}$,

$$\mathbf{E}\left[f(\hat{X}) \exp(bM(t) - \frac{1}{2}b^2 t)\right] = \tilde{\mathbf{E}}[f(\hat{X})]. \qquad (3.43)$$

By Lemma 64, under $\tilde{\mathbf{P}}$ the process $X(t) - \int_0^t \beta(s)\mathbf{1}ds$ solves the $\mathcal{A}_N^\theta$-martingale problem and as $\mathbf{1}$ is perpendicular to the plane, in which $\hat{X}$ lives, $\hat{X}$ has the same distribution under $\mathbf{P}$ and $\tilde{\mathbf{P}}$. Therefore for any continuous bounded function f

$$\tilde{\mathbf{E}}[f(\hat{X})] = \mathbf{E}[f(\hat{X})],$$

where the expectations are taken according to $\tilde{\mathbf{P}}$ and $\mathbf{P}$ respectively.

We also have

$$\mathbf{E}\left[\exp(bM(t) - \frac{1}{2}b^2 t)\right] = 1$$

for any solution to the $\mathcal{A}_N^\theta$-martingale problem. Hence for any solution

$$\mathbf{E}\left[f(\hat{X}) \exp(bM(t) - \frac{1}{2}b^2 t)\right] = \mathbf{E}[f(\hat{X})]\mathbf{E}\left[\exp(bM(t) - \frac{1}{2}b^2)t\right] \qquad (3.44)$$

This tells us that for fixed t, $M(t)$ is independent of the projected process $(\hat{X}(s); 0 \leq s \leq t)$.

To extend this to independence of the process M take $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq t$ and, for $i \in \{1, \dots n\}$, let $\beta_i(t) = \sum_{\pi \in P_N} \frac{b_i}{\sqrt{m(\pi)}} \mathbf{1}_{\{\pi(X(t))=\pi\}}$. Then

$$\mathbf{E}\left[f(\hat{X}) \exp\left(\sum_{i=1}^{n} b_i(M(t_i) - M(t_{i-1}))\right)\right]$$
$$= \mathbf{E}\left[f(\hat{X}) \exp\left(\sum_{i=1}^{n-1} b_i(M(t_i) - M(t_{i-1}))\right) \exp(b_n(M(t_n) - M(t_{n-1})))\right]$$

We know that the conditional process $\left(X(t_{n-1} + u); u \geq 0\right)$ given $\left(X(u); u \leq t_{n-1}\right)$ solves the $\mathcal{A}_N^\theta$-martingale problem, therefore the above expectation is equal to

$$\mathbf{E}\left[\mathbf{E}[f(\hat{X})|\mathcal{F}_{t_{n-1}}] \exp\left(\sum_{i=1}^{n-1} b_i(M(t_i) - M(t_{i-1}))\right)\right] \exp\left(\frac{1}{2} b_n^2 (t_n - t_{n-1})\right)$$

Repeating this process we end up with

$$\mathbf{E}\left[f(\hat{X}) \exp\left(\sum_{i=1}^{n} b_i(M(t_i) - M(t_{i-1}))\right)\right] = \mathbf{E}[f(\hat{X})] \exp\left(\frac{1}{2} \sum_{i=1}^{n} b_i^2 (t_i - t_{i-1})\right).$$

Thus

$$\mathbf{E}\left[f(\hat{X}) \exp\left(\sum_{i=1}^{n} b_i(M(t_i) - M(t_{i-1}))\right)\right]$$
$$= \mathbf{E}\left[f(\hat{X})\right] \mathbf{E}\left[\exp\left(\sum_{i=1}^{n} b_i(M(t_i) - M(t_{i-1}))\right)\right].$$

As $(b_i; i \geq 1)$ are arbitrary and it follows that $\hat{X}$ and the process M are independent.

Finally we can recover X from $\hat{X}$ and M by the following

$$\begin{aligned}
&x_1 + \int_0^t \sum_{\pi \in P_N} \mathbf{1}_{\{\pi(X(s))=\pi\}} \frac{1}{\sqrt{m(\pi)}} dM(s) \\
&+ \int_0^t \sum_{\pi \in P_N} \frac{1}{m(\pi)} \mathbf{1}_{\{\pi(X(t))=\pi\}} \left[\sum_{i=2}^N \frac{1}{m_i(\pi)} dX_1 - \sum_{i=2}^N \frac{1}{m_i(\pi)} dX_i \right] \\
&= x_1 + \int_0^t \sum_{\pi \in P_N} \mathbf{1}_{\{\pi(X(s))=\pi\}} dX_1(s) \\
&= X_1(t)
\end{aligned}$$

The expression $\sum_{i=2}^N \frac{1}{m_i(\pi)} X_1 - \sum_{i=2}^N \frac{1}{m_i(\pi)} X_i$ is a function of $\hat{X}$ only, as is $\mathbf{1}_{\{\pi(X(s))=\pi\}}$. Therefore we can recover X_1, and similarly we can recover the other components of X. □

So far these arguments have all been with the assumption of hypothesis (3.27). To complete the proof of the uniqueness statement in Theorem 32 we use an induction argument on the dimension $N \geq 1$ as suggested at the beginning of this section. We are still assuming that $\theta(0:1) = \theta(1:0) = 0$, which we can do by Lemma 34. Thus for $N = 1$ the $\mathcal{A}_N^\theta$-martingale problems reduces to Lévy's martingale characterisation of Brownian motion. Then assuming that uniqueness in law holds for any $n \leq N - 1$, Proposition 48 tells us that the hypothesis 3.27 holds. Then Proposition 65 gives us that the uniqueness-in-law property holds for dimension N.

3.6 An approximation scheme and existence

In this section we construct a sequence of Markov chains, which ,when appropriately scaled, has a limit which solves the $\mathcal{A}_N^\theta$-martingale problem.

We start with a family of non-negative parameters $p = (p(k:l); k, l \geq 0)$, which satisfy the consistency condition

$$p(k:l) = p(k+1:l) + p(k:l+1) \quad \text{for all } k,l \geq 0. \tag{3.45}$$

We consider a continuous time Markov chain $(Y(t); t \geq 0)$ with state space being the integer lattice $\mathbb{Z}^N$ that has generator given by

$$\mathcal{G}_N^p f(x) = \sum_{v \in \mathcal{V}(x)} p(v)\{f(x+v) - f(x)\} \tag{3.46}$$

for any measurable $f : \mathbb{R} \mapsto \mathbb{R}$. Here $p(v) = p(|I|, |J|)$, where $v = v_{IJ}$ in the same way as (3.1) in Section 3.1. Note that if ρ is a permutation of $\{1, 2, \ldots, N\}$ then if $(Y(t); t \geq 0)$ is a Markov chain with generator $\mathcal{G}_N^p$, $(\rho(Y(t)); t \geq 0)$ is also a Markov chain with generator $\mathcal{G}_N^p$.

The following proposition shows that Markov chains with generator $\mathcal{G}_N^p$ form a consistent family in N.

Proposition 66. *Suppose that X is a Markov chain with generator $\mathcal{G}_N^p$ and let Y be the process consisting of the first $N-1$ components of X. Then Y is a Markov chain with generator $\mathcal{G}_{N-1}^p$.*

Proof. The proof of follows in the same way as the proof of proposition 31. □

From now on we assume that p satisfies $p(0:0) = 1$ and $p(1:0) = p(0:1) = \frac{1}{2}$. Then with the help of the preceding proposition we see that each component Y_i of Y is a simple symmetric random walk on $\mathbf{Z}$ with zero drift. In particular, $\mathcal{G}_1^p f(x) = \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) - f(x)$, thus

$$Y_i(t) \quad \text{and} \quad Y_i(t)^2 - t \qquad \text{are martingales} \tag{3.47}$$

relative to the natural filtration of Y. Similarly, by expanding $\mathcal{G}_2^p$ explicitly, we can consider any pair of components (Y_i, Y_j) and we find that they evolve independently from each other when apart, but with a tendency to move together when they meet. In fact

$$Y_i(t)Y_j(t) - (1 - 4p(1:1)) \int_0^t \mathbf{1}(Y_i(s) = Y_j(s))ds, \qquad t \geq 0 \qquad (3.48)$$

and

$$|Y_i(t) - Y_j(t)| - 4p(1:1) \int_0^t \mathbf{1}(Y_i(s) = Y_j(s))ds, \qquad t \geq 0 \qquad (3.49)$$

are both martingales.

Fix the integer $N \geq 1$. Let $(p_n; n \geq 1)$ be a sequence of families of parameters, all satisfying (3.45), $p_n(1:0) = p_n(0:1) = \frac{1}{2}$, and such that as n tends to infinity,

$$n^{1/2}\left(p_n(k:l) - \tfrac{1}{2}\mathbf{1}(k=0) - \tfrac{1}{2}\mathbf{1}(l=0)\right) \to \theta(k:l), \qquad (3.50)$$

uniformly for all $0 \leq k, l \leq N$, where $(\theta(k:l); k, l \geq 0)$ satisfies the consistency and positivity conditions (3.3) and (3.7).

Lemma 67. *For each family of parameters $(\theta(k:l); k, l \geq 0)$ satisfying* (3.3) *and* (3.7) *and $\theta(0:0) = \theta(0:1) = \theta(1:0) = 0$ there exists a sequence of families of non-negative parameters $(p_n; n \geq 1)$ each satisfying* (3.45), *and $p_n(1:0) = p_n(0:1) = \frac{1}{2}$ such that* (3.50) *holds.*

Proof. Let

$$p'_n(k:l) = \frac{\theta(k:l)}{n^{1/2}} + \frac{1}{2}\mathbf{1}_{\{k=0\}} + \frac{1}{2}\mathbf{1}_{\{l=0\}}$$

then, by Lemma 33 and as the family θ satisfies (3.3) $(p'_n(k;l); k, l \geq 0)$ satisfies (3.45) for each n. It is also clear that $p'_n(0:1) = p'_n(1:0) = 1/2$ for all n and as θ satisfies (3.7), $p'_n(k:l) \geq 0$ for all $k, l \geq 1$. $p'_n(k:0)$ and $p'_n(0:l)$ are not necessarily non-negative but there exists N' such that, for all $0 \leq k, l \leq N$, and for all $n \geq N'$, $p'_n(k:0)$ and $p'_n(0:l)$ are non-negative. Then simply let $p_n(k:l) = p'_n(k:l)$ for all $n \geq N'$ and let for $n < N'$ let $p_n(k:l)$ be any family of non negative parameters satisfying (3.45) and $p_n(1:0) = p_n(1:0) = 1/2$ then $(p_n; n \geq 1)$ is a sequence of families of non-negative parameters each satisfying (3.45), $p_n(1:0) = p_n(0:1) = \frac{1}{2}$, and (3.50) holds. □

Let $(x_n; n \geq 0)$ be a sequence of points in $\mathbf{R}^N$ converging to a point x, with $x_n \in n^{-1/2}\mathbf{Z}^N$ for every n. For $n \geq 1$, let Y^n be the scaled process given by $Y^n(t) = n^{-1/2}Y(nt)$ for $t \geq 0$, where Y is a Markov chain with generator $\mathcal{G}_N^{p_n}$ starting from $n^{1/2}x_n$. In the following convergence in law means weak convergence of probability measures on the Skorokhod space $\mathbf{D}([0,\infty), \mathbf{R}^N)$. The Skorokhod space is complete and separable with respect to the Skorkhod topology, see [EK86] or [Bil99]. This implies that the space of Borel probability measure on $\mathbf{D}([0,\infty), \mathbf{R}^N)$ is itself complete and separable under the weak topology. So that any relatively compact sequence of Borel probability measures on $\mathbf{D}([0,\infty), \mathbf{R}^N)$ has a convergent subsequence.

Proposition 68. *Suppose that the sequence of processes $(Y^n(t); t \geq 0)$ converges in law to a process $(X(t); t \geq 0)$. Then for each $i \neq j$*

$$X_i(t)X_j(t) - \int_0^t 1_{\{X_i(s)=X_j(s)\}} ds$$

is a martingale relative to the natural filtration of X.

Proof. Firstly we note, by Donsker's theorem, in the limit each coordinate process $(X_i(t) : t \geq 0)$ is a standard Brownian motion. Indeed as Y_i^n and $Y_i^n(t)^2 - t$ are both martingales, we will show that in turn this gives that X_i and $X_i(t)^2 - t$ are martingales.

Fix $s \leq t$ and let $g : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ be bounded, continuous, non-negative and measurable with respect to $\mathcal{D}_s$ where $(\mathcal{D}_t; t \geq 0)$ is the filtration generated by the coordinate process. Then

$$\mathbf{E}[g(Y^n)(Y_i^n(t) - Y_i^n(s))] = 0$$

and

$$\mathbf{E}[g(Y^n)(Y_i^n(t)^2 - Y_i^n(s)^2)] = E[g(Y^n)](t-s)$$

For $\alpha \in C([0,\infty),\mathbb{R}^N)$ the coordinate mapping $\alpha \mapsto \alpha_i(t)$ is continuous with respect to the uniform topology and in the limit X is almost surely continuous. Thus, by the continuous mapping theorem, $g(Y^n)(Y_i^n(t) - Y_i^n(s)) \Rightarrow g(X)(X_i(t) - X_i(s))$ and $g(Y^n)(Y_i^n(t)^2 - Y_i^n(s)^2) \Rightarrow g(X)(X_i(t)^2 - X_i(s)^2)$. Uniform integrability comes from $\mathbf{E}[Y_i^n(t)^2] = t$ and $\mathbf{E}[Y_i^n(t)^4] = 3t^2$ for all $n \geq 1$. This together with the fact that g is bounded gives that

$$\mathbf{E}[g(X)(X_i(t) - X_i(s))] = \lim_{n\to\infty} \mathbf{E}[g(Y^n)(Y_i^n(t) - Y_i^n(s))] = 0$$

and

$$\begin{aligned}\mathbf{E}[g(X)(X_i(t)^2 - X_i(s)^2)] &= \lim_{n\to\infty} \mathbf{E}[g(Y^n)(Y_i^n(t)^2 - Y_i^n(s)^2)] \\ &= \lim_{n\to\infty} \mathbf{E}[g(Y^n)](t-s) = \mathbf{E}[g(X)](t-s).\end{aligned}$$

As both of these hold for any bounded, continuous $\mathcal{D}_s$ measurable function g and for any $s \leq t$, we have that X_i and $(X_i(t)^2 - t; t \geq 0)$ are both martingales.

Define for $\epsilon > 0$ the bounded continuous function $f_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ by $f_\epsilon(z) = \max(0, 1-|z|/\epsilon)$. We note that $f_\epsilon(z) \geq \mathbf{1}_{\{z=0\}}$ for all z, then this fact combined with (3.48) tells us that

$$Y_i^n(t)Y_j^n(t) - \int_0^t f_\epsilon(Y_i^n(s) - Y_j^n(s))ds, \qquad t \geq 0$$

is a supermartingale. Thus for any g as described above

$$\mathbf{E}\left[g(Y^n)\left(Y_i^n(t)Y_j^n(t) - Y_i^n(s)Y_j^n(s) - \int_s^t f_\epsilon(Y_i^n(u) - Y_j^n(u))du\right)\right] \leq 0$$

We note the function $\alpha \mapsto \int_0^t f_\epsilon(\alpha_i(s) - \alpha_j(s))ds$ is continuous with respect to the uniform topology and hence is continuous with respect to the Skorokhod topology on the subspace $\mathbf{C}([0,\infty), \mathbf{R}^N)$, hence by the continuous mapping theorem we have

$$g(Y^n)\left(Y_i^n(t)Y_j^n(t) - \int_0^t f_\epsilon(Y_i^n(s) - Y_j^n(s))ds\right)$$
$$\Rightarrow g(X)\left(X_i(t)X_j(t) - \int_0^t f_\epsilon(X_i(s) - X_j(s))ds\right),$$

where uniform integrability comes from the fact f_ϵ is bounded and $\mathbf{E}[(Y_i^n(t)Y_j^n(t))^2] \leq \mathbf{E}[Y_i^n(t)^4] + \mathbf{E}[Y_j^n(t)^4] = 6t^2$ for all n, and so

$$\mathbf{E}\left[g(X)\left(X_i(t)X_j(t) - X_i(s)X_j(s) - \int_s^t f_\epsilon(X_i(u) - X_j(u))du\right)\right] \leq 0.$$

As this hold for any bounded, continuous $\mathcal{D}_s$ measurable g, and for any $s \leq t$, we have that

$$X_i(t)X_j(t) - \int_0^t f_\epsilon(X_i(s) - X_j(s))ds, \qquad t \geq 0$$

is a supermartingale. This is true for all $\epsilon > 0$ so by the monotone convergence theorem, letting ϵ tend down to zero we get that

$$X_i(t)X_j(t) - \int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds, \qquad t \geq 0$$

is a supermartingale.

We also have from (3.48) that

$$Y_i^n(t)Y_j^n(t), \qquad t \geq 0$$

is a submartingale and in a similar style to the above it follows that

$$X_i(t)X_j(t), \qquad t \geq 0$$

is also a submartingale. $\langle X_i, X_j \rangle$ must be an increasing process such that $t \mapsto \langle X_i, X_j \rangle_t - \int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds$ is decreasing. This implies that

$$\int_0^t \mathbf{1}_{\{X_i(s) \neq X_j(s)\}} d\langle X_i, X_j \rangle_s = 0$$

and so

$$\int_0^t \mathbf{1}_{\{X_i(s) \neq X_j(s)\}} d\langle X_i - X_j \rangle_s = 2\int_0^t \mathbf{1}_{\{X_i(s) \neq X_j(s)\}} ds.$$

The occupation times formula tells us that

$$\int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}} d\langle X_i - X_j \rangle_s = \int_{-\infty}^{\infty} \mathbf{1}_{\{a=0\}} L_t^a(X_i - X_j) da = 0$$

and so we must have

$$\langle X_i - X_j \rangle_t = 2\int_0^t \mathbf{1}_{\{X_i(s) \neq X_j(s)\}} ds$$

and hence

$$\langle X_i, X_j \rangle_t = \int_0^t \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds.$$

□

Proposition 69. *Suppose that the sequence of processes $(Y^n(t); t \geq 0)$ converges in law to a process $(X(t); t \geq 0)$. Then for each $i \neq j$*

$$|X_i(t) - X_j(t)| - 2\theta X_i(t) X_j(t)$$

is a martingale with respect to the natural filtration of X.

Proof. Firstly, if Y is a Markov chain with generator $\mathcal{G}_N^{p_n}$, we note that (3.48) and (3.49) together give us that

$$|Y_i(t) - Y_j(t)| - 4p_n(1:1)Y_i(t)Y_j(t) - (4p_n(1:1))^2 \int_0^t \mathbf{1}_{\{Y_i(s)=Y_j(s)\}} ds$$

is a martingale and therefore so is

$$|Y_i(nt) - Y_j(nt)| - 4p_n(1:1)Y_i(nt)Y_j(nt) - (4p_n(1:1))^2 \int_0^{nt} \mathbf{1}_{\{Y_i(s)=Y_j(s)\}} ds$$

which is equal to

$$|Y_i(nt) - Y_j(nt)| - 4p_n(1:1)Y_i(nt)Y_j(nt) - n(4p_n(1:1))^2 \int_0^t \mathbf{1}_{\{Y_i(ns)=Y_j(ns)\}} ds$$

then multiplying throughout by $n^{-1/2}$ gives us that

$$\begin{aligned} |n^{-1/2}Y_i(nt) - n^{-1/2}Y_j(t)| &- 4\sqrt{n}p_n(1:1)n^{-1/2}Y_i(nt)n^{-1/2}Y_j(nt) \\ &- \sqrt{n}(4p_n(1:1))^2 \int_0^t \mathbf{1}_{\{Y_i(ns)=Y_j(ns)\}} ds \end{aligned}$$

is a martingale. Then as $Y_i(ns) = Y_j(ns)$ if and only if $n^{-1/2}Y_i(ns) =$

$n^{-1/2}Y_j(ns)$, we have that

$$|Y_i^n(t)-Y_j^n(t)|-4\sqrt{n}p_n(1:1)Y_i^n(t)Y_j^n(t)-\sqrt{n}(4p_n(1:1))^2\int_0^t \mathbf{1}_{\{Y_i^n(s)=Y_j^n(s)\}}ds$$

is a martingale for all $n \geq 1$. We know that $\sqrt{n}p_n(1:1) \to \theta(1:1)$ as $n \to \infty$ and so $\sqrt{n}(4p_n(1:1))^2 \to 0$ as $n \to \infty$. Therefore for any $\epsilon > 0$ there exists N' such that for all $n \geq N'$

$$\theta - \epsilon < \sqrt{n}p_n(1:1) < \theta + \epsilon$$

and

$$0 \leq \sqrt{n}(4p_n(1:1))^2 < \epsilon.$$

This means that for all $n \geq N'$

$$h_t(Y^n) = |Y_i^n(t) - Y_j^n(t)| - 4(\theta(1:1)+\epsilon)Y_i^n(t)Y_j^n(t) - \epsilon t$$

is a supermartingale and

$$h_t'(Y^n) = |Y_i^n(t) - Y_j^n(t)| - 4(\theta(1:1)-\epsilon)Y_i^n(t)Y_j^n(t)$$

is a submartingale. $h_t : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ and $h_t' : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ are both continuous functions with respect to the uniform topology and hence are continuous with respect to the Skorokhod topology on the subspace $C([0,\infty),\mathbb{R}^N)$.

Let $g : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ be any bounded continuous non-negative $\mathcal{D}_s$ measurable function. Then $\alpha \mapsto g(\alpha)(h_t(\alpha) - h_s(\alpha))$ is continuous function on

$C([0,\infty),\mathbb{R}^N)$. So by the continuous mapping theorem

$$g(Y^n)(h_t(Y^n) - h_s(Y^n)) \Rightarrow g(X)(h_t(X) - h_s(X))$$

and a similar result holds for h'.

We assume for now that $\{h_t(Y^n) : n \geq 1\}$ and $\{h'_t(Y^n) : n \geq 1\}$are uniformly integrable which implies that $\{(h_t(Y^n) - h_s(Y^n)) : n \geq 1\}$ and $\{(h'_t(Y^n) - h'_s(Y^n)) : n \geq 1\}$ are uniformly integrable also, therefore this together with g being bounded gives us

$$E\left[g(X)(h_t(X) - h_s(X))\right] = \lim_{n\to\infty} \mathbf{E}\left[g(Y)(h_t(Y^n) - h_s(Y^n))\right].$$

Now $h_t(Y^n)$ is a supermartingale for all $n \geq N'$ which implies

$$\mathbf{E}\left[g(Y^n)(h_t(Y^n) - h_s(Y^n))\right] \leq 0 \text{ for all } n \geq N' \tag{3.51}$$

for any $s \leq t$. Therefore

$$E\left[g(X)(h_t(X) - h_s(X))\right] \leq 0. \tag{3.52}$$

This hold for any bounded, continuous, non-negative $\mathcal{D}_s$ measurable function g, and so implies that $h_t(X)$ is a supermartingale. Thus

$$|X_i(t) - X_j(t)| - 4(\theta(1:1) + \epsilon)X_i(t)X_j(t) - \epsilon t$$

is a supermartingale for all $\epsilon > 0$ which implies by the dominated convergence theorem that

$$|X_i(t) - X_j(t)| - 4\theta(1:1)X_i(t)X_j(t) \tag{3.53}$$

is a supermartingale.

In contrast $h'_t(Y^n)$ is a submartingale for all $n \geq N'$ so

$$\mathbf{E}\left[g(Y^n)(h'_t(Y^n) - h'_s(Y^n))\right] \geq 0 \text{ for all } n \geq N' \tag{3.54}$$

therefore it follows that

$$E\left[g(X)(h'_t(X) - h'_s(X))\right] \geq 0. \tag{3.55}$$

for any bounded continuous non-negative $\mathcal{D}_s$ measurable function g. Thus $h'_t(X)$ is a submartingale. Thus we have that

$$|X_i(t) - X_j(t)| - 4(\theta(1:1) - \epsilon)X_i(t)X_j(t)$$

is a submartingale for all $\epsilon > 0$ which implies that

$$|X_i(t) - X_j(t)| - 4\theta(1:1)X_i(t)X_j(t)$$

is a submartingale which together with (3.53) and the fact that $\theta = 2\theta(1:1)$ give us that

$$|X_i(t) - X_j(t)| - 2\theta X_i(t)X_j(t)$$

is a martingale.

We are left with showing that $\{h_t(Y^n) : n \geq 1\}$ and $\{h'_t(Y^n) : n \geq 1\}$ are uniformly integrable. It is enough to show that $E[h_t(Y^n)^2] < \infty$ and $E[h'_t(Y^n)^2] < \infty$ for all n. Now $2xy \leq x^2 + y^2$ and $(x+y)^2 \leq 2(x^2+y^2)$ Thus uniform integrability follows from $\mathbf{E}[Y_i^n(t)^2] = t < \infty$ and $\mathbf{E}[Y_i^n(t)^4] = 3t^2 < \infty$. □

The above propositions, together with Proposition 71 below, prove that if the sequence of processes $(Y^n(t); t \geq 0)$ converges in law to a process $(X(t); t \geq$

0), then $(X(t); t \geq 0)$ is a solution to the $\mathcal{A}_N^\theta$-martingale problem. To prove Proposition 71 we will need the following lemma, the proof of which can be found in the appendix of [Bic02].

Lemma 70. *Assume S is a separable metric space. Let X be some other S-valued random variable defined on some probability space. Let $(Y^n; n \geq 1)$ be a sequence of S-valued random variable each defined on some (not necessarily the same) probability space, such that $Y^n \Rightarrow X$. Let $h : S \mapsto \mathbb{R}$ be a function and let S' be the set of points in S such that h is lower (upper) semicontinuous. If under the law of the X, $\mathbf{P}(X \in S') = 1$ and $\{h(Y^n) : n \geq 1\}$ are uniformly integrable. Then $h(X)$ is integrable and*

$$\liminf_{n \to \infty} \mathbf{E}[h(Y^n)] \geq \mathbf{E}[h(X)]$$

$$(\limsup_{n \to \infty} \mathbf{E}[(h(Y^n)] \leq \mathbf{E}[h(X)])$$

Proposition 71. *Suppose that the sequence of processes $(Y^n(t); t \geq 0)$ converges in law to a process $(X(t); t \geq 0)$. Then for all $f \in L_N$*

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds, \quad t \geq 0$$

is a martingale relative to the natural filtration of X.

Proof. Let Y be a Markov process with generator $\mathcal{G}_N^{p_n}$, thus we have that

$$f(Y(t)) - \int_0^t \sum_{v \in \mathcal{V}(Y(s))} p_n(v)\{f(Y(s) + v) - f(Y(s))\}ds$$

is a martingale. Functions in the space L_N are linear within cells and continuous at the boundary of cells. Thus for $f \in L_N$ and for $x \in \mathbb{Z}$ we have the equality

$f(x+v) - f(x) = \nabla_v f(x)$, therefore, for all $f \in L_N$,

$$f(Y(t)) - \int_0^t \sum_{v \in \mathcal{V}(Y(s))} p_n(v) \nabla_v f(Y(s)) ds, \quad t \geq 0$$

is a martingale and therefore so is

$$f(Y(nt)) - \int_0^{nt} \sum_{v \in \mathcal{V}(Y(s))} p_n(v) \nabla_v f(Y(s)) ds, \quad t \geq 0$$

which is equal to

$$f(Y(nt)) - n \int_0^t \sum_{v \in \mathcal{V}(Y(ns))} p_n(v) \nabla_v f(Y(ns)) ds, \quad t \geq 0.$$

For all $f \in L_N$, $f(ax) = af(x)$, thus

$$f(n^{-1/2}Y(nt)) - \int_0^t \sum_{v \in \mathcal{V}(Y(ns))} \sqrt{n} p_n(v) \nabla_v f(Y(ns)) ds, \quad t \geq 0$$

is a martingale. Then, as $\nabla_v f(ax) = \nabla_v f(x) ds$, we have that

$$f(n^{-1/2}Y(nt)) - \int_0^t \sum_{v \in \mathcal{V}(Y(ns))} \sqrt{n} p_n(v) \nabla_v f(n^{-1/2}Y(ns)) ds, \quad t \geq 0$$

is a martingale.

Clearly $\pi(Y(nt)) = \pi(n^{-1/2}Y(nt))$ and hence $\mathcal{V}(Y(nt)) = \mathcal{V}(n^{-1/2}Y(nt))$, thus

$$f(Y^n(t)) - \int_0^t \sum_{v \in \mathcal{V}(Y^n(s))} \sqrt{n} p_n(v) \nabla_v f(Y^n(s)) ds$$

is a martingale.

Also for $f \in L^N$, we have, as we have seen before, that for all $v \in \mathcal{V}_0(x)$,

$\nabla_v f(x) = -\nabla_{-v} f(x)$. This then implies that for any $x \in \mathbb{R}^N$,

$$\sum_{v\in\mathcal{V}_0(x)} \nabla_v f(x) = 0$$

hence

$$\sum_{v\in\mathcal{V}_0(Y^n(s))} \frac{1}{2}\sqrt{n}\nabla_v f(Y^n(s)) = 0$$

and so

$$f(Y^n(t)) - \int_0^t \sum_{v\in\mathcal{V}(Y^n(s))} \sqrt{n}p_n(v)\nabla_v f(Y^n(s))ds$$

$$f(Y^n(t)) - \int_0^t \sum_{v\in\mathcal{V}_+(Y^n(s))} \sqrt{n}p_n(v)\nabla_v$$

$$- \int_0^t \sum_{v\in\mathcal{V}_0(Y^n(s))} \sqrt{n}(p_n(v) - 1/2)\nabla_v f(Y^n(s))ds.$$

The above can be written in the form

$$f(Y^n(t)) - \int_0^t \sum_{v\in\mathcal{V}(Y^n(s))} \sqrt{n}(p(v) - 1/2\mathbf{1}_{\{k=0\}} - 1/2\mathbf{1}_{\{l=0\}})\nabla_v f(Y^n(s))ds$$

where k and l are determined by v, via $v = v_{IJ}$, $k = |I|$ and $l = |J|$.

$\sqrt{n}(p_n(v) - 1/2\mathbf{1}_{\{k=0\}} - 1/2\mathbf{1}_{\{l=0\}}) \to \theta(v)$ as $n \to \infty$, so for any $\epsilon > 0$ there exists N' such that for all $n \geq N'$

$$\theta(v) - \epsilon < \sqrt{n}(p_n(v) - 1/2\mathbf{1}_{\{k=0\}} - 1/2\mathbf{1}_{\{l=0\}}) < \theta(v) + \epsilon \tag{3.56}$$

for all possible v.

As before E_N is the collection of cells in $\mathbb{R}^N$ and remember that for any

$f \in L^N$ we can write f in the form

$$f(x) = \sum_{E \in E_N} \sum_{i=1}^{m} a_i(E) x_i.$$

Let

$$K_f = \max_{\substack{0 \le i \le N \\ E \in E_N}} |a_i(E)|, \tag{3.57}$$

then it is easy to see that $\nabla_v f(x) \le K_f$. Thus, with (3.56) in mind, we have that for all $n \ge N'$

$$\begin{aligned} & f(Y^n(t)) - \int_0^t \sum_{v \in \mathcal{V}(Y^n(s))} \theta(v) \nabla_v f(Y^n(s)) ds - \epsilon K_f t, \qquad t \ge 0 \\ &= f(Y^n(t)) - \int_0^t \mathcal{A}_N^\theta f(Y^n(s)) ds - \epsilon K_f t, \qquad t \ge 0 \end{aligned}$$

is a supermartingale and similarly

$$f(Y^n(t)) - \int_0^t \mathcal{A}_N^\theta f(Y^n(s)) ds + \epsilon K_f t$$

is a submartingale.

We wish to show that the above supermartingale and submartingale are persistent in the limit. Unfortunately however $\alpha \mapsto \int_0^t \mathcal{A}_N^\theta f(\alpha(s)) ds$ is not a continuous function, and neither is it semicontinuous. We have that the mapping

$$\alpha \mapsto \sum_{i \ne j} \int_s^t \mathbf{1}_{\{\alpha_i(u) = \alpha_j(u)\}} du,$$

for $s \le t$, is upper semicontinuous on $C([0,\infty), \mathbb{R}^N)$ with respect to the uniform topology. From this it follows that for any $f \in L_N$ there exists $\delta > 0$ such that

$$\alpha \mapsto \sum_{i \ne j} \int_s^t \mathbf{1}_{\{\alpha_i(u) = \alpha_j(u)\}} du \pm \delta \int_s^t \mathcal{A}_N^\theta f(\alpha(u)) du$$

is also upper semicontinuous.

Now let $f^*(x) = \sum_{i\neq j}|x_i - x_j|$, then $\mathcal{A}_N^\theta f^*(x) = \sum_{i\neq j}\mathbf{1}_{\{x_i=x_j\}}$. Thus, letting $h_t^+ : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ be defined by

$$\begin{aligned} h_t^+(\alpha) = f^*(\alpha(t)) - \int_0^t \theta(1:1)\sum_{i\neq j}\mathbf{1}_{\{\alpha_i(s)=\alpha_j(s)\}}ds \\ + \delta f(\alpha(t)) - \delta\int_0^t \mathcal{A}_N^\theta f(\alpha(s))ds - \epsilon(1+K_f)t, \qquad (3.58) \end{aligned}$$

we have that $(h_t^+(Y^n); t\geq 0)$ is a supermartingale and

$$\alpha \mapsto h_t(\alpha) - h_s(\alpha)$$

is lower semicontinuous on $C([0,\infty),\mathbb{R}^N)$. Similarly, if we let $h_t^- : D([0,\infty),\mathbb{R}^N) \mapsto$ $\mathbb{R}$ be defined by

$$\begin{aligned} h_t^-(\alpha) = f^*(\alpha(t)) - \int_0^t \theta(1:1)\sum_{i\neq j}\mathbf{1}_{\{\alpha_i(s)=\alpha_j(s)\}}ds + \\ - \delta f(\alpha(t)) + \delta\int_0^t \mathcal{A}_N^\theta f(\alpha(s))ds - \epsilon(1+K_f)t, \qquad (3.59) \end{aligned}$$

then $(h_t^-(Y^n); t\geq 0)$ is also a supermartingale and

$$\alpha \mapsto h_t^-(\alpha) - h_s^-(\alpha)$$

is lower semicontinuous on $C([0,\infty),\mathbb{R}^N)$.

For now ,assume that $\{h_t^\pm(Y^n) : n\geq 1\}$ are uniformly integrable. Let $g : D([0,\infty),\mathbb{R}^N) \mapsto \mathbb{R}$ be bounded, continuous, non-negative and measurable with respect to $\mathcal{D}_s$ where $(\mathcal{D}_t; t\geq 0)$ is the filtration generated by the coordinate process. We are assuming that $\{h_t(Y^n) : n\geq 1\}$ are uniformly integrable which implies that $\{(h_t(X) - h_s(X)) : n\geq 1\}$ are uniformly integrable also. Then, using lemma 70, we have

$$E\left[g(X)(h_t^{\pm}(X)-h_s^{\pm}(X))\right]$$
$$\leq \liminf_{n\to\infty} \mathbf{E}\left[g(Y^n)(h_t^{\pm}(Y^n)-h_s^{\pm}(Y^n))\right]. \tag{3.60}$$

Now $h_t(Y^n)$ is a supermartingale for all $n \geq N'$, which implies

$$\mathbf{E}\left[g(Y^n)(h_t(Y^n)-h_s(Y^n))\right] \leq 0 \text{ for all } n \geq N' \tag{3.61}$$

therefore

$$E\left[g(X)(h_t^{\pm}(X)-h_s^{\pm}(X))\right] \leq 0. \tag{3.62}$$

This holds for all bounded, continuous, non-negative $\mathcal{D}_s$ measurable functions g and for all $s \leq t$, which implies by that $(h_t^+(X); t \geq 0)$ and $(h_t^-(X); t \geq 0)$ are both supermartingales.

Recall the definitions of h_t^+, (3.58), and h_t^-, (3.59), and letting $\epsilon \downarrow 0$ we have that

$$f^*(X(t)) - \int_0^t \theta(1:1) \sum_{i\neq j} \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds$$
$$+\, \delta f(X(t)) - \delta \int_0^t \mathcal{A}_N^\theta f(X(s))ds, \qquad t \geq 0 \tag{3.63}$$

and

$$f^*(X(t)) - \int_0^t \theta(1:1) \sum_{i\neq j} \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds +$$
$$-\, \delta f(X(t)) + \delta \int_0^t \mathcal{A}_N^\theta f(X(s))ds, \qquad t \geq 0 \tag{3.64}$$

are both supermartingales.

From propositions 69 and 68 we have

$$f^*(X(t)) - \int_0^t (\theta(1:1)) \sum_{i \neq j} \mathbf{1}_{\{X_i(s)=X_j(s)\}} ds, \qquad t \geq 0$$

is a martingale which means that

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s)) ds, \qquad t \geq 0$$

must be a supermartingale from (3.63), and must be a submartingale from (3.64). Thus

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s)) ds, \qquad t \geq 0$$

is a martingale.

We are left with showing that $\{h_t^+(Y^n) : n \geq 1\}$ and $\{h_t^-(Y^n) : n \geq 1\}$ are uniformly integrable. It is enough to show $\mathbf{E}[h_t^\pm(Y^n)^2] < \infty$ for all n.

$$\int_0^t \theta(1:1) \sum_{i \neq j} \mathbf{1}_{\{X_i(s)=X_j(s)\}} \pm \delta \mathcal{A}_N^\theta f(X(s)) ds$$

is bounded so we are left with checking $\mathbf{E}[(f^*(Y^n(t)) \pm \delta f(Y^n(t)))^2] < \infty$. $f^* \pm \delta f \in L^N$, so we check $\mathbf{E}[f(Y^n(t))^2] < \infty$ for all $f \in L^N$.

For all $f \in L_N$ it is straight forward to show that $f(x) \leq K_f N ||x||$, where $||.||$ is the Euclidean norm and K_f is given in (3.57). Thus

$$\mathbf{E}[f(Y^n(t))^2] \leq K_f^2 N^2 \sum_{i=1}^N \mathbf{E}[Y_i^n(t)^2] = K_f^2 N^3 t < \infty$$

□

We have shown that we can construct a sequence of processes whose limit if it exists solves the $\mathcal{A}_N^\theta$-martingale problem. Existence of the limit process comes for the fact that the Skorokhod space $D([0,\infty),\mathbb{R})$ with the Skorokhod topology is complete and separable. From this it follows that the space of Borel measures on the Skorokhod space is complete and separable with respect to the weak topology. This means that if (Y^n) is our sequence of processes and $\{Y^n; n \geq 1\}$ is relatively compact then there exists a subsequence $(n_k; k \geq 1)$ such that Y^{n_k} converges in law as $k \to \infty$. As the space $D([0,\infty),\mathbb{R})$ with the Skorokhod topology is complete and separable relative compactness is equivalent to tightness and tightness comes from Lemma 22 which tells us that because the marginals of the process are tight the process itself is tight. The sequence of laws of each marginal is tight as they are simply the laws of scaled simple symmetric random walks converging to Brownian motion. Then by Proposition 71, (Y^{n_k}) must converge in law to X and as this is true for any such subsequence the sequence (Y^n) itself must converge in law to X. Thus the limit process, which solves the $\mathcal{A}_N^\theta$-martingale problem, exists and the existence part of Theorem 32 is proven.

3.7 A stochastic flow of kernels

In chapter 1 we described a stochastic flow on a measurable space $(E,\mathcal{E})$ being a double indexed family $(K_{s,t}; s \leq t)$ of random $E \times \mathcal{E}$ transition kernels satisfying the flow property, and properties of stationary and independent increments. The flow property is given by

$$K_{s,u}(x,A) = \int_E K_{s,t}(x,dy)K_{t,u}(y,A) \qquad x \in E, A \in \mathcal{A}$$

almost surely for all $s \leq t \leq u$. The stationary and independent increments property is given as $K_{t_1,t_2}, K_{t_2,t_3}, \ldots, K_{t_{n-1},t_n}$ are independent for all choices of $t_1 < t_2 \cdots < t_n$ and $K_{s,t} \stackrel{d}{=} K_{s+k,t+h}$ for all h and $s < t$. Given a stochastic flow

K we can construct the N-point motion of the flow, which is an N-dimensional Markov process. The semigroup of this N-point motion is given by

$$P_t^N(x, A) = \mathbf{E}[K_{0,t}(x_1, A_1)K_{0,t}(x_2, A_2), \dots, K_{0,t}(x_N, A_N)],$$

for all $x = (x_1, x_2, \dots, x_n) \in E^N$ and $A = A_1 \times \cdots A_N \in \mathcal{E}^N$. We can see that this N-dimensional process has stationary independent increments. We also see that family of N-point motions has the consistency property in that any M coordinates taken from the N-dimensional process are distributed as the M-point motion of the family. We have that the stochastic flow uniquely determines a consistent family of N-point motion semigroups $((P_t^N; t \geq 0); N \geq 1)$. As discussed in chapter 1, it is possible to get complete information of the flow K from the N-point motions. In [LJR04a] they prove that whenever the space E is a locally compact separable metric space and $((P_t^N; t \geq 0) : N \geq 1)$ is a consistent family of Feller semigroups on this space then there exists a stochastic flow of kernels K whose N-point motion is given by $(P_t^N; t \geq 0)$ for each $N \geq 1$ and the law of K is uniquely determined in the sense of finite dimensional distributions.

If X is a solution to the $\mathcal{A}_N^\theta$ martingale problem started at $x \in \mathbb{R}^N$ then let $P_t^{N,\theta}$ be defined by

$$P_t^{N,\theta} f(x) = \mathbf{E}_x[f(X(t))]$$

for continuous, bounded f. This can be defined for any $x \in \mathbb{R}^N$ and any $N \geq 1$. Then $((P_t^{N,\theta}; t \geq 0) : N \geq 1)$ is a consistent family of semigroups. In order to show that there exists a flow $(K_{s,t}^\theta; s \leq t)$ whose N-point motions are given by $(P_t^{N,\theta}; t \geq 0)$ for each $N \geq 1$ we need to first show that we have that $(P_t^{N,\theta}; t \geq 0)$ has the Feller property. By the Feller property we mean that $x \mapsto P_t^{N,\theta}(x)$ maps the space of continuous bounded functions into itself and

$\lim_{t\downarrow 0} P_t f(x) = f(x)$.

Proposition 72. *For each $N \geq 1$ the semigroup $(P_t^{N,\theta}; t \geq 0)$ has the Feller property.*

Proof. Let $\hat{X}$ be a solution to the $\mathcal{A}_{2N}^{\theta}$-martingale problem started at $(x_1, \ldots, x_N, y_1 \ldots y_N)$. Let X and Y be two N-dimensional process such that $Y_i = X_i = \hat{X}_i$ for $2 \leq i \leq N$. We let $X_1 = \hat{X}_1$ whereas we define Y_1 by

$$Y_1(t) = \begin{cases} \hat{X}_1(t) & t \leq T \\ \hat{X}_{N+1}(t) & t > T, \end{cases}$$

where $T = \inf\{t \geq 0 : \hat{X}_1(t) = \hat{X}_{N+1}(t)\}$. Then by Proposition 31 and the strong Markov property X and Y are both governed by the semigroup $(P_t^{N,\theta}; t \geq 0)$. Letting $x = (x_1, \ldots, x_N)$ and $y = (y_1, x_2, \ldots, x_N)$ we have

$$\begin{aligned} |P_t^{N,\theta}(f(x)) - P_t^{N,\theta}(f(y))| &= |\mathbf{E}[f(X(t)) - f(Y(t))]| \\ &= |\mathbf{E}[(f(X(t)) - f(Y(t)))\mathbf{1}_{\{t \leq T\}}]| \\ &\leq 2||f||_\infty \mathbf{P}(T \geq t) \\ &= 2||f||_\infty \int_0^{|x_1 - y_1|} \frac{1}{\sqrt{\pi t}} \exp(-y^2/4t) \\ &\leq \frac{2||f||_\infty}{\sqrt{\pi t}} |x_1 - y_1|. \end{aligned}$$

The penultimate equality coming from the distribution of Brownian hitting times and (X_1, X_{N+1}) being a pair of θ-coupled Brownian motions. Now let $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$ and $y^i = (y_1, \ldots, y_i, x_{i+1}, \ldots, x_N)$ for $i =$

$1, \ldots, N$. Then it follows from the above that

$$\begin{aligned}|P_t^{N,\theta}(f(x)) - P_t^{N,\theta}(f(y))| &\leq \sum_{i=1}^{N} |P_t^{N,\theta}(f(y^{i-1})) - P_t^{N,\theta}(f(y^i))| \\ &\leq \frac{C}{\sqrt{\pi t}} \sum_{i=1}^{N} |x_i - y_i|\end{aligned}$$

It follows that $x \mapsto P_t^{N,\theta} f(x)$ is continuous. $\lim_{t \downarrow 0} P_t^{N,\theta} f(x) = f(x)$ follows from X being (right) continuous at 0. □

Consider now the system of weighted arrows described in chapter 1, with the environment given by the weights of the arrows $(Q_{n,k}; (n,k) \in L)$. We described a stochastic flow of kernels constructed from these weights. Here we construct a continuous time version still on integers $\mathbb{Z}$.

Let Λ be a poisson point process on $\mathbb{R} \times \mathbb{Z}$ and for each point $(t,x) \in \Lambda$ we attach an independent random variable $Q(t,x)$ with distribution μ, where μ is a random variable on $[0,1]$. We consider a particle moving in $\mathbb{Z}$ whose motioned is governed by $(Q(t,x); (t,x) \in \Lambda)$ as follows. The particle jumps at and only at space time points $(t,x) \in \Lambda$. When the particle does jump, it jumps from (t,x) to $(t,x+1)$ with probability $Q(t,x)$ and from (t,x) to $(t,x-1)$ with probability $1 - Q(t,x)$. We define $K_{s,t}^{\mu}(x,A)$ to be the conditional probability given the environment, $(\Lambda, (Q(t,x); (t,x) \in \Lambda))$, that the particle when started at $x \in \mathbb{Z}$ and time $s \in \mathbb{R}$ is located within the set $A \subset \mathbb{Z}$ at time $t > s$. Then $K_{s,t}^{\mu}(x,A)$ is stochastic flow of kernels on $\mathbb{Z}$.

Proposition 73. *The N-point motion of the flow K^{μ} is a Markov chain on $\mathbf{Z}^N$ with generator $\mathcal{G}_N^p$ given by* (3.46) *where*

$$p(k:l) = \int_0^1 x^k (1-x)^l \mu(dx).$$

Proof. Let A be the generator of the N-point motion of K^{μ}, then by definition

$$Af(x) = \lim_{t \downarrow 0} \frac{1}{t} \left[\mathbf{E}_x[f(X(t))] - f(x) \right]$$

We need therefore to calculate $\mathbf{E}_x[f(X(t))]$. Fix an initial some configuration of N-particles, $x \in \mathbb{Z}^N$. Let $\pi(x)$ the partition of $\{1, \ldots, N\}$, as described at the beginning of Section 3.1. Let $m(\pi)$ be the number of components of partition π and $m_i(\pi)$ be the size of the component of π that contains i.

A jump occurs at a point of the poisson process $(y, t) \in \Lambda$ if there exists $i \in \{1, 2, \ldots, N\}$ such that $X_i(t) = y$. Then we have

$$\begin{aligned} \mathbf{P}_x(0 \text{ jumps by time } t) = e^{-m(\pi(x))t} = 1 - m(\pi(x))t + O(t^2) \\ \mathbf{P}_x(1 \text{ jump by time } t) = m(\pi(x))te^{-m(\pi(x))t} = m(\pi(x))t + O(t^2) \\ \mathbf{P}_x(2 \text{ or more jumps by time } t) = O(t^2) \end{aligned} \tag{3.65}$$

Clearly

$$E_x[f(X_t)|0 \text{ jumps }] = f(x). \tag{3.66}$$

Given that there is one jump, the probability it occurs on the integer x_i is simply $\frac{1}{m(\pi(x))}$. Then if $m_i(\pi(x)) = k + l$ then the probability that k of these go up and l of these go down is given by

$$Q(t', x_i)^k (1 - Q(t', x_i))^l.$$

Where t' is the time that the jump occurs. Therefore, for $v \in \mathcal{V}(x)$

$$\mathbf{P}_x(X_t = x + v | 1 \text{ jump }, Q(t', x_i) = z) = \frac{1}{m(\pi(x))} z^k (1 - z)^l$$

where k and l are the number of components of v that are 1 and -1 respectively. We know that $\mathbf{P}_x(Q(t', x_i) \in dz) = \mu(dz)$ hence

$$\mathbf{P}_x(X_t = x + v | 1 \text{ jump }) = \frac{1}{m(\pi(x))} \int_0^1 z^k (1-z)^l \mu(dz) = \frac{1}{m(\pi(x))} p(v).$$

and

$$\mathbf{E}_x[f(X_t) | 1 \text{ jump }] = \sum_{v \in \mathcal{V}(x)} f(x+v) \frac{1}{m(\pi(x))} p(v).$$

Then putting this together with (3.66) and (3.65) we have

$$\mathbf{E}_x[f(X_t)] = f(x)(1 - m(\pi(x))t) + m(\pi(x))t \sum_{v \in V(x)} f(x+v) \frac{1}{m(\pi(x))} p(v) + O(t^2)$$

and therefore it follows that

$$\lim_{t \downarrow 0} \frac{1}{t} \left[\mathbf{E}_x[f(X_t)] - f(x) \right] = -m(\pi(x)) f(x) + \sum_{v \in V(x)} f(x+v) p(v).$$

By the definition of $p(k:l)$ we have for any i

$$\sum_{k+l = m_i((\pi(x))} p(k:l) = 1.$$

From this is follows that

$$\sum_{v \in V(x)} p(v) = m(\pi(x)).$$

Therefore we are left with

$$\lim_{t \downarrow 0} \frac{1}{t} \left[\mathbf{E}_x[f(X_t)] - f(x) \right] = \sum_{v \in V(x)} (f(x+v)p(v) - f(x)) = \mathcal{G}_N^p f(x).$$

□

Now suppose we have a sequence of probability measures $(\mu_n \geq 1)$ on $[0,1]$. For each n we can associate a stochastic flow of kernels K^{μ_n} on the integers $\mathbb{Z}$ as above. We perform a diffusive scaling. Let $\tilde{K}^{\mu_n}$ be the flow of kernels on the scaled lattice $n^{-1/2}\mathbb{Z}$ given by

$$\tilde{K}^{\mu_n}_{s,t}(x, A) = K^{\mu_n}_{ns,nt}(n^{1/2}x, n^{1/2}A)$$

Definition 74. We say that a sequence of flows of Kernels K^n on $n^{1/2}\mathbb{Z}$ converges in distribution to a flow of kernels K on $\mathbb{R}$ if for any $x \in \mathbb{R}^N$ and any sequence $(x_n; n \geq 0)$ with $x_n \in n^{-1/2}\mathbf{Z}^N$ such that $x_n \to x$ as n tends to infinity. Then, for any N, the N-point motion of K^n started from x_n converges in distribution to the N-point motion of K started from x.

Let $(x_n; n \geq 0)$ be a sequence of points in $\mathbf{R}^N$ converging to a point x, with $x_n \in n^{-1/2}\mathbf{Z}^N$ for every n. For $n \geq 1$, let Y^n be the scaled process given by $Y^n(t) = n^{-1/2}Y(nt)$ for $t \geq 0$, where Y is a Markov chain with generator $\mathcal{G}^{p_n}_N$ starting from $n^{1/2}x_n$.

Theorem 75. *Suppose we have a sequence of probability measure $(\mu_n; n \geq 1)$ in that for each n, $\int_0^t x\mu(dx) = 1/2$, and suppose as n tends to infinity*

$$\sqrt{n}x(1-x)\mu_n(dx) \text{ converges weakly to } \nu(dx)$$

where ν is some finite measure on $[0,1]$. Then as n tends to infinity the sequence of flows $\tilde{K}^{\mu_n}$ converges in distribution to a flow K^θ on $\mathbb{R}$ whose N-point motions are given by the solution to the $\mathcal{A}^\theta_N$-martingale problem, with the family of parameters $(\theta(k:l); k, l \geq 0)$ determined by

$$\theta(k:l) = \int_0^1 x^{k-l}(1-x)^{l-1}\nu(dx)$$

and $\theta(0:1) = \theta(1:0) = 0$.

Proof. For any $x \in \mathbb{R}^N$ choose a sequence $(x_n; n \geq 0)$ with $x_n \in n^{-1/2}\mathbf{Z}^N$ such that $x_n \to x$. We define the family of parameters $p_n(k:l) = \int_0^1 x^k(1-x)^l \mu^n(dx)$ then by proposition the 73 the N-point motion of $\tilde{K}^{\mu_n}$ started at x_n is given by Y^n where $Y^n(t) = n^{-1/2}Y(nt)$ for $t \geq 0$ and Y is a Markov chain with generator $\mathcal{G}_N^{p_n}$ starting from $n^{1/2}x_n$.

For $k, l \geq 1$,

$$
\begin{aligned}
&n^{1/2}(p_n(k,l) - \tfrac{1}{2}\mathbf{1}_{\{k=0\}} - \tfrac{1}{2}\mathbf{1}_{\{l=0\}}) \\
&\quad = n^{1/2}\int_0^1 x^k(1-x)^l \mu^n(dx) \to \int_0^1 x^{k-l}(1-x)^{l-1}\nu(dx) \\
&\quad = \theta(k:l)
\end{aligned}
$$

and $n^{1/2}(p_n(0,1) - \frac{1}{2}\mathbf{1}_{\{k=0\}} - \frac{1}{2}\mathbf{1}_{\{l=0\}}) = n^{1/2}(p_n(1,0) - \frac{1}{2}\mathbf{1}_{\{k=0\}} - \frac{1}{2}\mathbf{1}_{\{l=0\}}) = 0$ for all $n \geq 1$ as for each n, μ_n is centred. For each $k, l \geq 0$, $p_n(k,l)$ and $\theta(k:l)$ satisfy the relation (3.50). By the results of Section 3.6 we have that there exists a limit in distribution to the sequence $(Y^n : n \geq 1)$ and such a limit, Y solves the $\mathcal{A}_N^\theta$-martingale problem started from x. The law of Y is uniquely determined and has the law of the N-point motion of K^θ. $\square$

Chapter 4

θ-coupled Brownian webs

Recall from the introduction the lattice of points $L = \{(k,n) \in \mathbf{Z}^2 : k + n \text{ is even}\}$, and the family of independent random stationary processes $(\xi_{k,n}; (k,n) \in L)$, such that for each $(k,n) \in L$, $(\xi_{k,n}(u); u \geq 0)$ is a stationary Markov process on $\{1, -1\}$ with unit rate of jumping between states. As each process is stationary, at any fixed time u we have $\mathbf{P}(\xi_{k,n}(u) = 1) = \mathbf{P}(\xi_{k,n}(u) = -1) = \frac{1}{2}$.

At any fixed time u we can construct a family of coalescing simple symmetric random walks $\mathcal{S}(u)$ as in the introduction, see figure 1.1. As in the introduction we let $\mathcal{S}^\epsilon(u)$ be the collection of paths of $\mathcal{S}(u)$ after a diffusive scaling (time multiplied by a factor of ϵ and space by a factor of $\sqrt{\epsilon}$). Then from [FINR04], $\mathcal{S}^\epsilon(u)$ converges to a Brownian Web in the sense of weak convergence of probability measures on the metric space they describe, $(\mathcal{H}, d_{\mathcal{H}})$.

For $u_1 \neq u_2$ the pair $(\mathcal{S}^\epsilon(\sqrt{\epsilon}u_1), \mathcal{S}^\epsilon(\sqrt{\epsilon}u_2))$ converges to a pair of Brownian webs such that a pair of paths (one from each system) converges in law to a pair of θ-coupled Brownian motions, where $\theta = |u_1 - u_2| > 0$. θ-coupled Brownian motions are described in Proposition 15. These properties essentially characterise the law of a pair $(\mathcal{W}, \mathcal{W}')$, which we shall call a pair of θ-coupled

Brownian webs. It is reasonable then to suppose that there exists a stationary Markov process $(\mathcal{W}(u); u \geq 0)$ such that for each u, $\mathcal{W}(u)$ is a Brownian web and for any pair of times $u_1 \neq u_2$, the law of $(\mathcal{W}(u_1), \mathcal{W}(u_2))$ is that of a pair of θ-coupled Brownian webs, where $\theta = |u_2 - u_1|$. The main achievement of this chapter is to characterise a pair of θ-coupled Brownian webs, this is given as Theorem 86. Material from this chapter appears in [HW07]. We begin by studying processes in Euclidean space before we move on to webs.

4.1 Sticky Coalescing Systems

Our aim in this section is to construct a system on $m+n$ paths started from any $m+n$ fixed points in $\mathbb{R}^2$. These paths will have the properties that the first m paths, labelled blue, behave as a system of coalescing Brownian motions, as do the remaining n paths, labelled red. However, observing the motion of any one red path and any one blue path, the two paths behave as a pair of θ-coupled Brownian motions. The purpose of this is to then extend the blue paths to a Brownian web and the red paths to another Brownian web, the pair then having the joint distribution we require.

4.1.1 Construction of a system of coalescing Brownian motions

We describe here the usual construction of a finite system of coalescing Brownian motions. Similar constructions of this type can be found in [Arr79],[TW98] and [FINR02] among other places.

Let $B = (B_1, \ldots, B_n)$ be a standard n-dimensional Brownian motion on $\mathbb{R}^n$ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $((x_1, t_1), \ldots, (x_n, t_n))$ be some deterministic set of n points in $\mathbb{R}^2$ and let W_j be defined as follows:

$$W_j(t) = x_j + B_j(t - t_j)$$

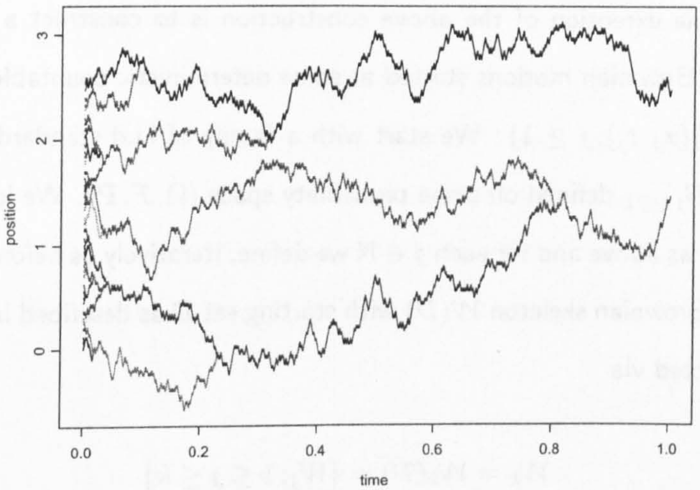

Figure 4.1: System of coalescing Brownian motions

for $t \geq t_j$, so that W_j is a Brownian path started at position x_j and time t_j.

We now specify some coalescing rules. Let $\tilde{W}_1 = W_1$, then for $j > 1$ we construct iteratively the values τ_j, k_j and the path $\tilde{W}_j$. Let

$$\tau_j = \min_{i<j} \inf\{t \geq 0 : W_i(t) = \tilde{W}_j(t)\}$$

$$k_j = \min\{k \in \{1, \ldots, j-1\} : W_j(\tau_j) = \tilde{W}_k(\tau_j)\}$$

and we let

$$\tilde{W}_j(t) = \begin{cases} W_j(t) & t \leq \tau_j \\ \tilde{W}_{k_j}(t) & t \geq \tau_j. \end{cases}$$

We then say that $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_n)$ is a system of coalescing Brownian motions starting from $((x_1, t_1), \ldots, (x_n, t_n))$.

Note that as a consequence of the strong Markov property of Brownian motion, the law of W is independent of the order in which the coalescence is performed. Detailed discussions of this property can be found in [Arr79].

A simple extension of the above construction is to construct a system of coalescing Brownian motions started at some deterministic countable subset of $\mathbb{R}^2$, $\mathcal{D} = \{(x_j, t_j); j \geq 1\}$. We start with a family of i.i.d standard Brownian motions $(B_j)_{j\geq 1}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We let $(W_j)_{j\geq 1}$ be defined as above and for each $j \in \mathbb{N}$ we define, iteratively as before, the path $\tilde{W}_j$. The Brownian skeleton $\mathcal{W}(\mathcal{D})$ with starting set $\mathcal{D}$ as described in [FINR02] is constructed via

$$\mathcal{W}_k = \mathcal{W}_k(\mathcal{D}) = \{\tilde{W}_j; 1 \leq j \leq k\}$$
$$\mathcal{W} = \mathcal{W}(\mathcal{D}) = \bigcup_k \mathcal{W}_k.$$

We note that the Brownian skeleton can be thought of as a subset of the metric space (Π, d) described in [FINR04]. We describe this metric space later in Section 4.2.

4.1.2 Sticky coalescing system from single starting time

A sticky coalescing system (SCS) started from a single starting time is an $M+N$ dimensional diffusion, whose components can be thought as modelling M red particles and N blue particles each moving in $\mathbb{R}$. The red particles when considered on their own behave as M coalescing Brownian motions, and similarly the blue particles considered on their own behave as N coalescing Brownian motions. We then have the further condition that if $(X(t); t \geq 0)$ is the trajectory of any red particle and $(Y(t); t \geq 0)$ is the trajectory of any blue particle then the pair of components $(X(t), Y(t); t \geq 0)$ is a pair of θ-coupled Brownian motions. The theorem below shows that the law of SCS with a single starting point is uniquely specified by the pairwise distribution of paths and a co-adaption property.

Theorem 76. *Fix $M, N \geq 1$. Let $(Z(t); t \geq 0) = ((X_1(t), \ldots X_M(t), Y_1(t), \ldots Y_N(t)) : t \geq 0)$ be an $\mathbb{R}^{M+N}$-valued stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ with the properties*

$(X_i(t); t \geq 0)$ *is an $\mathcal{F}_t$-Brownian motion started at x_i for all $i \in \{1, \ldots, M\}$,*

$(Y_i(t); t \geq 0)$ *is an $\mathcal{F}_t$-Brownian motion started at y_j for all $j \in \{1, \ldots, N\}$,*

$((X_i(t), X_j(t)); t \geq 0)$ *is a pair of coalescing Brownian motions for all $i \neq j$,*

$((Y_i(t), Y_j(t)); t \geq 0)$ *is a pair of coalescing Brownian motions for all $i \neq j$*

$((X_i(t), Y_j(t)); t \geq 0)$ *is a pair of θ-coupled Brownian motions for all i and j.*

Then such a process exists and its law is uniquely determined.

Assuming Theorem 76, we have the following consistency lemma.

Lemma 77. *Let $(Z(t); t \geq 0) = ((X_1(t), \ldots X_M(t), Y_1(t), \ldots Y_N(t)) : t \geq 0)$ be an $\mathbb{R}^{M+N}$-valued stochastic process as given in Theorem 76 with starting values $\{x_1, \ldots, x_M, y_1, \ldots, y_N\}$ then the process Z' given by*

$$Z' = ((X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_M, Y_1, \ldots Y_N))$$

is equal in law to the $\mathbb{R}^{M-1,N}$-valued process of Theorem 76, started at

$$((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_M, y_1, \ldots y_N))$$

and the process Z'' given by

$$Z' = ((X_1, \ldots X_M, Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots Y_N))$$

is equal in law to the $\mathbb{R}^{M,N-1}$*-valued process of Theorem 76, started at*

$$((x_1, \ldots x_M, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N)).$$

Proof. Clearly each of the processes Z' and Z'' automatically satisfy the properties given in Theorem 76 with the appropriate starting values and as these properties uniquely specify the law of the process the lemma is proven. □

Lemma 78. *Let* $(Z(t); t \geq 0) = ((X_1(t), \ldots X_M(t), Y_1(t), \ldots Y_N(t)) : t \geq 0)$ *be an* $\mathbb{R}^{M+N}$*-valued stochastic process as given in Theorem 76 defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ *with starting values* $\{x_1, \ldots, x_M, y_1, \ldots, y_N\}$*. For any* $\mathcal{F}_t$*-stopping time* $\tau \geq 0$ *we define the process* $Z' = (X_1', \ldots X_M', Y_1', \ldots Y_N')$ *by*

$$Z'(t) = Z(t + \tau).$$

Then the conditional distribution of Z' *given* $\mathcal{F}_\tau$ *is equal to the law of the* $\mathbb{R}^{M,N}$*-valued process given in Theorem 76 started at*

$$(X_1(\tau), \ldots X_M(\tau), Y_1(\tau), \ldots Y_N(\tau))$$

Proof. As the law of Z can be given as a solution to a time homogeneous martingale problem and as, in proving Theorem 76, we show that the martingale problem is well posed, then the solution Z has the strong Markov property, see [SV79]. □

4.1.3 Proof of Theorem 76

The statements of Theorem 76 can be expressed in the following terms. The $N+M$ dimensional process $Z = (X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ has initial values given by $Z(0) = z = (x_1, \ldots, x_M, y_1, \ldots, y_N)$ and the following processes are all martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$:

$$\begin{array}{lll}
X_i(t) & (X_i(t))^2 - t & \\
Y_i(t) & (Y_i(t))^2 - t & \\
|X_i(t) - X_j(t)| & |Y_i(t) - Y_j(t)| & i \neq j \\
(X_i(t) - X_j(t))^2 - 2\int_0^t \mathbf{1}_{\{X_i(s) \neq X_j(s)\}} ds & & i \neq j \\
(Y_i(t) - Y_j(t))^2 - 2\int_0^t \mathbf{1}_{\{Y_i(s) \neq Y_j(s)\}} ds & & i \neq j \\
|X_i(t) - Y_j(t)| - \int_0^t 2\theta \mathbf{1}_{\{X_i(s) = Y_j(s)\}} ds & & \\
(X_i(t) - Y_j(t))^2 - \int_0^t 2\mathbf{1}_{\{X_i(s) \neq Y_j(s)\}} ds \ . & &
\end{array} \tag{4.1}$$

We call the set of these processes $\mathcal{M}_\theta^{M,N}$ and if $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ is a $M+N$ dimensional process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ such that for each $\Psi \in \mathcal{M}_\theta^{M,N}$, Ψ is an $(\mathcal{F}_t, \mathbf{P})$-martingale then we say that Z is a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem.

Our strategy for proving Theorem 76 involves first of all showing that if Z is any solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem started at $z \in \mathbb{R}^{M+N}$ then the law of the process $(Z(t \wedge \zeta_0); t \geq 0)$ is uniquely specified, where ζ_0 is the first time two red particles or two blue particles meet. Thus we have the following proposition.

Proposition 79. *There exists a process $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ started ar $z = (x_1, \dots, x_M, y_1, \dots, y_N)$ such that for each $\Psi \in \mathcal{M}_\theta^{M,N}$ the process $\Psi(t \wedge \zeta_0)$ is a martingale, where*

$$\zeta_0 = \min(\inf_{i \neq j} \inf\{t \geq 0 : X_i(t) = X_j(t)\}, \inf_{i \neq j} \inf\{t \geq 0 : Y_i(t) = Y_j(t)\}).$$

Moreover the law of the process $(Z(t \wedge \zeta_0) : t \geq 0)$ is uniquely specified.

We proceed by finding processes that are characterised by martingales, which coincide with the processes in $\mathcal{M}_\theta^{M,N}$ on certain subsets of the state space. We then use a localisation technique of the type found in [SV79]. Without loss of generality assume $M \leq N$. Define the set of pairings P to be the set of injective maps $p : \{1, \dots, M\} \mapsto \{1, \dots, N\}$. So if $M = N$ then P is just the set of permutations of $\{1, 2, \dots, M\}$.

Proposition 80. *For some $p \in P$, let $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ be an $\mathbb{R}^{M+N}$-valued stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with the properties*

$(X_i(t); t \geq 0)$ *is an $\mathcal{F}_t$-Brownian motion started at x_i for all $i \in \{1, \dots, M\}$*

$(Y_i(t); t \geq 0)$ *is an $\mathcal{F}_t$-Brownian motion started at y_j for all $j \in \{1, \dots, N\}$*

$((X_i(t), X_j(t)); t \geq 0)$ *and* $((Y_i(t), Y_j(t)); t \geq 0)$ *are pairs of independent Brownian motions for all $i \neq j$*

$((X_i(t), Y_j(t)); t \geq 0)$ *is a pair of independent Brownian motions for all $j \neq p(i)$.*

$((X_i(t), Y_{p(i)}(t); t \geq 0)$ *is a pair of θ-coupled Brownian motions for all $i \in \{1, \dots, M\}$.*

Then such a process exists and its law is uniquely determined.

Proof. Assume without loss of generality that $M \leq N$, and that the pairing p is such that $p(i) = i$ for $i \in \{1, \dots M\}$. To prove existence, we let (X_i, Y_i) be a pair of θ-coupled Brownian motions, as given in Proposition 15, for each $i \in \{1, \dots M\}$ defined on a common probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Furthermore let the pairs $\{(X_i, Y_i), i \in \{1, \dots, M\}\}$ be mutually independent. If $N > M$ then let $(Y_{M+1} \dots, Y_N)$ be mutually independent Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ that are also mutually independent of

$\{(X_i, Y_i), i \in \{1, \dots, M\}\}$. Let $Z = (X_1, \dots, X_M, Y_1, \dots Y_N)$. Then it follows that Z satisfies all the properties required in Proposition 80.

If $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ is any process satisfying the properties of Proposition 80 then each pair (X_i, Y_i) is a pair of θ-coupled Brownian motions and $(Y_{M+1}, \dots, Y_N)$ is a collection of Brownian motions. Therefore in order to prove uniqueness in law it only remains to prove their mutual independence. For each $i \in \{1, \dots, M\}$ let $U_i = X_i - Y_i$ and $V_i = X_i + Y_i$. It is then easy to see that $\langle U_i, V_i \rangle \equiv 0$. It is also possible to show, using the pairwise independence statements, that $\langle U_i, U_j \rangle = \langle V_i, V_j \rangle = \langle U_i, V_j \rangle \equiv 0$ for all $i \neq j$ and $\langle U_i, Y_j \rangle = \langle V_i, Y_j \rangle = \langle Y_j, Y_k \rangle \equiv 0$ for all $i \in \{1, \dots, M\}$, $j, k \in \{M+1, \dots, N\}$ with $j \neq k$.

The quadratic variation of U_i and V_i is given by $\langle U_i \rangle_t = 2 \int_0^t \mathbf{1}_{\{U_i(s) \neq 0\}} ds$ and $\langle V_i \rangle_t = 2t + 2 \int_0^t \mathbf{1}_{\{U_i(s) = 0\}} ds$ respectively and by an argument seen in the proof of Proposition 6, $\langle U_i \rangle_\infty = \langle V_i \rangle_\infty$ almost surely. It is therefore possible to apply Knight's Theorem so that if $\alpha_t^i = \inf\{u : \langle U_i \rangle_u > t\}$ and $\beta_t^i = \inf\{u : \langle V_i \rangle_u > t\}$ then the process

$$((U_1(\alpha_t^1), \dots, U_M(\alpha_t^M), V_1(\beta_t^1), \dots, V_M(\beta_t^M), Y_{M+1}(t), \dots, Y_N(t)); t \geq 0)$$

is an $M+N$-dimensional Brownian motion. Calling this Brownian motion $B = (B_1, \dots, B_{M+N})$. It is possible to show (compare with the proofs of propositions 6, and 16) that $2\alpha_t^i = t + \frac{1}{\theta} L_t(B_i)$ and $\beta_t^i = \alpha^i_{{A^{\theta/2}}^i_t}$ where ${A^{\theta/2}}^i_t = \inf\{u \geq 0 : u + \frac{2}{\theta} L_u(B_i) > t\}$. As α^i and β^i only depend on the process B_i, it follows that each of the pairs $((U_i, V_i); i \in \{1, \dots, M\})$ and the Brownian motions $(Y_{M+1}, \dots, Y_N)$ are mutually independent. Thus each of the pairs $((X_i, Y_i); i \in \{1, \dots, M\})$ and the Brownian motions $(Y_{M+1}, \dots, Y_N)$ are mutually independent. $\square$

Proof of Proposition 79. We note that the statements in Proposition 80 can also be described in the following terms. Z is a $M+N$ dimensional process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ with initial value given by $Z(0) = z = (x_1, \dots, x_M, y_1 \dots y_N)$ and the following processes are martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. For $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$,

$$\begin{array}{lll} X_i(t) & (X_i(t))^2 - t & \\ Y_i(t) & (Y_i(t))^2 - t & \\ (X_i(t) - X_j(t))^2 - 2t & (Y_i(t) - Y_j(t))^2 - 2t & i \neq j \\ (X_i(t) - Y_j(t))^2 - 2t & & j \neq p(i) \\ |X_i(t) - Y_{p(i)}(t)| - \int_0^t 2\theta \mathbf{1}_{\{X_i(s) = Y_{p(i)}(s)\}} ds & & \\ (X_i(t) - Y_{p(i)}(t))^2 - 2\int_0^t \mathbf{1}_{\{X_i(s) \neq Y_{p(i)}(s)\}} ds & & \end{array} . \tag{4.2}$$

We call this set of functionals $\mathcal{M}_{\theta,p}^{M,N}$. If $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ is a $M+N$ dimensional process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ such that for each $\Psi \in \mathcal{M}_{\theta,p}^{M,N}$, Ψ is an $(\mathcal{F}_t, \mathbf{P})$-martingale then we say that Z is a solution to the $\mathcal{M}_{\theta,p}^{M,N}$-martingale problem.

Now for each pairing $p \in P$ we define an open set V_p to be $\{z = (x_1, \dots, x_M, y_1, \dots, y_N) \in \mathbb{R}^{M+N}\}$ such that

1. $x_i \neq y_j$ for all $j \neq p(i)$
2. $x_i \neq x_j$ for all $i \neq j$
3. $y_i \neq y_j$ for all $i \neq j$.

Fix $p \in P$ and $z \in V_p$. Let Z be any solution to the $\mathcal{M}_{\theta}^{M,N}$-martingale problem started at $z \in \mathbb{R}^{M+N}$, and let $\tau_p = \inf\{t \geq 0 : Z(t) \notin V_p\}$. Then it is possible to show that for each process Ψ in $\mathcal{M}_{\theta,p}^{M,N}$ the process $(\Psi(t \wedge \tau_p); t \geq 0)$ is a martingale. Conditional on Z, let Z' be a solution to the $\mathcal{M}_{\theta,p}^{M,N}$-martingale problem started at $Z(\tau_p)$. Letting Z'' be given by

$$Z''(t) = \begin{cases} Z(t) & t \le \tau_p \\ Z'(t-\tau_p) & t > \tau_p. \end{cases}$$

Then it follows, by splicing arguments of [SV79], that Z'' solves the $\mathcal{M}^{M,N}_{\theta,p}$-martingale problem started at z. Thus the law of $(Z''(t \wedge \tau_p); t \ge 0)$ and hence the law of $(Z(t \wedge \tau_p) : t \ge 0)$ is uniquely specified.

For $\epsilon > 0$, we define Γ_ϵ be the open set $\{z \in \mathbb{R}^{M+N}\}$ such that

1. $|x_i - x_j| > 2\epsilon \ \forall i \ne j$
2. $|y_i - y_j| > 2\epsilon \ \forall i \ne j$
3. $|x_i| < 1/\epsilon$ for all i.

Then, for each $z \in \Gamma_\epsilon$ there exists a pairing p_z such that V_{p_z} contains the ball $\{z' \in R^{M+N} : ||z' - z|| < \epsilon/2\}$ and hence the collection $\{V_p : p \in P\}$ forms an overlapping cover of Γ_ϵ.

Fix $z \in \Gamma_\epsilon$ and suppose Z solves the $\mathcal{M}^{M,N}_\theta$-martingale problem starting from z. Let $\tau_0 = 0$ and let

$$\tau_{i+1} = \inf\{t \ge \tau_i; Z(t) \notin V_{p_i} \cap \Gamma_\epsilon\}$$

where $p_i = p_{X(\tau_i)}$. Let $\zeta_\epsilon = \inf\{t \ge 0; Z(t) \notin \Gamma_\epsilon\}$, then this stopping time is almost surely finite as each coordinate behaves as Brownian motion and

$$\zeta_\epsilon \le \min_i(\min(\inf\{t \ge 0; |X_i| \ge 1/\epsilon\}, \inf\{t \ge 0; |Y_i| \ge 1/\epsilon\})).$$

Then by the continuity of paths of Z, $\tau_i = \zeta_\epsilon$ for sufficiently large i with probability one. Now consider the conditional distribution of $(Z((\tau_i + t) \wedge \tau_{i+1}); t \ge 0)$

given $\mathcal{F}_{\tau_i}$. From results in [SV79], this conditional law is almost surely a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem started from $Z(\tau_i)$ and stopped upon the first exit of the set $V_{p_i} \cap \Gamma_\epsilon$. By the arguments given above, for each i the conditional law of $(Z((\tau_i + t) \wedge \tau_{i+1}); t \geq 0)$ given $\mathcal{F}_{\tau_i}$ is uniquely specified. It follows, therefore by a splicing argument of the type found in [SV79] that the law of the process $(Z(t \wedge \tau_i) : t \geq 0)$ is uniquely determined for all i, then as $\tau_i = \zeta_\epsilon$ for sufficiently large i with probability 1 the law of the process $(Z(t \wedge \zeta_\epsilon); t \geq 0)$ is uniquely specified.

Let Γ_0 be the open set $\{z = (x_1, \dots, x_M, y_1, \dots, y_N) \in \mathbb{R}^{M+N}\}$ such that

1. $x_i \neq x_j$ for all $i \neq j$

2. $y_i \neq y_j$ for all $i \neq j$

and we have $\zeta_0 = \inf\{t \geq 0 : x(t) \notin \Gamma_0\}$.

We have that $\bigcup_{\epsilon>0} \Gamma_\epsilon = \Gamma_0$ hence, by continuity of paths of Z, $\zeta_\epsilon \to \zeta_0$ as $\epsilon \downarrow 0$ and hence the law of $(Z(t \wedge \zeta_0); t \geq 0)$ is uniquely specified.

We now set about proving existence of a process such that $\Psi(t \wedge \zeta_0)$ is a martingale for all $\Psi \in \mathcal{M}_\theta^{M,N}$ in (4.1). For some $\epsilon > 0$, fix $z \in \Gamma_\epsilon$ and let Z^1 be a solution to the $\mathcal{M}_{\theta,p_z}^{M,N}$-martingale problem started at z whose law is characterised by Proposition 80. Let Z^{k+1} conditional on $(Z^i; i \leq k)$ be the process given in proposition 80 started at $Z^k(\tau_k^*)$ and with $p = p_k$. Here $\tau_0^* = 0$ and $\tau_{k+1}^* = \inf\{t \geq 0; Z^{k+1} \notin V_{p_k} \cap \Gamma_\epsilon\}$ and $p_k = p_{Z^k(\tau_k^*)}$. Now let $\tau_0 = 0$ and $\tau_{n+1} = \tau_n + \tau_{n+1}^*$ and define a process Z by

$$Z(t) = Z^{n+1}(t - \tau_n) \quad \tau_n \leq t \leq \tau_{n+1}$$

for $n \geq 0$. We now see that the sequence of stopping times τ_n is given by

$\tau_0 = 0$ and for $n \geq 1$,

$$\tau_{n+1} = \inf\{t \geq \tau_n : Z(t) \notin V_{p_{Z(\tau_n)}} \cap \Gamma_\epsilon\}.$$

For each k, $Z^k = (X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ has the property that for each $\Psi \in \mathcal{M}_{\theta,p}^{M,N}$, of (4.2), Ψ is a martingale. This implies that for each $\Psi \in \mathcal{M}_\theta^{M,N}$ of (4.1), $(\Psi(t \wedge \tau_k^*) : t \geq 0)$ is a martingale. If follows, from splicing lemmas as found in [SV79], that if Z is constructed as above and $Z = (X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ then for each $\Psi \in \mathcal{M}_\theta^{M,N}$, $\Psi(t \wedge \tau_n)$ is a martingale for all n. Thus $\Psi(t \wedge \zeta_\epsilon)$ is a martingale for all $\Psi \in \mathcal{M}_\theta^{M,N}$. Letting ϵ tend down to zero it is then possible to see that there exists a process $Z = (X_1, \ldots, X_M, Y_1, \ldots, Y_N)$ such that $\Psi(t \wedge \zeta_0)$ is a martingale for all $\Psi \in \mathcal{M}_\theta^{M,N}$. $\square$

To complete the proof of Theorem 76 we use an induction argument on the dimension $N + M$. Firstly if $M = N = 1$ the process in question reduces to a 2-dimensional process (X_1, Y_1) being a pair of θ-coupled Brownian motions. The existence and uniqueness in law of such a process is proved in proposition 15.

Now assume that for all $k \in \{1, \ldots, M-1\}$ and $l \in \{1, \ldots, N-1\}$ and also for $(k,l) = (M-1, N)$ and $(k,l) = (M, N-1)$ there exists a solution to $\mathcal{M}_\theta^{k,l}$ martingale problem and the law of such a solution is uniquely specified. To prove that there exists a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem we note that by Proposition 79 there exists a process $Z' = (X'_1, \ldots, X'_M, Y'_1, \ldots, Y'_N)$ started at $z \in \mathbb{R}^{M+N}$ such that for any $M \in \mathcal{M}_\theta^{M,N}$, $\Psi(t \wedge \zeta_0)$ is a martingale where

$$\zeta_0 = \min(\inf_{i \neq j} \inf\{t \geq 0 : X'_i(t) = X'_j(t)\}, \inf_{i \neq j} \inf\{t \geq 0 : Y'_i(t) = Y'_j(t)\}).$$

Without loss of generality suppose that $X_1'(\zeta_0) = X_2'(\zeta_0)$. Then conditional on Z' let $Z'' = (X_1'', \dots, X_M'', Y_1'', \dots, Y_N'')$ be an $\mathbb{R}^{M+N}$-dimensional process such that $(X_2'' \dots, X_M'', Y_1'', \dots, Y_N'')$ is a solution to the $\mathcal{M}_\theta^{M-1,N}$-martingale problem started at $(X_2'(\zeta_0), \dots, X_M'(\zeta_0), Y_1'(\zeta_0), \dots, Y_N'(\zeta_0))$ and $X_1'' = X_2''$. It follows, by splicing lemmas in [SV79] again, that the process Z given by

$$Z(t) = \begin{cases} Z'(t) & 0 \le t \le \zeta_0 \\ Z''(t-\zeta_0) & t > \zeta_0 \end{cases}$$

has the property that Ψ is a martingale for all $\Psi \in \mathcal{M}_\theta^{M,N}$.

To show that the law of a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem is uniquely specified, first note that if $Z = (X_1, \dots, X_M, Y_1, \dots, Y_N)$ is a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem then from proposition 79 the law of a $(Z(t \wedge \zeta_0); t \ge 0)$ is uniquely determined. Assume again without loss of generality that $X_1(\zeta_0) = X_2(\zeta_0)$. The conditional law of $Z(t+\zeta_0)$ given $\sigma(Z(t \wedge \zeta_0); t \ge 0)$ is a solution to the $\mathcal{M}_\theta^{M,N}$-martingale problem. As in Lemma 77 the conditional law of $(X_2(t+\tau), \dots, X_M(t+\tau), Y_1(t+\tau), \dots, Y_N(t+\tau))$ given $\sigma(Z(t \wedge \zeta_0); t \ge 0)$ is a solution to the $\mathcal{M}_\theta^{M-1,N}$ martingale problem and therefore, by the induction hypothesis, this conditional distribution is uniquely specified. All that remains to show is that $X_1(t+\zeta_0) = X_2(t+\zeta_0)$ for all $t \ge 0$, but as $|X_1 - X_2|$ is a non-negative martingale and $|X_1(\zeta_0) - X_2(\zeta_0)| = 0$ we must have that $|X_1(t+\zeta_0) - X_2(t+\zeta_0)| = 0$ for all $t \ge 0$.

4.1.4 Construction and properties of a sticky coalescing system (SCS)

Let $(x_1, t_1), \dots, (x_n, t_n), (y_1, u_1), \dots, (y_m, u_m)$ be $n+m$ fixed points in $\mathbb{R}^2$. Assume without loss of generality that $t_1 \le t_2 \le \dots \le t_n$ and $u_1 \le u_2 \le$

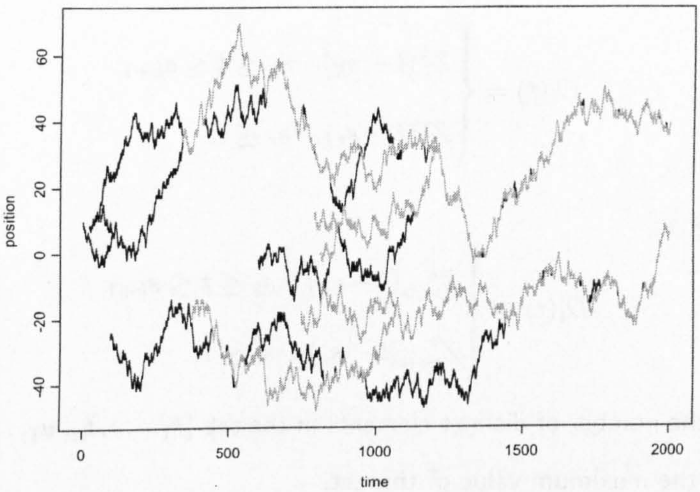

Figure 4.2: Sticky coalescing system

$\cdots \le u_m$. We order the set $\{t_1, \dots, t_n, u_1, \dots, u_m\}$ and we remove duplicated values, then we let s_i be the ith element in this ordered set. We define $k(i) = \max\{k; t_k \le s_i\}$ and $l(i) = \max\{k; u_k \le s_i\}$. As an example, if we have an ordering $t_1 = t_2 = u_1 < t_3 < u_2 \cdots$, then $s_1 = u_1 = t_1 = t_2$, $k(1) = 2$ and $l(1) = 1$.

The first step is constructed as follows: Let $(Z^1(t); t \ge 0)$ be distributed as the $\mathbb{R}^{k(1)+l(1)}$-valued processes given in Theorem 76, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and started at $(x_1, \dots, x_{k(1)}, y_1, \dots, y_{l(1)})$. Then for each $i \ge 2$ let Z^i be a $\mathbb{R}^{k(i)+l(i)}$ valued process whose conditional distribution given $(Z^j; j < i)$ is as the process given in Theorem 76 started at

$$(Z_1^{i-1}(s_i - s_{i-1}), Z_2^{i-1}(s_i - s_{i-1}), \dots, Z_{k(i-1)}^{i-1}(s_i - s_{i-1}), x_{k(i-1)+1}, \dots, x_{k(i)},$$
$$Z_{k(i)+1}^{i-1}(s_i - s_{i-1}), Z_{k(i)+2}^{i-1}(s_i - s_{i-1}), \dots, Z_{l(i-1)}^{i-1}(s_i - s_{i-1})), y_{l(i-1)+1}, \dots, y_{l(i)}).$$

Finally we construct a collection of $n+m$ paths $C = (C_1, \dots, C_n, D_1, \dots, D_m)$

started at $((x_1,t_1),\ldots,(x_n,t_n),(y_1,u_1),\ldots,(y_m,u_m))$ as follows: For $t \geq t_i$

$$C_i(t) = \begin{cases} Z_i^k(t-s_k) & s_k \leq t \leq s_{k+1} \\ Z_i^r(t-s_r) & s_r \leq t, \end{cases}$$

and for $t \geq u_i$

$$D_i(t) = \begin{cases} Z_{i+n}^k(t-s_k) & s_k \leq t \leq s_{k+1} \\ Z_{i+n}^r(t-s_r) & s_r \leq t. \end{cases}$$

Here r is the number of distinct elements in the set $\{t_1,\ldots,t_n,u_1,\ldots,u_m\}$, so that s_r is the maximum value of this set.

We say that any collection of paths that has the same law as C is an SCS with starting values $((x_1,t_1),\ldots,(x_n,t_n),(y_1,u_1),\ldots,(y_m,u_m))$.

We have the following two lemmas which describe some useful properties of an SCS. The first one tells us that the laws of SCSs have some consistency as we vary the number of starting points. The second lemma tells us that the first n paths or the last m paths of an SCS, when viewed on their own, behave as a system of coalescing Brownian motions.

Lemma 81. *If $C = (C_1,\ldots,C_n,D_1,\ldots,D_m)$ is an SCS with starting values $((x_1,t_1),\ldots,(x_n,t_n),(y_1,u_1),\ldots,(y_m,u_m))$ then*

$$C' = (C_1,\ldots,C_{j-1},C_{j+1},\ldots,C_n,D_1,\ldots,D_m)$$

is an SCS started at

$$((x_1,t_1),\ldots,(x_{j-1},t_{j-1}),(x_{j+1},t_{j+1}),\ldots,(x_n,t_n),(y_1,u_1),\ldots,(y_m,u_m)).$$

and

$$C'' = (C_1,\ldots,\ldots,C_n,D_1,\ldots,D_{k-1},D_{k+1},\ldots,D_m)$$

is an SCS started at

$$((x_1,t_1),\dots,(x_n,t_n),(y_1,u_1),\dots,(y_{k-1},u_{k-1}),(y_{k+1},u_{k+1}),\dots,(y_m,u_m)).$$

Proof. It is only necessary to prove the first case because of the symmetry of the construction. The removed path C_j, is started at (x_j,t_j). Suppose that $t_j = s_i$, the ith ordered time in the construction. We can construct the paths $(C_1,\dots,C_{j-1},C_{j+1},C_n,D_1,\dots,D_m)$ using the above construction with the jth coordinate removed from each Z^l with $l \geq i$. The result then follows from Lemma 77. We note there is a special case when there does not exist $k \neq j$ such that $t_k = t_j$ and there does not exist k such that $u_k = t_j$. In this case the construction of the SCS started at

$$((x_1,t_1),\dots,(x_{j-1},t_{j-1}),(x_{j+1},t_{j+1}),\dots,(x_n,t_n),(y_1,u_1),\dots,(y_m,u_m))$$

does not include the processes Z^i from the original construction. However this exception is overcome by the fact, by Lemma 78, that the conditional distribution of $(Z^{i-1}(s_i - s_{i-1} + t); t \geq 0)$ given $(Z^j; j < i-1)$ and $\sigma(Z^{i-1}(t - s_{i-1}); 0 \leq t \leq s_i)$ has the law of the process given in Theorem 76 started at $Z^{i-1}(s_i - s_{i-1})$. □

Corollary 82. *An immediate consequence of Lemma 81 is that the process.*

$$((C_1(t+s_i),\dots,C_{k(i)}(t+s_i),D_1(t+s_i),\dots,D_{l(i)}(t+s_i));t \geq 0)$$

is equal in distribution to $Z^i(t)$, an $\mathbb{R}^{k(i)+l(i)}$ valued process, as given in Theorem

76, started at

$$(C_1(s_i), \ldots, C_{k(i-1)}, x_{k(i-1)+1}, \ldots, x_{k(i)},$$
$$D_1(s_i), \ldots, D_{l(i-1)}, y_{l(i-1)+1}, \ldots, y_{l(i)}).$$

This can be seen just by using Lemma 81 to remove all starting values (x_j, t_j) *with* $s_i < t_j$.

Lemma 83. *If* $C = (C_1, \ldots, C_n, D_1, \ldots, D_m)$ *is an SCS with starting values* $((x_1, t_1), \ldots, (x_n, t_n), (y_1, u_1), \ldots, (y_m, u_m))$ *then*

$$C' = (C_1, \ldots, C_n)$$

is equal in law to a system of coalescing Brownian motions started from

$$((x_1, t_1), \ldots, (x_n, t_n)),$$

and

$$D' = (D_1, \ldots, D_m)$$

is equal in law to a system of coalescing Brownian motions started from

$$((y_1, u_1), \ldots, (y_m, u_m)).$$

(A system of coalescing Brownian motions is a collection of paths as constructed in 4.1.1.)

Proof. By the symmetry of the problem, we only need to consider the first of the two cases given in the lemma. Remove the last $l(i)$ coordinates from Z^i for each i. The construction of SCS then becomes equivalent to the construction given in Section 4.1.1. □

4.2 The Brownian Web

The Brownian web, as found in [FINR04], is a new characterisation of a random network consisting of the paths of coalescing Brownian motions starting from every point in $\mathbb{R} \times \mathbb{R}$, space and time. More precisely this means that paths behave as independent one dimensional Brownian motions until the first time any two paths meet, from this point in time the two paths behave as the same Brownian motion.

In the paper by Fontes et al. [FINR04] they extend the earlier work of Arratia [Arr79] and of Tóth and Werner [TW98]. Arratia was the first to study a system of coalescing Brownian motions starting from every point in $\mathbb{R}$ at time 0. He was motivated by the limiting behaviour of some nearest neighbour interacting particle systems on the one dimensional lattice, such as coalescing random walks, annihilating random walks, and voter models. In [TW98] they study a system of coalescing Brownian motions started from every point in $\mathbb{R} \times \mathbb{R}^+$ motivated by the problem of constructing a continuum "self-repelling motions".

It is relatively straightforward to define a system of coalescing Brownian motions starting from a finite collection of points in $\mathbb{R} \times \mathbb{R}$ and then to extend this to a system of coalescing Brownian motions starting from a countable dense subset of $\mathbb{R} \times \mathbb{R}$, see Section 4.1.1. A question arises about what to do with the remaining starting points. If $C_{x,t_0}(t)$ is the position of the Brownian motion started at (x, t_0) at time t, then from [TW98] or [Arr79] the method would be, in a sense, to apply some right (or left) continuity condition to $x \mapsto C(x, t_0)(t)$. Discussions of different regularity conditions can be found in [TW98].

The characterisation in [FINR04] attacks the problem from a different angle by defining a metric space of paths with starting points in $\mathbb{R}^2$. To construct the paths started from outside some countable dense subset of $\mathbb{R}^2$ the closure is taken in this metric space.

Effectively by taking the closure in this metric space of paths we are allowing limits to be taken from below and above a starting point. For any deterministic starting point this does not make a difference, with probability one, to the resulting path starting from that point, but, for some non-deterministic points, the Brownian web construction leads to the possibility of two different paths starting from the same point.

The main advantage of the Brownian web construction is that it exists as a random point in a certain metric space, which allows the use of certain weak convergence results and will give us the ability to construct a Markov chain on the space itself.

4.2.1 The metric spaces

We start with the metric space $(\bar{\mathbb{R}}^2, \rho)$ where $\bar{\mathbb{R}}^2$ is the completion of $\mathbb{R}^2$ under the metric ρ, which is given by

$$\rho((x_1,t_1),(x_2,t_2)) = \left|\frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|}\right| \vee |\tanh(t_1) - \tanh(t_2)|. \quad (4.3)$$

Other metrics could equally have been used to give the same topological structure at the extreme points of the space. For any (x,t) belonging to a finite ball around some fixed point in $\mathbb{R}^2$ the metric is topologically equivalent to usual Euclidean metric. We also note that any subset of $\bar{\mathbb{R}}^2$ is bounded under this metric.

From this metric space we build a second metric space. For $t_0 \in [-\infty,\infty]$ let $C[t_0]$ be the set of functions f from $[t_0,\infty]$ to $[-\infty,\infty]$ such that the map

$t \mapsto (f(t), t)$ is continuous w.r.t. the metric ρ. We then define the space Π as

$$\Pi = \bigcup_{t_0 \in [-\infty,\infty]} C[t_0] \times \{t_0\}.$$

For $(f, t_0) \in \Pi$ we let $\hat{f}$ be the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for all $t < t_0$. The metric d, on π, is defined as

$$d((f_1, t_1), (f_2, t_2)) = \left(\sup_t \left| \frac{\tanh(\hat{f}_1(t))}{1 + |t|} - \frac{\tanh(\hat{f}_2(t))}{1 + |t|} \right| \right) \vee |\tanh(t_1) - \tanh(t_2)|.$$

Our random objects are going to be compact subsets of the metric space (Π, d). We let $\mathcal{H}$ be the set of compact subsets of (Π, d) and let $d_{\mathcal{H}}$ be the induced Hausdorff metric defined as follows:

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2).$$

Our random object will be an element of the metric space $(\mathcal{H}, d_{\mathcal{H}})$. Let $\mathcal{F}_{\mathcal{H}}$ be the Borel sigma-algebra associated with $(\mathcal{H}, d_{\mathcal{H}})$. We have the following theorem which gives us a characterisation of the Brownian web:

Theorem 84 (Theorem 2.1 in [FINR04]). *There is a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\overline{\mathcal{W}}$ whose distribution is uniquely determined by the following three properties:*

(o) From any deterministic point (x, t) in $\mathbb{R}^2$, there is almost surely a unique path $W_{x,t}$ starting from (x,t).

(i) For any deterministic $n, (x_1, t_1), \ldots, (x_n, t_n)$, the joint distribution of $W_{x_1,t_1}, \ldots, W_{x_n,t_n}$ is that of coalescing Brownian motions.

(ii) For any deterministic dense countable subset of $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\overline{\mathcal{W}}$ is the closure in (Π, d) of $\{W_{x,t} : (x, t) \in \mathcal{D}\}$

4.2.2 Constructing a Brownian web

We start with $\mathcal{D} = ((x_j, t_j), j \geq 1)$ an ordered countable dense subset of $\mathbb{R}^2$, then we take i.i.d. Brownian paths starting at each point in $\mathcal{D}$, and supported on some underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We apply an iterative coalescing rule, as described in Section 4.1.1 to give what is called a Brownian skeleton, $\mathcal{W}(\mathcal{D})$. We take the closure in the space (Π, d) to give $\overline{\mathcal{W}}(\mathcal{D})$.

To prove Theorem 84 it is shown in [FINR04] that $\overline{\mathcal{W}}(\mathcal{D})$ satisfies the properties (o) , (i) and (ii) of Theorem 84 and that the distribution of $\overline{\mathcal{W}}(\mathcal{D})$ does not depend on the choice of $\mathcal{D}$. It also needs to be shown that $\overline{\mathcal{W}}(\mathcal{D})$ is indeed $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued, that is $\overline{\mathcal{W}}(\mathcal{D})$ is a compact subset of (Π, d) and that the map from the Ω to $\mathcal{H}$, $\omega \mapsto \overline{\mathcal{W}}(\mathcal{D})(\omega)$ is $\mathcal{F}/\mathcal{F}_{\mathcal{H}}$-measurable. This is shown in the appendices of [FINR04], we outline the measurability argument here.

In Appendix A of [FINR04] it is shown that $\mathcal{F}_{\mathcal{H}}$ is equal to the σ-algebra generated by sets of the form

$$C^{t_0}_{I_1,\ldots,I_n} = \{K \in \mathcal{H} : \text{ there exists } (f,t) \in K \text{ with } t > t_0 \text{ such that } (f,t) \text{ goes through } I_1, \ldots, I_n\}, \quad (4.4)$$

where I_k is a horizontal segment of $\mathbb{R}^2$, that is $I_k = I'_k \times \{t_k\}$, where I'_k is some, not necessarily finite, open interval. It then remains to show that for any set of the form $C^{t_0}_{I_1,\ldots,I_n}$, $\overline{\mathcal{W}}(\mathcal{D})^{-1}(C^{t_0}_{I_1,\ldots,I_n}) \in \mathcal{F}$. That is we want to show that the event

$$\{ \text{ there exists } (f,t) \in \overline{\mathcal{W}}(\mathcal{D}) \text{ with } t \geq t_0 \text{ such that } (f,t) \text{ goes through } I_1, \ldots, I_n\} \quad (4.5)$$

is an element of $\mathcal{F}$. From the construction of $\mathcal{W}(\mathcal{D})$ it is clear that events of the form

$$\{ \text{ there exists } (f,t) \in \mathcal{W}(\mathcal{D}) \text{ with } t \geq t_0 \text{ such that } (f,t) \text{ goes through } I_1^i, \ldots, I_n^i \} \tag{4.6}$$

are elements of $\mathcal{F}$ and then it is straight forward to show that events of the form (4.5) can be written as countable intersections of events of the form of (4.6).

We note that what is actually shown in [FINR04] is that $\overline{W}(\mathcal{D})$ is only compact almost surely. That is $\overline{W}(\mathcal{D})$ belongs to some space larger than $\mathcal{H}$, $\{\omega : \overline{W}(\mathcal{D}) \in \mathcal{H}\} \in \mathcal{F}$ and $\mathbf{P}(\{\omega : \overline{W}(\mathcal{D}) \in \mathcal{H}\}) = 1$. This however does not pose a problem as we can simply redefine $\overline{W}(\mathcal{D})$ on the null set $N = \{\omega : \overline{W}(\mathcal{D}) \notin \mathcal{H}\}$ so that for all $\omega \in N$, $\overline{W}(\mathcal{D})(\omega)$ is some arbitrary compact subset of (Π, d). For example we could use the empty set. Then $\overline{W}(\mathcal{D})(\omega) \in \mathcal{H}$ for all ω and $\overline{W}(\mathcal{D})$ is $\mathcal{F}/\mathcal{F}_{\mathcal{H}}$ measurable.

It is important to note that there are many $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables that satisfy (o) and (i) but not (ii). For example if we start with a Brownian web $\overline{\mathcal{W}}$ then take a randomly chosen point in $(x_0, t_0) \in \mathbb{R}^2$ chosen with some distribution which is absolutely continuous with respect to Lebesgue measure and let $f(t) = x_0$ for all $t \geq t_0$. Clearly the random object $\overline{\mathcal{W}} \cup (f, t_0)$ is $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ valued and (o) and (i) are satisfied but (ii) is not. The Brownian web is in a sense the minimal $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable that satisfies (o) and (i) in that other random variables that satisfy (o) and (i) contain extra paths. This idea is reinforced by the alternative characterisations of the Brownian web given in theorems 3.1 and 4.1 of [FINR04]. Theorem 3.1 replaces (ii) with a property that says the web must be lowest in terms of a stochastic ordering $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables satisfying (o) and (i). Theorem 4.1 replaces (ii) with a property which specifies the distribution of the number of distinct points at a fixed time, which come from paths that passed through some specified interval at a fixed earlier time.

4.3 θ-coupled Brownian webs

If $\overline{\mathcal{W}}$ is a Brownian web and (x,t) is a deterministic point in $\mathbb{R}^2$, let $W_{x,t}$ be a path belonging to $\overline{\mathcal{W}}$ and started from (x,t), which by Theorem 84 is almost surely unique. To clarify, we have $W_{x,t} = f$ for some $(f,t) \in \overline{\mathcal{W}}$ with $f(t) = x$ and there is almost surely one such $(f,t) \in \overline{\mathcal{W}}$ with $f(t) = x$. From here on we let $W_{x,t}$ be the almost surely unique path belonging to $\overline{\mathcal{W}}$ and started from some deterministic point $(x,t) \in \mathbb{R}^2$ and, for a second Brownian web $\overline{\mathcal{W}}'$, we let $W'_{y,u}$ be the almost surely unique path belonging to $\overline{\mathcal{W}}'$ and started from some deterministic point $(y,u) \in \mathbb{R}^2$.

We wish to describe a coupling of Brownian webs $\overline{\mathcal{W}}$ and $\overline{\mathcal{W}}'$ whose correlation is given by the distribution of pairs of paths with one taken from each of the webs. As with the coupling of many stochastic processes we need to have some co-adaptive property for the pair. We define what we mean by co-adapted in the context of Brownian webs.

Definition 85. A pair of Brownian webs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ are said to be co-adapted if, there exists a family of sub σ-algebras $(\mathcal{F}_t)_{t\in\mathbb{R}}$ of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s < t$, and for any set of $n+m$ deterministic points in $\mathbb{R}^2$, $(x_1,t_1),\ldots,(x_n,t_n),(y_1,u_n),\ldots,(y_m,u_m)$, the following properties hold:

- For all $1 \le i \le n$ and for $s > t_i$, $W_{x_i,t_i}(s)$ is $\mathcal{F}_s$ measurable and the process $(W_{x_i,t_i}(s+v) - W_{x,t}(s); v \ge 0)$ is independent of $\mathcal{F}_s$.
- For all $1 \le j \le m$ and for $s > t_j$, $W'_{y_j,u_j}(s)$ is $\mathcal{F}_s$ measurable and the process $(W'_{y_j,u_j}(s+v) - W'_{y_j,u_j}(s); v \ge 0)$ is independent of $\mathcal{F}_s$.

Our aim is to prove the following theorem:

Theorem 86. *There exists a* $(\mathcal{H}\times\mathcal{H}, \mathcal{F}_{\mathcal{H}}\otimes\mathcal{F}_{\mathcal{H}})$ *valued random variable* $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ *defined on the some probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *whose law is uniquely determined by the following properties:*

(i) $\overline{\mathcal{W}}$ *is a Brownian web and* $\overline{\mathcal{W}}'$ *is a Brownian web.*

(ii) $\overline{\mathcal{W}}$ *and* $\overline{\mathcal{W}}'$ *are co-adapted.*

(iii) For any pair of deterministic points in $\mathbb{R}^2$, (x,t) *and* (y,u)*, the distribution of the pair of paths* $((W_{x,t}(s+t\vee u), W'_{y,u}(s+t\vee u)); s\geq 0)$ *is that of a pair of* θ*-coupled Brownian motions.*

The random object $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ is then called a pair of θ-coupled Brownian webs.

Proof. Firstly to prove existence of such an object we construct a process with the desired properties.

Let $\mathcal{D} = \{(x_i, t_i, y_i, u_i); i \geq 1\}$ be a countable dense subset of $\mathbb{R}^4$ and let $\mathcal{C}_i = C([t_i,\infty),\mathbb{R}) \times C([u_i,\infty),\mathbb{R})$. For any finite subset of the natural numbers, F, with $|F|$ elements, let μ_F be the law of the an SCS as constructed in Section 4.1.4 starting from $((x_i,t_i),(y_i,u_i); i\in F)$. It is easy to see, by lemma 81, that the family of measures $\{\mu_F : F\subset \mathbb{N}, F \text{ finite}\}$ is consistent in the sense that for any two finite subsets of the natural numbers $F_2 \subset F_1$ and for all $A\in\mathcal{B}(\prod_{i\in F_2}\mathcal{C}_i)$

$$\mu_{F_2}(A) = \mu_{F_1}(\pi^{-1}_{F_1F_2}(A))$$

where $\pi_{F_1F_2} : \prod_{i\in F_1}\mathcal{C}_i \mapsto \prod_{i\in F_2}\mathcal{C}_i$ is the projection mapping $\pi((x_i; i\in F_1)) = (x_i; i\in F_2)$.

The Kolmogorov consistency theorem, see for example [Par67] Theorem 5.1, tells us that there exists a unique measure μ on the space

$$\prod_{i=1}^{\infty}(\mathcal{C}_i, \mathcal{B}(\mathcal{C}_i))$$

such that for any $A \in \mathcal{B}(\prod_{i\in F}\mathcal{C}_i)$, $\mu(A) = \mu_F(\pi_F^{-1}(A))$ where the mapping $\pi_F : \prod_{i=1}^{\infty}\mathcal{C}_i \mapsto \prod_{i\in F}\mathcal{C}_i$ is given by $\pi_F((x_i; i \in \mathbb{N})) = (x_i; i \in F)$.

There is continuous map h from the space $\mathcal{C}_i$ with the uniform topology to the metric space $(\Pi \times \Pi, d+d)$ given by the following: If (x, y) is an element of $\mathcal{C}_i$ then $h((x, y)) = ((f, t_i), (g, u_i))$ is an element of $\Pi \times \Pi$ with $f(t) = x(t)$ for all $t_i \le t < \infty$ and $g(t) = y(t)$ for all $u_i \le t < \infty$ and $f(\infty) = \lim_{t\to\infty} f(t)$ and $g(\infty) = \lim_{t\to\infty} g(t)$, where the limit is taken with respect to the metric ρ, see (4.3). Thus the limit in both cases is the point in $\bar{\mathbb{R}}^2$, which is the identified points of the form (x, ∞). That is, $\rho((f(t), t), (x, \infty)) \to 0$ as $t \to \infty$ for any f and any x, but in $\bar{\mathbb{R}}^2$ all points of the form (x, ∞) are identified.

Let $X \in \prod_{i=1}^{\infty}\mathcal{C}_i$ be a random object defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ whose law is given by the measure μ from above. The random object $h(X)$ is an $\mathcal{F}$-measurable random subset of the metric space $(\Pi \times \Pi, d+d)$ and we call this object $(\mathcal{W}(\mathcal{D}), \mathcal{W}'(\mathcal{D}))$ with $\mathcal{W}(\mathcal{D})$ being the collection of paths starting from $\{(x_i, t_i); i \ge 1\}$ and $\mathcal{W}'(\mathcal{D})$ being the collection of paths starting from $\{(y_i, u_i); i \ge 1\}$. It is clearly true that $\mathcal{D}^1 = \{(x_i, t_i); i \ge 1\}$ and $\mathcal{D}^2 = \{(y_i, u_i); i \ge 1\}$ are both countable dense sets in $\mathbb{R}^2$. It therefore follows, from Lemma 83, that $\mathcal{W}(\mathcal{D})$ and $\mathcal{W}'(\mathcal{D})$ are both Brownian skeletons, with starting sets $\mathcal{D}^1$ and $\mathcal{D}^2$ respectively, as defined in [FINR04], and whose construction is given as part of Section 4.1.1.

Let $\overline{\mathcal{W}}(\mathcal{D})$ be the closure of $\mathcal{W}(\mathcal{D})$ in the metric space (Π, d) and let $\overline{\mathcal{W}}'(\mathcal{D})$ be the closure of $\mathcal{W}'(\mathcal{D})$ in the metric space (Π, d). By propositions 3.1 - 3.3

in [FINR04], $\overline{\mathcal{W}}(\mathcal{D})$ and $\overline{\mathcal{W}}'(\mathcal{D})$ are both Brownian webs. This covers property (i) of the theorem.

Next we show that $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}'(\mathcal{D}))$ satisfies property (iii). For any deterministic choice of $(x,t,y,u) \in \mathbb{R}^4$, let $W_{x,t}$ (resp. $W'_{y,u}$) be a path in $\overline{\mathcal{W}}(\mathcal{D})$ (resp. $\overline{\mathcal{W}}'(\mathcal{D})$) started at (x,t) (resp. (y,u)), which by Theorem 84 is almost surely unique. We wish to show that the pair of processes $((W_{x,t}(s + u \vee t), W'_{y,u}(s + u \vee t)); s \geq 0)$ is equal in distribution to a pair of θ-coupled Brownian motions.

By Lemma 81, and corollary 82, the process $((W_{x,t}(s + u \vee t), W'_{y,u}(s + u \vee t)); s \geq 0)$ is a pair of θ-coupled Brownian motions for any choice of $(x,t,y,u) \in \mathcal{D}$. The result for any choice of $(x,t,y,u) \in \mathbb{R}^4$ follows by applying methods from the proof of Theorem 2.1 in [TW98]. Fix some point $(x,t,y,u) \in \mathbb{R}^4$. Then by lemma 8.1 of [TW98] there exists a sequence of points in $\mathcal{D}$, $((x_n, t_n, y_n, u_n); n \geq 1)$ such that for each n, $t_n < t$, $u_n < u$ and $\lim_{n\to\infty}(x_n, t_n, y_n, u_n) = (x,t,y,u)$, converging exponentially fast. Thus for all $\epsilon > 0$ there exists a constant $C < \infty$ such that for all $n \geq 1$,

$$\begin{aligned} &\mathbf{P}\left(W_{x,t}(s) = W_{x_n,t_n}(s) \text{ for all } s \geq t+\epsilon \right.\\ &\qquad\qquad \left. \text{and } W'_{y,u}(s) = W'_{y_n,u_n}(s) \text{ for all } s \geq u+\epsilon\right) \geq 1 - C\frac{2^{-n}}{\sqrt{\epsilon}}. \end{aligned} \tag{4.7}$$

Similarly if $C^n = (C_1^n, C_2^n, D_1^n, D_2^n)$ is an SCS with starting points $((x,t),(x_n,t_n),(y,u),(y_n,u_n))$ then by Lemma 83 and by elementary estimates on the distribution of Brownian motion hitting times there exists a constant

$C' < \infty$ with

$$\mathbf{P}\left(C_1^n(s) = C_2^n(s) \text{ for all } s \geq t + \epsilon \right.$$
$$\left. \text{and } D_1^n(s) = D_2^n(s) \text{ for all } s \geq u + \epsilon\right) \geq 1 - C'\frac{2^{-n}}{\sqrt{\epsilon}}. \quad (4.8)$$

For fixed integers $l, k \geq 1$ we fix k points in 'time' with $t_i' > t$ and k intervals (a_i, b_i), $i = 1, \dots k$ and we fix l points in 'time' with $u_j' > t$ and l intervals (c_j, d_j), $j = 1, \dots l$. We define the events

$$\begin{aligned}
E &= \left\{W_{x,t}(t_i') \in (a_i, b_i) \text{ and } W_{y,u}'(u_j') \in (c_j, d_j) : i = 1, \dots k,\ j = 1, \dots, l\right\} \\
E_n &= \left\{W_{x_n,t_n}(t_i') \in (a_i, b_i) \text{ and } W_{y_n,u_n}'(u_j') \in (c_j, d_j) : i = 1, \dots k,\ j = 1, \dots, l\right\} \\
\tilde{E}_n &= \left\{C_1^n(t_i') \in (a_i, b_i) \text{ and } D_1^n(u_j') \in (c_j, d_j) : i = 1, \dots k,\ j = 1, \dots, l\right\}.
\end{aligned} \quad (4.9)$$

We have the following triangle inequality:

$$|\mathbf{P}(E) - \mathbf{P}(\tilde{E}_n)| \leq |\mathbf{P}(E) - \mathbf{P}(E_n)| + |\mathbf{P}(E_n) - \mathbf{P}(\tilde{E}_n)|. \quad (4.10)$$

Fix $\epsilon > 0$ such that $\epsilon < \min\{t_i' - t, u_j' - u_j : i = 1 \dots, k, j \dots, l\}$ and then by (4.7), (4.8) and (4.10)

$$|\mathbf{P}(E) - \mathbf{P}(\tilde{E}_n)| \leq (C + C')\frac{2^{-n}}{\sqrt{\epsilon}}. \quad (4.11)$$

Note that $\mathbf{P}(\tilde{E}_n)$ does not depend on n and by lemma 81 and corollary 82, the process $((C_1^n(s + u \vee t), D_1^n(s + u \vee t)); s \geq 0)$ is a pair of θ-coupled Brownian motions for all n. Let $\mathbf{P}(\tilde{E}) = \mathbf{P}(\tilde{E}_n)$ and then letting $n \uparrow \infty$, (4.11) implies that

$$\mathbf{P}(E) = \mathbf{P}(\tilde{E})$$

which gives us that the finite dimensional distributions of $(W_{x,t}(.), W'_{y,u}(.))$ are equal to the finite dimensional distributions of $(C_1^n(.), D_1^n(.))$. These are the same for all n and so we have that

$$((W_{x,t}(s+u \vee t), W'_{y,u}(s+u \vee t)); s \geq 0)$$

is a pair of θ-coupled Brownian motions, and this now holds for any choice of $(x,t,y,u) \in \mathbb{R}^4$. This gives us property (iii) and indeed a similar argument to the above tells us that for any $n+m$ deterministic points in $\mathbb{R}^2$, $((x_1,t_1), \ldots, (x_n,t_n), (y_1,u_1), \ldots, (y_m,u_m))$, the collection of paths

$$(W_{x_1,t_1}, \ldots, W_{x_n,t_n}, W'_{y_1,u_1}, \ldots, W'_{y_m,u_m})$$

is an SCS with starting points $((x_1,t_1), \ldots, (x_n,t_n), (y_1,u_1), \ldots, (y_m,u_m))$. Then, letting

$$\mathcal{F}_s = \sigma(W_{x_1,t_1}(u \vee t_1), \ldots, W_{x_n,t_n}(u \vee t_n), W'_{y_1,u_1}(u \vee u_1), \ldots, W'_{y_m,u_m}(u \vee u_m); u \leq s)$$

, property (ii) follows from Markov property of an SCS.

We have just shown that there exists a process with properties (i), (ii) and (iii) in Theorem 86. To show that the law of such a process is uniquely determined first assume we have a second $(\mathcal{H} \times \mathcal{H}, \mathcal{F}_{\mathcal{H}} \otimes \mathcal{F}_{\mathcal{H}})$-valued random variable $(\overline{\mathcal{U}}, \overline{\mathcal{U}}')$ which satisfies properties (i),(ii) and (iii) in Theorem 86. Let $\mathcal{D}$ be the same countable subset of $\mathbb{R}^4$ as used in the construction of $(\mathcal{W}(\mathcal{D}), \mathcal{W}'(\mathcal{D}))$. Let $((x_1,t_1,y_1,u_1), \ldots, (x_n,t_n,y_n,u_n))$ be some deterministic finite subset of $\mathcal{D}$. For each i, let U_{x_i,t_i} be a path belonging to $\overline{\mathcal{U}}$ starting at (x_i,t_i), which by Theorem 84 is almost surely unique and we let U'_{y_i,u_i} be a path belonging to $\overline{\mathcal{U}}'$ starting at (y_i,u_i), which is also almost surely unique.

We wish to show that the properties of $(\overline{U}, \overline{U'})$ given in Theorem 86 imply that

$$U = (U_{x_1,t_1}, \ldots, U_{x_n,t_n}, U'_{y_1,u_1}, \ldots, U_{y_n,u_n})$$

is an SCS, of Section 4.1.4, started at

$$((x_1, t_1), \ldots, (x_n, t_n), (y_1, u_1), \ldots, (y_n, u_n)).$$

By property (i) of Theorem 86, and thus by property (i) of Theorem 84, for each i, the process

$$(U_{x_i,t_i}(t + t_i); t \geq 0)(\text{ resp. } (U'_{y_i,u_i}(y + u_i); t \geq 0))$$

is a Brownian motion started at x_i, (y_i) and by property (ii) of Theorem 86, there exists a family of σ-algebras $(\mathcal{F}_t; t \in \mathbb{R})$ such that

$$(U_{x_i,t_i}(t_i + t); t \geq 0)(\text{ resp. } (U'_{y_i,u_i}(u_i + t); t \geq 0))$$

is an $\mathcal{F}_{t+t_i}(\mathcal{F}_{t+u_i})$-Brownian motion.

By the Markov property for Brownian motion for any $j, k \geq i$ the process

$$(U_{x_i,t_i}(t + (t_j \vee u_k)); t \geq 0)(\text{ resp. } (U'_{y_i,u_i}(t + (t_j \vee u_k)); t \geq 0))$$

is an $\mathcal{F}_{t+(t_j \vee u_k)}$-Brownian motion started at $U_{x_i,t_i}(t_j \vee u_k)$, $((U'_{y_i,u_i}(t_j \vee u_k))$. Also, for $(i \leq j, k)$, property (i) of Theorem 84, together with the Markov property give us that the process

$$((U_{x_i,t_i}(t + t_j \vee u_k), U_{x_j,t_j}(t + t_j \vee u_k)); t \geq 0)$$
$$(\text{ resp. } ((U'_{y_i,u_i}(t + t_j \vee u_k), U'_{y_k,u_k}(t + t_j \vee u_k)); t \geq 0))$$

is equal in distribution to a pair of coalescing Brownian motions. By property (iii) of Theorem 86, the process $((U_{x_i,t_i}(t+t_j \vee u_k), U'_{x_j,t_j}(t+t_j \vee u_k)); t \geq 0)$ is equal in distribution to a θ-coupled Brownian motion. Therefore

$$(U_{x_1,t_1}(t+t_j \vee u_k), \ldots, U_{x_j,t_j}(t+t_j \vee u_k), U'_{y_1,u_1}(t+t_j \vee u_k), \ldots, U'_{y_j,u_j}(t+t_j \vee u_k))$$

satisfies all the properties of Theorem 76 required to uniquely specify its law. Hence, if s_i, $k(i)$ and $l(i)$ are as defined in 4.1.4 then, for each i, the process

$$(U_{x_1,t_1}(t+s_i), \ldots, U_{x_{k(i)},t_{k(i)}}(t+s_i), U'_{y_1,u_1}(t+s_i), \ldots, U'_{y_{l(i)},u_{l(i)}}(t+s_i))$$

is equal in distribution to the process given in Theorem 76 started at

$$(U_{x_1,t_1}(s_i), \ldots, U_{x_{k(i-1)},t_{k(i-1)}}(s_i), x_{k(i-1)+1}, \ldots, x_{k(i)},$$
$$U'_{y_1,u_1}(s_i), \ldots, U'_{y_{l(i-1)},u_{l(i-1)}}, y_{l(i-1)+1}, \ldots, y_{l(i)})$$

and so $U = (U_{x_1,t_1}, \ldots, U_{x_n,t_n}, U'_{y_1,u_1}, \ldots, U_{y_n,u_n})$ is an SCS.

Let $\mathcal{D}$ now be ordered as before in the construction of $(\mathcal{W}(\mathcal{D}), \mathcal{W}'(\mathcal{D}))$ $\mathcal{D} = \{(x_i, t_i, y_i, u_i); i \geq 1\}$. For any finite subset of the natural numbers, F, the process $(U_{x_i,t_i}, U'_{y_i,u_i}; i \in F)$ is equal in distribution to an SCS with starting values $((x_i,t_i),(y_i,u_i); i \in F)$. By the Kolmogorov extension theorem $(\mathcal{U},\mathcal{U}') = \{U_{x,t}, U_{y,u}; (x,t,y,u) \in \mathcal{D}\}$ is equal to the law of $(\mathcal{W}(\mathcal{D}), \mathcal{W}'(\mathcal{D}))$. Finally by property (ii) of Theorem 84, $\overline{\mathcal{U}}$ is the closure of U and $\overline{\mathcal{U}}'$ is the closure of U' hence $(\overline{\mathcal{U}},\overline{\mathcal{U}}')$ is equal in law to $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}'(\mathcal{D}))$.

$\square$

4.4 (p, n)-coupled Brownian webs and convergence

In this section we construct a pair of coupled Brownian webs whose interaction is given by the fact that if we choose one path in each of the Brownian webs, then the pair of paths together behave as a pair of (p, n)-coupled Brownian motions as given in proposition 23. We call such a pair of webs a pair of (p, n)-coupled Brownian webs. Choose p to vary with n such that $p = p(n)$ satisfies $\lim_{n\to\infty} 2\sqrt{\frac{n}{\pi}}p(n) = \theta \in (0, \infty)$. The a pair of (p, n)-coupled Brownian webs converges in law to a pair of θ-coupled Brownian webs.

4.4.1 Constructing a pair of (p, n)-coupled Brownian webs

Let $(x_1, t_1), \ldots, (x_N, t_N), (y_1, u_1), \ldots, (y_M, u_M)$ be $N+M$ deterministic points in $\mathbb{R}^2$. Assume without loss of generality that $t_1 \leq t_2 \leq \cdots \leq t_N$ and $u_1 \leq u_2 \leq \cdots \leq u_M$. Fix $n \geq 1$. For $i \in \mathbb{Z}$ let $s_i = \frac{i}{n}$, $k(i) = \max\{k; t_k \leq s_i\}$ and $l(i) = \max\{k; u_k \leq s_i\}$.

Let $(Y_i; i \in \mathbb{Z})$ be a collection of i.i.d. Bernoulli random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$. That is

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1 - p. \end{cases}$$

For each $i \in \mathbb{Z}$ we construct $k(i)+l(i)$ paths, $\Upsilon^i = (C_1^i, \ldots, C_{k(i)}^i, D_1^i, \ldots, D_{l(i)}^i)$ by the following rules:

If $Y_i = 0$, let Υ^i,conditional on $(\Upsilon^j; j < i)$, be distributed as a system of coalescing Brownian motions started from

$$((C_1^{i-1}(s_i), s_i), \ldots, (C_{k(i-1)}^{i-1}(s_i), s_i), (x_{k(i-1)+1}, t_{k(i-1)+1}), \ldots, (x_{k(i)}, t_{k(i)})$$
$$(D_1^{i-1}(s_i), s_1), \ldots, (D_{l(i-1)}^{i-1}(s_i), s_i), (y_{l(i-1)+1}, u_{l(i-1)+1}), \ldots, (y_{l(i)}, u_{l(i)}))$$

as described in Section 4.1.1.

If $Y_i = 1$, let $(C_1^i, \ldots, C_{k(i)}^i)$,conditional on $(\Upsilon^j; j < i)$, be distributed as a system of coalescing Brownian motions started from

$$((C_1^{i-1}(s_i), s_i), \ldots, (C_{k(i-1)}^{i-1}(s_i), s_i), (x_{k(i-1)+1}, t_{k(i-1)+1}), \ldots, (x_{k(i)}, t_{k(i)})$$

and let $(D_1^i, \ldots, D_{l(i)}^i)$,conditional on $(\Upsilon^j; j < i)$, be distributed as a system of coalescing Brownian motions, started from

$$((D_1^{i-1}(s_i), s_1), \ldots, (D_{l(i-1)}^{i-1}(s_i), s_i), (y_{l(i-1)+1}, u_{l(i-1)+1}), \ldots, (y_{l(i)}, u_{l(i)})$$

and let $(D_1^i, \ldots, D_{l(i)}^i)$ and $(C_1^i, \ldots, C_{k(i)}^i)$ be conditionally independent given $(\Upsilon^j; j < i)$.

We now construct a collection of $N+M$ paths $\Upsilon = (C_1, \ldots, C_N, D_1, \ldots, D_M)$ started at $((x_1, t_1), \ldots, (x_N, t_N), (y_1, u_1), \ldots, (y_M, u_M))$.

For $k \in \mathbb{Z}$ and for $t \geq t_i$

$$C_i(t) = C_i^k(t - (k-1)/n), \qquad \frac{k-1}{n} \leq t \leq \frac{k}{n}.$$

For $k \in \mathbb{Z}$ and for $t \geq u_i$

$$D_i(t) = D_i^k(t - (k-1)/n), \qquad \frac{k-1}{n} \leq t \leq \frac{k}{n}.$$

It is possible to see that $(C_1, \ldots, C_N)$ is a system of coalescing Brownian motions started from $((x_1, t_1), \ldots, (x_N, t_N))$ and $(D_1, \ldots, D_M)$ is a system of a coalescing Brownian motions started at $((y_1, u_1), \ldots, (y_M, u_M))$. The following lemma shows the collection of paths constructed above have a consistency property as we vary the number of starting points.

Lemma 87. *For fixed $n \geq 1$ and $p \in [0,1]$, if $\Upsilon = (C_1, \ldots, C_N, D_1, \ldots, D_M)$ is a collection of $N + M$ paths as constructed above then*

$$\Upsilon' = (C_1, \ldots, C_{j-1}, C_{j-1}, C_N, D_1, \ldots, D_M)$$

is equal in law to a collection of $N-1+M$ paths as constructed above started at $((x_1, t_1), \ldots, (x_{j-1}, t_{j-1}), (x_{j+1}, t_{j+1}), \ldots, (x_N, t_N), (y_1, u_1), \ldots, (y_M, u_M))$ and

$$\Upsilon'' = (C_1, \ldots, C_N, D_1, \ldots, D_{k-1}, D_{k+1}, \ldots, D_M)$$

is equal in law to a collection of $N+M-1$ paths as constructed above started at $((x_1, t_1), \ldots, (x_N, t_N), (y_1, u_1), \ldots, (y_{k-1}, u_{k-1}), (y_{k+1}, u_{k+1}), \ldots, (y_M, u_M))$.

Proof. By the symmetry of the construction we only need to consider the first case. Fix the collection of Bernoulli trials $(Y_i; i \in \mathbb{Z})$, then for each i such that $Y_i = 0$, $(C_1^i, \ldots, C_{j-1}^i, C_{j-1}^i, C_{k(i)}^i, D_1^i, \ldots, D_{l(i)}^i)$ is equal to a system of coalescing Brownian motions. This can easily be seen using Lemma 81. For each i such that $Y_i = 1$, $(C_1^i, \ldots, C_{j-1}^i, C_{j+1}^i, \ldots C_{k(i)}^i)$ is a system of coalescing Brownian motions again by Lemma 81 and of course $(D_1^i, \ldots, D_{l(i)}^i)$ is a system of coalescing Brownian motions. Finally $(C_1^i, \ldots, C_{j-1}^i, C_{j+1}^i, \ldots C_{k(i)}^i)$ and $(D_1^i, \ldots, D_{l(i)}^i)$ are conditionally independent given $(C^j; j < i)$. $\square$

Lemma 88. *If $(C_1, \ldots, C_N, D_1, \ldots, D_M)$ is a collection of $N + M$ paths as constructed above defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t; t \in R)$*

is a collection of sub σ-algebras of $\mathcal{F}$ with $\mathcal{F}_t = \sigma(C_1(s \vee t_1), \dots, C_N(s \vee t_N), D_1(s \vee u_1), \dots, D_M(s \vee u_M); s \leq t)$, then for each i and for all t, $C_i(t)$ is $\mathcal{F}_t$ measurable and $(C_i(t+s) - C_i(t); s \geq 0)$ is independent of $\mathcal{F}_t$.

Proof. Measurability is obvious and independence follows by observing the conditional nature of the construction of the collection of paths, the Markov property of a system of coalescing Brownian motions and the independent increments property of Brownian motion. □

Let $\mathcal{D} = \{(x_i, t_i, y_i, u_i); i \geq 1\}$ be an ordered countable dense subset of $\mathbb{R}^4$. Comparing with the construction of $(\mathcal{W}(\mathcal{D}), \mathcal{W}'(\mathcal{D}))$ given in the proof of Theorem 86, it is possible to see, using the above properties of
$\Upsilon = (C_1, \dots, C_N, D_1, \dots, D_M)$ and the Kolmogorov consistency theorem, that there exists a random object $(\mathcal{W}(\mathcal{D}), \mathcal{W}^n(\mathcal{D}))$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ that is a subset of the space $(\Pi \times \Pi, d + d)$ with the following properties: $\mathcal{W}(\mathcal{D})$ is a collection of paths started from $\mathcal{D}_1 = \{(x_i, t_i); i \geq 1\}$, $\mathcal{W}^{(n)}(\mathcal{D})$ is the collection of paths started at $\mathcal{D}_2 = \{(y_i, u_i); i \geq 1\}$ and for any finite subset F of $\mathbb{N}$ with $|F|$ elements, the collection of paths in $(\mathcal{W}(\mathcal{D}), \mathcal{W}^{(n)}(D))$ started at $((x_i, t_i), (y_i, u_i); i \in F)$ has the same law as $\Upsilon = (C_1, \dots, C_{|F|}, D_1, \dots, D_{|F|})$, the collection of paths constructed above started at $((x_i, t_i), (y_i, u_i); i \in F)$.

We now let $\overline{\mathcal{W}}(\mathcal{D})$ be the closure of $\mathcal{W}(\mathcal{D})$ in (Π, d) and we let $\overline{\mathcal{W}}^{(n)}(\mathcal{D})$ be the closure of $\mathcal{W}^{(n)}(\mathcal{D})$. So that $\overline{\mathcal{W}}(\mathcal{D})$ and $\overline{\mathcal{W}}^{(n)}(\mathcal{D})$ are both equal in law to the Brownian web. We call any $(\mathcal{H} \times \mathcal{H}, \mathcal{F}_{\mathcal{H}} \otimes \mathcal{F}_{\mathcal{H}})$-valued random variable whose law is equal to that of the random object $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ a pair of (p, n)-coupled Brownian webs.

4.4.2 Convergence to θ-coupled Brownian motions

Theorem 89. *Let $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ be a pair of $(p(n), n)$-coupled Brownian webs, with $p(n)$ satisfying $\lim_{n\to\infty} 2\sqrt{\frac{n}{\pi}}p(n) = \theta \in (0,\infty)$. Then the law of $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ converges as n tends down to zero to the law of a pair of θ-coupled Brownian webs.*

Proof. It has been shown in [FINR04] that the metric space $(\mathcal{H}, d_{\mathcal{H}})$ is complete and separable, therefore any measure on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ is a tight measure, see [Bil99]. Lemma 22 in chapter 2 shows that any family of measures $(\mu_n; n \geq 1)$ on a product space is tight if both families of marginal distributions are tight. As the marginal distributions of $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ are tight for all $n \geq 1$ then the family of distributions $((\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D})); n \geq 1)$ is indeed tight.

Assume that we have some subsequence $(n_k; k \geq 1)$ such that $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n_k)}(\mathcal{D}))$ converges in distribution to $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ for some $(\mathcal{H}\times\mathcal{H}, \mathcal{F}_{\mathcal{H}}\times \mathcal{F}_{\mathcal{H}})$-valued random variable $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$.

The function which maps $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ to $((W_{x,t}(s), W^{(n)}_{y,u}(s)); s \geq 0)$ is measurable with respect to $\mathcal{F}_{\mathcal{H}}$ and $\mathcal{B}(C([t,\infty),\mathbb{R}) \times C([u,\infty),\mathbb{R}))$ hence we can apply a useful lemma which can be found in [RY99] (lemma 0.5.7). This gives us that if $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n_k)}(\mathcal{D}))$ converges in law to $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ then $((W_{x,t}(s), W^{(n_k)}_{y,u}(s)); s \geq 0)$ converges in law to $((W_{x,t}(s), W'_{y,u}(s)); s \geq 0)$.

Using events of a form similar to those in (4.9) and methods used in the proof of Theorem 86 it is possible to show that for any deterministic point $(x,t,y,u) \in \mathbb{R}^4$ the distribution of $((W_{x,t}(s+t\vee u), W^{(n)}_{y,u}(s+t\vee u)); s \geq 0)$ is equal to that of a pair of $(p(n), n)$-coupled Brownian motions, as described in proposition 23. Indeed it is possible to show that for any $N+M$ deterministic points in $\mathbb{R}^2$, $((x_1,t_1),\ldots,(x_N,t_N),(y_1,u_1),\ldots,(y_M,u_M))$,

$$(W_{x_1,t_1},\ldots,W_{x_N,t_N},W^{(n)}_{y_1,u_1},\ldots,W^{(n)}_{y_1,u_1})$$

is a collection of paths as constructed at the beginning of Section 4.4.1. Thus by lemma 88, $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ is co-adapted with $\mathcal{F}_s = \sigma(W_{x_1,t_1}(u \vee t_1), \ldots, W_{x_N,t_N}(u \vee t_N), W^{(n)}_{y_1,u_1}(u \vee u_1), \ldots, W^{(n)}_{y_M,u_M}(u \vee u_m); u \leq s)$. It follows that $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}'(\mathcal{D}))$ is co-adapted with $\mathcal{F}_s = \sigma(W_{x_1,t_1}(u \vee t_1), \ldots, W_{x_N,t_N}(u \vee t_N), W'_{y_1,u_1}(u \vee u_1), \ldots, W'_{y_M,u_M}(u \vee u_M); u \leq s)$ and hence $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ satisfies property (ii) of Theorem 86.

By Proposition 23 the law of a pair of $(p(n), n)$-coupled Brownian motions converges to the law of a pair of θ-coupled Brownian motions hence $((W_{x,t}(s), W'_{y,u}(s)); s \geq 0)$ is equal in law to a pair of θ-coupled Brownian motions. As this is true for any deterministic choice of $(x, t, y, u) \in \mathbb{R}^4$, $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ satisfies property (iii) of Theorem 86.

For every n the marginals of $(\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D}))$ are both Brownian webs so it immediately follows that $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ satisfies property (i) of Theorem 86.

Hence every subsequence of $((\overline{\mathcal{W}}(\mathcal{D}), \overline{\mathcal{W}}^{(n)}(\mathcal{D})); n \geq 1)$ that converges in law at all, must converge to the law of a pair of θ-coupled Brownian webs, hence by the corollary of Theorem 5.1 of [Bil99] the entire sequence must converge in law to a pair of θ-coupled Brownian motions. □

4.5 Brownian web triples

Consider three Brownian Webs $\overline{\mathcal{W}}$, $\overline{\mathcal{W}}'$ and $\overline{\mathcal{W}}^{(n)}$ such that $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^{(n)})$ is distributed as a pair of (p, n)-coupled Brownian webs, $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ is distributed as a pair of θ_1-coupled Brownian webs and $\overline{\mathcal{W}}'$ and $\overline{\mathcal{W}}^{(n)}$ are independent given $\overline{\mathcal{W}}$. The pair $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^{(n)})$ converges to a pair of θ_2-coupled Brownian webs by Theorem 89. Consider an almost surely unique path in $\overline{\mathcal{W}}'$ started at some fixed point $(x, t) \in \mathbb{R}^2$,$W' = W'_{x,t}$ and a path in $\overline{\mathcal{W}}^{(n)}$ started at (y, u), $W^{(n)} = W^{(n)}_{y,u}$. Over each interval $k/n \leq t \leq (k+1)/n$ the pair $((W^{(n)}(t), W'(t)) : k/n \leq t \leq (k+1)/n)$ behaves as a pair of θ_1-coupled Brownian motions with probability p

and a pair of coalescing Brownian motions with probability $1-p$. If $p=p(n)$, with $p(n)$ satisfying $\lim_{n\to\infty} 2\sqrt{\frac{n}{\pi}}p(n) = \theta \in (0,\infty)$ then, from Proposition 26, it follows that the pair $(W', W^{(n)})$ converges in distribution to a pair of $\theta_1+\theta_2$-coupled Brownian motions. Similarly $(\overline{\mathcal{W}}', \overline{\mathcal{W}}^{(n)})$ converges in law to a pair of $\theta_1+\theta_2$ coupled Brownian webs. Moreover the triple $(\overline{\mathcal{W}}, \overline{\mathcal{W}}', \overline{\mathcal{W}}^{(n)})$ converges in law to $(\overline{\mathcal{W}}, \overline{\mathcal{W}}', \overline{\mathcal{W}}'')$, where $\overline{\mathcal{W}}''$ and $\overline{\mathcal{W}}'$ are conditionally independent given $\overline{\mathcal{W}}$. Thus we have the following theorem, a complete proof of which is given in [HW07].

Theorem 90. *Let $\overline{\mathcal{W}}$, $\overline{\mathcal{W}}'$ and $\overline{\mathcal{W}}''$ be three Brownian webs such that*

- $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ *is distributed as a pair of θ_1-coupled Brownian webs,*
- $(\overline{\mathcal{W}}, \overline{\mathcal{W}}'')$ *is distributed as a pair of θ_2-coupled Brownian web,*
- $\overline{\mathcal{W}}'$ *and* $\overline{\mathcal{W}}''$ *are conditionally independent given* $\overline{\mathcal{W}}$.

Then $(\overline{\mathcal{W}}', \overline{\mathcal{W}}'')$ is distributed as a pair of $\theta_1+\theta_2$-coupled Brownian webs.

Let $(\overline{\mathcal{W}}, \overline{\mathcal{W}}', \overline{\mathcal{W}}'')$ be three Brownian webs as described in the theorem above, with $\theta_1=\theta_2=\theta$. As discussed in the introduction we can define a stochastic flow of kernels via

$$K_{s,t}(x,A) = \mathbf{P}(W'_{x,s}(t) \in A|\overline{\mathcal{W}}),$$

where $W'_{x,s}$ is the almost surely unique path in $\overline{\mathcal{W}}'$ started from (x,s). The N-point motion of the stochastic flow K is a Markov process on $\mathbb{R}^N$ with transition semigroup given by

$$P_t^N(x,A) = \mathbf{E}[K_{0,t}(x_1,A_1)K_{0,t}(x_2,A_2)\cdots K_{0,t}(x_N,A_N)]$$

for $x=(x_1,x_2,\dots x_N)\in\mathbb{R}^N$ and $A=A_1\times A_2\times\cdots A_N\in\mathcal{B}(\mathbb{R}^N)$. The one point motion is given by

$$P_t^1(x,A) = \mathbf{E}[K_{0,t}(x,A)] = \mathbf{P}(W'_{x,0}(t) \in A).$$

Thus the one-point motion of K is a Brownian motion on $\mathbb{R}$.

Note that as the pairs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ and $(\overline{\mathcal{W}}, \overline{\mathcal{W}}'')$ are both θ-coupled Brownian webs, $K_{s,t}(x,A) = \mathbf{P}(W''_{x,s}(t) \in A|\overline{\mathcal{W}})$. Thus the two point motion of K is given by

$$\begin{aligned} P_t^2(x,A) &= \mathbf{E}[K_{0,t}(x_1,A_1)K_{0,t}(x_2,A_2)] \\ &= \mathbf{E}\left[\mathbf{P}(W'_{x_1,0}(t) \in A_1|\overline{\mathcal{W}})\mathbf{P}(W''_{x_2,0}(t) \in A_2|\overline{\mathcal{W}})\right] \\ &= \mathbf{P}(\{W'_{x_1,0}(t) \in A_1\} \cap \{W''_{x_2,0}(t) \in A_2\}), \end{aligned}$$

the last equality coming from the conditional independence of $\overline{\mathcal{W}}'$ and $\overline{\mathcal{W}}''$ given $\overline{\mathcal{W}}$. Thus the two-point motion of K is that of a pair of 2θ-coupled Brownian motions.

Now define N Brownian webs $\overline{\mathcal{W}}^1, \ldots, \overline{\mathcal{W}}^N$ such that for each i the pair $(\overline{\mathcal{W}}, \overline{\mathcal{W}}^i)$ is a pair of θ-coupled Brownian webs and the webs $(\overline{\mathcal{W}}^i : i \in \{1, \ldots, N\})$ are conditionally independent of each other given the $\overline{\mathcal{W}}$. The N-point motion of K is then given by

$$P_t^N(x,A) = \mathbf{P}\left(\bigcap_{i=1}^N \{W^i{}_{x_i,0}(t) \in A_i\}\right).$$

By considering (p,n)-coupled webs it is shown in [HW07] that the N-point motion of K is given by the solution of the $\mathcal{A}_N^\theta$-martingale problem of chapter

3 with the family of parameters θ given by

$$\theta(k:l)=\begin{cases}\theta & k=l=1\\ \frac{\theta}{2} & k=1,l\geq 2 \text{ or } k\geq 2,l=1\\ -(k-1)\frac{\theta}{2} & k\geq 1,l=0\\ -(l-1)\frac{\theta}{2} & k=0,l\geq 1\\ 0 & \text{otherwise.}\end{cases} \tag{4.12}$$

We call the stochastic flow of kernels, K which is characterised by this $\mathcal{A}_N^\theta$-martingale problem an erosion flow, as any group of $n\geq 2$ particles can only split, instantaneously, into a single particle and a group of $n-1$ particles. To give a heuristic reasoning behind this the occurrence of this particular $\mathcal{A}_N^\theta$-martingale problem when conditioning one Brownian web on its θ-coupled partner, we consider the discrete time arrow process described in chapter 1.

Let $\mathcal{S}(u_1)$ and $\mathcal{S}(u_2)$ be two systems of coalescing simple symmetric random walks as described in chapter 1. At any particular point $(k,n)\in L$,

$$\begin{aligned}\mathbf{P}(\xi_{k,n}(u_2)=1|\xi_{k,n}(u_1)=1)=\mathbf{P}(\xi_{k,n}(u_2)=-1|\xi_{k,n}(u_1)=-1)\\ =\frac{1}{2}\left(1+e^{-2|u_2-u_1|}\right),\end{aligned} \tag{4.13}$$

whereas

$$\begin{aligned}\mathbf{P}(\xi_{k,n}(u_2)=-1|\xi_{k,n}(u_1)=1)=\mathbf{P}(\xi_{k,n}(u_2)=1|\xi_{k,n}(u_1)=-1)\\ =\frac{1}{2}\left(1-e^{-2|u_2-u_1|}\right).\end{aligned} \tag{4.14}$$

Each pair of arrows $(\xi_{k,n}(u_1),\xi_{k,n}(u_2))$ is mutually independent of every other pair and $\mathbf{P}(\xi_{k,n}(u_1)=1)=\mathbf{P}(\xi_{k,n}(u_1)=-1)=\frac{1}{2}$. Thus the conditioned system of arrows, $\mathcal{S}(u_2)|\mathcal{S}(u_1)$, can be seen to be equivalent to a system of weighted arrows with random weights given by

$$Q_{n,k} = \begin{cases} \frac{1}{2}\left(1 - e^{-2|u_2-u_1|}\right) & \text{with probability } \frac{1}{2} \\ \frac{1}{2}\left(1 + e^{-2|u_2-u_1|}\right) & \text{with probability } \frac{1}{2}. \end{cases}$$

Let $\mathcal{S}^\epsilon(u_1)$, be the collection of paths $\mathcal{S}(u_1)$ under a diffusive scaling (time scaled by ϵ and space scaled by $\sqrt{\epsilon}$). We define $\mathcal{S}^\epsilon(u_2)$ similarly. Let $|u_2-u_1| = \theta$, then as described in chapter 1 we would expect the pair $(\mathcal{S}^\epsilon(\sqrt{\epsilon}u_1), \mathcal{S}^\epsilon(\sqrt{\epsilon}u_2))$ to converge to a pair of θ-coupled Brownian webs. Under this scaling the weighted system of arrows produced from conditioning has weights given by

$$Q_{n,k} = \begin{cases} \theta\sqrt{\epsilon} & \text{with probability } \frac{1}{2} \\ 1 - \theta\sqrt{\epsilon} & \text{with probability } \frac{1}{2} \end{cases} \tag{4.15}$$

for small ϵ, where $Q_{n,k}$ is the weight of the arrow pointing upwards from $(n,k) \in L$.

In this system of weighted arrows if we have $k+l$ particles at some point $(n,k) \in L$, the probability that these $k+l$ particles separate with k particles moving up and l particles moving down is given by $\mathbf{E}[Q_{n,k}^k(1-Q_{k,n})^l]$. It follows from this that in the limit the rate of separation of $k+l$ particles into k up and l down is given by

$$\theta(k:l) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{\epsilon}}\mathbf{E}[Q_{n,k}^k(1-Q_{k,n})^l]. \tag{4.16}$$

Applying this to (4.15) gives a family $(\theta(k:l) : k,l \geq 1)$ which satisfies (4.12). The remaining parameters can be found via the consistency property (3.3) and the fact that no drift implies that $\theta(0:1) = \theta(1:0) = 0$. Compare this concept with the approximation scheme of Section 3.6 and in particular Theorem 75.

4.6 Extensions

Proposition 16 of chapter 2 tells us that we can characterise a pair of θ-coupled Brownian motions (X, X') started from $(x_1, x_2) \in \mathbb{R}^2$, with the extra property that for some β_1 and β_2 such that $|\beta_1 - \beta_2| \leq 2\theta$, X has a drift β_1 and X' has drift β_2. Using Proposition 16 and the same methods as used in Section 4.3 we can characterise a pair of Brownian webs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}') \in \mathcal{H} \times \mathcal{H}$ such that each path in $\overline{\mathcal{W}}$ started from some deterministic point $(x, t) \in \mathbb{R}^2$ is Brownian motion started at (x, t) with drift β_1, each path in $\overline{\mathcal{W}}$ started from some deterministic point is Brownian motion with drift β_2 and a pair of paths, one from each web, has the distribution of the pair of paths given in Proposition 16.

We consider the N-point motion of the stochastic flow K given by

$$K_{s,t}(x, A) = \mathbf{P}(W'_{x,s}(t) \in A|\overline{\mathcal{W}}).$$

Going back to the pair of coalescing systems of random walks $(\mathcal{S}, \mathcal{S}')$, we consider the direction of the pair $(\xi_{n,k}, \xi'_{n,k})$ of arrows that point from some point $(n, k) \in L$. We want the distribution of a single path in $\mathcal{S}^\epsilon$ to converge to a Brownian motion with drift β_1. This corresponds to the distribution of each arrow being given by

$$\mathbf{P}(\xi^\epsilon_{n,k} = 1) = 1 - \mathbf{P}(\xi^\epsilon_{n,k} = -1) = \frac{1}{2} + \frac{\beta_1}{2}\sqrt{\epsilon}.$$

Similarly we must have

$$\mathbf{P}({\xi'}^\epsilon_{n,k} = 1) = 1 - \mathbf{P}({\xi'}^\epsilon_{n,k} = -1) = \frac{1}{2} + \frac{\beta_2}{2}\sqrt{\epsilon}$$

asymptotically as $\epsilon \downarrow 0$. In order to achieve the θ-coupled property of a path in each system we must have $\lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{\epsilon}}\mathbf{P}(\xi^{\epsilon}_{n,k} \neq \xi'^{\epsilon}_{n,k}) = \theta$. These three conditions lead to the following conditional distributions

$$1 - \mathbf{P}(\xi'_{n,k} = 1 | \xi_{n,k} = 1) = \mathbf{P}(\xi'_{n,k} = -1 | \xi_{n,k} = 1) \approx \frac{\sqrt{\epsilon}}{2}(2\theta + \beta_1 - \beta_2)$$

whereas

$$\mathbf{P}(\xi'_{n,k} = 1 | \xi_{n,k} = -1) = 1 - \mathbf{P}(\xi'_{n,k} = -1 | \xi_{n,k} = -1) \approx \frac{\sqrt{\epsilon}}{2}(2\theta + \beta_2 - \beta_1)$$

Note that in order that these probabilities are valid we must have the same restriction that is enforced in Proposition 16, that is $|\beta_1 - \beta_2| \leq 2\theta$. The conditioned arrow system, $\mathcal{S}'|\mathcal{S}$, can be considered as a system of weighted arrows with the weight of the arrow pointing upwards from $(n,k) \in L$ given by

$$Q_{n,k} = \begin{cases} 1 - \frac{\sqrt{\epsilon}}{2}(2\theta + \beta_1 - \beta_2) & \text{with prob. } \frac{1}{2} + \frac{\beta_1}{2}\sqrt{\epsilon} \\ \frac{\sqrt{\epsilon}}{2}(2\theta + \beta_2 - \beta_1) & \text{with prob. } \frac{1}{2} - \frac{\beta_1}{2}\sqrt{\epsilon}. \end{cases}$$

Then using (4.16) we conjecture that the distribution of the N-point motion of K solves the $\mathcal{A}^{\theta}_{N}$-martingale problem with the family of parameters $(\theta(k : l); k, l \geq 1)$ given by

$$\theta(k:l) = \begin{cases} \frac{2\theta+\beta_1-\beta_2}{4} & k > 1, l = 1 \\ \frac{2\theta+\beta_2-\beta_1}{4} & k > 1, l = 1 \\ \frac{\theta}{2} & k = l = 1 \\ 0 & k, l \geq 2. \end{cases}$$

The assumption that $\theta(0:0) = 0$ means that $\theta(1:0) = -\theta(0:1) = \frac{\beta_2}{2}$. Then

$\theta(k:0)$ and $\theta(0:l)$ can be found from the consistency property (3.3). Thus

$$\theta(k:0) = \frac{\beta_2}{2} - \theta - (k-2)\frac{2\theta + \beta_1 - \beta_2}{4}, \qquad k \geq 2$$

and

$$\theta(0:l) = -\frac{\beta_2}{2} - \theta - (l-2)\frac{2\theta + \beta_1 - \beta_2}{4}, \qquad l \geq 2.$$

This process then has the property that when $n \geq 3$ particles are together they can separate asymmetrically meaning it is more likely that one particle moves up and $n-1$ move down rather than the other way round. The extreme case being when $|\beta_1 - \beta_2| = 2\theta$. For example when $\beta_1 = -\beta_2 = \theta$ then groups of $n \geq 3$ particles can only separate in a way such that $n-1$ particles move up and the remaining particle moves down.

Let us return to the triple of webs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}', \overline{\mathcal{W}}'')$ with $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ being a pair of θ_1-coupled webs, $(\overline{\mathcal{W}}, \overline{\mathcal{W}}'')$ a pair of θ_2-coupled Brownian webs, and $\overline{\mathcal{W}}'$ and $\overline{\mathcal{W}}''$ conditionally independent given $\overline{\mathcal{W}}$. We can then investigate the motion of N-point motion of a stochastic flow K defined by

$$K_{s,t}(x, A) = \mathbf{P}(W_{x,s}(t) \in A | \overline{\mathcal{W}}', \overline{\mathcal{W}}'').$$

We use the method, as before, of observing the behaviour of the conditioned system of arrows to give a heuristic answer. In order that the systems of arrows

$(\mathcal{S}, \mathcal{S}', \mathcal{S}'')$ would have the properties described above in the limit we need

$$\begin{aligned}
\mathbf{P}(\xi'_{n,k} = \xi''_{n,k}) &\approx \frac{1}{2} - \frac{\sqrt{\epsilon}}{2}(\theta_1 + \theta_2) \\
\mathbf{P}(\xi_{n,k} = \xi'_{n,k} | \xi'_{n,k} = \xi''_{n,k}) &= 1 - O(\epsilon) \\
\mathbf{P}(\xi_{n,k} = 1 | \xi'_{n,k} = 1 \neq \xi''_{n,k}) &\approx \frac{2\theta_2}{2(\theta_1 + \theta_2)} \\
\mathbf{P}(\xi_{n,k} = 1 | \xi''_{n,k} = 1 \neq \xi'_{n,k}) &\approx \frac{2\theta_1}{2(\theta_1 + \theta_2)}.
\end{aligned}$$

.

This leads to expectation that the N-point motion of K solves the $\mathcal{A}_N^\theta$ martingale problem with $(\theta(k:l) : k, l \geq 1)$ given by

$$\theta(k:l) = \frac{(2\theta_1)^k(2\theta_2)^l + (2\theta_1)^l(2\theta_2)^k)}{(2(\theta_1 + \theta_2))^{k+l-1}}$$

and $\theta(k:0)$ and $\theta(0:l)$ found via the consistency property with $\theta(1:0) = \theta(0:1) = 0$.

Letting $\theta_1 = \theta$ and letting θ_2 tend to infinity returns us to the family of parameters given in (4.12). Letting $\theta_1 = \theta_2 = \theta$ gives us a family of paths associated with the Brownian net of [SS06].

It is natural to ask if it is possible to achieve the solutions of any $\mathcal{A}_N^\theta$-martingale problem via the motions of paths in conditioned webs. We believe that this may be possible by selecting the parameters θ_1 and θ_2 according to some distribution. Then conditional on θ_1 and θ_2 the webs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}', \overline{\mathcal{W}}'')$ are as described before.

Chapter 5

Duality

In this section we describe a process which we call alternating Brownian motion or alternating Brownian motion of rate θ. This process behaves as Brownian motion while away from zero and is reflected at zero until an exponential local time at zero is reached, at which point the sign of the process changes. Let $(B(t); t \geq 0)$ be a Brownian motion , and $(L_t; t \geq 0)$ be the local time at zero of B. Let $(N(t); t \geq 0)$ be an independent poisson process of rate θ. We construct an alternating Brownian motion of rate θ by

$$\hat{X}(t) = \begin{cases} |B(t)| & N(L_t) \text{ even} \\ -|B(t)| & N(L_t) \text{ odd .} \end{cases} \tag{5.1}$$

We show that alternating Brownian motion has a certain duality relationship with sticky Brownian motion. The duality we describe is one which often occurs when studying stochastic flows running forwards in time together with their "dual" flows running backwards. An explanation of the duality and the relationship with stochastic flows is given in Section 5.2.

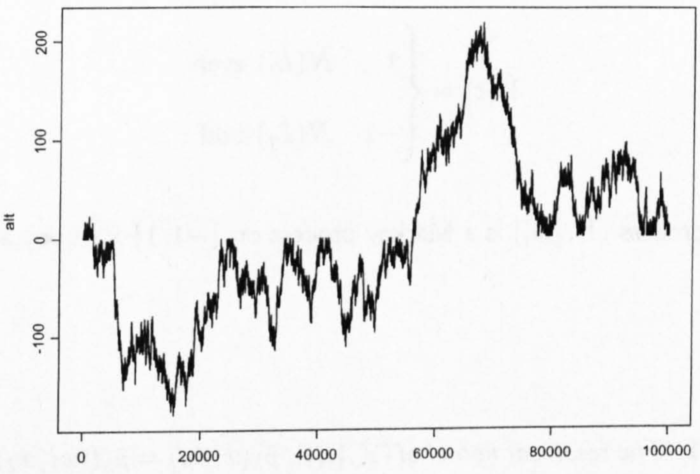

Figure 5.1: Alternating Brownian motion

5.1 Alternating Brownian motion

For a given Brownian motion $(B(t); t \geq 0)$, and a poisson process of rate θ, $(N(t); t \geq 0)$, we define an alternating Brownian motion as above. Thus,

$$\hat{X}(t) = \begin{cases} |B(t)| & N(L_t) \text{ even} \\ -|B(t)| & N(L_t) \text{ odd} . \end{cases}$$

Note that we have a process started at zero and also starting in the positive half line. We could equivalently have a process starting in the negative half line. We say that the former is an alternating Brownian motion started at 0^+ and the latter is an alternating Brownian motion started at 0^-. Because of this $\hat{X}$

is not Markov. We define a process $\hat{Y}$ on the space $\{-1,1\}$ by

$$\hat{Y}(t) = \begin{cases} 1 & N(L_t) \text{ even} \\ -1 & N(L_t) \text{ odd} . \end{cases}$$

Then the process $(\hat{Y}, |B|)$ is a Markov process on $\{-1,1\} \times [0,\infty)$ and $\hat{X}(t) = \hat{Y}(t)B(t)$.

Lemma 91. *The resolvent kernel of* $(\hat{Y}, |B|)$, $\hat{p}_\lambda(x,dy) = \hat{p}_\lambda((x_1,x_2),(y_1,dy_2))$, *is given by*

$$\hat{p}_\lambda(x,dy) = \frac{e^{-\gamma|x_2-y_2|}}{\gamma} + \frac{x_1 y_1 e^{-\gamma(x_2+y_2)}}{2\theta+\gamma} dy_2,$$

where $\gamma = \sqrt{2\lambda}$

Proof. With $B(0) = 0$, we have

$$\mathbf{P}(|B(t)| \in dy_2, L_t \in dl) = \sqrt{\frac{2}{\pi t^3}}(y_2+l)e^{\frac{(y+l)^2}{2t}} dy_2 dl.$$

The Laplace transform of the above is given by

$$\int_0^\infty e^{-\lambda t}\mathbf{P}(|B(t)| \in dy_2, L_t \in dl)dt = e^{-\gamma(y_2+l)}dy_2 dl.$$

The number of sign changes of $\hat{Y}$ by time t is given by a Poisson distribution of mean θL_t, therefore we have

$$\int_0^\infty e^{-\lambda t}\mathbf{P}(|B(t)| \in dy_2, \hat{Y} \text{ has changed } n \text{ times by time } t)dt$$
$$= 2dy_2 \int_0^\infty e^{-\gamma(y_2+l)}\frac{(\theta l)^n}{n!}e^{-\theta l}dl.$$

Then evaluating the integral gives

$$
\begin{aligned}
2\int_0^\infty e^{-\gamma(y_2+l)}\frac{(\theta l)^n}{n!}e^{-\theta l}dl &= 2e^{-\gamma y_2}\frac{\theta^n}{n!}\int_0^\infty l^n e^{-(\theta+\gamma)l}dl \\
&= e^{-\gamma y_2}\frac{\theta^n}{n!}\frac{n!}{(\theta+\gamma)^{n+1}} \\
&= \frac{2}{\theta+\gamma}e^{-\gamma y_2}\left(\frac{\theta}{\theta+\gamma}\right)^n.
\end{aligned}
$$

Now we can find the kernel for $y_1 = x_1$ by summing over even values of n

$$
\begin{aligned}
\hat{p}_\lambda((x_1,0),(y_1,dy_2)) &= \left(\frac{2}{\theta+\gamma}\right)e^{-\gamma y_2}\sum_{k=0}^\infty\left(\frac{\theta}{\theta+\gamma}\right)^{2k}dy_2 \\
&= e^{-\gamma y_2}\left(\frac{2}{\theta+\gamma}\right)\frac{1}{1-\left(\frac{\theta}{\theta+\gamma}\right)^2}dy_2 \\
&= e^{-\gamma y_2}\left(\frac{2}{\theta+\gamma}\right)\left(\frac{(\theta+\gamma)^2}{2\theta\gamma+\gamma^2}\right)dy_2 \\
&= \frac{\theta+\gamma}{\theta\gamma+\lambda}e^{-\gamma y_2}dy_2,
\end{aligned}
$$

and we can find the kernel for $y_1 \neq x_1$ by summing over odd values of n

$$
\begin{aligned}
\hat{p}_\lambda((x_1,0),(y_2,dy_2)) &= \left(\frac{2}{\theta+\gamma}\right)e^{-\gamma y_2}\sum_{k=0}^\infty\left(\frac{\theta}{\theta+\gamma}\right)^{2k+1}dy_2 \\
&= \left(\frac{2}{\theta+\gamma}\right)\left(\frac{\theta}{\theta+\gamma}\right)e^{-\gamma y_2}\sum_{k=0}^\infty\left(\frac{\theta}{\theta+\gamma}\right)^{2k}dy_2 \\
&= \left(\frac{\theta}{\theta+\gamma}\right)\left(\frac{\theta+\gamma}{\theta\gamma+\lambda}\right)e^{-\gamma y_2}dy_2 \\
&= \frac{\theta}{\theta\gamma+\lambda}e^{-\gamma y_2}dy_2.
\end{aligned}
$$

Now we have

$$
\frac{\theta+\gamma}{\theta\gamma+\lambda} = \frac{1}{\gamma}+\frac{1}{2\theta+\gamma}
$$

and

$$\frac{\theta}{\theta\gamma+\lambda} = \frac{1}{\gamma} - \frac{1}{2\theta+\gamma}$$

so it follows that

$$\hat{p}_\lambda((x_1, 0), (y_1, dy_2)) = \frac{e^{-\gamma y_2}}{\gamma} + \frac{x_1 y_1 e^{-\gamma y_2}}{2\theta+\gamma} dy_2.$$

Using the general resolvent, (2.12), from chapter 2 we get

$$\hat{p}_\lambda(x, dy) = \frac{e^{-\gamma|x_2-y_2|}}{\gamma} + \frac{x_1 y_1 e^{-\gamma(x_2+y_2)}}{2\theta+\gamma} dy_2.$$

□

Note that we can also define a Markov process, $\hat{Z}$, on the space $\mathbb{R}/\{0\} \cup \{0^-, 0^+\}$ by

$$\hat{Z} = \begin{cases} \hat{Y}(t)|B(t)| & |B(t)| > 0 \\ 0^+ & |B(t)| = 0, Y(t) = 1 \\ 0^- & |B(t)| = 0, Y(t) = -1. \end{cases}$$

Then $\hat{X}$ is equivalent to $\hat{Z}$ with 0^+ and 0^- identified. Letting $\mathrm{sgn}(0^+) = -\,\mathrm{sgn}(0^-) = 1$ we have a resolvent kernel for $\hat{Z}$ of

$$\hat{p}_\lambda(x, dy) = \frac{e^{-\gamma|x-y|}}{\gamma} + \frac{\mathrm{sgn}(x)\ \mathrm{sgn}(y) e^{-\gamma(|x|+|y|)}}{2\theta+\gamma} dy. \tag{5.2}$$

5.2 Duality

The duality we are interested in is of the "H-dual" type, as described in [Lig85]. Two processes X and $\hat{X}$ are H-dual if

$$E_x[H(X(t), y)] = E_y[H(x, \hat{X}(t))].$$

In our case we have $H(x,y) = \mathbf{1}_{\{x<y\}}$ (compare with the examples given in [Lig85]). So for two processes X and $\hat{X}$ the duality is given by

$$\mathbf{P}_x(X(t) < y) = \mathbf{P}_y(\hat{X}(t) > x). \tag{5.3}$$

The relationship of this duality to stochastic flows can be seen if we have the property that paths in the flow are non-crossing. If $(X_{s,t}(x); 0 \le s \le t, x \in \mathbb{R})$ is a family of random variables with the property $X_{s,t}(x) \le X_{s,t}(y)$ whenever $x \le y$ then each $X_{s,t}$ can be viewed as a random increasing function of the starting point x. We can therefore take its right continuous inverse. $X_{t,s}^{-1}(y) = \inf\{x; X_{s,t}(x) > y\}$. The events $\{X_{s,s+t}(x) > y\}$ and $\{X_{s+t,s}^{-1}(y) < x\}$, typically differ by a null set. In this case we have the relationship

$$\mathbf{P}(X_{s,s+t}(y) > x) = \mathbf{P}(X_{s+t,s}^{-1}(x) < y).$$

As an example of how this works, consider a flow of non-crossing paths each behaving as reflecting Brownian motion constructed via Skorokhod reflection from a single standard Brownian motion B:

$$X_{s,t}(x) = \max(x + B(t) - B(s), B(t) - \inf_{s \le u \le t} B_u). \tag{5.4}$$

Clearly $X_{s,t}(x) \le X_{s,t}(y)$ whenever $x \le y$ and for each $x \ge 0$, $(X_{0,t}(x); t \ge 0)$ is a reflecting Brownian motion started at x.

Inverting (5.4) gives

$$X_{t,s}^{-1}(y) = (x + B(s) - B(t))\mathbf{1}_{\{x \ge B(t) - \inf_{s \le u \le t} B_u\}}.$$

$X_{t,s}^{-1}(x) \le X_{t,s}^{-1}(y)$ for all $x \le y$ and for all $y \ge 0$, $(X_{t,t-s}^{-1}; s \ge 0)$ is an absorbing Brownian motion, i.e. it behaves as Brownian motion until the first

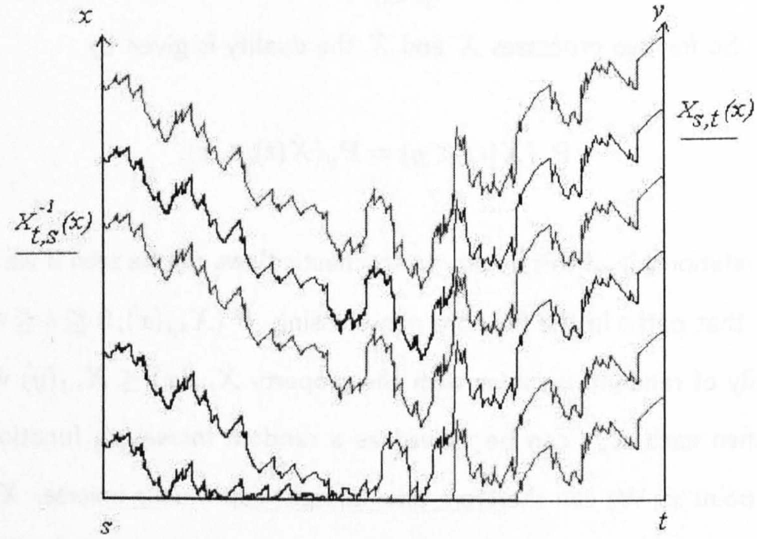

Figure 5.2: Duality between reflecting and absorbing Brownian motion

time it hits zero at which point it remains there indefinitely. Therefore if X is a reflecting Brownian motion then the dual of this process $\hat{X}$, satisfying the relation (5.3) is an absorbing Brownian motion. Dualities between absorbed and reflecting boundary conditions for general diffusion are given in [WW04].

In a similar way it is possible to construct a flow of non-crossing paths based on a single driving Brownian motion such that each path behaves as a one sided sticky Brownian motion with parameter θ and thus we find that the dual process is what is known as an elastic Brownian motion: a Brownian motion which is absorbed after an exponential (rate θ) amount of local time at zero has passed.

It is not possible, or at least very difficult, to construct a non-crossing flow based on a single driving Brownian motion where every path behaves as a two sided sticky Brownian motion. Nevertheless, when X is two sided sticky Brownian motion, the duality condition (5.3) still holds for some process $\hat{X}$ and this process turns out to be the alternating Brownian motion described in the previous section.

The equation (5.3) can be expressed in terms of transition kernels, when they exist

$$\int_x^\infty p_t(y,dz) = \int_{-\infty}^y \hat{p}_t(x,dz),$$

where due to the strict inequalities in (5.3) the integrals are assumed to be exclusive of the end points. This is important, as for sticky Brownian motion the transition kernel, $p_t(y,dz)$, has an atom at zero. An equivalent condition involving resolvent kernels is

$$\int_x^\infty p_\lambda(y,dz) = \int_{-\infty}^y \hat{p}_\lambda(x,dz). \tag{5.5}$$

Proposition 92. *The resolvent kernel of a two sided sticky Brownian motion, given in* (2.9) *and the resolvent kernel of* $\hat{Z}$, *given in* (5.2) *satisfy the relationship* (5.5). *Letting* $\mathbb{R}/\{0\} \cup \{0^+,0^-\}$ *have usual ordering on* $\mathbb{R}$, *with* 0^+ *and* 0^- *identified to* 0, *it follows that sticky Brownian motion and alternating Brownian motion satisfy the duality given in* (5.3).

Proof. Consider the left hand side of (5.5)

$$\int_x^\infty p_\lambda(y,dz) = \int_x^\infty \frac{e^{-\gamma|y-z|}}{\gamma} - \frac{e^{-\gamma(|y|+|z|)}}{2\theta+\gamma} + \frac{e^{-\gamma|y|}}{\theta\gamma+\lambda}\delta_0(dz). \tag{5.6}$$

Firstly

$$\begin{aligned}
\int_x^\infty e^{-\gamma|z-y|}dz &= \int_{x-y}^\infty e^{-\gamma|u|}du \\
&= \begin{cases} \int_{x-y}^0 e^{\gamma u}du + \int_0^\infty e^{-\gamma u}du & x<y \\ \int_{x-y}^\infty e^{-\gamma u}du & y \le x \end{cases} \\
&= \begin{cases} \frac{1}{\gamma}(1-e^{-\gamma(y-x)}) + \frac{1}{\gamma} & x<y \\ \frac{1}{\gamma}e^{-\gamma(x-y)} & y \le x \end{cases} \\
&= \frac{2}{\gamma}\mathbf{1}_{\{x<y\}} + \frac{\mathrm{sgn}(x-y)}{\gamma}e^{-\gamma|x-y|}
\end{aligned}$$

where here $\mathrm{sgn}(0)=1$. Next we have

$$\begin{aligned}
\int_x^\infty e^{-\gamma(|z|+|y|)}dz &= \begin{cases} \int_x^\infty e^{-\gamma(z+|y|)}dz & x \ge 0 \\ \int_x^0 e^{-\gamma(-z+|y|)}dz + \int_0^\infty e^{-\gamma(z+|y|)}dz & x<0 \end{cases} \\
&= \begin{cases} \frac{1}{\gamma}e^{-\gamma(x+|y|)} & x \ge 0 \\ \frac{1}{\gamma}(e^{-\gamma|y|} - e^{-\gamma(-x+|y|)}) + \frac{1}{\gamma}e^{-\gamma|y|} & x<0 \end{cases} \\
&= \frac{2}{\gamma}e^{-\gamma|y|}\mathbf{1}_{\{x<0\}} + \frac{\mathrm{sgn}(x)}{\gamma}e^{-\gamma(|x|+|y|)}
\end{aligned}$$

where again $\mathrm{sgn}(0)=1$. Finally

$$\int_x^\infty e^{-\gamma|y|}\delta_0(dz) = e^{-\gamma|y|}\mathbf{1}_{\{x<0\}}.$$

These results together with (5.6) give us

$$\begin{aligned}
\int_x^\infty p_\lambda(y,dz) &= \frac{1}{\lambda}\mathbf{1}_{\{x<y\}} + \frac{\mathrm{sgn}(x-y)}{2\lambda}e^{-\gamma|x-y|} \\
&\quad - \frac{1}{\theta\gamma+\lambda}e^{-\gamma|y|}\mathbf{1}_{\{x<0\}} - \frac{\mathrm{sgn}(x)}{\gamma(2\theta+\gamma)}e^{-\gamma(|x|+|y|)} \\
&\quad + \frac{1}{\theta\gamma+\lambda}e^{-\gamma|y|}\mathbf{1}_{\{x<0\}} \\
&= \frac{1}{\lambda}\mathbf{1}_{\{x<y\}} + \frac{\mathrm{sgn}(x-y)}{2\lambda}e^{-\gamma|x-y|} - \frac{\mathrm{sgn}(x)}{\gamma(2\theta+\gamma)}e^{-\gamma(|x|+|y|)}. \qquad (5.7)
\end{aligned}$$

Now let us consider the right side of (5.5), using the resolvent kernel for $\hat{Z}$ given in (5.2).

$$\int_{-\infty}^y \hat{p}_\lambda(x,dz) = \int_{-\infty}^y \frac{e^{-\gamma|x-z|}}{\gamma} + \frac{\mathrm{sgn}(x)\ \mathrm{sgn}(z)e^{-\gamma(|x|+|z|)}}{2\theta+\gamma}dz \qquad (5.8)$$

Firstly we have,

$$\begin{aligned}
\int_{-\infty}^y e^{-\gamma|x-z|}dz &= \int_{-\infty}^{y-x} e^{-\gamma|u|}du \\
&= \begin{cases} \int_0^{y-x} e^{-\gamma u}du + \int_{-\infty}^0 e^{\gamma u}du & x<y \\ \int_{-\infty}^{y-x} e^{\gamma u}du & y \le x \end{cases} \\
&= \begin{cases} \frac{1}{\gamma}(1-e^{-\gamma(y-x)}) + \frac{1}{\gamma} & x<y \\ \frac{1}{\gamma}e^{-\gamma(x-y)} & y \le x \end{cases} \\
&= \frac{2}{\gamma}\mathbf{1}_{\{x<y\}} + \frac{\mathrm{sgn}(x-y)}{\gamma}e^{-\gamma|x-y|}
\end{aligned}$$

where sgn(0) $= 1$, and $0^+ - 0^- = 0$. Next,

$$\int_{-\infty}^{y} \mathrm{sgn}(x)\,\mathrm{sgn}(z) e^{-\gamma(|x|+|z|)} dz$$
$$= \begin{cases} -\int_{-\infty}^{y} \mathrm{sgn}(x) e^{\gamma(z-|x|)} dz & y < 0 \\ \int_0^y \mathrm{sgn}(x) e^{-\gamma(|x|+z|)} dz - \int_{-\infty}^{0} \mathrm{sgn}(x) e^{\gamma(z-|x|)} dz & y \geq 0 \end{cases}$$
$$= \begin{cases} -\frac{\mathrm{sgn}(x) e^{-\gamma(|x|+|y|)}}{\gamma} & y < 0 \\ -\frac{\mathrm{sgn}(x) e^{-\gamma(|x|+|y|)}}{\gamma} + \frac{\mathrm{sgn}(x) e^{-\gamma|x|}}{\gamma} - \frac{\mathrm{sgn}(x) e^{-\gamma|x|}}{\gamma}, & y \geq 0 \end{cases}$$
$$= -\frac{\mathrm{sgn}(x) e^{-\gamma(|x|+|y|)}}{\gamma}$$

These results together with (5.8) gives us

$$\int_{-\infty}^{y} \hat{p}_\lambda(x, dz) = \frac{1}{\lambda}\mathbf{1}_{\{x<y\}} + \frac{\mathrm{sgn}(x-y)}{2\lambda} e^{-\gamma|x-y|} - \frac{\mathrm{sgn}(x)}{\gamma(2\theta+\gamma)} e^{-\gamma(|x|+|y|)} \tag{5.9}$$

Comparing (5.7) and (5.9) we have the result for $x \neq 0$. Checking the case when $x = 0$ we have

$$\mathbf{P}_y(X(t) = 0) = \lim_{\epsilon \downarrow 0} (\mathbf{P}_y(X(t) > \epsilon) - \mathbf{P}_y(X(t) > -\epsilon)) = \frac{2}{\gamma(2\theta+\gamma)} e^{-\gamma|y|}$$

and

$$\lim_{\epsilon \downarrow 0} \Big(\mathbf{P}_\epsilon(\hat{X}(t) < y) - \mathbf{P}_{-\epsilon}(\hat{X}(t) < y)\Big) = \mathbf{P}_{0^+}(\hat{Z}(t) < y) - \mathbf{P}_{0^-}(\hat{Z}(t) < y)$$
$$= \frac{2}{\gamma(2\theta+\gamma)} e^{-\gamma|y|}.$$

This proves the result. $\square$

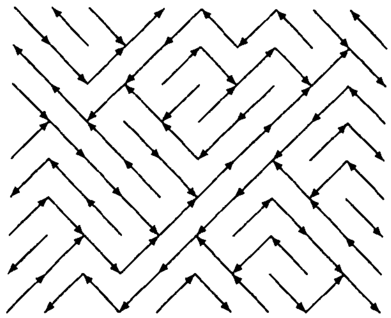

Figure 5.3: Coalescing random walks with dual

5.2.1 The relation to flows

We described a system of coalescing random walks $\mathcal{S}$ in chapter 1, which is constructed by joining independent random arrows which go between points in $L = \{(n,k) : n+k \text{ is even}\}$. We also described a natural dual to this system $\mathcal{S}'$ by placing arrows in between points in $L' = \{(n,k) : n+k \text{ is odd}\}$. See figure 5.3.

Under a diffusive scaling $\mathcal{S}$ converges to a Brownian web. whereas $\mathcal{S}'$ converges to a Brownian web that is rotated by 180 degrees so that time is running backwards. A pair of Brownian webs $(\overline{\mathcal{W}}, \overline{\mathcal{W}}')$ with this duality is described in [FINR04] and [FN06]. Suppose that we observe a path $W_{x,0}$ in $\overline{\mathcal{W}}$ starting from a fixed point $(x,0) \in \mathbb{R}^2$. The process $(W_{x,0}(t) : t \geq 0)$ has the law of a Brownian motion started at x. Conditional on this path we observe a path $W'_{x,T}$ in the dual web $\overline{\mathcal{W}}'$ started at fixed point (x,T). Then the conditional law of $(W'_{x,T}(T-t) : 0 \leq t \leq T)$ given $W_{x,0}$ is the law of a Brownian motion that is reflected when on the path $W_{x,0}$ in the Skorokhod sense. This result can be found in several places, one example is [STW00].

Observing the system of dual systems of coalescing random walks $(\mathcal{S}, \mathcal{S}')$ we can see that a path in $\mathcal{S}$ and a path in $\mathcal{S}'$ will behave independently when apart and when the two paths meet (within 1 unit above or below) the motions are restricted because the two paths are never allowed to cross.

Suppose now that $\mathcal{S}$ is a system of weighted arrows discussed in chapter 1. Thus for each $(n, k) \in L$ the arrow pointing upwards from (n, k) has a weight of $Q_{n,k}$ and the arrow pointing downwards has a weight of $1 - Q_{n,k}$, where $Q_{n,k}$ is some random variable on $[0, 1]$ and the family $(Q_{n,k} : (n, k) \in L)$ is mutually independent and identically distributed. Let $\mathcal{S}'$ be the natural dual to the system as described in chapter 1. Thus an arrow in $\mathcal{S}'$ from $(n+1, k) \in L'$ to $(n, k-1) \in L'$ has the same weight as the arrow in $\mathcal{S}$ from $(n, k) \in L$ to $(n+1, k+1) \in L$, which is given by $Q_{n,k}$. A path $(S(t) : t \in \mathbb{Z})$ in $\mathcal{S}$ behaves as a simple symmetric random walk. Considering a path S in $\mathcal{S}$ and a path S' in $\mathcal{S}'$ with time run backwards. Each path behaves as a simple symmetric random walk when apart. When apart, the paths behave independently. When the path S' meets the path S (1 unit above or below), say this occurs at some point $(n+1, k) \in L'$, so that S' is above S. So we have $S'(n+1) = k$ and $S(n+1) = k - 1$. The probability that $S'(n) > S(n)$ is given by $\mathbf{E}[Q_{n,k}^2 + (1 - Q_{n,k})^2]$ whereas the probability that the paths cross so that $S'(n) < S(n)$ is given by $\mathbf{E}[Q_{n,k}(1 - Q_{n,k})]$. This chance of crossing occurs independently every time the paths S' meets S.

Apply a diffusive scaling (time by ϵ and space by $\sqrt{\epsilon}$) to the systems $\mathcal{S}$ and $\mathcal{S}'$, and also scaling $Q_{n,k}$ such that $\frac{1}{\sqrt{\epsilon}}\mathbf{E}[Q_{n,k}(1 - Q_{n,k})] \to \theta$. It would seem plausible that in the limit S and S' behaves as Brownian motions, independent when apart and conditional on S, S' is reflected (in a Skorokhod sense) off S until $L_t(S - S')$ reaches an exponentially distributed value and which point S' crosses S. Thus it is possible to see that there is a relation between the distribution of these paths and the alternating Brownian motion described above.

5.3 Balls and Boxes

In [DEF+00] a duality is observed between coalescing Brownian motions and a set valued process. The sets consist of finite disjoint unions of intervals of the real line. The endpoints of the intervals perform Brownian motion until they meet another endpoint, at which instant they annihilate each other. If the endpoints were from the same interval the interval disappears. If the endpoints were from different intervals then the two intervals merge to become one interval. Call this set valued process U^B, starting at $B = \bigcup_{i=1}^{m}(v_{2i-1}, v_{2i}]$, for $\{v_1, \dots, v_{2m}\} \in \mathbb{R}^{2m}$. Let the process W^A be n coalescing Brownian motions , started at $A = \{w_1, w_2 \dots, w_n\} \in \mathbb{R}^n$. The result we are considering says that

$$\mathbf{P}(W^A(t) \subseteq B) = \mathbf{P}(A \subseteq U^B). \tag{5.10}$$

In the paper they argue that this is true by a known duality between the discrete versions of the processes and using known scaling limits. In this section we will show that (5.5) can be seen to be true, using the idea that the process W^A can be thought of as a selection of paths from a Brownian web $\overline{\mathcal{W}}$. Whereas the endpoints of interval process U^B can be thought of as a selection of paths in the dual Brownian web, $\overline{\mathcal{W}}'$, of $\overline{\mathcal{W}}$, as described in [FINR04] and [FN06]. We will also give an argument for a similar result in the sticky case.

Let $w_1 \leq w_2 \leq \dots \leq w_n$ be n fixed points in $\mathbb{R}$. For $i \in \{1, \dots n\}$ let $W^i = W_{w_i,0}$ be the almost surely unique path in $\overline{\mathcal{W}}$ started at $(w_i, 0) \in \mathbb{R}^2$, so that the paths $(W^1, W^2 \dots, W^n)$ is a system of coalescing Brownian motions started at $w_1 \leq w_2 \leq \dots \leq w_n$. Let $v_1 \leq v_2 \leq \dots \leq v_{2m}$ be $2m$ fixed point in $\mathbb{R}$. For each $i \in \{1, \dots, 2m\}$, let $V^i = W_{v_i,T}$, be an almost surely unique path started at $(v_i, T) \in \mathbb{R}^2$ in $\overline{\mathcal{W}}'$ with time running backwards. Then $(V^1(T-t), V^2(T-t), \dots, V^{2m}(T-t); t \geq 0)$ is a system of coalescing Brownian motions started at $v_1 \leq v_2 \leq \ldots \leq v_{2m}$ in the backward flow. At time $s \in [0,t]$ we have $n+2m$ points in $\mathbb{R}$ given by $W^1(s), W^2(s), \ldots, W^n(s), V^1(s), V^2(s), \ldots, V^{2m}(s)$, the ordering of which is preserved for all $s \in [0,t]$. For each $s \in [0,t]$, we define a set $\hat{U}(s)$ by $\hat{U}(s) = \bigcup_{i=1}^m (V^i(s), V^{i+1}(s)]$. If $\{W^1(s), W^2(s), \ldots W^n(s)\} \subseteq \hat{U}(s)$ for some $0 \leq s \leq t$, $\{W^1(s), W^2(s), \ldots W^n(s)\} \subseteq \hat{U}(s)$ for all $0 \leq s \leq t$. In particular

$$\{W^1(t), W^2(t), \ldots W^n(t)\} \subseteq \hat{U}(t) \Leftrightarrow \{W^1(0), W^2(0), \ldots W^n(0)\} \subseteq \hat{U}(0).$$

Clearly $\{W^1(t), W^2(t), \ldots W^n(t)\}$ is equal in distribution to W^A. $(\hat{U}(T-t) : t \geq 0)$ is a set valued process, where the sets are finite unions of intervals. The end points of which are behaving as coalescing Brownian motion. When two endpoints meet they stay together from then on. If the endpoints are of from the same interval, the interval disappears and if endpoints are of different intervals the intervals merge. Thus $(\hat{U}(T-t) : t \geq 0)$ is equal in distribution to U^B and we have (5.10).

5.3.1 The sticky case

If W is a one dimensional Brownian motion started at x then

$$\mathbf{P}(W(t) \in B) = \mathbf{P}(x \in U^B(t))$$

where $B = \bigcup_{i=1}^m (v_{2i-1}, v_{2i}]$, for $\{v_1, \ldots, v_{2m}\} \in \mathbb{R}^{2m}$, as before. This is a particular case of (5.10) but can also be proved directly using the transition probabilities of Brownian motion and the reflection principle. From this we can see that if $W = (W_1, \ldots W_n)$ are n independent Brownian motions started at

$x = (x_1, \ldots, x_n)$ then

$$\mathbf{P}(X \in B) = \prod_{i=1}^{n} \mathbf{P}(x_i \in U_i^B(t)),$$

where $(U_i^B : i \in \{1, \ldots, n\})$ are i.i.d copies of U^B. It is this concept that we wish to generalise in order to find a result for the sticky case, as we consider particles that are sampled independently given some environment.

Recall that if $(K_{s,t}; s \leq t)$ is a stochastic flow of kernels, as described in chapter 1, then the N-point of motion of K is a Markov process with transition semigroup given by

$$P_t^N(x, A) = \mathbf{E}[K_{0,t}(x_1, A_1)K_{0,t}(x_2, A_2) \cdots K_{0,t}(x_N, A_N)]$$

for all $x = (x_1, x_2, \ldots x_N) \in \mathbb{R}^N$ and $A = A_1 \times A_2 \times \cdots A_N \in \mathcal{B}(\mathbb{R}^N)$. Define a dual flow of kernels $\hat{K}$ by

$$\hat{K}_{s,t}(x, (-\infty, y]) = K_{s,t}(y, [x, \infty)) \tag{5.11}$$

Let the one-point motion of K, given by $P_t^N(x, A) = \mathbf{E}[K_{0,t}(x, A)]$, be a Brownian motion. Then the one-point motion of $\hat{K}$ given by

$$P_t^N(y, [x, \infty)) = \mathbf{E}[K_{0,t}(y, [x\infty))]\mathbf{E}[K_{0,t}(x, (-\infty, y])] = P_t^N(x, (-\infty, y])$$

is also a Brownian motion. Let $X = (X_1, X_2, \ldots, X_n)$ be the n-point motion of K and started at $(x_1, \ldots, x_n)$. Let $Y = (Y_1^i, \ldots, Y_{2m}^i : i \in \{1, \ldots n\})$ be the $2mn$-point motion of $\hat{K}$.

For each i, let $(V_1^i, \ldots, V_{2m}^i)$ be the process which follows the paths of $(Y_1^i, \ldots, Y_{2m}^i)$ but with the coalescing rule that whenever two paths meet both paths follow the path of the lower index. Thus, for each $i \in \{1,\ldots,n\}$, $(V_1^i,\ldots,V_{2m}^i)$ are coalescing Brownian motions whereas, for each $j \in \{1,\ldots,n\}$, $(V_j^1,\ldots,V_j^n)$ is an n-point motion of $\hat{K}$.

For each i let U_i^B be the interval valued process given by

$$U_i^B(t) = \bigcup_{j=1}^{m} [V_{2j-1}^i(t), V_{2j}^i(t)).$$

The starting value of this process is $B = \bigcup_{j=1}^m [y_{2j-1}, y_{2j})$ for all i. Note that as $(V_1^i,\ldots,V_{2m}^i)$ are coalescing Brownian motions then for each i, U_i^B has the distribution of U^B described above.

Proposition 93. *With X and $(U_i^B : 1 \leq i \leq N)$ as described above, the following equality holds*

$$\mathbf{P}_x(X(t) \subset B) = \mathbf{P}\left(\bigcap_{i=1}^{n} \{x_i \in U_i^B(t)\}\right) \tag{5.12}$$

where $x = (x_1,\ldots,x_n)$.

Proof. We have that

$$\mathbf{P}_x(X(t) \subset B) = \mathbf{E}\left[K_{0,t}(x_1, B) \cdots K_{0,t}(x_n, B)\right]. \tag{5.13}$$

Then for each i

$$\begin{aligned}
K_{0,t}(x_i, B) &= \sum_{j=1}^{m} K_{0,t}(x_i, (y_{2j-1}, y_{2j}]) \\
&= \sum_{j=1}^{m} K_{0,t}(x_i, (-\infty, y_{2j}]) - K_{0,t}(x_i, (-\infty, y_{2j-1}]) \\
&= \sum_{i=1}^{m} \hat{K}_{0,t}(y_{2j}, [x_i, \infty)) - \hat{K}_{0,t}(y_{2j-1}, [x_i, \infty)).
\end{aligned}$$

We can consider the flow of kernels K as representing a random environment.

Let the σ-algebra $\mathcal{G}$ be the information given in the environment which is also the information given in the environment represented by $\hat{K}$. Thus

$$\begin{aligned}\sum_{j=1}^{m} &\hat{K}_{0,t}(y_{2j},[x_i,\infty)) - \hat{K}_{0,t}(y_{2i-1},[x_i,\infty)) \\ &= \sum_{j=1}^{m} \mathbf{P}(Y_{2j}^i(t) \geq x_i|\mathcal{G}) - \mathbf{P}(Y_{2j-1}^i(t) \geq x_i|\mathcal{G}).\end{aligned}$$

By considering the value of $\sum_{j=1}^{m} \hat{K}_{s,t}(Y_{2j}^i(s),[x_i,\infty)) - \hat{K}_{s,t}(Y_{2j-1}^i(s),[x_i,\infty))$ at collision times of Y it follows that

$$\begin{aligned}\sum_{j=1}^{m} &\hat{K}_t(y_{2j},[x_i,\infty)) - \hat{K}_t(y_{2j-1},[x_i,\infty)) \\ &= \sum_{j=1}^{m} \mathbf{P}(V_{2j}^i(t) \geq x_i|\mathcal{G}) - \mathbf{P}(V_{2j-1}^i(t) \geq x_i|\mathcal{G}) \\ &= \sum_{j=1}^{m} \mathbf{P}(x_i \in (V_{2j-1}^i(t), V_{2j}^i(t)]|\mathcal{G}) \\ &= \mathbf{P}(x_i \in U_i^B(t)|\mathcal{G}).\end{aligned}$$

This together with (5.13) gives (5.12). □

Let K be a flow of kernels such that the N-point motion of K is a solution to the $\mathcal{A}_N^{\theta}$-martingale problem. If $\hat{K}$ is the flow of kernels given by (5.11), we conjecture that the N-point motion of $\hat{K}$ is a solution to the $\mathcal{A}_N^{\theta'}$-martingale problem, where the family of parameters θ' is given by $\theta'(k:l) = \theta(l:k)$, for $k,l \geq 0$.

Then (5.12) holds when $X = (X_1, X_2, \ldots, X_n)$ is a solution to the $\mathcal{A}_n^{\theta}$-martingale problem motion started at $(x_1, \ldots, x_n)$, $Y = (Y_1^i, \ldots, Y_{2m}^i : i \in \{1, \ldots n\})$ is a solution to the $\mathcal{A}_{2nm}^{\theta'}$-martingale problem, and U_i^B is constructed from Y as before. We note that taking the case when $n = 2$ the distribution of

$(V_1^1, \ldots, V_{2m}^1, V_1^2, \ldots, V_{2m}^2)$ is that of a sticky coalescing system as constructed in Section 4.1.4. Also note that taking the extreme case of $\theta(k : l) = 0$. for each k and l yields $U_1^B = \cdots = U_n^B$, and so (5.12) reduces to (5.10).

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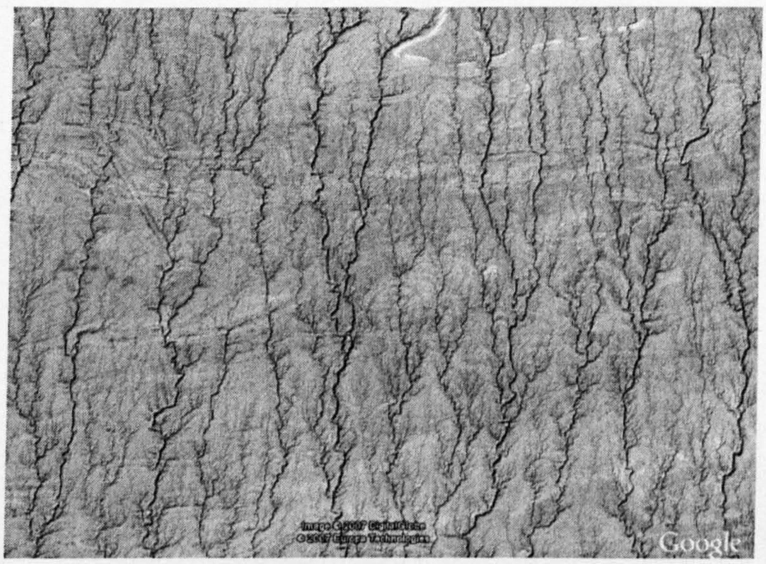

Figure 5.4: Somewhere in the world, ©Google Earth