Three essays on Bargaining

On Refutability of the Nash Bargaining Solution.

On Inter- and Intra-Party Politics.

A Bargaining Model with Strategic Generosity.

by

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Contents

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Abstract

This dissertation is a collection of three essays that share one common feature: all three of them relate to the literature on Bargaining. The first and second essay are joint work with my supervisor, Professor Andrés Carvajal.

In our first essay we investigate the testable implications of the Nash bargaining solution. We develop polynomial tests of the NBS under different hypothesis about the default levels. For instance, with, and without observation from the outside econometrician of the levels of utility that the individuals would have obtained outside the negotiation. We use the Tarski-Seindenberg algorithm to characterize rationalizable data as those that satisfy a finite system of polynomial inequalities.

In our second essay we introduce a new equilibrium concept for games of political competition. We model electoral competition within each party, assuming inner-party members have somewhat conflicting preferences. By using the bargaining protocol à la Baron and Ferejohn (1989) we explicitly model party members' strategic interactions, their incentives and their decision of whom to elect. Our equilibrium concept attempts to model each member's decision as if each player were uncertain about, (i) the faction that will eventually dominate the decision made by the other party and (ii) the faction that will dominate in the party's nomination.

In the last essay I focus on one of the classical problems in bargaining:

the divide the dollar problem. In our framework we assume players' utility functions mirror selfish and Rawlsian preferences. We derive the set of subgame perfect equilibria for different arrangements of player types and study why strategic generosity emerges under the bargaining protocol we assume.

Chapter 1

Essay one

On Refutability of the Nash Bargaining Solution.

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1.1. Introduction

Among the most prominent solution concepts to bargaining problems is the one proposed by Nash in 1953. This solution has been used to describe a variety of decision processes—for instance, to model the firm's decision process as a bargaining game between a itself and its union, the share of profits in a cartel or job-search models. The purpose of this paper is to investigate the testable restrictions that the NBS has on the allocations of an aggregate resource (i.e, sharing of a pie) from a revealed preference analysis. We develop polynomial tests for the NBS, under different hypotheses about the behaviour of disagreement levels, and use the Tarski-Seindenberg algorithm to characterize rationalizable data as those that satisfy a finite system of polynomial inequalities.

Related Literature: The studies that have investigated the empirical content of the NBS can be divided in two main streams: the differential approach and the revealed preference analysis. In the former, Manser and Brown (1980) studied the empirical content of household decision making, (i.e, how to allocate resources and gains). They step away from Becker's (1973, 1974) bargaining rule that the household maximizes one individual's utility function and explicitly allow for different utility functions within a household inhabited by two individuals. They define gains in a marriage to exist if the maximum utility that each individual can attain lies inside the utility possibility frontier. If this occurs, the household must decide on a distribution of resources and distribution of gains. They invoke two bargaining solution concepts, NBS and Kalai and Smorodinsky's (1975) solution concept, and also consider, what they call the Pareto optimal solution, which is the dictatorial case. In all cases they assume that the two individuals have von Neuman-Morgenstern utility functions and maximize subject to the corresponding constraints, which in turn yield the demand functions. The comparative static properties of the demand functions are compared with maximizing a single utility function, using the Slutsky conditions. A drawback of their work is the presence of a bargaining rule like the one allowed by Becker (1974) in order to guarantee the resolution of conflict. Along the same vein, McElroy and Horney (1981 and 1990), derive a Nash generalization of the Slutsky conditions for household demand function and explain behavior consistent with Nash behavior and the so-called neoclassical individual utility maximizer.[1]

For some time, the literature on collective household decision processes borrowed solution concepts, such as NBS, Kalai-Smorodinsky and Nash-Zeuthen[2], however, Chiappori (1988) suggested that "these conditions are not restricitve, unless the agents' premaritial preferences are known." Instead, Chiappori (1992) derived, from a collective setting, a set of testable restrictions on observable behaviour, using labour supply, under the form of partial differential equations. Contrary to the Slutsky

[1] For studies that refer to the differential approach, but use different solution concepts, see, Lunderberg and Pollack (1993), Chen and Woolley (2001). In non-strategic settings, the differential approach has allowed economists to argue that non-observed fundamentals can be unambiguously recovered from observed equilibrium outcomes; for a survey of this literature, and also of early literature on the revealed preference approach, see Carvajal et al (2004).

[2] See, Svejnar (1980).

equations obtained from the traditional models, he suggested an alternative way of deriving structural conditions (i.e, conditions on parameters) on the functional forms for demand or labour supply functions. Chiappori and Doni (2006) extends this analysis to the case of the NBS.

The second strand of literature follows the discussion of Samuelson (1938), Afriat (1967) and Varian (1982) which focuses on revealed preference theory of individual behaviour to characterize rationalizabilty using a finite set of consumption data, that is, prices and quantities. Only until recently has the literature obtained testable implications on data for game theoretical solution concepts. For example, Sprumont (2000) considers a non-cooperative game played by a finite number of players, each of whom can choose a strategy from a finite set, and identifies general necessary and sufficient conditions for Nash-rationalizability. Ray and Zhou (2001) focus on extensive form games and derive a set of necessary and sufficient conditions for sub-game Nash rationalizability. Other related literature includes Carvajal (2010), who studies whether Nash-Walras equilibrium imposes testable restrictions on the equilibrium prices of economies with externalities, and Carvajal et al (2012), who develop the revealed preference analysis of the Cournot model of oligopolistic competition.

Following the revealed preference approach, we do not impose any specific functional form, thus we do not test consistency of observed behaviour conditional on a specific functional form. For instance, our results do not rely on empirical work that uses parametric specifications or preferences, or other restrictive assumptions of the parameters of the model. In this respect, Chambers and Echenique (2011) is closely related in the sense that the they too focus on the allocation of a single-dimensional resource. The analyst has available data on how money is divided amongst a fixed number of agents, but has no information on the individuals' preferences and the protocol that leads to the division. They select three theories that could possibly explain the division of money, those of the utilitarian, Nash and egalitarian max-min models (assuming the observed disagreement utility levels are fixed) and show that all three models are observationally equivalent. A main difference is that we characterize rationalizable data as those that satisfy a system of quadratic inequalities and apply the Tarski-Seindenberg algorithm to obtain a test for the NBS (under various hypotheses about the behaviour of the default utility levels), unlike them who use a dual characterization of the problem that satisfies a system of polynomial inequalities. Under the hypothesis that default utility levels vary they apply the Positivstellensatz to construct tests for Nash bargaining and the utilitarian model. Also, Chambers and Echenique only consider a subset of the scenarios that we study here.

Cherchye, Demuynck and De Rock (2011) also investigate the empirical content of the NBS, but they focus on a different setting than ours: they consider a model where a pair of agents bargain over a consumption bundle and allow agents to have the option of making consumption purchases on their own, therefore they assume disagreement points vary endogenously, whereas we bargain over the share of a pie (e.g. divide the dollar) and default utility levels are exogenously determined. They provide necessary and sufficient conditions to solve the system of inequalities that must be satisfied should data be rationalizable. Finally, they design and conduct an experiment allowing them to obtain data on (i) individuals default utility levels of consumption bundles and (ii) bargaining outcomes so that they could verify the consistency of the pair of consumptions bundles with the Nash bargaining solution.

Outline of the paper: In section 2, we define the NBS under the assumption that the outside econometrician can observe both, the allocation of an aggregate endowment, and the utility levels individuals can obtain provided they do not reach an agreement, and derive the necessary and sufficient conditions for a data set to be rationalizable. Then, in section 3, we drop the strong assumption of default utilities levels being observable, and find that the hypothesis has no scientific meaning, from a Popperian viewpoint. We restore falsifiability when we impose some conditions on the unobservable default utility levels. After showing that the solution concept can be corroborated, or refuted, by imposing conditions on the unobserved default levels, in section 4 and 5 we assume, instead, the econometrician has information about the behaviour of income levels, individuals can attain outside the cooperative agreement, and test if a data set is rationalizable. Due to the constricted nature of our tests (i.e. the data either satisfies the optimisation hypothesis, or it does not) if a test isn't rationalizable, how do we know it isn't the case that by applying a small perturbation to the data, the test of rationalizability may pass? "If some data fail the tests, but only by a small amount, we might be tempted to attribute this failure to measurement error, left out variables, or other sorts of stochastic influences rather than to reject the hypothesis outright" Varian (1985). Therefore, in section 6, and in the tradition of Varian (1985) we measure the magnitude of departure from the optimisation hypothesis, i.e. we construct a statistical version of the test of rationalizability. In section 7 we generalize the setting of section 2 for an arbitrary number of players and asymmetric bargaining powers. Finally, in section 8 we conclude.

1.2. Nash bargaining model

Suppose that we observe, for a finite number of situations, the way in which two people split a common endowment: for each t in the set $\{1, \ldots, T\}$, we observe the allocation $(x_t^1, x_t^2) \in \mathbb{R}^2$, of an aggregate resource $X_t \in \mathbb{R}$. When can we guarantee that this information can be modelled by the Nash bargaining solution, given that we cannot observe the individuals' utility functions?

Let us denote by $u^i(x)$ the utility level that individual i attains if she consumes x units of the resource. If the collective decision of the two people is consistent with the Nash solution, each observation must solve the program

$$\max_{x^1,x^2} \left\{ [u^1(x^1) - v_t^1][u^2(x^2) - v_t^2] : x^1 + x^2 = X_t \text{ and } u^i(x^i) > v_t^i \right\}, \tag{1.1}$$

where v_t^1 and v_t^2 are exogenously determined utility levels that the players can obtain by themselves if they break up the negotiations. Note that we assume, as in the original work of Nash, that there is a feasible allocation that leaves both individuals strictly better off than if they do withdraw from the negotiations. Now, suppose that all an outside econometrician can observe is the data

$$\left\{ (x_t^1, x_t^2, X_t) : t = 1, \ldots, T \right\}, \tag{1.2}$$

but he does not know the utility functions u^1 and u^2. What restrictions does the structure of Program (1.1) impose on data set (1.2)?[3]

1.2.1. *Under Observation of Default Utility Levels*

Suppose, for the moment, that the analyst also has information about the utility levels that both individuals would attain if they did not agree on how to share the resource.[4] In this case, the data set would be of the form

$$\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \dots, T\right\}, \tag{1.3}$$

where we assume that $x_t^1 + x_t^2 = X_t$ at all observations.[5]

We say that data set (1.3) *is rationalizable* if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1.1).[6]

[3] It is important to note that we test the hypothesis that the observed outcome is derived from the Nash bargaining solution under a restricted domain of problems: those corresponding to a given pair of individuals, for a finite set of aggregate resources and default payoffs. This is in contrast to the axiomatic analysis of the solution, where the entire universe of bargaining problems is considered. In this sense, our results do not offer an axiomatization of the solution, but just the set of all the necessary conditions that the data have to satisfy if they are derived from the solution in the finite set of observations of the individuals.

[4] Surely, later we will relax the assumption that these *status quo* utility levels are observable.

[5] We comment, below, why we do not allow for waste of the aggregate resource.

[6] Since we impose strict monotonicity of preferences, it is an immediate testable implication of the rationalizability definition that the individuals do not want to waste resources. Of course, there are other solution concepts in which the latter is not true, and the assumption that $x_t^1 + x_t^2 = X_t$

1.2.2. *A test of rationalizability: necessary conditions*

PROPOSITION 1. *If a data set of form* (1.3) *is rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the following system: for all t *and* t',

$$\mu_{t'}^i \leqslant \mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x_{t'}^i - x_t^i), \tag{1.4}$$

with strict inequality if $x_t^i \neq x_{t'}^i$*; for all* i *and all* t,

$$\mu_t^i > \nu_t^i; \tag{1.5}$$

and for all t,

$$\lambda_t > 0. \tag{1.6}$$

Proof: Under strict monotonicity, we can re-write Program (1.1) as

$$\min_x \left\{f_t(x) : u^1(x) > \nu_t^1 \text{ and } u^2(X_t - x) > \nu_t^2\right\}, \tag{1.7}$$

where

$$f_t(x) := [\nu_t^1 - u^1(x)][u^2(X - x) - \nu_t^2].$$

Since both utility functions are concave, they are Lipschitz continuous and, hence, a necessary condition for x_t^1 to solve Program (1.7) is that $0 \in \partial f_t(x_t^1)$.[7] By construction, $0 \in \partial f_t(x_t^1)$ only if there exist $\delta_t^1 \in \partial u^1(x_t^1)$ and $\delta_t^2 \in \partial u^2(x_t^2)$ such that

$$\delta_t^1[u^2(X_t - x_t^1) - \nu_t^2] = \delta_t^2[u^1(x_t^1) - \nu_t^1].$$

would be restrictive if we were studying those solutions. For the focus of our paper, the question is what restrictions, other than the fact that there is no waste, are implied by the solution hypothesis.

[7] We use ∂g to denote the subgradient of any function g.

Since function u^1 is strictly increasing, if we define the number

$$\lambda_t := \frac{\delta_t^1}{u^1(x_t^1) - v_t^1} > 0,$$

we get that, for both individuals, necessarily, $\delta_t^i = \lambda_t[u^i(x_t^i) - v_t^i]$. Since both u^1 and u^2 are strictly concave and $\delta_t^i \in \partial u^i(x_t^i)$, the latter implies that, for all $x \in \mathbb{R}$, $x \neq x_t^i$,

$$u^i(x) < u^i(x_t^i) + \lambda_t[u^i(x_t^i) - v_t^i](x - x_t^i).$$

Thus, we can define the numbers $\mu_t^i := u^i(x_t^i) > v_t^i$. *Q.E.D.*

A comment on the implications of this proposition is in order.[8] It is well known that the Nash bargaining solution treats the preferences of the individuals as cardinal objects. It may then seem paradoxical that our revealed preferences approach is applicable in this context, since this approach is normally of ordinal nature. But it is then important to note that the necessary condition (1.4) imposes a common factor to the term that appears on the right-hand side of the usual Afriat expansion: λ_t is common to both individuals. This commonality gives the whole system cardinal content: the λ_t that solves Eq. (1.4) is *not* robust to arbitrary transformations of the preferences of the individuals, even if they preserve their ordinal content.

While this result provides a necessary condition for rationalizability, by itself it does not constitute a test of the hypothesis, for in principle it could be that the necessary condition is tautological, in which case any data set would be rationalizable and the hypothesis would not be refutable.

[8] We thank Alejandro Saporiti for this observation.

1.2.3. *Power of the test: sufficiency and non-tautology of the necessary condition*

Our next claim is that the existence of a solution to the system of inequalities defined by Eqs. (1.4) to (1.6) exhausts the necessary conditions of the hypothesis that a data set is rationalizable, as this condition is also sufficient for the hypothesis.

PROPOSITION **2.** *Given a data set of form* (1.3), *suppose that there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of inequalities (1.4) to (1.6), defined in Proposition 1. Then, the data set is rationalizable, and the utility functions that rationalize it can be constructed in the class $\mathbf{C}^2$.

Proof: Given a solution $\left\{(\mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$ to the system of inequalities, construct the following utility functions: for each player i,

$$u_0^i(x) := \min\left\{\mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x - x_t^i) : t = 1, \ldots, T\right\},$$

mapping $\mathbb{R}$ into $\mathbb{R}$. These functions are continuous, concave and strictly increasing, and are $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$.

With these constructions, and for any observation t, note that $u_0^i(x_t^i) \leqslant \mu_t^i$. If this inequality was strict, then for some other observation t' we would have that

$$\mu_{t'}^i + \lambda_{t'}(\mu_{t'}^i - \nu_{t'}^i)(x_t^i - x_{t'}^i) < \mu_t^i,$$

which contradicts Eq. (1.4),[9] so we must conclude that $u_0^i(x_t^i) = \mu_t^i$.

[9] By Eqs. (1.4) the linear approximation must be the tangent line to the general function, i.e. $u_0^i(x_t^i) = u^i(x_t^i)$.

Now, consider any pair $(x^1, x^2) \neq (x^1_t, x^2_t)$ that is feasible in the program

$$\max_{x^1, x^2} \left\{ [u^1_0(x^1) - \nu^1_t][u^2_0(x^2) - \nu^2_t] : x^1 + x^2 \leqslant X_t \text{ and } u^i_0(x^i) > \nu^i_t \right\}. \tag{1.8}$$

By definition and construction,

$$0 < u^i_0(x^i) - \nu^i_t \leqslant \mu^i_t + \lambda_t(\mu^i_t - \nu^i_t)(x^i - x^i_t) - \nu^i_t$$

for both $i = 1, 2$. Multiplying, we thus get that

$$\begin{aligned} [u^1_0(x^1) - \nu^1_t][u^2_0(x^2) - \nu^2_t] &\leqslant [\mu^1_t + \lambda_t(\mu^1_t - \nu^1_t)(x^1 - x^1_t) - \nu^1_t][\mu^2_t + \lambda_t(\mu^2_t - \nu^2_t)(x^2 - x^2_t) - \nu^2_t] \\ &= (\mu^1_t - \nu^1_t)(\mu^2_t - \nu^2_t)[1 + \lambda_t(x^1 - x^1_t)][1 + \lambda_t(x^2 - x^2_t)] \\ &= (\mu^1_t - \nu^1_t)(\mu^2_t - \nu^2_t)[1 + \lambda_t(x^1 + x^2 - X_t) + (\lambda_t)^2(x^1 - x^1_t)(x^2 - x^2_t)], \end{aligned}$$

where, in the last line, we have used the fact that $x^1_t + x^2_t = X_t$. Now, consider each of the terms on the right-hand side of the latter expression: first, by feasibility of (x^1, x^2), we have that

$$\mu^1_t - \nu^1_t > 0 \text{ and } \mu^2_t - \nu^2_t > 0;$$

also, by Eq. (1.6) and feasibility, we have that

$$\lambda_t(x^1 + x^2 - X_t) \leqslant 0;$$

and, finally, we have that

$$(x^1 - x^1_t)(x^2 - x^2_t) \leqslant 0,$$

since again, by feasibility, $x^1 + x^2 \leqslant X_t = x^1_t + x^2_t$. Since $(x^1, x^2) \neq (x^1_t, x^2_t)$, the previous two inequalities cannot hold with equality at the same time. This implies that

$$1 + \lambda_t(x^1 + x^2 - X_t) + (\lambda_t)^2(x^1 - x^1_t)(x^2 - x^2_t) < 1,$$

and hence that

$$(\mu_t^1 - \nu_t^1)(\mu_t^2 - \nu_t^2)[1 + \lambda_t(x^1 + x^2 - X_t) + (\lambda_t)^2(x^1 - x_t^1)(x^2 - x_t^2)] < (\mu_t^1 - \nu_t^1)(\mu_t^2 - \nu_t^2).$$

We conclude, hence, that

$$[u_0^1(x^1) - \nu_t^1][u_0^2(x^2) - \nu_t^2] < (\mu_t^1 - \nu_t^1)(\mu_t^2 - \nu_t^2) = [u_0^1(x_t^1) - \nu_t^1][u_0^2(x_t^2) - \nu_t^2], \tag{1.9}$$

and since, by Eq. (1.5), $u_0^i(x_t^i) > \nu_t^i$ for both $i = 1, 2$, and $x_t^1 + x_t^2 = X_t$, we conclude that (x_t^1, x_t^2) solves Program (1.8).

To complete the proof, we ought to show that u_0^1 and u_0^2 can be deformed into functions u^1 and u^2 that are strictly concave and smooth. This can be done, using a convolution, since the inequalities of Eq. (1.4) and Eq. (1.9) are all strict whenever $x_t^i \neq x_{t'}^i$, and since the number of observations is finite. The details of this construction are deferred to Appendix A. *Q.E.D.*

An implication of the proposition is that, as is commonly the case in the revealed preference literature, an analyst would need at least two observations in order to be able to reject the hypothesis of the Nash bargaining solution: if $T = 1$, then condition 1.4 is vacuous and the whole system is always satisfied.[10]

[10] An explicit construction that rationalizes any single observation $(x_1, x_2, X, \nu_1, \nu_2)$ is as follows: assume, with no loss of generality, that $x_2 > x_1 > 0$, and let $u^1(x) = \ln x + \alpha$ and $u^2(x) = \ln x + \beta$, with $\beta > \nu_2 + \max\{0, -\ln x_2\}$ and

$$\alpha = \nu^1 - \ln x_1 + \frac{x_2}{x_1}(\ln x_2 + \beta - \nu_2).$$

More importantly, together with Proposition 1, this latter result allows us to claim that the hypothesis of rationalizability is testable, and to state the type of test an analyst can develop.

PROPOSITION 3. *There exists a non-tautological condition that a data set of form* (1.3) *satisfies if, and only if, it is rationalizable. Moreover, this condition is a finite set of polynomial inequalities on* $\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \ldots, T\right\}$.

Proof: The set of values of

$$\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2, \mu_t^1, \mu_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that satisfy the system of inequalities defined by Eqs. (1.4) to (1.6) is, by definition, a semi-algebraic set. By the Tarski-Seidenberg algorithm, the projection of this set into the space of data (that is, of values of $\left\{(x_t^1, x_t^2, X_t, v_t^1, v_t^2) : t = 1, \ldots, T\right\}$) is semi-algebraic as well, which means that it can be characterized by a finite set of polynomial inequalities. Then, by Propositions 1 and 2, a data set is rationalizable if, and only if, it satisfies these polynomial inequalities (or, put another way, if it lies in the latter projected set).

To see that such system of polynomial inequalities is not tautological, it suffices to find a non-rationalizable data set. To see this, consider a set of the form (1.3) where, for a pair of observations t and t' one has that $v_t^1 = v_{t'}^1$, $v_t^2 = v_{t'}^2$, $X_t < X_{t'}$ and $x_t^1 > x_{t'}^1$. If such set were rationalizable, there would exist $\delta_t^i \in \partial u^i(x_t^i)$ and $\delta_{t'}^i \in \partial u^i(x_{t'}^i)$, for $i = 1, 2$, such that

$$\delta_t^1[u^2(x_t^2) - v_t^2] = \delta_t^2[u^1(x_t^1) - v_t^1] \tag{1.10}$$

and

$$\delta^1_{t'}[u^2(x^2_{t'}) - v^2_{t'}] = \delta^2_{t'}[u^1(x^1_{t'}) - v^1_{t'}]. \tag{1.11}$$

By concavity of the utility functions, $\delta^1_t < \delta^1_{t'}$ and $\delta^2_t > \delta^2_{t'}$, given that $x^2_t < x^2_{t'}$ since $X_t < X_{t'}$. By their monotoniciy, similarly, $u^1(x^1_t) > u^1(x^1_{t'})$ and $u^2(x^2_t) < u^2(x^2_{t'})$. It then follows that

$$\begin{aligned}
\delta^1_t[u^2(x^2_t) - v^2_t] &< \delta^1_{t'}[u^2(x^2_{t'}) - v^2_t] \\
&= \delta^1_{t'}[u^2(x^2_{t'}) - v^2_{t'}] \\
&= \delta^2_{t'}[u^1(x^1_{t'}) - v^1_{t'}] \\
&= \delta^2_{t'}[u^1(x^1_{t'}) - v^1_t] \\
&< \delta^2_t[u^1(x^1_t) - v^1_t] \\
&= \delta^1_t[u^2(x^2_t) - v^2_t],
\end{aligned}$$

where the second equality follows from Eq. (1.11), and the last one from (1.10). This is obviously impossible, so we conclude that the data set cannot be rationalized. *Q.E.D.*

In fact, it is useful to complement the proof of this proposition with an analysis of the comparative statics of Program (1.1), for the case when the utility functions are differentiable twice. In such case, note that we can write the first-order condition of program

$$\max_{x^1,x^2} \left\{ [u^1(x^1) - v^1][u^2(X - x^1) - v^2] : u^1(x^1) > v^1 \text{ and } u^2(X - x^1) > v^2 \right\},$$

as

$$\partial u^1(x^1)[u^2(X - x^1) - v^2] = \partial u^2(X - x^1)[u^1(x^1) - v^1].$$

If we totally differentiate this equation, we get that, over the manifold of solutions to the program, dx^1 equals

$$\frac{\left\{\partial^2 u^2(x^2)[u^1(x^1) - v^1] - \partial u^1(x^1)\partial u^2(x^2)\right\} dX - \partial u^2(x^2)dv^1 + \partial u^1(x^1)dv^2}{\partial^2 u^1(x^1)[u^2(x^2) - v^2] - 2\partial u^1(x^1)\partial u^2(x^2) + \partial^2 u^2(x^2)[u^1(x^1) - v^1]}. \tag{1.12}$$

Since both utility functions are strictly increasing and strictly concave, and since $u^1(x^1) > v^1$ and $u^2(X - x^1) > v^2$, it follows that the denominator in this equation is negative. For the same reasons, the term that multiplies dX in the numerator is negative too, while the terms that multiply dv^1 and dv^2 are both positive. As a consequence, it follows that if $dX > 0$, $dv^1 > 0$ and $dv^2 < 0$, then, unambiguously, $dx^1 > 0$. It then follows that a data set of form (1.3) is *not* rationalizable if it contains a pair of distinct observations, t and t', such that

$$X_t > X_{t'}, v_t^1 > v_{t'}^1, v_t^2 < v_{t'}^2 \text{ and } x_t^1 < x_{t'}^1.$$

Subsequently, this observation will allow us to claim that, under differentiable utility functions, the hypothesis of rationalizability is refutable on the basis of a test.

1.3. Without Observation of Default Utility Levels

Suppose now that the observer has no information about the default utility levels of the two individuals, so that the data set reduces to (1.2).[11]

[11] We maintain the assumption that $x_t^1 + x_t^2 = X_t$ at all observations.

1.3.1. No assumptions on the default utility levels: unfalsifiability

Suppose that the analyst is not willing to impose any conditions on the (unobserved) utility levels that the individuals could have obtained by withdrawing from the negotiation. In this setting, we shall say that the data set of form (1.2) *is rationalizable* if there exist default utility levels v_t^1 and v_t^2, for $t = 1, \dots, T$, and individual utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, and are such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1.1).

Our next result shows that in this case the hypothesis of rationalizability becomes irrefutable.

Proposition 4. *Any data set of the form* (1.2) *is rationalizable. Moreover, in the rationalization both utility functions can be constructed in the class* $\mathbf{C}^2$.

Before proving the proposition, we introduce a lemma that will be useful for the result.

Lemma 1. *For any finite set of numbers* $\{x_s : s = 1, \dots, S\}$, *there exists an array of pairs of numbers,* $\{(\mu_s, \delta_s) : s = 1, \dots, S\}$, *such that* $\delta_s > 0$ *for all* s, *and*

$$\mu_{s'} < \mu_s + \delta_s(x_{s'} - x_s)$$

for all s *and all* $s' \neq s$.

Proof: With no loss of generality, let us assume that $x_1 > x_2 > \dots > x_S$. We can re-write the desired conditions as

$$\mu_{s'} - \mu_s + \delta_s(x_s - x_{s'}) < 0,$$

for all s and all $s' \neq s$, with $-\delta_s < 0$ for all s.

If such system has no solution, then, by the Theorem of the Alternative,[12] we can find a double array of non-negative numbers $\{\alpha_{s,s'} : s, s' = 1, \ldots, S, s' \neq s\}$ and an array of non-negative numbers $\{\beta_s : s = 1, \ldots, S\}$ such that, for all s,

$$\sum_{s' \neq s} \alpha_{s,s'} = \sum_{s' \neq s} \alpha_{s',s} \tag{1.13}$$

and

$$\sum_{s' \neq s} \alpha_{s,s'}(x_s - x_{s'}) = \beta_s, \tag{1.14}$$

with at least one of the numbers in these two arrays being different from zero.

Aggregating Eq. (1.14) across observations, $\sum_s \sum_{s' \neq s} \alpha_{s,s'}(x_s - x_{s'}) = \sum_s \beta_s$. By Eq. (1.13), the left-hand side of the latter expression is null, which in turn implies that $\beta_s = 0$ for all s. Using this, Eq. (1.14) implies that $\sum_{s' \neq 1} \alpha_{1,s'}(x_1 - x_{s'}) = 0$. But note that, since $x_1 > x_s$ for all $s \neq 1$, this equality is possible only if $\alpha_{1,s'} = 0$ for all $s' \neq 1$. Then, (1.13) implies that, moreover, $\alpha_{s',1} = 0$ for all $s' \neq 1$. But then, for $s = 2$, (1.14) implies that

$$\sum_{s \geqslant 3} \alpha_{2,s'}(x_2 - x_{s'}) = \sum_{s' \neq 2} \alpha_{2,s'}(x_2 - x_{s'}) = 0.$$

Again, since $x_2 > x_s$ for all $s \geqslant 3$, the latter implies that $\alpha_{2,s'} = 0$ for all $s' \neq 2$, and (1.13) again implies that $\alpha_{s',2} = 0$ for all $s' \neq 2$. Continuing in this fashion, we obtain that $\alpha_{s,s'} = 0$ for all s and all $s' \neq s$, which contradicts the fact that at least one of the numbers in the two arrays is different from zero. *Q.E.D.*

[12] See, for example, R. Tyrell Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

This lemma is the key step in the proof of Proposition 4:

Proof of Proposition 4: Given the data set, by Lemma 1 we have that for each individual $i = 1, 2$, we can find numbers μ_t^i and $\delta_t^i > 0$ for all t, such that

$$\mu_{t'}^i \leqslant \mu_t^i + \delta_t^i (x_{t'}^i - x_t^i),$$

with strict inequality if $x_{t'}^i \neq x_t^i$.

Now, fix an arbitrary array of numbers $\left\{\nu_t^1 : t = 1, \dots, T\right\}$ such that $\nu_t^1 < \mu_t^1$ at all t. Using these numbers, define, for each t,

$$\lambda_t := \frac{\delta_t^1}{\mu_t^1 - \nu_t^1}$$

and

$$\nu_t^2 := \mu_t^2 - \frac{\delta_t^2}{\lambda_t}.$$

By construction, $\lambda_t > 0$ and $\nu_t^2 < \mu_t^2$, while it is immediate that

$$\mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x_{t'}^i - x_t^i) = \mu_t^i + \delta_t^i(x_{t'}^i - x_t^i) \geqslant \mu_{t'}^i.$$

with strict inequality if $x_{t'}^i - x_t^i$. By Proposition 2, it follows that there are individual utility functions u^1 and u^2 that satisfy the desired properties and are such that at each observation t, the pair (x_t^1, x_t^2) solves Program (1.1). *Q.E.D.*

Proposition 4 tells the analyst that if he is not willing to make any assumptions on the levels of utility that the two individuals could secure for themselves by breaking up the negotiations, or on the evolution of these levels, the hypothesis that the data can be explained using the Nash bargaining solution is not refutable and, hence, the hypothesis itself is unscientific from a Popperian perspective. In fact, a careful look at the proof of that proposition shows that even if the analyst had observations of the default utility levels of one of the individuals, but not on the levels of the other one, then the hypothesis of Nash bargaining solution would be unfalsifiable: this is, indeed, what the proof actually shows. Moreover, with a small modification of the proof one can show that the player for whom the analyst has observed the default utility levels need not be the same at all observations: if, for each observation, one observes ν_t^i but not $\nu_t^{\neg i}$, the argument continues to hold by defining

$$\lambda_t := \frac{\delta_t^i}{\mu_t^i - \nu_t^i} > 0$$

and

$$\nu_t^{\neg i} := \mu_t^{\neg i} - \frac{\delta_t^{\neg i}}{\lambda_t} < \mu_t^{\neg i};$$

even in this case, the hypothesis continues to be irrefutable.

1.3.2. *Some assumptions on the default utility levels: falsifiability restored*

We now know that, in order to have a testable theory, the analyst has to impose conditions on the evolution of default utility levels. These conditions will now form part of the hypothesis being tested,[13] and rejection of this joint test does not inform which of the hypotheses drove the rejection. Still, if in a given exercise the analyst

[13] As much as, for instance, the conditions that define the class of utility functions that are being allowed by our definitions.

can reasonably impose structure on these unobserved variables, if this structure is 'enough' it may restore the refutability of the hypothesis.

For instance, say that a data set of the form (1.2) *is rationalizable with invariant default utility levels* if there exist two numbers v^1 and v^2, and individual utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x^1_t, x^2_t) solves program

$$\max_{x^1,x^2} \left\{ [u^1(x^1) - v^1][u^2(x^2) - v^2] : x^1 + x^2 \leqslant X_t \text{ and } u^i(x^i) > v^i \right\}.$$

Our next result is that this extra assumption on the unobserved utility levels strengthens the hypothesis and renders it refutable again.

Proposition 5. *There exists a non-tautological condition that a data set of form* (1.2) *satisfies if, and only if, it is rationalizable with invariant default utility levels. Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{ (x^1_t, x^2_t, X_t) : t = 1, \dots, T \right\}.$$

Proof: Since the details of the argument are similar to previous ones, we shall omit them. First, as in the proof of Proposition 1 if the data set is rationalizable with invariant default utility levels, there must exist a pair of numbers (v^1, v^2) and an array of numbers

$$\left\{ (\mu^1_t, \mu^2_t, \lambda_t) : t = 1, \dots, T \right\}$$

that solves the following system:

$$\mu^i_{t'} \leqslant \mu^i_t + \lambda_t(\mu^i_t - v^i)(x^i_{t'} - x^i_t), \tag{1.15}$$

with strict inequality if $x^i_{t'} \neq x^i_t$,

$$\mu^i_t > \nu^i, \tag{1.16}$$

and

$$\lambda_t > 0. \tag{1.17}$$

On the other hand, it follows from Proposition 2 that the existence of this solution to the system is sufficient for the data set to be rationalizable, and since the pair (ν^1, ν^2) remains constant across all observations, the data set is, *a fortiori*, rationalized with invariant default utility levels.

Now, as in the proof of Proposition 3, the system of inequalities (1.15), (1.16) and (1.17) defines a semi-algebraic set, so its projection into the space of data is semi-algebraic as well, and is characterized by a finite set of polynomial inequalities. In order to see that such condition is non-tautological, it suffices to observe that, as in Proposition 3, any data set in which x^i_t and X_t are not co-monotone cannot be rationalized with invariant default utilities.[14] *Q.E.D.*

Invariance of the unobserved utility levels is not the only case in which the analyst can recover refutability: our argument can be extended to argue that the hypothesis is refutable if it imposes that the unobserved utility levels $\left\{(\nu^1_t, \nu^2_t) : t = 1, \dots, T\right\}$, while not being observed, satisfy that, ν^1_t is weakly co-monotone with X_t and ν^2_t is weakly anti-co-monotone with X_t. Indeed, in this case the necessary and sufficient

[14] Variables y_t and z_t are said to be co-monotone if $y_t \geqslant y_{t'}$ occurs when, and only when, $z_t \geqslant z_{t'}$. They are anti-co-monotone if $y_t \geqslant y_{t'}$ occurs when, and only when, $z_t \leqslant z_{t'}$.

system of inequalities is defined on the array

$$\left\{(\mu_t^1, \mu_t^2, \nu_t^1, \nu_t^2, \lambda_t) : t = 1, \dots, T\right\},$$

and it includes Eqs. (1.4), (1.5) and (1.6), as well as the following two requirements, which are immediate from the new assumptions: for every t and every t'

$$(\nu_t^1 - \nu_{t'}^1)(X_t - X_{t'}) \geqslant 0,$$

while

$$(\nu_t^2 - \nu_{t'}^2)(X_t - X_{t'}) \leqslant 0.$$

Since these extra inequalities are polynomial, the set of solutions remains semi-algebraic and our analysis of its projection is still valid. On the other hand, it follows immediately from Eq. (1.12) that in this case x_t^1 and X_t must be co-monotonic, so the system that characterizes the projection is again non-tautological.[15]

1.4. Default Income Levels

The model under consideration, and the way in which we have dealt with it so far, specifies utility levels that the individuals can attain by withdrawing from the negotiation without reaching an agreement. We have seen that even if utility levels are observed, or at least are restricted by the econometrician when they are unobserved, makes a significant difference in terms of the empirical implications of the model. However, it is more realistic to assume that the analyst has some information, or hypothesis, about the behavior of the *income* levels that the individuals can obtain

[15] Now, though, we cannot say anything about the co-variation of x_t^2 and X_t.

outside the cooperative agreement as opposed to the *utility* they derive from these income levels. So, suppose now that the data set is of the form

$$\left\{(x_t^1, x_t^2, X_t, y_t^1, y_t^2) : t = 1, \dots, T\right\}, \tag{1.18}$$

where y_t^i represents the default income level that individual i could have obtained at observation t if an agreement had not been reached. We continue to assume that $x_t^1 + x_t^2 = X_t$, and additionally, suppose that $x_t^i > y_t^i$ at all observations.

We will say that a data set of the form (1.18) *is rationalizable* if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are strictly increasing and strictly concave, such that at each observation t, the pair (x_t^1, x_t^2) solves the program

$$\max_{x^1, x^2} \left\{[u^1(x^1) - u^1(y_t^1)][u^2(x^2) - u^2(y_t^2)] : x^1 + x^2 \leqslant X_t \text{ and } u^i(x^i) > u^i(y_t^i)\right\}. \tag{1.19}$$

The next result, which is analogous to Proposition 1, says that a necessary condition for rationalizability is the existence of a solution to a system of polynomial inequalities. The system required for this new setting, however, is slightly more complicated as we now need to account for more points in the domain of the utility functions.

Proposition 6. *If a data set of the form* (1.18) *is rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \nu_t^1, \nu_t^2, \delta_t^1, \delta_t^2, \lambda_t) : t = 1, \dots, T\right\}$$

that solves the following system:

$$\mu_{t'}^i \leqslant \mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x_{t'}^i - x_t^i), \tag{1.20}$$

with strict inequality if $x^i_{t'} \neq x^i_t$;

$$\nu^i_{t'} \leqslant \nu^i_t + \delta^i_t(y^i_{t'} - y^i_t), \tag{1.21}$$

with strict inequality if $y^i_{t'} \neq y^i_t$;

$$\mu^i_{t'} \leqslant \nu^i_t + \delta^i_t(x^i_{t'} - y^i_t), \tag{1.22}$$

with strict inequality if $x^i_{t'} \neq y^i_t$;

$$\nu^i_{t'} \leqslant \mu^i_t + \lambda_t(\mu^i_t - \nu^i_t)(y^i_{t'} - x^i_t) \tag{1.23}$$

with strict inequality if $y^i_{t'} \neq x^i_t$;

$$\mu^i_t > \nu^i_t \tag{1.24}$$

and

$$\delta^i_t > 0 \tag{1.25}$$

for all i, *all* t *and all* $t' \neq t$*; and*

$$\lambda_t > 0 \tag{1.26}$$

for all t.

Proof: As in the proof of Proposition 1, the first-order necessary condition of Program (1.19) require that for some $\delta^1_t \in \partial u^1(x^1_t)$ and $\delta^2_t \in \partial u^2(x^2_t)$, we have that

$$\delta^1_t[u^2(x^2_t) - u^2(y^2_t)] = \delta^2_t[u^1(x^1_t) - u^1(y^1_t)].$$

Since u^1 is strictly increasing, defining the number

$$\lambda_t = \frac{\delta^1_t}{u^1(x^1_t) - u^1(y^1_t)} > 0,$$

we get that

$$\delta_t^i = \lambda_t[u^i(x_t^i) - \nu_t^i].$$

Now, if we also pick $\delta_t^i \in \partial u^i(y_t^i) > 0$, $\mu_t^i = u^i(x_t^i)$ and $\nu_t^i = u^i(y_t^i)$, we get Eqs. (1.20) to (1.23) from the fact that u^1 and u^2 are strictly concave, while we get Eq. (1.24) from their strict monotonicity, given that $x_t^i > y_t^i$ by assumption. The other two conditions, Eqs. (1.25) and (1.26), also follow from monotonicity. *Q.E.D.*

Since we now need to account for more points in the domains of the utility functions, the system of inequalities in this setting is more complex. The first condition in the system, Eq. (1.20), compares the utility level at the attained income of observation t' with its first-order approximation around the attained income of observation t. This was the same comparison that we had in the system of Proposition 1. Now, we also need to compare the utility level at the default income of observation t' with its linear approximation around the default income of observation t, which is done by Eq. (1.21); the utility level at the attained income of observation t' with its linear approximation around the default income of observation t, which is Eq. (1.22); and the utility at the default income of observation t' with its approximation around the attained income of observation t, namely Eq. (1.23). Importantly, while the hypothesis of rationalizability imposes a condition on the derivatives of the utility functions at the attained income levels (i.e., the equality given by the first-order condition of Program (1.19)), their derivatives at the default income levels are only constrained to be positive.

These extra conditions have to be imposed as part of the system, for otherwise we cannot guarantee its sufficiency, which we obtain in the following result.

Proposition 7. *Given a data set of the form* (1.18)*, suppose that there exists an array of numbers*

$$\left\{(\mu_t^1, \mu_t^2, \nu_t^1, \nu_t^2, \delta_t^1, \delta_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that solves the system of inequalities defined in Proposition 6. Then, the data set is rationalizable and, moreover, the utility functions that rationalize it can be constructed in the class $\mathbf{C}^2$.

Proof: The argument resembles the proof of Proposition 2. Given a solution

$$\left\{(\mu_t^1, \mu_t^2, \nu_t^1, \nu_t^2, \delta_t^1, \delta_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

to the system of inequalities, construct the functions

$$u_0^i(x) := \min\left\{\min\left\{\mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x - x_t^i), \nu_t^i + \delta_t^i(x - y_t^i)\right\} : t = 1, \ldots, T\right\}.$$

which satisfy that $u_0^i(x_t^i) = \mu_t^i$ and $u_0^i(y_t^i) = \nu_t^i$.[16]

Taking, as before, any pair $(x^1, x^2) \neq (x_t^1, x_t^2)$ that is feasible in the program

$$\max_{x^1, x^2}\left\{[u_0^1(x^1) - \nu_t^1][u_0^2(x^2) - \nu_t^2] : x^1 + x^2 \leqslant X_t \text{ and } u_0^i(x^i) > \nu_t^i\right\},$$

we still get that

$$0 < u_0^i(x^i) - \nu_t^i \leqslant \mu_t^i + \lambda_t(\mu_t^i - \nu_t^i)(x - x_t^i) - \nu_t^i.$$

[16] And which are continuous, concave and strictly increasing, and are $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$.

Since this is what was required for the argument of Proposition 2, we can omit the remaining details. *Q.E.D.*

Importantly, the conditions that are added to the system are still polynomial inequalities, so the set of arrays

$$\left\{(x_t^1, x_t^2, X_t, y_t^1, y_t^2, \nu_t^1, \nu_t^2, \mu_t^1, \mu_t^2, \delta_t^1, \delta_t^2, \lambda_t) : t = 1, \ldots, T\right\}$$

that satisfy the system is semi-algebraic, and so is, therefore, its projection into the space of data. As before, this set is, thus, characterized by a finite set of polynomial inequalities, which constitute the strongest possible test of the hypothesis of rationalizability in this setting. Assuming that the utility functions that rationalize the data are $\mathbf{C}^2$, we can see that the test is not a tautology by replacing $d\nu^i = \partial u^i(y^i)dy^i$ in Eq. (1.12), in order to obtain comparative statics for the case that we are considering: dx^1 now equals the ratio of

$$\begin{aligned}\left\{\partial^2 u^2(x^2)[u^1(x^1) - u^1(y^1)] - \partial u^1(x^1)\partial u^2(x^2)\right\} dX \\ -\partial u^2(x^2)\partial u^1(y^1)dy^1 + \partial u^1(x^1)\partial u^2(y^2)dy^2\end{aligned}$$

and

$$\partial^2 u^1(x^1)[u^2(x^2) - u^2(y^2)] - 2\partial u^1(x^1)\partial u^2(x^2) + \partial^2 u^2(x^2)[u^1(x^1) - u^1(y^1)].$$

Under monotonicity, the latter implies that no set that contains a pair of observations for which

$$X_t > X_{t'}, y_t^1 > y_{t'}^1, y_t^2 < y_{t'}^2 \text{ and } x_t^1 < x_{t'}^1$$

can be rationalized. Thus, the following proposition summarizes these observations.

Proposition 8. *There exists a non-tautological condition that a data set of form* (1.18) *satisfies if, and only if, it is rationalizable by utility functions in the class* $\mathbf{C}^2$. *Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{(x_t^1, x_t^2, X_t, y_t^1, y_t^2) : t = 1, \ldots, T\right\}.$$

1.5. Bounds on Unobserved Default Income Levels

The analysis of the case when default income levels are observed allows us to address a weakness that our previous results display. Consider again the setting of Section 1.2.1, where we have data of the form (1.3). Our definition of rationalizability in that section did not require the existence of an unobserved default income level at which the individuals would obtain the observed default utility levels under our construction of the utility functions: indeed, the definition of rationalizability does not require that there exist numbers

$$\{(y_t^1, y_t^2) : t = 1, \ldots, T\}$$

such that $u^1(y_t^1) = v_t^1$ and $u^2(y_t^2) = v_t^2$ at all observations. And while our construction in Proposition 2 delivers us this extra requirement, for the functions u^i constructed in the proof are unbounded below, it may still be the case that in such construction the implicit y_t^i for which $u^i(y_t^i) = v_t^i$ is not plausible from the point of view of the analyst, for some i and some t – for instance, it could be the case that such income level has to be allowed to take negative values.

This problem can be addressed by extending the system of Proposition 1 in the same way as Eqs. (1.20) to (1.26). For instance, let $Y_t^i \subseteq \mathbb{R}$ be a nonempty set of default income levels that the analyst considers possible for individual i at observation t, and denote by Y the collection of all these constraints:

$$Y := \left\{ (Y_t^1, Y_t^2) : t = 1, \ldots, T \right\}.$$

Say that a data set of the form 1.3 is *rationalizable with respect to* Y if there exist utility functions $u^1 : \mathbb{R} \to \mathbb{R}$ and $u^2 : \mathbb{R} \to \mathbb{R}$, both of which are $\mathbf{C}^2$, strictly increasing and strictly concave, and default income levels

$$\left\{ (y_t^1, y_t^2) : t = 1, \ldots, T \right\}$$

such that (i) at each observation t, the pair (x_t^1, x_t^2) solves Program (1.1); (ii) at each observation t, the default income levels are feasible, in the sense that $y_t^1 \in Y_t^1$ and $y_t^2 \in Y_t^2$; and (iii) the default income levels deliver the observed default utilities: for all t, $u^1(y_t^1) = v_t^1$ and $u^2(y_t^2) = v_t^2$.

The following analysis strengthens the results of Section 1.2.1 for this definition of rationalizability. In addition to the assumptions introduced there, we here assume that all the sets Y_t^i contain at least one $y < x_t^i$. As the results can be proven by arguments similar to the ones given in Section 1.4, we state them without a detailed proof.

Proposition 9. *Fix the collection* Y *of constraints. A data set of the form* (1.18) *is rationalizable with respect to* Y *if, and only if, there exists an array of numbers*

$$\left\{ (\mu_t^1, \mu_t^2, y_t^1, y_t^2, \delta_t^1, \delta_t^1, \lambda_t) : t = 1, \ldots, T \right\}$$

that solves the system of equations (1.20) *to* (1.26), *with* $y_t^i \in Y_t^i$ *and* $y_t^i < x_t^i$ *for all* i *and all* t. *If, moreover, all the constraints in* Y *are semi-algebraic, then there exists a* *non-tautological condition that a data set satisfies if, and only if, it is rationalizable with respect to* Y*. This condition is a finite set of polynomial inequalities on*

$$\left\{(x_t^1, x_t^2, X_t, \nu_t^1, \nu_t^2) : t = 1, \ldots, T\right\}.$$

Proof: Only the fact that the condition imposed by the hypothesis of rationalizability with respect to Y is non-tautological requires a comment. The reason why this is the case is, simply, that if a set is rationalizable with respect to Y then it is rationalizable (in the sense of Section 1.2.1). But since there are data sets that cannot be rationalized, then there are sets that cannot be rationalizable with respect to Y and it follows that the projection of the set of solutions to the system defined in the proposition into the space of data has to be a proper subset of that space. Since we are assuming that all the sets in Y are semi-algebraic, the fact that the condition is non-tautological is implied by the Tarski-Seidenberg Theorem. *Q.E.D.*

1.6. A Statistical Test

A possible weakness of the results we have obtained so far is the lack of a measure of how strong a rejection of the hypothesis of rationalizability is: that is, the application of our tests is dichotomic in the sense that a data set is rationalizable or not, but when a test is not rationalizable we still do not know how big or small a perturbation to the data would make it consistent with the hypothesis. If a very 'small' perturbation to one of the observations sufficed for the data set to pass the test of rationalizability, the analyst may want to consider the data consistent with the hypothesis, attributing the 'small error' to causes like, for instance, an error in the collection of the data.

This criticism is common to all the basic literature on revealed preferences, but can be addressed by extending the analysis in the direction of the construction of statistical versions of the test (as is done in that literature as well) following, Varian (1985).

For the most plausible framework, suppose that the observed data consists of

$$\left\{(x_t^1, x_t^2, y_t^1, y_t^2) : t = 1, \dots, T\right\},$$

as in Section 1.4, only with the caveat that we now let X_t be constructed as $x_t^1 + x_t^2$, instead of it being observed.[17] The analyst may believe that the real income of individual i at observation t was $x_t^i + \varepsilon_{i,t}^x$, and that her default income was $y_t^i + \varepsilon_{i,t}^y$, accounting for measurement error by the (unobserved) perturbations

$$\varepsilon := \left\{(\varepsilon_{1,t}^x, \varepsilon_{2,t}^x, \varepsilon_{1,t}^y, \varepsilon_{2,t}^y) : t = 1, \dots, T\right\}$$

In this setting, as in Varian (1985), we can construct a statistical version of the test of rationalizability. First, let $\mathcal{D}$ be the set of all values of

$$\left\{(\breve{x}_t^1, \breve{x}_t^2, \breve{y}_t^1, \breve{y}_t^2) : t = 1, \dots, T\right\}$$

that satisfy the condition given by Proposition 3, using $\breve{X}_t = \breve{x}_t^1 + \breve{x}_t^2$ (or, equivalently for which there exist a solution to the system of conditions defined in Eqs. (1.20) to (1.26)). Second, for the array

$$e := \left\{(e_{1,t}^x, e_{2,t}^x, e_{1,t}^y, e_{2,t}^y) : t = 1, \dots, T\right\},$$

[17] The reason for this will be apparent momentarily.

define χ as the minimum value of the sum of squared errors, $e \cdot e$, subject to the constraint that

$$(x_t^1 + e_{1,t}^x, x_t^2 + e_{2,t}^x, y_t^1 + e_{1,t}^y, y_t^2 + e_{2,t}^y)_{t=1}^T \in \mathcal{D}.$$

Under the assumption that the perturbations vector ε follows a normal distribution with mean $(0, \ldots, 0)$ and variance-covariance matrix $\sigma\mathbb{I}$, it is immediate that χ/σ follows the χ^2 distribution with $4T$ degrees of freedom. This statistic can be used to test the hypothesis of rationalizability, using the null hypothesis that $\chi = 0$.

1.7. Generalization: Asymmetric Bargaining and Arbitrary Number of Players

The results we have obtained so far can be generalized to an arbitrary numbers of players whose bargaining powers may differ (but are assumed to be constant). For simplicity of presentation, we concentrate on the setting where default utility levels are observed.

That is, suppose that the analyst has observed data of the form

$$\left\{(x_t^i, v_t^i) : i = 1, \ldots, I \text{ and } t = 1, \ldots, T\right\}, \tag{1.27}$$

and let $X_t := \sum_i x_t^i$. Fix a vector $\alpha = (\alpha_1, \ldots, \alpha_I) \gg 0$, and say that one such data set *is α-rationalizable* if there exist utility functions $u^i : \mathbb{R} \to \mathbb{R}$, for $i = 1, \ldots, I$, all of which are $\mathbf{C}^2$, strictly increasing and strictly concave, such that each vector $(x_t^1, \ldots, x_t^I)$ solves the program

$$\max_{(x^1, \ldots, x^I)} \left\{\textstyle\prod_i [u^i(x^i) - v_t^i]^{\alpha_i} : \sum_i x^i \leqslant X_t \text{ and } u^i(x^i) > v_t^i\right\}, \tag{1.28}$$

where the coefficient $\alpha_i > 0$ represents individual i's bargaining power.[18]

As in Section 1.2.1, the first-order conditions of this problem are that for all individuals we have that

$$\frac{\alpha_i \partial u^i(x_t^i)}{u^i(x_t^i) - v_t^i} = \lambda_t, \tag{1.29}$$

for some $\lambda_t > 0$.[19] For all individuals, then,

$$\partial u^i(x_t^i) = \frac{1}{\alpha_i}\lambda_t[u^i(x_t^i) - v_t^i].$$

Since u^i is strictly concave, the latter implies that, for all $x \in \mathbb{R}$, $x \neq x_t^i$,

$$u^i(x) < u^i(x_t^i) + \frac{1}{\alpha_i}\lambda_t[u^i(x_t^i) - v_t^i](x - x_t^i),$$

and, therefore, defining the numbers $\mu_t^i := u^i(x_t^i)$, we have proved the following proposition:

Proposition **10**. *Let $\alpha \gg 0$. If a data set of form* (1.27) *is α-rationalizable, then there exists an array of numbers*

$$\left\{(\mu_t^1, \dots, \mu_t^I, \lambda_t) : t = 1, \dots, T\right\}$$

that solves the following system:

$$\mu_{t'}^i \leqslant \mu_t^i + \frac{1}{\alpha_i}\lambda_t(\mu_t^i - v_t^i)(x_{t'}^i - x_t^i), \tag{1.30}$$

[18] Note that the vector α is part of the hypothesis being tested. That is, the researcher must specify the bargaining powers that are to be tested. We do not consider the weaker hypothesis of whether there exist some bargaining powers under which the data can be rationalized, which would amount to the introduction of an existential quantifier for α in the system below.

[19] With a slight abuse of notation, we now use ∂u^i to denote the derivative of function u^i, given that we have assumed that these functions are all differentiable.

with strict inequality if $x^i_{t'} \neq x^i_t$,

$$\mu^i_t > \nu^i_t, \tag{1.31}$$

and

$$\lambda_t > 0. \tag{1.32}$$

Perhaps it is not surprising, but it is important that the necessary condition we just proposed is sufficient as well.

Proposition 11. *Given a data set of form* (1.27) *and a vector* $\alpha \gg 0$, *suppose that there exists an array of numbers*

$$\left\{(\mu^1_t, \dots, \mu^I_t, \lambda_t) : t = 1, \dots, T\right\}$$

that solves the system of inequalities defined in Proposition 10. Then, the data set is α*-rationalizable.*

Proof: Given the array $\left\{(\mu^1_t, \dots, \mu^I_t, \lambda_t) : t = 1, \dots, T\right\}$ that solves the system of inequalities, construct, for each player i,

$$u^i_0(x) := \min \left\{ \mu^i_t + \frac{1}{\alpha_i} \lambda_t (\mu^i_t - \nu^i_t)(x - x^i_t) : t = 1, \dots, T \right\},$$

mapping $\mathbb{R}$ into $\mathbb{R}$. This functions gives $u^i_0(x^i_t) = \mu^i_t$ at all t, is continuous, concave and strictly increasing, and is $\mathbf{C}^\infty$ at all but a finite number of points in $\mathbb{R}$. Since the inequalities are strict, using the argument presented in Appendix A, the function can be transformed into a $\mathbf{C}^2$, strictly concave and strictly increasing function u^i such that $u^i(x^i_t) = u^i_0(x^i_t)$ and $\partial u^i(x^i_t) = \partial u^i_0(x^i_t)$.

With these constructions, since Program (1.28) is log-concave, it suffices to show that vector $(x_t^1, \dots, x_t^I)$ satisfies its first-order conditions, namely Eq. (1.29), for it to be its solution. But this is immediate, by construction, for

$$\frac{\alpha_i \partial u^i(x_t^i)}{u^i(x_t^i) - v_t^i} = \frac{\alpha_i \partial u_0^i(x_t^i)}{u_0^i(x_t^i) - v_t^i} = \lambda_t$$

for all i. *Q.E.D.*

Also, the comparative statics of Eq. (1.12) generalize to the current setting. For instance, keeping for simplicity the assumption that there are only two players but letting their bargaining powers be α_1 and α_2, we get that

$$dx^1 = \frac{\left\{\alpha_2 \partial^2 u^2(x^2)[u^1(x^1) - v^1] - \alpha_1 \partial u^1(x^1) \partial u^2(x^2)\right\} dX - \alpha_2 \partial u^2(x^2) dv^1 + \alpha_1 \partial u^1(x^1) dv^2}{\alpha_1 \partial^2 u^1(x^1)[u^2(x^2) - v^2] - (\alpha_1 + \alpha_2) \partial u^1(x^1) \partial u^2(x^2) + \alpha_2 \partial^2 u^2(x^2)[u^1(x^1) - v^1]},$$

which suffices to imply that no data set can be α-rationalized if it contains a pair of distinct observations, t and t', such that

$$X_t > X_{t'}, v_t^1 > v_{t'}^1, v_t^2 < v_{t'}^2 \text{ and } x_t^1 < x_{t'}^1.$$

(In the case of I players, if there are t and t', such that

$$X_t > X_{t'}, v_t^1 > v_{t'}^1, v_t^i < v_{t'}^i \text{ for all } i \neq 1 \text{ and } x_t^1 < x_{t'}^1.)$$

In consequence, we have proven the following result.

Proposition 12. *Let $\alpha \gg 0$. There exists a non-tautological condition that a data set of form* (1.27) *satisfies if, and only if, it is α-rationalizable. Moreover, this condition is a finite set of polynomial inequalities on*

$$\left\{(x_t^i, v_t^i) : i = 1, \dots, I \textit{ and } t = 1, \dots, T\right\}.$$

1.8. Conclusions

We study the empirical implications of the Nash bargaining solution, assuming that the outside analyst observes a finite set of bargaining outcomes, under different hypotheses about the behaviour of the disagreement levels. We first consider a model where two agents bargain over the division of an aggregate endowment, and follow a revealed preference approach to verify the empirical validity of the Nash bargaining solution. In addition, we extend our study for the case of n agents, whose bargaining power may be asymmetric (but constant).

We derive a revealed preference characterization of the Nash bargaining solution for the cases where default levels are observed, and when the analyst has some information on the behaviour of the disagreement levels, for instance, invariance of unobserved default utility levels. Furthermore, we recover refutability under the hypothesis that the disagreement point, while not being observed or invariant, is weakly co-montone with the aggregate resource, in the case of one agent and weakly anti-comonotone with the aggregate resource for the other.

We use the Tarski-Sidenberg algorithm to construct the type of test an observer can develop to claim the hypothesis that the data can be explained by using the Nash bargaining solution. We also construct a statistical version of the rationalizability test, to give specific content as to when we can attribute the failure of the hypothesis of rationalizability to measurement error.

Chapter 2

Essay two

On Inter- and Intra-Party Politics.

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2.1. Introduction

The extent to which political science and economics have modeled electoral competition, is ample, and in many cases diametrically opposed. Downs (1957) used Hotelling's (1929) model of spatial competition to explain a competition between two candidates whose motivation for practising politics is to enjoy benefits and power of governing. A key assumption is that, voters care about policies and therefore they would vote for the candidate (or party) who was closest to his or her political standpoint, yet candidates are solely motivated by the desire to win office as an opportunity to enjoy a successful career. This, in turn, drove parties to adopt the political standpoint of the median voter. The history of democracy has showed us that political parties are both formed and influenced by interest and advocacy groups which have cared about policies. So, to propose a model in which candidates do not care about policies was viewed to be historically inaccurate.

Contrary to Downs, Wittman (1973) proposed a model of electoral competition in which parties have policy goals, which does not mean that politicians are ideologically dogmatic or lethargic about winning elections, or even value the position of their platform as an end in itself, but, like voters, candidates are solely interested in policy implementation. Although Wittman managed to overcome the ahistorical feature of the Downs model, that is, he modeled political competition between parties that have policy preferences, he could not escape the conclusion that political equilibria consists of both parties playing the same policy.[1]

[1] This common result hods for the case when both parties know for certain the distribution of

Together with the assumptions about candidates' motivation and parties not having complete information about voters' preferences Calvert (1985) concluded that, in equilibrium, as long as candidates share the same beliefs about the electorate both parties propose the same policies. Using the same changes in the traditional assumptions as Wittman (1977), Calvert gradually departed from the former results.[2] Nonetheless, both strong policy orientation and uncertainty will lead to candidate divergence, as Wittman had claimed. But, if policy orientation and uncertainty are limited then only small perturbations of candidate convergence emerge.[3] Put differently, the median policy result is quite robust to changes in its key assumptions: candidate motivation or uncertainty.

A major drawback of models in electoral competition with a unidimensional policy space, is that they predict a unique equilibrium in which party platforms are identical, which is seldom observed. A possible explanation for these findings is that they fail to recognize voters' uncertainty about the candidates. In other words, they neglect the importance of voters' expectations as a determinant of votes. Among those that offered an alternative approach are Berndhardt and Ingerman (1985)

voters types in a unidimensional policy space. However, if candidates are uncertain about vote behaviour, in other words, if they know the probability distribution over the possible distribution of voter types in a one dimensional policy space, Downs continued to predict that, in equilibrium, both candidates propose identical policies, whereas Wittman predicted differentiated equilibria.

[2] If candidates are not motivated by the desire to win office, but, by the quality of the policies that follow from an election, in a k-issue space, both candidates propose equilibrium policies that will diverge, Wittman (1983).

[3] If both these non-standard assumptions are made together, the convergence result no longer holds, see Calvert (1985).

who introduced the idea that voters' choices are influenced by their expectations about the candidates. One can argue that by allowing voters to treat candidates' positions as lotteries over uncertain outcomes, classical models had, indeed, introduced voter's uncertainty about the candidates. Although the motivation of these models had been to study if a candidate preferred, or not, a more or less noisy lottery, they also suggest that voters' subjective probabilistic valuation of post-election outcomes, given the election of candidate a as candidate a had changed his announced position, could explain why (i) incumbents were difficult to defeat and (ii) the lack of political platform convergence to the median platform. They construct a spatial model to explain political competition between two candidates where voters' expectations are treated explicitly. They found that a candidate can win election depending on (i) how well the incumbents past announcements reflect the electorates' current position and (ii) how relatively risky voters perceive the challenger to be. Their predictions, appeared to be consistent with the persistent findings in empirical studies. More importantly, they move away from the median policy result and find that, in equilibrium, candidates choose different platforms.

Contrary to the existence of equilibria in the one-dimensional policy space, an important result of models with multidimensional issue spaces and party certainty about voter behaviour is that equilibria, both in the Downs and Wittman[4] models, typically fail to exist. Without failing to recognize the importance of this result, parties seldom compete in an environment of complete certainty, hence, an alternative

[4] Only trivial interior Wittman equilibria exist generically in the multidimensional case, whereas an interior Downs equilibrium exists only nongenerically.

by which researchers have gotten around the non-existence of Nash equilibria in pure strategies in games of electoral competition under uncertainty when the policy space is multidimensional[5], has been to change the game played between parties from a simultaneous one to a sequential game, in which case, the Stackelberg equilibrium generally exists in the multidimensional, two-party game. But, who should be the one to move first? The incumbent or the challenger? Moreover, to define a natural order or an imperative argument by which one can label one of the two parties a natural leader[6] has been open for debate.

Another important contribution in the multidimensional set-up has been when answering the question as to why parties adopt progressive income tax rules, for which it is necessary to augment the issue space to more than one dimension so that, at least, progressive and regressive tax income policies be represented. Based on the evidence of the twentieth-century European political history of inner-party struggle over the policy space, Roemer (1999), introduced a new equilibrium concept called, *party unanimity Nash equilibrium* (*PUNE*) to overcome the non-existence of Nash equilibria. He assumed that there exist three types of politicians in each party, and each party must reach inner-party unanimity before the platform is announced, hence, before it is ready to be revealed in the electoral contest. He used the criterion of Pareto efficiency from the point of view of each of the types within the party, given the other party's policy, to solve the problem of existence in a multidimensional setting. Furthermore, despite introducing factional conflict within

[5] See Coughlin (1992), Lindbeck and Weibull (1987)

[6] See, Roemer (2001).

parties,[7] Roemer's framework does not explicitly model the strategic interactions taking place within each party.

Along the same vein, we adopt Roemer's institutional assumption that in reality, parties seldom act as unitary actors; on the contrary, members have different and somewhat conflicting preferences and each party consists of three types of politicians. This paper presents a new equilibrium concept for games of political competition where candidates, who belong to intra-party factions, must unanimously agree on which platform to announce in the electoral contest. More importantly, we investigate electoral competition within each party and their choice of political platforms by using a specific bargaining protocol to model party members strategic interactions and incentives. The heart of our equilibrium concept consists of invoking Baron and Ferejohn (1989) bargaining protocol (hereafter, B.F), to explicitly model what happens within each party's internal-decision process. Contrary to Roemer's formulation, where each faction member, [8] in an attempt to secure its best outcome, evaluates whether to deviate, or not, from a tax policy to another, when the other faction is playing a policy t, our solution concept entails an additional

[7] In order to study why, typically, both the left and right parties propose progressive income tax schemes in political competition. For related literature on political competition in non-unitary parties, internal party organization and the effect of the interaction between parties and candidates on platforms, see, Caulliad and Tirole (2002), Testa (2003), Castanheira *et al.* (2010) and Perisco *et al.* (2007).

[8] In Roemer (2001), one of the three factions called the reformists, play no active role in his solution concept, therefore the political equilibrium arises from a inner-party struggle between the remaining two factions

source of uncertainty. Despite the similarities between Roemer's setting and ours, we model each members' decision as if he were uncertain about, (i) the faction that will eventually dominate the decision made by the other party and (ii) the faction that will dominate in the party's nomination. While Roemer models each party members' decision as if he were certain of the type of politician that represents the opposing party.

From a theoretical standpoint, what if Pareto efficiency may not be sustainable. Now, even though Roemer's solution concept was meant to overcome the problem of existence of Nash equilibria in a multidimensional setup, we will assume a unidimensional policy space since our motivation is not to offer an alternative for solving the problem of existence in a multidimensional framework, but, mainly to model the strategic incentives of faction members in a party and their choice of a political platform. Evidence shows that due to different and somewhat conflicting preferences members face an inter-party struggle over the policy space and so by modelling what goes on behind close doors within each we find that Roemer's equilibrium concept may result inadequate for political competition in certain settings, for example, those where primary debates are held. Put differently, we show that Roemer's solution concept may not serve as an appropriate analytical description of any political competition context.

This chapter proceeds as follows. In section 2.2, we present the fundamental setting in which our results will be obtained, introduce some necessary assumptions and borrow Roemer's definitions for each faction. In Section 2.3, we present Roemer's solution concept, introduce an alternative specification of intra-party bargaining that is consistent with Roemer's results and present some additional new assumptions and definitions we need for our solution concept. In section 2.4, we present a complete characterization of our equilibrium concept, *intra-party voting Nash equilibrium*. In Section 2.5, which is the main section of this chapter, we report the basic Nash equilibria, *party unanimity Nash equilibria*, Nash bargaining solution and our *intra-party voting Nash equilibria* for the game of electoral competition and present part of our main results. In section 2.6, we claim the reason intra-party unanimity Nash equilibria fails to exercise the efficiency property that Roemer's solution concept displays, lies on the fact that each party's factions are choosing uncertain about the policy that the other party will propose. In section 2.7, we conclude.

2.2. Political Environment and Basic Assumptions

In this section we establish the setting in which all our results will be obtained, and introduce some assumptions that will be maintained throughout the paper. These assumptions are rather restrictive, but they suffice for us to make the main observations of the paper in the simplest possible setting. Of course, the assumptions are not needed for the definitions we use in the paper – neither for the ones we borrow from the literature, nor for the one that we introduce here.

2.2.1. Fundamentals of the political contest

Suppose that two parties, denoted by L and R, are competing to gain the support of some constituency in a given election. Let the policy space be the set A, and suppose that each party, $i \in \{L, R\}$, has an ideology function $v_i : A \to \mathbb{R}$, which represents how much the party values different policies, according to its political principles.

The population of voters is represented by the distribution of political preferences. This distribution is summarized by the function $\pi : A \times A \to [0, 1]$, which measures the popular support for party L: if (a_L, a_R) is the pair of policies chosen by the two parties, then $\pi(a_L, a_R)$ measures the proportion of the constituency that supports party L. We will assume that there is no abstention in the constituency, so that under the same pair of policies the support for party R is $1 - \pi(a_L, a_R)$.

2.2.2. Basic assumptions

We assume that the policy space, A, is a subset of the one-dimensional Euclidean space $\mathbb{R}$, endowed with its natural pre-order and its natural topology, and that both ideology functions, v_L and v_R, are continuous. Justifying our choice of jargon, so that we can refer to party L as *the left* and to party R as *the right*, we will assume that there exist policies $a_*, a^* \in A$ such that (i) $a_* \leqslant a^*$; (ii) function v_L is decreasing on $[a_*, \infty) \cap A$; and (iii) function v_R is increasing on $(\infty, a^*] \cap A$.[9]

[9] If both numbers exist, this assumption implies that $\text{argmax}_a v_R(a) \geqslant \text{argmax}_a v_L(a)$.

Also, we assume that function π is continuous at all (a_L, a_R) with $a_L \neq a_R$, and impose that if $a_R > a_L$, then π is non-decreasing in both arguments; while if $a_L > a_R$ then π is non-increasing in both arguments. Finally, for all our results we will assume that the voters have no party affiliation, in the sense that

$$\pi(a', a'') = 1 - \pi(a'', a'),$$

for any pair of policies $a', a'' \in A$.[10]

2.2.3. Types of Politicians

As in Roemer (1999), in each party there are three types of politicians: those who only care about the ideological value of the policy chosen by their party, those who only care about the popular support that their party obtains in the election, and those who care about the ideological value of the policy that is actually implemented as a result of the election.

Militant

The first type of politician, which will be referred to as *militant*, only cares about the policy chosen by its party, measured according to the party's ideology function: if the pair of policies chosen is (a_L, a_R), then the values given by the militant politicians of each party are $v_L(a_L)$ and $v_R(a_R)$.[11]

[10] This assumption implies that $\pi(a, a) = 1/2$ for any policy a.

[11] Refer to Appendix B.1 for proof of the existence of a Nash equilibrium between militants.

Opportunist

The second type of politician, which shall be called *opportunist*, values a pair of policies (a_L, a_R) by the support that this pair entails to his own party: in the left, the opportunists care about $\pi(a_L, a_R)$, while in the right they care about $1 - \pi(a_L, a_R)$.[12]

Let $F : A \to [0,1]$ be an increasing and continuous function such that

$$\lim_{a \to \inf A} F(a) = 0$$

and

$$\lim_{a \to \sup A} F(a) = 1,$$

and suppose that

$$\pi(a', a'') = \begin{cases} F(\frac{a'+a''}{2}), & \text{if } a'' > a'; \\ \frac{1}{2}, & \text{if } a' = a''; \\ 1 - F(\frac{a'+a''}{2}), & \text{if } a'' < a'. \end{cases}$$

Pragmatist

Finally, a politician shall be called a *pragmatist* if he cares about the policy that will be implemented as a result of the election, valued according to his party's ideology.[13]

[12] See Appendix B.1 for the existence of a Nash equilibrium between opportunists.

[13] In the original language of Roemer (1999), this type of politician is called a *reformist*. We are not using this term, in order to preempt confusion between the "right" party and the "reformists" of either party, in mnemonics that will be used later on in the paper. An alternative denomination for this type of politician could be *non-extremists*, in the sense that they do not care simply about the extremes of either pure ideology, or pure popular support, as the other two types of politicians do.

Given the distribution of votes resulting from a pair (a_L, a_R) of policies, in the left the pragmatist politicians care about

$$\mu_L(a_L, a_R) := \pi(a_L, a_R)\nu_L(a_L) + [1 - \pi(a_L, a_R)]\nu_L(a_R), \tag{2.1}$$

while in the right they care about

$$\mu_R(a_L, a_R) := \pi(a_L, a_R)\nu_R(a_L) + [1 - \pi(a_L, a_R)]\nu_R(a_R). \tag{2.2}$$

It will later be useful to observe that each function μ_i is continuous in a_i at all (a_L, a_R), in spite of the possible discontinuity of function π when $a_L = a_R$.

2.3. Party-Unanimity Nash Equilibrium

Roemer (1999) introduced a concept of equilibrium that presumes the fact that the three types of politician are present in the membership of each party, and that a consensus must be obtained from these three factions of the party before the party's policy can be adopted. In Roemer's formulation, the intra-party decision process is unmodeled. Meanwhile, he adopts the view that whatever internal process is adopted by each party in order to choose its platform will lead to an efficient choice, using the criterion of Pareto efficiency for the membership of the party.

2.3.1. The concept of equilibrium

A *party-unanimity Nash equilibrium* is a pair of policies (a_L, a_R) such that:

(i) $\nexists a \in A$ for which,

$$\nu_L(a) \geqslant \nu_L(a_L), \pi(a, a_R) \geqslant \pi(a_L, a_R) \text{ and } \mu_L(a, a_R) \geqslant \mu_L(a_L, a_R),$$

with at least one of the three inequalities being strict;

(ii) $\nexists a \in A$ such that,

$$\nu_R(a) \geqslant \nu_R(a_R), \pi(a_L, a) \leqslant \pi(a_L, a_R) \text{ and } \mu_R(a_L, a) \geqslant \mu_R(a_L, a_R),$$

with, again, at least one of these inequalities being strict.

It is worthy of mentioning that at a party-unanimity Nash equilibrium, each party takes as given the policy platform chosen by its opposition, and chooses for itself a platform that is Pareto efficient platform from the point of view of its membership. In this sense, all three factions of the left will consider deviating from policy a_L to a, when the right is playing a_R, if all three factions from the left are not worse off by playing a and at least one of them is strictly better off. Conversely, the same may be said for the case of the right. In the remainder of the paper, we shall also use the term *Roemer equilibrium* to refer to a Party Unanimity Nash Equilibrium.

In order to simplify the application of Roemer's concept, the following claim is true: There does not exist a Roemer equilibrium (a_L, a_R) such that $\nu_L(a_R) > \nu_L(a_L)$, or that $\nu_R(a_L) > \nu_R(a_R)$. As shown in Appendix B.2, we prove our claim.

An important property of Roemer's equilibrium solution concept, which he makes explicit in Roemer (2001), is that the pragmatists of each party are inconsequential for the equilibrium choice of policies. To see that this is indeed the case, refer to Appendix B.3.

2.3.2. *The Nash bargaining solution and Roemer equilibrium*

Presumably, the outcome of an intra-party decision process is not independent of the importance that each of the factions has in the party. Along this line, the fact that pragmatist politicians play no role in the choice of their party's policy under a Roemer equilibrium may thus appear as a shortcoming of the concept. Like we mentioned earlier, by offering a natural intra-party decision process under which *PUNE* results inadequate as a solution concept for political competition, we provide an alternative equilibrium concept for political games when Pareto efficiency is not sustainable. What do we mean by offering a *natural* decision process? Although in most presidential systems, the president is elected by popular vote, it isn't a surprise that party factions engage in negotiations, despite the common interest in agreement, they still have conflicting preferences among each other. For instance, United States use caucuses and nationwide state level primaries to elect a candidate for office that will narrow the field of candidates before the general election. But, what if Pareto efficiency isn't not sustainable when party members decide over what party platform to present. Furthermore, what if, instead, faction members negotiate over the terms of the party platform to present, say, in the primaries. And as a result, factions harm one another within the party as evidence suggests happens during electoral races.

But before addressing our theoretical curiosity, we first introduce an alternative specification of intra-party politics that is consistent with Roemer's idea.

For each party, let $\alpha_i, \beta_i \geqslant 0$ be two numbers such that $\alpha_i + \beta_i \leqslant 1$. Roemer

(2001) defines a *Nash bargaining solution with weights* $((\alpha_L, \beta_L), (\alpha_R, \beta_R))$ to be a pair (a_L, a_R) such that policy a_L solves program

$$\max_a \{\pi(a, a_R)^{\alpha_L} [v_L(a) - v_L(a_R)]^{\beta_L}\}, \tag{2.3}$$

while policy a_R solves program

$$\max_a \{[1 - \pi(a_L, a)]^{\alpha_R} [v_R(a) - v_R(a_L)]^{\beta_R}\}. \tag{2.4}$$

A pair (a_L, a_R) is said to be a *Nash bargaining solution*, if there exist numbers $((\alpha_L, \beta_L), (\alpha_R, \beta_R))$ such that (a_L, a_R) is a Nash bargaining solution with the weights $((\alpha_L, \beta_L), (\alpha_R, \beta_R))$.

This concept has a natural interpretation. Use α_i to denote the proportion of members of party i who are militant, and use β_i for the proportion that are opportunistic; the remaining proportion, $1 - \alpha_i - \beta_i$, are, implicitly, pragmatist politicians. The definition corresponds to the assumption that, given the policy chosen by its opposition, each party solves its decision process according to a weighted Nash bargaining problem between the militant and opportunist factions, under the assumption that both factions believe that if they break their discussion without some agreed policy, then the other party will win the election for sure and will impose its policy: for the opportunists, the "default" payoff implicit in these Nash problems is 0, as if the other party won the election for sure in the absence of an agreed policy from their own party; and for the militants the default payoff is $v_i(a_{\neg i})$, as if the break-up of the party's negotiations were tantamount to the adoption of the other party's policy.

It is straightforward that if a pair of policies (a_L, a_R) is a Nash bargaining solution with implicit weights $((\alpha_L, \beta_L), (\alpha_R, \beta_R))$ that are all strictly positive, then it is also a Roemer equilibrium: otherwise, it would be possible to increase the payoff of one of the factions in one of the parties without decreasing the payoff of the other faction of the same party; as long as the first faction's weight in that party is strictly positive, this would imply that one can increase to maximand of that party's problem in one of the two expressions, (2.3) or (2.4), thus contradicting the premise that the pair of policies was a Nash bargaining solution. Under some technical requirements on the ideology functions and the support function (π), Roemer (2001) further argues that any party-unanimity Nash equilibrium where no party wins the election for sure is also a Nash bargaining solution.

2.3.3. The role of pragmatist politicians

We can use the characterization of Roemer equilibria via Nash bargaining solutions to revisit the issue of the irrelevance of pragmatist politicians in Roemer's analysis. In principle, the definition of Nash bargaining solution has dismissed the existence of the pragmatist faction of each party, and is such that the outcome of the political competition only depends on the internal composition of the parties' memberships up to the ratio of the fractions of militants to opportunists. One attempt to bring the pragmatist politicians back into the fray would be to include their values and their weight into the Nash bargaining problem. That is, let us consider the problem of the left, modified to

$$\max_{a}\{\pi(a,a_R)^{\alpha_L}[\nu_L(a)-\nu_L(a_R)]^{\beta_L}[\mu_L(a,a_R)-\mu_L]^{1-\alpha_L-\beta_L}\},$$

for some default value μ_L, which may depend on a_R and represents the value that a pragmatist politician from the left would attach to the failing of his own party's internal decision process. If one wants to be consistent with the idea that all politicians render a failure in the internal negotiations of their parties as a capitulation in the political competition, then the value that should be given to the default μ_L should be precisely $\nu_L(a_R)$, for in such case the practical politicians in the left party believe that the right is going to implement policy a_R for sure. Now, in this case, using, namely Equation (2.1),

$$\pi(a,a_R)^{\alpha_L}[\nu_L(a)-\nu_L(a_R)]^{\beta_L}[\mu_L(a,a_R)-\mu]^{1-\alpha_L-\beta_L}=\pi(a,a_R)^{1-\beta_L}[\nu_L(a)-\nu_L(a_R)]^{1-\alpha_L},$$

so it follows, once again, that the practical politicians play no significant role: the outcome of the left party's decision process is going to depend on the distribution of types only up to the ratio of $1-\alpha_L$ to $1-\beta_L$.

2.4. Intra-Party Voting Nash Equilibrium

The relationship between party-unanimity Nash equilibria and Nash bargaining solutions is seen by Roemer as a micro-foundation for the premise that each party's platform should be efficient from the point of view of its membership, given the other party's policy.[14] In contexts in which such premise is acceptable, one can

[14] Indeed, in Roemer's applications of his equilibrium concept, he uses the observed policy proposals to back out, under the assumption that the pair of policies is a party-unanimity Nash equilibrium,

take Roemer's equilibrium concept as an appropriate analytical description of the political context, and may use the Nash bargaining solution as its microfoundation of what happens within each of the competing parties. We do not, however, believe that all political situations are amenable to the premise of inter-party Pareto efficiency. Consider, for instance, the case of presidential systems in which pre-candidates from each party first compete for their party's nomination in a primary vote. The political debate before the primaries is often heated, and the criticism that the eventual winner receives from fellow pre-candidates is later on used by the other party's nominee in the presidential race. This cannot be Pareto efficient: each party would prefer, given the eventual choice of a nominee, to have had a less damaging debate in its primary.

Also, in the Roemer-Nash framework, each party is modeled as choosing a policy platform *given the platform of the other party*. In the presence of different factions in each party, we find this objectionable, for it is as if each party knew which of the factions is eventually going to dominate the decision made by the other party. In the language of the presidential primary elections, it is as if each of the pre-candidates of a party knew who is going to be the eventual nominee of the other party – indeed, an untenable assumption.

We now propose a concept of equilibrium that tries to overcome these two problems. We believe that this concept is appropriate for contests with significant pri-

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the normalized weights of the militant and opportunist factions of each party. At a theoretical level, Roemer argues that the set of party-unanimity Nash equilibria constitutes a manifold, and that the profiles of weights $((\alpha_L, \beta_L), (\alpha_R, \beta_R))$ are a parameterization of it.

mary debates, under the premise that the discussion behind the intra-party decisions is binding for the posterior inter-party competition. We maintain the assumption that the proportion of militant politicians in party i is α_i, the proportion of opportunists is β_i, and the remaining members, $1-\alpha_i-\beta_i$, are pragmatist. For simplicity, we will denote the type of politician by $t \in \{m, o, p\}$, with obvious mnemonics.

In each party, each type of politician will have a policy proposal, which we denote by $a_{i,t} \in A$. We distinguish a policy that party i may adopt, from the triple of policies adopted by its factions, by using $a_i \in A$ for the former while $\vec{a}_i = (a_{i,m}, a_{i,o}, a_{i,p}) \in A^3$ denotes the latter. We will also denote the pair of policy triples by $\vec{a} = (\vec{a}_L, \vec{a}_R) \in A^3 \times A^3$.

2.4.1. Objective functions under uncertainty

Consider a politician from party L. From his point of view, the type of politician of party R against whom he will compete for votes, should he win the nomination from his party, is unknown: he believes that he will face a militant with probability α_R, an opportunist with probability β_R and a pragmatist with probability $1-\alpha_R-\beta_R$. Now, suppose that party L is considering the adoption of a policy a; if the policy of party R is a_R, then the expected value of this politician is $v_L(a)$, $\pi(a, a_R)$ or

$$\mu_L(a, a_R) := \pi(a, a_R)v_L(a) + [1-\pi(a, a_R)]v_L(a_R),$$

depending on whether he is of type m, o or p. But, as we said before, this politician does not know who is going to win party R's nomination, so, given a triple $\vec{a}_R$, his values are as follows, depending on his type: if he is a militant, it is $V_L(a, \vec{a}_R) :=$

$v_L(a)$, for he does not care about R's policy; if he is an opportunist, his value is

$$\Pi_L(a, \vec{a}_R) := \alpha_R \pi(a, a_{R,m}) + \beta_R \pi(a, a_{R,o}) + (1 - \alpha_R - \beta_R)\pi(a, a_{R,p});$$

and, similarly, if he is a pragmatist, his value is

$$M_L(a, \vec{a}_R) := \alpha_R \mu_L(a, a_{R,m}) + \beta_R \mu_L(a, a_{R,o}) + (1 - \alpha_R - \beta_R)\mu_L(a, a_{R,p}).$$

Of course, the definitions of objective functions for the factions of party R, which we will denote by μ_R, Π_R and M_R are similar, except for the fact that its opportunists care about $1 - \pi$:

$$\Pi_R(\vec{a}_L, a) := \alpha_L[1 - \pi(a_{L,m}, a)] + \beta_L[1 - \pi(a_{L,o}, a)] + (1 - \alpha_L - \beta_L)[1 - \pi(a_{L,p}, a)]$$

The other details are omitted.

2.4.2. *Objective functions over own party's policy triple*

In the expected value functions just defined, the only uncertainty is over the type of politician that will represent the opposing party, but the policy considered by the politician's own party is given. For reasons that will be clear momentarily, we will need to define the expected value of each type of politician, *ex-ante* to the determination of his own party's nominee. That is to say, fix a pair of policy triples $\vec{a}$, and consider again a politician from party L; if he believes that his party is going to be represented by one of its militants with probability α_L, by one of its opportunists with probability β_L, and by one of its pragmatists with probability $1 - \alpha_L - \beta_L$, then his values are as follows: if he himself is a militant, it is

$$V_L^e(\vec{a}) = \alpha_L V_L(a_{L,m}, \vec{a}_R) + \beta_L V_L(a_{L,o}, \vec{a}_R) + (1 - \alpha_L - \beta_L)V_L(a_{L,p}, \vec{a}_R);$$

if he is an opportunist, his value is

$$\Pi_L^e(\vec{a}) = \alpha_L \Pi_L(a_{L,m}, \vec{a}_R) + \beta_L \Pi_L(a_{L,o}, \vec{a}_R) + (1 - \alpha_L - \beta_L)\Pi_L(a_{L,p}, \vec{a}_R);$$

and, in the same vein, it is

$$M_L^e(\vec{a}) = \alpha_L M_L(a_{L,m}, \vec{a}_R) + \beta_L M_L(a_{L,o}, \vec{a}_R) + (1 - \alpha_L - \beta_L) M_L(a_{L,p}, \vec{a}_R)$$

if he is a pragmatist.

As before, the definitions of ex-ante objective functions for the politicians of party R are identical.

2.4.3. A protocol for the intra-party contest

The key difference between Roemer's equilibrium concept and ours is the adoption of an explicit bargaining protocol to model what happens in each party's internal decision process: we assume that the policy triple of each party is adopted at a subgame-perfect Nash equilibrium of the protocol introduced by Baron and Ferejohn (89), given the triple of the other party. We next describe this protocol.

In each party, i, the internal discussion takes place by a series of rounds. In each round, a politician is chosen at random and he makes a policy proposal; after the proposal is made, the members of the party vote to accept or reject the proposal. The internal statutes of the party specify a proportion $\gamma_i \geqslant 1/2$ of «accept» votes that are needed for a policy to be adopted by the party. If the round's proposal passes this threshold, it becomes the party's policy. Otherwise, a new round begins, again by randomly selecting, with replacement.

We assume that party i's politicians (uniformly) discount the value of an agreed policy by a discount factor of $\delta_i < 1$, for each round of negotiation that has passed before this agreement has been reached. This impatience may reflect, for instance, the loss of reputation that the party may suffer when the constituency realizes its internal disagreement.

We use Baron and Ferejohn (1989) to introduce our concept of equilibrium. For the sake of simplicity and tractability, we assume that in each party, all politicians of a given type follow the same strategy, and focus on stationary subgame-perfect equilibria of the intra-party game, so that delay in reaching an agreement does not occur at equilibrium. Also, we assume implicitly that each politician votes to accept the policy that he would propose should he be elected to do so.

2.4.4. Passing proposals

As before, for the sake of a simpler presentation, we concentrate on the case of party L, and take as given the policy triple of party R, namely $\vec{a}_R$. The analysis of party R is of course the same, *mutatis mutandis*.

Given $a_{L,m}$ and $a_{L,p}$, we shall say that L*'s militants vote to accept the proposal of the opportunists*, $a_{L,o}$, if

$$V_L(a_{L,o}, \vec{a}_R) \geqslant \delta_L V_L^e(\vec{a}_L, \vec{a}_R),$$

with $\vec{a}_R = (a_{L,m}, a_{L,o}, a_{L,p})$. This means that the militants are no worse off under the current proposal of the opportunists than if they voted to reject it, given what they expect as value for a further round of negotiation, discounted. Similarly, given $a_{L,m}$

and $a_{L,o}$, L*'s militants vote to accept the proposal of the pragmatists,* $a_{L,p}$, if

$$V_L(a_{L,p}, \vec{a}_R) \geqslant \delta_L V_L^e(\vec{a}_L, \vec{a}_R).$$

For the other types of politicians of party L the definitions are analogous, but we provide them explicitly, for the sake of completeness. Given $a_{L,o}$ and $a_{L,p}$, we shall say that L*'s opportunists vote to accept the proposal of the militants,* $a_{L,m}$, if

$$\Pi_L(a_{L,m}, \vec{a}_R) \geqslant \delta_L \Pi_L^e(\vec{a}_L, \vec{a}_R),$$

and, given $a_{L,m}$ and $a_{L,o}$, that L*'s opportunists vote to accept the proposal of the pragmatists,* $a_{L,p}$, if

$$\Pi_L(a_{L,p}, \vec{a}_R) \geqslant \delta_L \Pi_L^e(\vec{a}_L, \vec{a}_R).$$

Also, given $a_{L,o}$ and $a_{L,p}$, we shall we say that L*'s pragmatists vote to accept the proposal of the militants,* $a_{L,m}$, if

$$M_L(a_{L,m}, \vec{a}_R) \geqslant \delta_L M_L^e(\vec{a}_L, \vec{a}_R),$$

and, given $a_{L,m}$ and $a_{L,p}$, that L*'s pragmatists vote to accept the proposal of the opportunists,* $a_{L,o}$, if

$$M_L(a_{L,o}, \vec{a}_R) \geqslant \delta_L M_L^e(\vec{a}_L, \vec{a}_R).$$

Given these definitions, we can define the constraints that the politicians of each faction face when choosing a policy proposal. Again, we maintain fixed a given triple, $\vec{a}_R$, for party R. Given $a_{L,o}$ and $a_{L,p}$, we say that *the proposal of the militants,* $a_{L,m}$, *passes* if one of the following four conditions holds:

(a) $\alpha_L \geqslant \gamma_L$; or

(b) the opportunists vote to accept $a_{L,m}$ and $\alpha_L + \beta_L \geqslant \gamma_L$; or

(c) the pragmatists vote to accept $a_{L,m}$ and $1 - \beta_L \geqslant \gamma_L$; or

(d) both the opportunists and the pragmatists vote to accept $a_{L,m}$.

In words, the proposal of the militants passes if the number of party members who prefer to adopt it, in comparison with letting the negotiation proceed for one further round, is above the threshold required by the party's statutes. This can occur when the militants are sufficiently many to impose their policies without the support of any other faction of the party (case a), when they can convince the opportunists and together they pass that threshold (case b), when they can convince the pragmatists and together they pass the threshold (case c), or when there is unanimous support for the proposal (case d).

This idea extends to the other two factions of the party, straightforwardly; we give the formal definitions next. Given $a_{L,m}$ and $a_{L,p}$, we say that *the proposal of the opportunists*, $a_{L,o}$, *passes* if one of the following conditions holds:

(a) $\beta_L \geqslant \gamma_L$; or

(b) the militants vote to accept $a_{L,m}$ and $\alpha_L + \beta_L \geqslant \gamma_L$; or

(c) the pragmatists vote to accept $a_{L,m}$ and $1 - \alpha_L \geqslant \gamma_L$; or

(d) both the militants and the pragmatists vote to accept $a_{L,m}$.

Similarly, given $a_{L,m}$ and $a_{L,o}$, we say that *the proposal of the pragmatists*, $a_{L,p}$, *passes* if one of the following conditions holds:

(a) $1 - \alpha_L - \beta_L \geqslant \gamma_L$; or

(b) the militants vote to accept $a_{L,p}$ and $1 - \beta_L \geqslant \gamma_L$; or

(c) the opportunists vote to accept $a_{L,p}$ and $1-\alpha_L \geqslant \gamma_L$; or

(d) both the militants and the opportunists vote to accept $a_{L,p}$.

2.4.5. The concept of equilibrium

Under the definitions previously introduced, we can introduce our concept of equilibrium. Intuitively, we exploit the idea of Baron and Ferejohn (89), under which each party reaches an agreement without delay, at a stationary subgame-perfect equilibrium of the intra-party negotiation game, while we model the situation as a simultaneous-move inter-party game, where we are interested in straightforward Nash equilibria.

We say that a pair of policy profiles $\vec{a} = (\vec{a}_L, \vec{a}_R)$ is an **intra-party voting Nash equilibrium** (with thresholds (γ_L, γ_R)), if the following six conditions hold:

- **Militants in party L are rational**: platform $a_{L,m}$ solves the following optimization problem:

$$\max \{V_L(a, \vec{a}_R) : a \text{ passes, given } (a_{L,o}, a_{L,p}) \text{ and } \vec{a}_R\}. \tag{2.5}$$

- **Opportunists in party L are rational**: platform $a_{L,o}$ solves the following optimization problem:

$$\max \{\Pi_L(a, \vec{a}_R) : a \text{ passes, given } (a_{L,m}, a_{L,p}) \text{ and } \vec{a}_R\}. \tag{2.6}$$

- **Pragmatists in party L are rational**: platform $a_{L,p}$ solves the following optimization problem:

$$\max \{M_L(a, \vec{a}_R) : a \text{ passes, given } (a_{L,m}, a_{L,o}) \text{ and } \vec{a}_R\}. \tag{2.7}$$

- **Militants in party R are rational**: platform $a_{R,m}$ solves the following optimization problem:

$$\max\{V_R(\vec{a}_L, a) : a \text{ passes, given } (a_{R,o}, a_{R,p}) \text{ and } \vec{a}_L\}. \tag{2.8}$$

- **Opportunists in party R are rational**: platform $a_{R,o}$ solves the following optimization problem:

$$\max\{\Pi_R(\vec{a}_L, a) : a \text{ passes, given } (a_{R,m}, a_{R,p}) \text{ and } \vec{a}_L\}. \tag{2.9}$$

- **Pragmatists in party R are rational**: platform $a_{R,p}$ solves the following optimization problem:

$$\max\{M_R(\vec{a}_L, a) : a \text{ passes, given } (a_{R,m}, a_{R,o}) \text{ and } \vec{a}_L\}. \tag{2.10}$$

Intuitively, at an intra-party voting Nash equilibrium, no faction of either party would prefer to unilaterally deviate from the policy it is adopting, given the policies of the other two factions of its own party and the three factions of the opposing party. They would not find it beneficial to adopt a different policy, in the sense that the policies that would be supported by sufficiently many fellow members of the party, the chosen policy is the one that is best according to the objective of that faction. Implicit in the definition are two more elements of the strategy of all politicians which are related to how they vote if it happens to be another member of their own party who is chosen to make a proposal: first, that they support the proposal that they themselves would make; and, second, that for any other proposal, they vote in its support if, and only if, that policy gives them at least as much payoff, according to their own interest, as the expected payoff they would get from rejecting the policy and running a new round of negotiations (where they would all make the same proposals as in the previous round). These properties are important, because they rule out trivial equilibria of standard majority games, where negligible players vote in favor (or against) a proposal just because they do not have the power to change the outcome of the vote.

Importantly, in our concept of equilibrium all politicians are selfish, in the sense that they want to impose the policy that is best according to their own interests, while none tries to ensure that the resulting party profile is efficient from the point of view of their parties, at least not by design of the game. It is precisely this lack of concern for the party's «optimality»that we want to emphasize, for this is the feature of Roemer's equilibrium concept that, we believe, can often be untenable. In order to capture the ideas of Roemer, however, we first specify a class of games in which our concept is particularly amenable to comparison to his definition.

2.4.6. Intra-party unanimous voting Nash equilibrium

We shall say that an intra-party voting Nash equilibrium $\vec{a} = (\vec{a}_L, \vec{a}_R)$ **requires unanimity**, if the thresholds of the game are such that no coalition of just two factions of a party can pass a policy proposal: for both $i = L, R$, the following is true:

$$\alpha_i + \beta_i < \gamma_i,\ 1 - \beta_i < \gamma_i \text{ and } 1 - \alpha_i < \gamma_i.$$

Under this condition, a faction's proposal in either party can only pass if the other two factions of that party vote to accept it.[15]

2.5. Inefficiency of Intra-Party Voting Equilibrium Policies

Our aim is to show that Roemer's solution concept may fail to capture some aspects of the political game, for instance, when intra-party competition is significant, for instance, the case of presidential elections. The idea of Roemer is to capture the property of unanimity through the application of intra-party efficiency of the chosen platform, but without explicitly modeling the strategic interactions taking place within each party. Conversely, we model the decision process of each party, without the need for the unanimity property. However, in order to capture Roemer's premises, from now this point on we will restrict attention to intra-party voting Nash equilibria that require unanimity.

Of course, between the two concepts there is a formal difference that we need to address in order for the comparison to be meaningful: while a party unanimity Nash equilibrium is a point in $A \times A$, an intra-party voting Nash equilibrium is one in $A^3 \times A^3$. It would not make sense to say that an outcome of the game is an equilibrium under both definitions. At the same time, party-unanimity Nash equilibria are often in multiplicity, and, as mentioned before, under simple assumptions they constitute a manifold of policy pairs, parameterized by the values of (α_L, β_L) and

[15] That is, if the reason why the proposal passes is given by condition (d) in the definitions of passing proposals.

(α_R, β_R). Since in our concept of intra-party voting equilibrium we maintain the latter parameters fixed, and in order to deal with the higher dimensionality of our concept of equilibrium, we study whether the nine pairs of policies that constitute the support of an intra-party voting Nash equilibrium, for given values of (α_L, β_L) and (α_R, β_R), are necessarily party-unanimity Nash equilibria of the game with the same fundamentals.[16] By means of an example, we show that this is not the case.

For the sake of completeness in our comparisons, we compute in Appendix B.4, all the basic Nash equilibria, P.U.N.E and the Nash bargaining equilibria for the particular game we consider. Next, we compute the intra-party unanimous voting equilibria and prove the following proposition.

Proposition 13. *There exist games in which the support of intra-party unanimous voting Nash equilibria is not a subset of the set of P.U.N.E.*

2.5.1. Fundamentals

Suppose that the policy space is $A = [0, 1]$. The ideology functions of the two parties are

$$v_L(a) = 1 - a \text{ and } v_R(a) = a.$$

The linearity of these functions will make our computations as simple as possible, and for the same reason we consider the following piecewise linear function to

[16] But where, again, (α_L, β_L) and (α_R, β_R) play no role.

represent the support received by party L as a function of the two platforms chosen:

$$\pi(a_L, a_R) = \begin{cases} \max\{\min\{b(a_L + a_R) - C, 1\}, 0\}, & \text{if } a_L < a_R; \\ \frac{1}{2}, & \text{if } a_L = a_R; \\ \min\{\max\{1 - b(a_L + a_R) + C, 0\}, 1\}, & \text{if } a_L > a_R. \end{cases}$$

Here, the parameter $b > 0$ and the constant $C > 0$ are fixed, and we do not try to offer a micro-foundation for the function. It is important, however, to note that function π satisfies all the assumptions imposed on it in §2.2.2. In particular, it obeys that $\pi(a', a'') = 1 - \pi(a'', a')$, which gives us an easy formula for the computation of the support for party R. This latter assumption, however, does not impose symmetry for the inter-party game: it does not imply, for example, that $\pi(0, 1) = 1/2$.

In our computations, in Appendix B.4, we assume, furthermore, that $1/2 < b \leqslant 3/2$, $C < 1$ and $2b - 2 \leqslant C < 2b - 1$. Figure 2.1 shows the area of parameters allowed by these conditions. These assumptions simply help us to solve some ambiguities in our computations. The game is symmetric between parties when $C = b - 1/2$; later on, the particular case when $b = 1$ and $C = 1/2$, will be considered. For arbitrary values of b and C, we can illustrate the levels of the support function π as in Figure 2.2.

Since the values of (α_L, β_L), (α_R, β_R), δ_L and δ_R are not needed for all the computations, we do not yet give values for these parameters.

In the next section, ***we compute intra-party unanimous voting Nash equilibrium for the simplified version of the game, and prove the Proposition 13***.

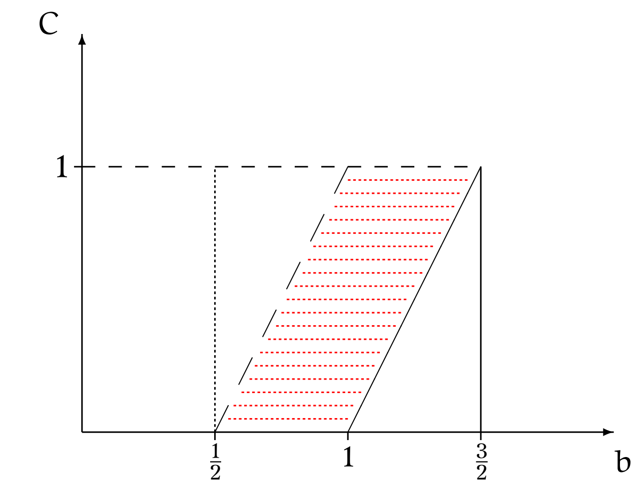

Figure 2.1: Valid values for parameters b and C (the area shadowed in red).

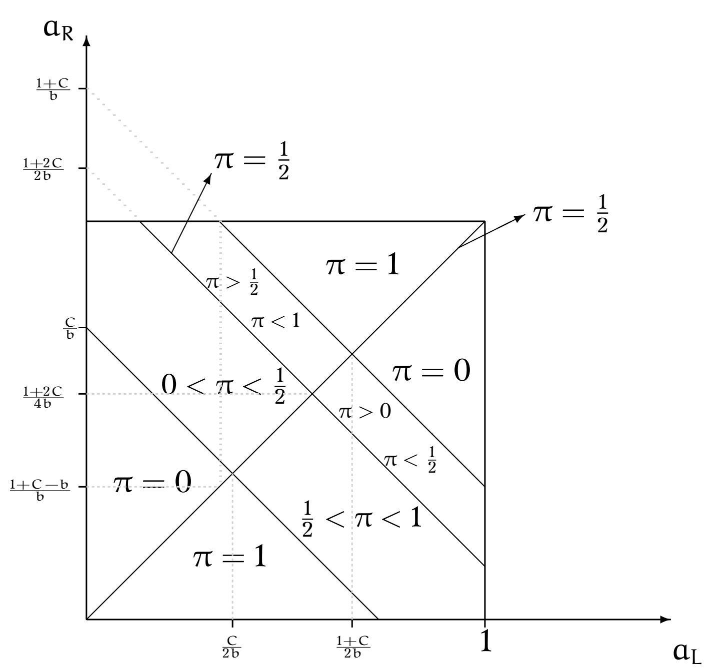

Figure 2.2: The levels of function π.

2.5.2. Intra-party voting Nash Equilibria

Since this task is mathematically much more complicated than the computation of the equilibria we have found so far, we need to appeal to numerical methods and, thus, need to impose specific values to the parameters under consideration. For similar reasons, we simplify the game further by imposing parametric values that make it symmetric between the two parties.[17] Also, in order to make the requirements of our concept of equilibrium as close as possible as Roemer's motivation for the concept of party-unanimity Nash equilibrium, we will concentrate on intra-party voting Nash equilibrium that require unanimity for both parties.

Specifically, the game that we are going to consider will have $b = 1$, $C = 1/2$, $\delta_L = \delta_R$, $0 < \alpha_L = \alpha_R$, $0 < \beta_L = \beta_R$, $1 - \alpha_L - \beta_L < 1$ and $\gamma_L = \gamma_R = 1$. The first two conditions suffice to make the sets of party-unanimity Nash equilibria and of Nash bargaining solutions symmetric. Using our results from Appendix E, we present the set of Roemer equilbria in Figure 2.3. The other assumptions, which are of no relevance for Roemer equilibria, will allow us to search for symmetric intra-party voting Nash equilibria, where $a_{L,t} = 1 - a_{R,t}$ for each of the three types, $t \in \{m, o, p\}$.

Consistency of Roemer equilibria and intra-party voting Nash equilibria

As mentioned earlier, a Roemer equilibrium is a pair of policies in $(a_L, a_R) \in A \times A$, while an intra-party voting Nash equilibrium is a pair of policy profiles $(\vec{a}_L, \vec{a}_R) \in A^3 \times A^3$; therefore, we need to specify a sense in which these two objects can be compared. At the same time, the fact that we have a whole set of Roemer equilibria, which does not depend on the parameters of the intra-party voting problem (namely δ_i, α_i, β_i and γ_i), allows us to establish the following criterion: given a pair

[17] By this, we mean that the functions and optimization problems of the Right party are the rotation of those of the Left party, in the sense that $\nu_R(a) = \nu_L(1 - a)$ and $\pi(a, a') = 1 - \pi(a', a)$.

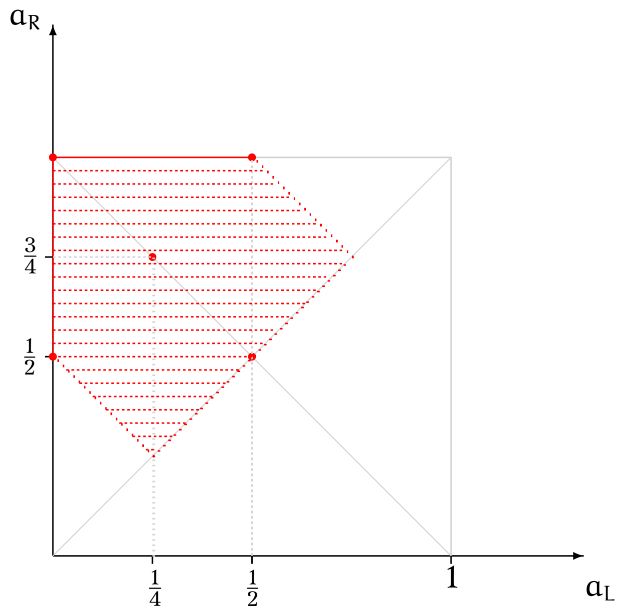

Figure 2.3: The set of Roemer equilibria of the simple game, with $b = 1$ and $C = 1/2$ is the shaded area. It only includes its boundaries where these are drawn in a continuous line, and it contains, in particular, the five dots (of these dots, $(0, 1)$ is the Nash equilibrium between militants, $(\frac{1}{4}, \frac{3}{4})$ is the Nash equilibrium between pragmatists, and $(\frac{1}{2}, \frac{1}{2})$ is the Nash equilibrium between opportunists).

$(\vec{a}_L, \vec{a}_R) \in A^3 \times A^3$, define the **support** of this pair as the set

$$\{a_{L,m}, a_{L,o}, a_{L,p}\} \times \{a_{R,m}, a_{R,o}, a_{R,p}\};$$

then, we would say that **intra-party voting Nash equilibrium is consistent with Roemer equilibrium** if for any value of the parameters δ_i, α_i, β_i and γ_i, for $i = L, R$, a pair of policy profiles $(\vec{a}_L, \vec{a}_R)$ is a intra-party voting Nash equilibrium and pair (a_L, a_R) lies in the support of this equilibrium, then (a_L, a_R) is a Roemer equilibrium.

Intuitively, consistency would require that at every intra-party voting Nash equilibrium, in each party each faction proposes a policy that is Pareto efficient from the point of view of its own party, given each of the three policies that are chosen by the factions of the other party. This definition is demanding, since in our definition of equilibrium each faction chooses a policy under uncertainty about the policy that the opposing party will play, for it does not know which one of that party's faction will win its intra-party game – this is an *ex-ante* choice, while, under this uncertainty, the concept of consistency would require *ex-post* efficiency of all the choices. Our main point in this paper is precisely that: we are going to argue that intra-party voting Nash equilibrium is not consistent with Roemer equilibrium. This means that in a situation in which the primary intra-party contests are important, the concept of Roemer equilibrium is misleading, for it requires *ex-post* efficiency of a series of choices that in reality are made *ex-ante*. We will later prove that, indeed, this divergence of criteria is the reason for the inconsistency we claim.

Computation of intra-party voting Nash equilibrium

In order to compute intra-party voting Nash equilibria, we apply the following numerical algorithm, given particular values of the parameters $\delta_L = \delta_R$, $0 < \alpha_L = \alpha_R$, $0 < \beta_L = \beta_R$, $1 - \alpha_L - \beta_L < 1$ and $\gamma_L = \gamma_R = 1$. In the algorithm $G[0,1]$ represents a discretization of the interval $[0,1]$ containing 1,000 points.

1. Fix a strategy $\vec{a}_R = (a_{R,m}, a_{R,o}, a_{R,p})$ for party R.

2. Define strategy $\vec{a}_L = (1 - a_{R,m}, 1 - a_{R,o}, 1 - a_{R,p})$ for party R.

3. For every $a \in G[0,1]$, compute the values of functions

$$\nu_L(a), \pi(a, a_{R,m}), \pi(a, a_{R,o}), \pi(a, a_{R,p}), \mu_L(a, a_{R,m}), \mu_L(a, a_{R,o}) \text{ and } \mu_L(a, a_{R,p}).$$

4. For every $a \in G[0,1]$, compute the expected value functions

$$V_L(a), \Pi_L(a, \vec{a}_R) \text{ and } M_L(a, \vec{a}_R).$$

5. For every $a \in G[0,1]$, compute the discounted continuation values of the other types:

   (a) for militants

$$\delta_L \Pi_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R] \text{ and } \delta_L M_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R].$$

   (b) for opportunists

$$\delta_L V_L^e[(a_{L,m}, a, a_{L,p})] \text{ and } \delta_L M_L^e[(a_{L,m}, a, a_{L,p}), \vec{a}_R].$$

   (c) for pragmatists

$$\delta_L V_L^e[(a_{L,m}, a_{L,o}, a)] \text{ and } \delta_L \Pi_L^e[(a_{L,m}, a_{L,o}, a), \vec{a}_R].$$

6. For every $a \in G[0,1]$, determine if it passes:

   (a) for militants, a passes if

$$\Pi_L(a, \vec{a}_R) \geqslant \delta_L \Pi_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R] \text{ and } M_L(a, \vec{a}_R) \geqslant \delta_L M_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R].$$

   (b) for opportunists, a passes if

$$V_L(a) \geqslant \delta_L V_L^e[(a_{L,m}, a, a_{L,p})] \text{ and } M_L(a, \vec{a}_R) \geqslant \delta_L M_L^e[(a_{L,m}, a, a_{L,p}), \vec{a}_R].$$

   (c) for pragmatists, a passes if

$$V_L(a) \geqslant \delta_L V_L^e[(a_{L,m}, a_{L,o}, a)] \text{ and } \Pi_L(a, \vec{a}_R) \geqslant \delta_L \Pi_L^e[(a_{L,m}, a_{L,o}, a), \vec{a}_R].$$

7. Determine optimal policies for each faction:

   (a) for militants, find $\hat{a}_m$ that makes $v(a)$ maximum amongst all those that pass.

   (b) for opportunists, find $\hat{a}_o$ that makes $\Pi_L(a, \vec{a}_R)$ maximum amongst all those that pass.

   (c) for pragmatists, find $\hat{a}_p$ that makes $M_L(a, \vec{a}_R)$ maximum amongst all those that pass.

8. If

$$\hat{a}_m = a_{L,m}, \hat{a}_o = a_{L,o} \text{ and } \hat{a}_p = a_{L,p},$$

   stop. Else, perturb strategy $\vec{a}_R = (a_{R,m}, a_{R,o}, a_{R,p})$ and iterate to 2.

From the output of the algorithm, we get a symmetric intra-party voting Nash equilibrium $(\vec{a}_L, \vec{a}_R)$.

Inefficiency of intra-party voting Nash equilibrium

In addition to the values of $b = 1$ and $C = 1/2$, consider the game with the following values for the exogenous parameters: $\delta_L = \delta_R = 0.55$, $\alpha_L = \alpha_R = 0.01$ and $\beta_L = \beta_R = 0.98$. After running the first seven steps of the algorithm above, for $\vec{a}_R = (a_{R,m}, a_{R,o}, a_{R,p}) = (0.974, 0.412, 0.75)$, we construct the three figures in Appendix C [18].

[18] Which correspond to the Graphs of Pragmatists, Militants and Opportunists, respcetively

For militants: Consider the second figure in Appendix C. The straight, downward sloping line is the objective function of the militants of the Left, $V_L(a)$ (which is measured on the axis to the right). The other two functions are the difference between the immediate expected payoff and the discounted continuation value for the other two factions of the party: the smoother curve measures

$$M_L(a, \vec{a}_R) - \delta_L M_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R],$$

while the more irregular, piecewise linear function represents

$$\Pi_L(a, \vec{a}_R) - \delta_L \Pi_L^e[(a, a_{L,o}, a_{L,p}), \vec{a}_R].$$

The former curve is above 0 for values of a between 0 and 0.727, but the latter is below 0 for values of a below 0.026 and above 0.75, which means that, the Left's pragmatists would vote to support policies $0 \leqslant a \leqslant 0.727$ proposed by the party's militants, while its opportunists would only support policies $0.026 \leqslant a \leqslant 0.75$. Of these «feasible» values, the one that maximizes V_L is $\hat{a}_m = 0.026$.

For Opportunists: Consider now third figure in Appendix C. The straight line measures

$$V_L(a) - \delta_L V_L^e[(a_{L,m}, a, a_{L,p})],$$

while the smoother curve measures

$$M_L(a, \vec{a}_R) - \delta_L M_L^e[(a_{L,m}, a, a_{L,p}), \vec{a}_R].$$

The militants would vote to support any policy where the former line is above 0, while the pragmatists would vote to support policies where the latter curve takes positive values. Under unanimity, the only policies that pass are those where both of these conditions hold, which are any $a \leqslant 0.588$. The Left's opportunists will try to maximize $\Pi_L(a, \vec{a}_R)$, which is the piecewise linear function, subject to $a \leqslant 0.588$. Maximization, thus, occurs precisely at the corner $\hat{a}_o \leqslant 0.588$.

For Pragmatists: Finally, consider the first figure in Appendix C. The straight line measures

$$V_L(a) - \delta_L V_L^e[(a_{L,m}, a_{L,o}, a)],$$

while the piecewise linear function, which is measured in the right axis, depicts

$$\Pi_L(a, \vec{a}_R) - \delta_L \Pi_L^e[(a_{L,m}, a_{L,o}, a), \vec{a}_R].$$

Again, for the pragmatists to obtain the support of the other two factions, these two functions must take non-negative values, which occurs for $a \leqslant 0.761$. The pragmatists seek to maximize $M_L(a, \vec{a}_R)$, namely the smooth function, over that subset of policies, so their optimal policy is $\hat{a}_p = 0.25$.

Equilibrium: The computations above give us a profile of policies where each faction in the Left party is maximizing its expected value, given the policies chosen by the other two factions of the same party, and by the three factions of the Right. That is, vector $\vec{a} = (0.026, 0.588, 0.25)$ gives an equilibrium of the intra-party game for the Left. For us to obtain an intra-party voting Nash equilibrium, we would need to further argue that the three factions of the Right are also maximizing their expected values, given the policies chosen by the other factions of the Right, and by the three factions of the Left. By symmetry, however, it suffices to observe that $\vec{a} = 1 - \vec{a}_R$, to conclude that this is indeed the case, and, hence that

$$(\vec{a}_L, \vec{a}_R) = [(0.026, 0.588, 0.25), (0.974, 0.412, 0.75)]$$

is an intra-party voting Nash equilibrium. This means that along the policy space $A = [0, 1]$ the left party will announce three different platforms, depending on which faction wins the internal elections. If the militants, opportunist or pragmatist win the internal elections, then they will place themselves in the position 0.026, 0.588 or 0.25 along the policy space. The same can be said for the right party.

Inefficiency of equilibrium policies: Now that we have computed the intra-party unanimous Nash equilibria, what remains to prove proposition 13 is to show that the support of the entire intra-party equilibria is not in the set of Roemer equilibria. Figure 2.4 depicts the support of our equilibria and compares it with the set of P.U.N.E. The main finding of this exercise is that there are points in the support that do not constitute a party-unanimity Nash equilibrium. There are three such pairs: $(a_{L,m}, a_{R,o})$, $(a_{L,o}, a_{R,m})$ and $(a_{L,o}, a_{R,o})$. The pairs $(a_{L,m}, a_{R,o})$ and $(a_{L,o}, a_{R,m})$ fail to be Roemer equilibrium for the same reason: in both cases, the militants of a party are picking a policy that is so «extreme» in terms of that party's ideology, that when such policy is confronted with the opposing party's opportunists, which are in turn taking a policy quite far from their ideological extremes, the first party fails the intra-party efficiency condition required by Roemer. Indeed, in both cases the party of the militants has zero probability of winning, yet it is not adopting the ideal policy for its militants. The reason why the militants adopted this position was that, ex-ante,[19] their parties' opportunists would not have supported an even more extreme platform.

The last pair of policies, $(a_{L,o}, a_{R,o})$, also fails to be a Roemer equilibrium, but

[19] That is, before knowing which of the opposing party's factions they were going to face.

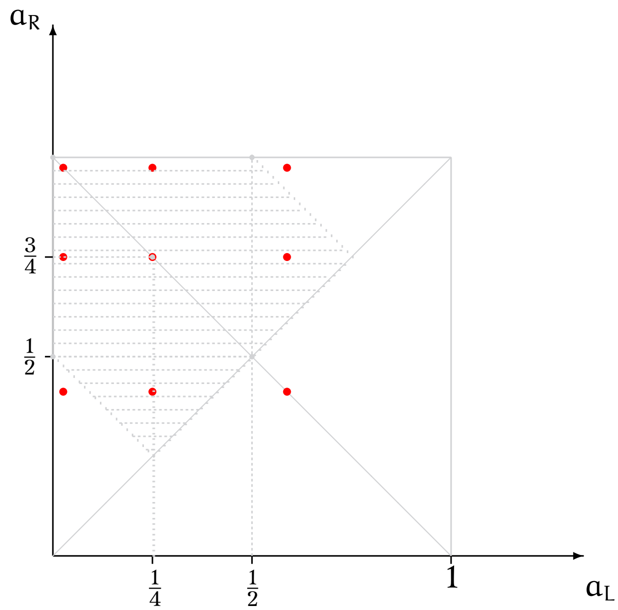

Figure 2.4: The nine thick dots constitute the support of the equilibrium pair of policies. The shaded area is the set of party-unanimity Nash equilibria.

for a different, and perhaps more interesting reason. In this case, both parties' opportunists would be competing in the election, and the Right party would have adopted a more «leftist» platform than the Left party. This is not internally efficient for any of the parties: given the policy chosen by the opposition, all the factions of a given party would have been better off had they chosen a policy closer to the one of the opposition. In this case, the militants and pragmatists of each party would have supported one such change in policy if it had been proposed by their fellow opportunists, but it was not in the latter's interest to do so: given the probability that the opposition would adopt the policy proposed by its pragmatists, ex-ante the opportunists choose the most extreme policy that their fellow militants are willing to support.

In the three points we have mentioned, the interpretation of an observed pair of policies as a Roemer equilibrium would be quite misleading. In the first two pairs, the analyst would need to mis-specify the voters preferences to account for them. In the latter, the analyst would have to mis-specify the parties' ideology functions. These errors would arise from the utilization of a concept in which the intra-party contest plays little strategic role: by imposing intra-party efficiency, Roemer's concept is, *a fortiori* dismissing all the externalities that a party's factions impose on each other during primary elections. The point of our next section is to argue that, indeed, the reason why intra-party voting Nash equilibria may fail to display the properties required by Roemer's party-unanimity Nash equilibrium lies mainly on the fact that each party's factions are choosing under uncertainty about the policy that the other party will propose. This uncertainty is due to the lack of knowledge of its party members about which of its factions will prevail in its internal contest.

2.6. Intra-party inefficiency in intra-party voting Nash equilibrium

In order to argue that it is due to the inherent uncertainty that each party faces about the outcome in its opposition's primary vote, we will now consider the fictitious situation in which the three factions of a party (the Left) have to solve the intra-party contest under the assumption that the opposing party's policy is given.

Proposition 14. *Fix a policy* $a_R \in [0,1]$, *and suppose that the three factions of the Right are going to follow that policy:* $\vec{a}_R = (a_R, a_R, a_R)$. *Let* $\vec{a}_L$ *be such that Conditions*

(2.5), (2.6) and (2.7) are satisfied, with $\gamma_L = 1$. Then, the three factions of the Left choose policies to the left of a_R:

$$a_{L,m} \leqslant a_R,\ a_{L,o} \leqslant a_R \ \textit{and}\ a_{L,p} \leqslant a_R.$$

Proof: Each of the three inequalities can be argued separately. The most interesting case, though, is the second inequality, for it is apparent that if the opportunists are to the left of a_R, then so will the other two factions of the Left.

To argue this inequality, suppose, by way of contradiction, that $a_{L,o} > a_R$. By assumption, $v_L(a_R) > v_L(a_{L,o})$, which suffices to imply that $V_L(a_R) > V_L(a_{L,o})$, and hence that

$$V_L(a_R) > \delta_L V_L^e(a_{L,m}, a_R, a_{L,p}).$$

For the same reason, $\mu_L(a_R, a_R) > \mu_L(a_{L,o}, a_R)$, which implies that $M_L(a_R, \vec{a}_R) > M_L(a_{L,o}, \vec{a}_R)$, and therefore that

$$M_L(a_R, \vec{a}_R) > \delta_L M_L^e[(a_{L,m}, a_R, a_{L,p}), \vec{a}_R],$$

since $\delta_L < 1$ and $\beta_L \leqslant 1$. The latter implies that both the militants and the pragmatists of the Left would vote to support a_R if it were proposed by their fellow opportunists, so that such policy would pass, even when $\gamma_L = 1$. By continuity, if the opportunists proposed $a = a_R + \varepsilon$, for $\varepsilon > 0$ small enough, such policy would also pass, but this is impossible because, in such case, $\Pi_L(a, \vec{a}_R) > \Pi(a_{L,o}, \vec{a}_R)$, which contradicts Condition (2.6). **Q.E.D.**

This proposition addresses the point we posed at the end of the previous section, that is, in the absence of uncertainty about the rival's policy, the strong intra-party inefficiency displayed by the policy pair $(a_{L,o}, a_{R,o})$ in our previous example would not occur. An analogous result holds for the type of intra-party inefficiency exhibited by the other two pairs, but in a weaker sense: if the solution to an intra-party contest displays such inefficiency in the absence of uncertainty about the policy proposed by the opposing party, then there is a policy triple that also solves the intra-party contest and is, at least, weakly Pareto superior.

Proposition 15. *Fix a policy $a_R \in [0,1]$, and suppose that the three factions of the Right are going to follow that policy: $\vec{a}_R = (a_R, a_R, a_R)$. Let $\vec{a}_L$ be such that Conditions (2.5), (2.6) and (2.7) are satisfied, with $\gamma_L = 1$. Then,*

1. *if $\pi(a_{L,m}, a_R) = 0$, then $a_{L,m} = 0$;*
2. *if $\pi(a_{L,o}, a_R) = 0$, then $a = 0$ passes and $\Pi_L(a_{L,o}, \vec{a}_R) = \Pi_L(0, \vec{a}_R)$; and*
3. *if $\pi(a_{L,p}, a_R) = 0$, then $a = 0$ passes and $M_L(a_{L,o}, \vec{a}_R) = M_L(0, \vec{a}_R)$.*

Proof: Again, the three results can be proved independently, but the most interesting is the second one. To see that the second claim is true, suppose that $\pi(a_{L,o}, a_R) = 0$. Since $\vec{a}_L$ satisfies Condition (2.6), it must be true that the Left's pragmatis vote to accept $a_{L,o}$. Since $a_{L,o} \geqslant 0$, it must be true that

$$0 \leqslant \pi(0, a_R) \leqslant \pi(a_{L,o}, a_R) = 0,$$

which suffices to imply that $M_L(0, \vec{a}_R) = M_L(a_{L,o}, \vec{a}_R)$. The latter implies that the pragmatists also vote to accept $a = 0$ when proposed by the opportunists. By definition of function ν_L, so do the militants, which implies that $a = 0$ passes. To see that $\Pi_L(a_{L,o}, \vec{a}_R) = \Pi_L(0, \vec{a}_R)$, observe, again, that $\pi(0, a_R) = \pi(a_{L,o}, a_R)$, which suffices for the result. **Q.E.D.**

2.7. Conclusions

We present a two-party model of electoral competition in which each party must elect a type of politician that will subsequently compete with the elected member of the opposing party.

We use Baron and Ferejohn's bargaining protocol to explicitly model the decision process by which each party elects a type of political actor, and offer an alternative solution concept, *intra-party unanimity Nash equilibria* for games of electoral competition.

We design an example to show that not all the policy pairs in the support of an intra-party Nash equilibrium are necessarily Roemer equilibria and prove the intra-party inefficiency captured by a triplet of policy pairs occurs due to the uncertainty each party has about the policy the contender will propose.

Chapter 3

Essay three

A Bargaining Model with Strategic Generosity.

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3.1. Introduction

A question in economics which, not seldom, has been investigated is how people negotiate over how to divide a dollar and what will the agreed partition be. The two fundamental approaches within the literature have been the axiomatic and the strategic approach. While the former, associated with Nash (1950), attempts to design a set of axioms that a "fair" and "reasonable" division should satisfy, and identify the division with these properties; the latter, associated with Rubinstein (1982), provides both an explicit description of an entire bargaining situation and a precise notion of the players' rationality which yield a solution concept. Although, numerous studies within the latter approach embrace salient features such as, timing of the offers (i.e. how frequent offers may be)[1] and the institutional structure (i.e. sequence of moves)[2], not much has been explored for the case where agents' preferences, within a specific bargaining environment, are represented by heterogenous utility functions.

Suppose individuals in a bargaining game negotiate over how to divide a fixed amount of money, say a dollar. Their preferences are represented by different utility functions, namely, they represent selfish and egalitarian individuals preferences. The first type of player derives utility solely from the shares they can obtain for themselves, whereas the second type derives utility, both from his individual shares and those allocated to the remaining players. The objective of this paper is to propose a model of endogenous formation of strategic generosity. Furthermore, in

[1] See Sákovics (1993), Perry and Reny (1993) and Admati and Perry (1987)

[2] See Baron and Ferejohn (1989) and Sutton (1986)

absence of any bargaining protocol all two types will stay with the entire dollar for themselves, regardless of the other regarding preferences of the egalitarian player. Nonetheless, the presence of a negotiation protocol à la Rubinstein allows strategic generosity to endogenously emerge. The basic setup analyzed in this paper explicitly describes the procedure by which the proposer, when having to decide how much of the dollar to offer the seconder in order for the latter to accept, strategically considers (in accordance with his preferences) the consequences of changes in the distribution of types on his payoffs. Similarly, we describe the procedure by which the seconder, when deciding either to accept or reject an offer, strategically considers the consequences of variations in the profile of types.

Essentially, we concentrate on the role played by the heterogeneity of players preferences and changes in the profile of types on the endogenous formation of strategic generosity. To keep the analysis tractable, we follow Rubinstein (1982) in that we exogenously specify a fixed time between offers, one offer per time unit, but rather than depending upon offers made sequentially, the order by which players are selected to make a proposal is random. We allow for players preferences to be represented by different utility functions and derive the set of subgame perfect equilibrium for different arrangements of player types and show that strategic generosity comes into scene endogenously under the chosen negotiation protocol.

Related Literature: We face a bargaining problem when multiple parties have before them several possible contractual agreements that offer, if they coordinate their actions, an opportunity to gain but their interests are not entirely identical. Over a century ago, Edgeworth (1881) considered negotiation between individuals to be the most fundamental problem in economics. Since then, studying the problem from a non-cooperative approach requires both an explicit description of the entire bargaining situation, as well as a precise notion of the players' rationality. This is commonly known as the strategic approach to the bargaining problem. Assuming parties behave rationally, the question this situation poses is what will the agreed contract be?

Nash (1950 and 1953) was one of the first to study a class of two-person bargaining situations. He offered two independent derivations of the solution to the problem. The first is a set of several properties that determine the solution uniquely. The second was an effort to complement the axiomatic approach where he reduced the cooperative game to a non-cooperative game and proved that the solution is the limit of a sequence of equilibria of smoothed games. It was Nash himself who expressed that ...."the two approaches to the problem, via the negotiation model or via the axioms, are complementary; each helps to justify and clarify".[3]

Occasionally, the strategic approach has been criticized on the grounds that the predictions of the model are highly dependent and sensitive to the exact description of the extensive form. Aumann (1987) shared this same concern when expressing the reasons why cooperative games came to be treated separately: ....."when one

[3] He also claims that the situation to be treated is more general than that in the bargaining problem mentioned above since the solution is characterized by the maximization of the Nash product, that is the differences between the values of the game and their default utilities (i.e. utility levels that players can obtain if they break up negotiations: if they don't cooperate). As shown if Nash (1953).

does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers, and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail...... But problems of negotiation are usually more amorphous; it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak. Another reason is that even when the procedures are specified, non- cooperative analyses of a cooperative game often lead to highly non-unique results, so that they are often quite inconclusive."[4]

Although Rubinstein (1982) too recognized that the main difficulty with the strategic approach has been the need to specify the moves of the game, specially when not all bargaining situations have a unique procedure. Rubinstein's (1982) bargaining game is an example of the sensitivity to the exact description the extensive forms, as it turn out it is the order and timing of the offers the main force driving the predictions of the model. Nonetheless, he also expressed some reservations concerning the axiomatic approach stressing that.... (1) some of the axioms are not easily defended in abstract and that ....(2) additional information, such as the negotiation time preferences, seems to be relevant to the solution".[5] Our model

[4] This chapter originally appeared in The New Palgrave: A Dictionary of Economics, Vol. 2, edited by J.Eatwell, M. Miligate and P. Newman, pp. 460-482. Macmillan London (1987). Reprinted with permission.

[5] See Rubinstein (1985)

is a strategic model, and as such it embodies a detailed description of a special procedure which we will specify in detail further into the paper.

This paper is organized as follows. We develop the basic framework and define the concept of sub-game perfect Nash equilibrium in the next section. In section 3.3 we solve the game, present and analyze our results. My concluding remarks and some directions for further research appear in section 3.4.

3.2. The Bargaining Model

3.2.1. The protocol

Let the set of players be $\mathcal{N} = \{1, 2, 3\}$. At each stage of the game only one randomly chosen player can make an offer. For the game to end, that is, for players to reach an agreement, the offer must be accepted by the majority of players. There are no bounds on the number of rounds of negotiation. In the event that an agreement isn't reached, places no restriction on the offers subsequently made, for example, there are no rules binding players to any previous offers they have made. The continuation game does not depend on the actions of the players in the current period. Last, we don't assume players are impatient about enjoying the fruits of an agreement, in other words, they do not face costly time lapses between bargaining rounds. So, three players have the opportunity to reach agreement on how to divide a dollar and to do so a player is randomly selected, with probability 1/3, to propose an allocation from the (compact and convex) set of all feasible divisions of a dollar,

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \mid \sum_{i=1}^{3} x_i = 1\}$$

Once a proposal has been made, the remaining two players vote whether to accept or reject it. For an agreement to be reached we have determined the voting rule to be that of the majority: understanding that the player who makes a proposal supports it, if one of the remaining players votes to accept it, an agreement has been reached.

If the players reach an agreement, they immediately receive the allocation and the game ends. If they do not agree, a new player is randomly selected (with replacement) to propose, and the game continues as before. In the case that players never agree on an allocation of the dollar, the outcome is the disagreement event and the payoff for each individual is zero.

3.2.2. Individual preferences

We allow for the players' utility functions to depend, not solely, on their individual payments, but we consider agents who also value the overall allocation of the dollar. In this sense, players also take into account the social allocation of the aggregate resource so that their utility functions mirror selfish and Rawlsian principles. Hence, player i is selfish if her utility function is

$$u_i(x) = x_i;$$

and she is Rawlsian if it is

$$u_i(x) = x_i + \beta \min_{j,k \neq i} \{x_j, x_k\},$$

again for some $\beta \in (0, 1)$. Across rounds of negotiation, individuals do not discount their payoffs.

Rawlsian type players are “altruistic", in the sense that they derive utility from the income that is allocated to others. We will assume that $\beta < 1$ and it this sense ensure that, by itself, altruism does not induce “generosity". For instance, if a player can simply impose the allocation of the dollar without negotiation then regardless of their type, both will choose the “selfish" allocation; which means that each type will choose to stay with the entire dollar for themselves. But once types abide to a negotiation protocol we ask ourselves whether players respond strategically by sharing less to generously sharing much more. Can *"strategic generosity"* emerge as a result of the bargaining process?

We will solve the game considering different arrangements of types: (i) three players of the same type, for each of the types; (ii) two selfish and one Rawlsian player; and (iii) two Rawlsian and one selfish player.

3.2.3. Actions, pure strategies and pure-ish strategies

We refer to a player's actions according to his standpoint. In other words, what player i can offer, when he's been randomly selected, is $x_i = (1 - x_{i,j} - x_{i,k}, x_{i,j}, x_{i,k})$ where $x_{i,j}$ and $x_{i,k}$ correspond to the shares players j and k receive from player i.

In addition, when a player (different from i) has made an offer to i, the latter's response will be: $d_i \in \{A, R\}$, where A and R denote whether i accepts, or rejects the offer, respectively.

In this order of ideas, a pure strategy for player i must specify the following: for each period t, after each possible history of actions observed up to round $t-1$, an offer x_{i_t} that she makes -if selected to propose-; and for each t, after each possible history of actions observed up to round $t-1$, and for each possible proposal x_{j_t} that that player j can make with $j \neq i$,, a decision d_{i_t} that i enunciates.

If we are to consider mixed strategies, then we must specify, under the same contingencies, a probability distribution $P_{i,t}$ over set X, and another over the set $\{A, R\}$. The former would determine the proposal $x_{i,t}$, the latter the decision $d_{i,t}$.

Since the game is stationary, we will concentrate our attention on equilibria where strategies are stationary too. Therefore, the actions of each player at round t will be independent of the history of actions up to round $t-1$. So, at equilibrium we can denote the strategies by dropping the subindex t.

Whenever possible, we will look for equilibrium in pure strategies, but this type of equilibrium does not always exist. If the game has no equilibrium in pure strategies, we will consider a simple case of mixed strategies, that we may call "pure-ish": we will not allow players to randomize on their decisions whether to accept or reject a proposal, and we will only allow the proposer to randomize between two points in X. That is, in our equilibrium strategies:

1. When player i is chosen to propose, she chooses two points $x_{i,j}, x_{i,k} \in X$ and

two probabilities $p_{i,j}$ and $p_{i,k}$ such that $p_{i,j} + p_{i,k} = 1$ $\forall j$ and $k \neq i$ and $j \neq k$; player i will propose $x_{i,j}$ with probability $p_{i,j}$, and $x_{i,k}$ with probability $p_{i,k}$.

2. When player i isn't selected to propose, her decision to accept or reject is determined by the following rule: for some number v_i, player i will accept a proposal $x_{j,i}$ if, and only if, $u_i(x_{j,i}) \geqslant v_i$ for $j \neq i$, regardless of who makes the proposal. Moreover, the utility level v will have to be the utility level that player i obtains in the game, at equilibrium.

The first condition simply seeks to keep the strategies as "pure" as possible, while at the same time it gives the players space for randomization. The property is appealing, under the following interpretation: when player i has been chosen to propose, she only needs to get the approval of one of the other two players; she will try to get the approval of player j with probability $p_{i,j}$, and that of player k with probability $p_{i,k}$. The second property seeks to dismiss trivial majority equilibria where the players that have not been chosen to propose, approve any proposal just because that is what the other player is doing.

3.2.4. Subgame-perfect Nash equilibrium

Denote the one-dimensional unit simplex by Δ. Under the provisions above, a stationary, pure-ish strategy for player i can be written as a pair (f_i, v_i), where

$$f_i = ((p_{i,j}, p_{i,k}), x_{i,j}, x_{i,k}) \in \Delta \times X \times X$$

and $v_i \in [0,1]$.

Definition 1. *A stationary, subgame-perfect Nash equilibrium in pure-ish strategies will be an array*

$$((f_1, v_1), (f_2, v_2), (f_3, v_3))$$

such that, for each player i*, the following conditions hold true:*

$$p_{i,j} > 0 \textit{ implies that } x_{i,j} \in \textit{argmax}_{x_i}\{u_i(x_i) : u_j(x_i) \geqslant v_j\}, \tag{3.1}$$

$$p_{i,j} > 0 \textit{ implies that } u_i(x_{i,j}) \geqslant \max_{x_i}\{u_i(x_i) : u_k(x_i) \geqslant v_k\}, \tag{3.2}$$

$\forall j \neq i$ *and*

$$v_i = \frac{1}{3}[\sum_{j=1}^{3} \sum_{k \neq j} p_{j,k} u_i(x_{j,k})]. \tag{3.3}$$

Eqs. (3.1) and (3.2) state that player i is rational at the time of making offers, for it requires that: (i) when trying to get the support of player j, she does it by choosing, among the proposals that j would accept, the one that is most convenient for herself; and (ii) if she is going to try to convince player j, it must be that she cannot be better off by convincing player $k \neq j$. Eq. (3.3) states that player i is rational when deciding whether to accept or reject an offer.

Moreover, we will only look at symmetric equilibria, where two players of the same type follow the same strategy.

3.3. Bargaining Equilibrium

Let us first consider the case where all three players are selfish. An array $((f_1, v_1), (f_2, v_2), (f_3, v_3))$ is subgame perfect if it satisfies Eqs. (3.1) to (3.3), that is,

$$x_3^1 \geqslant \frac{1}{3}[x_3^1 + (1 - x_1^2)],$$

$$x_1^2 \geqslant \frac{1}{3}[x_1^2 + (1 - x_2^3)],$$

and

$$x_2^3 \geqslant \frac{1}{3}[x_2^3 + (1 - x_3^1)]$$

correspondingly. Since the game is stationary and symmetric: $x_1^2 = x_2^3 = x_3^1$. By solving the system of inequalities we get that, at any stage of the game, when it's player $1'$s turn to propose she will offer player 2, $x_{1,2} = (\frac{2}{3}, \frac{1}{3}, 0)$ with probability $p_{1,2} = 1$ and player 2 will accept. Under the conditions above Player $1'$s degenerate pure-ish stationary strategy can be written as, $(1, 0, (\frac{2}{3}, \frac{1}{3}, 0), \frac{1}{3})$

Because the game is stationary and symmetric: without loss of generality, if instead of player 1 being randomly selected to propose, it is player 2, who choses player 3 to bargain with in order to reach an reach agreement, and if instead of player 2 being the proposer, it is player 3 who chooses among the proposals player 1 will accept to reach agreement, the stationary subgame-perfect Nash equilibrium in pure strategies will be,

For player 1.

$$((1, 0), (\frac{2}{3}, \frac{1}{3}, 0), \frac{1}{3}),$$

For player 2.

$$((1, 0), (0, \frac{2}{3}, \frac{1}{3}), \frac{1}{3}),$$

For player 3.

$$((1, 0), (\frac{1}{3}, 0, \frac{2}{3}), \frac{1}{3}).$$

Definition 2. *A selfish offer (I) from player* i *is an offer such that,* $x_{i,j} = \frac{1}{3}$, $x_{i,k} = 0$ *and* $(1 - x_{i,j}) = \frac{2}{3}$.

Proposition 16. *Whenever all three players present in the game are selfish, regardless of whose turn it is to make a proposal, all of them make selfish offers (I).*

Proof:

Let's consider the case where all three players are selfish. Say player 2 accepts a proposal, from 1, $x^{*}_{1,2} = (\frac{2}{3} + \epsilon, \frac{1}{3} - \epsilon, 0)$. Accepting the offer, by Eqs. (3.3), implies she is behaving rationally, but $u_2(x^{*}_1) < v_2$ which cannot be.[6] On the other hand, because player 1 requires the support of only one player for the game to end, say she tries to convince player 2 to accept a proposal for the game to reach agreement such that $x^{**}_{1,2} = (\frac{2}{3} - \epsilon, \frac{1}{3} + \epsilon, 0)$. Player 1 must have have chosen, among the proposals that 2 will accept, the one that maximizes her utility. So, by not choosing among the proposals that player 3 would have accepted, implies player 1 cannot increase her utility beyond the level she will achieve when proposing $x^{**}_{1,2}$. However, by symmetry, player 1 can convince player 3 of accepting proposal, $x_{1,3} = (\frac{2}{3}, 0, \frac{1}{3})$. Therefore, by choosing among the proposals player 2 accepts, player 1 isn't maximizing her utility, since, $u_1(x^{**}_{1,2}) < u_1(x_{1,3})$. She could be better off if she gains support from 3 by offering, $x_{1,3}$.[7] *Q.E.D.*

[6] Refer to Figure 4, (case 1) of Appendix C.2 where we have plotted a selfish players expected utility, for different levels of β. In this case player $2'$s continuation payoff equals $\frac{1}{3}$ which is below her cut-off point, therefore she will not accept.

[7] See Appendix C.2, Figure 4 (case 1) to see that player $3'$s continuation payoff equals her utility payoff of player $1'$s proposal, therefore 3 will indeed accept $1'$s offer.

Now, let's consider the case where players 1 and 2, are selfish and 3 is Rawlsian. Since this arrangement has no subgame perfect Nash equilibrium in pure strategies, to look for the stationary equilibrium in mixed strategies[8] Eqs. (3.1) to (3.3) must be satisfied. To find the stationary pure-ish strategy for player i, it must be that she is indifferent between making a proposal to player j and k, for all $i, j and k \in \{1, 2, 3\}$ rand all $i \neq j \neq k$. In which case, $u_1(x_{1,2}) = u_1(x_{1,3})$, $u_2(x_{2,1}) = u_2(x_{2,3})$ and $u_3(x_{3,1}) = u_3(x_{3,2})$ which implies, :$x_2^1 = x_2^3$ and $x_1^2 = x_1^3$. Thus, by symmetry, $x_2^1 = x_1^2$, $x_1^3 = x_2^3$, $p_{1,2} = p_{2,1}$, consequently,

$$x_j^1 \geqslant \frac{2}{7 - 2p_{1,2}},$$

for $j = (2, 3)$

$$x_j^2 \geqslant \frac{2}{7 - 2p_{2,1}}$$

for $j = (1, 3)$ and

$$x_i^3 \geqslant \frac{1}{2 + 2p_{i,j} - 2p_{i,j}\beta}$$

for i and $j = (1, 2)$.

By solving the system of inequalities: player 1 offers: $x_{1,2} = (\frac{6-5\beta}{9-7\beta}, \frac{3-2\beta}{9-7\beta}, 0)$ and $x_{1,3} = (\frac{6-5\beta}{9-7\beta}, 0, \frac{3-2\beta}{9-7\beta})$ with probability $p_{1,2} = \frac{3}{6-4\beta}$ and $p_{1,3} = \frac{3-4\beta}{6-4\beta}$, respectively. By symmetry, $x_{2,1} = (\frac{3-2\beta}{9-7\beta}, \frac{6-5\beta}{9-7\beta}, 0)$ with $p_{2,1} = \frac{3}{6-4\beta}$ and (ii) $x_{2,3} = (0, \frac{6-5\beta}{9-7\beta}, \frac{3-2\beta}{9-7\beta})$ with probability $p_{2,3} = \frac{3-4\beta}{6-4\beta}$.

[8] To see why there exists no equilibrium in pure strategies, refer to Appendix C.2.

Whenever it's a selfish type's turn to offer, if $\beta \geqslant \frac{3}{4}$ he places a offer to her same type only, hence, $p_{1,2} = p_{2,1} = 1$. By symmetry, same for player 2, thus among the offers 1 accepts, the latter accepts the one that maximises her utility. To convince 1 of accepting, it must be that 2 cannot be better off by making an offer to player 3. When β is high making a proposal to a Rawlsian type that the latter will accept is too costly compared to what it costs to convince a player of his same type. Since the Rawlsians' continuation payoff is greater than the selfish type's, the latter type will solely propose to his same type.[9]

Now, if $\beta < \frac{3}{4}$, by symmetry: $p_{1,2} = p_{2,1}$ then $\frac{1}{2} \leqslant p_{1,2} \leqslant 1$. In line with what has been said above, as β increases the probability that a selfish player makes an offer to his same type increases. In essence, a greater (lower) β translates into a greater (lower) expected utility for the Rawlsian player, discouraging (encouraging), say player 2 to make a proposal to player 3.

Last, when player 3 makes an offer, she flips a fair coin and offers: $x_{3,1} = (\frac{3-2\beta}{9-7\beta}, 0, \frac{6-5\beta}{9-7\beta})$ and $x_{3,2} = (0, \frac{3-2\beta}{9-7\beta}, \frac{6-5\beta}{9-7\beta})$. Furthermore, irrespective of the value of $\beta \in (0,1)$ he turns out making the same offer the selfish type does.

The stationary subgame perfect Nash equilibrium in pure-ish strategies will be,

For player 1.

$$((\frac{3}{6-4\beta}, \frac{3-4\beta}{6-4\beta}), (\frac{6-5\beta}{9-7\beta}, \frac{3-2\beta}{9-7\beta}, 0), (\frac{6-5\beta}{9-7\beta}, 0, \frac{3-2\beta}{9-7\beta}))\frac{3-2\beta}{9-7\beta}),$$

if $\beta < \frac{3}{4}$.

[9] See Appendix C.2 and compare cases 4 and 1 of Figure 4 and 6.

$$((1,0),(\frac{6-5\beta}{9-7\beta},\frac{3-2\beta}{9-7\beta},0),(\frac{6-5\beta}{9-7\beta},0,\frac{3-2\beta}{9-7\beta}))\frac{3-2\beta}{9-7\beta}),$$

if $\beta \geqslant \frac{3}{4}$

For player 2.

$$((\frac{3}{6-4\beta},\frac{3-4\beta}{6-4\beta}),(\frac{3-2\beta}{9-7\beta},\frac{6-5\beta}{9-7\beta},0),((0,\frac{6-5\beta}{9-7\beta},\frac{3-2\beta}{9-7\beta}),\frac{3-2\beta}{9-7\beta}),$$

if $\beta < \frac{3}{4}$.

$$((1,0),(\frac{3-2\beta}{9-7\beta},\frac{6-5\beta}{9-7\beta},0),((0,\frac{6-5\beta}{9-7\beta},\frac{3-2\beta}{9-7\beta}),\frac{3-2\beta}{9-7\beta}),$$

if $\beta \geqslant \frac{3}{4}$.

For player 3.

$$((\frac{1}{2},\frac{1}{2}),(\frac{3-2\beta}{9-7\beta},0,\frac{6-5\beta}{9-7\beta}),(0,\frac{3-2\beta}{9-7\beta},\frac{6-5\beta}{9-7\beta}),\frac{3-2\beta}{9-7\beta})$$

Definition 3. *A semi-selfish offer from player i is an offer such that* $x_{i,j} = \frac{3-2\beta}{9-7\beta}$, $x_{i,k} = 0$ *and* $(1 - x_{i,j}) = \frac{6-5\beta}{9-7\beta}$.

Proposition 17. *When two out of three players present in the game are selfish, irrespective of the value of* β, *all three players will make semi-selfish offers.*

Proof: Provided $\beta \geqslant \frac{3}{4}$ and since player 1 and 2 make proposals amongst each other, if Player 2 proposes, $x^*_{2,1} = (\frac{3-2\beta}{9-7\beta} - \epsilon, \frac{6-5\beta}{9-7\beta} + \epsilon, 0)$ player 1's continuation payoff is greater than the utility she will receive from $x^*_{2,1}$, therefore, she will not accept.

If she proposes $x^{**}_{2,1} = \frac{3-2\beta}{9-7\beta} + \epsilon, \frac{6-5\beta}{9-7\beta} - \epsilon, 0)$, for $\epsilon > 0$ she isn't choosing the proposal that is most convenient for her since, $u_2(x_{2,1}) > u_2(x^{**}_{2,1})$.

Player 2 makes a proposal, $x^{**}_{2,3} = (0, \frac{6-5\beta}{9-7\beta} - \epsilon, \frac{3-2\beta}{9-7\beta} + \epsilon)$ it must be that, $u_2(x^{**}_{2,3}) > u_((x_{2,3})$, but this cannot be since: $\frac{6-5\beta}{9-7\beta} - \epsilon < \frac{6-5\beta}{9-\beta}$ for $\epsilon > 0$. Oppositely, when player 2 chooses to make a proposal to player 1 and offers, $x^{**}_{2,1} = (\frac{3-2\beta}{9-7\beta} + \epsilon, \frac{6-5\beta}{9-7\beta} - \epsilon, 0)$ she is be better off, at any stage of the game, gaining support from player 3, in other words, proposing, among the offers the latter accepts, $x_{2,3} = (0, \frac{6-5\beta}{9-7\beta}, \frac{3-2\beta}{9-7\beta})$ since $u_2(x^{**}_{2,1}) < u_2(x_{2,3})$.

If it's player $3'$s turn to propose, as β increases, it's too costly for a Rawlsian type to convince the other type, the best the former can do is to propose what he accepts when the other type makes an offer to him, that is, $x^3_i = \frac{3-2\beta}{9-7\beta}\ \forall i = (1, 2)$. In this sense, the presence of a "*altruistic*" type is of no importance for the social allocation of the aggregate resource. *Q.E.D.*

Let's invert the labelling of players so that players 1 and 2 are Rawlsian and 3 is selfish. Particularly, the game has a subgame perfect Nash equilibrium in pure-ish strategies if,

$$u_1(x^1_3) = u_1(x^1_2) = \frac{1}{3}[u_1(x^1_2) + u_1(x^2_1) + \frac{1}{2}(u_1(x^1_3) + u_1(x^2_3))]$$

and

$$u_2(x^2_3) = u_2(x^2_1) = \frac{1}{3}[u_2(x^2_1) + u_2(x^1_2) + \frac{1}{2}(u_2(x^2_3) + u_2(x^1_3))]$$

By symmetry, $x^2_1 = x^1_2$, $x^3_2 = x^3_1$ and $x^1_3 = x^2_3$ and solving the system of inequalities we get three general findings: (i) irrespective of the value of $\beta \in (0, 1)$ a Rawlsian type always stays with the greatest amount of shares, (ii) for $\beta < \frac{1}{2}$ the Rawlsian only offer his type a positive amount of shares; conversely, if $\beta > \frac{1}{2}$ he offers positive amounts to both the remaining players and (iii) for $\beta \geqslant 0.645$ the selfish player gives up the greatest amount of shares to give it to whomever she chooses to negotiate with. Therefore, the stationary subgame perfect Nash equilibrium in pure-ish strategies will be,

For player 1.

$$((1,0),(\frac{5-2\beta-\beta^2}{7-\beta^2},\frac{2+2\beta}{7-\beta^2},0),\frac{2+2\beta}{7-\beta^2}),$$

for $\beta \leqslant \frac{1}{2}$.

$$((1,0),(\frac{3-\beta^2}{7-\beta^2},\frac{2}{7-\beta^2},\frac{2}{7-\beta^2}),\frac{2+2\beta}{7-\beta^2}),$$

for $\frac{1}{2} < \beta < 0.645$.

$$((1,0),(\frac{3+\beta^2}{7+2\beta+\beta^2},\frac{2+\beta}{7+2\beta+\beta^2},\frac{2+\beta}{7+2\beta+\beta^2}),\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2}),$$

for $0.645 < \beta \leqslant 1$.

For player 2.

$$((1,0),(\frac{2+2\beta}{7-\beta^2},\frac{5-2\beta-\beta^2}{7-\beta^2},0),\frac{2+2\beta}{7-\beta^2}),$$

for $\beta \leqslant \frac{1}{2}$.

$$((1,0),(\frac{2}{7-\beta^2},\frac{3-\beta^2}{7-\beta^2},\frac{2}{7-\beta^2}),\frac{2+2\beta}{7-\beta^2}),$$

for $\frac{1}{2} < \beta < 0.645$.

$$((1,0),(\frac{2+\beta}{7+2\beta+\beta^2},\frac{3+\beta^2}{7+2\beta+\beta^2},\frac{2+\beta}{7+2\beta+\beta^2}),\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2}),$$

for $0.645 < \beta \leqslant 1$.

For player 3.

$$((\frac{1}{2},\frac{1}{2}),(\frac{2+2\beta}{7-\beta^2},0,\frac{5-2\beta-\beta^2}{7-\beta^2}),(0,\frac{2+2\beta}{7-\beta^2},\frac{5-2\beta-\beta^2}{7-\beta^2}),\frac{5-2\beta-\beta^2}{7-\beta^2}),$$

for $\beta \leqslant 0.645$

$$((\frac{1}{2},\frac{1}{2}),(\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2},0,\frac{5-\beta}{7+2\beta+\beta^2}),(0,\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2},\frac{5-\beta}{7+2\beta+\beta^2}),\frac{5-\beta}{7+2\beta+\beta^2}),$$

for $0.645 < \beta \leqslant 1$

Definition 4. *A greedy offer from players* 1 *or* 2 *is an offer where,* $x_{1,2} = x_{2,1} = \frac{2+2\beta}{7-\beta^2}$, $x_{1,3} = x_{2,3} = 0$ *and* $(1-x_{1,2}) = (1-x_{2,1}) = \frac{5-2\beta-\beta^2}{7-\beta^2}$ *when* $\beta < \frac{1}{2}$.

Definition 5. *A generous offer from players* 1 *or* 2 *is an offer such that,* $x_{1,2} = x_{1,3} = x_{2,1} = x_{2,3} = \frac{2}{7-\beta^2}$ *and* $(1-2x_{1,2}) = (1-x_{2,1}) = \frac{3-\beta^2}{7-\beta^2}$ *for* $\frac{1}{2} \leqslant \beta < 0.645$

Definition 6. *An even more generous proposal from players* 1 *or* 2 *is an offer where,* $x_{1,2} = x_{1,3} = x_{2,1} = x_{2,3} = \frac{2+\beta}{7+2\beta+\beta^2}$ *and* $(1-2x_{1,2}) = (1-2x_{2,1}) = \frac{3+\beta^2}{7+2\beta+\beta^2}$ *when* $0.645 \leqslant \beta < 1$.

Definition 7. *A greedy proposal from the selfish type is an offer such that,* $x_{3,j} = \frac{2+2\beta}{7-\beta^2}$ *and* $(1-x_{3,j}) = \frac{5-2\beta-\beta^2}{7-\beta^2}$ *when* $\beta < \frac{1}{2}$. *And* $x_{3,j} = \frac{2+2\beta}{7-\beta^2}$ *and* $(1-x_{3,j}) = \frac{5-2\beta-\beta^2}{7-\beta^2}$ *when* $\frac{1}{2} \leqslant \beta < 0.645$ *for* $j = 1$ *or* 2.

Definition 8. *A one-to-one self sacrifice offer from a selfish type is an offer where,* $x_{3,j} = \frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2}$ *and* $(1-x_{3,j}) = \frac{5-\beta}{7+2\beta+\beta^2}$ *when* $0.645 \leqslant \beta < 1$, *for* $j = 1$ *or* 2.

Definitions 4 and 7 imply that whoever the proposer is, he will only make an offer to one of the other players, but never to both. For instance, in definition 4 players 1 and 2 only make offers among themselves, whereas in definition 7, player 3 tosses a coin to choose which of the two *altruistic* players he will make an offer to. Additionally, definitions 5 and 6 imply that both *altruistic* players choose their same type to bargain with by offering positive, and equal, amounts to both the remaining players. Because β is still relatively small, trying to convince a *altruistic* player to accept a proposal is relatively cheap. This is why a *altruistic* proposer makes a less generous offer to the remaining players, i.e, definition 5. As β increases this makes players 1 and 2 more difficult to convince. Therefore, bargaining with an *altruistic* player is more costly, thus offers are more generous, i.e, definition 6. Along this line of thought, whenever it's a selfish types' turn to make an offer, he knows that in order to convince either one of the other *altruistic* types he will have to offer the larger share to whomever he chooses to bargain with and stays with the smaller share of the dollar, which leads us to definition 8.

PROPOSITION 18. *Suppose there are two Rawlsian players in the game, if the following conditions hold,*

$$\beta < \frac{1}{2}, \tag{3.4a}$$

$$\frac{1}{2} \leqslant \beta < 0.645, \tag{3.4b}$$

$$0.645 \leqslant \beta < 1 \tag{3.4c}$$

then both of them make (i) a greedy offer, (ii) a generous offer and (iii) an even more generous offer, respectively.

Proof: If $\beta < \frac{1}{2}$, we know that by solving the system of inequalities so that $x_i^3 = 0$ for $i \in \{1, 2\}$, player 1 will offer player 2, $x_{1,2} = (\frac{5-2\beta-\beta^2}{7-\beta^2}, \frac{2+2\beta}{7-\beta^2}, 0)$ and player 2 accepts. Oppositely, player 1 will accept, player $2'$s offer, $x_{2,1} = (\frac{2+2\beta}{7-\beta^2}, \frac{5-2\beta-\beta^2}{7-\beta^2}, 0)$. If instead, player 1 proposes to player 2, $x^*_{1,2} = (\frac{5-2\beta-\beta^2}{7-\beta^2}, \frac{2}{7-\beta^2}, \frac{2\beta}{7-\beta^2})$, the former will gain $\beta(\frac{2\beta}{7-\beta^2})$ and loose $(1-\beta)(\frac{2\beta}{7-\beta^2})$. Nevertheless, if he sticks to, $x_{1,2}$ player 1 will gain $(1-\beta)(\frac{2\beta}{7-\beta^2})$ and loose $\beta(\frac{2\beta}{7-\beta^2})$. In addition, player $2'$s gains and losses cancel out, therefore, since $\beta < \frac{1}{2}$ player 1 will always propose $x_{1,2}$ and player 2 will accept. Due to symmetry when it's player $2'$s turn to propose, she will also propose $x_{2,1}$ and player 1 will always accept. On the other hand, if player 1 chooses to gain support from player 3 and proposes $x^*_{1,3} = (\frac{5-2\beta-\beta^2}{7-\beta}, 0, \frac{2+2\beta}{7-\beta})$, because the latter's minimum willingness to accept a proposal is strictly greater than the former's maximum willingness to make one, $x^*_{1,3}$ is not among the proposals that player 3 will accept in order for player 1 to gain support, therefore proposal, $x^*_{1,3}$ cannot be.

When $\frac{1}{2} \leqslant \beta < 0.645$, player 1 proposes, $x_{1,2} = (\frac{3-\beta^2}{7-\beta^2}, \frac{2}{7-\beta^2}, \frac{2}{7-\beta^2})$ and player 2 accepts. Similarly, if it's player 2's turn to propose, he offers player 1 $x_{2,1} = (\frac{2}{7-\beta^2}, \frac{3-\beta^2}{7-\beta^2}, \frac{2}{7-\beta^2})$ and player 1 accepts. The only reason why player 1 would deviate so that $x_1^2 > x_1^3 \geqslant 0$, were if $1 - x_1^2 - x_1^3 + \beta x_1^3 > \frac{3-\beta^2}{7-\beta^2}$ and the same goes for player 2 in the sense that, $x_1^2 + \beta x_1^3 > \frac{2+2\beta}{7-\beta^2}$. However, if $\beta \geqslant \frac{1}{2}$ the minimum player 2 is willing to receive from player 1 in order to reach an agreement conditional on $x_1^3 = 0$ is greater than the maximum player 1 is willing to offer.[10]. Player 1 will only have an incentive to deviate so that $x_1^2 > x_1^3 = 0$ if $\beta < \frac{1}{2}$, which cannot be.

Last, when $0.645 \leqslant \beta < 1$, player 1 will will offer $x_{1,2} = (\frac{3+\beta^2}{7+2\beta+\beta^2}, \frac{2+\beta}{7+2\beta+\beta^2}, \frac{2+\beta}{7+2\beta+\beta^2})$ to player 2 and the latter accepts. When selected to propose, player 2, offers, $x_{2,1} = (\frac{3+\beta^2}{7+2\beta+\beta^2}, \frac{2+\beta}{7+2\beta+\beta^2}, \frac{2+\beta}{7+2\beta+\beta^2})$ and player 1 accepts. Again, if player 1 proposes, $x^*_{1,2} = (\frac{3+\beta^2}{7+2\beta+\beta^2} - \epsilon, \frac{2+\beta}{7+2\beta+\beta^2} + \epsilon, 0)$ to player 2, the latter will accept since his gains, $\beta(\frac{2+\beta}{7+2\beta+\beta^2})$, cancel out with his losses. However, player 1's losses, $\beta(\frac{2+\beta}{7+2\beta+\beta^2})$ outweigh his gains, $(1-\beta)(\frac{2+\beta}{7+2\beta+\beta^2})$, making it impossible for player 1 to propose $x^*_{1,2}$.

Q.E.D.

Proposition **19**. *Suppose there is only one selfish player, if,*

$$\beta \leqslant 0.645, \tag{3.5a}$$

$$\beta > 0.645, \tag{3.5b}$$

provided its the selfish players' turn to propose, he will make (i) a greedy offer and (ii) a one-to-one self sacrifice offer, respectively.

[10] Look at Appendix C.3, graph 1 and we can see that there is no intersection point.

Proof: If $\beta \leqslant 0.645$, a selfish type will offer, with probability $p_{3,1} = p_{3,2} = \frac{1}{2}$, $x_{3,1} = (\frac{2+2\beta}{7-\beta^2}, 0, \frac{5-2\beta-\beta^2}{7-\beta^2})$, $x_{3,2} = (0, \frac{2+2\beta}{7-\beta^2}, \frac{5-2\beta-\beta^2}{7-\beta^2})$ and player 1 and 2 accept respectively. The selfish type, behaving rationally, will never offer an *altruistic* player, say 1, $x_{3,1} = (\frac{2+2\beta}{7-\beta^2} - \epsilon, 0, \frac{5-2\beta-\beta^2}{7-\beta} + \epsilon)$, since $u_1(x_{3,1}) < v_1$, thus, 1 will reject the proposal.

But if $0.645 < \beta < 1$, player 3 will offer with probability $p_{3,1} = p_{3,2} = \frac{1}{2}$, $x_{3,1} = (\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2}, 0, \frac{5-\beta}{7+2\beta+\beta^2})$ and $x_{3,2} = (0, \frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2}, \frac{5-\beta}{7+2\beta+\beta^2})$. When the selfish type makes the following proposal to player 1, $x_{3,1} = (\frac{(2+\beta)(1+\beta)}{7+2\beta+\beta^2} - \epsilon, 0, \frac{5-\beta}{7+2\beta+\beta^2} + \epsilon)$, since $u_1(x_{3,1}) < v_1$ the *altruistic* player will reject the proposal. When $\beta > \frac{1}{2}$ the selfish's continuation payoff increases since it becomes cheaper for either one of the Rawlsian players to convince its same type. This is what is causing the non-monotonicity of a selfish players expected utility.[11] Due to symmetry we can say the same for player 2. *Q.E.D.*

In a bargain free environment, when $\beta < 1$ players choose to stay with the entire dollar for themselves, regardless of their type. However, two salient aspects of this specific profile of players is that, for high values of β, strategic generosity arises.[12]. In other words, high values of β makes it costly to convince a *altruistic* player, thus the selfish type will offer a large individual payment to whomever he decides to convince. [13] Furthermore, since β is high, a Rawlsian will allocate the whole dollar

[11] See Appendix C.2, Figure 3, Case 5.

[12] See Appendix C.2, Figure 2, Case 2

[13] In Appendix C.2, Figure 4, case 5 we see that the function is increasing in β meaning that for high levels of β the selfish player has to sacrifice a larger share if his individual payment to give it too any of the other two players if he wants to try and convince the other to accept his proposal.

amongst all three, in other words, high values of β allows the *altruistic* proposer to exploit his awareness towards the least advantaged. This is precisely what explains the non-monotonicity of the offers at equilibrium for the Rawlsian type.[14]

Last, irrespective of the value of β, both *altruistic* players never place an offer to the selfish type since it will always be the case that the Rawlsian's maximum amount he is willing to offer is less than the minimum amount of shares the selfish player is willing to receive. Hence, an *altruist* never chooses to bargain with the selfish player.[15]. For instance, say it's player $1'$s turn to propose and $\frac{1}{2} \leqslant \beta < 0.645$. The minimum amount player $3'$s is willing to receive is $\underline{x}_1^3 \geqslant \frac{5-2\beta-\beta^2}{7-\beta^2}$. Thus, player 1 will deviate and make an offer only to player 3 if $1 - x_1^3 \geqslant \frac{3-\beta^2+2\beta}{7-\beta^2}$. But since the maximum amount player 1 is willing to offer is less than minimum $3'$s is willing to receive, the former will not bargain with player 3. We can argue the same for the two remaining two cases, that is, when $\beta < \frac{1}{2}$ and $0.645 \leqslant \beta < 1$.

Let us finally consider our last case, that is, when all three players are *altruistic*. In particular, this player profile has a subgame perfect Nash equilibrium in pure-ish strategies if,

$$u_1(x_{i,1}) \geqslant \frac{1}{3}[u_1(x_{2,1}) + u_1(x_{3,1}) + u_1(x_{1,i})],$$

for $i = \{2, 3\}$.

[14]In Appendix C.2, Figure 2, Case 2.

[15] Refer to Appendix G.

$$u_2(x_{i,2}) \geqslant \frac{1}{3}[u_2(x_{1,2}) + u_2(x_{3,1}) + u_2(x_{2,i})],$$

for $i = \{1, 3\}$.

$$u_3(x_{i,3} \geqslant \frac{1}{3}[u_3(x_{1,3}) + u_3(x_{2,3}) + u_3(x_{3,i})],$$

for $i = \{1, 2\}$.

By symmetry, $x_1^2 = x_2^3 = x_3^1$ and $x_1^3 = x_2^1 = x_3^2$. Let, $1 - x_i^j - x_i^k \geqslant x_i^j \geqslant x_i^k$ for $i, j, k = \{1, 2, 3\}$ for $i \neq j \neq k$, therefore,

$$x_i^j \geqslant \frac{1 - \beta x_i^k}{3 - \beta}$$

and

$$x_i^k \geqslant \frac{1 + x_i^j(\beta - 3)}{\beta}$$

By solving the system of inequalities: when $\beta < \frac{1}{2}$ and, say it's player $1'$s turn to make a proposal, she chooses to gain support from player 2 and offers, $x_{1,2} = (\frac{2-\beta}{3-\beta}, \frac{1}{3-\beta}, 0)$. In the case it's player $2'$s turn she offers, $x_{2,3} = (0, \frac{2-\beta}{3-\beta}, \frac{1}{3-\beta}$ to player 3. Last, player 3 will propose, $x_{3,1} = (\frac{1}{3-\beta}, 0, \frac{2-\beta}{3-\beta})$.

Conversely, if $\beta > \frac{1}{2}$, whoever it's turn it is to make a proposal, she chooses to offer the egalitarian allocation of the dollar, hence $x_{1,2} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $x_{2,3} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $x_{3,1} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and players 2, 3 and 1 accept, respectively.

So, the stationary subgame-perfect Nash equilibrium in pure strategies will be, *For player 1.*

$$((1,0),(\frac{2-\beta}{3-\beta},\frac{1}{3-\beta},0),\frac{1}{3}),$$

if $\beta \leqslant \frac{1}{2}$ and,

$$((1,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3}),\frac{1}{3}),$$

if $\frac{1}{2} < \beta \leqslant 1$.

For player 2.

$$((1,0),(0,\frac{2-\beta}{3-\beta},\frac{1}{3-\beta}),\frac{1}{3}),$$

if $\beta \leqslant \frac{1}{2}$ and,

$$((1,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3}),\frac{1}{3}),$$

if $\frac{1}{2} < \beta \leqslant 1$.

For player 3.

$$((1,0),(\frac{1}{3-\beta},0,\frac{2-\beta}{3-\beta}),\frac{1}{3}),$$

if $\beta \leqslant \frac{1}{2}$ and,

$$((1,0),(\frac{1}{3},\frac{1}{3},\frac{1}{3}),\frac{1}{3}),$$

if $\frac{1}{2} < \beta \leqslant 1$.

Definition **9**. *An egalitarian proposal from player* i *is an offer such that, all three players* $i, j,$ *and* k *receive, from* i*, the same individual payment, that is,* $x_{i,j} = x_{i,k} = (1 - x_{i,j} - x_{i,k}) = \frac{1}{3}$ *when* $\frac{1}{2} \leqslant \beta < 1$.

Definition **10**. *A selfish offer (II) from player* i *is an offer such that,* $x_{i,j} = \frac{1}{3-\beta}$, $x_{i,k} = 0$ *and* $(1 - x_{i,j}) = \frac{2-\beta}{3-\beta}$.

Proposition **20**. *Suppose all three players are Rawlsian, if the following hold,*

$$\beta < \frac{1}{2} \tag{3.6a}$$

$$\beta > \frac{1}{2} \tag{3.6b}$$

any one of the Rawlsian players make (i) a selfish offer (II) and (ii) an egalitarian offer, respectively.

Proof: If player 1 proposes, $x^*_{1,2} = (\frac{2-\beta}{3-\beta} - \epsilon, \frac{1}{3-\beta}, \epsilon)$ for $\epsilon > 0$, player 2 will accept since his gains, $\beta\epsilon$ will cancel out with his losses. However, player $1's$ losses, $(1-\beta)\epsilon$, exceed his gains, $\beta\epsilon$, hence, $u_1(x^*_{1,2}) < u_1(x_{1,2})$, which cannot be; by symmetry, neither player 2 or 3 will deviate.

Why will the proposer, regardless of whose turn it is to propose, jump to the egalitarian allocation when $\beta > \frac{1}{2}$? Say its player $1's$ turn to propose, and she offers $x^*_{1,2} = (\frac{2-\beta}{3-\beta} - \epsilon, \frac{1}{3-\beta}, \epsilon)$ as opposed to the egalitarian allocation. Her utility payoff is, $1 - (x^2_1 - \beta\epsilon) - \epsilon + \beta\epsilon = \frac{2-\beta}{3-\beta} + (2\beta - 1)\epsilon$. Since $\beta > \frac{1}{2}$ and players are not impatient to close a deal whoever is chosen to reach agreement will reject any offer $\frac{1}{3} - \epsilon$ therefore she's better off proposing the egalitarian allocation. *Q.E.D.*

3.4. Conclusions

Why does strategic generosity emerge only for certain player profiles? Is strategic generosity solely motivated by a players desire, regardless of their utility representation, to help others? Of course it isn't the pursuit of other player's well being and happiness what the proposer is trying to achieve, but the satisfaction his own desire for it.

Even though all types of players who make a proposal under the absence of negotiations choose the selfish allocation, by considering different arrangements of players we observe that, strategic generosity arises when each proposer, acting in her own interest, makes a proposal so that her decision is in accordance with a willingness to act in consideration of other players. For some composition of types, it wasn't obvious what was driving the proposers' actions to be in consonance with the interests of another.

We found the following preliminary results. First, when two Rawlsian players and one selfish player face negotiations, if $\beta < \frac{1}{2}$ a Rawlsian's proposal is aligned with that of a pure selfish player. Surely the proposer gains, if he were to offer symmetrical individual payments to the remaining players, isn't enough to compensate the costs he would have to pay for making such an offer, that is, $(1-\beta)x_1^2 > \beta x_1^2$ and $(1-\beta)x_2^1 > \beta x_2^1$, respectively. Second, as β increases and surpasses the threshold of $\frac{1}{2}$ both *altruistic* players become more costly to convince. We can understand this simply by flipping the previous argument. In which case, a Rawlsian proposer offers symmetrical individual payments to both players which explains the non-monotonicity present in Figure 3, Case 5. Moreover, for $\beta > 0.645$ the selfish player expected utility increases making it too costly for the other type to make him an offer. Therefore, provided it's the selfish types turn to place an offer he can no longer convince the the other type to accept, either a selfish (I) proposal, or a greedy proposal. It is by acting in her own interests that he takes into consideration other the *altruistic's* interests. Strangely enough, the selfish player turns out offering a greater amount of shares compared to what he stays for herself. In the first case, Rawlsian players behave seemingly in a selfish way, whereas in the second, selfish players do seemingly behave selflessly. The first finding explains the non-monotonicity of Figure 2, Case 2 of Appendix C.2. The last explains the behaviour of the offers at equilibrium of the selfish type in Figure 3, Case 5 of Appendix C.2.

Our model is a strategic model, and as such it embodies a detailed description of a special procedure which we have already specified, but of course, such procedure is only one of possibly many. Furthermore, we know our results may possibly depend vert strongly on the specification of our bargaining protocol, therefore, in an effort to overcome these potential weaknesses, a possible solution is to consider different bargaining protocols, as a first attempt to investigate if the bargaining procedure we have chosen is responsible of driving agents to behave generously. This, unfortunately we will leave for further research. While we have preliminary results for the the three player setup without discounting, further work is still to be done. For instance, to (i) introduce discounting, (ii) apply other bargaining protocols to the game in order identify if our results are strongly sensitive to the enforcement procedures (iii) generalize the model for n players and (iv) characterize the set of subgame perfect pure-ish Nash equilibrium.

Appendix A

On Refutability of the Nash Bargaining Solution

A.1. Appendix A: Strict Concavity and Differentiability of Preferences

The purpose of this appendix is to illustrate how the piecewise linear preferences that were constructed in the results of the paper can be transformed into strictly concave $\mathbf{C}^2$ functions. For the sake of simplicity in our presentation, we concentrate on the system of inequalities obtained in the first section of the paper, but the reader can readily check that the analysis extends to the rest of the cases considered. Also, since the construction is made individual by individual, here we simplify our notation by ignoring the super-index that identifies the agents that are considered in the paper.

That is, suppose that an analyst is given an array of numbers

$$\{(x_t, \nu_t, \mu_t, \lambda_t) : t = 1, \ldots, T\}$$

that satisfies the following system:

$$\mu_{t'} \leqslant \mu_t + \lambda_t(\mu_t - \nu_t)(x_{t'} - x_t), \tag{A.1}$$

with strict inequality if $x_{t'} \neq x_t$, $\mu_t > \nu_t$,and $\lambda_t > 0$.

As in Theorem 2 in Matzkin and Richter (1991), take a strictly convex function $h(x) \geqslant 0$ such that $h(x) = 0$ only at $x = 0$, and whose derivative is always less than 1. Given Eq. (1.30) and the fact that the number of observations is finite, one can construct functions

$$\nu_t(x) = \mu_t + \lambda_t(\mu_t - \nu_t)(x - x_t) - \epsilon_t h(x_t - x),$$

where ϵ_t is small enough that

$$\mu_{t'} \leqslant \mu_t + \lambda_t(\mu_t - \nu_t)(x_{t'} - x_t) - \epsilon_t h(x_t - x_{t'}),$$

with strict inequality whenever $x_t \neq x_{t'}$.

Now, define

$$u_0(x) = \min\{\nu_t(x) : t = 1, \ldots, T\}.$$

This function is continuous, strictly concave and strictly monotone. By construction, it also satisfies that $u_0(x_t) = \mu_t$ and that

$$u_0(x) - \nu_t \leqslant \mu_t + \lambda_t(\mu_t - \nu_t)(x - x_t) - \nu_t,$$

which is the inequality that is critical for the argument that the functions constructed in Section 1.2.1 rationalize the observed data.

It only remains to to show that the function can be further perturbed to smooth out its (finitely many) kinks. For this one can use a deformation like the one proposed by Chiappori and Rochet (1987). By construction, one can fix $\varepsilon > 0$ such that, for all t, $u(x) = \nu_t(x)$ whenever $|x - x_t| < \varepsilon$. Then, define the function $\rho : \mathbb{R} \to \mathbb{R}$ be defined as

$$\rho(\psi) = \begin{cases} \exp\left(-\frac{1}{\psi^2-1}\right)\left[\int_{\mathbb{R}} \exp\left(-\frac{1}{\mu^2-1}\right) d\mu\right]^{-1}, & \text{if } |\psi| \leqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the following mapping, which takes the convolution of u_0 and ρ:

$$u(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}} u_0(x - \psi)\rho\left(\frac{\psi}{\epsilon}\right) d\psi.$$

Chiappori and Rochet show that this function is $\mathbf{C}^{\infty}$, strictly concave and strictly increasing, and rationalizes the same maxima as u_0.

Appendix B

On Inter and Intra-Party Politics

B.1. Appendix B: Basic Nash Equilibria

Using the definitions of the different types of politicians in section 2, we will refer to a pair (a_L, a_R) as a *Nash equilibrium between militants* if

$$a_L \in \text{argmax}_a \nu_L(a) \text{ and } a_R \in \text{argmax}_a \nu_R(a).$$

The following result is an immediate implication of Weierstrass's Theorem, given that both ideology functions are continuous.

Proposition. *Suppose that the policy space, A, is compact. Then, a Nash equilibrium between militants exists.*

We say a *Nash equilibrium between opportunists* is a pair (a_L, a_R) of policies such that

$$a_L \in \text{argmax}_a \pi(a, a_R) \text{ and } a_R \in \text{argmax}_a [1 - \pi(a_L, a)].$$

The following result casts the prominent result of Duncan (1948) in the language of our paper. We state it without a proof.

Proposition (The Median Voter Theorem). *In this case, the (only) Nash equilibrium between opportunists is the pair $(a_L, a_R) = (a, a)$, for $a := F^{-1}(1/2)$.*

A *Nash equilibrium between pragmatist politicians* is a pair (a_L, a_R) of policies such that

$$a_R \in \text{argmax}_a \mu_R(a_L, a) \text{ and } a_L \in \text{argmax}_a \mu_L(a, a_R).$$

B.2. Appendix: No P.U.N.E with overshooting

Lemma 2. *There does not exist a party-unanimity Nash equilibrium (a_L, a_R) such that $\nu_L(a_R) > \nu_L(a_L)$, or that $\nu_R(a_L) > \nu_R(a_R)$.*

Proof: Suppose, by way of contradiction to our claim, that $v_L(a_R) > v_L(a_L)$. Consider first the case when $a_L < a_R$. For a small enough $\varepsilon > 0$, let $a = a_R - \varepsilon$. It follows from our assumptions that

$$v_L(a) > v_L(a_L), \pi(a, a_R) \geqslant \pi(a_L, a_R) \text{ and } \mu_L(a, a_R) > \mu_L(a_L, a_R),$$

where the first inequality follows by continuity of v_L, the second one from the fact that $a > a_L$, and the last one follows from the fact that

$$\lim_{\varepsilon \uparrow 0} \mu_L(a_R - \varepsilon, a_R) = v_L(a_R) > \mu_L(a_L, a_R),$$

given that function μ_L is continuous. These three inequalities contradict the fact that (a_L, a_R) is a party-unanimity Nash equilibrium.

When, on the other hand, $a_L > a_R$, if we let $a = a_R + \varepsilon$ for a small enough $\varepsilon > 0$, it is true that

$$v_L(a) > v_L(a_L), \pi(a, a_R) \geqslant \pi(a_L, a_R) \text{ and } \mu_L(a, a_R) > \mu_L(a_L, a_R),$$

where the latter inequality follows from the fact that

$$\lim_{\varepsilon \downarrow 0} \mu_L(a_R + \varepsilon, a_R) = v_L(a_R) > \mu_L(a_L, a_R).$$

Again, this is impossible.

The argument for the claim that $v_R(a_L) > v_R(a_R)$ cannot occur at equilibrium is similar. **Q.E.D.**

B.3. Appendix: The role of pragmatist politicians I

Note that one can re-write the expected value function considered by this faction of the left party, namely Equation (2.1), as

$$\mu_L(a_L, a_R) \equiv \pi(a_L, a_R)[v_L(a_L) - v_L(a_R)] + v_L(a_R). \tag{B.1}$$

By Lemma 2, for party-unanimity Nash equilibria we need only consider (a_L, a_R) such that $v_L(a_L) \geqslant v_L(a_R)$, and, then, for any policy $a \in A$ such that

$$v_L(a) \geqslant v_L(a_L) \text{ and } \pi(a, a_R) \geqslant \pi(a_L, a_R),$$

one immediately obtains that, in addition,

$$\mu_L(a, a_R) \geqslant \mu_L(a_L, a_R).$$

A similar argument applies to party R, using Equation (2.2), which allows us to state the following result.

Proposition 21. *A pair of policies (a_L, a_R) is a party-unanimity Nash equilibrium if, and only if,*
(i) there does not exist a policy $a \in A$ such that

$$v_L(a) \geqslant v_L(a_L) \text{ and } \pi(a, a_R) \geqslant \pi(a_L, a_R),$$

with at least one of the two inequalities being strict; and
(ii) there does not exist a policy $a \in A$ for which

$$v_R(a) \geqslant v_R(a_R) \text{ and } \pi(a_L, a) \leqslant \pi(a_L, a_R),$$

with at least one strict inequality.

B.4. Appendix: Nash Equilibria, PUNE and N.B.S

Nash Equilibria

For the sake of completeness, here we report the Nash equilibria that the game would have if each of the parties consisted of only one type of politicians (the same type in both parties).

The only Nash equilibrium between Militants is straightforward: in each party, it is a strictly dominant strategy to play its most extreme policy, so the possible Nash equilibrium is the policy pair $(a_L, a_R) = (0, 1)$.

Although the support function π that we are now using is not founded by an underlying distribution of voters' preferences, an analogous to the Median Voter Theorem nonetheless holds. To see that this is the case, note first that there can be no Nash equilibrium between opportunists' where the two parties play different policies. If this were the case, namely is $a_L \neq a_R$, one of the two parties (at least) would be making suboptimal decisions: if $\pi(a_L, a_R) = 1$, then party R would prefer, for instance to play policy $a = a_L$; if $\pi(a_L, a_R) = 0$, analogously, party L would prefer $a = a_R$; if $0 < \pi(a_L, a_R) < 1$, since $b > 0$, both parties would prefer to play policies that are closer to the one of the opposing party.

It follows that in order to find Nash equilibria between opportunists we only need to consider policy pairs of the type $(a_L, a_R) = (\tilde{a}, \tilde{a})$. Now, notice that if $2b\tilde{a} - C < 1/2$, we have that each party would be better off by deviating to a different policy: for instance, if $a_R = a$, then, if party L deviates to $a = \tilde{a} + \varepsilon$, for small enough $\varepsilon > 0$,

$$\pi(a, a_R) = 1 - 2b\tilde{a} - b\varepsilon + C > \frac{1}{2} = \pi(a_L, a_R).$$

A similar argument allows the dismissal of pairs $(a_L, a_R) = (\tilde{a}, \tilde{a})$ such that $2b\tilde{a} - C > 1/2$, so the only possible equilibrium is at

$$(a_L, a_R) = \left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right).$$

Now, that this is, indeed, a Nash equilibrium between opportunists is straightforward again: function $\pi(a, \tilde{a})$, which is continuous as a function of a, attains its global maximum at $a = \tilde{a}$.

The only Nash equilibrium between pragmatists is the policy pair

$$(a_L, a_R) = \left(\frac{C}{2b}, \frac{1+C}{2b}\right).$$

For the sake of presentation, we argue this statement as proof of the following proposition.

Proposition 22. *In the simplified version of the game we are now considering, the only Nash equilibrium between Militants is the policy pair* $(a_L, a_R) = (0, 1)$, *the only equilibrium between opportunists is*

$$(a_L, a_R) = \left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right),$$

and the only equilibrium between pragmatists is

$$(a_L, a_R) = \left(\frac{C}{2b}, \frac{1+C}{2b}\right).$$

Proof: It only remains to prove the third statement. Note first that there can be no equilibrium (a_L, a_R) where $\pi(a_L, a_R) > 0$ and $a_L > a_R$, for, in such case, if the left played $a = a_R$

$$\mu_L(a, a_R) = 1 - a_R > \pi(a_L, a_R)(1 - a_L) + [1 - \pi(a_L, a_R)](1 - a_R) = \mu_L(a_L, a_R).$$

We can similarly rule out (a_L, a_R) where $\pi(a_L, a_R) < 1$ and $a_L > a_R$, using an argument for the Right party.

Next, consider the case when $(a_L, a_R) = (\tilde{a}, \tilde{a})$. Suppose first that $\tilde{a} > C/2b$. Then, by using $a = \tilde{a} - \varepsilon$, for $\varepsilon > 0$ small enough, the Left party would have $\pi(a, a_R) > 0$ and

$$\mu_L(a, a_R) = \pi(a, \tilde{a})(1 - a) + [1 - \pi(a, \tilde{a})](1 - \tilde{a}) > 1 - \tilde{a} = \pi(a_L, a_R).$$

If, on the other hand, $\tilde{a} \leqslant C/2b$, then, playing $a = \tilde{a} + \varepsilon$, for $\varepsilon > 0$ small enough, the Right party would have

$$\mu_R(a_L, a) = \pi(\tilde{a}, a)\tilde{a} + [1 - \pi(\tilde{a}, a)]a > \tilde{a} = \pi(a_L, a_R).$$

Now, consider the case in which $a_L < a_R$ and $\pi(a_L, a_R) = 0$. Suppose first that $a_R > C/2b$. In this case, for any

$$\frac{c}{b} - a_R < a < a_R,$$

we would have that $\pi(a, a_R) > 0$, and, hence, that

$$\mu_L(a, a_R) = \pi(a, a_R)(1 - a) + [1 - \pi(a, a_R)](1 - a_R) > 1 - a_R = \mu_L(a_L, a_R),$$

so the Left party would not be best-responding. On the other hand, note that if $a_R \leqslant C/2b$, then $a_L < C/2b$ and then, letting $a = C/b - a_L$ we would have that $a > a_R > a_L$ and $\pi(a_L, a) = 0$, so that

$$\mu_R(a_L, a) = a > a_R = \mu_R(a_L, a_R),$$

so the Right party would not be best-responding.

As before, a similar argument allows us to rule out the possibility that $a_L < a_R$ and $\pi(a_L, a_R) = 1$ at equilibrium.

The only policy profiles that can be Nash equilibrium between pragmatists, then, must have $0 < \pi(a_L, a_R) < 1$ and $a_L \leqslant a_R$. With $0 < \pi(a, a_R) < 1$ and considering $a < a_R$ only, we can write

$$\mu_L(a, a_R) = [b(a + a_R) - C](a_R - a) + (1 - a_R),$$

which is maximized at $a_L = C/2b$. Similarly, if $0 < \pi(a_L, a) < 1$ and considering $a > a_L$ only,

$$\mu_R(a_L, a) = [b(a_L + a) - C](a_L - a) + a,$$

which is maximized at $a_R = (1 + C)/2b$. It follows that the only potential candidate for Nash equilibrium is the pair

$$(a_L, a_R) = \left(\frac{C}{2b}, \frac{1 + C}{2b}\right).$$

That these two policies constitute a pair of mutual *local* best responses is immediate from their definitions. To see that these are actually global best responses, note first that if $\pi(a, a_R) = 0$, or when $a = a_R$, then

$$\mu_L(a, a_R) = \frac{2b - 1 - C}{2b} < \frac{1}{2} = \mu_L(a_L, a_R),$$

where the inequality holds because, by assumption, $C \geqslant 2b-2 > b-1$. Obviously, if $a > a_R$, then $\mu_L(a, a_R) < 1/2 = \mu_L(a_L, a_R)$. For the Right, note that if $\pi(a_L, a) = 1$, or when $a = a_L$, then

$$\mu_R(a_L, a) = \frac{C}{2b} < \frac{1}{2} = \mu_R(a_L, a_R),$$

where the inequality comes from the fact that, by assumption, $C < 2b-1$ and $C < 1$, which suffices to imply that $C < b$. Again, if $a < a_L$, then $\mu_R(a_L, a) < 1/2 = \mu_R(a_L, a_R)$. **Q.E.D.**

Figure B.1 illustrates the positions of the three basic equilibria in the space of policy pairs. A particular feature of the functional forms we have chosen is that in both Nash equilibria played between opportunists and between pragmatists the two parties obtain one half of the popular support:

$$\pi\left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right) = b\left(\frac{1+2C}{4b} + \frac{1+2C}{4b}\right) - C = \frac{1}{2},$$

while

$$\pi\left(\frac{C}{2b}, \frac{1+C}{2b}\right) = b\left(\frac{C}{2b} + \frac{1+C}{2b}\right) - C = \frac{1}{2}.$$

The same need not be true in the Nash equilibrium between militants, as $\pi(0, 1) = b - C$.

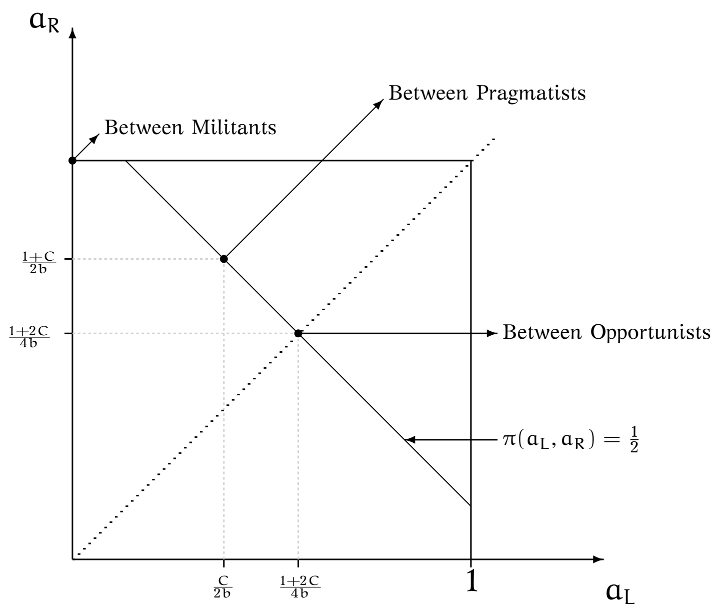


Figure B.1: The basic Nash equilibria of the simple game.

Party-unanimity Nash equilibria

We now compute the set of *all* Roemer equilibria for the simplified version of the game. Some previous results allow us to simplify this task: by Lemma 2 (See Appendix B.2), we know that we only need to consider policy pairs (a_L, a_R) where $a_L \leqslant a_R$; by Proposition 21, given a policy pair (a_L, a_R) we only need to check that for no a it is true that $\nu_L(a) \geqslant \nu_L(a_L)$ and $\pi(a, a_R) \geqslant \pi(a_L, a_R)$, with at least one strict inequality, nor $\nu_R(a) \geqslant \nu_R(a_R)$ and $\pi(a_L, a) \leqslant \pi(a_L, a_R)$, with one strict inequality at least.

Lemma 3. *The only party-unanimity Nash equilibrium at which* $\pi(a_L, a_R) = 0$ *is*

$$(a_L, a_R) = \left(0, \frac{C}{b}\right).$$

Proof: We first argue that $(a_L, a_R) = (0, C/b)$ is indeed an equilibrium. As 0 is the only solution to program

$$\max_{a \in A} v_L(a),$$

it follows that there can be no $a \neq 0$ for which $v_L(a) \geqslant v_L(a_L)$. As for the Right, any $a < C/b$ would imply $v_R(a) < v_R(a_R)$, while for any $a > C/b$ one would have

$$\pi(a_L, a) = \pi(0, a) = ba - C > b\frac{C}{b} - C = 0 = \pi(a_L, a_R).$$

Now, consider any pair $(a_L, a_R) \neq (0, C/b)$ such that $\pi(a_L, a_R) = 0$. If $a_L > 0$, letting $a = 0$ we get $v_L(a) > v_L(a_L)$ and $\pi(a, a_R) \geqslant \pi(a_L, a_R)$, so if this policy pair is going to be an equilibrium it must be that $a_L = 0$. If $a_R < C/b$, then, by playing $a = C/b$ the Right would have $v_R(a) > v_R(a_R)$ and $\pi(a_L, a) = 0 \leqslant \pi(a_L, a_R)$. Alternatively, it must be true that $a_R > C/b$, but this is impossible, since in such case

$$\pi(a_L, a_R) = \pi(0, a_R) = ba_R - C > b\frac{C}{b} - C = 0,$$

which is a contradiction. **Q.E.D.**

A similar argument, *mutatis mutandis*, gives us the following result.

Lemma 4. *The only party-unanimity Nash equilibrium at which* $\pi(a_L, a_R) = 1$ *is*

$$(a_L, a_R) = \left(\frac{1 + C - b}{b}, 1\right).$$

Proof: In the same vein we argue that $\left(\frac{1+C-b}{b}, 1\right)$ is an equilibrium. Since 1 is the only solution to program

$$\max_{a \in A} v_R(a)$$

there can be no $a \neq 1$ for which $v_R(a) > v_R(a_R)$. In the case of party L, any $a > \frac{1+C-b}{b}$ would imply that $v_L(a) < v_L(a_L)$, and for $a < \frac{1+C-b}{b}$ we have,

$$\pi(a, a_R) = \pi(a, 1) = b(1 + a) - C < 1 = \pi(a_L, a_R)$$

Now, let us consider any pair $(a_L, a_R) \neq (\frac{1+C+b}{b}, 1)$ such that $\pi(a_L, a_R) = 1$. If $a_R < 1$, letting $a = 1$, we get that $v_R(a) > v_R(a_R)$ and $\pi(a_L, a) \leqslant \pi(a_L, a_R)$, so, if the policy pair is an equilibrium, it must be that $a_R = 1$. If $a_L > \frac{1+C-b}{b}$, then by L playing $a = \frac{1+C-b}{b}$, the party would get $v_L(a) > v_L(a_L)$ and $\pi(a, a_R) \geqslant \pi(a_L, a_R)$. So, it must be true that $a_L < \frac{1+C-b}{b}$, but,

$$\pi(a_L, a_R) = \pi(a_L, 1) = b(a_L + 1) - C < b\left(1 + \frac{1 + C + b}{b}\right) - C = 1$$

which is a contradiction. **Q.E.D.**

Lemma 5. *The only party-unanimity Nash equilibrium at which* $a_L = a_R$ *is*

$$(a_L, a_R) = \left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right).$$

Proof: Note first that $(a_L, a_R) = (0, 0)$ cannot be an equilibrium: for $\varepsilon > 0$ small enough, since $C > 0$, letting $a = \varepsilon$ the Right

$$\pi(a_L, a) = \max\{0, b\varepsilon - C\} = 0 < \frac{1}{2} = \pi(a_L, a_R),$$

while $\nu_R(a) > \nu_R(a_R)$. A similar argument allows us to rule out the possibility that $(a_L, a_R) = (1, 1)$ be an equilibrium.

Now, fix any $0 < a < 1$ and consider the profile $(a_L, a_R) = (a, a)$. If $\lim_{a\uparrow a} \pi(a, a) > 1/2$, then, by playing $a = a - \varepsilon$, for $\varepsilon > 0$ small enough, the Left would have that $\pi(a, a_R) > 1/2 = \pi(a_L, a_R)$ while $\nu_L(a) > \nu_L(a_L)$, so the pair cannot be an equilibrium. Alternatively, if $\lim_{a\uparrow a} \pi(a, a) < 1/2$, we must have, by construction, that $\lim_{a\downarrow a} \pi(a, a) < 1/2$, in which case, by playing $a = a + \varepsilon$, for $\varepsilon > 0$ small enough, the Right would have that $\pi(a_L, a) < 1/2 = \pi(a_L, a_R)$ while $\nu_R(a) > \nu_R(a_R)$.

The only case left is when $\lim_{a\to a} \pi(a, \tilde{a}) = 1/2$. The only point at which this occurs is

$$\tilde{a} = \frac{1+2C}{4b}.$$

As in the case of the Nash equilibrium between Opportunists, if we observe that, as a function of a, $\pi(a, \tilde{a})$ attains its maximum precisely at $\tilde{a}$, we conclude that $(a_L, a_R) = (\tilde{a}, \tilde{a})$ is a Roemer equilibrium: any other a would cause $\pi(a, a_R) < 1/2 = \pi(a_L, a_R)$ and $\pi(a_L, a) > 1/2 = \pi(a_L, a_R)$. **Q.E.D.**

These three lemmas have considered «corner» policy pairs. We can rule out pairs with $a_L > a_R$ as equilibria, either by appealing to the general claim made in Lemma 2 (see Appendix C), or by the following straightforward observations: if $a_L > a_R$ and $\pi(a_L, a_R) \leqslant 1/2$, then by playing $a = a_R$, the Left would get $\nu_L(a) > \nu_L(a_L)$ and $\pi(a, a_R) = 1/2 \geqslant \pi(a_L, a_R)$; if, alternatively, $a_L > a_R$ and $\pi(a_L, a_R) > 1/2$, then, by playing $a = a_L$, the Right would get $\nu_R(a) > \nu_R(a_R)$ and $\pi(a_L, a) = 1/2 < \pi(a_L, a_R)$.

The following lemma completes the analysis.

Lemma 6. *Any policy pair* (a_L, a_R) *such that* $a_L < a_R$ *and* $0 < \pi(a_L, a_R) < 1$ *is a party-unanimity Nash equilibrium.*

Proof: Consider any such policy pair (a_L, a_R), and take any $a \in A$. If $a < a_L$, then $\pi(a, a_R) < \pi(a_L, a_R)$, while if $a > a_L$, $\nu_L(a) < \nu_L(a_L)$. Similarly, $a > a_R$, then $\pi(a_L, a) > \pi(a_L, a_R)$, while if $a < a_R$, $\nu_R(a) < \nu_R(a_R)$. **Q.E.D.**

For the sake of clarity in our presentation, we summarize these results in the following proposition, and illustrate the set of Roemer equilibria in Figure B.2.

Proposition 23. *In the simplified version of the game we are now considering, the only party-unanimity Nash equilibria are the policy pairs*

$$\left(0, \frac{C}{b}\right), \left(\frac{1+C-b}{b}, 1\right) \text{ and } \left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right),$$

as well as any policy pair (a_L, a_R) *such that*

$$a_L < a_R \ \textit{and} \ 0 < \pi(a_L, a_R) < 1.$$

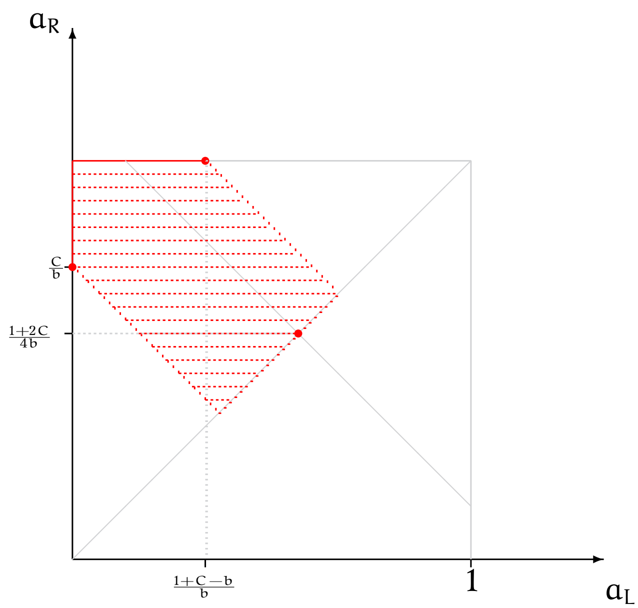


Figure B.2: The set of party-unanimity Nash equilibria of the simple game is the shaded area. It only includes its boundaries where these are drawn in a continuous line, and it contains, in particular, the three dots.

It is immediate, either from comparison of Propositions 22 and 23 or from comparison of Figures B.1 and B.2, that each of the simple Nash equilibria of the game is also a Roemer equilibrium.

Nash bargaining solutions

Using the observation of §2.3.3, we replace Eqs. (2.3) and (2.4) by the following definition: we say that a policy pair (a_L, a_R) is a Nash bargaining solution if there exist nonnegative numbers α_L, β_L, α_R, β_R such that $\alpha_L + \beta_L \leqslant 1$ and $\alpha_R + \beta_R \leqslant 1$, for which policy a_L solves program

$$\max_a [\pi(a, a_R)^{1-\beta_L}(a_R - a_L)^{1-\alpha_L}],$$

while policy a^R solves program

$$\max_a \{[1 - \pi(a_L, a)]^{1-\beta_R}(a_R - a_L)^{1-\alpha_R}\}.$$

Focusing on policy pairs that satisfy $a_L \leqslant a_R$, we can rewrite these programs by requiring that a_L solve

$$\max_a [\pi(a, a_R)^{\rho_L}(a_R - a_L)], \tag{B.2}$$

and that a^R solve

$$\max_a \{[1 - \pi(a_L, a)]^{\rho_R}(a_R - a_L)\}, \tag{B.3}$$

where

$$\rho_L := \frac{1-\beta_L}{1-\alpha_L} \text{ and } \rho_R := \frac{1-\beta_R}{1-\alpha_R}.$$

In this simpler notation, the values of ρ_i that we can allow are any number in $\mathbb{R}_+$, as well as ∞. If $\rho_i = 0$, the militants of party i have no say in the negotiations of their party. When it is the opportunists that have no bargaining power, we use $\rho_i = \infty$ to denote that the party's program is, respectively,

$$\max_a \pi(a, a_R) \text{ or } \max_a [1 - \pi(a_L, a)]. \tag{B.4}$$

Interior equilibria

Consider first the case in which $0 < \rho_i < \infty$ for both parties. We want to find all the equilibria for which $a_L < a_R$ and $0 < \pi(a_L, a_R) < 1$, to which we refer as «interior».

Ignore, for a moment, the constraints that $a_L \geqslant 0$ and $a_R \leqslant 1$, which we shall impose momentarily. If $\rho_L \geqslant 1$, the best-response function defined by Program (B.2) is, by direct computation,

$$a_L(a_R) = \frac{C}{b(\rho_L+1)} + \frac{\rho_L - 1}{\rho_L + 1} a_R, \tag{B.5}$$

for all $0 < a_R \leqslant 1$. When $\rho_L < 1$, the constraint that $a_L < a_R$ may bind, and the best-response function is

$$a_L(a_R) = \begin{cases} 0, & \text{if } a_R < \frac{C}{b(1-\rho_L)}; \\ \frac{C}{b(\rho_L+1)} + \frac{\rho_L-1}{\rho_L+1} a_R, & \text{otherwise.} \end{cases} \tag{B.6}$$

Similarly, the best-response function defined by Program (B.3), when $\rho_R \geqslant 1$ is

$$a_R(a_L) = \frac{1+C}{b(\rho_R+1)} + \frac{\rho_R - 1}{\rho_R + 1} a_L,$$

for all $0 \leqslant a_L < 1$. Or, if $\rho_R < 1$, it is

$$a_R(a_L) = \begin{cases} 1, & \text{if } a_L > \frac{1+C-b(\rho_R+1)}{b(1-\rho_R)}; \\ \frac{1+C}{b(\rho_R+1)} + \frac{\rho_R-1}{\rho_R+1} a_L, & \text{otherwise.} \end{cases}$$

Consider first the case where $\rho_L > 1$ and $\rho_R > 1$. Using the two best-response functions given above, we find that their intersection occurs when

$$a_L = \frac{C}{2b} + \frac{\rho_L - 1}{2b(\rho_L + \rho_R)} \text{ and } a_R = \frac{1+C}{2b} + \frac{1-\rho_R}{2b(\rho_L+\rho_R)}. \tag{B.7}$$

Now, for this policy pair to be a bona fide equilibrium, it must satisfy the constraints that $a_L \geqslant 0$ and $a_R \leqslant 1$. By direct computation, these constraints are satisfied if the following condition holds:

$$\rho_R \geqslant \max\left\{\frac{1}{C} - \rho_L \frac{1+C}{C}, \frac{1}{2b-C} + \rho_L\left(\frac{1}{2b-C} - 1\right)\right\}. \tag{B.8}$$

Figure B.3 illustrates the values of (ρ_L, ρ_R) for which Condition (B.8) is satisfied. It also defines some areas and points in the space of (ρ_L, ρ_R) that will be useful to express all the equilibria of the game.

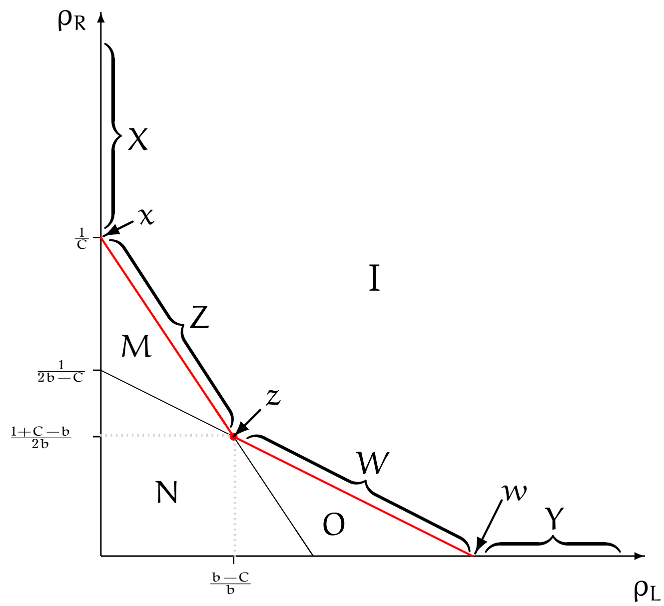


Figure B.3: Condition B.8 holds at any point in the area above **both** downward-sloping lines (that is, at the union of areas I, X, Z, W and Y).

It is immediate from Eq. (B.7) that these interior Nash bargaining solutions amount to perturbations

$$\frac{1}{2b}(\Delta_L, \Delta_R)$$

to the Nash equilibrium between pragmatists, these perturbations being of the form

$$\Delta_L = \frac{\rho_L - 1}{\rho_L + \rho_R} \text{ and } \Delta_R = \frac{1 - \rho_R}{\rho_L + \rho_R}.$$

In order to recover the set of all these solutions, we can then study the set of pairs (Δ_L, Δ_R) that are induced by (ρ_L, ρ_R) in the union of areas I, Z and W in Figure B.3 – namely, pairs (ρ_L, ρ_R) that satisfy Condition (B.8). This space of (Δ_L, Δ_R) is illustrated in Figure B.4, where we also associate regions of the space with the areas of (ρ_L, ρ_R) that induce them.

In order to find the set of interior Nash bargaining solutions, now we simply need to add these perturbations (rescaled by 1/2b) to the policy pair

$$(a_L, a_R) = \left(\frac{C}{2b}, \frac{1+C}{2b}\right).$$

This is done in Figure B.5.

By direct computation, if we consider pairs (ρ_L, ρ_R) that violate Condition B.8, then the constraints that $a_L \geqslant 0$ and $a_R \leqslant 1$ bind, and the associated Nash bargaining solutions lie in the boundary of the space of interior solutions constructed before. These cases correspond to the areas marked M, N and O in Figure B.3, and the set of solutions generated by them is illustrated in Figure B.5, by marking the areas they induce.

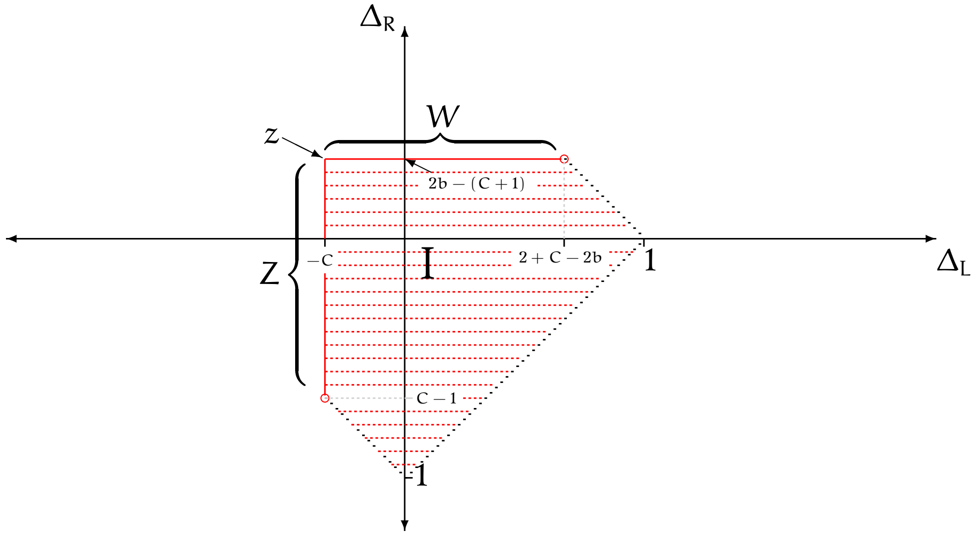


Figure B.4: The space of perturbations (Δ_L, Δ_R) induced by (ρ_L, ρ_R) in the union of areas I, Z and W in Figure B.3 is the shaded area. The labels indicate the area of Figure B.3 that induces each region.

Importantly, comparison of Figures B.2 and B.5 shows that all the interior Nash bargaining solutions are party-unanimity Nash equilibria, which verifies the observations of §3.3.2.

Corner solutions

We now compute the Nash bargaining solutions for the cases when some ρ_i is 0 or ∞, as well as for the case where $a_L = a_R$ at the solution. We refer to these cases as «corner» solutions. We treat all these cases independently.

Case 1: If $\rho_L = \rho_R = 0$. In this case, the Nash bargaining solution corresponds to the unique Nash equilibrium between militants, namely $(a_L, a_R) = (0, 1)$.

Case 2: If $\rho_L = 0$ **and** $\rho_R \in \mathbb{R}_{++}$. In this case, the only solution to Program (B.2) is $a_L = 0$. On the other hand, given $a_L = 0$, the solution to Program (B.3) is

$$a_R = \begin{cases} \min\left\{\frac{1+C}{b(\rho_R+1)}, 1\right\}, & \text{if } \rho < \frac{1}{C}; \\ \frac{C}{b}, & \text{otherwise.} \end{cases}$$

The second condition in the expression above corresponds to point x and area X in Figure B.3; all these points generate the policy pair $(a_L, a_R) = (0, C/b)$ as Nash bargaining solution. The first condition in the expression generates the set $\{0\} \times (C/b, 1]$ of policy pairs. Of course, the case $\rho_L \in \mathbb{R}_{++}$ **and** $\rho_R = 0$ is analogous, and generates the set

$$\left[0, \frac{1+C-b}{b}\right] \times \{1\}$$

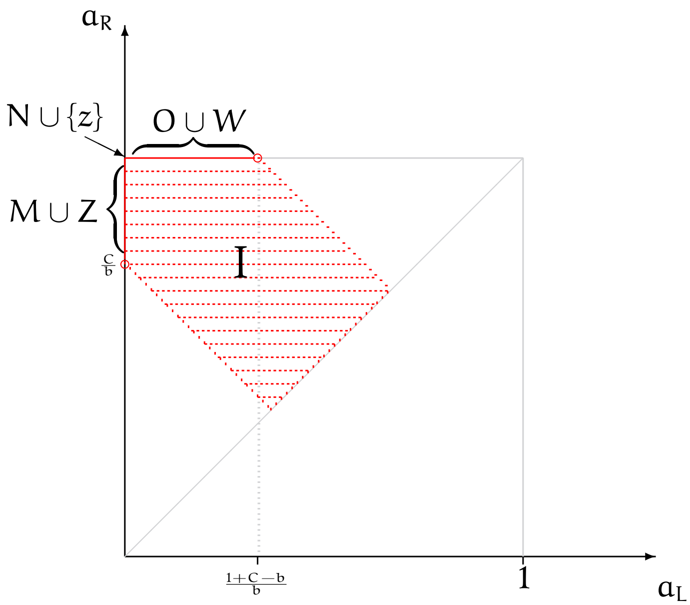


Figure B.5: The set of interior Nash bargaining solutions of the simple game is the shaded area. It only includes its boundaries where these are drawn in a continuous line; it does not contain the two dots. The labels indicate the area of Figure B.3 that induces each region.

of policy pairs as Nash bargaining solutions.[1]

Case 3: If $\rho_L = \infty$ and $\rho_R = \infty$. In this case, the only Nash bargaining solution is the Nash equilibrium between opportunists,

$$(a_L, a_R) = \left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right).$$

It follows also by construction that this policy pair is the only solution with $a_L = a_R$.

As in the case of interior solutions, the Nash bargaining solutions generated by pairs (ρ_L, ρ_R) are all Roemer equilibria. If we add these solutions to the set of interior solutions obtained before, we enrich Figure B.5 as in Figure B.6. Importantly, all these solutions are, at the same time, party unanimity Nash equilibria. The same property will not hold for the two remaining cases, which we consider next.

Case 4: If $\rho_L = 0$ and $\rho_R = \infty$. In this case, for the Left the only solution to Program (B.2) is to play policy $a_L = 0$, whatever the Right does. Given this, the set of solutions to Program (B.4) for the Right is any $a_R \in (0, c/b]$, so that the set $\{0\} \times (0, c/b]$ of policy pairs

[1] Note that the correspondence defined by the Nash bargaining solution as one varies the value of (ρ_L, ρ_R) is not upper hemi-continuous: the sequence of policy pairs generated by a sequence of pairs (ρ_L, ρ_R) that converges to a point in area X of Figure B.3 would converge to some point in the downward-sloping dashed line that starts at point $(0, C/b)$ in Figure B.5; any point in area X, on the other hand, would generate $(0, C/b)$.

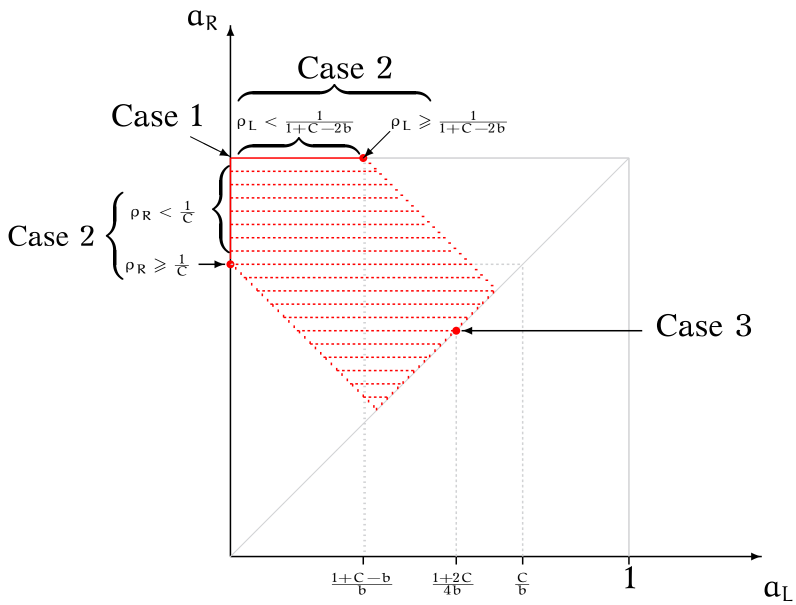


Figure B.6: The set of corner Nash bargaining solutions of the simple game, corresponding to cases 1-3, adds the three dots to the shaded area. The boundaries at $a_L = 0$ and $a_R = 1$ are generated by further values of (ρ_L, ρ_R) (that is, in addition to those that gave these boundaries as interior solutions).

also constitutes Nash bargaining solutions for the game. Similarly, the case when $\rho_L = \infty$ and $\rho_R = 0$ generates the set

$$\left[\frac{1}{1+C-2b}, 1\right) \times \{1\}$$

of policy pairs as Nash bargaining solutions. These sets, as well as those solutions generated by the next case, are illustrated in Figure B.8.

Case 5: If $\rho_L \in \mathbb{R}_{++}$ **and** $\rho_R = \infty$. In this case, the Right is represented only by opportunists, and the relevant maximization program is given by Eq. (B.4). By direct computation, the correspondence defined by the solution to that program is

$$a_R(a_L) = \begin{cases} \left[a_L, \frac{C}{b} - a_L, \right), & \text{if } a_L \leqslant \frac{C}{2b}; \\ \emptyset, & \text{if } \frac{C}{2b} < a_L \geqslant \frac{1+2C}{4b}; \\ \left\{\frac{1+2C}{4b}\right\}, & \text{if } a_L = \frac{1+2C}{4b}; \\ \emptyset, & \text{if } \frac{1+2C}{4b} < a_L < \frac{1+C}{2b}; \\ \left[\frac{1+C}{b} - a_L, a_L\right), & \text{if } a_L \geqslant \frac{1+C}{2b}. \end{cases} \tag{B.9}$$

For the Left, the relevant program is given by Eq. (B.2), and its maximizer by Eqs. (B.5) or (B.6), depending on the value of ρ_L. These correspondences are depicted in Figure B.7. It is immediate from the Figure that the set

$$\{(a_L, a_R) : 0 \leqslant a_L < a_R \leqslant \frac{C}{2b}\}$$

of policy pairs is the set of Nash bargaining solutions generated by this case. The analogous case when $\rho_L = \infty$ and $\rho_R \in \mathbb{R}_{++}$ generates the set

$$\{(a_L, a_R) : \frac{1+C}{2b} \leqslant a_L < a_R \leqslant 1\}$$

as Nash bargaining solutions.

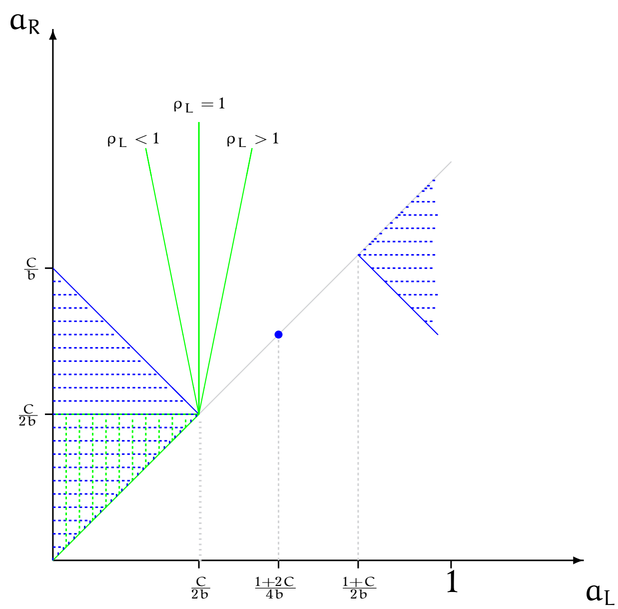


Figure B.7: The maximizer correspondences (B.9) and (B.5) or (B.6). The intersection of the two graphs gives the Nash bargaining solutions corresponding to Case 5.

Figure B.8 depicts the Nash bargaining solutions generated by Cases 4 and 5. Importantly, most of these policy pairs are not Roemer equilibrium. For case 4, one could refine the Nash bargaining solutions, by requiring that each party's policy satisfy the «efficiency» conditions of Proposition 23, in which case the only surviving policy pairs would be

$$\left(0, \frac{C}{b}\right) \text{ and } \left(\frac{1+C-b}{b}, 1\right),$$

which are, indeed, Roemer equilibria. But in Case 5, it follows that the same refinement would fail to generate a solution, for the refinements of correspondences (B.9) and (B.5) or (B.6) would not intersect.

We summarize all these findings in the following proposition.

Proposition 24. *In the simplified version of the game we are now considering,*

1. *the set of all the Roemer equilibria is the set of Nash bargaining solutions generated by finite ρ_L and ρ_R;*
2. *when $\rho_i = \infty$ for one of the parties, the induced Nash bargaining solutions need not be Roemer equilibria;*

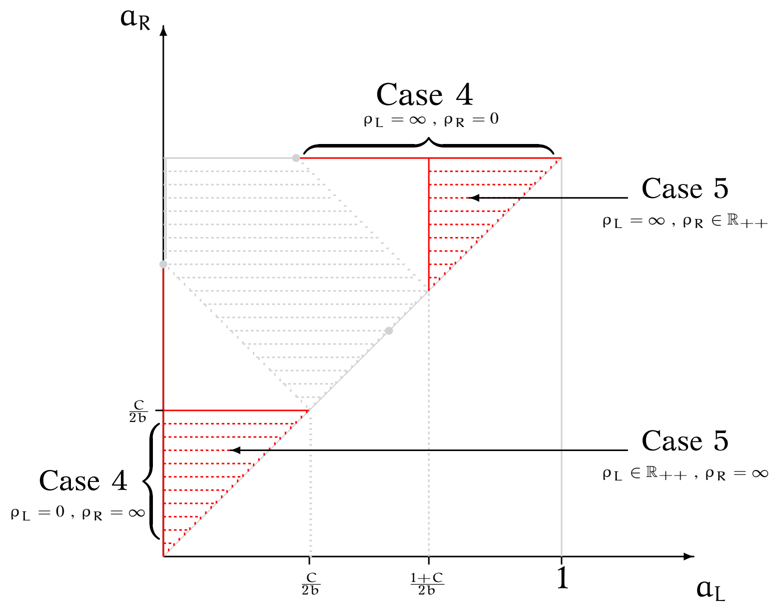


Figure B.8: The set of corner Nash bargaining solutions of the simple game corresponding to cases 4 and 5 adds the shaded area.

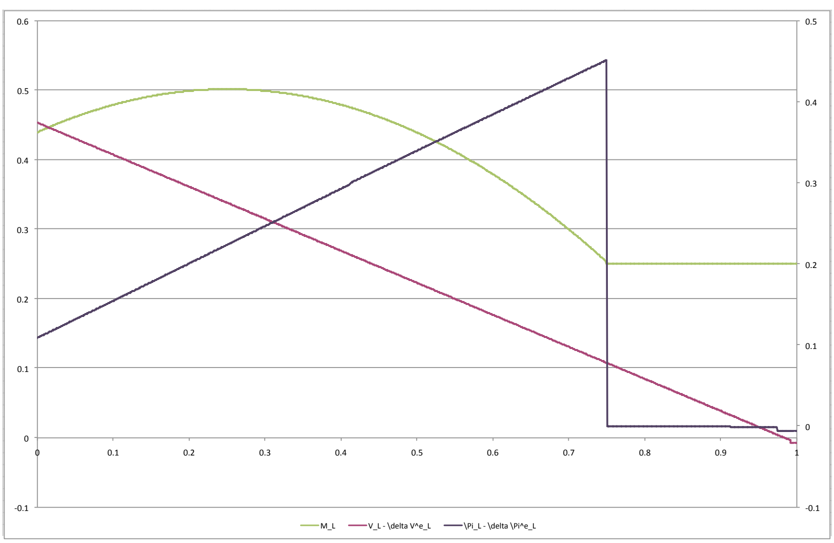
3. *in any case,* $a_L < a_R$ *holds at all Nash bargaining solutions, with the exception of policy pair*

$$\left(\frac{1+2C}{4b}, \frac{1+2C}{4b}\right).$$

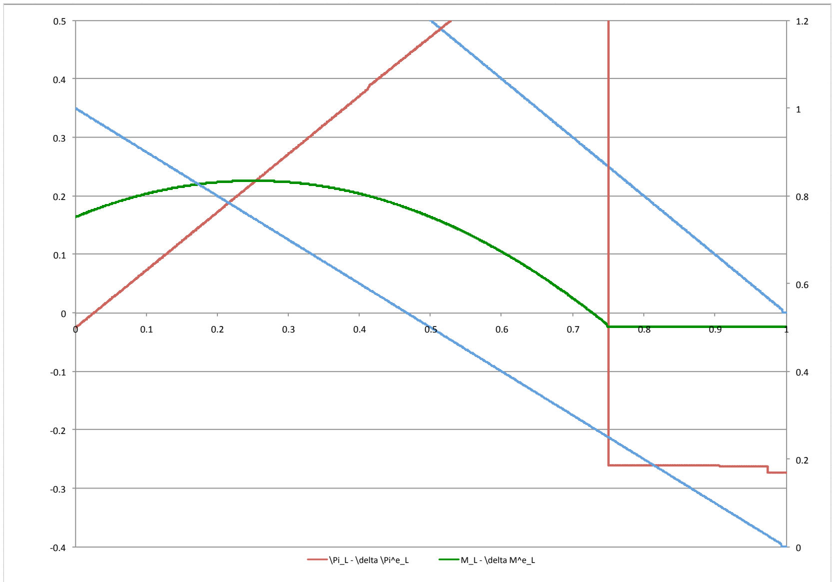
Appendix C

A bargaining model with Strategic Generosity

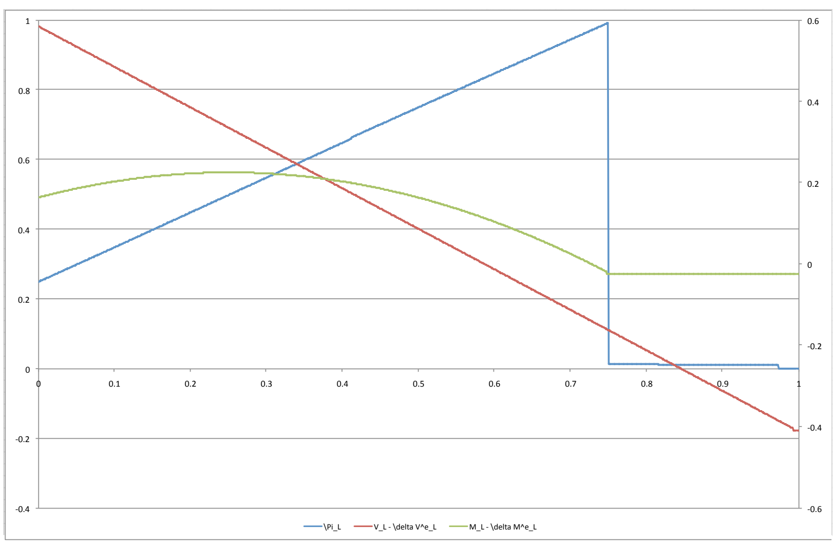
C.1. Appendix: Pragmatists, Militants and Opportunists



Graph for Pragmatists



Graph for Militants



Graph for Opportunists

C.2. Appendix: Offers at equilibrium

Figure. 1

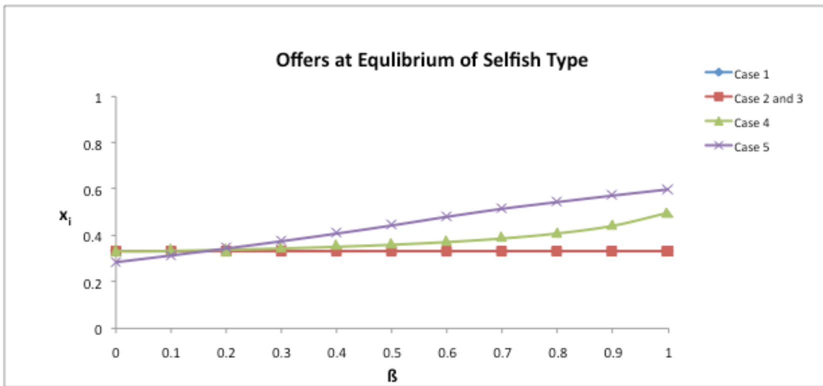


Figure. 2

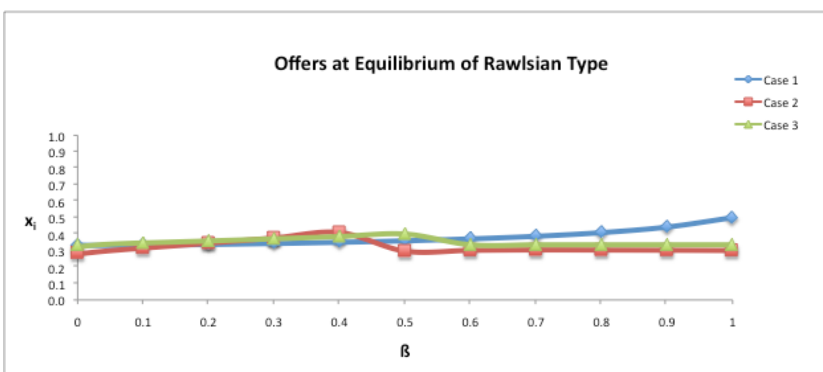


Figure. 3

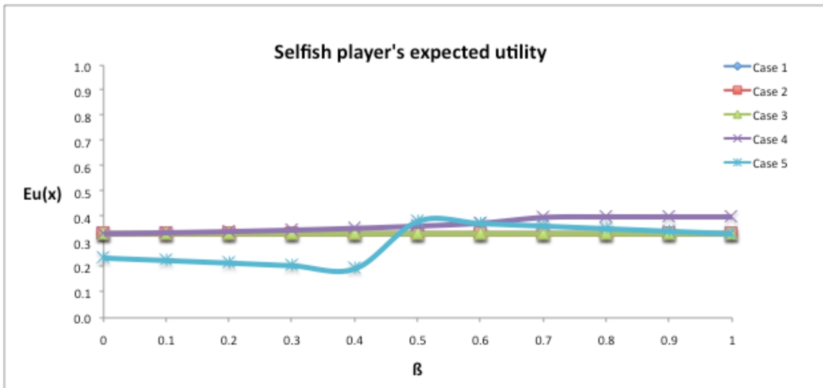
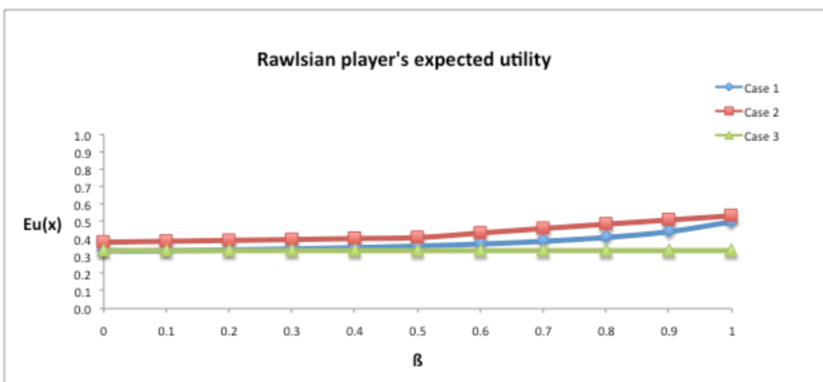


Figure. 4



C.3. Appendix: Maximum and minimum willingness to pay and receive, respectively

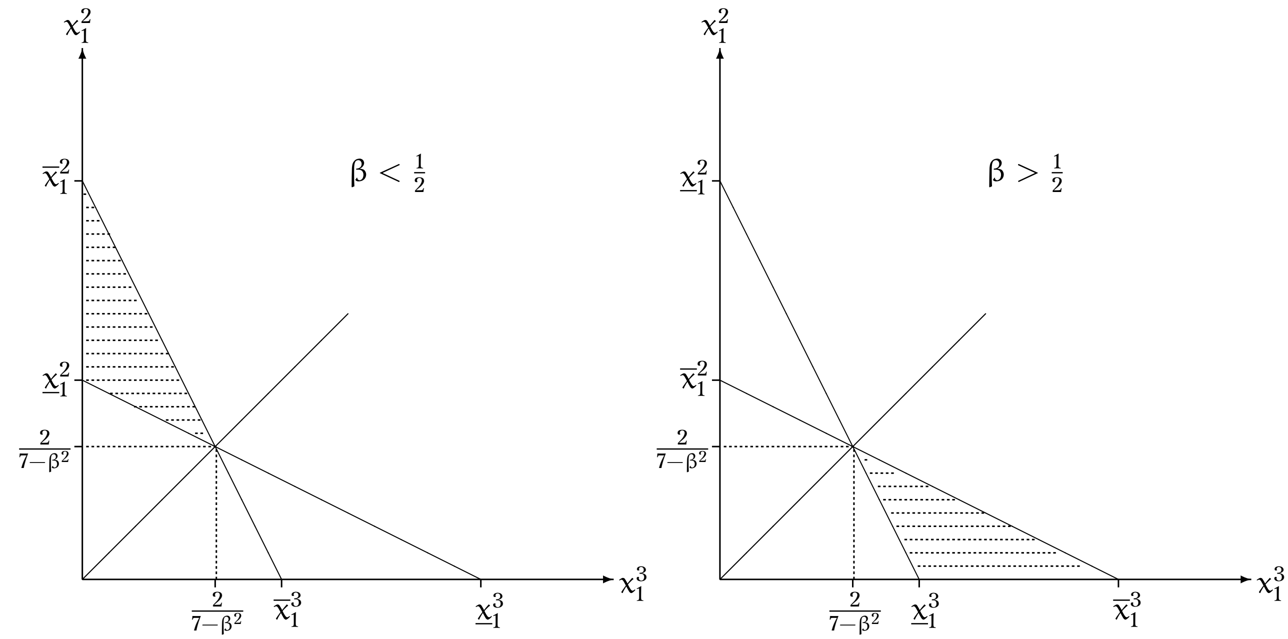


Figure C.1: Maximum willingness to pay and Minimum willingness to receive

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