One Essay on Time-Inconsistent Preferences and Competitive Equilibrium and Two Essays on Optimal Monetary Policy

by

Nikolaos Kokonas

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University of Warwick, Department of Economics

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Declaration

I declare that the material contained in this thesis has not been used or published before. This thesis is my own work and it has not been submitted for another degree or at another university.

Summary

The first part of the thesis investigates the characterization of asset prices and investor's behavior under time-inconsistent preferences. For the latter type of preferences, we assume myopia or hyperbolic-discounting (HD). We consider an infinite horizon economy under certainty with two heterogeneous CRRA individuals, one good and one long-lived asset. The question of survival in the market arises when individuals are HD maximizers or myopic with wrong expectations about equilibrium asset prices. We provide sufficient conditions such that more myopic individuals dominate over less myopic ones and also sophisticated HD maximizers with intertemporal elasticity of substitution (IES) equal to one, log-utilities, dominate over HD maximizers with IES higher than one. Thus, individuals that vanish in the long-run will not have an impact on asset prices. On the other hand, asset prices are characterized by extreme dynamics if the economy is populated by myopic individuals only, who have perfect foresight about equilibrium asset prices. We show that even though the dividends of the long-lived asset are constant over time, there exist asset price dynamics that resemble an ever-expanding asset price bubble.

The second part of the thesis investigates the characterization of optimal monetary policy under two different scenaria. In the first scenario we consider a two-period monetary economy with inside and outside money and an environment with fix prices and excess capacities in equilibrium. If unemployment is of a keynesian nature, a Friedman rule argument characterizes optimal monetary policy whereas if unemployment is of a more classical nature, high real wages, optimal policy requires positive nominal rates. In the second scenario we consider an economy with idiosyncratic risk and credit frictions. Monetary policy provides missing insurance due to credit frictions through the distribution of non-contingent seignorage transfer across states.

Part

Bounded Rationality, Time-Inconsistency and Financial Markets

- Chapter 1 -

Myopia, Time-Inconsistency and Financial Markets

§ 1.1 Introduction

Experimental studies have shown that individuals tend to be more impatient with respect to short-run decisions compared to decisions taken in the distant future. Laibson (1997) formalized the trade-off between short-run impatience vs long-run patience with discount functions that take an approximately hyperbolic form. This form of discounting implies that the preferences of individuals change over time. Individuals are time-inconsistent. To formalize the idea of time-inconsistency due to short-run impatience, we will consider two different forms of time-inconsistent preferences. The first form is hyperbolic discounting (HD) where individuals have discount functions that take an approximately hyperbolic form. The second form is myopic behavior. Individuals are myopic when they are able to plan only for finite periods in the future. Every time they wake up in a new period they realize another period in the future and recalculate their plan[1]. As a consequence, they behave in a time-inconsistent way[2].

The motivation that underlies this work is to characterize the competitive equilibrium of a simple financial economy when individuals are time-inconsistent and to argue that certain phenomena can be explained when individual's preferences are described by the previous two forms of time-inconsistency. In particular, we are going to analyze the following two

[1] We have adopted the convention that individuals realize one more period in the future every time they wake up in a new planning horizon. See Lovo and Polemarchakis (2010) and Spiro (2012)

[2] Myopia can be thought as an extreme form of hyperbolic discounting. Individuals care about n periods in the future and put zero weight in their intertemporal utility function after $n+1$ periods.

phenomena. The first relates to the idea of survival in the market when individual's preferences are described by the previous two forms of time-inconsistency. In the discussion that follows we will distinguish between non-survival and bankruptcy. The second phenomenon in question relates to fluctuations of asset prices that are not explained by fluctuations in the dividends of the assets. Before elaborating further on the importance of these two phenomena, it is necessary to contrast the approach followed here with the literature in behavioral finance. This literature argues that some financial phenomena can be better understood using models in which some agents are not fully rational[3]. Deviation from rationality means two things: Violation of expected utility and/or violation of bayesian updating. Time-inconsistent individuals do not violate the previous two tenets of individual rationality. Myopic or HD individuals *may* not have perfect foresight of equilibrium prices[4].

The aforementioned phenomena intersect at one point: the characterization of asset prices. Fluctuations of asset prices that are not explained by fluctuations in the dividends of the assets is a pattern well-known in the literature[5]. We argue that myopic behavior *could*[6] be an explanation as to why asset prices fluctuate more than dividends. On the other hand, the literature on survival has analyzed the long-run dynamics of economies populated by individuals with correct or incorrect beliefs about the true probability distribution of states of nature and different rates of impatience. There is no issue of time-inconsistency. The question of survival in the market translates into the question of whether all individuals have an impact on asset prices in the long-run. Asset prices reflect only the preference parameters of surviving individuals. Blume and Easley (1992) showed that, controlling for saving rates, individuals whose beliefs are closest to the truth and maximize the expected logarithm of next period output[7] (MEL) are going to be the most prosperous in the market compared to ones who do not follow MEL but have correct beliefs. Sandroni (2000) and Blume and Easley (2006) showed that in dynamically complete markets, the *only* determinants for survival is the accuracy of beliefs and the differences in discount factors. Attitudes towards risk are irrelevant. Lastly, Beker and Chattopadhyay (2010) take up the issue of survival in a dynamically incomplete market environment. An example with one log-guy and one guy with arbitrary CRRA demonstrates that even with homogenous beliefs and identical discount factors, the log-guy will not survive. The issue of survival depends critically on the

[3] Barberis and Thaler (2002)

[4] Perfect foresight of equilibrium prices is not incompatible with myopic or time-inconsistent (HD) behavior.

[5] The classic reference is Shiller (1981).

[6] The generality of the result is an issue we need to tackle.

[7] Individuals with log-utilities always follow the MEL rule.

assumption of dynamically complete/incomplete markets.

In this work we argue that myopia or HD can be an explanation for the two phenomena mentioned before. We show that under certain conditions, myopic or HD maximizers vanish in the long-run. For myopic individuals we consider the case of symmetric and asymmetric planning horizons. The latter case is more closely in spirit to the survival literature because different planning horizons translates into differences in rates of impatience. We provide sufficient conditions such that more myopic individuals dominate over less myopic ones. For the case of hyperbolic discounting we consider individuals with homogenous hyperbolic parameters, homogenous rates of impatience and different instantaneous utilities. We provide conditions such that even if individuals are sophisticated, they realize their time-inconsistency problem, individuals with intertemporal elasticity of substitution (IES) equal to one dominate over individuals with (IES) greater than one. This case is not entirely in the spirit of the survival literature because individuals differ only in instantaneous utilities. The more interesting case is to analyze economies populated by time-consistent and time-inconsistent (HD) individuals. Going back to the literature on survival, the latter case can be thought as the analog of an economy populated by individuals with correct (rational guys) and incorrect (irrational guys) beliefs. We do not pursue this interesting case in this paper. Nevertheless, there are two good reasons why we should focus *first* on the case where HD individuals differ only in instantaneous utilities. The first reason is that even in the case where HD individuals differ only in instantaneous utilities, long-run survival in the market is not guaranteed. We must analyze in detail this case if we want to have a better understanding of more complicated economies populated by time-consistent and time-inconsistent individuals. The second reason is related to the literature on time-inconsistency and competitive equilibrium. Herings and Rohde (2008) analyze an economy with heterogenous time-inconsistent individuals. They study existence of competitive equilibria but do not provide any characterization. Krusell et al. (2002), Luttmer and Mariotti (2003) characterize the competitive equilibrium of an economy with a *representative* individual under time-inconsistent preferences. The characterization of competitive equilibrium under time-inconsistent and heterogenous preferences has not been analyzed extensively.

To tackle the problem we focus on a simple framework. We consider an infinite horizon economy under certainty. There are two individuals with identical discount factors[8] and heterogenous CRRA preferences. There exist a long-lived asset with a dividend pattern that produces the aggregate output every period. *Importantly*, any fluctuations in the dividends of the

[8]Identical hyperbolic parameters in the case of HD.

asset are deterministic and follow a stationary pattern[9]. We will consider modifications to this set-up by introducing two long-lived assets.

Myopic individuals can be classified into two types depending on the definition of myopic equilibrium we choose to work. Myopic individuals who mispredict equilibrium asset prices in the future and myopic individuals who have perfect foresight about equilibrium asset prices. For economies populated by the former types, the question of long-run survival arises whereas for economies populated by the latter types, asset prices are characterized by extreme dynamics. In an economy with one long-lived asset, the only determinant for long-run survival is the saving behavior of individuals. We provide sufficient conditions such that for symmetric myopic individuals, the individual with IES close/or equal to one, log-utilities, will dominate over the individual with arbitrary CRRA preferences with IES not close/or equal to one. The sufficient condition requires enough fluctuations in the dividends of the asset which translates into enough variability in the saving rates. Myopic individuals with IES close/or equal to one dominate because their saving rates fluctuate less than the saving rates of myopic individuals with IES not close to one. Since individuals interchange roles in the asset market, they are buyers for some periods and sellers for other periods, low variability in the saving rate translates into higher savings intertemporally. The crux of the argument is that individuals reoptimize every period in the light of new information and the saving rate of one individual fluctuates less than the saving rate of the other individual. Interestingly enough, for economies populated by asymmetric myopic individuals, we find cases where the more myopic individual with IES sufficiently close to one dominate over the less myopic one with IES sufficiently far from one. The reason is similar to the symmetric case.

Modify the previous set-up by introducing another long-lived asset. In an economy with two long-lived assets, survival in the market depends on saving rates and investment behavior. We construct an interesting example in the spirit of Blume and Easley (1992). There are two extremely[10] myopic investors with identical log-utilities and homogenous discount factors. They have identical saving rules but follow different portfolio strategies. One of them invest only in the more productive asset every period and the other holds a more diversified portfolio. We show that the investor which invests only in the more productive asset will vanish in the long-run. The interesting part is that we require both myopia and portfolio rules. In the non-myopic

[9]This has to be contrasted with the literature in finance, Yan (2008), Branger et al. (2011) and the references therein, which assume that there is growth in the aggregate endowment of the economy. As a consequence attitudes towards risk are important determinants for survival under complete markets. Non-survival in the long-run is possible even when individuals have homogeneous beliefs and equal discount factors.

[10]They are able to plan only for one period in the future.

case (benchmark) the non-survival result disappears. Along these lines we can distinguish between bankruptcy and non-survival. Following the previous case, there are two long-lived assets and two investors with arbitrary CRRA preferences. One asset has a constant return, fix it to one for simplicity, and the other asset has a returns that fluctuates every period. We assume that the asset with constant return is traded by everybody and the other asset is an endowment of only one investor. We allow the investor with the extra endowment to short-sell. In the spirit of Longstaff (2009), one investor has illiquid wealth every period and the asset that is traded by everybody is the liquid asset. To avoid confusion, the investor with the extra endowment can trade this endowment since he can short-sell the asset with the constant return. We simply assume that the illiquid wealth can be traded through short-sales of the liquid asset. The dynamics of asset trades can make the investor with illiquid wealth in his portfolio to go bankrupt. Bankruptcy in that set-up means that the investor hits her nonnegativity constraint on consumption because the value of her debt in a given planning horizon is so high and the value of her endowments is not enough to pay back. Bankruptcy may happen in finite time.

The second phenomenon mentioned before relates to the existence of asset price fluctuations that are not explained by fluctuations of asset returns. To investigate this possibility we will focus on an economy populated by the second type of myopic individuals. In particular, we will define a *perfect foresight myopic spot equilibrium* (PFMSE). Individuals are myopic but they have perfect foresight for the future prices they forecast in a planning horizon. When they wake up in a new planning horizon they have already forecasted correctly the spot price of that period and all the remaining future prices in the planning horizon. The crucial part is that individuals always observe one more price in the future since they realize that there is one more period in the future. This new price is key because it is going to clear the spot market every period. PFMSE is a weak definition of equilibrium because we require only spot market clearing in every planning horizon. We will demonstrate that existence of equilibrium is a complicated task. Nevertheless, we prove existence for certain cases and show that equilibrium asset prices are strictly increasing even though the dividends of the asset are constant over time. Asset prices explode in the long-run.

We claimed in the beginning that an economy populated by HD maximizers is characterized by complicated dynamics in the sense that there is the possibility of non-survival in the long-run. To analyze this claim we split HD maximizers into two main categories: naive and sophisticated individuals. The former do not realize that their preferences are going to change in the future whereas the latter do realize it. Since naive individuals do not realize their preference reversal we will use a similar argument as in the myopic case to analyze non-survival in the long-run. Naive individuals are always surprised by realizing that their preferences have changed. They wake up next period, face a different set of prices and reoptimize. The conclusion is similar with myopic case. On the other hand, sophisticated individuals behave in a completely different way. They realize that their preferences are going to change in the future and take this into account when they maximize today. In order to solve for the problem of sophisticated individuals we split the individual into different selves indexed by the time period. Each self chooses how much to consume today and how many units of the asset to hold for tomorrow given the equilibrium behavior of future selves. We construct an equilibrium where an individual with IES equal to one, log-utilities, dominates over an individual with IES greater than one. This argument does not require deterministic fluctuations in the dividends of the asset since we fix the latter to one every period.

In section 2 we will analyze two illustrative examples. The first example is the aforementioned case of two extremely myopic individuals who follow different portfolio rules. The second example analyzes an economy of two extremely myopic individuals with different instantaneous utilities. The individual with IES equal to one, log-utility, dominates over the individual with IES equal to two. In section 3 we analyze the case of myopia. Lastly, in section 4 we move to hyperbolic-discounting and the distinction between naive and sophisticated individuals.

§ 1.2 Two Illustrative Examples

The following examples illustrate some points that we will touch upon in later sections. We start with the assumption that individuals are myopic and make only finite plans for the future. The first example illustrates the idea that myopic behavior coupled with specific investment rules drives one individual out of the market eventually. The second example illustrates the fact that in an economy populated by myopic individuals who follow different saving rules, the individual with IES equal to one, log-utility, dominates over an individual with IES equal to two, $u(c) = 2\sqrt{c}$.

1.2.1 Myopic behavior, Survival and Portfolio Rules

Consider an economy under certainty that extends to infinity, $t = 1, 2, \ldots..$ The economy is populated by two investors with identical preferences, log-utilities, and identical rates of impatience, $\beta < 1$. Total output every period is produced by two long-lived assets, (z, y), with the following pattern of dividends,

$$z \to \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, ...$$
$$y \to \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, ...$$

and the aggregate output is equal to one every period. Suppose also that each investor owns one of the assets in the beginning.

Before analyzing the case of myopia, it is important to analyze first the benchmark case where both investors are fully farsighted. The preferences of investors in the benchmark case are as follows,

$$U^i = \sum_{t=1}^{\infty} \beta^{t-1} \log(x_t^i)$$

and their budget constraint every period is as follows,

$$x_t^i + q_t^z z_t^i + q_t^y y_t^i = y_{t-1}^i(d_t^y + q_t^y) + z_{t-1}^i(d_t^z + q_t^z), \ \forall t \tag{1.1}$$

where $d_t + q_t$ is the dividend return plus the capital value of the asset. Since investors invest in two assets every period, the no-arbitrage condition takes the following form,

$$\frac{d_{t+1}^z + q_{t+1}^z}{q_t^z} = \frac{d_{t+1}^y + q_{t+1}^y}{q_t^y} \tag{1.2}$$

which means that the two assets have equal returns. Combining (1),(2), we can collapse the period by period constraints into an intertemporal constraint and maximize lifetime utility subject to the intertemporal constraint for each investor. We are interested only in the qualitative properties of the equilibrium in the benchmark case. In particular, since we have considered a *frictionless* economy under certainty with no aggregate risk, the aggregate output is one every period, each investor will equalize her consumption intertemporally. Since each investor starts her life with positive initial wealth and equalizes her consumption intertemporally, they will consume a positive fraction of the aggregate output every period. As a consequence, both investors survive in the long-run. In particular, suppose individual one is endowed with tree z in the beginning and individual two is endowed with tree y. We can easily verify that the consumption of each investor is as follows:

$$x^1 = \frac{\frac{1}{3} + \beta\frac{2}{3}}{1+\beta}, \ x^2 = \frac{\frac{2}{3} + \beta\frac{1}{3}}{1+\beta}$$

Let us move away from the benchmark case and introduce myopic behavior. We assume that investors are able to plan for only one period in the future. They are extremely myopic. When they wake up in a new period they realize one more period in the future. It is important to state the period one problem of each investor,

$$\max \left[\log(x_1^1) + \beta \log(x_2^1)\right], \ s.t \qquad \max \left[\log(x_1^2) + \beta \log(x_2^2)\right], \ s.t$$

$$x_1^1 + q_1^z z_1^1 + q_1^y y_1^1 = \frac{1}{3} + q_1^z \qquad x_1^2 + q_1^z z_1^2 + q_1^y y_1^2 = \frac{2}{3} + q_1^y$$

$$x_2^1 + q_2^z z_2^1 + q_2^y y_2^1 = (\frac{1}{3} + q_2^y) y_1^1 + \qquad x_2^2 + q_2^z z_2^2 + q_2^y y_2^2 = (\frac{1}{3} + q_2^y) y_1^2 +$$

$$(\frac{2}{3} + q_2^z) z_1^1 \qquad (\frac{2}{3} + q_2^z) z_1^2$$

$$y_2^1 = z_2^1 = 0 \qquad z_2^2 = y_2^2 = 0$$

where investor one is endowed with asset y in the beginning and investor two is endowed with asset z. Investors start their life in period one and are able to plan only up to period two. As a consequence we have to impose a transversality condition at the end of period two[11]. From the viewpoint of period one, equilibrium asset prices in period two are equal to zero, $q_2^z = q_2^y = 0$. Investors wake up in period two and realize that there is another period in the future. They realize that equilibrium asset prices, q_2^z, q_2^y, are not zero as they expected yesterday and face a problem similar to the one stated above. The same reasoning continues *ad infinitum*.

From the maximization problem of each investor in period one we realize that we can not solve uniquely for the asset demands of investors. Using the no-arbitrage condition (2) we can collapse the two constraints into an intertemporal and solve for the real allocation. This leaves the asset holdings indeterminate. The *indeterminacy of asset holdings* is the key to this example. If we do not break this indeterminacy we can not get uniquely determined asset demands and thus solve recursively for each myopic horizon.

Consider the following rule of thumb that allow us to get uniquely determined dynamics in this case,

Portfolio rule. *Investor one invest only in the more productive asset every period whereas investor two holds a more diversified portfolio.*

The intuition behind this rule is quite simple. Whenever investor one observes that asset z is more productive tomorrow than asset y, $\frac{2}{3} > \frac{1}{3}$, she invests only in asset z and the opposite when asset y is more productive

[11] At the optimum solution to the investor's problem the constraint should be binding and that is why we state the transversality condition with equality.

than asset z. Lastly, we must define the equilibrium concept that we use. We require that investors maximize utility subject to the constraints given q_t and from the viewpoint of period t, markets clear as follows:

$$\sum_i x^i_{t+j} = 1,\ \sum_i z^i_{t+j} = 1,\ \sum_i y^i_{t+j} = 1,\ \forall j,\ j = 0, 1$$

Consider following proposition,

Proposition I. *Investor one will not survive in the market, $x^1_t \to 0$ as $t \to \infty$.*

Proof. Investor's one initial wealth in period one is as follows,

$$w^1_1 = (\frac{1}{3} + q^z_1)y^z_0$$

where we have assumed she is endowed with asset z initially. As discussed before, we can use the no-arbitrage condition (2) and collapse the constraints into an intertemporal one and maximize utility subject to the intertemporal constraint. The optimal demand for period one is as follows:

$$x^1_1 = (1 - s^1_1)w^1_1$$

where s^1_1 is the saving rate of investor one. From the assumption of log-utilities, $s^1_1 = s^1_t = s^1 = \frac{\beta}{1+\beta}$. The portfolio of asset holdings of investor one is as follows,

$$q^z_1 z^1_1 + q^y_1 y^1_1 = s^1 w^1_1 \tag{1.3}$$

and since there is no uncertainty, expression (3) simply states that asset holdings are indeterminate. Applying the investment rule discussed before we can eliminate the indeterminacy and get uniquely determined dynamics. In particular, investor one will invest only in asset z because it the most productive according to the productivities in period two, $q^z_1 z^1_1 = s^1 w^1_1$. The wealth of investor one in period two can be written as follows,

$$w^1_2 = r^z_2 s^1 w^1_1 \tag{1.4}$$

where $r^z_2 = \frac{\frac{2}{3}+q^z_2}{q^z_1}$ is the return of asset z. We can rewrite expression (4) for any period t as follows,

$$w_{t+1}^1 = r_{t+1}^H s^1 w_t^1, \ \forall t \tag{1.5}$$

where $r_{t+1}^H = \frac{\frac{2}{3}+q_2^H}{q_1^z}$, is the return of the more (High) productive asset every period.

The problem of the second investor is similar. The optimum portfolio of assets for the second investor is as follows,

$$q_{t+1}^z z_{t+1}^2 + q_{t+1}^y y_{t+1}^2 = s^2 w_t^2$$

where s^2 is the saving rate of investor two and from the assumption of log-utilities, $s^2 = \frac{\beta}{1+\beta}$. According to the previous investment rule, investor two holds a more diversified portfolio since she invests in both assets. The wealth dynamics of investor two can be written as follows,

$$w_{t+1}^2 = \Big(\alpha_t r_{t+1}^H + (1-\alpha_t) r_{t+1}^L\Big) s^2 w_t^2 \tag{1.6}$$

where α_{t+1} is the share of investor's two initial wealth the she invests in the more productive asset and $1-\alpha_{t+1}$ the share that she invests in the less productive asset.

Expressions (4),(6) are the key relations we need in order to show that the statement in proposition I is true. Consider the ratio of wealth levels between investor two and investor one,

$$\frac{w_{t+1}^2}{w_{t+1}^1} = \frac{s^2 w_t^2}{s^1 w_t^1}\left(\alpha_{t+1} + (1-\alpha_{t+1})\frac{r_{t+1}^L}{r_{t+1}^H}\right)$$

which can be rewritten as follows,

$$\frac{w_{t+1}^2}{w_{t+1}^1} = \frac{w_1^2}{w_1^1}\prod_{s=1}^{t}\left(\underbrace{\alpha_{s+1} + (1-\alpha_{s+1})\frac{r_{s+1}^L}{r_{s+1}^H}}\right)$$

If $\frac{w_{t+1}^2}{w_{t+1}^1} \to \infty$ as $t \to \infty$, then $w_{t+1}^1 \to 0$ and as a consequence $x_t^1 \to 0$. In other words, we must show that the term in the underbrace is greater than one for all t. For the term in the underbrace to be greater than one for all t, it is sufficient to show that $\frac{r_{t+1}^L}{r_{t+1}^H} > 1$ for all t.

The return on investor's wealth at $t+1$ from the viewpoint of period t for each asset is as follows,

$$r^H_{t|t+1} = \frac{\frac{2}{3}}{q^H_t} = \frac{\frac{2}{3}}{\beta\frac{2}{3}} = \frac{1}{\beta},\ r^L_{t|t+1} = \frac{\frac{1}{3}}{q^L_t} = \frac{\frac{1}{3}}{\beta\frac{1}{3}} = \frac{1}{\beta} \tag{1.7}$$

and the price of the asset at $t+1$ from the viewpoint of period t is equal to zero. Moreover, the equilibrium asset prices at period t are as follows: $q^H_t = \beta\frac{2}{3}$, $q^L_t = \beta\frac{1}{3}$. Once investors wake up in period $t+1$, they observe that the more productive asset becomes the less productive according to the productivities of $t+2$. The returns in (7) modify as follows,

$$r^H_{t+1} = \frac{\frac{2}{3} + \beta\frac{1}{3}}{\beta\frac{2}{3}} < r^L_{t+1} = \frac{\frac{1}{3} + \beta\frac{2}{3}}{\beta\frac{1}{3}}$$

Going back to the wealth ratio, the term in the product in the right hand side is greater than one,

$$\frac{r^L_{t+1}}{r^H_{t+1}} = \frac{r^L}{r^H} > 1 \Rightarrow \alpha_{t+1} + (1-\alpha_{t+1})\frac{r^L}{r^H} > 1$$

and $\frac{w^2_{t+1}}{w^1_{t+1}} \to \infty$ as $t \to \infty$. □

The key to the previous result is the myopic behavior of investors and the specific investment rule we have imposed. Investor one invests only in the more productive asset from the viewpoint of period t. When she wakes up in $t+1$ she realizes that her investment is the less productive from the viewpoint of $t+1$ and the return on wealth invested at period t is lower than the return of the other asset. Investor two dominates in the long-run because she invests in a portfolio with higher returns.

This example is in the spirit of Blume and Easley (1992). They show that if investors follow the same saving rules, then an investor with log-utilities dominates regardless of the portfolio rules used by any other investor. In this example we showed that even in the case of identical preferences, log-utilities, and identical saving rules, there are investment rules that dominate the market.

1.2.2 Myopic behavior, Survival and Saving Rules

The main set-up is similar to the previous example with some modifications. There are two individuals with different instantaneous utilities,

$$u^1(x) = 2\sqrt{x},\ u^2(x) = \log(x)$$

and identical rates of impatience, $\beta < 1$. There is one long-lived asset with the following pattern of dividends,

$$\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \ldots..$$

which means that aggregate output is either $\frac{1}{3}$ or $\frac{2}{3}$.

Let us comment first on the benchmark case before moving to myopic behavior. The preferences of individual i in the fully farsighted case are as follows

$$U^i = \sum_{t=1}^{\infty} \beta^{t-1} u^i(x_t^i)$$

and the period by period budget constraint is as follows

$$x_t^i + q_t y_t^i = y_{t-1}^i (d_t + q_t), \ \forall t$$

We are interested in the qualitative properties of the equilibrium and not in an explicit solution. Consider the following three facts. First, markets are complete so that individuals maximize lifetime utility subject to the intertemporal constraint. Second, individuals have identical rates of impatience. Lastly, the dividends of the asset follow a cyclical deterministic pattern. There is no growth in the aggregate output. Combining the previous three facts we can conclude that both individuals survive in the long-run.

Consider the case of myopia. Suppose individuals are extremely myopic and are able to plan only for one period in the future. Their preferences are described by the following expressions,

$$u_t^1(x_t^1, x_{t+1}^1) = 2\sqrt{x_t^1} + 2\beta\sqrt{x_{t+1}^1}$$
$$u_t^2(x_t^2, x_{t+1}^2) = \log(x_t^2) + \beta \log(x_{t+1}^2)$$

The following diagram describes how this economy evolves through time,

$$(t, t+1), \ (t+1, t+2), \ (t+2, t+3), \ (t+3, t+4), \ldots.$$

At period t individuals can only perceive and plan up to period $t+1$. When they wake up at period $t+1$, they get a surprise and realize there is another period, $t+2$. As a consequence they reoptimize. When they wake up at $t+2$, they realize there is again another period, $t+3$ and reoptimize again.

The typical maximization problems of individuals in every sequence of two period economies are as follows,

$$\begin{aligned}
&\max u_t^1,\ s.t && \max u_t^2,\ s.t\\
&x_t^1 + q_t y_t^1 = y_{t-1}^1(d_t + q_t) && x_t^2 + q_t y_t^2 = y_{t-1}^2(d_t + q_t)\\
&x_{t+1}^1 + q_{t+1} y_{t+1}^1 = (d_{t+1} + q_{t+1}) y_t^1 && x_{t+1}^2 + q_{t+1} y_{t+1}^2 = (d_{t+1} + q_{t+1}) y_t^2\\
&y_{t+1}^1 \geqslant 0 && y_{t+1}^2 \geqslant 0
\end{aligned}$$

At the optimum solution to the above problems, $y_{t+1}^i = 0$. The definition of equilibrium is the same as in the previous example. Thus, in equilibrium $q_{t+1} = 0$.

The optimal demands of individual one in every sequence of two period economies are as follows,

$$\begin{aligned}
x_t^1 &= \frac{y_{t-1}^1(d_t + q_t)}{1 + \beta^2 \frac{d_{t+1}}{q_t}}\\
x_{t+1}^1 &= \frac{\beta^2}{\left(\frac{q_t}{d_{t+1}}\right)^2} \frac{y_{t-1}^1(d_t + q_t)}{1 + \beta^2 \frac{d_{t+1}}{q_t}}
\end{aligned}$$

and for individual two,

$$\begin{aligned}
x_t^2 &= \frac{y_{t-1}^2(d_t + q_t)}{1 + \beta}\\
x_{t+1}^2 &= \frac{\beta}{\frac{q_t}{d_{t+1}}} \frac{y_{t-1}^2(d_t + q_t)}{1 + \beta}
\end{aligned}$$

From market clearing, $x_t^1 + x_t^2 = d_t$, we compute the price of the asset, q_t, as follows,

$$q_t = \frac{-(d_t y_{t-1}^1 + \frac{y_{t-1}^2}{1+\beta}(d_t + \beta^2 d_{t+1}) - d_t)}{2(y_{t-1}^1 + \frac{y_{t-1}^2}{1+\beta})} + \frac{\sqrt{(d_t y_{t-1}^1 + \frac{y_{t-1}^2}{1+\beta}(d_t + \beta^2 d_{t+1}) - d_t)^2 + 4(y_{t-1}^1 + \frac{y_{t-1}^2}{1+\beta})\beta^2 d_t d_{t+1}(1 - \frac{y_{t-1}^2}{1+\beta})}}{2(y_{t-1}^1 + \frac{y_{t-1}^2}{1+\beta})} \tag{1.8}$$

Assume also that $\beta = 0.9$ and individuals start with initial units of the asset, $\delta^i = 0.5$. We will show that the following proposition is true,

Proposition II. *Individual one will not survive in the market, $x_t^1 \to 0$ as $t \to \infty$.*

Consider first the following table which shows the dynamics of asset prices and the dynamics of asset holdings of the log-guy,

Table 1.1: *Market Dynamics*

x_t^2	q_t	y_t^2
0.187057	0.377517	0.445944
0.266701	0.469823	0.510895
0.19079	0.376233	0.456395
0.273331	0.471282	0.521977
0.194572	0.374911	0.467083
0.280111	0.472792	0.533214
.	.	.
.	.	.

and this table will become useful in a few paragraphs.

In order to prove proposition II, we have to analyze the savings behavior of both individuals. Consider the asset demands of each individual

$$y_t^2 = y_{t-1}^2 \frac{d_{t,k} + q_t}{q_t} \underbrace{\frac{\beta}{1+\beta}}_{s_{t,j}^2} \tag{1.9}$$

$$y_t^1 = y_{t-1}^1 \frac{d_{t,k} + q_t}{q_t} \underbrace{\frac{\beta^2 \left(\frac{q_t}{d_{t+1,j}}\right)^{-\frac{1}{2}}}{1+\beta^2 \left(\frac{q_t}{d_{t+1,j}}\right)^{-\frac{1}{2}}}}_{s_{t,j}^1}, \; k \neq j, \; k, j = H, L \tag{1.10}$$

where the subscripts k, j keeps track the cyclicality of dividends. Whether it is H or L, $\frac{2}{3}$ or $\frac{1}{3}$ respectively. Also, $s_{t,j}^i$ is the saving rate of each individual. Solving recursively (9),(10) we get

$$y_\infty^i = \frac{1}{2} \prod_{t=1}^{\infty} \frac{d_{t,k} + q_t}{q_t} \frac{d_{t+1,j} + q_{t+1}}{q_{t+1}} s_{t,k}^i s_{t+1,j}^i \tag{1.11}$$

and $\frac{1}{2}$ are the initial units of the asset. Consider the ratio of asset holdings of individual two over individual one,

$$\frac{y_\infty^2}{y_\infty^1} = \prod_{t=1}^{\infty} \frac{s_{t,k}^2 s_{t+1,j}^2}{s_{t,k}^1 s_{t+1,j}^1} \tag{1.12}$$

To prove proposition II, we have to show that $\frac{y^2_\infty}{y^1_\infty} \to \infty$. If $\frac{y^2_\infty}{y^1_\infty} \to \infty$, then by market clearing, $y^1_t + y^2_t = 1, \forall t$, $y^1_\infty \to 0$ and $y^2_\infty \to 1$. If $y^1_\infty \to 0$ then $x^1_t \to 0$ as $t \to \infty$.

Rewritte the term in the product of the right-hand side of (12) as follows,

$$\underbrace{\frac{\beta^2}{(1+\beta)^2}}_{s^2_{t,k}s^2_{t+1,j}} - \underbrace{\frac{\beta^2\left(\frac{q_t}{d_{t+1,H}}\right)^{-\frac{1}{2}}}{1+\beta^2\left(\frac{q_t}{d_{t+1,H}}\right)^{-\frac{1}{2}}} \frac{\beta^2\left(\frac{q_{t+1}}{d_{t+2,L}}\right)^{-\frac{1}{2}}}{1+\beta^2\left(\frac{q_{t+1}}{d_{t+2,L}}\right)^{-\frac{1}{2}}}}_{s^1_{t,k}s^1_{t+1,j}} = \epsilon_{t,t+1} \tag{1.13}$$

If $\epsilon_{t,t+1} > 0, \forall t$, then $\frac{s^2_{t,k}s^2_{t+1,j}}{s^1_{t,k}s^1_{t+1,j}} > 1, \forall t$. If we can show that the dynamics of this economy imply that $\frac{s^2_{t,k}s^2_{t+1,j}}{s^1_{t,k}s^1_{t+1,j}} > 1, \forall t$, then the RHS of (12) diverges to infinity and we will have proven proposition II.

Consider the following figure which graphs condition (13) for all values of y^2_H, y^2_L[12],

[12] We have substituted expression (8) of equilibrium asset price in (13). The LHS of (13) becomes a function of initial asset holdings that each individual starts each new planning horizon, y^i_H, y^i_L. From market clearing, $\sum_i y^i = 1$, we can express the LHS of (13) as function of initial asset holdings of individual two, y^2_H, y^2_L.

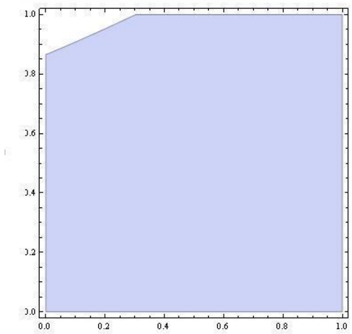


Figure 1.1

where in the horizontal axis we have y_L^2 and in the vertical y_H^2. In the shaded light blue region, $\epsilon_{t,t+1} > 0$. Go back to table 1. Consider the first two points of asset holdings in low and high periods respectively, $(0.445944, 0.510895)$. We start somewhere in the middle of the box inside the shaded blue region. Since we started from low and then high, the next period is low again. From (13) and the figure above we know that the log-guy is going to end up with more than 0.445944 units of the tree he had in the low period he started. We get 0.456395. Next, consider $(0.510895, 0.456395)$ which are the asset holdings of high and low periods. The next period is going to be high. Since we are inside the shaded region, the log-guy is going to end up with more that 0.510895 units of the asset. We get 0.521977. Continuing with this algorithm we see that the movements of asset holdings move from the middle to the upper RHS of the box.

Although the above figure shows that in the blue region $\epsilon_{t,t+1} > 0$, we have to do more work in order to prove proposition II. In particular, we have to show that as $t \to \infty$, $\epsilon_{t,t+1}$ does not converge to zero. If $\epsilon_{t,t+1} > 0 \; \forall t$ but converges to zero, then from expression (12), $\frac{y_\infty^2}{y_\infty^1} \nrightarrow \infty$, and both individuals survive in the long-run. Consider the following figure,

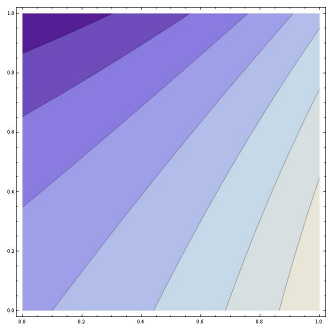


Figure 1.2

Figure 2 depicts the contour plots of $\epsilon_{t,t+1}$. As we move from the LHS to the RHS of the box, $\epsilon_{t,t+1}$ increases[13]. Since we start form the middle of the box and move to the right, $\epsilon_{t,t+1}$ is always positive and increases. It will never converge to zero.

The key to the above result is the *saving* behavior of individuals and the *time-inconsistency* of optimal plans. Individual one is eliminated in the long-run because she saves less than the log-guy. By construction, the log-individual is a seller of the asset in planning horizons where the dividend pattern is $(t, t+1) = (\frac{1}{3}, \frac{2}{3})$ and a buyer in planning horizons where the dividend pattern is $((\frac{2}{3}, \frac{1}{3})$. The dominance of the log-individual over individual two is not obvious. The important aspect is that the log-guy follows a constant saving rule. She saves $\frac{\beta}{1+\beta}$ fraction of her initial wealth every period. In the planning horizons where the log-guy is a buyer, she acquires more units of the asset than individual one but in the horizons where she is a seller she does not sell enough because she follows a constant saving rule. The net effect is that $\frac{s^2_{t,k} s^2_{t+1,j}}{s^1_{t,k} s^1_{t+1,j}} > 1, \forall t$.

The fact that the saving behavior of individuals drives the non-survival result is because of myopia[14]. Myopic behavior implies time-inconsistency of optimal plans. Every time individuals wake up in a new planning horizon they reoptimize. They perceive one more period in the future and they realize that the price of the asset in the market is not zero as they thought according to yesterday's plan. This is the difference with the benchmark case and that is why some saving rules may dominate over others.

1.2.3 A comment on the two examples

The previous examples illustrated two mechanisms under which some myopic investors/individuals are eliminated in the long-run. In the first example investors followed identical saving rules but invested in different portfolios whereas in the second example individuals invested in the same portfolio, one asset, but followed different saving rules. We will analyze these two cases in the next section.

§ 1.3 Myopic Behavior

Let us analyze first myopic behavior and then proceed to the other form of time-inconsistency which is hyperbolic discounting. Myopic individuals can be split into two main categories: myopic individuals who have perfect foresight about equilibrium and myopic individuals who make wrong forecasts about equilibrium prices. The following tree describes the cases we analyze in this section.

[13] The colors in the figure become lighter

[14] Extreme myopia in that case.

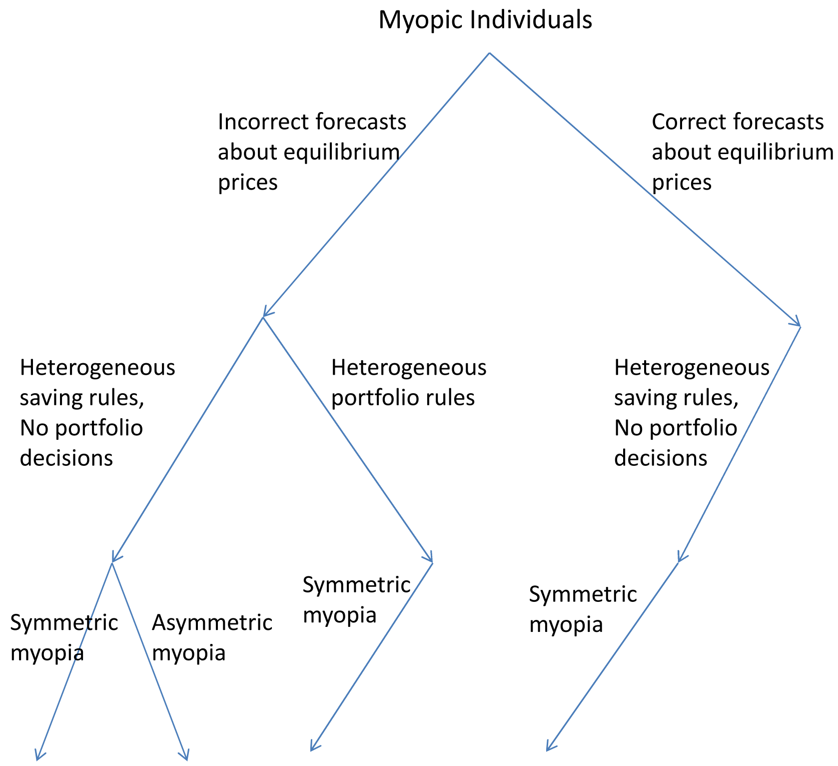


Figure 1.3

1.3.1 Myopic individuals with incorrect forecasts

The analysis in this section follows closely the two examples analyzed before. In particular, we start first with the environment described in the second example. Individuals do not have a portfolio decision to make but follow different saving rules. We focus attention to the case when individuals have different planning horizons, asymmetric myopia. We provide sufficient conditions such that more myopic individuals dominate over less myopic ones. Next, we go back to the first example and consider the case where two investors follow different portfolio rules. We show that certain portfolio rules lead some investors to bankruptcy. We will compare the phenomenon of non-survival in the long-run with that of bankruptcy.

1.3.1.1 Myopic behavior, Survival and Saving Rules

The main set-up is similar to that of the second example. There are two individuals with heterogenous CRRA preferences,

$$u^1(x) = \frac{x^{1-a^1}}{1-a^1}$$
$$u^2(x) = \frac{x^{1-a^2}}{1-a^2}$$

where $a^1, a^2 > 0$ and $a^1 \neq a^2$. There is one long-lived asset with a dividend structure, $d_t,\ t = 1, 2, ..$, that follows a cyclical deterministic pattern[15]. Aggregate output every period is equal to d_t.

Let us analyze again the benchmark case where individuals are fully farsighted. The lifetime utility of individuals is as follows,

$$U^i = \sum_{t=1}^{\infty} \beta^{t-1} \frac{x_t^{1-a^i}}{1-a^i}$$

and the period by period budget constraint is as follows

$$x_t^i + q_t y_t^i = y_{t-1}^i (d_t + q_t),\ \forall t \tag{1.14}$$

We can take the constraints in (14) and collapse them into an intertemporal constraint. Individuals maximize utility subject to the intertemporal constraint. The FOCs of individuals are as follows,

$$\beta^{t-1} \left(x_t^i \right)^{-\alpha^i} = \lambda^i R_t,\ i = 1, 2 \tag{1.15}$$

where R_t is the present-value price of the good at period t from the viewpoint of period one,

$$R_t = 1 \times \frac{q_1}{d_2 + q_2} \times \frac{q_2}{d_3 + q_3} \times \times \times \frac{q_t}{d_{t+1} + q_{t+1}},\ R_1 = 1$$

Combine individuals FOCs as follows

$$\frac{(x_t^1)^{-\alpha^1}}{(x_t^2)^{-\alpha^2}} = \frac{\lambda^1}{\lambda^2} \tag{1.16}$$

[15] For example, there are periods that the dividends take high values and period that take low values and this pattern keeps repeating.

and good's market clearing is as follows

$$x_t^1 + x_t^2 = d_t, \ \forall t \tag{1.17}$$

Combine (16),(17) we get

$$x_t^1 + (x_t^1)^{\frac{\alpha^1}{\alpha^2}} \left(\frac{\lambda^1}{\lambda^2}\right)^{\frac{1}{\alpha^2}} = d_t, \ or \tag{1.18}$$

$$x_t^2 + (x_t^2)^{\frac{\alpha^2}{\alpha^1}} \left(\frac{\lambda^2}{\lambda^1}\right)^{\frac{1}{\alpha^1}} = d_t \tag{1.19}$$

From (18),(19) we can conclude that *both individuals survive in the long-run.* Expression (18) or (19) are time-dependent only if the $d_t's$ are time-dependent. Since we have assumed that the $d_t's$ follow a cyclical deterministic pattern, individual i will consume the same amount of the good every time the $d's$ are the same.

Let us move to the case of myopia. We focus first on the case of symmetric myopia in order to explain the main methodology and then we move to the more interesting case of asymmetric myopia. Suppose both individuals plan for T periods in the future. Every time they wake up in a new period, they realize another period in the future and reoptimize. Individuals revise their price forecasts that were held in the previous planning horizon and make a different plan than the one made yesterday.

Consider the problem of each individual in every planning horizon, $t \to t + T$,

$$\begin{aligned}
&\max \left[\sum_{s=t}^{t+T} \beta^{s-t} \frac{x_s^{1-a^i}}{1-a^i} \right], \ s.t \\
&x_t^i + q_t y_t^i = y_{t-1}^i (d_t + q_t) \\
&. \\
&. \\
&x_{t+T}^i + q_{t+T} y_{t+T}^i = (q_{t+T} + d_{t+T}) y_{t+T-1}^i \\
&y_{t+T}^i = 0
\end{aligned}$$

Individuals make a plan from the viewpoint of period t up to period $t + T$. They forecast prices up to period $t + T$ and decide how many units of the asset to buy in each period. The crucial assumption is the retrading argument implicit in the maximization above. Individuals will never stick to the plan they make in a particular planning horizon. They wake up in a new planning horizon and revise their consumption and asset purchases made in the previous planning horizon.

To solve the previous problem we either consider the intertemporal constraint or substitute x_t^i from the period by period constraints into the objective function. The FOC between $t, t+1$ is as follows,

$$q_t(x_t^i)^{-\alpha^i} = \beta(d_{t+1} + q_{t+1})(x_{t+1}^i)^{-\alpha^i}$$

The optimal demands from the viewpoint of period t are as follows,

$$x_t^i = \frac{y_{t-1}^i(d_t + q_t)}{1 + \sum_{j=1}^T \beta^{\frac{j}{a^i}} R_{t+j}^{\frac{a^i-1}{a^i}}}, \; x_{t+j}^i = x_t^i \frac{\beta^{\frac{j}{\alpha^i}}}{R_{t+j}^{1/\alpha^i}}, \; j = 1, .., T \tag{1.20}$$

where R_{t+j} are the present-value prices from the viewpoint of t,

$$R_{t+j} = 1 \times \frac{q_t}{d_{t+1} + q_{t+1}} \times \frac{q_{t+1}}{d_{t+1} + q_{t+2}} \times \times \times \frac{q_{t+j}}{d_{t+j+1} + q_{t+j+1}}$$

More importantly, the optimal asset holdings of each individual in every actual period t are as follows

$$y_t^i = y_{t-1}^i \frac{q_t + d_t}{q_t} \underbrace{\frac{\sum_{j=1}^T \beta^{\frac{j}{a^i}} R_{t+j}^{\frac{a^i-1}{a^i}}}{1 + \sum_{j=1}^T \beta^{\frac{j}{a^i}} R_{t+j}^{\frac{a^i-1}{a^i}}}}_{s_t^i} \tag{1.21}$$

where s_t^i is the saving rate of individual i in every actual period t.

Before moving to the main part of this section, consider the following definition of an *Incorrect Forecasts Myopic Equilibrium (IFME)*

IFME. *Fix a planning horizon, $t \to t + T$. A pair $((q_{t+k}^*)_{k=0}^{T-1}, (x_{t+j}^*)_{j=0}^{j=T})$ constitutes an Incorrect Forecasts Myopic Equilibrium if*

- *Individuals* $\max U_t^i$ *s.t b.c given* $(q_{t+k}^*)_{k=0}^{T-1}$
- *Markets clear:* $\sum_i x_{t+j}^i = d_{t+j}$, $\sum_i y_{t+k}^i = 1$, *for all* $j = 0, .., T$, $k = 0, 1, ..T - 1$

The previous definition of equilibrium has an *unappealing* feature: individuals revise their price forecasts every period. We will demonstrate that myopic behavior and perfect foresight requires more work for the existence of equilibrum.

In the example of section 2.2 we proved that individual one will vanish in the long-run because the ratio of the product of saving rates between low and high periods tended to infinity, expression (12). To analyze the conditions for non-survival we will focus on expression (21), which describes the asset demands of each individual in every actual period t. The difficulty with expression (21) is that it depends on asset prices. In particular, on present-value prices. If we knew something about the behavior of equilibrium present-value prices, we could compare the saving rates of individuals as in the example of section 2.2. Thus, the first issue we have to tackle is to characterize the movements of equilibrium present-value prices for the previous simple economy.

Consider the following claim,

Claim I. *Equilibrium present value prices, from the viewpoint of period t, belong to the following intervals,*

$$R_{t+j} \in \left[\beta^j \left(\frac{d_t}{d_{t+j}} \right)^{\alpha^{-i}}, \beta^j \left(\frac{d_t}{d_{t+j}} \right)^{\alpha^i} \right], \ j = 1, ..., T$$

Proof. Suppose the previous economy is populated by only one of the two individuals. We can compute the no-trade equilibrium prices, and present-value prices, of the representative individual economy. From the optimal demands in (20) and the good's market clearing every period, we get that

$$R^i_{t+j} = \beta^j \left(\frac{d_t}{d_{t+j}} \right)^{\alpha^i}$$

where the subscript i in the present-value price means that we solve for the representative individual economy with each individual i.

Let us go back to the original economy with two individuals. Suppose an equilibrium exist in every planning horizon. Consider the good's market clearing in period $t+j$ using the optimal demands in (20),

$$\left(x^1_t \right)^* \frac{\beta^{\frac{j}{\alpha^1}}}{\left(R^{1/\alpha^1}_{t+j} \right)^*} + \left(x^2_t \right)^* \frac{\beta^{\frac{j}{\alpha^2}}}{\left(R^{1/\alpha^2}_{t+j} \right)^*} = d_{t+j} \tag{1.22}$$

where $\left(x^i_t \right)^*$ is the equilibrium consumption demand in period t and R^*_{t+j} is the equilibrium present-value price for consumption in $t+j$ from the viewpoint of period t. Rewrite (22) as follows,

$$\left(x_t^1\right)^* \frac{\beta^{\frac{j}{\alpha^1}}}{\left(R_{t+j}^{1/\alpha^1}\right)^*} + \left(x_t^2\right)^* \frac{\beta^{\frac{j}{\alpha^2}}}{\left(R_{t+j}^{1/\alpha^2}\right)^*} = \frac{d_{t+j}}{d_t}\left[\underbrace{\left(x_t^1\right)^* + \left(x_t^2\right)^*}_{=d_t}\right] \tag{1.23}$$

which is an equivalent way of writing (22). Rewrite (23) as follows,

$$\left(x_t^1\right)^* \underbrace{\left[\frac{\beta^{\frac{j}{\alpha^1}}}{\left(R_{t+j}^{1/\alpha^1}\right)^*} - \frac{d_{t+j}}{d_t}\right]} + \left(x_t^2\right)^* \underbrace{\left[\frac{\beta^{\frac{j}{\alpha^2}}}{\left(R_{t+j}^{1/\alpha^2}\right)^*} - \frac{d_{t+j}}{d_t}\right]} = 0 \tag{1.24}$$

Consider the terms in the underbrace. Since we have assumed that an equilibrium exist, it must be that the terms in the underbrace have different signs. If present-value prices do not belong to the intervals mentioned in the claim above, then (24) is not true and this contradicts equilibrium in each planning horizon which we have assumed initially. For present-value prices that do not belong to the intervals mentioned before, the terms in the underbrace have the same sign. □

The previous claim give us enough information to characterize the consumption dynamics in this economy. Consider the saving rate of each individual in every actual period t,

$$s_t^i = \frac{\sum_{j=1}^T \beta^{\frac{j}{a^i}} R_{t+j}^{\frac{a^i-1}{a^i}}}{1 + \sum_{j=1}^T \beta^{\frac{j}{a^i}} R_{t+j}^{\frac{a^i-1}{a^i}}} \tag{1.25}$$

Since the saving rate of each individual depends on the present-value prices, we can exploit the fact that equilibrium present-value prices are bounded. In particular, we can use the endpoints of the intervals in the above claim and consider the smallest or the highest possible value that (25) can take. To understand the usefulness of the latter statement, suppose that the dividends of the asset follow a pattern that repeats every $m + m^{'}$ periods,

$$\underbrace{d_1, d_2, ..., d_m}_{m\ periods}, \underbrace{d_{m+1}, d_{m+2}, ..., d_{m'}}_{m'\ periods}, \underbrace{d_1, d_2, ..., d_m}_{m\ periods}, ... \tag{1.26}$$

We will prove that there exists an $\epsilon > 0$ such that the following inequality is true,

$$\frac{y_{t+m+m'}^1 / y_t^1}{y_{t+m+m'}^2 / y_t^2} = \frac{\prod_t^{t+m+m'} s_t^1}{\prod_t^{t+m+m'} s_t^2} > 1 + \epsilon, \ \forall t \tag{1.27}$$

Since the dividend pattern repeats as in (26), each individual at period t will face the same dividend structure at $t+m+m^{'}$. If (27) is true and we consider also asset market clearing, $y_t^1 + y_t^2 = 1$, the following is also true,

$$y^1_{t+m+m^{'}} > y^1_t,\ y^2_{t+m+m^{'}} < y^2_t,\ \forall t \tag{1.28}$$

Given the previous discussion, consider the following proposition,

Proposition III. *If there exists an $\epsilon > 0$ such that (27) is true, then individual 2 will vanish in the long-run.*

Proof. We want to show that the ratio of actual saving rates between individual 1 and 2 tends to infinity,

$$\frac{y^1_\infty}{y^2_\infty} = \frac{y^1_1}{y^2_1}\prod_{t=1}^{\infty}\frac{s^1_t}{s^2_t} \to \infty$$

and from asset market clearing, $y^2_\infty \to 0$. If (27) is true, we know that the ratio of the product of saving rates every $m+m^{'}$ periods is bounded below by $1+\epsilon$ for all t. We can rewrite the previous infinite product as follows,

$$\underbrace{\frac{s^1_t \times \times s^2_{t+m+m^{'}}}{s^2_t \times \times s^2_{t+m+m^{'}}}}_{>1+\epsilon}\ \underbrace{\frac{s^1_{t+1+m+m^{'}} \times \times s^2_{t+1+2(m+m^{'})}}{s^2_{t+1+m+m^{'}} \times \times \times\ s^2_{t+1+2(m+m^{'})}}}_{>1+\epsilon} \times\times$$

Since the terms that repeat every $m+m^{'}$ periods are bounded below by $1+\epsilon$, then $\frac{y^1_\infty}{y^2_\infty} \to \infty$.

□

To show that (27) is true, we will take advantage of the fact that present-value prices are bounded in any equilibrium of the above economy. Focusing first on symmetric myopia, consider the following utility functions for both individuals,

$$u^1(x) = \log(x),\ u^2(x) = \frac{x^{1-\alpha^2}}{1-a^2},\ \alpha^2 > 0,\ \alpha^2 \neq 1$$

Consider the dividend pattern of (26). The actual saving rate of the log-guy every period is constant

$$s^1_t = \frac{\sum_{j=1}^{T}\beta^j}{1+\sum_{j=1}^{T}\beta^j}$$

whereas the saving rate of individual two depends on present-value prices,

$$s_t^2 = \frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{t+j}^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{t+j}^{\frac{a^2-1}{a^2}}} \tag{1.29}$$

From Claim I, we can consider the highest present-value prices in any equilibrium and compute the highest possible value that the saving rate of individual two in (29) can take as follows,

$$\frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_t}{d_{t+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_t}{d_{t+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}} > \frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{t+j}^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{t+j}^{\frac{a^2-1}{a^2}}}, \ \forall t, \ i = 1 \ or \ 2 \tag{1.30}$$

where the LHS of (30) is the highest possible value that the saving rate of individual two can take in any equilibrium and the RHS is the actual value that the saving rate of individual two takes in any myopic equilibrium. The next step is to compare the product of the saving rates of each individual for $m + m'$ periods,

$$\underbrace{\frac{\sum_{j=1}^{T} \beta^j}{1 + \sum_{j=1}^{T} \beta^j} \times \times \times \frac{\sum_{j=1}^{T} \beta^j}{1 + \sum_{j=1}^{T} \beta^j}}_{m+m' \ times} > \prod_{k=1}^{m+m'} \frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}} >$$
$$\prod_{k=1}^{m+m'} \frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{k+j}^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{T} \beta^{\frac{j}{a^2}} R_{k+j}^{\frac{a^2-1}{a^2}}} \tag{1.31}$$

where the subscript k keeps track of the cyclical pattern of dividends whereas the subscript j measures the length of the planning horizon. If (31) is true, then there exists an $\epsilon > 0$ such that (27) is true and as a consequence proposition III applies.

Finally, we have to find conditions such that (31) is true. Suppose first that $\alpha^2 > 1$. Suppose also that there exists at least one planning horizon where $\frac{d_k}{d_{k+j}} < 1, \ j = 1, .., T$. This means that the dividend of the asset today is lower that the dividends of the asset in all the future periods of the planning horizon from the viewpoint of today. Observe also that all the terms in (31) enter multiplicative and the saving rate of each individual can not exceed one. Injecting enough fluctuations in the dividend pattern of the asset, we reduce $\frac{d_k}{d_{k+j}}$ as much as we want for all j in the planning horizon in question. Thus, we can lower as much as we want the highest possible value the saving rate of individual 2 can take in that planning horizon and as a consequence satisfy the first inequality in (31).

Fix $0 < \alpha^2 < 1$. Suppose again that there exists at least one planning horizon where $\frac{d_k}{d_{k+j}} > 1$, $j = 1, .., T$. This means that the dividend of the asset today is higher that the dividends of the asset in all the future periods of the planning horizon from the viewpoint of today. Injecting enough fluctuations in the dividend pattern of the asset, we increase $\frac{d_k}{d_{k+j}}$ as much as we want for all j in the planning horizon in question. The previous argument applies in that case as well.

To sum up, suppose we have a dividend pattern that repeats as in (26). Suppose also we have periods that the dividend returns are low followed by periods that returns are high and periods that the returns are high followed by periods that returns are low. This can be thought as periods where the economy is booming and periods when the economy is in a contraction. We can inject enough fluctuations between low and high periods and satisfy (31). Lastly, we have assumed that individual one has log-utilities, intertemporal elasticity of substitution (IES) equal to one. Expression (31) holds true even if individual one has IES not equal but close to one. The following proposition summarizes the previous discussion

Proposition IV. *Consider the case of symmetric myopia. Suppose individual one has IES close or equal to one, log-utility, and individual two has IES different than one. If either one of the following two conditions apply:*

- *β is high and we inject enough fluctuations between low and high periods in the dividend pattern of the asset.*
- *β is low and we require more fluctuations between low and high periods in the dividend pattern of the asset than the previous case.*

then there exists an $\epsilon > 0$ such that (31) is true.

Let us proceed to the case of ***asymmetric myopia***. In the previous paragraphs we analyzed the long-run dynamics of the economy when individuals have the same planning horizons. We have reduced the characterization of equilibrium into a simple condition, expression (31). The latter approach will prove to be useful in the case of asymmetric myopia.

To motivate the issue of asymmetric myopia, consider the following scenario:

$$\begin{array}{ll}
\max \sum_{s=t}^{t+1} \beta^{s-t} \frac{x_s^{1-\alpha^1}}{1-\alpha^1}, \ s.t & \max \sum_{s=t}^{t+2} \beta^{s-t} \frac{x_s^{1-\alpha^2}}{1-\alpha^2}, \ s.t \\
x_t^1 + q_t y_t^1 = y_{t-1}^1 (d_t + q_t) & x_t^2 + q_t y_t^2 = y_{t-1}^2 (d_t + q_t) \\
x_{t+1}^1 + q_{t+1} y_{t+1}^1 = y_t^1 (d_{t+1} + q_{t+1}) & x_{t+1}^2 + q_{t+1} y_{t+1}^2 = y_t^2 (d_{t+1} + q_{t+1}) \\
y_{t+1}^1 \geq 0 & x_{t+2}^2 + q_{t+2} y_{t+2}^2 = (q_{t+2} + d_{t+2}) y_{t+1}^2 \\
 & y_{t+2}^2 \geq 0
\end{array}$$

Individual one is extremely myopic whereas individual two plans for two periods in the future. Before analyzing this case further, we have to comment on the terminal conditions in the above maximization problems. The crucial part for existence of an *Incorrect Forecast Asymmetric Myopic Equilibrium (IFAME)* is the terminal condition we have imposed to the extreme myopic individual. We do not allow individual one to die with debt at the end of her horizon. In particular, if individual one could die with debt at the end of every planning horizon then there could be no equilibrium. The terminal conditions we require in the asymmetric myopic case are not different than the transversality conditions we require in any finite horizon economy where individuals accumulate assets every period. In the latter economy, if we do not impose a condition which states that individuals do not die with debt, we do not get existence of equilibrium.

Let us go back to the two maximization problems above. The terminals conditions are binding and as a consequence $q_{t+2} = 0$ in equilibrium from the viewpoint of t. The question that we seek to answer in that case is the following:

Question. *Do there exist parameter values which imply that individual two will not survive in the long-run even though she is more farsighted than individual one?*

To understand why the previous question is not trivial, consider the saving rates of each individual every actual period t,

$$s_t^1 = \frac{\beta^{\frac{1}{\alpha^1}} R_{t+1}^{\frac{\alpha^1-1}{\alpha^1}}}{1 + \beta^{\frac{1}{\alpha^1}} R_{t+1}^{\frac{\alpha^1-1}{\alpha^1}}}$$

$$s_t^2 = \frac{\beta^{\frac{1}{\alpha^2}} R_{t+1}^{\frac{\alpha^2-1}{\alpha^2}} + \beta^{\frac{2}{\alpha^2}} R_{t+2}^{\frac{\alpha^2-1}{\alpha^2}}}{1 + \beta^{\frac{1}{\alpha^2}} R_{t+1}^{\frac{\alpha^2-1}{\alpha^2}} + \beta^{\frac{2}{\alpha^2}} R_{t+2}^{\frac{\alpha^2-1}{\alpha^2}}}$$

where $R_{t+1} = \frac{q_t}{d_{t+1}+q_{t+1}}$, $R_{t+2} = \frac{q_t}{d_{t+1}+q_{t+1}} \frac{q_{t+1}}{d_{t+2}}$, are the present-value prices from the viewpoint of period t. The saving rate of individual two reflects the second period that she plans in the future. Thus, her saving rate has an additional term compared to the saving rate of individual one. For this reason, the previous question is not trivial.

Following the same argument as in claim I, the equilibrium present-value prices from the viewpoint of period t belong to the following intervals,

$$R_{t+1} \in \left[\beta\left(\frac{d_t}{d_{t+1}}\right)^{\alpha^{-i}}, \beta\left(\frac{d_t}{d_{t+1}}\right)^{\alpha^i}\right],\ R_{t+2} \in \left[0, \beta^2\left(\frac{d_t}{d_{t+2}}\right)^{\alpha^2}\right],\ i = 1, 2$$

where the zero in the second interval comes from the fact that if we solve for the representative individual economy with individual one, $R_{t+2} = 0$ because individual one is extremely myopic. Lastly, consider the following dividend pattern which allows explicit computations,

$$\begin{aligned} &\underbrace{\gamma, \delta, \epsilon}_{L}, \underbrace{\gamma^{'}, \delta^{'}, \epsilon^{'}}_{H}, \underbrace{\gamma, \delta, \epsilon}_{L}, \underbrace{\gamma^{'}, \delta^{'}, \epsilon^{'}}_{H}, \ldots. \\ &\gamma^{'} > \delta^{'} > \epsilon^{'} > \epsilon > \delta > \gamma > 0 \end{aligned} \tag{1.32}$$

The following proposition states sufficient conditions such that the more farsighted individual will vanish in the long-run,

Proposition V. *Fix $\alpha^2 > \alpha^1 = 1$ and $\beta = 1$. If α^2 is sufficiently high and we inject enough fluctuations across low and high periods in (32), then there exists an $\varepsilon > 0$ such that the following is true*

$$\frac{\prod_t^{t+5} s_t^1}{\prod_t^{t+5} s_t^2} > 1 + \varepsilon,\ \forall t$$

Proof. Since the dividend pattern repeats every six periods, consider the following expression which is analogous to (31),

$$\underbrace{\frac{1}{2} \times \times \frac{1}{2}}_{6\ times} > \prod_{k=1}^{6} \frac{\sum_{j=1}^{2}\left(\left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}}{1 + \sum_{j=1}^{2}\left(\left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}} \tag{1.33}$$

and suppose for $k = 1$ we start from γ in the dividend pattern of (32). Also, the RHS of (32) is the maximum value that the saving rate of individual two can potentially take. Consider the following figure which shows that (32) is true

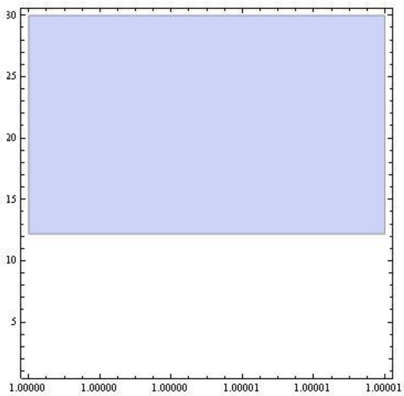


Figure 1.4

where in the horizontal axis we depict $\alpha^1 = 1$ and in the vertical axis α^2. Also, we have considered the following values for the dividend of the asset

$$\gamma = 0.1,\ \delta = 0.5,\ \epsilon = 1,\ \gamma^{'} = 14,\ \delta^{'} = 11,\ \epsilon^{'} = 9$$

and we can play around with the dividend pattern. In the blue region, (32) holds true. If (32) holds true, then the RHS of (32) is always higher than the actual equilibrium product of saving rates for any six periods in a row. This proves the proposition.

□

The previous proposition implies that the more farsighted individual vanishes in the long-run and as a consequence only the more myopic individual survives. The next step is to discuss the sufficient conditions we imposed in proposition V. To understand the latter conditions consider the more general formula of expression (33),

$$\underbrace{\frac{\sum_{j=1}^{T^{'}} \beta^j}{1+\sum_{j=1}^{T^{'}} \beta^j} \times \times \times \frac{\sum_{j=1}^{T^{'}} \beta^j}{1+\sum_{j=1}^{T^{'}} \beta^j}}_{m+m^{'}\ times} > \prod_{k=1}^{m+m^{'}} \frac{\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}}{1+\sum_{j=1}^{T} \beta^{\frac{j}{a^2}} \left(\beta^j \left(\frac{d_k}{d_{k+j}}\right)^{\alpha^i}\right)^{\frac{a^2-1}{a^2}}} \tag{1.34}$$

and $T^{'} < T$. We have set $\alpha^1 = 1$ and we fix again $\alpha^2 > 1$. Individual one, the log-guy, plans for $T^{'}$ periods in the future whereas individual two plans for T periods in the future. The sufficient conditions we require such that the more farsighted individual vanishes in the long-run are analogous to the ones in proposition IV with some important differences. Firstly, we focus on the case where $\alpha^2 > 1$. The reason is that on the RHS of (34), we consider the maximum value that the saving rate of individual two can take in any period. For $\alpha^2 < 1$, in order to get the maximum possible value of the RHS of (34) we have to set the present-value prices for some periods at their smallest values they can take in any equilibrium. The smallest value that equilibrium present-value prices could take in some periods is simply zero. This comes from the fact that equilibrium present value prices are bounded between the no-trade present value prices when we consider that the economy is populated by only one of the two individuals. The following proposition describes the sufficient conditions we require such that the more farsighted individual will vanish in the long-run,

Proposition VI. *Consider the case of asymmetric myopia. Suppose* $\alpha^1 = 1$ *and* $\alpha^2 > 1$. *If either one of the following two conditions apply:*

- β *is high,* α^2 *is sufficiently high and we require more fluctuations between low and high periods in the dividend pattern of the asset than proposition IV*
- β *is low,* α^2 *is sufficiently high and we require more fluctuations between low and high periods in the dividend pattern of the asset than the previous bullet point.*

then expression (34) holds true.

1.3.1.2 Myopic behavior, Survival/Bankruptcy and Investment Rules

In the previous section we showed that in an economy with no role for investment in assets, heterogenous saving behavior could drive more farsighted individuals out of the market eventually. Although this is interesting enough, the fact that individuals do not invest in various assets is a limitation. Certain investment rules may have interesting implications for the long-run dynamics in myopic economies as illustrated in example 2.1. In this section we consider a similar set-up as in the example of 2.1 but we abstract from log-utilities and consider arbitrary CRRA preferences.

Suppose there are two assets in the economy, (z, y), with the following pattern of dividends,

$$\begin{aligned} y &\rightarrow \; 1, 1, 1, 1, 1, ... \\ z &\rightarrow \; d_L, d_H, d_L, d_H, ... \end{aligned}$$

We will state two *equivalent* interpretations for the set-up in example 2.1. The *first interpretation* follows closely the argument of the example in section 2.1. There are two long-lived assets in the economy with the previous pattern of returns. The maximization problem of the two investors in period one are as follows

$$\begin{aligned}
&\max \Big[\sum_{s=t}^{t+1} \beta^{s-t} \frac{x_s^{1-\alpha^1}}{1-\alpha^1}\Big], \ s.t \\
&x_1^1 + q_1^z z_1^1 + q_1^y y_1^1 = y_0^1(1+q_1^y) \\
&x_2^1 + q_2^z z_2^1 + q_2^y y_2^1 = z_1^1(d_H + q_2^z)+ \\
&y_1^1(1+q_2^y) \\
&z_2^1 = y_2^1 = 0
\end{aligned}$$

$$\begin{aligned}
&\max \Big[\sum_{s=t}^{t+1} \beta^{s-t} \frac{x_s^{1-\alpha^2}}{1-\alpha^2}\Big], \ s.t \\
&x_1^2 + q_1^z z_1^2 + q_1^y y_1^2 = d_L + q_1^z + \\
&y_0^2(1+q_1^y) \\
&x_2^2 + q_2^z z_2^2 + q_2^y y_2^2 = z_1^2(d_H + q_2^z)+ \\
&y_1^2(1+q_2^y) \\
&z_2^2 = y_2^2 = 0
\end{aligned}$$

where we have assumed that investor two starts her life with all the units of the asset z and y_0^2 units of asset y. Investor one starts her life with y_0^1 units of asset y. Following the same argument as in example 2.1, we need to specify a rule to break the indeterminacy of portfolio holdings in order to get uniquely determined dynamics. Suppose we interpret asset y as the safe asset and asset z as the risky asset[16]. In this section we will analyze the implications of the following portfolio rule

Investment Rule. *Investor one invests only in the safe asset whereas investor two invests in a portfolio which includes the risky asset as well.*

Given this rule, we can analyze the dynamics of the previous economy. Unlike the example in 2.1, consider the *second interpretation* of the previous economy which is given by the following maximization problems of the two investors in period one

$$\begin{aligned}
&\max \Big[\sum_{s=t}^{t+1} \beta^{s-t} \frac{x_s^{1-\alpha^1}}{1-\alpha^1}\Big], \ s.t \\
&x_1^1 + q_1 y_1^1 = y_0^1(1+q_1) \\
&x_2^1 + q_2 y_2^1 = y_1^1(1+q_2) \\
&y_2^1 = 0
\end{aligned}$$

$$\begin{aligned}
&\max \Big[\sum_{s=t}^{t+1} \beta^{s-t} \frac{x_s^{1-\alpha^2}}{1-\alpha^2}\Big], \ s.t \\
&x_1^2 + q_1 y_1^2 = y_0^2(1+q_1) + d_L \\
&x_2^2 + q_2 y_2^2 = y_1^2(1+q_2) + d_H \\
&y_1^2 \geq -d_H, \ y_2^2 = 0
\end{aligned}$$

[16] The distinction between safe and risky assets is sloppy because there is no risk in the standard sense.

The second interpretation states that there is one long-lived asset in the economy and that investor two is endowed with extra units of the good every period. These two economies give equivalent allocations. Consider the first economy. The no-arbitrage condition between asset y and z is as follows,

$$q_t^z = d_k q_t^y, \ k = H, L$$

and using the no-arbitrage and the previous investment rule, we get the intertemporal constraints of both investors in each planning horizon as follows,

$$\begin{aligned} x_t^1 + q_t^y x_{t+1}^1 &= y_{t-1}^1(1 + q_t) \\ x_t^2 + q_t^y x_{t+1}^2 &= y_{t-1}^2(1 + q_t) + d_k + \underbrace{q_t^y d_{k'}}_{=q_t^z}, \ k, k^{'} = H, L \end{aligned}$$

Consider the second economy. The intertemporal constraints, given that the debt constraints are not binding, are as follows,

$$\begin{aligned} x_t^1 + q_t x_{t+1}^1 &= y_{t-1}^1(1 + q_t) \\ x_t^2 + q_t x_{t+1}^2 &= y_{t-1}^2(1 + q_t) + d_k + q_t d_{k'}, \ k, k^{'} = H, L \end{aligned}$$

and we immediately observe that the intertemporal constraints for the two economies are the same. Thus, the allocations are equivalent. The previous distinction is useful because we can justify the economy where we need an ad hoc rule to break the indeterminacy of portfolio holdings with a simpler economy with one asset and endowments.

For the rest of the talk in this section let us focus on the second economy. Before moving to more details, we can give another interpretation to the previous investment rule. Suppose asset z is the illiquid asset in the economy and asset y is the liquid asset[17]. The illiquid asset can be traded *only* through short-sales of the liquid asset. The following results can be interpreted under the distinction liquid/illiquid assets or risky/safe assets.

Let us rewrite the maximization problem of investors given that we focus on the second economy,

[17] For instance, the liquid asset represents corporate equity whereas the illiquid asset represent non corporate equity such as returns from privately owned firms. The distinction between liquid and illiquid assets follows Longstaff (2009).

$$\max\Big[\sum_{s=t}^{t+1}\beta^{s-t}\frac{x_s^{1-\alpha^1}}{1-\alpha^1}\Big],\ s.t \qquad \max\Big[\sum_{s=t}^{t+1}\beta^{s-t}\frac{x_s^{1-\alpha^2}}{1-\alpha^2}\Big],\ s.t$$

$$x_t^1+q_t y_t^1=y_{t-1}^1(1+q_t) \qquad x_t^2+q_t y_t^2=y_{t-1}^2(1+q_t)+d_k$$

$$x_{t+1}^1+q_{t+1}y_{t+1}^1=y_t^1(1+q_{t+1}) \qquad x_2^2+q_{t+1}y_2^2=y_t^2(1+q_{t+1})+d_{k'}$$

$$y_{t+1}^1=0 \qquad y_t^2\geq -d_{k'},\ y_{t+1}^2=0$$

and since in equilibrium[18] $q_{t+1}=0$, the debt constraint takes the previous form. The optimal asset demands are as follows,

$$y_t^1=y_{t-1}^1\frac{1+q_t}{q_t}\frac{\beta^{\frac{1}{\alpha^1}}q_t^{\frac{\alpha^1-1}{\alpha^1}}}{1+\beta^{\frac{1}{\alpha^1}}q_t^{\frac{\alpha^1-1}{\alpha^1}}} \tag{1.35}$$

$$y_t^2=\frac{\beta^{\frac{1}{\alpha^2}}q_t^{\frac{\alpha^2-1}{\alpha^2}}}{1+\beta^{\frac{1}{\alpha^2}}q_t^{\frac{\alpha^2-1}{\alpha^2}}}\Big[y_{t-1}^2\frac{1+q_t}{q_t}+\frac{d_k-\beta^{-\frac{1}{\alpha^2}}q_t^{\frac{1}{\alpha^2}}d_{k'}}{q_t}\Big] \tag{1.36}$$

To proceed with the analysis further, we need to say something about the characterization of equilibrium asset prices as we did before. Consider the following claim,

Claim II. *The equilibrium asset prices belong to the following intervals*

$$q_t\in\left[\beta\left(\frac{1+d_k}{1+d_{k'}}\right)^{a^{-i}},\beta\left(\frac{1+d_k}{1+d_{k'}}\right)^{a^i}\right],\ \forall t,\ \ k,k^{'}=H,L \tag{1.37}$$

Proof. To prove that the bounds of the intervals in (37) are the ones stated above, consider the following thought experiment: Suppose we consider a representative investor's economy where the representative investor is endowed with illiquid wealth, d_k, and there is a liquid asset that she can invest every period. The no-trade asset price of the liquid asset in every planning horizon is as follows

$$q_t=\beta\left(\frac{1+d_k}{1+d_{k'}}\right)^{a^i}\ \ \forall t,\ \ k,k^{'}=H,L$$

[18] because we have imposed that $y_{t+1}^2=0$

Let us go back to the economy with two investors. The optimal demands in every planning horizon are as follows

$$x_t^1 = \frac{y_{t-1}^1(1+q_t)}{1+\beta^{\frac{1}{a^1}} q_t^{\frac{a^1-1}{a^1}}}, \; x_{t+1}^1 = x_t^1 \frac{\beta^{\frac{1}{\alpha^1}}}{q_t^{\frac{1}{\alpha^1}}}$$
$$x_t^2 = \frac{y_{t-1}^2(1+q_t)+d_k+q_t d_{k'}}{1+\beta^{\frac{1}{a^2}} q_t^{\frac{a^2-1}{a^2}}}, \; x_{t+1}^2 = x_t^2 \frac{\beta^{\frac{1}{\alpha^2}}}{q_t^{\frac{1}{\alpha^2}}}$$

and we have assumed that the debt constraint of investor two is not binding. Suppose that there exists an equilibrium in every planning horizon. Consider the good's market clearing in $t+1$ for the planning horizon $t \to t+1$,

$$x_t^1 \frac{\beta^{\frac{1}{\alpha^1}}}{q_t^{\frac{1}{\alpha^1}}} + x_t^2 \frac{\beta^{\frac{1}{\alpha^2}}}{q_t^{\frac{1}{\alpha^2}}} = \frac{1+d_{k'}}{1+d_k} \underbrace{(x_t^1 + x_t^2)}_{=1+d_k}$$

and it can be rewritten as follows

$$x_t^1 \left(\frac{\beta^{\frac{1}{\alpha^1}}}{q_t^{\frac{1}{\alpha^1}}} - \frac{1+d_{k'}}{1+d_k} \right) + x_t^2 \left(\frac{\beta^{\frac{1}{\alpha^2}}}{q_t^{\frac{1}{\alpha^2}}} - \frac{1+d_{k'}}{1+d_k} \right) = 0$$

and for equilibrium asset prices outside the intervals in (37), the terms in parenthesis of the above market clearing have the same sign and are different from zero. Thus, the LHS is not equal to zero and as a consequence this violates market clearing in every planning horizon.

□

Consider also the following two assumptions,

Assumption I. *Individuals are enough patient, β close to one.*

Assumption II. *The fluctuations of the illiquid asset are low, the d's are small enough and the following inequalities are true,*

$$1 > d_H > d_L > 0$$

Assumption I simplifies the analysis whereas assumption II makes the problem more interesting as we will demonstrate below. In the previous sections we analyzed conditions where some individuals/investors vanish in the long-run. In this section we want to analyze a phenomenon which is similar to the one studied before. Investors become *bankrupt* if they accumulate enough debt such that they can not repay back. To formalize this idea, let us go back to the optimal asset demands in (35),(36). Expression (36) is complicated because investor two holds illiquid wealth in her portfolio. On the other hand, expression (35) is very simple because investor one does not hold illiquid wealth in her portfolio. Suppose we know that there exists an $\epsilon > 0$, such that in any equilibrium the following is true,

$$\frac{y_t^1}{y_{t-2}^1} > 1 + \epsilon, \ \forall t \tag{1.38}$$

and the reason we consider the ratio of asset holdings every two periods is because the returns of the illiquid asset every two periods are the same. If the inequality in (38) is true, then the asset holdings of investor one are strictly increasing in every planning horizon. Suppose both investors start with initial holdings of the liquid asset, $y_0^1 > 0, 1 - y_0^1 > 0$. If (38) is true, then in any equilibrium, $y_t^1 + y_t^2 = 1$, investor two will start short-selling after a specific planning horizon and as a consequence she will start accumulating more and more debt. The usefulness of assumption II is to guarantee that we do not get trivial results. If we inject enough fluctuations in the illiquid asset, investor two is endowed with enough units of the good in some planning horizons. She takes short positions in the market because of the difference in her endowments today and tomorrow in a given planning horizon. On the other hand, if the returns of the illiquid asset are less than the liquid one and there are no fluctuations in the returns of the illiquid asset, equilibrium asset prices equal β in every planning horizon and expression (38) equals one. In that case after trading in one planning horizon the equilibrium is the same in every other planning horizon. There are no dynamics. We consider small fluctuations in the return of the illiquid asset in order to get dynamics in the economy and also not to give a high incentive to investor two to take short positions in the market.

The following lemma describes conditions such that there exists an $\epsilon > 0$ which satisfies condition (38),

Lemma I. *There exists an $\epsilon > 0$ such that (38) is true if either one of the following conditions is true*

$$\alpha^2 > \alpha^1 > 1 \textit{ or } \alpha^1 > \alpha^2 > 1$$
$$1 > \alpha^2 > \alpha^1 > \frac{1}{2} \textit{ or } 1 > \alpha^1 > \alpha^2 > \frac{1}{2}$$

and there is low heterogeneity, the difference between α^1, α^2 is small enough.

Proof. Consider the ratio in (38),

$$\frac{y_t^1}{y_{t-2}^1} = \frac{1+q_t}{q_t}\frac{1+q_{t-1}}{q_{t-1}}\frac{\beta^{\frac{1}{\alpha^1}} q_t^{\frac{\alpha^1-1}{\alpha^1}}}{1+\beta^{\frac{1}{\alpha^1}} q_t^{\frac{\alpha^1-1}{\alpha^1}}}\frac{\beta^{\frac{1}{\alpha^1}} q_{t-1}^{\frac{\alpha^1-1}{\alpha^1}}}{1+\beta^{\frac{1}{\alpha^1}} q_{t-1}^{\frac{\alpha^1-1}{\alpha^1}}} \tag{1.39}$$

we want to show that (39) is bounded below by a number greater than one. Consider the equilibrium asset price intervals from claim II. Make the RHS of (39) the smallest possible by using the appropriate bounds from the intervals in (37). If the smallest possible value of the RHS of (39) is greater than one, then expression (38) is satisfied.

It turns that low heterogeneity is the key. Consider the following graph which depicts the smallest value of (39) for the case where $\alpha^1 > \alpha^2 > 1$,

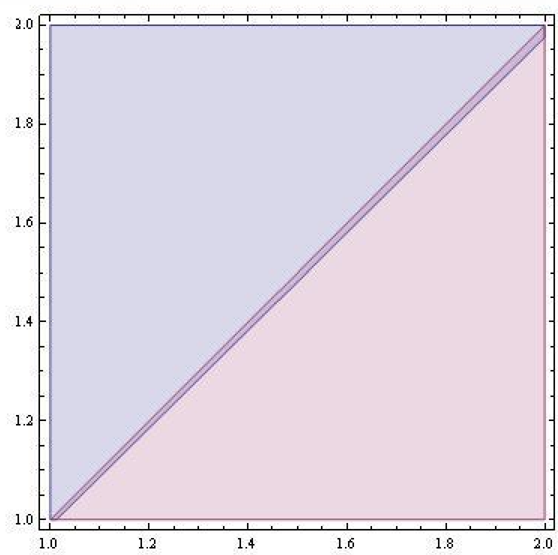


Figure 1.5

Where in the vertical axis we depict α^2 and in the horizontal, α^1. Also for simplicity we have fixed, $d_L = 0.2, d_H = 0.23, \beta = 1$. In the dark purple region close to the 45 degree line, the smallest value of (39) is greater than one. For the other cases, the graph is similar to the one above.

□

The previous lemma showed that the asset holdings of investor one are strictly increasing in every planning horizon. This is sufficient for the argument we want to make. Nevertheless, consider the asset demands of the second investor in every planning horizon[19]. Consider the expression (36) and also the ratio of asset holdings of investor two every two periods

[19] This argument is redundant.

$$\frac{y_t^2}{y_{t-2}^2} = \underbrace{\frac{1+q_t}{q_t}\frac{1+q_{t-1}}{q_{t-1}}s_{2,t},s_{2,t-1}}+$$

$$s_{2,t}\frac{1+q_t}{q_t}\frac{1}{y_{t-2}^2}\left[\underbrace{s_{2,t-1}\frac{e_{t-1}-\beta^{-\frac{1}{\alpha^2}}q_{t-1}^{\frac{1}{\alpha^2}}e_t}{q_{t-1}}+\frac{e_t-\beta^{-\frac{1}{\alpha^2}}q_t^{\frac{1}{\alpha^2}}e_{t+1}}{1+q_t}}\right] \tag{1.40}$$

where e_t is the illiquid wealth of investor two and

$$s_{2,t} = \frac{\beta^{\frac{1}{\alpha^2}}q_t^{\frac{\alpha^2-1}{\alpha^2}}}{1+\beta^{\frac{1}{\alpha^2}}q_t^{\frac{\alpha^2-1}{\alpha^2}}}$$

We can show following the argument in the previous lemma that the first term in the underbrace is always greater than one. Thus, the second term in the underbrace must be negative and the ratio $\frac{y_t^2}{y_{t-2}^2}$ must be bounded above by a number less than one when investor two holds positive claims of the liquid asset. When investor two holds negative claims of the liquid asset, short-sells, $\frac{y_t^2}{y_{t-2}^2}$ must be greater than one because by the previous lemma she will accumulate more and more debt. Using the intervals in (37) we get the following fact

$$q_{t,L}^{\frac{1}{\alpha^2}} > \frac{d_L}{d_H},\; q_{t,H}^{\frac{1}{\alpha^2}} < \frac{d_H}{d_L} \tag{1.41}$$

where the intervals were constructed under the assumption that[20] $\alpha^2 > \alpha^1$. We can rewritte (40) depending on the shock every period t as follows,

$$\frac{y_t^2}{y_{t-2}^2} = \frac{1+q_{t,L}}{q_{t,L}}\frac{1+q_{t-1,H}}{q_{t-1,H}}s_{2,t,L}s_{2,t-1,H}-$$

$$\frac{1}{y_{t-2}^2}s_{2,t,L}\frac{1+q_{t,L}}{q_{t,L}}\left[\underbrace{\frac{d_H q_{t,L}^{1/\alpha^2}-d_L}{1+q_{t,L}}-s_{2,t-1,H}\frac{d_H-d_L q_{t-1,H}^{1/\alpha^2}}{q_{t-1,H}}}\right]$$

$$\frac{y_t^2}{y_{t-2}^2} = \frac{1+q_{t,L}}{q_{t,L}}\frac{1+q_{t-1,H}}{q_{t-1,H}}s_{2,t,L}s_{2,t-1,H}-$$

$$\frac{1}{y_{t-2}^2}s_{2,t,H}\frac{1+q_{t,H}}{q_{t,H}}\left[\underbrace{s_{2,t-1,L}\frac{-d_L+d_H q_{t-1,L}^{1/\alpha^2}}{q_{t-1,L}}-\frac{-d_L q_{t,H}^{1/\alpha^2}+d_H}{1+q_{t,H}}}\right]$$

[20]We could consider also the case where $\alpha^2 < \alpha^1$ and get the same results since we focus on the case of sufficiently low heterogeneity.

and combining sufficiently low fluctuations of illiquid wealth, sufficiently low heterogeneity and (41), we can show that the terms in the underbraces are positive. In order to do this, consider the first and the second term in each of the terms in underbraces. Given the previous conditions, we can make the first term the smallest possible and the second the highest possible from the intervals in (37) and show that the difference is positive. If this difference is positive then the terms in the underbraces are always positive under the previous conditions. Since the terms in the underbarce are positive we can move one step further and show that, $\frac{y_t^2}{y_{t-2}^2} < 1-\epsilon$, for $\epsilon > 0$ when investor two holds positive claims of the liquid asset. Simply make the first term of $\frac{y_t^2}{y_{t-2}^2}$ the highest possible and the second term the smallest possible. Also, in the RHS of $\frac{y_t^2}{y_{t-2}^2}$, we have the asset holdings of period $t-2$. Suppose we treat $t-2$ as the initial period and y_{t-2} as initial asset holdings. We can show that under the previous conditions, sufficiently low fluctuations of illiquid wealth, sufficiently low heterogeneity and (41), $\frac{y_t^2}{y_{t-2}^2} < 1-\epsilon$, $\forall t$. Thus, investor two will sell the positive claims of the liquid asset and she will start and accumulating debt.

Consider the following proposition

Proposition VII. *Investor two will go bankrupt in finite time.*

Consider first a *heuristic* argument. The two key concepts are the myopic behavior of investors and the fact that the asset holdings of investor one are strictly increasing, condition (38). Consider the following diagram which splits the economy in separate planning horizons,

$$\underbrace{(d_L, d_H)}_{L}, \underbrace{(d_H, d_L)}_{H}, \underbrace{(d_L, d_H)}_{L}, \underbrace{(d_H, d_L)}_{H}, ...$$

where the first entry is the the return of the illiquid wealth in every actual period and the second entry is the return of the illiquid wealth tomorrow. This diagram is informative if we combine it with condition (38). From (38) we know that investor two will start going in debt after a certain point. Also, from the previous diagram, we can infer something simple. The evolution of debt is not going to be smooth. In L planning horizons individuals can hold more debt because they expect more productive illiquid wealth tomorrow. In the H horizons the opposite happens. From (38) we know that debt is going to be strictly increasing. There must be a period T^*, where investor two enters a H horizon with so much debt from yesterdays plan such that there can not be a well-defined equilibrium where she can consume a positive amount of the aggregate output. Lastly, when investor two wakes up in a new planning horizon she realizes that the debt she owes increases by the capital value in the market which was zero from yesterdays plan. Thus, bankruptcy is the only possibility.

Proof. Consider the asset demand of investor one at period t. It can be written as follows,

$$y_t^1 = y_1^1 \prod_{k=1}^{t-1} \frac{1+q_k}{q_k} \frac{1+q_{k+1}}{q_{k+1}} s_{1,k} s_{1,k+1}$$

from (38) we know that the product term is bounded below as follows,

$$\prod_{k=1}^{t-1} \frac{1+q_k}{q_k} \frac{1+q_{k+1}}{q_{k+1}} s_{1,k} s_{1,k+1} > (1+\epsilon)^{t-1}$$

for a fixed $\epsilon > 0$. By market clearing we know also that $y_t^1 < 1+d_L < 1+d_H$. The first bound applies to H horizons where $y_t^2 \geq -d_L$ whereas the second to L horizons where $y_t^2 \geq -d_H$. Combining these two facts we get

$$(1+\epsilon)^{t-1} < \prod_{k=1}^{t-1} \frac{1+q_k}{q_k} \frac{1+q_{k+1}}{q_{k+1}} s_{1,k} s_{1,k+1} < \frac{1+d_L}{y_1^1} < \frac{1+d_H}{y_1^1}$$

but as t increases, the lower bound will increase and it will be greater than the first upper bound, $\frac{1+d_L}{y_1^1}$, which is a fixed number. This will happen for a finite t^*.

□

An Example with log-preferences. To get a better picture of the above result, consider the same institutional set-up as before but endow investors with identical preferences, log-utilities. Without repeating much of the details, the equilibrium asset demands of investors are as follows

$$y_t^1 = y_{t-1}^1 \frac{\beta}{1+\beta} \frac{1+e_{t+1}+\beta(1+e_t)}{\beta(1+e_t)}, \; y_t^2 = 1 - y_t^1$$

where individual one has no illiquid wealth and only individual two is endowed with illiquid wealth every period.

Fix, $1 > d_H > d_L > 0$, and $d's$ are sufficiently small. Even if $d's$ are sufficiently small, the following is true

$$\frac{y_t^1}{y_{t-2}^1} = \frac{\beta}{1+\beta} \frac{1+d_H+\beta(1+d_L)}{\beta(1+d_L)} \frac{\beta}{1+\beta} \frac{1+d_L+\beta(1+d_H)}{\beta(1+d_H)} > 1, \; \forall t$$

which means that $\frac{y_t^1}{y_{t-2}^1}$ is bounded below by a number greater than one. From market clearing, $\frac{y_t^2}{y_{t-2}^2}$ is bounded above by a number less than one.

Suppose we know that there exists a small $\epsilon > 0$ such that

$$\frac{y_t^1}{y_{t-2}^1} < 1 - \epsilon, \ \forall t \tag{1.42}$$

If (42) is true, then bankruptcy is not possible. Investor one will never hold more than one unit of the liquid asset and as a consequence investor two will never start short-selling. The following lemma gives sufficient conditions for (42) to hold.

Lemma II. *There exists an $\epsilon > 0$ such that (42) is true if either one of the following conditions is true,*

$$\alpha^2 > \alpha^1, \ \alpha^1, \alpha^2 < \frac{1}{2}, \ \textit{or}$$
$$\alpha^1 > \alpha^2, \ \alpha^1, \alpha^2 < \frac{1}{2}$$

and there is low heterogeneity, the difference between α^1, α^2 is small enough.

Proof. The proof is similar to the previous lemma. We consider the highest possible value of $\frac{y_t^1}{y_{t-2}^1}$ and show that it is less than one. Fix $\alpha^2 \in [0.1, 0.5)$, $\alpha^1 \in [0.1, 0.5)$ and $\beta = 1$. We get a similar graph as in the previous lemma. $\square$

1.3.2 Myopic individuals with perfect foresight

In the previous sections we analyzed the case where myopic individuals have incorrect forecast about equilibrium asset prices in every planning horizon. We treated each planning horizon as an autonomous economy. In this section we will introduce an alternative equilibrium concept where myopic individuals have perfect foresight about equilibrium asset prices in the future. This alternative equilibrium concept is based on Lovo and Polemarchakis (2010).

1.3.2.1 Extreme Asset Prices

Suppose again there are two individuals with CRRA preferences that can plan up to two periods in the future[21],

[21]Individuals have symmetric planning horizons

$$
\begin{aligned}
&x_t^i + \overline{q}_t y_t^i = y_{t-1}^i(d_t + \overline{q}_t) \\
&x_{t+1}^i + \widetilde{q}_{t+1} y_{t+1}^i = y_t^i(d_{t+1} + \widetilde{q}_{t+1}) \\
&x_{t+2}^i + \widetilde{q}_{t+2} y_{t+2}^i = y_{t+1}^i(d_{t+2} + \widetilde{q}_{t+2}) \\
&y_{t+2}^i = 0
\end{aligned}
$$

where $\overline{q}$ denotes the realized price every period and $\widetilde{q}$ denotes the individual's expectation about the future price. Consider the following definition of a *Perfect Foresight Myopic Spot Equilibrium* (PFMSE)

PFMSE. *A PFMSE is a sequence of asset prices $\{q_t\}_{t=1}^{\infty}$ such that (i) individuals maximize utility s.t the b.c's given q_t (ii) the* ***spot*** *markets for the good and the asset are in equilibrium,*

$$\sum_i x_t^i = d_t, \ \sum_i y_t^i = 1$$

and individual's price expectations are correct.

We require a weaker equilibrium concept which allows market clearing only for the spot market of every actual period. Individuals forecasts *correctly* $\widetilde{q}_{t+1}$ and believe that $\widetilde{q}_{t+2}$ is zero from the viewpoint of period t. They wake up in $t+1$ and they already know the spot price of that period because it was forecasted correctly yesterday. The crucial part is that the price they thought it was zero from the viewpoint of period t, $\widetilde{q}_{t+2}$, it is not zero from the viewpoint of $t+1$. From the viewpoint of $t+1$, they forecast correctly the equilibrium price of $t+2$ and believe that the price of $t+3$ is zero. This argument repeats *ad infinitum*.

For the rest of the analysis in this section fix $\beta = 1$ and $d_t = 1$. The optimal demands of individual i in every actual period t are as follows,

$$x_t^i = \frac{y_{t-1}^i(1+q_t)}{1 + \left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} + \left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} q_{t+1}^{\frac{\alpha^i-1}{\alpha^i}}}$$

$$y_t^i = y_{t-1}^i \frac{1+q_t}{q_t} \frac{\left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} + \left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} q_{t+1}^{\frac{\alpha^i-1}{\alpha^i}}}{1 + \left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} + \left(\frac{q_t}{1+q_{t+1}}\right)^{\frac{\alpha^i-1}{\alpha^i}} q_{t+1}^{\frac{\alpha^i-1}{\alpha^i}}}$$

Consider the following example,

Parametric example. Fix

$$\alpha^1 = 0.4,\ \alpha^2 = 0.5,\ y_0^1 = 0.8,\ y_0^2 = 0.2 \tag{1.43}$$

Consider the following result,

Extreme Asset Prices. *Fix parameters as in (37). The* ***only*** *asset price sequence that satisfy spot market clearing every period has the following property: $q_{t+1} > q_t\ \forall t$. Asset prices are strictly increasing through time.*

The above result refers obviously to the specific parameters in (37). The really interesting part of the argument is that ***only a strictly increasing asset price sequence satisfies spot market clearing every period.***

The dynamics of equilibrium asset prices can be visualized in the following table

Table 1.2

q_t
2.231
2.2315
2.2323
2.2343
2.2409
2.265
2.3547
2.6852
3.8924
.

If we have employed the previous equilibrium concept where both agents mispredict the prices all the time, the equilibrium asset prices in every planning horizon would be $(2,1),(2,1),(2,1),(2,1),..$, where the first entry of each vector refers to the realized price every period and the second entry refers to the forecasted price tomorrow. The number 2 is what we will call the *myopic fundamental value of the tree* when we treat each planning horizon as an autonomous economy[22]. Trying to find appropriate initial values in order to construct an equilibrium price sequence, all the initial values have to be above 2.

Eventually the asset prices are going to be very large and they will continue increasing. *The economy can not reach a steady state.* Nevertheless, the allocation in that case converges as follows,

[22] When prices are revised all the time in order to ensure market clearing at each date of every planning horizon.

$$x_t^1 \to 0,\ x_t^2 \to 1$$
$$y_t^1 \to 0.865917,\ y_t^2 \to 0.1341$$

Let us try to understand the mechanism behind the previous example. The interesting part of the previous argument was that only a strictly increasing price sequence of assets could guarantee spot market clearing for every t.

Fix CRRA preferences again with $0 < a^i < 1$ and suppose the dividends of the tree are fixed to one every period. The optimal demand of individual i at period t can be written in the following form

$$x_t^i = \frac{y_{t-1}^i(1+q_t)}{1+\beta^{\frac{1}{\alpha^i}}\left(\frac{1+q_{t+1}}{q_t}\right)^{\frac{1-\alpha^i}{\alpha^i}}\left[1+\beta^{\frac{1}{\alpha^i}}/q_{t+1}^{\frac{1-\alpha^i}{\alpha^i}}\right]}$$

Before analyzing the mechanics of the previous example, it is useful to prove the following two points

- There exist an initial condition of prices, (q_1, q_2), such that the markets at date 1 clear.
- Given the initial condition, spot market equilibrium at subsequent dates is not guaranteed.

Let us demonstrate first that there exist an initial condition of prices that clears the markets at date 1. Let us focus on the good's market because from Walra's law the asset market at date 1 clears. Market clearing at date 1 can be written as follows,

$$\underbrace{\frac{y_0^1(1+q_1)}{1+\beta^{\frac{1}{\alpha^1}}\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+\beta^{\frac{1}{\alpha^1}}/q_2^{\frac{1-\alpha^1}{\alpha^1}}\right]}}_{x_1^1} + \underbrace{\frac{y_0^2(1+q_1)}{1+\beta^{\frac{1}{\alpha^2}}\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+\beta^{\frac{1}{\alpha^2}}/q_2^{\frac{1-\alpha^2}{\alpha^2}}\right]}}_{x_1^2} = 1$$

Fix a positive value for $q_2 > 0$. Then, we can always find a value for q_1 that clears the markets because of the following conditions

$$q_1 \to 0 \Rightarrow x_1^i \to 0$$
$$q_1 \to \infty \Rightarrow x_1^i \to \infty$$

and since the aggregate excess demand changes continuously with q_1, there exist $q_1 > 0$ that clears the market at date 1.

Let us go to the second point. Given the initial condition, (q_1, q_2), market clearing in period two is as follows

$$\underbrace{\frac{y_1^1(1+\overline{q}_2)}{1+\beta^{\frac{1}{\alpha^1}}\left(\frac{1+q_3}{\overline{q}_2}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+\beta^{\frac{1}{\alpha^1}}/q_3{}^{\frac{1-\alpha^1}{\alpha^1}}\right]}}_{x_2^1}+\underbrace{\frac{y_1^2(1+\overline{q}_2)}{1+\beta^{\frac{1}{\alpha^2}}\left(\frac{1+q_3}{\overline{q}_2}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+\beta^{\frac{1}{\alpha^2}}/q_3{}^{\frac{1-\alpha^2}{\alpha^2}}\right]}}_{x_2^2}=1$$

and $\overline{q}_2$ is fixed because it is already known from yesterday. We need to use the new price, q_3, to equilibrate the market in period two. The following conditions demonstrate that finding a new price that clears the market is not obvious,

$$\begin{aligned} q_3 \to 0 &\Rightarrow x_2^i \to 0 \\ q_3 \to \infty &\Rightarrow x_2^i \to 0 \end{aligned}$$

The reason why market clearing is not obvious is the following: *we could fall in a region of excess supply in the market and according to the conditions above there might be no price to reduce the excess supply and clear the market.* On the other hand, if there is an excess demand, there is always a price that clears the market.

Let us state the above conditions as a rule because it will become useful later on,

Market Clearing Rule (MKR). Fix an initial condition of market clearing prices, (q_{t-1}, q_t). The new price that individuals observe should satisfy the conditions below,

$$\begin{aligned} q_{t+1} \to 0 &\Rightarrow x_t^i \to 0 \\ q_{t+1} \to \infty &\Rightarrow x_t^i \to 0 \end{aligned}$$

Let us go to discuss the mechanics of the previous example and then try to see the more general picture.

To find the initial condition we do something simple. We know that the price combination, $(q_1, q_2) = (2, 1)$, clears all the markets if we view each planning horizon as consisting of autonomous economies every period. Fix $q_2 = 1$ and $q_1 < 2$. There is an excess supply in the market for any $q_1 < 2$. Fix $q_1 < 2$ and vary q_2. There is still an excess supply for any q_1, q_2. For $q_1 < 2$ we have fallen in the region of excess supply for any q_2. Fix $q_1 > 2$ and $q_2 = 1$. There is an excess demand given initial asset holdings. From the MKR we can either increase or decrease q_2 given $q_1 > 2$ and clear the market. It turns out that for any $q_1 > 2$, we need to decrease q_2 below 2 in order to clear the market. This is not good because according to the previous argument we will not be able to clear the market tomorrow with a spot price $q_2 < 2$. The initial condition we require is $q_2 > q_1 > 2$. In the example above we have that $(q_1, q_2) = (2.231, 2.2315)$.

Next, individuals wake up in period two and take as given q_2 which they know already from their correct forecast yesterday. In addition to that, they see another period in the future and a new price, q_3, that they could not take into account yesterday. Given the new initial asset holdings carried from yesterday, *fix initially* $\overline{q}_2 = q_3$ and check whether there is an excess supply or excess demand in the market. It turns out that in the example we get an excess demand in the market. Reducing the price is not good because it has to go below 2 to clear the market. The only possibility is to increase the price above $\overline{q}_2$ to clear the market and be able to clear the spot market when individuals wake up in a new planning horizon. The same procedure continues *ad infinitum*.

The previous case constitutes only an example. Let us try to do an ϵ generalization of the previous case. Proving that there exist *only* a strictly sequence of asset prices that is consistent with spot market clearing is not straightforward. We can show first that there exist a strictly increasing sequence of asset prices which is consistent with spot market clearing.

Fix $\beta = 1, d_t = 1$ and $0 < \alpha^i < 1$. Consider also the following definition of the myopic fundamental value (MFV) of the asset,

MFV. *The MFV of the asset at every initial period t of each planning horizon, q_t^{MFV}, is the price that results from treating each planning horizon as an autonomous economy and requiring that prices clear all the markets within the planning horizon from the viewpoint of t.*

and in our simple case[23], $q_t^{MFV} = 2 \ \forall t$.

The next lemma gives sufficient conditions for a date one spot market clearing initial condition with the following property to exist: $q_2 > q_1 > q_1^{MFV} = 2$,

Lemma III. *Fix $\beta = 1$, $d_t = 1$ and $0 < \alpha^i < 1$. If α^i is sufficiently close to one and initial asset prices are above q^{MFV} or if initial asset prices are* ***sufficiently higher*** *than q^{MFV} and α^i are not close to one, then there exist a date one market clearing condition with the following properties: $q_2 > q_1 > q_1^{MFV} = 2$.*

Proof. Consider the date one good's market clearing,

[23] For $\beta = 1, d_t = 1$.

$$\underbrace{\frac{y_0^1(1+q_1)}{1+\beta^{\frac{1}{\alpha^1}}\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+\beta^{\frac{1}{\alpha^1}}/q_2{}^{\frac{1-\alpha^1}{\alpha^1}}\right]}}_{x_1^1}+\underbrace{\frac{y_0^2(1+q_1)}{1+\beta^{\frac{1}{\alpha^2}}\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+\beta^{\frac{1}{\alpha^2}}/q_2{}^{\frac{1-\alpha^2}{\alpha^2}}\right]}}_{x_1^2}=1$$

and suppose individuals start with initial asset holdings as follows: y_0^1, y_0^2 and $y_0^1+y_0^2=1$. Rewrite the market clearing as follows,

$$(1-y_0^2)\left[\frac{(1+q_1)}{1+\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+1/q_2{}^{\frac{1-\alpha^1}{\alpha^1}}\right]}-1\right]+$$
$$y_0^2\left[\frac{(1+q_1)}{1+\left(\frac{1+q_2}{q_1}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+1/q_2{}^{\frac{1-\alpha^2}{\alpha^2}}\right]}-1\right]=0$$

The idea of the proof is to create an excess demand in the market by making each term in the brackets positive and then show that in order to reduce the excess demand and clear the market we need initial prices to be as in the statement of the lemma.

Fix initially $q_1>2$ and $q_2=q_1=q$. Each term in the brackets is positive if the following inequality is satisfied,

$$q^{\frac{1}{\alpha^i}}>(1+q)^{\frac{1-\alpha^i}{\alpha^i}}\left[1+1/q^{\frac{1-\alpha^i}{\alpha^i}}\right] \tag{1.44}$$

but we can easily show that given the conditions in the lemma, the above inequality is satisfied. Consider the graph of (44),

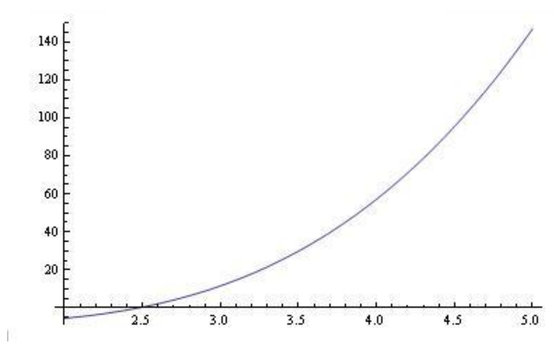


Figure 1.6

where we have fixed $\alpha^i=0.3$. In the horizontal axis we have q and in the vertical axis expression (44).

Since initially we required $q_2 = q_1 = q > 2$ by the MKR we know that we can always increase q_2 further, reduce the excess demand in the market and clear the market. $\square$

Consider also the following proposition

Proposition VIII. *Given the conditions of Lemma III are satisfied, there exist a strictly increasing sequence of asset prices that clears the spot markets every period.*

Proof. The proof follows the proof of Lemma III. Write again the market clearing of period two,

$$(1-y_1^2)\left[\frac{(1+q_2)}{1+\left(\frac{1+q_3}{q_2}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+1/q_3^{\frac{1-\alpha^1}{\alpha^1}}\right]}-1\right]+$$
$$y_1^2\left[\frac{(1+q_2)}{1+\left(\frac{1+q_3}{q_2}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+1/q_3^{\frac{1-\alpha^2}{\alpha^2}}\right]}-1\right]=0$$

and q_2 we know it from yesterdays forecast. Fix again $q_3 = q_2 = q$. Again we can make each term in the brackets positive if (44) is satisfied. But from the graph of (44) in the previous lemma we can repeat the same argument. $\square$

The previous argument guarantees the existence of a strictly increasing sequence of asset prices but it does not say that this is the only way to clear the spot markets *ad infinitum*. Consider a simple diagrammatic argument that illustrates the previous point. Consider again the market clearing equation,

$$(1-y_{t-1}^2)\left[\frac{(1+q_t)}{1+\left(\frac{1+q_{t+1}}{q_t}\right)^{\frac{1-\alpha^1}{\alpha^1}}\left[1+1/q_{t+1}^{\frac{1-\alpha^1}{\alpha^1}}\right]}-1\right]+$$
$$y_{t-1}^2\left[\frac{(1+q_t)}{1+\left(\frac{1+q_{t+1}}{q_t}\right)^{\frac{1-\alpha^2}{\alpha^2}}\left[1+1/q_{t+1}^{\frac{1-\alpha^2}{\alpha^2}}\right]}-1\right]=0$$

the following graph depicts the terms in brackets in the space of (q_t, q_{t+1}) for give values of $\alpha's$. For the following graph, fix $\alpha^1 = 0.4$, $\alpha^2 = 0.5$ and $q_t, q_{t+1} \in (0, 5]$.

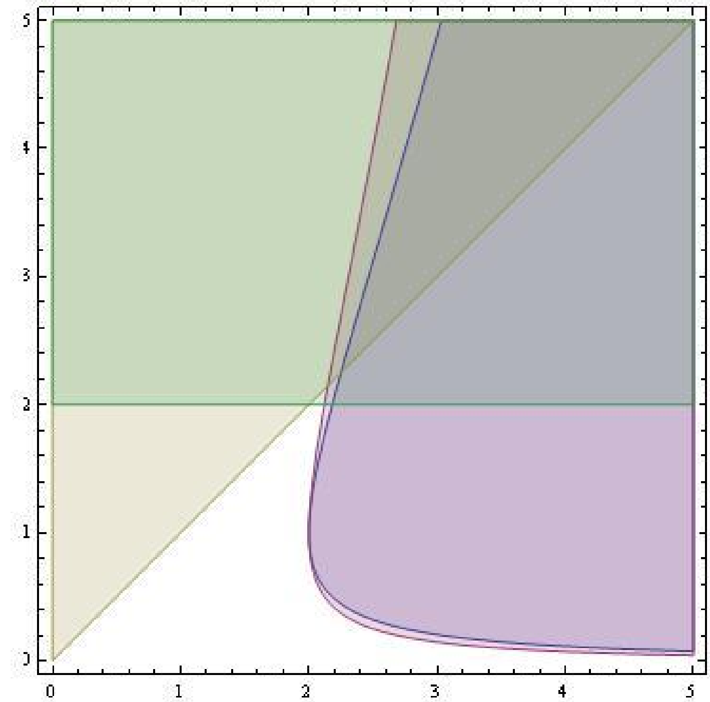


Figure 1.7

in the horizontal axis we have q_t and in the vertical axis q_{t+1}. The area on the right of the blue curve is the first term in brackets, individual one, and depicts the points where the term in brackets is positive. The envelope red curve above the blue graph is the second term in brackets, individual two, and shows again the points where it is positive. Thus, the region on the right of the blue curve is the aggregate excess demand region and the region to the left of the envelope one is the region of aggregate excess supply. The region between the two curves is the region where we should look for market clearing prices.

From the graph we see that for $q_t < 2$ there is no price q_{t+1} to reduce the excess supply in the market and restore equilibrium. For $q_t > 2$ we may be able to find an equilibrium for $q_{t+1} < 2$, but we will not be able to clear the market next period since we will fall in the excess supply region. The idea of the previous proof can be visualized in the above graph. If we start from a set of points on the 45 degree line and to the right of the blue graph, then we have an aggregate excess demand in the market. It is clear form the graph that reducing q_{t+1} is not an option because the price has to fall close to zero in order to clear the market. We will not be able to clear markets next period. The price has to increase even further. We can play with $\alpha's$ and consider various cases. The main message coming from the previous analysis and the above graph is unchanged.

§ 1.4 Hyperbolic Discounting and Survival in the Market

Suppose individuals have hyperbolic discount functions and they discount the short-run at a higher rate. Consider the following class of preferences,

$$U_t^i = \frac{x_t^{1-a^i}}{1-a^i} + \beta \frac{x_{t+1}^{1-a^i}}{1-a^i} + \zeta \sum_{j=2}^{\infty} \beta^j \frac{x_{t+j}^{1-a^i}}{1-a^i} \tag{1.45}$$

$$U_t^i = \frac{x_t^{1-a^i}}{1-a^i} + \zeta\beta \frac{x_{t+1}^{1-a^i}}{1-a^i} + \zeta \sum_{j=2}^{\infty} \beta^j \frac{x_{t+j}^{1-a^i}}{1-a^i} \tag{1.46}$$

where $\zeta \leqslant 1$, is the hyperbolic parameter. Consider first expression (45). For $\zeta = 0$ we are back to the extreme myopic case and for $\zeta = 1$ we go back to the benchmark case. With preferences as in (45), we can make the comparison between the case of myopia discussed before and that of hyperbolic discounting where individuals perceive the entire horizon. Expression (46) represents the standard present-biased preferences considered in the literature.

Following the previous analysis, we will split individuals into two types: naive and sophisticated individuals. The former behave in a myopic way and do not realize that their preferences are going to change in the future. The latter do realize that their preferences are going to change.

1.4.1 Exponential Discounting (Benchmark Case)

The benchmark case is similar to the one in section 3.1. Let us briefly repeat it. Consider an infinite horizon economy under certainty and two individuals with the following instantaneous utilities,

$$u^1(c) = \log(c), \ u^2(c) = 2\sqrt{c}$$

and time-separable intertemporal utilities. Individuals have identical discount rates. Also there is one long-lived asset with a constant dividend pattern equal to one across time. The budget constraint every period is as follows

$$c_t^i + q_t y_t^i = y_{t-1}^i(1+q_t)$$

From the assumption of identical discount rates, the fact that we consider an environment of certainty and most importantly aggregate output is constant over time, it is one every period, it is not difficult to show that in equilibrium each individual equalizes its consumption intertemporally. In equilibrium, $c_t^i = c^i$, $\forall t$.

The FOC for individual i between periods t, $t+1$ is as follows,

$$q_t u_c^i(c_t^i) = \beta(1+q_{t+1})u_c^i(c_{t+1}^i)$$

Substitute $c_t^i = c_{t+1}^i$ in the FOC in order to get

$$\frac{q_t}{1+q_{t+1}} = \beta$$

and solving recursively, $q = \frac{\beta}{1-\beta}$. We can easily verify that the latter is an equilibrium price which implies that individuals consume the following amount of the good every period,

$$c^1 = y_0^1, \; c^2 = 1 - y_0^1$$

where $(y_0^1, 1-y_0^1)$ are their initial shares of the asset. The previous allocation implies that individuals do not trade in equilibrium and just consume their initial endowments. In the next section we argue that hyperbolic-discounting modifies this no-trade result.

1.4.2 Sophisticated individuals

Following the literature on time-inconsistency[24] we model each sophisticated individuals as consisting of different selves that play a game with each other self. The self of individual t takes as given the strategies of future selves and simply chooses consumption and asset holdings at period t. The economy consists of two individuals with the following instantaneous utilities,

$$u^1(c) = \log(c), \; u^2(c) = 2\sqrt{c}$$

and the intertemporal utility takes the following form,

$$U_t^i = u^i(c_t) + \zeta\beta u^i(c_{t+1}) + \zeta\sum_{j=2}^{\infty}\beta^j u^i(c_{t+j})$$

[24]Phelps and Pollak (1968), Laibson (1996), Laibson (1997)

where $\zeta \in (0,1)$ is the quasi-hyperbolic parameter. The rest of the economy is as before with the exception that the dividend process of the tree is constant, $d_t = d = 1$.

To solve the problem for each individual we will combine stuff from Luttmer and Mariotti (2003) and Harris and Laibson (2001) to derive the generalized Euler equation. In particular, we will use the heuristic derivation of the generalized Euler equation from Harris and Laibson (2001).

Define the following functions,

$$F_t^i(w_t) = u^i(c_t(w_t)) + \zeta \sum_{j=1}^{\infty} \beta^j u^i(c_{t+j}(w_t))$$
$$V_t^i(w_t) = u^i(c_t(w_t)) + \sum_{j=1}^{\infty} \beta^j u^i(c_{t+j}(w_t))$$

where $c_{t+j}(w_t)$ is the implied consumption of self $t+j$ in the subgame in which self t starts with wealth w_t^i. The current value function $F_t^i(w_t)$ is the utility of the self t in the subgame where the self t starts with initial wealth w_t^i. The continuation value $V_t^i(w_t^i)$ is the utility perceived by self $t-1$ for the subgame of self t. The budget constraint of period t can be written in a more convenient form,

$$c_t^i + R_{t+1} w_{t+1}^i = w_t^i$$

where $w_t^i = y_{t-1}^i(1+q_t)$ and $R_{t+1} = \frac{q_t}{1+q_{t+1}}$.

The problem of self t becomes as follows,

$$F_t^i(w_t^i) = \max_{c_t, w_{t+1} \geq 0} \{u(c_t^i) + \beta\zeta V_{t+1}^i(w_{t+1}^i)\}, \ s.t$$
$$c_t^i + R_{t+1} w_{t+1}^i = w_t^i$$

and the above problem can be rewritten as follows,

$$F_t^i(w_t^i) = \max_{w_{t+1} \geq 0} \{u(w_t^i - R_{t+1} w_{t+1}^i) + \beta\zeta V_{t+1}^i(w_{t+1}^i)\} \tag{1.47}$$

the FOC is as follows,

$$R_{t+1} u^{'}(w_t^i - R_{t+1} w_{t+1}^i) = \beta\zeta \frac{dV_{t+1}^i(w_{t+1}^i)}{dw_{t+1}}$$

Combine the FOC with the envelope condition, assuming that we have a solution of the form $w_{t+1} = g(w_t)$, we get

$$\frac{dF_t^i(w_t^i)}{dw_t} = u^{'}(c_t(w_t)) \tag{1.48}$$

The problem as it is stated in (47) is not recursive because the function F and V are different. From the definition of these functions we know that the following relation is true,

$$\zeta V_{t+1}^i(w_{t+1}) = F_{t+1}^i(w_{t+1}) - (1-\zeta)u(c_{t+1}(w_{t+1})) \tag{1.49}$$

substitute (49) into (47) to get

$$F_t^i(w_t^i) = u^i(c_t^i) + \beta \underbrace{\Big[F_{t+1}^i(w_{t+1}^i) - (1-\zeta)u^i(c_{t+1}^i(w_{t+1}))\Big]} \tag{1.50}$$

and successive substitutions of F, yields the right objective function for self t. From expression (50) we can see the difference with naive individuals. The self t of individual i will choose consumption and assets at t taking as given the second term in the underbrace of the RHS. Individuals are sophisticated because they can take into account the preference reversal of future selves, $F_{t+1}^i(w_{t+1}^i)$. Successive substitutions of F_{t+1}^i in (50) yields the right objective function for self t.

Finally, differentiating (49) we get,

$$\zeta\frac{dV_{t+1}^i(w_{t+1}^i)}{dw_{t+1}} = \frac{dF_{t+1}^i(w_{t+1}^i)}{dw_{t+1}} - (1-\zeta)u^{'}(c_{t+1}(w_{t+1}))c_{t+1}^{'}(w_{t+1}) \tag{1.51}$$

Substituting (48),(51) into the FOC, we get the *generalized Euler equation*(GEE) as follows,

$$R_{t+1}u^{'}(w_t^i - R_{t+1}w_{t+1}^i) = \beta u^{'}(c_{t+1}(w_{t+1}))\Big[1-(1-\zeta)c_{t+1}^{'}(w_{t+1})\Big] \tag{1.52}$$

The last piece in the puzzle that we require is to say something about $c_{t+1}(w_{t+1})$. Since we have assumed homothetic preferences, we can conjecture the following form for the consumption function of individuals,

$$c_t^i = (1-s_t^i)w_t^i \tag{1.53}$$

where s_t^i is the saving rate of self t at date t. *Importantly, the saving rate does not depend on wealth.*

Combining (52),(53) and the instantaneous utilities for each individual, the generalized Euler equation for each individual can be written as follows,

$$R_{t+1}(w_t^2 - R_{t+1}w_{t+1}^2)^{-0.5} = \Big((1 - s_{t+1}^2)w_{t+1}^2\Big)^{-0.5}\Big[\beta\zeta(1 - s_{t+1}^2) + \beta s_{t+1}^2\Big]$$
$$R_{t+1}(w_t^1 - R_{t+1}w_{t+1}^1)^{-1} = \Big((1 - s_{t+1}^1)w_{t+1}^1\Big)^{-1}\Big[\beta\zeta(1 - s_{t+1}^1) + \beta s_{t+1}^1\Big]$$

and the term in brackets is the effective discount factor for each individual in that case.

Manipulating the Euler equations above we end up in the following formula for the wealth dynamics of each individual,

$$w_{t+1}^i = \frac{w_t^i}{R_{t+1}} s_t^i$$

where

$$s_t^1 = \frac{\frac{\left(\beta\zeta(1-s_{t+1}^1)+\beta s_{t+1}^1\right)}{1-s_{t+1}^1}}{1 + \frac{\left(\beta\zeta(1-s_{t+1}^1)+\beta s_{t+1}^1\right)}{1-s_{t+1}^1}},\ s_t^2 = \frac{R_{t+1}^{-1}\frac{\left(\beta\zeta(1-s_{t+1}^2)+\beta s_{t+1}^2\right)^2}{1-s_{t+1}^2}}{1 + \frac{R_{t+1}^{-1}\left(\beta\zeta(1-s_{t+1}^2)+\beta s_{t+1}^2\right)^2}{1-s_{t+1}^2}} \tag{1.54}$$

and (54) gives the best-response functions for each individual in the subgame that self t participates. These best-responses can be rewritten as follows,

$$s_t^1 = \phi(s_{t+1}^1),\ s_t^2 = f(s_{t+1}^2)$$

Before advancing further, let us state the definition of an intarpersonal and of a competitive equilibrium for this economy. The definition is adapted from Luttmer and Mariotti (2003),

Intrapersonal and Competitve Equilibrium. An *intarpersonal equilibrium* for the subgame that self t participates is a strategy profile of consumption and next period wealth, $\{c_t, w_{t+1}\}_{t=1}^{\infty}$, that self t chooses taking as given the sequence of prices $\{q_t\}_{t=1}^{\infty}$, and the strategies of date s selves, $s > t$.

A *competitive equilibrium* is a strategy profile, $\{c_t, w_{t+1}\}_{t=1}^{\infty}$, and a sequence of asset prices, $\{q_t\}_{t=1}^{\infty}$, such that (i) $\{c_t, w_{t+1}\}_{t=1}^{\infty}$ is an intrapersonal equilibrium at prices $\{q_t\}_{t=1}^{\infty}$, and (ii) good's and asset markets clear at every period,

$$\sum_{i=1}^{2} c_t^i = d_t = 1,\ \sum_{i=1}^{2} y_t^i = 1, \forall t$$

In this section we show that we can construct an equilibrium with the following characteristics,

Non-Survival Equilibrium. *Fix $\beta = 0.99$, $\zeta = 0.008$. We will construct an intarpersonal competitive equilibrium where the saving rate of individual two is sufficiently low and always lower than the saving rate of individual one. Individual two vanishes in the long-run and asset prices converge to $q \to \frac{\zeta\beta}{1-\beta} = 0.792$.*

We *construct* the aforementioned equilibrium using the following four steps,

- Step 1: Low saving Trap
- Step 2: Characteristics of the equilibrium
- Step 3: An economy with two log-individuals
- Step 4: An explicit construction

Step 1: Low saving Trap.

The crucial step in the construction of the previous equilibrium is the best-response functions in (54). Consider the best-response of the log-guy. We assume that the log-guy plays symmetric strategies, $s^1_t = s^1_{t+1}$, and saves a constant fraction of her wealth every period,

$$s^1 = \frac{\beta\zeta}{1-\beta(1-\zeta)} \tag{1.55}$$

The analysis of the log-guy is relatively simple because her saving rate does not depend on prices. On the other hand, the saving rate of individual two depends on prices.

Consider the following thought experiment: Fix present-value prices, R_{t+1}, and compute the fix points of the individual's two best response function in (54), $s^2 = f(.., R, s^2)$. The result is the following two expressions:

$$s^2_t = \frac{R_{t+1} + 2\beta^2(-1+\zeta)\zeta + \sqrt{R_{t+1}}\sqrt{R_{t+1} + 4\beta^2(-1+\zeta)\zeta}}{2\beta^2(-1+\zeta)^2} \tag{1.56}$$

$$s^2_t = \frac{R_{t+1} + 2\beta^2(-1+\zeta)\zeta - \sqrt{R_{t+1}}\sqrt{R_{t+1} + 4\beta^2(-1+\zeta)\zeta}}{2\beta^2(-1+\zeta)^2} \tag{1.57}$$

As we will demonstrate in the next step, we want to construct an equilibrium where asset prices and present-value prices converge to $q^* = \frac{\beta\zeta}{1-\beta} = 0.792$ and $R^* = \frac{\beta\zeta}{1-\beta(1-\zeta)} = 0.442$. These are the *no-trade prices* when individual one is the *representative* individual in the market.

At R^*, expressions (56),(57) take the following values respectively:

$$s^2 = 0.442 = \frac{\beta\zeta}{1-\beta(1-\zeta)} = R^* = s^1$$
$$s^2 = 1.472 \times 10^{-4}$$

It is evident from the two fix points above that the different selves of individual two could coordinate to either situation. I define the fix point where all the selves of individual two save relatively low compare to individual one as a *Low Saving Trap.* In particular, consider the following definition

Low Saving Trap. *A Low Saving Trap is a situation where the saving rate of individual two is initially lower than that of individual one and is also strictly decreasing in the future until it converges to a finite point.*

In the next steps we argue that the phenomenon of a *Low Saving Trap* for individual two emerges as part of the equilibrium we construct.

Step 2: Characteristics of the equilibrium.

The usefulness of this step is to describe the characteristics of the equilibrium we want to construct before we start with the construction.

The equilibrium sequence of asset prices and present-value prices should have the following characteristics

1. Initially, $q_1 < \frac{\beta}{1-\beta} = 0.792$, $R_2 < \frac{\beta\zeta}{1-\beta(1-\zeta)} = 0.442$
2. $\{q_t\}_{t=1}^{\infty}$, strictly increasing with a decreasing rate
3. $\{R_{t+1}\}_{t=1}^{\infty}$, strictly increasing
4. $q_t \to 0.792$, $R_{t+1} \to 0.442$ as $t \to \infty$
5. Find $\{q_t\}_{t=1}^{\infty}$, $\{R_{t+1}\}_{t=1}^{\infty}$ such that $\{s_t^2\}_{t=1}^{\infty}$ is *sufficiently close* to $s^2 = 1.472 \times 10^{-4}$

Since we want to construct an equilibrium where individual two vanishes in the long-run, characteristics 1-4 simply describe some properties of the equilibrium price sequence. It should converge to the no-trade prices as if individual one was the representative individual in the market. On the other hand, the last characteristic, number 5, is more elaborate and we will comment on it in the next paragraph.

The most crucial step in the construction is to construct a sequence of saving rates for individual two such that he falls in a Low Saving Trap as defined before. In particular, the sequence of saving rates of individual would have the following property:

$$s_1^2 > s_2^2 > s_3^2 > \cdots > s_t^2 \to s_\infty^2 = 1.472 \times 10^{-4},\ as\ t \to \infty \tag{1.58}$$

Going back to the last characteristic enumerated before, number 5, the equilibrium price sequence must make s_1^2 sufficiently close to s_∞^2 initially and also must be consistent with property (58).

Step 3: An economy with two logarithms.

Consider an economy of two individuals with the following intertemporal utilities[25]

$$U_t^1 = \log(c_t) + \zeta \sum_{j=1}^{\infty} \beta_1^j \log(c_{t+j})$$
$$U_t^2 = \log(c_t) + \delta \sum_{j=1}^{\infty} \beta_2^j \log(c_{t+j})$$

where $\beta_1 = 0.99$ and $\zeta = 0.008$. Thus, individual one has identical preferences as in the original economy. From (55) we get that $s^1 = 0.442$. Let us assume that $\beta_1 > \beta_2$ and $\zeta > \delta$. The saving rate of individual two is as follows,

$$s^2 = \frac{\beta_2 \delta}{1 - \beta_2(1-\delta)} \tag{1.59}$$

[25]The main set-up is similar to the that of the original economy.

Suppose we choose values for (β_2, δ)[26] such that $s^2 = 1.472 \times 10^{-4}$. Thus, $s^1 > s^2$, by construction. Consider the good's market clearing every period t,

$$\underbrace{y^1_{t-1}(1+q_t)(1-s^1)}_{c^1_t} + \underbrace{(1-y^1_{t-1})(1+q_t)(1-s^2)}_{c^2_t} = 1 \tag{1.60}$$

and to get the consumption demands we have used (53). From (60), we can solve explicitly for asset prices every period since the saving rates of individuals are constant[27],

$$q_t = \frac{s^2 + y^1_{t-1}(s^1 - s^2)}{(1-s^2) - y^1_{t-1}(s^1 - s^2)} \tag{1.61}$$

Since $s^1 > s^2$, individual two will continuously sell shares of the tree and individual one will continuously buy shares of the tree. In other words, $y^1_t \to 1$ and $y^2_t \to 0$ as $t \to \infty$. For any profile of initial asset holdings, $0 < y^i_1 < 1$, it is evident from (61) that

$$q_t \to \frac{s^1}{1-s^1} = \frac{\beta\zeta}{1-\beta} = 0.792, \ \textit{as } t \to \infty \tag{1.62}$$

The equilibrium asset price sequence, $\{q_t\}_{t=1}^{\infty}$, is strictly increasing and converges as in (62). The next step is to argue that it increases with a decreasing rate.

To show that $\{q_t\}_{t=1}^{\infty}$ increases with a decreasing rate at least after a certain point in time, consider the following thought experiment: Suppose individuals are endowed with equal asset holdings initially. Since $s^1 > s^2$, individual two will continuously keep selling shares of the tree and individual one will continuously keep buying them. The asset price initially is lower than 0.792 but $\{q_t\}_{t=1}^{\infty}$ increases and converges as in (62). Since individual two will keep selling shares of the tree, after a certain point in time his shares will become sufficiently low and he will want to continue selling shares of the tree with a decreasing rate. The reason for the latter effect has to do with the fact that individual two will prefer to save even an infinitesimal amount every period rather than zero. His marginal utility at zero is infinite. Since

[26] One combination is $\beta_2 = 0.024, \delta = 0.006$.

[27] From the assumption of log-utilities and the fact that individuals do not have other endowments.

individual two contemplates the previous scenario taking equilibrium prices as given, equilibrium prices should reflect the fact that after a certain point in time, individual two prefers to sell shares of the tree with a decreasing rate. It is evident from (62), that if y^1_{t-1} increases with a decreasing rate, then $\{q_t\}_{t=1}^{\infty}$ increases with a decreasing rate.

According to the previous argument, there exists a period T^*, such that individual two starts with sufficiently low initial asset holdings, $y^2_{T^*-1}$, and from T^* onwards asset holdings of individual one and asset prices increase with a decreasing rate. To sum up, consider the following properties of the equilibrium asset price sequence

Properties of the equilibrium asset price sequence. Let us assume that individual two starts with sufficiently low initial asset holdings, $y^2_0 > 0$ but close to zero. He is initially extremely poor. This assumption guarantees that $\{q_t\}_{t=1}^{\infty}$ increases with a decreasing rate and converges as in (62) and moreover it guarantees that q_1 is lower but sufficiently close to 0.792. According to the previous assumption on initial asset holdings, all the dynamic changes in the economy for asset demands, consumption demands, asset prices, are infinitesimally small.

Step 4: An explicit construction.

Let us go back to the original economy: $u^1(c) = \log(c)$, $u^2(c) = 2\sqrt{c}$ and $\beta = 0.99$, $\zeta = 0.008$. The main argument is that we will use the equilibrium price sequence from the previous economy, $\{q_t\}_{t=1}^{\infty}$, with the properties discussed at the end of the previous step, in order to construct *explicitly* an equilibrium for the original economy.

In order to construct an equilibrium for the original economy explicitly, we will use the following three steps

- Saving rates of individual two as decreasing steps
- Euler Equations
- Market Clearing

Saving behavior of individual two.

One of the main difficulties in the construction of an equilibrium for the original economy is the fact that the saving behavior of individual two is relatively complicated compared to the saving behavior of individual one. For ease of exposition and since it is the main focus of this step, let us repeat once more the best-response function of individual two

$$s_t^2 = \frac{R_{t+1}^{-1} \frac{\left(\beta\zeta(1-s_{t+1}^2)+\beta s_{t+1}^2\right)^2}{1-s_{t+1}^2}}{1 + \frac{R_{t+1}^{-1}\left(\beta\zeta(1-s_{t+1}^2)+\beta s_{t+1}^2\right)^2}{1-s_{t+1}^2}} \tag{1.63}$$

The best-response of individual two depends on his future self best-response and importantly on the present-value prices today. To analyze individual's two saving behavior we must figure out how his savings depend on present-value prices. We will construct a sequence of saving rates for individual two using the equilibrium asset price sequence computed in the previous step.

Consider the asset price sequence computed in step 3, $\{q_t\}_{t=1}^{\infty}$. We said in step 3 that if the initial asset holdings of individual two are sufficiently low, then $\{q_t\}_{t=1}^{\infty}$ is strictly increasing with a decreasing rate and q_1 is lower but sufficiently close to $q^* = 0.792$. This property turns out to be extremely important as we will demonstrate right away.

Since saving rates depend on present-value prices, R_{t+1}, we need to specify a sequence of present-value prices which will help us construct a sequence of saving rates. Present-value prices are defined as follows

$$R_{t+1} = \frac{q_t}{1 + q_{t+1}}$$

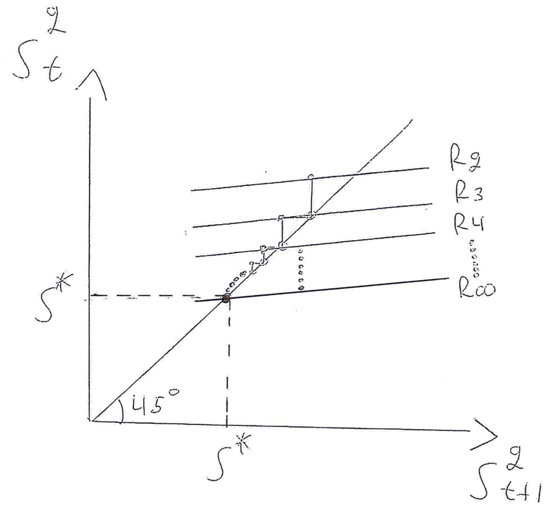
Consider the equilibrium price sequence from step 3. For this price sequence, the sequence of present-value prices, $\{R_{t+1}\}_{t=1}^{\infty}$, is strictly increasing with a decreasing rate and converges to $R^* = 0.442$. Since $\{q_t\}_{t=1}^{\infty}$ is strictly increasing with a decreasing rate, then we get the following implication

$$R_{t+2} > R_{t+1} \Leftrightarrow \underbrace{\frac{q_{t+1} - q_t}{q_t}} > \underbrace{\frac{q_{t+2} - q_{t+1}}{q_{t+1}}} \frac{q_{t+1}}{1 + q_{t+1}} \tag{1.64}$$

but from the properties of the equilibrium price sequence analyzed at the end of step 3, (64) is satisfied. In particular, the terms in the underbrace represent the growth rates of asset prices from $(t, t+1)$ and $(t+1, t+2)$ respectively. If $\{q_t\}_{t=1}^{\infty}$ increases with a decreasing rate then (64) is satisfied.

Lastly, since q_1 is lower but sufficiently close to $q^* = 0.792$, R_2 is lower but sufficiently close to $R^* = 0.442$. Thus, all the increases in $\{R_{t+1}\}_{t=1}^{\infty}$ are infinitesimal.

The next step is to use the sequence of $\{R_{t+1}\}_{t=1}^{\infty}$ in order to construct a sequence of saving rates for individual two, $\{s_t^2\}_{t=1}^{\infty}$. ***This is the crucial part of the construction of the equilibrium.*** The next graph illustrates the main mechanisms behind the construction of individual's two saving rates.



Saving rates as decreasing steps.

The previous figure graphs the best-response function of individual two for different values of R_{t+1}. The s^* in the figure is a fix point of (63) at $R^* = 0.442$, $s^* = 1.472 \times 10^{-4}$. The best-response function of individual two can be written as $s_t^2 = f(R_{t+1}, s_{t+1}^2)$. We showed that $\{R_{t+1}\}_{t=1}^{\infty}$ is strictly increasing, R_2 is lower but sufficiently close to R^* and $R_{t+1} \to R^*$. Fixing s_t^2, s_{t+1}^2, we plot the graph of f for different values of R_{t+1} around the fix point s^*. It is evident from (63) that if R_{t+1} increases then the best-response function should move downwards. It does so in a parallel fashion as the figure illustrates because increases of R_{t+1} do not change the slope of the best-response function. Also, the graph of f in a neighborhood of s^* looks like a straight line with a slope very close to zero. The reason is that in a neighborhood of s^*, differences in saving behavior between different selves are infinitesimal.

From the sequence of $\{R_{t+1}\}_{t=1}^{\infty}$ that we constructed before, we can construct a countable infinite number of graphs which decrease in a parallel fashion and eventually cross the fix point s^*. Associate the first graph with R_2, the second with R_3, the third with R_4 and so forth until R_∞ is the graph that crosses s^*.

So far we have not said anything about the saving behavior of individual's two selves. Our goal is to construct a sequence of saving rates for individual two with the following two properties: (i) s_1^2 is higher but sufficiently close to s^*, (ii) and

$$s_1^2 > s_2^2 > s_3^2 > \cdots > s_t^2 \to s_\infty^2 = 1.472 \times 10^{-4}, \ as\ t \to \infty$$

We will show that there exists a ***unique*** sequence $\{s_t^2\}_{t=1}^{\infty}$, which satisfies the previous two properties. The proof evolves along the following lines: Let us start from a graph of (63) associated with R_T, for T sufficiently high. In other words, start from a period far away in the future. Pick a point on this graph which is sufficiently close and above the 45^0 line. From this point, project a horizontal line parallel to the x axis in order to hit the 45^0 line. The point where you hit the 45^0 line is below the graph R_T because the graph of the best-response function has a nonzero positive slope. From this point, project a vertical line parallel to the y axis in order to hit the graph R_{T-1} which is above the graph R_T. Following the previous argument, we construct a sequence of increasing steps, a sequence of increasing saving rates, until we hit the initial graph R_1. Importantly, we can repeat the previous argument for any large T we want. Thus, we can start as close as we want to s^* and construct a sequence of increasing steps in order to go back to the initial graph, R_1.

The steps that we construct are infinitesimal steps because $\{R_{t+1}\}_{t=1}^{\infty}$ changes infinitesimally. This important because if we start from a point close to s^* and we go up the steps, we hit the R_1 graph at s_1^2 which is higher but sufficiently close to s^*. The R_1 graph intersects the 45^0 line at a point which is higher but sufficiently close to s^* by construction since R_1 is sufficiently close to R^*. From the previous argument we know that we will hit R_1 at a point sufficiently close but above the 45^0 line. It follows that s_1^2 has to be higher but sufficiently close to s^*. Lastly, given $\{R_{t+1}\}_{t=1}^{\infty}$, the sequence of increasing (decreasing) steps we construct is unique. The reason is that if we start from a point in the distant future, there is only one path of increasing steps to take us back to the beginning for the given price sequence.

To sum up, we have shown that we can construct a countable number of decreasing steps with the properties discussed before. This is important because the saving behavior of all selves of individual two are similar to the saving behavior of all selves of individual two in the economy considered in step 3.

Euler Equations.

The idea of this step is to construct the Euler equations for each individual. Let us start with individual one. His generalized Euler equation is as follows

$$R_{t+1}(w_t^1 - R_{t+1}w_{t+1}^1)^{-1} = \Big((1 - s_{t+1}^1)w_{t+1}^1\Big)^{-1}\Big[\beta\zeta(1 - s_{t+1}^1) + \beta s_{t+1}^1\Big]$$

which can be simplified as follows

$$w_{t+1}^1 = \frac{w_t^1}{R_{t+1}}s^1 \; or \;\; y_t^1 = y_{t-1}^1\frac{1+q_t}{q_t}s^1 \tag{1.65}$$

where expression (65) is undoubtedly simpler to use. Consider again the asset price sequence computed in step 3, $\{q_t\}_{t=1}^{\infty}$. This price sequence was computed under the assumption that individual two (individual one) starts with sufficiently low (high) initial asset holdings. We assume that individuals start with the same initial asst holdings as in the economy of step 3. Construct his Euler equations recursively as follows,

$$y_1^1 = y_0^1 \frac{1+q_1}{q_1} s^1$$
$$y_2^1 = y_1^1 \frac{1+q_2}{q_2} s^1$$
$$\vdots$$

The Euler equations of individual one are identical with the Euler equations of individual one in the economy of step 3. Let us do the same for individual two. Consider the asset price sequence of step 3, $\{q_t\}_{t=1}^{\infty}$, the sequence of saving rates constructed before, $\{s_t^2\}_{t=1}^{\infty}$ and suppose also that individual two starts with the same initial asset holdings as in the economy of step 3. Construct his Euler equations recursively as follows:

$$y_1^2 = y_0^2 \frac{1+q_1}{q_1} s_1^2$$
$$y_2^2 = y_1^2 \frac{1+q_2}{q_2} s_2^2$$
$$\vdots$$

It is not very difficult to see that the asset holdings of individual two will tend to zero. The term $\frac{1+q_t}{q_t} s_t^2$ is always less than one and tends to a positive number as $t \to \infty$.

Market Clearing.

So far we have used the equilibrium asset price sequence from step 3 in order to construct a sequence of saving rates for individual two and the Euler equations for both individuals in the original economy. The last step is to argue that this equilibrium price sequence is indeed an equilibrium price sequence for the *original* economy.

Consider the asset market clearing every period

$$\underbrace{y_{t-1}^1 \frac{1+q_t}{q_t} s^1}_{y_t^1} + \underbrace{y_{t-1}^2 \frac{1+q_t}{q_t} s_t^2}_{y_t^2} = 1 \tag{1.66}$$

Let us start from period one,

$$y_0^1 \frac{1+q_1}{q_1} s^1 + y_0^2 \frac{1+q_1}{q_1} s_1^2 = 1 \tag{1.67}$$

When we constructed the sequence of saving rates for individual two, we argued that s_1^2 is higher but sufficiently close to s^*. Thus, period one market clearing for the original economy, expression (67), and period one market clearing for the economy of step 3, are equivalent. Although for period one we are done, it is not obvious how to guarantee market clearing in that set-up for the following reason: We can not check market clearing every period because the horizon is infinite and thus it is not obvious how to guarantee that any potential mistakes in market clearing will not propagate over time.

Rewrite period one asset market clearing as follows:

$$y_1^2 - (1 - y_1^1) = 0$$

and observe that the asset demands of individual one are identical with the asset demands in the economy of step 3. Thus, from the market clearing equations in step 3 we know that $1 - y_1^1 = (y_1^2)^{Log-Econ}$. We can rewrite period one asset market clearing as the difference of the following objects,

$$\underbrace{(y_1^2)^{Orig-Econ} - (y_1^2)^{Log-Econ}}(= 0) \tag{1.68}$$

by construction, the term in the underbrace is nonnegative. We have assumed that individual two is extremely poor initially. This means that both terms in the underbrace are sufficiently close to zero. Define the following sufficiently small and fix number ϵ,

$$\epsilon = (y_1^2)^{Orig-Econ} \tag{1.69}$$

Going back to period one market clearing as expressed in (68), we get the following inequality

$$(y_1^2)^{Orig-Econ} - (y_1^2)^{Log-Econ} < \epsilon \tag{1.70}$$

The idea is to place a uniform bound on the asset market clearing. This is the idea encapsulated by the inequality above for period one. The next step is to consider asset market clearing for any period t. We can rewrite it as in (68),

$$\underbrace{(y_t^2)^{Orig-Econ} - (y_t^2)^{Log-Econ}} (= 0)$$

By construction, the difference in the underbarce is nonnegative. Also, individual's two asset holdings are strictly decreasing over time. Thus,

$$(y_t^2)^{Orig-Econ} - (y_t^2)^{Log-Econ} < \epsilon, \quad \forall t$$

We have managed to place a uniform and sufficiently small bound on asset market clearing. This guarantees that any mistakes will not exceed ϵ[28].

The previous argument showed that we can construct an equilibrium where individual two vanishes in the long-run because his effective discount factor every period is lower than the effective discount factor of individual one. Thus, her saving rate is lower than the saving rate of individual one.

Lastly, we need to specify a transversality condition (TC). The model we considered before is a deterministic version of the Lucas asset pricing model. To derive the TC, we will solve recursively the generalized euler equation for each individual. Before doing this, define the following term,

$$\gamma_t^i = 1 - (1 - \zeta)(1 - s_t^i)$$

Solving recursively the generalized Euler equation for each individual we get the following:

$$q_t u^{'}(c_t) = \underbrace{\beta \gamma_t u^{'}(c_{t+1}) + \beta^2 \gamma_t \gamma_{t+1} u^{'}(c_{t+2}) + ...} +$$
$$\underbrace{\lim_{T \to \infty} \beta^T \Big(\prod_t^T \gamma_t \Big) u^{'}(c_T) q_T}$$

and the last term in the underbrace is the transversality condition (TC). Divide the previous expression by $u^{'}(c_t)$ and the TC becomes

$$\lim_{T \to \infty} \beta^T \Big(\prod_t^T \gamma_t \Big) \frac{u^{'}(c_T)}{u^{'}(c_t)} q_T$$

[28]This idea is related to the ϵ-equilibrium concept discussed in Kubler and Schmedders (2005)

From the Euler equations of the two individuals, the term $\beta^T \Big(\prod_t^T \gamma_t \Big) \frac{u^{'}(c_T)}{u^{'}(c_t)}$ is common and equal to the product of present-value prices up to period T. Thus, to check if the (TC) is satisfied, it suffices to check it for one individual. According to the previous construction, $q_t \to 0.792$, and the consumption of individual one tended to one. Thus,

$$\lim_{T\to\infty} \beta^T \Big(\prod_t^T \gamma_t \Big) \frac{u^{'}(c_T)}{u^{'}(c_t)} q_T \to 0$$

The previous equilibrium is not the only one. Suppose we fix asset prices to $q = 0.792$ every period and suppose also that all the selves of individual two play the following strategy: $s_t^2 = s_{t+1}^2 = s_{t+2}^2 = \cdot\cdot = 0.442 = s^1$. It is not very hard to show that this satisfies all the conditions for an intrapersonal competitive equilibrium where each individual consumes the following amount of the good every period

$$c^1 = y_0^1,\ c^2 = 1 - y_0^1$$

where $y_0^1, 1 - y_0^1$ are the initial shares of the tree.

1.4.3 Naive Individuals

Naive individuals do not realize that their preference are going to change in the future. They think that their preferences are described by those that they perceive at the current period. We will stick to the equilibrium concept described in Herings and Rohde (2008) for the naive case and sequential complete markets. In particular, naive individuals set up their plan today taking the sequence of future prices as given. The difference between naive and sophisticated individuals is that the former wake up next period and realize that their preferences have changed. Thus, they trade again in the market facing a different set of prices than the ones forecasted yesterday.

Let us first examine the previous economy when individuals are naive HD maximizers. Each naive individual solves the following problem every planning period,

$$\max \Big[u^i(c_t) + \zeta \beta u^i(c_{t+1}) + \zeta \sum_{j=2}^{\infty} \beta^j u^i(c_{t+j}) \Big], s.t$$
$$c_t^i + q_t y_t^i = y_{t-1}^i (1 + q_t)$$

The FOCs for each planning horizon and each individual are as follows

$$
\begin{aligned}
&q_t u_c^i(c_t) = \beta\zeta(1 + q_{t+1})u_c^i(c_{t+1}) \\
&q_{t+1} u_c^i(c_{t+1}) = \beta(1 + q_{t+2})u_c^i(c_{t+2}) \\
&. \\
&.
\end{aligned}
$$

We can verify that $c^i = y_0^i$, $\forall t$ and at the start of each planning horizon for each actual period t, the asset prices are as follows:

$$
\frac{q_t}{1 + q_{t+1}} = \beta\zeta, \; \frac{q_{t+1}}{1 + q_{t+2}} = \beta, \; \frac{q_{t+2}}{1 + q_{t+3}} = \beta, ...
$$

which means that the *actual* price level every period is $q_t = \frac{\beta\zeta}{1-\beta}$, for every planning horizon. The naive equilibrium and the second equilibrium for the sophisticated we mentioned before give the same consumption for each individual.

The way we solve for a naive competitive equilibrium is similar to the way we solve for an *Incorrect Forecast Myopic Equilibrium* (IFME) analyzed in the previous sections. It turns out that we can establish a clear connection for economies populated by myopic individuals and economies populated by naive HD maximizers. Consider the following type of preferences,

$$
U_t^i = \frac{c_t^{1-a^i}}{1 - a^i} + \beta \frac{c_{t+1}^{1-a^i}}{1 - a^i} + \zeta \sum_{j=2}^{\infty} \beta^j \frac{c_{t+j}^{1-a^i}}{1 - a^i}
$$

which have been discussed in the beginning of section 4.

Consider the dividend pattern of (26) discussed in the case of myopia. Solve the above economy as if each individual is the representative in the market. The way we do it is to collapse all the constraints into an intertemporal one and work again with the present value prices. From assumption I we know that dividends are bounded and the dividend pattern repeats every six periods. Solving for the representative individual case we get the no-trade present value prices from the viewpoint of every actual period as follows: $R_{t+1}^i = \beta(\frac{d_t}{d_{t+1}})^{\alpha^i}$,$R_{t+j}^i = \zeta\beta^j(\frac{d_t}{d_{t+j}})^{\alpha^i}$, $j = 2, ..$, and the subscript i in the present value prices refers to the individual we pick to solve for the representative case. Following claim I, the equilibrium present value prices in the economy with two individuals are going to be bounded between the present value prices of the representative individual cases. Thus asset prices in this economy have to be bounded.

Following a similar argument as in the finite myopic horizon case, we can compute the saving rate of individual i every actual period t as follows,

$$s_t^i = \frac{\beta^{\frac{1}{\alpha^i}} R_t^{\frac{\alpha^i-1}{\alpha^i}} + \zeta^{\frac{1}{\alpha^i}} \sum_{j=1}^{\infty} \beta^{\frac{j+1}{\alpha^i}} R_{t+j}^{\frac{\alpha^i-1}{\alpha^i}}}{1 + \beta^{\frac{1}{\alpha^i}} R_t^{\frac{\alpha^i-1}{\alpha^i}} + \zeta^{\frac{1}{\alpha^i}} \sum_{j=1}^{\infty} \beta^{\frac{j+1}{\alpha^i}} R_{t+j}^{\frac{\alpha^i-1}{\alpha^i}}} \tag{1.71}$$

Consider the following Proposition,

Proposition IX. *Suppose individuals are enough patient, high β, and suppose we inject enough fluctuations in the dividends of the asset. If ζ is sufficiently close to zero and either one of the following cases apply,*

- *there exist a region of α's such that $\alpha^2 > \alpha^1$, $1 \leq \alpha^1$ and α^1 sufficiently close to one.*
- *there exist a region of α's such that $1 \geq \alpha^1 > \alpha^2 > 0$ and α^1 sufficiently close to one.*

then individual two vanishes in the long-run.

Proof. For ζ sufficiently close to zero, the analysis follows from the discussion in the case of myopia.

□

The previous result follows from the discussion for the myopic case. For ζ close to zero, the previous results on the long-run survival of symmetric myopic individuals apply also for naive HD maximizers. The same is true for the case of asymmetric myopia. For sufficiently low values of ζ, more present-biased individuals dominate over less present-biased ones.

§ 1.5 Concluding Remarks

In this work we argued that particular forms of dynamically-inconsistent preferences can give rise to equilibrium dynamics that may help us explain certain phenomena observed in financial markets. The main result discussed extensively in the previous paragraphs concerns the long-run survival of some individuals. The reason we are interested in the long-run non-survival of some individuals is twofold. First, we get a clear characterization of equilibrium asset prices. The second and more important reason involves the behavior of asset trades. The benchmark case implies that trade in assets follow the pattern of dividends we specified. In other words, individuals trade the same number of shares every time they enter in the same aggregate state. The introduction of time-inconsistency modifies this result. There is *continuous trade in assets* that does not vanish in the long-run and does not necessarily depend on the aggregate state of the economy. The other interesting point about time-inconsistent preferences involves the case of myopia and the dynamics of equilibrium asset prices. We showed cases where economies populated by myopic individuals are characterized by extreme asset price fluctuations even though dividends of the assets are constant over time.

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Part

Optimal Monetary Policy

- Chapter 2 -

Unemployment Equilibria in a Monetary Economy

§ 2.1 Introduction

In this paper we characterize optimal monetary policy when two different types of unemployment may prevail in the market, *Keynesian Unemployment* and *Classical Unemployment*. We want to analyze how optimal monetary policy prescriptions depend on the type of unemployment that prevails in the market. Unlike models of Blanchard and Gali (2008), Trigari (2005), unemployment in that set-up is not a result of search and matching frictions. Unemployment in equilibrium is possible because *some* prices are fixed and different from the Walrasian ones which imply zero unemployment. Given this way of introducing unemployment in the economy, we able to analyze different unemployment regimes depending on the direction we fix some prices with respect to their Walrasian counterparts. Given the possibility of different unemployment regimes in the market, the characterization of monetary policy across these different unemployment regimes is the main question we want to address.

Following Malinvaud (1982), we define as *Keynesian Unemployment* the situation where because of a low real wage individuals reduce their demands of goods and as a consequence firms face a demand constraints when they decide how much to produce. This spills over to the labor market where firms decide to hire less because of the demand constraint they face in the goods market. The other type which we will call *Classical Unemployment* is a situation where firms hire less because the real wage is so high such that their profitability reduces.

We employ a three period economy with operative cash-in-advance constraints. The last period is added for accounting purposes. The timing of transactions is similar to Lucas and Stokey (1987) and the monetary economy is based on Nakajima and Polemarchakis (2005). There are two types of individuals, employed and unemployed and there is only one consumption good each period. The timing of transactions is as follows: First the asset market opens where individuals hold money balances, trade a riskless bond and they are endowed with initial wealth-outside money. After asset market closes, good's market opens where individuals buy the good with initial money balances brought from the asset market. They also supply labor elastically at the prevailing wage rate. The timing of transactions is similar in period two except that they do not receive outside money. Money balances are injected into the economy through open market operations by the monetary-authority. The monetary authority fixes also the policy parameters which are the nominal interest rates in period one and period two. Lastly, the production side of the economy consist of two firms: a non-investment and an investment firm. The former hires labor in period one and produces total output of period one while the latter invest in period one and hires labor in period two in order to produce the aggregate output of period two. Both firms have to acquire money balances from the asset market in order to finance inputs and issue profits.

Unemployment prevails only in period one because we assume that the price of the good and that of labor are fixed exogenously. Prices are fully flexible in period two and clear markets in the usual sense. The case where prices of period one are allowed to adjust freely we will refer to as the *Walrasian* case. Since prices are fixed and different from the Walrasian ones, we have a notion of equilibrium involving quantities. This means that excess capacities are possible in the market. Since excess capacities are possible, a rationing scheme determines how these excess capacities are distributed among the agents of the economy. The notion of equilibrium that we use in period one is described by the following conditions

1. Voluntary exchange, no one is forced to trade more than he wishes
2. There can be no rationed sellers and rationed buyers in the same market
3. Trade balances

Let us analyze first the Keynesian Unemployment regime. As mentioned before, this regime is characterized by excess supply in the good's market and excess supply in the labor market because of a real wage in period one that is lower from the Walrasian one. The excess supply in the good's market originates from the fact that non-investment firm faces a demand constraint when it maximizes profits in period one. This spills over to the labor market where the non-investment firm hires less than the Walrasian case of full employment. The rationing scheme takes a simple form in that case. Individuals can be fully employed or unemployed and non-investment firm faces a binding demand constraint when it maximizes profits. Investment firm is not constrained in any market. Given this set-up, the policy question that is of interest to us can be summarized as follows: *What is the characterization of optimal monetary policy when the monetary-authority can manipulate nominal interest rates in order to reduce unemployment in the economy and make everybody at least not worse-off?*

Nominal interest rates represent a cost to liquidity because individuals have to borrow in order to acquire money balances in the asset market every period. The reason for unemployment in equilibrium, in the Keynesian regime, is a lower real wage than the Walrasian one which induces individuals to reduce the demand in the market. A policy designed to reduce unemployment in that regime should be able to boost demand in period one and as a consequence reduce unemployment. If the monetary-authority sets nominal rates close to zero, then it can achieve the lowest possible unemployment level given fixed prices of period one and make employed and unemployed individuals better-off.

We mentioned before that the only cause of unemployment in equilibrium is that period one prices are fixed and different from the Walrasian ones. There is a possibility that the behavior of monetary-authority can be one of the main causes of excessive unemployment. Suppose period one prices are *very* close to the Walrasian ones. The excess capacities in the market should not very large. If the monetary-authority sets nominal rates higher than a certain threshold, it can be calibrated to be around 1.5, then for prices *very* close to the Walrasian ones the unemployment rate is sufficiently high. If the nominal rate is below that threshold, then unemployment is close to zero. High nominal rates in period one imply that the effective labor supply of employed individuals is sufficiently higher than labor demand. Labor market clearing requires high levels of unemployment.

Consider the other type of unemployment, *Classical Unemployment*. In that regime firms are not constrained in any market. Only individuals are constrained in period one. Unemployed individuals are excluded from the labor market in period one as before. The main difference with the Keynesian regime is the behavior of employed people. We assume that employed individuals face a constraint in the good's market in period one. They buy less than they would have bought if the constraint was not binding. As a consequence they consume more leisure in period one since more labor hours would be devoted to less consumption today. Their consumption pattern is shifted towards more consumption in period two. Unemployment is possible in equilibrium because the real wage of period one is higher than the Walrasian one and the non-investment firm will hires less. To sum up, this regime is characterized by excess demand in the good's market and excess supply in the labor market of period one.

As before let us examine the behavior of equilibrium close to Walrasian prices given that conditions for excess demand and excess supply are satisfied. If the nominal interest rate of period one is higher than a certain threshold, then for period one prices close to the Walrasian ones there can be sufficiently high unemployment but also the unemployment rate can exceed one which makes the equilibrium not well-defined. This threshold can be computed to be close to zero. The reason is as follows: Since employed individuals face a constraint in the good's market their labor supply in period is reduced. Higher nominal rates in period one decrease consumption and labor supply of period one even further. But initially employed individuals were implicitly constrained to supply less hours because they were constrained in the good's market. Higher period one nominal rates amplify this effect. To exclude this anomaly and have a well-defined equilibrium the monetary-authority should fix the nominal rate of period one close to zero.

We said before that employed individuals are forced to consume more tomorrow because they face a binding constraint in the good's market today. An optimal policy should restore consumption smoothing for employed people, reduce unemployment and make the remaining unemployed individuals not worse-off. The optimal policy in that case is as follows: If the nominal rate of period two is expected to be *sufficiently* high *initially*, then announcing a reduction of the nominal rate tomorrow decreases unemployment and make everybody better-off. Employed individuals expecting a high nominal rate tomorrow are able to shift their consumption pattern towards more consumption today because investment and consumption demand of unemployed individuals in period one are low. An announcemnt of lower nominal rates tomorrow will make them better-off because they would like to sacrifice consumption and labor supply today in order to consume and supply more hours in period two. Unemployment decreases because employed individuals reduce their labor supply today since the labor demand does not depend on the nominal rate of period two [1]. Unemployed individuals become better-off after an announcement of lower rates tomorrow because they increase consumption today since they can borrow more given that the cost of liquidity is reduced tomorrow and they can increase consumption and labor supply tomorrow since the real wage of period two increases and the price of the good falls. The monetary authority should target the inflation rate initially by fixing high nominal rates in period two. If this is the case, then monetary policy can improve upon the initial allocation.

To sum up, we have built a stylized monetary economy featuring two

[1] The total supply and labor demand of the non-investment firm depends only on period one prices and the nominal rate of period one.

different unemployment regimes. The policy conclusions are revised to a large extend when we talk about different unemployment types. The very nature of unemployment is of great interest when we design policies that aim to reduce unemployment. Also, when the conditions in the market do not justify high unemployment, prices close to the Walrasian ones, monetary policy can be the cause of excessive unemployment.

In section 2 we analyze the benchmark economy. Section 3 is devoted to the construction of unemployment regimes when the supply of labor is inelastic. Section 4 deals with the issue of optimal monetary policy. Lastly, section 5 deals with elastic labor supply.

§ 2.2 A MONETARY ECONOMY

2.2.1 The basic model

The basic ingredients of our simple monetary economy are listed below:

1. There are three periods: $t = 1, 2, 3$. The last period is added for an accounting purpose. There are no stochastic shocks in the economy.
2. There is one consumption good each period, x_t, and individuals are endowed with time that they supply inelastically to firms for a wage income in return- $w_t\bar{l}_t$.
3. $(p_t, w_t)_{t=1}^{2}$ denotes the commodity and labor prices respectively.
4. There are two firms in the economy. One produces the entire output of period one whereas the other invests in period one and produces period's two output with period one investment.
5. Money is the sole medium of exchange. It is valued through a cash-in-advance constraint.
6. A monetary-fiscal authority supplies balances, charging or paying interest on account balances.
7. Individuals are endowed with initial nominal wealth δ. It is a form of outside money. This corresponds to the initial public liability.

2.2.2 Individuals

Assume there is a large number of identical individuals. The timing of transactions is as follows:

1. The asset markets open first where individuals exchange bonds with money balances and receive their initial wealth.

2. After asset markets are closed, goods markets open where individuals buy consumption goods and sell their endowment of time.

3. End of period money balances are carried to the next period.

4. At the beginning of the next period the timing of transactions is similar.

5. Lastly, in period three individuals redeem their debt.

Individuals visit the asset market in order to exchange bonds and money balances according to the constraint

$$\widehat{m}_1 + \frac{b_1}{1+r_1} = \delta$$

where b_1 denotes bond units that are exchanged with the monetary-fiscal authority, $\widehat{m}_1$ denotes initial money balances and δ is initial wealth.

Once the asset market is closed individuals visit the good's market that opens next. With the initial balances acquired before they buy the consumption good according to the following constraint

$$p_1 x_1 \leq \widehat{m}_1$$

and accumulate end-of-period money balances through receipts from sales of labor time

$$m_1 = \widehat{m}_1 - p_1 x_1 + w_1 \bar{l}_1$$

The constraints of period one reduce to

$$\begin{aligned} & p_1 x_1 + m_1 + \frac{b_1}{1+r_1} \leq \delta + w_1 \bar{l}_1 \\ & m_1 \geq w_1 \bar{l}_1 \end{aligned}$$

With a similar argument, the budget constraints of period two are as follows

$$\begin{aligned} & p_2 x_2 + m_2 + \frac{b_2}{1+r_2} \leq m_1 + b_1 + w_2 \bar{l}_2 \\ & m_2 \geq w_2 \bar{l}_2 \end{aligned}$$

Lastly, at the beginning of period three individuals repay their debt to the monetary-authority

$$m_2 + b_2 \geq 0$$

Given this debt constraint, the flow budget constraints reduce to the intertemporal constraint

$$p_1x_1 + \frac{1}{1+r_1}p_2x_2 + \frac{r_1}{1+r_1}m_1 + \frac{r_2}{(1+r_1)(1+r_2)}m_2 \leq \delta + w_1\bar{l}_1 + \frac{w_2\bar{l}_2}{1+r_1}$$

The cash constraints can be written as

$$\frac{r_1}{1+r_1}m_1 = \frac{r_1}{1+r_1}w_1\bar{l}_1$$
$$\frac{r_2}{1+r_2}m_2 = \frac{r_2}{1+r_2}w_2\bar{l}_2$$

because if $r_1, r_2 > 0$ the cash constraint binds. If $r_1, r_2 = 0$ both side of the equation are zero. Substituting these back to the intertemporal constraint, we get

$$p_1x_1 + \frac{1}{1+r_1}p_2x_2 \leq \delta + \frac{w_1\bar{l}_1}{1+r_1} + \frac{w_2\bar{l}_2}{(1+r_1)(1+r_2)} = \Pi$$

The representative individual solves the following problem[2]

$$\max_{x_1,x_2} \left[\log x_1 + \beta \log x_2\right]$$
$$s.t \quad p_1x_1 + \frac{1}{1+r_1}p_2x_2 = \Pi$$

The respective demands are as follows

$$x_1 = \frac{1}{p_1(1+\beta)}\Pi$$
$$x_2 = \frac{\beta(1+r_1)}{p_2(1+\beta)}\Pi$$

[2]the intertemporal constraint is binding at the optimum. Also the previous transversality condition is binding

$$m_2 + b_2 = 0$$

2.2.3 Firms

There are two firms in the economy. There is a firm in period one that produces the total output of that period and another firm that invest in period one and produces output in period two[3]. The crucial point is that firms also visit the asset market of the economy. They acquire funds for buying inputs and financing profits.

2.2.3.1 Non-investment firm

The non-investment (NI) firm produces the aggregate output of period one with the following simple technology

$$y_1 = \sqrt{l_1}$$

where l_1 is the labor demand of period one.

The timing of transactions is similar to that of an individual. NI firm visit the asset market to exchange bonds for money balances in the amount needed to pay for inputs and issue profits

$$\widehat{n}_1^{NI} + \frac{1}{1+r_1} b_1^{NI} = 0$$

and then buys inputs and issues profits[4]

$$\widehat{n}_1^{NI} = w_1 l_1 + \pi_1$$

It then accumulates cash balances through receipts for sale of output

$$n_1^{NI} = p_1 y_1$$

and uses these terminal balances to repay its debt to the bank

$$n_1^{NI} + b_1^{NI} = 0$$

This defines the dividend policy as

$$\pi_1 = \frac{p_1 y_1}{1+r_1} - w_1 l_1$$

[3] the reason for this separation will become clear afterwards.

[4] Write these constraints with equality to simplify the presentation.

Finally, the NI firm's problem becomes

$$\max_{l_1}\left[\frac{p_1 y_1}{1+r_1} - w_1 l_1\right]$$
$$s.t \quad y_1 = \sqrt{l_1}$$

At the optimum we get

$$y_1 = \frac{p_1}{2(1+r_1)w_1}$$
$$l_1 = \frac{p_1^2}{4(1+r_1)^2 w_1^2}$$
$$\pi_1 = \frac{p_1^2}{4(1+r_1)^2 w_1}$$

2.2.3.2 Investment firm

The investment (I) firm of our story buys part of the output in period one and holds it as investment in order to augment the productivity of labor in period two. Period's two production function takes the following form

$$y_2 = F(l_2, I_1) = \sqrt{l_2}\sqrt{I_1}$$

The timing of transactions is analogous to that of the individual. The investment firm visits the asset market

$$\widehat{n}_1^I + \frac{b_1^I}{1+r_1} = 0$$

The good's market opens next. The investment firm buys investment opportunities, part of period's one output, with the cash acquired in the asset market

$$\widehat{n}_1^I = p_1 I_1$$

At the beginning of period two, the investment firm visits the asset market once more

$$\widehat{n}_2^I + \frac{b_2^I}{1+r_2} = b_1^I$$

acquires cash balances, $\widehat{n}_2^I$, in exchange for bonds, b_2^I, and receives the proceeds of earlier transactions, b_1^I.
With the cash acquired in the asset market, the investment firm buys inputs and issues profits

$$\widehat{n}_2^I = w_2 l_2 + \pi_2$$

It accumulates cash through the receipts for sale of output

$$n_2^I = p_2 y_2$$

and uses these cash balances to repay its debt

$$n_2^I + b_2^I = 0$$

This defines the dividend policy of the investment firm as follows

$$\pi_2 = \frac{p_2 y_2}{1 + r_2} - w_2 l_2 - (1 + r_1) p_1 I_1$$

The profit maximization problem of the investment firm calls for maximizing discounted profits

$$\begin{aligned} &\max_{I_1, l_2} \left[\frac{p_2 y_2}{(1 + r_2)(1 + r_1)} - \frac{w_2 l_2}{(1 + r_1)} - p_1 I_1 \right] \\ &s.t \quad y_2 = F(l_2, I_1) \end{aligned}$$

The first-order conditions with respect to I_1, l_2 are as follows

$$\begin{aligned} \frac{p_2}{p_1} &= \frac{2(1 + r_1)(1 + r_2)}{\sqrt{l_2}} \sqrt{I_1} \\ \frac{w_2}{p_2} &= \frac{1}{2(1 + r_2)} \frac{\sqrt{I_1}}{\sqrt{l_2}} \end{aligned}$$

At the optimum, profits are equal to zero.

2.2.4 The monetary-fiscal authority

The last agent in our economy is the monetary-fiscal authority (MFA). Its main role is to determine the monetary policy rule, money supply or interest rate rule, that will be followed and supply money balances into the economy in exchange for bonds.

The flow budget constraint of period one can be written as follows

$$M_1 + \frac{1}{1+r_1} B_1 + \pi_1 = \delta$$

For period two, we get

$$M_2 + \frac{1}{1+r_2} B_2 + \pi_2 = M_1 + B_1$$

and at the beginning of period three

$$M_2 + B_2 = 0$$

Some notation:

M_1 : stands for the total money balances the MFA supplies to individuals and firms in period one

B_1 : total bonds that are exchanged with individuals and firms for money balances in period one

M_2 : total money balances supplied to individuals and firms in period two

B_2 : total bonds exchanged with individuals and firms for money balances in period two

δ : initial public liability- initial wealth of individuals.

From the flow budget constraints we observe that profits appear in the left hand side of these constraints. What we really mean is encapsulated in the following assumption,

Assumption 2.2.4.1. *Profits are taxed by the monetary-fiscal authority.*

The previous assumption will be quite useful for the rest of the analysis. The motivation for it is to be able to guarantee that a fully determinate monetary equilibrium is well-defined whenever interest rates are zero and also simplify the analysis when we talk about unemployment.

The next thing to specify is the policy rule that the MFA follows.

Monetary policy. *The MFA fixes the nominal interest rates,* $r_1, r_2 \geq 0$ *and accommodates the money demand in the market.*

2.2.5 Equilibrium

Equilibrium in the goods and labor markets require the following:

$$x_1 + I_1 = y_1 = \sqrt{\bar{l}_1} \tag{2.1}$$

$$x_2 = F(\bar{l}_2, I_1) \tag{2.2}$$

$$\bar{l}_1 = l_1 = \frac{p_1^2}{4(1+r_1)^2 w_1^2} \tag{2.3}$$

$$\bar{l}_2 = l_2 \tag{2.4}$$

the money market clears *a fortiori*, since the MFA accommodates the money demand

$$m_1 + n_1^{NI} = M_1$$
$$m_2 + n_2^{I} = M_2$$

Bond market clearing is as follows

$$b_1 + b_1^{NI} + b_1^{I} = B_1$$
$$b_2 + b_2^{I} = B_2$$

The last condition that we must take into account is the intertemporal constraint of MFA,

$$\frac{r_1}{1+r_1} M_1 + \frac{r_2}{(1+r_1)(1+r_2)} M_2 + \pi_1 + \frac{1}{1+r_1}\pi_2 = \delta \tag{2.5}$$

since δ is treated as outside money, condition (5) provides an additional restriction for the determination of equilibrium prices.

Conditions (1)-(4), the investing firm's first-order conditions and condition (5), give us sufficient equations to determine prices and investment opportunities in equilibrium. Combine (1),(2) to get

$$\frac{p_2}{p_1} = \beta(1+r_1)\frac{\sqrt{\bar{l}_1} - I_1}{F(\bar{l}_2, I_1)} \tag{2.6}$$

Combining (6) together with the FOCs of the investing firm, we determine investment opportunities in equilibrium as follows:

$$I_1 = \frac{\beta\sqrt{\bar{l}_1}}{2(1+r_2)+\beta}$$

Condition (5) can be written as

$$\frac{r_1}{1+r_1}\Big(\frac{w_1\bar{l}_1}{p_1}+y_1\Big)+\frac{r_2}{(1+r_1)(1+r_2)}\Big(\frac{w_2\bar{l}_2}{p_2}\frac{p_2}{p_1}+\frac{p_2}{p_1}y_2\Big)+\frac{\pi_1}{p_1}=\frac{\delta}{p_1} \tag{2.7}$$

and since relative prices are determined in equilibrium we can compute the equilibrium price level of period one-p_1. After a bit of algebra we end up in the following expression for the price level

$$p_1=\frac{\delta}{\sqrt{\bar{l}_1}\left[\frac{4r_1+2r_1^2+1}{2(1+r_1)^2}+\frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}\right]}$$

The nominal wages are determined from (3) and the second FOC of the investment firm. Finally, p_2 is determined from (6). We get a fully *determinate* equilibrium.

2.2.5.1 A note on the (in)determinacy result

In the previous discussion δ is interpreted as outside money. This is the *only* reason the equilibrium is determinate in that case. In order to understand this point we turn first our attention to the money and bond market clearing conditions and then to the case of distribution of nominal transfers every period.

Under an interest rate peg, the MFA accommodates the money demand, the money market clearing equations become identities. Bond market clearing implies that the flow budget constraint of the MFA holds every period. No restrictions are added to the equilibrium system. This argument does not depend on the distribution of transfers every period.

Suppose that individuals receive nominal transfers every period, h_1,h_2, instead of endowed with fixed initial wealth. The individual's intertemporal constraint is modified as follows:

$$p_1x_1+\frac{1}{1+r_1}p_2x_2=h_1+\frac{h_2}{1+r_1}+\frac{w_1\bar{l}_1}{1+r_1}+\frac{w_2\bar{l}_2}{(1+r_1)(1+r_2)} \tag{2.8}$$

and the MFA's intertemporal constraint is as follows

$$\frac{r_1}{1+r_1}M_1+\frac{r_2}{(1+r_1)(1+r_2)}M_2+\pi_1+\frac{1}{1+r_1}\pi_2=H_1+\frac{H_2}{1+r_1} \tag{2.9}$$

with

$$h_1 = H_1$$
$$h_2 = H_2$$

Since h_1,h_2 are considered as transfers, expression (9) becomes an identity. Nominal transfers accommodate any price change by adjusting accordingly. Since (9) becomes an identity, we can no longer solve for period's one price level. The intertemporal constraint in (8) is homogenous of degree zero following an identical change of all prices and transfers. The degree of indeterminacy is one since there is no uncertainty[5]. The indeterminacy is purely nominal under an interest rate peg and a policy of distribution of nominal transfers every period. The introduction of outside money instead of transfers makes equation (9) an additional equilibrium restriction. This will become important in later analysis.

2.2.5.2 A note on individual's bond holdings

Going back to the asset market budget constraints, it is evident that the firms of our economy will sell bonds to the MFA in exchange for money balances-$b_1^{NI}, b_1^I, b_2^I < 0$.

The case of individuals is quite different though. From the asset market constraint of period one, the equilibrium bond holdings are computed as follows:

$$b_1 = (1 + r_1)\delta \left[1 - \frac{2(1 + r_2)}{(2(1 + r_2) + \beta) \left[\frac{4r_1 + 2r_1^2 + 1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)} \right]} \right]$$

which is *positive* unless nominal interest rates are close to/or zero. When nominal interest rates are close to/or zero, the real value of the initial wealth decreases whereas the good's price of period one increases. Individuals sell bonds to the MFA, $b_1 < 0$, in order to obtain more money balances for transactions purposes. The derivative of b_1 with respect to nominal interest rates is positive, verifying the previous argument.

2.2.6 Pareto Optimality

The equilibrium allocation of individuals is as follows

[5] Conditions (1),(2) are not independent equations for determining equilibrium prices. We can only determine relative prices in equilibrium.

$$x_1 = \frac{2(1+r_2)}{2(1+r_2)+\beta}\sqrt{\bar{l}_1}$$
$$x_2 = \sqrt{\bar{l}_2}\sqrt{\frac{\beta\sqrt{\bar{l}_1}}{2(1+r_2)+\beta}}$$

The equilibrium allocation does not depend on r_1. The reason is that output in period one is fixed since labor is supplied inelastically. It depends on r_2, since individuals who are the net demanders of output in period two and investment firm who is the net supplier would face different relative prices[6].

Differentiating the utility of the individual with respect to r_2, we get:

$$\frac{\partial u}{\partial r_2} = -\frac{\beta r_2}{2(1+r_2)^2+\beta(1+r_2)} < 0$$

so that reducing the nominal interest rate of period two would induce a Pareto improvement upon the initial allocation. The initial equilibrium allocation with positive interest rates is suboptimal.

§ 2.3 Unemployment equilibria

2.3.1 Introductory remarks

The monetary economy described above serves as a useful benchmark for analyzing monetary policy in equilibrium. The main goal of this paper is to study the role of monetary policy when different types of unemployment prevail in the market. In order to do this we must modify the above story considerably.

Suppose there are three periods again. Define the first period as the immediate short-run and period two as the medium-run. Period three serves only for accounting purposes. During period one quantities adjust faster than prices. Prices are kept fixed in the short-run. During period two prices adjust to their new equilibrium level. Since prices are fixed in the short-run and we are dealing with quantity adjustments only, we must specify the regime that prevails in the market. In this paper we deal with the following two regimes:

- *Keynesian Unemployment*, excess supply of labor and goods.
- *Classical Unemployment*, excess demand of goods and excess supply of labor.

[6]Compare the solution of p_2/p_1 from the investment firm's FOC and expression (6).

Its regime specifies a different way individuals and firms are rationed in the market. We need to be more explicit about the way individuals and firms are rationed.

Before going to comment on the way individuals and firms are rationed in the market, we will assume that rationing schemes satisfy the following two properties:

- *Voluntary exchange*, no one is forced to purchase more than he demands or to sell more than he supplies.
- *Market efficiency*, this implies that one will not find rationed demanders and rationed suppliers in the same market.

2.3.1.1 Individuals and Rationing Schemes

If excess supply prevails in the market for labor in period one, some individuals supply their labor inelastically while some others are rationed from the market. Rationed individuals become unemployed and do not earn labor income. The fraction of the unemployed individuals is u and satisfies the following restriction, $0 \leq u \leq 1$.

The maximization problem of unemployed individuals is as follows:

$$\max_{x_1, x_2} (\log x_1 + \beta \log x_2)$$
$$s.t$$
$$p_1 x_1 + \frac{1}{1+r_1} p_2 x_2 = \delta + \frac{w_1 \bar{l}_1}{1+r_1} + \frac{w_2 \bar{l}_2}{(1+r_1)(1+r_2)}$$
$$\bar{l}_1 = 0$$

The respective optimal demands are as follows:

$$x_1 = \frac{1}{p_1(1+\beta)} \widetilde{\Pi} \tag{2.10}$$

$$x_2 = \frac{\beta(1+r_1)}{p_2(1+\beta)} \widetilde{\Pi} \tag{2.11}$$

where

$$\widetilde{\Pi} = \delta + \frac{w_2 \bar{l}_2}{(1+r_1)(1+r_2)}$$

Unemployed individuals are *effectively rationed* in their supply of labor. Removing the constraint in the labor market during period one will increase their utility level.

Rationing on the good's market will be assumed to follow a very simple scheme: *employed* individuals will be equally rationed and *unemployed* individuals will not be rationed on this market.

The maximization problem of employed individuals is as follows:

$$\max_{x_1,x_2}(\log x_1 + \beta \log x_2)$$
$$s.t$$
$$p_1 x_1 + \frac{1}{1+r_1} p_2 x_2 = \delta + \frac{w_1 \bar{l}_1}{1+r_1} + \frac{w_2 \bar{l}_2}{(1+r_1)(1+r_2)}$$
$$x_1 = \overline{x}_1$$

The respective demands are:

$$x_1 = \overline{x}_1 \tag{2.12}$$
$$x_2 = \frac{(1+r_1)}{p_2}\Big(\Pi - p_1 \overline{x}_1\Big) \tag{2.13}$$

The employed individuals aggregate excess demand for period one consumption is defined as follows:

$$D = (1-u)\Big[\frac{\Pi}{p_1(1+\beta)} - \overline{x}_1\Big] \tag{2.14}$$

If condition (14) is positive then employed individuals are forced to reduce consumption today and increase consumption tomorrow. This consumption profile contradicts their optimal plan when there is no rationing in the good's market. Employed individuals are *effectively rationed* in their demand for period one consumption. Unemployed individuals are not rationed in the good's market. Their consumption plan is given from (10),(11).

Since voluntary exchange is assumed, a comparison between employed's and unemployed's utility levels becomes very important. In the *Keynesian equilibrium* the answer is immediate. Employed's utility is higher than the unemployed's one. In the *Classical unemployment* case is not immediate.It will become clear why it is so when we lay down the equilibrium conditions and talk about the characterization of equilibrium.

2.3.1.2 Firms and Rationing Schemes

Since we are considering two different market regimes, Keynesian or Classical, firms will be rationed only if excess supply prevails in the good's market. To be more precise, only the non-investing firm will be rationed in the good's market.

The NI firm's profit maximization is modified as follows:

$$\max\left[\frac{p_1 y_1}{1+r_1} - w_1 l_1\right]$$
$$s.t$$
$$y_1 \leq \sqrt{l_1}$$
$$y_1 \leq \overline{y}_1$$

where $\overline{y}_1$ is the sales constraint coming from the good's market. In the case of binding constraints, maximum profits are as follows:

$$\widetilde{\pi}_1 = \frac{p_1 \overline{y}_1}{1+r_1} - w_1 \overline{y}_1^2$$

In the Classical unemployment regime non-investing firm supplies output and hires labor according to the Walrasian plan specified before and repeated here:

$$y_1 = \frac{p_1}{2(1+r_1)w_1}$$
$$l_1 = \frac{p_1^2}{4(1+r_1)^2 w_1^2}$$
$$\pi_1 = \frac{p_1^2}{4(1+r_1)^2 w_1}$$

2.3.2 Keynesian Unemployment

Equilibrium conditions in goods and labor markets respectively are as follows:

$$(1-u)\frac{1}{p_1(1+\beta)}\Pi + u\frac{1}{p_1(1+\beta)}\widetilde{\Pi} + I_1 = \overline{y}_1 \tag{2.15}$$
$$(1-u)\frac{\beta(1+r_1)}{p_2(1+\beta)}\Pi + u\frac{\beta(1+r_1)}{p_2(1+\beta)}\widetilde{\Pi} = F(\bar{l}_2, I_1) \tag{2.16}$$
$$(1-u)\bar{l}_1 = f^{-1}(\overline{y}_1) = \overline{y}_1^2 \tag{2.17}$$
$$\bar{l}_2 = l_2 \tag{2.18}$$

where $\overline{y}_1$, the demand-determined output, is allocated between the new level of investment demand, I_1, and consumption of employed and unemployed individuals. Besides rationed in the labor market during period one, individuals are identical in all other respects. Taking this into account, the interpretation of (16),(17),(18) is immediate.

The money market clears *a fortiori* since the MFA accommodates the money demand:

$$m_1^T + n_1^{NI} = M_1 \tag{2.19}$$
$$m_2^T + n_2^I = M_2 \tag{2.20}$$

where m_1^T, m_2^T are the total,(T), money demands of individuals. They are equal to:

$$\begin{aligned} m_1^T &= (1-u)w_1\bar{l}_1 \\ &= w_1\overline{y}_1^2 \end{aligned} \tag{2.21}$$
$$m_2^T = (1-u)w_2\bar{l}_2 + uw_2\bar{l}_2 = w_2\bar{l}_2 \tag{2.22}$$

where (21) follows from the fact that unemployed individuals do not receive labor income in period one.

The bond market clearing is as follows:

$$(1-u)b_1^E + ub_1^{UN} + b_1^{NI} + b_1^I = B_1 \tag{2.23}$$
$$(1-u)b_2^E + ub_2^{UN} + b_2^I = B_2 \tag{2.24}$$

Also the following conditions are true:

$$b_1^{UN} > b_1^E \tag{2.25}$$
$$b_2^{UN} = b_2^E = b_2 \tag{2.26}$$

Condition (25) follows from the asset market's budget constraints in period one and the fact that employed's consumption is greater than unemployed's one. Condition (26) follows from the individuals transversality condition and the fact that end-of-period money balances of period two are equal across individuals.

Lastly, the MFA's intertemporal constraint is as follows:

$$\frac{r_1}{1+r_1}M_1 + \frac{r_2}{(1+r_1)(1+r_2)}M_2 + \widetilde{\pi}_1 + \frac{1}{1+r_1}\widetilde{\pi}_2 = \delta \tag{2.27}$$

We rewrite it as follows:

$$\frac{r_1}{1+r_1}\Big(\frac{w_1}{p_1}\overline{y}_1^2 + \overline{y}_1\Big) + \frac{r_2}{(1+r_1)(1+r_2)}\Big(\frac{w_2\bar{l}_2}{p_2}\frac{p_2}{p_1} + \frac{p_2}{p_1}y_2\Big) + \frac{\widetilde{\pi}_1}{p_1} = \frac{\delta}{p_1} \tag{2.28}$$

Following the same argument with the flexible price economy, the new level of investment demand is equal to:

$$I_1 = \frac{\beta}{2(1+r_2)+\beta}\overline{y}_1$$

Substituting for relative prices and investment demand, (28) becomes:

$$-\frac{w_1\overline{y}_1^2}{1+r_1} + Ap_1\overline{y}_1 - \delta = 0 \tag{2.29}$$

where

$$A = 1 + \frac{\beta r_2(3+2r_2)}{2(1+r_2)^2+\beta(1+r_2)}$$

Solving (29) with respect to $\overline{y}_1$ we get:

$$\overline{y}_1 = \frac{Ap_1(1+r_1) \pm \sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)}}{2w_1} \tag{2.30}$$

Expression (30) is very important for the rest of the analysis in the Keynesian unemployment regime. It is here that the assumption of taxation of profits by the MFA will prove to be useful. Its usefulness lies in the simplicity of expression (30). The importance of this simplicity is twofold. First, it will be easy to prove that the bigger root is not accepted as a solution when we specify the condition required for the characterization of the Keynesian Unemployment equilibrium. Second, the comparative statics of the smaller root with respect to r_1, r_2 give us unambiguous results that will prove useful. Differentiating the smaller root of (30) with respect to r_1, r_2 we get:

$$\frac{\partial\overline{y}_1}{\partial r_1} = \frac{Ap_1\sqrt{A^2p_1^2(1+r_1)^2-4w_1\delta(1+r_1)} - A^2p_1^2(1+r_1) + 2w_1\delta}{2w_1\sqrt{A^2p_1^2(1+r_1)^2-4w_1\delta(1+r_1)}} \tag{2.31}$$

$$\frac{\partial\overline{y}_1}{\partial r_2} = \frac{\partial A}{\partial r_2}\frac{p_1(1+r_1)}{2w_1}\left[\underbrace{1 - \frac{Ap_1(1+r_1)}{\sqrt{A^2p_1^2(1+r_1)^2-4w_1\delta(1+r_1)}}}_{(<0)}\right] \tag{2.32}$$

The numerator of (31) is always negative[7]:

[7] in order to prove this, use the following property: If $\alpha, b \geq 0$ and $\alpha^2 < b^2$, then $\alpha < b$.

$$Ap_1\sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)} < A^2p_1^2(1+r_1) - 2w_1\delta$$

so that the derivative in (31) is always negative.
The derivative in (32) is always negative as well because the term in brackets is negative and the following is true:

$$\frac{\partial A}{\partial r_2} = \frac{\beta\left(2r_2^2(1+\beta) + 3(2+\beta) + 4r_2(2+\beta)\right)}{(1+r_2)^2(2+2r_2+\beta)^2} > 0$$

The main message stemming from the signs of (31),(32) is that reductions of the nominal interest rates will boost aggregate demand in period one and as a consequence will increase demand-determined output.

2.3.2.1 Characterization of Keynesian Unemployment

Before looking at the welfare implications of an interest rate policy, we must examine the fix price domain that characterizes the Keynesian Unemployment regime. The separation between non-investment and investment firms proves to be quite useful.

In order for the above equilibrium to represent excess supply in the markets during period one, the following two conditions must apply:

$$\frac{p_1}{2(1+r_1)w_1} > \overline{y}_1 \tag{2.33}$$

$$\sqrt{\overline{l}_1} > \overline{y}_1 \tag{2.34}$$

where (33) expresses the fact that there is excess supply in the good's market in period one and (34) that there is less than full employment. The usefulness of having two firms in the economy is that conditions (33),(34) are easy to manipulate and characterize the domain of Keynesian Unemployment.

Consider first the bigger root of (30). It will not satisfy the condition (33) even if the term in the square root is positive:

$$\frac{p_1}{1+r_1} > Ap_1(1+r_1) + \sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)}$$

Applying this property we get the following:

$$A^2p_1^2\Big[A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)\Big] < A^4p_1^4(1+r_1)^2 + 4w_1^2\delta^2 - 4A^2p_1^2(1+r_1)w_1\delta$$

which reduces to

$$4w_1^2\delta^2 > 0$$

which can not be true. The only possible candidate for a solution is the smaller root of (30). The fix price domain characterizing the Keynesian Unemployment regime should satisfy the following two inequalities:

$$\frac{p_1}{1+r_1} > Ap_1(1+r_1) - \sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)}$$
$$\sqrt{\bar{l}_1} > \frac{Ap_1(1+r_1) - \sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)}}{2w_1}$$

which is a system of two inequalities in p_1, w_1. Manipulating the previous inequalities we get[8]:

$$p_1^2 > \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.35}$$
$$p_1 > \frac{w_1\bar{l}_1 + \delta(1+r_1)}{A(1+r_1)\sqrt{\bar{l}_1}} \tag{2.36}$$

These curves intersect at two points: the Walrasian equilibrium (p_1^*, w_1^*) and at $(\bar{p}_1, \bar{w}_1)$ which satisfies:

$$\bar{p}_1 > p_1^*, \quad \bar{w}_1 > w_1^*$$

so that the slope of (36) close to the Walrasian equilibrium is higher than (35).

2.3.2.2 An intermediate case

Suppose we fix only the period one nominal wage rate. All other prices are allowed to adjust to clear markets. Only the labor market in period one does not clear in the usual sense.

Consider the FOC of the non-investment firm:

$$p_1 = 2(1+r_1)w_1\overline{y}_1 \tag{2.37}$$

Combining (28),(37) we compute the equilibrium value of period one output as follows[9]:

$$\overline{y}_1 = \sqrt{\frac{\delta}{w_1}\frac{1}{2(1+r_1)\left(\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}\right)}}$$

[8] If condition (35) holds true, then the discriminant of (30) is always positive.

[9] Decreasing r_1 or r_2 will increase period one output and decrease the unemployment level in that case as well.

There is unemployment in period one for values of w_1 that satisfy the following condition:

$$\overline{y}_1 < \sqrt{\bar{l}_1}$$

or

$$w_1 > w_1^*$$

where w_1^* is the nominal wage schedule in the Walrasian equilibrium.

The equilibrium value of period one price level is given by the following relation:

$$p_1^2 = \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}}$$

which is identical to (35).

2.3.3 Classical Unemployment

Consider the other type of unemployment where only individuals are rationed in the good's and labor markets during period one. The respective equilibrium conditions are written as follows:

$$(1-u)\overline{x}_1 + u\frac{1}{p_1(1+\beta)}\widetilde{\Pi} + I_1 = \frac{p_1}{2(1+r_1)w_1} \tag{2.38}$$

$$(1-u)\frac{(1+r_1)}{p_2}\Big(\Pi - p_1\overline{x}_1\Big) + u\frac{\beta(1+r_1)}{p_2(1+\beta)}\widetilde{\Pi} = F(\bar{l}_2, \widetilde{I}_1) \tag{2.39}$$

$$(1-u)\bar{l}_1 = \frac{p_1^2}{4(1+r_1)^2 w_1^2} = l_1^d \tag{2.40}$$

$$\bar{l}_2 = l_2^d \tag{2.41}$$

The money market clears *a fortiori* since the MFA accommodates the money demand:

$$m_1^T + n_1^{NI} = M_1 \tag{2.42}$$

$$m_2^T + n_2^I = M_2 \tag{2.43}$$

where m_1^T, m_2^T are equal to:

$$\begin{aligned} m_1^T &= (1-u)w_1\bar{l}_1 \\ &= w_1 l_1^d = \frac{p_1^2}{4(1+r_1)^2 w_1} \end{aligned} \tag{2.44}$$

$$m_2^T = (1-u)w_2\bar{l}_2 + uw_2\bar{l}_2 = w_2\bar{l}_2 \tag{2.45}$$

The bond market clearing is analogous to the Keynesian case. It will not be repeated again.

The MFA's constraint is written as follows:

$$\frac{r_1}{1+r_1}\Big(\frac{w_1}{p_1}l_1^d + y_1\Big) + \frac{r_2}{(1+r_1)(1+r_2)}\Big(\frac{w_2\bar{l}_2}{p_2}\frac{p_2}{p_1} + \frac{p_2}{p_1}y_2\Big) + \frac{\pi_1}{p_1} = \frac{\delta}{p_1} \tag{2.46}$$

Combing the investment's firm first order conditions and (46), we compute the new level of investment demand

$$I_1 = \frac{(1+r_2)}{r_2(3+2r_2)}\left[\frac{\delta}{p_1} - \frac{p_1}{4w_1}\frac{4r_1+2r_1^2+1}{(1+r_1)^3}\right] \tag{2.47}$$

and p_2, w_2 are computed from the investment's firm first order conditions and $u, \overline{x}_1$ from (40),(38) respectively.

Employed individuals violate their optimal plan and consume more in period two since they are rationed in the good's market during period one. Removing the constraint from the good's market will allow them to smooth consumption across periods and achieve a higher level of utility. The crucial point is to show that employed individuals achieve a higher level of utility than the unemployed achieve. Since exchange is voluntary, we need to guarantee that employed individuals *participate* in the exchange, namely that they supply labor in period one.

If the aggregate excess demand, condition (14), is positive then employed's period two consumption demand is always higher than the unemployed's one,

$$\frac{(1+r_1)}{p_2}\Big(\Pi - p_1\overline{x}_1\Big) > \frac{\beta(1+r_1)}{p_2(1+\beta)}\widetilde{\Pi}$$

which reduces to

$$\frac{\Pi}{p_1(1+\beta)} + \frac{\beta}{p_1(1+\beta)}\frac{w_1\bar{l}_1}{1+r_1} > \overline{x}_1$$

For period one consumption demand we need to show that the following inequality is true[10],

$$\overline{x}_1 \geq \frac{1}{p_1(1+\beta)}\widetilde{\Pi}$$

which is true for parameter values that satisfy the following condition:

$$p_1^2 \geq \frac{4\delta(1+r_1)^3[2(1+r_2)^2+\beta(1+r_2)]w_1}{2r_2(3+2r_2)(1+r_1)^2(1+\beta)+(4r_1+2r_1^2+1)(1+(1+r_2)(1+\beta))} \tag{2.48}$$

For simplicity, let us denote with $\Lambda(w_1, r_1, r_2, .)$ the right hand side of (48).

Participation constraint. *If the aggregate excess demand is positive,(14), and condition (48) is satisfied, then employed individuals achieve a higher level of utility than the unemployed ones.*

2.3.3.1 Characterization of Classical unemployment

Since only employed individuals are rationed in the good's market in period one, the employed's aggregate excess demand should be positive

$$D = (1-u)\left[\frac{\Pi}{p_1(1+\beta)} - \overline{x}_1\right] > 0$$

Similarly, since a fraction u of individuals are rationed in the labor market, positive unemployment means that the following must be true

$$u = 1 - \frac{p_1^2}{4(1+r_1)^2 w_1^2}\frac{1}{\bar{l}_1} > 0$$

The fix price domain characterizing Classical unemployment regime should satisfy the following inequalities:

$$p_1^2 < \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2}+\frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.49}$$

$$p_1^2 < 4(1+r_1)^2 w_1^2 \bar{l}_1 \tag{2.50}$$

[10] A comparison of utility levels between employed and unemployed would have been sufficient for the participation constraint argument.

where (49) implies excess demand in the good's market and (50) unemployment in the labor market.

Comparing the conditions that characterize the two unemployment regimes, we see that condition (49) is identical to (35)- only that the inequality is reversed. Substitute for zero aggregate excess demand, $D = 0$, into the equilibrium conditions of the Classical unemployment regime. They become identical with the equilibrium conditions in the Keynesian unemployment regime. Thus there is one common curve that separates the two regimes.

In order to complete the characterization argument we must ensure that the participation constraint restriction, (48), does not contradict conditions (49),(50).

Classical unemployment with *full participation* requires the following conditions:

$$\Lambda(w_1, r_1, r_2, .) \leq p_1^2 < \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.51}$$

and also condition (50) must hold.

The interval in (51) is well defined since

$$\frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} > \Lambda(w_1, r_1, r_2, .) \Rightarrow 1 > 0$$

2.3.3.2 A note on the characterization of Classical unemployment

From (47) we see that equilibrium investment demand becomes infinite at $r_2 = 0$. The reason is the following: If we fix $r_2 = 0$ initially, then from (46) we can no longer solve for the investment demand simply because the middle term in that equilibrium condition vanishes.

We can solve this problem by postulating that the investment firm's technology displays decreasing returns to scale. Thus, profits are positive in equilibrium. If investment firm's profits appear in (46), then the previous argument when $r_2 = 0$ does not apply. Working in a classical unemployment regime we will assume that initially r_2 is strictly positive.

2.3.3.3 Over-investment

The new level of investment demand, I_1, is given in (47). In this section we will lay down conditions such that there is *over-investment* compared to the Walrasian equilibrium.

Consider the following inequality

$$I_1 > \frac{\beta\sqrt{\bar{l}_1}}{2(1+r_2)+\beta}$$

It is always true if and only if

$$p_1 < \Omega(w_1, r_1, r_2, .) \tag{2.52}$$

where

$$\Omega(w_1, r_1, r_2, .) = \frac{\sqrt{\left(\frac{\beta w_1}{2(1+r_2)+\beta}\sqrt{\bar{l}_1}\right)^2 + \frac{(1+r_2)^2}{r_2^2(3+2r_2)^2}\frac{(4r_1+2r_1^2+1)}{(1+r_1)^3}\delta w_1} - \frac{\beta w_1}{2(1+r_2)+\beta}\sqrt{\bar{l}_1}}{\frac{(1+r_2)}{r_2(3+2r_2)}\frac{(4r_1+2r_1^2+1)}{2(1+r_1)^3}}$$

The graphs of (49),(52), intersect at the Walrasian equilibrium, (p_1^*, w_1^*), and at zero. For points close[11] to the Walrasian equilibrium condition (52) is implied from (49).

§ 2.4 Welfare

In the first part of the paper we showed that if the price level is free to adjust to clear markets, the equilibrium allocation is *not* Pareto optimal. This inefficiency is implied by the cash-in-advance technology that we have assumed. In the second part we extended this framework to incorporate two different types of unemployment into the analysis.

The question we want to answer in this part and in this paper is the following: What are the welfare implications of a nominal interest rate perturbation given the unemployment regime that prevails in the market?

2.4.1 Keynesian Unemployment

From (31),(32) we determine the sign of investment's demand comparative statics with respect to r_1, r_2,

$$\frac{\partial I_1}{\partial r_1} = \frac{\beta}{2(1+r_2)+\beta}\frac{\partial \overline{y}_1}{\partial r_1} < 0$$
$$\frac{\partial I_1}{\partial r_2} = \frac{\beta}{2(1+r_2)+\beta}\frac{\partial \overline{y}_1}{\partial r_2} - \frac{2\beta}{(2(1+r_2)+\beta)^2}\overline{y}_1 < 0$$

[11]At $w_1 < w_1^*$ the slope of (52) is higher than that of (49). For all (p_1, w_1), (52) is implied by (49),(50). At $w_1 > w_1^*$ the slope of (52) decreases and becomes lower than the slope of (49). This happens at $\overline{w}_1 = w_1^* + \epsilon$, $\epsilon > 0$. At $w_1 > \overline{w}_1$, the parameter region consistent with Classical unemployment splits in two regions. One that (52) is implied from (49) and another one that (52) is not implied from (49). The former region is much bigger that the latter.

where $\overline{y}_1$ is determined in (30).

Consider the inflation rate which is computed from the investment firm's first order conditions,

$$\frac{p_2}{p_1} = \frac{2(1+r_1)(1+r_2)}{\sqrt{l_2}}\sqrt{I_1}$$

The derivative of the inflation rate with respect to r_1 is as follows

$$\frac{\partial(p_2/p_1)}{\partial r_1} = \frac{\sqrt{I_1}}{\sqrt{l_2}}(1+r_1)(1+r_2)\left[\frac{2}{1+r_1} + \frac{\partial \overline{y}_1}{\partial r_1}\frac{1}{\overline{y}_1}\right] \tag{2.53}$$

and the sign of it depends on the term in the underbrace.

The term in brackets is positive if the following condition apply

$$-\frac{\partial \overline{y}_1}{\partial r_1}\frac{1}{\overline{y}_1} < \frac{2}{1+r_1} \Rightarrow p_1^2 > \frac{25 w_1 \delta}{6A^2(1+r_1)} \tag{2.54}$$

The second inequality in (54) is implied by (35). Reductions of period one nominal rate, reduce the inflation rate.

The derivative of the inflation rate with respect to r_2 is as follows

$$\frac{\partial(p_2/p_1)}{\partial r_2} = \frac{\sqrt{I_1}}{\sqrt{l_2}}(1+r_1)(1+r_2)\left[\frac{2}{1+r_2} + \frac{\partial I_1}{\partial r_2}\frac{1}{I_1}\right] \tag{2.55}$$

The term in brackets is positive if

$$-\frac{\partial I_1}{\partial r_2}\frac{1}{I_1} < \frac{2}{1+r_2} \Rightarrow p_1^2 > \frac{\left(\frac{2(1+r_2)+2\beta}{2(1+r_2)^2+\beta(1+r_2)}\right)^2 4w_1\delta(1+r_1)}{(1+r_1)^2\left(A^2\left(\frac{2(1+r_2)+2\beta}{2(1+r_2)^2+\beta(1+r_2)}\right)^2 - (\partial A/\partial r_2)^2\right)} \tag{2.56}$$

and the denominator is always positive.

The second inequality in (56) is always implied by (35) for $r_2 > 0.81$ even if r_1 is close to zero[12].

We analyze the comparative statics of the inflation rate with respect to perturbations in r_1, r_2 because it is going to be useful to analyze welfare implications in the next section.

[12]If $r_2 \leq 0.81$ then (35) implies (56) for all $r_1 > \Phi(r_2 \leq 0.81)$, where

$$\Phi(r_2) = \frac{1}{2-\beta+4r_2(1+\beta)+2r_2^2(1+\beta)}\Bigg[-2+\beta-4r_2(1+\beta)-2{r_2}^2(1+\beta)+$$
$$\sqrt{2}\sqrt{(1+r_2)\left(2+\beta-\beta^2+2{r_2}^3(1+\beta)+2{r_2}^2\left(3+4\beta+\beta^2\right)+r_2\left(6+7\beta+4\beta^2\right)\right)}\Bigg]$$

2.4.1.1 Unemployed individuals

The *equilibrium* allocation of unemployed individuals is as follows:

$$x_1 = \frac{1}{1+\beta}\left(\frac{\delta}{p_1} + \frac{I_1}{1+r_2}\right)$$
$$x_2 = \frac{\beta}{(1+\beta)(1+r_2)}\left(\frac{\delta\sqrt{\bar{l}_2}}{2p_1\sqrt{I_1}} + \frac{\sqrt{\bar{l}_2}\sqrt{I_1}}{2(1+r_2)}\right)$$

where

$$I_1 = \frac{\beta}{2(1+r_2)+\beta}\overline{y}_1$$
$$\overline{y}_1 = \frac{p_1(1+r_1)A - \sqrt{A^2p_1^2(1+r_1)^2 - 4w_1\delta(1+r_1)}}{2w_1}$$
$$A = 1 + \frac{\beta r_2(3+2r_2)}{2(1+r_2)^2+\beta(1+r_2)}$$

Perturbing r_1 we compute the effect on the unemployed's equilibrium allocation as follows:

$$\frac{\partial x_1}{\partial r_1} = \frac{1}{(1+\beta)(1+r_2)}\frac{\partial I_1}{\partial r_1} < 0 \tag{2.57}$$
$$\frac{\partial x_2}{\partial r_1} = \frac{\beta}{(1+\beta)(1+r_2)}\frac{\sqrt{\bar{l}_2}}{4\sqrt{I_1}}\left[\frac{1}{1+r_2} - \frac{\delta}{p_1I_1}\right]\frac{\partial I_1}{\partial r_1} \tag{2.58}$$

The sign of (58) depends on the term in brackets. It is positive if

$$\frac{1}{1+r_2} - \frac{\delta}{p_1I_1} > 0 \Rightarrow \overline{y}_1 > \frac{p_1(1+r_1)}{w_1}\left(A - \frac{\beta}{2(1+r_2)^2+\beta(1+r_2)}\right) \tag{2.59}$$

where we have used (29) to derive the second inequality above. From the existence argument we know that condition (33) has to hold. Comparing

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At $r_1 \geq \Phi(r_2)$, (35) is greater or equal to (56). Condition $r_1 > \Phi(r_2 \leq 0.81)$ is not very restrictive. If $r_2 = 0$, the maximum value of Φ is $\Phi(r_2 = 0, \beta = 1) = 1$. Otherwise, Φ is a small number.

If $r_2 > 0.81$ then (56) is implied from (35) for any non-negative value of r_1 because $\Phi(r_2 > 0.81) < 0$.

(33),(59) we end up in a contradiction[13]. The derivative in (58) is always positive.

The effect on unemployed's utility is computed as follows:

$$\frac{\partial u}{\partial r_1} = \frac{\partial I_1}{\partial r_1}\left[\frac{I_1}{1+r_2} - \frac{\beta}{2+\beta}\frac{\delta}{p_1}\right]\frac{2(2+\beta)(1+r_2)p_1 I_1}{\delta(1+r_2)+p_1 I_1}$$

The term in brackets is positive if

$$\frac{I_1}{1+r_2} - \frac{\beta}{2+\beta}\frac{\delta}{p_1} > 0 \Rightarrow \overline{y}_1 > \frac{p_1(1+r_1)}{w_1}\left[A - \frac{2+\beta}{2(1+r_2)^2+\beta(1+r_2)}\right] \tag{2.60}$$

where we have used (29) again to derive the second inequality in (60). Comparing again (33),(60) we end up in the following condition for a Pareto improvement,

Condition I.

In order for a reduction in the nominal interest rate of period one to make unemployed individuals better off, the nominal rate of period two should be close to/or zero initially such that (60) is satisfied[14].

If $r_2 = 0$, then (60) is trivially satisfied. Since r_2 must be very close to zero, we can also work with the following *stricter* form of the above condition

[13]From (33),(59) the following must be true

$$\frac{p_1(1+r_1)}{w_1}\left(A - \frac{\beta}{2(1+r_2)^2+\beta(1+r_2)}\right) < \overline{y}_1 < \frac{p_1}{2w_1(1+r_1)}$$

which cannot hold because

$$\frac{p_1(1+r_1)}{w_1}\left(A - \frac{\beta}{2(1+r_2)^2+\beta(1+r_2)}\right) > \frac{p_1}{2w_1(1+r_1)}$$

for *all* parameter values.

[14]From (33),(60) the following must be true

$$\frac{p_1(1+r_1)}{w_1}\left(A - \frac{2+\beta}{2(1+r_2)^2+\beta(1+r_2)}\right) < \overline{y}_1 < \frac{p_1}{2w_1(1+r_1)}$$

which is a well-defined interval if

$$\frac{p_1}{2w_1(1+r_1)} > \frac{p_1(1+r_1)}{w_1}\left(A - \frac{2+\beta}{2(1+r_2)^2+\beta(1+r_2)}\right)$$

This inequality holds if r_2 is a very small number or $r_2 = 0$ for any r_1.

Condition I (Strict Form).

In order for a reduction in the nominal interest rate of period one to make unemployed individuals better off, the nominal rate of period two should be equal to zero initially.

Considering perturbations of r_2, the effect on unemployed's equilibrium allocation is as follows:

$$\frac{\partial x_1}{\partial r_2} = \frac{1}{1+\beta}\left(- \frac{I_1}{(1+r_2)^2} + \frac{1}{1+r_2}\frac{\partial I_1}{\partial r_2}\right) < 0 \tag{2.61}$$

$$\frac{\partial x_2}{\partial r_2} = \frac{-\beta\sqrt{\bar{l}_2}\sqrt{I_1}}{2(1+\beta)(1+r_2)^3} - \frac{\beta\delta\sqrt{\bar{l}_2}}{4p_1\sqrt{I_1}(1+\beta)(1+r_2)}\left(\frac{2}{1+r_2} + \frac{1}{I_1}\frac{\partial I_1}{\partial r_2}\right) +$$
$$\frac{\beta}{(1+\beta)(1+r_2)}\left(\frac{\sqrt{\bar{l}_2}}{4(1+r_2)\sqrt{I_1}}\frac{\partial I_1}{\partial r_2} - \frac{\sqrt{\bar{l}_2}\sqrt{I_1}}{2(1+r_2)^2}\right) \tag{2.62}$$

Condition II.

If condition (56) is satisfied then the derivative in (62) is always negative. As a consequence, reductions of r_2 make unemployed individuals better off.

Condition II is unnecessary. If we consider perturbations of r_2 on unemployed's utility we can verify that reductions of r_2 always increase utility[15]. Thus, we are led to the following condition,

Condition II. (**revisited**)

Decreases of r_2 always make unemployed individuals better off.

2.4.1.2 Employed individuals

The equilibrium allocation of employed individuals is as follows:

$$x_1 = \frac{1}{1+\beta}\left[\frac{\delta}{p_1} + \frac{w_1\bar{l}_1}{p_1(1+r_1)} + \frac{I_1}{1+r_2}\right]$$
$$x_2 = \frac{\beta}{(1+\beta)(1+r_2)}\left[\frac{\delta\sqrt{\bar{l}_2}}{2p_1\sqrt{I_1}} + \frac{w_1\bar{l}_1}{p_1}\frac{\sqrt{\bar{l}_2}}{2(1+r_1)\sqrt{I_1}} + \frac{\sqrt{\bar{l}_2}\sqrt{I_1}}{2(1+r_2)}\right]$$

Perturbing r_1 we compute the effect on the unemployed's equilibrium allocation as follows:

[15] We can show this by doing some numerical calibrations involving the parameters of the model.

$$\frac{\partial x_1}{\partial r_1} = \frac{1}{(1+\beta)}\left[-\frac{w_1\bar{l}_1}{p_1(1+r_1)^2}+\frac{1}{1+r_2}\frac{\partial I_1}{\partial r_1}\right]<0$$

$$\frac{\partial x_2}{\partial r_1} = \frac{\beta}{(1+\beta)(1+r_2)}\left[-\frac{\delta\sqrt{\bar{l}_2}}{4p_1I_1\sqrt{I_1}}\frac{\partial I_1}{\partial r_1}+\frac{\sqrt{\bar{l}_2}}{4(1+r_2)\sqrt{I_1}}\frac{\partial I_1}{\partial r_1}-\right.$$
$$\left.\frac{w_1\bar{l}_1\sqrt{\bar{l}_2}}{4p_1(1+r_1)I_1\sqrt{I_1}}\frac{\partial I_1}{\partial r_1}-\frac{w_1\bar{l}_1\sqrt{\bar{l}_2}}{2p_1(1+r_1)^2\sqrt{I_1}}\right]$$

The effect on employed's utility is computed as follows:

$$\frac{\partial u}{\partial r_1}=\frac{1}{\Pi}\left[-\frac{\beta w_1\bar{l}_1}{2p_1(1+r_1)}\left(\frac{2(1+\beta)}{\beta(1+r_1)}+\frac{1}{\overline{y}_1}\frac{\partial\overline{y}_1}{\partial r_1}\right)+\frac{\partial I_1}{\partial r_1}\left(\frac{2+\beta}{2(1+r_2)}-\frac{\beta\delta}{2p_1I_1}\right)\right] \tag{2.63}$$

We do not have to make further restrictions. The first term in parenthesis is implied by (53),(54), because

$$\frac{2(1+\beta)}{\beta(1+r_1)}>\frac{2}{1+r_1}$$

The second term in parenthesis is implied by condition I. Thus, decreases of r_1 make employed individuals better off according to the previous conditions.

Lastly, consider perturbations of r_2. The effect on the equilibrium allocation is as follows:

$$\frac{\partial x_1}{\partial r_2}=\frac{1}{1+\beta}\left[\frac{1}{1+r_2}\frac{\partial I_1}{\partial r_2}-\frac{I_1}{(1+r_2)^2}\right]<0 \tag{2.64}$$

$$\frac{\partial x_2}{\partial r_2}=-\frac{\beta\sqrt{\bar{l}_2}}{4(1+\beta)(1+r_2)p_1\sqrt{I_1}}\left[\frac{2}{1+r_2}+\frac{1}{I_1}\frac{\partial I_1}{\partial r_2}\right]\left[\delta+\frac{w_1\bar{l}_1}{1+r_1}\right]-$$
$$\frac{\beta\sqrt{\bar{l}_2}\sqrt{I_1}}{2(1+\beta)(1+r_2)^3}-\frac{\beta\sqrt{\bar{l}_2}}{(1+\beta)(1+r_2)\sqrt{I_1}}\left[\frac{I_1}{2(1+r_2)^2}-\frac{\partial I_1/\partial r_2}{4(1+r_2)}\right] \tag{2.65}$$

Again the same argument applies as in the case of unemployed individuals. Reductions of r_2 always make employed individuals better-off.

2.4.1.3 Keynesian Unemployment and the Friedman Rule

The main message from the previous discussion is that optimal monetary policy is characterized by zero interest rates. In a Keynesian Unemployment

Equilibrium output is demand determined in period one. Nominal rates represent a cost to liquidity under the previous cash-in-advance set-up. Zero nominal rates imply costless borrowing in the asset markets which boost the aggregate demand in period one.

For condition I we do not require r_1 to be positive initially. We required only r_2 to be zero initially. This means that we could have fixed $r_1 = r_2 = 0$ and argue that the monetary-authority should not deviate from this rule. For condition II, the revisited version, we do not require nominal rates to be positive initially. We could have fixed them at zero and consider the derivatives with respect to r_2. From footnote 12 we showed that for low initial values of r_1, r_2, the good's price of period two reduces when r_2 is reduced. This is sufficient to make employed and unemployed better-off according to condition II, not the revisited one. Since we can show numerically that this sufficient condition is redundant, it is not surprising that we can fix nominal rates at zero and conclude that it is not optimal to increase r_2.

In order to understand the argument behind *condition I*, it is useful to take a look at the asset market constraints of unemployed individuals[16]. Following a decrease of r_1, demand-determined output is increased in period one. Unemployed individuals increase their holdings of initial money balances in their portfolio in order to increase consumption and reduce their holdings of bonds. Since their bond holdings, savings, in period one decrease, they will enter with less resources in the asset market in period two. Thus, setting initially $r_2 = 0$, makes it costless to borrow money balances in the asset markets of period two, given that they transfer less resources from period one.

When we consider perturbations only of r_2 we can show numerically for employed and unemployed that r_2 should be zero at the optimum. The reason is that r_2 is a cost on liquidity and reductions of it boost aggregate demand when unemployment is of a Keynesian nature. Condition II is redundant. We can verify numerically that the revised form of condition II is true for unemployed and employed individuals. Thus, condition II is a corollary of the strict form of condition I.

2.4.2 Classical unemployment

The comparative statics of investment demand with respect to r_1, r_2 are as follows:

[16] The argument for employed individuals is analogous.

$$\frac{\partial I_1}{\partial r_1} = \frac{p_1}{4w_1}\frac{1+r_2}{r_2(3+2r_2)}\frac{2(r_1+2.22)(r_1-0.22)}{(1+r_1)^4}$$
$$\frac{\partial I_1}{\partial r_2} = -\left[\frac{\delta}{p_1} - \frac{p_1}{4w_1}\frac{(4r_1+2r_1^2+1)}{(1+r_1)^3}\right]\frac{(3+4r_2+2r_2^2)}{r_2^2(3+2r_2)^2} < 0$$

and for initial values of r_1 above 0.22[17], the investment demand is increasing in r_1.

The derivative of the inflation rate with respect to r_1 is as follows:

$$\frac{\partial(p_2/p_1)}{\partial r_1} = \frac{\sqrt{I_1}}{\sqrt{l_2}}(1+r_1)(1+r_2)\left[\frac{2}{1+r_1} + \frac{\partial I_1}{\partial r_1}\frac{1}{I_1}\right]$$

always positive for $r_1 > 0.22$.

The derivative of the inflation rate with respect to r_2 is as follows:

$$\frac{\partial(p_2/p_1)}{\partial r_2} = \frac{\sqrt{I_1}}{\sqrt{l_2}}(1+r_1)(1+r_2)\left[\frac{2}{1+r_2} + \frac{\partial I_1}{\partial r_2}\frac{1}{I_1}\right]$$

and for $r_2 \gtrless 0.82$ the derivative above is either positive or negative.

2.4.2.1 Unemployed individuals

The equilibrium allocation of unemployed individuals is as follows

$$x_1 = \frac{1}{1+\beta}\left(\frac{\delta}{p_1} + \frac{I_1}{1+r_2}\right)$$
$$x_2 = \frac{\beta}{(1+\beta)(1+r_2)}\left(\frac{\delta\sqrt{l_2}}{2p_1\sqrt{I_1}} + \frac{\sqrt{l_2}\sqrt{I_1}}{2(1+r_2)}\right)$$

where

$$I_1 = \frac{(1+r_2)}{r_2(3+2r_2)}\left[\frac{\delta}{p_1} - \frac{p_1}{4w_1}\frac{4r_1+2r_1^2+1}{(1+r_1)^3}\right]$$

The derivative of the equilibrium allocation with respect to r_1 is as follows:

[17] This threshold value will play a role when we discuss the elastic labor supply case. For the rest of the analysis in this section I will assume that initially $r_1 > 0.22$ whenever we consider *perturbations* with respect to r_1.

$$\frac{\partial x_1}{\partial r_1} = \frac{1}{(1+\beta)(1+r_2)}\frac{\partial I_1}{\partial r_1} \tag{2.66}$$

$$\frac{\partial x_2}{\partial r_1} = \frac{\beta}{(1+\beta)(1+r_2)}\frac{\sqrt{l_2}}{4\sqrt{I_1}}\left[\frac{1}{1+r_2} - \frac{\delta}{p_1 I_1}\right]\frac{\partial I_1}{\partial r_1} \tag{2.67}$$

The effect on unemployed's utility is:

$$\frac{\partial u}{\partial r_1} = \frac{\partial I_1}{\partial r_1}\left[\frac{I_1}{1+r_2} - \frac{\beta}{2+\beta}\frac{\delta}{p_1}\right]\frac{2(2+\beta)(1+r_2)p_1 I_1}{\delta(1+r_2)+p_1 I_1} \tag{2.68}$$

The term in parentheses determines the sign of (68). A *sufficient* condition for this term to be negative and for the unemployed to become better off is:

$$\beta r_2(3+2r_2) \geq 2+\beta \tag{2.69}$$

A *necessary* and *sufficient* condition for the term in parentheses to be negative and for the unemployed to become better off is:

$$p_1^2 + \frac{w_1\delta}{\frac{4r_1+2r_1^2+1}{4(1+r_1)^3}}\left[\frac{\beta r_2(3+2r_2)-(2+\beta)}{2+\beta}\right] > 0 \tag{2.70}$$

which is automatically satisfied whenever (69) is true. Condition (70) should apply whenever (69) is not true. Without loss of generality, consider the following sufficient condition

Condition III.

If the nominal interest rate of period two is expected to be sufficiently high initially, condition (69), then reducing the nominal rate of period one will make unemployed individuals better off.

The derivative of the equilibrium allocation with respect to r_2 is as follows:

$$\frac{\partial x_1}{\partial r_2} = \frac{1}{1+\beta}\left(-\frac{I_1}{(1+r_2)^2} + \frac{1}{1+r_2}\frac{\partial I_1}{\partial r_2}\right) < 0$$

$$\frac{\partial x_2}{\partial r_2} = \frac{-\beta\sqrt{l_2}\sqrt{I_1}}{2(1+\beta)(1+r_2)^3} - \frac{\beta\delta\sqrt{l_2}}{4p_1\sqrt{I_1}(1+\beta)(1+r_2)}\left(\frac{2}{1+r_2} + \frac{1}{I_1}\frac{\partial I_1}{\partial r_2}\right) +$$
$$\frac{\beta}{(1+\beta)(1+r_2)}\left(\frac{\sqrt{l_2}}{4(1+r_2)\sqrt{I_1}}\frac{\partial I_1}{\partial r_2} - \frac{\sqrt{l_2}\sqrt{I_1}}{2(1+r_2)^2}\right)$$

The following sufficient condition guarantees that unemployed individuals become better off after a decrease of r_2,

Condition IV.

If the nominal interest rate of period two is expected to be sufficiently high initially, $r_2 > 0.82$*, then a reduction of it will make unemployed individuals better off.*

2.4.2.2 Employed individuals

In order to determine the effect of an interest rate perturbation on the employed's equilibrium allocation, we need to determine the sign of the following derivatives: $\partial \overline{x}_1/\partial r_1, \partial \overline{x}_1/\partial r_2$.

Considering perturbations of r_1, the following is true:

$$\frac{\partial \overline{x}_1}{\partial r_1} < 0 \Leftrightarrow p_1^2 < \frac{\sqrt{B^2 + 4\Delta\Gamma} - B}{2\Gamma} \tag{2.71}$$

where

$$\begin{aligned}
B &= 1 + \left[\frac{1}{1+\beta} + 1 + r_2\right] \frac{3 + 4r_1 + 2r_1^2}{2(1+r_1)^2 r_2(3+2r_2)} \\
\Gamma &= \frac{(r_1 + 2.22)(r_1 - 0.22)}{4w_1^2 \overline{l}_1 (1+\beta)(1+r_1)^4 r_2(3+2r_2)} \\
\Delta &= \frac{4(1+r_1)(1+r_2)w_1\delta}{r_2(3+2r_2)} + \frac{4(1+r_1)w_1\delta}{1+\beta}\left[1 + \frac{1}{r_2(3+2r_2)}\right]
\end{aligned}$$

Condition (71) looks complicated but we can show that it is implied from (49).

The effect on employed's utility is as follows:

$$\begin{aligned}
\frac{\partial u}{\partial r_1} = & \frac{-\beta}{(\Pi - p_1\overline{x}_1)} \frac{1}{2I_1} \frac{\partial I_1}{\partial r_1}\Big[\Pi - p_1\overline{x}_1\Big] + \frac{\partial \overline{x}_1}{\partial r_1} \frac{p_1(1+\beta)}{(\Pi - p_1\overline{x}_1)\overline{x}_1}\Big[\frac{\Pi}{p_1(1+\beta)} - \overline{x}_1\Big] + \\
& \frac{\beta}{(\Pi - p_1\overline{x}_1)} \frac{1}{4w_1(1+r_1)^4}\left[\underbrace{\frac{2r_1^2 + 4r_1 - 1}{r_2(3+2r_2)} p_1^2 - 4w_1^2(1+r_1)^2 \overline{l}_1}_{(?)}\right]
\end{aligned} \tag{2.72}$$

The sign of the above expression is negative if the term in the underbrace is negative. The term in the underbrace is negative when we consider condition (50) from the characterization argument if the following is true:

$$\frac{2r_1^2 + 4r_1 - 1}{r_2(3+2r_2)} \leq 1$$

and it is satisfied whenever r_2 is sufficiently higher than r_1. This lead us to the following condition

Condition V.

If monetary policy is sufficiently tighter in period two than period one initially, $r_2 > r_1$, then reducing the nominal interest rate today will make employed individuals better off.

The effect on employed's equilibrium allocation from a perturbation of r_2 is as follows:

$$\frac{\partial \overline{x}_1}{\partial r_2} > 0 \tag{2.73}$$

$$\frac{\partial x_2}{\partial r_2} = \frac{-\sqrt{\tilde{l}_2}}{2p_1(1+r_2)\sqrt{\widetilde{I}_1}}\left(\frac{1}{2I_1}\frac{\partial I_1}{\partial r_2} + \frac{1}{1+r_2}\right)\left(\Pi - p_1\overline{x}_1\right) - \frac{\sqrt{\tilde{l}_2}}{2p_1(1+r_2)\sqrt{I_1}}\left(-\frac{p_1 \partial I_1/\partial r_2}{1+r_2} + \frac{p_1 I_1}{(1+r_2)^2} + p_1\frac{\partial \overline{x}_1}{\partial r_2}\right) \tag{2.74}$$

where the sign of (73) follows from the good's market clearing of period one and the sign of (74) is always negative whenever $r_2 > 0.82$.

The effect on the employed's utility is

$$\frac{\partial u}{\partial r_2} = \frac{1}{\overline{x}_1}\frac{\partial \overline{x}_1}{\partial r_2} - \frac{\beta}{\Pi - p_1\overline{x}_1}\left[\left(\frac{1}{2I_1}\frac{\partial I_1}{\partial r_2} + \frac{1}{1+r_2}\right)\left(\Pi - p_1\overline{x}_1\right) + \left(-\frac{p_1 \partial I_1/\partial r_2}{1+r_2} + \frac{p_1 I_1}{(1+r_2)^2} + p_1\frac{\partial \overline{x}_1}{\partial r_2}\right)\right] \tag{2.75}$$

Since the above expression is rather complicated, we can determine the sign of it using an *informal* argument. Suppose we focus attention to fix price combinations *close* to the Walrasian equilibrium but strictly inside the region of Classical unemployment implied by (49),(50). This means that the constraint in the good's market in period one is *not severe*. If the interest rate in period two is expected to fall, then individuals will save more in the asset market in period one and increase their indebtedness in the asset market tomorrow. Since the constraint in period one is not severe, the increase in consumption tomorrow outweighs the decrease in consumption today so that total utility is increased.

Condition VI.

If the nominal interest rate of period two is expected to be sufficiently high initially, $r_2 > 0.82$, and we restrict attention to period one price-wage configurations close to the Walrasian ones which satisfy (49),(50), then a reduction of r_2 will make the employed individuals better off.

2.4.2.3 The formal argument behind Condition VI

A *formal* argument requires to manipulate expression (75). Reducing the nominal rate of period two will make employed individuals better off if the following expression is positive:

$$\Psi(p_1, w_1, \cdot) = -\frac{\partial I_1}{\partial r_2}(p_1, w_1, \cdot)\frac{p_1}{(1+r_2)[\Pi(p_1, w_1, \cdot) - p_1\overline{x}_1(p_1, w_1, \cdot)]}\Bigg[\beta - \frac{(1+\beta)(1+r_2) + u(p_1, w_1, \cdot)}{(1-u(p_1, w_1, \cdot))^2\overline{x}_1(p_1, w_1, \cdot)}D(p_1, w_1, \cdot)\Bigg] + \frac{p_1 I_1(p_1, w_1, \cdot)}{(1+r_2)^2[\Pi(p_1, w_1, \cdot) - p_1\overline{x}_1(p_1, w_1, \cdot)]}\Bigg[\beta - \frac{u(p_1, w_1, \cdot)}{(1-u(p_1, w_1, \cdot))^2\overline{x}_1(p_1, w_1, \cdot)}D(p_1, w_1, \cdot)\Bigg] + \frac{\beta}{2}\left[\frac{2}{1+r_2} - \frac{3+4r_2+2r_2^2}{r_2(3+2r_2)(1+r_2)}\right] \quad (2.76)$$

Proving that $\Psi(p_1, w_1, \cdot) > 0$ is really involved. Instead, I will characterize the sign of $\Psi(p_1, w_1, \cdot)$ at price-wage configurations close to the Walrasian one[18]. Suppose the price-wage configuration of period one is given by the following relations,

$$p_1^n = p_1^* - \epsilon_1^n,\ 0 < \epsilon_1 < 1$$
$$w_1^n = w_1^* + \epsilon_2^n,\ 0 < \epsilon_2 < 1,\ n = 1, 2, 3, ...$$

and

$$\lim_{n\to\infty} p_1^n = p_1^*, \quad \lim_{n\to\infty} w_1^n = w_1^*$$

It is not difficult to find sufficiently small values of ϵ_1, ϵ_2 such that conditions (49),(50) from the characterization argument are satisfied and as $n \to \infty$, the price-wage configuration tend to the Walrasian one but from points strictly inside the region characterizing Classical Unemployment. Using the above sequences we can show that if prices are close to the Walrasian ones but strictly inside the region of Classical Unemployment, then the sign of $\Psi(p_1, w_1, \cdot)$ is positive. In order to do that, we need to show that the following limit is strictly positive,

[18] Satisfying always (49),(50)

$$\lim_{n\to\infty} \Psi(p_1^n, w_1^n, \cdot)$$

This can be easily proved since we can determine the following limits

$$\begin{aligned}
&\lim_{n\to\infty} u(p_1^n, w_1^n, \cdot) = 0\\
&\lim_{n\to\infty} D(p_1^n, w_1^n, \cdot) = 0\\
&\lim_{n\to\infty} \Big(-\frac{\partial I_1}{\partial r_2}(p_1^n, w_1^n, \cdot)\Big) = \textit{positive constant}\\
&\lim_{n\to\infty} \overline{x}_1(p_1^n, w_1^n, \cdot) = \frac{\Pi(p_1^*, w_1^*, \cdot)}{p_1^*(1+\beta)}
\end{aligned}$$

Given the above limit values and $r_2 > 0.82$, $\lim_{n\to\infty} \Psi(p_1^n, w_1^n, \cdot)$, is always positive.

2.4.2.4 Classical Unemployment and Optimal Monetary Policy

Let us start with the unemployed individuals. The effects of perturbations of r_1 is different than the Keynesian case. Reducing r_1 will reduce consumption demand of unemployed in period one because the real wage of period two falls and as a consequence the intertemporal income of unemployed individuals is reduced. Unemployed individuals are better-off because the good's price level in period two falls after a reduction of r_1 and this outweigh the fall of the real wage in period after a reduction of r_1. This happens because r_2 is sufficiently high initially. Consider perturbations of r_2. A sufficient condition for reductions of r_2 to make unemployed individuals better-off is the good's price level to fall in period two. This requires $r_2 > 0.82$ initially.

Consider the employed individuals. Reductions of r_1 increase employed's constrained demand in period one because investment demand and consumption demand of unemployed falls. The good's price in period two falls. Employed's net discounted income, $\Pi - p_1\overline{x}_1$, falls as well after a reduction of r_1. Condition V guarantees that the latter effect does not dominate the former effects on employed's utility. Consider perturbations of r_2. Reducing r_2 is welfare improving for employed individuals if r_2 is sufficiently high initially. The intuition is that since employed individuals are constrained in the good's market in period one, they are forced to consume more in period two. If r_2 is sufficiently high initially, then the constraint of employed in the good's market is not so severe because unemployed individuals demand in period one and equilibrium investment are sufficiently low. Reducing r_2 in that case will make employed individuals reduce consumption in period one and increase consumption in period two.

§ 2.5 Elastic labor supply

2.5.1 Flexible prices

Let us start again with the case of flexible prices.

The problem of each individual is modified as follows:

$$\max_{x_1,x_2,l_1,l_2}\left[\log(x_1)+\log(\bar{l}_1-l_1)+\beta\log(x_2)+\beta\log(\bar{l}_2-l_2)\right]$$

$$s.t$$

$$p_1x_1+\frac{p_2x_2}{1+r_1}=\delta+\frac{w_1l_1}{1+r_1}+\frac{w_2l_2}{(1+r_1)(1+r_2)}$$

The respective demands and supplies are as follows:

$$\begin{aligned}
x_1 &= \frac{1}{2p_1(1+\beta)}\Pi\\
x_2 &= \frac{\beta(1+r_1)}{2p_2(1+\beta)}\Pi\\
l_1 &= \bar{l}_1-\frac{1+r_1}{2w_1(1+\beta)}\Pi\\
l_2 &= \bar{l}_2-\frac{\beta(1+r_1)(1+r_2)}{2w_2(1+\beta)}\Pi
\end{aligned}$$

where

$$\Pi=\delta+\frac{w_1\bar{l}_1}{1+r_1}+\frac{w_2\bar{l}_2}{(1+r_1)(1+r_2)}$$

The equilibrium for this case is as follows,

$$x_1+I_1=y_1=\frac{p_1}{2w_1(1+r_1)} \tag{2.77}$$

$$x_2=F(l_2,I_1) \tag{2.78}$$

$$l_1=\frac{p_1^2}{4w_1^2(1+r_1)^2}=l_1^d \tag{2.79}$$

$$l_2=l_2^d \tag{2.80}$$

$$\begin{aligned}
&\frac{r_1}{1+r_1}\Big(w_1l_1^d+p_1y_1\Big)+\frac{r_2}{(1+r_1)(1+r_2)}\Big(w_2l_2^d+p_2y_2\Big)+\pi_1+\\
&\frac{1}{1+r_1}\pi_2=\delta
\end{aligned} \tag{2.81}$$

and the first-order-conditions from the investing firm's problem are as follows:

$$\frac{w_2}{p_2} = \frac{1}{1+r_2}\frac{\sqrt{I_1}}{2\sqrt{l_2^d}} \tag{2.82}$$

$$\frac{p_2}{p_1} = 2(1+r_1)(1+r_2)\frac{\sqrt{I_1}}{\sqrt{l_2^d}} \tag{2.83}$$

Combining (77),(78) we get:

$$\frac{p_2}{p_1} = \beta(1+r_1)\frac{y_1 - I_1}{F(l_2, I_1)} \tag{2.84}$$

and from (83),(84) we compute I_1 as follows:

$$I_1 = \frac{\beta}{2(1+r_2)+\beta}y_1 \tag{2.85}$$

Combining (81),(82),(83),(85) we end up in the following relation:

$$\frac{p_1^2}{w_1}A = \delta \tag{2.86}$$

where

$$A = \frac{4r_1 + 2r_1^2 + 1}{4(1+r_1)^3} + \frac{\beta r_2(3+2r_2)}{2(1+r_1)(1+r_2)[2(1+r_2)+\beta]}$$

Substituting (86) into (79) we compute the equilibrium value of w_1 as follows:

$$w_1^* \bar{l}_1 \frac{1+\beta}{2+\beta} = \frac{1+r_1}{2(1+\beta)}\Big[\delta + \delta\frac{\beta}{2+\beta} + \frac{\beta(1+\beta)\delta}{A(2+\beta)(1+r_1)(1+r_2)[2(1+r_2)+\beta]}\Big] + \frac{\delta}{4A(1+r_1)^2} \tag{2.87}$$

a rather complicated solution.

The equilibrium value of period one labor supply is as follows

$$l_1 = \frac{\delta}{4A(1+r_1)^2 w_1^*}$$

and of period two labor supply,

$$l_2 = \frac{\bar{l}_2}{3+4r_2+2r_2^2}$$

which is independent of r_1.

2.5.2 Keynesian Unemployment

The equilibrium conditions are as follows:

$$(1-u)\frac{\Pi}{2p_1(1+\beta)} + u\frac{\widetilde{\Pi}}{2p_1(1+\beta)} + I_1 = \overline{y}_1 \tag{2.88}$$

$$(1-u)\frac{\beta(1+r_1)\Pi}{2p_2(1+\beta)} + u\frac{\beta(1+r_1)\widetilde{\Pi}}{2p_2(1+\beta)} = F(l_2, I_1) \tag{2.89}$$

$$(1-u)l_1 = f^{-1}(\overline{y}_1) = \overline{y}_1^2 \tag{2.90}$$

$$(1-u)l_2 + u\widetilde{l}_2 = l_2^d \tag{2.91}$$

$$\frac{r_1}{1+r_1}\Big(w_1\overline{y}_1^2 + p_1\overline{y}_1\Big) + \frac{r_2}{(1+r_1)(1+r_2)}(w_2 l_2^d + p_2 y_2) +$$
$$\pi_1 + \frac{1}{1+r_1}\pi_2 = \delta \tag{2.92}$$

where

$$\widetilde{\Pi} = \delta + \frac{w_2 \bar{l}_2}{(1+r_1)(1+r_2)}$$
$$\widetilde{l}_2 = \bar{l}_2 - \frac{\beta(1+r_1)(1+r_2)}{2w_2(1+\beta)}\widetilde{\Pi}$$

Combining (88),(89) we get:

$$\frac{p_2}{p_1} = \beta(1+r_1)\frac{\overline{y}_1 - I_1}{F(l_2, I_1)}$$

and

$$I_1 = \frac{\beta}{2(1+r_2)+\beta}\overline{y}_1$$

From (92) we get:

$$\overline{y}_1 = \frac{p_1(1+r_1)\Delta \pm \sqrt{(1+r_1)^2\Delta^2 p_1^2 - 4w_1\delta_0(1+r_1)}}{2w_1}$$

where[19]

[19] only the smaller root is accepted as a solution as it is shown in the characterization argument below.

$$\Delta = 1 + \frac{\beta r_2(3 + 2r_2)}{(1 + r_2)(2(1 + r_2) + \beta)}$$

From the investment firm's first order conditions we get that:

$$\frac{w_2 l_2^d}{1 + r_1} = p_1 I_1 \tag{2.93}$$

and from (91),(93) we get:

$$\frac{w_2 \bar{l}_2}{(1 + r_1)(1 + r_2)} = \frac{2(1 + \beta)}{2 + \beta}\frac{p_1 I_1}{1 + r_2} + \frac{\beta}{2 + \beta}\delta + (1 - u)\frac{\beta}{2 + \beta}\frac{w_1 \bar{l}_1}{1 + r_1} \tag{2.94}$$

Labor market clearing of period one, (92), can be written as follows:

$$u^2 D + u(E - 2D) - (E - B - D) = 0$$

where

$$\begin{aligned}
E &= \frac{w_1 \bar{l}_1}{1 + r_1} - \frac{1}{2(1 + \beta)}\left[\delta + \frac{w_1 \bar{l}_1}{1 + r_1} + \frac{2(1 + \beta)}{2 + \beta}\frac{p_1 I_1}{1 + r_2} + \frac{\beta}{2 + \beta}\delta\right] \\
B &= \frac{w_1 \overline{y}_1^2}{1 + r_1} \\
D &= \frac{\beta}{2(2 + \beta)(1 + \beta)}\frac{w_1 \bar{l}_1}{1 + r_1}
\end{aligned}$$

Solving for the unemployment rate we get:

$$u = \frac{-(E - 2D) \pm \sqrt{(E - 2D)^2 + 4D(E - B - D)}}{2D} \tag{2.95}$$

From (95) there is the possibility of two solutions. The characterization argument is more challenging in that case since we need to look for conditions that allows us to work with only one of them.

2.5.2.1 Characterization of Keynesian Unemployment

The solutions for the demand-determined output and the unemployment rate are as follows,

$$
\begin{aligned}
\overline{y}_1 &= \frac{p_1(1+r_1)\Delta \pm \sqrt{(1+r_1)^2\Delta^2 p_1^2 - 4w_1\delta_0(1+r_1)}}{2w_1} \\
u &= \frac{-(E-2D) \pm \sqrt{(E-2D)^2 + 4D(E-B-D)}}{2D}
\end{aligned}
$$

For the demand-determined output solution the argument in the inelastic labor case applies. We accept only the smaller root. For the unemployment solution we need to do more work.

If $E-B-D>0$, then

$$
u = \frac{-(E-2D) + \sqrt{(E-2D)^2 + 4D(E-B-D)}}{2D} \tag{2.96}
$$

is accepted as a solution.

Condition $E-B-D>0$, can be equivalently written as follows:

$$
l_1^W - \overline{y}_1^2 > 0 \tag{2.97}
$$

where l_1^W is the Walrasian labor supply plan. At the Walrasian price-wage combination, condition (97) becomes zero. Manipulating (97) we end up in the following inequality,

$$
p_1\Theta\sqrt{(1+r_1)^2\Delta^2 p_1^2 - 4w_1\delta(1+r_1)} > p_1^2(1+r_1)\Delta\Theta - \left(\frac{w_1^2\bar{l}_1}{1+r_1}\frac{1+\beta}{2+\beta} + \frac{w_1\delta(1+\beta)}{2+\beta}\right)
$$

where the right hand side is positive for a sufficiently large neighborhood of points close to the Walrasian prices. Thus, the above expression becomes

$$
p_1^2 > \frac{\left(\frac{w_1^2\bar{l}_1}{1+r_1}\frac{1+\beta}{2+\beta} + \frac{w_1\delta(1+\beta)}{2+\beta}\right)^2}{2(1+r_1)\Delta\Theta\left(\frac{w_1^2\bar{l}_1}{1+r_1}\frac{1+\beta}{2+\beta} + \frac{w_1\delta(1+\beta)}{2+\beta}\right) - 4w_1\delta(1+r_1)\Theta^2} \tag{2.98}
$$

where

$$
\begin{aligned}
\Theta &= \frac{\beta}{2(2+\beta)(2(1+r_2)^2 + \beta(1+r_2))} + \frac{\Delta}{2} \\
\Delta &= 1 + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}
\end{aligned}
$$

Expression (98) has an *asymptote* at

$$\hat{w}_1 = \frac{(1+r_1)(2+\beta)\delta}{\Delta(1+\beta)\bar{l}_1}\left[2\Theta - \frac{(1+\beta)\Delta}{2+\beta}\right]$$

and also $w_1^* > \hat{w}_1$ for all parameter values.

The next step is to verify that if the price-wage combination of period one is equal to the Walrasian one, then the unemployment rate tends is zero. We said before that if the price-wage combination of period one is equal to the walrasian one, then

$$E - B - D = 0 \tag{2.99}$$

Since we are working with the bigger root from the solution above, if prices equal the Walrasian ones and condition (99) applies, a *necessary* condition for the unemployment rate to be zero is

$$E - 2D > 0 \Leftrightarrow l_1^W > \frac{\beta\bar{l}_1}{2(2+\beta)(1+\beta)} \tag{2.100}$$

which means that the Walrasian labor supply plan has to be bounded below. Combining (97),(100), we see that (100) is implied by (97) if the following is true,

$$\overline{y}_1^2 > \frac{\beta\bar{l}_1}{2(2+\beta)(1+\beta)} \tag{2.101}$$

Condition (101) reduces to the following requirement

$$\sqrt{(1+r_1)^2\Delta^2 p_1^2 - 4w_1\delta(1+r_1)} < p_1(1+r_1)\Delta - 2w_1\sqrt{\frac{\beta\bar{l}_1}{2(2+\beta)(1+\beta)}}$$

and the right hand side is strictly positive close to the Walrasian prices. It can be simplified as follows,

$$p_1 < \frac{\frac{\beta w_1\bar{l}_1}{2(1+\beta)(2+\beta)} + \delta(1+r_1)}{(1+r_1)\Delta\sqrt{\frac{\beta\bar{l}_1}{2(1+\beta)(2+\beta)}}} \tag{2.102}$$

To sum up, conditions for a *well-defined*[20] Keynesian unemployment equilibrium when we accept the big root from (95) are as follows

[20] well-defined in the sense that whenever prices tend to the Walrasian ones, the unemployment rate tends to zero.

$$p_1^2 > \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.103}$$

$$p_1 < \frac{\frac{\beta w_1 \bar{l}_1}{2(1+\beta)(2+\beta)} + \delta(1+r_1)}{(1+r_1)\Delta\sqrt{\frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}}} \tag{2.104}$$

$$p_1^2 > \frac{\left(\frac{w_1^2 \bar{l}_1}{1+r_1}\frac{1+\beta}{2+\beta} + \frac{w_1\delta(1+\beta)}{2+\beta}\right)^2}{2(1+r_1)\Delta\Theta\left(\frac{w_1^2 \bar{l}_1}{1+r_1}\frac{1+\beta}{2+\beta} + \frac{w_1\delta(1+\beta)}{2+\beta}\right) - 4w_1\delta(1+r_1)\Theta^2} \tag{2.105}$$

Condition (103) implies that there is excess supply in the good's market in period one. Conditions (104),(105) imply that the unemployment rate is positive and is close to zero in a neighborhood close to the Walrasian equilibrium prices.

In order to complete the argument we must guarantee that the above inequalities, (103)-(105), do not lead to a contradiction. We must prove that the graph of (104) is above the graph of (103) and as a consequence above the graph of (105). The graphs of (103),(104) cross at

$$\bar{w}_1 = \frac{(X^2 - 2A\Gamma) \pm X\sqrt{X^2 - 4A\Gamma}}{2A^2}$$

where

$$X = \sqrt{\frac{2(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}}}$$

$$A = \frac{\frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}}{(1+r_1)\Delta\sqrt{\frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}}}$$

$$\Gamma = \frac{\delta(1+r_1)}{(1+r_1)\Delta\sqrt{\frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}}}$$

where we will not pay attention to the bigger root because it refers to points very far from the Walrasian equilibrium price-wage configuration. If $\bar{w}_1 > w_1^*$, then the graph of (104) is above the graph of (103). We need to show that $\bar{w}_1 > w_1^*$. For this we need to place the following restriction on the policy parameters,

$$r_1 < \Phi(r_2, \beta, \delta, \bar{l}_1) \tag{2.106}$$

where $\Phi(\cdot)$ is a rather complicated expression. If we calibrate[21] the parameters of the model, we end up in the following restriction

$$r_1 < 1.5 \tag{2.107}$$

Discussion of condition (107) will be given in the following section.

If $E - B - D < 0$, $E - 2D < 0$, then we could accept both solutions. Since this multiplicity of equilibria is not very helpful in doing comparative statics, we will not pursue it further.

2.5.2.2 Monetary policy as the main cause of excessive unemployment: Keynesian Unemployment regime

The solution that we decided to accept and work is the following:

$$u = \frac{-(E-2D) + \sqrt{(E-2D)^2 + 4D(E-B-D)}}{2D}$$

This solution has an interesting feature. According to the previous discussion, two different situations can arise in a neighborhood of the walrasian equilibrium,

Low Unemployment	$E - 2D > 0$
High Unemployment	$E - 2D < 0$

The Keynesian equilibrium is well-defined in a neighborhood of the Walrasian equilibrium when $E - 2D > 0$, such that the unemployment rate is close to zero whenever prices tend to the Walrasian ones.

There is also a possibility that the unemployment rate is sufficiently high whenever prices of period one are very close, but not equal, to the Walrasian ones. This is interesting because the principal reason for any kind of excess capacities in equilibrium is that prices of period one are different from the Walrasian ones. Thus, when the fundamentals in the market, period one prices, do not justify high rates of unemployment, the actions of the monetary-authority can create excessive unemployment.

Condition (107) is a sufficient condition to rule out abnormally high levels of unemployment for price-wage configurations close to the Walrasian

[21]Consider the following values for the structural parameters of the economy,

β	0.9
$\bar{l}_2$	1
$\bar{l}_1$	1
δ	0.5

equilibrium one. It simply states that if monetary policy in period one is not sufficiently tight, then Keynesian equilibrium is well-defined for price-wage configurations close to the Walrasian equilibrium one.

The interesting point is that if condition (107) does not apply and monetary policy is sufficiently tight in period one, then high levels of unemployment are possible close for price-wage configurations close to the Walrasian equilibrium one. Effective labor market clearing in period one can be written as follows,

$$(1-u)\underbrace{\Big(l_1^W + u\frac{\beta\bar{l}_1}{2(2+\beta)(1+\beta)}\Big)}_{effective\ labor\ supply} = \overline{y}_1^2 \tag{2.108}$$

If condition (109) is not satisfied then we can not guarantee that $E-2D>0$. In that case the following situation may arise,

$$\frac{\beta\bar{l}_1}{2(2+\beta)(1+\beta)} > l_1^W > \overline{y}_1^2 \tag{2.109}$$

From (108),(109) we see that the effective labor supply is sufficiently higher than labor demand when monetary policy is very tight in period one. The only way for labor market to clear is the unemployment rate to be also sufficiently high.

To sum up, monetary policy can be the *main cause of excessive unemployment in the economy if condition (107) is violated.* This is interesting because it happens in a neighborhood of period one prices close to the Walrasian equilibrium ones. This means that although the fundamentals in the economy do not justify high rates of unemployment, the unemployment level can be sufficiently high.

The following numerical examples illustrate the previous argument,

- $$r_1 = 1,\ r_2 = 1$$
  $$p_1^* = 1.02842,\ w_1^* = 0.705545$$

(p_1, w_1)	Condition (102)	Condition (103)	Unempl. rate
$(1.0286, 0.7055)$	+	+	0.000522861
$(1.03, 0.7053)$	+	+	0.0064
$(1.05, 0.697)$	+	+	0.011
$(1.04, 0.705)$	+	+	0.05
$(1.07, 0.693)$	+	+	0.07

•

$$r_1 = 1.8, \;\; r_2 = 1$$
$$p_1^* = 1.33109, \;\; w_1^* = 0.882998$$

(p_1, w_1)	Condition (102)	Condition (103)	Unempl. rate
$(1.34, 0.8815)$	negative	negative	0.1334
$(1.35, 0.88)$	negative	negative	0.16
$(1.38, 0.875)$	negative	negative	0.2
$(1.334, 0.8826)$	negative	negative	0.125

We can observe by comparing the two cases that for points close to (p_1^*, w_1^*), the unemployment level in the second case is not close to zero whereas in the first it is. The second case violates condition (109) since monetary policy is sufficiently tight in period one.

2.5.3 Classical Unemployment

Consider the other type of unemployment which would appear if firms were not rationed but individuals were rationed in both markets. The set up is similar to the one before.

The problem of employed[22] individuals is as follows:

$$\max_{x_1, x_2, l_1, l_2} \left[\log(x_1) + \log(\bar{l}_1 - l_1) + \beta \log(x_2) + \beta \log(\bar{l}_2 - l_2) \right]$$
$$s.t$$
$$p_1 x_1 + \frac{p_2 x_2}{1 + r_1} = \delta + \frac{w_1 l_1}{1 + r_1} + \frac{w_2 l_2}{(1 + r_1)(1 + r_2)}$$
$$x_1 = \bar{x}_1$$

The respective demands and supplies are as follows:

$$x_2 = \frac{\beta(1 + r_1)}{p_2(1 + 2\beta)} \left[\Pi - p_1 \bar{x}_1 \right]$$
$$l_1 = \bar{l}_1 - \frac{1 + r_1}{w_1(1 + 2\beta)} \left[\Pi - p_1 \bar{x}_1 \right]$$
$$l_2 = \bar{l}_2 - \frac{\beta(1 + r_1)(1 + r_2)}{w_2(1 + 2\beta)} \left[\Pi - p_1 \bar{x}_1 \right]$$

[22] The problem of unemployed individuals is similar to the Keynesian case.

2.5.3.1 Equilibrium

$$(1-u)\bar{x}_1 + u\frac{1}{2p_1(1+\beta)}\widetilde{\Pi} + I_1 = \frac{p_1}{2(1+r_1)w_1} \tag{2.110}$$
$$(1-u)\frac{\beta(1+r_1)}{p_2(1+2\beta)}\Big[\Pi - p_1\bar{x}_1\Big] + u\frac{\beta(1+r_1)}{2p_2(1+\beta)}\widetilde{\Pi} = F(l_2^d, I_1) \tag{2.111}$$
$$(1-u)\Big[\bar{l}_1 - \frac{1+r_1}{w_1(1+2\beta)}\Big[\Pi - p_1\bar{x}_1\Big]\Big] = \frac{p_1^2}{4(1+r_1)^2 w_1^2} = l_1^d \tag{2.112}$$
$$(1-u)l_2 + u\tilde{l}_2 = l_2^d \tag{2.113}$$
$$\frac{r_1}{1+r_1}\Big(w_1 l_1^d + p_1 y_1\Big) + \frac{r_2}{(1+r_1)(1+r_2)}\Big(w_2 l_2^d + p_2 y_2\Big) + \widetilde{\pi}_1 + \frac{1}{1+r_1}\widetilde{\pi}_2 = \delta \tag{2.114}$$

where

$$\widetilde{\Pi} = \delta + \frac{w_2\bar{l}_2}{(1+r_1)(1+r_2)}$$
$$\tilde{l}_2 = \bar{l}_2 - \frac{\beta(1+r_1)(1+r_2)}{2w_2(1+\beta)}\widetilde{\Pi}$$

The investment firm's first order conditions are:

$$\frac{w_2}{p_2} = \frac{1}{1+r_2}\frac{1}{2\sqrt{l_2^d}}\sqrt{I_1} \tag{2.115}$$
$$\frac{p_2}{p_1} = \frac{2(1+r_1)(1+r_2)}{\sqrt{l_2^d}}\sqrt{I_1} \tag{2.116}$$

From (114),(115),(116) we get:

$$I_1 = \frac{1+r_2}{r_2(3+2r_2)}\Big(\frac{\delta}{p_1} - \frac{p_1}{4w_1}\frac{4r_1+2r_1^2+1}{(1+r_1)^3}\Big) \tag{2.117}$$

Combining (110),(113),(115),(116),(117) we get the following expression:

$$\frac{w_2\bar{l}_2}{(1+r_1)(1+r_2)} = \frac{1+2\beta}{(1+\beta)}\frac{p_1 I_1}{1+r_2} + \frac{\beta}{1+\beta}p_1 I_1 - \frac{\beta}{1+\beta}\frac{p_1^2}{2(1+r_1)w_1} + \frac{\beta}{1+\beta}\delta + \frac{\beta(1-u)}{1+\beta}\frac{w_1\bar{l}_1}{1+r_1} \tag{2.118}$$

Labor market clearing in period one, expression (112), can be written as follows:

$$Zu - Vu^2 + B = 0 \tag{2.119}$$

where

$$\begin{aligned}
B =& \frac{w_1\bar{l}_1}{1+r_1}\frac{\beta}{1+\beta} - \frac{\delta}{1+\beta} - p_1 I_1 \frac{2+r_2}{(1+\beta)(1+r_2)} + \\
& \frac{p_1^2}{2(1+r_1)(1+\beta)w_1} - \frac{p_1^2}{4(1+r_1)^3 w_1} \\
V =& \frac{\beta}{2(1+\beta)^2}\frac{w_1\bar{l}_1}{1+r_1} \\
Z =& \delta\frac{1+2\beta}{2(1+\beta)^2} - \frac{w_1\bar{l}_1}{1+r_1}\frac{\beta(1+2\beta)}{2(1+\beta)^2} + p_1 I_1 \frac{1+2\beta+\beta(1+r_2)}{2(1+\beta)^2(1+r_2)} - \\
& \frac{\beta p_1^2}{4(1+r_1)(1+\beta)^2 w_1}
\end{aligned}$$

Solving (119) we get:

$$u = \frac{-Z \pm \sqrt{Z^2 + 4VB}}{-2V} \tag{2.120}$$

As in the case of Keynesian equilibrium, we get two solutions for the unemployment rate. The next section is devoted to the analysis of (120) and the characterization of the classical equilibrium.

2.5.3.2 Characterization of Classical Equilibrium

If $B > 0$, then only the following solution is accepted:

$$u = \frac{Z + \sqrt{Z^2 + 4VB}}{2V}$$

In order for B to be greater than zero, the following must be true:

$$\begin{aligned}
&\left[\frac{(2+r_2)(4r_1+2r_1^2+1)}{4r_2(3+2r_2)(1+\beta)(1+r_1)^3} + \frac{1}{2(1+r_1)(1+\beta)} - \frac{1}{4(1+r_1)^3}\right] p_1^2 > \\
&w_1\delta\left[\frac{1}{1+\beta} + \frac{2+r_2}{r_2(3+2r_2)(1+\beta)}\right] - \frac{\beta}{1+\beta}\frac{w_1^2\bar{l}_1}{1+r_1}
\end{aligned} \tag{2.121}$$

For points close to the Walrasian equilibrium, the RHS of (121) must be positive. This requires the following restriction:

$$w_1 < \delta\left[\frac{1}{1+\beta} + \frac{2+r_2}{r_2(3+2r_2)(1+\beta)}\right]\frac{(1+r_1)(1+\beta)}{\beta\bar{l}_1} = \hat{w}_1$$

and also $\hat{w}_1 > w_1^*$.

The unemployment rate is zero if the price-wage combination equals the Walrasian one if and only if

$$\sqrt{Z^2 + 4VB} = -Z$$

which requires that $Z < 0$. This is true if and only if

$$\left[\frac{(1+2\beta+\beta(1+r_2))(4r_1+2r_1^2+1)}{8r_2(3+2r_2)(1+\beta)^2(1+r_1)^3} + \frac{\beta}{4(1+r_1)(1+\beta)^2}\right]p_1^2 >$$
$$w_1\delta\left[\frac{1+2\beta}{2(1+\beta)^2} + \frac{1+2\beta+\beta(1+r_2)}{2r_2(3+2r_2)(1+\beta)^2}\right] - \frac{w_1^2\bar{l}_1}{1+r_1}\frac{\beta(1+2\beta)}{2(1+\beta)^2} \tag{2.122}$$

The RHS is positive if and only if

$$w_1 < \delta\left[\frac{1+2\beta}{2(1+\beta)^2} + \frac{1+2\beta+\beta(1+r_2)}{2r_2(3+2r_2)(1+\beta)^2}\right]\frac{2(1+r_1)(1+\beta)^2}{\beta(1+2\beta)\bar{l}_1} = \bar{w}_1$$

also $\bar{w}_1 > w_1^*$ and $\hat{w}_1 > \bar{w}_1$.

Only employed individuals are constrained in the good's market. The respective excess demand is as follows

$$D = (1-u)\Big[\frac{\Pi}{2p_1(1+\beta)} - \bar{x}_1\Big] = (1-u)D^{'}$$

and is positive for

$$p_1^2 < \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.123}$$

Condition (123) is identical with the Keynesian case. If we substitute $D = 0$ to the equilibrium conditions they become identical with that of the Keynesian unemployment regime.

The equilibrium must be well-defined close to the Walrasian price-wage combination. The graph of (122) must be below the graph of (121). Otherwise, $Z > 0$ and the unemployment rate is not zero whenever prices equal the Walrasian ones. The graphs of (122),(123) cross at

$$\tilde{w}_1 = \Big[(1+r_1)(1+r_2)\left(2+3\beta+6\beta^2+8r_1(1+r_2)\left(1+2\beta+2\beta^2\right)+\right.$$
$$\left.4r_1^2(1+v)\left(1+2\beta+2\beta^2\right)+r_2\left(2+4\beta+8\beta^2\right)\right)\delta\Big]\Big/\Big[\bar{l}_1\beta(1+2\beta)(2+\beta+$$
$$r_2^2(2+4\beta)+r_2(4+7\beta)+4r_1\left(2+\beta+4r_2(1+\beta)+2r_2^2(1+\beta)\right)+$$
$$2r_1^2\left(2+\beta+4r_2(1+\beta)+2r_2^2(1+\beta)\right))\Big]$$

For $w_1^* > \tilde{w}_1$, consider the following restriction on policy parameters[23],

$$r_1 < 0.22 \tag{2.124}$$

If condition (124) applies, then the graph of (122) is below the graph of (121). The reason is simple: the graphs of (121),(123) cross at the walrasian prices, (p_1^*, w_1^*). From condition (124), $w_1^* > \tilde{w}_1$ which implies that the graph of (122) is below the graph of (121) and the equilibrium is well-defined close to the walrasian equilibrium.

The last thing to prove is that the unemployment level is less than one- $u < 1$. This is true if and only if

$$p_1^2 < \frac{w_1\delta\left(\frac{1}{1+\beta}-\frac{1+2\beta}{2(1+\beta)^2}+\frac{2+r_2}{r_2(3+2r_2)(1+\beta)}-\frac{1+2\beta+\beta(1+r_2)}{r_2(3+2r_2)2(1+\beta)^2}\right)}{\frac{(2+r_2)(4r_1+2r_1^2+1)}{4r_2(3+2r_2)(1+\beta)(1+r_1)^3}+\frac{1}{2(1+r_1)(1+\beta)}-\frac{1}{4(1+r_1)^3}-\frac{(1+2\beta+\beta(1+r_2))(4r_1+2r_1^2+1)}{8r_2(3+2r_2)(1+\beta)^2(1+r_1)^3}-\frac{\beta}{4(1+r_1)(1+\beta)^2}}$$

The above inequality is implied by (126).

To sum up, the followings conditions are required in order to have a *well-defined* classical equilibrium in a neighborhood of the walrasian equilibrium,

[23] Again we set the fundamentals of the economy to the following values,

β	0.9
$\bar{l}_2$	1
$\bar{l}_1$	1
δ	0.5

$$p_1^2 < \frac{2w_1(1+r_1)\delta}{\frac{4r_1+2r_1^2+1}{2(1+r_1)^2} + \frac{\beta r_2(3+2r_2)}{(1+r_2)(2(1+r_2)+\beta)}} \tag{2.125}$$

$$\left[\frac{(2+r_2)(4r_1+2r_1^2+1)}{4r_2(3+2r_2)(1+\beta)(1+r_1)^3} + \frac{1}{2(1+r_1)(1+\beta)} - \frac{1}{4(1+r_1)^3}\right]p_1^2 >$$

$$w_1\delta\left[\frac{1}{1+\beta} + \frac{2+r_2}{r_2(3+2r_2)(1+\beta)}\right] - \frac{\beta}{1+\beta}\frac{w_1^2\bar{l}_1}{1+r_1} \tag{2.126}$$

$$\left[\frac{(1+2\beta+\beta(1+r_2))(4r_1+2r_1^2+1)}{8r_2(3+2r_2)(1+\beta)^2(1+r_1)^3} + \frac{\beta}{4(1+r_1)(1+\beta)^2}\right]p_1^2 >$$

$$w_1\delta\left[\frac{1+2\beta}{2(1+\beta)^2} + \frac{1+2\beta+\beta(1+r_2)}{2r_2(3+2r_2)(1+\beta)^2}\right] - \frac{w_1^2\bar{l}_1}{1+r_1}\frac{\beta(1+2\beta)}{2(1+\beta)^2} \tag{2.127}$$

Condition (125) implies excess demand in the good's market in period one. Conditions (126),(127) simply state that there is positive unemployment and the unemployment rate is close to zero in a neighborhood of the walrasian equilibrium.

2.5.3.3 Monetary policy as the main cause of excessive unemployment: Classical Unemployment regime

In order for the equilibrium to be well-defined we need to place some restrictions on the policy parameters as in the case of keynesian equilibrium. The difference with the keynesian case is that condition (124) requires monetary policy in period one to be very loose. The nominal rate of period one must be close to zero.

The reason that monetary policy in period one has to be very loose in order for the equilibrium to be well-defined is closely connected with the construction of the classical unemployment regime. Since consumption of employed individuals is constrained in the good's market in period one, employed individuals will reduce their labor supply in period one compared to the walrasian case,

$$l_1^W > l_1^{Cl} \tag{2.128}$$

and (128) is true if and only if $\frac{\Pi}{2p_1(1+\beta)} - \bar{x}_1 > 0$, which is true by the definition of classical equilibrium. Employed individuals will decide to supply less labor in period one since extra income from supplying more labor can be spent in less units of the consumption good in period one. If the nominal rate of period one is sufficiently high, the constraints in the good's market of period one is tighter so that employed individuals reduce labor supply even further. The equilibrium is not well-defined close to walrasian prices because the unemployment rate can exceed one and labor supply goes negative. Condition (124) rules out this possibility.

§ 2.6 Welfare

2.6.1 Keynesian Unemployment

Before analyzing the effects of monetary policy on individual's welfare, it is very interesting to consider first the effect of monetary policy on the unemployment rate.

2.6.1.1 Effects of interest rate perturbations on unemployment

Consider the effect of perturbations of r_2 on unemployment given any initial value of r_1:

$$\frac{du}{dr_2} = \frac{1}{2D\sqrt{E^2 - 4BD}} \left[\frac{dE}{dr_2} \left(E - \sqrt{E^2 - 4BD} - 4D\frac{dB}{dr_2} \right) \right] > 0$$

where

$$\frac{dE}{dr_2} > 0, \ \frac{dB}{dr_2} < 0$$

The answer that we get is unambiguous. Announcing a reduction of the nominal rate tomorrow, will reduce the unemployment rate today.

The comparative statics with r_1 do not give us an unambiguous answer as before. Decompose the effective labor supply of employed individuals in to two parts,

$$l_1 = l_1^W + u\frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}$$

where the effective labor supply of employed individuals is the sum of two terms: the walrasian labor supply plan and a second term which indicates that the effective labor supply plan is higher than the walrasian one since a measure of individuals is rationed in the labor market.

The derivative of unemployment with respect to r_1 is as follows:

$$\frac{\partial u}{\partial r_1} = \frac{-2\overline{y}_1 \frac{\partial \overline{y}_1}{\partial r_1} + (1-u)\frac{\partial l_1^W}{\partial r_1}}{l_1^W + u\frac{\beta \bar{l}_1}{(1+\beta)(2+\beta)} - \frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}} \tag{2.129}$$

where

$$\frac{\partial l_1^W}{\partial r_1} = \frac{-1}{2w_1(1+\beta)}\left[\frac{2(1+\beta)\delta}{2+\beta} + \frac{2(1+\beta)}{2+\beta}\frac{p_1(1+r_1)I_1}{1+r_2}\left[\underbrace{\frac{1}{1+r_1} + \frac{1}{I_1}\frac{\partial I_1}{\partial r_1}}_{(>0)}\right]\right] < 0$$

$$\frac{\partial \overline{y}_1}{\partial r_1} < 0$$

The sign of (129) depends solely on the sign of the term in the numerator. We will fix interest rate close to zero initially and according to the argument given in the previous section, the denominator is always positive,

$$l_1^W > \frac{\beta \bar{l}_1}{2(1+\beta)(2+\beta)}$$

2.6.1.2 Numerical analysis of (129)

In the analysis that follows we will fix the nominal rates at $\{r_1 = 0, r_2 = 0.3\}$ initially, and examine whether it is optimal for the monetary-authority to deviate from that rule. At these initial values, the Walrasian prices are as follows

$$p_1^* = 1.10465,\ w_1^* = 0.870804$$

Consider the following numerical results,

(p_1, w_1)	du/dr_1
$(1.07, 0.62 - 0.79)$	+
$(1.07, 0.47 - 0.61)$	-
$(1.09, 0.63 - 0.83)$	+
$(1.09, 0.46 - 0.62)$	-
$(1.1, 0.64 - 0.84)$	+
$(1.1, 0.46 - 0.63)$	-
$(1.05, 0.61 - 0.73)$	+
$(1.05, 0.49 - 0.6)$	-

(p_1, w_1)	du/dr_1
$(1.2, 0.7 - 1)$	+
$(1.2, 0.41 - 0.69)$	-
$(1.15, 0.67 - 0.92)$	+
$(1.15, 0.43 - 0.66)$	-
(.)	.
(.)	.
(.)	.
(.)	.

We can observe from the above tables that when the real wage initially is sufficiently low and monetary policy is loose initially, then increasing the nominal rate of period one reduces the unemployment level in the economy. Going back to (129), in order for the unemployment rate to reduce when the nominal rate of period one increases, the numerator of (129) must be negative,

$$-2\overline{y}_1 \frac{\partial \overline{y}_1}{\partial r_1} < (1-u) \left| \frac{\partial l_1^W}{\partial r_1} \right| \tag{2.130}$$

which means that the differential change of labor demand after a perturbation of r_1 should be less than the differential change of the walrasian labor supply plan in the percentage of employed individuals. Employed individuals have less incentive to supply labor when the real wage is relatively low. If on top of that the monetary authority increases the nominal rate in period one, they would want to substitute labor with leisure even more. This is depicted in condition (130).

The main message from the previous argument is relatively simple. When nominal rates are initially close to zero and the real wage of period one is sufficiently low, then increasing the nominal rate of period one will decrease the unemployment rate. This happens because the increase of r_1 will decrease effective labor supply more than it decreases labor demand such that unemployment has to fall in order for employment to increase and clear the labor market.

2.6.1.3 Individual welfare

Consider a differential change of unemployed's and employed's indirect utility from simultaneous perturbations of r_1, r_2:

$$dU^{UN} = \frac{\partial U^{UN}}{\partial r_1} dr_1 + \frac{\partial U^{UN}}{\partial r_2} dr_2$$
$$dU^{E} = \frac{\partial U^{E}}{\partial r_1} dr_1 + \frac{\partial U^{E}}{\partial r_2} dr_2$$

Let us continue with our numerical exercises. Consider the following cases starting with the utility of ***employed*** individuals,

(p_1, w_1)	dr_1, dr_2
$(1.07, 0.62 - 0.79)$	$dr_1 = 0, dr_2 < 0$
$(1.07, 0.47 - 0.61)$	$dr_1 = 0, dr_2 < 0$
$(1.09, 0.63 - 0.83)$	$dr_1 = 0, dr_2 < 0$
$(1.09, 0.46 - 0.62)$	$dr_1 = 0, dr_2 < 0$
$(1.1, 0.64 - 0.84)$	$dr_1 = 0, dr_2 < 0$
$(1.1, 0.46 - 0.63)$	$dr_1 = 0, dr_2 < 0$
$(1.05, 0.61 - 0.73)$	$dr_1 = 0, dr_2 < 0$
$(1.05, 0.49 - 0.6)$	$dr_1 = 0, dr_2 < 0$

(p_1, w_1)	dr_1, dr_2
$(1.2, 0.7 - 1)$	$dr_1 = 0, dr_2 < 0$
$(1.2, 0.41 - 0.69)$	$dr_1 = 0, dr_2 < 0$
$(1.15, 0.67 - 0.92)$	$dr_1 = 0, dr_2 < 0$
$(1.15, 0.43 - 0.66)$	$dr_1 = 0, dr_2 < 0$
$(1.12, 0.8 - 0.89)$	$dr_1 = 0, dr_2 < 0$
$(1.13, 0.83 - 0.9)$	$dr_1 = 0, dr_2 < 0$
$(1.11, 0.79 - 0.87)$	$dr_1 = 0, dr_2 < 0$

For ***unemployed*** individuals consider the following comparative statics,

(p_1, w_1)	dr_1, dr_2
$(1.07, 0.72 - 0.79)$	$dr_1 = 0, dr_2 < 0$
$(1.07, 0.47 - 0.71)$	$dr_1 > 0, dr_2 < 0$
$(1.09, 0.75 - 0.83)$	$dr_1 = 0, dr_2 < 0$
$(1.09, 0.46 - 0.74)$	$dr_1 > 0, dr_2 < 0$
$(1.1, 0.77 - 0.84)$	$dr_1 = 0, dr_2 < 0$
$(1.1, 0.46 - 0.76)$	$dr_1 > 0, dr_2 < 0$
$(1.05, 0.68 - 0.73)$	$dr_1 = 0, dr_2 < 0$
$(1.05, 0.49 - 0.67)$	$dr_1 > 0, dr_2 < 0$

(p_1, w_1)	dr_1, dr_2
$(1.2, 0.95 - 1)$	$dr_1 = 0, dr_2 < 0$
$(1.2, 0.41 - 0.95)$	$dr_1 > 0, dr_2 < 0$
$(1.15, 0.86 - 0.92)$	$dr_1 = 0, dr_2 < 0$
$(1.15, 0.43 - 0.85)$	$dr_1 > 0, dr_2 < 0$
$(1.12, 0.8 - 0.89)$	$dr_1 = 0, dr_2 < 0$
$(1.13, 0.83 - 0.9)$	$dr_1 = 0, dr_2 < 0$
$(1.11, 0.79 - 0.87)$	$dr_1 = 0, dr_2 < 0$

Given the above comparative statics, consider the following monetary policy rule

Monetary policy rule. *If we restrict attention to period one price-wage configurations in a neighborhood of the Walrasian equilibrium one, (p_1^*, w_1^*), but strictly inside the region characterizing Keynesian Unemployment, then the optimal monetary policy is to set nominal rates at values close to zero.*

Restricting attention to points in a neighborhood of the Walrasian equilibrium give us a more clear answer about optimal policy. Optimal monetary policy in that case calls for a Freidman rule argument. Nominal rates must be set optimally close to zero. Since the nature of unemployment in that case is a lack of demand in period one, low nominal rates will boost aggregate demand and reduce unemployment. Employed individuals will benefit from low nominal rates since the cost of consumption is low. They will borrow more money balances in the asset markets and will supply more labor. Unemployed individuals benefit from low nominal rates also since the cost of consumption is low and the excess supply constraints are not severe in period one. The previous comparative statics do not give us a clear answer for the welfare effects of unemployed individuals and that is why we restrict attention to points close to the Walrasian equilibrium.

2.6.2 Classical Unemployment

We said before that in order to have a well-defined classical equilibrium close to the Walrasian one, the nominal rate of period one must be close to zero. In the comparative statics that follow we will fix $r_1 = 0.1$ and do comparative statics only with r_2.

Let us start with the effects of perturbations of r_2 on the unemployment rate,

$$\frac{\partial u}{\partial r_2}(2V - Z) = \frac{\partial B}{\partial r_2} + u\frac{\partial Z}{\partial r_2}$$

where

$$\frac{\partial B}{\partial r_2} = -\left[p_1 \frac{2 + r_2}{(1+\beta)(1+r_2)}\frac{\partial I_1}{\partial r_2} - \frac{p_1 I_1}{(1+\beta)(1+r_2)^2}\right] > 0$$
$$\frac{\partial Z}{\partial r_2} = p_1 \frac{1 + 2\beta + \beta(1+r_2)}{2(1+\beta)^2(1+r_2)}\frac{\partial I_1}{\partial r_2} - p_1 I_1 \frac{1+2\beta}{2(1+\beta)^2(1+r_2)^2} < 0$$

and

$$\frac{\partial B}{\partial r_2} > \left|\frac{\partial Z}{\partial r_2}\right|$$

so that

$$\frac{\partial u}{\partial r_2} > 0$$

2.6.2.1 Individual welfare

Consider the following numerical comparative statics for both individuals,

- $$r_1 = 0.1, \ r_2 = 0.6$$
  $$p_1^* = 0.911327, \ w_1^* = 0.704059$$

(p_1, w_1)	$\partial u^{UN}/\partial r_2$	$\partial u^{EM}/\partial r_2$
$(0.9, 0.75 - 0.78)$	< 0	$+$
$(0.89, 0.78 - 0.8)$	< 0	$+$
$(0.91, 0.72 - 0.74)$	< 0	$+$
$(0.88, 0.81 - 0.83)$	< 0	$+$

- $$r_1 = 0.1,\ r_2 = 0.8$$
  $$p_1^* = 0.860467,\ w_1^* = 0.670177$$

(p_1, w_1)	$\partial u^{UN}/\partial r_2$	$\partial u^{EM}/\partial r_2$
$(0.84, 0.715 - 0.78)$	< 0	$+$
$(0.8, 0.78 - 0.82)$	< 0	$+$
$(0.83, 0.735 - 0.75)$	< 0	$+$
$(0.82, 0.75 - 0.78)$	< 0	$+$

- $$r_1 = 0.1,\ r_2 = 1$$
  $$p_1^* = 0.824585,\ w_1^* = 0.646537$$

(p_1, w_1)	$\partial u^{UN}/\partial r_2$	$\partial u^{EM}/\partial r_2$
$(0.81, 0.67 - 0.69)$	< 0	< 0
$(0.8, 0.69 - 0.695)$	< 0	$+$
$(0.78, 0.71 - 0.74)$	< 0	$+$
$(0.79, 0.7 - 0.73)$	< 0	$+$

- $$r_1 = 0.1,\ r_2 = 1.5$$
  $$p_1^* = 0.768828,\ w_1^* = 0.610403$$

(p_1, w_1)	$\partial u^{UN}/\partial r_2$	$\partial u^{EM}/\partial r_2$
$(0.75, 0.63 - 0.66)$	< 0	< 0
$(0.74, 0.64 - 0.67)$	< 0	< 0
$(0.73, 0.65 - 0.68)$	< 0	< 0
$(0.72, 0.66 - 0.68)$	< 0	< 0

The above tables suggest that if the nominal rate of period two is sufficiently high *initially*, then announcing a reduction of it will make everybody better-off. Thus, we led to the following policy rule

Monetary policy rule. *Restrict attention to period one price-wage configurations in a neighborhood of the Walrasian equilibrium one,* (p_1^*, w_1^*), *but strictly inside the region characterizing Classical Unemployment. If the* *nominal rate of period two, r_2, is sufficiently high initially, then the optimal policy of the monetary authority is to announce a reduction of r_2 in period one. This policy reduces the unemployment level in the economy and makes the remaining unemployed and the rest of the employed individuals better-off.*

Optimal monetary policy takes a completely different form in that case since the nature of unemployment is different form the keynesian equilibrium. Let us focus our attention to the actions of employed individuals since the results in that case are driven mostly from their behavior.

Since employed individuals are constrained in the good's market in period one they are forced to save more today and consume more tomorrow. As a consequence they will reduce their supply of labor in period one as was discussed before. We also discussed the reasons why the nominal rate of period one should be close to zero. Suppose the nominal rate of period two is expected to be initially sufficiently high. In other words, monetary policy tomorrow is expected to be sufficiently tight. If employed individuals expect tight monetary policy tomorrow, their constraint in the good's market in period one is relaxed because unemployed individuals consume less and there is less investment. Thus, if r_2 is sufficiently high initially, then there is a redistribution of consumption from period two to period one for employed individuals.

If the initial situation is as before, then announcing a reduction of r_2 will be make everybody better-off in the economy. Unemployed individuals benefit from the low cost of borrowing tomorrow, borrow more money balances today, consume more and transfer more debt, or less savings, in period two. Since the cost of borrowing tomorrow has been reduced, they are able to borrow more money balances in period and increase their supply of labor. Employed individuals also benefit from the lower cost of borrowing tomorrow. Their constraint in the market today becomes tighter so that they can consume less today and supply less labor. This is optimal since initially r_2 was sufficiently high and they were able to transfer enough units of consumption from period two. They borrow more money balances in the asset market tomorrow and supply more labor.

§ 2.7 Conclusion

In this paper we tried to show that given the unemployment regime that prevails in the market, monetary policy can make individuals better off by using the nominal rate as its policy instrument. The interesting point is that interest rate policy implies different policy restrictions when the unemployment regime in the market is different. Monetary policy has to take into account the type of unemployment that prevails in the market.

Extensions to the above framework include the case of elastic labor and economies with uncertainty and an incomplete asset market. With an incomplete asset market, the effect of an interest rate policy on unemployment *might* be different across states of nature. Improving interventions will be very interesting to characterize.

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- Chapter 3 -

Entrepreneurial risk and Optimal Monetary Policy

§ 3.1 Introduction

The main objective of this paper is to characterize optimal monetary policy in the face of entrepreneurial risk and credit frictions in the asset markets. We want to argue that optimal monetary policy is characterized by *strictly positive* interest rates and optimal monetary policy depends on the magnitude of entrepreneurial risk in the economy.

The issue of entrepreneurial risk has been analyzed by Angeletos (2007). He introduced firm-specific risk in a neoclassical growth model where each household owns a constant returns to scale technology. He showed that the equilibrium is characterized by lower interest rates and underaccumulation of capital when the elasticity of intertemporal substitution is higher than the fraction of private equity to total wealth. Panousi (2010) built on Angeletos (2007) and showed that capital taxation may stimulate capital accumulation. Gottardi et al. (2012), analyzed a two period general equilibrium model with entrepreneurial and labor risk. They dealt with the issue of optimal tax rates on capital and labor. Akyol (2004) characterized optimal monetary policy in an economy faced with uninsured endowment risk. He argued that the optimal monetary policy is to have an inflation tax of 10 percent although the welfare gains are small compared to the Friedman rule. Money in that framework serves as an additional asset and it is valued through a timing friction in the bond market. Also capital accumulation is absent. As we will show the effects of idiosyncratic risk on individual's return of capital can have interesting implication for the conduct of optimal monetary policy.

Firstly, we analyze extensively a two-period economy where all entrepreneurs are ex-ante identical. Our construction has similarities with Gottardi et al. (2012). We will analyze various cases starting from a simple entrepreneurial economy in the first section and moving to economies with elastic labor and an aggregate firm producing the total output in the economy. The main message is that optimal monetary policy requires positive interest rates. This implies positive seignorage profits for the monetary-authority which in turn is one of the main insurance instruments the monetary-authority has at its disposal. A non-contingent distribution of seignorage across states can be beneficial in an environment of idiosyncratic contingencies and credit frictions.

An alternative policy of the monetary-authority is to follow money supply rules. We consider a simple entrepreneurial economy, and instead of interest rates we analyze money-supply rules. Compute the optimal interest rate under an interest rate rule policy, i^*. Then compute the optimal money supply growth that is implied from this interest rate rule, m^*. We will construct the economy with money supply rules in such a way such that if the monetary-authority fixes the growth rate at m^*, then at least one equilibrium will gives us i^*. We say at least because with money supply rules there may be more than one equilibria at m^*. Interestingly enough, the economy with insurable risk, complete markets, will give us a unique equilibrium in the case of money supply rules under the specific construction we employ. The main message from this part is that moving from interest rate to money supply rules may have important consequences for optimal policy. In particular, at m^*, computed in the case of interest rate rules, moving to money supply rules we may get another equilibrium which is far from being optimal.

Lastly, we extend the two-period economy to three periods. Given that we employ the same monetary structure, we will be able to get closed form solutions in the three period economy under log-utilities. It is interesting to do this because optimal policy in that case means setting the interest rates of different periods at different values. In particular, we will argue that the monetary-authority should set higher interest rates in the long-run than in the medium-run. The long-run in that case is the third period. In that period there are entrepreneurs which have received a bad shock twice in a row. The latter require insurance in the form of seignorage distribution. The only way for the monetary-authority to provide insurance to these group is to set the nominal rate at that period at sufficiently high values. This implies that rich entrepreneurs are taxed by increasing the inflation rate from period two to period three. In order for the monetary-authority not to impose a big tax on the rich entrepreneurs, which will be suboptimal, it should set nominal rates at period two at lower values and make the opportunity cost of holding money balances lower than that of period three.

In section 2 we analyze a two-period economy with entrepreneurial risk and interest rate rules. We are going to modify the set-up in various ways and characterize optimal monetary policy. In section 3 we deal with money-supply rules. Lastly, in section 4 we extend the two-period set-up in three periods. We analyze the optimal response of the monetary-authority across periods.

§ 3.2 The basic environment

There are three periods: $t = 1, 2, 3$. A stochastic shock, $s \in S = \{H, L\}$, realizes at the beginning of period two. The high realization of the uncertainty, (H), occurs with probability μ. The last period is added only for accounting purposes.

3.2.1 Entrepreneurs

The economy is populated by a continuum of entrepreneurs of size one. Each of them is endowed with a constant-returns to scale technology which transforms capital held as investment in period one to consumption units in period two. The stochastic shock is purely idiosyncratic and affects the productivity of entrepreneur's technology,

$$y_{2,s} = \gamma_s k$$

Entrepreneurs are identical from an ex-ante perspective, but differ in the realization of the idiosyncratic shock in period two. A fraction μ of them will be more productive (H) and a fraction $1-\mu$ will be less productive (L). Without loss of generality, assume $\gamma_H > \gamma_L > 0$. Assume also that there is no aggregate risk: $\mu\gamma_H + (1-\mu)\gamma_L = 1$.

Preferences are represented by the following CRRA specification

$$U = \frac{x_1^{1-\alpha}}{1-\alpha} + \mu\frac{x_{2,H}^{1-\alpha}}{1-\alpha} + (1-\mu)\frac{x_{2,L}^{1-\alpha}}{1-\alpha},\ \alpha > 0$$

Concerning the timing of transactions we assume that at each date-event the asset market opens before the goods market. As a consequence, cash obtained from sales of output or endowments have to be carried over next period.

Entrepreneurs enter in the asset market in period one to trade cash, a riskless bond and receive transfers from the monetary authority,

$$\hat{m}_1 + \frac{b_1}{1+i_1} \leq h_1 \tag{3.1}$$

When the asset market closes, the goods market open. Entrepreneurs buy consumption and invest in physical capital with the initial money balances acquired in the asset market,

$$p_1 x_1 + p_1 k \leq \hat{m}_1 \tag{3.2}$$

They are also endowed with one unit of the good in period one and receive money balances from selling their endowment in the market. The end-of-period money balances that transfer next period are as follows

$$m_1 = \hat{m}_1 - (p_1 x_1 + p_1 k) + p_1 \tag{3.3}$$

Manipulating (2),(3) we end up in an equivalent form of the cash-in-advance constraint

$$m_1 \geq p_1 \tag{3.4}$$

Substituting (3) into (1) yields the flow budget constraint of period one

$$p_1 x_1 + p_1 k + m_1 + \frac{b_1}{1+i_1} \leq h_1 + p_1$$

The idiosyncratic shock realizes in the beginning of period two, after capital is installed but before entrepreneurs enter in the asset market. High productive entrepreneurs enter the asset market,

$$\hat{m}_{2,H} + \frac{b_{2,H}}{1+i_2} \leq h_2 + b_1 + m_1$$

In the good's market in period two they buy consumption according to the following cash constraint

$$p_2 x_{2,H} \leq \hat{m}_{2,H}$$

and receive end-of-period cash balances by selling the proceeds of their production

$$m_{2,H} = \hat{m}_{2,H} - p_2 x_{2,H} + p_2 y_{2,H}$$

According to the previous argument the cash constraint can be written as follows:

$$m_{2,H} \geq p_2 y_{2,H}$$

and the flow constraint is as follows

$$p_2x_{2,H} + \frac{b_{2,H}}{1+i_2} + m_{2,H} \leq h_2 + b_1 + m_1 + p_2y_{2,H}$$

In the beginning of period three, high productive entrepreneurs redeem their debt

$$m_{2,H} + b_{2,H} \geq 0$$

For the low productive entrepreneurs the timing is similar and will not be repeated.

A representative entrepreneur solves the following *ex-ante* problem

$$\begin{aligned}
&\max\left[\frac{x_1^{1-\alpha}}{1-\alpha} + \mu\frac{x_{2,H}^{1-\alpha}}{1-\alpha} + (1-\mu)\frac{x_{2,L}^{1-\alpha}}{1-\alpha}\right],\ s.t\\
&p_1x_1 + p_1k + \frac{b_1}{1+i_1} + m_1 \leq h_1 + p_1\\
&m_1 \geq p_1\\
&p_2x_{2,H} + m_{2,H} + \frac{b_{2,H}}{1+i_2} \leq h_2 + b_1 + m_1 + p_2y_{2,H}\\
&m_{2,H} \geq p_2y_{2,H}\\
&m_{2,H} + b_{2,H} \geq 0\\
&p_2x_{2,L} + m_{2,L} + \frac{b_{2,L}}{1+i_2} \leq h_2 + b_1 + m_1 + p_2y_{2,L}\\
&m_{2,L} \geq p_2y_{2,L}\\
&m_{2,L} + b_{2,L} \geq 0
\end{aligned}$$

Assume that the cash constraints bind and we know that at the optimum allocation all the rest of the constraints should bind as well. The unconstrained problem of the entrepreneur can be written as follows:

$$\max_{k,b_1}\Bigg[\frac{(h_1-p_1k-\frac{b_1}{1+i_1})^{1-\alpha}}{1-\alpha}\frac{1}{p_1^{1-\alpha}}+\\
\frac{\mu}{p_2^{1-\alpha}}\frac{(h_2+b_1+p_1+\frac{p_2y_{2,H}}{1+i_2})^{1-\alpha}}{1-\alpha}+\\
\frac{(1-\mu)}{p_2^{1-\alpha}}\frac{(h_2+b_1+p_1+\frac{p_2y_{2,L}}{1+i_2})^{1-\alpha}}{1-\alpha}\Bigg]$$

The first order conditions with respect to bond and capital holdings respectively, are as follows

$$\Big(h_1-p_1k-\frac{b_1}{1+i_1}\Big)^{-\alpha}=\\
\frac{\mu(1+i_1)}{(p_2/p_1)^{1-\alpha}}\Big(h_2+b_1+p_1+\frac{p_2k\gamma_H}{1+i_2}\Big)^{-\alpha}+\\
\frac{(1-\mu)(1+i_1)}{(p_2/p_1)^{1-\alpha}}\Big(h_2+b_1+p_1+\frac{p_2k\gamma_L}{1+i_2}\Big)^{-\alpha} \qquad (3.5)$$

$$\Big(h_1-p_1k-\frac{b_1}{1+i_1}\Big)^{-\alpha}=\\
\frac{\mu\gamma_H}{1+i_2}\Big(\frac{p_2}{p_1}\Big)^{\alpha}\Big(h_2+b_1+p_1+\frac{p_2k\gamma_H}{1+i_2}\Big)^{-\alpha}+\\
\frac{(1-\mu)\gamma_L}{1+i_2}\Big(\frac{p_2}{p_1}\Big)^{\alpha}\Big(h_2+b_1+p_1+\frac{p_2k\gamma_L}{1+i_2}\Big)^{-\alpha} \qquad (3.6)$$

3.2.2 Monetary-Fiscal Authority

The flow constraints of the monetary-fiscal authority (MFA) are as follows:

$$
\begin{aligned}
&M_1 + \frac{1}{1+i_1} B_1 = H_1 \\
&M_2 + \frac{1}{1+i_2} B_2 = H_2 + M_1 + B_1 \\
&M_2 + B_2 = 0
\end{aligned}
$$

where M_1, M_2 are money supplies, B_1, B_2 are bonds traded by the monetary authority and H_1, H_2 are transfers to individuals. Consider the intertemporal constraint of the MFA,

$$\frac{i_1}{1+i_1} M_1 + \frac{i_2}{(1+i_1)(1+i_2)} M_2 = H_1 + H_2 \frac{1}{1+i_1}$$

the following rule for the distribution of seignorage simplifies our analysis considerably,

Distribution of Seignorage. *The MFA distributes seignorage according to the following rule,*

$$H_1 = \frac{i_1}{1+i_1} M_1, \;\; H_2 = \frac{i_2}{1+i_2} M_2$$

Lastly, the MFA fixes monetary policy,

Interest rate rules. *The monetary authority specifies an implicit rule for nominal interest rates setting them at non-negative values:* $i_1 \geq 0, i_2 \geq 0$*, accommodating the money demand in the market.*

3.2.3 Equilibrium

The equilibrium conditions of the previous economy are as follows:

$$
\begin{aligned}
&x_1 + k = 1 \\
&\mu x_{2,H} + (1-\mu)x_{2,L} = y_2 \\
&m_1 = M_1,\ \mu m_{2,H} + (1-\mu)m_{2,L} = M_2 \\
&b_1 = B_1,\ \mu b_{2,H} + (1-\mu)b_{2,L} = B_2 \\
&h_1 = H_1 = \frac{i_1}{1+i_1}M_1,\ h_2 = H_2 = \frac{i_2}{1+i_2}M_2
\end{aligned}
$$

and $y_2 = \mu y_{2,H} + (1-\mu)y_{2,L}$.

Combine conditions (5),(6) and the following version of equilibrium conditions

$$b_1 = -p_1,\ h_1 = \frac{i_1}{1+i_1}p_1,\ h_2 = \frac{i_2}{1+i_2}p_2 k$$

to get

$$\frac{p_2}{p_1} = (1+i_1)(1+i_2)\frac{\mu(\gamma_L+i_2)^\alpha + (1-\mu)(\gamma_H+i_2)^\alpha}{\mu\gamma_H(\gamma_L+i_2)^\alpha + (1-\mu)\gamma_L(\gamma_H+i_2)^\alpha} \tag{3.7}$$

$$\left(\frac{k}{1-k}\right)^\alpha = \frac{\mu\gamma_H(\gamma_L+i_2)^\alpha + (1-\mu)\gamma_L(\gamma_H+i_2)^\alpha}{(\gamma_L+i_2)^\alpha(\gamma_H+i_2)^\alpha}\left(\frac{1}{1+i_2}\right)^{1-\alpha} \tag{3.8}$$

3.2.4 Nominal effects of i_1 , Market Incompleteness and Cash-in-Advance Frictions

It is evident from equilibrium relation (8), that the real allocation will not depend on i_1, only on i_2. The nominal rate of period one has *only* nominal effects according to (7). The *only* reason that the nominal rate in period one has no real effects is because the supply of output in period one is fixed. Suppose on the contrary that the supply of period one was endogenous. Individuals must sacrifice some of their leisure time in order to produce output in period one. If nominal rates are sufficiently high in period one, individuals will substitute away consumption and capital holdings since the effective cost of buying them is high. They will reduce the supply of output by holding less end-of-period money balances since the opportunity cost of holding them is also high and consume more leisure time. This argument collapses when output is supplied inelastically.

Since i_1 does not affect the real allocation, we can reinterpret the monetary policy rule by saying that the monetary authority *commits to a uniform rule*:

$$i_1 = i_2 = i$$

or simply fix $i_1 = 0$. Doing either of the two gives the same answer.

Let us go back to the ex-ante problem of the representative entrepreneur. It can be written in the following form,

$$\begin{aligned}
&\max \left[\frac{x_1^{1-\alpha}}{1-\alpha} + \mu \frac{x_{2,H}^{1-\alpha}}{1-\alpha} + (1-\mu)\frac{x_{2,L}^{1-\alpha}}{1-\alpha}\right], \ s.t \\
&p_1 x_1 + p_1 k + \frac{b_1}{1+i_1} = h_1 \\
&p_2 x_{2,H} = h_2 + b_1 + p_1 + \frac{p_2 k}{1+i_2}\gamma_H \\
&p_2 x_{2,L} = h_2 + b_1 + p_1 + \frac{p_2 k}{1+i_2}\gamma_L
\end{aligned}$$

It is useful to rewrite the problem in that form in order to understand what is the main friction that drives the result in this paper. Markets are not incomplete in the usual sense, less assets more states and the return matrix not full rank. The friction that drives the results is that entrepreneurs in period one can buy the output of period one through credit. Observe that in the budget constraints of period two we have an extra term, p_1. This is the value of period one output. Entrepreneurs supply inelastically one unit of time and produce the total output of period one. They receive the wage from supplying labor inelastically as end-of-period money balances, $m_1 = p_1 \cdot 1$. But end-of-period money balances can not be reinvested in the asset market because asset markets are closed when the good's market opens. Entrepreneurs have to carry end-of-period money balances in the next period. They consume period one output only through credit.

In equilibrium, the ex-ante indirect utility of the representative entrepreneur can be written

$$V = \log(1-k^*) + \mu \log\left(\frac{i_2}{1+i_2}k^* + \frac{k^*}{1+i_2}\gamma_H\right) + (1-\mu)\log\left(\frac{i_2}{1+i_2}k^* + \frac{k^*}{1+i_2}\gamma_L\right)$$

and $*$ denotes equilibrium objects. Writing the indirect utility in the above form we have used the previous version of equilibrium conditions. Taking a closer look at the indirect utility, it is *as if* in the economy there was only one asset in period one, capital.

3.2.5 Equilibrium (continued)

Since we have considered the class of CRRA preferences, let us first examine the case of log-utilities which corresponds to the case of $\alpha = 1$.

Equilibrium inflation is as follows:

$$\frac{p_2}{p_1} = \frac{(1+i)^2(i + \mu\gamma_L + (1-\mu)\gamma_H)}{i + \gamma_H\gamma_L}$$

which can be written in an equivalent form

$$\frac{p_2}{p_1} = (1+i)^2 \Big[1 + \frac{\sigma_\gamma^2}{i + \gamma_H \gamma_L} \Big]$$

where σ_γ^2 represents the volatility of entrepreneurial risk and the second term in brackets is the risk premium on private equity. The role of the risk premium is to provide sufficient incentives to the entrepreneurs to invest in the risky project in period one.

Equilibrium capital holdings are as follows

$$k = \frac{i + \gamma_H \gamma_L}{(1+i)^2 + (2+i)\gamma_H \gamma_L + i\sigma_\gamma^2 - 1}$$

Entrepreneurs will *under-invest* compared to the benchmark case[1] unless the nominal rate goes to zero:

$$\frac{1}{2+i} - \frac{i + \gamma_H \gamma_L}{(1+i)^2 + (2+i)\gamma_H \gamma_L + i\sigma_\gamma^2 - 1} \geq 0 \Rightarrow i\sigma_\gamma^2 \geq 0$$

The first part represents capital holdings in the benchmark case.

3.2.6 Optimal monetary policy

The optimal monetary policy is summarized in the next proposition,

Proposition 1. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rate at strictly positive values in order to maximize aggregate welfare.*

Proof. Consider the derivative of the entrepreneur's ex-ante indirect utility with respect to the nominal rate

[1]The volatility of entrepreneurial risk is zero.

$$\left.\frac{dU}{di}\right|_{i=\bar{i}} = \frac{(1-2k^*)}{k^*(1-k^*)}\left.\frac{dk}{di}\right|_{i=\bar{i}} + \frac{\mu\gamma_L + (1-\mu)\gamma_H - \gamma_H\gamma_L}{(\bar{i}+\gamma_H)(\bar{i}+\gamma_L)} \tag{3.9}$$

where we evaluate (9) at $i = \bar{i} > 0$. We can not evaluate it at zero because the equilibrium is not defined for negative nominal rates. In the case of negative nominal rates, pure arbitrage profits are possible by selling short the riskless bond and holding money. Also,

$$k^* = \frac{\bar{i} + \gamma_H\gamma_L}{(1+\bar{i})^2 + (2+\bar{i})\gamma_H\gamma_L + \bar{i}\sigma_\gamma^2 - 1} \tag{3.10}$$

$$\frac{dk}{di} = -\frac{\bar{i}^2 + \gamma_H\gamma_L(2\bar{i} + \gamma_H + \gamma_L - 1)}{(\bar{i}^2 + 2\gamma_H\gamma_L + \bar{i} + \bar{i}(\gamma_H + \gamma_L))^2} < 0 \tag{3.11}$$

Consider the following limits of the right hand side of (9),

$$\lim_{\bar{i}\to 0}(k^*) = 0.5$$
$$\lim_{\bar{i}\to 0}\left(\frac{dk}{di}\right) = -\frac{\gamma_H + \gamma_L - 1}{4\gamma_H\gamma_L}$$
$$\lim_{\bar{i}\to 0}(1-2k^*) = 0$$
$$\lim_{\bar{i}\to 0}\frac{\mu\gamma_L + (1-\mu)\gamma_H - \gamma_H\gamma_L}{(\bar{i}+\gamma_H)(\bar{i}+\gamma_L)} = \frac{\mu\gamma_L + (1-\mu)\gamma_H - \gamma_H\gamma_L}{\gamma_H\gamma_L} > 0$$

The above limits show that as $\bar{i} \to 0$, the right hand side of (9) converges to a *positive* finite value. It is not optimal to set nominal rates to zero.

Consider the other extreme, $\bar{i} \to \infty$. We want to prove that as $\bar{i}$ increases, the right hand side of (9) becomes negative so that it is not optimal to set nominal rates at high values. There is a *unique* intermediate value which makes the right hand side of (9) zero. This is the optimal interest rate. Uniqueness of the optimal interest rate follows from the strict concavity of the utility function.

Consider the limit of the second term in the right hand side of (9),

$$\lim_{\bar{i}\to\infty} \frac{\mu\gamma_L + (1-\mu)\gamma_H - \gamma_H\gamma_L}{(\bar{i}+\gamma_H)(\bar{i}+\gamma_L)} = 0$$

Consider the limit of the first term in the right hand side of (9) as $i \to \infty$. Figure 1 shows that this limit goes to zero,

$$\lim_{\bar{i}\to\infty} \frac{(1-2k^*)\frac{dk}{di}}{k^*(1-k^*)} = 0$$

Consider also the following limits,

$$\lim_{\bar{i}\to\infty} k^* = 0$$
$$\lim_{\bar{i}\to\infty} \frac{dk}{di} = 0$$

and the reason that the above limits go to zero as $\bar{i} \to \infty$ is that the denominator increases faster than the numerator as can be seen from (10),(11). The first term of the right hand side of (9) goes to zero as $\bar{i} \to \infty$ because the numerator decreases faster than the denominator.

Lastly, we have to show that the second term in the right hand side of (9) converges *faster* to zero than the first term so that as the nominal rate increases, the right hand side of (9) becomes negative. The following graphs depict the value of each term in the RHS of (9) as the nominal rate becomes very large for some values of the idiosyncratic shock

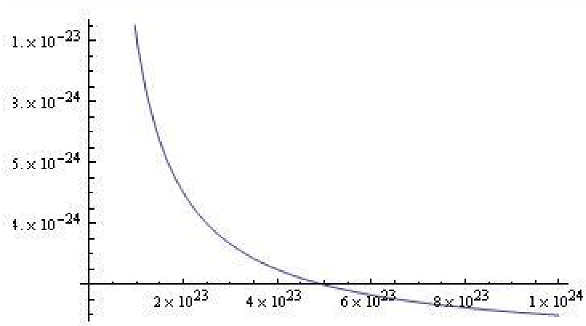

Figure 3.1: *First term of the RHS of (9)*

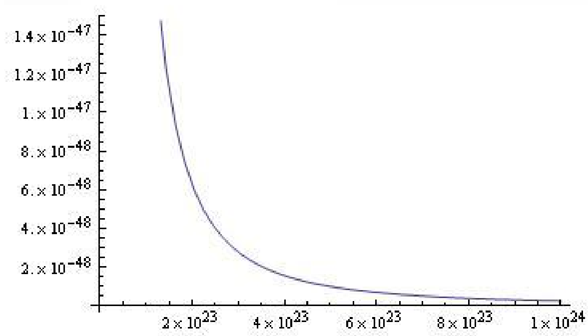

Figure 3.2: *Second term of the RHS of (9)*

As we can observe from the above graphs, the second term in the RHS of (9) converges faster to zero than the first term.

□

Optimal monetary policy balances the trade-off between consumption today and tomorrow because of the credit frictions in the asset markets. If the nominal rate is below the optimal one, entrepreneurs under-consume today, high productive entrepreneurs over-consume tomorrow and less productive ones consume less. If the nominal rate is above the optimal one the argument is analogous.

Consider also the following proposition

Proposition 2. *Given $\sigma_\gamma^2 > 0$, money growth under en-*

trepreneurial risk is higher compared to the benchmark case.

Proof. Money growth when the volatility of idiosyncratic risk is positive is as follows:

$$\frac{M_2}{M_1} = \frac{(i+\gamma_H\gamma_L)(1+i)^2\left[1+\frac{\sigma_\gamma^2}{i+\gamma_H\gamma_L}\right]}{(1+i)^2+(2+i)\gamma_H\gamma_L+i\sigma_\gamma^2-1} \tag{3.12}$$

whereas when the volatility of risk is zero, money growth is as follows:

$$\frac{M_2}{M_1} = \frac{(1+i)^2}{2+i} \tag{3.13}$$

As long as $\sigma_\gamma^2 > 0$, expression (12) is always greater than (13).

□

3.2.7 The case of $\alpha \neq 1$

Fix $\alpha \neq 1$.

Equilibrium inflation is as follows:

$$\frac{p_2}{p_1} = (1+i)^2 \frac{\mu(\gamma_L+i)^\alpha + (1-\mu)(\gamma_H+i)^\alpha}{\mu\gamma_H(\gamma_L+i)^\alpha + (1-\mu)\gamma_L(\gamma_H+i)^\alpha}$$

From expression (8) we get equilibrium capital holdings

$$k = \frac{\left(\frac{1}{1+i}\right)^{\frac{1-\alpha}{\alpha}}\left(\mu\gamma_H(\gamma_L+i)^\alpha + (1-\mu)\gamma_L(\gamma_H+i)^\alpha\right)^{\frac{1}{\alpha}}}{(\gamma_H+i)(\gamma_L+i) + \left(\frac{1}{1+i}\right)^{\frac{1-\alpha}{\alpha}}\left(\mu\gamma_H(\gamma_L+i)^\alpha + (1-\mu)\gamma_L(\gamma_H+i)^\alpha\right)^{\frac{1}{\alpha}}}$$

3.2.8 Optimal monetary policy

Consider the following proposition,

Proposition 3. *Fix $\alpha \neq 1$. Given $\sigma_\gamma^2 > 0$ and suppose also that the degree of heterogeneity is sufficiently high, optimal monetary policy should set the nominal rate at strictly positive values in order to maximize aggregate welfare.*

Proof. The argument is similar to the one given in the previous proposition.

Consider the derivative of ex-ante indirect utility with respect to the nominal rate evaluated at $i = \bar{i} > 0$,

$$\frac{dU}{di} = \frac{dk}{di}\Big[\mu(\gamma_L+\bar{i})^\alpha + (1-\mu)(\gamma_H+\bar{i})^\alpha\Big]\bar{i}$$
$$\Big[(\gamma_L+\bar{i})(\gamma_H+\bar{i}) +$$
$$(1+\bar{i})^{\frac{\alpha-1}{\alpha}}(\mu\gamma_H(\gamma_L+\bar{i})^\alpha + (1-\mu)\gamma_L(\gamma_H+\bar{i})^\alpha)\Big]^\alpha \Big/$$
$$(\gamma_L+\bar{i})^\alpha(\gamma_H+\bar{i})^\alpha(1+\bar{i})^{\alpha-1}\Big[\mu\gamma_H(\gamma_L+\bar{i})^\alpha +$$
$$(1-\mu)\gamma_L(\gamma_H+\bar{i})^\alpha\Big] +$$
$$\frac{(1+\bar{i})^\alpha}{k^{\alpha-1}}\frac{[(1-\mu)(1-\gamma_L)(\bar{i}+\gamma_H)^\alpha - \mu(\gamma_H-1)(\bar{i}+\gamma_L)^\alpha]}{(\gamma_L+\bar{i})^\alpha(\gamma_H+\bar{i})^\alpha} \tag{3.14}$$

where $dk/di < 0$ and

$$k = \frac{\left(\frac{1}{1+\bar{i}}\right)^{\frac{1-\alpha}{\alpha}}\Big(\mu\gamma_H(\gamma_L+\bar{i})^\alpha + (1-\mu)\gamma_L(\gamma_H+\bar{i})^\alpha\Big)^{\frac{1}{\alpha}}}{(\gamma_H+\bar{i})(\gamma_L+\bar{i}) + \left(\frac{1}{1+\bar{i}}\right)^{\frac{1-\alpha}{\alpha}}\Big(\mu\gamma_H(\gamma_L+\bar{i})^\alpha + (1-\mu)\gamma_L(\gamma_H+\bar{i})^\alpha\Big)^{\frac{1}{\alpha}}}$$

As $\bar{i} \to 0$ the first term in the derivative becomes zero and the second term is positive given the assumption of sufficiently high heterogeneity. Thus, the derivative in (14) is positive.

The assumption about sufficiently high heterogeneity guarantees that the second term of (14) is positive as $\bar{i} \to 0$,

$$\left(\frac{\gamma_H}{\gamma_L}\right)^{\alpha} > \frac{\mu(\gamma_H - 1)}{(1-\mu)(1-\gamma_L)}$$

□

3.2.9 Elastic labor in period one

Let us now relax the assumption about inelastic supply in period one.

Suppose entrepreneurs have access to a short-run technology in period one that transforms labor into consumption units. They are also endowed with one unit of leisure time.

The ex-ante problem of a representative entrepreneur is as follows:

$$\max\left[\frac{x_1^{1-\alpha}}{1-\alpha}+\frac{(1-y_1)^{1-\alpha}}{1-\alpha}+\mu\frac{x_{2,H}^{1-\alpha}}{1-\alpha}+(1-\mu)\frac{x_{2,L}^{1-\alpha}}{1-\alpha}\right],\ s.t$$

$$p_1x_1+p_1k+\frac{b_1}{1+i_1}+m_1\leq h_1+p_1y_1$$

$$m_1\geq p_1y_1$$

$$p_2x_{2,H}+m_{2,H}+\frac{b_{2,H}}{1+i_2}\leq h_2+b_1+m_1+p_2y_{2,H}$$

$$m_{2,H}\geq p_2y_{2,H}$$

$$m_{2,H}+b_{2,H}\geq 0$$

$$p_2x_{2,L}+m_{2,L}+\frac{b_{2,L}}{1+i_2}\leq h_2+b_1+m_1+p_2y_{2,L}$$

$$m_{2,L}\geq p_2y_{2,L}$$

$$m_{2,L}+b_{2,L}\geq 0$$

The unconstrained problem can be written as follows:

$$\max_{k,b_1,y_1}\Bigg[\frac{(h_1-p_1k-\frac{b_1}{1+i_1})^{1-\alpha}}{1-\alpha}\frac{1}{p_1^{1-\alpha}}+\frac{(1-y_1)^{1-\alpha}}{1-\alpha}+$$
$$\frac{\mu}{p_2^{1-\alpha}}\frac{(h_2+b_1+p_1y_1+\frac{p_2y_{2,H}}{1+i_2})^{1-\alpha}}{1-\alpha}+$$
$$\frac{(1-\mu)}{p_2^{1-\alpha}}\frac{(h_2+b_1+p_1y_1+\frac{p_2y_{2,L}}{1+i_2})^{1-\alpha}}{1-\alpha}\Bigg]$$

The FOCs with respect to b_1, k, y_1 respectively are as follows:

$$\left(h_1 - p_1 k - \frac{b_1}{1+i_1}\right)^{-\alpha} =$$
$$\frac{\mu(1+i_1)}{(p_2/p_1)^{1-\alpha}}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_H}{1+i_2}\right)^{-\alpha} +$$
$$\frac{(1-\mu)(1+i_1)}{(p_2/p_1)^{1-\alpha}}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_L}{1+i_2}\right)^{-\alpha} \tag{3.15}$$
$$\left(h_1 - p_1 k - \frac{b_1}{1+i_1}\right)^{-\alpha} =$$
$$\frac{\mu\gamma_H}{1+i_2}\left(\frac{p_2}{p_1}\right)^{\alpha}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_H}{1+i_2}\right)^{-\alpha} +$$
$$\frac{(1-\mu)\gamma_L}{1+i_2}\left(\frac{p_2}{p_1}\right)^{\alpha}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_L}{1+i_2}\right)^{-\alpha} \tag{3.16}$$
$$(1-y_1)^{-\alpha} = \frac{\mu}{(p_2/p_1)p_2^{-\alpha}}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_H}{1+i_2}\right)^{-\alpha} +$$
$$\frac{1-\mu}{(p_2/p_1)p_2^{-\alpha}}\left(h_2 + b_1 + p_1 y_1 + \frac{p_2 k \gamma_L}{1+i_2}\right)^{-\alpha} \tag{3.17}$$

From the equilibrium conditions we get the following:

$$b_1 = -p_1 y_1, \; h_1 = \frac{i_1}{1+i_1} p_1 y_1, \; h_2 = \frac{i_2}{1+i_2} p_2 k$$

Combining the equilibrium and the previous FOC conditions we get

$$k = \frac{A^{\frac{1}{\alpha}}}{1 + (1+i_1)^{\frac{1}{\alpha}} + A^{\frac{1}{\alpha}}}$$
$$y_1 = \frac{1 + (1+i_1)^{\frac{1}{\alpha}} \frac{A^{\frac{1}{\alpha}}}{1+(1+i_1)^{\frac{1}{\alpha}}+A^{\frac{1}{\alpha}}}}{1 + (1+i_1)^{\frac{1}{\alpha}}}$$

where

$$A = \frac{\mu\gamma_H(\gamma_L + i_2)^\alpha + (1-\mu)\gamma_L(\gamma_H + i_2)^\alpha}{(\gamma_H + i_2)^\alpha(\gamma_L + i_2)^\alpha}(1+i_2)^{\alpha-1}$$

3.2.10 Optimal monetary policy

Consider the following proposition

Proposition 4. *Given* $\sigma_\gamma^2 > 0$*, optimal monetary policy should set the nominal rate from period one to period two at zero,* $i_1 = 0$*, and the nominal rate from period two to period three at strictly positive values,* $i_2 > 0$*, in order to maximize aggregate welfare.*

Proof. Let us examine the case of log-utilities which is easier to analyze.

Consider the total differential change of the entrepreneur's ex-ante indirect utility from simultaneous perturbations on i_1, i_2:

$$dU = \frac{\partial U}{\partial i_1}di_1 + \frac{\partial U}{\partial i_2}di_2$$

where the partial derivatives with respect to i_1, i_2 are as follows:

$$\frac{\partial U}{\partial i_1} = \frac{1}{y_1 - k}\left(\frac{\partial y_1}{\partial i_1} - \frac{\partial k}{\partial i_1}\right) - \frac{1}{1-y_1}\frac{\partial y_1}{\partial i_1} + \frac{\mu(1+\bar{i}_2)}{k(\gamma_H + \bar{i}_2)}\frac{\partial k}{\partial i_1} +$$
$$\frac{(1-\mu)(1+\bar{i}_2)}{k(\gamma_L + \bar{i}_2)}\frac{\partial k}{\partial i_1} \tag{3.18}$$
$$\frac{\partial U}{\partial i_2} = \frac{1}{y_1 - k}\left(\frac{\partial y_1}{\partial i_2} - \frac{\partial k}{\partial i_2}\right) - \frac{1}{1-y_1}\frac{\partial y_1}{\partial i_2} +$$
$$\frac{\mu(1+\bar{i}_2)}{k(\gamma_H + \bar{i}_2)}\left[\frac{\partial k}{\partial i_2}\Big(\frac{\bar{i}_2}{1+\bar{i}_2} + \frac{\gamma_H}{1+\bar{i}_2}\Big) +\right.$$
$$\left. k\Big(\frac{1}{(1+\bar{i}_2)^2} - \frac{\gamma_H}{(1+\bar{i}_2)^2}\Big)\right] +$$
$$\frac{(1-\mu)(1+\bar{i}_2)}{k(\gamma_L + \bar{i}_2)}\left[\frac{\partial k}{\partial i_2}\Big(\frac{\bar{i}_2}{1+\bar{i}_2} + \frac{\gamma_L}{1+\bar{i}_2}\Big) +\right.$$
$$\left. k\Big(\frac{1}{(1+\bar{i}_2)^2} - \frac{\gamma_L}{(1+\bar{i}_2)^2}\Big)\right] \tag{3.19}$$

and also

$$\frac{\partial y_1}{\partial i_1} = \frac{(k + (1+\bar{i}_1)\frac{\partial k}{\partial i_1})(2+\bar{i}_1) - (1+\bar{i}_1)k}{(2+\bar{i}_1)^2}$$
$$\frac{\partial k}{\partial i_1} = -\frac{A}{(2+\bar{i}_1+A)^2}$$
$$\frac{\partial y_1}{\partial i_2} = \frac{\partial k}{\partial i_2}\frac{1+\bar{i}_1}{2+\bar{i}_1}, \quad \frac{\partial k}{\partial i_2} = \frac{\partial A}{\partial i_2}\frac{2+\bar{i}_1}{(2+\bar{i}_1+A)^2}$$
$$\frac{\partial A}{\partial i_2} = \frac{\gamma_H\gamma_L - \bar{i}_2^2 - \gamma_H\gamma_L(\gamma_H+\gamma_L) + 2\bar{i}_2\gamma_H\gamma_L}{(\gamma_H\gamma_L + \bar{i}_2(\gamma_H+\gamma_L) + \bar{i}_2^2)^2}$$

where all the derivatives are evaluated at $\bar{i}_1 > 0, \bar{i}_2 > 0$ and k, y_1 are fixed at their equilibrium values.

Let us repeat the previous experiment once more. Suppose that $\bar{i}_1 \to 0$, $\bar{i}_2 \to 0$. The partial derivatives of U are as follows:

$$\frac{\partial U}{\partial i_1} = -0.25 - \frac{\mu\gamma_L + (1-\mu)\gamma_H}{3\gamma_H\gamma_L}$$
$$\frac{\partial U}{\partial i_2} = \frac{\sigma_\gamma^2}{\gamma_H\gamma_L}$$

Going back to the total differential change of ex-ante utility, the optimal monetary policy is characterized as follows:

$$di_1 = 0, \ di_2 > 0$$

We proved that as nominal rates tend to zero, it is optimal to fix i_1 close to zero and i_2 at strictly positive values. To finish the proof we must show that if we increase i_2 above its optimal point, aggregate welfare decreases. Since it optimal to keep i_1 very close to zero, fix $\bar{i}_1$ close to zero and let $\bar{i}_2 \to \infty$. The limit values of the RHS terms of (18),(19) respectively, are as follows:

$$\lim_{\bar{i}_2 \to \infty} (RHS(18)) = -0.5, \ \lim_{\bar{i}_2 \to \infty} (RHS(19)) = 0$$

but $\lim_{\bar{i}_2 \to \infty}(RHS(19))$ tends to zero from negative values as the graph below shows.

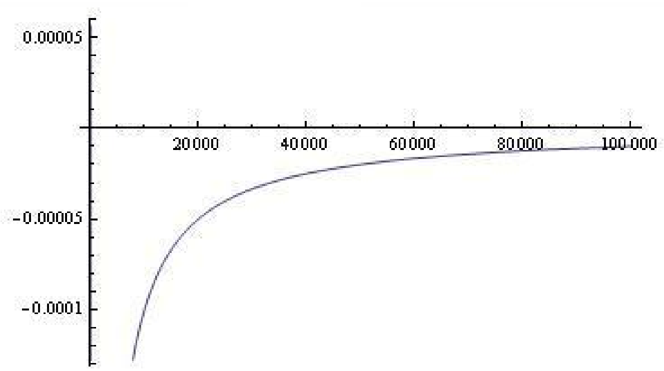

Figure 3.3: *RHS of (19)*

There is a unique optimal value of $\bar{i}_2$ such that further increases above it decrease aggregate welfare.

□

3.2.11 A modified set-up

The monetary economy is similar with the previous case. The only difference is that in period two there is an aggregate firm that hires capital and labor in order to produce aggregate output in period two. It solves the following problem

$$\max_{K,L}\left[F(K,L) - wL - qK\right]$$

where $F(K,L) = K^{\theta}L^{1-\theta}$, $0 < \theta < 1$.

The FOC with respect to K, L are

$$\frac{q}{p_2} = \theta\left(\frac{L}{K}\right)^{1-\theta}, \quad (1-\theta)\left(\frac{K}{L}\right)^{\theta} = \frac{w}{p_2}$$

A representative entrepreneur solves the following *ex-ante* problem

$$
\begin{aligned}
&\max\Big[\frac{x_1^{1-\alpha}}{1-\alpha}+\mu\frac{x_{2,H}^{1-\alpha}}{1-\alpha}+(1-\mu)\frac{x_{2,L}^{1-\alpha}}{1-\alpha}\Big],\ s.t\\
&p_1x_1+p_1k+\frac{b_1}{1+i_1}+m_1\leq h_1+p_1\\
&m_1\geq p_1\\
&p_2x_{2,H}+m_{2,H}+\frac{b_{2,H}}{1+i_2}\leq h_2+b_1+m_1+w+qk\gamma_H\\
&m_{2,H}\geq w+qk\gamma_H\\
&m_{2,H}+b_{2,H}\geq 0\\
&p_2x_{2,L}+m_{2,L}+\frac{b_{2,L}}{1+i_2}\leq h_2+b_1+m_1+w+qk\gamma_L\\
&m_{2,L}\geq w+qk\gamma_L\\
&m_{2,L}+b_{2,L}\geq 0
\end{aligned}
$$

The FOC of the unconstrained maximization with respect to b_1, k are

$$
\begin{aligned}
&\Big(h_1-p_1k-\frac{b_1}{1+i_1}\Big)^{-\alpha}=\\
&\frac{\mu(1+i_1)}{(p_2/p_1)^{1-\alpha}}\Big(h_2+b_1+p_1+\frac{w}{1+i_2}+\frac{qk\gamma_H}{1+i_2}\Big)^{-\alpha}+\\
&\frac{(1-\mu)(1+i_1)}{(p_2/p_1)^{1-\alpha}}\Big(h_2+b_1+p_1+\frac{w}{1+i_2}+\frac{qk\gamma_L}{1+i_2}\Big)^{-\alpha}\\
&\Big(h_1-p_1k-\frac{b_1}{1+i_1}\Big)^{-\alpha}=\\
&\frac{\mu\gamma_H}{1+i_2}\frac{q}{p_2}\Big(h_2+b_1+p_1+\frac{w}{1+i_2}+\frac{qk\gamma_H}{1+i_2}\Big)^{-\alpha}\left(\frac{p_2}{p_1}\right)^{\alpha}+\\
&\frac{(1-\mu)\gamma_L}{1+i_2}\frac{q}{p_2}\Big(h_2+b_1+p_1+\frac{w}{1+i_2}+\frac{qk\gamma_L}{1+i_2}\Big)^{-\alpha}\left(\frac{p_2}{p_1}\right)^{\alpha}
\end{aligned}
$$

The equilibrium conditions can written as follows:

$$b_1 = -p_1,\ h_2 = \frac{i_2}{1+i_2} p_2 K^{\theta},\ h_1 = \frac{i_1}{1+i_1} p_1$$

To facilitate computations, consider the simple case of $\alpha = 1$, $\theta = 0.5$.

The equilibrium capital holdings are as follows:

$$K = \frac{1 + 2i_2 + \gamma_H\gamma_L}{2 + 2(3i_2 + 2i_2^2) + 2\gamma_H\gamma_L + (1+2i_2)(\gamma_H + \gamma_L)}$$

The no-arbitrage condition between capital and bonds is as follows:

$$\frac{q}{(1+i_1)p_1} = (1+i_2)\left[1 + \frac{\sigma_\gamma^2}{1 + 2i_2 + \gamma_H\gamma_L}\right]$$

The above relation is the net return of the entrepreneur for investing in the risky project since $(1+i_1)p_1$ is the effective price of capital in period one and $q/(1+i_2)$ the return in period two. The second term in brackets is the risk premium that the market provides in order for the entrepreneur to hold capital in period one.

The introduction of labor income in period two, in each personal state, induces entrepreneurs to hold less capital in period one compared to the complete market solution. The effective wealth of entrepreneurs tomorrow increases with the introduction of labor income. Thus, the precautionary motive effect of holding more capital is dominated by the fact that risk averse entrepreneurs would hold less capital since they benefit from the market premium in terms of net return and in the bad state they get labor income with certainty.

It is interesting to characterize optimal monetary policy in this case. The following proposition argues that even if we the entrepreneur receives a certain amount of labor income in each state tomorrow and although he under-invest in period one, the monetary authority should set positive nominal rates in order to maximize aggregate welfare.

Proposition 5. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rate at strictly positive values, $i_2 > 0$, in order to maximize aggregate welfare.*

Proof. The idea of the proof follows from the previous sections. Consider the derivative of indirect utility with respect to i_2 evaluated at $\bar{i}_2 > 0$:

$$\begin{aligned}\frac{dU}{di_2} &= \frac{dK}{di_2}\frac{(1-3K)}{2K(1-K)}+ \\ &\frac{1}{1+\bar{i}_2}\Big[\frac{\mu(1-\gamma_H)}{2\bar{i}_2+1+\gamma_H}+\frac{(1-\mu)(1-\gamma_L)}{2\bar{i}_2+1+\gamma_L}\Big]\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}K &= \frac{1+2\bar{i}_2+\gamma_H\gamma_L}{2+2(3\bar{i}_2+2\bar{i}_2^2)+2\gamma_H\gamma_L+(1+2\bar{i}_2)(\gamma_H+\gamma_L)} \\ \frac{dK}{di_2} &= \frac{1}{(2+2(3\bar{i}_2+2\bar{i}_2^2)+2\gamma_H\gamma_L+(1+2\bar{i}_2)(\gamma_H+\gamma_L))^2} \\ &\Big[4+12\bar{i}_2+8\bar{i}_2^2+ \\ &4\gamma_H\gamma_L+2(\gamma_H+\gamma_L)+4\bar{i}_2(\gamma_H+\gamma_L)- \\ &(1+2\bar{i}_2+\gamma_H\gamma_L)(6+8\bar{i}_2+2(\gamma_H+\gamma_L))\Big]\end{aligned}$$

Suppose that $\bar{i}_2 \to 0$, then from (20) we get:

$$\frac{dU}{di_2} = \frac{\sigma_\gamma^2}{2+2\gamma_H\gamma_L+\sigma_\gamma^2}\Big[1-\frac{1+\gamma_H\gamma_L(2+\gamma_H\gamma_L+\sigma_\gamma^2)}{(3+3\gamma_H\gamma_L+\sigma_\gamma^2)(1+\gamma_H\gamma_L)}\Big] \tag{3.21}$$

□

3.2.11.1 A more general set-up

The purpose of this section is to show that the previous argument does not depend on the specific utility functions that we chose to work with. Consider the following two assumptions:

Assumption 3.2.11.1. *The flow utility function, $u : R_{++} \to R$, is continuously differentiable, strictly increasing and strictly concave. The following condition hold as well:*

$$\lim_{x \to 0} u_x(\cdot) = \infty$$

Assumption 3.2.11.2. *The production function, $F : R_+^2 \to R_+$, is continuously differentiable, strictly increasing, homogeneous of degree one and strictly quasi-concave. It satisfies also the following:*

$$F(0,l) = 0,\ F_K(K,l) > 0,\ F_l(K,l) > 0,\ all\ K,l > 0$$
$$\lim_{K \to 0} F_K(K,1) = \infty,\ \lim_{K \to \infty} F_K(K,1) = 0$$

The ex-ante problem of the entrepreneur can be written as follows:

$$
\begin{aligned}
&\max \Big[u(x_1) + \mu u(x_{2,H}) + (1-\mu)u(x_{2,L})\Big], \ s.t \\
&p_1 x_1 + p_1 k + \frac{b_1}{1+i_1} + m_1 \leq h_1 + p_1 \\
&m_1 \geq p_1 \\
&p_2 x_{2,H} + m_{2,H} + \frac{b_{2,H}}{1+i_2} \leq h_2 + b_1 + m_1 + w + qk\gamma_H \\
&m_{2,H} \geq w + qk\gamma_H \\
&m_{2,H} + b_{2,H} \geq 0 \\
&p_2 x_{2,L} + m_{2,L} + \frac{b_{2,L}}{1+i_2} \leq h_2 + b_1 + m_1 + w + qk\gamma_L \\
&m_{2,L} \geq w + qk\gamma_L \\
&m_{2,L} + b_{2,L} \geq 0
\end{aligned}
$$

The FOC with respect to consumption demands, bond holdings and capital respectively are as follows:

$$u_x(x_1) = \lambda_1 p_1, \ \mu u_x(x_{2,H}) = \lambda_H p_2, \ (1-\mu)u_x(x_{2,L}) = \lambda_L p_2 \tag{3.22}$$

$$\frac{\lambda_1}{1+i_1} = \lambda_H + \lambda_L \tag{3.23}$$

$$\lambda_1 p_1 = \frac{q}{1+i_2}(\gamma_H \lambda_H + \gamma_L \lambda_L) \tag{3.24}$$

Combining (22),(23),(24), we get the following two conditions:

$$u_x(x_1) = \frac{1+i_1}{p_2/p_1}\Big[\mu u_x(x_{2,H}) + (1-\mu)u_x(x_{2,L})\Big] \tag{3.25}$$

$$u_x(x_1) = \frac{q}{(1+i_2)p_2}\Big[\mu\gamma_H u_x(x_{2,H}) + (1-\mu)\gamma_L u_x(x_{2,L})\Big] \tag{3.26}$$

Condition (25) is a familiar condition which states that the marginal rate of substitution should equal the *real* interest rate at the optimum. Condition (26) should hold at the optimum in order for entrepreneurs to invest in the risky project in period one.

From (25),(26) the RHSs should be equal

$$\begin{aligned}&\frac{1+i_1}{p_2/p_1}\Big[\mu u_x(x_{2,H})+(1-\mu)u_x(x_{2,L})\Big]=\\&\frac{q}{(1+i_2)p_2}\Big[\mu\gamma_H u_x(x_{2,H})+(1-\mu)\gamma_L u_x(x_{2,L})\Big]\end{aligned}\tag{3.27}$$

but given the normalization we have imposed, $\mu\gamma_H+(1-\mu)\gamma_L=1$, we can show that the following is true

$$\begin{aligned}\mu u_x(x_{2,H})+&(1-\mu)u_x(x_{2,L})>\\&\mu\gamma_H u_x(x_{2,H})+(1-\mu)\gamma_L u_x(x_{2,L})\Leftrightarrow\\&(u_x(x_{2,L})-u_x(x_{2,H}))(1-\mu)(1-\gamma_L)>0\quad or\\&(u_x(x_{2,L})-u_x(x_{2,H}))\mu(\gamma_H-1)>0\end{aligned}\tag{3.28}$$

Given (28), the following should hold in order for (27) to be true

$$\frac{1+i_1}{p_2/p_1}<\frac{q}{(1+i_2)p_2}\tag{3.29}$$

Which implies

$$\frac{q/(1+i_2)}{(1+i_1)p_1}>1\tag{3.30}$$

Condition (30) requires that the net return from investing in the risky project in period one to be greater than the total productivity of aggregate investment which is equal to one by our normalization. The term $\frac{q}{1+i_2}$ is the market return in period two from selling capital to the aggregate firm whereas $(1+i_1)p_1$ is the effective price for investing in the risky project in period one. Condition (30) expresses the fact that in order for entrepreneurs to invest in a risky project in period one, the market must provide sufficient incentives to do so. It must provide entrepreneurs with a risk premium for investing in the risky project in period one,

$$\frac{q/(1+i_2)}{(1+i_1)p_1} = 1 + risk\ premium$$

Consider the derivative of indirect utility with respect to i_2

$$\frac{dU}{di_2} = - u_x(1-K)\frac{dK}{di_2} + \mu u_x(x^*_{2,H})\frac{dx_{2,H}}{di_2} + (1-\mu)u_x(x^*_{2,L})\frac{dx_{2,L}}{di_2} \tag{3.31}$$

where $x^*_{2,H}, x^*_{2,L}$ are equilibrium consumption demands.

Before writing down the derivatives of $x^*_{2,H}, x^*_{2,L}$ with respect to i_2, we know that since F is homogeneous of degree one, the following are true:

$$KF_K + lF_l = F, \; KF_{KK} + lF_{lK} = 0$$

The derivatives of $x^*_{2,H}, x^*_{2,L}$ with respect to i_2 are as follows:

$$\frac{dx_{2,s}}{di_2} = KF_K\frac{1-\gamma_s}{(1+i_2)^2} - KF_{KK}\frac{dK}{di_2}\frac{1-\gamma_s}{1+i_2} + \frac{dK}{di_2}\frac{F_K}{1+i_2}(i_2+\gamma_s) \quad s \in S = \{H, L\} \tag{3.32}$$

Substitute (26),(33) in (32) we get the following expression

$$\frac{dU}{di_2} = \frac{dK}{di_2}\Big[\frac{\bar{i}_2}{1+\bar{i}_2}F_K\Big(\mu u_x(x_{2,H}) + (1-\mu)u_x(x_{2,L})\Big) - \\ \mu\frac{KF_{KK}}{1+\bar{i}_2}\Big(u_x(x_{2,L}) - u_x(x_{2,H})\Big)(\gamma_H - 1)\Big] + \\ \mu\frac{KF_K}{(1+\bar{i}_2)^2}\Big(u_x(x_{2,L}) - u_x(x_{2,H})\Big)(\gamma_H - 1) \tag{3.33}$$

As $\bar{i}_2 \to 0$, (33) simplifies to

$$\frac{dU}{di_2} = \mu K\Big(u_x(x_{2,L}) - u_x(x_{2,H})\Big)(\gamma_H - 1)\Big[F_K - F_{KK}\frac{dK}{di_2}\Big] \tag{3.34}$$

It is always suboptimal to set the nominal rate to zero if the term in brackets is positive. To understand this intuitively, consider the following production function

$$F(K,L) = K^\theta L^{1-\theta},\ 0 < \theta < 1$$

The term in brackets is positive if and only if

$$\frac{dK}{di_2}\Big/K < \frac{1}{1-\theta} \tag{3.35}$$

which means that the rate of change of capital holdings after a perturbation of i_2 should be less than a number which is greater than one, $\frac{1}{1-\theta} > 1$.

To sum up, a sufficient condition for nominal rates to be strictly positive is the following,

$$F_K > F_{KK}\frac{dK}{di_2}$$

3.2.12 Elastic labor in period two

Suppose we incorporate labor-leisure decisions in period two. Entrepreneurs choose their supply of labor after they have learned the realization of their productivity shock.

The ex-ante problem is as follows:

$$\max\Big[\log(x_1)+\mu\log(x_{2,H})+(1-\mu)\log(x_{2,L})+$$
$$\mu\log(1-l_H)+(1-\mu)\log(1-l_L)\Big],\ s.t$$
$$p_1x_1+p_1k+\frac{b_1}{1+i_1}+m_1\le h_1+p_1$$
$$m_1\ge p_1$$
$$p_2x_{2,H}+m_{2,H}+\frac{b_{2,H}}{1+i_2}\le h_2+b_1+m_1+wl_H+qk\gamma_H$$
$$m_{2,H}\ge wl_H+qk\gamma_H$$
$$m_{2,H}+b_{2,H}\ge 0$$
$$p_2x_{2,L}+m_{2,L}+\frac{b_{2,L}}{1+i_2}\le h_2+b_1+m_1+wl_L+qk\gamma_L$$
$$m_{2,L}\ge wl_L+qk\gamma_L$$
$$m_{2,L}+b_{2,L}\ge 0$$

The optimal labor supplies, l_H, l_L, are as follows

$$l_H=0.5-\frac{1+i_2}{2w}\Big[h_2+b_1+p_1+\frac{qk}{1+i_2}\gamma_H\Big]$$
$$l_L=0.5-\frac{1+i_2}{2w}\Big[h_2+b_1+p_1+\frac{qk}{1+i_2}\gamma_L\Big]$$

Equilibrium labor demand and capital holdings are as follows:

$$
\begin{aligned}
L &= \frac{1}{3 + 2i_2} \\
K &= \frac{6 + 8i_2 + 2\gamma_H\gamma_L}{15 + 32i_2 + 16i_2^2 + (3 + 4i_2)(\gamma_H + \gamma_L) + 3\gamma_H\gamma_L}
\end{aligned}
$$

The no-arbitrage between capital and bonds are as follows

$$\frac{q}{(1 + i_1)p_1} = (1 + i_2)\left[1 + \frac{\sigma_\gamma^2}{3 + 4i_2 + \gamma_H\gamma_L}\right]$$

Lastly, the equilibrium labor supplies are as follows:

$$l_H = 0.5 - \frac{2i_2 + \gamma_H}{2(3 + 2i_2)},\ l_L = 0.5 - \frac{2i_2 + \gamma_L}{2(3 + 2i_2)}$$

Before going to optimal policy, there is a subtlety that we need to discuss. Consider the labor supply of the high productive entrepreneurs. If the realization of the high productivity shock is large, then high productive entrepreneurs will have less incentives to supply labor and not to consume more leisure. Labor supply becomes negative if the high shock is very large so that we have to exclude this possibility,

$$l_H \geq 0 \Rightarrow \gamma_H \leq 3 \tag{3.36}$$

Given condition (22), the derivative of l_H, l_L with respect to i_2 is:

$$\frac{dl_s}{di_2} = -\frac{3 - \gamma_s}{(3 + 2i_2)^2} \leq 0,\ s \in S = \{H, L\} \tag{3.37}$$

Reductions of the nominal rate induce entrepreneurs to supply more labor since the real wage in period two increases.

From (37) we see that after a reduction of i_2, low type entrepreneurs have a higher inclination to increase labor supply than than high types. High type entrepreneurs prefer to supply less labor and consume more leisure since they receive a high productivity shock whereas low types do the complete opposite.

According to the previous arguments, the role of the monetary-authority in an environment of incomplete insurance possibilities is to provide insurance by setting positive nominal rates. We saw before that the introduction of fixed labor income across states does not modify the objective of the monetary authority. The following proposition argues that the policy conclusions are not revised when we introduce labor-leisure decisions in period two.

Proposition 6. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rate at positive values, $i_2 > 0$, in order to maximize aggregate welfare.*

Proof. Consider the derivative of indirect utility with respect to i_2 evaluated at $\bar{i}_2 > 0$ initially and as before consider the case when $\bar{i}_2 \to 0$:

$$\frac{dU}{di_2} = \underbrace{\frac{dK}{di_2}\frac{1-3K}{2K(1-K)} - \frac{12+4\gamma_H\gamma_L}{3[12+4\gamma_H\gamma_L+3\sigma_\gamma^2]}}_{consumption\ part} + \underbrace{\frac{12+4\gamma_H\gamma_L+6\sigma_\gamma^2}{3[12+4\gamma_H\gamma_L+3\sigma_\gamma^2]}}_{leisure\ part} \tag{3.38}$$

where

$$\frac{dK}{di_2} = \frac{1}{(15 + 32\bar{i}_2 + 16\bar{i}_2^2 + (3 + 4\bar{i}_2)(\gamma_H + \gamma_L) + 3\gamma_H\gamma_L)^2}$$
$$\Big[8(15 + 32\bar{i}_2 + 16\bar{i}_2^2 +$$
$$(3 + 4\bar{i}_2)(\gamma_H + \gamma_L) + 3\gamma_H\gamma_L) -$$
$$(6 + 8\bar{i}_2 + 2\gamma_H\gamma_L)(32 + 32\bar{i}_2 + 4(\gamma_H + \gamma_L))\Big]$$

and as $\bar{i}_2 \to 0$

$$\frac{dK}{di_2} = -\frac{1}{(15 + 4(\gamma_H + \gamma_L) + 3\gamma_H\gamma_L)^2}$$
$$\Big[72 + 40\gamma_H\gamma_L + 8\gamma_H\gamma_L(1 + \gamma_H\gamma_L + \sigma_\gamma^2)\Big]$$

Expression (38) becomes

$$\frac{dU}{di_2} = \frac{-\sigma_\gamma^2\Big[72 + 40\gamma_H\gamma_L + 8\gamma_H\gamma_L(1 + \gamma_H\gamma_L + \sigma_\gamma^2)\Big]}{(12 + 4\gamma_H\gamma_L)(12 + 4\gamma_H\gamma_L + 3\sigma_\gamma^2)(6 + 2\gamma_H\gamma_L + \sigma_\gamma^2)} +$$
$$\frac{2\sigma_\gamma^2}{12 + 4\gamma_H\gamma_L + 3\sigma_\gamma^2} \tag{3.39}$$

The RHS of (39) is always positive. Setting the nominal rate to zero is always suboptimal.

□

The previous proposition stated that even with labor-leisure decisions, the monetary authority should engage in active policies by setting positive rates. Even though the qualitative conclusions do not change with respect to the previous models we discussed, the mechanism behind proposition 6 is different from the previous propositions.

In the previous models we analyzed the objective of the monetary authority as an insurance institution. The role of the monetary-authority was to balance the trade-off between consumption in period one and consumption in period two across states. Even in the case of under-investment in period one, reducing i_2 to zero was not optimal. The reason was that even though entrepreneurs over-consumed in period one, low values of i_2 implied very low consumption in the bad state which was ex-ante suboptimal.

When labor-leisure decisions are present the above story is modified. Consider the derivative of ex-ante utility with respect to i_2, expression (38). The first term in the underbrace represents the differential change of utility from consumption today and tomorrow across states. The second term in the underbrace is the differential change of utility from leisure. If there is under-investment compared to the first-best in period one then the first term in the underbrace is always negative. The second term is always positive. Suppose $i_2 > 0$ initially. Reductions of i_2 increase consumption in the good state but more importantly increase consumption in the bad state as well. The main reason as to why consumption in the bad state increases when i_2 falls is because low type entrepreneurs supply more labor since the real wage increases. Even though seignorage transfers fall, the increase in labor and capital income outweighs the decrease of seignorage transfers. Elastic labor is a form of insurance for entrepreneurs. This is to be contrasted with the previous models. The first term in (38) is always negative since after a decrease of i_2 the increase in consumption across states dominates the decrease in consumption in period one. This is because entrepreneurs over-consume initially. But even though entrepreneurs are willing to decrease consumption today and increase consumption across states after a decrease of i_2, they get disutility from supplying more labor. The reduction in leisure dominates the effect on consumption demands in utility terms.

Consider the following remark

Remark 1. *The introduction of **labor-leisure** decisions in period two combined with an **under-investment** effect in period one minimizes the intertemporal trade-off of consumption. After a reduction of i_2, entrepreneurs will optimally want to reduce consumption in period one, they over-consume initially, and will always increase consumption **even** in the bad state since they can increase the supply of labor. State-contingent labor supply acts as a form of **costly** insurance. Costly, in terms of the disutility it generates.*

Given the previous discussion, we want to prove that the monetary-authority has less incentives to engage in active policies if there are labor-leisure decisions and an under-investment effect in period one. To this end, consider the following utility specification

$$\begin{aligned} U = \log(x_1) + \mu\log(x_{2,H}) + (1-\mu)\log(x_{2,L}) + \\ \theta\mu\log(1-l_H) + \theta(1-\mu)\log(1-l_L) \end{aligned} \tag{3.40}$$

where for the moment assume only that $\theta > 0$. By active policies, I simply mean that the monetary-authority should set nominal rates at positive values in order to maximize ex-ante welfare. The incentive of the monetary-authority to engage in active policies is measured by the derivative of ex-ante indirect utility with respect to i_2, $\frac{dU}{di_2}$. If $\frac{dU}{di_2} = 0$, then there is no incentive to engage in active policies.

Solving the ex-ante problem of the entrepreneur using (40) we compute the new equilibrium. The no-arbitrage between capital and bonds is as follows:

$$\frac{q}{(1+i_1)p_1} = (1+i_2)\left[1 + \frac{\sigma_\gamma^2}{1+2i_2+\gamma_H\gamma_L+2\theta(1+i_2)}\right]$$

and equilibrium capital and labor demand are as follows:

$$\begin{aligned}
&L = \frac{1}{1+2\theta(1+i_2)} \\
&K = (1+\theta)(1+2i_2(1+\theta)+\gamma_H\gamma_L+2\theta)\Big/ \\
&\Big[2\theta^2\Big(3+5i_2+2i_2^2\Big)+ \\
&\theta\Big(7+16i_2+8i_2^2+2(1+i_2)(\gamma_H+\gamma_L)+\gamma_H\gamma_L\Big)+ \\
&2+6i_2+4i_2^2+2\gamma_H\gamma_L+ \\
&(\gamma_H+\gamma_L)(1+2i_2)\Big]
\end{aligned}$$

Equilibrium labor supplies are

$$l_s = \frac{1}{1+\theta} - \frac{\theta(2i_2+\gamma_s)}{(1+\theta)(1+2\theta(1+i_2))}, \;\; s \in S = \{H, L\}$$

and equilibrium labor supply in the high state is nonnegative if and only if

$$\gamma_H \leq \frac{1+2\theta}{\theta} \tag{3.41}$$

The next proposition argues that if entrepreneurs care little about leisure, inelastic labor supply, the incentives of the monetary-authority to pursue active policies are higher than when entrepreneurs attach a relatively high weight to leisure compared to consumption. The reason is the argument given in the previous paragraphs.

Proposition 7. *Given that entrepreneurs under-investment in period one, the incentive of the monetary-authority to engage in active policies is a decreasing function of θ.*

Proof. Consider the derivative of ex-ante indirect utility with respect to i_2, evaluated at $\bar{i}_2 > 0$. Consider the same derivative as $\bar{i}_2 \to 0$,

$$\frac{dU}{di_2} = \Big[(1+2\theta+\gamma_H)(1+2\theta+\gamma_L)(1+2\theta+\gamma_H\gamma_L)$$
$$(2+6\theta^2+\gamma_H+\gamma_L+2\gamma_H\gamma_L+\theta(7+2\gamma_L+\gamma_H(2+\gamma_L)))\Big]^{-1}$$
$$\Big[(1+2\theta)(-1+\gamma_H)(-1+\gamma_L)\Big(1+6\theta^3+\gamma_H^2\gamma_L+$$
$$\gamma_H\gamma_L(1+\gamma_L)+\theta^2(-7+5\gamma_H\gamma_L)+$$
$$\theta(-3+\gamma_H^2\gamma_L+\gamma_H(-3+\gamma_L)\gamma_L)\Big)\Big]+$$
$$\frac{2\theta(-1+\theta(-2+\gamma_L))(-1+\mu)}{(1+2\theta)(1+2\theta+\gamma_L)} - \frac{2\theta(-1+\theta(-2+\gamma_H))\mu}{(1+2\theta)(1+2\theta+\gamma_H)}+$$
$$\frac{(2\theta^2(1+2\theta)^{0.5}+(1+2\theta)^{1.5}(-1+\gamma_L)+\theta(1+2\theta)^{0.5}(1+\gamma_L))(-1+\mu)}{(1+2\theta)^{1.5}((1+2\theta)+\gamma_L)}+$$
$$\frac{(-2\theta^2+\theta(-1-\gamma_H)+(1+2\theta)(1-\gamma_H))\mu}{(1+2\theta)((1+2\theta)+\gamma_H)} = \Phi(\gamma_H,\gamma_L,\theta,\mu) \tag{3.42}$$

where $\Phi(.)$ represents the incentive of the monetary-authority to increase i_2 whenever it is initially close to zero. We must show that $\Phi(.)$ is a decreasing function of θ. Since $\Phi(.)$ is a complicated algebraic expression, we must resort to numerical calibrations to demonstrate our result. Before doing so, it useful to examine a simple case to see why proposition 7 holds true. Consider the derivative $\frac{d\Phi}{d\theta}$ evaluated at $\theta = 1$. To simplify algebra consider also the maximum degree of heterogeneity in period two, $\gamma_L = 0, \gamma_H = \frac{1}{\mu}$. Then the derivative $\frac{d\Phi}{d\theta}$ is as follows:

$$\frac{d\Phi}{d\theta} = \frac{-0.111111 - 3.22222\mu - 22.2222\mu^2 - 30.7778\mu^3 + 56.3333\mu^4}{(1 + 8\mu + 15\mu^2)^2} < 0$$

which is always negative. The incentive of the monetary-authority to engage in active policies increases when θ decreases below one. The problem with the previous argument is that it is a special case.

Since $\Phi(.)$ is a complicated expression, consider the following graphs:

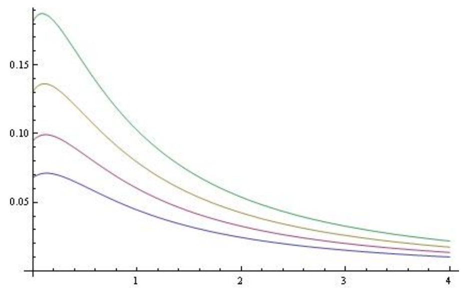

Figure 3.4: $\Phi(\theta)$, $\mu = 0.6$

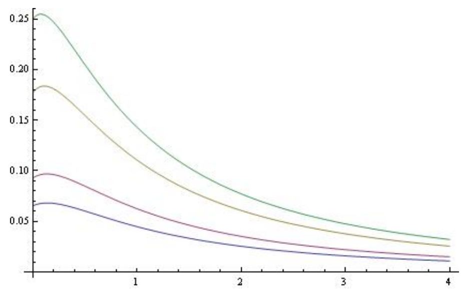

Figure 3.5: $\Phi(\theta)$, $\mu = 0.5$

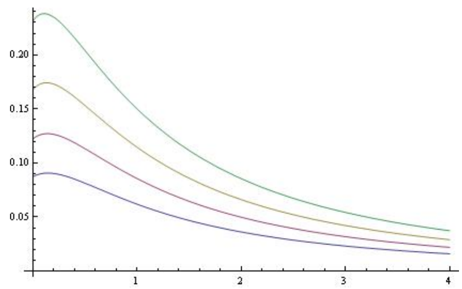

Figure 3.6: $\Phi(\theta)$, $\mu = 0.4$

The above graphs show that when θ tends to zero, the incentive to engage in active policies increases. On the other hand, when θ is increases, optimal policy is close to the Friedman rule argument.

$\square$

The next corollary is a direct implication of proposition 7,

Corollary 1. *Given that entrepreneurs under-investment in period one, if labor-leisure decisions are equally or more important than consumption decisions, then optimal monetary policy is very close to the Friedman rule argument. If labor-leisure decisions are less important, labor is supplied inelastically, then optimal monetary policy should deviate from the Friedman rule argument.*

A useful digression is to consider the case of idiosyncratic shocks on labor productivity.

3.2.12.1 Elastic labor supply: idiosyncratic shocks on labor productivity

The ex-ante problem of individuals is as follows

$$
\begin{aligned}
&\max\Big[\log(x_1) + \mu\log(x_{2,H}) + (1-\mu)\log(x_{2,L}) + \\
&\mu\log(1-l_H) + (1-\mu)\log(1-l_L)\Big],\ s.t \\
&p_1x_1 + p_1k + \frac{b_1}{1+i_1} + m_1 \leq h_1 + p_1 \\
&m_1 \geq p_1 \\
&p_2x_{2,H} + m_{2,H} + \frac{b_{2,H}}{1+i_2} \leq h_2 + b_1 + m_1 + wl_He_H + qk \\
&m_{2,H} \geq wl_He_H + qk \\
&m_{2,H} + b_{2,H} \geq 0 \\
&p_2x_{2,L} + m_{2,L} + \frac{b_{2,L}}{1+i_2} \leq h_2 + b_1 + m_1 + wl_Le_L + qk \\
&m_{2,L} \geq wl_Le_L + qk \\
&m_{2,L} + b_{2,L} \geq 0
\end{aligned}
$$

The equilibrium conditions in this case are as follows:

$$
\begin{aligned}
&x_1 + k = 1 \\
&\mu x_{2,H} + (1-\mu)x_{2,L} = F(K,L) \\
&\mu e_H l_H + (1-\mu)e_L l_L = L \\
&k = K \\
&h_1 = \frac{i_1}{1+i_1}M_1,\ h_2 = \frac{i_2}{1+i_2}M_2 \\
&M_1 = m_1,\ M_2 = \mu m_{2,H} + (1-\mu)m_{2,L} \\
&B_1 = b_1,\ B_2 = \mu b_{2,H} + (1-\mu)b_{2,L}
\end{aligned}
$$

Equilibrium capital, bond holdings and labor demand are

$$
\begin{aligned}
&K = \left[0.5 + i_2 + \frac{\mu e_L + (1-\mu)e_H}{2}(3+2i_2)\right] \Big/ \\
&\Big[0.75 + 2i_2 + i_2^2 + \\
&\frac{(e_H + e_L)(3+2i_2)(1+2i_2)}{4} + \\
&\frac{\mu e_L + (1-\mu)e_H}{2}(3+2i) + \frac{e_H e_L}{4}(3+2i_2)^2\Big] \\
&b_1 = -p_1,\ L = \frac{1}{3+2i_2}
\end{aligned}
$$

Equilibrium labor supplies are

$$
l_H = 0.5 - \frac{1+2i_2}{2e_H(3+2i_2)} \tag{3.43}
$$

$$
l_L = 0.5 - \frac{1+2i_2}{2e_L(3+2i_2)} \tag{3.44}
$$

Given our previous discussion on elastic labor, we need to guarantee that labor supplies are nonnegative. Labor supply for high types poses no problem. The supply of labor for low types is nonnegative if and only if

$$e_L \geq \frac{1+2i_2}{3+2i_2} \tag{3.45}$$

If the monetary-authority sets high nominal rates tomorrow, given that (45) is always satisfied, it will induce low type entrepreneurs to take more leisure and supply less labor. The main difference with the previous case of entrepreneurial risk is that high nominal values of i_2 can induce low types to supply zero labor.

Consider the following proposition

Proposition 8. *Given $\sigma_\gamma^2 > 0$ and condition (45), optimal monetary policy should set the nominal rate at positive values, $i_2 > 0$, in order to maximize aggregate welfare.*

Proof. Consider the derivative of indirect utility with respect to i_2 evaluated at $\bar{i}_2 > 0$ initially and as before consider the case when $\bar{i}_2 \to 0$:

$$\frac{dU}{di_2} = \underbrace{\frac{dK}{di_2}\frac{1-3K}{2K(1-K)} - \frac{4+12e_He_L-6\sigma_\gamma^2}{3[4+12e_He_L+3\sigma_\gamma^2]}}_{consumption\ part} + \underbrace{\frac{4+12e_He_L+12\sigma_\gamma^2}{3[4+12e_He_L+3\sigma_\gamma^2]}}_{leisure\ part} \tag{3.46}$$

where as $\bar{i}_2 \to 0$

$$\frac{dK}{di_2} = -\frac{8+60\sigma_\gamma^2+72e_H^2e_L^2+36(\sigma_\gamma^2)^2+108e_He_L\sigma_\gamma^2+48e_He_L}{[6+18e_He_L+9\sigma_\gamma^2]^2}$$

$$K = \frac{2+6e_He_L+6\sigma_\gamma^2}{18e_He_L+9\sigma_\gamma^2+6} > \frac{1}{3}$$

The derivative in (47) can be written as follows:

$$\frac{dU}{di_2} = \frac{(8 + 60\sigma_\gamma^2 + 72e_H^2 e_L^2 + 36(\sigma_\gamma^2)^2 + 108e_H e_L \sigma_\gamma^2 + 48e_H e_L)9\sigma_\gamma^2}{(4 + 12e_H e_L + 12\sigma_\gamma^2)(12e_H e_L + 3\sigma_\gamma^2 + 4)(6 + 18e_H e_L + 9\sigma_\gamma^2)} + \frac{18\sigma_\gamma^2}{3(12e_H e_L + 3\sigma_\gamma^2 + 4)} \tag{3.47}$$

□

Consider the following proposition for the case of inelastic labor supply and shocks to the productivity of labor,

Proposition 9. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rate at positive values, $i_2 > 0$, in order to maximize aggregate welfare.*

Proof. Consider the derivative of indirect utility with respect to i_2 evaluated at $\bar{i}_2 > 0$ initially and as before consider the case when $\bar{i}_2 \to 0$:

$$\frac{dU}{di_2} = \frac{2\sigma_\gamma^2\Big(\sigma_\gamma^2 + (1 + e_H e_L + \sigma_\gamma^2)^2\Big)}{(3 + 3e_H e_L + 2\sigma_\gamma^2)(1 + e_H e_L + \sigma_\gamma^2)(2 + 2e_H e_L + \sigma_\gamma^2)} + \frac{\sigma_\gamma^2}{2 + 2e_H e_L + \sigma_\gamma^2}$$

□

3.2.13 Volatility of risk and incentives to engage in active policies

The previous analysis focused on the fact that in the presence of incomplete insurance possibilities, the monetary-authority should set positive nominal rates in order to achieve the second-best allocation. In this section we want to see in *what way* the incentives to engage in active policies are affected by the volatility of risk.

Consider the following ***experiment***. Fix the probability distribution of shocks and consider the normalization we have imposed in the beginning, $\mu\gamma_H + (1-\mu)\gamma_L = 1$. We want to show that the incentives to engage in active policies increase when heterogeneity in period two increases, $\gamma_H - \gamma_L$ increases, so that σ_γ^2 increases. In other words, we want to show that the monetary-authority has a higher incentive to intervene actively in the market when the volatility of risk is high.

Proposition 10. *The incentives of the monetary-authority to engage in active policies is an increasing function of heterogeneity in period two and as a consequence increasing in risk volatility.*

Proof. The incentives of the monetary-authority to engage in active policies is measured by the derivative of ex-ante utility with respect to i_2, $\frac{dU}{di_2}|_{i_2=\bar{i}_2}$ where $\bar{i}_2 > 0$ initially. Suppose as before that we take the limit from above of $\frac{dU}{di_2}$, $\bar{i}_2 \to 0$.

Consider the model of section 2.6. Expression (9) is

$$\frac{dU}{di_2} = \frac{\sigma_\gamma^2}{\gamma_H \gamma_L}$$

and the right hand side increases when heterogeneity in period two increases because σ_γ^2 increases and $\gamma_H\gamma_L$ reduces. The incentives of the monetary-authority to engage in active policies increase with the degree of heterogeneity.

Consider the model of section 2.11 with inelastic labor supply. Expression (21) is as follows

$$\frac{dU}{di_2} =$$
$$\frac{\sigma_\gamma^2}{2+2\gamma_H\gamma_L+\sigma_\gamma^2}\left[1-\frac{1+\gamma_H\gamma_L(2+\gamma_H\gamma_L+\sigma_\gamma^2)}{(3+3\gamma_H\gamma_L+\sigma_\gamma^2)(1+\gamma_H\gamma_L)}\right]=\Upsilon$$

It is difficult in the above expression to see what happens when heterogeneity increases because both γ_H, γ_L change and as a consequence σ_γ^2 changes. Instead, suppose that we set $\gamma_H = \frac{1-(1-\mu)\gamma_L}{\mu}$ keeping μ fixed and take the derivative of Υ with respect to γ_L. We get

$$\begin{aligned}
&\frac{d\Upsilon}{d\gamma_L} = -((-1+\gamma_L)(5\gamma_L^9(-1+\mu)^5(-1+2\mu)+\\
&\gamma_L^8(-1+\mu)^4(-23+31\mu+14\mu^2)+\\
&\gamma_L^7(-1+\mu)^3(-39+33\mu+11\mu^2+30\mu^3)+\\
&\gamma_L^6(-1+\mu)^2(-25+12\mu+2\mu^2-35\mu^3+70\mu^4)+\\
&\mu(-1-4\mu-6\mu^2-2\mu^3+7\mu^4+6\mu^5)+\\
&\gamma_L\mu(1-12\mu-48\mu^2-14\mu^3+47\mu^4+26\mu^5)+\\
&\gamma_L^2(1-2\mu-9\mu^2-25\mu^3-92\mu^4+41\mu^5+86\mu^6)+\\
&\gamma_L^5(-5+14\mu-40\mu^2+70\mu^3+71\mu^4-238\mu^5+128\mu^6)+\\
&\gamma_L^3(-7+10\mu+59\mu^2-33\mu^3-136\mu^4-35\mu^5+142\mu^6)+\\
&\gamma_L^4(15-22\mu-4\mu^2+132\mu^3-121\mu^4-160\mu^5+160\mu^6)))/\\
&((1+\gamma_L)^2(1+\gamma_L(-1+\mu)+\mu)^2\\
&(\gamma_L+\gamma_L^2(-1+\mu)+\mu)^2(1+\gamma_L+\\
&2\gamma_L^2(-1+\mu)+2\mu+2\gamma_L\mu)^2)
\end{aligned}$$

The above derivative is a complicated expression. If we evaluate it numerically by fixing any probability distribution, $\mu \in (0,1)$, and choose to set initially any value $\gamma_L \in [0,1)$ then the above derivative is always negative.

□

§ 3.3 Money supply rules

Consider the other class of monetary policy rules: money supply policies. More specifically,

Money supply rules. *The monetary-authority fix the money supplies, $\bar{M}_1, \bar{M}_2 > 0$, and let the nominal rates, i_1, i_2, determine in equilibrium.*

In order to talk about money supply policies we need some qualification to the previous argument. In the following section we will analyze money supply policies in the simple framework of the entrepreneurial economy of section 2.1.

3.3.1 Multiple equilibria, money supply rules and optimal monetary policy

Consider the set-up of section 2.1. Combine the equilibrium conditions of section 2.3 together with the money market clearing, we get the following 4 equations,

$$\frac{p_2}{p_1} = \frac{(1+i_1)(1+i_2)}{i_2 + \gamma_H\gamma_L}\left[i_2 + \gamma_H\gamma_L + \sigma_\gamma^2\right] \tag{3.48}$$

$$k = \frac{i_2 + \gamma_H\gamma_L}{(1+i_2)^2 + (2+i_2)\gamma_H\gamma_L + i_2\sigma_\gamma^2 - 1} \tag{3.49}$$

$$p_1 = m_1 = \bar{M}_1 \tag{3.50}$$

$$\mu\gamma_H p_2 k + (1-\mu)\gamma_L p_2 k = m_2 = \bar{M}_2 \tag{3.51}$$

where we have used the previous rule for the distribution of seignorage,

$$H_1 = \frac{i_1}{1+i_1}\bar{M}_1, \;\; H_2 = \frac{i_2}{1+i_2}\bar{M}_2$$

Equations (48)-(51) are 4 four independent equations in 5 unknowns. There is one degree of real indeterminacy left. Unless we eliminate the indeterminacy we can not proceed further and claim something more. We need one more equation to the above system.

To solve this problem assume that entrepreneurs are endowed with outside money ***only*** in period one. The period one budget constraint of entrepreneurs in the asset market is as follows

$$\hat{m}_1 + \frac{b_1}{1+i_1} = \delta$$

where δ represents units of outside money.

The intertemporal constraint of the monetary-authority is as follows

$$\frac{i_1}{1+i_1}\bar{M}_1 + \frac{i_2}{(1+i_1)(1+i_2)}\bar{M}_2 = \delta + \frac{h_2}{1+i_1}$$

where δ is an initially public liability for the monetary-authority and h_2 are transfers to entrepreneurs in period two across states. The monetary-authority finances the initial public liability and transfers in period two through seignorage profits from printing money balances. From the previous rule for the distribution of seignorage we get,

$$\frac{i_1}{1+i_1}\bar{M}_1 = \delta \tag{3.52}$$

$$\frac{i_2}{1+i_2}\bar{M}_2 = H_2 \tag{3.53}$$

essentially we have picked a particular value for period one transfers. This way provide us with the additional equation we need in order to eliminate the real indeterminacy. From

(52) we can solve for i_1 and eliminate the degree of real indeterminacy,

$$i_1 = \frac{\delta}{\bar{M}_1 - \delta} \tag{3.54}$$

where $\bar{M}_1 - \delta > 0$ for $i_1 \geq 0$. Observe that we can choose the parameters in (54) to fix any nonnegative value for i_1. Since the *policy* variable for the monetary-authority would be the *growth rate* of money supply between period one and period two, we can choose any positive values for $\bar{M}_1, \delta$ such as to make i_1 very close to zero. Choosing $(\delta, \bar{M}_1)$, such that i_1 is very close to zero in equilibrium, we can safely ignore it from (48). Alternatively, we can assume that individuals do not receive transfers or outside money in period one and they receive transfers only in period two. Using the previous rule for the distribution of seignorage, i_1 should be zero in equilibrium.

Given the previous argument, combining (48)-(51) we end up in the following relation:

$$\frac{(1+i_2)(i_2+\gamma_H\gamma_L+\sigma_\gamma^2)}{(1+i_2)^2+\gamma_H\gamma_L(2+i_2)+i_2\sigma_\gamma^2-1} = \frac{\bar{M}_2}{\bar{M}_1}$$

Define $m = \bar{M}_2/\bar{M}_1 - 1$, as the growth rate of the money supply. The above relation can written in the following quadratic form:

$$i_2^2 m + i_2(1+2m+\gamma_H\gamma_L m+\sigma_\gamma^2 m) + \gamma_H\gamma_L(1+2m) - \sigma_\gamma^2 = 0 \tag{3.55}$$

Define

$$B = 1+2m+\gamma_H\gamma_L m+\sigma_\gamma^2 m, \;\; \Gamma = \gamma_H\gamma_L(1+2m) - \sigma_\gamma^2$$

Expression (56) gives the following two solutions:

$$i_2 = \frac{-B + \sqrt{B^2 - 4\Gamma m}}{2m} \tag{3.56}$$

$$i_2 = \frac{-B - \sqrt{B^2 - 4\Gamma m}}{2m} \tag{3.57}$$

Given the possibility of two solutions, we need to know which solution to accept and under what conditions. Let us start with the signs of B, Γ. $B \gtreqless 0$ if and only if

$$\frac{\bar{M}_2}{\bar{M}_1} \gtreqless \frac{1 + \gamma_H \gamma_L + \sigma_\gamma^2}{2 + \gamma_H \gamma_L + \sigma_\gamma^2} \tag{3.58}$$

$\Gamma \gtreqless 0$ if and only if

$$\frac{\bar{M}_2}{\bar{M}_1} \gtreqless \frac{\sigma_\gamma^2 + \gamma_H \gamma_L}{2\gamma_H \gamma_L} \tag{3.59}$$

Suppose that $m > 0$. Then,

- If $B > 0, \Gamma > 0$ then we reject both solutions.
- If $B > 0, \Gamma < 0$ then we accept (56) provided the discriminant is positive.

Suppose that $m < 0$. Then,

- If $B > 0, \Gamma > 0$ then accept (57).
- If $B < 0, \Gamma > 0$ then accept (57) again.
- If $B < 0, \Gamma < 0$ then reject both.
- If $B > 0, \Gamma < 0$ then ***accept both*** provided the discriminant is positive.

The reason we choose to fix parameters such as to make i_1 very close to zero is to be able to compare optimal monetary policy in the case of interest rate rules with optimal policies in the case of money supply rules. We know that i_1 does not affect the real allocation in the case of interest rate rules but affects the real allocation in the case of money supply rules. This introduces an asymmetry between the two cases. In the case of interest rate rules we can simply fix i_1 to zero since it does not affect the real allocation. In the case of money supply rules we can choose parameters together with the assumption on the distribution of seignorage such as to make the equilibrium i_1 very close to zero. Why we do this will be clarified in the next paragraph.

Start with the case of interest rate rules as before. From the previous analysis we know that given the volatility of risk, it is optimal for the monetary-authority to set $i_2^* > 0$, and maximize aggregate welfare. Take i_2^* and compute the optimal rate of money growth in that case, m^*. The data we have from interest rate rules are (i_2^*, m^*). Consider money supply rules. We would like to argue the following: *If the monetary-authority fixes the growth rate of money supply at m^*, then the equilibrium interest rate would equal i_2^* and as a consequence we would maximize aggregate welfare in that case as well.*

We can not establish a one-to-one mapping from optimal interest rate policies to optimal money supply policies. If i_2^* maximizes ex-ante welfare under an interest rate rule then m^* is the optimal rate of money growth. Under a money supply rule, m^* is the optimal policy *only if* the equilibrium is unique. If the equilibrium is not unique, then under m^* we get one solution which will be i_2^* but we get a second one, $i_2 \neq i_2^*$. It is interesting to know when optimal monetary policy implies a unique equilibrium because in the case of uniqueness, optimal interest rate policies and optimal money supply policies provide the same answer.

Consider the following numerical example which illustrates the fact that we accept two solutions,

Numerical example. Suppose $\gamma_s = 0.5$ and from the previous normalization, $\gamma_H + \gamma_L = 2$.

Consider the following table,

(γ_H, γ_L)	i_2^*	m^*	$\widehat{i_2}$
$(1.55, 0.45)$	0.3486	-0.3	0.098
$(1.6, 0.4)$	0.4	-0.2576	0.482319
$(1.65, 0.35)$	0.4412	-0.2230	1.04262
$(1.7, 0.3)$	0.4843	-0.1863	1.88442
$(1.75, 0.25)$	0.54	-0.1489	3.1747
$(1.8, 0.2)$	0.6077	-0.1125	5.27969
$(1.85, 0.15)$	0.6756	-0.076	9.5
$(1.9, 0.1)$	0.762	-0.0437	19.1

The first column represents the realizations of productivity shocks. The second column is the optimal interest rate policy that results in a second-best allocation when the monetary-authority follows an interest rate policy. The third column is the optimal growth rate of the money supply that results from the optimal policy i_2^*.

Let us conduct the following ***experiment*** that was mentioned before: Suppose the monetary-authority follows money supply rules. If it fixes the growth rate at m^*, then two equilibria are possible. One equilibrium which corresponds to the second-best policy computed in the case of interest rate policies, i_2^*, and another equilibrium, $\widehat{i_2}$. The latter equilibrium implies an allocation that can potentially be very far from the second-best one as we can observe from the table above.

Consider the benchmark case, $\sigma_\gamma^2 = 0$. The solutions of i_2 are as follows:

$$i_2 = \frac{-(1+3m) + \sqrt{(1+3m)^2 - 4m(1+2m)}}{2m}$$
$$i_2 = \frac{-(1+3m) - \sqrt{(1+3m)^2 - 4m(1+2m)}}{2m}$$

and only the second solution is accepted.

3.3.2 Model of section 2.11

Consider the model of section 2.11 with inelastic labor supply. The argument is similar to the previous one. We get the following two solutions

$$i_2 = \frac{-B + \sqrt{B^2 - 16m\Gamma}}{8m} \tag{3.60}$$
$$i_2 = \frac{-B - \sqrt{B^2 - 16m\Gamma}}{8m} \tag{3.61}$$

where

$$B = 2 + m(8 + 2\gamma_H\gamma_L + 2\sigma_\gamma^2)$$
$$\Gamma = 1 + \gamma_H\gamma_L - \sigma_\gamma^2 + m(3 + 3\gamma_H\gamma_L + \sigma_\gamma^2)$$

$B \gtreqless 0$ iff

$$\frac{\bar{M}_2}{\bar{M}_1} \gtrless \frac{3 + \gamma_H\gamma_L + \sigma_\gamma^2}{4 + \gamma_H\gamma_L + \sigma_\gamma^2}$$

and $\Gamma \gtreqless 0$ iff

$$\frac{\bar{M}_2}{\bar{M}_1} \gtrless \frac{2 + 2\gamma_H\gamma_L + 2\sigma_\gamma^2}{3 + 3\gamma_H\gamma_L + \sigma_\gamma^2}$$

§ 3.4 Finite horizon

Suppose we start again from the set-up of section 2.1 and consider the case of three periods. Idiosyncratic shocks are assumed to be i.i.d. In the following paragraphs we will use a dynamic programming argument in finite horizon to solve for the optimal plan of entrepreneurs.

Let us start from the last period and move backwards. In period three, the objective is to maximize

$$V_{3,s,s'} = \max[\log(x_{3,s,s'})], \;\; s, s^{'} \in S = \{H, L\} \tag{3.62}$$

where s is the state that realized in period two and $s^{'}$ the state that realized in period three. The budget constraints are as follows

$$p_3 x_{3,s,s'} + \frac{b_{3,s'}}{1+i_3} + m_{3,s,s'} \leq h_3 + b_{2,s} + m_{2,s} + p_3 k_{3,s} \gamma_{s'} \tag{3.63}$$

$$m_{3,s,s'} \geq p_3 k_{3,s} \gamma_{s'} \tag{3.64}$$

$$m_{3,s,s'} + b_{3,s'} \geq 0 \tag{3.65}$$

We will assume that the cash constraints will always bind, constraint (64). The only choice variable for entrepreneurs is to choose $b_{3,s'}$ in order to decide the optimal consumption in the last period. At an optimum solution, constraints (63), (65) should bind. As a result the value function in (62) becomes

$$V_{3,s,s'} = \log\left(h_3 + b_{2,s} + m_{2,s} + \frac{p_3 k_{3,s}}{1+i_3}\gamma_{s'}\right) - \log(p_3) \tag{3.66}$$

At period two, the problem becomes

$$V_{2,s} = \max_{b_{2,s},k_{3,s}} \Big[\log \left(w_2 - \frac{b_{2,s}}{1+i_2} - p_2 k_{3,s} \right) - \log(p_2) + \mu V_{3,s,H} + (1-\mu) V_{3,s,L} \Big] \tag{3.67}$$

where

$$V_{3,s,H} = \log \left(\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3} \gamma_H \right) - \log(p_3)$$
$$V_{3,s,L} = \log \left(\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3} \gamma_L \right) - \log(p_3)$$
$$w_2 = h_2 + b_1 + p_1 k_1$$
$$\tilde{w}_{3,s} = h_3 + p_2 k_2 \gamma_s$$

The FOCs with respect to $b_{2,s}, k_{3,s}$ respectively, are as follows

$$\frac{1}{w_2 - \frac{b_{2,s}}{1+i_2} - p_2 k_{3,s}} = \frac{\mu(1+i_2)}{\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3}\gamma_H} + \frac{(1-\mu)(1+i_2)}{\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3}\gamma_L} \tag{3.68}$$

$$\frac{1}{w_2 - \frac{b_{2,s}}{1+i_2} - p_2 k_{3,s}} = \frac{\mu \frac{p_3}{p_2} \frac{\gamma_H}{1+i_3}}{\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3}\gamma_H} + \frac{(1-\mu) \frac{p_3}{p_2} \frac{\gamma_L}{1+i_3}}{\tilde{w}_{3,s} + b_{2,s} + \frac{p_3 k_{3,s}}{1+i_3}\gamma_L} \tag{3.69}$$

Combining (68),(69) we get

$$b_{2,s} + \tilde{w}_{3,s} = \frac{p_3 k_{3,s}}{1+i_3} \zeta_2 \tag{3.70}$$

where

$$\zeta_2 = \frac{(1+i_2)(1+i_3)A - \gamma_H \gamma_L \frac{p_3}{p_2}}{\frac{p_3}{p_2} - (1+i_2)(1+i_3)}$$
$$A = \mu\gamma_L + (1-\mu)\gamma_H$$

The optimal demands for bonds and capital are as follows:

$$b_{2,s} = \frac{1+i_2}{1+i_3}\Phi_2\zeta_2\Big(w_2 + \frac{\tilde{w}_{3,s}}{1+i_2}\Big) - \tilde{w}_{3,s} \tag{3.71}$$
$$k_{3,s} = \frac{1+i_2}{p_3}\Phi_2\Big(w_2 + \frac{\tilde{w}_{3,s}}{1+i_2}\Big) \tag{3.72}$$

where

$$\Phi_2 = \frac{\zeta_2 + A}{\frac{(\zeta_2+\gamma_H)(\zeta_2+\gamma_L)}{1+i_3} + (\frac{1+i_2}{p_3/p_2} + \frac{\zeta_2}{1+i_3})(\zeta_2 + A)}$$

The optimal value of $V_{2,s}$ is as follows

$$\begin{aligned} V_{2,s} = &\Big\{ \log\Big(1 - \frac{\Phi_2(1+i_2)}{p_3/p_2} - \frac{\Phi_2\zeta_2}{1+i_3}\Big) - \log(p_2) - \log(p_3) + \\ &\mu\log\Big((\zeta_2+\gamma_H)\frac{\Phi_2(1+i_2)}{1+i_3}\Big) + \\ &(1-\mu)\log\Big((\zeta_2+\gamma_L)\frac{\Phi_2(1+i_2)}{1+i_3}\Big)\Big\} + \\ &2\log\Big(w_2 + \frac{\tilde{w}_{3,s}}{1+i_2}\Big) \end{aligned}$$

It can be rewritten as

$$V_{2,s} = B + 2\log\Big(\tilde{w}_2 + b_1 + \frac{p_2 k_2}{1+i_2}\gamma_s\Big)$$

where B is the term in brackets and

$$\tilde{w}_2 = p_1 k_1 + h_2 + \frac{h_3}{1+i_2}$$

Since in period one entrepreneurs are identical, we assume that everybody is endowed with one unit of the good. As a consequence, we get the following *initial condition*: $k_1 = 1$.

In period one the problem is as follows:

$$V_1 = \max_{b_1, k_2} \Big[\log\Big(w_1 - p_1 k_2 - \frac{b_1}{1+i_1}\Big) - \log(p_1) + \\ \mu V_{2,H} + (1-\mu) V_{2,L} \Big]$$

and $w_1 = h_1$.

The FOCs with respect to b_1, k_2 respectively, are as follows

$$\frac{1}{w_1 - p_1 k_2 - \frac{b_1}{1+i_1}} = \frac{2\mu(1+i_1)}{\tilde{w}_2 + b_1 + \frac{p_2 k_2}{1+i_2}\gamma_H} + \frac{2(1-\mu)(1+i_1)}{\tilde{w}_2 + b_1 + \frac{p_2 k_2}{1+i_2}\gamma_L} \tag{3.73}$$

$$\frac{1}{w_1 - p_1 k_2 - \frac{b_1}{1+i_1}} = \frac{2\mu \frac{p_2}{p_1}\frac{\gamma_H}{1+i_2}}{\tilde{w}_2 + b_1 + \frac{p_2 k_2}{1+i_2}\gamma_H} + \frac{2(1-\mu)\frac{p_2}{p_1}\frac{\gamma_L}{1+i_2}}{\tilde{w}_2 + b_1 + \frac{p_2 k_2}{1+i_2}\gamma_L} \tag{3.74}$$

Combining (73),(74) we get

$$b_1 + \tilde{w}_2 = \frac{p_2 k_2}{1+i_2}\zeta_1 \tag{3.75}$$

where

$$\zeta_1 = \frac{(1+i_1)(1+i_2)A - \gamma_H \gamma_L \frac{p_2}{p_1}}{\frac{p_2}{p_1} - (1+i_1)(1+i_2)}$$

The optimal demands for bonds and capital are as follows:

$$b_1 = \frac{1+i_1}{1+i_2}\Phi_1\zeta_1\Big(w_1 + \frac{\tilde{w}_2}{1+i_1}\Big) - \tilde{w}_2 \tag{3.76}$$

$$k_2 = \frac{1+i_1}{p_2}\Phi_1\Big(w_1 + \frac{\tilde{w}_2}{1+i_1}\Big) \tag{3.77}$$

and

$$\Phi_1 = \frac{2(\zeta_1 + A)}{\frac{(\zeta_1+\gamma_H)(\zeta_1+\gamma_L)}{1+i_2} + 2(\frac{1+i_1}{p_2/p_1} + \frac{\zeta_1}{1+i_2})(\zeta_1 + A)}$$

The optimal V_1 is as follows

$$\begin{aligned}
&V_1 = B + \log\Big(1 - \frac{1+i_1}{p_2/p_1}\Phi_1 - \frac{\Phi\zeta_1}{1+i_2}\Big) + \\
&2\mu \log\Big(\frac{1+i_1}{1+i_2}\Phi_1(\zeta_1+\gamma_H)\Big) + \\
&2(1-\mu)\log\Big(\frac{1+i_1}{1+i_2}\Phi_1(\zeta_1+\gamma_L)\Big) + \\
&3\log\Big(w_1 + \frac{\tilde{w}_2}{1+i_1}\Big) - \log(p_1)
\end{aligned}$$

3.4.1 Equilibrium

The equilibrium conditions are as follows

$$x_1 + k_2 = 1 \tag{3.78}$$

$$\mu x_{2,H} + (1-\mu)x_{2,L} + K_3 = K_2 \tag{3.79}$$

$$\begin{aligned}
&\mu(\mu x_{3,H,H} + (1-\mu)x_{3,H,L}) + \\
&(1-\mu)(\mu x_{3,L,H} + (1-\mu)x_{3,L,L}) = K_3
\end{aligned} \tag{3.80}$$

$$h_1 = \frac{i_1}{1+i_1}p_1,\ h_2 = \frac{i_2}{1+i_2}p_2K_2,\ h_3 = \frac{i_3}{1+i_3}p_3K_3$$

$$K_2 = \mu\gamma_H k_2 + (1-\mu)\gamma_L k_2 = k_2,\ K_3 = \mu k_{3,H} + (1-\mu)k_{3,L}$$

We will use only (78),(79) because (80) clears as a residual. Equilibrium conditions (78),(79) simplify as follows

$$b_1 = -p_1 \tag{3.81}$$

$$\frac{i_2}{1+i_2}K_2 - \frac{1}{p_2(1+i_2)}(\mu b_{2,H} + (1-\mu)b_{2,L}) = K_2 \tag{3.82}$$

Optimality conditions (71),(72),(76),(77) respectively, can be written as follows

$$k_{3,s} = \frac{1+i_2}{p_3}\Big[\frac{i_2}{1+i_2}p_2K_2 + \frac{i_3}{(1+i_2)(1+i_3)}p_3K_3 + \frac{p_2K_2}{1+i_2}\gamma_s\Big]\Phi_2 \tag{3.83}$$

$$b_{2,s} = \frac{1+i_2}{1+i_3}\Phi_2\zeta_2\Big[\frac{i_2}{1+i_2}p_2K_2 + \frac{i_3}{(1+i_2)(1+i_3)}p_3K_3 + \frac{p_2K_2}{1+i_2}\gamma_s\Big] - \frac{i_3}{1+i_3}p_3K_3 - p_2K_2\gamma_s \tag{3.84}$$

$$K_2 = \frac{1+i_1}{p_2}\Phi_1\Big[\frac{i_1}{1+i_1}p_1 + \frac{p_1}{1+i_1} + \frac{i_2p_2K_2}{(1+i_1)(1+i_2)} + \frac{i_3p_3K_3}{(1+i_1)(1+i_2)(1+i_3)}\Big] \tag{3.85}$$

$$\frac{i_2}{1+i_2}p_2K_2 + \frac{i_3}{(1+i_2)(1+i_3)}p_3K_3 = \frac{1+i_1}{1+i_2}\Phi_1\zeta_1\Big[\frac{i_1}{1+i_1}p_1 + \frac{p_1}{1+i_1} + \frac{i_2p_2K_2}{(1+i_1)(1+i_1)} + \frac{i_3p_3K_3}{(1+i_1)(1+i_2)(1+i_3)}\Big] \tag{3.86}$$

Aggregating (83),(84) respectively, we get

$$K_3\Big(1 - \frac{i_3}{1+i_3}\Phi_2\Big) = \frac{(1+i_2)K_2\Phi_2}{p_3/p_2} \tag{3.87}$$

$$\mu\frac{b_{2,H}}{p_2} + (1-\mu)\frac{b_{2,L}}{p_2} =$$

$$\frac{1+i_2}{1+i_3}\Big[K_2 + \frac{i_3}{(1+i_2)(1+i_3)}\frac{p_3}{p_2}K_3\Big]\Phi_2\zeta_2 -$$

$$\frac{i_3}{1+i_3}\frac{p_3}{p_2}K_3 - K_2 \tag{3.88}$$

Substitute (91) in (92) and again substitute the resulting relation in equilibrium condition (86) to solve for the equilibrium inflation rate from period two to period three,

$$\frac{p_3}{p_2} = (1+i_2)(1+i_3)\Big(1 + \frac{\sigma_\gamma^2}{\gamma_H\gamma_L + i_3}\Big) \tag{3.89}$$

To compute the inflation rate from period one to period two, divide (85),(86) and use also (87). We get

$$\frac{p_2}{p_1} = (1+i_1)(1+i_2)\Big(1 + \frac{\sigma_\gamma^2}{\Gamma + \gamma_H\gamma_L}\Big) \tag{3.90}$$

where

$$\Gamma = i_2 + \frac{i_3}{1+i_3}\frac{(1+i_2)\Phi_2}{1 - \frac{i_3}{1+i_3}\Phi_2}$$

$$\Phi_2 = \frac{(1+i_3)(\sigma_\gamma^2 + \gamma_H\gamma_L + i_3)}{2\gamma_H\gamma_L(1+i_3) + 2i_3(1+\sigma_\gamma^2) + 2i_3^2}$$

The equilibrium value of K_2 is as follows

$$K_2 = \frac{2(\Gamma + \gamma_H\gamma_L)}{\Gamma^2 + \gamma_H\gamma_L\Gamma + 3\gamma_H\gamma_L + \sigma_\gamma^2\Gamma + 3\Gamma} \tag{3.91}$$

and the equilibrium value of K_3 is computed from (92).

3.4.2 Optimal monetary policy

Let us start with the simplest case of uniform interest rate policies across periods. The monetary authority sets the same value for the nominal rate in every period[2].

Consider the following proposition

Proposition 11. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rate at positive values, $i > 0$, in order to maximize aggregate welfare.*

Proof. Consider again the derivative of V_1 with respect to i, evaluated at $\bar{i} > 0$. This expression is rather complicated. Consider the limit from above as $\bar{i} \to 0$,

$$\frac{dV_1}{di} = \frac{\sigma_\gamma^2 + 4\gamma_H\gamma_L}{\gamma_H^2\gamma_L^2}\sigma_\gamma^2$$

□

Consider the more interesting situation where monetary policy is different across periods. The following proposition argues that optimal monetary policy should set at least one of $\{i_2, i_3\}$ at positive values,

Proposition 12. *Given $\sigma_\gamma^2 > 0$, optimal monetary policy should set the nominal rates, $\{i_2, i_3\}$, at positive values in order to maximize aggregate welfare.*

Proof. Consider the total derivative of V_1,

$$dV_1 = \frac{\partial V_1}{\partial i_2} di_2 + \frac{\partial V_1}{\partial i_3} di_3$$

where the all derivatives are evaluated at $\bar{i}_2, \bar{i}_3 > 0$. Consider the case where $\bar{i}_2, \bar{i}_3 \to 0$,

[2] A useful simplification to consider.

$$\frac{\partial V_1}{\partial i_2} = \frac{2\sigma_\gamma^2}{\gamma_H \gamma_L}, \; \frac{\partial V_1}{\partial i_3} = \frac{\sigma_\gamma^2 + 2\gamma_H\gamma_L}{\gamma_H^2\gamma_L^2}\sigma_\gamma^2$$

The total derivative of V_1 becomes

$$dV_1 = \frac{2\sigma_\gamma^2}{\gamma_H\gamma_L} di_2 + \frac{\sigma_\gamma^2 + 2\gamma_H\gamma_L}{\gamma_H^2\gamma_L^2}\sigma_\gamma^2 di_3 \tag{3.92}$$

□

The previous propositions established the fact that the monetary-authority should have an active role in the presence of entrepreneurial risk. In the analysis that follows we will characterize the behavior of optimal monetary policy intertemporally.

3.4.2.1 Characterization of optimal monetary policy

In this section we want to characterize optimal policy in a global fashion. The following statement describes optimal policy in that case

Optimal monetary policy rule I. *Given $\sigma_\gamma^2 > 0$, the monetary-authority should pursue the following policy in order to maximize aggregate welfare: set $i_2^*, i_3^* > 0$ and $i_3^* > i_2^*$.*

In order to show that the above policy prescription is valid, consider *first* the following *numerical examples* where we compute the optimal policy by *fixing* the prob. distribution and playing with the realization of productivity shocks.

Consider the cases where $\mu = 0.5, \mu = 0.4, \mu = 0.6$ respectively,

(γ_H, γ_L)	(i_2^*, i_3^*)	(γ_H, γ_L)	(i_2^*, i_3^*)	(γ_H, γ_L)	(i_2^*, i_3^*)
$(1.3, 0.7)$	(0.12,0.1)	$(1.45, 0.7)$	(0.12,0.2)	$(1.2, 0.7)$	(0,0.1)
$(1.4, 0.6)$	(0.12,0.3)	$(1.6, 0.6)$	(0.16,0.3)	$(1.2667, 0.6)$	(0.12,0.2)
$(1.5, 0.5)$	(0.2,0.35)	$(1.75, 0.5)$	(0.21,0.4)	$(1.33, 0.5)$	(0.15,0.3)
$(1.6, 0.4)$	(0.25,0.4)	$(1.9, 0.4)$	(0.23,0.5)	$(1.4, 0.4)$	(0.2,0.3)
$(1.7, 0.3)$	(0.25,0.5)	$(2.05, 0.3)$	(0.27,0.6)	$(1.466, 0.3)$	(0.24,0.4)
$(1.8, 0.2)$	(0.28,0.6)	$(2.2, 0.2)$	(0.31,0.7)	$(1.533, 0.2)$	(0.27,0.5)
$(1.9, 0.1)$	(0.35,0.7)	$(2.35, 0.1)$	(0.36,0.8)	$(1.6, 0.1)$	(0.3,0.7)

Consider also the cases where $\mu = 0.3, \mu = 0.2, \mu = 0.1$ respectively,

(γ_H, γ_L)	(i_2^*, i_3^*)	(γ_H, γ_L)	(i_2^*, i_3^*)	(γ_H, γ_L)	(i_2^*, i_3^*)
$(1.7, 0.7)$	(0.15,0.2)	$(2.2, 0.7)$	(0.16,0.3)	$(3.7, 0.7)$	(0.21,0.3)
$(1.9333, 0.6)$	(0.2,0.3)	$(2.6, 0.6)$	(0.25,0.3)	$(4.6, 0.6)$	(0.24,0.4)
$(2.1666, 0.5)$	(0.24,0.4)	$(3, 0.5)$	(0.24,0.5)	$(5.5, 0.5)$	(0.25,0.5)
$(2.4, 0.4)$	(0.27,0.5)	$(3.4, 0.4)$	(0.3,0.5)	$(6.4, 0.4)$	(0.27,0.5)
$(2.6333, 0.3)$	(0.29,0.6)	$(3.8, 0.3)$	(0.3,0.6)	$(7.3, 0.3)$	(0.21,0.6)
$(2.8666, 0.2)$	(0.31,0.7)	$(4.2, 0.2)$	(0.3,0.8)	$(8.2, 0.2)$	(0.21,0.7)
$(3.1, 0.1)$	(0.4,1)	$(4.6, 0.1)$	(0.4,1.1)	$(9.1, 0.1)$	(0.27,1)

The above numerical examples confirm, roughly speaking, the previous discussion about optimal policy rules.

3.4.2.2 Discussion

Let us understand the intuition of the previous rule. We saw from the previous numerical analysis that the optimal nominal rate at period three should be higher than that of period two. The reason is that at period three there are 4 different types of entrepreneurs. There are entrepreneurs that had received a bad shock in period two and three. If heterogeneity is sufficiently high, the monetary authority should provide enough insurance to these entrepreneurs at period three. This means that i_3 should be sufficiently high in order for seignorage transfers in period three to be sufficiently high. The monetary-authority effectively taxes rich entrepreneurs by increasing the inflation rate from period two to period three in order to provide insurance to poor entrepreneurs. Given this effect, setting i_2 sufficiently higher than i_3 means that the monetary-authority taxes rich entrepreneurs suboptimally high because it increases even more the inflation rate from period two to period three and makes money balances expensive to hold at period two. The optimal policy is to use i_3 to provide enough insurance at period three to poor entrepreneurs and keeping the cost of borrowing money balances at period two at relatively lower levels.

3.4.3 Shocks to labor and capital productivity

Let us go back to section 2.11. Suppose that idiosyncratic shocks affect the productivity of capital and that of labor. Assume for simplicity that the same shock hits both capital and labor productivity. The three-period argument is analogous to the previous one.

In period three, the value function is as follows

$$V_{3,s,s'} = \log\left[\widetilde{v}_{3,s,s'} + b_{2,s} + \frac{q_3 k_{3,s}}{1+i_3}\gamma_{s'}\right] - \log(p_3)$$

where

$$\widetilde{v}_{3,s,s'} = h_3 + w_2\gamma_s + \frac{w_3}{1+i_3}\gamma_{s'} + q_2 k_2 \gamma_s$$

In period two, the problem is as follows

$$\max_{b_{2,s},k_{3,s}} \left\{ \log\left(v_2 - \frac{b_{2,s}}{1+i_2} - p_2 k_{3,s}\right) - \log(p_2) + \right.$$
$$\left. \mu V_{3,s,H} + (1-\mu)V_{3,s,L}\right\}$$

where

$$v_2 = h_2 + b_1 + p_1$$

From the FOC we get

$$\begin{aligned}
b_{2,s} + v_{3,s} &= \zeta_2 \left(\frac{w_3}{1+i_3} + \frac{q_3 k_{3,s}}{1+i_3} \right) \\
q_3 k_{3,s} &= (1+i_2)\Phi_2 \Big(v_2 + \frac{v_{3,s}}{1+i_2} \Big) - \frac{w_3}{1+i_3} Z_2 \\
b_{2,s} &= \zeta_2 \Phi_2 \frac{1+i_2}{1+i_3} \left(\frac{w_3}{q_3/p_2} + v_2 + \frac{v_{3,s}}{1+i_2} \right) - v_{3,s}
\end{aligned}$$

where

$$\begin{aligned}
v_{3,s} &= h_3 + w_2 \gamma_s + q_2 k_2 \gamma_s \\
\zeta_2 &= \frac{(1+i_2)(1+i_3)A - \gamma_H \gamma_L \frac{q_3}{p_2}}{\frac{q_3}{p_2} - (1+i_2)(1+i_3)} \\
\Phi_2 &= \frac{\zeta_2 + A}{\frac{(\zeta_2+\gamma_H)(\zeta_2+\gamma_L)}{1+i_3} + (\frac{1+i_2}{q_3/p_2} + \frac{\zeta_2}{1+i_3})(\zeta_2 + A)} \\
Z_2 &= \frac{(\zeta_2+\gamma_H)(\zeta_2+\gamma_L) + \zeta_2(\zeta + A)}{\frac{(\zeta_2+\gamma_H)(\zeta_2+\gamma_L)}{1+i_3} + (\frac{1+i_2}{q_3/p_2} + \frac{\zeta_2}{1+i_3})(\zeta_2 + A)}
\end{aligned}$$

The value function becomes

$$\begin{aligned}
V_{2,s} = \Bigg\{ &\log \Big(1 - \frac{\zeta_2 \Phi_2}{1+i_3} - \frac{(1+i_2)\Phi_2}{q_3/p_2} \Big) - \log(p_2) - \log(p_3) + \\
&\mu \log \Big(\frac{(1+i_2)}{1+i_3} \Phi_2 (\zeta_2 + \gamma_H) \Big) + \\
&(1-\mu) \log \Big(\frac{(1+i_2)}{1+i_3} \Phi_2 (\zeta_2 + \gamma_L) \Big) \Bigg\} + \\
&2 \log \Big(v_2 + \frac{v_{3,s}}{1+i_2} + \frac{w_3}{q_3/p_2} \Big)
\end{aligned}$$

Equivalently, it can be written as

$$V_{2,s} = B + 2\log\Big(\widetilde{v}_{2,s} + b_1 + \frac{q_2 k_2}{1+i_2}\gamma_s\Big)$$

where

$$\widetilde{v}_{2,s} = h_2 + p_1 + \frac{h_3}{1+i_2} + \frac{w_2}{1+i_2}\gamma_s + \frac{w_3}{q_3/p_2}$$

In period one the problem is as follows

$$\max_{b_1,k_2}\Big\{\log\Big(h_1 - \frac{b_1}{1+i_1} - p_1 k_2\Big) - \log(p_1) +$$
$$\mu V_{2,H} + (1-\mu)V_{2,L}\Big\}$$

From the FOC we get

$$b_1 + \widetilde{v}_2 = \zeta_1\Big(\frac{q_2 k_2}{1+i_2} + \frac{w_2}{1+i_2}\Big)$$
$$q_2 k_2 = (1+i_1)\Phi_1\Big(h_1 + \frac{\widetilde{v}_2}{1+i_1}\Big) - \frac{w_2}{1+i_2}Z_1$$
$$b_1 = \zeta_1\Phi_1\frac{1+i_1}{1+i_2}\Big(\frac{w_2}{q_2/p_1} + h_1 + \frac{\widetilde{v}_2}{1+i_1}\Big) - \widetilde{v}_2$$

where

$$\widetilde{v}_2 = h_2 + p_1 + \frac{h_3}{1+i_2} + \frac{w_3}{q_3/p_2}$$
$$\zeta_1 = \frac{(1+i_1)(1+i_2)A - \gamma_H\gamma_L\frac{q_2}{p_1}}{\frac{q_2}{p_1} - (1+i_1)(1+i_2)}$$
$$\Phi_1 = \frac{2(\zeta_1 + A)}{\frac{(\zeta_1+\gamma_H)(\zeta_1+\gamma_L)}{1+i_2} + 2(\frac{1+i_1}{q_2/p_1} + \frac{\zeta_1}{1+i_2})(\zeta_1 + A)}$$
$$Z_1 = \frac{(\zeta_1+\gamma_H)(\zeta_1+\gamma_L) + 2\zeta_1(\zeta_1 + A)}{\frac{(\zeta_1+\gamma_H)(\zeta_1+\gamma_L)}{1+i_1} + 2(\frac{1+i_1}{q_2/p_1} + \frac{\zeta_1}{1+i_2})(\zeta_1 + A)}$$

The optimal value function in period one is as follows

$$V_1 = \Bigg\{ B - \log(p_1) + \log\left(1 - \frac{(1+i_1)\Phi_1}{q_2/p_1} - \frac{\zeta_1\Phi_1}{1+i_2}\right) +$$
$$2\mu \log\left(\Phi_1 \frac{1+i_1}{1+i_2}(\zeta_1 + \gamma_H)\right) +$$
$$2(1-\mu)\log\left(\Phi_1 \frac{1+i_1}{1+i_2}(\zeta_1 + \gamma_L)\right)\Bigg\} +$$
$$3\log\left(h_1 + \frac{\widetilde{v}_2}{1+i_1} + \frac{w_2}{q_2/p_1}\right)$$

3.4.3.1 Equilibrium

The equilibrium return from investing in capital from period two to period three is

$$\frac{q_3}{p_2} = (1+i_2)(1+i_3)\left(1 + \frac{\sigma_\gamma^2}{i_3 + \gamma_H\gamma_L}\right)$$

and the equilibrium return from investing in capital from period one to period two is

$$\frac{q_2}{p_1} = (1+i_1)(1+i_2)\left(1 + \frac{\sigma_\gamma^2}{\Gamma + \gamma_H\gamma_L}\right)$$

where

$$\Gamma = i_2 + \frac{\Phi_2^*(1+i_2)}{\Delta}\left[\frac{2i_3}{1+i_3} + \frac{i_3 + \gamma_H\gamma_L}{(1+i_3)(i_3 + \gamma_H\gamma_L + \sigma_\gamma^2)}\right]$$
$$\Delta = 2 - \Phi_2^*\left[\frac{2i_3}{1+i_3} + \frac{i_3 + \gamma_H\gamma_L}{(1+i_3)(i_3 + \gamma_H\gamma_L + \sigma_\gamma^2)}\right]$$

and Φ_2^* is the equilibrium value of Φ_2.

3.4.3.2 Characterization of optimal monetary policy

Consider the following optimal policy rule where we exclude the case of extreme heterogeneity, $\gamma_L = 0, \gamma_H = \frac{1}{\mu}$,

Optimal monetary policy rule II. *Given $\sigma_\gamma^2 > 0$, the monetary-authority should pursue the following policy in order to maximize aggregate welfare: set $i_2^* = 0$ and $i_3^* \geq 0$. The equality with zero or strict inequality for i_3^* depends on the volatility of risk.*

In order to show that the above policy prescription is valid, consider *first* the following *numerical examples* where we compute the optimal policy by *fixing* the prob. distribution and playing with the realization of productivity shocks.

Consider the cases where $\mu = 0.5, \mu = 0.4, \mu = 0.3$ respectively,

(γ_H, γ_L)	(i_2^*, i_3^*)
$(1.3, 0.7)$	.
$(1.4, 0.6)$	.
$(1.5, 0.5)$	.
$(1.6, 0.4)$	(0,0)
$(1.7, 0.3)$	(0,0.1)
$(1.8, 0.2)$	(0,0.2)
$(1.9, 0.1)$	(0,0.5)

(γ_H, γ_L)	(i_2^*, i_3^*)
$(1.45, 0.7)$	.
$(1.6, 0.6)$	.
$(1.75, 0.5)$	.
$(1.9, 0.4)$	(0,0)
$(2.05, 0.3)$	(0,0.1)
$(2.2, 0.2)$	(0,0.3)
$(2.35, 0.1)$	(0,0.6)

(γ_H, γ_L)	(i_2^*, i_3^*)
$(1.7, 0.7)$	.
$(1.9333, 0.6)$	.
$(2.1666, 0.5)$	(0,0)
$(2.4, 0.4)$	(0,0.1)
$(2.6333, 0.3)$	(0,0.2)
$(2.8666, 0.2)$	(0,0.4)
$(3.1, 0.1)$	(0.05,0.7)

Consider also the cases where $\mu = 0.2, \mu = 0.1$, respectively,

(γ_H, γ_L)	(i_2^*, i_3^*)
$(2.2, 0.7)$	.
$(2.6, 0.6)$	.
$(3, 0.5)$	(0,0)
$(3.4, 0.4)$	(0,0.1)
$(3.8, 0.3)$	(0,0.2)
$(4.2, 0.2)$	(0,0.4)
$(4.6, 0.1)$	(0.07,0.8)

(γ_H, γ_L)	(i_2^*, i_3^*)
$(3.7, 0.7)$	.
$(4.6, 0.6)$	.
$(5.5, 0.5)$	(0,0)
$(6.4, 0.4)$	(0,0.1)
$(7.3, 0.3)$	(0,0.2)
$(8.2, 0.2)$	(0,0.3)
$(9.1, 0.1)$	(0,0.5)

§ 3.5 Conclusions

In this paper we argued that idiosyncratic risk and credit frictions implied from the cash-in-advance structure imply that optimal monetary policy is characterized by positive nominal rates and positive seignorage. Credit frictions in the asset market create an insurance problem under idiosyncratic risk. As a consequence seignorage can provide insurance to individuals that have received a bad shock. Positive seigniorage transfers require positive nominal rates. Moving to money-supply rules the picture changes abruptly because we get a multiplicity of equilibria. Under the previous construction, optimal monetary policy, optimal money growth rate, can achieve a second-best solution but the same optimal rule can give also an allocation that is very far from the second-best. Lastly, in a three period economy optimal policy calls for higher nominal rates in period three than period two.

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