THE UNIVERSITY OF WARWICK

University of Warwick institutional repository: http://go.warwick.ac.uk/wrap

A Thesis Submitted for the Degree of PhD at the University of Warwick

http://go.warwick.ac.uk/wrap/57258

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.

THE UNIVERSITY OF WARWICK

Library Declaration and Deposit Agreement

1. STUDENT DETAILS

Please complete the following:

Full name: …………………………………………………………………………………………….

University ID number: ………………………………………………………………………………

2. THESIS DEPOSIT

2.1 I understand that under my registration at the University, I am required to deposit my thesis with the University in BOTH hard copy and in digital format. The digital version should normally be saved as a single pdf file.

2.2 The hard copy will be housed in the University Library. The digital version will be deposited in the University's Institutional Repository (WRAP). Unless otherwise indicated (see 2.3 below) this will be made openly accessible on the Internet and will be supplied to the British Library to be made available online via its Electronic Theses Online Service (EThOS) service.
[At present, theses submitted for a Master's degree by Research (MA, MSc, LLM, MS or MMedSci) are not being deposited in WRAP and not being made available via EthOS. This may change in future.]

2.3 In exceptional circumstances, the Chair of the Board of Graduate Studies may grant permission for an embargo to be placed on public access to the hard copy thesis for a limited period. It is also possible to apply separately for an embargo on the digital version. (Further information is available in the *Guide to Examinations for Higher Degrees by Research*.)

2.4 *If you are depositing a thesis for a Master's degree by Research, please complete section (a) below. For all other research degrees, please complete both sections (a) and (b) below:*

(a) Hard Copy

I hereby deposit a hard copy of my thesis in the University Library to be made publicly available to readers (please delete as appropriate) EITHER immediately OR after an embargo period of ………................... months/years as agreed by the Chair of the Board of Graduate Studies.

I agree that my thesis may be photocopied. YES / NO (*Please delete as appropriate*)

(b) Digital Copy

I hereby deposit a digital copy of my thesis to be held in WRAP and made available via EThOS.

Please choose one of the following options:

EITHER My thesis can be made publicly available online. YES / NO (*Please delete as appropriate*)

OR My thesis can be made publicly available only after…..[date] (Please give date)
YES / NO (*Please delete as appropriate*)

OR My full thesis cannot be made publicly available online but I am submitting a separately identified additional, abridged version that can be made available online.
YES / NO (*Please delete as appropriate*)

OR My thesis cannot be made publicly available online. YES / NO (*Please delete as appropriate*)

3. GRANTING OF NON-EXCLUSIVE RIGHTS

Whether I deposit my Work personally or through an assistant or other agent, I agree to the following:

Rights granted to the University of Warwick and the British Library and the user of the thesis through this agreement are non-exclusive. I retain all rights in the thesis in its present version or future versions. I agree that the institutional repository administrators and the British Library or their agents may, without changing content, digitise and migrate the thesis to any medium or format for the purpose of future preservation and accessibility.

4. DECLARATIONS

(a) I DECLARE THAT:

- I am the author and owner of the copyright in the thesis and/or I have the authority of the authors and owners of the copyright in the thesis to make this agreement. Reproduction of any part of this thesis for teaching or in academic or other forms of publication is subject to the normal limitations on the use of copyrighted materials and to the proper and full acknowledgement of its source.

- The digital version of the thesis I am supplying is the same version as the final, hard-bound copy submitted in completion of my degree, once any minor corrections have been completed.

- I have exercised reasonable care to ensure that the thesis is original, and does not to the best of my knowledge break any UK law or other Intellectual Property Right, or contain any confidential material.

- I understand that, through the medium of the Internet, files will be available to automated agents, and may be searched and copied by, for example, text mining and plagiarism detection software.

(b) IF I HAVE AGREED (in Section 2 above) TO MAKE MY THESIS PUBLICLY AVAILABLE DIGITALLY, I ALSO DECLARE THAT:

- I grant the University of Warwick and the British Library a licence to make available on the Internet the thesis in digitised format through the Institutional Repository and through the British Library via the EThOS service.

- If my thesis does include any substantial subsidiary material owned by third-party copyright holders, I have sought and obtained permission to include it in any version of my thesis available in digital format and that this permission encompasses the rights that I have granted to the University of Warwick and to the British Library.

5. LEGAL INFRINGEMENTS

I understand that neither the University of Warwick nor the British Library have any obligation to take legal action on behalf of myself, or other rights holders, in the event of infringement of intellectual property rights, breach of contract or of any other right, in the thesis.

Please sign this agreement and return it to the Graduate School Office when you submit your thesis.

Student's signature: ....…… Date: ....

Structure of measures in Lipschitz differentiability spaces

by

David Bate

Thesis

Submitted to The University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

July 2013

Contents

Acknowledgments

Firstly, I would like to thank Prof. David Preiss for our invaluable conversations throughout my time as his student, his dedicated reading of this manuscript and for sharing his immense passion for mathematics.

I also thank the EPSRC for their financial support throughout my PhD. Without this support, I simply would not have been able to pursue a career in mathematics.

Last, but certainly not least, I would like to thank my family; my parents Karen and Steve for giving me the possibility to take full advantage of all of the opportunities I have had; my grandparents, Bev, Ray, Floss and Gord for their encouragement and support; and my partner Ana for her constant help and motivation.

Declarations

This thesis appears as it was when recently submitted for publication, except for minor restructuring and an expanded introduction. A preprint has also been made available [Bat12].

With the exception of those results clearly attributed to other authors, all of the material contained within this thesis is the original work of the author.

This thesis has not been submitted for a degree at another university.

Abstract

We prove the equivalence of two seemingly very different ways of generalising Rademacher's theorem to metric measure spaces. The first was introduced by Cheeger and is based upon differentiation with respect to another, fixed, chart function. The second approach is new for this generality and originates in some ideas of Alberti. It is based upon forming partial derivatives along a very rich structure of Lipschitz curves, analogous to the differentiability theory of Euclidean spaces. By examining this structure further, we naturally arrive to several descriptions of *Lipschitz differentiability spaces*.

Chapter 1

Introduction

1.1 Metric measure spaces and Lipschitz functions

We begin by defining the primary objects discussed within this thesis.

Definition 1.1.1. A *metric measure space* is a complete, separable metric space (X, d) equipped with a finite measure μ which is Borel regular (that is, μ is an outer measure such that all Borel subsets of X are measurable and each subset of X is contained within a Borel set of the same measure).

Analysis in metric measure spaces has become an important subject in modern mathematics. Historically, metric measure spaces appeared as suitable, non-smooth limits of smooth structures, such as Riemannian manifolds, and so they are very important in various areas of mathematics such as the theory of partial differential equations, differential geometry and harmonic analysis. Under such non-smooth limit, the natural volume may converge to a measure that is unrelated to the limit metric. Therefore, there is a great benefit to studying such spaces in their most abstract form. Moreover, by studying analytic and geometric concepts in such generality, the fundamental reasons why such concepts are true may become far more apparent.

Central to the study of metric measure spaces is the notion of a Lipschitz function.

Definition 1.1.2. A function $f\colon (X, d) \to (Y, \rho)$ between two metric spaces is said to be *L-Lipschitz*, for some $L \geq 0$, if

$$\rho(f(x), f(y)) \leq Ld(x, y)$$

for each $x, y \in X$ (or simply *Lipschitz* whenever such an L exists). If, in addition,

f is invertible onto its image with L-Lipschitz inverse, then f is said to be *L-bi-Lipschitz* (or simply *bi-Lipschitz*).

Bi-Lipschitz functions are the morphisms of metric spaces and so are very important to the study of metric measure spaces. In particular, bi-Lipschitz functions may be used to impose the rich metric structure of Euclidean space onto an arbitrary metric space. For example, we may generalise rectifiable sets, and hence rectifiable measures, to metric spaces by replacing the continuously differentiable functions required in their definition with bi-Lipschitz functions (see below). This gives an equivalent definition in Euclidean space.

A priori, metric measure spaces have no smooth structure and so common theme in their study is to find general conditions under which a theory of the first order calculus exists. This has included a rich theory of upper gradients (functions that mimic the behaviour of the absolute value of the derivative in Euclidean space) and from this a generalisation of Sobolev space theory, together with the existence of related inequalities such as the Poincaré and Sobolev inequalities.

In Euclidean space, the notions of measure, Lipschitz functions and derivative are connected by the famous theorems of Lebesgue and Rademacher (see [Rad19]).

Theorem 1.1.3 (Lebesgue $n = 1$, Rademacher $n > 1$)**.** *Every Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable (Lebesgue) almost everywhere.*

Therefore, a very natural pursuit within analysis on metric measure spaces is to to seek generalisations of Rademacher's theorem that replace the domain with a wider class of metric measure spaces.

One of the goals of this thesis is to show the equivalence of two seemingly very different generalisations of Rademacher's theorem. The first stems from the work of Cheeger [Che99] and further developed into the concept of a *Lipschitz differentiability space* following the work of several authors, most notably Keith [Kei04]. This generalisation is based upon the notion of differentiability with respect to another, fixed, Lipschitz function $\varphi\colon X \to \mathbb{R}^n$. The second originates in some ideas of Alberti [Alb93] and Alberti, Csörnyei and Preiss [ACP] (see [ACP10] for an announcement) and is based on the immediate consequence of Lebesgue's theorem that one may always differentiate a Lipschitz function almost everywhere along a Lipschitz curve. For measures represented by integration over such curves, this leads to a notion akin to partial differentiability μ almost everywhere and, if μ has more such representations, can lead to a notion of differentiability.

1.2 Lipschitz differentiability spaces

Our first notion of differentiability in metric measure spaces was introduced by Cheeger [Che99] and later refined by Keith [Kei04]. Of our two generalisations, it is the most direct generalisation of differentiability in Euclidean space and requires the local linear approximation of a function with respect to another fixed function. More precisely, given $U \subset X$ Borel and $\varphi\colon X \to \mathbb{R}^n$ Lipschitz, differentiability of a real valued function f at $x_0 \in U$ with respect to (U, φ) (which is termed an *n-dimensional chart*) requires the existence of a linear $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = f(x_0) + L(\varphi(x) - \varphi(x_0)) + o(d(x, x_0))$$

as $X \ni x \to x_0$. A Lipschitz differentiability space simply assumes the conclusion of Rademacher's theorem as a hypothesis, using this notion of derivative.

Definition 1.2.1. A *Lipschitz differentiability space* is a metric measure space (X, d, μ) for which there exists a countable decomposition into charts such that every real valued Lipschitz function is differentiable at μ almost every point of every chart.

A precise definition of this notion of a derivative and of a Lipschitz differentiability space is given in Definition 2.3.1).

Perhaps the best known non-trivial example of Lipschitz differentiability spaces is the Heisenberg group (see [Hei01]). This fact follows from the following theorem of Cheeger. For simplicity, we defer the details of the hypotheses of this theorem until Section 7.1.

Theorem 1.2.2 ([Che99], Theorem 4.38)**.** *Any doubling metric measure space that satisfies the Poincaré inequality is a Lipschitz differentiability space.*

Using this differentiability theory, Cheeger also obtained a very natural, necessary condition for a doubling Lipschitz differentiability space that satisfies the Poincaré inequality to be bi-Lipschitz embeddable into Euclidean space. Although we will not state his condition within this thesis, we note that it is not satisfied by many of the interesting doubling spaces that satisfy the Poincaré inequality, for example the Heisenberg group, giving a "non-embeddability" theorem for a very general class of metric measure spaces. This condition is obtained by exploiting the fact that the derivative of a bi-Lipschitz embedding has much stronger properties than the original embedding.

Such embeddability results give independent motivation to studying the derivative within metric measure spaces. This motivation mirrors that of generalising Rademacher's theorem to Banach spaces and its application to solving similar problems of bi-Lipschitz embeddability.

1.3 Alberti representations

Based upon the ideas of Alberti [Alb93], we now define our second generalisation of Rademacher's theorem to metric measure spaces. Contrary to differentiability with respect to charts, we shall see that the notion of almost everywhere differentiability is inherent to the definition of an Alberti representation.

In Definition 2.2.1 we will define $\Gamma(X)$ to be the set of Lipschitz curves into X (whose domains are not necessarily connected) and equip it with a metric similar to the metric of uniform convergence. Therefore, we are able to consider Borel measures on $\Gamma(X)$ and hence define an Alberti representation.

Definition 2.1.2. Let (X, d, μ) be a metric measure space, $\mathbb{P}$ a Borel probability measure on $\Gamma(X)$ and, for each $\gamma \in \Gamma(X)$, let μ_γ be a Borel measure on X that is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \operatorname{Im} \gamma$. We say that $(\mathbb{P}, \{\mu_\gamma\})$ is an *Alberti representation* of μ if, for every Borel $Y \subset X$, $\gamma \mapsto \mu_\gamma(Y)$ is Borel measurable and

$$\mu(Y) = \int_{\Gamma(X)} \mu_\gamma(Y) \mathrm{d}\mathbb{P}(\gamma).$$

Further, in Definition 2.2.5, we will define the *direction* of an Alberti representation and use it to distinguish different Alberti representations. This concept can be summarised with the following definition.

Definition 1.3.1. Alberti representations $(\mathbb{P}_1, \{\mu_\gamma^1\}), \ldots, (\mathbb{P}_n, \{\mu_\gamma^n\})$ of a metric measure space (X, d, μ) are *independent* if there exists a Lipschitz function $\varphi\colon X \to \mathbb{R}^n$ and linearly independent cones $C_1, \ldots, C_n \subset \mathbb{R}^n$ such that

$$(\varphi \circ \gamma)'(t) \in C_i$$

for each $1 \leq i \leq n$, $\mathbb{P}_i$-almost every $\gamma \in \Gamma(X)$ and $\mathcal{L}^1$-almost every t in the domain of γ.

Fubini's theorem provides a simple example of an Alberti representation: it represents the n-dimensional Lebesgue measure as the integral of one dimensional Lebesgue measures on lines parallel to the x-axis. For any metric measure space

(X, d, μ) and any Lipschitz $f\colon X \to \mathbb{R}$, Lebesgue's theorem says that the derivative $(f \circ \gamma)'(t)$ exists for every $\gamma \in \Gamma(X)$ and almost every t. If μ has an Alberti representation, as for the example using Fubini's theorem, we define the notion of a *partial derivative* of f in the direction of this representation at almost every point using $(f \circ \gamma)'(t)$. Therefore, if a measure has n independent Alberti representations, there exists n independent partial derivatives. This will lead to our new notion of the derivative of a Lipschitz function in a metric measure space.

This approach of obtaining a derivative from representations of the underlying measure by integration over curves originated in $\mathbb{R}^n$ within some ideas of Alberti [Alb93] and was studied further by Alberti, Csörnyei and Preiss [ACP]. For example, the following theorem is of particular importance to us (note, however, that the results of [ACP10] are presented using an entirely different language).

Theorem 1.3.2 ([ACP10], Theorem 1.18). *Let μ be a measure on $\mathbb{R}^n$. Then $(\mathbb{R}^n, \mu)$ is a Lipschitz differentiability space if and only if μ has n independent Alberti representations.*

Most importantly, the statement that μ has n independent Alberti representations is proved by assuming the converse and using it to find a subset S of positive μ measure for which it is possible to construct a Lipschitz function that is not differentiable μ-almost everywhere on S. Therefore, $(\mathbb{R}, \mu)$ is not a Lipschitz differentiability space.

1.4 Outline and summary of main results

Inspired by results such as Theorem 1.3.2, it becomes highly desirable to describe Lipschitz differentiability spaces by Alberti representations. The possibility of results similar to Theorem 1.3.2 for Lipschitz differentiability spaces began with an observation of Preiss. He observed that, from a theorem of Cheeger and Kleiner [CK09, Theorem 4.2], one may obtain an Alberti representation of any doubling Lipschitz differentiability space that satisfies the Poincaré inequality. We significantly strengthen this observation with the following theorem.

Theorem 4.2.3. *Let (U, φ) be an n dimensional chart in a Lipschitz differentiability space. Then there exists a countable Borel decomposition $U = \cup_i U_i$ such that each $\mu \llcorner U_i$ has n independent Alberti representations.*

This gives a natural realisation of the possibility of describing Lipschitz differentiability spaces by partial derivatives which was first pointed out in [CK09] (see Corollary 4.2.4).

In addition, by introducing the notion of a *universal* collection of Alberti representations (see Definition 5.1.1), we are able to completely describe those metric measure spaces that are Lipschitz differentiability spaces via Alberti representations.

Theorem 5.1.8. *A metric measure space (X, d, μ) is a Lipschitz differentiability space if and only if there exists a countable Borel decomposition $X = \cup_i U_i$ such that each $\mu \llcorner U_i$ has a universal collection of Alberti representations of size $n_i \in \mathbb{N}$.*

In this case, for each $i \in \mathbb{N}$ there exists a $\varphi_i \colon X \to \mathbb{R}^{n_i}$ such that (U_i, φ_i) is a chart of dimension n_i.

For this reason, we will not consider the derivative obtained from a collection of Alberti representations as a separate notion, but rather as a description of the usual notion of Lipschitz differentiability spaces. One advantage of this is that it naturally leads to several descriptions of Lipschitz differentiability spaces, and emphasises the key point that such spaces possess a very rich structure of curves. Indeed, we introduce classes of subsets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ (see Definitions 3.4.3, 5.2.5 and 6.7 respectively) of a metric space, each explicitly defined by the geometry of Lipschitz curves, and obtain the following theorem.

Theorem 5.2.11 and 6.8. *A metric measure space (X, d, μ) is a Lipschitz differentiability space if and only if every porous subset and every $\widetilde{A}$, respectively $\widetilde{B}$, respectively $\widetilde{C}$, subset of X is μ-null.*

Since our only hypothesis on a Lipschitz differentiability space is that Lipschitz functions are differentiable almost everywhere, we prove the existence of such Alberti representations by first reducing the problem to showing that a certain class of sets must all have measure zero. Then, for each set S in this class, we construct a Lipschitz function that is differentiable almost nowhere on S, so that S has measure zero. Our definition of this class of sets, and our ideas for constructing such a non-differentiable Lipschitz function, are inspired by the constructions of [ACP] in Euclidean space.

1.5 Notation

We will use the following notation throughout this thesis.

For (X, d) a metric space, $Y \subset X$, $x \in X$ and $r > 0$, we denote by $B(x, r)$ and $\overline{B}(x, r)$ the open and closed balls centred at x of radius r and by $B(Y, r)$ and $\overline{B}(Y, r)$ the open and closed r-neighbourhood of Y of radius r.

For a measurable set $A \subset X$ we let $\mu \llcorner A$ denote the restriction of μ to A.

For two measures μ and ν on X, we say that $\mu \ll \nu$ if μ is absolutely continuous with respect to ν.

For $n \in \mathbb{N}$ we write $\mathcal{L}^n$ for Lebesgue measure on $\mathbb{R}^n$.

For $k \in \mathbb{N}$ we write $\mathcal{H}^k$ for the k-dimensional Hausdorff measure on (X, d). We say that $A \subset X$ is k-rectifiable if there exists Borel sets $A_i \subset \mathbb{R}^k$ and bi-Lipschitz functions $f_i \colon A_i \to X$ such that

$$\mathcal{H}^k\left(X \setminus \bigcup_{i \in \mathbb{N}} f_i(U_i)\right) = 0.$$

Further, we say that μ is k-rectifiable if there exists a k-rectifiable $A \subset X$ such that $\mu \ll \mathcal{H}^k \llcorner A$.

If f is Lipschitz and $x \in X$, we write $\operatorname{Lip}(f, x)$ for the *pointwise Lipschitz constant* of f at x given by

$$\operatorname{Lip}(f, x) := \limsup_{r \to 0} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : 0 < d(x, y) < r \right\}.$$

Further, since a derivative is not defined at an isolated point, we will suppose that the set of isolated points in any metric measure space has measure zero. However, in Remark 3.2.5, we will see that this assumption is not, a priori, necessary in a Lipschitz differentiability space.

Finally, a standard argument shows that the measure of a metric measure space must be a Radon measure. Although we assume a metric measure space to be a complete, separable metric space with a finite measure, we note that by applying the main results proved here to any compact subset, the main results hold for any metric space equipped with a Radon measure.

Chapter 2

Differentiability in metric measure spaces

Within this chapter we discuss further our two notions of generalising Rademacher's theorem to metric measure spaces. After defining the required concepts we will prove some of the basic results regarding each generalisation. In particular, in Corollary 2.2.8, we will show how the existence of independent Alberti representations allows us to define the *gradient* of a Lipschitz function at almost every point. Further, in Lemma 2.3.4, we will introduce a condition for non-differentiability within Lipschitz differentiability spaces.

2.1 Alberti representations and their basic properties

Before defining an Alberti representation, we define the set of curves that form a representation.

Definition 2.1.1. Let (X,d) be a metric space. We denote by $\Gamma(X)$ the set of bi-Lipschitz functions

$$\gamma\colon \operatorname{Dom}\gamma \to X$$

with $\operatorname{Dom}\gamma \subset \mathbb{R}$ non-empty and compact. For a function $f\colon Y \to Z$ we denote

$$\operatorname{Graph} f = \{(y,z) \in Y \times Z : f(y) = z\}.$$

Finally, for Y non-empty and compact, $\operatorname{Graph} f$ is non-empty and compact and so we may equip $\Gamma(X)$ with the Hausdorff metric

$$d(\gamma,\gamma') = \operatorname{Hausdorff}(\operatorname{Graph}\gamma, \operatorname{Graph}\gamma'),$$

and hence consider Borel sets in $\Gamma(X)$. We also define $\Pi(X)$ to be the set of all $\gamma \in \Gamma(X)$ with $\operatorname{Dom}\gamma$ a closed interval.

We will refer to elements of $\Gamma(X)$ as *Lipschitz curves*. Note that $\Pi(X)$ may only consist of the Lipschitz curves whose domain is a single point.

Suppose that (X, d) is a metric space, $f\colon X \to \mathbb{R}^n$ is Lipschitz and $\gamma \in \Gamma(X)$. Then $f \circ \gamma\colon \operatorname{Dom}\gamma \subset \mathbb{R} \to \mathbb{R}^n$ is Lipschitz and so may be extended to a Lipschitz function F defined on an interval. Therefore, by Lebesgue's theorem, F is differentiable almost everywhere. However, at any non-isolated point t_0 of $\operatorname{Dom}\gamma$, if $F'(t_0)$ exists then it's value is determined by the value of $f \circ \gamma$ near to t_0, independently of the extension F. We may therefore define the derivative of $f \circ \gamma$ almost everywhere in $\operatorname{Dom}\gamma$ to be equal to the derivative of any Lipschitz extension to an interval.

With these constructions, we define an Alberti representation as follows.

Definition 2.1.2. Let (X, d, μ) be a metric measure space, $\mathbb{P}$ a Borel probability measure on $\Gamma(X)$ and, for each $\gamma \in \Gamma(X)$, let μ_γ be a Borel measure on X that is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \operatorname{Im}\gamma$, the restriction of $\mathcal{H}^1$ to $\operatorname{Im}\gamma$. For measurable $A \subset X$ we say that $(\mathbb{P}, \{\mu_\gamma\})$ is an Alberti representation of $\mu \llcorner A$ if, for every Borel $Y \subset A$, $\gamma \mapsto \mu_\gamma(Y)$ is Borel measurable and

$$\mu(Y) = \int_{\Gamma(X)} \mu_\gamma(Y) \mathrm{d}\mathbb{P}(\gamma).$$

Given an Alberti representation $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$ we will write "almost every $\gamma \in \mathcal{A}$" to mean "$\mathbb{P}$-almost every $\gamma \in \Gamma(X)$" and given a curve $\gamma \in \Gamma(X)$ we write "almost every $t \in \operatorname{Dom}\gamma$" to mean "$\gamma_{\#}^{-1}(\mu_\gamma)$-almost every $t \in \operatorname{Dom}\gamma$".

First observe that, if $\mu \llcorner A$ has an Alberti representation $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$ and $f\colon X \to \mathbb{R}$ is a positive simple function, then

$$\int_A f \mathrm{d}\mu = \int_{\Gamma(X)} \int_{\operatorname{Im}\gamma} f \mathrm{d}\mu_\gamma \mathrm{d}\mathbb{P}(\gamma).$$

Therefore, by the monotone convergence theorem, this formula holds for any positive Borel function $f\colon X \to \mathbb{R}$. Also note that, if $N \subset X$ is μ-null, then $\mathcal{A}$ is also an Alberti representation of $\mu \llcorner A \cup N$. Finally, given a measurable set $B \subset A$, we may define $\mathcal{A} \llcorner B$, the restriction of $\mathcal{A}$ to B, given by $(\mathbb{P}, \{\mu_\gamma \llcorner B\})$.

We may also form new Alberti representations from existing representations.

Lemma 2.1.3. *Let (X, d, μ) be a metric measure space and $A \subset X$ measurable. Suppose that there exists a finite measure $\mathbb{M}$ on $\Gamma(X)$ and for each $\gamma \in \Gamma(X)$ a* *measure* $\nu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\gamma$ *such that* $\mu \llcorner A$ *is absolutely continuous with respect to the Borel measure*

$$\nu(B) = \int_{\Gamma(X)} \nu_\gamma(B) d\mathbb{M}$$

and such that ν *is finite. Then* $\mu \llcorner A$ *has an Alberti representation.*

Proof. By dividing by $\mathbb{M}(\Gamma(X))$ we may suppose that $\mathbb{M}$ is a probability measure. Since μ and ν are both finite, by the Radon-Nikodym Theorem there exists a Borel measurable $F\colon A \to \mathbb{R}$ such that, for each Borel $B \subset A$,

$$\mu(B) = \int_B F \mathrm{d}\nu = \int_{\Gamma(X)} \int_\gamma \mathbf{1}_B F \mathrm{d}\nu_\gamma \mathrm{d}\mathbb{M}.$$

Therefore, if we define $\mathrm{d}\mu_\gamma = F\mathrm{d}\nu_\gamma$,

$$\mu(B) = \int_{\Gamma(X)} \mu_\gamma(B) \mathrm{d}\mathbb{P}$$

and so $\mu \llcorner A$ has an Alberti representation. □

Lemma 2.1.4. *Let* (X, d, μ) *be a metric measure space and for each* $k \in \mathbb{N}$ *let* $A_k \subset X$ *be measurable such that* $\mu \llcorner A_k$ *has an Alberti representation. Then* $\mu \llcorner \cup_k A_k$ *has an Alberti representation.*

Proof. For each $k \in \mathbb{N}$ let $\mathcal{A}_k = (\mathbb{P}_k, \{\mu_{\gamma,k}\})$ be an Alberti representation of $\mu \llcorner A_k$ and set $\mathbb{P} = \sum_k 2^{-k}\mathbb{P}_k$, a Borel probability measure on $\Gamma(X)$. For each $\gamma \in \Gamma(X)$ and each $k \in \mathbb{N}$ let $F_{\gamma,k}$ be the Radon-Nikodym derivative of $\gamma^{-1}(\mu_\gamma)$ with respect to Lebesgue measure and set

$$S_\gamma = \{t \in \operatorname{Dom}\gamma : \exists\, k \in \mathbb{N},\ F_{\gamma,k}(t) > 0\}.$$

Then S_γ is a bounded subset of $\mathbb{R}$ of positive measure and so we may define the measure

$$\nu_\gamma(Y) := \begin{cases} \mathcal{H}^1 \llcorner \gamma(S_\gamma)/\mathcal{H}^1(\gamma(S_\gamma)) & \text{if } \mathcal{H}^1(\gamma(S_\gamma)) > 0 \\ 0 & \text{otherwise} \end{cases}$$

on X. Further, we may define the finite measure ν on X given by

$$\nu(Y) = \int_{\Gamma(X)} \nu_\gamma(Y) \mathrm{d}\mathbb{P}$$

for each Borel $Y \subset X$.

Now let $Y \subset X$ be Borel with $\nu(Y) = 0$ and let $k \in \mathbb{N}$. Then for $\mathbb{P}_k$-a.e.

$\gamma \in \Gamma(X)$ we have $\nu_\gamma(Y) = 0$. However, by definition, $\mu_{\gamma,k} \llcorner S_\gamma \ll \nu_\gamma$ for every $\gamma \in \Gamma(X)$. Therefore, for $\mathbb{P}_k$-a.e. $\gamma \in \Gamma(X)$, $\mu_{\gamma,k}(Y) = 0$. Since $\mu \llcorner A_k$ has the above Alberti representation, we must have $\mu(A_k \cap Y) = 0$. In particular $\mu(Y \cap \cup_k A_k) = 0$ and so $\mu \llcorner \cup_k A_k$ is absolutely continuous with respect to ν. By applying the previous Lemma we obtain the required Alberti representation $(\mathbb{P}, \{\mu_\gamma\})$ of $\mu \llcorner \cup_k A_k$. $\square$

The previous two Lemmas preserve many of the properties of the Alberti representations in their hypotheses. More precisely, suppose under the hypotheses of Lemma 2.1.4 that $\tilde{\Gamma} \subset \Gamma(X)$ is a set of full $\mathbb{P}_k$-measure for each $k \in \mathbb{N}$ and that, for each $\gamma \in \tilde{\Gamma}$, $\tilde{\gamma} \subset \operatorname{Im}\gamma$ is a set of full $\mu_{\gamma,k}$-measure, for each $k \in \mathbb{N}$. Then $\tilde{\Gamma}$ is a set of full $\mathbb{P}$-measure and for each $\gamma \in \tilde{\Gamma}$, $\tilde{\gamma}$ is a set of full μ_γ-measure. We will make particular use of this fact in the following form. Suppose that $f \colon X \to \mathbb{R}$ is Lipschitz such that, for each $k \in \mathbb{N}$, almost every $\gamma \in \mathcal{A}_k$ and almost every $t \in \operatorname{Dom}\gamma$, $(f \circ \gamma)'(t) > 0$. Then for almost every $\gamma \in \mathcal{A}$ and almost every $t \in \operatorname{Dom}\gamma$, $(f \circ \gamma)'(t) > 0$.

In the definition of an Alberti representation we only require the representation to hold for all Borel subsets of X. However, as we will now see, it is easy to extend this representation to all μ-measurable subsets of X.

Lemma 2.1.5. *Let (X, d, μ) be a metric measure space and $A \subset X$ measurable such that $\mu \llcorner A$ has an Alberti representation $(\mathbb{P}, \{\mu_\gamma\})$. Then for every μ-measurable $Y \subset A$, $\gamma \mapsto \mu_\gamma(Y)$ is $\mathbb{P}$-measurable and*

$$\mu(Y) = \int_{\Gamma(X)} \mu_\gamma(Y)\, d\mathbb{P}.$$

Proof. Let $Y \subset A$ be μ-measurable and $B \subset Y \subset C$ be Borel with $\mu(B) = \mu(C)$. Then for every $\gamma \in \Gamma(X)$,

$$\mu_\gamma(B) \le \mu_\gamma(Y) \le \mu_\gamma(C).$$

However, since $\mu(B)$ and $\mu(C)$ are given by the Alberti representation, we know that $\mu_\gamma(B) = \mu_\gamma(C)$ for almost every $\gamma \in \mathcal{A}$. Therefore $\gamma \mapsto \mu_\gamma(Y)$ is $\mathbb{P}$-measurable and

$$\begin{aligned}\mu(Y) = \mu(B) &= \int_{\Gamma(X)} \mu_\gamma(B) \mathrm{d}\mathbb{P} \\ &\leq \int_{\Gamma(X)} \mu_\gamma(Y) \mathrm{d}\mathbb{P} \\ &\leq \int_{\Gamma(X)} \mu_\gamma(C) \mathrm{d}\mathbb{P} = \mu(C) = \mu(Y).\end{aligned}$$

Thus $\mu(Y)$ is given by the Alberti representation. □

2.2 Alberti representations and differentiability

We now demonstrate how an Alberti representation provides a metric measure space with a notion of almost everywhere differentiability of Lipschitz functions.

Definition 2.2.1. For a metric space (X, d) we define $H(X)$ to be the collection of non-empty compact subsets of $\mathbb{R} \times X$ with the Hausdorff metric, so that $H(X)$ is complete and separable. We also identify $\Gamma(X)$ with it's isometric image in $H(X)$ via $\gamma \mapsto \operatorname{Graph} \gamma$ and set

$$A(X) = \{(x, \gamma) \in X \times \Gamma(X) : \exists\, t \in \operatorname{Dom} \gamma,\ \gamma(t) = x\}.$$

Finally, for any $K \subset X$, we define the set

$$DP(K) := \{(x, \gamma) \in A(X) : \gamma^{-1}(x) \text{ is a density point of } \gamma^{-1}(K)\}.$$

Lemma 2.2.2. *Let (X, d) be a metric space and $L > 0$. Then the set of all L-bi-Lipschitz $\gamma \in \Gamma(X)$ is a closed subset of $H(X)$, and $\Gamma(X)$ is a Borel subset of $H(X)$. Further, $A(X)$ is a Borel subset of $X \times H(X)$. Finally, for any $a_1 = (\gamma_1(t_1), \gamma_1)$ and $a_2 = (\gamma_2(t_2), \gamma_2)$ in $A(X)$,*

$$|t_1 - t_2| \leq (2 \min_{i=1,2} \operatorname{biLip} \gamma_i + 1) d(a_1, a_2)).$$

Proof. For any $L > 0$ suppose that $\gamma_m \in \Gamma(X)$ are L-bi-Lipschitz and $\gamma_m \to \gamma$, for some $\gamma \in H(X)$. We will show that γ is the graph of some L-bi-Lipschitz function.

Indeed, let (s, x) and (t, y) belong to γ and, for each $m \in \mathbb{N}$, (s_m, x_m) and $(t_m, y_m) \in \gamma_m$ with $(s_m, x_m) \to (s, m)$ and $(t_m, y_m) \to (t, x)$. Then since each γ_m is

L-bi-Lipschitz,

$$d(x,y) = \lim_{m\to\infty} d(\gamma_m(s_m),\gamma_m(t_m)) \leq \lim_{m\to\infty} L|s_m - t_m| = L|s-t|$$

and similarly

$$d(x,y) \geq |s-t|/L.$$

Therefore γ is the graph of some L-bi-Lipschitz function $\gamma\colon \operatorname{Dom}\gamma \to X$. In particular, the set of L-bi-Lipschitz elements of $\Gamma(X)$ is a closed subset of $H(X)$. By taking a union over $L \in \mathbb{N}$ we conclude that $\Gamma(X)$ is a Borel subset of $H(X)$.

Now suppose that $(x,\gamma) \in X \times \Gamma(X)$ does not belong to $A(X)$. Then since $\operatorname{Im}\gamma$ is a compact set there exists a $\delta > 0$ such that $d(x, \operatorname{Im}\gamma) > \delta$. In particular $B((x,\gamma),\delta/2)$ is disjoint from $A(X)$ and so $A(X)$ is a closed subset of $X \times \Gamma(X)$.

Finally, given $a_1 = (\gamma_1(t_1),\gamma_1), a_2 = (\gamma_2(t_2),\gamma_2) \in A(X)$, suppose that

$$L = \operatorname{biLip}\gamma_1 \leq \operatorname{biLip}\gamma_2.$$

There exists a $t \in \operatorname{Dom}\gamma_1$ with

$$\max\{|t_2 - t|, d(\gamma_1(t),\gamma_2(t_2))\} \leq d(a_1,a_2)$$

and so

$$\begin{aligned}
|t_1 - t_2| &\leq |t_1 - t| + |t - t_2| \\
&\leq Ld(\gamma_1(t_1),\gamma_1(t)) + |t - t_2| \\
&\leq Ld(\gamma_1(t_1),\gamma_2(t_2)) + Ld(\gamma_2(t_2),\gamma_1(t_1)) + |t - t_2| \\
&\leq (2L+1)d(a_1,a_2)
\end{aligned}$$

as required. □

We will describe an Alberti representation using a Borel set $B \subset A(X)$ whose projection into $\Gamma(X)$ is a set of full measure. We now demonstrate the particular Borel sets that we will use.

Lemma 2.2.3. *Let (X,d,μ) be a metric measure space.*

1. *For $g\colon X \to \mathbb{R}$ continuous, the function $f\colon A(X) \to \mathbb{R}\cup\{\infty\}$ defined by*

$$f(x,\gamma) = \begin{cases} (g\circ\gamma)'(\gamma^{-1}(x)) & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

is Borel measurable.

2. *For any compact $K \subset X$, $DP(K)$ is a Borel subset of $A(X)$.*

Proof. Let $\delta, \epsilon > 0$ and $q \in \mathbb{R}$. Then the set of $(\gamma(t_0), \gamma) \in A(X)$ with

$$|g(\gamma(t)) - g(\gamma(t_0)) - q(t - t_0)| \leq \epsilon |t - t_0|$$

for all $t \in \operatorname{Dom} \gamma \cap B(t_0, \delta)$, is closed. Therefore, after taking suitable countable intersections and unions of such sets, the set where $(g \circ \gamma)'(\gamma^{-1}(x))$ exists and belongs to some open subset of $\mathbb{R}$ is Borel. Thus the set of points where the derivative does not exist is Borel as is the set of points where the derivative belongs to some given Borel set.

To prove the second statement, let $\gamma_m \to \gamma$ in $H(X)$, $t \in \mathbb{R}$ and $r > 0$. Then $\operatorname{Dom} \gamma_m \to \operatorname{Dom} \gamma$ in the Hausdorff metric. In particular, for every $\epsilon > 0$, $\operatorname{Dom} \gamma_m \subset \overline{B}(\operatorname{Dom} \gamma, \epsilon)$ for sufficiently large m. Therefore

$$\begin{aligned} \limsup_{m \to \infty} \mathcal{L}^1(\overline{B}(t, r) \cap \operatorname{Dom} \gamma_m) &\leq \lim_{\epsilon \to 0} \mathcal{L}^1(\overline{B}(t, r) \cap \overline{B}(\operatorname{Dom} \gamma, \epsilon)) \\ &= \mathcal{L}^1(\overline{B}(t, r) \cap \operatorname{Dom} \gamma) \qquad (2.2.1) \end{aligned}$$

since $\operatorname{Dom} \gamma$ is compact.

Now let $K \subset X$ be compact. We first show that

$$\gamma \mapsto \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on $\Gamma(X)$. For this let $\gamma_m \to \gamma$ in $H(X)$ such that each $\gamma_m^{-1}(K)$ is non-empty and for every $m \in \mathbb{N}$ define $\operatorname{Dom} \tilde{\gamma}_m = \gamma_m^{-1}(K)$ and $\tilde{\gamma}_m$ to be the restriction of γ_m to $\operatorname{Dom} \tilde{\gamma}_m$. Then since $\gamma_m \to \gamma$ we know that all of the $\operatorname{Dom} \tilde{\gamma}_m$ belong to some compact subset A of $\mathbb{R}$. In particular each $\tilde{\gamma}_m$ is contained within $A \times K$, a compact set, inside which the Hausdorff metric on its non-empty subsets is also compact. Therefore, for any sequence $m(k) \to \infty$ there exists a subsequence $m(k_i) \to \infty$ such that the $\gamma_{m(k_i)}$ converge to some non-empty compact $B \subset A \times K$.

However, since $\gamma_m \to \gamma$, we know that B must be a subset of γ restricted to K and so, by equation (2.2.1),

$$\limsup_{i \to \infty} \mathcal{L}^1(\overline{B}(t, r) \cap \operatorname{Dom} \tilde{\gamma}_{m(k_i)}) \leq \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K)).$$

This is true for any sequence $m(k) \to \infty$ and so

$$\gamma \mapsto \mathcal{L}^1(\overline{B}(t,r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on $\Gamma(X)$.

Further, if $(\gamma_m(t_m), \gamma_m) \to (\gamma(t), \gamma)$ in $A(X)$, then by Lemma 2.2.2 we know that $|t_m - t| \to 0$ and so

$$\begin{aligned}\limsup_{m\to\infty} \mathcal{L}^1(\overline{B}(t_m,r) \cap \gamma_m^{-1}(K)) &\le \limsup_{m\to\infty} \mathcal{L}^1(\overline{B}(t,r) \cap \gamma_m^{-1}(K)) + 2|t_m - t| \\ &\le \mathcal{L}^1(\overline{B}(t,r) \cap \gamma^{-1}(K)).\end{aligned}$$

In particular

$$(x,\gamma) \mapsto \mathcal{L}^1(B(\gamma^{-1}(x),r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on $A(X)$. By taking a suitable countable collection of intersections and unions we see that $DP(K)$ is a Borel subset of $A(X)$. □

By combining these results we now show, for almost every point x in a metric measure space with an Alberti representation, the existence of a Lipschitz curve of which x is a density point. Moreover, such a Lipschitz curve may be taken from a set of full measure, with respect to the integrating measure of the Alberti representation.

Proposition 2.2.4. *Let (X, d, μ) be a metric measure space and $M \subset X$ measurable such that $\mu \llcorner M$ has an Alberti representation $\mathcal{A}$. Suppose that (Y, ρ) is a complete, separable metric space, $B \subset Y$ is Borel and $f \colon A(X) \to Y$ is a Borel function such that, for almost every $\gamma \in \mathcal{A}$ and almost every $t \in \operatorname{Dom}\gamma$, $f(\gamma(t), \gamma) \in B$. Then the set*

$$P(M) := \{x \in M : \exists\, \gamma \in \Gamma(X),\ (x,\gamma) \in DP(M),\ f(x,\gamma) \in B\}$$

is a set of full measure in M and for each $x \in P(M)$ there exists a $\gamma^x \in \Gamma(X)$ that satisfies the conditions given in the definition of $P(M)$ such that $x \mapsto f(x, \gamma^x)$ is measurable.

Proof. We first prove the statement under the additional hypotheses that M is compact. If so, by Lemma 2.2.3 (2), the set

$$G := \{(f(x,\gamma), x, \gamma) \in B \times A(X) : x \in M,\ (x,\gamma) \in DP(M)\}$$

is the graph of a Borel function restricted to a Borel set, and so is Borel. In particular, it's projection onto M is a Suslin set and so is measurable. This projection equals $P(M)$. Further, the projection of G onto $B \times M$ given by

$$\{(y,x) \in B \times M : \exists\, \gamma \in \Gamma(X),\ (x,\gamma) \in DP(M),\ y = f(x,\gamma) \in B\}$$

is also a Suslin set. This set is the graph of a function defined on M and so, by the Jankov-von Neumann Selection Theorem (see [Kec95], Theorem 18.1), for every $x \in P(M)$ there exists a $\gamma^x \in \Gamma(X)$ such that $(x,\gamma^x) \in DP(M)$, $f(x,\gamma^x) \in B$ and such that $x \mapsto f(x,\gamma^x)$ is measurable.

Suppose that $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$ and let us consider $C = M \setminus P(M)$, a measurable set. Then for any $\gamma \in \Gamma(X)$ and $t \in \operatorname{Dom}\gamma$ with $\gamma(t) \in C$, since $(\gamma(t),\gamma) \in A(X)$, we must have either $f(\gamma(t),\gamma) \notin B$ or t not a density point of $\gamma^{-1}(C)$. Therefore, from our hypotheses on $\mathcal{A}$, there exists a $\mathbb{P}$-null set N such that either $\gamma \in N$ or $\mathcal{L}^1(\gamma^{-1}(C)) = 0$. By Lemma 2.1.5,

$$\mu(C) = \int_{\Gamma(X)\setminus N} \mu_\gamma(C)\mathrm{d}\mathbb{P} + \int_N \mu_\gamma(C)\mathrm{d}\mathbb{P} = 0$$

so that $P(M)$ is a set of full measure in M.

Now suppose that $M \subset X$ is measurable and let N be a μ-null set such that $M \setminus N$ is a countable union of compact sets K_i. Then by the above, $\cup_i P(K_i)$ is a set of full measure in M. Moreover, for every $i \in \mathbb{N}$, $K_i \subset M$ and so $P(K_i) \subset P(M)$. In particular $\mu(M \setminus P(M)) = 0$ so that such a γ^x exists for almost every $x \in M$. □

In particular we may use this Proposition to find a *partial derivative* of a Lipschitz function at almost every point of a metric measure space with an Alberti representation. In fact, we may find a *gradient* of partial derivatives, whenever a metric measure space has many *independent* Alberti representations, each distinguished in the following way.

Definition 2.2.5. For $w \in \mathbb{S}^{n-1}$ and $0 < \theta < 1$, define the *cone of width θ centred on w* to be the set

$$C(w,\theta) = \{v \in \mathbb{R}^n : v \cdot w \geq (1-\theta)\|v\|\}$$

and the *open cone of width θ centred on w* to be

$$C^\circ(w,\theta) = \{v \in \mathbb{R}^n : v \cdot w > (1-\theta)\|v\|\}.$$

Also, for a cone C we denote the open cone with the same centre and width by C°.

Now let (X,d,μ) be a metric measure space and $\varphi\colon\ X \to \mathbb{R}^n$ Lipschitz. For a cone $C \subset \mathbb{R}^n$ we say that a Lipschitz curve $\gamma \in \Gamma(X)$ is in the *φ-direction of C* if, for almost every $t \in \operatorname{Dom}\gamma$,

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\}.$$

Further, we say that an Alberti representation $\mathcal{A}$ is in the *φ-direction of C* if almost every $\gamma \in \mathcal{A}$ is in the φ-direction of C.

Finally, we say that closed cones $C_1, \ldots, C_m \subset \mathbb{R}^n$ are *independent* if, for any choice of $v_i \in C_i \setminus \{0\}$, the v_i are linearly independent and that a collection $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of Alberti representations is *φ-independent* if there exists independent cones $C_1, \ldots, C_m$ such that each $\mathcal{A}_i$ is in the φ-direction of C_i.

Definition 2.2.6. Let (X, d, μ) be a metric measure space, $x_0 \in X$ and $f \colon X \to \mathbb{R}$ and $\varphi \colon X \to \mathbb{R}^n$ Lipschitz. Suppose that $\gamma_1, \ldots, \gamma_n \in \Gamma(X)$ such that each $\gamma_i^{-1}(x_0) = 0$ is a density point of $\operatorname{Dom}\gamma_i$ and that the $(\varphi \circ \gamma_i)'(0)$ exist and form a linearly independent set. We define the *gradient of f at x_0 with respect to φ and $\gamma_1, \ldots, \gamma_n$* to be the unique $\nabla f(x_0) \in \mathbb{R}^n$ such that

$$(f \circ \gamma_i)'(0) = \nabla f(x_0) \cdot (\varphi \circ \gamma_i)'(0).$$

Further, we say that $\nabla f(x_0) \in \mathbb{R}^n$ is *a gradient of f at x_0 with respect to φ* if there exist such $\gamma_1, \ldots, \gamma_n \in \Gamma(X)$ such that $\nabla f(x_0)$ is the gradient of f at x_0 with respect to φ and $\gamma_1, \ldots, \gamma_n$.

Remark 2.2.7. A very easy construction of a Lipschitz function on the plane with both partial derivatives equal to zero and a non-zero directional derivative at the origin shows that a gradient with respect to a fixed φ need not be unique.

We obtain the following Corollary of Proposition 2.2.4.

Corollary 2.2.8. *Let (X, d, μ) be a metric measure space and $\varphi \colon X \to \mathbb{R}^n$ Lipschitz such that μ has n φ-independent Alberti representations. Then for any Lipschitz $f \colon X \to \mathbb{R}$ there exists a gradient ∇f of f almost everywhere with respect to φ. Further, this gradient may be chosen such that $x_0 \mapsto \nabla f(x_0)$ is measurable.*

2.3 Lipschitz differentiability spaces and a condition for non-differentiability

We begin by recalling the notion of differentiability in metric spaces introduced by Cheeger [Che99] and Keith [Kei04].

Definition 2.3.1. Let (X, d) be a metric space and $n \in \mathbb{N}$. We say that a Borel set $U \subset X$ and a Lipschitz function $\varphi\colon X \to \mathbb{R}^n$ form a *chart of dimension* n, (U, φ) and that a function $f\colon X \to \mathbb{R}$ is *differentiable at* $x_0 \in U$ with respect to (U, φ) if there exists a unique $Df(x_0) \in \mathbb{R}^n$ (the *derivative of* f *at* x_0) such that

$$\limsup_{X \ni x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

Further, we say that a metric measure space (X, d, μ) is a *Lipschitz differentiability space* if there exists a countable decomposition of X into charts such that any Lipschitz function $f\colon X \to \mathbb{R}$ is differentiable at almost every point of every chart.

Whenever it will not cause confusion, we will say "a chart in a Lipschitz differentiability space" to mean a chart in a Lipschitz differentiability space with respect to which every real valued Lipschitz function is differentiable almost everywhere.

For a survey on the existing theory of Lipschitz differentiability spaces, the reader is referred to the primer written by Kleiner and Mackay, [KM11]. (Note that, in this paper and the papers of Cheeger and Keith, a Lipschitz differentiability space is referred to as a *metric measure space that admits a measurable differentiable structure.*)

Remark 2.3.2. One may wonder about the notion of a *zero dimensional chart*, where every Lipschitz function has derivative zero almost everywhere. In Corollary 3.2.4 we see that such a chart must have measure zero.

We first give a characterisation of the uniqueness of derivatives in a chart that will lead us to necessary conditions for a function to be differentiable at a point. This characterisation first appeared in [BS13].

Lemma 2.3.3 ([BS13], Lemma 2.1)**.** *Let (U, φ) be an n-dimensional chart in a metric measure space (X, d, μ) and $x_0 \in U$. The following are equivalent:*

1. *There exists a $\lambda > 0$ and $X \ni x_m \to x_0$ such that, for any $v \in \mathbb{S}^{n-1}$,*

$$\liminf_{m \to \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \geq \lambda. \tag{2.3.1}$$

2. *There exists a $\lambda > 0$ such that, for any $v \in \mathbb{S}^{n-1}$,*

$$\limsup_{x \to x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} \geq \lambda.$$

3. For any $f\colon X \to \mathbb{R}$, if there exists a $Df(x_0) \in \mathbb{R}^n$ such that

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0,$$

then it is unique.

In particular, the fact that such a $Df(x_0)$ is unique depends only upon the chart and is independent of f.

Proof. Equation (1) implies (2). Now suppose that (2) holds and for a function $f\colon X \to \mathbb{R}$ there exist $Df(x_0)$ and $Df'(x_0) \in \mathbb{R}^n$ that satisfy the hypotheses of (3). Then by the triangle inequality

$$\limsup_{x \to x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot (D - D')|}{d(x, x_0)} = 0$$

and so $\|D - D'\| = 0$.

Finally suppose that (3) holds for some $f\colon X \to \mathbb{R}$. Then for any $v_0 \in \mathbb{S}^{n-1}$,

$$\limsup_{x \to x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v_0|}{d(x, x_0)} > 0.$$

Therefore, there exists $x_m^0 \to x_0$ and $w_0 \in \overline{B}(0, \operatorname{Lip} \varphi)$ such that $w_0 \cdot v_0 > 0$ and

$$\lim_{m \to \infty} \frac{\varphi(x_m^0) - \varphi(x_0)}{d(x_m^0, x_0)} = w_0.$$

For each $1 \leq i < n$ inductively choose $v_i \in \mathbb{S}^{n-1}$ such that $v_i \cdot w_j = 0$ for each $0 \leq j < i$ and let $x_m^i \to x_0$ and $w_i \in \overline{B}(0, \operatorname{Lip} \varphi)$ such that $w_i \cdot v_i > 0$ and

$$\lim_{m \to \infty} \frac{\varphi(x_m^i) - \varphi(x_0)}{d(x_m^i, x_0)} = w_i.$$

Then for any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, suppose that $\alpha_0 w_0 + \ldots + \alpha_{n-1} w_{n-1} = 0$. By taking the inner product with w_{n-1} we see that $\alpha_{n-1} = 0$. Repeating we see that $\alpha_i = 0$ for each $0 \leq i < n$ and so the w_i form a basis of $\mathbb{R}^n$. In particular, there exists a $\lambda > 0$ such that, for each $v \in \mathbb{S}^{n-1}$, there exists a $0 \leq j < n$ with $|w_j \cdot v| \geq \lambda$.

Therefore, if we set $x_{mn+i} = x_m^i$ for each $m \in \mathbb{N}$ and $0 \leq i < n$, for any $v \in \mathbb{S}^{n-1}$ there exists a $0 \leq j < n$ such that

$$\liminf_{m \to \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \geq \lim_{m \to \infty} \frac{|(\varphi(x_m^j) - \varphi(x_0)) \cdot v|}{d(x_m^j, x_0)} \geq \lambda$$

as required. ◻

Using this we may give a necessary condition for a function to be differentiable at a point in a chart.

Lemma 2.3.4. *Let* (U,φ) *be an* n*-dimensional chart in a metric space* (X,d), $x_0 \in U$ *and* $x_m \to x_0$ *satisfying* (2.3.1) *for some* $\lambda > 0$. *Then for any* $f\colon X \to \mathbb{R}$ *that is differentiable at* x_0 *we have*

$$\|Df(x_0)\| \leq \operatorname{Lip}(f,x_0)/\lambda$$

and

$$\frac{\lambda}{\operatorname{Lip}\varphi}\operatorname{Lip}(f,x_0) \leq \liminf_{m\to\infty} \max_{0\leq i<n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i},x_0)}.$$

Proof. If $f\colon X \to \mathbb{R}$ is differentiable at $x_0 \in X$ then by the triangle inequality

$$\begin{aligned}\liminf_{m\to\infty} \max_{0\leq i<n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i},x_0)} &\geq \liminf_{m\to\infty} \max_{0\leq i<n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0))\cdot D|}{d(x_{mn+i},x_0)} \\ &\geq \lambda\|D\|.\end{aligned}$$

This proves the first inequality. Secondly, by another application of the triangle inequality,

$$\begin{aligned}\operatorname{Lip}(f,x_0) &= \limsup_{x\to x_0} \frac{|f(x)-f(x_0)|}{d(x,x_0)} \\ &\leq \|Df(x_0)\| \limsup_{x\to x_0} \frac{\|\varphi(x)-\varphi(x_0)\|}{d(x,x_0)} \\ &\leq \|Df(x_0)\| \operatorname{Lip}\varphi.\end{aligned}$$

Combining this with the first inequality gives the required result. ◻

A very easy application of this Lemma gives the following Corollary on singletons in a Lipschitz differentiability space. This will be used without any specific reference.

Corollary 2.3.5. *Any singleton in a Lipschitz differentiability space has measure zero.*

Proof. Let (U,φ) be any n-dimensional chart in a Lipschitz differentiability space (X,d,μ). For almost any $x_0 \in U$ let $x_m \to x_0$ and $\lambda > 0$ be obtained from Lemma 2.3.3 and by taking a suitable subsequence if necessary, we may suppose that

$$\max_{0\leq i<n} d(x_{mn+i},x_0) \leq \min_{0\leq i<n} d(x_{(m+1)n+i},x_0)/2$$

for each $m \in \mathbb{N}$. Then the 1-Lipschitz function

$$f(x) = \sum_{m\in 2\mathbb{N}} \max_{0\leq i<n} (d(x_{mn+i}, x_0)/4 - d(x_{mn+i}, x))^+$$

satisfies

$$\frac{f(x_{mn+i}) - f(x_0)}{d(x_{mn+i}, x_0)} = \begin{cases} 1/4 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

for each $0 \leq i < n$. In particular, the condition given in Lemma 2.3.4 does not hold at x_0 and so f is not differentiable at x_0, as required. □

When constructing a non-differentiable Lipschitz function, we will ensure that

$$\liminf_{m\to\infty} \max_{0\leq i<n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i}, x_0)}$$

is sufficiently small by first bounding $d(x_{mn+i}, x_0)$ from below, for a fixed m, all $0 \leq i < n$ and all x_0 in a certain set of large measure. We now do this uniformly across the chart to simplify the construction.

Definition 2.3.6. Let (X, d) be a metric space and $\lambda > 0$. We say that $U \subset X$ and a Lipschitz function $\varphi\colon X \to \mathbb{R}^n$ form a λ-*structured chart* of dimension n if U is compact and there exists a $U' \subset U$ of full measure such that, for every $R > 0$ there exists an $r > 0$ and for every $x_0 \in U'$ points $x_1, \ldots, , x_n \in U$ with each $r < d(x_i, x_0) < R$ and

$$\max_{1\leq i\leq n} \frac{|(\varphi(x_i) - \varphi(x_0)) \cdot v|}{d(x_i, x_0)} \geq \lambda$$

for every $v \in \mathbb{S}^{n-1}$.

We say that (U, φ) is a structured chart if it is a λ-structured chart for some $\lambda > 0$.

Note that a structured chart is also a chart and so we may consider the derivative of a real valued function with respect to a structured chart. The following Lemma shows that we may just consider structured charts when working with a Lipschitz differentiability space.

Lemma 2.3.7. *Let (U, φ) be an n-dimensional chart in a metric measure space (X, d, μ) such that*

$$\mathrm{Lip}(v \cdot \varphi, x_0) > 0$$

for every $v \in \mathbb{S}^{n-1}$ and almost every $x_0 \in U$. Then for any $\epsilon > 0$ there exists a $U' \subset U$ with $\mu(U \setminus U') < \epsilon$ and a countable set $N \subset X$ such that $(U' \cup N, \varphi)$ is a *structured chart.*

In particular, we may decompose any Lipschitz differentiability space into a μ-null set and a countable collection of structured charts with respect to which any real valued Lipschitz function is differentiable almost everywhere.

Proof. For $x_0, x_1, \ldots, x_n \in X$ let us denote by $P(x_0; x_1, \ldots, x_n)$ the property

$$\max_{1 \le i \le n} \frac{|(\varphi(x_i) - \varphi(x_0)) \cdot v|}{d(x_i, x_0)} \ge \lambda \text{ for every } v \in \mathbb{S}^{n-1}.$$

Then for any $R, \lambda, r > 0$, the set of $x_0 \in U$ for which there exist $x_1, \ldots, x_n \in X$ with $r < d(x_i, x_0) < R$ for each $1 \le i \le n$ and such that $P(x_0; x_1, \ldots, x_n)$ holds is an open set. Therefore, the set of those $x_0 \in U$ that satisfy (2.3.1) for a given $\lambda > 0$ is a Borel set. In particular, for any $\epsilon > 0$ there exists a compact $U' \subset U$ with $\mu(U \setminus U') < \epsilon$, so that (U', φ) is still a chart, and a $\lambda > 0$ such that every point of U' satisfies (2.3.1) for λ.

Now let $R > 0$ and λ be as found above. The function

$$x_0 \mapsto \sup \left\{ \min_{1 \le i \le n} d(x_i, x_0) : d(x_i, x_0) \le R \text{ and } P(x_0; x_1, \ldots, x_n) \right\}$$

is positive and lower semicontinuous on U'. Therefore, there exists an $r > 0$ such that this function is bounded below by r on U'. Let $y^1, \ldots, y^M$ be a finite $\lambda r / 2 \operatorname{Lip} \varphi$-net of U' and for each $1 \le j \le M$ let $x_1^j, \ldots, x_n^j \in X$ with $r \le d(x_i^j, y^j) \le R$ for each $1 \le i \le n$ and such that $P(y^j; x_1^j, \ldots, x_n^j)$ for each $1 \le j \le M$. We set

$$N_R = \{x_i^j : 1 \le i \le n,\ 1 \le j \le M\}.$$

Then for any $x_0 \in U'$ and $v \in \mathbb{S}^{n-1}$, there exists a y^j with $d(x_0, y^j) < \lambda r / 2 \operatorname{Lip} \varphi$ and a $1 \le i \le n$ such that

$$\begin{aligned} \max_{1 \le i \le n} |(\varphi(x_i^j) - \varphi(x_0)) \cdot v| &\ge |(\varphi(x_i^j) - \varphi(y^j)) \cdot v| - \|\varphi(y^j) - \varphi(x_0)\| \\ &\ge \lambda d(x_i^j, y^j) - \lambda d(x_i^j, y^j)/2 \\ &\ge \lambda d(x_i^j, x_0)/(2 + \lambda / \operatorname{Lip} \varphi) \\ &:= \lambda' d(x_i^j, x_0). \end{aligned}$$

Therefore, if we define $N = \cup_i N_{1/i}$ and $V = U' \cup N$, for any $x_0 \in U'$ there exists a sequence $V \ni x_m \to x_0$ that satisfies (2.3.1) for λ'. Moreover, since each $N_{1/k}$ is a finite subset of $B(U', \lambda \operatorname{Lip} \varphi / 2k)$, for any sequence $x_m \in V$ either there exists a constant subsequence or $d(x_m, U') \to 0$. In particular, V is compact and so (V, φ)

is a structured chart.

Finally, in a Lipschitz differentiability space N is a μ-null set and so (V, φ) is a chart with respect to which any Lipschitz function is differentiable almost everywhere in V. □

To conclude this section we highlight a key fact that will be used when investigating Alberti representations in Lipschitz differentiability spaces. This result should be compared to the concept of a gradient (see Definition 2.2.6). We will use this without any specific reference.

Lemma 2.3.8. *Let (U, φ) be a chart in a metric measure space (X, d, μ) and $\gamma \in \Gamma(X)$. Suppose that for some non-isolated $t_0 \in \operatorname{Dom}\gamma$ and $f\colon X \to \mathbb{R}$, f is differentiable with respect to (U, φ) at $\gamma(t_0)$ and that $(\varphi \circ \gamma)'(t_0)$ exists. Then*

$$(f \circ \gamma)'(t_0) = Df(x_0) \cdot (\varphi \circ \gamma)'(t_0).$$

Proof. If γ is L Lipschitz and $\gamma(t_0) = x_0$, use the triangle inequality and the fact that

$$\limsup_{\operatorname{Dom}\gamma \ni t \to t_0} \frac{|P(\gamma(t)) - P(\gamma(t_0))|}{|t - t_0|} \leq L \limsup_{\operatorname{Im}\gamma \ni x \to x_0} \frac{|P(x) - P(x_0)|}{d(x, x_0)}$$

for any $P\colon X \to \mathbb{R}$. □

Chapter 3

Constructions

In this chapter we present the constructions required for proving the existence of n independent Alberti representations of any n dimensional chart of a Lipschitz differentiability space. Note, however, that these constructions do not use the fact that we work inside a Lipschitz differentiability space, only that we have a chart structure.

Section 3.1 is devoted to the following construction of a non-differentiable Lipschitz function.

Proposition 3.1.3. *Let (U,φ) be a structured chart in a metric measure space (X,d,μ), $S \subset U$ be Borel and $L,\beta > 0$. Suppose that there exists a sequence of L-Lipschitz functions $f_m \colon X \to \mathbb{R}$ such that:*

1. *For every $x_0 \in S$ there exists an $M \in \mathbb{N}$ such that, for each $m \geq M$, there exists an $x \in X$ with $0 < d(x,x_0) < 1/m$ and*

$$|f_m(x) - f_m(x_0)| \geq \beta d(x,x_0).$$

2. *There exists a $\rho_m > 0$ such that, for every $x_0 \in S$ and any $y,z \in B(x_0,\rho_m)$,*

$$|f_m(y) - f_m(z)| \leq d(y,z)/m.$$

Then there exists a Lipschitz function $F \colon X \to \mathbb{R}$ that is differentiable μ-almost nowhere on S. In particular, if X is a Lipschitz differentiability space, then S is μ-null.

We give some simple consequences of this construction in Section 3.2. We will not make use of the results contained there but provide them for a more detailed description of Lipschitz differentiability spaces.

Independently of this, in Section 3.3, we will use a generalisation of the Lebesgue decomposition theorem, the Gliksberg-König-Seever decomposition theorem (see Theorem 3.3.1), to give a very natural characterisation of those metric measure spaces with an Alberti representation in the direction of a cone.

Following this, in Section 3.4, we define a class of subsets of any metric measure space, $\mathcal{A}(\varphi)$ (for any Lipschitz $\varphi\colon X \to \mathbb{R}^n$), and prove the following theorem.

Theorem 3.4.6. *Let (U, φ) be a λ-structured chart of dimension n in a metric measure space (X, d, μ). Then there exists a countable Borel decomposition $U = \cup_i U_i$ such that each $\mu \llcorner U_i$ has n φ-independent Alberti representations if and only if, for every $S \in \widetilde{A}(\varphi)$ with $S \subset U$, $\mu(S) = 0$.*

In Chapter 4, we will combine these results by constructing the sequence of functions required for Proposition 3.1.3 for any $S \in \mathcal{A}(\varphi)$, for any structured chart (U, φ).

3.1 Construction of a non-differentiable Lipschitz function

We will construct a non-differentiable Lipschitz function from a given sequence of Lipschitz functions. The first step will be to modify each function in such a sequence in the following way.

Lemma 3.1.1. *Let (X, d, μ) be a metric measure space, $h > 4\epsilon > 0$ and $L > 0$. Then for any L-Lipschitz $f\colon X \to \mathbb{R}$ and Borel $S \subset X$ there exists an L-Lipschitz $\tilde{f}\colon X \to \mathbb{R}$ and Borel $\tilde{S} \subset S$ with $\mu(\tilde{S}) \geq (1 - 4\epsilon/h)\mu(S)$ such that:*

1. *The support of $\tilde{f}$ is contained within $B(S, 2h/L)$.*
2. $0 \leq \tilde{f} \leq h$.
3. *For every $x, y \in B(S, h/L)$ with $x \neq y$,*
$$\frac{|\tilde{f}(x) - \tilde{f}(y)|}{d(x, y)} \leq \frac{|f(x) - f(y)|}{d(x, y)}.$$
4. *For every $x_0 \in \tilde{S}$ and $x \in X$ with $0 < d(x, x_0) \leq \epsilon/L$,*
$$\frac{|\tilde{f}(x) - \tilde{f}(x_0)|}{d(x, x_0)} = \frac{|f(x) - f(x_0)|}{d(x, x_0)}.$$

Proof. Let M be the greatest integer less than $h/4\epsilon$ and for each $0 \leq k < M$ define $Z_k(t) = d(t, 2\epsilon k + 2h\mathbb{Z})$ and $\hat{f}_k = Z_k \circ f \colon X \to \mathbb{R}$. Then each $\hat{f}_k$ satisfies (2) and (3). Now define

$$F_k = \{x_0 \in S : d(f(x_0), 2\epsilon k + 2h\mathbb{Z}) < \epsilon\}.$$

Then for integer $0 \leq k < M$, the F_k are disjoint Borel subsets of S. Indeed, if $x_0 \in F_k \cap F_{k'}$ then there exists an $n \in \mathbb{N}$ with $|k-k'+hn/\epsilon| < 1$. Since $|k-k'| < h/4\epsilon$ and $h/\epsilon > 4$, this can only happen if $n = 0$ and $k = k'$ and so such F_k and $F_{k'}$ are either disjoint or equal. Therefore, there exists an m with

$$\mu(F_m) \leq \mu(S)/M \leq 4\epsilon\mu(S)/h.$$

We set $\tilde{S} = S \setminus F_m$ so that, for $x_0 \in \tilde{S}$ and $x \in X$ with $d(x, x_0) \leq \epsilon/L$,

$$|\hat{f}_m(x) - \hat{f}_m(x_0)| < \epsilon \text{ and } d(f(x_0), 2\epsilon m + 2h\mathbb{Z}) \geq \epsilon.$$

In particular

$$|\hat{f}_m(x) - \hat{f}_m(x_0)| = |f(x) - f(x_0)|$$

and so (4) holds.

Finally we define

$$\tilde{f} \colon B(S, h/L) \cup X \setminus B(S, 2h/L) \to \mathbb{R}$$
$$\tilde{f}(x) = \begin{cases} \hat{f}_m & x \in B(S, h/L) \\ 0 & x \in X \setminus B(X, 2h/L). \end{cases}$$

Then, since $0 \leq \tilde{f} \leq h$, $\tilde{f}$ is L-Lipschitz and so we may extend it to a function on the whole of X that satisfies the required properties. □

We now give the main Lemma required for the construction of a non-differentiable Lipschitz function. It takes a set S and a sequence of Lipschitz functions that, in some neighbourhood of S, witness both large and small difference quotients at all points of S and combines them to construct a Lipschitz function that witnesses such difference quotients in all neighbourhoods of S. Later in this section we will see that such a function cannot be differentiable at any point of S.

Lemma 3.1.2. *Let (X, d, μ) be a metric measure space, $S \subset X$ Borel and $L, \beta > 0$. Suppose that there exists a sequence of L-Lipschitz functions $f_m \colon X \to \mathbb{R}$ such that:*

1. *For every $x_0 \in S$ there exists an $M \in \mathbb{N}$ such that for each $m \geq M$ there exists*

an $x \in X$ with $0 < d(x, x_0) < 1/m$ and

$$|f_m(x) - f_m(x_0)| \geq L\beta d(x, x_0).$$

2. *For every $m \in \mathbb{N}$ there exists a $\rho_m > 0$ such that, for every $x_0 \in S$ and any $y, z \in B(x_0, \rho_m)$,*

$$|f_m(y) - f_m(z)| \leq d(y, z)/m.$$

Then for any $\alpha > 0$ and $R_i > r_i > 0$ such that $R_i, r_i \to 0$, there exists $i(k) \to \infty$ and a Lipschitz function $F \colon X \to \mathbb{R}$ such that:

- *For almost every $x_0 \in S$, $\operatorname{Lip}(F, x_0) \geq L\beta - \alpha$.*
- *For every $k \in \mathbb{N}$, $x_0 \in S$ and $x \in X$ with $r_{i(k)} < d(x, x_0) < R_{i(k)}$,*

$$\frac{|f(x) - f(x_0)|}{d(x, x_0)} \leq \alpha.$$

Proof. By dividing by L if necessary, we may suppose that $L = 1$ and by taking a suitable subsequence we may suppose that $R_i \leq r_{i-1}$ for each $i \in \mathbb{N}$. We define sequences $m(k)$, $i(k) \to \infty$ inductively as follows, choosing $m(0) = i(0) = 1$. Given $m(k)$ and $i(k)$ choose $i(k+1) > i(k)$ such that

$$R_{i(k+1)} < \rho_{m(k)} \text{ and } r_{i(k+1)} \leq 2^{-(k+1)}\alpha r_{i(j)}$$

for every $j \leq k$. Then choose $m(k+1)$ such that

$$\frac{1}{m(k+1)} \leq 2^{-(k+1)} r_{i(k+1)} \text{ and } m(k+1) \geq 2^{k+1}/\alpha.$$

Note that these conditions imply, for every $j \in \mathbb{N}$,

$$\sum_{k>j} r_{i(k)} \leq \alpha r_{i(j)} \text{ and } \sum_{k \in \mathbb{N}} 1/m(k) \leq \alpha.$$

We define 1-Lipschitz functions $g_k \colon X \to \mathbb{R}$ and a Borel set $S_k \subset S$ by applying Lemma 3.1.1 to $f_{m(k)}$ with $h = r_{i(k)}/2$ and $\epsilon = 1/m(k)$. Then $\mu(S_k) \geq (1 - 2^{3-k})\mu(S)$ and the g_k have the following properties:

1. The support of g_k is contained within $B(S, r_{i(k)})$.
2. $0 \leq g_k \leq r_{i(k)}/2$.

3. For every $x_0 \in S_k$ there exists an x with $0 < d(x, x_0) < 1/m(k)$ such that

$$\frac{|g_k(x) - g_k(x_0)|}{d(x, x_0)} \geq \beta.$$

4. For every $x_0 \in S$ and any $x, y \in B(x_0, R_{i(k)})$ with $x \neq y$,

$$\frac{|g_k(x) - g_k(y)|}{d(x, y)} \leq \frac{1}{m(k)}.$$

Finally we define $F = \sum_k g_k$ and

$$S' = \bigcap_{m \in \mathbb{N}} \bigcup_{k \geq m} S_k,$$

a set of full measure in S.

We first show that F is Lipschitz. Let $x \neq y \in X$ with

$$r_{i(k+1)} \leq d(x, y) < r_{i(k)}$$

and suppose that, for some $k' \leq k$,

$$|g_{k'}(x) - g_{k'}(y)| > d(x, y)/m_{k'}.$$

Then one of x or y must necessarily belong to $B(S, r_{i(k)})$. However,

$$d(x, y) < r_{i(k)} \leq r_{i(k')}$$

and so there exists an $x_0 \in S$ such that

$$x, y \in B(x_0, 2r_{i(k')}) \subset B(x_0, R_{i(k'-1)}).$$

Since the $R_{i(k)}$ strictly decrease we have, for all $j < k'$,

$$|g_j(x) - g_j(y)| \leq d(x, y)/m(j).$$

In particular this can only happen for at most one value of $k' \leq k$. If such a k' does not exist set $k' = k$. Then we have

$$\begin{aligned}
|F(x)-F(y)| &\leq \sum_{k'\neq j\leq k} |g_j(x)-g_j(y)| + |g_{k'}(x)-g_{k'}(y)| + \sum_{j\geq k+1} |g_j(x)-g_j(y)| \\
&\leq \sum_{k'\neq j\leq k} d(x,y)/m(j) + 2d(x,y) + \sum_{j>k+1} r_{i(j)}/2 \\
&\leq (\alpha+2)d(x,y) + \alpha r_{i(k+1)} \\
&\leq 2(\alpha+1)d(x,y)
\end{aligned}$$

and so $F\colon X \to \mathbb{R}$ is Lipschitz.

Now let $x_0 \in S'$ and $K \in \mathbb{N}$ such that $x_0 \in S_k$ for each $k \geq K$. Then for each $k \geq K$ there exists an $x \in X$ with $0 < d(x,x_0) < 1/m(k)$ such that

$$\frac{|g_k(x)-g_k(x_0)|}{d(x,x_0)} \geq \beta.$$

Note that we also have $x \in B(S, R_{i(j)})$ for any $j < k$ and so

$$\begin{aligned}
|F(x)-F(x_0)| &\geq |g_k(x)-g_k(x_0)| - \sum_{j<k} |g_j(x)-g_j(x_0)| - \sum_{j>k} |g_j(x)-g_j(x_0)| \\
&\geq \beta d(x,x_0) - \sum_{j<k} d(x,x_0)/m(j) - \sum_{j>k} r_{i(j)}/2 \\
&\geq \beta d(x,x_0) - \alpha d(x,x_0) - \alpha r_{i(k)} \\
&\geq (\beta - 2\alpha) d(x,x_0).
\end{aligned}$$

Such an x exists for each $k \geq K$ and so $\mathrm{Lip}(F, x_0) \geq \beta - 2\alpha$.

Now let $x_0 \in S$ and for any $k \in \mathbb{N}$ let $x \in X$ with $r_{i(k)} < d(x,x_0) < R_{i(k)}$. Then

$$\begin{aligned}
|F(x_i)-F(x_0)| &\leq \sum_{j\leq k} |g_j(x)-g_j(x_0)| + \sum_{j>k} |g_j(x)-g_j(x_0)| \\
&\leq \alpha d(x,x_0) + \sum_{j>k} r_{i(j)}/2 \\
&\leq 2\alpha d(x,x_0).
\end{aligned}$$

Therefore, F satisfies the conclusion of the Lemma for 2α. □

By combining this construction with the definition of a structured chart, we show that the constructed function is differentiable almost nowhere on such a set.

Proposition 3.1.3. *Let (U, φ) be a structured chart in a metric measure space (X, d, μ), $S \subset U$ be Borel and $L, \beta > 0$. Suppose that there exists a sequence of L-Lipschitz functions $f_m \colon X \to \mathbb{R}$ such that:*

1. *For every $x_0 \in S$ there exists an $M \in \mathbb{N}$ such that, for each $m \geq M$, there exists an $x \in X$ with $0 < d(x, x_0) < 1/m$ and*

$$|f_m(x) - f_m(x_0)| \geq \beta d(x, x_0).$$

2. *There exists a $\rho_m > 0$ such that, for every $x_0 \in S$ and any $y, z \in B(x_0, \rho_m)$,*

$$|f_m(y) - f_m(z)| \leq d(y, z)/m.$$

Then there exists a Lipschitz function $F \colon X \to \mathbb{R}$ that is differentiable μ-almost nowhere on S. In particular, if X is a Lipschitz differentiability space, then S is μ-null.

Proof. For $\lambda > 0$ let (U, φ) be a λ-structured chart and choose $\alpha < \lambda\beta/(\operatorname{Lip}\varphi + 1)$. Then by the definition of a structured chart, there exist positive $R_i > r_i \to 0$ such that, for almost every $x_0 \in U$ and every $i \in \mathbb{N}$, there exist $x_1, \ldots, x_n \in U$ with $r_i < d(x_j, x_0) < R_i$ and

$$\max_{1 \leq j \leq n} \frac{|(\varphi(x_j) - \varphi(x_0)) \cdot v|}{d(x_j, x_0)} \geq \lambda$$

for every $v \in \mathbb{S}^{n-1}$. We apply Lemma 3.1.2 to obtain a Lipschitz function $F \colon X \to \mathbb{R}$ and $i(k) \to \infty$ such that, for almost every $x_0 \in S$, $\operatorname{Lip}(F, x_0) \geq \beta - \alpha$ and for every $k \in \mathbb{N}$ and $x \in X$ with $r_{i(k)} < d(x, x_0) < R_{i(k)}$,

$$\frac{|F(x) - F(x_0)|}{d(x, x_0)} \leq \alpha.$$

In particular, there exists a sequence $x_m \to x_0$ satisfying (2.3.1) for λ such that

$$\liminf_{m \to \infty} \max_{0 \leq i < n} \frac{|(f(x_{mn+i}) - f(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \leq \alpha.$$

Therefore, by Lemma 2.3.4 and our choice of α, F is not differentiable at almost every point of S. □

The following result first appeared in [BS13], however we now give a short proof as this result will be required when establishing our characterisations of Lipschitz differentiability spaces. We first require some notation.

Definition 3.1.4. Let (X, d) be a metric space, $S \subset X$ Borel and $x_0 \in S$. We say that S is *porous at* x_0 if there exists an $\eta > 0$ and $X \ni x_m \to x_0$ such that

$$B(x_m, \eta d(x_m, x_0)) \cap S = \emptyset.$$

Further, we say that S is *porous* if it is porous at x_0 for each $x_0 \in S$.

Corollary 3.1.5 ([BS13], Theorem 2.4)**.** *Porous sets in a Lipschitz differentiability space* (X, d, μ) *have measure zero.*

Proof. For any porous set $S \subset X$ there exists a countable Borel decomposition $S = \cup_i S_i \cup N$ and a sequence $\eta_i > 0$ such that $\mu(N) = 0$, each S_i is compact and contained within a structured chart and for each $i \in \mathbb{N}$ and $x_0 \in S_i$, there exists $x_m \to x_0$ such that

$$B(x_m, \eta_i d(x_m, x_0)) \cap S_i = \emptyset.$$

Then for any $r > 0$ and $i \in \mathbb{N}$, the function

$$x_0 \mapsto \sup\{d(x, x_0) < r : d(x, S_i) > \eta_i d(x, x_0)/2\}$$

is well defined, strictly positive and lower semicontinuous on S_i and so there exists an $\epsilon_{r,i} > 0$ such that it is bounded below by $\epsilon_{r,i}$. Then the functions

$$f_m(x) = \min\{d(x, S_i) - \epsilon_{1/m,i}, 0\}$$

satisfy the hypotheses of Proposition 3.1.3 for S_i and so S_i is μ-null. In particular, S is also μ-null. □

3.2 Remarks on Lipschitz differentiability spaces

We give several simple consequences of Proposition 3.1.3.

Corollary 3.2.1 ([BS13], Corollary 2.7)**.** *For any measurable subset* Y *of a Lipschitz differentiability space* (X, d, μ)*,* (Y, d, μ) *is a Lipschitz differentiability space with respect to the same chart structure.*

Moreover, the derivative of a real valued Lipschitz function in Y *agrees with the derivative of any Lipschitz extension of* f *to* X*, almost everywhere.*

Proof. Let $f\colon Y \to \mathbb{R}$ be Lipschitz and (U, φ) a λ-structured chart of dimension n in X. We may extend f to a Lipschitz function defined on X and so, for almost every $x_0 \in Y \cap U$, there exists a unique $Df(x_0)$ such that

$$\limsup_{X \ni x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

This $Df(x_0)$ will also be a derivative for the metric measure space (Y, d, μ), provided it is unique. As seen in Lemma 2.3.3 such a $Df(x_0)$ is not unique if and only if there exists a $v \in \mathbb{S}^{n-1}$ such that

$$\limsup_{Y \ni x \to x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} = 0.$$

However, since (U, φ) is a λ-structured chart in X, there exist $X \ni x_m \to x_0$ with

$$\lim_{m \to \infty} \frac{|(\varphi(x_m) - \varphi(x_0)) \cdot v|}{d(x_m, x_0)} \geq \lambda.$$

By combining these two relations and using the fact that φ is Lipschitz, we see that

$$B(x_m, \lambda d(x_m, x_0)/2 \operatorname{Lip} \varphi) \cap Y = \emptyset$$

for sufficiently large m. In particular, for each $v \in \mathbb{S}^{n-1}$, the set of such x_0 is a porous set in X. By taking the union of such x_0 over a countable dense subset of $\mathbb{S}^{n-1}$, we see that the set of those x_0 where $Df(x_0)$ is not unique is μ-null. Therefore the derivative of f in X is also the derivative of f in Y, almost everywhere. □

We also show how we may apply the construction of a non-differentiable Lipschitz function when we only have control over the infinitesimal behaviour of such a sequence of Lipschitz functions.

Corollary 3.2.2. *Let (X, d, μ) be a Lipschitz differentiability space, $S \subset X$ Borel and $L, \beta > 0$. Suppose that there exists a sequence of L-Lipschitz functions $f_m\colon X \to \mathbb{R}$ such that, for almost every $x_0 \in S$:*

- *There exists an $M \in \mathbb{N}$ such that, for each $m \geq M$, there exists an $x \in X$ with $0 < d(x, x_0) < 1/m$ and*

$$|f_m(x) - f_m(x_0)| \geq \beta d(x, x_0).$$

- *For each $m \in \mathbb{N}$*

$$\operatorname{Lip}(f_m, x_0) \leq 1/m.$$

Then S is μ-null.

Proof. By Lemma 2.3.7, it suffices to prove the result whenever S is contained within a structured chart (U, φ). For this, we will apply Proposition 3.1.3 to a suitable subset of S.

For any $\epsilon > 0$ there exist a Borel $S' \subset S$ with $\mu(S') \geq \mu(S) - \epsilon$, an $M \in \mathbb{N}$ and a sequence $\rho_m > 0$ such that:

- For every $m \geq M$, $x_0 \in S'$ and $y \in X$ with $0 < d(y, x_0) < \rho_m$,
$$\frac{|f_m(y) - f_m(x_0)|}{d(y, x_0)} \leq 2/m.$$
- For every $m \geq M$ and $x_0 \in S'$ there exists an $x \in X$ with $0 < d(x, x_0) < 1/m$ and
$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \geq \beta.$$

In particular, for any $x_0 \in S'$ and $y, z \in B(x_0, \rho_m) \cap S'$,

$$\frac{|f_m(y) - f_m(z)|}{d(y, z)} \leq 2/m.$$

Also observe that, for any $x_0 \in S'$, there either exists an $M' \geq M$ and for each $m \geq M'$ an $x \in S'$ with

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \geq \beta/2$$

or there exists $X \ni x_j \to x_0$ such that $B(x_j, \beta d(x_j, x_0)/2L) \cap S' = \emptyset$. Therefore, there exists a Borel decomposition $S' = S'' \cup P$ where P is a porous subset of X and S'' satisfies the hypotheses of Proposition 3.1.3 for the Lipschitz differentiability space (S', d, μ). Therefore S'' and so S' are μ-null. Finally, since $\epsilon > 0$ was arbitrary, S is also μ-null. □

Using the previous Corollary, we give an example of metric spaces that can only be Lipschitz differentiability spaces when they have zero measure.

Corollary 3.2.3. *Let (X, d, μ) be a Lipschitz differentiability space, $Y \subset X$ Borel and for each $m \in \mathbb{N}$ let N_m be a $1/m$-net of Y. Suppose that, for each $m \in \mathbb{N}$ and $q \in N_m$,*

$$\limsup_{X \ni x \to x_0} \frac{|d(x, q) - d(x_0, q)|}{d(x, x_0)} = 0$$

for almost every $x_0 \in Y \setminus \cup_m N_m$*. Then* $\mu(Y) = 0$*.*

In particular, for any metric measure space (X, d, μ) *and* $0 < \delta < 1$, (X, d^δ, μ) *is a Lipschitz differentiability space if and only if* $\mu(X) = 0$*.*

Proof. Suppose that (U, φ) is a structured chart in X and $S \subset Y \cap U \setminus \cup_m N_m$ is compact. Then for each $m \in \mathbb{N}$ there exists a finite $N'_m \subset N_m$ such that $B(N'_m, 1/m) \supset S$. We define the 1-Lipschitz function

$$f_m(x) = d(x, N'_m) = \min\{d(x, q) : q \in N'_m\}.$$

Then by the hypothesis on the N_m, $\operatorname{Lip}(f_m, x_0) = 0$ for almost every $x_0 \in S$.

Further, for any $x_0 \in S$ let $q \in N'_m$ with $d(x_0, q)$ minimal. Then $d(x_0, q) < 1/m$ and

$$\frac{|f_m(x_0) - f_m(q)|}{d(x, q)} = 1.$$

Therefore the hypotheses of Corollary 3.2.2 are satisfied for S', so that S' and hence S are μ-null. Since $S \subset U \setminus \cup_m N_m$ was an arbitrary compact set and each N_m is separated, $Y \cap U$ must also be μ-null.

Now let (X, d, μ) be a metric measure space and $0 < \delta < 1$. For any $a > 0$, $|(a + r)^\delta - a^\delta|/r^\delta \to 0$ as $r \to 0$. In particular, for any $x_0, z \in (X, d^\delta)$ with $x_0 \neq z$, we may use the triangle inequality in (X, d) to deduce

$$\limsup_{x \to x_0} \frac{|d^\delta(x, z) - d^\delta(x_0, z)|}{d^\delta(x, x_0)} \leq \frac{|(d(x_0, z) + d(x, x_0))^\delta - d^\delta(x_0, z)|}{d^\delta(x, x_0)} = 0.$$

Therefore, by choosing N_m to be any $1/m$-net of X, if (X, d^δ, μ) is a Lipschitz differentiability space we must have $\mu(X) = 0$. □

Finally, we address the notion of a zero dimensional chart in a Lipschitz differentiability space.

Corollary 3.2.4. *Let* (X, d, μ) *be a metric measure space and* $U \subset X$ *Borel. Suppose that, for any Lipschitz* $f \colon X \to \mathbb{R}$,

$$\limsup_{X \ni x \to x_0} \frac{|f(x) - f(x_0)|}{d(x, x_0)} = 0$$

for almost every $x_0 \in U$*. Then* $\mu(U) = 0$*.*

Proof. First observe that, for any $x_0 \in X$, the function $x \mapsto d(x, x_0)$ satisfies $\operatorname{Lip}(f, x_0) > 0$ and so singletons must have measure zero in U. In particular, almost every point of U is a limit point of U.

Now, for any $m \in \mathbb{N}$ let N_m be a $1/m$-net of U, $S \subset U \setminus N_m$ be compact and $N'_m \subset N_m$ be finite with $B(N'_m, 1/m) \supset S$. We define

$$f_m(x) = d(x, N'_m) = \min\{d(x,q) : q \in N'_m\}$$

and let $S' \subset S$ be a compact and $\rho > 0$ such that

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \leq 1/m$$

for each $x_0 \in S'$ and $0 < d(x, x_0) < \rho$. In particular, for any $x_0 \in S'$ and $y, z \in B(x_0, \rho) \cap S'$,

$$\frac{|f_m(y) - f_m(z)|}{d(y,z)} \leq 1/m.$$

Moreover, for any $x_0 \in S'$ there exists a $x \in U$ with $0 < d(x, x_0) < 1/m$ and

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} = 1.$$

We apply Lemma 3.1.2 to obtain a Lipschitz function $F \colon X \to \mathbb{R}$ with $\mathrm{Lip}(F, x_0) > 0$ for almost every $x_0 \in S'$. Therefore S' and hence S and X are μ-null. □

Remark 3.2.5. Suppose we choose to define the lim sup of a function at an isolated point to equal zero (as is the case in [Kei04]). Then this Corollary shows that a set in a Lipschitz differentiability space is a chart of dimension zero if and only if almost every point is isolated. This answers the question raised in Remark 2.1.3 of [Kei04] on the nature of charts of dimension zero.

3.3 Construction of a single Alberti representation

We now give a characterisation of those metric measure spaces with an Alberti representation in the direction of a given cone (see Definition 2.2.5). Our method relies upon the Gliksberg-König-Seever Decomposition Theorem, which we state first.

Theorem 3.3.1 ([Rud08], Theorem 9.4.4)**.** *Let (X, d, μ) be a compact metric measure space and K a weak*-compact, convex and non-empty set of regular Borel probability measures on X. Then there exists a Borel decomposition $X = A \cup S$ such that $\mu \llcorner A \ll \nu$ for some $\nu \in K$ and $k(S) = 0$ for every $k \in K$.*

We would like to apply this Theorem with K a set of normalised $\mathcal{H}^1$ measures supported on elements of a compact subset of $\Gamma(X)$ (see Definition 2.1.1). However, this set need not be compact. Instead, in the following Lemma, we apply it with K a set of normalised $\mathcal{H}^1$ measures supported on elements of a compact subset of $\Pi(X)$ (see Definition 2.1.1). Following this, we will use this Lemma to obtain a conclusion in terms of $\Gamma(X)$.

Lemma 3.3.2. *Let (X, d, μ) be a compact metric measure space and $J \subset \Pi(X)$ closed. Then there exists a Borel decomposition $X = A \cup S$ and an Alberti representation $\mathcal{A}$ of $\mu \llcorner A$ such that:*

1. *Almost every $\gamma \in \mathcal{A}$ belongs to J.*
2. *For every $\gamma \in J$, $\mathcal{H}^1(\gamma \cap S) = 0$.*

Remark 3.3.3. If μ gives measure zero to any set S that satisfies (2) then $\mathcal{A}$ is an Alberti representation of μ. Observe that, from the definition of an Alberti representation, this condition is also necessary for a representation of μ satisfying (1) to exist. Therefore, this Lemma gives a characterisation of the existence of such an Alberti representation.

Proof. If J is empty then setting $A = \emptyset$ and $\mathcal{A} = (0, \{\mathcal{H}^1 \llcorner \gamma\})$ completes the proof. Otherwise, for each $k \in \mathbb{N}$ let J_k be the set of k-bi-Lipschitz $\gamma \in J$ with $\operatorname{Dom}\gamma \subset [-k, k]$ and $\mathcal{L}^1(\operatorname{Dom}\gamma) \geq 1/k$. Then by Lemma 2.2.2, since $J \subset \Pi(X)$ is closed, each J_k is isometrically equivalent to a closed subset of the compact subsets of $[-k, k] \times X$ with the Hausdorff metric, and so is compact.

Now fix $k \in \mathbb{N}$ and for each $\gamma \in J_k$ let $\mathcal{H}_\gamma$ be the pushforward of the Lebesgue measure on $\operatorname{Dom}\gamma$ under γ, multiplied by a suitable scalar such that $\mathcal{H}_\gamma$ is a probability measure. Then since J_k is compact, the set of such measures is weak*-compact and so K, its weak*-closed convex hull, satisfies the hypothesis of Theorem 3.3.1. Moreover, probability measures on J_k are weak*-compact and so any measure $\nu \in K$ is of the form

$$\nu(A) = \int_{\Pi(X)} \mathcal{H}_\gamma(A) \mathrm{d}\mathbb{P}'(\gamma),$$

for some Borel probability measure $\mathbb{P}'$ on $\Pi(X)$ with $\mathbb{P}'(J_k) = 1$.

Therefore, Theorem 3.3.1 gives a decomposition $X = A_k \cup S_k$ where $\mu \llcorner A_k$ is absolutely continuous with respect to some $\nu \in K$ and $\mathcal{H}_\gamma(S_k) = 0$ for every $\gamma \in J_k$. In particular, an application of Lemma 2.1.3 constructs an Alberti representation $\mathcal{A}_k$ of $\mu \llcorner A_k$ from ν such that almost every $\gamma \in \mathcal{A}_k$ belongs to J_k and $\mathcal{H}^1(\gamma \cap S_k) = 0$ for every $\gamma \in J_k$. Finally, if we define $A = \cup_k A_k$ and $S = \cap_k S_k$, then $X = A \cup S$ is a Borel decomposition of X such that, by Lemma 2.1.4, $\mu \llcorner A$ has an Alberti representation $\mathcal{A}$ such that almost every $\gamma \in \mathcal{A}$ belongs to J, and such that S

satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in J$ with $\mathcal{L}^1(\operatorname{Dom}\gamma) > 0$. Since $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in J$ with $\mathcal{L}(\operatorname{Dom}\gamma) = 0$, this decomposition is of the required form. □

We summarise the exact setting in which we will use the above Lemma. Not only does the resulting Alberti representation have a direction but we may also control the magnitude of the partial derivatives of a finite number of Lipschitz functions. Later (in Definition 3.3.7), this will be developed into the notion of the *speed* of an Alberti representation.

Corollary 3.3.4. *Let (X, d, μ) be a compact metric measure space and, for some $n \in \mathbb{N}$, let $\varphi, \psi \colon X \to \mathbb{R}^n$ be Lipschitz. Then for any cone $C \subset \mathbb{R}^n$ and $\delta_1, \ldots, \delta_n > 0$, there exists a Borel decomposition $X = A \cup S$ such that:*

- *There exists an Alberti representation $\mathcal{A}$ of $\mu \llcorner A$ in the φ-direction of C such that, for almost every $\gamma \in \mathcal{A}$, γ belongs to $\Pi(X)$ and*
$$\|(\psi_i \circ \gamma)'(t_0)\| \geq \delta_i \operatorname{Lip}(\gamma, t_0)$$
for almost every $t_0 \in \operatorname{Dom}\gamma$ and every $1 \leq i \leq n$.
- *$\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Pi(X)$ in the φ-direction of C with*
$$\|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))\| \geq \delta_i \|\gamma(t) - \gamma(t')\| \tag{3.3.1}$$
for every $t, t' \in \operatorname{Dom}\gamma$ and every $1 \leq i \leq n$.

Proof. Let $\gamma_m \in \Pi(X)$ be a sequence of Lipschitz curves in the φ-direction of C and, for some $\gamma \in \Pi(X)$, suppose that $\gamma_m \to \gamma$. Then for any $t, t' \in \operatorname{Dom}\gamma$ and for each $m \in \mathbb{N}$ there exists $t_m, t'_m \in \operatorname{Dom}\gamma_m$ such that
$$\gamma_m(t_m) \to \gamma(t) \text{ and } \gamma_m(t'_m) \to \gamma(t').$$

Then
$$\begin{aligned}(\varphi(\gamma(t)) - \varphi(\gamma(t'))) \cdot w &= \lim_{m\to\infty} (\varphi(\gamma_m(t_m)) - \varphi(\gamma_m(t_m))) \cdot w \\ &\geq \lim_{m\to\infty} (1-\theta)\|\varphi(\gamma_m(t_m)) - \varphi(\gamma_m(t_m))\| \\ &= (1-\theta)\|\varphi(\gamma(t)) - \varphi(\gamma(t'))\|\end{aligned}$$

so that γ is in the φ-direction of C. Similarly, if each γ_m satisfies (3.3.1) for every $t, t' \in \operatorname{Dom}\gamma$ and every $1 \leq i \leq n$, then so does γ. Therefore, the set J of all

$\gamma \in \Pi(X)$ in the φ-direction of C that satisfy (3.3.1) for every $1 \le i \le n$ and every $t, t' \in \operatorname{Dom}\gamma$ is closed. Moreover, for any $\gamma \in J$,

$$\|(\psi_i \circ \gamma)'(t_0)\| \ge \delta_i \operatorname{Lip}(\gamma, t_0)$$

for almost every $t_0 \in \operatorname{Dom}\gamma$ and each $1 \le i \le n$. By applying Lemma 3.3.2 to J we obtain a decomposition $X = A \cup S$ of the required form. □

Observe that, if X is a metric space such that $\Pi(X)$ only contains Lipschitz curves γ with $\operatorname{Dom}\gamma$ a single point, then the decomposition $A = \emptyset$, $S = X$ satisfies the conclusion of the previous Lemma. To obtain a meaningful result, we will instead apply the Lemma to the closed, convex hull of an isometric copy of X within a Banach space $\mathcal{B}$. In this case $\Pi(\mathcal{B})$ is very rich and so we obtain a much stronger conclusion.

So that we may describe such an Alberti representation purely in the language of the metric space, we now investigate Lipschitz curves in the singular set S.

Lemma 3.3.5. *Let $\mathcal{B}$ be a Banach space, $X \subset \mathcal{B}$ closed and convex and let $\varphi, \psi\colon \mathcal{B} \to \mathbb{R}^n$ be Lipschitz. For a cone $C \subset \mathbb{R}^n$ and $\delta_1, \ldots, \delta_n > 0$ suppose that a Borel set $S \subset \mathcal{B}$ satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for all $\gamma \in \Pi(X)$ in the φ-direction of C with*

$$\|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))\| \ge \delta_i \|\gamma(t) - \gamma(t')\|$$

for every $t, t' \in \operatorname{Dom}\gamma$ and each $1 \le i \le n$. Then for any measurable $D \subset \mathbb{R}$ and Lipschitz $\gamma\colon D \to X$ the set

$$\{t_0 \in D : (\varphi \circ \gamma)'(t_0) \in C^\circ \text{ and } |(\psi_i \circ \gamma)'(t_0)| > \delta_i \operatorname{Lip}(\gamma, t_0)\ \forall\ 1 \le i \le n\}$$

is Lebesgue null.

Proof. Suppose that the conclusion does not hold for some measurable $D \subset \mathbb{R}$ of positive measure and $\gamma\colon D \to X$. We will construct a new function $\widetilde{\gamma} \in \Pi(X)$ that agrees with γ on a set of positive measure and satisfies the conditions given in the hypotheses of the Lemma, producing a contradiction.

By standard measure theoretic techniques, there exist $\Phi, \Psi \in \mathbb{R}^n$, $b, \xi > 0$ and $\delta_i' > \delta_i$ with

$$\overline{B}(\Phi, \xi) \subset C \text{ and each } |\Psi_i| \ge \delta_i' b$$

such that, for every $\epsilon > 0$, there exists a bounded $D' \subset D$ of positive measure with

$$\|(\varphi \circ \gamma)'(t_0) - \Phi\| < \epsilon,\ \ |(\psi_i \circ \gamma)'(t_0)| > |\Psi_i| \text{ and } \operatorname{Lip}(\gamma, t_0) < b$$

for every $1 \leq i \leq n$ and $t_0 \in D'$. Let D' be such a set for some choice of

$$0 < \epsilon < \min\{\|\Phi\|, \xi\}.$$

Further, let $R > 0$ and $D'' \subset D'$ have positive measure such that, for every $t_0 \in D''$ and every $t \in \operatorname{Dom} \gamma$ with $|t - t_0| \leq R$,

$$\|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - \Phi(t - t_0)\| \leq \epsilon |t - t_0|,$$

for each $1 \leq i \leq n$

$$|\psi_i(\gamma(t)) - \psi_i(\gamma(t_0))| \geq |\Psi_i(t - t_0)|,$$

and

$$\|\gamma(t) - \gamma(t_0)\| \leq b|t - t_0|.$$

We set D_0 to be the intersection of D'' with an interval of diameter R chosen so that D_0 has positive measure, and I to be the smallest closed interval containing D_0. Finally, we define $\widetilde{\gamma}$ to equal γ on D_0 and extend $\widetilde{\gamma}$ to I whilst maintaining the Lipschitz constant by first extending to the closure of D_0 and then linearly on the connected components of the complement.

First observe that, since X is closed and convex, $\operatorname{Im} \widetilde{\gamma} \subset X$. Further, since $\gamma|_{D_0}$ is in the φ-direction of C, $\mathcal{H}^1(\gamma(D_0)) > 0$. Now, for any connected component (c, d) of $I \setminus \overline{D}_0$, let $c_m, d_m \in D_0$ with $c_m \to c$ and $d_m \to d$. Then

$$\begin{aligned}\|\varphi(\widetilde{\gamma}(c)) - \varphi(\widetilde{\gamma}(d)) - \Phi(c - d)\| &= \lim_{m \to \infty} \|\varphi(\widetilde{\gamma}(c_m)) - \varphi(\widetilde{\gamma}(d_m)) - \Phi(c_m - d_m)\| \\ &\leq \epsilon |c - d|.\end{aligned}$$

Similarly

$$|\psi_i(\widetilde{\gamma}(c)) - \psi_i(\widetilde{\gamma}(d))| \geq |\Psi_i(c - d)|$$

for each $1 \leq i \leq n$ and

$$\|\widetilde{\gamma}(c) - \widetilde{\gamma}(d)\| \leq b|c - d|.$$

In particular, since $\widetilde{\gamma}$ is extended linearly to (c, d), for almost every $t_0 \in I$,

$$\|(\varphi \circ \widetilde{\gamma})'(t_0) - \Phi\| \leq \epsilon$$

so that $\tilde{\gamma}$ is in the φ-direction of C. Similarly

$$|(\psi_i \circ \widetilde{\gamma})'(t_0)| \geq |\Psi_i|$$

for each $1 \leq i \leq n$ and, for every $t, t' \in I$,

$$\|\widetilde{\gamma}(t) - \widetilde{\gamma}(t')\| \leq b|t - t'|.$$

Therefore, for any $t \leq t' \in I$,

$$\begin{aligned}
\|\widetilde{\gamma}(t) - \widetilde{\gamma}(t')\| &\geq \|\varphi(\widetilde{\gamma}(t)) - \varphi(\widetilde{\gamma}(t'))\| \\
&= \left\| \int_{[t,t']} (\varphi \circ \widetilde{\gamma})'(t_0) \right\| \\
&\geq (\|\Phi\| - \epsilon)|t - t'|
\end{aligned}$$

so that $\widetilde{\gamma}$ is bi-Lipschitz and so belongs to $\Pi(X)$. Similarly, for any $1 \leq i \leq n$,

$$\begin{aligned}
|\psi_i(\widetilde{\gamma}(t)) - \psi_j(\widetilde{\gamma}(t'))| &= \left| \int_{[t,t']} (\psi_i \circ \widetilde{\gamma})'(t_0) \right| \\
&\geq \delta_i' b|t - t_0| \\
&\geq \delta_i \|\widetilde{\gamma}(t) - \widetilde{\gamma}(t')\|.
\end{aligned}$$

Therefore $\widetilde{\gamma}$ satisfies the hypotheses of the Lemma and so

$$0 = \mathcal{H}^1(\tilde{\gamma} \cap S) \geq \mathcal{H}^1(\tilde{\gamma}(D_0)) > 0,$$

a contradiction. □

To apply these results to a metric measure space, we now define an embedding into a Banach space that exposes additional structure given by some Lipschitz functions defined on the metric space.

Definition 3.3.6. For $n \in \mathbb{N}$ we define $\mathcal{B}$ to be the Banach space

$$\mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n \times \ell_\infty$$

with norm

$$\|(u, v, s)\| = \max\{\|u\|_{\mathbb{R}^n}, \|v\|_{\mathbb{R}^n}, \|s\|_{\ell_\infty}\}.$$

Further, suppose that (X, d) is a metric space and $\varphi, \psi \colon X \to \mathbb{R}^n$ Lipschitz. We fix a bi-Lipschitz embedding $\iota \colon X \to \ell_\infty$ and define the bi-Lipschitz embedding $\iota^* \colon X \to \mathcal{B}$ by

$$x \mapsto (\varphi(x), \psi(x), \iota(x)).$$

We identify X with its image in $\mathcal{B}$ under ι^* and note that the projection P_1 onto the first factor (respectively P_2 onto the second factor) in $\mathcal{B}$ agrees with φ (respectively ψ) on X. In particular, for any $\gamma \in \Gamma(X)$, the derivative $(\varphi \circ \gamma)'$ (respectively $(\psi \circ \gamma)'$) agrees with $(P_1 \circ \gamma)'$ (respectively $(P_2 \circ \gamma)'$) almost everywhere.

Finally, we define an additional property of an Alberti representation. The *speed* of an Alberti representation quantitatively describes how the partial derivative of a Lipschitz function with respect to the Alberti representation compares to the infinitesimal behaviour of such a function. This notion will have a particular importance when characterising Lipschitz differentiability spaces when we will be interested in Alberti representations with speed independent of the Lipschitz function.

Definition 3.3.7. Let (X, d) be a metric space, $\varphi \colon X \to \mathbb{R}^n$ Lipschitz and $\delta > 0$. We say that $\gamma \in \Gamma(X)$ has *speed* δ (or φ*-speed* δ whenever the implied Lipschitz function is not clear) if for almost every $t_0 \in \operatorname{Dom}\gamma$,

$$\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \operatorname{Lip}(\varphi, \gamma(t_0)) \operatorname{Lip}(\gamma, t_0).$$

Further, we say that an Alberti representation $\mathcal{A}$ has *speed* δ (or φ-speed δ) if almost every $\gamma \in \mathcal{A}$ has speed δ.

Corollary 3.3.8. *Let (X, d, μ) be a metric measure space and $\varphi, \psi \colon X \to \mathbb{R}^n$ Lipschitz such that $\operatorname{Lip}(\psi_i, x_0) > 0$ for each $1 \leq i \leq n$. Then, for any cone $C \subset \mathbb{R}^n$ and $\delta_1, \ldots, \delta_n > 0$, there exists a Borel decomposition $X = A \cup S$ such that:*

- *There exists an Alberti representation of $\mu \llcorner A$ in the φ-direction of C with ψ_i-speed δ_i for each $1 \leq i \leq n$.*
- *$\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C° with ψ_i-speed strictly greater than δ_i, for each $1 \leq i \leq n$.*

In particular, for any cone $C \subset \mathbb{R}^n$ there exists a Borel decomposition $X = A \cup S$ where $\mu \llcorner A$ has an Alberti representation in the φ-direction of C and $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C.

Proof. Fix $\lambda > 1$, $k_1, \ldots, k_n \in \mathbb{Z}$ and let $Y \subset X$ be compact with

$$\lambda^{-k_i} \geq \operatorname{Lip}(\psi_i, x_0) \geq \lambda^{-k_i - 1}$$

for every $x_0 \in Y$ and every $1 \leq i \leq n$. Since Y is compact, it's closed convex hull $\widetilde{Y}$ is also compact. By applying Corollary 3.3.4 to $(\widetilde{Y}, d, \mu)$ we obtain a Borel decomposition $Y = Y_a \cup Y_s$ such that:

- There exists an Alberti representation $\mathcal{A}$ of $\mu \llcorner Y_a$ in the φ-direction of C such that
$$|(\psi_i \circ \gamma)'(t_0)| \geq \delta_i \lambda^{k_i} \operatorname{Lip}(\gamma, t_0)$$
for almost every $\gamma \in \mathcal{A}$, almost every $t_0 \in \operatorname{Dom} \gamma$ and every $1 \leq i \leq n$.

- $\mathcal{H}^1(\gamma \cap Y_s) = 0$ for every $\gamma \in \Pi(\widetilde{Y})$ in the φ-direction of C with
$$|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))| \geq \delta_i \lambda^{k_i} \|\gamma(t) - \gamma(t')\|$$
for each $t, t' \in \operatorname{Dom} \gamma$ and every $1 \leq i \leq n$.

By our choice of λ, $\mathcal{A}$ has ψ_i-speed δ_i for each $1 \leq i \leq n$. Further, by Lemma 3.3.5, for any $\gamma \in \Gamma(Y)$ in the φ-direction of C° with each ψ_i-speed strictly greater than $\delta_i \lambda$, $\mathcal{H}^1(\gamma \cap S) = 0$.

By varying the $k_i \in \mathbb{Z}$, there exists a countable cover of X by such Y except for a μ-null set and so, after combining the respective representations using Lemma 2.1.4, for each $\lambda > 1$ there exists a Borel decomposition $X = A_\lambda \cup S_\lambda$ where A_λ has an Alberti representation in the φ-direction of C with ψ_i-speed δ_i for each $1 \leq i \leq n$ and such that S satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every γ in the φ-direction of C° with each ψ_i-speed greater than $\delta_i \lambda$. Writing $A = \cup_\lambda A_\lambda$ and $S = \cap_\lambda S_\lambda$ (where the union and intersection are taken over $\lambda \in \mathbb{Q},\ \lambda > 1$), Lemma 2.1.4 completes the proof. □

3.4 Construction of multiple Alberti representations

We now improve upon Corollary 3.3.8 by producing multiple independent Alberti representations of a metric measure space (X, d, μ).

Suppose that $\varphi \colon X \to \mathbb{R}^n$ is Lipschitz and that $C_1, \ldots, C_n$ is a collection of independent cones in $\mathbb{R}^n$. Then, by multiple applications of Corollary 3.3.8, there exists a Borel decomposition $X = A \cup S$ such that $\mu \llcorner A$ has n φ-independent Alberti representations and S has a Borel decomposition $S = S_1 \cup \ldots \cup S_n$ such that, for each $1 \leq i \leq n$, $\mathcal{H}^1(\gamma \cap S_i) = 0$ for each $\gamma \in \Gamma(X)$ in the φ-direction of C_i.

However, as we will see in the next section, the method of constructing a non-differentiable Lipschitz function on S requires, for any $\epsilon > 0$, a decomposition as above such that each C_i has width $1 - \epsilon$. Of course, for sufficiently small ϵ, these cones will not be independent. So that we may produce independent Alberti representations and also satisfy this condition for S, we introduce a method that reduces the width of a cone that defines the direction of an Alberti representation, at the expense of a countable decomposition.

Corollary 3.4.1. *Let (X, d, μ) be a metric measure space, $A \subset X$ Borel and $\varphi, \psi \colon X \to \mathbb{R}^n$ Lipschitz. Suppose that, for some cone $C = C(w, \theta) \subset \mathbb{R}^n$, $\mu \llcorner A$ has an Alberti representation $\mathcal{A}$ in the φ-direction of C such that $|(\psi_i \circ \gamma)'(t_0)| > 0$ for almost every $\gamma \in \mathcal{A}$, almost every $t_0 \in \operatorname{Dom} \gamma$ and every $1 \leq i \leq n$. Then, for any countable collection of cones C_m with*

$$\bigcup_{m \in \mathbb{N}} C_m^\circ \supset C \setminus \{0\},$$

there exists a countable Borel decomposition $A = \cup_k A_k$ such that each $\mu \llcorner A_k$ has an Alberti representation $\mathcal{A}_k$ in the φ-direction of some C_m with ψ_i-speed strictly greater than $1/k$, for each $1 \leq i \leq n$.

Moreover, if for some $\delta_1, \ldots, \delta_n > 0$, $\mathcal{A}$ has ψ_i-speed strictly greater than δ_i for each $1 \leq i \leq n$, then so does each $\mathcal{A}_k$.

Proof. By applying Corollary 3.3.8 using each cone C_m in the hypotheses and each $\delta_i = 1/k$, we obtain a countable Borel decomposition of A into a sets A_k such that each $\mu \llcorner A_k$ has an Alberti representation of the required form and a set S that satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for each $\gamma \in \Gamma(X)$ in the φ-direction of some C_m° and with positive ψ_i-speed for each $1 \leq i \leq n$. Therefore, since the C_m° cover $C \setminus \{0\}$, S satisfies this for all γ in the φ-direction of C with positive ψ_i-speed for each $1 \leq i \leq n$. Since $\mu \llcorner A$ is represented by $\mathcal{A}$, $\mu(S) = 0$.

Now suppose that $\delta_i' > \delta_i$ and that $\mathcal{A}$ has ψ_i-speed δ_i' for each $1 \leq i \leq n$. Then if we repeat the same process but for each $1 \leq i \leq n$ take each $\mathcal{A}_k$ to have ψ_i-speed between δ_i' and δ_i, we obtain the same decomposition but with S satisfying $\mathcal{H}^1(\gamma \cap S) = 0$ for each γ in the φ-direction of C with ψ_i-speed δ_i' for each $1 \leq i \leq n$. Again, since $\mu \llcorner A$ is represented by $\mathcal{A}$, $\mu(S) = 0$. □

Definition 3.4.2. We will refer to the above process of obtaining new Alberti representations with the same properties as an existing representation but in the direction of thinner cones as *refining* an Alberti representation.

The requirement in the previous Corollary that both φ and ψ take values in the same Euclidean space is not necessary. Indeed, suppose that (X, d, μ) is a metric measure space satisfying the hypotheses of Corollary 3.4.1 for $\varphi \colon X \to \mathbb{R}^n$ and $\psi \colon X \to \mathbb{R}^m$ Lipschitz. If $m < n$ then by repeating any $n - m + 1$ coordinate functions of ψ, we obtain a Lipschitz function $\widetilde{\psi}$ into $\mathbb{R}^n$. Similarly, if $n < m$, we may define a Lipschitz function $\widetilde{\varphi}$ into $\mathbb{R}^m$ and $\widetilde{w} \in \mathbb{R}^m$ by $\widetilde{\varphi}_i = \varphi_i$ and $\widetilde{w}_i = w_i$ for $1 \leq i \leq n$ and $\widetilde{\varphi}_i = \widetilde{w}_i = 0$ for $n < i \leq m$. Then a Lipschitz curve is in the φ-direction of $C(w, \theta)$ if and only if it is in the $\widetilde{\varphi}$-direction of $C(\widetilde{w}, \theta)$. In both cases, the original hypotheses of Corollary 3.4.1 are satisfied and it's conclusion gives us Alberti representations of the required form.

We will require refinements of Alberti representations that maintain the speed of a vector valued function. Suppose an Alberti representation $\mathcal{A}$ of (X, d, μ) is in the φ-direction of a cone C, so that $\|(\varphi \circ \gamma)'(t_0)\| > 0$ for almost every $\gamma \in \mathcal{A}$ and almost every $t_0 \in \gamma$. Then we may refine $\mathcal{A}$ into Alberti representations, each with positive φ-speed. Further, suppose that $\mathcal{A}$ has φ-speed strictly greater than δ, for some $\delta > 0$. Then there exists a countable Borel decomposition $X = \cup_i X_i$ and for each i, rational $\delta_1 \ldots, \delta_n > 0$ with $\delta_1^2 + \ldots + \delta_n^2 > \delta^2$ such that $\mathcal{A} \llcorner X_i$ has φ_i-speed strictly greater than δ_i, for each $1 \leq i \leq n$. Therefore, we may take arbitrary refinements of $\mathcal{A}$, each with φ-speed strictly greater than δ.

As in previous constructions, we will find multiple independent Alberti representations of a metric measure space (X, d, μ) by decomposing X into two sets; one set on which μ has many independent Alberti representations and another, singular, set. We now define the general form that such a singular set takes. In fact, we will later see that such sets play a fundamental role when determining whether a metric measure space is a Lipschitz differentiability space. These sets, and the methods involving them in the following section, are a natural generalisation of those considered in [ACP] in which the authors investigate measures on Euclidean space with respect to which Rademacher's Theorem holds.

Definition 3.4.3. Let (X, d) be a metric space, $\varphi \colon X \to \mathbb{R}^n$ Lipschitz and $\delta, \theta, \lambda > 0$. We define $\widetilde{A}(\varphi; \delta, \theta, \lambda)$ to be the set of $S \subset X$ such that:

- For every $x_0 \in S$,

$$\operatorname{Lip}(v \cdot \varphi, x_0) > \lambda \operatorname{Lip}(\varphi, x_0) \ \forall v \in \mathbb{S}^{n-1}.$$

- There exists a countable Borel decomposition $S = \cup_k S_k$ and a countable collection of closed cones $C_k \subset \mathbb{R}^n$ of width $1 - \theta$ such that each S_k satisfies $\mathcal{H}^1(\gamma \cap S_k) = 0$ for each $\gamma \in \Gamma(X)$ in the φ-direction of C_k with φ-speed δ.

Further, we define $\widetilde{A}(\delta, \theta, \lambda)$ to be the set of $S \subset X$ for which there exists a countable Borel decomposition $S = \cup_k S_k$ and a sequence of Lipschitz functions $\varphi_k \colon X \to \mathbb{R}^{n_k}$ such that $S_k \in \widetilde{A}(\varphi_k; \delta, \theta, \lambda)$ for each $k \in \mathbb{N}$. Finally, we define $\widetilde{A}(\varphi)$, respectively $\widetilde{A}$, to be the set of $S \subset X$ for which there exists a $\lambda > 0$ such that $S \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$, respectively $S \in \widetilde{A}(\delta, \theta, \lambda)$, for every $\delta, \theta > 0$.

Remark 3.4.4. Observe that $\widetilde{A}(\varphi; \delta, \theta, \lambda) \subset \widetilde{A}(\varphi; \delta', \theta', \lambda')$ whenever $\delta \leq \delta'$, $\theta \geq \theta'$ and $\lambda \leq \lambda'$, and hence $\widetilde{A}(\delta, \theta, \lambda) \subset \widetilde{A}(\delta', \theta', \lambda')$.

By combining the previous results of this section we may construct many independent Alberti representations of a measure, one-by-one.

Proposition 3.4.5. *Let (U, φ) be a λ-structured chart of dimension n in a metric measure space (X, d, μ). Then for any $0 < \delta, \theta < 1$ there exists a countable Borel decomposition*

$$U = S \cup \bigcup_{i \in \mathbb{N}} U_i$$

such that each $\mu \llcorner U_i$ has n φ-independent Alberti representations with φ-speed strictly greater than δ and S belongs to $\widetilde{A}(\varphi; \delta, \theta, \lambda / \operatorname{Lip} \varphi)$.

Proof. Observe that it suffices to prove the result for $\operatorname{Lip} \varphi = 1$. In this case, by applying Corollary 3.3.8 using a cone of width $1 - \theta$, there exists a Borel decomposition $U = U' \cup S$ such that $\mu \llcorner U'$ has an Alberti representation with φ-speed strictly greater than δ and $S \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$.

Now suppose that, for some $1 \leq m < n$, there exists a countable Borel decomposition $U = \cup_i U_i \cup S$ such that each $\mu \llcorner U_i$ has m independent Alberti representations with φ-speed strictly greater than δ and $S \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$. Then, by refining if necessary, we may suppose that the representations of each $\mu \llcorner U_i$ are in the φ-direction of cones of width α, for some $0 < \alpha < \sqrt{1 - \theta^2}$.

Now fix an $i \in \mathbb{N}$ and let $w_1, \ldots, w_m \in \mathbb{S}^{n-1}$ such that $C(w_1, \alpha), \ldots, C(w_m, \alpha)$ are independent and $\mu \llcorner U_i$ has an Alberti representation in the φ-direction of $C(w_i, \alpha)$, for each $1 \leq i \leq m$. We choose $w_{m+1} \in \mathbb{S}^{n-1}$ orthogonal to $w_1, \ldots, w_m$ so that, by our choice of α, $C(w_1, \alpha), \ldots, C(w_m, \alpha), C(w_{m+1}, \theta)$ are independent cones in $\mathbb{R}^n$. By applying Corollary 3.3.8 using $C(w_{m+1}, \theta)$, there exists a Borel decomposition $U_i = U_i' \cup S_i$ such that $\mu \llcorner U_i'$ has $m + 1$ independent Alberti representations with φ-speed strictly greater than δ and $S_i \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$.

Since $S \cup S_1 \cup S_2 \cup \ldots \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$, we see that there exists a countable Borel decomposition $X = \cup_i U_i' \cup S'$ such that each $\mu \llcorner U_i'$ has $m + 1$ independent Alberti representations and $S' \in \widetilde{A}(\varphi; \delta, \theta, \lambda)$. By repeating this process $n - 1$ times we obtain the required decomposition. □

Theorem 3.4.6. *Let (U, φ) be a λ-structured chart of dimension n in a metric measure space (X, d, μ). Then there exists a countable Borel decomposition $U = \cup_i U_i$ such that each $\mu \llcorner U_i$ has n φ-independent Alberti representations if and only if, for every $S \in \widetilde{A}(\varphi)$, $\mu(S) = 0$.*

Proof. First suppose that, for every $S \in \widetilde{A}(\varphi)$, $\mu(S \cap U) = 0$. By the previous Proposition, for each $m \in \mathbb{N}$ there exists a Borel decomposition $U = U_m \cup S_m$ such that U_m

has a countable decomposition of the required form and $S_m \in \widetilde{A}(\varphi; 1/m, 1/m, \lambda)$. This gives a Borel decomposition $U = S \cup U_1 \cup U_2 \cup \ldots$ where each U_i is of the required form and $S = \cap_m S_m \in \widetilde{A}(\varphi)$, as required.

Conversely, suppose that such a decomposition exists. Then by refining each Alberti representation if necessary, for each U_j we may suppose that there exists a $\delta > 0$ and independent cones $C_1, \ldots, C_n$ such that for each $1 \leq i \leq n$, $\mu \llcorner U_j$ has an Alberti representation in the φ-direction of C_i with φ-speed δ. For such a U_j there exists an $m \in \mathbb{N}$ such that $1/m < \delta$ and such that any cone of width $1 - 1/m$ must completely contain one of the C_i. Since $\mu \llcorner U_j$ has the above Alberti representations, for any $S \in \widetilde{A}(\varphi; 1/m, 1/m, \lambda)$, $\mu(S \cap U_j) = 0$. In particular, for any $S \in \widetilde{A}(\varphi)$, $\mu(S \cap U_j) = 0$ and hence $\mu(S \cap U) = 0$. □

Chapter 4

Alberti representations of Lipschitz differentiability spaces

By combining our results from the previous chapter, we now prove the existence of independent Alberti representations in Lipschitz differentiability spaces (Theorem 4.2.3). In light of the previous chapter, the main focus here will be on constructing the Lipschitz functions required to apply Proposition 3.1.3 to any $S \in \widetilde{A}(\varphi)$, for any structured chart (U, φ) in a metric measure space.

4.1 Constructions in Banach spaces

We will again embed our metric space into a Banach space. This provides many Lipschitz curves that we may use to define the Lipschitz functions required for Proposition 3.1.3.

In this section we fix the following notation for simplicity.

Notation 4.1.1. For $n \in \mathbb{N}$ we let $\mathcal{B}$ to be the Banach space defined in Definition 3.3.6, let $\varphi \colon \mathcal{B} \to \mathbb{R}^n$ be the projection onto the first factor and fix $w \in \mathbb{S}^{n-1}$. We also fix X a compact subset of $\mathbb{R}^n \times \{0\} \times \ell_\infty \subset \mathcal{B}$, $0 < \delta, \theta < 1$ and $S \subset X$ closed with $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Gamma(X)$ in the φ-direction of $C(w, \theta)$ with φ-speed δ.

For each $t \in \mathbb{R}$ we define $\mathcal{P}_t \colon \mathbb{R}^n \to \mathbb{R}^n$ to be the orthogonal projection onto

$$\{v \in \mathbb{R}^n : (v - tw) \cdot w = 0\}$$

and define $P_t \colon \mathcal{B} \to \mathcal{B}$ by

$$P_t(u, v, s) = (\mathcal{P}_t(u), v, s).$$

We also define $\mathcal{P} = \mathcal{P}_0$, $\varphi_0 = \inf\{\varphi(x)\cdot w : x \in X\}$, $P = P_{\varphi_0}$ and Ω to be the closed, convex hull of $P(X) \cup X$, a compact, convex set.

Further, for any measurable $I \subset \mathbb{R}$ and Lipschitz $\gamma\colon I \to \Omega$, we express

$$\gamma = (\varphi \circ \gamma, 0, \gamma^*)$$

and write $\mathcal{V}_\gamma(t) = \mathrm{Lip}(\gamma^*, t)$, so that

$$\mathrm{Lip}(\gamma, t) = \max\{\|(\varphi \circ \gamma)'(t)\|, \mathcal{V}_\gamma(t)\}$$

for almost every $t \in I$.

In addition, for any Borel $V \subset \mathcal{B}$ and $\epsilon > 0$, we define

$$Q(V,\gamma,\epsilon) = \int_{\gamma^{-1}(\Omega\setminus V)} (\varphi\circ\gamma)'\cdot w + \int_{\gamma^{-1}(\Omega)} \big(K(\theta)\|\mathcal{P}((\varphi\circ\gamma)')\| + \delta\mathcal{V}_\gamma\big) + \epsilon\mathcal{H}^1(\gamma),$$

where $K(\theta) = (1-\theta)/\sqrt{\theta(2-\theta)}$.

Lemma 4.1.2. *For any $\epsilon > 0$ there exists a $\mathcal{B}$-open set $V \supset S$ such that, for any compact interval I and Lipschitz $\gamma\colon I \to \Omega$ with $(\varphi\circ\gamma)'\cdot w \geq 0$ almost everywhere,*

$$Q(V,\gamma,\epsilon) \geq (\varphi(\gamma_e) - \varphi(\gamma_s))\cdot w - \epsilon,$$

where γ_s and γ_e are the endpoints of γ.

Proof. Suppose that the conclusion is false. Then there exists an $\epsilon > 0$ and for each $m \in \mathbb{N}$ a Lipschitz γ_m that violates the inequality for the open set $V_m = B(S, 1/m)$. Since Ω is compact, by replacing ϵ by $\epsilon/2$ if necessary, we may suppose that each γ_m has the same endpoints, γ_s and γ_e. In particular, for each $m \in \mathbb{N}$,

$$\mathcal{H}^1(\gamma_m) \leq Q(V_m,\gamma_m,\epsilon)/\epsilon \leq (\varphi(\gamma_e)) - \varphi(\gamma_s))\cdot w/\epsilon.$$

Therefore, there exists an $L > 0$ and a reparametrisation of each γ_m such that each is a 1-Lipschitz function defined on $[0, L]$. Further, by the Arzelà-Ascoli theorem and after possibly choosing a suitable subsequence, there exists a 1-Lipschitz $\gamma\colon [0,L] \to \Omega$ such that $\gamma_m \to \gamma$ uniformly. Moreover, since $(\varphi\circ\gamma_m)' \geq 0$ almost everywhere for each $m \in \mathbb{N}$, $(\varphi\circ\gamma)' \geq 0$ almost everywhere. We will show that the image of γ intersects S in a set of positive measure from which we deduce a contradiction to our hypothesis.

Fix an $m \in \mathbb{N}$ and $\eta > 0$. Since $\beta = \gamma^{-1}(\Omega \setminus \overline{V_m})$ is an open subset of $\mathbb{R}$ of finite measure, there exist disjoint compact intervals $\beta_1, \ldots, \beta_N \subset \beta$ such that

$\mathcal{L}^1(\beta \setminus \cup_i \beta_i) \leq \eta$. Therefore,

$$\int_{\gamma^{-1}(\Omega\setminus\overline{V_m})} (\varphi\circ\gamma)'\cdot w \leq \int_{\cup_i\beta_i} (\varphi\circ\gamma)'\cdot w + \eta \operatorname{Lip}\varphi.$$

Moreover, by the lower semicontinuity of the total variation of Lipschitz functions under uniform convergence,

$$\int_{\cup_i\beta_i} (\varphi\circ\gamma)'\cdot w \leq \liminf_{k\to\infty} \int_{\cup_i\beta_i} (\varphi\circ\gamma_k)'\cdot w.$$

Now, since $\gamma_k \to \gamma$ uniformly and each $\gamma(\beta_i)$ is compact, there exists a $K\in\mathbb{N}$ such that, for all $k \geq K$ and $1\leq i\leq N$, $\gamma_k(\beta_i)\subset \Omega\setminus\overline{V_m}$. Therefore, since $(\varphi\circ\gamma_k)'\cdot w\geq 0$ almost everywhere for each k,

$$\int_{\gamma^{-1}(\Omega\setminus\overline{V_m})} (\varphi\circ\gamma)'\cdot w \leq \liminf_{k\to\infty} \int_{\gamma_k^{-1}(\Omega\setminus\overline{V_m})} (\varphi\circ\gamma_k)'\cdot w + \eta \operatorname{Lip}\varphi.$$

Further, by again using the lower semicontinuity of total variation and of $\alpha\mapsto\mathcal{H}^1(\alpha)$,

$$Q(\overline{V_m},\gamma,\epsilon) \leq \liminf_{k\to\infty} Q(\overline{V_m},\gamma_k,\epsilon) + \eta\operatorname{Lip}\varphi.$$

Finally, since $(\varphi\circ\gamma)'\geq 0$ almost everywhere and $V_k\subset V_m$ for every $k\geq m$,

$$Q(\overline{V_m},\gamma,\epsilon) \leq (\varphi(\gamma_e)-\varphi(\gamma_s))\cdot w - \epsilon + \eta\operatorname{Lip}\varphi.$$

This is true for all $m\in\mathbb{N}$ and $\eta>0$. Moreover, since S is closed, $\cap_m\overline{V_m}=S$ and so

$$Q(S,\gamma,\epsilon) \leq (\varphi(\gamma_e)-\varphi(\gamma_s))\cdot w - \epsilon.$$

In particular, by the fundamental theorem of calculus,

$$\epsilon \leq \int_{\gamma^{-1}(S)} (\varphi\circ\gamma)'\cdot w - \int_{\gamma^{-1}(\Omega)} \big(K(\theta)\|\mathcal{P}((\varphi\circ\gamma)')\| + \delta\mathcal{V}_\gamma\big) - \epsilon\mathcal{H}^1(\gamma). \tag{4.1.1}$$

On the other hand, when γ is restricted to

$$D := \{t_0 : (\varphi\circ\gamma)'(t_0)\cdot w > (1-\theta)\|(\varphi\circ\gamma)'(t_0)\| \geq (1-\theta)\delta\operatorname{Lip}(\gamma,t_0)\},$$

it is a Lipschitz function in the φ-direction of C with φ-speed δ. Moreover, $\operatorname{Lip}(\gamma,t_0)>0$ for every $t_0\in D$ and so we may decompose D into a Lebesgue null set and a countable collection of compact sets K_i on which γ is bi-Lipschitz. In particular, each $\gamma|_{K_i}\in\Gamma(X)$ and so $D\cap\gamma^{-1}(S)$ is Lebesgue null. Therefore, for almost every

$t_0 \in \gamma^{-1}(S)$, either

$$(\varphi\circ\gamma)'(t_0) \le (1-\theta)\|(\varphi\circ\gamma)'(t_0)\| \le K(\theta)\|\mathcal{P}(\varphi\circ\gamma)'(t_0)\|,$$

or

$$\|(\varphi\circ\gamma)'(t_0)\| \le \delta\,\mathrm{Lip}(\gamma,t_0) \text{ and so } \mathrm{Lip}(\gamma,t_0)=\mathcal{V}_\gamma(t_0).$$

Thus

$$\int_{\gamma^{-1}(S)} (\varphi\circ\gamma)'\cdot w \le \int_{\gamma^{-1}(S)} \left(K(\theta)\|\mathcal{P}((\varphi\circ\gamma)')\| + \delta\mathcal{V}_\gamma\right). \qquad (4.1.2)$$

By combining equations (4.1.1) and (4.1.2) we obtain $\epsilon \le 0$, giving the required contradiction. □

We use this result to construct a Lipschitz function on Ω defined via Lipschitz curves connecting points in Ω. We will see that functions of this form have the properties required to apply Proposition 3.1.3.

Lemma 4.1.3. *For any $\epsilon > 0$ there exists a $\mathcal{B}$-open set $V \supset S$ and a $(1+K(\theta)+\delta+\epsilon)$-Lipschitz function $f\colon \Omega \to \mathbb{R}$ such that:*

1. *For every $x, x_0 \in \Omega$ with $(\varphi(x)-\varphi(x_0))\cdot w \ge 0$,*

$$f(x) - f(x_0) \ge (\varphi(x)-\varphi(x_0))\cdot w - \epsilon.$$

2. *For every y, z contained in a ball $B \subset V$,*

$$|f(y)-f(z)| \le K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (\delta+\epsilon)\|y-z\|.$$

Proof. For $\epsilon > 0$ let $V \supset S$ be the Ω-open set obtained from an application of Lemma 4.1.2. By our definition of Ω, for every $x \in \Omega$ the straight line segment joining $P(x)$ to x lies in Ω. Therefore we may define a function $f\colon \Omega \to \mathbb{R}$ by

$$f(x) = \inf Q(V,\gamma,\epsilon)$$

where the infimum is taken over all $l \ge 0$ and all Lipschitz $\gamma\colon [0,l] \to \Omega$ with $(\varphi\circ\gamma)'\cdot w \ge 0$ almost everywhere such that $\gamma(0) \in P(\Omega)$ and $\gamma(l) = x$. We will call such a curve *admissible for x*.

We now use the conclusion of Lemma 4.1.2 to deduce the required properties of f. We first show that f is Lipschitz and satisfies (2). Indeed, let $y, z \in \Omega$ with $(\varphi(z)-\varphi(y))\cdot w \ge 0$ and let $\gamma\colon [0,l] \to \Omega$ be any admissible curve for y. Define

$\widetilde{\gamma}\colon [0, l+1] \to \mathcal{B}$ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t) & t \in [0, l] \\ y + (t-l)(z-y) & t \in (l, l+1]. \end{cases}$$

Then since $(\varphi(z)-\varphi(y))\cdot w \geq 0$, $(\varphi\circ\widetilde{\gamma})' \geq 0$ almost everywhere and so $\widetilde{\gamma}$ is admissible for z. Therefore

$$\begin{aligned} f(z) &\leq f(y) + Q(V, \widetilde{\gamma}|_{[l,l+1]}) \\ &\leq f(y) + \mathcal{H}^1(\widetilde{\gamma}([l,l+1]) \setminus V) + K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (\delta+\epsilon)\mathcal{H}^1(\widetilde{\gamma}([l,l+1])) \\ &\leq f(y) + K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (\delta+\epsilon)\|y-z\| \text{ if } y, z \in \text{Ball} \subset V \qquad (4.1.3) \\ &\leq f(y) + K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (1+\delta+\epsilon)\|y-z\| \text{ otherwise.} \qquad (4.1.4) \end{aligned}$$

To bound $f(y)$, let $\gamma\colon [0, l] \to \Omega$ be admissible for z and set

$$t_0 = \inf\{t \in I : \varphi(\gamma(t)) \cdot w \geq \varphi(y) \cdot w\}.$$

Then since $(\varphi \circ \gamma)' \geq 0$ almost everywhere, $(\varphi(\gamma(t)) - \varphi(y)) \cdot w \geq 0$ for all $t \geq t_0$. We define $\widetilde{\gamma}$ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq t_0 \\ P_{\varphi(y)\cdot w}(\gamma(t)) & t_0 < t \leq l \\ \widetilde{\gamma}(l) + (t-l)(y - \widetilde{\gamma}(l)) & t \in (l, l+1]. \end{cases}$$

Then $\widetilde{\gamma}(l+1) = y$ and $(\varphi \circ \widetilde{\gamma})'(t) \cdot w = 0$ for almost every $t \in [t_0, l+1]$, so that $\widetilde{\gamma}$ is admissible for y. Further, for almost every $t \in [t_0, l]$,

$$\|\mathcal{P}((\varphi \circ \widetilde{\gamma})'(t))\| \leq \|\mathcal{P}((\varphi \circ \gamma)'(t))\| \text{ and } \mathcal{V}_{\widetilde{\gamma}}(t) = \mathcal{V}_{\gamma}(t).$$

Therefore

$$Q(V, \widetilde{\gamma}|_{[t_0,l]}) \leq Q(V, \gamma|_{[t_0,l]})$$

and so

$$f(y) \leq f(z) + K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (\delta+\epsilon)\|y-z\|. \qquad (4.1.5)$$

By combining equations (4.1.3), (4.1.4) and (4.1.5) we see that f is a $(1+K(\theta)+\delta+\epsilon)$-Lipschitz function such that, for every y, z belonging to a ball contained in V,

$$|f(y) - f(z)| \leq K(\theta)\|\mathcal{P}(\varphi(y)-\varphi(z))\| + (\delta+\epsilon)\|y-z\|.$$

To prove (1) let $x, x_0 \in \Omega$ with $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$. By the definition of Ω and P, the straight line segment γ joining $P(x_0)$ to x_0 lies entirely within Ω and satisfies $(\varphi \circ \gamma)' \geq 0$ almost everywhere and so is admissible for x_0. Further, $\mathcal{P}((\varphi \circ \gamma)'(t_0)) = 0 = \mathcal{V}_\gamma(t_0)$ for almost every t_0. Therefore,

$$f(x_0) \leq (\varphi(x_0) - \varphi(P(x_0))) \cdot w + \epsilon \|x_0 - P(x_0)\|.$$

Further, by the conclusion of Lemma 4.1.2,

$$f(x) \geq (\varphi(x) - \varphi(P(x))) \cdot w - \epsilon.$$

Therefore, since $(\varphi(P(x)) - \varphi(P(x_0))) \cdot w = 0$,

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \epsilon(1 + \operatorname{diam} X).$$

This proves the result for $(1 + \operatorname{diam} X)\epsilon$, which suffices. □

4.2 Application to Lipschitz differentiability spaces

We first apply the construction from the previos section to subsets of metric measure spaces.

Lemma 4.2.1. *Let (X, d, μ) be a metric measure space, $U \subset X$ compact and $\varphi \colon X \to \mathbb{R}^n$ Lipschitz. Suppose that, for some $w \in \mathbb{S}^{n-1}$, $0 < \theta < 1$ and $\delta > 0$, $S \subset U$ is closed and satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of $C(w, \theta)$ with φ-speed δ. Then for any $\epsilon > 0$ there exists a $(1 + K(\theta) + \delta + \epsilon) \operatorname{Lip} \varphi$-Lipschitz function $f \colon U \to \mathbb{R}$ and a $\rho > 0$ such that:*

- *For every $x_0 \in S$ and $x \in U$ with $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$,*

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \epsilon$$

- *For every $x_0 \in S$ and $y, z \in B(x_0, \rho)$,*

$$|f(y) - f(z)| \leq K(\theta) \|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon) \operatorname{Lip} \varphi d(y, z).$$

Here $\mathcal{P} \colon \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto the hyperplane orthogonal to w passing through the origin.

Proof. Let us identify U with its image in $\mathcal{B}$ via the bi-Lipschitz isomorphism ι^* defined in Definition 3.3.6 (for $\psi = 0$). Note that φ agrees with the projection onto the first factor on U and so we may extend φ to all of $\mathcal{B}$ by defining it to be this projection. Then S is also a subset of $\mathcal{B}$ that satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Gamma(U)$ in the φ-direction of $C(w,\theta)$ with φ-speed δ. We denote by $\|x-y\|$ the distance between x and y in $\mathcal{B}$ and by $d(x,y)$ the original distance between x and y in U, so that

$$d(x,y) \leq \|x-y\| \leq \operatorname{Lip}\varphi d(x,y).$$

Define f to be the Lipschitz function and V the open set obtained from an application of Lemma 4.1.3. Since $U \subset V$ is compact there exists a $\rho > 0$ such that, for every $x_0 \in S$, $B(x_0,\rho) \subset V$. This function has the required properties. $\square$

We now combine the functions constructed in the previous Lemma to satisfy the hypotheses of Proposition 3.1.3 for arbitrarily large subsets of any $S \in \widetilde{A}(\varphi)$, for any structured chart (U,φ).

Lemma 4.2.2. *Let (U,φ) be a structured chart in a metric measure space (X,d,μ) and suppose that $S \in \widetilde{A}(\varphi)$ with $S \subset U$. Then there exists a $\beta > 0$ and, for any $\delta > 0$, a 1-Lipschitz function $f\colon X \to \mathbb{R}$, a compact $S' \subset S$ with $\mu(S') \geq \mu(S) - \delta$ and a $\rho > 0$ such that:*

- *For every $x_0 \in S'$ there exists an $x \in U$ with $0 < d(x,x_0) < \delta$ and*

$$|f(x) - f(x_0)| \geq \beta d(x,x_0).$$

- *For every $x_0 \in S$ and $y,z \in B(x_0,\rho)$,*

$$|f(y) - f(z)| \leq \delta d(y,z).$$

Proof. Note that it suffices to prove the result for any 1/2-Lipschitz φ and $0 < \delta < 1/2$. If so, let $\lambda > 0$ and $0 < \theta < 1$ such that (U,φ) is a λ-structured chart and $K(\theta) \leq \delta/2$. Then there exists a countable Borel decomposition $S = \cup_i S_i$ and for each $i \in \mathbb{N}$ a $w_i \in \mathbb{S}^{n-1}$ such that $\mathcal{H}^1(\gamma \cap S_i) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of $C(w_i,\theta)$ with φ-speed δ.

For every $i \in \mathbb{N}$, let $Q_i \subset S_i$ be disjoint and compact and let $N \in \mathbb{N}$ such that

$$\mu(Q_1 \cup \ldots \cup Q_N) > \mu(S) - \delta.$$

Then there exists a $h > 0$ such that the $B(Q_i, 2h)$ are disjoint. Further, let $0 < 4R < \min\{h,\delta\}$ such that $(1 - 4R/h)\mu(Q_1 \cup \ldots \cup Q_N) > \mu(S) - \delta$. Then, since (U,φ) is a structured chart, there exists an $r > 0$ such that, for every $x_0 \in U$, there exists an $x \in X$ with $r < d(x, x_0) < R$ and

$$\frac{|(\varphi(x) - \varphi(x_0)) \cdot w_i|}{d(x, x_0)} \geq \lambda.$$

We also set $\epsilon = \lambda r/2$ and, for each $1 \leq i \leq N$, define

$$g_i \colon B(Q_i, h) \to \mathbb{R}$$

to be the 1-Lipschitz function obtained from Lemma 4.2.1 restricted to $B(Q_i, h)$ for this choice of ϵ, θ and w_i. Further, we let $f_i \colon B(Q_i, h) \to \mathbb{R}$ and $P_i \subset Q_i$ be obtained from applying Lemma 3.1.1 to g_i and Q_i with the choice of $\epsilon = R$. We extend each f_i to a 1-Lipschitz function defined on X with value zero outsize $B(Q_i, 2h)$.

Then for any $x_0 \in P_i$ there exists an $x \in X$ with $r < d(x, x_0) < R < \delta$ and

$$\begin{aligned}
|f_i(x) - f_i(x_0)| &= |g_i(x) - g_i(x_0)| \\
&\geq |(\varphi(x) - \varphi(x_0)) \cdot w_i| - \epsilon \\
&\geq \lambda d(x, x_0) - \lambda r/2 \\
&\geq \lambda d(x, x_0)/2.
\end{aligned}$$

Further, we let $\rho > 0$ such that, for any $y, z \in \text{Ball} \subset B(Q_i, \rho)$,

$$\begin{aligned}
|f_i(y) - f_i(z)| &\leq |g_i(y) - g_i(z)| \\
&\leq K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)d(y, z) \\
&\leq 2\delta d(y, z).
\end{aligned}$$

Finally we set $S' = P_1 \cup \ldots \cup P_N$ so that

$$\mu(S') \geq (1 - 4R/h)\mu(Q_1 \cup \ldots \cup Q_N) \geq \mu(S) - \delta$$

and

$$f = \sum_{1 \leq i \leq N} g_i.$$

Then since the g_i have disjoint support, f and S' satisfy the conclusion of the Lemma for 2δ and $\beta = \lambda/2$. □

We now combine all of our previous results to obtain our first statement on the structure of measures in Lipschitz differentiability spaces.

Theorem 4.2.3. *Let (U, φ) be an n-dimensional chart in a Lipschitz differentiability* *space (X,d,μ). Then any $\widetilde{A}(\varphi)$ subset of U is μ-null. Therefore, there exists a countable Borel decomposition*

$$U = \bigcup_{k\in\mathbb{N}} U_k$$

such that, for each $k \in \mathbb{N}$, $\mu \llcorner U_k$ has n φ-independent Alberti representations.

Moreover, for any Lipschitz $\psi \colon X \to \mathbb{R}^m$ and $U' \subset U$ of positive measure, if U' has m ψ-independent Alberti representations then $m \leq n$.

Proof. By Lemma 2.3.7 it suffices to prove that any $\widetilde{A}(\varphi)$ subset S of a structured chart (U,φ) is μ-null. If S is such a set, for any $\epsilon > 0$ and each $m \in \mathbb{N}$ let $0 < \delta_m < 1/m$ such that $\sum_m \delta_m < \epsilon$ and let $f_m \colon X \to \mathbb{R}$ and $S_m \subset S$ be obtained by applying Lemma 4.2.2 with δ_m. Then $S' := \cap_m S_m$ satisfies $\mu(S') \geq \mu(S) - \epsilon$ and the f_m satisfy the hypotheses of Proposition 3.1.3 for S'. Therefore there exists a Lipschitz function differentiable μ-almost nowhere on S'. Thus S' and hence S are μ null.

In particular, any $\widetilde{A}(\varphi)$ subset of U is μ-null and so, by Theorem 3.4.6, there exists the required collection of Alberti representations.

Now suppose that (U,φ) is any chart, $\psi \colon X \to \mathbb{R}^m$ is Lipschitz and $U' \subset U$ has m ψ-independent Alberti representations. Then for almost every $x_0 \in U'$ we have $\operatorname{Lip}(v \cdot \psi, x_0) > 0$ for each $v \in \mathbb{S}^{m-1}$. However, for almost every $x_0 \in U$, each $D\psi_i(x_0)$ exists. Therefore, if $m > n$, there exists a $v \in \mathbb{S}^{m-1}$ such that $\sum_i v_i D\psi_i(x_0) = 0$. In particular we have $\operatorname{Lip}(v \cdot \psi, x_0) = 0$ and so U' must be μ-null. □

We now apply our theory of Alberti representations to charts, relating the behaviour of Lipschitz differentiability spaces to the existing differentiability theory of Euclidean spaces. Recall the notion of a gradient given in Definition 2.2.6.

Corollary 4.2.4. *Let (U,φ) be an n-dimensional chart in a Lipschitz differentiability space (X,d,μ). Then for almost every $x \in U$ there exists $\gamma_1^x, \ldots, \gamma_n^x \in \Gamma(X)$ such that each $(\gamma_i^x)^{-1}(x) = 0$ is a density point of $(\gamma_i^x)^{-1}(U)$ and the $(\varphi \circ \gamma_i^x)'(0)$ are linearly independent.*

Moreover, for any such γ_i^x, for any Lipschitz $f \colon X \to \mathbb{R}$ and almost every $x \in U$, the gradient of f at x with respect to φ and $\gamma_1^x, \ldots, \gamma_n^x$ equals $Df(x)$.

Proof. By the previous Theorem there exists a countable Borel decomposition $U = \cup_i U_i$ of U into sets with n φ-independent Alberti representations. Therefore, by applying Proposition 2.2.4 to each representation, for almost every x in any U_i there exists such $\gamma_1^x, \ldots, \gamma_n^x$.

Moreover, if $Df(x)$ exists, then since

$$(f \circ \gamma_i^x)'(0) = Df(x) \cdot (\varphi \circ \gamma_i^x)'(0),$$

$Df(x)$ equals the gradient of f at x with respect to φ and $\gamma_1^x, \ldots, \gamma_n^x$. □

The previous Corollary should be compared to [CK09], Theorem 3.3. This theorem asserts that, for any n dimensional chart in a doubling Lipschitz differentiability space that satisfies the Poincaré inequality, and for any collection $f_1, f_2 \ldots$ of Lipschitz functions, for almost every $x \in U$ there exist $\gamma_1^x, \ldots, \gamma_n^x \in \Gamma(X)$ (whose domains are in fact intervals) such that, for each $i \in \mathbb{N}$, the gradient of f_i at x with respect to φ and $\gamma_1^x, \ldots, \gamma_n^x$ equals $Df_i(x)$.

We now use the derivative of a Lipschitz function to show that any $\widetilde{A}$ subset of a Lipschitz differentiability space has measure zero. For this, we first investigate how the direction of a Lipschitz curve varies with respect to different Lipschitz functions in a Lipschitz differentiability space.

Lemma 4.2.5. *Let (U, φ) be an n-dimensional λ-structured chart in a Lipschitz differentiability space (X, d, μ), $S \subset U$ Borel and $\eta > 0$. Suppose that $\psi \colon X \to \mathbb{R}^m$ is Lipschitz and*

$$\operatorname{Lip}(v \cdot \psi, x_0) > \eta \operatorname{Lip}(\psi, x_0)$$

for every $x_0 \in S$ and $v \in \mathbb{S}^{m-1}$. Then for any $\widetilde{w} \in \mathbb{S}^{m-1}$ and $0 < \epsilon < \theta < 1$, there exists a countable Borel decomposition $S = \cup_i S_i \cup N$ where $\mu(N) = 0$ and, for each $i \in \mathbb{N}$, there exists a $w_i \in \mathbb{S}^{n-1}$ such that, for any $\delta > 0$, any $\gamma \in \Gamma(S_i)$ in the φ-direction of $C(w_i, \theta - \epsilon)$ with φ-speed δ is in the ψ-direction of $C(\widetilde{w}, 1 - (1-\theta)\lambda\eta/\operatorname{Lip}\varphi)$ with ψ-speed $\delta\eta$.

Proof. We use the derivative of each ψ_i to transform the direction of such a Lipschitz curve. Indeed, for $x_0 \in S$ suppose that the derivative, $D\psi_i(x_0)$, of each component of ψ exists at x_0. If we write $D\psi \colon \mathbb{R}^m \to \mathbb{R}^n$ for the linear map whose columns are the $D\psi_i(x_0)$, then for each $v \in \mathbb{R}^m$,

$$\operatorname{Lip}(D\psi(v) \cdot \varphi, x_0) = \operatorname{Lip}(v \cdot \psi, x_0) \geq \eta \|v\| \operatorname{Lip}(\psi, x_0)$$

and so

$$\|D\psi(v)\| \operatorname{Lip}(\varphi, x_0) \geq \eta \|v\| \operatorname{Lip}(\psi, x_0). \tag{4.2.1}$$

We also obtain

$$\lambda \|D\psi(v)\| \leq \|v\| \operatorname{Lip}(\psi, x_0). \tag{4.2.2}$$

In particular, $D\psi$ is injective and it's inverse $D\psi^{-1}\colon \operatorname{Im} D\psi \to \mathbb{R}^n$ satisfies the inequalities corresponding to (4.2.1) and (4.2.2).

Now let $w \in \mathbb{S}^{n-1} \cap \operatorname{Im} D\psi$, $0 < \theta < 1$ and suppose that $\gamma \in \Gamma(X)$ satisfies $\gamma(t_0) = x_0$ and that both $(\varphi \circ \gamma)'(t_0)$ and $(\psi \circ \gamma)'(t_0)$ exist. Then, for every $v \in \mathbb{R}^m$,

$$v \cdot (\psi \circ \gamma)'(t_0) = D\psi(v) \cdot (\varphi \circ \gamma)'(t_0).$$

Therefore, if $(\varphi \circ \gamma)'(t_0) \in C(w, \theta)$, by equation (4.2.2),

$$\begin{aligned} D\psi(D\psi^{-1}(w)) \cdot (\varphi \circ \gamma)'(t_0) &= w \cdot (\varphi \circ \gamma)'(t_0) \\ &\geq (1-\theta)\|(\varphi \circ \gamma)'(t_0)\| \\ &= (1-\theta)\|D\psi^{-1} \cdot (\psi \circ \gamma)'(t_0)\| \\ &\geq (1-\theta)\lambda\|(\psi \circ \gamma)'(t_0)\| / \operatorname{Lip}(\psi, x_0) \end{aligned}$$

If we let $\widetilde{w} \in \mathbb{R}^m$ be a scalar multiple of $D\psi^{-1}(w)$ with norm 1, then since

$$\|D\psi^{-1}\| \leq \operatorname{Lip}(\varphi, x_0)/\eta \operatorname{Lip}(\psi, x_0),$$

the previous inequality gives

$$\begin{aligned} \widetilde{w} \cdot (\psi \circ \gamma)'(t_0) &\geq \frac{(1-\theta)\lambda\|(\psi \circ \gamma)'(t_0)\|}{\|D\psi^{-1}(w)\| \operatorname{Lip}(\psi, x_0)} \\ &\geq \frac{(1-\theta)\lambda\eta\|(\psi \circ \gamma)'(t_0)\|}{\operatorname{Lip}(\varphi, x_0)}. \end{aligned}$$

Therefore $(\psi \circ \gamma)'(t_0) \in C(\widetilde{w}, 1 - (1-\theta)\lambda\eta / \operatorname{Lip} \varphi)$.

Similarly, if $\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \operatorname{Lip}(\varphi, x_0) \operatorname{Lip}(\gamma, t_0)$, then by equation (4.2.1) there exists a $v \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} \|(\varphi \circ \gamma)'(t_0)\| &= |v \cdot (\varphi \circ \gamma)'(t_0)\| \\ &= |D\psi^{-1}(v) \cdot (\psi \circ \gamma)'(t_0)| \\ &\leq \|(\psi \circ \gamma)'(t_0)\| \operatorname{Lip}(\varphi, x_0)/\eta \operatorname{Lip}(\psi, x_0). \end{aligned}$$

Therefore

$$\|(\psi \circ \gamma)'(t_0)\| \geq \delta\eta \operatorname{Lip}(\psi, x_0) \operatorname{Lip}(\gamma, t_0).$$

Let $S = \cup_i S_i \cup N$ be a countable Borel decomposition where $\mu(N) = 0$ and such that, for each $i \in \mathbb{N}$, there exists a $w_i \in \mathbb{S}^{n-1}$ with

$$\left\| \frac{D\psi(x_0)(\widetilde{w})}{\|D\psi(x_0)(\widetilde{w})\|} - w_i \right\| < \epsilon.$$

Then if $\gamma \in \Gamma(S_i)$ is in the φ-direction of $C(w_i, \theta - \epsilon)$ with φ-speed δ, for almost every $t_0 \in \operatorname{Dom} \gamma$,

$$(\varphi \circ \gamma)'(t_0) \in C(w_i, \theta - \epsilon) \subset C\left(\frac{D\psi(x_0)(\widetilde{w})}{\|D\psi(x_0)(\widetilde{w})\|}, \theta\right)$$

and

$$\|(\psi \circ \gamma)'(t_0)\| \geq \eta\delta \operatorname{Lip}(\psi, x_0) \operatorname{Lip}(\gamma, t_0).$$

Therefore, by the above estimates, γ is in the ψ-direction of $C(\widetilde{w}, 1-(1-\theta)\lambda\eta/\operatorname{Lip}\varphi)$ with ψ-speed $\delta\eta$. □

As a consequence, we obtain the following Theorem.

Theorem 4.2.6. *Any $\widetilde{A}$ subset of a Lipschitz differentiability space has measure zero.*

Proof. For $\lambda > 0$ let (U, φ) be a λ-structured chart and $U = \cup_i U_i$ be a countable Borel decomposition such that, for each $i \in \mathbb{N}$, there exists independent cones $C_1, \ldots, C_n$ and a $\delta > 0$ such that $\mu \llcorner U_i$ has Alberti representations in the φ-direction of each C_i with φ-speed δ. Further, let $0 < \epsilon < \theta < 1$ such that any cone in $\mathbb{R}^n$ of width $\theta - \epsilon$ completely contains one of the C_i.

We work with a fixed U_i. Suppose that for some Lipschitz $\psi \colon X \to \mathbb{R}^m$ and $\widetilde{w} \in \mathbb{S}^{m-1}$, $S \subset U_i$ satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Gamma(X)$ in the ψ-direction of $C(\widetilde{w}, 1 - (1-\theta)\lambda\eta/\operatorname{Lip}\varphi)$ with ψ-speed $\delta\eta$. Then by the previous Lemma there exists a countable Borel decomposition $S = \cup_i S_i \cup N$ and for each $i \in \mathbb{N}$ cones C_i of width $\theta - \epsilon$ such that $\mu(N) = 0$ and $\mathcal{H}^1(\gamma \cap S_i) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C_i with speed δ. However, one of the Alberti representations of $\mu \llcorner U_i$ is in the φ-direction of this cone with φ-speed δ and so $\mu(S_i) = 0$ and hence $\mu(S) = 0$.

Finally, let $S' \in \widetilde{A}$ and $\eta > 0$ such that $S' \in \widetilde{A}(\delta', \theta', \eta)$ for any $0 < \delta', \theta' < 1$. In particular $S \in \widetilde{A}(\eta\delta, 1 - (1-\theta)\lambda\eta/\operatorname{Lip}\varphi, \eta)$. Therefore, there exists a countable decomposition $S' = \cup_j S_j$ and for each $j \in \mathbb{N}$ a Lipschitz $\psi_j \colon X \to \mathbb{R}^{n_j}$ such that each S_j has the form of S above. Therefore, each S_j has measure zero. In particular, any $\widetilde{A}$ subset of U_i and hence of (X, d, μ) must be μ-null. □

Finally, as a consequence of the existence of the above Alberti representations, we give a partial, positive answer to [Che99], Conjecture 4.63 regarding the image of a chart under the chart map.

Corollary 4.2.7. *For $n = 1$ or 2 let (U, φ) be an n-dimensional chart in a Lipschitz differentiability space (X, d, μ). Then the pushforward $\varphi_*(\mu \llcorner U)$ is absolutely continuous with respect to n-dimensional Lebesgue measure.*

In particular, if $\mu(U) > 0$ then $\mathcal{L}^n(\varphi(U)) > 0$.

Proof. Suppose that a measure ν has an Alberti representation in the φ-direction of a cone $C \subset \mathbb{R}^n$. Then an easy application of Corollary 3.3.8 shows that $\varphi_*\nu$ also has an Alberti representation in the direction of C. By Theorem 4.2.3 there exists a countable Borel decomposition $U = \cup_m U_m$ such that each $\mu \llcorner U_m$ has n-independent Alberti representations, so that each $\varphi_*(\mu \llcorner U_m)$ has n independent Alberti representations.

Any measure on $\mathbb{R}$ with an Alberti representation is absolutely continuous with respect to Lebesgue measure. Further, results from [ACP] show that any measure on $\mathbb{R}^2$ with two independent Alberti representations must also be absolutely continuous with respect to Lebesgue measure. Therefore, in either case, $\varphi_*(\mu \llcorner U)$ is absolutely continuous with respect to Lebesgue measure. □

This improves the known cases proved by Keith ($n = 1$, see [Kei04], page 282) and Gong ($n = 2$, see [Gon11], Theorem 1.4) who prove Corollary 4.2.7 for Lipschitz differentiability spaces that satisfy the Poincaré inequality and possess a doubling measure (see Definition 7.1.1).

Remark 4.2.8. Using a recent announcement of Csörnyei and Jones (see `www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf`, pages 15-23) one may show that, for any $n \in \mathbb{N}$, any measure on $\mathbb{R}^n$ with n independent Alberti representations is absolutely continuous with respect to Lebesgue measure. Therefore, the previous argument also proves the statements of Corollary 4.2.7 for any $n \in \mathbb{N}$.

One may also use this announcement to generalise the techniques of Keith and Gong to prove Corollary 4.2.7 for all $n \in \mathbb{N}$ for Lipschitz differentiability spaces that satisfy the Poincaré inequality and possess a doubling measure.

Chapter 5

Characterisations of Lipschitz differentiability spaces

We now show how the Alberti representations found in the previous chapter characterise the differentiability properties of a metric measure space. These characterisations have the following two distinct flavours.

First, we introduce the notion of a *universal* collection of Alberti representations and prove that a measure has a universal collection of Alberti representations of size n if and only if it is supported by a chart of dimension n with respect to which every Lipschitz function is differentiable almost everywhere. It is easy to see that the Alberti representations found in the previous chapter are universal and so the majority of Section 5.1 is devoted to proving that the gradient obtained from a universal collection of Alberti representations is in fact the derivative with respect to a chart.

Secondly, we saw in Theorem 3.4.6 that the existence of many independent Alberti representations is equivalent to certain subsets of the metric measure space having measure zero. By taking this idea further, in Section 5.2 we describe classes of subsets of any metric measure space, defined by the geometry of Lipschitz curves within that space, and prove that a metric measure space is a Lipschitz differentiability space if and only if each of these sets have measure zero.

5.1 A characterisation via Alberti representations

We first introduce the notion of a *universal* set of Alberti representations for when a set of representations describe all of the Lipschitz functions on a metric measure space.

Definition 5.1.1. Let (U, φ) be a chart in a metric measure space (X, d, μ) and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be a collection of φ-independent Alberti representations of $\mu \llcorner U$, each with strictly positive φ-speed. For $\rho > 0$ we say that this collection of Alberti representations is *ρ-universal* (or just *universal* if such a ρ exists) if, for any Lipschitz $f \colon X \to \mathbb{R}$, there exists a finite Borel decomposition $X = X_1 \cup \ldots \cup X_n$ such that the Alberti representation of $\mu \llcorner X_i$ induced by $\mathcal{A}_i$ has f-speed ρ.

Remark 5.1.2. A universal collection of Alberti representations is a much stronger concept than a *maximal* collection of representations (i.e. a collection for which there are no other independent representations). For example, any purely unrectifiable metric measure space has a maximal (empty) collection of Alberti representations. However, by Corollary 3.2.4, this collection is not universal as there exists a Lipschitz function with positive pointwise Lipschitz constant on a set of positive measure. (For a slightly less trivial example, one may consider the Cartesian product of $\mathbb{R}$ and a purely unrectifiable space.)

We will see that a universal collection of Alberti representations is precisely the required concept so that the gradient (see Definition 2.2.6) ∇f of a Lipschitz function f forms a derivative. We prove this in a very natural way: observe that the derivative of $f - \nabla f \cdot \varphi$ along the Lipschitz curves defining the gradient is zero. Further, if the Alberti representations are sufficiently refined, then the derivative along almost every curve in the representation is also sufficiently small. Therefore, after a suitable limit of such gradients obtained from a sequence of Alberti representations, each refining the previous, we obtain a derivative.

We begin with a simple result that bounds the magnitude of the gradient using properties of Alberti representations. For this we must quantitatively describe an independent collection of Alberti representations.

Definition 5.1.3. For $\xi > 0$ we say $v_1, \ldots, v_m \in \mathbb{R}^n$ are *ξ-separated* if, for any $\lambda \in \mathbb{R}^m \setminus \{0\}$,

$$\left\| \sum_{i=1}^{m} \lambda_i v_i \right\| > \xi \max_{1 \leq i \leq m} \|\lambda_i v_i\|$$

and that closed cones $C_1, \ldots, C_m$ are ξ-separated if any choice of $v_i \in C_i \setminus \{0\}$ are ξ-separated. Further, we say that Alberti representations $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are *ξ-separated* if there exists ξ separated cones $C_1, \ldots, C_m$ such that each $\mathcal{A}_i$ in the φ-direction of C_i.

Observe that cones $C_1, \ldots, C_m$ are ξ-separated if and only if, for every $1 \leq i \leq m$, the distance of $C_i \cap \mathbb{S}^{n-1}$ from those v in the symmetric convex hull of the

C_j (for $j \neq i$) with $\|v\| \leq 1$ is strictly greater than ξ. In particular, any independent cones are ξ-separated for some $\xi > 0$.

Suppose that a metric measure space (X, d, μ) has Alberti representations in the φ-direction of ξ-separated cones $C(w_1, \theta), \ldots, C(w_m, \theta)$. Then there exists an $\epsilon > 0$ such that $C(w_1, \theta+\epsilon), \ldots, C(w_m, \theta+\epsilon)$ are also ξ-separated and a finite cover of each $C(w_i, \theta)$ by cones $C_1^i, \ldots, C_{N_i}^i$ of width ϵ that are contained within $C(w_i, \theta+\epsilon)$. By applying Corollary 3.4.1 using the C_j^i we obtain *arbitrary* refinements of these Alberti representations that are also ξ-separated. Moreover, if for some Lipschitz $\psi \colon X \to \mathbb{R}^n$ and $\delta_1, \ldots, \delta_n > 0$, the original Alberti representations have ψ_i-speed strictly greater than δ_i for each $1 \leq i \leq N$, then so do the refinements.

For this section we fix the following notation.

Notation 5.1.4. We fix a metric measure space (X, d, μ) and for $\rho, \delta, \lambda, \xi > 0$ let (U, φ) be an n-dimensional λ-structured chart in (X, d, μ) such that $\mu \llcorner U$ has a ρ-universal collection of n, ξ-separated Alberti representations with φ-speed strictly greater than δ.

Using the notion of separated Alberti representations, we may bound the magnitude of a gradient.

Lemma 5.1.5. *Let $f \colon X \to \mathbb{R}$ be Lipschitz, $x_0 \in U$ and $\gamma_1, \ldots, \gamma_n \in \Gamma(X)$ such that the $(\varphi \circ \gamma_i)'(0)$ are ξ-separated and, for each $1 \leq i \leq n$, $\gamma_i^{-1}(x_0) = 0$ is a density point of $\operatorname{Dom} \gamma_i$ and*

$$(\varphi \circ \gamma_i)'(0) \geq \delta \operatorname{Lip}(\varphi, x_0) \operatorname{Lip}(\gamma_i, 0).$$

Then the gradient $\nabla f(x_0)$ of f at x_0 with respect to φ and $\gamma_1, \ldots, \gamma_n$ satisfies

$$\|\nabla f(x_0)\| \leq n \operatorname{Lip}(f, x_0)/\xi\delta\lambda.$$

Proof. For any ξ-separated $v_1, \ldots, v_n$ and $G \in \mathbb{R}^n$, there exists a $1 \leq i \leq n$ such that

$$\xi\|G\|/n \leq |G \cdot v_i|.$$

Therefore,

$$\begin{aligned}
\operatorname{Lip}(f, x_0) \operatorname{Lip}(\gamma_i, 0) &\geq |(f \circ \gamma_i)'(0)| \\
&= |\nabla f(x_0) \cdot (\varphi \circ \gamma_i)'(0)| \\
&\geq \xi \|\nabla f(x_0)\| \|(\varphi \circ \gamma_i)'(0)\|/n \\
&\geq \xi\delta \|\nabla f(x_0)\| \operatorname{Lip}(\varphi, x_0) \operatorname{Lip}(\gamma_i, 0)/n.
\end{aligned}$$

That is,

$$\|\nabla f(x_0)\| \leq \operatorname{Lip} f n/\xi\delta \operatorname{Lip}(\varphi, x_0)$$

as required. $\square$

Next we refine a universal collection of Alberti representations whilst maintaining their speed with respect to a finite collection of Lipschitz functions.

Lemma 5.1.6. *Let $\mathcal{F}$ be a finite collection of real valued Lipschitz functions defined on X, $f\colon X \to \mathbb{R}$ Lipschitz and $\epsilon > 0$. Then there exists a finite Borel decomposition $X = X_1 \cup \ldots \cup X_N$ and for each $1 \leq i \leq N$, Alberti representations $\mathcal{A}'_1, \ldots, \mathcal{A}'_n$ of $\mu \llcorner X_i$ such that:*

1. *The $\mathcal{A}'_k$ are ξ-separated and have φ-speed strictly greater than δ.*
2. *For every $g \in \mathcal{F}$ there exists an $\mathcal{A}'_k$ with g-speed ρ.*
3. *If we write $\Phi\colon X \to \mathbb{R}^{n+1}$ to be the function obtained by appending f to φ, then each $\mathcal{A}'_k$ is in the Φ-direction of a cone of width ϵ.*

Proof. The collection $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is ρ-universal and so there exists a finite Borel decomposition $X = Z_1 \cup \ldots \cup Z_L$, such that, for any $g \in \mathcal{F}$ and $1 \leq i \leq L$, there exists a $1 \leq k \leq n$ such that $\mathcal{A}_k \llcorner Z_i$ has g-speed ρ.

We fix $1 \leq i \leq L$ and for each $1 \leq k \leq n$ let $\mathcal{F}(k)$ be the set of $g \in \mathcal{F}$ such that $\mathcal{A}_k \llcorner Z_i$ has g-speed ρ. Then by Corollary 3.4.1 we may refine each $\mathcal{A}_k \llcorner Z_i$ to obtain a finite decomposition $Z_i = Y_k^1 \cup \ldots \cup Y_k^{M_k}$ and Alberti representations $\mathcal{A}_k^1, \ldots, \mathcal{A}_k^{M_k}$ of $\mu \llcorner Y_k^1, \ldots, \mu \llcorner Y_k^{M_k}$ respectively, in the Φ-direction of cones of width ϵ and with φ-speed strictly greater than δ, and such that each $\mathcal{A}_k^i$ has g-speed ρ, for each $g \in \mathcal{F}(k)$. Moreover, we may make these refinements such that the representations are ξ-separated, with respect to φ. Therefore, for these representations, both (1) and (3) are satisfied.

By taking the intersection of Y_k^m for $1 \leq k \leq n$, we obtain a finite Borel decomposition $Z_i = X_1^i \cup \ldots \cup X_{N_i}^i$ such that each X_j^i is a subset of some Y_k^m, for each $1 \leq k \leq n$. Therefore, for each $1 \leq m \leq N_i$ and any $g \in \mathcal{F}$, there exists a $1 \leq k \leq n$ such that $\mathcal{A}_k \llcorner X_m^i$ has g-speed ρ. In particular, (2) is satisfied and so $X = \cup_{i,m} X_m^i$ is a decomposition of the required form. $\square$

Finally, we use this refinement to show that the gradient obtained from such Alberti representations is an ϵ-derivative at almost every point.

Lemma 5.1.7. *Let $f\colon X \to \mathbb{R}$ be Lipschitz. Then there exists an $M \in \mathbb{R}$ such that, for every $\epsilon > 0$ and almost every $x_0 \in U$, there exists a $\nabla f(x_0) \in \mathbb{R}^n$ with*

$$\operatorname{Lip}(f - \nabla f(x_0) \cdot \varphi, x_0) \leq M\sqrt{\epsilon}.$$

Proof. Fix $\epsilon > 0$ and let $\mathcal{Q}$ be a finite ϵ-net of $\overline{B}(0, \operatorname{Lip} f n/\xi\delta\lambda) \subset \mathbb{R}^n$ and

$$\mathcal{F} = \{f - D \cdot \varphi : D \in \mathcal{Q}\}.$$

Then by the previous Lemma there exists a countable decomposition $U = \cup_i U_i$ such that each $\mu \llcorner U_i$ has Alberti representations $\mathcal{A}'_1, \ldots, \mathcal{A}'_n$ satisfying the properties of the previous Lemma for ϵ, f and $\mathcal{F}$. We fix such a U_i and define $\Phi\colon X \to \mathbb{R}^{n+1}$ to be the Lipschitz function obtained by appending f to φ and let $C_1, \ldots, C_n \subset \mathbb{R}^{n+1}$ have width ϵ such that each $\mathcal{A}_k$ is in the Φ-direction of C_k.

By combining Proposition 2.2.4 and Lemma 5.1.5, for almost every $x_0 \in U_i$ there exist $\gamma_1, \ldots, \gamma_n \in \Gamma(X)$ in the Φ-direction of $C_1, \ldots, C_n$ respectively such that the gradient $\nabla f(x_0)$ of f at x_0 with respect to φ and $\gamma_1, \ldots, \gamma_n$ satisfies

$$\|\nabla f(x_0)\| \leq n \operatorname{Lip}(f, x_0)/\xi\delta\lambda.$$

In particular, there exists a $D \in \mathcal{Q}$ with $\|D - \nabla f(x_0)\| < \epsilon$. We write $\mathbf{D}$ and $\nabla \mathbf{f}(x_0) \in \mathbb{R}^{n+1}$ for the vectors obtained by appending -1 to D and $\nabla f(x_0)$ respectively.

Suppose that $v, v' \in \mathbb{R}^{n+1}$ belong to a cone of width ϵ. Then

$$\left\| \frac{v}{\|v\|} - \frac{v'}{\|v'\|} \right\| \leq \sqrt{\epsilon(2-\epsilon)}$$

and so, for any $a \in \mathbb{R}^{n+1}$,

$$|a \cdot v| \leq (|a \cdot v'|/\|v'\| + \sqrt{\epsilon(2-\epsilon)})\|v\|.$$

For almost every $x_0 \in U$, $\nabla f(x_0)$ is the gradient of f with respect to φ and $\gamma_1, \ldots, \gamma_n$, and so $\nabla \mathbf{f}(x_0) \cdot (\Phi \circ \gamma_i)'(0) = 0$ for each $1 \leq i \leq n$. Therefore, if $\widetilde{\gamma} \in \Gamma(X)$ is in the Φ-direction of some C_i with $\widetilde{\gamma}(t_0) = x_0$ such that $(\Phi \circ \widetilde{\gamma})'(t_0)$ exists, then

$$|\nabla \mathbf{f}(x_0) \cdot (\Phi \circ \widetilde{\gamma})'(t_0)| \leq \sqrt{\epsilon(2-\epsilon)})\|\nabla \mathbf{f}(x_0)\|\|(\Phi \circ \widetilde{\gamma})'(t_0)\|.$$

In particular,

$$|\mathbf{D} \cdot (\Phi \circ \widetilde{\gamma})'(t_0)| \leq \sqrt{\epsilon(2-\epsilon)}\|\mathbf{D}\|\|(\Phi \circ \widetilde{\gamma})'(t_0)\| + 2\epsilon(\Phi \circ \widetilde{\gamma})'(t_0).$$

However, for each $1 \leq i \leq n$, almost every $\widetilde{\gamma} \in \mathcal{A}_i$ is of this form. Further, since $D \cdot \varphi - f \in \mathcal{F}$, one of the $\mathcal{A}'_k$ has $D \cdot \varphi - f$ speed ρ. Therefore,

$$\rho \operatorname{Lip}(D \cdot \varphi - f, x_0) \leq \sqrt{\epsilon(2-\epsilon)}\|\mathbf{D}\| \operatorname{Lip} \Phi + 2\epsilon \operatorname{Lip} \Phi$$

for almost every $x_0 \in U$. Finally, since

$$\|\mathbf{D}\| \leq 1 + \|\nabla f(x_0)\| \leq 1 + n \operatorname{Lip} f/\xi\delta\lambda,$$

we have

$$\operatorname{Lip}(f - \nabla f(x_0) \cdot \varphi, x_0) \leq (\operatorname{Lip} \varphi + \operatorname{Lip} f)(2\epsilon + \sqrt{\epsilon(2-\epsilon)}(1 + n \operatorname{Lip} f/\xi\delta\lambda))$$

almost everywhere in U, as required. □

Using this construction we may give our first characterisation of Lipschitz differentiability spaces. We now work without the fixed quantities given in Notation 5.1.4.

Theorem 5.1.8. *A metric measure space (X, d, μ) is a Lipschitz differentiability space if and only if there exists a countable Borel decomposition $X = \cup_i U_i$ such that each $\mu \llcorner U_i$ has a finite universal collection of Alberti representations.*

In this case, for each $i \in \mathbb{N}$ let $\varphi_i \colon X \to \mathbb{R}^{n_i}$ be Lipschitz such that the Alberti representations of $\mu \llcorner U_i$ are φ_i-independent. Then each (U_i, φ_i) is a chart and the derivative of a Lipschitz function $f \colon X \to \mathbb{R}$ is given by any gradient of f with respect to φ_i at almost every point.

Proof. We first show that the condition is necessary for a metric measure space to be a Lipschitz differentiability space. For any $i \in \mathbb{N}$ let $\varphi_i \colon X \to \mathbb{R}^{n_i}$ be Lipschitz such that the Alberti representations of $\mu \llcorner U_i$ described in the hypotheses are φ_i-independent. Then by applying Proposition 2.2.4, for almost every $x_0 \in U_i$ there exist $\gamma_1, \ldots, \gamma_{n_i} \in \Gamma(X)$ such that the $(\varphi \circ \gamma_i)'(0)$ form a basis of $\mathbb{R}^{n_i}$ and, for each $1 \leq i \leq n_i$, $\gamma_i^{-1}(x_0) = 0$ is a density point of $\operatorname{Dom} \gamma_i$. Therefore, for almost every $x_0 \in U_i$, there exists a $\lambda(x_0) > 0$ such that, for any $v \in \mathbb{S}^{n_i - 1}$,

$$\limsup_{X \ni x \to x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} \geq \lambda(x_0).$$

In particular, by Lemma 2.3.3, any possible derivative of a function at x_0 is unique.

Now let $\delta, \xi > 0$ such that the Alberti representations of $\mu \llcorner U_i$ are ξ-separated with respect to φ_i and have φ_i-speed strictly greater than δ. By taking a further decomposition (and allowing for a possible μ-null subset of X), we may suppose that (U_i, φ_i) is λ-structured, for some $\lambda > 0$. Then we are in the situation described by Notation 5.1.4 and so, for any Lipschitz $f\colon X \to \mathbb{R}$, by Lemma 5.1.7 there exists an $M > 0$ such that, for any $\epsilon > 0$ and almost every $x_0 \in U$, there exists a $\nabla f(x_0) \in \mathbb{R}^{n_i}$ with $\|\nabla f(x_0)\| \leq n_i \operatorname{Lip} f/\xi\delta\lambda$ and

$$\operatorname{Lip}(f - \nabla f(x_0) \cdot \varphi_i, x_0) \leq M\sqrt{\epsilon}.$$

Applying this with some sequence $\epsilon_m \to 0$, for almost every $x_0 \in U_i$ we obtain a bounded sequence of such $\nabla f(x_0)$ and so, after passing to a convergent subsequence, we obtain a $Df(x_0) \in \mathbb{R}^{n_i}$ that is a derivative of f at x_0 with respect to (U_i, φ_i). Further, by Corollary 4.2.4, this derivative is given by any gradient of f with respect to φ_i.

Conversely, let (U, φ) be an n-dimensional, λ-structured chart in a Lipschitz differentiability space. Then by Theorem 4.2.3 and Corollary 3.4.1, there exists a countable Borel decomposition $U = \cup_i U_i$ and for every $i \in \mathbb{N}$ a $\delta > 0$ such that, for each $i \in \mathbb{N}$, $\mu \llcorner U_i$ has n φ-independent Alberti representations with φ-speed δ.

Fix such a U_i and suppose such representations $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mu \llcorner U_i$ are in the φ-direction of ξ-separated cones $C_1, \ldots, C_n$. Then for any Lipschitz $f\colon X \to \mathbb{R}$ and almost every $x_0 \in U_i$, $Df(x_0)$ exists. Moreover, there exists a $1 \leq j \leq n$ such that, for any $v \in C_j$,

$$\xi \|Df(x_0)\| \|v\|/n \leq |Df(x_0) \cdot v|.$$

For $1 \leq j \leq n$ let V_j be the set of x_0 that satisfy this for C_j. Therefore, if $x_0 \in V_j$ and $\gamma \in \Gamma(X)$ with $\gamma(t_0) = x_0$, $\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \operatorname{Lip}(\varphi, x_0) \operatorname{Lip}(\gamma, t_0)$ and $(\varphi \circ \gamma)'(t_0) \in C_j$ then

$$\begin{aligned}
(f \circ \gamma)'(t_0) &= |Df(x_0) \cdot (\varphi \circ \gamma)'(t_0)| \\
&\geq \|Df(x_0)\| \|(\varphi \circ \gamma)'(t_0)\| \\
&\geq \delta \operatorname{Lip}(f, x_0) \operatorname{Lip}(\varphi, x_0) \operatorname{Lip}(\gamma, t_0) \\
&\geq \delta\lambda \operatorname{Lip}(f, x_0) \operatorname{Lip}(\gamma, t_0).
\end{aligned}$$

That is, $\mathcal{A}_j \llcorner V_j$ has f-speed $\delta\lambda$. Therefore, the $\mathcal{A}_j$ are universal. □

5.2 Characterisations via null sets

We now present other characterisations that prescribe a class of sets that determine if a metric measure space is a Lipschitz differentiability space. Such sets will be the singular sets found earlier when constructing Alberti representations of a measure. Therefore, in a metric measure space where such sets have measure zero, there exist many Alberti representations. Further, if these representations are sufficiently different, for almost every x_0, particular points of the Lipschitz curves obtained from the representations form a separated subset of balls centred at x_0. By combining this with a doubling condition on the metric space, we obtain a bound on the total number of such different Alberti representations. In turn, this leads to the existence of a derivative of a Lipschitz function at almost every point.

We begin by giving a method that constructs a chart structure in a metric measure space.

Lemma 5.2.1. *Let (X, d, μ) be a metric measure space, $Y \subset X$ Borel and $N \in \mathbb{N}$. Then the following are equivalent:*

- *For any Lipschitz $\psi \colon X \to \mathbb{R}^{N+1}$, for almost every $x_0 \in Y$ there exists an $a(x_0) \in \mathbb{S}^N$ such that*
$$\operatorname{Lip}(a(x_0) \cdot \psi, x_0) = 0.$$
- *There exists a countable Borel decomposition $Y = \cup_i U_i$ and Lipschitz functions $\varphi_i \colon X \to \mathbb{R}^{n_i}$ such that each (U_i, φ_i) is a chart of dimension at most N, with respect to which any Lipschitz $f \colon X \to \mathbb{R}$ is differentiable at almost every point of U_i.*

Proof. First suppose that the second statement is true, (U_i, φ_i) a chart of the decomposition and $\psi \colon X \to \mathbb{R}^{N+1}$ Lipschitz. Then for almost every $x_0 \in U_i$ and each $1 \leq j \leq N+1$, the derivative of ψ_j exists at x_0 and belongs to $\mathbb{R}^{n_i}$, for $n_i < N+1$. Therefore, if we write $D\psi$ for the matrix whose columns are the $D\psi_i(x_0)$, there exists an $a(x_0) \in \mathbb{S}^N$ such that $D\psi \cdot a(x_0) = 0$. Then by the definition of a derivative, $\operatorname{Lip}(a(x_0) \cdot \psi, x_0) = 0$.

Now suppose that the first statement holds and let $U \subset Y$ be a Borel set of positive measure. First observe that either any Lipschitz function $\psi \colon X \to \mathbb{R}$ satisfies $\operatorname{Lip}(\varphi, x_0) = 0$ for almost every $x_0 \in U$ in which case U is a chart of dimension zero (as described in Corollary 3.2.4) or there exists a Borel set $U' \subset U$ of positive measure and a Lipschitz $\varphi \colon X \to \mathbb{R}$ such that $\operatorname{Lip}(\varphi, x_0) > 0$ for every $x_0 \in U'$.

By our hypotheses on Y, there exists a maximal $n \in \mathbb{N}$ such that there exists a Lipschitz $\varphi\colon X \to \mathbb{R}^n$ and a $U' \subset U$ of positive measure with $\operatorname{Lip}(v \cdot \varphi, x_0) > 0$ for every $x_0 \in U'$ and every $v \in \mathbb{S}^{n-1}$. Further, we have $n \leq N$. Then for such an n, φ and U', for any Lipschitz $f\colon X \to \mathbb{R}$ and almost every $x_0 \in U'$ there exists a $Df(x_0) \in \mathbb{R}^n$ such that $\operatorname{Lip}(f - Df(x_0) \cdot \varphi, x_0) = 0$. Moreover, by Lemma 2.3.3, the condition $\operatorname{Lip}(v \cdot \varphi, x_0) > 0$ for every $v \in \mathbb{S}^{n-1}$ is equivalent to the uniqueness of such a $Df(x_0)$.

Therefore, within any U of positive measure, there exists a chart of positive measure with dimension less than N, with respect to which any real valued Lipschitz function is differentiable almost everywhere. Since μ is finite we obtain a countable decomposition of U into charts with respect to which Lipschitz functions are differentiable almost everywhere. □

We now introduce additional properties of metric measure spaces that will allow us to satisfy the hypotheses of Lemma 5.2.1.

Definition 5.2.2. We say that a metric measure space (X, d, μ) is *pointwise doubling* if

$$\limsup_{r \to 0} \frac{\mu(B(x_0, r))}{\mu(B(x_0, r/2))} < \infty$$

for almost every $x_0 \in X$.

Further, for $0 < \delta < 1$ and $\eta \geq 1$, we define $M(\delta, \eta) = \eta^{2 - \log_2 \delta/5}$ and D_η to be the set of Borel subsets Y of X such that, for each $x_0 \in Y$, $0 < \delta < 1$ and $r > 0$, there exist $x_1, \ldots, x_M \in Y$ with

$$B(x_0, r) \cap Y \subset \bigcup_{i=1}^{M} B(x_i, \delta r),$$

for some $M \in \mathbb{N}$ less than $M(\delta, \eta)$.

Lemma 5.2.3. *For any pointwise doubling metric measure space (X, d, μ), there exists a Borel decomposition*

$$X = N \cup \bigcup_{m=1}^{\infty} Y_m$$

where $\mu(N) = 0$ and for each $m \in \mathbb{N}$ there exists an $\eta \geq 1$ such that $Y_m \in D_\eta$.

Proof. For $\eta \geq 1$ and $R > 0$ let

$$X_{R,\eta} := \{x_0 \in X : \eta \mu(B(x_0, r/2)) > \mu(B(x_0, r)) > 0, \ \forall\, 0 < r < R\}.$$

For any $x_0 \in X$ and $r > 0$,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, r/2))} = \lim_{r_m \nearrow r} \frac{\mu(B(x_0, r_m))}{\mu(B(x_0, r_m/2))}$$

and so, in the definition of $X_{R,\eta}$, it is equivalent to satisfy the condition for rational $0 < r < R$. Further, for a fixed $r > 0$, $x_0 \mapsto \mu(B(x_0, r))$ is lower semicontinuous. Therefore, each $X_{R,\eta}$ is a Borel set.

Now fix $\eta \geq 1$, $R > 0$ and $x_0 \in X_{R,\eta}$. Then for any $m \in \mathbb{N}$ and $0 < r < R$,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, 2^{-m}r))} \leq \eta^m$$

and so, for any $0 < \delta < 1$,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \leq \eta^M,$$

for M the least integer greater than $-\log_2 \delta$.

Now let $0 < r < R/2$. Then

$$B(x_0, r) \cap X_{R,\eta} \subset \bigcup \{B(x, \delta r/5) : x \in B(x_0, r) \cap X_{R,\eta}\}$$

and so there exists $F \subset B(x_0, r) \cap X_{R,\eta}$ such that

$$B(x_0, r) \cap X_{R,\eta} \subset \bigcup \{B(x, \delta r) : x \in F\}$$

and such that the $B(x_0, \delta r/5)$ are disjoint. Firstly, since $\mu(B(x_0, 2r))$ is finite and for $x \in F$ the $B(x, \delta r/5)$ are disjoint subsets of $B(x_0, 2r)$ with positive measure, F is countable. Further,

$$\eta^M \mu(B(x_0, r)) \geq \eta^M \sum_{x \in F} \mu(B(x, \delta r/5)) \geq \sum_{x \in F} \mu(B(x, 2r)) \geq \sum_{x \in F} \mu(B(x_0, r)),$$

for M the least integer greater than $2 - \log_2 \delta/5$. In particular, the cardinality of F is less than $M(\delta, \eta)$. Therefore, for any $y \in X$, $X_{R,\eta} \cap B(y, R/4)$ belongs to D_η.

Finally, for $R_i \to 0$ and $\eta_i \to \infty$, the X_{R_i,η_i} cover almost all of X and so there exists a decomposition of X of the required form. □

Metric measure spaces in which all porous sets (see Definition 3.1.4) have measure zero are pointwise doubling (for example, see [MMPZ03], Theorem 3.6). Therefore, we obtain the following consequence of Corollary 3.1.5.

Corollary 5.2.4 ([BS13], Corollary 2.6)**.** *Any Lipschitz differentiability space is* *pointwise doubling.*

In addition, we also require concepts related to an $\widetilde{A}$ set (see Definition 3.4.3).

Definition 5.2.5. Let (X,d) be a metric space and $\delta>0$. We define $\widetilde{B}_\delta$ to be the set of $S\subset X$ for which there exist a countable Borel decomposition $S=\cup_i S_i$ and Lipschitz functions $f_i\colon X\to\mathbb{R}$ such that, for every $i\in\mathbb{N}$,

$$S_i\subset\{x_0\in X:\operatorname{Lip}(f_i,x_0)>0\}$$

and $\mathcal{H}^1(\gamma\cap S_i)=0$ for every $\gamma\in\Gamma(X)$ with f_i-speed δ. Further, we define $\widetilde{B}$ to be the set of $S\subset X$ that belong to $\widetilde{B}_\delta$ for every $\delta>0$.

Remark 5.2.6. Note that for any $0<\delta\le\delta'$ we have $\widetilde{B}_\delta\subset\widetilde{B}_{\delta'}$.

Observe that $\widetilde{B}_\delta\subset\widetilde{A}(\delta,\theta,\lambda)$ for any $0<\delta,\theta,\lambda<1$ and so $\widetilde{B}\subset\widetilde{A}$. In particular, any $\widetilde{B}$ subset of a Lipschitz differentiability space has measure zero. Also, by Corollary 3.3.8, metric measure spaces in which $\widetilde{B}_\delta$ sets are μ-null have many Alberti representations with speed δ. We now show how the curves obtained from such Alberti representations interact with the doubling properties of a metric measure space. This is done by creating a separated set from points belonging to curves obtained from such representations.

Lemma 5.2.7. *Let (X,d,μ) be a metric measure space, $\varphi\colon X\to\mathbb{R}^n$ Lipschitz, $Y\subset X$ Borel and $\delta>0$. Suppose that $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are Alberti representations of $\mu\llcorner Y$ and $w_1,\ldots,w_n\in\mathbb{S}^{n-1}$ such that, for every $1\le i\ne j\le n$, either*

$$\frac{|w_i\cdot(\varphi\circ\gamma_i)'(t_i)|}{\operatorname{Lip}(\gamma_i,t_i)}>\frac{|w_i\cdot(\varphi\circ\gamma_j)'(t_j)|}{\operatorname{Lip}(\gamma_j,t_j)}+\delta\operatorname{Lip}(w_i\cdot\varphi,\gamma_i(t_i))>0,$$

or the corresponding inequalities with i and j exchanged. Then, for almost every $x_0\in X$ and any $r>0$, there exist $x_1,\ldots,x_n\in B(x_0,r)$ such that

$$d(x_i,x_j)\ge\delta r-\phi(r)$$

for each $1\le i\ne j\le n$, where $\phi(r)/r\to 0$ as $r\to 0$.

In particular, if $Y\subset W\in D_\eta$, for some $\eta\ge 1$, then $n\le M(\delta/2,\eta)$.

Proof. Let $\mathcal{Q}$ be a countable dense subset of $\mathbb{S}^{n-1}$. Then by standard techniques, there exists a Borel decomposition

$$Y=N\cup\bigcup_{m\in\mathbb{N}}Y_m$$

where $\mu(N) = 0$ and for each $m \in \mathbb{N}$ and $q \in \mathcal{Q}$, $\operatorname{Lip}(q \cdot \varphi)$ is continuous on each Y_m. Therefore, for any $\epsilon, \rho > 0$, $m \in \mathbb{N}$ and $q \in \mathcal{Q}$, there exists a $Z \subset Y_m$ with $\mu(Y_m \setminus Z) < \rho$ and an $R > 0$ such that

$$|q \cdot (\varphi(x) - \varphi(y))| \leq (\operatorname{Lip}(q \cdot \varphi, x_0) + \epsilon) d(x, y) \tag{5.2.1}$$

for each $0 < r < R$, $x_0 \in Z$, $y \in Z \cap B(x_0, r)$ and $x \in B(y, r)$. Therefore, by decomposing each Y_m further, we may suppose that for every $\epsilon > 0$, $q \in \mathcal{Q}$, $m \in \mathbb{N}$ there exists an $R > 0$ such that (5.2.1) holds for each $0 < r < R$, $x_0 \in Y_m$, $y \in Z \cap B(x_0, r)$ and $x \in B(y, r)$.

We now work with a fixed Y_m. By Proposition 2.2.4, for almost every $x_0 \in Y_m$ there exists $\gamma_1, \ldots, \gamma_n \in \Gamma(X)$ such that, for each $1 \leq i \leq n$, $\gamma_i^{-1}(x_0) = t_0$ is a density point of $\gamma_i^{-1}(Y_m)$ and $\operatorname{Lip}(\gamma_i, t_0) = 1$. Further, there exists $w_1, \ldots, w_n \in \mathcal{Q}$ such that, for each $1 \leq i \neq j \leq n$, either

$$|w_i \cdot (\varphi \circ \gamma_i)'(t_0)| > |w_i \cdot (\varphi \circ \gamma_j)'(t_0)| + \delta \operatorname{Lip}(w_i \cdot \varphi, \gamma_i(t_0)) > 0, \tag{5.2.2}$$

or the corresponding inequalities with i and j exchanged.

Let $\epsilon > 0$. Then there exists an $R_1 > 0$ such that, for any $0 < r < R_1$, if t belongs to each $\operatorname{Dom} \gamma_i$ with $0 < |t - t_0| < r$ then, after setting $x_k = \gamma_k(t)$,

$$\begin{aligned} \operatorname{Lip}(w_i \cdot \varphi, x_0) d(x_i, x_j) &\geq |w_i \cdot (\varphi(x_i) - \varphi(x_j))| - \epsilon r \\ &\geq |w_i \cdot (\varphi(x_i) - \varphi(x_0))| - |w_i \cdot (\varphi(x_j) - \varphi(x_0)| - \epsilon r \end{aligned}$$

for each $1 \leq i, j \leq n$. Further, there exists a $0 < R_2 < R_1$ such that, for any $0 < r < R_2$, $1 \leq k \leq n$ and $x_k = \gamma_k(t)$,

$$\|(\varphi \circ \gamma)'(t_0)(t_k - t_0) - \varphi(x_k) - \varphi(x_0)\| \leq \epsilon r.$$

In particular,

$$\operatorname{Lip}(w_i \cdot \varphi, x_0) d(x_i, x_j) \geq |w_i \cdot (\varphi \circ \gamma_i)'(t_0)(t - t_0)| - |w_i \cdot (\varphi \circ \gamma_j)'(t_0)(t - t_0)| - 3\epsilon r.$$

Further, since t_0 is a density point of each $\operatorname{Dom} \gamma_k$, there exists a $0 < R_3 < R_2$ such that, for any $0 < r < R_3$ and $1 \leq k \leq n$, there exists a t in each $\operatorname{Dom} \gamma_k$ such that $\gamma_k(t) \in B(x_0, r)$ and $|t - t_0 - r| \leq \epsilon r$. Therefore, if $x_k = \gamma_k(t)$ for each $1 \leq k \leq n$,

$$\operatorname{Lip}(w_i \cdot \varphi, x_0) d(x_i, x_j) \geq r |w_i \cdot (\varphi \circ \gamma_i)'(t_0)| - r |w_i \cdot (\varphi \circ \gamma_j)'(t_0)| - 5\epsilon r.$$

By exchanging i and j if necessary, we may apply (5.2.2) as stated so that

$$\operatorname{Lip}(w_i \cdot \varphi, x_0) d(x_i, x_j) \geq \delta r \operatorname{Lip}(w_i \cdot \varphi, x_0) - 5\epsilon r.$$

Finally, also by our initial hypotheses, $\operatorname{Lip}(w_i \cdot \varphi, \gamma_i(t_0)) > 0$ and so we deduce that there exists $\phi \colon \mathbb{R} \to \mathbb{R}$ such that $\phi(r)/r \to 0$ as $r \to 0$ and

$$d(x_i, x_j) \geq \delta r - \phi(r).$$

The Y_m cover almost all of Y and so such $x_1, \ldots, x_n$ may be found for almost every $x_0 \in Y$, as required.

Now suppose that for some $\eta \geq 1$, $Y \subset W \in D_\eta$. Then for any $0 < \epsilon < \delta$, if $R > 0$ such that $|\phi(r)| < \epsilon r$ for each $0 < r < R$, the x_i found above all belong to $B(x_0, r)$ but are separated by a distance of at least $(\delta - \epsilon) r$. Therefore no two x_k can belong to the same ball of radius $(\delta - \epsilon) r/2$. In particular $n \leq N((\delta - \epsilon)/2, \eta)$ for all $0 < \epsilon < \delta$ and so $n \leq N(\delta/2, \eta)$. □

We use this construction to satisfy the hypotheses of Lemma 5.2.1.

Proposition 5.2.8. *Let (X, d, μ) be a metric measure space, $\delta > 0$ and, for $\eta \geq 1$, let Y be a Borel subset of some $W \in D_\eta$. Suppose that, for any Lipschitz $f \colon X \to \mathbb{R}$, there exists an Alberti representation of $\mu \llcorner Y$ with f-speed δ. Then, for any $N > M(\delta/2, \eta)$ and Lipschitz $\varphi \colon X \to \mathbb{R}^N$, for almost every $x_0 \in Y$ there exists an $a(x_0) \in \mathbb{R}^N$ such that*

$$\operatorname{Lip}(a(x_0) \cdot \varphi, x_0) = 0.$$

Proof. Suppose that $\varphi \colon X \to \mathbb{R}^N$ is Lipschitz and, for some Borel $Z \subset Y$ of positive measure, $\operatorname{Lip}(v \cdot \varphi, x_0) > 0$ for every $x_0 \in Z$ and every $v \in \mathbb{S}^{N-1}$. By taking a countable decomposition of Z if necessary, we may suppose that there exists a $\lambda > 0$ such that

$$\operatorname{Lip}(v \cdot \varphi, x_0) \geq \lambda \operatorname{Lip}(\varphi, x_0)$$

for every $v \in \mathbb{S}^{N-1}$ and $x_0 \in Z$. We will prove the Proposition by proving $N \leq M(\delta, \eta)$, using the assumption that there exists an Alberti representation of $\mu \llcorner Z$ with $v \cdot \varphi$-speed δ, for any $v \in \mathbb{S}^{N-1}$.

Let $\epsilon > 0$ and $\mathcal{A}_1$ be such a representation for an arbitrary choice of $v_1 \in \mathbb{S}^{N-1}$. Inductively, suppose that $m \leq N$ and that there exists a countable Borel decomposition $Z = \cup_i Z_i$ such that each $\mu \llcorner Z_i$ has $m-1$ Alberti representations $\mathcal{A}_1^i, \ldots, \mathcal{A}_{m-1}^i$. Then, by refining the representations if necessary, we may suppose that there exists $w_1^i, \ldots, w_{m-1}^i \in \mathbb{S}^{N-1}$ such that $\mathcal{A}_k^i$ is in the φ-direction of $C(w_k^i, \epsilon)$

for each $1 \le k < m$. We choose w_m^i orthogonal to each w_j^i and let $\mathcal{A}_m^i$ be an Alberti representation of $\mu \llcorner Z_i$ with $w_m^i \cdot \varphi$-speed δ. Then w_m^i satisfies

$$w_m^i \cdot (\varphi \circ \gamma_j)'(t) \le \frac{\sqrt{\epsilon(2-\epsilon)}}{\lambda} \operatorname{Lip}(\varphi, \gamma_j(t_j)) \operatorname{Lip}(\gamma_j, t_j)$$

for almost every $\gamma_j \in \mathcal{A}_j^i$ and almost every $t_j \in \operatorname{Dom}\gamma_j$, for each $1 \le j < m$. In particular,

$$\frac{w_m^i \cdot (\varphi \circ \gamma_m)'(t_m)}{\operatorname{Lip}(\gamma_m, t_m)} > \frac{w_m^i \cdot (\varphi \circ \gamma_j)'(t_j)}{\operatorname{Lip}(\gamma_j, t_j)} + \left(\delta - \frac{2\sqrt{\epsilon(2-\epsilon)}}{\lambda}\right) \operatorname{Lip}(w_m^i \cdot \varphi, \gamma_m(t_m))$$

for every $1 \le j < m$, almost every $\gamma_m \in \mathcal{A}_m^i$ and $\gamma_j \in \mathcal{A}_j^i$ and almost every $t_m \in \gamma_m$ and $t_j \in \gamma_j$.

By repeating this process $N-1$ times, we obtain N Alberti representations that satisfy the hypotheses of Lemma 5.2.7 and so $N \le M(\delta/2 - \sqrt{\epsilon(2-\epsilon)}/\lambda, \eta)$. This is true for every $\epsilon > 0$ and so $N \le M(\delta/2, \eta)$, as required. □

Using this result we may also bound the dimension of charts in Lipschitz differentiability spaces.

Corollary 5.2.9. *Let (U, φ) be a chart in a Lipschitz differentiability space and $\eta \ge 1$. Suppose that $Y \subset U$ has positive μ measure, is contained within some $W \in D_\eta$ and that any $\widetilde{B}_\delta$ subset of Y is μ-null. Then the dimension of (U, φ) is bounded above by $M(\delta/2, \eta)$.*

Proof. Suppose that (U, φ) is a chart of dimension n. Then by Lemma 2.3.3, $\operatorname{Lip}(v \cdot \varphi, x_0) > 0$ for all $v \in \mathbb{S}^{n-1}$ and almost every $x_0 \in U$. Therefore, by Proposition 5.2.8, $n \le M(\delta/2, \eta)$. □

Using the previous construction we may give our characterisation of Lipschitz differentiability spaces via null sets.

Theorem 5.2.10. *For (X, d, μ) a metric measure space, the following are equivalent:*

1. *(X, d, μ) is a Lipschitz differentiability space.*
2. *Any $\widetilde{A}$ subset of X is μ-null and X is pointwise doubling.*
3. *Any $\widetilde{B}$ subset of X is μ-null and X is pointwise doubling.*
4. *There exists a countable Borel decomposition $X = \cup_i X_i$ and sequences $\eta_i \ge 1$ and $\delta_i > 0$ such that:*

- *For any Lipschitz $f\colon X \to \mathbb{R}$ and $i \in \mathbb{N}$, there exists an Alberti representation of $\mu \llcorner X_i$ with f-speed δ_i.*
- *For every $i \in \mathbb{N}$ there exists a $W \in D_{\eta_i}$ such that $\mu(X_i \setminus W) = 0$.*

Proof. We will prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Firstly, by Theorem 4.2.6, any $\widetilde{A}$ subset of a Lipschitz differentiability space is μ-null. Further, by Corollary 5.2.4, any Lipschitz differentiability space is pointwise doubling, giving the first implication. The second implication is true since $\widetilde{B} \subset \widetilde{A}$ for any metric measure space.

To prove (3) $\Rightarrow$ (4) observe that, for any $\delta > 0$, $\widetilde{B}_\delta$ is closed under taking countable unions. Therefore there exists a countable Borel decomposition $X = \cup_i X_i \cup N$ where $N \in \widetilde{B}$ and such that, for each $i \in \mathbb{N}$, every $\widetilde{B}_{1/i}$ subset of X_i has measure zero. In particular, given a Lipschitz function $f\colon X \to \mathbb{R}$, we may apply Corollary 3.3.8 to obtain an Alberti representation of $\mu \llcorner X_i$ with f-speed $1/i$. Further, by Lemma 5.2.3, for each $i \in \mathbb{N}$ there exists a countable Borel decomposition $X_i = \cup_j (X_i^j \cup N_i)$ where N_i is μ-null and for each $j \in \mathbb{N}$ there exists a $W \in D_{1/j}$ such that $X_i^j \subset W$. Therefore, the decomposition $X = \cup_{i,j}(X_i^j \cup N_i)$ is of the required form.

For the final implication, by applying Lemma 5.2.1 and Proposition 5.2.8, we see that (X, d, μ) is a Lipschitz differentiability space. □

Finally, as mentioned above, metric measure spaces in which porous sets have measure zero are pointwise doubling. Moreover, by Corollary 3.1.5, porous sets in Lipschitz differentiability spaces have measure zero. Therefore, we obtain an additional characterisation of Lipschitz differentiability spaces entirely via null sets.

Theorem 5.2.11. *For (X, d, μ) a metric measure space, the following are equivalent:*

- *(X, d, μ) is a Lipschitz differentiability space.*
- *Any $\widetilde{A}$ and any porous set in X is μ-null.*
- *Any $\widetilde{B}$ and any porous set in X is μ-null.*

Chapter 6

Arbitrary Alberti representations

We now extend the results from the previous chapters to find Alberti representations in the direction of an arbitrary cone C in a chart (U, φ) of a Lipschitz differentiability space (X, d, μ). This also leads to a further characterisation of Lipschitz differentiability spaces via related null sets.

As before, we will produce this representation using Corollary 3.3.8 and so we are required to show that any Borel $S \subset U$ that satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C has measure zero. This is achieved by giving a decomposition $S = \cup_i S_i$ such that each S_i satisfies $\mathcal{H}^1(\gamma \cap S_i) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of a cone C_i that defines the direction of an existing Alberti representation.

For this we will need to deduce properties of the derivative of a function obtained from Lemma 4.2.1. In particular, we require a bound on the directional derivative of the function in the direction w, for w as in the hypotheses of the Lemma. This is possible for any x_0 for which there exists $x_m \to x_0$ such that $\varphi(x_m) - \varphi(x_0)$ is parallel to w.

To begin we concatenate the curves obtained from our existing Alberti representations, in suitable ratios, to show the existence of such a sequence. Recall the notion of separated Alberti representations given in Definition 5.1.3.

Lemma 6.1. *Let (X, d, μ) be a metric measure space, $\varphi\colon X \to \mathbb{R}^n$ Lipschitz and $\delta, \xi > 0$. Suppose that X has n ξ-separated Alberti representations with speed strictly greater than δ. Then there exists an $\eta > 0$ such that, for any measurable $S \subset X$*

and for almost every $x_0 \in S$*, given any* $v \in \mathbb{S}^{n-1}$ *there exists* $S \ni x_m \to x_0$ *with*

$$\limsup_{m\to\infty} \left\| \frac{\varphi(x_m)-\varphi(x_0)}{d(x_m,x_0)} - v\frac{\|\varphi(x_m)-\varphi(x_0)\|}{d(x_m,x_0)} \right\| = 0$$

and

$$\limsup_{m\to\infty} \frac{\|\varphi(x_m)-\varphi(x_0)\|}{d(x_m,x_0)} \geq \eta \operatorname{Lip}(\varphi,x_0).$$

Proof. Let $S \subset X$ be measurable, $w \in \mathbb{S}^{n-1}$ and $0<\epsilon<1$. By refining if necessary, we may suppose that the above Alberti representations are in the φ-direction of ξ-separated cones $C_1 = C(w_1,\epsilon),\ldots,C_n = C(w_n,\epsilon)$. Further, for each $1 \leq i \leq n$ let $C_i' \supset C_i$ be open cones that are also ξ-separated.

For $1 \leq i \leq n$ let Γ^i be the set of $\gamma \in \Gamma(X)$ in the φ-direction of C_i with speed δ. Further, for each $0 < R < \epsilon$, let $G_R^i(S)$ be the set of $x_0 \in S$ for which there exist a $\gamma \in \Gamma^i$ and a $t_0 \in \operatorname{Dom}\gamma$ with $\gamma(t_0) = x_0$ such that, for every $0 < r < R/\|(\varphi\circ\gamma)'(t_0)\|$ and every $t \in \operatorname{Dom}\gamma$ with $|t-t_0| \leq 5R/\|(\varphi\circ\gamma)'(t_0)\|$, the following conditions hold:

1. $\mathcal{L}^1(\gamma^{-1}(S)\cap B(t_0,2r)) \geq 4r(1-\epsilon)$.
2. $\|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - (\varphi\circ\gamma)'(t_0)\| \leq \epsilon\|(\varphi\circ\gamma)'(t_0)\||t-t_0|$.
3. $\|(\varphi\circ\gamma)'(t_0)\||t-t_0| \leq 2\|\varphi(\gamma(t))-\varphi(\gamma(t_0))\|$.
4. $\varphi(\gamma(t)) - \varphi(\gamma(t_0)) \in C_i'$.
5. $\|(\varphi\circ\gamma)'(t_0)\||t-t_0| \geq \delta \operatorname{Lip}(\varphi,\gamma(t_0)) d(\gamma(t),\gamma(t_0))/2$.

Since μ has an Alberti representation in the φ-direction of each C_i with speed δ, by Proposition 2.2.4 each $G_R^i(S)$ is measurable and monotonically increases to a set of full measure in S as $R \searrow 0$.

Now let $x_0 \in G_R^i(S)$, let $\gamma \in \Gamma^i$ be as in the definition of G_R^i for x_0 and write $w = \sum_i \lambda_i w_i$. Then for any $0 < r < R/\max\lambda_i$,

$$r\lambda_i/\|(\varphi\circ\gamma)'(t_0)\| \leq R/\|(\varphi\circ\gamma)'(t_0)\|$$

and so, by (1), there exists a $t \in \operatorname{Dom}\gamma$ with

$$0 \leq t - t_0 - \frac{r\lambda_i}{\|(\varphi\circ\gamma)'(t_0)\|} \leq \frac{4r\epsilon}{\|(\varphi\circ\gamma)'(t_0)\|} \leq 4\epsilon(t-t_0).$$

In particular

$$|t-t_0| \leq 5R/\|(\varphi\circ\gamma)'(t_0)\|$$

and

$$\left| \|(\varphi \circ \gamma)'(t_0)\| - \frac{r\lambda_i}{|t - t_0|} \right| \leq 4\epsilon.$$

Note also that, since $(\varphi \circ \gamma)'(t_0)$ belongs to C_i,

$$\left\| \frac{(\varphi \circ \gamma)'(t_0)}{\|(\varphi \circ \gamma)'(t_0)\|} - w_i \right\| \leq \sqrt{2}\epsilon.$$

Therefore, by the triangle inequality, (2) and (3),

$$\begin{aligned} \|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - r\lambda_i w_i\| &\leq 10\sqrt{\epsilon}\|(\varphi \circ \gamma)'(t_0)\||t - t_0| \\ &\leq 20\sqrt{\epsilon}\|\varphi(\gamma(t)) - \varphi(\gamma(t_0))\| \end{aligned}$$

Moreover, by (5),

$$\delta \operatorname{Lip}(\varphi, \gamma(t_0)) d(\gamma(t), \gamma(t_0)) \leq 5R \leq 5\epsilon.$$

That is, for every $x_0 \in G_R^i$, there exists an $x \in S$ with

$$\|\varphi(x) - \varphi(x_0) - r\lambda_i w_i\| \leq 20\sqrt{\epsilon}\|\varphi(x) - \varphi(x_0)\| \tag{6.0.1}$$

and

$$(x, x_0) \leq 5\epsilon / \delta \operatorname{Lip}(\varphi, \gamma(t_0)). \tag{6.0.2}$$

Now define $S_R^0 = S$ and for each $0 \leq i < n$ define

$$S_R^{i+1} = G_R^{i+1}\left(S_R^i\right).$$

Then for any $x_0 \in S_R^n$ and $0 \leq i < n$ there exists $x_i \in S_R^{n-i}$ such that the relations (6.0.1) and (6.0.2) hold between x_i and x_{i+1}. Therefore

$$\|\varphi(x_n) - \varphi(x_0) - rw\| \leq 20\sqrt{\epsilon} \sum_{0 \leq i < n} \|\varphi(x_i) - \varphi(x_{i+1})\|$$

and so, by the triangle inequality,

$$\|\varphi(x_n) - \varphi(x_0) - \|\varphi(x_n) - \varphi(x_0)\|w\| \leq 40\sqrt{\epsilon} \sum_{0 \leq i < n} \|\varphi(x_i) - \varphi(x_{i+1})\|.$$

However each γ above was chosen so that $\varphi(x_i) - \varphi(x_{i-1}) \in C_i'$ for each $0 \leq i < n$.

Since the C'_i form a collection of ξ-separated cones,

$$\begin{aligned}\|\varphi(x_0)-\varphi(x_n)\| &= \left\|\sum_{0\le i<n}\varphi(x_i)-\varphi(x_{i+1})\right\| \\ &\ge \xi \max_{0\le i<n}\|\varphi(x_i)-\varphi(x_{i+1})\| \\ &\ge \xi\delta \operatorname{Lip}(\varphi,x_0)\max_{0\le i<n} d(x_i,x_{i+1})/4 > 0.\end{aligned}$$

Therefore

$$\|\varphi(x_n)-\varphi(x_0)-\|\varphi(x_n)-\varphi(x_0)\|w\| \le 40\sqrt{\epsilon}n \operatorname{Lip}\varphi d(x_n,x_0)/\xi$$

and by the triangle inequality

$$\|\varphi(x_n)-\varphi(x_0)\| \ge \xi\delta \operatorname{Lip}(\varphi,x_0) d(x_n,x_0)/4n.$$

Note also that we must have $x_0 \neq x_n$ and

$$d(x_n,x_0) \le \sum_{0\le i<n} d(x_i,x_{i+1}) \le 5n\epsilon.$$

Finally, $\cup_{m\in\mathbb{N}} S^n_{1/m}$ is a set of full measure in S and so for almost every $x_0 \in S$ there exists an $x_n \in S$ with the above properties, for a given $0<\epsilon<1$. Taking a countable intersection over $\epsilon \in (0,1)\cap\mathbb{Q}$ completes the proof for $\eta = \xi\delta/n$. □

Corollary 6.2. *Let (U,φ) be a λ-structured chart in a Lipschitz differentiability space and for some $\xi,\delta>0$, let $V\subset U$ be Borel such that $\mu \llcorner V$ has n ξ-separated Alberti representations with speed strictly greater than δ. Suppose that, for some $w \in \mathbb{S}^{n-1}$ and $0<\epsilon<1$, $S\subset V$ is closed and satisfies $\mathcal{H}^1(\gamma\cap S)=0$ for any $\gamma\in\Gamma(X)$ in the φ-direction of $C(w,\epsilon)$. Then there exists an $\eta>0$ and, for any $\zeta>0$, a $(K(\epsilon)+1+\zeta)\operatorname{Lip}\varphi$-Lipschitz function $f\colon U\to\mathbb{R}$ such that*

1. *For every $x_0\in S$ and $x\in U$ with $(\varphi(x)-\varphi(x_0))\cdot w \ge 0$,*

$$f(x)-f(x_0) \ge (\varphi(x)-\varphi(x_0))\cdot w - \zeta.$$

2. *For almost every $x_0 \in S$*

$$Df(x_0)\cdot w \le \zeta/\eta\lambda.$$

Proof. Let $f\colon U\to\mathbb{R}$ be the Lipschitz function obtained from an application of Lemma 4.2.1 (for $\delta = \zeta\min\{1, 1/\operatorname{diam} U, 1/\operatorname{Lip}\varphi\}$), so that f automatically satisfies all of the required properties (for 2ζ) except for (2). Further, there exists a $\rho > 0$ such that, for every $x_0 \in S$ and $y, z \in B(x_0, \rho)$,

$$|f(y) - f(z)| \leq K(\epsilon)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + \zeta d(y, z)$$

for $K(\epsilon) = (1-\epsilon)/\sqrt{\epsilon(2-\epsilon)}$ and $\mathcal{P}\colon \mathbb{R} \to \mathbb{R}$ the orthogonal projection onto the plane orthogonal to w passing through the origin.

Now let $\eta > 0$ such that the conclusion of Lemma 6.1 holds. Then for almost every $x_0 \in V$, $Df(x_0)$ exists and there exist $U \ni x_m \to x_0$ with $\|\varphi(x_m) - \varphi(x_0)\| \geq \eta d(x_m, x_0)$ such that

$$\frac{\|\varphi(x_m) - \varphi(x_0) - \|\varphi(x_m) - \varphi(x_0)\|w\|}{d(x_m, x_0)} \to 0.$$

In particular, by the triangle inequality,

$$\limsup_{m\to\infty} \frac{\|\mathcal{P}(\varphi(x_m) - \varphi(x_0))\|}{d(x_m, x_0)} \leq \|\mathcal{P}(w)\| \limsup_{m\to\infty} \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} = 0$$

and so

$$\limsup_{m\to\infty} \frac{|f(x_m) - f(x_0)|}{d(x_m, x_0)} \leq \zeta.$$

Therefore

$$\begin{aligned}
\eta \operatorname{Lip}(\varphi, x_0) Df(x_0) \cdot w &\leq \limsup_{m\to\infty} \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} Df(x_0) \cdot w \\
&\leq \limsup_{m\to\infty} \frac{Df(x_0) \cdot (\varphi(x_m) - \varphi(x_0))}{d(x_m, x_0)} \\
&= \limsup_{m\to\infty} \frac{|f(x_m) - f(x_0)|}{d(x_m, x_0)} \\
&\leq \zeta.
\end{aligned}$$

Since (U, φ) is a λ-structured chart, $\operatorname{Lip}(\varphi, x_0) \geq \lambda$ and so dividing by $\eta\lambda$ completes the proof. □

Given a set S as in the hypotheses of Corollary 6.2, we will decompose it using the following Lemma. In our application, the D_m in the hypotheses will be the derivatives of Lipschitz functions obtained from Corollary 6.2.

Lemma 6.3. *Let (X, d, μ) be a metric measure space, $U \subset X$ Borel and $w \in \mathbb{S}^{n-1}$. Suppose that, for some $\beta \geq 0$ and $E > 0$, there exists a sequence of measurable* *functions* $D_m\colon X \to \mathbb{R}^n$ *with essential supremum* E *such that*

$$\limsup_{m\to\infty} D_m(x)\cdot w + \beta \le w\cdot w = 1$$

for almost every $x \in U$*. Suppose further that, for some* $\xi > 0$ *and* ξ*-separated* $w_1, \ldots, w_k \in \mathbb{S}^{n-1}$*,* w *belongs to the convex cone of the* w_i*. Then there exists a Borel decomposition*

$$S = N \cup \bigcup_{j\in\mathbb{N}} S_j$$

with $\mu(N) = 0$ *and, for every* $j \in \mathbb{N}$*, a* $1 \le i \le n$ *such that, for every* $0 < \theta < 1$*,*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{1\le k\le m} D_k(x)\cdot v + \frac{\|v\|\beta\xi}{n} \le w\cdot v + \sqrt{2\theta}(E+1)\|v\|$$

uniformly for $x \in S_j$ *and* $v \in C(w_i, \theta)$*.*

Proof. Since each D_m is bounded, there exists a $g \in L^2(U)$ such that, after passing to a subsequence, $D_m \to g$ weakly. Then, by the Banach-Saks theorem, there exists a further subsequence such that the functions

$$\widetilde{D}_m := \frac{1}{m} \sum_{1\le j\le m} D_j$$

converge pointwise almost everywhere to g. Moreover, since

$$\limsup_{m\to\infty} D_m(x)\cdot w + \beta \le w\cdot w,$$

$g(x)\cdot w + \beta \le w \cdot w$ almost everywhere.

Let

$$S = N \cup \bigcup_{j\in\mathbb{N}} S_j$$

be a Borel decomposition of S where N is μ-null and such that $\widetilde{D}_m \to g$ uniformly on each S_j. Further, since the w_i are ξ-separated and w lies in the convex cone of the w_i, there exists $0 \le \lambda_i \le 1/\xi$ with $w = \sum_i \lambda_i w_i$. Therefore, for each x_0 in some S_j, there exists a $1 \le i \le n$ with

$$g\cdot w_i + \beta\xi/n \le w\cdot w_i.$$

We may therefore decompose each S_j into the sets

$$S_j^i = \{x_0 \in S_j : g \cdot w_i + \beta\xi/n \leq w \cdot w_i\}$$

for $1 \leq i \leq k$. By taking a further decomposition, we may suppose that each S_j^i is compact.

Finally, for any $0 < \theta < 1$, $j \in \mathbb{N}$, $1 \leq i \leq n$ and $v \in C(w_i, \theta)$,

$$\begin{aligned} g(x) \cdot v + \|v\|\beta\xi/n &= \|v\|(g(x) \cdot w_i + \beta\xi/n) + g(x) \cdot (v - \|v\|w_i) \\ &\leq \|v\|(w \cdot w_i \sqrt{2\theta} E) \\ &\leq w \cdot v + \sqrt{2\theta}(E+1)\|v\|. \end{aligned}$$

Since $\widetilde{D}_m \to g$ uniformly on each S_j^i, this decomposition is of the required form. □

Finally, we will show that a set with certain properties intersects a Lipschitz curve γ in a set of measure zero by applying the following Lemma to the domain of γ.

Lemma 6.4. *Let $S \subset \mathbb{R}$ be measurable and $L > 0$. Suppose that there exists a sequence of L-Lipschitz functions $f_m \colon S \to \mathbb{R}$ and measurable functions $\Phi, \Psi \colon S \to \mathbb{R}$ such that, for almost every $t_0 \in S$, $\Phi(t_0) < \Psi(t_0)$ and:*

1. *There exists $m_j \to \infty$ such that, for every $t \in S$ with $t \geq t_0$,*

$$f_{m_j}(t) - f_{m_j}(t_0) \geq \Psi(t_0)(t - t_0) - 1/m_j.$$

2. *There exists an $M \in \mathbb{N}$ such that, for every $m \geq M$,*

$$Df_m(t_0) \leq \Phi(t_0).$$

Then S is Lebesgue null.

Proof. Suppose that such an S has positive Lebesgue measure. Then there exists an $M \in \mathbb{N}$, $\alpha < \beta \in \mathbb{R}$ and $S' \subset S$ of positive measure such that (1) and

$$Df_m(t_0) \leq \Phi(t_0) \leq \alpha < \beta \leq \Psi(t_0)$$

for all $t_0 \in S'$ and $m \geq M$. Let t_0 be a density point of S' and $R > 0$ such that, for every $t \in (t_0, t_0 + R)$,

$$\frac{\mathcal{L}^1(S' \cap (t_0, t))}{t} \geq 1 - (\beta - \alpha)/2L.$$

Then, for any $m \geq M$ and $t \in (t_0, t_0 + R) \cap S$,

$$\begin{aligned} f_m(t) - f_m(t_0) &= \int_{(t_0,t)\cap S'} Df_m + \int_{(t_0,t)\setminus S'} Df_m \\ &\leq \alpha(t - t_0) + (t - t_0)L(\beta - \alpha)/2L. \\ &\leq (\beta + \alpha)(t - t_0)/2. \end{aligned}$$

However, if $m_j \geq M$ such that (1) holds for t_0, then for any $t \in (t_0, t_0 + R) \cap S$,

$$(\beta + \alpha)(t - t_0)/2 \geq f_{m_j}(t) - f_{m_j}(t_0) \geq \beta(t - t_0) - 1/m_j.$$

In particular, $|t - t_0| \leq 2/(\beta - \alpha)m_j$ for $m_j \to \infty$ and so $t = t_0$, contradicting our assumption that t_0 is a density point of S. □

We now apply these results to a chart in a Lipschitz differentiability space.

Theorem 6.5. *Let (U, φ) be an n-dimensional chart in a Lipschitz differentiability space (X, d, μ), $w \in \mathbb{S}^{n-1}$ and $0 < \epsilon < 1$. Then there exists an Alberti representation of $\mu \llcorner U$ in the φ-direction of $C(w, \epsilon)$.*

Proof. Since any chart in a Lipschitz differentiability space has a countable decomposition into a μ-null set and structured charts, and since we may combine Alberti representations using Lemma 2.1.4, it suffices to prove the result for (U, φ) a λ-structured chart, for some $\lambda > 0$. Further, by Corollary 3.3.8, there exists a decomposition $U = A \cup S$ where $\mu \llcorner A$ has an Alberti representation in the φ-direction of $C(w, \epsilon)$ and S satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of $C(w, \epsilon)$. Finally, by Theorem 4.2.3, there exists a countable decomposition $U = \cup_m U_m$ such that each $\mu \llcorner U_m$ has n φ-independent Alberti representations. By refining these representations if necessary, we may suppose that for any $m \in \mathbb{N}$ there exist $w_1, \ldots, w_n \in \mathbb{S}^{n-1}$ and $0 < \theta, \xi, \delta < 1$ such that the Alberti representations of $\mu \llcorner U_m$ are in the φ-direction of ξ-separated cones $C(w_1, \theta), \ldots, C(w_n, \theta)$, with speed δ. By refining these representations further, we may suppose that

$$\frac{\sqrt{2\theta}(2 + K(\epsilon) + \lambda)}{\lambda} \leq \frac{\xi}{4n}$$

for $K(\epsilon) = (1 - \epsilon)/\sqrt{\epsilon(2 - \epsilon)}$. Therefore, we are required to show that, for any $m \in \mathbb{N}$, any compact $S' \subset S \cap U_m$ is μ-null.

We apply Corollary 6.2 (with $\zeta = 1/m$) to obtain a sequence of $(2 + K(\epsilon))$-Lipschitz functions $f_m \colon U \to \mathbb{R}$ such that:

- For every $x_0 \in S'$ and $x \in U$ with $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$,

$$f_m(x) - f_m(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - 1/m.$$

- For almost every $x_0 \in S'$,

$$\limsup_{m \to \infty} Df_m(x_0) \cdot w + 1 \leq w \cdot w.$$

Note that, since (U, φ) is a λ-structured chart, by Lemma 2.3.4,

$$\|Df_m(x_0)\| \leq (2 + K(\epsilon))/\lambda.$$

Therefore, by Lemma 6.3, there exists a countable Borel decomposition

$$S = N \cup \bigcup_{j \in \mathbb{N}} S_j$$

where $\mu(N) = 0$ and for each $j \in \mathbb{N}$ there exists a $1 \leq i \leq n$ and an $M \in \mathbb{N}$ such that, for every $m \geq M$ and $x_0 \in S_j$,

$$\frac{1}{m} \sum_{1 \leq k \leq m} Df_k(x_0) \cdot v + \frac{\xi}{n}\|v\| \leq w \cdot v + \frac{\xi}{2n}\|v\| \tag{6.0.3}$$

for all $v \in C(w_i, \theta)$. Define, for each $m \in \mathbb{N}$

$$F_m = \frac{1}{m} \sum_{1 \leq k \leq m} f_k.$$

For each $j \in \mathbb{N}$ and almost every $x_0 \in S_j$ each $Df_k(x_0)$ exists and so

$$DF_m = \frac{1}{m} \sum_{1 \leq k \leq m} Df_k.$$

Further, for every $x_0 \in S$ and $x \in U$ with $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$,

$$F_m(x) - F_m(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - 1/m.$$

We fix $j \in \mathbb{N}$ and let $1 \leq i \leq n$ and $M \in \mathbb{N}$ such that (6.0.3) holds for every $m \geq M$ and $x_0 \in S_j$. It suffices to prove that $\mu(S_j) = 0$. Given the above Alberti representations of $\mu \llcorner U_m$, it suffices to prove that $\mathcal{H}^1(\gamma \cap S_j) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of $C(w_i, \theta)$.

To show this, fix $\gamma \in \Gamma(X)$ in the φ-direction of $C(w_i, \theta)$ and define, for each $m \in \mathbb{N}$, the $(2+K(\epsilon)) \operatorname{Lip}\varphi \operatorname{Lip}\gamma$-Lipschitz function

$$g_m = F_m \circ \gamma \colon \operatorname{Dom}\gamma \to \mathbb{R}.$$

Then, for any $t_0 \in \operatorname{Dom}\gamma$, if $(\varphi \circ \gamma)'(t_0)$ exists and $\gamma(t_0) \in S_j$,

$$Dg_m(t_0) = DF_m(\gamma(t_0)) \cdot (\varphi \circ \gamma)'(t_0).$$

Therefore, if $(\varphi \circ \gamma)'(t_0) \in C(w_i, \theta)$,

$$\begin{aligned} Dg_m(t_0) &\leq w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{2n} \|(\varphi \circ \gamma)'(t_0)\| \\ &\leq w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{2n} |w \cdot (\varphi \circ \gamma)'(t_0)| \\ &:= \Phi(t_0). \end{aligned}$$

Also, for any $t, t_0 \in \operatorname{Dom}\gamma$,

$$g_m(t) - g_m(t_0) \geq (\varphi(\gamma(t)) - \varphi(\gamma(t_0))) \cdot w - 1/m.$$

Suppose that $\gamma^{-1}(S_j)$ has positive measure. Then there exists an $R > 0$ and a $T \subset \gamma^{-1}(S_j)$ of positive measure such that, for every $t_0 \in T$ and $t \in \operatorname{Dom}\gamma$ with $|t - t_0| \leq R$, $(\varphi \circ \gamma)'(t_0) \in C(w_i, \theta)$ and

$$\begin{aligned} (\varphi(\gamma(t)) - \varphi(\gamma(t_0))) \cdot w &\geq \left(w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{4n} |w \cdot (\varphi \circ \gamma)'(t_0)| \right) (t - t_0) \\ &:= \Psi(t_0)(t - t_0). \end{aligned}$$

We choose $s \in T$ such that $T' := T \cap B(s, R)$ has positive measure. Then for almost every $t_0 \in T'$, $\Psi(t_0) > \Phi(t_0)$ and:

- For any $m \geq M$ and every $t \in T'$ with $t \geq t_0$,

$$g_m(t) - g_m(t_0) \geq \Psi(t_0)(t - t_0) - 1/m.$$

- For every $m \geq M$,

$$Dg_m(t_0) \leq \Phi(t_0).$$

Therefore, by Lemma 6.4, T' is Lebesgue null, a contradiction. □

We may use this Theorem to improve our description of the local structure of a Lipschitz differentiability space.

Corollary 6.6. *Let (U, φ) be an n-dimensional chart in a Lipschitz differentiability space (X, d, μ). Then for almost every $x \in U$ and any cone $C \subset \mathbb{R}^n$, there exists a $\gamma^x \in \Gamma(X)$ such that $(\gamma^x)^{-1}(x) = 0$ is a density point of $(\gamma^x)^{-1}(U)$ and such that $(\varphi \circ \gamma^x)'(0) \in C$.*

We also obtain another characterisation of Lipschitz differentiability spaces corresponding to arbitrary Alberti representations.

Definition 6.7. For a metric measure space (X, d, μ) we define $\widetilde{C}$ to be the set of Borel $S \subset X$ for which there exist $0 < \epsilon, \eta < 1$ and, for every $\delta > 0$, there exist an $n \in \mathbb{N}$, a Lipschitz $\psi \colon X \to \mathbb{R}^n$ and a cone $C \subset \mathbb{R}^n$ of width ϵ such that:

- For every $x_0 \in S$,
$$\operatorname{Lip}(v \cdot \psi, x_0) > \eta \operatorname{Lip}(\psi, x_0).$$

- For every $\gamma \in \Gamma(X)$ in the ψ-direction of C with ψ-speed δ, $\mathcal{H}^1(\gamma \cap S) = 0$.

Theorem 6.8. *For a metric measure space (X, d, μ) the following are equivalent:*

- *(X, d, μ) is a Lipschitz differentiability space.*
- *Every $\widetilde{C}$ subset of X is μ-null and X is pointwise doubling.*
- *Every $\widetilde{C}$ subset of X and every porous subset of X is μ-null.*

Proof. These two conditions are sufficient since there exists a countable decomposition of any $\widetilde{A}$ subset of X into $\widetilde{C}$ sets. Therefore, if any $\widetilde{C}$ subset of X is μ-null, so is any $\widetilde{A}$ subset.

Conversely, for any $S \in \widetilde{C}$ that is contained within a structured chart of a Lipschitz differentiability space, by Lemma 4.2.5 there exists a $0 < \epsilon < 1$ and a sequence of cones C_m of width ϵ such that $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C_m with speed $1/m$. Then there exists a cone C of width $\epsilon/2$ and $m_j \to \infty$ such that $C \subset C_{m_j}$ for each $j \in \mathbb{N}$. Therefore $\mathcal{H}^1(\gamma \cap S) = 0$ for any $\gamma \in \Gamma(X)$ in the φ-direction of C and so, by Theorem 6.5, S is μ-null. □

Chapter 7

Relationship with other works

As an example of the use of our theory, we give alternate proofs of some of the existing results within the area of Lipschitz differentiability spaces.

7.1 Cheeger's differentiation theorem

We begin by introducing the notion of a Poincaré inequality in a metric measure space.

Definition 7.1.1. Let (X, d) be a metric space and $f\colon X \to \mathbb{R}$ Lipschitz. We say that a measurable function $\rho\colon X \to \mathbb{R}$ is an *upper gradient of* f if, for any $\gamma \in \Pi(X)$ that is parametrised by arc length,

$$|f(\gamma_e) - f(\gamma_s)| \leq \int_{\operatorname{Dom}\gamma} \rho \circ \gamma \, \mathrm{d}\mathcal{L}^1$$

where γ_e and γ_s are the end points of γ.

Further, for $p, P \geq 1$ we say that a metric measure space (X, d, μ) satisfies the p-Poincaré inequality with constant P if, for every closed ball $B = \overline{B}(x_0, r) \subset X$, $\mu(B) > 0$ and

$$⨍_B |f - f_B| \leq Pr \left(⨍_{PB} \rho^p \right)^{1/p}.$$

Here

$$f_B = ⨍_B f := \frac{1}{\mu(B)} \int_B f.$$

Finally, for $C \geq 1$, we say that a metric measure space (X, d, μ) is C-*doubling* if

$$\mu(\overline{B}(x_0, r)) \leq C\mu(\overline{B}(x_0, r/2))$$

for every $x_0 \in X$ and every $r > 0$.

Theorem 7.1.2 (Cheeger [Che99])**.** *Any C-doubling metric measure space (X, d, μ) that satisfies the p-Poincaré inequality with constant P is a Lipschitz differentiability space. Moreover, each chart is of a dimension bounded above by an integer depending only upon C and P, independent of the metric measure space.*

Proof. Since a C-doubling metric measure space is a D_η set, for some $\eta \geq 1$ depending only upon C, by Theorem 5.2.11 and Corollary 5.2.9 it suffices to prove the existence of a $\delta > 0$ depending only upon C and P such that any $\widetilde{B}_\delta$ subset of X is μ-null.

Firstly, by Proposition 4.3.3 of [Kei04], for any C-doubling metric measure space, there exists a C' depending only upon C such that, for any Lipschitz $f\colon X \to \mathbb{R}$,

$$\operatorname{Lip}(f, x_0) \leq C' \lim_{r\to 0} \frac{1}{r} \fint_{\overline{B}(x_0,r)} |f - f_B|$$

for almost every $x_0 \in X$. Therefore, if ρ is an upper gradient of f, by using the Poincaré inequality and applying Lebesgue's differentiation theorem to ρ, we see that there exists a $C'' > 0$ depending only upon C and P such that

$$\rho(x_0) \geq C'' \operatorname{Lip}(f, x_0) \tag{7.1.1}$$

for almost every $x_0 \in X$. We set $\delta = C''/2$.

Now suppose that $f\colon X \to \mathbb{R}$ is Lipschitz and $S \subset \{x_0 : \operatorname{Lip}(f, x_0) > 0\}$ satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Gamma(X)$ with f-speed δ. Then for any $\gamma \in \Gamma(X)$ and almost every $t_0 \in \gamma^{-1}(S)$,

$$(f \circ \gamma)'(t_0) \leq \delta \operatorname{Lip}(f, \gamma(t_0)) \operatorname{Lip}(\gamma, t_0).$$

In particular this is true for any $\gamma \in \Pi(X)$ that is parametrised by arc length and so

$$\rho(x) = \begin{cases} \delta \operatorname{Lip}(f, x) & x \in S \\ \operatorname{Lip}(f, x) & \text{otherwise} \end{cases}$$

is an upper gradient of f. However, for almost every $x \in S$, by equation (7.1.1),

$$\delta \operatorname{Lip}(f, x) = \rho(x) \geq 2\delta \operatorname{Lip}(f, x) > 0.$$

Therefore S must be μ-null. In particular, any $\widetilde{B}_\delta$ subset of X is μ-null, as required. □

As pointed out in the introduction, the existence of an Alberti representation of any doubling Lipschitz differentiability space (X, d, μ) that satisfies the Poincaré inequality may be deduced from a theorem of Cheeger and Kleiner. To see this, suppose that S is compact, contained within a chart and satisfies $\mathcal{H}^1(\gamma \cap S) = 0$ for every $\gamma \in \Pi(X)$. Then by applying [CK09], Theorem 4.2 (and adopting it's terminology) with f a component of the chart map and the negligible set $N = \{(\gamma, t) : \gamma(t) \in S\}$, we see that the minimal upper gradient of f equals zero almost everywhere in S. However, the minimal upper gradient of f equals $\operatorname{Lip}(f, .) > 0$ almost everywhere and so S must be μ-null. An application of Lemma 3.3.2 gives the required Alberti representation. In fact, almost every curve in this Alberti representation is defined on an interval.

7.2 Keith's Lip-lip condition

In [Kei04], Keith introduced the Lip-lip condition (see below) on a metric measure space and showed that any doubling metric measure space with a Lip-lip condition is a Lipschitz differentiability space (via the Poincaré inequality). We now use our theory to give an alternate proof of this fact and to prove the converse statement. This gives a characterisation of Lipschitz differentiability spaces via the Lip-lip condition.

Definition 7.2.1. We say that a metric measure space (X, d, μ) satisfies a *Lip-lip* condition if there exists a countable Borel decomposition $X = \cup_i X_i$ and for each $i \in \mathbb{N}$ a $\delta_i > 0$ such that, for any Lipschitz $f \colon X \to \mathbb{R}$,

$$\begin{aligned}\delta_i \operatorname{Lip}(f, x_0) &\leq \liminf_{r \to 0} \sup \left\{ \frac{|f(x) - f(x_0)|}{r} : 0 < d(x, x_0) < r \right\} \\ &:= \operatorname{lip}(f, x_0)\end{aligned}$$

for almost every $x_0 \in X_i$.

Theorem 7.2.2. *A metric measure space (X, d, μ) satisfies a Lip-lip condition if and only if every $\widetilde{B}$ subset of X is μ-null.*

Proof. First suppose that any $\widetilde{B}$ subset of X is μ-null. Then by Theorem 5.2.11, there exists a countable Borel decomposition $X = \cup_i X_i$ and for each $i \in \mathbb{N}$ a $\delta_i > 0$ such that, for any Lipschitz $f \colon X \to \mathbb{R}$ and every $i \in \mathbb{N}$, there exists an Alberti representation of $\mu \llcorner X_i$ with speed δ_i. In particular, for every $i \in \mathbb{N}$ and almost every $x_0 \in X_i$, there exist $\gamma \in \Gamma(X)$ and t_0 a density point of $\operatorname{Dom} \gamma$ such that $\gamma(t_0) = x_0$ and

$$\delta_i \operatorname{Lip}(f, x_0) \leq (f \circ \gamma)'(t_0).$$

However, since t_0 is a density point of $\operatorname{Dom}\gamma$,

$$(f \circ \gamma)'(t_0) \le \operatorname{lip}(f, x_0)$$

and so X satisfies a Lip-lip condition.

Conversely, suppose that $S \subset X$ belongs to $\widetilde{B}$, so that S belongs to $\widetilde{A}$. Then for any $\epsilon > 0$ we may apply Lemma 4.2.1 to construct a sequence of functions that satisfy the hypotheses of Lemma 3.1.2 on some Borel $S' \subset S$ with $\mu(S') \ge \mu(S) - \epsilon$. Therefore, by applying Lemma 3.1.2 with positive sequences $R_i, r_i \to 0$ such that $r_i/R_i \to 0$ as $i \to \infty$, for any $\delta > 0$ we obtain a Lipschitz function $f \colon X \to \mathbb{R}$ with

$$\delta \operatorname{Lip}(f, x_0) > \operatorname{lip}(f, x_0)$$

for almost every $x_0 \in S'$. In particular, if X satisfies a Lip-lip condition, S' and hence S are μ-null. □

By combining this Theorem with Theorem 5.2.11, we obtain the following Corollary. Very shortly after the first preprint of these results appeared, Gong gave a second, independent proof of this Corollary in [Gon12].

Corollary 7.2.3. *A metric measure space (X, d, μ) is a Lipschitz differentiability space if and only if it satisfies a Lip-lip condition and is pointwise doubling.*

Bibliography

[ACP] Giovanni Alberti, Marianna Csörnyei, and David Preiss, *Structure of null sets, differentiability of Lipschitz functions, and other problems*, preprint.

[ACP10] Giovanni Alberti, Marianna Csörnyei, and David Preiss, *Differentiability of Lipschitz functions, structure of null sets, and other problems*, Proceedings of the International Congress of Mathematicians. Volume III (New Delhi), Hindustan Book Agency, 2010, pp. 1379–1394. MR 2827846 (2012g:26016)

[Alb93] Giovanni Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 2, 239–274. MR 1215412 (94g:49100)

[Bat12] David Bate, *Structure of measures in Lipschitz differentiability spaces*, arXiv **1208.1954** (2012), 61.

[BS13] David Bate and Gareth Speight, *Differentiability, porosity and doubling in metric measure spaces*, Proc. Amer. Math. Soc. **141** (2013), no. 3, 971–985. MR 3003689

[Che99] Jeff Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517. MR 1708448 (2000g:53043)

[CK09] Jeff Cheeger and Bruce Kleiner, *Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon-Nikodým property*, Geom. Funct. Anal. **19** (2009), no. 4, 1017–1028. MR 2570313 (2011c:30138)

[Gon11] Jasun Gong, *Rigidity of derivations in the plane and in metric measure spaces*, arXiv **1110.4282** (2011), 33.

[Gon12] ———, *The Lip-lip condition on metric measure spaces*, arXiv **1208.2869** (2012), 31.

[Hei01] Juha Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. MR 1800917 (2002c:30028)

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[Kei04] Stephen Keith, *A differentiable structure for metric measure spaces*, Adv. Math. **183** (2004), no. 2, 271–315. MR 2041901 (2005g:46070)

[KM11] Bruce Kleiner and John Mackay, *Differentiable structures on metric measure spaces: A primer*, arXiv **1108.1324** (2011), 23.

[MMPZ03] M. Eugenia Mera, Manuel Morán, David Preiss, and Luděk Zajíček, *Porosity, σ-porosity and measures*, Nonlinearity **16** (2003), no. 1, 247–255. MR 1950786 (2003m:28003)

[Rad19] Hans Rademacher, *Über partielle und totale differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale*, Math. Ann. **79** (1919), no. 4, 340–359. MR 1511935

[Rud08] Walter Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Classics in Mathematics, pp. xiv+436, Springer-Verlag, Berlin, 2008. MR 2446682 (2009g:32001)