Scaling properties of one-dimensional cluster-cluster aggregation with Lévy diffusion

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We present a study of the scaling properties of cluster-cluster aggregation with a source of monomers in the stationary state when the spatial transport of particles occurs by Lévy flights. We show that the transition from mean-field statistics to fluctuation-dominated statistics which, for the more commonly considered case of diffusive transport, occurs as the spatial dimension of the system is tuned through two from above, can be mimicked even in one dimension by varying the characteristic exponent, β, of the the Lévy jump length distribution. We also show that the two-point mass correlation function, responsible for the flux of mass in the stationary state, is strongly universal: its scaling exponent is given by the mean field value independent of the spatial dimension and independent of the value of β. Finally we study numerically the two point spatial correlation function which characterises the structure of the depletion zone around heavy particles in the diffusion limited regime. We find that this correlation function vanishes with a non-trivial fractional power of the separation between particles as this separation goes to zero. We provide a scaling argument for the value of this exponent which is in reasonable agreement with the numerical measurements.

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I. INTRODUCTION TO CLUSTER-CLUSTER AGGREGATION (CCA)

Cluster-cluster aggregation (CCA) refers to the statistical dynamics of a large collection of particles (or clusters) which move around under the action of some transport process and which, in addition, have a probability to coalesce irreversibly when they collide. The transport is usually stochastic, diffusion being a commonly studied case, and coalescence of clusters conserves mass. In general, both the rate of coalescence and the transport coefficient are functions of cluster mass. Such dynamics are relevant to many physical phenomena. Commonly cited applications include the study of aerosols [1], the development of structure in the expanding universe [2], clustering of algae [3] and cloud droplet coalescence [4]. From a theoretical perspective, CCA can be modelled using relatively simple interacting particle systems and can thus provide a testing ground for statistical mechanics theories of complex systems. The theoretical objective is to understand the statistical properties of the mass distribution of the clusters.

CCA is an intrinsically non-equilibrium process. Aggregation is irreversible so there is no possibility for the statistical dynamics to have detailed balance. In some circumstances it may be appropriate to consider fragmentation of clusters in addition to aggregation. In such cases, detailed balance may be achievable. We do not consider such situations here however. Two different scenarios may be studied which we refer to as the decay and forced cases. In the decay case, the more common of the two, the system is initially populated with a large number of small particles (monomers) which are considered to be the fundamental unit for the aggregation process and whose mass can accordingly be taken to be one. The size distribution subsequently spreads in the space of cluster masses as larger clusters are generated by the aggregation of smaller ones. This time evolution proceeds for all time (we do not consider models exhibiting a gelation transition here) often in a self-similar fashion. In the forced case, the system initially has no clusters but we continually deposit monomers into the system at a constant rate from an external source. The size distribution again spreads in the mass space but, due to the constant replenishment of the monomers by the external source, eventually reaches a stationary state at any given cluster size. In this stationary state, the depletion of clusters of a given size by aggregation events is exactly balanced by the formation of such clusters by the aggregation of smaller ones. This results in a constant flux of mass in the mass space. An analogy may be drawn with the flux of energy through scales in the Kolmogorov cascade of hydrodynamic turbulence [5]. We shall be concerned primarily with the stationary forced case.

The most basic statistical measure is the average mass density, $N(m,t)$, giving the average density of clusters of mass m at time t. At the mean-field level, $N(m,t)$ satisfies the Smoluchowski equation [6]. For many applications, the collision kernel in the Smoluchowski equation, which reflects the mass dependence of the underlying microscopic coalescence rates, is a homogeneous function

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whose degree we shall denote by ζ. For a broad class of such kernels, the Smoluchowski equation has self-similar solutions which describe scaling behaviour of $N(m,t)$ in both m and t. In particular, $N(m,t)$ has power law asymptotics for $N(m,t)$ as a function of mass. In the decay case, we often find $N(m,t) \sim m^{-(\zeta+1)}$. In the forced case, the stationary state is given by $N(m) \sim m^{-(\zeta+3)/2}$ [5, 7]. The reviews by by Aldous [8] and Leyvraz [9] provide comprehensive discussions of the Smoluchowski equation and its scaling solutions.

One of the most interesting aspects of CCA is that this mean field description can fail under certain conditions. This results in a transition to a fluctuation dominated regime with completely different scaling properties for the large m asymptotics. The fundamental assumption underpinning mean field theory is the absence of spatial correlations. In particular, the joint distribution of clusters having masses m_1 and m_2 at a given point, $\mathbf{x}$, which is linked to the probability of collision, factorises as the product of the one-point distributions of clusters with masses m_1 and m_2. Likewise for higher order joint distributions. This works well in the so-called reaction limited regime when the density of clusters is sufficiently high that each aggregating cluster has a large number of potential partners to with which to aggregate. In the most commonly considered case, in which the transport process is simple diffusion, this assumption turns out to be inconsistent for large masses if the spatial dimension is less than or equal to two. The reason is that in two dimensions and below random walks are recurrent so that clusters which encounter each other once are highly probable to do so infinitely many times and hence to aggregate. Heavy clusters thereby generate an effective depletion zone around themselves. The dynamics thus becomes diffusion limited and dominated by diffusive fluctuations making the system much more difficult to analyse. The only case for which a systematic study of the diffusion limited regime has been done is the case of constant kernel in which the aggregation rate is independent of mass ($\zeta = 0$). For the forced case in $d < 2$, the diffusion limited stationary mass density can be shown to scale as $N(m) \sim m^{-(2d+2)/(d+2)}$ which is shallower than the corresponding mean-field prediction, $N(m) \sim m^{-3/2}$. Note that the diffusion limited scaling exponent coincides with the reaction limited one when $d = 2$ which reflects the fact that $d = 2$ is the critical dimension for this model. In addition to having a different value for the scaling exponent of $N(m)$, the statistical properties of the diffusion limited regime are also structurally different: higher order correlation functions exhibit multi-scaling in the sense that they do not scale as powers of the corresponding single-point densities as they do in the mean field regime.

In this article we focus in detail on this transition from mean field to fluctuation dominated regime in CCA. The fact that the parameter controlling the transition is the physical dimension of the model is inconvenient for both numerical exploration and physical applicability. We remark that the important ingredient for the application of mean field theory is not the physical dimension per se but rather whether or not the transport mechanism has the ability to break spatial correlations between particles. We show that by replacing diffusive transport with Lévy flights it is possible to continuously tune between fluctuation-dominated behaviour and mean-field behaviour *even in one dimension.* The tuning parameter is the characteristic exponent of the Lévy distribution underlying the jump size distribution for particle hops. The physical intuition is that correlations between particles are broken even in low dimensions if there are sufficiently many long range hops to effectively mix the clusters. We also explore the special role played in the forced case by the second order mass correlation function whose scaling exponent is strongly universal as a result of the so-called Constant Flux Relation (CFR) [10]. Its scaling exponent is, perhaps surprisingly, independent of both the spatial transport mechanism and the physical dimension and is always given by its mean field value.

The outline of the article is as follows. We begin by defining the stochastic model used in studying CCA, as well as a brief overview of the statistical field theory used in deriving the Lee-Cardy equation for this model (Section 2). Then, we summarise the scaling properties of the multi-particle correlation functions or arbitrary order as applied to both reaction-limited and diffusion-limited regimes (Section 3). Using the Constant Flux Relation (CFR) we then derive the scaling invariance of the flux-carrying correlation function, demonstrating that this result is independent of the dimension of the system and of the mechanism of spatial transport (Section 4). Implementing Lévy flights for the one-dimensional model, we then characterise the suggested form of the mass density as a function of the Lévy exponent, which we validate against numerical simulation together with taking direct measurements for the integrated forms of the mass correlation functions (Section 5). Here we verify the scaling independence of the flux-carrying correlation function as we alter the spatial transport mechanism. We finish the paper by determining a predicted form for the structure of the depletion zones as a function of the Lévy exponent (Section 6), which we validate numerically before providing a conclusion.

II. THE LEE-CARDY EQUATION AND STOCHASTIC SMOLUCHOWSKI EQUATION

We now describe a simple stochastic model, first introduced without injection of particles in [11] and with injection of particles in [12], which can be used to study the phenomenon of CCA. Discrete particles, each carrying a mass index, m, occupy a d-dimensional square lattice, $\mathbb{L}$, with no restrictions on the number of particles occupying each lattice site. We take the mass index to be a non-negative integer multiple of a fundamental mass, m_0. This means that all particles are composed of clusters of a single basic monomer unit. The state of the system at time t is characterised by the set of occupation numbers, $N_{\mathbf{x}\,m\,t}$, specifying the number of particles of each mass m at each point, $\mathbf{x}$ of the lattice. The occupation numbers evolve in time according to the following microscopic dynamical transitions which occur in continuous time with exponentially distributed waiting times:

- Diffusion: A particle at any site may hop to an adjacent site. These hops occur at rate $\widetilde{D}$, taken to be independent of particle mass.
- Injection of monomers: A new monomer may appear at any site on $\mathbb{L}$. The injection of monomers takes place at rate $\widetilde{J}$.
- Aggregation: Two particles having masses m_1 and m_2 which are at the same lattice point, $\mathbf{x}$, may aggregate to create a single new particle having mass $m_1 + m_2$. This occurs at rate $\widetilde{\lambda} N_{\mathbf{x}\,m_1\,t}(N_{\mathbf{x}\,m_2\,t} - \delta_{m_1\,m_2})$ where $\widetilde{\lambda}$ is the reaction rate, taken to be independent of mass, and the $N_{\mathbf{x}\,m\,t}$ factors encode the pairwise nature of the aggregation process (the Kronecker symbol, $\delta_{m_1\,m_2}$, ensures that the rate is correct when two particles of the same mass aggregate).

The "rates" in these rules refer to the corresponding waiting times. These rules define a Markov chain which can be easily simulated using the Gillespie algorithm [13]. All numerical results presented in this paper have been obtained in this fashion.

This model has a Master Equation whose solution can be used to compute averages with respect to the microscopic dynamics. Such averages, which we shall denote by $\langle . \rangle_{\mathbb{L}}$, are equivalent to averages with respect to the effective action of an associated statistical field theory. This effective action is obtained from the dynamical rules by the well known Doi-Peliti construction. For details of the procedure for this specific model see [14]. For many microscopic models describing interacting particle systems, averages with respect to this effective action are further equivalent to averages with respect to solutions of a stochastic rate equation [15, 16] known as the Lee-Cardy equation. The Lee-Cardy equation specific to our CCA model, first derived in [17], takes the form of a stochastic Smoluchowski equation. In the continuous limit it takes the form:

$$\begin{aligned} \partial_t \phi_{\mathbf{x}\,t\,m} &= D\Delta\phi_{\mathbf{x}\,t\,m} + \frac{J}{m_0}\delta(m - m_0) \\ &+ \lambda \int_0^m \phi_{\mathbf{x}\,m'\,t}\,\phi_{\mathbf{x}\,m-m'\,t}\,dm' \\ &- 2\lambda\,\phi_{\mathbf{x}\,m\,t}\int_0^\infty \phi_{\mathbf{x}\,m'\,t}\,dm' \\ &+ i\sqrt{\lambda}\,\phi_{\mathbf{x}\,m\,t}\,\xi_{\mathbf{x}\,t} \end{aligned} \qquad (1)$$

where $\xi_{\mathbf{x}\,t}$ is Gaussian white noise and Δ is the d-dimensional Laplacian. D, λ and J are respectively the diffusion constant, reaction rate and mass injection rate obtained from an appropriate rescaling of the corresponding microscopic rates with the lattice spacing in the continuous limit [14]. We shall denote averages with respect to the solutions of Eq. (1) by $\langle . \rangle_\xi$.

One of the most striking aspects of the Lee-Cardy equation is that the noise term is imaginary indicating that the field $\phi_{\mathbf{x}\,m\,t}$ cannot be literally interpreted as the density of particles, $N_{\mathbf{x}\,m\,t}$, however tempting such an identification may seem. Imaginary noise encodes the fact that the particles in the system are anti-correlated [15, 18]. Physically, this anti-correlation comes about since the process of aggregation tends, on average, to produce a depletion zone around each particle. We shall discuss these depletion zones in detail in Sec. VI.

Careful study of the construction leading to Eq. (1) leads to the conclusion [14] that single point *moments* of $\phi_{\mathbf{x}\,m\,t}$ correspond to *factorial moments* of the physical density, $N_{\mathbf{x}\,m\,t}$:

$$\langle \phi^n_{\mathbf{x}\,m\,t} \rangle_\xi = \frac{1}{n!}\langle N_{\mathbf{x}\,m\,t}(N_{\mathbf{x}\,m\,t} - 1)\dots(N_{\mathbf{x}\,m\,t} - n + 1)\rangle_{\mathbb{L}}. \qquad (2)$$

Multi-point moments translate directly. For example,

$$\langle \phi_{\mathbf{x}\,m_1\,t}\phi_{\mathbf{x}\,m_2\,t}\rangle_\xi = \langle N_{\mathbf{x}\,m_1\,t}N_{\mathbf{x}\,m_2\,t}\rangle_{\mathbb{L}} \text{ if } m_1 \neq m_2. \qquad (3)$$

These relations allow us to translate correlation functions of Eq. (1) which are most suitable for theoretical analysis into corresponding correlation functions in the original lattice model which are most suitable for numerical measurements. In relation to the interpretation of the field $\phi_{\mathbf{x}\,m\,t}$ alluded to above, Eq. (2) tells us that $\phi_{\mathbf{x}\,m\,t}$ is equal to the density, $N_{\mathbf{x}\,m\,t}$, on average but differs in distribution.

If the occupation numbers are large, we can, to leading order, ignore the distinction between moments and factorial moments on the right hand side of Eq. (2). Furthermore, if we could neglect correlations between particles, higher order correlation functions could be factorised into products of densities:

$$\langle N_{\mathbf{x}\,m_1\,t}N_{\mathbf{x}\,m_2\,t}\rangle_{\mathbb{L}} \approx \langle N_{\mathbf{x}\,m_1\,t}\rangle_{\mathbb{L}}\langle N_{\mathbf{x}\,m_2\,t}\rangle_{\mathbb{L}}. \qquad (4)$$

This is the essence of the mean field approximation for CCA. If the density exhibits scaling with some physical quantity such as the mass or the source strength, it follows from the mean field approximation that the higher order correlation functions exhibit simple scaling. That is to say that if the density has a scaling exponent, γ_1, then the corresponding scaling exponent of an order n correlation function is $\gamma_n = n\,\gamma_1$.

We have already mentioned that the interactions between particles resulting from aggregation provides a natural mechanism for generating anti-correlations between particles. The extent to which the approximation (4) is a good one therefore depends on the strength of these anti-correlations. It has been understood that in low dimensions the large time, large mass behaviour is always diffusion limited resulting in anticorrelations between particles which are sufficiently strong to invalidate the mean field approximation. Low dimensions in this case means $d < 2$ with the case $d = 2$ being marginal. One expects to observe a violation of mean field theory accompanied by the onset of multiscaling as one varies the dimension of the lattice $\mathbb{L}$. This is indeed true with strong multiscaling observed in both the decaying [17] and forced [14, 19] cases in $d = 1$ which mellows to logarithmic corrections to the mean-field scaling in $d = 2$. Triggering this transition from mean-field to non-mean-field behaviour by varying the spatial dimension is not very easy in practice. In Sec. V of this article we show that introducing long-range diffusion can break the anti-correlations between particles resulting in behaviour which is very reminiscent of mean-field behaviour *even in one dimension*. Before this however we will review the scaling properties of CCA with regular diffusion.

III. SCALING PROPERTIES OF CCA WITH A SOURCE OF MONOMERS: REACTION LIMITED AND DIFFUSION LIMITED REGIMES

A constant injection of monomers into the system allows for the creation of a non-equilibrium stationary state, which is characterised by a constant flux of mass from smaller to larger masses under aggregation. However, since there is no upper mass limit defined on the system, the creation of arbitrarily large clusters can go on indefinitely. In this sense, the stationary state is only defined for some inertial range of masses. Any stationary solution to the Stochastic Smoluchowski Equation describing the mass density must therefore be of Kolmogorov type [5]. That is, these solutions exhibit scaling evocative of the Kolmogorov spectrum found in forced hydrodynamic turbulence, with a cascade defined by a constant flux of mass from smaller to larger clusters mediated by the aggregation of particles. From this point on, we shall assume statistical homogeneity and drop the $\mathbf{x}$ index.

The mass density has dimension $[N_m] = L^{-d} M^{-1}$. The dimensions of the physical parameters appearing in Eq. (1) are as follows $[D] = L^2 T^{-1}$, $[\lambda] = L^d T^{-1}$ and $[J] = M L^{-d} T^{-1}$. In order to describe a cascade, the stationary density should depend on the mass flux, J. There are two natural ways to construct an expression with the correct dimension using only one additional parameter in addition to the flux:

$$N_m \propto \sqrt{\frac{J}{\lambda}}\, m^{-\frac{3}{2}}, \tag{5}$$

$$N_m \propto \left(\frac{J}{D}\right)^{\frac{d}{d+2}} m^{-\frac{2d+2}{d+2}}. \tag{6}$$

The first corresponds to the reaction-limited regime in which diffusion plays no role. The second corresponds to the diffusion-limited regime in which the reaction rate plays no role since reactions are effectively instantaneous compared to the time required for particles to find each other by diffusion.

If we average Eq. (1) in the statistically homogeneous case and make the mean field approximation, Eq. (4), then we obtain the standard Smoluchowski coagulation equation:

$$\begin{aligned}\partial_t N_m(t) \;=\; & \lambda \int_0^\infty dm_1 dm_2 N_{m_1} N_{m_2} \delta(m - m_1 - m_2) \\ & - \lambda \int_0^\infty dm_1 dm_2 N_m N_{m_1} \delta(m_2 - m - m_1) \\ & - \lambda \int_0^\infty dm_1 dm_2 N_m N_{m_2} \delta(m_1 - m_2 - m) \\ & + (J/m_0)\, \delta(m - m_0).\end{aligned} \tag{7}$$

The exact stationary solution of this equation is [5, 7]

$$N_m = \sqrt{\frac{J}{2\pi\lambda}}\, m^{-\frac{3}{2}}, \tag{8}$$

confirming that the reaction-limited scaling, Eq. (5), corresponds to the mean field scaling. In one dimension, exact calculations [12, 20] showed that for the case of instantaneous reactions, $\lambda \to \infty$,

$$N_m \sim \Gamma(1/3)^{-1} \left(\frac{4J}{9D}\right)^{\frac{1}{3}} m^{-\frac{4}{3}}, \tag{9}$$

for $m \gg 1$, which agrees with the diffusion-limited scaling, Eq. (6), with $d = 1$. Furthermore the dynamical scaling properties of the time-relaxation to these steady states in both the mean-field and one-dimensional cases were computed exactly in [21]. It was subsequently shown in [22] that the diffusion-limited scaling, Eq. (6), is actually exact for all $d < 2$. The fact that the exponents in Eqs. (5) and (6) become equal when $d = 2$ reflects that the critical dimension for this system is 2.

The presence of multiscaling in $d < 2$ can be detected by measuring the higher order mass correlation functions:

$$c_n(m_1, \ldots, m_n, t) = \langle \phi_{\mathbf{x}\, m_1\, t} \cdots \phi_{\mathbf{x}\, m_n\, t} \rangle_\xi. \tag{10}$$

The case $n = 1$ is just the mass density as discussed above. In the statistically homogeneous case, these correlation functions do not depend on the spatial point, $\mathbf{x}$, at which they are measured. For aggregation with source, $c_n(m_1, \ldots, m_n, t)$ becomes independent of t as $t \to \infty$ and is a homogeneous function of mass for large masses:

$$c_n(h\, m_1, \ldots, h\, m_n) \sim h^{\gamma_n} c_n(m_1, \ldots, m_n). \tag{11}$$

In [19], the exponents γ_n were calculated perturbatively in $\epsilon = 2 - d$ using dynamical renormalisation group:

$$\gamma_n = \left(\frac{2d+2}{d+2}\right) n + \frac{n\,(n-1)}{2\,(d+2)}\epsilon + O(\epsilon^2), \tag{12}$$

and found to agree remarkably well with Monte Carlo simulations in $d = 1$ despite the fact that, formally, $\epsilon = 1$ in this case. The epsilon expansion written in this way, with d appearing explicitly, demonstrates that the effect of fluctuations is to correct the *simple* scaling $\gamma_n = n\,\gamma_1$ (recall that γ_1 is given exactly by $\gamma_1 = \frac{2d+2}{d+2}$ [22]), thereby demonstrating the presence of multi-scaling in the model. Full details are given in [14]. Direct measurement of the exponents γ_n from numerics is quite difficult for $n > 1$ due to the strong fluctuations in the diffusion-limited regime. In practice, for $n > 1$, better statistics are obtained by measuring the integrated correlation functions,

$$C_n(m) = \int_m^\infty dm_1 \ldots \int_m^\infty dm_{n-1}\, c_n(m, m_1 \ldots, m_{n-1}). \tag{13}$$

$C_n(m)$ can be expressed in terms of site occupation numbers using Eqs. (2) and (3) and scales with exponent $\gamma_n - n + 1$. Notice from Eq. (12) that $\gamma_2 = 3$, independent of the spatial dimension, to first order in the ϵ-expansion. Furthermore, this is exactly the exponent one would expect from mean-field theory even though Eq. (12) specifically describes the non-mean-field regime. In fact, γ_2 is exactly given by its mean-field value in all dimensions regardless of whether mean-field theory applies or not. This is a specific case of a general property of many non-equilibrium stationary states which carry a flux of some conserved quantity: strong universality of the flux-carrying correlation function [10]. The fact that conservation laws determine the scaling of the flux-carry correlation function exactly is well know in turbulence as Kolmogorov's $\frac{4}{5}$-Law (see [23] for a review and wide-ranging discussions). We now study this important feature in more detail.

IV. CONSTANT FLUX RELATION

We have already remarked that in the forced case, CCA reaches a quasi-stationary state characterised by a constant flux of mass from small to large masses. The Constant Flux Relation (CFR) [10] expresses the fact that this constant flux of mass exactly determines the scaling of the flux-carrying correlation function in the stationary state, a common feature of most driven non-equilibrium systems which reach stationary states dominated by a constant flux of a conserved quantity.

The Lee-Cardy equation, Eq. (1), is equivalent to the underlying effective field theory, a fact which sometimes calls into question the usefulness of the Lee-Cardy formalism. The derivation of the CFR, however, is enormously easier using the Lee-Cardy equation than it would be otherwise. At stationarity, and assuming spatial homogeneity, Eq. (1) can be written

$$\begin{aligned} 0 &= \lambda \int_0^m \langle \phi(m')\phi(m-m')\rangle_\xi dm' \\ &- 2\lambda \int_0^\infty \langle \phi(m)\phi(m')\rangle_\xi dm', \end{aligned} \tag{14}$$

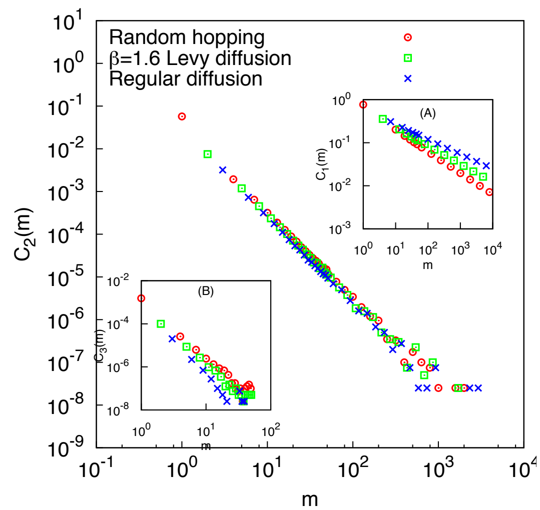

FIG. 1: Scaling universality of the second order integrated mass correlation function, $C_2(m)$ as given by Eq. (13), with mass, m, as shown for three different transport mechanisms. In comparison, scaling for the first and third order integrated mass correlation functions are shown in insets (A) and (B) respectively.

where we have taken averages with respect to the noise term ξ and let $m > m_0$. Let us assume that for large masses, the correlation function $\langle\phi(m_1)\phi(m_2)\rangle_\xi$ takes the form

$$C(m_1, m_2) = (m_1 m_2)^{-x}\psi\left(\frac{m_1}{m_2}\right), \tag{15}$$

where we have introduced the dimensionless scaling function ψ. By the symmetry of C, ψ has the property $\psi(x) = \psi(x^{-1})$. This defines our flux-carrying correlation function. Re-writing Eq. (14) in the symmetric form

$$\begin{aligned} 0 &= \lambda \int_0^\infty \int_0^\infty C(m_1, m_2)\delta(m - m_1 - m_2) dm_1 dm_2 \\ &- \lambda \int_0^\infty \int_0^\infty C(m, m_1)\delta(m_2 - m - m_1) dm_1 dm_2 \\ &- \lambda \int_0^\infty \int_0^\infty C(m, m_2)\delta(m_1 - m_2 - m) dm_1 dm_2, \end{aligned} \tag{16}$$

and applying the Zakharov transformations [24]

$$\begin{aligned} (m_1, m_2) &\longrightarrow \left(\tfrac{m m_1}{m_2}, \tfrac{m^2}{m_2}\right), \\ (m_1, m_2) &\longrightarrow \left(\tfrac{m^2}{m_1}, \tfrac{m m_2}{m_1}\right) \end{aligned} \tag{17}$$

to the second and third integrals of Eq. (16) respectively, results in an expression in which each delta-function has the same argument. This gives the condition

$$\begin{aligned} 0 &= \int_0^\infty \int_0^\infty (m_1 m_2)^{-x}\psi\left(\frac{m_1}{m_2}\right)(m^y - m_2^y - m_1^y) \\ &\qquad \times \delta(m - m_2 - m_1) dm_1 dm_2, \end{aligned} \tag{18}$$

which can only be satisfied for $y := 2x - 2 = 1$, or equivalently, $x = \frac{3}{2}$. Therefore, the flux-carrying correlation function must be of the form

$$C(m_1, m_2) \sim (m_1 m_2)^{-\frac{3}{2}} \psi\left(\frac{m_1}{m_2}\right) \qquad (19)$$

for sufficiently large masses, which is a homogeneous function of degree -3. We have implicitly assumed that the corresponding integrals in Eq. (16) converge for this exponent. This can be demonstrated explicitly for the constant kernel case but, for general kernels must be verified numerically [25]. An immediate corollary of this result applies to the specific case $m_1 = m_2$: using the factorial moments relationship in Eq. (2) we have that

$$\frac{1}{2}\langle N_m(N_m - 1)\rangle_{\mathbb{L}} \sim m^{-3}. \qquad (20)$$

Namely, that the stationary average density of pairs must scale with exponent -3 for sufficiently large m.

In the mean field regime, higher order correlation functions factorise into products of densities as expressed in Eq. (4). It immediately follows from Eq. (8) that $\gamma_2 = 3$ in mean field theory. Thus the scaling exponent, γ_2, is given by its mean field value in all dimensions. We note also that the dependence on the transport process disappears on averaging Eq. (1) so that the exponent γ_2 is strongly universal, as suggested by figure (1). In the next section we consider CCA with Lévy flights. The choice of the characteristic exponent, β, of the Lévy flights strongly affects the scaling of general correlation functions in the model. We shall see, however, that the value of γ_2 is insensitive to the choice of β.

V. ONE-DIMENSIONAL CCA WITH LÉVY DIFFUSION

CCA with Lévy diffusion is of some interest in certain surface growth phenomena [27, 28] but our interest stems from the theoretical point of view that it allows us to mimic the transition to mean-field behaviour even in one dimension. From now on we focus entirely on the one dimensional case.

We modify the model introduced earlier by allowing particles to make long range jumps. The waiting times between particle hops remain exponentially distributed with a fixed rate. Rather than simply hopping to nearest neighbours, however, the length of each jump is independently sampled from a probability distribution with a heavy tail. In our numerical experiments, we took this distribution to be a symmetric Lévy distribution with scale factor 1 and scaling exponent $\beta \in (0, 2]$. Jump lengths were rounded to the nearest integer to remain consistent with our lattice formulation. Such hops are referred to as Lévy *flights* [29] and should not be confused with the (often more physically realistic) case of Lévy *walks* in which the waiting time distribution and

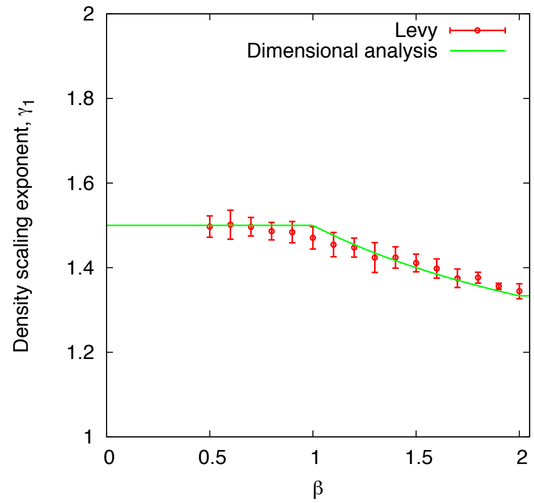

FIG. 2: Comparison of the numerically measured exponent γ_1 against the prediction derived from dimensional analysis, Eq. (23), as plotted against the Lévy characteristic, β . Data was taken from stationary simulations on a lattice of size 10^5. The error bars on the exponents represent two standard deviations as estimated by bootstrapping a least squares estimator with the data measuring the integrated density, $C_1(m)$, as defined by Eq. (13), over 4 decades of m.

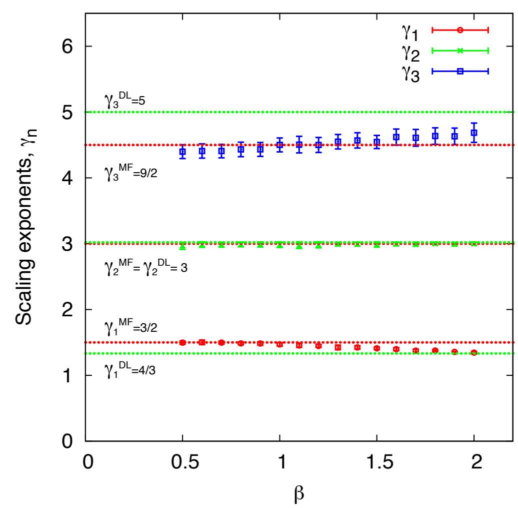

FIG. 3: Scaling exponents for the first three mass correlation functions as functions of Lévy characteristic β. The exponents γ_2 and γ_3 were estimated using a maximum likelihood estimator [26] for a power law distribution of pairs and triplets entering Eq. (13). Theoretically predicted values for both the mean field and diffusion limited cases are included for each exponent. Error bars correspond to two standard deviations calculated by bootstrapping the estimator.

the jump length distribution are not independent to account for the fact that for particles of finite mass long jumps should take more time. The probability distribution, $P(x, t)$ of the position of a particle starting at $x = 0$ at $t = 0$ and exhibiting one dimensional Lévy flights with exponent β satisfies a fractional diffusion equation:

$$\frac{\partial P}{\partial t} = D_\beta \frac{\partial^\beta P}{\partial x^\beta} \tag{21}$$

where the generalised (or anomalous) diffusion coefficient, D_β, has dimension $L^\beta T^{-1}$ and the fractional derivative is defined through its Fourier space representation. See [29] for details. For $\beta = 2$ the Lévy distribution reduces to the Gaussian distribution and we recover the case of regular diffusion. In this case, the mean square displacement of a particle grows linearly in time $\langle x^2(t)\rangle = Dt$ providing a natural way to define the diffusion length, $l_D = \sqrt{D\,t}$. For $\beta < 2$, $\langle x^2(t)\rangle$ is infinite for any finite time and we require a different approach to defining the anomalous diffusion length. One reasonable way to do this uses the fact that Eq. (21) has a self-similar solution describing the evolution of $P(x,t)$ in which the self-similar variable is $z = \frac{|x|}{(D_\beta\, t)^{1/\beta}}$ [29]. It therefore makes sense, as one might expect from dimensional considerations, to define the anomalous diffusion length as $l_\beta = (D_\beta\, t)^{1/\beta}$ despite the fact that the mean square displacement is divergent.

One may repeat the dimensional argument outlined in Sec. III in the diffusion-limited regime taking the anomalous diffusion coefficient, D_β, instead of D and arrive for $d = 1$ at

$$N_m \propto \left(\frac{J}{D_\beta}\right)^{\frac{1}{1+\beta}} m^{-\frac{2+\beta}{1+\beta}}. \tag{22}$$

For $\beta = 2$ we recover the known diffusion limited scaling, Eq. (9). For $\beta = 1$, the scaling becomes that of the mean-field answer, Eq. (8), with D_β replacing the reaction rate, λ. This suggests that at $\beta = 1$, the Lévy flights have become sufficiently long range to break all local anti-correlations between particles. If β is decreased below 1, we do not expect that the scaling properties would change any further since particles are now effectively independent. Thus we have the following dimensional prediction for the scaling of the mass density in the presence of Lévy flights:

$$\gamma_1 = \begin{cases} \frac{3}{2} & \beta < 1 \\ \frac{2+\beta}{1+\beta} & 1 \leq \beta \leq 2 \end{cases} \tag{23}$$

Fig. 2 compares the results of numerical measurements of γ_1 with the predictions of Eq. (23) and indicates fair agreement. If we are correct in interpreting Eq. (23) as describing the breaking of correlations by the long-range hops then it should follow that multiscaling described by Eq. (12) should also be lost as β is tuned from 2 to 1 and simple scaling should be restored for the higher order correlation functions. This is illustrated in Fig. 3 which shows the measured scaling exponents of the first three higher order correlation functions as a function of β. We see that within error, simple scaling is restored in one dimension as $\beta \to 1$. This is not because we force the system to become reaction-limited by increasing the number of particles per site (the average number of particle per site in all simulations shown here was about 0.8 so we remain in the low density regime). Rather the transport process breaks correlations between particles allowing the mean field behaviour to become observable again. It was pointed out to us that a somewhat similar transition in the scaling exponents of CCA as a additional scaling parameter is varied was demonstrated within mean field theory in [7]. There, an additional scaling parameter was introduced by the inclusion of exogeneous growth of clusters on top of the basic model dynamics.

As an aside, we remark that the dimensional argument leading to Eq. (23) suggests $\gamma_1 = \frac{2d+\beta}{d+\beta}$ when d is left arbitrary. Setting $\gamma_1 = \frac{3}{2}$ might lead one to conjecture that the critical dimension for CCA with Lévy diffusion is $d_c = \beta$. While intriguing, this suggestion of a fractional critical dimension lacks the exact expression for γ_1 which allowed a convincing argument to be made for $d_c = 2$ in the case of regular diffusion ($\beta = 2$) [22].

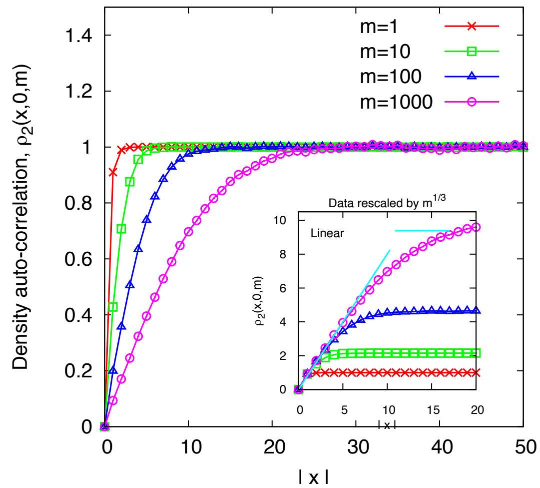

FIG. 4: The structure of the depletion zone as given by Eq. (27) in the case of regular diffusion. These plots have been normalised against the single-point densities corresponding to each choice of mass, m. The inset shows plausible evidence in support of the predicted form given by Eq. (27) after an appropriate rescaling of the data by $m^{1/3}$.

VI. STRUCTURE OF THE DEPLETION ZONE

The breaking of anti-correlations between particles by Lévy flights should also be seen in the structure of the depletion zones surrounding particles. We now turn our attention to characterising this effect in one dimension. To detect the presence of depletion zones around particles in CCA we need to calculate the multipoint correlation functions between densities at different spatial points:

$$\rho_n(\mathbf{x}_1, \ldots \mathbf{x}_n, t) = \langle \phi_{\mathbf{x}_1\, m_1\, t} \ldots \phi_{\mathbf{x}_n\, m_n\, t}\rangle_\xi. \tag{24}$$

A closely related 2-point correlation function was calculated exactly in one dimension for the case of infinite reaction rate in [22] and demonstrated clearly the tendency for clusters to anti-correlate in the diffusion-limited regime. It is natural to normalise such correlation functions with the mass density so that the ratio goes to 1 when particles are uncorrelated. To avoid excessively clumsy notation, we suppress the mass dependence of the spatial correlations for the time being.

If we forget about the mass dependence of the particles for the moment we are left with the coagulation process $A+A \to A$. One can study how the analogous correlation functions in this system behave. For the $A+A \to A$ system without source it was predicted in [30] and later proven rigorously in [31] that in one dimension:

$$\rho_n(\mathbf{x}_1, \dots \mathbf{x}_n, t) \sim \left(\frac{1}{\sqrt{D\,t}}\right)^n \prod_{1\leq i<j\leq n} \frac{|\mathbf{x}_i - \mathbf{x}_j|}{\sqrt{D\,t}}, \qquad (25)$$

where $\sim$ denotes the short-distance asymptotic behaviour. The first term on the right hand side is simply the nth power of the density. The second term accounts for correlations, producing multiscaling of the time-decay exponents for the $A+A \to A$ problem in $d=1$. The fact that $\rho_n(\mathbf{x}_1, \dots \mathbf{x}_n, t)$ vanishes as any two points are brought together reflects the fact that particles are anti-correlated.

Since the mechanism for the generation of anticorrelations in the stationary CCA model is the same as for the $A+A \to A$, one can use Eq. (25) to guess the corresponding formula for the CCA model in the stationary state [32]. In Eq. (25) the diffusion length for decaying coagulation appears as $l_D = \sqrt{D\,t}$. The corresponding diffusion length for particles of mass m in *stationary* CCA is, from dimensional considerations,

$$l_D = \left(\frac{D\,m}{J}\right)^{\frac{1}{3}}. \qquad (26)$$

Upon substitution into Eq. (25) we arrive at the following suggestion for the asymptotic behaviour, now in mass, of the multi-point correlation functions in stationary CCA:

$$\rho_n(\mathbf{x}_1, \dots \mathbf{x}_n, m) \sim \left(\frac{J}{D}\right)^{\frac{n}{3}} m^{-\frac{4}{3}n} \prod_{1\leq i<j\leq n} \frac{|\mathbf{x}_i - \mathbf{x}_j|}{(\frac{D\,m}{J})^{1/3}}. \qquad (27)$$

In the absence of a complete calculation, the plausibility of this argument should be assessed by comparison with numerics. This is done for $n=2$ in Fig. 4. Eq. (27) implies that after rescaling with $m^{1/3}$, $\rho_2(\mathbf{x}_1, \mathbf{x}_2, m)/\rho_1(m)^2$ for different values of m should collapse onto the same curve which vanishes linearly as the separation, $|\mathbf{x}_1 - \mathbf{x}_2|$, tends to 0. Fig. 4 clearly shows that the depletion zone is larger for the heavier particles. The inset provides plausible evidence in support of the expected scaling.

We may now ask what happens in the case of Lévy flights. Fig. VI shows that as β is varied from 2 to 1, the depletion zone disappears. For comparison, the corresponding correlation functions for the limiting cases of completely local hops and completely random hops are shown. The results for Lévy flights lie somewhere in between. Interestingly, the correlation between particles does not seem to vanish linearly with separation in the Lévy case. See, for example, the main panel of Fig. VI.

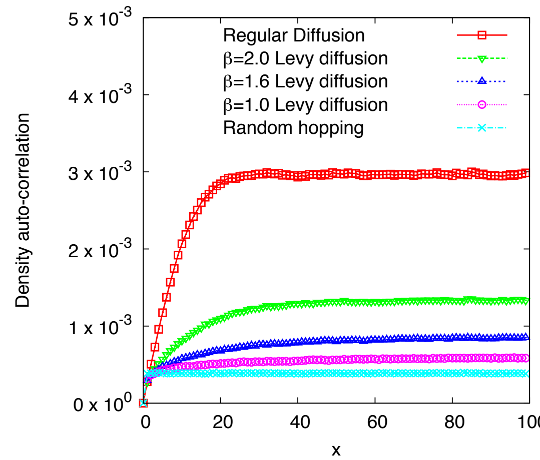

FIG. 5: The depletion zone as shown for different transport mechanisms without normalisation against the single-point densities, for a fixed choice of mass, $m = 1000$. As the Lévy characteristic, β is varied from 2 down to 1, the anticorrelation void disappears, with the limiting cases of random lattice hops and regular diffusion also being shown.

In the case of stationary CCA with Lévy flights, the anomalous diffusion length for particles of mass m in one dimension is, dimensionally,

$$l_{D_\beta} = \left(\frac{D_\beta\,m}{J}\right)^{\frac{1}{1+\beta}}. \qquad (28)$$

The fact that the correlation between particles vanishes linearly in the case of regular diffusion follows from the calculations performed in [30] and is connected to the fact that the spatial transport mechanism is diffusion. Dimensionally, any power of $|\mathbf{x}_i - \mathbf{x}_j|/l_D$ in Eq. (25) or Eq. (27) would be consistent so there is no reason to expect that this power should remain unity in the Lévy case when the transport mechanism is changed:

$$\rho_n(\mathbf{x}_1, \dots \mathbf{x}_n, m) \sim \left(\frac{J}{D}\right)^{\frac{n}{1+\beta}} m^{-\frac{2+\beta}{1+\beta}n} \times \prod_{1\leq i<j\leq n} \left[\frac{|\mathbf{x}_i - \mathbf{x}_j|}{(\frac{D_\beta\,m}{J})^{1/(1+\beta)}}\right]^\alpha. \qquad (29)$$

We put forward a scaling argument to fix the power of α. From the discussion of the Constant Flux Relation in Sec. IV, we know that the second order correlation function at a single spatial point is theoretically proportional to the mass flux, J, in the stationary state regardless of the transport mechanism. This fact is strongly

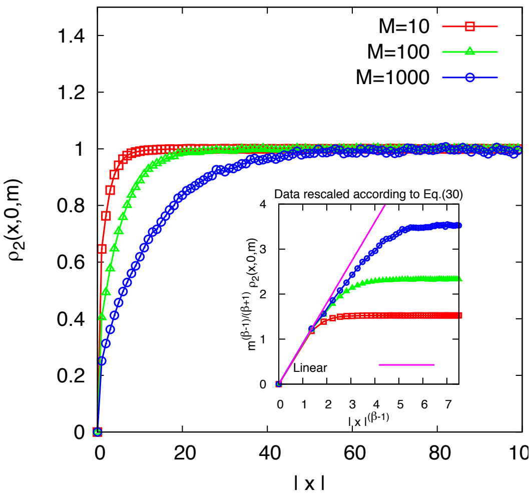

FIG. 6: Shape of the depletion zone as given by the prediction Eq. (30) for the Lévy characteristic $\beta = 3/2$. Each plot corresponds to a different choice of mass, m, and have been normalised against the corresponding single-point densities. The inset provides plausible evidence for this prediction through the data collapse suggested by Eq. (30).

supported for the case of Lévy flights by the numerical measurements presented in Fig. 3. If one accepts that this scaling remains true when the single point correlation function is split onto two nearby points, then one would expect that the power of J in Eq. (29) is unity for $n = 2$ (or equally, the power of m should be -3) which then requires that the exponent α should be equal to $\beta - 1$. This is consistent with Eq. (27) when $\beta = 2$ and consistent with the disappearance of the depletion zone when $\beta = 1$. For $1 < \beta \leq 2$ we therefore propose the following shape for the depletion zone as measured by the two-point function:

$$\rho_2(\mathbf{x}_1, \mathbf{x}_2, m) \sim \left(\frac{J}{D}\right)^{\frac{2}{1+\beta}} m^{-\frac{4+2\beta}{1+\beta}} \frac{|\mathbf{x}_1 - \mathbf{x}_2|^{\beta-1}}{(\frac{D_\beta\, m}{J})^{(\beta-1)/(1+\beta)}}. \tag{30}$$

This prediction is tested against the numerical measurements for $\beta = 3/2$ in Fig. VI. The inset shows the measured values of $\rho_2(\mathbf{x}_1, \mathbf{x}_2, m)/\rho_1(m)^2$ for several different values of m rescaled by $m^{(\beta-1)/(1+\beta)}$ plotted as a function of $|\mathbf{x}_1 - \mathbf{x}_2|^{\beta-1}$. According to Eq. (30), the data should collapse to a single curve vanishing linearly for small separations. The collapsed curve seems to rule out the possibility of linear scaling as one might expect by eye from the main panel of Fig. VI and makes the suggested scaling, Eq. (30), seem plausible. Further investigations, preferably supported by analytic calculations, will be required in order to draw a definitive conclusion. In particular, it is important to properly characterise the "internal" mass structure of correlation functions like Eq. (27) for different masses, particularly for $n = 2$. It might be expected that the denominators of the $|x_i - x_j|$ terms have mass dependences of the form $(m_i m_j)^{1/6} f(m_i/m_j)$ where $f(x)$ is a homogeneous function of degree 0 which has been ignored in our scaling arguments. The legitimacy or otherwise of this neglect is likely to be somehow related to the question of the locality of the mass cascade (see [25] for some discussion). This is a tricky issue in general and will require further work.

VII. CONCLUSIONS

To summarise, we have studied in detail, the scaling properties of one dimensional stationary CCA with Lévy flights. We have demonstrated that the transition from mean field statistics to fluctuation dominated statistics usually observed as the physical dimension is tuned through two from above, can be mimicked in one dimension by varying the characteristic exponent, β, of the jump size distribution of the Lévy flights. This is physically reasonable since the introduction of long range hops provides a mechanism to weaken correlations between particles in the system and may erase them entirely if they are sufficiently frequent. Our predicted values for the scaling exponent of the mass density as a function of β based on dimensional arguments agreed well with numerical simulations of the underlying stochastic particle system.

We also provided a direct demonstration of the strong universality of the scaling exponent of the mass-flux-carrying correlation function in the stationary state. It is independent of both the physical dimension and the value of β as expected from theoretical considerations.

Finally we performed detailed investigations of the spatial structure of the depletion zones surrounding heavy particles in stationary CCA with both regular diffusion and Lévy flights. Our results indicate that in the case of regular diffusion, the exact results of [30] describing the multiscaling of correlation functions in the decaying $A + A \to A$ model, have direct analogues for stationary CCA. In the case of Lévy flights, our numerical studies suggest that the two-point correlation function measuring the anti-correlation between particles in the system vanishes with a non-trivial fractional power of the separation between the particles as this separation decreases to zero. Using our knowledge of the exact scaling for the flux-carrying correlation function we put forward a scaling argument that this power should be $\beta - 1$. This scaling is consistent with numerical observations but further efforts will be required to make it definitive.

It is not immediately evident how to extend the methods employed in [14, 30] to account for Lévy flights since the fractional Laplacian entering the propagator of effective field theory seems difficult to obtain from the operator representation of the master equation. A direct calculation from the master equation seems equally daunting. Nevertheless we expect that some progress in this direction could and should be made. This will greatly clarify the scaling arguments presented in this article.

Several interesting extensions could be made to the model in order to make it more physically realistic. One would be to introduce mass-dependent diffusion, for example, and allowing the diffusion rate to decrease as a power of the mass, $D(m) \sim m^{-a}$, as one would expect for real particles. Another would be to replace the Lévy flights with Lévy walks to account for the fact that physical particles cannot instantaneously make long range jumps. We know that neither of these modifications would affect the scaling exponent, γ_2 of the two-point function which is fixed by the CFR alone. The value of γ_1 and indeed γ_n for $n > 2$ are, however, expected to be sensitive to such modifications and would considerably complicate the picture.

Another natural extension would be to ask similar questions about the structure of anticorrelations in the so-called charge model with Lévy diffusion. The charge model [21, 33] is a closely related model in which masses (or charges) can take positive and negative values. When the rates of injection of positive and negative charges are equal, the flux J is zero on average so that the scaling arguments presented here fail. Instead of an exact conservation law for charge one instead has a statistical conservation law for charge squared. This was used to derive the CFR ($\gamma_2 = 4$) for the charge model in [10]. Repeating the dimensional arguments leading to Eq. (22) in arbitrary dimension taking J to be the average flux of m^2 gives $\gamma_1 = \frac{3d+\beta}{d+\beta}$ as a first guess for the charge model scaling. In $d = 1$ one has $\gamma_1 = \frac{3+\beta}{1+\beta}$ which gives the diffusion limited result $\gamma_1 = 5/3$ when $\beta = 2$ and the mean-field answer $\gamma_1 = 2$ when $\beta = 1$, both of which are known from [33]. If this dimensional prediction were substantiated by an additional analytic or numerical study it could be used to generalise the remaining arguments.

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