On Area Comparison and Rigidity Involving the Scalar Curvature

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Contents

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This thesis is dedicated to my father, without whose help this work would have not been possible.

Declarations

I declare that, to best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

The results of section 3.2 and chapter 4 are joint work with Mario Micallef and comprise the paper [MM] to appear in Proceedings of the American Mathematical Society. The results from the rest of the thesis have not yet been submitted, either online or to a journal.

This thesis was type in LaTeX and a modified version of the style package *warwick-thesis* was used.

Abstract

In this thesis we study the effects of lower bounds for the curvature of a Riemannian manifold M on the geometry and topology of closed, minimal hypersurfaces. We will prove an area comparison theorem for totally geodesic surfaces which is an optimal analogue of the Heintze-Karcher-Maeada Theorem in the context of 3-manifolds with lower bounds on scalar curvature (Theorem 3.8). The optimality of this result will be addressed by explicitly constructing several counterexamples in dimensions $n \geq 4$. This area comparison theorem turns out that it provides a unified proof of three splitting and rigidity theorems for 3-manifolds with lower bounds on the scalar curvature that were first proved, independently, by Cai-Galloway, Bray-Brendle-Neves and Nunes (Theorem 4.7 (a)-(c)). In the final part of this thesis we will address some natural higher dimensional generalisations of these splitting and rigidity results and emphasise some connections with the Yamabe problem.

Chapter 1

Introduction

1.1 Comparison Geometry

The aim of Comparison Geometry is to understand the structure of a general Riemannian manifold (M, g) whose geometry is bounded by that of a given complete Riemannian manifold of constant sectional curvature k, i.e. space form. The effect of a lower or an upper curvature bound on the geometry of M usually translates into upper or lower bounds for other geometric quantities of M such as, for example, volume, diameter or injectivity radius. Very often the equality case in these bounds corresponds to a critical case and, once equality is attained in these bounds, some sort of rigidity phenomenon arises.

This is very well illustrated by the pioneering result of Bonnet, going back to the roots of Comparison Geometry. In 1855 he gave an upper bound on the length of a minimising geodesic in terms of a positive, lower bound on the curvature of a given surface [Bon58]. (See also [Dar94, Livre 6, Ch. 5] and [Pet06, Ch.6.4.1].) It follows that any complete surface M, with curvature greater or equal to $k > 0$, has diameter $d(M) \leq \pi/\sqrt{k}$ with equality if and only if M is isometric to $\mathbb{S}^2_k$. This result was later generalised in 1926 by Synge to higher dimensions [Syn26] [1].

A major breakthrough came in the 1950's with Rauch's comparison theorem [Rau51]. This result compares the length of geodesics in a manifold M, with curvature $\geq k$, with the length of geodesics in a space form of constant curvature k. Soon after Rauch's result, Toponogov [Top59] proved a similar comparison theorem for geodesic triangles. In all of these mentioned results the geodesics emanate from a fixed point. Subsequent work was concerned with generalisations of some of these results to higher dimensional submanifolds and to weaker curvature conditions such as Ricci curvature lower bounds. For example, it was shown by Myers [Mye41] that the above-mentioned theorem by Bonnet and Synge can actually be generalised to

[1] Actually, the equality case was not addressed by Bonnet nor by Synge or Myers. Myers extended Synge's result to the positive Ricci curvature case. The equality case was addressed only three decades later by Cheng in [Che75].

positive Ricci curvature. On the other hand, Toponogov's triangle comparison theorem fails for Ricci curvature lower bounds.

In this direction of generalising some of the above-mentioned results to a weaker curvature condition, Bishop proved a strong volume comparison theorem [BC64], where the volume of geodesic balls in n-dimensional manifolds M with Ricci curvature $\geq (n-1)k$ is shown to be less than or equal to the volume of balls, with the same radius, in the space of constant curvature k [2]. Bishop's proof still considers geodesics emanating form a single point, namely the centre of the ball.

A different approach was taken by Berger in [Ber62]. In order to give a more efficient proof of the Toponogov Triangle Comparison Theorem, he considered a family of geodesics normal to a given geodesic and not just form a single point.

1.2 Area Comparison for Hypersurfaces

The starting point of our research is a very general area comparison theorem from 1978 by Heintze and Karcher. They considered geodesic emanating not only from a given geodesic, like in the case of Berger's result, but actually from a closed submanifold. Thus with case (b) of their theorem [HK78, Theorem 3.2] they generalise the above-mentioned result of Berger. From case (d) of the same theorem we have the following area comparison theorem discovered also, independently, by Maeda [Mae78].

Theorem A (Heintze-Karcher-Maeda Area Comparison Theorem)**.** *Let M be a complete Riemannian n-manifold of non-negative Ricci curvature and let Σ be a closed, two-sided, minimal hypersurface. Let*

$$\Sigma_t := \{exp_x(t\nu(x)) : x \in \Sigma\}, \quad |t| < C,$$

where ν is the unit normal vector field to Σ and C is the distance to the focal point closest to Σ in the direction ν. Then, for all $|t| < C$,

$$A(t) \leq A(0), \tag{1.1}$$

where $A(t) := Area(\Sigma_t)$.

Easy counterexamples show that only a lower bound on the scalar curvature in Theorem A is not sufficient to ensure the same conclusion; not even under the

[2] Due to the improvement of this result by Gromov, this theorem is now called the Bishop-Gromov Volume Comparison Theorem. For a detailed discussion we refer to [Pet06, Ch. 9.1]

stronger assumption of Σ being totally geodesic. Indeed, we can take the product $M := \mathbb{S}^2 \times (-\varepsilon, \varepsilon)$ of a round 2-sphere with a small interval, equipped with the warped metric $(1 + t^{2k})g + dt^2,\ k \geq 1$. The normal Ricci curvature of M will be non-positive. However, for small enough $\varepsilon > 0$ and for a large enough $k \in \mathbb{N}$, the positive curvature of the leaves will dominate and hence M will have positive scalar curvature.

Therefore, in order to ensure that area non-increases in the case of a lower bound on the scalar curvature, additional geometric assumptions must be imposed on Σ. It's not obvious what would be a "natural" analogue of Heintze-Karcher-Maeda theorem for manifolds with lower bounds on the scalar curvature. It turns out that one answer comes form the study of stable minimal surfaces in 3-manifolds with scalar curvature bounded from below.

We will investigate the optimal assumptions required under which a Heintze-Karcher-Maeda type theorem could hold for manifolds with scalar curvature bounded below. By using different methods than the ones used by Heintze-Karcher and Maeda, we will prove the following area comparison theorem.

Theorem B (Area Comparison Involving the Scalar Curvature [MM])**.** *Let M be a complete 3-manifold with scalar curvature $S \geq S_0$, where $S_0 \in \mathbb{R}$. Let $\Sigma \subset M$ be an immersed, two-sided, closed, surface of genus γ such that*

(i) Σ is totally geodesic,

(ii) the normal Ricci curvature of M vanishes all along Σ and

(iii) $S = S_0$ at every point of Σ.

Let $\{\Sigma_t\}$, $t \in (-\varepsilon, \varepsilon)$, be a constant mean curvature foliation [3] in a neighbourhood of Σ and denote by $A(t)$ the area of Σ_t. Then there exists $0 < \delta < \varepsilon$ such that

$$\text{for } |t| < \delta, \quad A(t) \leq A(0).$$

Moreover, Σ has constant Gauss curvature equal to $\frac{1}{2}S_0$ and therefore, by Gauss-Bonnet theorem, $|S_0|A(0) = 8\pi(\gamma - 1)$, if S_0 is non-zero.

Although the proof of Theorem B relies heavily on the Gauss-Bonnet theorem, the restriction to two-dimensional surfaces Σ is no mere matter of technical issues, nor are the very restrictive assumptions (i)-(iii). As we will see later on, these assumptions are optimal in this setting. This very "rigid" geometry we have to have

[3] For the existence of such a foliation see Proposition 3.12.

along Σ in order to ensure the conclusion is yet just another reflection of how weak the scalar curvature is when compared with the Ricci one.

1.3 Area Bounds for Stable Minimal Surfaces and Splitting of 3-Manifolds

The link between the area comparison theorems A and B is provided by the geometry of stable minimal surfaces in manifolds with scalar- or Ricci curvature bounded below.

In [Sim68, Corollary 3.6.1] Simons observed that there are no closed, stable, minimal, two-sided hypersurfaces Σ in a manifold of positive Ricci curvature. He regarded this statement as an extension of the classical Synge Theorem where the strictly positive sectional curvature of the ambient manifold prevents any closed geodesic from being locally of least length. An easy, but unstated, extension of Simons's observation is that a closed, stable, minimal, two-sided hypersurface Σ in a manifold M of non-negative Ricci curvature is necessarily totally geodesic and the normal Ricci curvature of M must vanish all along Σ. Therefore, since M has non-negative Ricci curvature, the normal Ricci curvature of M attains its minimum along Σ. As we will describe in the following, when only the scalar curvature of an ambient 3-manifold is assumed to be bounded from below then M has a very similar geometry along certain stable, minimal surfaces contained in M.

In a celebrated paper from 1979 Schoen and Yau discovered a deep connection between the topology of stable minimal surfaces and the scalar curvature S of the ambient 3-manifold M [SY79]. Namely, by using the second variation of area formula, they showed that any closed, two-sided, stable, minimal surface in a 3-manifold of positive scalar curvature must have genus zero. Soon after, Fischer-Colbrie and Schoen studied the case $S \geqslant 0$ and proved in [FCS80] that, in this case, the genus of Σ must be zero or one, and if it is one, then Σ is totally geodesic and flat and both the normal Ricci curvature and the scalar curvature S of M vanish all along Σ. Since M has non-negative scalar curvature it means that S attains its minimum along this totally geodesic torus. As we will see in the following the torus is by no means a special case.

A closer look at the proof of Schoen and Yau reveals that a lower bound on the scalar curvature S of the ambient 3-manifold provides a bound on the area of a stable minimal surface contained in it. More precisely, as observed in [SZ97], if $S \geq S_0$, then the area of any closed, stable minimal surface Σ with genus $\gamma \neq 1$, satisfies

$$\begin{cases} \mathrm{A}(\Sigma) \leq 4\pi & \text{if } S_0 = 2 \\ \mathrm{A}(\Sigma) \geq 4\pi(\gamma - 1) & \text{if } S_0 = -2 \text{ and } \gamma \geq 2. \end{cases} \tag{1.2}$$

Using an analysis similar to that used by Fischer-Colbrie and Schoen in the genus one case, it was observed in [BBN10] for $S_0 > 0$ and in [Nun] for $S_0 < 0$ that the critical case of (1.2) corresponds to an infinitesimal splitting of the ambient manifold along Σ. More precisely

(i) Σ is totally geodesic,

(ii) the normal Ricci curvature of M vanishes all along Σ and

Furthermore we have

(iii) $S = S_0$ at every point of Σ.

These are our assumptions (i)-(iii) in Theorem B. It therefore follows from this theorem that the area of Σ non-increases as Σ moves in the normal direction inside M. This means that Σ is not *strictly* area-minimising. The same conclusion holds for the genus one case. Therefore, if one additionally assumes Σ to be area-minimising then one can further show that, in this case, the infinitesimal splitting of M along Σ actually propagates to an entire neighbourhood of Σ and hence the ambient 3-manifold M is locally isometric to a product. This was previously proven by Bray, Brendle and Neves [BBN10] for $S_0 > 0$, by Cai and Galloway [CG00b] for $S_0 = 0$ and by Nunes [Nun] for $S_0 < 0$.

Theorem C (Splitting of 3-Manifolds)**.** *Let (M, g) be a complete Riemannian 3-manifold with scalar curvature $S \geq S_0$ where $S_0 \in \mathbb{R}$. Assume that M contains a closed, embedded, oriented, two-sided, area-minimising surface Σ.*

(a) Suppose [4] that $S_0 = 2$ and that $A(\Sigma) = 4\pi$. Then Σ has genus zero and it has a neighbourhood which is isometric to the product $g_1 + dt^2$ on $S^2 \times (-\delta, \delta)$ where g_1 is the metric on the Euclidean two-sphere of radius 1.

(b) Suppose that $S_0 = 0$ and that Σ has genus one. Then Σ has a neighbourhood which is flat and isometric to the product $g_0 + dt^2$ on $\mathbb{T}^2 \times (-\delta, \delta)$ where g_0 is a flat metric on the 2-torus $\mathbb{T}^2$.

[4] By scaling the metric if necessary we can assume, without the loss of generality, that $S_0 = 2,\ 0$ and -2 in the cases (a), (b) and (c), respectively.

(c) *Suppose that $S_0 = -2$ and that Σ has genus $\gamma \geqslant 2$ and $A(\Sigma) = 4\pi(\gamma - 1)$. Then Σ has a neighbourhood which is isometric to the product $g_{-1} + dt^2$ on $\Sigma \times (-\delta, \delta)$ where g_{-1} is a metric of constant Gauss curvature equal to -1 on Σ.*

The original proofs of these three cases are very different in nature and, with one exception only, the techniques used seem to be specialised for each case individually. For this reason, after analysing the original proofs, it might not be obvious that, in each case, the splitting is actually caused by the same geometric phenomenon. This, however, will become more transparent in the light of the above-mentioned area comparison Theorem B.

1.4 Area Bounds for Stable Minimal Hypersurfaces and Splitting of n-Manifolds

Since the proof of the above-mentioned area bounds (1.2) rely on the Gauss-Bonnet theorem and hence are given in terms of the Euler characteristic of Σ, they do not directly extend to higher dimensional hypersurfaces. Therefore, if one wants to look for possible generalisations of (1.2) then one has to look at other topological invariants which generalise the Euler characteristic to higher dimensions.

In the final part of this thesis we will study area bounds for closed, stable, minimal hypersurfaces Σ in manifolds with scalar curvature bounded below. It turns out that, in this setting, a good generalisation for the two-dimensional Euler characteristic is provided by the σ-constant, denoted by $\sigma(\Sigma)$, which in two dimensions is just a multiple of the Euler characteristic. As we will see later in Chapter 5, this is a fairly weak invariant, defined in terms of the total scalar curvature of Σ and which, furthermore, is insensitive to one-dimensional "fibres", having therefore that $\sigma(\mathbb{S}^{n-1} \times \mathbb{S}^1) = \sigma(\mathbb{S}^n)$.

Despite its weakness, the σ-constant has the key property of detecting when a closed manifold admits a metric of positive scalar curvature. And, like the Euler characteristic in two dimensions, it divides the family of compact manifolds into three classes, $\sigma(\Sigma) > 0$, $\sigma(\Sigma) = 0$ and $\sigma(\Sigma) < 0$, whether Σ admits a metric of positive, zero or negative scalar curvature, respectively.

For our purpose the σ-constant turns out to be the "right" generalisation of the two-dimensional Euler characteristic in the case of negative scalar curvature. Indeed, it was showed by Cai and Galloway [CG00a] that a closed, stable, minimal hypersurface Σ with $\sigma(\Sigma) < 0$ in a manifold with scalar curvature $S \geq S_0$, where $S_0 < 0$, must necessarily have area bounded from below with a bound given by

$$A(\Sigma)^{\frac{2}{n-1}} \geq \frac{\sigma(\Sigma)}{S_0}. \tag{1.3}$$

When $n = 3$ in (1.3), $\sigma(\Sigma) = 4\pi\chi(\Sigma)$ and $S_0 = -2$. Hence we recover the case of genus $\gamma \geq 2$ of inequality (1.2).

We will investigate the equality case in (1.3) and show that, as in the three-dimensional case, the critical case corresponds to an infinitesimal splitting of the ambient manifold along the hypersurface Σ.

Theorem D (Infinitesimal Splitting of n-Manifolds)**.** *Let M be a n-manifold with scalar curvature $S^M \geq S_0$, where $S_0 \in (-\infty, 0)$. Let Σ be a closed, two-sided, stable, minimal hypersurface with $\sigma(\Sigma) < 0$ and area satisfying*

$$S_0 A(\Sigma)^{\frac{2}{n-1}} = \sigma(\Sigma).$$

Then Σ is totally geodesic and the normal Ricci curvature of M vanishes at every point of Σ, i.e. M splits infinitesimally along Σ. Furthermore, Σ is an Einstein manifold and the scalar curvature S^M of M equals S_0 at every point of Σ.

In the end of the final chapter we will discuss a conjectural picture in which this infinitesimal splitting of the ambient manifold extends to an entire neighbourhood of an *area-minimising* hypersurface.

1.5 Outline of the Chapters

This thesis consists in five chapters and an Appendix, all of them strongly connected with each other. In this **first chapter** we introduced the reader to the main results of the thesis emphasising briefly how these results fit in the broader context of comparison geometry.

In the **second chapter** we will fix our notations and terminology and introduce the main geometrical tools required along the thesis. The final part of this chapter consists in a brief description of the geometry of the scalar curvature.

Chapter 3 deals almost entirely with area comparison for surfaces and hypersurfaces in manifolds with lower bounds on the scalar- and Ricci curvature, respectively. After describing an area comparison result by Heintze-Karcher and Maeda, we will prove an optimal analogue of this result in the context of 3-manifolds with scalar curvature bounded below (Theorem 3.8). In the final section of the third chapter we will address the optimality of our result and construct several examples for this purpose.

In **Chapter 4** we will discuss three splitting and rigidity results by Bray-Brendle-Neves, Cai-Galloway and Nunes (Theorem 4.7). We will show how our area comparison theorem form Chapter 3 can be used to provide a unified and more elementary proof of these three results.

Chapter 5 is concerned with the generalisation to higher dimensions of some of the area comparison and rigidity results from the previous two chapters. We will prove an infinitesimal splitting result for stable, minimal hypersurfaces in manifolds with scalar curvature bounded from below by a negative constant (Theorem 5.10). This theorem extends a previous result by Schoen and Yau. Furthermore we will explain how the proof of a splitting theorem by Cai (Theorem 5.12) can be used to provide a generalisation of Theorem B in the context of manifolds of non-negative scalar curvature, . In the end of this chapter we will discuss a conjectural local splitting property which makes the subject of our current research.

To keep the exposition as fluent as possible we have omitted the proof of several theorems and propositions from the text. However, in order to make the thesis as self-contained as possible, we have gathered these proofs in the **Appendix**.

Convention. We will use the "traditional“ abbreviation q.e.d to mark the end of a proof, while the symbol □ will be used to mark the end of an example or of a remark.

Chapter 2

Fundamentals

2.1 Notations and Terminology

Throughout this thesis, if not stated otherwise, (M, g) will denote a smooth, connected, n-dimensional Riemannian manifold and ∇ will denote the Levi-Civita connection corresponding to a Riemannian metric g. Occasionally we will write the metric as ds^2. For a point $x \in M$ we denote by T_xM the tangent space of M at the point x. The set of smooth vector fields on M will be denoted by $\mathfrak{X}(M)$.

2.1.1 The Intrinsic Geometry

For $X, Y, Z, W \in \mathfrak{X}(M)$ the *Riemannian curvature tensor* of type $(1, 3)$, denoted by R, is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the curvature tensor of type $(0, 4)$, also denoted by R, is given by

$$R(X, Y, Z, W) := g(R(X, Y)Z, W).$$

If we denote by $X \wedge Y$ the two-plane spanned by two pointwise linearly independent vector fields X, Y then the *sectional curvature* of M, for the section $X \wedge Y$, is defined as

$$K(X \wedge Y) := \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

Let $x \in M$ and let $\{E_1, ..., E_n\}$ be an orthonormal basis of T_xM. The *Ricci curvature* of M of type $(0, 2)$ is defined as the trace of R and is given by

$$\begin{aligned} Ric(X, Y) &:= \operatorname{trace}\{Z \to R(Z, X)Y\} \\ &= \sum_{i=1}^{n} R(E_i, X, Y, E_i). \end{aligned}$$

Equivalently, the Ricci curvature can also be defined as the symmetric $(1,1)$-tensor

$$Ric(X) = \sum_{i=1}^{n} R(X, E_i)E_i.$$

Tracing the Ricci curvature tensor of type $(1,1)$ we obtain a smooth function on M, denoted by S, called the *scalar curvature* of M and given by

$$\begin{aligned} S(x) : &= \mathrm{trace}\big\{Y(x) \to Ric(Y)(x)\big\} \\ &= \sum_{i=1}^{n} g(Ric(E_i), E_i)(x) \\ &= 2\sum_{i<j} K(E_i \wedge E_j)(x). \end{aligned}$$

2.1.2 The Extrinsic Geometry

Let Σ be a complete, immersed, two-sided hypersurface in M. For every $x \in \Sigma$ let $T_x\Sigma \subset T_xM$ denote the tangent space of Σ at the point x and let $\nu \in \mathfrak{X}(\Sigma)^{\perp} \subset \mathfrak{X}(M)$ denote the unit normal vector field of Σ. The *Weingarten map* (or the *shape operator*) of Σ is defined as

$$\mathcal{W}(X) := \nabla_X \nu, \quad \text{for any } X \in \mathfrak{X}(\Sigma).$$

The *second fundamental form* of Σ is a symmetric bilinear form on $\mathfrak{X}(\Sigma)$ defined as

$$\begin{aligned} B(X,Y) : &= g(\mathcal{W}(X), Y) \\ &= g(\nabla_X \nu, Y). \end{aligned}$$

We define the *mean curvature* of Σ at a point $x \in \Sigma$ as the trace of the Weingarten map:

$$\begin{aligned} H(x) : &= \mathrm{trace}\big\{Y(x) \to \nabla_Y \nu(x)\big\} \\ &= \sum_{i=1}^{n-1} B(E_i, E_i)(x) \\ &= \sum_{i=1}^{n-1} \kappa_i(x), \end{aligned}$$

where κ_i denote the principal curvatures of Σ. With our convention, the n-dimensional round sphere $\mathbb{S}^n(1)$ of radius one has mean curvature n in the Euclidean space.

2.1.3 The Gauss Equation

Let $X, Y, Z, W \in \mathfrak{X}(\Sigma)$. The relationship between the curvature of the ambient manifold M and the curvature, both intrinsic and extrinsic, of a hypersurface Σ is given by the *Gauss equation*

$$R^M(X,Y,Z,W) = R^\Sigma(X,Y,Z,W) - B(X,W)B(Y,Z) + B(X,Z)B(Y,W), \quad (2.1)$$

where R^M and R^Σ denote the Riemannian curvature of M and Σ, respectively. If Σ is a hypersurface in M then, by tracing twice the previous equation, we have

$$2Ric^M(\nu,\nu) = S^M - S^\Sigma + H^2 - |B|^2, \quad (2.2)$$

where $Ric^M(\nu,\nu)$ is the Ricci curvature of M in the normal direction of Σ, $|B|$ denotes the norm of the second fundamental form of Σ and S^M, S^Σ denote the scalar curvature of M and Σ, respectively.

2.1.4 Normal Neighbourhoods

Let $f : \Sigma \to M$ be an immersion of a closed, oriented $(n-1)$-dimensional manifold into an n-dimensional manifold M and for $\varepsilon > 0$ let $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth real function with $w(x,0) = 0$ for all $x \in \Sigma$. For every $t \in (-\varepsilon, \varepsilon)$ let

$$f_t(x) := \exp_x(w(x,t)\nu(x)), \quad x \in \Sigma,\ t \in (-\varepsilon, \varepsilon).$$

Thus $f_0 = f$. We denote $\Sigma_t := f_t(\Sigma)$ and define the *lapse function* $\rho_t \colon \Sigma \to \mathbb{R}$ by

$$\rho_t(x) := g\Big(\nu_t(x), \frac{\partial}{\partial t} f_t(x)\Big), \quad (2.3)$$

where ν_t is a unit normal to Σ_t, chosen so as to be continuous in t. The lapse function satisfies the following evolution equation. (See, for e.g., [HP99], Theorem 3.2.)

$$\frac{\partial}{\partial t} H_t = -\Delta_{\Sigma_t}\rho_t - (Ric(\nu_t,\nu_t) + |B_t|^2)\rho_t, \quad (2.4)$$

where Δ_{Σ_t} is the Laplacian on Σ_t, and B_t and H_t is the second fundamental form and the mean curvature of Σ_t, respectively.

Remark 2.1. The lapse function ρ_t can be viewed as the speed by which Σ needs to be moved in the normal direction in order to obtain Σ_t. In the special case when $w(x,t) = t$, the lapse function becomes $\rho_t \equiv 1$ which means that Σ is moved with constant speed in the normal direction. In this case the evolution equation (2.4) becomes

$$\frac{\partial}{\partial t}H_t = -Ric(\nu_t, \nu_t) - |B_t|^2. \tag{2.5}$$

2.2 Stability of Minimal Hypersurfaces

With notations as in 2.1.4 denote by $A(t)$ the area of Σ_t. The *first variation of area* formula is given by

$$A'(t) = \int_\Sigma H_t \rho_t d\mu_t, \tag{2.6}$$

where $H_t = H(x,t)$ denotes the mean curvature of Σ_t and $d\mu_t$ denotes the area element of Σ_t with respect to the induced metric.

Definition 1. *A hypersurface Σ is called* minimal *if Σ is a critical point for the area functional; i.e.*

$$A'(0) = 0. \tag{2.7}$$

Equivalently, Σ is minimal, if and only if $H(x,0) = 0$ for all $x \in \Sigma$.

From (2.6) we obtain the *second variation of area* formula

$$A''(t) = -\int_\Sigma \Big\{\rho_t \Delta_{\Sigma_t}\rho_t + \Big(Ric(\nu_t,\nu_t) + |B_t|^2 - H_t^2\Big)\rho_t^2\Big\} d\mu_t, \tag{2.8}$$

where Δ_Σ denotes the Laplacian on Σ with respect to the induced metric.

Definition 2. *A minimal hypersurface Σ is called* stable *if*

$$A''(0) \geq 0. \tag{2.9}$$

Equivalently, integrating by parts in the equality (2.8), Σ is stable if and only if

$$\int_\Sigma (Ric(\nu,\nu) + |B|^2)\rho^2 d\mu \leq \int_\Sigma |\nabla\rho|^2 d\mu, \tag{2.10}$$

where by (2.3) $\rho = g\big(\nu, \frac{\partial f_t}{\partial t}\big|_{t=0}\big)$.

Definition 3. *A closed hypersurface Σ is called* area minimising *if Σ has least area among all hypersurfaces homotopic to Σ.*

Remark 2.2. From Definition 2 we see that a stable minimal hypersurface locally minimises the area up to second order. □

2.3 The Geometry of Warped Products

Let $M := M_1 \times ... \times M_k \times B$ be a product manifold where $B,\ M_1,\ ...,\ M_k$ are smooth manifolds and let $u_i,\ i = 1, ..., k$, be k smooth, positive, real functions on B.

Definition 4. *A (multiple)* warped product *metric on M is a Riemannian metric of the form*

$$ds^2 = \sum_{i=1}^{k} u_k^2(t) ds_i^2 + ds_0^2,$$

where ds_i^2 denotes, for all $i = 1, ..., k$, the Riemannian metric on M_i and ds_0^2 the Riemannian metric on B.

We will describe the geometry of (M, g) in terms of the geometry of B, the "warped manifolds" M_i and the warping functions u_i. For our current purpose we will specialise to the situation when the *base manifold* B is a subset $I \subseteq \mathbb{R}$ and, furthermore, when $k = 1$ (called *warped products*) or when $k = 2$ (called *doubly warped products*). In the following we will present the geometric formulae for warped manifolds required along this thesis.

Proposition 2.3. *Let P be a p-dimensional Riemannian manifold and let u be a positive, smooth, real function on $\mathbb{R}$. For $I \subseteq \mathbb{R}$, let $M := P \times I$ be a warped product manifold with the metric given by*

$$ds^2 = u^2(t) d\sigma^2 + dt^2,$$

where $d\sigma^2$ denotes the metric on P. For any vector fields $X, Y \in \mathfrak{X}(P)$ and $\partial_t \in \mathfrak{X}(P)^\perp \subset \mathfrak{X}(M)$, the non-zero components of the (3,1) curvature tensor R of M are given by

(a) $R(X, \partial_t)\partial_t = -\frac{\ddot{u}}{u} X$

(b) $R(\partial_t, X)X = -\ddot{u}u\partial_t$

(c) $R(X, Y)Y = R^P(X, Y)Y - \frac{\dot{u}^2}{u^2} X$

From the definition of the Ricci and scalar curvature, we obtain the following two corollaries.

Corollary 2.4. *For any vector field $X \in \mathfrak{X}(P)$ and $\partial_t \in \mathfrak{X}(P)^\perp$, we have*

(a) $Ric(\partial_t) = -p\frac{\ddot{u}}{u}\partial_t$

(b) $Ric(X) = Ric^P(X) - \Big(\frac{\ddot{u}}{u} + (p-1)\frac{\dot{u}^2}{u^2}\Big)X$

Corollary 2.5. *The scalar curvature of the warped product M is given by*

$$S = \frac{1}{u^2}S^P - 2p\frac{\ddot{u}}{u} - p(p-1)\frac{\dot{u}^2}{u^2}, \tag{2.11}$$

where S^P is the scalar curvature of P.

For the proof of the previous proposition, as well as for the two corollaries, we refer to [O'N83], Chapter 7: Proposition 42, Corollary 43 and Exercise 13(b).

The next proposition and corollaries describe the curvatures of a doubly warped product manifold.

Proposition 2.6. *Let M_1 and M_2 be a p-dimensional and a q-dimensional Riemannian manifold, respectively and let u and w be two positive, smooth, real functions on $\mathbb{R}$. For $I \subseteq \mathbb{R}$, let $M := M_1 \times M_2 \times I$ be the doubly warped product with the metric given by*

$$ds^2 = u^2(t)ds_1^2 + w^2(t)ds_2^2 + dt^2,$$

where ds_1^2 and ds_2^2 denote the metric on M_1 and M_2, respectively. For any vector fields $X, Y \in \mathfrak{X}(M_1)$, $U, V \in \mathfrak{X}(M_2)$ and $\partial_t \in \mathfrak{X}(M_1 \times M_2)^\perp \subset \mathfrak{X}(M)$, the non-zero components of the (3,1) curvature tensor R of M are given by

(a) $R(X, \partial_t)\partial_t = -\frac{\ddot{u}}{u}X$

(b) $R(\partial_t, X)X = -\ddot{u}u\partial_t$

(c) $R(U, \partial_t)\partial_t = -\frac{\ddot{w}}{w}U$

(d) $R(\partial_t, U)U = -\ddot{w}w\partial_t$

(e) $R(X, Y)Y = R_1(X, Y)Y - \dot{u}^2X$

(f) $R(U, V)V = R_2(U, V)V - \dot{w}^2U$

(g) $R(X, U)U = -\frac{\dot{u}w\dot{w}}{u}X$

(h) $R(U, X)X = -\frac{u\dot{u}\dot{w}}{w}U,$

where $\dot{(\ )} = \frac{d}{dt}(\ )$ and R_1 and R_2 denote the curvature tensors of ds_1^2 and ds_2^2, respectively.

Again, by the definition of the Ricci and scalar curvature, we obtain the following two corollaries.

Corollary 2.7. *For any vector fields* $X \in \mathfrak{X}(M_1)$, $U \in \mathfrak{X}(M_2)$ *and* $\partial_t \in \mathfrak{X}(M_1 \times M_2)^{\perp}$, *we have*

(a) $Ric(\partial_t) = -\Big(p\dfrac{\ddot{u}}{u} + q\dfrac{\ddot{w}}{w}\Big)\partial_t$

(b) $Ric(X) = Ric_1(X) - \Big(\dfrac{\ddot{u}}{u} + (p-1)\dfrac{\dot{u}^2}{u^2} + q\dfrac{\dot{u}\dot{w}}{uw}\Big)X$

(c) $Ric(U) = Ric_2(U) - \Big(\dfrac{\ddot{w}}{w} + (q-1)\dfrac{\dot{w}^2}{w^2} + p\dfrac{\dot{u}\dot{w}}{uw}\Big)U,$

where $Ric_i(\cdot)$ *denotes the Ricci curvature of the metric* ds_i^2, *for* $i = 1, 2$.

Corollary 2.8. *The scalar curvature of the doubly warped product* M *is given by*

$$\begin{aligned} S = \frac{1}{u^2}S_1 + \frac{1}{w^2}S_2 - 2\Big(p\frac{\ddot{u}}{u} + q\frac{\ddot{w}}{w}\Big) - p(p-1)\frac{\dot{u}^2}{u^2} \\ - q(q-1)\frac{\dot{w}^2}{w^2} - 2pq\frac{\dot{u}\dot{w}}{uw}, \end{aligned} \tag{2.12}$$

where S_1 *and* S_2 *are the scalar curvatures of* ds_1^2 *and* ds_2^2, *respectively.*

We omit the proofs of these proposition and corollaries, being slight modifications of the single warped product case and the doubly warped product of spheres discussed in [Pet06] Ch.3, Section 2.4.

2.4 The Geometric Meaning of the Scalar Curvature

In a sense, all the results presented in this thesis are concerned with the effect that the scalar curvature has on the geometry of a manifold and on its hypersurfaces and, to some extent, with the difference in "weakness" between the scalar and Ricci curvature. Very often these differences are straightforward, being emphasised by easy examples. Sometimes, however, differences can be more subtle as we will see in Section 3.3.

For this reason we include a brief account on the geometric meaning of the scalar curvature. In what follows we will mainly follow [Gro96], §1.

Let M be a n-dimensional Riemannian manifold with scalar curvature S and let

$B^M(x,r)$ be the ball of radius $r > 0$ and centered at $x \in M$. The geometric meaning of scalar curvature is probably best suggested by the following formula.

$$\mathrm{Vol}B^M(x,r) = \mathrm{Vol}B(r)\Big\{1 - \frac{S(x)}{6(n+2)}r^2 + \mathcal{O}(r^2)\Big\}, \tag{2.13}$$

where $B(r)$ is the ball of same radius r in the flat space $\mathbb{E}^n$ and $\frac{\mathcal{O}(r^2)}{r^2} \to 0$ as $r \to 0$. (A fairly detailed proof of (2.13) can be found in [GHL04], Ch. 3.H.4.)
It follows form (2.13) that positive scalar curvature decreases the volume of balls at a "microscopic" level; i.e. if $S > 0$ and $r = \varepsilon > 0$ is small enough, then

$$\mathrm{Vol}B^M(x,\varepsilon) < \mathrm{Vol}B(\varepsilon). \tag{2.14}$$

Inequality (2.14) also holds for Ricci curvature lower bounds and, in this case, we actually have a volume comparison theorem at a "macroscopic" level. Indeed, this is the content of the Bishop-Gromov Volume Comparison Theorem, already mentioned in the introduction. (See, for e.g., [Pet06], Ch.9, Section 1.2.). However, this result no longer holds for scalar curvature lower bounds. The manifold $M = \mathbb{S}^2 \times \mathbb{R}$ has $S > 0$ but $\mathrm{Vol}(M) = \infty$. Nevertheless, in his thesis Bray showed that a Bishop-Gromov theorem still holds for scalar curvature lower bounds, as long as one considers only metrics "close" to the the initial one in a suitable sense. (See [Bra97], Theorem 18.)

Chapter 3

Area Comparison for Surfaces

From both the evolution equation (2.5) and from the first variation of area formula (2.6) we see that, when a closed hypersurface Σ is moved with constant speed in the normal direction, the Ricci curvature of the ambient manifold M controls the rate of change of the area of Σ. In particular, when M has non-negative Ricci curvature, we expect that the area of Σ_t will not increase. This is indeed the case and it will be the content of the area comparison theorem of Heintze-Karcher and Maeda (Theorem 3.3) which will be discussed in the first section of this chapter.

In the remainder of this chapter we will prove an area comparison theorem for certain totally geodesic surfaces in 3-manifolds with scalar curvature bounded from below. This theorem is an optimal analogue of the Heintze-Karcher-Maeda theorem. We will emphasise the optimality of this result by explicitly constructing several examples for dimensions greater than or equal to four.

3.1 Area Comparison in n-Manifolds with Ricci Curvature Bounded Below

In 1936, by using the second variation of arc-length, Synge proved the following remarkable topological result.

Theorem 3.1. ([Syn36]) *If M is a compact, even-dimensional, orientable, Riemannian manifold of positive sectional curvature then M is simply connected* [1]*.*

For a discussion of Synge's theorem, as well as for further developments related to this result, we refer to the survey by Petersen [Pet03].

Sketch of the proof. Suppose, for a contradiction, that M is not simply connected and let $x \in M$. Then M contains a homotopically non-trivial, simple, closed, minimising geodesic $\gamma : [0,1] \to M$, with $\gamma(0) = \gamma(1) = x$, which in particular is a

[1] In [Syn36] the odd-dimensional case is also addressed: If M is compact, odd-dimensional of positive sectional curvature then M is orientable.

"stable", minimal, closed curve. Moreover, the parallel transport in M is an orientation preserving isometry of $(T_{\gamma(0)}M)^{\perp}$. Therefore, there exists a closed parallel vector field $X(t)$ along $\gamma(t)$ and perpendicular to $\dot{\gamma}(t)$. Since $X(t)$ is parallel and $\gamma(t)$ is closed, by the second variation of arc length we have

$$\begin{aligned} \frac{d^2L}{ds^2}\Big|_{s=0} &= \int_0^1 \Big\{|\dot{X}(t)|^2 - |X(t)|^2 K(\dot{\gamma}(t)\wedge X(t))\Big\} ds \\ &= -\int_0^1 |X(t)|^2 K(\dot{\gamma}(t)\wedge X(t) ds \\ &< 0, \end{aligned} \tag{3.1}$$

where the second inequality follows since $X(0) = X(1)$ and the last one since M has strictly positive sectional curvature. Therefore the length of $\gamma(t)$ decreases in the direction of $X(t)$ and hence there must be nearby closed curves that are shorter than γ. This is a contradiction since γ was assumed to be length-minimising. q.e.d.

By looking at the second variation of length formula (3.1) and comparing it with the second variation of area formula (2.8), one might expect a similar geometric phenomenon as in the proof of Synge's theorem if one replaces "positive sectional curvature" with "positive Ricci curvature" and "stable geodesic" with "stable minimal hypersurface" [2]. This is indeed the case and it was first pointed out by Simons in [Sim68, Corollary 3.6.1]. Namely, by taking $\rho \equiv 1$ in (2.10), it follows immediately from Definition 2 that there are no stable, two-sided, minimal hypersurfaces in manifolds of positive Ricci curvature.

An easy, but unstated, extension of Simons's observation is that a closed, stable, minimal, 2-sided hypersurface Σ in a manifold M of *non-negative* Ricci curvature is necessarily totally geodesic and, furthermore, the normal Ricci curvature of M must vanish all along Σ. Therefore, since M is of non-negative Ricci curvature, the normal Ricci curvature attains its minimum along the stable minimal hypersurface Σ. Furthermore, since Σ is also totally geodesic, the ambient manifold M splits infinitesimally as a product along Σ. However the splitting need not be local, as illustrated by the following example.

Example 3.2. The metric $(1-t^4)g+dt^2$ on $\Sigma\times(-\varepsilon,\varepsilon)$, $1>\varepsilon>0$, has non-negative Ricci curvature if g has non-negative Ricci curvature. The surface $\Sigma\times\{0\}$ is stable, totally geodesic and the normal Ricci curvature $Ric(\partial_t,\partial_t)$ of M vanishes along Σ. Hence M splits infinitesimally along Σ, but not in a neighborhood of Σ. □

[2] A "similar geometric phenomenon" only from the point of view of the second variation of length and area, respectively. The distinction between even and odd dimensional ambient manifolds, so crucial in Synge's theorem, becomes vacuous in the case of hypersurfaces.

Notice however that, in this example, $\Sigma \times \{0\}$ does not minimise area since $A(\Sigma \times \{t\}) < A(\Sigma \times \{0\})$ for all $0 < |t| < \varepsilon$. Therefore, one might surmise whether the existence of a closed, *area-minimising*, 2-sided hypersurface Σ in a manifold M of non-negative Ricci curvature implies that the metric of M must split as a product near Σ. This is indeed the case and is the content of Corollary 3.5 below. This corollary follows from the following area comparison theorem discovered by Heintze and Karcher and, independently, by Maeda.

Theorem 3.3. [HK78, Theorem 3.2(d)], [Mae78, Lemma 2 (1)]. *Let M be a complete Riemannian n-manifold of non-negative Ricci curvature and let Σ be a closed, two-sided, minimal hypersurface. Let*

$$\Sigma_t := \{exp_x(t\nu(x)) : x \in \Sigma\}, \quad |t| < C,$$

where ν is the unit normal vector field to Σ and C is the distance to the focal point closest to Σ in the direction ν. Then, for all $|t| < C$,

$$A(t) \leq A(0), \tag{3.2}$$

where $A(t) := Area(\Sigma_t)$ and, in particular, $A(0)$ is the area of Σ.

Remark 3.4. The way we stated it, Theorem 3.3 does not appear in [HK78] nor in [Mae78]. It does however follow immediately from [HK78, Corollary 3.3.2] and [Mae78, Lemma 2(1)]. Our statement is closer to [Gra82, Lemma 6.3 (iii)]. □

Corollary 3.5. *Let M be a complete Riemannian n-manifold of non-negative Ricci curvature. If M contains a closed, two-sided, area-minimising hypersurface Σ then M splits isometrically as a product in a neighbourhood of Σ.*

Remark 3.6. The original proof of Heintze-Karcher-Maeda Theorem consists in a comparison argument between index forms and the minimising property of Jacobi fields. (See also [Sak96, Chapter IV (1-3)].) The proof we will present here is slightly different, although we will still make use of Jacobi fields along normal geodesics emanating from Σ. The proof relies on a Riccati inequality satisfied by the Weingarten map, being therefore more along the lines of [Kar89], [Esc87] and [Gra82]. We believe this argument is more in the spirit of our further generalisations. □

Proof of Theorem 3.3. Let $x \in \Sigma$ and let $c : [0, 1] \to M$ be a geodesic in M orthogonal to Σ at $x = c(0)$. Let $\delta > 0$ and let $\gamma : (-\delta, \delta) \to \Sigma$ be a geodesic segment in Σ with $\gamma(0) = c(0)$. Finally, let $c(t,s) := \exp_{\gamma(s)}(t\nu(s))$ be a variation of $c(t)$ such that all curves $c(\cdot, s) = c_s(\cdot)$ are geodesics orthogonal to Σ for all $|s| < \delta$. Since it comes from a variation through geodesics, the vector field

$$Y(t) := \frac{\partial}{\partial s} c(t,s)\Big|_{s=0}$$

is a Jacobi field along $c(t) = c(t,0)$ and is completely determined by the initial conditions:

$$\begin{aligned} Y(0) &= \gamma'(0) && (3.3) \\ Y'(0) &= \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} c(t,s)\Big|_{s=0} \\ &= \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} c(t,s)\Big|_{s=0} \\ &= \nabla_{Y(0)} \nu(s)\Big|_{s=0} \\ &= \mathcal{W}(Y(0)), && (3.4) \end{aligned}$$

where $\mathcal{W}$ is the Weingarten map of Σ. The calculation in (3.4) is independent of t and therefore (3.4) holds along any level surface Σ_t for any $|t| < C$:

$$\begin{aligned} Y'(t) :&= \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} c(t,s)\Big|_{s=0} \\ &= \nabla_{Y(t)} \nu_t \\ &= \mathcal{W}(Y(t)), \end{aligned}$$

where now $\mathcal{W}$ is the Weingarten map of Σ_t at the point $c(t)$. Therefore from the previous calculation we have

$$\begin{aligned} \mathcal{W}'(Y(t)) :&= \frac{\partial}{\partial t}\Big(\nabla_{\frac{\partial}{\partial t}} Y(t)\Big) - \nabla_{\nu_t}\big(\nabla_{\nu_t} Y(t)\big) \\ &= Y''(t) - \mathcal{W}^2(Y(t)). && (3.5) \end{aligned}$$

On the other hand $Y(t)$ is a Jacobi field and it therefore satisfies the Jacobi equation

$$Y''(t) = -R(Y(t), \nu_t)\nu_t, \qquad (3.6)$$

(cf., for e.g., [Pet06, Ch. 2.5.2]). Hence, from (3.5) and (3.6) we get the following Riccati equation for the Weingarten map

$$\mathcal{W}'(Y(t)) = -R(Y(t), \nu_t)\nu_t - \mathcal{W}^2(Y(t)). \qquad (3.7)$$

Next, consider $n-1$ linear independent such Jacobi fields $Y_i(t)$ along $c(t)$ for $i = 1, .., n-1$. Taking the trace in (3.7) we get the following, already mentioned, evolution equation

$$\frac{\partial}{\partial t} H_t = -Ric(\nu_t, \nu_t) - |B_t|^2. \tag{3.8}$$

By Schwarz inequality $(n-1)|B_t|^2 \geq H_t^2$ and, by assumption, the Ricci curvature of M is non-negative. Recall that $x \in \Sigma$ is fixed. Then, if we let $h(t) := H(x,t)$ be the mean curvature of Σ_t at $c(t)$, equation (3.8) gives

$$\begin{aligned} h'(t) &\leq -Ric(\nu_t, \nu_t) - \frac{1}{n-1} h^2(t) \\ &\leq 0. \end{aligned} \tag{3.9}$$

Therefore, since Σ is minimal, $h(0) = 0$ and hence, by (3.9), $h(t) = H(x,t) \leq 0$ for all $t < C$. The area inequality (3.2) now follows from the first variation of area formula (2.6). q.e.d.

Remark 3.7. For an argument based on Jacobi fields, as in the one we just encountered, information about the sectional- or Ricci curvature of M, away from the hypersurface, is required. In the absence of such information, Jacobi field techniques seem no longer suitable and therefore a new approach must be taken. This is the case of the area comparison theorem we will prove in the next section. □

3.2 Area Comparison in 3-Manifolds with Scalar Curvature Bounded Below

In this section we will discuss the main result of this chapter and prove an analogue of Heintze-Karcher-Maeda area comparison theorem for totally geodesic surfaces in 3-manifolds of scalar curvature bounded from below.

Theorem 3.8 (Area Comparison Involving the Scalar Curvature [MM]). *Let M be a complete 3-manifold with scalar curvature $S \geq S_0$, where $S_0 \in \mathbb{R}$. Let $\Sigma \subset M$ be an immersed, two-sided, closed, surface of genus γ such that*

(i) Σ is totally geodesic,

(ii) the normal Ricci curvature of M vanishes all along Σ and

(iii) $S = S_0$ at every point of Σ.

Let $\{\Sigma_t\}$, $t \in (-\varepsilon, \varepsilon)$, be a constant mean curvature foliation [3] *in a neighbourhood of Σ and denote by $A(t)$ the area of Σ_t. Then there exists $0 < \delta < \varepsilon$ such that*

$$\text{for } |t| < \delta, \quad A(t) \leq A(0).$$

Moreover, Σ has constant Gauss curvature equal to $\frac{1}{2}S_0$ and therefore, by Gauss-Bonnet theorem, $|S_0|A(0) = 8\pi|\gamma - 1|$, if S_0 is non-zero.

Remark 3.9. Theorem 3.8 can be loosely restated as following: If a 3-manifold with scalar curvature $S \geq S_0$ splits infinitesimally along a closed surface Σ and $S \equiv S_0$ along Σ, then Σ can not be strictly area-minimising inside M. □

Remark 3.10. Assumptions (i)-(iii) of Theorem 3.8 are optimal in the following sense. The 3-manifold $M := \mathbb{S}^2 \times (-\varepsilon, \varepsilon)$ equipped with the metric $(1+t^4)ds^2 + dt^2$, where ds^2 is round, satisfies $S \geq 0$ and also properties (i) and (ii). But the scalar curvature of M decreases away from $\mathbb{S}^2 \times \{0\}$ and therefore (iii) is not satisfied. This is easily seen from Corollary 2.8. We also see that the area of $\mathbb{S}^2 \times \{t\}$ is given by $A(t) = (1 + t^4)A(0)$ and hence $A(t)$ increases as $|t| < \varepsilon$ increases away from zero. Furthermore, the assumption in Theorem 3.8 on the dimension of M is also optimal. This, however, is a more subtle issue and it will be addressed in detail in the Section 3.3 and Chapter 5.4. □

Remark 3.11. It is worth pointing out an important difference between the proof of Theorem 3.8 and Theorem 3.3. In both proofs one has to perturb Σ in a way which decreases its area. In the case of a lower bound on Ricci curvature, the proof of Heintze-Karcher-Maeda theorem shows that a suitable perturbation is obtained by moving Σ with constant speed in the normal direction. However, in the case of a scalar curvature lower bound we do not have a priori knowledge on the Ricci tensor, away from Σ, and therefore this is not a suitable perturbation. It turns out that the right perturbation is to move Σ so that it still has constant mean curvature. This turns out to be possible by the following proposition which guarantees the existence of a constant mean curvature foliation in a neighbourhood of the initial, totally geodesic surface. □

Proposition 3.12. *Let ν be a unit normal field on a closed, oriented, two-sided minimal hypersurface Σ and let $\varepsilon > 0$. If the constant functions are Jacobi fields* [4]

[3] See Remark 3.11 and Proposition 3.12 below.

[4] A good picture is provided by the Sol geometry. If M is an oriented 3-manifold equipped with the Sol geometry then M admits a foliation by minimal tori, all of equal area and none of which are totally geodesic. See, for e.g., [Sco83], pg. 470.

on Σ then there exists $\varepsilon > \varepsilon_1 > 0$ and a smooth function $w\colon \Sigma \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ such that, for all $t \in (-\varepsilon_1, \varepsilon_1)$, the hypersurfaces

$$\Sigma_t := \{\exp_x(w(x,t)\nu(x)) : x \in \Sigma\}$$

have constant mean curvature $H(t)$. Moreover the function w satisfies

$$w(x,0) = 0, \qquad \frac{\partial}{\partial t} w(x,t)\Big|_{t=0} = 1, \qquad \textit{and} \qquad \int_\Sigma (w(\cdot,t) - t)\, d\mu = 0,$$

for all $x \in \Sigma$ and $t \in (-\varepsilon_1, \varepsilon_1)$.

To our knowledge, a complete proof for Proposition 3.12, based on the implicit function theorem, first appeared in [Nun]. Fairly detailed proofs appeared previously in [Cai02] and [ACG08]. For completeness we include in the Appendix the proof from [Nun].

Having now all the ingredients required, we proceed with the proof of the area comparison theorem 3.8.

Proof of Theorem 3.8. With the notation as in Proposition 3.12, let

$$f_t(x) := \exp_x(w(x,t)\nu(x)), \quad x \in \Sigma, \ t \in (-\varepsilon, \varepsilon).$$

Thus, $f_0 := f$ is the given totally geodesic embedding. We define the *lapse function* $\rho_t\colon \Sigma \to \mathbb{R}$ as in Section 2.1.4. Then ρ_t satisfies the evolution equation (2.4) we now restate

$$H'(t) = -\Delta_{\Sigma_t}\rho_t - (Ric(\nu_t, \nu_t) + |B_t|^2)\rho_t, \tag{3.10}$$

where B_t is the second fundamental form of Σ_t.

Remark 3.13. We denote $H' = \frac{\partial}{\partial t}H$ since, in this case, the mean curvature of Σ_t is constant along Σ_t and therefore is a function depending on t only. □

Using the Gauss equation (2.2), equation (3.10) becomes

$$H'(t) = -\Delta_{\Sigma_t}\rho_t - \Big(\frac{1}{2}S_t - K_t + \frac{1}{2}H^2(t) + \frac{1}{2}|B_t|^2\Big)\rho_t. \tag{3.11}$$

where $S_t(x) := S(f_t(x))$ and $K_t(x) := K(f_t(x))$ is the Gauss curvature of Σ_t at $f_t(x)$. Since the function $w(x,t)$ satisfies the three properties from Proposition 3.12 it follows, by the definition of the lapse function, that $\rho_0(x) = 1$ for all $x \in \Sigma$. Therefore we can assume, by decreasing ε if necessary, that $\rho_t(x) > 0$ for all $x \in \Sigma$

and $|t| < \varepsilon$. Hence we can divide (3.11) by ρ_t to obtain

$$H'(t)\frac{1}{\rho_t} = -\frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t - \tfrac{1}{2}S_t + K_t - \tfrac{1}{2}H^2(t) - \tfrac{1}{2}|B_t|^2. \tag{3.12}$$

The hypotheses (i), (ii), (iii) and $S \geq S_0$ imply, via the Gauss equation (2.2), that

$$K_0 \equiv \tfrac{1}{2}S_0 \leq \tfrac{1}{2}S_t(x) \qquad \forall\, x \in \Sigma,\ t \in (-\varepsilon, \varepsilon). \tag{3.13}$$

Therefore, (3.12) can be rewritten as

$$\begin{aligned} H'(t)\frac{1}{\rho_t} &= -\frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t + \tfrac{1}{2}(S_0 - S_t) + (K_t - K_0) - \tfrac{1}{2}H^2(t) - \tfrac{1}{2}|B_t|^2 \\ &\leq -\frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t + (K_t - K_0). \end{aligned} \tag{3.14}$$

Integrating (3.14) over Σ_t and by parts in the first term on the right we get

$$\begin{aligned} \int_\Sigma H'(t)\frac{1}{\rho_t}\, d\mu_t &\leq -\int_\Sigma \frac{|\nabla_t \rho_t|^2}{\rho_t^2}\, d\mu_t + \int_\Sigma (K_t - K_0)\, d\mu_t \\ &\leq \int_\Sigma (K_t - K_0)\, d\mu_t, \end{aligned} \tag{3.15}$$

where $d\mu_t$ denotes the area element of Σ_t with respect to $f_t^* g$.

By construction Σ_t has constant mean curvature and therefore its mean curvature H does not depend on the space variable x. On the other hand, by the Gauss-Bonnet theorem, the integral of the Gaussian curvature K is a topological invariant, being therefore independent of the time variable t. As such, inequality (3.15) becomes

$$H'(t)\int_\Sigma \frac{1}{\rho_t} d\mu_t \leq 4\pi(1-\gamma) - K_0 A(t). \tag{3.16}$$

Claim 1. There exists a positive real number $\delta < \varepsilon$ such that $H(t) \leq 0$ for all $t \in [0, \delta)$.

Proof of Claim 1. There are three cases to consider depending on the sign of S_0, the lower bound on the scalar curvature of M.

Case 1. $S_0 > 0$. By scaling, we can arrange $S_0 = 2$. Then by (3.13), we have that $K_0 \equiv 1$, and Σ has genus zero and $A(0) = 4\pi$. Therefore inequality (3.16) becomes

$$H'(t)\int_\Sigma \frac{1}{\rho_t}\, d\mu_t \leq 4\pi - A(t) = A(0) - A(t) = -\int_0^t A'(s)\, ds.$$

By the first variation of area formula (2.6), and using again that Σ_t has constant mean curvature, the last inequality becomes

$$H'(t)\int_\Sigma \frac{1}{\rho_t}\,d\mu_t \le -\int_0^t \Big\{H(s)\int_\Sigma \rho_s\,d\mu_s\Big\}\,ds \tag{3.17}$$

Let $\phi(t) := \int_\Sigma \frac{1}{\rho_t}\,d\mu_t$ and $\xi(t) := \int_\Sigma \rho_t\,d\mu_t$. Since ϕ is strictly positive for all $t \in (-\varepsilon,\varepsilon)$, inequality (3.17) becomes the following Gronwall type inequality [5]

$$H'(t) \le -\frac{1}{\phi(t)}\int_0^t H(s)\xi(s)\,ds. \tag{3.18}$$

As mentioned above, $\rho_0 \equiv 1$ and, by continuity, we may assume that $\frac{1}{2} < \rho_t(x) < 2$, $\forall\, t \in (-\varepsilon,\varepsilon)$ and $x \in \Sigma$. Integrating over Σ_t yields $\frac{1}{2}A(t) < \xi(t) < 2A(t)$. On the other hand, by choosing $\varepsilon > 0$ small enough, we may assume that $\frac{1}{2}A(0) < A(t) < 2A(0)$ and hence, $\frac{1}{4}A(0) < \xi(t) < 4A(0)\ \forall\, t \in (-\varepsilon,\varepsilon)$. A similar argument holds for $\phi(t)$. In particular we have

$$\frac{1}{\phi(t)} < \frac{4}{A(0)} \quad \text{and} \quad \xi(t) < 4A(0),\ \forall\, t \in (-\varepsilon,\varepsilon). \tag{3.19}$$

Suppose, for a contradiction, that there exists $t_+ \in (0,\delta)$ such that $H(t_+) > 0$. By continuity, $\exists\ t_- \in [0,t_+)$ such that $H(t_-) \le H(t)\ \forall\, t \in [0,t_+]$. Note that by (3.18) we must have $H(t_-) < 0$ since otherwise Claim 1 is proved. By the mean value theorem, $\exists\ t_1 \in (t_-,t_+)$ such that

$$H'(t_1) = \frac{H(t_+) - H(t_-)}{t_+ - t_-}.$$

So, by (3.18) and (3.19), we have:

$$\frac{H(t_+) - H(t_-)}{t_+ - t_-} = H'(t_1) \le -\frac{4}{A(0)}H(t_-)\big(4A(0)\big)t_1\,.$$

It follows that

$$H(t_+) \le H(t_-)(1 - 16\delta^2)$$

which is a contradiction if $0 < \delta < \frac{1}{4}$ because $H(t_+) > 0$ and $H(t_-) < 0$.

[5] The integral form of the classical Gronwall inequality reads: if $h(t)$ is a non-negative integrable function on $[0,a] \subset \mathbb{R}$ which satisfies the integral inequality $h(t) \le C_0\int_0^t h(s)ds + C_1$, for some constants $C_0,\ C_1 \ge 0$, then $h(t) \le C_1(1 + C_0te^{C_0t})$. In our situation, as explained in the proof, the functions φ and ξ are positive functions which are bounded away from zero. Therefore equation (3.18) resembles the classical Gronwall inequality being one derivative "stronger" than the classical one.

Case 2. $S_0 = 0$. By (3.13), we have that $K_0 \equiv 0$ and Σ has genus one. So, inequality (3.16) becomes $H'(t) \leq 0 \; \forall\, t \in [0,\varepsilon)$ and therefore, since $H(0) = 0$, $H(t) \leq 0 \; \forall\, t \in [0,\varepsilon)$.

Case 3. $S_0 < 0$. By scaling, we can arrange $S_0 = -2$. Then by (3.13), we have that $K_0 \equiv -1$, and Σ has genus $\gamma > 1$ and $A(0) = 4\pi(\gamma - 1)$. Therefore inequality (3.16) becomes

$$H'(t) \int_\Sigma \frac{1}{\rho_t}\, d\mu_t \leq A(t) - A(0) = \int_0^t A'(s)\, ds = \int_0^t \{H(s) \int_\Sigma \rho_s \, d\mu_s\}\, ds. \tag{3.20}$$

Assume, for a contradiction, that there exists $t_0 \in (0,\delta)$ such that $H(t_0) > 0$ and let

$$I := \{t \in [0,t_0] : H(t) \geqslant H(t_0)\}.$$

Claim 2: $\inf I = 0$.

Proof of Claim 2. Let $t^* := \inf I$ and assume, again for a contradiction, that $t^* > 0$. By the mean value theorem, $\exists\, t_1 \in (0,t^*)$ such that

$$H(t^*) = H'(t_1)t^*, \tag{3.21}$$

since $H(0) = 0$. From (3.20), (3.19) and (3.21) we have

$$H(t^*) \leq \frac{t^*}{\phi(t_1)} \int_0^{t_1} H(s)\xi(s)\, ds \tag{3.22}$$

$$\leq \frac{t^*}{\phi(t_1)} \int_0^{t_1} H(t^*)\xi(s)\, ds \leq \frac{4t^*}{A(0)} H(t^*)(4A(0)t_1)$$

$$< 16 H(t^*)\delta^2 \tag{3.23}$$

which is a contradiction if $\delta < \frac{1}{4}$ and Claim 2 has been proved. q.e.d.

Since $\inf I = 0$, it follows from the definition of I that $H(0) \geqslant H(t_0)$ and since, by assumption, $H(t_0) > 0$, we conclude that $H(0) > 0$. This contradicts the hypothesis that Σ is totally geodesic and the proof of Claim 1 is complete. q.e.d.

We can now easily complete the proof of Theorem 3.8. By Claim 1 we have that $H(t) \leq 0 \; \forall\, t \in [0,\delta)$ and therefore, the first variation of area formula (2.6) implies that $A'(t) \leq 0$. Hence $A(t) \leq A(0) \; \forall\, t \in [0,\delta)$. We can argue similarly for $t \in (-\delta, 0]$ to complete the proof of Theorem 3.8. q.e.d.

3.3 Strictly Area-Minimising Hypersurfaces in Manifolds with Scalar Curvature Bounded Below

It is natural to ask if Theorem 3.8 is true in dimensions higher than three, as is the case of the Heintze-Karcher-Maeda theorem. In Chapter 5 we will see that, under one additional topological assumption on Σ, the area comparison theorem 3.8 can be extended to higher dimensional manifolds of non-negative scalar curvature. However, it turns out that, without this additional topological assumption, Theorem 3.8 is true in dimension three only. We have the following Proposition.

Proposition 3.14. *There exist n-dimensional manifolds (M, ds^2), with $n \geq 4$ and scalar curvature S, that contain a closed, two-sided hypersurface Σ such that the following hold:*

(a) $S \geq S_0$, for some $S_0 \in \mathbb{R}$.

(b) Σ is strictly area minimising with respect to the induced metric and

(c) properties (i)-(iii) of Theorem 3.8 hold.

The proposition will be proven by explicitly constructing the metric ds^2 on M. Before doing so, let us first describe the intuition behind our construction. For this purpose, let $\mathscr{S}$ be a closed, oriented surface of any genus $\gamma \geq 0$ equipped with the metric ds_1^2 of constant Gaussian curvature and let $\mathbb{S}^1$ be unit circle with the metric ds_2^2. Furthermore, let $\Sigma := \mathscr{S} \times \mathbb{S}^1$. We aim to construct on $M = \Sigma \times (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, a doubly warped metric

$$ds^2 = u^2(t)ds_1^2 + w^2(t)ds_2^2 + dt^2, \tag{3.24}$$

where u and w are both smooth functions on $(-\varepsilon, \varepsilon)$ with $u(0) = w(0) = 1$.

In order to prove Proposition 3.14 we need to find two functions u and w such that both the area of the leaves $\Sigma_t := \Sigma \times \{t\}$ and the scalar curvature S_t of M, at points on the leaves, increase as $|t|$ increases away from zero. As we will see in the following, finding these functions is a rather delicate task since one needs to compensate for the negative sectional curvature that one brings in by increasing the area of the leaves Σ_t.

To see this consider the following picture. By choosing an increasing function u in (3.24), the area of $\mathscr{S}$ will increase. If, additionally, ds_1^2 is of negative curvature, then the scalar curvature of $\mathscr{S}_t := \mathscr{S} \times \{t\}$ will increase as well. However, the scalar curvature S_t of the manifold M will not necessary increase. Indeed, by Proposition 2.6 (a), if we let $u(t) := 1 + t^4$ then, for $X \in \mathfrak{X}(\mathscr{S})$, the sectional curvature for the section $X \wedge \partial_t$ is given by $K(X \wedge \partial_t) = -12t^2 + \mathcal{O}(t^6)$. Hence this will decrease the scalar curvature S_t of M at point on the leaves Σ_t.

We can compensate for these negative sectional curvatures by choosing an appropriate function w which will decrease the length of $\mathbb{S}^1 \times \{t\}$ and hence will bring in positive sectional curvature for sections containing ∂_t and a tangent vector to $\mathbb{S}^1 \times \{t\}$. This second step, however, has the drawback that it decreases the length of $\mathbb{S}^1 \times \{t\}$ and hence the area of the entire leaf Σ_t.

This picture suggests that one needs to find suitable warping functions u and w such that each of them will compensate for the "drawbacks" of the other. That is, in order to increase both the area of the leaves Σ_t and the scalar curvature of M at points on the leaves, the warping functions u and w will have to depend on each other. This dependence is probably best suggested by the following 3-dimensional example that resembles the Sol geometry.

Example 3.15. On the 3-torus $N = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ we put the doubly warped product metric $f^2(t)ds_1^2 + f^{-2}(t)ds_2^2 + dt^2$ where f is a smooth function on $\mathbb{S}^1$ with $f(0) = 1$. With this metric, N is a non-flat 3-torus foliated by flat, minimal 2-tori $T_t := \mathbb{S}^1 \times \mathbb{S}^1 \times \{t\}$, at least two of which are totally geodesic and all of which have equal area. Indeed, the Weingarten map of the leaf T_t is given by $\nabla_X \partial_t = (2f'/f)X$ and $\nabla_Y \partial_t = -(2f')(f^3)Y$, where X and Y are tangent to the "first" and "second" $\mathbb{S}^1$ in N, respectively (c.f. [O'N83] Ch.7, Proposition 35 (2)). Since the function f is smooth and defined on the compact domain $\mathbb{S}^1$, there exist at least two t_1 and t_2 such that $f'(t_i) = 0$, for $i = 1, 2$. Therefore at t_1 and t_2 the Weingarten map vanishes and hence T_{t_1} and T_{t_2} are totally geodesic tori.

Notice that, by construction, the leaves T_t have all equal area and therefore T_0 is not *strictly* area-minimising.

Furthermore, the normal Ricci curvature $Ric^N(\partial_t, \partial_t)$ of N involves second order derivatives of f (cf. Corollary 2.7 (a)). Therefore if, for example, $f(t) := 1 + t^{2k}$, for $k \geq 2$ we expect $Ric^N(\partial_t, \partial_t)$ to be of order t^{2k-2}. However, by the way the two warping functions f and f^{-1} depend on each other, the normal Ricci curvature of N will actually be of order t^{4k-2}.

This observation suggests that the normal Ricci curvature of a new "perturbed" metric $dt^2 + (f(t) + t^{2m})^2 ds_1^2 + f^{-2}(t)ds_2^2$ will still be of order t^{4k-2} for $m > k$ large enough, while the area of the leaves T_t will increase like t^{2m}. In particular $T \times \{0\}$

will be *strictly* area-minimising. □

With this example in mind we return to our construction.

Proof of Proposition 3.14. There are three cases to consider depending on the sign of the lower bound on the scalar curvature of M.

Case 1: $S_0 > 0$. Without the loss of generality we can assume $S_0 = 2$. For this case let $\Sigma = \mathbb{S}^2 \times \mathbb{T}^{n-3}$, where $\mathbb{S}^2$ is the 2-sphere, $\mathbb{T}^{n-3}$ is the $(n-3)$-dimensional torus and when $n = 4$, $\mathbb{T}^1$ is just the unit circle $\mathbb{S}^1$. On $M := \Sigma \times (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, we put the doubly warped product metric $ds^2 = u^2(t)ds_1^2 + w^2(t)ds_2^2 + dt^2$, where ds_1^2 is of constant curvature equal to 1 and ds_2^2 is flat. The warping functions u and w will be defined as follows:

$$\begin{cases} u(t) & := (1 + \dim(\mathbb{T}^{n-3})t^4)^{-1} \\ & = (1 + (n-3)t^4)^{-1} \\ w(t) & := 1 + \dim(\mathbb{S}^2)(t^4 + t^8) \\ & = 1 + 2t^4 + 2t^8. \end{cases}$$

Remark 3.16. As we will see below, the coefficients in the above expressions of u and w are such as to guarantee that the normal Ricci curvature of M vanishes to a high enough order. □

Elementary calculus gives

$$\begin{cases} \dot{u}(t) = -4(n-3)t^3\Big(1 + (n-3)t^4\Big)^{-2} \\ \dot{w}(t) = 8t^3 + 16t^7 \\ \ddot{u}(t) = \Big(-12(n-3)t^2 + 20(n-3)^2t^6\Big)\Big(1 + (n-3)t^4\Big)^{-3} \\ \ddot{w}(t) = 24t^2 + 112t^6. \end{cases} \tag{3.25}$$

We will first show that $\Sigma_0 := \Sigma \times \{0\}$ is totally geodesic. Indeed, if $X \in \mathfrak{X}(\mathbb{S}^2)$, then by (3.25) we have that

$$\nabla_{\partial_t} X\big|_{t=0} = \frac{u'(0)}{u(0)} X = 0$$

and for $U \in \mathfrak{X}(\mathbb{T}^{n-3})$ that

$$\nabla_{\partial_t} U\big|_{t=0} = \frac{w'(0)}{w(0)} U = 0.$$

Therefore, at $t = 0$, the Weingarten map vanishes identically implying that Σ_0 is totally geodesic and hence property (i) of Theorem 3.8 is satisfied.

By (3.25) and Corollary 2.7 (a), the normal Ricci curvature of M satisfies

$$\begin{aligned} Ric(\partial_t, \partial_t) &= -\Big(2\frac{\ddot{u}}{u} + (n-3)\frac{\ddot{w}}{w}\Big) \\ &= \frac{-c(n)t^6 + \mathcal{O}(t^8)}{(1+(n-3)t^4)^2(1+2t^4+2t^8)}, \end{aligned}$$

where $c(n)$ is a positive integer depending on n. Obviously $Ric(\partial_t, \partial_t)$ vanishes at $t = 0$ and hence property (ii) of Theorem 3.8 is satisfied. The scalar curvature of M is given by Corollary 2.8 and, by using (3.25), we see that it satisfies

$$\begin{aligned} S_t &= \frac{1}{u^2}S_1 - \Big(4\frac{\ddot{u}}{u} + 2(n-3)\frac{\ddot{w}}{w}\Big) - 2\frac{\dot{u}^2}{u^2} - (n-3)(n-4)\frac{\dot{w}^2}{w^2} - 4(n-3)\frac{\dot{u}\dot{w}}{uw} \\ &= \frac{1}{u^2}S_1 + \mathcal{O}(t^6), \end{aligned}$$

where $S_1 = 2$ is the scalar curvature of the round metric ds_1^2 of $\mathbb{S}^2$. Since by (3.25) we have that, at $t = 0$, $S_0 = S_1$, the last inequality gives

$$\begin{aligned} S_t - S_0 &= 2\Big(\frac{1}{u^2} - 1\Big) + \mathcal{O}(t^6) \\ &= 4(n-3)t^4 + \mathcal{O}(t^6). \end{aligned}$$

Therefore $S_t \geq S_0$ for sufficiently small $\varepsilon > 0$, which proves that (a) holds. Moreover, since $S \equiv S_0$ at $t = 0$, condition (iii) of Theorem 3.8 also holds.

Finally, the area element μ_t of Σ_t satisfies

$$\begin{aligned} d\mu_t &= u^2(t)w^{n-3}(t)d\mu_0 \\ &= \left(1 + \frac{(n-3)^2t^8 + \mathcal{O}(t^{12})}{1+2(n-3)t^4+(n-3)^2t^8}\right)d\mu_0 \\ &\geq \mu_0, \end{aligned} \tag{3.26}$$

where $d\mu_0$ is the area element of Σ_0. Therefore, after integrating the last quality over Σ we have that $A(0) < A(t)$ for $0 < t < \varepsilon$, which shows that Σ_0 has least area among all leaves Σ_t.

Finally, to show that Σ_0 is *strictly* area-minimising in M, and hence to prove property (b), we have to show that there are no hypersurfaces with area less than or equal to Σ_0 and which are not leaves.

Claim: For any smooth positive, non-constant function u on Σ with $0 < u(x) < \varepsilon$

for all $x \in \Sigma$, the hypersurface

$$\Sigma_u := \{\exp_x(u(x)\nu(x)) : x \in \Sigma\}, \quad \nu \in \mathfrak{X}(\Sigma)^{\perp}$$

has area strictly greater than the area of Σ.

Proof of the claim:. Let $A(\Sigma_u)$ be the area of Σ_u and let ν_u be the unit normal vector field along Σ_u. For all points $p \in \Sigma_u$, there exists $t \in [0, \varepsilon)$ and $x \in \Sigma$, such that $p = (x,t) \in \Sigma_t$. If ν_t is the unit normal vector field of Σ_t then

$$g(\nu_u(x), \nu_t(x)) \leq 1, \quad \forall p \in \Sigma_u, \tag{3.27}$$

with equality on an open set if and only if Σ_u and Σ_t coincide at this open set for some fixed value of t; that is, if and only if Σ_u is a leaf.

Let Ω be the region in M bounded by Σ and Σ_u. Then we have that

$$\begin{aligned}
\int_{\Omega} \mathrm{div}_{\Omega}(\nu_t) dV &= \int_{\Sigma_u} g(\nu_t, \nu_u) d\mu_u - \int_{\Sigma} g(\nu, \nu) d\mu \\
&= \int_{\Sigma_u} g(\nu_t, \nu_u) d\mu_u - A(\Sigma) \\
&\leq A(\Sigma_u) - A(\Sigma),
\end{aligned} \tag{3.28}$$

where the last inequality follows from (3.27). On the other hand $\mathrm{div}_{\Omega}(\nu)(p) = H(p)$, the mean curvature of Σ_t at the point $\exp_x(t\nu(x))$. By a direct calculation using (3.25), (3.26) and that $\frac{\partial}{\partial t} d\mu_t = H(x,t) d\mu_t$, we have

$$\begin{aligned}
H(x,t) &= \frac{2\dot{u}}{u} + (n-3)\frac{\dot{w}}{w} \\
&= \frac{16(n-3)^2 t^7 + \mathcal{O}(t^{11})}{1 + \mathcal{O}(t^4)} \\
&> 0,
\end{aligned}$$

for all $0 < t < \varepsilon$ and $x \in \Sigma$. Hence by (3.28) we conclude that $A(\Sigma) < A(\Sigma_u)$. This completes the proof of the claim and hence of the Case 1. q.e.d.

The remaining two cases are, to some extent, similar to the first one and, for this reason, we will omit some of the details.

Case 2: $S_0 < 0$. In this case we will define $\Sigma := N^{n-2} \times \mathbb{S}^1$, where N is a $(n-2)$-dimensional, closed, hyperbolic manifold and $\mathbb{S}^1$ is the unit circle. Then on $M := \Sigma \times (-\varepsilon, \varepsilon)$ we will put again a doubly warped product metric $ds^2 = u^2(t) ds_1^2 + w^2(t) ds_2^2 + dt^2$, where the functions u and w are given by $u(t) := 1 + t^4 + t^8$ and

$w(t) := (1 + (n-2)t^4)^{-1}$.

Case 3: $S_0 = 0$. In this case we let $\Sigma := N^{n-3} \times \mathbb{S}^2(r)$, where N^{n-3} is a closed, hyperbolic, $(n-3)$-dimensional manifold with scalar curvature $S_1 = -(n-3)(n-4)$ and $\mathbb{S}^2(r)$ is the two-sphere of radius $r := \sqrt{2/(n-3)(n-4)}$ equipped with the round metric ds_2^2. Hence ds_2^2 has scalar curvature $S_2 = 2/r^2 = (n-3)(n-4)$. Finally, the warping functions u and w are given by $u(t) := 1 + 2t^4 + 2t^8$ and $w(t) := (1 + (n-3)t^4)^{-1}$.

Remark 3.17. By letting the radius of the two-sphere be $r < \sqrt{2/(n-3)(n-4)}$ or $r > \sqrt{2/(n-3)(n-4)}$, the scalar curvature of M will be strictly positive or strictly negative, respectively. Thus providing several more examples in the cases 1 and 2 for all $n \geq 5$. $\square$

This completes the proof of Proposition 3.14. q.e.d.

Chapter 4

Splitting of 3-Manifolds

4.1 Existence of Metrics of Positive Scalar Curvature

It's been a long quest to understand the relationship between the sign of the scalar curvature of a Riemannian metric and the topology of the underlying manifold.

At a first glance, it might seem that the sign of the scalar curvature of a Riemannian manifold is not related with the topology of the underlying manifold. Indeed, if N is any Riemannian manifold with $S > -\infty$ and $\mathbb{S}^2(\varepsilon)$ is a round 2-sphere of radius $\varepsilon > 0$, then the manifold $M := N \times \mathbb{S}^2(\varepsilon)$, with the product metric, has positive scalar curvature for sufficiently small $\varepsilon > 0$. Yet the geometry and the topology of M will be as least as complicated as that of N.

It was a major breakthrough when, in 1960s, Lichnerowicz proved [Lic63] the existence of closed manifolds which admit no metric of positive scalar curvature [1]. His discovery was the starting point of understanding to what extent does the underlying topological structure of a manifold determine the sign of the scalar curvature. This is a very subtle problem and great progress has been made during the second half of the 20$^{\text{th}}$ century. For a survey on this topic see, for e.g., [Ber98], [Ber03, Ch.12.3.3] and [Kaz85]. At least "half" of the question was answered by Aubin in 1970.

Theorem 4.1. ([Aub70, p.400]) *On every closed Riemannian manifold of dimension $n \geq 3$ there exists a metric of negative scalar curvature.*

The result most relevant to our thesis is the major result of Schoen and Yau from 1979. In the celebrated paper [SY79], they discovered an obstruction to positive scalar curvature on 3-manifolds in terms of the topology of closed, stable, minimal surfaces.

[1] An example is provided by the Fermat surface, defined by the equation $x^4 + y^4 + z^4 + w^4 = 0$ in $\mathbb{CP}^3$. This is further an example of a K3 surface which, by Lichnerowicz's result, do not admit metrics of positive scalar curvature. See, for e.g., [Bes87, 6.72].

Theorem 4.2 ([SY79]). *Let M be an oriented 3-manifold whose fundamental group $\pi_1(M)$ contains a subgroup isomorphic to the fundamental group of a surface Σ of genus $\gamma \geq 1$. Then M admits no metric of positive scalar curvature.*

In the process of proving this result the following theorem is also obtained.

Theorem 4.3 ([SY79]). *Let M be a complete, oriented 3-manifold of positive scalar curvature S. Then M contains no compact, immersed, stable, minimal surface Σ of genus $\gamma \geq 1$.*

Proof. Since Σ is stable the second variation of area is non-negative for any normal variation $\rho \in \mathcal{C}_0^\infty(\Sigma)$ (see Definition 2, Ch. 2.3). We let $\rho \equiv 1$ and therefore inequality (2.10) becomes

$$\int_\Sigma (Ric(\nu,\nu) + |B|^2) d\mu \leq 0. \tag{4.1}$$

Using the Gauss equation (2.2), the minimality of Σ and the Gauss-Bonnet theorem, from the last inequality we have

$$\begin{aligned} \frac{1}{2}\int_\Sigma S d\mu &\leq \int_\Sigma (K - \frac{1}{2}|B|^2) d\mu \\ &\leq \int_\Sigma K d\mu = 2\pi\chi(\Sigma), \end{aligned} \tag{4.2}$$

where K is the Gauss curvature and $\chi(\Sigma) = 4\pi(1-\gamma)$ is the Euler characteristic of Σ, respectively. Since M has positive scalar curvature, it follows from (4.2) that the Euler characteristic of Σ must be positive and therefore its genus γ must be zero. q.e.d.

4.2 Area Bounds for Stable Minimal Surfaces and Infinitesimal Splitting of the Ambient Manifold

Soon after these results were proven, Fischer-Colbrie and Schoen studied the non-negative scalar curvature case and proved in [FCS80] that, in this case, the genus of Σ must be zero or one, and if it is one, then Σ is totally geodesic and flat and both the normal Ricci curvature and the scalar curvature S of M vanish all along Σ. Therefore M splits infinitesimally as a product along Σ. It turns out that this situation persists even for surfaces of genus different than one.

A closer look at the proof of Theorem 4.3 reveals that lower bounds on the scalar curvature of the 3-manifold M provide area bounds for closed, stable, minimal surfaces contained in M. Indeed, as we already mentioned in the Introduction, it was observed by Shen and Zhu [SZ97] that if the scalar curvature of M satisfies $S \geq S_0$, for some non-zero $S_0 \in \mathbb{R}$, then the area $A(\Sigma)$ of any closed, stable, minimal surface Σ with genus $\gamma \neq 1$, satisfies

$$\begin{cases} \mathrm{A}(\Sigma) \leq 4\pi & \text{if } S_0 = 2 \\ \mathrm{A}(\Sigma) \geq 4\pi(\gamma - 1) & \text{if } S_0 = -2 \text{ and } \gamma \geq 2. \end{cases} \tag{4.3}$$

The proof follows immediately from (4.2). The genus one case is excluded since no area bounds are possible for stable minimal tori. This is easily illustrated by stable two-dimensional tori in flat three-dimensional tori.

In the light of our discussion from the Introduction, the critical case of (4.3) might reveal some rigidity phenomenon. Indeed, this was pointed out by Bray, Brendle and Neves [BBN10] for $S_0 > 0$ and by Nunes [Nun] for $S_0 < 0$. It was shown that if equality is attained in (4.3) then, as in the case of stable minimal tori, the ambient 3-manifold M splits infinitesimally along Σ. More precisely we have the following proposition which includes the genus one case as well.

Proposition 4.4 ([FCS80], [BBN10] and [Nun]). *If Σ is a closed, stable, minimal torus in a 3-manifold of non-negative scalar curvature or, if Σ is a closed, stable, minimal surface attaining equality in* (4.3), *then*

(i) Σ is totally geodesic,

(ii) the normal Ricci curvature of M vanishes at every point of Σ and

(iii) the scalar curvature S of M is equal to S_0 all along Σ.

Remark 4.5. The proof of Proposition 4.4 is very similar in all three cases, being a slight modification of the genus one case proved in [FCS80, Theorem 3]. We will discuss only the cases when Σ attains equality in (4.3). This very same argument will later be used in Chapter 5 to prove part of an infinitesimal splitting theorem for stable minimal hypersurfaces (Theorem 5.10). $\square$

Proof of Proposition 4.4. The condition of Σ being stable can be rephrased in analytical terms as follows (cf. [FCS80], Theorem 1). A minimal surface Σ is stable if and only if the first eigenvalue λ_1 of the Jacobi operator $L_\Sigma := \Delta_\Sigma + (Ric(\nu,\nu) + |B|^2)$ is nonnegative; i.e.

$$0 \leq \lambda_1 = \inf_f \Big\{ \int_\Sigma \Big(|\nabla_\Sigma f|^2 - (Ric(\nu,\nu) + |B|^2) f^2 \Big) d\mu \ : \int_\Sigma f^2 d\mu = 1 \Big\}.$$

Recall that (4.3) follows from (4.1) and (4.2). Therefore, since Σ attains equality in (4.3), inequalities (4.1) and (4.2) become also equalities. If follows from equality in (4.2) that Σ is totally geodesic and hence (i) is proved. From equality in (4.1) follows that $\int_\Sigma (Ric(\nu,\nu) + |B|^2) d\mu = 0$. Therefore $\lambda_1 = 0$ and hence the constant functions lie in the kernel of L_Σ which implies that $Ric(\nu,\nu) + |B|^2 = 0$, $\forall x \in \Sigma$. This, together with (i), proves (ii). Finally, from equality in (4.2), we have that $\int_\Sigma (S - S_0) d\mu = 0$. And since, by assumption $S - S_0 \geq 0$, it follows that $S \equiv S_0$ along Σ. This proves (iii) and hence the Proposition. q.e.d.

Remark 4.6. We will see in the next section that if Σ is assumed to be *area minimising* and not just stable (see Definition 3) then this infinitesimal splitting of the ambient 3-manifold provided by Proposition 4.4 will actually propagate to an entire neighbourhood of Σ. □

In the end of this section let us point out that the area bounds (4.3) also appear in the literature of General Relativity. A classic result of Hawking [Haw72] asserts that the boundary of a black hole, in a (3+1)-dimensional asymptotically flat space-time which satisfies the dominant energy condition, must be topologically a sphere. Moreover, the area of the horizon Σ satisfies

$$A(\Sigma) < \frac{4\pi}{\Lambda},$$

where $\Lambda > 0$ is the cosmological constant.

In (3+1)-dimensional space-times which are asymptotically anti-de Sitter and which, in particular, have negative cosmological constant, the boundary of a black hole might be a surface with genus $\gamma \geq 2$. It was observed by Gibbons [Gib99] in the time symmetric case and by Woolger [Woo99] in the general case that in such a situation the area of the boundary Σ of a black hole satisfies

$$A(\Sigma) \geq \frac{4\pi(\gamma - 1)}{|\Lambda|}. \tag{4.4}$$

4.3 Rigidity of Area-Minimising Surfaces and Local Splitting of the Ambient Manifold

As we saw in the previous section, if a stable minimal surface Σ attains equality in (4.3) then, by Proposition 4.4, Σ is totally geodesic, the normal Ricci curvature of M vanishes along Σ and the scalar curvature of M attains its minimum value S_0 at every point of Σ. It therefore follows from the area comparison theorem 3.8 that Σ

is not *strictly* area-minimising. The same conclusion holds for the genus one case even though there are no area bounds for stable minimal tori.

Therefore, if one additionally assumes Σ to be area-minimising then one can further show that, in this case, the ambient 3-manifold M is actually isometrically to a Riemannian product in an entire neighbourhood of Σ. This was first proven by Bray, Brendle and Neves for $S_0 > 0$, by Cai and Galloway for $S_0 = 0$ and by Nunes for $S_0 < 0$.

Theorem 4.7 (Splitting of 3-Manifolds, [BBN10], [CG00b] and [Nun])**.** *Let (M, g) be a complete Riemannian 3-manifold with scalar curvature $S \geq S_0$ where $S_0 \in \mathbb{R}$. Assume that M contains a closed, embedded, oriented, two-sided, area-minimising surface Σ.*

(a) Suppose that $S_0 = 2$ and that $A(\Sigma) = 4\pi$. Then Σ has genus zero and it has a neighbourhood which is isometric to the product $g_1 + dt^2$ on $S^2 \times (-\delta, \delta)$ where g_1 is the metric on the Euclidean two-sphere of radius 1.

(b) Suppose that $S_0 = 0$ and that Σ has genus one. Then Σ has a neighbourhood which is flat and isometric to the product $g_0 + dt^2$ on $\mathbb{T}^2 \times (-\delta, \delta)$ where g_0 is a flat metric on the 2-torus $\mathbb{T}^2$.

(c) Suppose that $S_0 = -2$ and that Σ has genus $\gamma \geqslant 2$ and $A(\Sigma) = 4\pi(\gamma - 1)$. Then Σ has a neighbourhood which is isometric to the product $g_{-1} + dt^2$ on $\Sigma \times (-\delta, \delta)$ where g_{-1} is a metric of constant Gauss curvature equal to -1 on Σ.

The original proofs of these three cases are very different in nature. With one exception only, namely the positive scalar curvature case, the techniques used seem specialised for each case individually. For this reason it's not obvious that, in each case, the rigidity is actually "triggered" by the same geometric phenomenon: once a manifold M with $S \geq S_0$ splits infinitesimally along a closed surfaces Σ and $S \equiv S_0$ along Σ, then Σ can not be *strictly* area-minimising inside M. However, this geometric phenomenon becomes more transparent in the light of the area comparison Theorem 3.8 and its restatement from Remark 3.9.

4.3.1 On the Proof of the Splitting Theorem 4.7

Buried within the original proof of the splitting theorem 4.7 there are different proofs of the area comparison theorem 3.8 which are different from ours (Section 3.2). In order to emphasise this observation more clearly we will first sketch the original area comparison argument as it appears in [CG00b], [BBN10] and [Nun], while in the next subsection we will prove the actual splitting theorem 4.7 by making use of the area comparison theorem 3.8.

Case (a) Since $S_0 = 2$, by Proposition 4.4, $K_0 \equiv 1$, Σ has genus zero and $A(0) = 4\pi$. With the notation as in Proposition 3.12 and as in the proof of Theorem 3.8, let Σ_t be a constant mean curvature foliation in a neighbourhood of Σ, for all $t \in (-\varepsilon, \varepsilon)$, and let

$$f_t(x) := \exp_x(w(x,t)\nu(x)), \quad x \in \Sigma,\ t \in (-\varepsilon, \varepsilon),$$

where $w(x,t)$ is given by Proposition 3.12. The lapse function $\rho_t \colon \Sigma \to \mathbb{R}$ is defined as in (2.3).

Let $\overline{\rho}_t := A(t)^{-1} \int_\Sigma \rho_t d\mu_t$. Then, since there exist an uniform constant $c > 0$ such that

$$\int_\Sigma |\nabla \rho_t|^2 d\mu_t \geq c \int_\Sigma (\rho_t - \overline{\rho}_t)^2 d\mu_t$$

and since by Proposition 4.4 $Ric(\nu_t, \nu_t) + |B_t|^2 \to 0$ as $|t| \to 0$, we conclude that

$$\int_\Sigma \Big(|\nabla \rho_t|^2 d\mu_t - (Ric(\nu_t, \nu_t) + |B_t|^2)(\rho_t - \overline{\rho}_t)^2 \Big) d\mu_t \geq 0, \tag{4.5}$$

for sufficiently small $|t| > 0$. Next notice that, since Σ is area-minimising with area $A(0) = 4\pi$ and $S_0 = 2$, by the Gauss equation (2.2) we have that

$$\begin{aligned} 8\pi &= A(0)S_0 \leq A(t)S_0 \\ &\leq \int_\Sigma (S_t + |B_t|^2) d\mu_t \\ &= \int_\Sigma (2K_t + 2Ric(\nu_t, \nu_t) + 2|B_t|^2) d\mu_t \\ &= 8\pi + 2\int_\Sigma (Ric(\nu_t, \nu_t) + |B_t|^2) d\mu_t. \end{aligned} \tag{4.6}$$

Where in the last inequality we have used the Gauss-Bonnet theorem and that Σ has genus zero.

Remark 4.8. At this stage it is worth pointing out that in the above inequalities (4.6) we make essential use of the area-minimising property of Σ and the first inequality in (4.6) is precisely the inequality which fails in case (c) when $S_0 < 0$. In case (b), when $S_0 = 0$, inequalities (4.6) still hold but notice however that, in this case, the first inequality becomes vacuous. This is a first hint that the area-minimising property of Σ might not be necessary to prove that Σ can not be *strictly* area-minimising. $\square$

After multiplying the inequality (4.6) with $\overline{\rho}_t^2$, we have

$$\overline{\rho}_t^2 \int_\Sigma (Ric(\nu_t, \nu_t) + |B_t|^2) d\mu_t \geq 0, \tag{4.7}$$

for sufficiently small t. Adding the inequalities (4.5) and (4.7) we have

$$\int_\Sigma \Big\{ |\nabla \rho_t|^2 d\mu_t + (Ric(\nu_t, \nu_t) + |B_t|^2)\rho_t(2\overline{\rho}_t - \rho_t) \Big\} d\mu_t \geq 0.$$

The lapse function ρ_t satisfies the evolution equation (2.4) and therefore, after multiplying this equation with $2\overline{\rho_t} - \rho_t$, we get from the previous inequality that

$$\begin{aligned} 0 &\leq \int_\Sigma \Big\{ |\nabla \rho_t|^2 d\mu_t + (Ric(\nu_t, \nu_t) + |B_t|^2)\rho_t(2\overline{\rho}_t - \rho_t) \Big\} d\mu_t \\ &= -H'(t) \int_\Sigma (2\overline{\rho}_t - \rho_t) d\mu_t \\ &= -H'(t) \int_\Sigma \rho_t d\mu_t. \end{aligned}$$

Therefore, by the first variation of area formula (2.6)

$$A'(t) = H(t) \int_\Sigma \rho_t d\mu_t \leq 0,$$

where we have used that the mean curvature H of Σ_t is a function of t only. Hence $A(t) \leq A(0)$ and since Σ is area-minimising it follows that $A(t) = A(0)$, for sufficiently small $|t| < \varepsilon$.

Case (b). Let $N_\varepsilon := \Sigma \times (-\varepsilon, \varepsilon)$ be a normal neighbourhood of Σ with respect to the metric g of M. Let $g_n := u^2 g$ be a conformal deformation of g where $u(t,n) = e^{-2n^{-1}t^2}$, for $n \in \mathbb{N}$. Then the scalar curvature of (M, g_n) is given by

$$S_n = e^{2n^{-1}t^2}(S + 8n^{-1}(1 + tH - n^{-1}t^2)),$$

where H denotes the mean curvature of $\Sigma_t = \Sigma \times \{t\}$. We can see that, for large enough n and for small enough $\varepsilon > 0$, $S_n > 0$. The first observation is that Σ cannot be locally strictly area-minimising. If it were, then under the perturbation above, Σ would still be area-minimising. In particular Σ would be stable. However, since the genus of Σ is one, this would contradict Theorem 4.3.

The next step of the proof consists in constructing a normal neighbourhood $N_\varepsilon := \Sigma \times [-\varepsilon, \varepsilon]$ of Σ, as in Section 2.1.4. Then, by [HS88, Theorem 5.1], there is an area-minimising torus in N_ε on both sides of Σ. By cutting out the region bounded by these tori and pasting it to a second copy, one obtains a smooth 3-torus with non-negative scalar curvature. By [SY79], this torus must be flat which completes the proof.

Case (c). For some $\varepsilon > 0$ consider the manifold with boundary $N_\varepsilon := \Sigma \times [0, \varepsilon]$ with the pull-back metric $g_\varepsilon = \phi^* g$, where $\phi|_{N_\varepsilon} : N_\varepsilon \to M$ is an immersion of N_ε into M. The manifold N_ε has scalar curvature $S \geq S_0$ and the area of Σ equals $4\pi(\gamma - 1)$. As in the case (a), Proposition 4.4 implies that Σ is totally geodesic, the normal Ricci curvature of M vanishes along Σ and S attains it minimum S_0 at every point of Σ. Therefore, by Proposition 3.12, we can assume each leaf Σ_t to be of constant mean curvature. In particular, the boundary of N_ε consists of two disjoint components: Σ satisfying the properties (i)-(iii) from Proposition 4.4 and Σ_ε having constant mean curvature. The key ingredient in the proof is Escobar's solution of the Yamabe problem for manifolds with boundary [Esc92].

By Escobar's result, there exists a smooth function $u > 0$ on N_ε such that the new conformally deformed metric $\overline{g} = u^2 g_\varepsilon$ has constant negative scalar curvature, and the boundary of N_ε are minimal surfaces. Then the proof goes by contradiction. If there exists a positive number $t_0 \in (0, \varepsilon)$ such that the leaf Σ_{t_0} has mean curvature $H(t_0) > 0$, then, by the maximum principle and by the Hopf's boundary point lemma [GT01, Chapter 3], u must be strictly less than 1. Therefore,

$$A(\Sigma, \overline{g}) < A(\Sigma, g_\varepsilon) = 4\pi(\gamma - 1).$$

Finally, using [HS88, Theorem 5.1], there exists a compact embedded surface $\hat{\Sigma}$ with least area among all surfaces isotopic with Σ. In particular $\hat{\Sigma}$ is stable in N_ε and therefore

$$A(\hat{\Sigma}, \overline{g}) \leq A(\Sigma, \overline{g}) < 4\pi(\gamma - 1).$$

This contradicts Proposition 4.4. q.e.d.

4.3.2 A Different Perspective on the Splitting Theorem 4.7

In this section we explain how Theorem 4.7 follows from the area comparison Theorem 3.8. Our contribution consists in finding, via Theorem 3.8, a unified and more elementary proof of the splitting Theorem 4.7, emphasising perhaps better that, in all three cases (a)-(c) of Theorem 4.7, we are actually dealing with the same geometric phenomenon, captured in Remark 3.9. To some extent, this geometric phenomenon reassembles what we already encountered in the Heintze-Karcher-Maeda theorem, where the non-negative Ricci curvature of the ambient manifold was preventing any closed, minimal hypersurface form being *strictly* area-minimising.

In our situation, having "only" a lower bound on the scalar curvature of the ambient 3-manifold M, the link between the three cases (a)-(c) of Theorem 4.7 is provided by Proposition 4.4. This key proposition states that we have in M the "same" geometry along the area-minimising surface Σ, regardless the sign of the lower bound on the scalar curvature. The most important of these three properties is (iii) which asserts that the scalar curvature of the ambient 3-manifold can not decrease away form Σ, along normal geodesics emanating from Σ. This means that, once Σ start to move in its normal direction inside M it will experience the effect of the non-decreasing scalar curvature of M. By Theorem 3.8 this effect translates into the non-increasing area of Σ.

If Σ is an area minimising torus or an area minimising surface which attains equality in (5.5) then, by Proposition 4.4, it satisfies the hypothesis of the area comparison theorem 3.8. Therefore, in all three cases (a)-(c) of Theorem 4.7, inequality (3.18) holds and can be rewritten as

$$H'(t) \leqslant -\frac{S_0}{2\phi(t)} \int_0^t H(s)\xi(s)\, ds. \tag{4.8}$$

The analysis of this inequality differs slightly in each case (a)-(c), depending on the sign of the lower bound on scalar curvature S_0. However it does not require different techniques for each case individually. This analysis was already addressed in detail in the proof of Theorem 3.8.

Proof of Theorem 4.7. The conclusion of Theorem 3.8 and the assumption that Σ is area-minimising imply that, for the constant mean curvature family of surfaces Σ_t provided by Proposition 3.12, $A(t) = A(0),\ \forall\, t \in (-\delta, \delta)$. In particular, each Σ_t is area-minimising and, if $\gamma \neq 1$, the area of each Σ_t is equal to $4\pi|\gamma - 1|$. It follows, from Proposition 4.4 that each Σ_t is totally geodesic and that $Ric(\nu_t, \nu_t) = 0$ along Σ_t. This holds when $\gamma = 1$ as well. The evolution equation (3.10) then reduces to

$$\Delta_{\Sigma_t}\rho_t = 0,$$

which tells us that the lapse function ρ_t is harmonic, and therefore, since Σ is closed, is constant on Σ_t, i.e. ρ_t is a function of t only. In order to prove that M splits isometrically in a neighborhood of Σ we need to show that the normal vector field ν_t is parallel.

Claim: The vector field ν_t is parallel.

Proof of the Claim. The proof of the claim is a slight variation of the argument that appeared in [BBN10] and [Nun]. The surface Σ_t is totally geodesic and hence the Weingarten map vanishes for every t; i.e. $\nabla_{\frac{\partial f_t}{\partial x^i}} \nu_t = 0$. Therefore, since ρ_t is independent of the space variable $x \in \Sigma$, we have

$$\begin{aligned}
0 = \frac{\partial}{\partial x^i}\rho_t &= \langle \nabla_{\frac{\partial f_t}{\partial x^i}} \nu_t, \frac{\partial f_t}{\partial t}\rangle + \langle \nu_t, \nabla_{\frac{\partial f_t}{\partial x^i}} \frac{\partial f_t}{\partial t}\rangle \\
&= \langle \nu_t, \nabla_{\frac{\partial f_t}{\partial x^i}} \frac{\partial f_t}{\partial t}\rangle \qquad (\Sigma_t \text{ is totally geodesic}) \\
&= \frac{\partial}{\partial t}\langle \nu_t, \frac{\partial f_t}{\partial x^i}\rangle - \langle \nabla_{\frac{\partial f_t}{\partial t}} \nu_t, \frac{\partial f_t}{\partial x^i}\rangle \\
&= -\langle \nabla_{\frac{\partial f_t}{\partial t}} \nu_t, \frac{\partial f_t}{\partial x^i}\rangle.
\end{aligned}$$

Hence $\nabla_{\frac{\partial f_t}{\partial t}} \nu_t = 0$. This, together with the fact that Σ_t is totally geodesic, implies that the vector field ν_t is parallel. This completes the proof of the claim. q.e.d.

We now complete the proof of Theorem 4.7. It follows that the integral curves of ν_t are geodesics and that the flow Φ of ν_t is just the exponential map, i.e. $\Phi(t, x) = \exp_x(t\nu(x))$, $\forall\, x \in \Sigma$. Furthermore, since ν_t is, in particular, a Killing field, this exponential map $\exp_{(\cdot)}(t\nu(\cdot))$ is an isometry for all $t \in (-\delta, \delta)$. In other words, if g_Σ is the restriction of g to Σ then the exponential map of the δ-neighbourhood $\Sigma \times (-\delta, \delta)$ of the zero section of the normal bundle of Σ in M with the metric $g_\Sigma + dt^2$ is an isometry onto its image. q.e.d.

In the end of this section we point out that recently, our unified approach presented in this section was used to prove several different and more general splitting theorems. In this direction we mention the result of Espinar [Esp] for area minimising surfaces in 3-manifolds with density and the result of Ambrozio [Amb] for area-minimizing free boundary surfaces in mean convex three-manifolds.

Chapter 5

Splitting of n-Manifolds

In this chapter we will take a step forward towards generalising some of the results from the previous chapters to higher dimensions. The results of Chapter 4 rely on the Gauss-Bonnet Theorem which provides the essential link between the topology of a closed surface and the total Gaussian (i.e. scalar) curvature of the surface. If one wants to generalise some of these results to higher dimensions, then one needs to look at other topological invariants which generalise the Euler characteristic. It turns out that, for our purpose, a good generalisation is given by the σ-constant, introduced independently by Schoen and Kobayashi in relation with the Yamabe problem. We will see that, at least for the negative scalar curvature case, the area of a closed, stable, minimal hypersurface is bounded below in terms of its σ-constant and a negative lower bound on the scalar curvature of the ambient manifold. Moreover, the equality case will correspond to an infinitesimal splitting of the ambient manifold along the hypersurface. Whether is not only infinitesimal but actually propagates to an entire neighbourhood of the stable hypersurfaces is still an open question which will be discussed in the end of the chapter.

5.1 Conformal Deformation of Metrics to Constant Scalar Curvature and Yamabe Invariants

"It is the geometers dream to find a canonical metric g_{best} on a given smooth manifold M so that all the topology of M will be captured by the geometry" [1].

From the differential geometric point of view, the "best" metric represents a metric of constant sectional-, Ricci- or scalar curvature. In two dimensions, all these notions of curvature coincide with the Gauss curvature and the Uniformisation Theorem guarantees that for any compact, Riemannian 2-manifold (M, g) there exists a smooth function $u : M \to \mathbb{R}$ such that the conformally deformed metric $\overline{g} = u^2 g$ has constant Gaussian (and hence scalar) curvature.

[1] This line is attributed by M. Gromov to Heinz Hopf. See [Gro00, p. 138].

In higher dimensions, the existence of a metric $\overline{g}$ on a compact n-dimensional manifold M, conformal to g and having constant scalar curvature, was addressed by Yamabe [Yam60] and it became known as the Yamabe conjecture: Given a n-dimensional Riemannian manifold (M, g), $n \geq 3$, does there exist a metric $\overline{g}$, conformal to g, having constant scalar curvature [2]? This problem was solved affirmatively by Aubin for $n \geq 6$ and (M, g) not locally conformally flat [3] and by Schoen for all the remaining cases. For a complete discussion of the Yamabe problem we refer to survey by Lee and Parker [LP87] and, for more recent developments, to the survey by Brendle and Marques [BM11].

5.1.1 Yamabe Invariants

For a compact Riemannian manifold (M, g) consider the following functional, called the Einstein-Hilbert action

$$\mathcal{Y}(g) := \frac{\int_M S(g) d\mu}{\mathrm{Vol}(M)^{(n-2)/n}}, \tag{5.1}$$

where $S(g)$ is the scalar curvature of (M, g). Writing $\overline{g} = u^{\frac{4}{n-2}} g$ for a positive function u on M, the functional (5.1) becomes

$$\mathcal{Y}_g(u) = \frac{\int_M \left\{ \frac{4(n-1)}{n-2} |\nabla u|^2 + S(g) u^2 \right\} d\mu}{\left(\int_M u^{2n/(n-2)} d\mu \right)^{\frac{n-2}{n}}}. \tag{5.2}$$

The resolution of the above-mentioned Yamabe problem by Trudinger, Aubin and Schoen guarantees that the infimum in (5.2) over all $u > 0$ exists and, furthermore, is achieved by a metric of constant scalar curvature. We therefore define the *Yamabe invariant* by

$$Q_g(M) := \inf_{u>0} \mathcal{Y}_g(u), \tag{5.3}$$

which depends only on M and the conformal class of g. It was observed by Aubin [Aub76a] that the Yamabe invariant has the following fundamental property

Proposition 5.1. *For every compact Riemannian manifold* (M, g) *we have*

$$Q_g(M) \leq Q(\mathbb{S}^n),$$

[2] Yamabe claimed to have proved affirmatively this conjecture. His proof, however, contained an error discovered eight years later by Trudinger [Tru68]. Trudinger was able to repair Yamabe's original proof under additional assumptions on the manifold. For more details see [LP87] and [Aub76b].

[3] A manifold is called "locally conformally flat" if its Weyl tensor vanishes identically.

where $\mathbb{S}^n$ is the round n-sphere.

Proof. For the proof we refer to the Appendix. q.e.d.

Therefore the supremum in (5.3) over all conformal classes exists and is bounded above by $Q(\mathbb{S}^n)$. This led Schoen [Sch89] and Kobayashi [Kob87] to introduce a new differential-topological invariant defined by the following min-max procedure

Definition 5. *For every compact Riemannian manifold (M,g) we define the σ-* constant *of M as*

$$\begin{aligned}\sigma(M) :&= \sup_{[g]\in\mathcal{C}}\{\inf_{[g]}\mathcal{Y}(g)\}\\ &= \sup_{[g]\in\mathcal{C}} Q_g(M),\end{aligned} \tag{5.4}$$

where $\mathcal{C}$ is the space of conformal classes on M.

Proposition 5.2 ([Sch89])**.** *For any closed n-dimensional Riemannian manifold M we have $\sigma(M) \leq \sigma(\mathbb{S}^n)$, where $\sigma(\mathbb{S}^n) = n(n-1)\,Vol(\mathbb{S}^n)^{2/n}$.*

Proof. The proof follows immediately from Proposition 5.1 and Definition 5. q.e.d.

By the Gauss-Bonnet theorem, a compact surface Σ has genus $\gamma \geq 1$ if and only if it does not admit a metric of positive curvature. This property is perhaps the most important property the σ-constant shares with the Euler characteristic of a closed surface.

Proposition 5.3. ([Sch89], Lemma 1.2) *Let M be a smooth, closed, n-dimensional manifold. Then $\sigma(M) \leq 0$ if and only if M does not admit a metric of positive scalar curvature.*

Proof. For the proof we refer to the Appendix. q.e.d.

Therefore, in some sense, the σ-constant can be viewed as a generalisation of the Euler characteristic to higher dimensions. However, as we will see in the following, it is a much weaker invariant.

In the end of this section let us make a few remarks about the known values for the σ-constant.

5.1.2 Values for the σ-Constant

For a two dimensional manifold M of curvature -1, 0 or 1, the values of the σ-constant (i.e. twice the Euler characteristic) are given by the Gauss-Bonnet theorem $8\pi,\ 0,\ -8\pi,\ -16\pi,\ ...$ and hence are completely determined by the genus of M.

In higher dimensions the σ-constant is too weak an invariant to capture the entire topological richness the manifold M might have. This is quite clearly illustrated by the early result of Schoen [Sch89] who showed that the σ-constant seems to be insensitive to one-dimensional "fibers" in M, namely that

$$\sigma(\mathbb{S}^{n-1} \times \mathbb{S}^1) = \sigma(\mathbb{S}^n), \qquad n \geq 3,$$

where the σ-constant on the $\mathbb{S}^n$ is achieved by the round metric and therefore $\sigma(\mathbb{S}^n) = n(n-1)\mathrm{Vol}(\mathbb{S}^n(1))^{2/n}$.

If M is a closed 3-manifold then, by the resolution of the Geometrisation Conjecture, M can be written as a connected sum of irreducible 3-manifolds:

$$M = \#_{i=1}^{p}(H_i \cup G_i)\#_{j=0}^{m} L_j \#_{k=0}^{l}(\mathbb{S}^2 \times \mathbb{S}^1),$$

where H_i are hyperbolic 3-manifolds of finite volume, G_i are graph manifolds, L_j are closed manifold with finite fundamental group and the union $H_i \cup G_i$ is along embedded, incompressible 2-tori. It was shown by Anderson [And06] that Perelman's work on the Geometrisation Conjecture implies that, if $\sigma(M) \leq 0$, the σ-constant of M is realized by the volume of the hyperbolic part of M. Namely we have that

$$|\sigma(M)| = 6\Big(\sum_{i=1}^{p} \mathrm{Vol}(H_i)\Big)^{2/3},$$

which shows that the G-factors, the L-factors and the $\mathbb{S}^2 \times \mathbb{S}^1$-factors are all invisible to the σ-constant. See also [And97], [And99] and [And01].

Returning to the case of non-negative σ-constant, it follows from the work of Schoen and Yau [SY85] and Gromov and Lawson [GJ80] that the n-dimensional torus has

$$\sigma(\mathbb{T}^n) = 0.$$

See Remark 5.7 below.

By exploring inverse mean curvature techniques, Bray and Neves calculated the

σ-constant of the real projective 3-dimensional space and showed in [BN04] that

$$\begin{aligned}\sigma(\mathbb{RP}^3) &= \sigma(\mathbb{RP}^2 \times \mathbb{S}^1)\\ &= \sigma(\mathbb{RP}^3 \# (\mathbb{RP}^2 \times \mathbb{S}^1))\\ &= 4^{-1/3}\sigma(\mathbb{S}^3).\end{aligned}$$

This result was further generalised by Akutagawa and Neves who showed in [KA07] that

$$\sigma\big(\#_k(\mathbb{RP}^3)\#_l(\mathbb{RP}^2 \times \mathbb{S}^1)\#_m(\mathbb{S}^2 \times \mathbb{S}^1)\#_n(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1)\big) = \sigma(\mathbb{RP}^3),$$

if $k + l \geq 1$ and where $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ denotes the non-orientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$.

In four dimensions, LeBrun showed in [LeB97, Theorem 6] that the σ-constant of the complex projective plane is given by

$$\sigma(\mathbb{CP}^2) = 12\sqrt{2}\pi,$$

and, furthermore, that the σ-constant is realised by the Fubini-Study metric. A year later, in a joint work with Gursky [GL98], he extended this result and showed that

$$\sigma(\mathbb{CP}^2\#_m(\mathbb{S}^3 \times \mathbb{S}^1)) = \sigma(\mathbb{CP}^2), \ m \geq 0.$$

Remark 5.4. Notice that in all these previous results the $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^3 \times \mathbb{S}^1$ parts, when present, are invisible to the σ-constant. □

Finally, let us mention a surprising discovery due to Petean and LeBrun concerning the sign of the σ-constant of a n-dimensional manifolds. In [Pet00] Petean showed that all closed, simply connected n-dimensional manifolds M have $\sigma(\Sigma) \geq 0$ for $n \geq 5$. In contrast with this result, LeBrun [LeB03] discovered closed, 4-dimensional manifolds with $\sigma(M) < 0$ which have a finite cover $\tilde{M}$ with $\sigma(\tilde{M}) > 0$. Such an example is given by the manifold $M := X\#N$ where X is any non-spin, compact, complex surface (for e.g. a K3 surface) and $N := (\mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is given by the map $(x, y) \to (-x, -y)$. See also [LeB96] and [Pet98] in connection with this result.

5.2 Area Bounds for Stable Minimal Hypersurfaces and Infinitesimal Splitting of the Ambient Manifold

The proof of the area inequalities (4.3) for stable minimal surfaces relies on the Gauss-Bonnet theorem. For this reason these bounds, given in terms of the Euler characteristic, do not directly extend to higher dimensions. Therefore, if one wants to find possible area bounds for stable minimal hypersurfaces in manifolds of scalar curvature bounded below, then one needs to look at various generalisations for the Euler characteristic. In the light of the previous section, a good candidate is the σ-constant. Based on the crucial properties that the σ-constant shares with the Euler characteristic of surfaces, one might then surmise that the area bounds (4.3) might actually be generalised to stable minimal hypersurfaces.

At a first glance this approach seems not very promising since the inequality $A(\Sigma) \leq 4\pi$ cannot be generalised in this way. That is to say, the area of a closed, stable, minimal hypersurface is not necessary bounded above in terms of its σ-constant and a positive lower bound on the scalar curvature of the ambient manifold. This is illustrated by the following example.

Example 5.5. Let $\Sigma := \mathbb{S}^{n-2} \times \mathbb{S}^1(\ell)$, where $\mathbb{S}^{n-2}$ is the $(n-2)$-dimensional unit sphere and $\mathbb{S}^1(\ell)$ is the circle of radius ℓ. Let $M := \Sigma \times \mathbb{S}^1$ with the product metric. Then $S^M \equiv (n-2)(n-3) := S_0 > 0$ and Σ is a stable minimal hypersurface in M. As we already mentioned above, $\sigma(\Sigma) = \sigma(\mathbb{S}^{n-1})$ and moreover, $\sigma(\Sigma)$ is independent of both S_0 and ℓ. Therefore, by letting $\ell \to \infty$, the area of Σ becomes arbitrarily larger than $\sigma(\Sigma)$. $\square$

While, due to this example, there is no hope to bound the area of a stable minimal hypersurface in manifolds of scalar curvature bounded below by a positive constant, we can still investigate the case of a non-positive lower bound. In this direction we have a more promising perspective.

In [CG00a] Cai and Galloway showed that if Σ is a compact, stable, 2-sided, minimal hypersurface with $\sigma(\Sigma) < 0$ in a complete n-manifold M of scalar curvature S^M bounded below by a *negative* constant S_0, then the area of Σ satisfies

$$A(\Sigma)^{\frac{2}{n-1}} \geq \frac{\sigma(\Sigma)}{S_0}, \tag{5.5}$$

where the right-hand side is positive since, by assumption, both S_0 and $\sigma(\Sigma)$ are negative.

Remark 5.6. Notice that when $n = 3$ and $S_0 = -2$ in (5.5) we recover the higher genus case of inequality (4.3) since, in this case, $\sigma(\Sigma) = 4\pi\chi(\Sigma)$. Furthermore, the area bound (5.5) also generalises the Gibbons-Woolgar result (4.4). (See [CG00a] and [GM08] for further details.) □

Remark 5.7. By Schoen-Yau [SY85] and Gromov-Lawson [GJ80] the n-dimensional torus $\mathbb{T}^n$ admits no metrics of positive scalar curvature and therefore $\sigma(\mathbb{T}^n) \leq 0$. Furthermore, any scalar flat metric on $\mathbb{T}^n$ is flat. Therefore we actually have

$$\sigma(\mathbb{T}^n) = 0, \ n \geq 2.$$

From this follows that the case $\sigma(\Sigma) = 0$ resembles the genus one case for surfaces in three manifolds of non-negative scalar curvature and $(n-1)$-dimensional flat tori in n-dimensional flat tori show that no area bound are possible for stable minimal hypersurfaces with $\sigma(\Sigma) = 0$. □

Nevertheless, for non-negative scalar curvature, we have the following infinitesimal splitting result by Schoen and Yau which, by the previous remark, generalises the torus case of Proposition 4.4.

Theorem 5.8 ([SY85])**.** *Let M be a smooth, complete n-dimensional manifold with $S \geq 0$. If Σ is a closed, 2-sided, stable, minimal hypersurface in M with $\sigma(\Sigma) \leq 0$ then Σ is totally geodesic and the normal Ricci curvature of M vanish along Σ. i.e. M splits infinitesimally along Σ. Furthermore, the scalar curvature of M vanishes at every point of Σ.*

Remark 5.9. If Σ is the hypersurface from the previous theorem then, by the Gauss equation, its scalar curvature vanishes identically. Since, by assumption, $\sigma(\Sigma) \leq 0$, we conclude that actually $\sigma(\Sigma) = 0$.

Proof. The proof of Theorem 5.8 is included in the Appendix. q.e.d.

In the light of this result it is natural to investigate if equality in (5.5) also corresponds to an infinitesimal splitting of the ambient manifold, as in the higher genus case of (4.3) and Proposition 4.4. It turns out that this is indeed the case. The following theorem extends the Schoen-Yau Theorem 5.8 to manifolds with negative lower bounds on the scalar curvature.

Theorem 5.10. *Let M be a n-manifold with scalar curvature $S^M \geq S_0$, where $S_0 < 0$. Let Σ be a closed, two-sided, stable, minimal hypersurface with $\sigma(\Sigma) < 0$. Then the area of Σ satisfies inequality* (5.5) *and if equality is attained then Σ is* *totally geodesic and the normal Ricci curvature of M vanishes along Σ, i.e. M splits infinitesimally along Σ. Furthermore, the scalar curvature S^M of M equals S_0 at every point of Σ and Σ is an Einstein manifold.*

Remark 5.11. It follows immediately, by the Gauss equation or by the definition of Einstein metrics, that Σ has constant scalar curvature S_0. When $n = 3$ the condition of Σ being Einstein is equivalent with Σ having constant Gaussian curvature, as in Proposition 4.4. □

Proof of Theorem 5.10. As in [CG00a], we want to relate the Yamabe invariant (5.2) with the second variation formula formula (2.8). The hypersurface Σ is minimal, stable and therefore

$$0 \leq \int_\Sigma \Big\{|\nabla f|^2 - (Ric^M(\nu,\nu) + |B|^2)f^2\Big\}d\mu \tag{5.6}$$

and by Gauss equation (2.2):

$$\leq \int_\Sigma \Big\{2|\nabla f|^2 + (S^\Sigma - S^M - |B|^2)f^2\Big\}d\mu \tag{5.7}$$

$$\leq \int_\Sigma \Big\{2|\nabla f|^2 + (S^\Sigma - S^M)f^2\Big\}d\mu \tag{5.8}$$

and since $2 < \dfrac{4(n-2)}{n-3}$ for all $n \geq 4$ (5.9)

$$\leq \int_\Sigma \Big\{\frac{4(n-2)}{n-3}|\nabla f|^2 + S^\Sigma f^2\Big\}d\mu - \int_\Sigma S^M f^2 d\mu \tag{5.10}$$

by assumption $S^M \geq S_0$. Hence

$$\leq \int_\Sigma \Big\{\frac{4(n-2)}{n-3}|\nabla f|^2 + S^\Sigma f^2\Big\}d\mu - S_0\int_\Sigma f^2 d\mu \tag{5.11}$$

since $-S_0 > 0$, we apply the Hölder inequality to the last integral

$$\begin{aligned}&\leq \int_\Sigma \Big\{\frac{4(n-2)}{n-3}|\nabla f|^2 + S^\Sigma f^2\}d\mu - S_0\Big(\int_\Sigma d\mu\Big)^{\frac{2}{n-1}}\Big(\int_\Sigma f^{\frac{2(n-1)}{n-3}}d\mu\Big)^{\frac{n-3}{n-1}}\\ &= \int_\Sigma \Big\{\frac{4(n-2)}{n-3}|\nabla f|^2 + S^\Sigma f^2\}d\mu - S_0 A(\Sigma)^{\frac{2}{n-1}}\Big(\int_\Sigma f^{\frac{2(n-1)}{n-3}}d\mu\Big)^{\frac{n-3}{n-1}}.\end{aligned} \tag{5.12}$$

Dividing the last inequality by $\Big(\int_\Sigma f^{\frac{2(n-1)}{n-3}}d\mu\Big)^{\frac{n-3}{n-1}} > 0$ we have

$$S_0 A(\Sigma)^{\frac{2}{n-1}} \leq \frac{\int_\Sigma \Big\{\frac{4(n-2)}{n-3}|\nabla f|^2 + S^\Sigma f^2\}d\mu}{\Big(\int_\Sigma f^{\frac{2(n-1)}{n-3}}d\mu\Big)^{\frac{n-3}{n-1}}} \tag{5.13}$$

Since (5.13) holds for all $f \in \mathcal{C}^\infty(\Sigma)$, in particular, it holds for some $f > 0$ for which the infimum in the term on the right is achieved. Therefore from (5.13) and by (5.3) we have

$$S_0 A(\Sigma)^{\frac{2}{n-1}} \leq Q_g(f) \leq \sup_{[g]} Q_g(f) = \sigma(\Sigma), \tag{5.14}$$

where the last equality follows from the definition of the σ-constant 5.4. Therefore dividing (5.14) by $S_0 < 0$ we obtain the area bound (5.5).

If equality in (5.5) is attained then all inequalities in (5.14) become equalities. Therefore all inequalities (5.6) - (5.13) become also equalities. From equality between (5.7) and (5.8) it follows that Σ is totally geodesic. Next, since (5.9) is a strict inequality, it follows from equality between (5.8) and (5.10) that $|\nabla f|^2 = 0$ and hence that f is constant [4]. From equality between (5.10) and (5.11) we have that

$$\int_\Sigma (S^M - S_0) d\mu = 0$$

and since, by assumption $S^M - S_0 \geq 0$, it follows that $S^M = S_0$ along Σ. Finally, from equality in (5.6) we have that

$$\int_\Sigma (Ric^M(\nu,\nu) + |B|^2) d\mu = 0.$$

By the same argument used in the proof of Proposition 4.4, the constant functions lie in the kernel of the Jacobi operator $\Delta_\Sigma + Ric^M(\nu,\nu) + |B|^2$. Hence $Ric^M(\nu,\nu) + |B|^2 = 0$ and therefore, since Σ is totally geodesic, $Ric^M(\nu,\nu) \equiv 0$.

We next show that if Σ attains equality in (5.5) then Σ is Einstein. From equality in (5.14) it follows that $Q_g(f) = \sigma(\Sigma)$ for some $f > 0$. The result then follows from [Sch89, p. 126]. We include the argument for completeness. Since $\sigma(\Sigma) < 0$, by definition (5.4) of the σ-constant, $Q_g(f) < 0$. In this case there exist a unique function $f > 0$ such that $f^{2/(n-2)} g$ has constant negative scalar curvature equal to $Q_g(f)$. The existence follows from the resolution of the Yamabe problem [5] while the uniqueness from the maximum principle.

Let h be any trace-free (0,2) tensor on Σ and for some $\delta > 0$, let $g_s := g + sh,\ s \in (-\delta, \delta)$. Then, for all $|s| < \delta$, there exists a unique function $f_s > 0$ such that $f_s^{2/(n-2)} g_s$ has constant scalar curvature $Q_g(f_s) < 0$. The σ-constant is independent of s and therefore, for all $|s| < \delta$ we have $Q_g(f_s) \leq \sigma(\Sigma)$. That is, $\sigma(\Sigma)$ is a

[4] That f is constant also follows since the Hölder inequality becomes an equality in (5.11) and (5.12).

[5] We don't actually require the full solution of the Yamabe problem but only the "easy" case when $Q_g(M) \leq 0$.

maximum for $Q_g(f_s)$ seen as a function of s. Therefore

$$\frac{d}{ds}Q_g(f_s)\Big|_{s=0} = 0. \tag{5.15}$$

On the other hand, by [Top06, Proposition 2.3.9], we have that

$$\frac{\partial}{\partial s}S(g_s)\Big|_{s=0} = -g(Ric^{\Sigma}(g), h) + \text{ divergence terms.}$$

and by [Top06, Proposition 2.3.12] that

$$\frac{\partial}{\partial s}d\mu\Big|_{s=0} = \frac{1}{2}(Tr_g h)d\mu$$

Therefore, by divergence theorem and by (5.15), we have

$$\begin{aligned} 0 &= \frac{d}{ds}Q_g(f_s)\Big|_{s=0} \\ &= \frac{d}{ds}\int_{\Sigma} S(g_s)d\mu\Big|_{s=0} \\ &= -\int_{\Sigma} g\Big(Ric^{\Sigma}(g) - \frac{1}{2}S(g)g, h\Big)d\mu, \end{aligned}$$

for any trace-free symmetric (0,2) tensor h. Therefore $Ric^{\Sigma}(g) - \frac{1}{2}S(g)g = 0$ and hence Σ is Einstein. q.e.d.

5.3 Rigidity of Area-Minimising Hypersurfaces and Local Splitting of the Ambient Manifold

We saw in Chapter 4 that the infinitesimal splitting result of Proposition 4.4 actually propagates to an entire neighbourhood of a stable minimal torus Σ as long as Σ is assumed to be *area-minimising* and not just stable. This is case (b) of Theorem 4.7. It turns out that this is also the case for the higher dimensional infinitesimal splitting theorem 5.8 of Schoen and Yau. We have the following local splitting theorem by Cai

Theorem 5.12 ([Cai02]). *Let M be a smooth, complete n-manifold with $S \geq 0$. If Σ is a closed, 2-sided, area-minimising hypersurface in M with $\sigma(\Sigma) \leq 0$ then M splits isometrically as a product in a neighbourhood of Σ.*

Sketch of the proof. Since Σ is area-minimising, in particular Σ is stable and therefore, by the Schoen-Yau theorem 5.8, Σ is totally geodesic and the normal Ricci curvature of M vanishes along Σ. It follows by Proposition 3.12 that there exists a constant mean curvature foliation in the neighbourhood of the area-minimising surface. Hence the mean curvature of each leaf satisfies the evolution equation (3.10)

$$H'(t) = -\Delta_t \rho_t - (Ric^M(\nu_t, \nu_t) + |B_t|^2)\rho_t \tag{5.16}$$

Let $\overline{S_t}^\Sigma$ be the scalar curvature of Σ_t with the conformally deformed metric $\overline{g_t} = \rho_t^{2/(n-1)} g_t$. Then a direct calculation shows that the scalar curvature of $\overline{g_t}$ is given by

$$\overline{S_t}^\Sigma = \rho_t^{\frac{-n}{n-2}} \Big(S_t^\Sigma \rho_t - 2\Delta_t \rho_t + \frac{n-1}{n-2} \frac{|\nabla \rho_t|^2}{\rho_t^2} \Big). \tag{5.17}$$

The next step in the proof is to show that Σ is not *strictly* area-minimising. The proof goes by contradiction. Assume that there exists a $t_0 \in (0, \varepsilon)$ such that $H(t_0) > 0$ and hence that $A'(t_0) = \int_\Sigma H(t_0)\rho_{t_0} d\mu_{t_0} > 0$. Then by (5.16) and (5.17) we have at $t = t_0$ that

$$\overline{S}^\Sigma = \rho^{\frac{-2}{n-2}} \Big(2\rho^{-1} H'(t_0) + S^M + |B|^2 + H^2 + \frac{n-1}{n-2} \frac{|\nabla \rho|^2}{\rho^2} \Big).$$

Since, by assumption, $S^M \geq 0$ and $H'(t_0) > 0$, we conclude that $\overline{S}^\Sigma > 0$ and hence Σ admits a metric of positive scalar curvature which, by Proposition 5.3, is a contradiction since $\sigma(\Sigma) \leq 0$. Hence, since $H(0) = 0$, $H(t) \leq 0$ for all $t \in [0, \varepsilon)$ and the result follows by the first variation of area formula and since Σ is area-minimising. q.e.d.

Cai's proof of Theorem 5.12 establishes in particular the following generalisation of the area comparison theorem 3.8 in the case of manifolds of non-negative scalar curvature.

Theorem 5.13. *Let M be a complete n-manifold with scalar curvature $S \geq 0$. Let $\Sigma \subset M$ be an immersed, 2-sided, closed, hypersurface such that*

(i) Σ is totally geodesic,

(ii) the normal Ricci curvature of M vanishes all along Σ,

(iii) $S^M = 0$ at every point of Σ and

(iv) $\sigma(\Sigma) \leq 0$.

Let $\{\Sigma_t\}$, $t \in (-\varepsilon, \varepsilon)$, be a constant mean curvature foliation [6] *in a neighbourhood*

[6] The existence of such a foliation, as in Theorem 3.8, is guaranteed by Proposition 3.12.

of Σ and denote by $A(t)$ the area of Σ_t. Then there exists $0 < \delta < \varepsilon$ such that

$$\text{for } |t| < \delta, \quad A(t) \leq A(0).$$

Remark 5.14. Assumption (i)-(iii) are the same as in the area comparison theorem 3.8 for 3-manifolds. Concerning assumption (iv) is analogous with assuming genus one in the case of $n = 3$. Indeed, the case $S \geq 0$ of Theorem 3.8 was referring to 2-tori satisfying the same assumptions (i)-(iii). By the Gauss-Bonnet theorem the 2-tori have zero Euler characteristic or equivalently, by Remark 5.6, zero σ-constant. Therefore in the case $S_0 = 0$ of Theorem 3.8 condition (iv) was implicit by assuming the genus of Σ to be one. □

Remark 5.15. Furthermore notice that assumption (iv) of Theorem 5.13 can not be removed. This is illustrated by case (c) of Proposition 3.14 for $n = 5$ where $\Sigma := \mathscr{S} \times \mathbb{S}^2$ and $\mathscr{S}$ is a closed hyperbolic surface of genus $\gamma \geq 2$. By Proposition 3.14 there is a metric on $M := \Sigma \times (-\varepsilon, \varepsilon)$ such that $\Sigma \times \{0\}$ is strictly area-minimising and, furthermore, satisfies the properties (i)-(iii). However Σ does not satisfy condition (iv). Indeed, by putting on Σ the metric $ds^2 := ds_1^2 + \varepsilon ds_2^2$, where ds_1 is hyperbolic and ds_2 is round, we see that ds has positive scalar curvature for sufficiently small $\varepsilon > 0$ and therefore, by Proposition 5.3, we have $\sigma(\Sigma) > 0$. □

In the light of the local splitting theorem 5.12, one might surmise that the infinitesimal splitting of Theorem 5.10 might also be extended to a local result if, additionally, one assumes Σ to be area-minimising. While this still remains an open problem, our current research suggests that this might actually be the case. We are led to the following conjecture:

Conjecture 5.16. *Let M be a smooth, complete n-manifold with $S \geq S_0$, where $S_0 < 0$, and let Σ be a closed, 2-sided, area minimising hypersurface in M with $\sigma(\Sigma) < 0$. Then Σ satisfies*

$$|S_0| A(\Sigma)^{2/(n-1)} \geq |\sigma(\Sigma)| \tag{5.18}$$

and if equality is attained then M splits isometrically as a product in a neighbourhood of Σ.

Remark 5.17. Since the area bound 5.18 generalizes the area bound 4.3, Conjecture 5.16, if true, can be seen as a natural generalisation of the splitting theorem 4.7 (c). □

Remark 5.18. Notice that in the examples we have constructed in Section 3.3, all the underlying manifolds of Σ either don't have $\sigma(\Sigma) < 0$ or do not admit Einstein metrics, or both, as in the case of $\mathscr{S} \times \mathbb{S}^2$ and $\mathbb{T}^2 \times \mathbb{S}^2$. That $\mathscr{S} \times \mathbb{S}^2$ and $\mathbb{T}^2 \times \mathbb{S}^2$ don't admit Einstein metrics follows from an theorem due to Berger which states that 4-manifolds admitting Einstein metrics must have positive Euler characteristic [Bes87, 6.32]. $\square$

Appendix A

Collected Proofs

1 Proof of Proposition 3.12

Let $\alpha \in (0,1)$ and $\delta > 0$ small. We define the Banach spaces $X := \{u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u d\mu = 0\}$ and $Y = \{u \in C^{0,\alpha}(\Sigma) : \int_\Sigma u d\mu = 0\}$. Let $u \in X$ be such that $\|u\|_{C^{2,\alpha}} < \delta$ and denote by $\Sigma_u := \{\exp_x u(x)\nu(x) : x \in \Sigma\}$. Finally, let $H(u)$ be the mean curvature of Σ_u.

For some small $\varepsilon > 0$ consider the map $\Phi : X \times (-\varepsilon, \varepsilon) \to Y$ defined by

$$\Phi(u,t) := H(u+t) - \frac{1}{A(\Sigma)} \int_\Sigma H(u+t) d\mu.$$

Since, by assumption, Σ is minimal we have that $\Phi(0,0) = 0$.

We next calculate the linearisation of Φ at $(0,0)$. For some $v \in X$ we have

$$\begin{aligned} D\Phi(0,0) \cdot v &= \frac{d\Phi}{ds}(t, sv)\Big|_{(0,0)} \\ &= \frac{d}{ds}\Big(H(sv) - \frac{1}{A(\Sigma)} \int_\Sigma H(sv) d\mu \Big)\Big|_{s=0} \\ &= -\Delta_\Sigma v - (Ric(\nu,\nu) + |B|^2) v \\ &+ \frac{1}{A(\Sigma)} \int_\Sigma \Big(Ric(\nu,\nu) + |B|^2 \Big) v d\mu. \end{aligned} \tag{A.1}$$

By assumption, the constant functions are Jacobi fields on Σ and therefore they are contained in the kernel of the Jacobi operator $L_\Sigma = \Delta_\Sigma + Ric(\nu,\nu) + |B|^2$. Hence $Ric(\nu,\nu) + |B|^2 \equiv 0$ and from (A.1) we have

$$D\Phi(0,0) \cdot v = -\Delta_\Sigma v.$$

The operator $\Delta_\Sigma : X \to Y$ is a linear isomorphism and therefore, by the implicit function theorem, there exists a $0 < \varepsilon_1 < \varepsilon$ and $u(t) = u(\cdot, t) \in X$ such that, for all

$t \in (-\varepsilon_1, \varepsilon_1)$,

$$u(0) = 0 \text{ and } \Phi(u(t), t) = 0.$$

Hence, for $w(x,t) := t + u(x,t)$, the hypersurfaces $\Sigma_w = \Sigma_t$ have constant mean curvature for all $t \in (-\varepsilon_1, \varepsilon_1)$. This completes the proof of Proposition 3.12. q.e.d.

2 Proof of Proposition 5.1

In [Aub76a, Ch.2.14], Aubin showed that the optimal constant in the Sobolev inequality in $\mathbb{R}^n$ is given by $aQ(\mathbb{S}^n)^{-1}$, where $a := 4\frac{n-1}{n-2}$ and $Q(\mathbb{S}^n)$ is the Yamabe invariant defined by (5.3) and, moreover, this constant is attained by the family of functions

$$u_\alpha(x) := \Big(\frac{|x|^2 + \alpha^2}{\alpha}\Big)^{\frac{2-n}{2}}, \qquad \alpha \in \mathbb{R}. \tag{A.2}$$

Denoting by $r := |x|$ where $x \in \mathbb{R}^n$, we have

$$\partial_r u_\alpha = (2-n) r \alpha^{-1} \Big(\frac{|x|^2 + \alpha^2}{\alpha}\Big)^{-\frac{n}{2}}. \tag{A.3}$$

We therefore have

$$\begin{cases} u_\alpha \leq \alpha^{\frac{n-2}{2}} r^{2-n} \text{ and} \\ |\partial_r u_\alpha| \leq (n-2)\alpha^{\frac{n-2}{2}} r^{1-n}. \end{cases} \tag{A.4}$$

For $\varepsilon > 0$ let $B(\varepsilon)$ be the ball of radius ε in $\mathbb{R}^n$ centered at the origin and let $\eta \geq 0$ be a radial cut-off function in $B(2\varepsilon)$ such that

$$\begin{cases} \eta \equiv 1 & \text{in } B(\varepsilon), \\ \eta \leq 1 & \text{in } A_\varepsilon := B(2\varepsilon)\backslash B(\varepsilon) \text{ and} \\ \eta \equiv 0 & \text{outside of } B(2\varepsilon). \end{cases}$$

Let $\phi := \eta u_\alpha$ and let $C > 0$ be a constant. (In the following we will not keep track of constants and all of them will be denoted by C.) Then we have

$$\begin{aligned}
\int_{\mathbb{R}^n} a|\nabla\phi|^2 dx &= \int_{\mathbb{R}^n} \Big(a\eta^2|\nabla u_\alpha|^2 + 2a\eta u_\alpha\langle\nabla\eta, \nabla u_\alpha\rangle + au_\alpha^2|\nabla\eta|^2\Big)dx \\
&= \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + \int_{A_\varepsilon} a\Big(2\eta u_\alpha\langle\eta', \partial_r u_\alpha\rangle + u_\alpha^2|\eta'|^2\Big)dx \\
&\le \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + C\int_{A_\varepsilon} \Big(u_\alpha|\partial_r u_\alpha| + u_\alpha^2\Big)dx \\
&\text{and by (A.4) we have} \\
&\le \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + C\int_{A_\varepsilon} (n-2)\alpha^{n-2}r^{3-2n}dx \\
&= \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + \mathcal{O}(\alpha^{n-2}). \qquad \text{(A.5)}
\end{aligned}$$

Since the family of functions u_α attains equality in the Sobolev inequality with the optimal constant mentioned above, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx &= Q(\mathbb{S}^n)\Big(\int_{B(\varepsilon)} u_\alpha^p dx + \int_{\mathbb{R}^n\setminus B(\varepsilon)} u_\alpha^p dx\Big)^{\frac{2}{p}} \\
&\le Q(\mathbb{S}^n)\Big(\int_{B(2\varepsilon)} \phi^p dx + \int_{\mathbb{R}^n\setminus B(\varepsilon)} \alpha^n r^{-2n} dx\Big)^{\frac{2}{p}} \\
&\le Q(\mathbb{S}^n)\Big(\int_{B(2\varepsilon)} \phi^p dx\Big)^{\frac{2}{p}} + \mathcal{O}(\alpha^n) \\
&= Q(\mathbb{S}^n)\|\phi\|_{L^p}^2 + \mathcal{O}(\alpha^n). \qquad \text{(A.6)}
\end{aligned}$$

Therefore from (A.5) and (A.6) we have that the Einstein-Hilbert action of ϕ on $\mathbb{R}^n$ satisfies

$$\mathcal{Y}(\phi) \le Q(\mathbb{S}^n) + C\alpha^{n-2}. \qquad \text{(A.7)}$$

Let $x \in M$ and let $\{x^i\}$ be normal coordinates in a neighbourhood of x. Since in this coordinate system $g^{rr} = 1$ we see that $|\nabla\phi|^2 = |\partial_r\phi|^2$ as in the Euclidean case discussed above. Furthermore, the volume form of M in a neighbourhood of x satisfies $d\mu = (1 + \mathcal{O}(r))dx \le (1 + C\varepsilon)dx$, for some constant $C > 0$ and some $\varepsilon > 0$.

Therefore, from the previous calculation we have

$$\begin{aligned}
&\int_M \Big(a|\nabla\phi|^2 + S\phi^2\Big)d\mu \\
&\qquad\qquad \leq (1+C\varepsilon)\int_M \Big(a|\nabla\phi|^2 + S\phi^2\Big)dx \\
&\qquad\qquad \leq (1+C\varepsilon)\Big(Q(\mathcal{S}^n)\|\phi\|^2_{L^p} + C\alpha^{n-2} + \int_M S\phi^2 dx\Big) \\
&\qquad\qquad \leq (1+C\varepsilon)\Big(Q(\mathcal{S}^n)\|\phi\|^2_{L^p} + C\alpha^{n-2} + C\int_{\mathbb{S}^{n-1}(\varepsilon)}\int_0^{2\varepsilon} u_\alpha^2 r^{n-1} dr d\omega\Big)
\end{aligned} \tag{A.8}$$

Using the substitution $\beta := r\alpha^{-1}$ a direct calculation show that

$$\int_0^{2\varepsilon} u_\alpha^2 r^{n-1} dr \leq C\alpha,$$

for some constant $C > 0$. Therefore, from (A.8) we have

$$\frac{\int_M \Big(a|\nabla\phi|^2 + S\phi^2\Big)d\mu}{\|\phi\|^2_{L^p}} \leq (1+C\varepsilon)\Big(Q(\mathcal{S}^n) + C\alpha\Big).$$

By letting α and ε go to zero and by using the definition of the Yamabe invariant of M the result follows. q.e.d.

3 Proof of Proposition 5.3

Before we prove the proposition let us first make the following remark due to Yamabe.

Remark A.1. Let (M, g) be a closed Riemannian n-manifold, $n \geq 3$. If $u > 0$ is a smooth function on M then the scalar curvature $\overline{S}$ of the metric $\overline{g} := u^{4/(n-2)}g$ is given by

$$\begin{aligned}
\overline{S} &= -u^{\frac{n+2}{n-2}}\Big(\frac{4(n-1)}{n-2}\Delta_g u - Su\Big) \\
:&= -u^{\frac{n+2}{n-2}}\mathcal{L}u,
\end{aligned} \tag{A.9}$$

where the operator $\mathcal{L}$ is called the *conformal Laplacian.*

Therefore finding a metric $\overline{g}$ on M, conformal to g and having constant scalar curvature S_0 (i.e. a solution of the Yamabe problem), reduces to the solvability of the following elliptic equation

$$\frac{4(n-1)}{n-2}\Delta_g u - Su + S_0 u^{\frac{n+2}{n-2}} = 0.$$

□

Proof of Proposition 5.3. If $\sigma(M) > 0$ then there exists a unit volume metric $\overline{g}$ such that $Q(\overline{g}) > 0$. Hence the first eigenvalue of the conformal Laplacian $\mathcal{L}$ must be negative and therefore the lowest eigenfunction u_0 satisfies $\mathcal{L}u_0 < 0$. We conclude that the metric $\overline{g} := u_0^{4/(n-2)} g$ has scalar curvature $\overline{S} > 0$.

Conversely, if g has scalar curvature $S > 0$ then $Q(g) > 0$ and hence, by the definition of the σ-constant (5.4), $\sigma(M) > 0$. q.e.d.

4 Proof of Theorem 5.8

Since Σ is stable and minimal we have from (2.10) for $f := \rho$ that

$$0 \le \int_\Sigma \Big\{|\nabla f|^2 - (Ric^M(\nu,\nu) + |B|^2)f^2\Big\}d\mu$$

and by Gauss equation (2.2)

$$= \int_\Sigma \Big\{|\nabla f|^2 + \frac{1}{2}(S^\Sigma - S^M - |B|^2)f^2\Big\}d\mu \tag{A.10}$$

and since, by assumption, $-S^M \le 0$ the last inequality is

$$\le \int_\Sigma \Big(|\nabla f|^2 + \frac{1}{2}S^\Sigma f^2\Big)d\mu. \tag{A.11}$$

As in the proof of Proposition 5.3, consider the conformal Laplacian on Σ

$$\mathcal{L} := \Delta_\Sigma - \frac{n-3}{4(n-2)}S^\Sigma. \tag{A.12}$$

Claim 1: All the eigenvalues of $\mathcal{L}$ are non-negative.

Proof of Claim 1. Suppose not and let f be a non-zero solution of the following equation

$$\mathcal{L}f = -\lambda f, \quad \text{where } \lambda < 0.$$

Multiplying this equation with $\frac{4(n-2)}{n-3}f$, using (A.12), (A.11) and integration by parts we have

$$\begin{aligned}\frac{2(n-1)}{n-3}\int_\Sigma |\nabla f|^2 d\mu &= -\frac{1}{2}\int_\Sigma S^\Sigma f^2 d\mu + \frac{2\lambda(n-2)}{n-3}\int_\Sigma f^2 d\mu \\ &< -\frac{1}{2}\int_\Sigma S^\Sigma f^2 d\mu \qquad \text{(A.13)} \\ &\leq \int_\Sigma |\nabla f|^2 d\mu.\end{aligned}$$

Hence $\frac{2(n-2)}{n-3} < 1$ which is a contradiction and hence Claim 1 is proved. q.e.d.

Claim 2: Let λ_1 be the first eigenvalue of L. Then $\lambda_1 = 0$.

Proof of Claim 2. Suppose $\lambda_1 > 0$ and let u be the first eigenfunction of (A.12). Since the first eigenfunction of L does not change sign we may assume, without the loss of generality, that $u > 0$. Let $\overline{g} := u^{4/(n-3)}g$ be a new conformally deformed metric on Σ. Then the scalar curvature of $\overline{g}$ is given by

$$\begin{aligned}\overline{S}^\Sigma &= u^{-\frac{n+1}{n-3}}\Big(S^\Sigma u - \frac{4(n-2)}{n-3}\Delta_\Sigma u\Big) \\ &= \frac{4(n-2)}{n-3}u^{-\frac{n+1}{n-3}}\lambda_1 u \\ &> 0,\end{aligned}$$

where in the second equality we have used (A.12). Hence $\overline{S}^\Sigma > 0$ which implies that Σ admits a metric of positive scalar curvature. This is a contradiction since, by assumption, $\sigma(\Sigma) \leq 0$. This completes the proof of Claim 2. q.e.d.

Let f_1 be the first eigenfunction of L. Since $\lambda_1 = 0$ the inequality (A.13) becomes equality and therefore

$$\frac{2(n-1)}{n-3}\int_\Sigma |\nabla f_1|^2 d\mu \leq \int_\Sigma |\nabla f_1|^2 d\mu,$$

which implies that $|\nabla f_1| = 0$ and hence f is constant. Therefore (A.12) implies that $S^\Sigma = 0$. Inequality (A.10) now becomes

$$\int_\Sigma (S^M + |B|^2) d\mu \leq 0.$$

Since, by assumption, $S^M \geq 0$, it follows that Σ is totally geodesic and that $S^M = 0$ along Σ. Finally, from the Gauss equation (2.2) follows that $Ric^M(\nu,\nu) = 0$ along Σ. This completes the proof of Theorem 5.8. q.e.d.

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