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No 1017

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

Bidding Markets with Financial Constraints*

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January 24, 2013

Abstract

We develop a model of bidding markets with financial constraints a la Che and Gale (1998b) in which two firms optimally choose their budgets. First, we provide an alternative explanation for the dispersion of markups and "money left on the table" across procurement auctions. Interestingly, this explanation does not hinge on significant private information but on differences, both endogenous and exogenous, in the availability of financial resources. Second, we explain why the empirical analysis of the size of markups may be biased downwards or upwards with a bias positively correlated with the availability of financial resources when the researcher assumes that the data are generated by the standard auction model. Third, we show that large concentration and persistent asymmetries in market shares together with occasional leadership reversals can arise as a consequence of the firms internal financial decisions even in the absence of exogenous shocks.

JEL Classification Numbers: L13, D43, D44. Keywords: bidding markets, financial constraints, markups, money left on the table, market shares, industry dynamics.

*We are grateful to Luis Cabral, Dan Kovenock, Gilat Levy and Matthew Shum for their comments. We thank Fundación Ramón Areces for its financial support. Ángel Hernando-Veciana also thanks the financial support by the Spanish Ministry of Science through project ECO2009-10531.

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1 Introduction

Bidding markets are those in which trade is organized through bidding. The most commonly cited example is public procurement which amounts to between 10% and 20% of GDP in OECD countries. Other examples include procurement in the private sector and auctions both in the private and the public sector.[1]

The standard model of bidding implicitly assumes that the size of the projects is relatively small compared to the financial resources of the firms. Unfortunately, the current financial crisis has made evident that this assumption is not realistic for many bidding markets:

> "Offers submitted on Monday by Global Infrastructure Partners and a consortium led by Manchester Airport Group have been depressed [...] by the problems of raising the necessary bank finance."
>
> *Ferrovial receives depressed bids for Gatwick*, Financial Times, 28/Apr/2009

Che and Gale (1998b) show that the predictions of the standard model do not extend to the model where firms are financially constrained. The extent to which a firm is financially constrained in their model depends on its budget, *working capital* hereafter, which they assume to be exogenous. In reality, however, the firm's working capital is not exogenous but chosen out of the firm's internal financial resources, the *cash* hereafter, which in turn depends on the past performance of the firm. This paper shows how introducing this feature can help us understand different aspects of bidding markets.

First, we provide a new explanation for the dispersion of markups and "money left on the table"[2] across auctions observed in procurement. Interestingly, this explanation does not hinge on significant private information about working capitals and costs, but on differences in the availability of financial resources across auctions in a sense that we formalize later. This casts doubts about the usual interpretation for the dispersion of markups and "money left on the table" observed in procurement as indicative of incom-

[1] A detailed description of bidding markets and a wide range of examples can be found in Klemperer (2005), OECD (2006), OFT (2007) and Einav and Levin (2010).

[2] Money left on the table, also known as bid spreads, is defined as the difference between the lowest and the second lowest bids in first price procurement auctions.

plete information and large heterogeneity in production cost.[3] Second, we explain why the empirical analysis of the size of markups may be biased downwards or upwards with a bias positively correlated with the availability of financial resources when the researcher assumes that the data are generated by the standard model. Third, we show that large concentration and persistent asymmetries in market shares together with occasional leadership reversals can arise as a consequence of the firms internal financial decisions even in the absence of exogenous shocks.[4] This effect is greater for larger projects than for smaller projects, a prediction in line with the empirical evidence.[5]

Our model also provides a formal framework to analyze the conventional wisdom in economics that "auctions [still] work well if raising cash for bids is easy" (Aghion, Hart, and Moore (1992, p. 527))[6] as the standard model arises when the firms' working capitals are sufficiently abundant. Surprisingly, in our model firms keep little working capital in the long run, even when its opportunity cost is arbitrary low.

We are interested in markets in which only bids that have secured financing can be submitted, i.e. are *acceptable*.[7] For instance, this is the case of markets in which surety bonds are required.[8] We also follow Che and Gale's (1998b) insight that the set of ac-

[3] Indeed, as Weber (1981) pointed out:" Some authors have cited the substantial uncertainty concerning the extractable resources present on a tract, as a factor which makes large bid spreads [i.e. money left on the table] unavoidable." More recently, Krasnokutskaya (2011) noted that "The magnitude of the money left on the table variable [...] indicates that cost uncertainty may be substantial."

[4] Exogenous shocks give rise to similar predictions when they introduce randomness in the processes of either capacity accumulation, see Besanko and Doraszelski (2004), or learning by doing, see Besanko, Doraszelski, Kryukov, and Satterthwaite (2010). See also the discussion in page 6.

[5] Porter and Zona (1993) explain that "the market for large jobs [in procurement of highway maintenance] was highly concentrated. Only 22 firms submitted bids on jobs over \$1 million. On the 25 largest jobs, 45 percent of the 76 bids were submitted by the four largest firms."

[6] This conjecture has been recently questioned by Rhodes-Kropf and Viswanathan (2005) under the assumption that firms finance their bids by borrowing in a competitive financial market.

[7] Alternatively, we could have assumed that it was costly for the firm to submit a bid and not complying, e.g. the firm may bear a direct cost in case of default.

[8] In the U.S., the Miller Act regulates the provision of surety bonds for federal construction projects. State legislatures have enacted "Little Miller Acts" that establish similar requirements for state contracts. A surety bond plays two roles: first, it certifies that the proposed bid is not jeopardized by the technological and financial conditions of the firm, and second, it insures against the losses in case of non-compliance. Indeed, the Surety Information Office highlights that "the surety [...] may require a financial statement [that] [...] helps the surety company evaluate the working capital and overall financial condition of the

ceptable bids increases with the working capital. This feature is present in a number of settings in which firms have limited access to external financial resources. One example is an auction in which the price must be paid upfront, and hence the maximum acceptable bid increases with the firm's working capital. Another example is a procurement contest in which the firm must be able to finance the difference between its working capital and the cost of production. Because of this financing needs, the less the firm's working capital is, the larger its minimum acceptable bid must be if the external funds that are available to the firm increase with its bid or its profitability. The latter property arises when the sponsor pays in advance a fraction of the price,[9] a feature of the common practice of progress payments, or when the amount banks are willing to lend depends on the profitability of the project as it is usually the case.[10]

A representative example of the institutional details of the bidding markets we are interested in is highway maintenance procurement. As Hong and Shum (2002) pointed out "many of the contractors in these auctions bid on many contracts overtime, and likely derive a large part of their revenues from doing contract work for the state." Besides, Porter and Zona (1993) explain that "The set of firms submitting bids on large projects was small and fairly stable[...] There may have been significant barriers to entry, and there was little entry in a growing market."[11]

Motivated by these observations, we build a *static* model to explain our first two main results and a *dynamic* model to give a broader perspective on the first result and to explain the third one. In the static model, two firms endowed with some cash choose working capitals to compete in a first price auction for a procurement contract. We assume that the cost of complying is known and identical across firms, the minimum acceptable bid increases with the firm's working capital and only cash is publicly observable.[12] Since

company." See http://www.sio.org/html/Obtain.html#Financial and Calveras, Ganuza, and Hauk (2004).

[9]A numerical illustration can be found in Beker and Hernando-Veciana (2011).

[10]We show in Appendix B that this is also the theoretical prediction of a model inspired by the observation of Tirole (2006), page 114, that "The borrower must [...] keep a sufficient stake in the outcome of the project in order to have an incentive not to waste the money."

[11]Moreover, it can be shown that in a model with many firms and entry the natural extension of the equilibrium we study has the feature that only the two firms with more cash enter the market.

[12]The part of our first main result regarding markups and the third main results also hold in a version of our model with observable working capital, see Beker and Hernando-Veciana (2011).

using cash as working capital means postponing consumption, it is costly,[13] though all our results still hold true when that cost is arbitrarily low. Firms choose their working capitals and bids optimally. The dynamic model consists of the infinite repetition of the static model. The cash at the beginning of each period is equal to the last period unspent working capital plus the earnings in previous procurement contract and some exogenous cash-flow.

In our static model, to carry more working capital than strictly necessary to make the bid acceptable is strictly dominated because working capital is costly. Thus, the firm that carries more working capital wins the contract[14] and both firms incur in the cost of their working capital, i.e. the game verifies the *all pay auction structure*.

The strategic considerations that shape the equilibrium working capitals are the same as in the all pay auction with complete information.[15] Not surprisingly, in a version of our game with unlimited cash, there is a unique symmetric equilibrium in which firms randomize in a bounded interval with an atomless distribution. This is also the unique equilibrium in our game when the firms' cash is larger than the upper bound of the support of the equilibrium randomization. We call the scenario *symmetric* if this is the case, and *laggard-leader* otherwise. In this latter case, firms also randomize in a bounded interval, though the firm with less cash, the *laggard* hereafter, puts an atom at zero and the other firm, the *leader*, at the laggard's cash.

Our first two main results arise in the laggard-leader scenario. The dispersion of markups and "money left on the table" is due to heterogeneity across auctions in the availability of financial resources, i.e. either the firms' cash or the minimum acceptable

[13] Any other motivation for the cost of working capital would deliver similar results.

[14] This feature seems realistic in many procurement contracts:

> It is thought that Siemens' superior financial firepower was a significant factor in it beating Canada's Bombardier to preferred bidder status on Thameslink.

Minister blocks Boris Johnson's plan to fund 1bn Crossrail project, The Guardian, 11/Dec/2011.

[15] In particular, it resembles Che and Gale's (1998a) model of an all pay auction with caps in that working capitals are bounded by cash. Our model is more general in that they assume exogenous caps that are common to all agents. Another difference is that in our dynamic model the size of the prize varies with the rival's action. To the best of our knowledge, the literature on all pay auctions, see Kaplan, Luski, Sela, and Wettstein (2002) and Siegel (2009), has only considered the case of prizes that vary with the agent's action.

bids. Either of these two variables affect the equilibrium working capitals which determine the bids, and hence the markups and "money left on the table". Biases in the structural estimation of the size of markups can also arise if, as it is often the case, the econometrician does not observe costs. Imagine bid data from several auctions with identical financial conditions and suppose the data are generated by our model. On the one had, if the laggards have little cash, there is little money left on the table and large markups. However, a researcher that assumed the standard model would conclude that there is little cost heterogeneity and, as a consequence, small markups, i.e. the estimation would be biased downwards. On the other hand, if the laggards have relatively large cash, though not too large, there is sizable "money left on the table" and relatively low markups. However, a researcher that assumed the standard model would conclude that there is large cost heterogeneity and, as a consequence, large markups, i.e. the estimation would be biased upwards.

In our dynamic model, we consider the unique equilibrium that is the limit of the sequence of equilibria of models with an increasing number of periods.[16] Remarkably, the marginal continuation value of cash is equal to its marginal consumption value under a mild assumption about the minimum acceptable bid. Consequently, increasing the working capital while keeping constant the bid is suboptimal, as in the static model. One can also argue that firms do not carry more working capital than strictly necessary to make the bid acceptable and that the all pay auction structure arises.

On the equilibrium path, the frequency of each scenario depends on the severity of the financial constraint. We say that the financial constraint is *tight* (resp. *loose*) when the working capital that renders financial constraints irrelevant for bidding is large (resp. small) relative to the exogenous cash flow. The laggard-leader scenario occurs most of the time, as the cost of working capital becomes negligible, when the financial constraint is sufficiently tight. Another consequence is that the same firm tends to win consecutive procurement contracts.[17] On the contrary, firms win each contract with the same proba-

[16] The uniqueness result is proved in the supplementary material.

[17] To the extend that joint profits are larger in the laggard-leader scenario than in the symmetric scenario, our result is related to the literature on increasing dominance due to efficiency effects (see Budd, Harris, and Vickers (1993), Cabral and Riordan (1994) and Athey and Schmutzler (2001).) The novelty of our model is that the underlying static game displays neither strategic complementarity nor strategic substitutability.

bility when the financial constraints are so loose that the symmetric scenario occurs every period. This explains our prediction of greater market concentration and asymmetric market shares, together with occasional leadership reversals, for larger projects to the extend that one can associate the tightness of the financial constraint to the project's size.[18]

Che and Gale (1998b) and Zheng (2001) already showed that the dispersion of markups can reflect heterogeneity of working capital if this is assumed to be sufficiently scarce.[19] We show that scarcity is the typical situation once we allow firms to choose their working capital. Note, however, that whereas they assume that the distribution of working capitals is the same across firms, our results show that this is seldom the case. This difference is important because the lack of asymmetries in the distribution of working capitals precludes the possibility of large expected money left on the table when private information is small.

In Galenianos and Kircher (2008)'s model of monetary policy, firms also choose working capital before competing in an auction. In their equilibrium firms also randomize their working capital due to the all pay auction structure. However, since their working capital is not bounded by cash, the laggard-leader scenario does not arise.

Our paper contributes to the dynamic oligopoly literature "an area where much work needs to be done and much work can be done," as emphasized by Cabral (2012). In his terminology our model is a properly defined dynamic oligopoly model since cash acts as a "physical" link across periods. In particular, it contributes to a recent literature that explains how asymmetries in market shares arise and persist in otherwise symmetric models. In particular, Besanko and Doraszelski (2004), and Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) show that firm-specific shocks can give rise to a dynamic of market shares similar to ours. The difference, though, is that the dynamic in our model arises because firms randomize their working capital due to the all pay auction structure.

Our characterization of the dynamics resembles that of Kandori, Mailath, and Rob (1993) in that we study a Markov process in which two persistent scenarios occur infinitely

[18]As we do in Appendix B.

[19]Che and Gale (1996, 2000), DeMarzo, Kremer, and Skrzypacz (2005) and Rhodes-Kropf and Viswanathan (2005) also studied the effect of some given financial constraints in auctions and Pitchik and Schotter (1988), Maskin (2000), Benoit and Krishna (2001) and Pitchik (2009) how bidders distribute a fixed budget in a sequence of auctions. The latter is not a concern in our setup because the profits are realized before the beginning of the next auction.

often and we ask which of the two occurs most of the time as the randomness vanishes. We want to underscore that while the transition function of their stochastic process is exogenous, ours stems from the equilibrium strategies of the infinite horizon game. As in Cabral (2011), a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping, i.e. the system is very persistent but moves across extremely asymmetric states.

Other explanations for the persistency of markups are asymmetric information (i.e. the standard model), capacity constraints and collusion. Our model is empirically distinguishable from these models in that it predicts negative correlation between the laggard's cash and the bids (or the price). An alternative to distinguish our model from the standard model and the model of capacity constraints when the laggard's cash is not observable by the econometrician is to use as a proxy either the progress payments of the firms uncompleted contracts[20] or past bids.[21] These proxies do not explain the current bids in the standard model once one controls by costs or in the models of capacity constraints once one controls by backlog and costs, see Bajari and Ye (2003) and Jofre-Bonet and Pesendorfer (2003).

Another way in which our model is empirically distinguishable from the models of collusion is that the time average of the price may decrease with patience[22] and winning today increases the probability of winning tomorrow. Collusive models predict that patience increases the time average of the price, see Green and Porter (1984) and Athey and Bagwell (2001), and that winning today has either no effect on winning tomorrow, see Athey, Bagwell, and Sanchirico (2004), or decreases its probability, see McAfee and McMillan (1992), Athey and Bagwell (2001) and Aoyagi (2003).

Section 2 defines our canonical model of procurement with financial constraints. Section 3 analyzes the static model and Section 4 the dynamic model. Section 5 concludes. We also include an appendix with the more technical proofs (Appendix A) and an extension of our model to endogenize financial constraints (Appendix B).

[20] The California Department of Trasnportation makes this information available in http://www.dot.ca.gov/hq/asc/oap/payments/public/ctnums.htm

[21] The latter holds true because the laggard's past and current cash are positively correlated.

[22] This is what happens when the financial constraint is sufficiently loose.

2 A Reduced Form Model of Procurement with Financial Constraints

In this section, we describe a model of procurement that we later embed in the models of Sections 3 and 4. In this model, two firms[23] compete for a procurement contract of common and known cost c in a first price auction: each firm submits a bid, and the firm who submits the lower bid gets the contract at a price equal to its bid.[24] We assume that firms can only submit bids that have secured financing, i.e. *acceptable* bids. We assume that the minimum acceptable bid strictly decreases with the firm's working capital w and we denote it by $\pi(w) + c$.[25] We assume π to be continuously differentiable.

As we discuss in the Introduction, our assumption that firms can only submit acceptable bids captures a wide range of institutional arrangements whose aim is to preclude firms from submitting bids that they cannot comply, and in particular bids that cannot be financed.[26] Alternatively, the sponsor may provide incentives to guarantee that firms only submit bids they can comply with, for instance, by making them bear some of the costs of default. The monotonicity of the set of acceptable bids arises naturally in markets in which firms have limited access to external financial resources, as we discussed in the Introduction and as we illustrate in Appendix B.

We call the *markup* to the difference between the winning bid and the cost of the procurement contract c relative to this cost, and *money left on the table* to the absolute

[23]As in all pay auctions, see Baye, Kovenock, and de Vries (1996), assuming more than two firms rises the problem of multiplicity of equilibria. It may be shown that there is always an equilibrium in which two firms play the strategies we propose below for our two-firm model and the other firms choose zero working capital.

[24]A sale auction of a good with common and known value V can be easily encompassed in our analysis assuming that $c = -V < 0$ and bids are negative numbers.

[25]Thus, the model of auctions with budget constraints analyzed by Che and Gale (1998b) in Section 3.2 corresponds in our framework with $\pi(w) = V - w$ and the interpretation of our model as a sale auction, see Footnote 24.

[26]For instance, Meaney (2012) says that:

> As well as considering the financial aspects of bids, the DfT[the sponsor] assesses the deliverability and quality of the bidders proposals so as to be confident that the successful bidder is able to deliver on the commitments made in the bidding process.

value of the difference between the two bids relative to the cost of the procurement contract c. We say that the firm is *financially constrained* if its working capital is such that only bids strictly above the cost of the procurement contract are acceptable. We denote by θ the working capital that renders financial constraints irrelevant for bidding, i.e. $\theta = \pi(0)^{-1}$, and we assume that $\theta \in (0, \infty)$.

3 A Static Model

Firms start with some cash. We assume the firm's cash to be publicly observable. Each firm chooses simultaneously and independently how much of its cash to keep as working capital and an acceptable bid for a market as described in Section 2.

The firm maximizes the payments to the shareholders, its *consumption* hereafter, plus the discounted sum, at rate $\beta < 1$, of the working capital and the profits. Hence, a unit increase in working capital is costly in the sense that it reduces the current utility in one unit and increases the future utility in β. Throughout this paper, we say that the *cost of working capital becomes negligible* when β tends to 1.

We restrict to the case in which both firms start with different cash[27] and call *leader* to the firm that starts with more cash and *laggard* to the other firm. We assume that in case of a tie, the leader wins.[28]

We start by simplifying the strategy space. First, to carry more working capital than strictly necessary to make the bid acceptable is strictly dominated, in particular a bid b and a working capital w such that $b > \pi(w)+c$ is strictly dominated by the same bid b and a working capital $\tilde{w}$ such that $b = \pi(\tilde{w})+c$.[29] Next, zero working capital strictly dominates any working capital strictly larger than the working capital $\overline{\nu}^{\beta}$ for which the discounted procurement profits associated to the minimum acceptable bid $\beta\pi(\overline{\nu}^{\beta})$ are equal to the

[27]We endogenize the firms' cash in the model of next section and show that our interest in the case in which both firms start with different cash is motivated in that it holds almost surely along the equilibrium path because of the mixed strategies firms use every period.

[28]As it is usually the case in Bertrand games and all pay auctions, we deviate from the more natural uniformly random tie-breaking rule to guarantee the existence of an equilibrium. In our game, it applies the usual conjecture that a sufficiently fine discretization of the action space would overcome the existence problem and yield the same results but at the cost of a more cumbersome notation.

[29]The firm wins in the same occasions but saves on the cost of working capital.

implicit costs associated to postponing consumption $(1-\beta)\overline{\nu}^{\beta}$.

The game after the elimination of strictly dominated strategies verifies the *all pay auction structure* in the sense that the strategy space is unidimensional, i.e. each firm chooses a working capital and the corresponding minimum acceptable bid, and that the firm with higher working capital wins the procurement contract but carrying working capital is costly for both firms. As it is usually the case in all pay auctions, there is no pure strategy equilibrium. This can be easily understood when both firms' cash is weakly larger than $\overline{\nu}^{\beta}$. If both firms choose different working capitals, the one with more working capital has a strictly profitable deviation: to decrease marginally its working capital.[30] If both firms choose the same working capital w, there is also a strictly profitable deviation: to increase marginally its working capital if $w<\overline{\nu}^{\beta}$, and to choose zero working capital if $w=\overline{\nu}^{\beta}$.[31]

The usual indifference condition that holds in a mixed strategy equilibrium is verified in the margin if firms randomize with a cumulative distribution $\tilde{F}(w)$ that solves the differential equation,

$$1-\beta=\tilde{F}'(w)\beta\pi(w)+\tilde{F}(w)\beta\pi'(w). \tag{1}$$

This is because $\tilde{F}'(w)$ equalizes the marginal cost of increasing working capital around w with its marginal revenue. The former is equal to $1-\beta$ and the latter is equal to the sum of a positive and a negative effect. The positive effect $\tilde{F}'(w)\beta\pi(w)$ arises because the firm moves from losing to winning the procurement contract when the rival chooses a working capital close to w. The negative effect $\tilde{F}(w)\beta\pi'(w)$ arises because when the firm wins it makes lower profits.

We distinguish two scenarios depending on the amount of laggard's cash. We call the *symmetric scenario* to the case in which the laggard's cash is weakly greater than $\overline{\nu}^{\beta}$, and

[30]It saves on the cost of working capital and increases the profits from the procurement contract without affecting to the cases in which the firms wins.

[31]In the former case, the deviation is profitable because winning the procurement contract at $w<\overline{\nu}^{\beta}$ gives strictly positive profits and the deviation breaks the tie in favor of the deviating firm with an arbitrarily small increase in the cost of working capital and an arbitrarily small decrease in the profits from the procurement contract. In the latter case, $w=\overline{\nu}^{\beta}$ implies that one of the firms is winning with a probability strictly less one, and hence the definition of $\overline{\nu}^{\beta}$ means that this firms makes strictly lower payoffs than with zero working capital.

the *laggard-leader scenario* to the complementary case.

We call the *symmetric strategy* to the distribution function with support[32] $[0, \overline{\nu}^{\beta}]$ that verifies the differential Equation (1) with initial condition $\tilde{F}(0) = 0$. This distribution is equal to $\frac{(1-\beta)w}{\beta\pi(w)}$ for $w \in [0, \overline{\nu}^{\beta}]$.

Proposition 1. *There is a unique equilibrium in the symmetric scenario. In this equilibrium both firms play the symmetric strategy.*

This equilibrium verifies the usual property of all pay auctions that bidders without competitive advantage get their outside opportunity, i.e. the payoff of carrying zero working capital and losing the procurement contract.

Corollary 1. *In the symmetric scenario: (i) both the laggard and the leader have the same probability of winning the procurement contract, (ii) a marginal change in the initial distribution of cash has no effect on the equilibrium play.*

Besides, one can deduce by inspection of the symmetric strategy the following corollary:

Corollary 2. *In the symmetric scenario, each firm's working capital converges*[33] *to* θ *and both the markup and the money left on the table converge to zero, as the cost of working capital becomes negligible.*

In the standard auction model (Krishna (2002), Chapter 2), the price converges to the production cost and money left on the table vanishes as cost heterogeneity vanishes, i.e. as the firms' distribution of costs converges to the same value. This limit outcome also arises in the symmetric scenario, and in this sense financial constraints become irrelevant as the cost of working capital becomes negligible.

The above equilibrium strategy is not feasible for the laggard in the laggard-leader scenario. We denote the laggard's cash by m_l and we call the *laggard strategy* to the distribution function with support $[0, m_l]$ that verifies the differential Equation (1) and the condition $\tilde{F}(m_l) = 1$. This distribution is equal to $\frac{\beta\pi(m_l)-(1-\beta)(m_l-w)}{\beta\pi(w)}$ for $w \in [0, m_l]$. We call the *leader strategy* to the distribution function with support $[0, m_l]$ that verifies

[32] We use the definition of support of a probability measure in Stokey and Lucas (1999). According to their definition, the support is the smallest closed set with probability one.

[33] In what follows, convergence is always in distribution unless stated otherwise.

the differential Equation (1) with initial condition $\tilde{F}(0) = 0$. This distribution is equal to $\frac{(1-\beta)w}{\beta\pi(w)}$ for $w \in [0, m_l)$. Note that the laggard strategy puts an atom at zero and the leader strategy at m_l.[34]

Proposition 2. *There is a unique equilibrium in the laggard-leader scenario. In this equilibrium, the laggard plays the laggard strategy and the leader plays the leader strategy.*

Both firms put their atom of probability at points that do not upset the incentives of the rival to play its equilibrium randomization. There is only one such point for the laggard, whereas the leader's atom is at the minimum working capital which ensures that it wins the procurement contract. Interestingly, it can be shown that the laggard gets its outside opportunity, as in the symmetric scenario, whereas the leader gets an additive positive premium. The latter is a consequence of the leader's ability to undercut any acceptable bid of the laggard and the fact that any such bid is strictly profitable.

To discuss our first main result, we say that there is *at most ρ uncertainty* about a random vector when there exists a realization for which both agents put probability at least $1 - \rho$.[35]

Corollary 3. *In the laggard-leader scenario: (i) the leader wins the procurement contract with strictly greater probability than the laggard; (ii) a marginal increase in the laggard's cash changes the equilibrium outcome and, in particular, decreases the expected markup; (iii) the expected "money left on the table" is at least* $(1-\rho)\frac{\pi(0)-\pi(m_l)}{c}$ *and there is at most ρ uncertainty about the working capitals if* $\frac{\pi(m_l)}{\pi(0)}\left(1 - \frac{(1-\beta)m_l}{\beta\pi(m_l)}\right)^2 > 1 - \rho$.

Corollary 3 (i) follows from the comparison of the laggard and leader strategies, (ii) from the fact that an increase in the laggard's cash shifts to the right, in the sense of first order stochastic dominance, the laggard and leader strategies. To understand (iii) note that simple algebra shows that the left hand side of the inequality is the probability that both firms play at their mass points, and that the "money left on the table" when both firms play at their mass points is equal to $\frac{\pi(0)-\pi(m_l)}{c}$.

[34]Interestingly, this equilibrium has similar qualitative features as the equilibrium of an all pay auction in which both agents have the same cap but the tie-breaking rule allocates to one of the agents only. The latter model has been studied in an independent and simultaneous work by Szec (2010).

[35]A natural extension could handle continuous distributions.

Corollary 3 (ii)-(iii) captures our first main result. Point (ii) explains why the dispersion of markups and "money left on the table" observed across auctions can be explained by variations in the laggard's cash. Note that a similar argument also applies with respect to changes in π. Corollary 4 below shows that this result persist even as the cost of working capital becomes negligible. Point (iii) casts doubts about the usual interpretation of the dispersion of markups and money left on the table as indicative of incomplete information. For instance, in the linear specification of Appendix B an application of (iii) means that there is at most $\frac{m_l}{\theta}$ uncertainty and expected money left on the table at least $\left(1 - \frac{m_l}{\theta}\right) \frac{m_l}{c}$ as the cost of working capital vanishes. Thus, a sufficiently large θ implies almost no uncertainty together with sizable money left on the table.

In the model of Section 4, we show by means of numerical simulations that the endogenous distribution of the laggard's cash has sufficient variability to generate significant dispersion of markups and "money left on the table" across otherwise identical auctions. Interestingly, these results are provided for parameter values for which there is little uncertainty and the cost of working capital is small.

Corollary 4. *In the laggard-leader scenario, as the cost of working capital becomes negligible, the leader's equilibrium working capital converges to the laggard's cash, the probability that the leader wins the procurement contract converges to one, the markup converges to* $\frac{\pi(m_l)}{c}$*, and the expected money left on the table converges to* $\frac{\pi(m_l)}{c} \ln \frac{\pi(0)}{\pi(m_l)}$.

The corollary follows by inspection of the leader and laggard strategy. Intuitively, as the cost of working capital becomes negligible, the leader chooses its working capital to make sure it wins with probability one, whereas the laggard's distribution is such that the leader does not have incentives to reduce its working capital and thus the price that it pays.

Corollary 4 means that for β close to one, the money left on the table is increasing in the laggard's cash, if the latter is smaller than $\hat{m}$, and the markup is decreasing in the laggard's cash.[36] This is the basis for our second main result. To understand why some of the empirical analysis of the size of markups may be biased downwards or upwards with a bias positively correlated with the laggard's cash imagine bid data from several auctions

[36]Where $\hat{m}$ is the unique solution to $\ln \frac{\pi(0)}{\pi(\hat{m})} = 1$

with identical financial conditions and suppose the data are generated by our model. On the one hand, if the laggard has little cash there is little money left on the table and large markups. However, a researcher that assumed the standard model would conclude that there is little cost heterogeneity and, as a consequence, small markups, i.e. the estimation would be biased downwards. On the other hand, if the laggard has relatively large cash, though less than $\hat{m}$, there is sizable "money left on the table" and relatively low markups. However, a researcher that assumed the standard model would conclude that there is large cost heterogeneity and, as a consequence, large markups, i.e. the estimation would be biased upwards. Note that a similar argument can also be done for appropriate shifts in π.

4 The Dynamic Model

In this section, we endogenise the distribution of cash by assuming that it is derived from the past market outcomes. We provide conditions under which the laggard-leader scenario occurs most of the time. This approach provides a natural framework to study market shares and its dynamics, and to analyze the conventional wisdom in economics that "auctions [still] work well if raising cash for bids is easy." We provide numerical results that, on the one hand, complement the previous section analysis of our first main result and, on the other hand, shed some light on the concentration and persistency of market shares.

4.1 The Game

We consider the infinite repetition of the time structure of the game in the last section in which the cash in the first period is exogenous and afterwards it is equal to its working capital in the previous period plus the profits in the procurement contract and some exogenous cash flow $\underline{m} > 0$.[37] We assume that $\underline{m}$ is constant across time and firms, and interpret it as derived from other activities of the firm. The firm maximizes the sum of the discounted value of its expected consumptions. Hence, in any period t in which firms start with cash (m_t^1, m_t^2), choose working capitals (w_t^1, w_t^2), and Firm 1 wins the

[37] Our results can be extended to the case $\underline{m} = 0$.

procurement contract with profits Π_t, the next period distribution of cash is equal to $(m^1_{t+1}, m^2_{t+1}) = (w^1_t + \Pi_t + \underline{m}, w^2_t + \underline{m})$.

The following assumption is necessary in the proof of Proposition 3.

Assumption 1. $w + \pi(w) + \underline{m} \geq \theta$.

Since $\pi(w)$ is the minimum profit that a firm with working capital w can make when it wins the procurement contract, this assumption implies that the firm that wins the procurement contract one period, starts next period with cash at least θ, and hence it is not financially constrained. As we explain after Proposition 3, this assumption guarantees that firms do not want to carry more working capital than strictly necessary to make the bid acceptable.

A Markov strategy consists of a working capital distribution, σ, and a bid function, b. The working capital distribution of a firm with cash m that faces a rival with cash m' is a randomization over the feasible working capitals described by its cumulative distribution function $\sigma\left(.\middle|\, m, m'\right) \in \Delta(m)$, where $\Delta(m)$ denotes the set of cumulative distribution functions with support in $[0, m]$. The bid function of a firm with cash m and working capital w facing a rival with cash m' is an acceptable bid $b\left(w, m, m'\right) \geq c + \pi\left(w\right)$.

We refer to the beginning of the period lifetime payoff of a firm that has cash m when its rival has m' as the firm's value function and denote it by $W\left(m, m'\right)$.

We denote by $\phi(p, p', m, m')$ the procurement profits of a firm with cash m, and working capital w that bids p against a rival with cash $m'(\neq m)$ and working capital w' that bids p'. Formally:

$$\phi(p, p', m, m') \equiv \begin{cases} p - c & \text{if } p < p' \text{ or both } p = p' \text{ and } m > m' \\ 0 & \text{if } p > p' \text{ or both } p = p' \text{ and } m < m' \end{cases}$$

Definition: A (symmetric) Bidding and Investment (BI) equilibrium is a value function W, a bidding function b and a working capital distribution σ such that for every $m, m' \in \mathbb{R}_+$, $m \neq m'$, W is the value function and $\sigma\left(\cdot\middle|\, m, m'\right)$ and $b(\cdot, m, m')$ are the optimizers of the following Bellman equation:

$$\max_{\substack{\tilde{\sigma} \in \Delta(m) \\ \tilde{b}(w) \geq \pi(w) + c}} \int \left[m - w + \beta \int \tilde{W}_{\tilde{b}, b}\left(w, w', m, m'\right) \sigma\left(dw'\middle|\, m', m\right) \right] \tilde{\sigma}\left(dw\right),$$

where $\tilde{W}_{\tilde{b},b}(w, w', m, m')$ is equal to:

$$W(w + \underline{m} + \phi(\tilde{b}(w), b(w', m, m'), m, m'), w' + \underline{m} + \phi(b(w', m', m), \tilde{b}(w), m', m)).$$

Note that our equilibrium definition does not put any constraint in information sets in which both firms have the same amount of cash. This simplifies our analysis without upsetting our main results because in our proposed equilibrium firms do not have the same amount of cash with positive probability neither in the equilibrium path nor after a unilateral deviation.[38]

4.2 The Equilibrium Strategies

In what follows, we define a value function, a bidding function and a working capital distribution and show that they are a BI equilibrium. The strategies we propose correspond to a generalization of the equilibrium strategies in Section 3. The bid function is, as in the static model, the minimum acceptable bid which is profitable, this is $b^*(w, m, m') \equiv c + \pi(w)^+$.[39]

To define the working capital distribution, we use some auxiliary functions. First, let $\mathcal{P}$ be the set of continuous decreasing functions $\hat{\Psi} : \mathbb{R}_+ \to \left[0, \frac{\beta}{1-\beta}\pi(0)\right]$ that verify that $\hat{\Psi}(\theta) = 0$. Let $\overline{x} \in [(\theta - \underline{m})^+, \theta]$ be the unique solution to $\frac{1-\beta}{\beta}\frac{(\overline{x}-(\theta-\underline{m})^+)}{\pi(\overline{x})} = 1$. Lemma 2 in the Appendix characterizes the properties of the solutions to:

$$\frac{1-\beta}{\beta} = \hat{F}'(w)\left(\pi(w) + \hat{\Psi}(w + \underline{m})\right) + \hat{F}(w)\pi'(w), \tag{2}$$

for any $\hat{\Psi} \in \mathcal{P}$. There is a unique solution passing by each point in $[0, \overline{x}] \times [0, 1]$. The solution starting at $(0, 0)$ passes by a point $(\hat{\nu}^{\hat{\psi}}, 1)$ where $\hat{\nu}^{\hat{\psi}} \in [\overline{\nu}^{\beta}, \overline{x}]$. For any $m \geq \hat{\nu}^{\hat{\Psi}}$, we let $\hat{F}(\cdot, \hat{\Psi}, m)$ be a distribution function that between zero and $\hat{\nu}^{\hat{\Psi}}$ is equal to the solution of Equation (2) that passes by $(0, 0)$. For any $m < \hat{\nu}^{\hat{\Psi}}$, we let $\hat{F}(\cdot, \hat{\Psi}, m)$ be a distribution function that between zero and m is equal to the solution of Equation (2) passing by $(m, 1)$ and that puts the remaining probability at 0.

[38] We study in the supplementary material that the natural extension of our equilibrium that takes into account these information sets in a version of our model with finitely many periods has a unique equilibrium and it is symmetric. The equilibrium studied below corresponds to the limit of that equilibrium.

[39] We adopt the convention that $a^+ = a$ if $a \geq 0$ and $a^+ = 0$ otherwise.

Consider the following functional equation:

$$\hat{\Psi}(m) = \beta \hat{F}(0, \hat{\Psi}, m) \left(\pi(0) + \hat{\Psi}(\underline{m}) \right) \tag{3}$$

Lemma 3 in the Appendix applies Schauder Fixed-Point Theorem to show that this functional equation has a solution in $\mathcal{P}$ that we denote by Ψ^{β}.

Let $\nu^{\beta} = \hat{\nu}^{\Psi^{\beta}}$. For $m \geq \nu^{\beta}$, let $F^{\beta}_{l,m}$ and $F^{\beta}_{L,m}$ be both equal to $\hat{F}(\cdot, \Psi^{\beta}, \nu^{\beta})$. For $m \in [0, \nu^{\beta})$, let $F^{\beta}_{l,m}$ be equal to $\hat{F}(\cdot, \Psi^{\beta}, m)$ and let $F^{\beta}_{L,m}$ be equal to a truncation of $\hat{F}(\cdot, \Psi^{\beta}, \nu^{\beta})$ at m with support $[0, m]$ that puts the remaining probability at m. Our proposed working capital distribution is defined by:

$$\sigma^*(\cdot | m, m') = \begin{cases} F^{\beta}_{l,m}(\cdot) & \text{if } m < m' \\ F^{\beta}_{L,m'}(\cdot) & \text{if } m > m' \end{cases}$$

Let W^* be a value function strictly increasing in the first argument and weakly decreasing in the second one that verifies:

$$W^*\left(m, m'\right) = \begin{cases} m + \frac{\beta}{1-\beta}\underline{m} & \text{if } m < m' \\ \\ m + \frac{\beta}{1-\beta}\underline{m} + \Psi^{\beta}\left(m'\right) & \text{if } m > m'. \end{cases} \tag{4}$$

Thus, $\Psi^{\beta}(m')$ is an additive premium associated to being leader.

Proposition 3. (W^*, σ^*, b^*) *is a BI equilibrium.*

The intuition behind the proposition is based on our results in the static model. This is because the all pay auction structure is inherited from one period to the previous one in the following sense: if the payoffs of the reduced game in period t verify the all pay auction structure, the payoffs of the reduced game in period $t-1$ also verify it. To see why, note that the usual property of all pay auctions that bidders without competitive advantage get their outside opportunity implies that the laggard's continuation payoffs in period $t-1$ are equal to the discounted consumption value of its cash in period t. The leader's continuation payoffs have an additive premium which is a consequence of the leader's ability to carry sufficient working capital to undercut any acceptable bid of the laggard. This ability is independent of the amount of cash the leader has and so it is the premium. Consequently, the value of a marginal increase in cash in period t is equal to its discounted consumption value plus the value of switching from laggard to leader. Note that a marginal increase in cash switches the leadership when the cash of the firms is the same and, by Assumption 1, no less than θ. In this case, the premium is zero by definition of Ψ^{β}. We can thus conclude that a unit increase in working capital, keeping constant the bid, is costly in the sense that it reduces the current consumption in one unit but only increases the future utility in its discounted value β. This means, as in the static model, that it is not profitable to carry more working capital than strictly necessary to make the bid acceptable, and that in the corresponding unidimensional simplification of the strategy space, the firm that carries more working capital wins but carrying working capital is costly for both firms.[40]

We can also distinguish here between a symmetric and a laggard-leader scenarios, and it may be shown that an analogous version of Corollaries 1-4 holds true as well.

4.3 The Equilibrium Dynamics

To study the frequency of the symmetric and the laggard-leader scenarios, we study the stochastic process of the laggard's cash induced by our equilibrium.[41] Its state space is equal to $[\underline{m}, \nu^{\beta} + \underline{m}]$ because the procurement profits are non negative and none of the firms' working capitals is larger than ν^{β}. Moreover, the laggard's cash is determined by the equilibrium working capitals and bids in the previous period. The latter are a function of the former which have a distribution that only depends on the laggard's cash in the previous period. Thus, the laggard's cash follows a Markov process. Its transition

[40]Note that the property that firms do not want to carry more working capital than strictly necessary to make the bid acceptable is also a property of the unique equilibrium of the finite version of our model. This is because the recursive argument in the previous paragraph can be applied starting from the last one since the last period is the same game as the static model. A formal argument is provided in the supplementary material.

[41]To apply our analysis in the previous section, we restrict to information sets in which firms hold different amounts of cash. That is, we assume firms begin with different amounts of cash and so it is easy to see that the cash of the firms remain heterogeneous along the equilibrium path with probability one. Note also that the assumption of different initial cash holdings is without loss of generality. Indeed, it can be shown that if firms began with identical cash, their cash holdings would become heterogeneous after one period with probability one in any equilibrium that is the limit of the unique equilibrium of the finite horizon version of the model that we study in the supplementary material.

probabilities $Q^{\beta} : [\underline{m}, \nu^{\beta} + \underline{m}] \times \mathcal{B} \to [0,1]$, for $\mathcal{B}$ the Borel sets of $[\underline{m}, \nu^{\beta} + \underline{m}]$, can be easily deduced from the equilibrium. In particular, they are defined by:[42]

$$Q^{\beta}\left(m, [\underline{m}, x]\right) = 1 - \left(1 - F^{\beta}_{l,m}\left(x - \underline{m}\right)\right)\left(1 - F^{\beta}_{L,m}\left(x - \underline{m}\right)\right). \tag{5}$$

This expression is equal to one minus the probability that both the laggard and the leader's working capitals are strictly larger than $x - \underline{m}$.

A distribution $\mu : \mathcal{B} \to [0,1]$ is invariant if it verifies:

$$\mu\left(\mathcal{M}\right) = \int Q^{\beta}\left(m, \mathcal{M}\right)\mu\left(dm\right) \qquad \text{for all } \mathcal{M} \in \mathcal{B}. \tag{6}$$

Standard arguments[43] can be used to show that there exists a unique invariant distribution and it is globally stable. A suitable law of large numbers can be applied to show that the fraction of the time that the Markov process spends on any set $\mathcal{M} \in \mathcal{B}$ converges (almost surely) to $\mu(\mathcal{M})$.

To characterize the frequency of the laggard-leader and the symmetric scenarios, we distinguish two cases. We say that the *financial constraint is loose* when the ratio $\frac{\theta}{\underline{m}}$ is weakly less than one. Since $\nu^{\beta} \leq \theta$ for any $\beta < 1$, this condition guarantees that the symmetric scenario occurs every period. We also say that the *financial constraint is tight* when the ratio $\frac{\theta}{\underline{m}}$ is strictly greater than 4. This condition means that a firm that begins a period with cash $\underline{m}$ and does not win the next three procurement contracts is still financially constrained after three periods. We refer to the ratio $\frac{\theta}{\underline{m}}$ as a measure of the severity of the financial constraint.

Typically, the frequency of each scenario depends on a non trivial way on the transition probabilities. An exception is when the transition probabilities do not depend on the state which corresponds in our model to an exogenous cash flow sufficiently large to guarantee that only the symmetric scenario occurs, e.g. when the financial constraint is loose. In this case, the transition probabilities do not dependent on the laggard's current working capital and $\mu^{\beta}([\underline{m}, x)) = 1 - \left(1 - \hat{F}(x - \underline{m}, \Psi^{\beta}, \nu^{\beta})\right)^2$. A version of Corollaries 1 and 2 characterizes the properties of the equilibrium path.[44] More generally, the frequency of

[42] As a convention, we denote by $[\underline{m}, \underline{m}]$ the singleton $\{\underline{m}\}$.

[43] See Hopenhayn and Prescott (1992).

[44] In the more difficult case in which the transition probabilities depend on the state, the invariant distribution associated to the limit transition probabilities as the cost of working capital becomes negligible has

each scenario depends on the severity of the financial constraint as illustrated in Figure 1. In this figure, we plot the frequency of the symmetric scenario and the frequency of what we call the *extreme laggard-leader scenario*. This is when the laggard's cash is equal to $\underline{m}$.

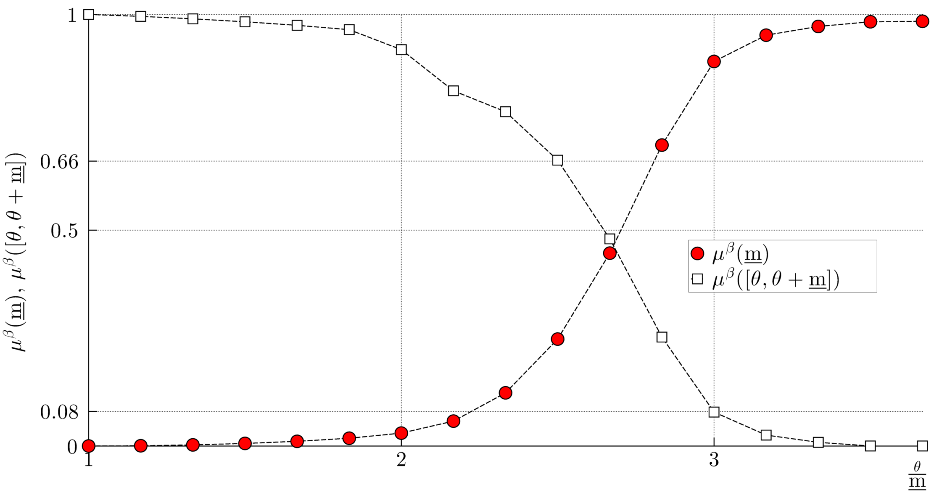

Figure 1: $\mu^\beta(\underline{m})$ and $\mu^\beta([\theta, \theta+\underline{m}])$ as a function of $\frac{\theta}{\underline{m}}$ for $\pi(x) = \theta - x$ and $\beta = 0.96$.

Note that the sum of the probability that μ^β puts on the extreme laggard-leader scenario and on the symmetric scenario is close to one for β close to one. This is because, as we show in Lemma 6 in the Appendix, the probability that stationary distribution puts outside these sets tends to zero as the cost of working capital becomes negligible.

We say that *the extreme laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible* when $\mu^\beta(\{\underline{m}\}) \to 1$.

Theorem 1. *If the financial constraint is tight and* $\lim_{\beta\to 1} \Psi^\beta(\underline{m}) = \infty$*, the extreme laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible.*

Next lemma gives a sufficient condition for $\lim_{\beta\to 1} \Psi^\beta(\underline{m}) = \infty$. Interestingly, the function π consistent with our model in Appendix B satisfies this sufficient condition

an easy characterization. This is because the transition probabilities become degenerate and concentrate its probability in one point only, either $\underline{m}$ or $\theta + \underline{m}$, and thus any distribution with support in $\{\underline{m}, \theta + \underline{m}\}$ is an invariant distribution. Since there are multiple invariant distributions, we cannot apply a continuity argument to characterize what happens when the cost of working capital is small.

when the financial constraint is tight.

Lemma 1. *If* $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$*, then* $\lim_{\beta\to 1} \Psi^{\beta}(\underline{m}) = \infty$*.*

The next corollary can be derived from Theorem 1 and the property of the extreme laggard-leader scenario, proved in Lemma 5 in the Appendix, that the laggard and the leader play with probability one at its atom when the cost of working capital becomes negligible.

Corollary 5. *If the financial constraint is tight and* $\lim_{\beta\to 1} \Psi^{\beta}(\underline{m}) = \infty$*, as the cost of working capital becomes negligible, the fraction of the time the following properties hold converges to one (almost surely): (i) the leader wins the procurement contract; (ii) the money left on the table is equal to* $\frac{\pi(0)-\pi(\underline{m})}{c}$ *and there is* 0 *uncertainty, and (iii) the markup is equal to* $\frac{\pi(\underline{m})}{c}$*.*

4.4 Numerical Solutions

In this section, we compute numerically the invariant distribution for empirically grounded values of the parameters[45] and $\pi(w) = \theta - w$, motivated by our model in Appendix B. Since π and, hence μ, are independent of c, any measure of markups provided is arbitrary unless we provide a relation of c with the rest of the variables of the model. We assume that $\frac{\theta}{c}$ is constant which is an implication of our analysis in Appendix B.

The left and right panel of Figure 2 show how the expected and standard deviations, respectively, of the markup and the money left on the table, change with the severity of the financial constraint.[46]

Figure 2 illustrates our first main result. The left panel quantifies our results in Corollary 3(ii) in the context of our dynamic model. It shows that exogenous differences in

[45] Early work of Hong and Shum (2002) suggests that firms that do highway maintenance typically bid in 4 contracts per year: "a data set of bids submitted in procurement contract auctions conducted by the NJDOT in the years 1989-1997 [...] firms which are awarded at least one contract bid in an average of 29-43 auctions." See also Porter and Zona (1993). Thus, we compute annual market shares for years of four periods. We also choose $\beta = 0.9602$ so that the annual discount rate is 0.85, slightly higher than the 0.80 used in Jofre-Bonet and Pesendorfer (2003), implying an annual expected cost of working capital of 0.15.

[46] It may be shown that one can obtain the graph corresponding to different values of $\frac{\theta}{c}$ simply by multiplying the values in the vertical axis in Figure 2 by $\frac{\theta}{c}$.

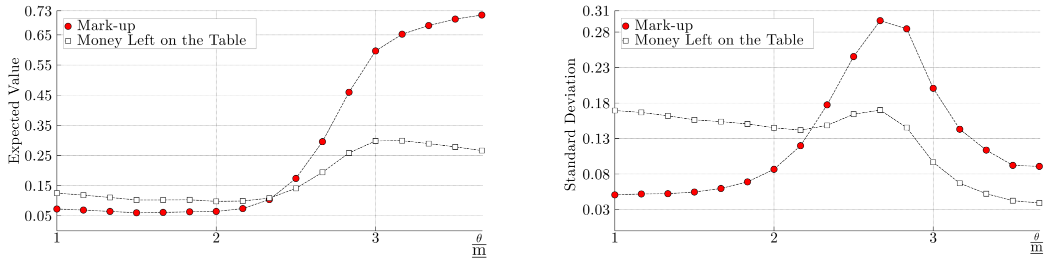

Figure 2: Expected and st. deviations of markup and money left on the table for $\frac{\theta}{c} = 1$.

the severity of the financial constraint explain differences in markups and money left on the table. The right panel shows that markups and money left on the table have significant volatility across auctions, for a given ratio $\frac{\theta}{m}$, due to the endogenous volatility of the firm's working capital and cash. Note that, by continuity, Corollary 5(ii) implies that uncertainty vanishes as the ratio $\frac{\theta}{m}$ approaches 4. Indeed, one can deduce from Lemma 5 in the Appendix that for any $\frac{\theta}{m}$ there is almost zero uncertainty for any reasonable measure of uncertainty.

Finally, we illustrate our third main result in Figure 3.

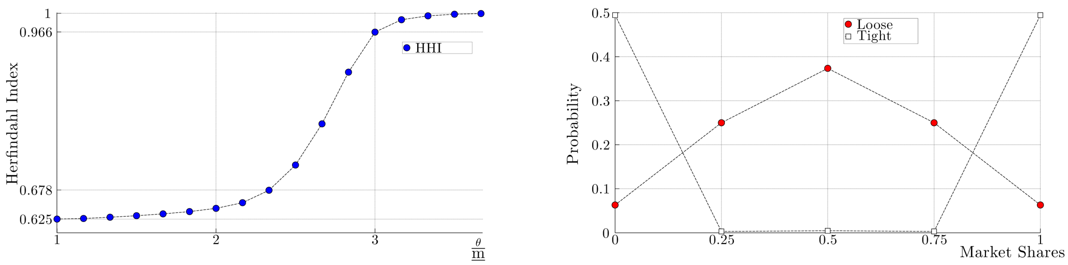

Figure 3: HHI and invariant distribution of annual market shares (1 year= 4 periods)

The left panel shows how the Herfindahl-Hirschman index (HHI) changes with the severity of the financial constraint. The firms internal financial decisions imply that concentration increases with the severity of the financial constraint. The right panel shows that this effect appears together with persistent asymmetries in market shares. It shows that in the case of tight financial constraints the same firm wins all the annual procurement contracts 98.92% of the years. On the contrary, in the case of a loose financial constraint there is little concentration in that each firms wins at least 25% of the annual procurement contracts in 87% of the years.

Similar conclusions can be derived with respect to the persistency of the leadership.

Figure 4 shows how the probability of a leadership reversal after 22 years changes with the severity of the financial constraint.[47]

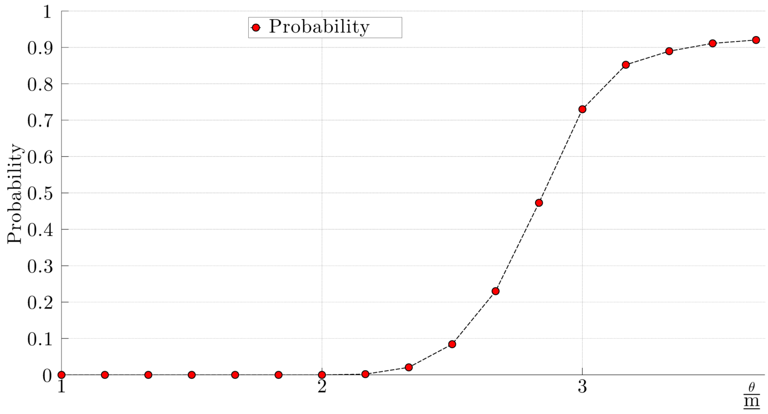

Figure 4: Lower bound for the probability of no leadership reversal in 22 years.

Therefore, a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping as in Cabral (2011), i.e. the system is very persistent but moves across extremely asymmetric states.

The above results explain our prediction of greater market concentration and asymmetric market shares, together with occasional leadership reversals, for larger projects to the extend that one can associate the severity of the financial constraint to the project's size, as we show in Appendix B.

We underscore that the concentration effects discussed in this section arise even in the absence of exogenous shocks unlike those obtained by Besanko and Doraszelski (2004), and Besanko, Doraszelski, Kryukov, and Satterthwaite (2010).

5 Conclusions

We have studied a model of bidding markets with financial constraints. A key element of our analysis is that the stage at which firms choose their working capitals resembles an all pay auction with caps.

The above features, and thus our results, seem pertinent for other models of investing under winner-take-all competition, like patent races. A natural extension should consider

[47]This time horizon allow the comparison of the predictions of our model with the stylized fact about unchanging industry leadership highlighted by Besanko and Doraszelski (2004).

alternative models of winner-take-all competition with financial constraints. Another interesting extension is to allow for private information about costs. This is the natural framework to test our model versus the standard one as it nest both of them. Existing results for all pay auctions and general contests[48] suggest these may be fruitful lines of future research.

Finally, our analysis points out a tractable way to incorporate the dynamics of liquidity in Galenianos and Kircher's (2008) analysis of monetary policy.

Appendix

A Proofs

Proof of Proposition 1

Let $\tilde{F}$ be the distribution function of the symmetric strategy. To see why the proposed strategy is an equilibrium note that the expected payoff of Firm i with cash m^i when it chooses working capital w and the other firm randomizes its working capital according to $\tilde{F}$ is equal to:

$$m^i - (1-\beta)w + \beta\pi(w)\tilde{F}(w),$$

which by definition of $\tilde{F}$ is equal to m^i for any $w \leq \overline{\nu}^\beta$, and strictly less than m^i otherwise, as required.

As we argue in the text, we can restrict to strategies with support in $[0, \overline{\nu}^\beta]$. We prove two properties that any equilibrium (σ^1, σ^2) must verify. Later, we show that the proposed strategy is the only one that verifies them. These two properties also hold true in the more general case in which we do not restrict to $m^1, m^2 \geq \overline{\nu}^\beta$.

Claim 1: *If $w \in (0, \overline{\nu}^\beta]$ belongs to the support of σ^i, then $\sigma^j([w-\epsilon, w]) > 0$ ($j \neq i$) for any $\epsilon > 0$.*

[48] Amann and Leininger (1996) study the relationship between the equilibrium of the all pay auction with and without private information and Alcalde and Dahm (2010) study the similarities between the equilibrium outcome in an all pay auction and in some other models of contests.

In order to get a contradiction suppose that $w \in (0, \overline{\nu}^{\beta}]$ belongs to the support of σ^i and $\sigma^j([w-\epsilon, w]) = 0$ for some $\epsilon > 0$. We shall argue that Firm i has a profitable deviation when Firm j plays σ^j. The contradiction hypothesis has two implications. (a) $w - \epsilon$ gives Firm i strictly greater expected payoffs than w since the former saves on the cost of working capital and increases the profit when winning without affecting the probability of winning. (b) Firm i's expected payoffs are continuous in its working capital at w since σ^j does not put an atom at w. (a) and (b) means that there exists an $\epsilon' \in (0, \epsilon)$ such that any working capital in $(w - \epsilon', w + \epsilon')$ gives strictly less expected payoffs than a working capital $w - \epsilon$. The fact that w belongs to the support of σ^i means that i puts strictly positive probability in $(w - \epsilon', w + \epsilon')$ and thus there is a profitable deviation.

Claim 2: If there exists some $w \in [0, \min\{\overline{\nu}^{\beta}, m^j\})$ such that $\sigma^j([w-\epsilon, w]) > 0$ for any $\epsilon > 0$, then $\sigma^i(\{w\}) = 0$ $(i \neq j)$.

By contradiction, suppose an $w \in [0, \min\{\overline{\nu}^{\beta}, m^j\})$ for which $\sigma^j([w-\epsilon, w]) > 0$ for any $\epsilon > 0$ and $\sigma^i(\{w\}) > 0$. For $\epsilon' > 0$ small enough, Firm j can improve by moving the probability that its puts in $[w - \epsilon', w]$, to a point slightly above w. This deviation affects marginally Firm j's cost of working capital and profit conditional on winning but allows the firm to win the procurement contract at a strictly positive profit if Firm i plays the atom at w.

Claim 1 and 2 imply that (i) the only points where there can be a mass point in the strategies is at zero or at $\overline{\nu}^{\beta}$, (ii) at most one of the firms' strategies can have an atom at zero, and (iii) the support of both σ^1 and σ^2 must be the same and equal to an interval $[0, \nu]$ for some $\nu > 0$. Conditions (i)-(iii) and the usual indifference condition that must hold in a mixed strategy equilibrium implies: (iv) that the distributions of each of the firms, σ^1 and σ^2, is equal to a continuous solution of Equation (1) in $(0, \nu)$. Suppose, first, that there is no atom at $\overline{\nu}^{\beta}$. The uniqueness of the solution of our differential equation, see Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that there is only one solution that passes by the point $(\nu, 1)$ for each $\nu \in [0, \overline{\nu}^{\beta}]$. Thus, both firms must use in equilibrium the same distribution. This together with (iv) means that this distribution must be the solution to our differential equation with initial condition $(0,0)$. This is our proposed distribution function. To conclude the proof we argue by contradiction that there is no atom at $\overline{\nu}^{\beta}$. Suppose there is an atom of probability $\gamma > 0$ in one of the distributions. By (iv), this distribution must be a solution to our differential equation that passes by $(\overline{\nu}^{\beta}, 1-\gamma)$. Since the solution with initial condition $(0,0)$ passes by $(\overline{\nu}^{\beta}, 1)$, the uniqueness of the solutions of our differential equation implies that the solution that passes by $(\overline{\nu}^{\beta}, 1-\gamma)$ must cross the horizontal axis strictly to the right of $(0,0)$ which is a contradiction with (iv). ■

Proof of Proposition 2

We start showing that the proposed strategies are an equilibrium. Let m_l be the laggard's cash and denote by $\tilde{F}_l$ and $\tilde{F}_L$ the laggard and the leader's solutions, respectively. The laggard's expected payoffs of a working capital $w \in [0, m_l)$ when the leader plays $\tilde{F}_L$ are equal to:

$$m^i - (1-\beta)w + \beta\pi(w)\tilde{F}_L(w),$$

which is constant and equal to m^i by definition of $\tilde{F}_L$. Recall that our tie breaking rule allocates the contract to the leader when both firms carry working capital m_l which guarantees that payoffs are continuous at $w = m_l$, and hence equal to m^i. Consequently, the laggard has no incentive to deviate. Similarly, the leader's expected payoffs of a working capital $w \in [0, \overline{x}]$ when the laggard plays $\tilde{F}_l$ are equal to:

$$m^i - (1-\beta)w + \beta\pi(w)\tilde{F}_l(w),$$

which, by definition of $\tilde{F}_l$, is constant and equal to $m^i - (1-\beta)m_l + \beta\pi(m_l)$ if $w \in [0, m_l]$ and equal to $m^i - (1-\beta)w + \beta\pi(w) < m^i - (1-\beta)m_l + \beta\pi(m_l)$ if $w \in (m_l, \overline{x}]$. Hence there are no incentives to deviate. Note that we are using that our tie breaking rule allocates the good to the leader when both firms choose zero working capital.

To prove that there is no other equilibrium, note that Claims 1 and 2 in the proof of Proposition 1 also hold true here. They imply here that (i) the laggard's strategy can have a probability mass only at zero and the leader's only at either zero or m_l, (ii) at most one of the firms' strategies can have an atom at zero, and (iii) the support of both σ^1 and σ^2 must be the same and equal to an interval $[0, \nu]$ for some $\nu \in [0, m_l]$. (i)-(iii) and the usual indifference condition that must hold in a mixed strategy equilibrium implies: (iv) that the distributions of each of the firms, σ^1 and σ^2, is equal to a continuous solution of Equation (1) in $(0, \nu)$. Since the solutions to Equation (1) do not cross in $[0, m_l]$ and the solution with initial condition $(0, 0)$ reaches one at $\overline{\nu}^{\beta} > m_l$, both firm's strategies must have atoms. (i) implies that the laggard's atom is at zero and (ii) that the leader's is at m_l. This together with (iv) imply our proposed strategies. ∎

Auxiliary Results of Section 4.2

Lemma 2. *For any $\hat{\Psi}, \hat{\Psi}' \in \mathcal{P}$,*

(i) For any $(x_0, y_0) \in [0, \overline{x}] \times [0, 1]$, there is a unique continuous solution to the differential equation (11) that passes by (x_0, y_0). All these solutions are locally increasing.

(ii) For any solution $\hat{F}$ to the differential equation (11) and $w > w'$:

$$\hat{F}(w)\pi(w) - \hat{F}(w')\pi(w') \leq \frac{1-\beta}{\beta}(w - w'),$$

with equality if $w, w' \in [\theta - \underline{m}, \overline{x}]$.

(iii) $\hat{\nu}^{\hat{\Psi}} \in [\overline{\nu}^{\beta}, \overline{x}]$.

(iv) $\hat{\Psi}' \leq \hat{\Psi}$ implies $\hat{F}(0, \hat{\Psi}', m) \leq \hat{F}(0, \hat{\Psi}, m)$.

(v) $\hat{F}(y, \hat{\Psi}, m)$ is continuous and decreasing in m for $y < m$.

Proof. Note that Equation (2) can be written as:

$$\hat{F}'(w) = \frac{\frac{1-\beta}{\beta} + (-\pi'(w))\hat{F}(w)}{\pi(w) + \hat{\Psi}(w + \underline{m})}. \tag{7}$$

The application of Theorem 7.1 in Coddington and Levinson (1984), pag. 22 to this expression implies the existence and uniqueness in (i). It also implies the continuity in (v). The monotonicity with respect to x follows from the fact that the right hand side of the above expression is strictly positive. The inequality of (ii) can be proved integrating both sides of the following inequality implied by Equation (2):

$$\hat{F}'(w)\pi(w) + \hat{F}(w)\pi'(w) \leq \frac{1-\beta}{\beta}.$$

The equality of (ii) follows because the above inequality holds with equality when $w \in [\theta - \underline{m}, \overline{x}]$ since $\hat{\Psi}(w + \underline{m}) = 0$. (iii) follows from the consequence of (i) that solutions are increasing and do not cross and two facts: (a) a solution with final condition $\hat{F}(\overline{\nu}^{\beta}) = 1$ verifies that $\hat{F}(0) \geq 0$, and (b) a solution with final condition $\hat{F}(\overline{x}) = 1$ verifies that $\hat{F}((\theta - \underline{m})^{+}) = 0$. (a) can be proved using that (ii) for $w = \overline{\nu}^{\beta}$ and $w' = 0$ implies that:

$$\hat{F}(0) \geq \frac{\pi(\overline{\nu}^{\beta}) - \frac{1-\beta}{\beta}\overline{\nu}^{\beta}}{\pi(0)},$$

and the definition of $\overline{\nu}^{\beta}$, whereas (b) can be shown using that (ii) for $w = \overline{x}$ and $w' = (\theta - \underline{m})^{+}$ implies that:

$$\hat{F}((\theta - \underline{m})^{+}) = \frac{\pi(\overline{x}) - \frac{1-\beta}{\beta}(\overline{x} - (\theta - \underline{m})^{+})}{\pi((\theta - \underline{m})^{+})},$$

and the definition of $\overline{x}$. (iv) uses that the right hand side of Equation (7) decreases when we increase $\hat{\Psi}$ and hence at any crossing point between a solution associated to $\hat{\Psi}'$ and a solution associated to $\hat{\Psi}$ the former has greater slope than the latter. Consequently the former solution can cross the latter only once and from below which implies our result. The monotonicity in (v) follows from the implication of (i) that the different solutions are increasing and do not cross. ■

Lemma 3. *Equation (3) has a solution in $\mathcal{P}$ that we denote by Ψ^{β}.*

Proof. Endow $\mathcal{P}$ with the sup-norm, that we denote by $|| \cdot ||$, and let T be an operator defined by a function that maps to each function $\hat{\Psi} \in \mathcal{P}$ a function equal to the right hand side of Equation (3).

We prove the lemma showing that the operator T meets all the conditions required by Schauder Fixed-Point Theorem, see Stokey and Lucas (1999), Theorem 17.4, page 520, in the subset $\hat{\mathcal{P}} \subset T$ of the functions $\hat{\Psi}$ such that $\hat{\Psi}(\overline{x}) = 0$.

Claim 1: $T(\hat{\mathcal{P}}) \subset \hat{\mathcal{P}}$. Lemma 2(v) implies that $T(\hat{\Psi})(m)$ is continuous and decreasing in m. By Lemma 2(iii) $\hat{\nu}^{\hat{\Psi}} \leq \overline{x}$ and hence $T(\hat{\Psi})(\overline{x}) = 0$. Finally, $\hat{\Psi}(\underline{m}) \leq \frac{\beta\pi(0)}{1-\beta}$ implies that $T(\hat{\Psi})(m) \leq \beta\hat{F}(0, \hat{\Psi}, m)\frac{\pi(0)}{1-\beta} \leq \frac{\beta}{1-\beta}\pi(0)$.

Claim 2: T **is continuous.** Take any convergent sequence $\{\hat{\Psi}_n\} \in \hat{\mathcal{P}}$ with limit $\hat{\Psi} \in \hat{\mathcal{P}}$. Let $\epsilon_n \equiv \sup_{\tilde{n} \geq n} ||\hat{\Psi}_{\tilde{n}} - \hat{\Psi}||$. Since $\hat{\Psi}_n \to \hat{\Psi}$, $\{\epsilon_n\}$ is a decreasing sequence converging to zero. We use:

$$\underline{\Psi}_n(m) \equiv \left(\hat{\Psi}(m) - \epsilon_n\right)^{+}.$$

and,

$$\overline{\Psi}_n(m) \equiv \min\left\{\hat{\Psi}(m)+\epsilon_n, \frac{\beta}{1-\beta}\pi(0)\right\}\left(1-\frac{(m-\overline{x})^+}{\theta-\overline{x}}\right),$$

By construction $\underline{\Psi}_n, \overline{\Psi}_n \in \mathcal{P}$, $\hat{\Psi}_n(m) \in [\underline{\Psi}_n(m), \overline{\Psi}_n(m)]$, $\{\underline{\Psi}_n\}$ is an increasing sequence and $\{\overline{\Psi}_n\}$ is a decreasing sequence, and $\underline{\Psi}_n$ and $\overline{\Psi}_n$ converge point-wise to $\hat{\Psi}$. Thus, $\{\hat{F}(0, \underline{\Psi}_n, m)\}$ and $\{\hat{F}(0, \overline{\Psi}_n, m)\}$ are increasing and decreasing, respectively sequences of continuous functions (in m), by Lemma 2(iv)-(v). Both sequences converge pointwise to $\hat{F}(0, \hat{\Psi}, m)$, by an adaptation of Theorem 7.1 in Coddington and Levinson (1984), pag. 22.[49] Thus, Theorem 7.13 in Rudin (1976), pag. 150, implies that the sequences of functions $\{\hat{F}(0, \overline{\Psi}_n, \cdot)\}$ and $\{\hat{F}(0, \underline{\Psi}_n, \cdot)\}$ convergence in the sup-norm to $F(0, \hat{\Psi}, \cdot)$. This implies the convergence of $T\hat{\Psi}_n$ to $T\hat{\Psi}$ since $\hat{F}(0, \hat{\Psi}_n, m) \in [\hat{F}(0, \underline{\Psi}_n, m), \hat{F}(0, \overline{\Psi}_n, m)]$ by Lemma 2(iv). This means that T is continuous as desired.

Claim 3: the family $T(\hat{\mathcal{P}})$ is equicontinuous. Since $T(\hat{\Psi})(\cdot)$ is decreasing, we shall show that there exists $\kappa > 0$ such that for any $m' < m$ and $\hat{\Psi} \in \mathcal{P}$, $T(\hat{\Psi})(m') - T(\hat{\Psi})(m) \leq \kappa(m - m')$. We start noting that $T(\hat{\Psi})(m') - T(\hat{\Psi})(m) = 0$ if $m', m \geq \hat{\nu}^{\hat{\Psi}}$, and that $T(\hat{\Psi})(m') - T(\hat{\Psi})(m) = T(\hat{\Psi})(m') - T(\hat{\Psi})(\hat{\nu}^{\hat{\Psi}})$ if $m' \leq \hat{\nu}^{\hat{\Psi}} \leq m$. Take now $m, m' \leq \hat{\nu}^{\hat{\Psi}}$.

$$\begin{aligned}
T(\hat{\Psi})(m') - T(\hat{\Psi})(m) &= \beta(\pi(0)+\hat{\Psi}(\underline{m}))(\hat{F}(0,\hat{\Psi},m') - \hat{F}(0,\hat{\Psi},m)) \\
&\leq \frac{\beta}{1-\beta}\pi(0)(\hat{F}(0,\hat{\Psi},m') - \hat{F}(0,\hat{\Psi},m)) \\
&= \frac{\beta}{1-\beta}\pi(0)\left(\left(1 - \int_0^{m'} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))\hat{F}(y,\hat{\Psi},m)}{\pi(y)+\hat{\Psi}(y+\underline{m})}dy\right.\right. \\
&\quad \left.\left. - \left(1 - \int_0^{m} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))\hat{F}(y,\hat{\Psi},m')}{\pi(y)+\hat{\Psi}(y+\underline{m})}dy\right)\right)\right. \\
&\leq \frac{\beta}{1-\beta}\pi(0)\int_{m'}^{m} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))\hat{F}(y,\hat{\Psi},m)}{\pi(y)+\hat{\Psi}(y+\underline{m})}dy \\
&\leq \frac{\beta}{1-\beta}\pi(0)\frac{\frac{1-\beta}{\beta}+\gamma}{\pi(\overline{x})}(m-m'),
\end{aligned}$$

[49] Just note that Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that the solution to the system of differential equations defined by:

$$\begin{aligned}
\hat{F}'(w) &= \frac{\frac{1-\beta}{\beta}+(-\pi'(w))\hat{F}(w)}{\pi(w)+\min\left\{\left(\hat{\Psi}(w+\underline{m})+\rho(w)\right)^+, \frac{\beta}{1-\beta}\pi(0)\right\}\left(1-\frac{(m-\overline{x})^+}{\theta-\overline{x}}\right)} \\
\rho'(w) &= 0,
\end{aligned}$$

with initial conditions $\hat{F}(x_0) = y_0$ and $\rho(x_0) = \epsilon$, is continuous in ϵ.

where we have used: in the first step, Equation (3); in the second step, that $\hat{\Psi}(\underline{m}) \leq \frac{\beta}{1-\beta}\pi(0)$; in the third step, that $\hat{F}(0, \hat{\Psi}, m')$ and $\hat{F}(0, \hat{\Psi}, m)$ verify Equation (11) below and that $\hat{F}(m', \hat{\Psi}, m') = \hat{F}(m, \hat{\Psi}, m) = 1$; in the fourth step, that $\hat{F}(y, \hat{\Psi}, m) \leq \hat{F}(y, \hat{\Psi}, m')$ by application of Lemma 2(v); and in the last step, that $F(y, \hat{\Psi}, m) \leq 1$, that $\hat{\Psi}(y+\underline{m}) \geq 0$, and that $-\pi'$ is continuous and hence bounded above in $[0, \theta]$ by some $\gamma \geq 0$ finite.

Hence, our κ is equal to $\frac{\beta}{1-\beta}\pi(0)\frac{\frac{1-\beta}{\beta}+\gamma}{\pi(\overline{x})} > 0$ as desired. ■

Proof of Proposition 3

To show that our bid function b^* maximizes the firm's Bellman equation, we prove the more general argument that for our continuation value W^*, and for any given bid and working capital of the rival, a working capital w and a bid $\tilde{b} > \pi(w)+c$ does strictly worse than the same bid $\tilde{b}$ and the minimum working capital that makes this bid acceptable, i.e. $\tilde{w}$ such that $\pi(\tilde{w})+c = \tilde{b}$. The argument is the same as in the static case: reducing today's working capital while keeping constant the bid increases today's utility in the amount of working capital reduced while it decreases tomorrow's continuation value in its discounted value. This is easy to deduce from the functional form of W^* when the reduction in today's working capital (keeping constant the bid) does not change the identity of tomorrow's leader. Otherwise, the result follows from the fact that the premium of being leader $\Psi^\beta(m')$ is equal to zero because the change in leadership can only occur when the other firm's cash is greater than θ. This is because the change does not affect to the cash that the other firm has and Assumption 1 implies that the cash of at least one firm must be larger than θ.

Next, we assume our continuation value W^* and that both firms use b^* and show that a firm cannot do better than using σ^* when the other firm uses σ^*. We start with the symmetric scenario. In this case, the other firm's randomization is equal to $\hat{F}(\cdot, \Psi^\beta, \nu^\beta)$, which we write $\check{F}$ to simplify the notation, and one can show after substituting the value of W^* and some algebra that the expected payoffs of choosing a working capital $w \in [0, m]$ are equal to:

$$m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta\int_0^{\min\{w,\nu^\beta\}} (\pi(w) + \Psi^\beta(\tilde{w} + \underline{m}))\check{F}'(\tilde{w})d\tilde{w}. \quad (8)$$

The differential of this expression with respect to w is equal to zero for $w \in [0, \nu^\beta]$ because $\check{F}$ verifies Equation (2), and negative for $w > \nu^\beta$. Thus, σ^* is optimal because it randomizes in the support $[0, \nu^\beta]$.

In the laggard-leader scenario, the support of σ^* is $[0, m]$ but otherwise, the argument is identical. In this case, we use that the expected payoffs of choosing a working capital $w \in [0, m]$ are equal to:

$$m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta \int_0^w (\pi(w) + \Psi^\beta(\tilde{w} + \underline{m}))(F^\beta_{L,m})'(\tilde{w})d\tilde{w}, \tag{9}$$

if $m < m'$, and to,

$$m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta F^\beta_{l,m}(0)(\pi(w) + \Psi^\beta(\underline{m})) + \\ \beta \int_0^{\min\{w,m'\}} (\pi(w) + \Psi^\beta(\tilde{w} + \underline{m}))(F^\beta_{l,m})'(\tilde{w})d\tilde{w}, \tag{10}$$

if $m > m'$. Note that although ties could occur with positive probability the tie breaking rule always allocates the contract to the leader.

Finally, to show that W^* is the value of our Bellman equation, note that the indifference condition discussed above and the fact that W^* is equal to each of the Equations (8)-(10) evaluated at $w = 0$. ■

Auxiliary Results Used in the Proof of Theorem 1

In the proof of Theorem 1, we use that the solutions $\hat{F}$ to Equation (2) verify:

$$\hat{F}(w) - \hat{F}(w') = \int_{w'}^{w} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))\hat{F}(y)}{\pi(y) + \Psi(y + \underline{m})} dy, \tag{11}$$

for $w \geq w'$. We also use that:

$$Q^\beta(m, [0, x]) = F^\beta_{l,m}(x - \underline{m}) + F^\beta_{L,m}(x - \underline{m}) - F^\beta_{l,m}(x - \underline{m})F^\beta_{L,m}(x - \underline{m}), \tag{12}$$

that:

$$Q^\beta(\theta, [0, x]) = F^\beta_{L,\theta}(x - \underline{m})(2 - F^\beta_{L,\theta}(x - \underline{m})), \tag{13}$$

since $F^\beta_{l,\theta} = F^\beta_{L,\theta}$, and the following lemmae.

Lemma 4.

(i) For any $m \geq \nu^\beta$ and $w \in [0, \overline{x}]$:

$$F^\beta_{l,m}(w) = F^\beta_{L,m}(w) \leq \frac{(1-\beta)w}{\beta\pi(w)}.$$

(ii) For any $m < \nu^\beta$, and $w \in [0, m)$:

$$F^\beta_{L,m}(w) \leq \frac{(1-\beta)w}{\beta\pi(w)} \text{ and } F^\beta_{l,m}(w) \geq \frac{\pi(m) - \frac{1-\beta}{\beta}(m-w)}{\pi(w)},$$

with equality if $w \geq \theta - \underline{m}$.

(iii) $\lim_{\beta\to 1} \nu^\beta = \theta$.

Proof. Lemma 2(ii) together with $F^\beta_{l,m}(0) = F^\beta_{L,m}(0) = 0$ for $m \geq \nu^\beta$, and $F^\beta_{l,m}(m) = 1$ $F^\beta_{L,m}(0) = 0$ for $m < \nu^\beta$ imply, respectively, (i) and (ii). To prove the last item we use Lemma 2(iii) and that $\lim_{\beta\to 1} \overline{\nu}^\beta = \theta$, and that $\overline{x} \leq \theta$. ■

Lemma 5. *Suppose* $\lim_{\beta\to 1} \Psi^\beta(\underline{m}) = \infty$.

- *If $\theta > 2\underline{m}$ then:*

$$\lim_{\beta\to 1} F^\beta_{l,m}(w) = \begin{cases} 1 & \text{if } m < \theta - \underline{m} \text{ and } w \in [0,m] \\ \frac{\pi(m)}{\pi(w)} & \text{if } m \in [\theta - \underline{m}, \theta) \text{ and } w \in [\theta - \underline{m}, m] \\ \frac{\pi(m)}{\pi(\theta-\underline{m})} & \text{if } m \in [\theta - \underline{m}, \theta) \text{ and } w \in [0, \theta - \underline{m}), \\ 0 & \text{if } m \geq \theta \text{ and } w \in [0,\theta), \end{cases}$$

$$\lim_{\beta\to 1} F^\beta_{L,m}(w) = \begin{cases} 1 & \text{if } w \geq \min\{m,\theta\} \\ 0 & \text{if } w < \min\{m,\theta\}. \end{cases}$$

- *If $\theta \geq 3\underline{m}$ then:*

$$\lim_{\beta\to 1}(1-\beta)\Psi^\beta(m) = \begin{cases} \pi(\underline{m}) & \text{if } m < \theta - \underline{m} \\ \pi(\underline{m})\frac{\pi(m)}{\pi(\theta-\underline{m})} & \text{if } m \in [\theta - \underline{m}, \theta). \end{cases}$$

Proof. Along this proof, we use that $\Psi^\beta(m)$, for $m < \theta$, also diverges to infinity. This is because: $\Psi^\beta(m) = \frac{F^\beta_{l,m}(0)}{F^\beta_{l,\underline{m}}(0)}\Psi^\beta(\underline{m})$ by Equation (3); $F^\beta_{l,\underline{m}}(0) \leq 1$; $F^\beta_{l,m}(0) \geq \frac{\pi(m) - \frac{1-\beta}{\beta}m}{\pi(0)}$ for $m < \nu^\beta$ by Lemma 4(ii); and Lemma 4(iii).

Now, note that $F^\beta_{l,m}(m) = 1$ for $m \leq \nu^\beta$ which together with Equation (11) means that:

$$F^\beta_{l,m}(w) = 1 - \int_w^m \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^\beta_{l,m}(y)}{\pi(y) + \Psi^\beta(y+\underline{m})}dy, \text{ if } m \leq \nu^\beta \text{ and } w \in [0,m].$$

Taking the limit[50] $\beta \to 1$ we get our result for $m < \theta - \underline{m}$ and $w \in [0, m]$ since the numerator is bounded and the denominator diverges.

The limit of $F^{\beta}_{l,m}$ when $m \in [\theta - \underline{m}, \theta)$ and $w \in [\theta - \underline{m}, m]$ follows directly from the fact that $F^{\beta}_{l,m}(m) = 1$ for $m \leq \nu^{\beta}$ and Lemma 4 (ii) and (iii).

This last result and the implication of Equation (11) that,

$$F^{\beta}_{l,m}(w) = F^{\beta}_{l,m}(\theta - \underline{m}) - \int_{w}^{\theta - m} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,m}(y)}{\pi(y) + \Psi^{\beta}(y + \underline{m})} dy, \text{ if } m \in [\theta - \underline{m}, \nu^{\beta}) \text{ and } w < \theta - m$$

implies our result for $m \in [\theta - \underline{m}, \theta)$ and $w \in [0, \theta - \underline{m})$ taking the limit $\beta \to 1$ and applying Lemma 4 (iii).

The remaining limits in the first item can be easily derived from Lemma 4.

We start the proof of the limit in the second item for the case $m = \underline{m}$. In this case, we have to show that $\lim_{\beta \to 1}(1 - \beta)\Psi^{\beta}(\underline{m}) = \pi(\underline{m})$. Equation (3) implies:

$$(1-\beta)\Psi^{\beta}(\underline{m}) = \beta F^{\beta}_{l,\underline{m}}(0)(\pi(0) + \Psi^{\beta}(\underline{m})) - \beta\Psi^{\beta}(\underline{m}) = \beta F^{\beta}_{l,\underline{m}}(0)\pi(0) - \beta\Psi^{\beta}(\underline{m})(1 - F^{\beta}_{l,\underline{m}}(0)),$$

where the first term in the last expression tends to $\pi(0)$ by application of the result in the first line of this lemma, and the last term is equal to:

$$\beta\Psi^{\beta}(\underline{m}) \int_{0}^{m} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,\underline{m}}(y)}{\pi(y) + \Psi^{\beta}(y + \underline{m})} dy = \beta \int_{0}^{m} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,\underline{m}}(y)}{\frac{\pi(y)}{\Psi^{\beta}(\underline{m})} + \frac{F^{\beta}_{l,y+\underline{m}}(0)}{F^{\beta}_{l,\underline{m}}(0)}} dy.$$

where we have used in the last equality the implication of Equation (3) that $\frac{\Psi^{\beta}(y+\underline{m})}{\Psi^{\beta}(\underline{m})} = \frac{F^{\beta}_{l,y+\underline{m}}(0)}{F^{\beta}_{l,\underline{m}}(0)}$. The limit of this last term when β tends to one is equal to $\pi(0) - \pi(\underline{m})$, as required, since $\Psi^{\beta}(\underline{m})$ diverges to infinity and, as we showed above, $F^{\beta}_{l,m}(w)$ tends to one for $w \leq m < \theta - \underline{m}$ and the lemma assumes that $2\underline{m} < \theta - \underline{m}$.

The result for a general m follows from the implication of Equation (3) that $\frac{\Psi^{\beta}(m)}{\Psi^{\beta}(\underline{m})} = \frac{F^{\beta}_{l,m}(0)}{F^{\beta}_{l,\underline{m}}(0)}$ and the limit results for $F^{\beta}_{l,m}$ in this lemma. ■

Lemma 6. *If* $\lim_{\beta \to 1} \Psi^{\beta}(\underline{m}) = \infty$ *and* $\theta > 2\underline{m}$*, then* $\lim_{\beta \to 1} \mu^{\beta}((\underline{m}, \theta]) = 0$.

Proof. The result can be deduced from the application of Equation (6) noting that,

$$Q^{\beta}(m, (\underline{m}, \theta]) = Q^{\beta}(m, [\underline{m}, \theta]) - Q^{\beta}(m, \{\underline{m}\})$$

[50] Here, and in what follows, we compute the limit of integrals applying, without mentioning it explicitly, the bounded convergence theorem (see Royden (1988), page 81).

and that the right hand side converges to zero for any $m \in [\underline{m}, \theta + \underline{m}]$ as can be deduced from Equation (12) and Lemma 5. ■

Proof of Theorem 1

For $\epsilon > 0$ and sufficiently small, we define the following sets $A \equiv \{\underline{m}\}$, $B \equiv (\underline{m}, \theta - 2\underline{m} - \epsilon]$, $C = (\theta - 2\underline{m} - \epsilon, \theta - \underline{m} - \epsilon)$, $D \equiv [\theta - \underline{m} - \epsilon, \theta - \epsilon)$, $E \equiv [\theta - \epsilon, \theta + \underline{m}]$.

Lemma 6 implies that it is sufficient to show that $\lim_{\beta \to 1} \mu^\beta(E) = 0$.

Note that since $Q(m, E) > 0$ only if $m \in D \cup E$:

$$\begin{aligned} \mu^\beta(E) &= \mu^\beta(D) \int_D Q^\beta(m, E) \frac{\mu^\beta(dm)}{\mu^\beta(D)} + \mu^\beta(E) \int_E Q^\beta(m, E) \frac{\mu^\beta(dm)}{\mu^\beta(E)} \\ &\leq \mu^\beta(D) + \left(\mu^\beta(E) \int_E Q^\beta(m, E) \frac{\mu^\beta(dm)}{\mu^\beta(E)} \right). \end{aligned}$$

Besides since $Q(m, D) > 0$ only if $m \in C \cup D \cup E$:,

$$\mu^\beta(D) = \mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)} + \mu^\beta(E) \int_E Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(E)}.$$

Substituting the latter equality in the former, using that $1 - Q^\beta(m, D) - Q^\beta(m, E) = Q^\beta(m, A \cup B \cup C)$ and solving for $\mu^\beta(E)$, one gets:

$$\mu^\beta(E) \leq \frac{\mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)}}{\int_E Q^\beta(m, A \cup B \cup C) \frac{\mu^\beta(dm)}{\mu^\beta(E)}} \leq \frac{\mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)}}{Q^\beta(\theta, A \cup B \cup C)}, \tag{14}$$

where the last inequality follows from the fact that $Q^\beta(m, [0, x])$ is increasing in $F^\beta_{l,m}(x)$ and $F^\beta_{L,m}(x)$ and that each of these two functions is greater than $F^\beta_{l,\theta}(x)$ and $F^\beta_{L,\theta}(x)$, respectively for all x. This last property follows from the definition of $F^\beta_{l,m}$ and $F^\beta_{L,m}$ and in the case of $F_{l,m}$ the monotonicity in Lemma 2(v).

We shall show that the right hand side of Equation (14) tends to zero as β tends to 1. We first argue that:

$$\begin{aligned} \lim_{\beta \to 1} \frac{Q^\beta(\theta, A \cup B \cup C)}{(1 - \beta)^2} &= 2 \lim_{\beta \to 1} \frac{F^\beta_{L,\theta}(\theta - 2\underline{m} - \epsilon)}{(1 - \beta)^2} \\ &= 2 \lim_{\beta \to 1} \int_0^{\theta - 2\underline{m} - \epsilon} \frac{\frac{1}{\beta} + (-\pi'(y)) \frac{F^\beta_{L,\theta}(y)}{1 - \beta}}{(1 - \beta)\pi(y) + (1 - \beta)\Psi^\beta(y + \underline{m})} dy \\ &= 2 \int_0^{\theta - 2\underline{m} - \epsilon} \frac{1}{\pi(\underline{m})} dy > 0, \end{aligned}$$

where we have used in the first step, Equation (13) and that $\lim_{\beta\to 1} F^{\beta}_{L,\theta}(\theta - 2\underline{m} - \epsilon) = 0$ by Lemma 5; we have used in the second step, Equation (11) and $F^{\beta}_{L,\theta}(0) = 0$; and we have used in the third step Lemma 5 and that for any $y < \min\{\theta - \underline{m}, m\}$:

$$\lim_{\beta\to 1} \frac{F^{\beta}_{L,m}(y)}{1-\beta} = \lim_{\beta\to 1} \int_0^y \frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^{\beta}_{L,m}(z)}{(1-\beta)\pi(z) + (1-\beta)\Psi^{\beta}(z+\underline{m})} dz = 0, \tag{15}$$

that can be deduced from Lemma 5.

Next, we show that for any $m < \theta$ (and hence any $m \in C \cup D$):

$$\begin{aligned}
\lim_{\beta\to 1} \frac{Q^{\beta}(m, D)}{1-\beta} &= \lim_{\beta\to 1} \frac{F^{\beta}_{l,m}(\theta - \underline{m} - \epsilon) - F^{\beta}_{l,m}(\theta - 2\underline{m} - \epsilon)}{(1-\beta)} \\
&= \lim_{\beta\to 1} \int_{\min\{m,\theta-2\underline{m}-\epsilon\}}^{\min\{m,\theta-\underline{m}-\epsilon\}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,m}(y)}{(1-\beta)\pi(y) + (1-\beta)\Psi^{\beta}(y+\underline{m})} dy \\
&\leq \int_{\min\{m,\theta-2\underline{m}-\epsilon\}}^{\min\{m,\theta-\underline{m}-\epsilon\}} \frac{-\pi'(y)}{\pi(\underline{m}) \min\left\{1, \frac{\pi(y+\underline{m})}{\pi(\theta-\underline{m})}\right\}} dy < \infty,
\end{aligned}$$

where we use in the first equality that,

$$\lim_{\beta\to 1} \frac{Q^{\beta}(m, [\underline{m}, x])}{1-\beta} = \lim_{\beta\to 1} \frac{F_{l,m}(x - \underline{m})}{1-\beta}$$

if $x < \theta$ because either $x - \underline{m} \geq m$ and then $Q^{\beta}(m, [\underline{m}, x]) = 1 = F_{l,m}(x - \underline{m})$ or $x - \underline{m} < m$ and then we can use equations (12) and (15); we use in the second equality, Equation (11); and we use in the inequality Lemma 5 and $F^{\beta}_{l,m}(y) \leq 1$.

We conclude the proof by showing that $\lim_{\beta\to 1} \frac{\mu^{\beta}(C\cup D)}{1-\beta} = 0$. To prove so, note that since $Q(m, C \cup D) = 0$ if $m < m^* \equiv \theta - 3\underline{m} - \epsilon$ and $Q^{\beta}(m, [0, x]) = Q^{\beta}(\theta, [0, x])$ if $m \in [\theta, \theta + \underline{m}]$:

$$\frac{\mu^{\beta}(C \cup D)}{1-\beta} = \int_{m^*}^{\theta} \frac{Q^{\beta}(m, C \cup D)}{1-\beta} \frac{\mu^{\beta}(dm)}{\mu^{\beta}((m^*, \theta))} \mu^{\beta}((m^*, \theta)) + \frac{Q^{\beta}(\theta, C \cup D)}{1-\beta} \mu^{\beta}([\theta, \theta + \underline{m}]),$$

The first term in the sum goes to zero because $Q^{\beta}(m, C \cup D) \leq Q^{\beta}(m, D)$ and $m^* > \underline{m}$ and we have already shown that $\lim_{\beta\to 1} \frac{Q^{\beta}(m,D)}{1-\beta} < \infty$ if $m < \theta$ and $\lim_{\beta\to 1} \mu^{\beta}((\underline{m}, \theta)) = 0$. That the second term also goes to zero can be deduced from equations (13) and (15) and the fact that $\mu^{\beta}([\theta, \theta + \underline{m}]) \leq 1$. ■

Proof of Lemma 1

Proof. The lemma follows from the following sequence of inequalities, that start from a transformation of Equation (3), taking the limit $\beta \to 1$:

$$\begin{aligned}
(1-\beta)\Psi^\beta(\underline{m}) &= \beta F_{l,\underline{m}}(0)\left(\pi(0)+\Psi^\beta(\underline{m})\right)-\beta\Psi^\beta(\underline{m}) \\
&= \beta F_{l,\underline{m}}(0)\left(\pi(0)-\frac{1-F_{l,\underline{m}}(0)}{F_{l,\underline{m}}(0)}\Psi^\beta(\underline{m})\right) \\
&\geq \beta F_{l,\underline{m}}(0)\left(\pi(0)-\frac{\frac{1-\beta}{\beta}\underline{m}+\pi(0)-\pi(\underline{m})}{F_{l,2\underline{m}}(0)}\right) \\
&\geq \beta F_{l,\underline{m}}(0)\pi(0)\frac{\pi(2\underline{m})+\pi(\underline{m})-\pi(0)-\frac{1-\beta}{\beta}3\underline{m}}{\pi(2\underline{m})-\frac{1-\beta}{\beta}2\underline{m}},
\end{aligned}$$

where we have used in the first inequality that,

$$\begin{aligned}
1-F_{l,\underline{m}}(0) &= \int_0^{\underline{m}}\frac{\frac{1-\beta}{\beta}+(-\pi'(y))F_{l,\underline{m}}(y)}{\pi(y)+\Psi^\beta(y+\underline{m})}dy \\
&\leq \int_0^{\underline{m}}\frac{\frac{1-\beta}{\beta}+(-\pi'(y))}{\Psi^\beta(2\underline{m})}dy \\
&= \frac{\frac{1-\beta}{\beta}\underline{m}+\pi(0)-\pi(\underline{m})}{\Psi^\beta(2\underline{m})}
\end{aligned}$$

and that by Equation (3) $\frac{\Psi^\beta(\underline{m})}{\Psi^\beta(2\underline{m})}=\frac{F_{l,\underline{m}}(0)}{F_{l,2\underline{m}}(0)}$, and in the second inequality Lemma 4 (ii) and (iii), since $2\underline{m}<\theta$. This last inequality is implied by $\pi(2\underline{m})>\pi(0)-\pi(\underline{m})\geq 0$. ■

B A Model of Financial Constraints

In this Appendix, we endogenize the function π in a model in which moral hazard and limited liability restrict the set of bids for which the firm can secure financing, i.e. the set of acceptable bids. In this model, the firm can borrow from a competitive banking sector but upon winning the procurement contract can divert a fraction of their total available funds at the cost of jeopardizing the success of the procurement contract. The main implication is that the minimum acceptable bid for a firm with working capital w is given by a function $\pi(w) \equiv \theta - w$, for some θ endogenously determined. Our analysis is for the case of the dynamic model of Section 4 although it it straightforward how to adapt it to the static model of Section 3.

We say that the bid of a firm has secured financing if either the firm's working capital w is larger than the cost of the procurement contract c or the firm can borrow the required funds $c-w$ to pay the cost of the procurement contract c. In the latter case, the necessary funds are provided by a competitive banking sector. We assume that the bank transfers the funds after the firm has won the contract. At that point, the firm can either: (a) use the total funds c, working capital plus bank lending, to pay the cost of the procurement contract and comply with the procurement contract and with the bank; or (b) divert an exogenously given fraction $\alpha \in (0, 1]$ of its total funds and comply neither with the procurement contract nor with the bank. In case (a), the firm's cash next period is equal to the sum of its working capital w, its *potential profits* Π, i.e. the revenue of the procurement contract minus its cost c, and the exogenous cash-flow $\underline{m}$, minus any net payment to the bank R. In case (b), limited liability implies that the firm gets expropriated of all non-diverted funds and starts the next period with cash equal to its diverted funds αc plus the exogenous cash-flow $\underline{m}$.

One consequence is that a firm with continuation value strictly increasing in its cash diverts funds if $w + \Pi - R < \alpha c$. This means that a bank that lends to a firm that diverts funds makes losses since its profits are equal to at most $(1 - \alpha)c - (c - w) < 0$, assuming no discounting between the moment in which the bank lends the funds and when it is paid back. Hence, banks do not lend to firms expected to divert funds, bank lending is risk free, and as a consequence banks charge a zero net interest rate and only lend if the potential profits of the firm are larger than $\alpha c - w$. Therefore (and no proof is required):

Proposition 4. *Only bids with potential profits larger than $\alpha c - w$ are acceptable for a firm with working capital w. Hence, $\pi(w) = \alpha c - w$, and $\theta = \alpha c$.*

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