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Nonlinear and Evolutionary Phenomena in Deterministic Growing Economies

by

Gui Pedro Araújo de Mendonça

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics and Complexity Science

University of Warwick, Warwick Mathematics Institute, Centre for Complexity Science

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Abstract

We discuss the implications of nonlinearity in competitive models of optimal endogenous growth. Departing from a simple representative agent setup with convex risk premium and investment adjustment costs, we define an open economy dynamic optimization problem and show that the optimal control solution is given by an autonomous nonlinear vector field in $\Re^3$ with multiple equilibria and no optimal stable solutions. We give a thorough analytical and numerical analysis of this system qualitative dynamics and show the existence of local singularities, such as fold (saddle-node), *Hopf* and Fold-Hopf bifurcations of equilibria. Finally, we discuss the policy implications of global nonlinear phenomena. We focus on dynamic scenarios arising in the vicinity of Fold-Hopf bifurcations and demonstrate the existence of global dynamic phenomena arising from the complex organization of the invariant manifolds of this system. We then consider this setup in a non-cooperative differential game environment, where asymmetric players choose open loop no feedback strategies and dynamics are coupled by an aggregate risk premium mechanism. When only convex risk premium is considered, we show that these games have a specific state-separability property, where players have optimal, but naive, beliefs about the evolution of the state of the game. We argue that the existence of optimal beliefs in this fashion, provides a unique framework to study the implications of the self-confirming equilibrium (SCE) hypothesis in a dynamic game setup. We propose to answer the following question. Are players able to concur on a SCE, where their expectations are self-fulfilling? To evaluate this hypothesis we consider a simple conjecture. If beliefs bound the state-space of the game asymptotically and strategies are *Lipschitz* continuous, then it is possible to describe SCE solutions and evaluate the qualitative properties of equilibrium. If strategies are not smooth, which is likely in environments where belief-based solutions require players to learn a SCE, then asymptotic dynamics can be evaluated numerically as a Hidden Markov Model (HMM). We discuss this topic for a class of games where players lack the relevant information to pursue their optimal strategies and have to base their decisions on subjective beliefs. We set up one of the games proposed as a multi-objective optimization problem under uncertainty and evaluate its asymptotic solution as a multi-criteria HMM. We show that under a simple linear learning regime there is convergence to a SCE and portray strong emergence phenomena as a result of persistent uncertainty.

Chapter 1

Introduction

When Thorstein Veblen coined the term evolutionary economics, in the late nineteenth century, he was questioning why economics was not an evolutionary science. Although Veblen was a strong supporter of an economic paradigm in line with Darwin's natural selection ideas, he could not have predicted that his radical interrogation was the starting point of one of the most controversial and long disputes on the foundations of economic theory. In very broad terms, the initial dispute between the neoclassical[1] and evolutionary schools of thought can be condensed in the following fashion. On one side, there was the neoclassical paradigm of rational choice decision and market clearing equilibrium and, at the other end of the spectrum, the concept of creative destruction fuelled by innovation and out of equilibrium markets in perpetual evolution, defended by Joseph Schumpeter and other evolutionary theorists. This debate was interrupted by the Great Depression, and the rise of Keynesian theories, which emphasized the stabilizing role of fiscal and monetary policy, as opposed to the *Laissez Faire* policy approach suggested by neoclassical theories. As Keynesian ideas gradually emerged as the main opposing force to neoclassical economics[2], evolutionary thinking, with some notable exceptions, faded from the economic mainstream and remained in the shadows during the post-war period. This was the result of an increasing orthodoxy in economic theory, which focused the analysis of economic phenomena on mathematical approaches. It is during this period that the neoclassical steady-state dogma was established, as the main dynamic paradigm in economic theory, first by Robert Sollow on exogenous economic growth, and later, in what

[1]The term neoclassical economics is also coined by Thorstein Veblen. See Colander [2000] for a discussion on this topic.

[2]Colander [2000] argues that the term, neoclassical economics, should be replaced by modern orthodox economics. Collander and other historians suggest a period of gradual evolution in neoclassical theory, during the 40's and the 50's, which led to a convergence towards a new modern orthodox paradigm.

become known as optimal growth theory, following the articles by Cass and Koopmans on the original intertemporal optimization problem of Ramsey[3]. This state of affairs lasted until the beginning of the 80's, when Richard Nelson and Sidney Winter[4] recovered the concept of out of equilibrium economic dynamics, following the mathematical developments on nonlinear dynamics and bifurcation theory. What become known as the steady change paradigm was the start of a new systems approach to economic dynamics. This new paradigm played an important role in the development of the new broad field of complexity economics, where other heterodox schools of thought converged. The recent financial crisis has helped to establish these inter-disciplinary approaches by putting further pressure on mainstream economics and policy, dominated until now by Keynesian and Neoclassical ideas. For better or worse, a reform process is now in place to replace the old dominant order by a new one. Still, a question remains. What should be salvaged from the old system of values, following the wreckage provoked by the financial sector excesses[5]?

This thesis proposes to evaluate this issue in a nonlinear optimal growth context. We ask the following question. Are orthodox approaches to economic modelling compatible with the modern notion of evolutionary phenomena, as defined by complex systems theory? Our departure point is endogenous optimal growth theory, introduced at the end of the eighties by Paul Romer and Robert Lucas. This is not an innocent choice. Endogenous growth models have shown some promise in this respect[6]. Our initial focus is on nonlinear phenomena arising in a low dimensional dynamic environment. We then extend this framework to a non-cooperative differential game setup with incomplete information. Our focus is now on emergence phenomena and co-evolutionary dynamics driven by player's belief based decisions. The approach proposed has its roots in modern mathematical literature. It is framed by Smale's eighth mathematical challenge[7], introducing dynamics in economics, where Stephen Smale stresses the need for a rigorous mathematical revaluation of the neo-

[3]Franck Ramsey solved the original problem of optimal savings and capital accumulation in 1928, using the calculus of variations. David Cass and Tjalling Koopmans confirmed Ramsey's results in the 60's, using optimal control methods, and provided several extensions to the original solution.

[4]See Nelson and Winter [1982].

[5]Hyman Minsky proposed in the 60's and the 70's theories about debt accumulation that explain the recent events in financial markets and credit institutions. The collapse of credit bubbles in this fashion is known as a Minsky moment. Among the wide spectrum of modern economic approaches, the forthcoming Minsky moment was correctly predicted by economists focused on balance sheet analysis. These practitioners forecast a long and painful balance sheet recession driven by households, firms and government deleveraging, after years of excess debt accumulation.

[6]The most famous example is the seminal proposal by Aghion and Howitt [1992] on endogenous growth through creative destruction.

[7]The full list of mathematical problems for the new century is found in Smale [1998].

classical general equilibrium problem in a dynamic context. Although Smale limits the challenge to this well known economics problem, we are of the opinion his call should be considered in a broad sense. The optimal growth setups we now start describing are small steps in this direction. As Smale so eloquently put it, *"The following problem is not one of pure mathematics, but lies on the interface of economics and mathematics"*.

Our first proposal deals with an economy populated by representative agents seeking to maximize consumption utility by taking optimal consumption and investment decisions. Agents face convex risk premium on bonds and investment adjustment costs in their budget constraint and accumulate productive capital linearly. This economic setup is correctly described by an aggregate intertemporal maximization problem, where the budget constraint describes the national account identity for an open economy. The optimal solution to the dynamic optimization problem is given by a nonlinear three-dimensional autonomous dynamical system. This framework is an optimal candidate to test evolutionary economic ideas in an modern orthodox decision setup. The low dimensional dynamics approach to economic phenomena is an old tradition in evolutionary economics. Early proposals on the coexistence of cycles and growth by Michal Kalecky, Nicholas Kaldor and Richard Goodwin, for example, modelled economies as two-dimensional vector fields. Our proposal goes a step further to deliver a system with multiple equilibria and several bifurcation phenomena, such as the fold-hopf bifurcation. This result is of particular interest. First, fold-hopf bifurcations have the potential to unleash a cascade of complex nonlinear phenomena. Second, to our knowledge, this proposal is the first in the field of economic growth applications to show the existence of this bifurcation. The fold-hopf bifurcation and associated nonlinear global phenomena has been gaining attention in several fields of applied mathematics. This is not yet the case in economic dynamics literature, despite the potential of this bifurcation scenario to explain empirically observed macroeconomic phenomena, such as structural change dynamics, as a result of complex global dynamics. We focus our analysis on this and other global meaningful conjectures, and give a thorough description of local dynamics and the global organization of the phase space for this economy. We also portray the existence of nonlinear phenomena arising from the complex organization of the system's invariant manifolds, and discuss the challenges posed to policy in this environment.

In chapters three and four, we extend this initial framework and consider now four competitive economies, populated by a discrete set of asymmetric players with incomplete information. We model these economies as open loop non-cooperative differential games, where decisions are coupled by risk premium dynamics, and players follow no feedback strategies. We consider that this last assumption is justified for games populated by a large set of agents. As in the previous proposal, this economic problem can be interpreted as a growing open economy. However, it is also reasonable to interpret this setup as a simple competitive portfolio game, where agents accumulate productive assets and decide if they want to leverage their wealth and accumulate debt, or diversify their portfolios and invest in financial assets. Each of the chapters describes two different economies. First, we consider games with institutionally determined individual risk premium. Then we introduce another nonlinearity and consider that individual risk premium evolves according to the players' financial balances. This setup has one major advantage. It allows for the definition of a general solution as an initial value problem. In chapter 3, we consider that agents do not face investment adjustment costs. The games described in this chapter are consistent with the existence of particular case of the state-separability property for open loop differential games. Optimal control conditions impose the existence of optimal asymmetric beliefs regarding the evolution of the state of the game. We argue that optimal solutions under these conditions, have to be consistent with the existence of a self-confirming equilibrium(SCE), following the proposal by Drew Fudenberg and David Levine. We then show that this problem can be solved by a two-step approach. First, if beliefs bound the state-space of the game asymptotically, then it is possible to evaluate the conditions for the existence of optimal SCE solution. Second, if the value and gradient of beliefs is known in the asymptotic state-space boundary and strategies are *Lipschitz* continuous, local dynamics can be evaluated qualitatively in the vicinity of the SCE. Otherwise, we can set up the asymptotic problem as a static multi-objective optimization problem under uncertainty and evaluate geometrically the existence of a SCE. In the first game discussed, we show that the existence of an optimal solution requires the assumption of less(or more) asymmetries among players. We then show that a numerical qualitative evaluation of SCE solutions is possible, when players commit to a unique investment strategy. Self-confirming solutions are locally stable for a wide range of parameter values. We then evaluate qualitatively state-separable solutions in the vicinity of SCE and give evidence of weak emergence phenomena. This approach is no longer possible when an additional nonlinearity is considered on the evolution of individual risk premium. Although beliefs bound the state-space of the game asymptotically, optimal solutions now require that players learn a SCE. We set up the asymptotic solution of this game as a static multi-objective expected maximization problem, and portray solutions consistent with SCE geometrically, for a range of feasible values of the state of the game. Further insight in this game's asymptotic solution requires the use of evolutionary optimization methods. We discuss the use of stochastic processes for this purpose in the final chapter.

In chapter 4, we reintroduce investment adjustment costs. Game's solutions are no longer consistent with the state-separability property, but reveal an interesting feature. To be able to pursue their optimal strategies, players require information about the state of the game. However, this information is not available to them. Players face a paradox. When choosing their strategies, agents disregarded crucial information that was necessary to pursue their optimal objectives. We assume that under these circumstances, players have to rely on subjective beliefs. We follow the same approach as previously, and again consider that these games are well posed, if beliefs bound the state-space of the game asymptotically. Optimal solutions are again defined as SCE solutions. For the game with institutionally determined individual risk premium, we show that a full description of strategic dynamics is possible, when naive beliefs consistent with smooth strategic dynamics are considered. However, qualitative analysis now shows that this solution is not locally stable for a wide range of parameter values. We argue that this is a reasonable outcome. Optimal self-confirming solutions impose real negative returns on foreign/financial assets and a positive premium on debt. Under uncertainty, players rather follow non-optimal strategies than trust in their naive beliefs. In the second game discussed in this chapter, beliefs impose asymptotic solutions consistent with infinitely many equilibria. Individual state outcomes are now a function of investment decisions. This relation is described by a hyperbola. However, the state-space of this game is still bounded by transversality conditions. Numerical simulations suggest that feasible solutions are defined only on the right hand side of the hyperbola. This is an interesting result. The relation between investment decisions and asset allocation, when a risk free asset is available, was first suggested by Robert Merton, on the efficient portfolio frontier[8]. Our result can be interpreted, in our opinion, as an extension of Merton's theoretical framework, as productive capital in this setup is consistent with the definition of a risk free asset. To evaluate asymptotic game outcomes, we set this game as a multi-objective expected maximization problem constrained by a bounded set of feasible investment strategies. The static version of this game is now described by a complex geometric problem. To tackle this issue, we propose to evaluate game outcomes as a Hidden Markov Model (HMM). The use of stochastic methods to evaluate multi-objective optimization problems under uncertainty is an inter-disciplinary approach

[8]This relation was originally introduced by Harry Markowitz for a portfolio choice model with risky assets. It is commonly known as the *"Markowitz Bullet"*.

that links concepts of evolutionary optimization, machine learning and game theory. This approach has two main advantages. It takes into account the co-evolutionary nature of our problem and allows the introduction of other decision criteria. On the other hand, it introduces a degree of subjectivity on possible game outcomes. Solutions will depend on how the incomplete information Markov learning process is designed. To balance these features, we propose to focus on the role of uncertainty, and assume that players have homogeneous beliefs about the evolution of the state of the game. When uncertainty is a linear function of the forecasting errors, we show that the Markov process reaches an absorbing state consistent with the definition of SCE. However, when uncertainty is persistent, there is evidence of strong emergence phenomena, and the system exhibits out of equilibrium dynamics driven by the decisions of a minority of players. This outcome is a consequence of decision under uncertainty. Subjectivity leads to unintended consequences, which drives strategic interactions among players and fuels a complex co-evolutionary process.

We finish this introduction with some comments regarding the organization of the thesis document. To limit the size of the document and still be able to give a detailed and accurate account, capable of reaching a broad scientific audience, on the inter-disciplinary topics discussed, we organize this thesis in the following fashion. In chapter 2, we put a greater emphasis on the model description and its features, and give a detailed survey of the relevant economic literature. In this chapter, our focus is on the implications of local bifurcation and global dynamic phenomena and its interpretation in an economic context. We relax the model description in the remaining chapters to focus the discussion on the game solutions and their interpretation. In these chapters, our first aim was the introduction of differential games as a potential framework to model competitive growing economies. We give a brief overview on the mathematical literature relevant to the class of games discussed and on the recent developments of this approach in the field of macroeconomic dynamics. Since both these chapters originated from a single project, the reader might find that some of the introductory discussion in chapter 4 resembles that of chapter 3. We decided to keep it this way, to maintain the coherence of the presentation and avoid misinterpretations, since the games proposed have subtle, but important differences. In the second game discussed in chapter 3, we describe optimal game solutions as outcomes of a complex geometric problem. Although an evaluation of possible game outcomes requires the use of sophisticated optimization techniques, we decided to leave this discussion to chapter 4. In this chapter, we introduce the application of stochastic processes to analyse the asymptotic outcomes of differential games. As mentioned previously, this is a novel inter-disciplinary research field. We provide a brief review on the growing literature of evolutionary multi-objective optimization and some examples of the use of these methods in modern economic applications. Finally, given that all the setups discussed are extensions of the framework proposed in chapter 2, the method to obtain the solution to the optimal control problems and its definition as an initial value problem, is similar for all models. We limit this presentation to the minimum necessary, but provide a description of the main results for each optimal control problem. This is necessary because these results are crucial to the derivation of sufficient conditions for the existence of optimal solutions.

Chapter 2

Nonlinear Phenomena in a Growing Economy with Convex Adjustment costs

2.1 Introduction

The aim of this chapter is to discuss the implications of global nonlinear phenomena for long run economic policy definition. We start from the hypothesis that in order to get further insight on long run macroeconomic phenomena, we have to extend our knowledge on nonlinear economic dynamics and the underlying global scenarios. Our approach is based on the following argument. The focus on local dynamics of economic systems leads to a short run policy focus. Therefore, in order to improve the effectiveness of policy in longer horizons, we have to improve our knowledge of environments where global stability conditions no longer apply. To demonstrate this hypothesis, we propose a model of endogenous optimal growth based on simple and well known economic assumptions. Endogenous growth theory was introduced by the seminal proposals of Romer [1986], Romer [1990], Uzawa [1965] and Lucas [1988]. We depart from a deterministic intertemporal optimization framework, following the optimal growth neoclassical framework of Ramsey [1928], Cass [1965] and Koopmans [1965], and set up this model as an open economy populated with N representative agents, assuming neoclassical market clearing micro foundations. Our framework is closely related to Romer [1986] proposal, as the growth engine of this economy is also driven by linear productive capital growth. Agents solve an optimal control consumption/investment problem in continuous time, following the seminal proposals of Merton [1970]. Although the investor problem has its roots on the field of financial mathematics, it is widely used for modeling open economies, given that on aggregate, the national income identity can be matched by the individual budget constraints[1]. We assume that agents in our economy face two nonlinear mechanisms, defined by convex risk premium on bonds and investment adjustment costs, following the well established proposals of Bardhan [1967] and Hayashi [1982], respectively. The paper by Eicher et al. [2008] is a recent example of an economic growth setup closely related to ours that assumes the existence of these two nonlinearities. Our main objective is to evaluate the conditions for existence of global optimal growth dynamics. We follow a straightforward technical analysis of our problem, based on local qualitative analysis, to show the existence of nonlinear phenomena, such as *Hopf*, fold (saddle-node) and fold-hopf bifurcations, consistent with economic feasible scenarios. A thorough numerical exploration of the parameter space does not reveal the existence of local stable solutions, following the *Routh-Hurwitz* stability criterion. Given this result, we focus our analysis on the definition of scenarios consistent with the existence of asymptotic optimal dynamic stable solutions. This is a reasonable objective for policy in a complex dynamic setup, where stable long run dynamics depend on the interaction of multiple equilibrium solutions. In order to define scenarios consistent with this criterion, we discuss several conjectures consistent with local bifurcation phenomena and the complex organization of this economy phase-space. The existence of fold-hopf bifurcations suggests the existence of solutions driven by heteroclinic and homoclinic orbits. Both these scenarios have economic interpretation and are meaningful for policy purposes, as they have the required dynamic properties to reproduce empirical evidence observed in economic aggregates, more concretely flights out of long run equilibrium and flights leading to a new equilibrium. We relate these conjectures to the hypothesis of endogenous structural change, since this phenomena can be produced by small changes of the model parameters. We then extend global analysis of this system and discuss the conditions for the existence of natural frontiers of the economic space, in the form of separatrix planes arising from the dynamics in the vicinity of the non-meaningful set of steady-states. This analysis suggests that the study of non feasible solutions in nonlinear economic models may provide meaningful insight for policy, in particular for economies facing dire institutional conditions[2].

Our proposal departs from Richard Goodwin's main paradigm. Goodwin considered that the extreme phenomena observed in economic data could only be

[1]Turnovsky [2002] provides a interesting mathematical discussion in continuous time modeling for open economy macrodynamic setups.

[2]By non feasible steady states, we refer to solutions of dynamic economic models that are not consistent with physical economic constraints.

explained by nonlinearity. Goodwin's seminal proposals, such as the nonlinear accelerator model, Goodwin [1951][3], are still the main benchmarks of evolutionary economic dynamics theory. Unfortunately, Goodwin's innovative proposal was largely dismissed in mainstream macroeconomic theory, on the grounds that the model's main nonlinear mechanism, a forced oscillator, had no justification in economic theory[4]. In order to avoid this criticism, we model our economy as an endogenous optimal growth model based on mainstream neoclassical assumptions and focus our analysis on the interpretation and evaluation of global dynamic phenomena. The discussion on global dynamic phenomena in theoretical growth economics has roots in mainstream literature that date back to the seventies decade. The focus then was on the definition of sufficient conditions guaranteeing global asymptotic stability. The papers by Brock and Scheinkman [1976], Cass and Shell [1976], Rockafellar [1976] and Nishimura [1981] remain some of the main proposals on this topic. Recent literature on global dynamic economics has focused on the existence of history dependence in nonlinear models of optimal growth with multiple equilibria. A thorough discussion on the mathematics central to many economic applications, along with a careful literature review on this topic can be found in Deissenberg et al. [2004].

Although there is no absolute and universal approach to economic phenomena, radical thinking has been consistently deterred in economic research due to the established orthodox approaches. Change comes slowly in economics and usually involves a long process of reform. In many cases this particular process of evolution led to the dismissal of many interesting ideas. When some of these ideas are able to establish themselves in academia, it does not mean that they are taken in consideration in the development of economic policy agenda. To justify this argument, we put forward two examples of this process that are related to the broad topic of evolutionary economic dynamics. In a recent book, Kirman [2010][5], provides evidence that the orthodox view on the *Marshallian* demand curve, as a microeconomics law, has been incorrectly extrapolated[6], from aggregate market data[7]. These polemics on

[3]The nonlinear accelerator model suggested the persistence of business cycles and periodic dynamics consistent with the long wave hypothesis. This conjecture was first put forward by the Russian economist Nicolai Kondratiev, in Kondratiev [1925]. For a discussion on the theoretical implications of Kondratiev proposal see Louçа [1999].

[4]There is an economic argument that interprets the countercyclical role of government spending as a source of forced oscillations in the economy. For a detailed discussion on this topic see Chian [2007].

[5]Chapter 3- Fish Markets: An Example of the Emergence of Aggregate Coordination

[6]A formal proof on the impossibility of deriving aggregate demand in markets with heteregeneous agents is given by Sonnenschein [1973]. A further mathematical discussion on the topic of market demand and excess demand functions can be found in Shafer and Sonnenschein [1993].

[7]Using qualitative and quantitative data on the Marseille fish market, Kirman develops a simple adaptive evolutionary model that can reasonably replicate aggregate market dynamics. The author

the evolutionary nature of economic phenomena have older roots in economic philosophy. Already in the nineteenth century, Veblen [1898] questions why economics is not an evolutionary science? Some decades later, the famous dispute between John Maynard Keynes and Franck Ramsey on the nature of probability in economic phenomena, paved the way for the introduction of the role of subjectivity in economic theory[8]. These initial discussions led to the development of game theory by John von Neumann and Oskar Morgenstern, which later influenced systems approach to social sciences and is presently one of the main paradigms in modern evolutionary theory. The concept of heterogeneous strategic behaviour under subjectivity, for example, is now a crucial paradigm in the field of financial economics. In two seminal papers, Brock and Hommes [1997] and Brock and Hommes [1998], show that adaptive evolutionary behaviour can arise in rational decision systems, where agents have heterogeneous beliefs (fundamentalists vs. chartists). The authors show the existence of homoclinic bifurcations and chaotic dynamics, arising as a consequence of adaptive beliefs. This hypothesis is considered a plausible justification for the existence of extreme events in financial markets, in particular exchange rate markets[9].

Our proposal draws from this last example and proposes to evaluate the implications of global dynamics in an endogenous growth framework with neoclassical assumptions. As previously described, the dynamics of this economy are determined by an autonomous nonlinear dynamical system in continuous time. According to Mackay [2008], complex systems research should focus on high dimensional dynamic phenomena arising in systems with many interdependent components. We agree with this interpretation, in the sense that nonlinear low dimensional dynamics does not involve many interdependent components. In chapter 3 and chapter 4 we extend this framework to a differential game environment consistent with the modern notion of a complex system. However, we believe that the exploration of global dynamic scenarios that are consistent with optimal control solutions, and the interpretation of such outcomes in growth models, still lie on the field of complex problems. This is particularly true for growth models with dynamics described by vector fields in $\Re^3$. In nonlinear dynamics literature one can find several applications that illustrate the complex challenges posed by such systems. An example of a system with similar

shows that the demand law holds on the aggregate. However, this aggregate behaviour arises from the interaction of heterogeneous agents that do not always have a demand curve with a negative slope.

[8]Keynes argued that in the future it would be possible to define economic phenomena in an objective probabilistic fashion, following the developments on theoretical physics in the first decades of the past century. On the other hand, Ramsey believed that there exists a degree of subjectivity driving economic decisions and any probabilistic approach to economic phenomena would have to take into account this feature.

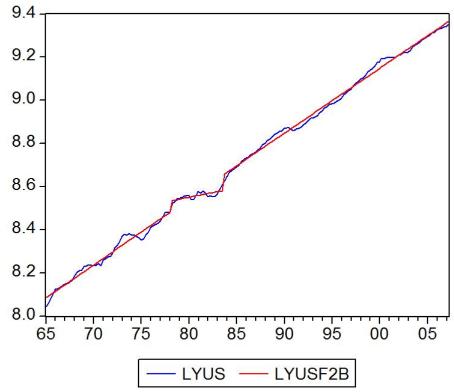
[9]A detailed discussion on this topic can be found in Hommes and Wagener [2008].

characteristics to the one proposed in this chapter is the Rabinovitch-Fabrikant system, following the proposal by Rabinovich and Fabrikant [1979]. Danca and Chen [2004] perform an extensive analysis of this vector field in $\Re^3$ and show that the global analysis of systems with quadratic and cubic terms is not straightforward. The authors also show that classical numerical integration methods are not reliable in this context.

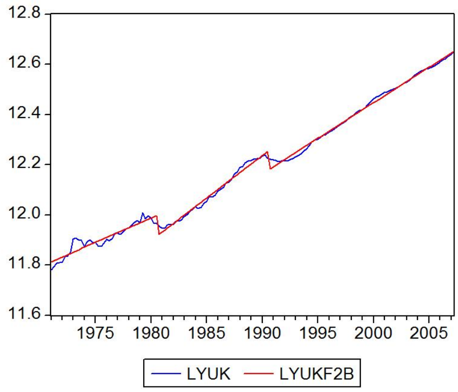
We argue that in nonlinear setups, it is crucial for policy definition to have a global perspective of the dynamics of a system. We thus propose to evaluate several global conjectures and the role of non feasible fixed points in a dynamic policy context. This argument is reinforced by the absence of stable economic meaningful solutions. In this environment, asymptotic stable solutions are only possible when we consider the existence of complex dynamics driven by the interaction of multiple equilibria. This outcome suggests a trade-off between stability and complexity in our system. Further, the existence of solutions undergoing fold-hopf bifurcations suggests that several complex global dynamic conjectures with dramatic policy implications are possible. Chen [2005] argues that this trade-off has to be considered, if we wish to study economic systems in a complex framework. The author also argues that the empirical evidence regarding endogenous structural change and nonlinear dynamics can only be tackled by evolutionary theory. The hypothesis of chronic macroeconomic instability has its roots in economics literature. Minsky [1992] put forward the financial instability hypothesis and suggested that business cycles are driven by financial decisions. Minsky's financial instability proposal is rooted in the *Schumpeterian* evolutionary hypothesis, which attributed an important role to financial intermediation as a driver of the long run cycle. In a recent book, Reinhart and Rogoff [2009] show that financial crisis and instability are a common feature in macroeconomic history. Unfortunately, Minsky was never able to translate his idea into a consistent mathematical dynamic setup[10]. The endogenous structural change hypothesis is supported by empirical data on real macroeconomic aggregates. Figure 2.1 shows evidence of structural change for US and UK $\log(GDP)$ quarterly data. What the modern growth literature has been unable to explain, are the mechanisms leading to structural change, depicted by the shifts in intercept and slope

[10] Keen [2011] suggests that Minsky's failure to devise such setup was linked to the use of the multiplier-accelerator model as a setup for his proposal. The financial instability hypothesis has recently be regaining a renewed attention following the events surrounding the recent financial crisis. Recent discussions propose a reinterpretation of Minsky's original hypothesis closer to the evolutionary long cycle hypothesis. Palley [2011] discusses the hypothesis of financial instability as a super-cycle. Keen [2011] follows the same lines and proposes a redefinition of the original setup based on the Goodwin [1967] nonlinear setup.

[11]. The reason for this shortfall on literature is, in our opinion, related to the systematic approach based on linear and quasi-linear dynamic optimization problems[12]. We firmly believe that the introduction of further nonlinearities in growth models may shed some light on the dynamics of structural change, which can be linked to the existence of global economic dynamic phenomena. The outcomes portrayed in Figure 2.1, for example, can be related to existence of heteroclinic and homoclinic dynamics, leading to permanent and temporary structural change phenomena, respectively[13].



(a) US log GDP and fitted trend with 2 breaks



(b) UK log GDP and fitted trend with 2 breaks

Figure 2.1: Evidence of trend dynamics and structural change on $\log(GDP)$ data.

We organize our presentation in the following fashion. We start with a description of the necessary and sufficient conditions for the existence of optimal solutions in the general optimal control problem. In section 2.3, we introduce the

[11]The data fitting model used in here is based in the well known methodology developed by Vogelsang and Perron [1998], Bai and Perron [1998] and Bai and Perron [2003] for estimation of multiple structural change events in data, following the structural change hypothesis by Perron [1989]. This method is based on a consistent error minimizing estimator. We use the simple Additive-Outlier (A-O) Crash/Change specification on this data and define models with 1 to 4 statistically significant structural changes and estimate Crash models for unemployment data at estimated break dates. The final specifications are chosen using a ranking method based on several data fitting statistical indicators. Similar outcomes are observed on additional estimations using data from seventeen OECD economies.

[12]In earlier versions of this project, we show that when only one nonlinearity is considered, the dynamic properties of these economies are consistent with the properties of linear systems.

[13]We consider evidence of homoclinic phenomena when the sum of changes in the series slope, portraying the long run growth rate, is statistically insignificant, and evidence of heteroclinic phenomena when aggregate changes are statistically significant. For the UK case we estimate two breaks, 1980:4 and 1990:4, and aggregate change in intercept and slope equal to -0.158927 and 0.002192, respectively. For the US case we estimate two breaks, 1978:2 and 1983:4, and aggregate change in intercept and slope equal to -0.01222375 and -0.000117, respectively.

representative agent setup and the intertemporal maximization problem. In section 2.4, we show how the solution to the optimal control problem can be derived via an autonomous dynamical system, and derive sufficient conditions for the existence of optimal growing solutions. In section 2.5 and section 2.6, we put forward the main conditions describing local dynamics and local bifurcations. In section 2.7, we discuss the role of non feasible solutions and their implications for policy in a complex dynamic setup. Finally, in section 2.8, we introduce our main global dynamic conjectures, discuss their economic interpretation, and their implications for policy definition in a nonlinear environment.

2.2 Necessary and Sufficient Conditions for the General Optimal Control Problem

In this section we give a general overview on necessary and sufficient conditions for the existence of optimal solutions in intertemporal maximization problems. For this purpose, we shall follow very closely Drazen [2007] notes on continuous time optimization and replicate here the main results for the general and discounted optimal control problems in both finite and infinite time. The reason for this presentation is twofold. First, Drazen [2007] notes follow closely the Arrow and Kurtz [1970] derivation of sufficient conditions for concave optimization problems with capital accumulation discussed in this thesis. Second, the general results discussed in this section allow us to simplify greatly the discussion presented in this chapter and more concretely in chapter 3 and chapter 4. To be coherent with Drazen [2007] notes, we replicate the author notation throughout this section, even though some of the notation collides with the notation used in this chapter. To avoid confusions regarding the notation used in this chapter and chapter 3 and chapter 4, we do not refer to the specific mathematical results described in the next paragraphs but only to the main results defined in this section.

The general mathematical intertemporal maximization problem in finite time, $T < \infty$, with a vector of s state variables, $\mathbf{x}(t)$, and a vector of n controls, $\mathbf{z}(t)$, can be stated formally as follows:

$$\begin{gathered}\underset{\mathbf{z}(t):0\leq t\leq\infty}{MAX}\int_{t=0}^{T} U\left[\mathbf{x}(t),\mathbf{z}(t)\right]dt + S\left[\mathbf{x}(T)\right] \\ \text{subject to } \dot{x}_i(t) = G_i\left[\mathbf{x}(t),\mathbf{z}(t)\right],\ i = 1,\ldots,s \\ \text{and } \mathbf{x}(0) = \mathbf{x}_0,\end{gathered} \tag{2.1}$$

where $S\left[\mathbf{x}(T)\right]$ represents the scrap value of the state variables at T. To derive the

Hamiltonian let $V(x_0)$ denote the solution to problem (2.1). Following the *Bellman's Principle of Optimality*, we can consider a small time interval $[t_0, t_0+]$ and rewrite the objective function as

$$V(\mathbf{x}(t_0),t_0) = \underset{\mathbf{z}(t):t_0\leq t\leq t_0+h}{MAX} \left\{ \int_{t=t_0}^{t_0+h} U\left[\mathbf{x}(t), \mathbf{z}(t), t\right] dt + V\left(\mathbf{x}(t_0+h), t_0+h\right) \right\}, \quad (2.2)$$

where $\mathbf{x}(t_0+h)$ is determined by the choice of controls and the state equations. Assume that $U[\mathbf{x}(t), \mathbf{z}(t)]$ is continuous in all its arguments and that the $\mathbf{z}(t)$ functions are continuous functions of time. Thus for small h continuity assumptions imply

$$U\left[\mathbf{x}(t), \mathbf{z}(t), t\right] \simeq U\left[\mathbf{x}(t_0), \mathbf{z}(t_0), t_0\right], \text{ for } t_0 \leq t \leq t_0+h. \quad (2.3)$$

Now assuming that $V(\mathbf{x}(t_0), t_0)$ is differentiable, we take a first order *Taylor* series expansion of V around $\mathbf{x}(t_0)$ and obtain

$$\begin{aligned} &V\left(\mathbf{x}(t_0+h), t_0+h\right) \simeq V\left(\mathbf{x}(t_0), t_0\right) \\ &+\sum_{i=1}^{s} V_{x_i}\left(\mathbf{x}(t_0), t_0\right) \cdot \left[x_i(t_0+h) - x_i(t_0)\right] + V_t h, \end{aligned} \quad (2.4)$$

where V_{x_i} and V_t are the partial derivatives of V with respect to x_i and t, evaluated at $(\mathbf{x}(t_0), t_0)$. Now we define

$$q_i = V_{x_i} \equiv \frac{\partial V(\mathbf{x}, t)}{\partial x_i}, \quad (2.5)$$

where q_i is the marginal contribution of the state variable x_i to the value of the optimal plan, which are usually referred as the co-state variables. A first order *Taylor* expansion of $x_i(t_0+h)$ around $x_i(t_0)$ yields

$$\begin{gathered} x_i(t_0+h) = x_i(t_0) + \left[(t_0+h) - t_0\right] \dot{x}_i(t_0) + O(2) \Leftrightarrow \\ \Leftrightarrow x_i(t_0+h) - x_i(t_0) \simeq hG_i\left[\mathbf{x}(t_0), \mathbf{z}(t_0)\right]. \end{gathered} \quad (2.6)$$

Combining (2.3), (2.4), (2.5) and (2.6), we can rewrite (2.2) as

$$\begin{aligned} V\left(\mathbf{x}(t_0), t_0\right) \simeq h \underset{\mathbf{z}(t_0)}{MAX} &\left\{ U\left[\mathbf{x}(t_0), \mathbf{z}(t_0), t_0\right] + \sum_{i=1}^{s} q_i G_i\left[\mathbf{x}(t_0), \mathbf{z}(t_0)\right] \right\} \\ &+V\left(\mathbf{x}(t_0), t_0\right) + V_t h. \end{aligned} \quad (2.7)$$

Cancelling $V(\mathbf{x}(t_0), t_0)$ in both sides and diving by h we obtain

$$-V_t = \underset{\mathbf{z}}{MAX}\, H \equiv H^*, \tag{2.8}$$

where H is defined by

$$H(\mathbf{x}, \mathbf{z}, \mathbf{q}, t) = U(\mathbf{x}, \mathbf{z}, t) + \sum_{i=1}^{s} q_i G_i[\mathbf{x}, \mathbf{z}]. \tag{2.9}$$

The function $H(\mathbf{x}, \mathbf{z}, \mathbf{q}, t)$ is called the *Hamiltonian* and (2.8) is a partial differential equation whose solution is a value function $V(\cdot)$.

To determine the optimal dynamic paths using the *Hamiltonian* in (2.9), we start by the definition of the necessary conditions for the existence of an optimal policy, described by $\mathbf{z}^*$, which must satisfy:

$$H_{z_k} = \frac{\partial H}{\partial z_k} \text{ for } k = 1, \ldots, n. \tag{2.10}$$

The co-state optimal path is obtained after differentiating the *Hamiltonian* evaluated at $\mathbf{z}^*$, $H^* = \mathbf{z}^*$, $H(\mathbf{x}, \mathbf{z}^*, \mathbf{q})$, with respect to each of the x_i, holding all other $\mathbf{x}$ and $\mathbf{q}$ constant to obtain

$$H^*_{x_i} = H_{x_i} + \sum_{k=1}^{n} H_{z_k} \frac{\partial z^*_k}{\partial x_i} = H_{x_i}, \tag{2.11}$$

where the last equality follows from the *Envelope Theorem.* Similarly,

$$H^*_{q_i} = H_{q_i}, \tag{2.12}$$

as the *Hamiltonian* is linear in each q_i it is simple to evaluate that

$$H^*_{q_i} = G_i[\mathbf{x}, \mathbf{z}^*] = \dot{x}_i. \tag{2.13}$$

The optimal path for q_i is obtained by taking the total derivative of (2.5) with respect to time to obtain

$$\dot{q}_i = \sum_{j=1}^{s} V_{x_i x_j} \dot{x}_j + V_{x_i t} \text{ where } V_{x_i x_j} \equiv \frac{\partial^2 V(\mathbf{x})}{\partial x_i \partial x_j} \tag{2.14}$$

To obtain $V_{x_i t}$, first note that H^* is a function of $\mathbf{x}$, $\mathbf{q}$ and t, but each of the q_i are in turn functions of $\mathbf{x}$ and t via (2.5). To determine $V_{x_i t}$ differentiate (2.8) with respect to x_i, holding everything constant to obtain

$$-V_{x_i t} = H^*_{x_i} + \sum_{j=1}^{s} H^*_{q_j} V_{x_i x_j}. \tag{2.15}$$

Substituting (2.11), (2.12) and (2.13) into (2.15), and then the resulting equation into (2.14), one derives

$$-V_{x_i t} = -H_{x_i} \equiv -\frac{\partial H}{\partial x_i}, \tag{2.16}$$

where the control $\mathbf{z}$ is chosen to maximize H. We are now able to put forward a definition for the *Pontryagin Maximum Principle.*

> *Let $z^*(t)$ be a choice of instruments that maximize $\int_{t=0}^{T} U[\mathbf{x}(t), \mathbf{z}(t)]\, dt$ subject to the conditions $\dot{x}_i(t) = G_i[\mathbf{x}(t), \mathbf{z}(t)]$ and $\mathbf{x}(0) = \mathbf{x}_0$. Then there exist auxiliary variables, $q_i(t)$, such that, for each period t, $z^*(t)$ maximizes $H(\mathbf{x}, \mathbf{z}, \mathbf{q}, t)$, for an interior solution, $\frac{\partial H}{\partial z_k} = 0$, where $H(\mathbf{x}, \mathbf{z}, \mathbf{q}, t) = U(\mathbf{x}, \mathbf{z}, t) + \sum_{i=1}^{s} q_i G_i[\mathbf{x}, \mathbf{z}]$ and the functions $q_i(t)$ satisfy the differential equations $\dot{q}_i = -\frac{\partial H}{\partial x_i}$ evaluated at $\mathbf{z}(t) = \mathbf{z}^*(t)$.*

The equations for the co-state and state variables define $2s$ first order differential equations. To bound solutions we require an equal number of boundary conditions. Initial values of the state variables provide s boundary conditions. Now assume that at T, $\mathbf{x}$ has no scrap value, that is $S[\mathbf{x}] = 0$ for any non negative values of x_i, then $x_i(T) = 0$. Otherwise one has

$$-V(\mathbf{x}(T), T) = S[\mathbf{x}(T)] \text{ so that } q_i(T) = \frac{\partial V(\mathbf{x}(T), T)}{\partial x_i(T)} = \frac{\partial S[\mathbf{x}(T)]}{\partial x_i(T)} \tag{2.17}$$

following the equality described at (2.5). Condition (2.17) provides the remaining s boundary conditions to our solution. These conditions are known as the transversality conditions.

With infinite horizon, the dynamics of the system are given by the same conditions but transversality conditions are different. The description above of finite horizon transversality conditions was based on the assumption that the state variables are always non-negative. Arrow and Kurtz [1970], consider some finite T and approximate this requirement by a scrap value function that imposes large penalties on negative values of x_i. An example of such scrap function could be

$$S\left[\mathbf{x}\left(T\right)\right]=\sum_{i=1}^{s}P_{i}\min\left(x_{i}\left(T\right),0\right), \tag{2.18}$$

where the vector of penalties, P_i, is composed of very large numbers. Now recall that: (i) when $x_i < 0$, we have $S(\mathbf{x}) = P_i x_i$ and $\frac{\partial S(\mathbf{x})}{\partial x_i} = P_i$; (ii) when $x_i > 0$, we have $S(\mathbf{x}) = 0$ and $\frac{\partial S(\mathbf{x})}{\partial x_i} = 0$; and finally (iii) when $x_i = 0$, we have $S(\mathbf{x}) = 0$ and $0 < \frac{\partial S(\mathbf{x})}{\partial x_i} < P_i$. It follows that for P_i sufficiently large, the terminal value of x_i would never be negative for all i, $x_i(T) \geq 0$. Combining these results one may write: (i) $x_i(T) > 0 \Rightarrow \frac{\partial S(\mathbf{x})}{\partial x_i} < P_i = 0$; and (ii) $x_i(T) = 0 \Rightarrow \frac{\partial S(\mathbf{x})}{\partial x_i} < P_i \geq 0$. Using the result in (2.17), one can write these conditions as: (i) $q_i(T) \geq 0$; and (ii) $q_i(T)\, x_i(T) = 0$. As $T \to \infty$ these last two conditions become

$$\lim_{T\to\infty} q_i\left(T\right) \geq 0 \ \text{ and } \lim_{T\to\infty} q_i\left(T\right) x_i\left(T\right), \tag{2.19}$$

which correspond to the infinite horizon transversality conditions.

We now consider the case with discounting, where the future outcomes are discounted by a factor that is a function of time. Let $\alpha(t)$ represent the discount factor. The objective function can now be written as

$$\int_{t=0}^{\infty}\alpha\left(t\right)U\left[\mathbf{x}\left(t\right),\mathbf{z}\left(t\right)\right]dt \tag{2.20}$$

and the *Hamiltonian* now comes

$$\alpha\left(t\right)H\left(\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{q}\left(t\right)\right)=\alpha\left(t\right)\left[U\left(\mathbf{x}\left(t\right),\mathbf{z}\left(t\right)\right)+\sum_{i=1}^{s}q_{i}\left(t\right)G_{i}\left[\mathbf{x}\left(t\right),\mathbf{z}\left(t\right)\right]\right]. \tag{2.21}$$

With discounting we shall now refer to H as the current value, or undiscounted, *Hamiltonian*, which is generally given by

$$H\left(\mathbf{x},\mathbf{z},\mathbf{q}\right)=U\left(\mathbf{x},\mathbf{z}\right)+\sum_{i=1}^{s}q_{i}G_{i}\left[\mathbf{x},\mathbf{z}\right], \tag{2.22}$$

and identical to (2.9). The necessary conditions for an optimum, analogous to (2.14), are now given by

$$\frac{d\left[\alpha\left(t\right)q_{i}\left(t\right)\right]}{dt}=-\frac{\partial\alpha\left(t\right)H}{\partial x_{i}}, \tag{2.23}$$

$$\dot{q}_{i}\left(t\right)=p\left(t\right)q_{i}\left(t\right)-\frac{\partial H}{\partial x_{i}},\ \text{where}\ p\left(t\right)=\frac{\dot{\alpha}\left(t\right)}{\alpha\left(t\right)^{3}}. \tag{2.24}$$

Note that a policy $\mathbf{z}(t)$ that maximizes H for every period t also maximizes $\alpha(t)\, H$

for every period of t. Thus with discounting, the definition of the *Pontryagin Maximum Principle* just requires some small modifications:

> *Let* $z^*(t)$ *be a choice of instruments that maximize* $\int_{t=0}^{\infty} \alpha(t) U[\mathbf{x}(t), \mathbf{z}(t)] dt$ *subject to the conditions* $\dot{x}_i(t) = G_i[\mathbf{x}(t), \mathbf{z}(t)]$ *and* $\mathbf{x}(0) = \mathbf{x}_0$. *Then there exist auxiliary variables,* $q_i(t)$*, such that, for each period* t*,* $z^*(t)$ *maximizes* $H(\mathbf{x}, \mathbf{z}, \mathbf{q}, t)$*, for an interior solution,* $\frac{\partial H}{\partial z_k} = 0$*, where* $H(\mathbf{x}, \mathbf{z}, \mathbf{q}) \equiv U(\mathbf{x}, \mathbf{z}) + \sum_{i=1}^{s} q_i G_i[\mathbf{x}, \mathbf{z}]$ *is the current value hamiltonian and the functions* $q_i(t)$ *satisfy the differential equations* $\dot{q}_i = p(t) q_i - \frac{\partial H}{\partial x_i}$ *evaluated at* $\mathbf{z}(t) = \mathbf{z}^*(t)$*, where* $p(t) \equiv \dot{\alpha}(t) / \alpha(t)^3$.

With discounting the infinite horizon transversality conditions are now given by the corresponding asymptotic conditions, following the result in (2.19),

$$\lim_{T \to \infty} \alpha(t) q_i(T) \geq 0 \quad \text{and} \quad \lim_{T \to \infty} \alpha(t) q_i(T) x_i(T). \tag{2.25}$$

The conditions derived in this section are necessary conditions for an optimum. These conditions are sufficient if $H^*(\mathbf{x}, \mathbf{q}) \equiv H(\mathbf{x}, \mathbf{z}^*, \mathbf{q})$ is concave in $\mathbf{x}$ given $\mathbf{q}$. In other words the *Hamiltonian Hessian* matrix composed by the second order partial derivatives of x_i, evaluated at the optimal policy $\mathbf{z}^*$, $H^*_{x_i x_j}$, has to be a semi-definite negative matrix. Sufficiency follows from the transversality conditions described in (2.25). See Arrow and Kurtz [1970] for this original proof. A description of this proof, which follows the notation used in this section, can also be found in the notes by Drazen [2007]. We discard this presentation because the problems we deal in this thesis assume the existence of two state variables, where one can be positive. In this set of control problems, it is not straightforward to check that the first transversality condition and the concavity of the *Hamiltonian* in the states, are fulfilled. However, for problems where the objective function is continuous differentiable and strictly concave, it is sufficient to check if solutions fulfil the second Arrow and Kurtz [1970] transversality condition asymptotically. This relaxation proves useful when dealing with applied optimal control problems with several state variables and a concave objective. For a discussion of this result in the general optimal control problem see Acemoglu [2009] chapter 7, section 5. For a discussion of this approach on applied optimal control problems similar to the ones discussed in this thesis see the books on economic growth by Turnovsky [1999] and Barro and Sala-i-Martin [2004]. Finally, the proof required for the existence of uniqueness of solutions to the general optimal control problem is a difficult proof that requires knowledge of sophisticated mathematical techniques, such as measure theoretic notions, and imposes several constraints on the problem. Acemoglu [2009], in chapter 7 section 6 of his book, provides an accessible sketch of this proof and describes the conditions for the existence and uniqueness of solutions for the general optimal control problem discussed in the beginning of this section.

2.3 The representative agent economy

We consider an open competitive economy populated by N representative agents (identical individuals) that live infinitely for $t \in [0, T]$, where $T = \infty$. Households invest in domestic and foreign capital in exchange for returns on these assets, purchase goods for consumption, and save by accumulating domestic and foreign assets. We further consider that agents can resort to debt accumulation to finance investment in domestic assets and/or consumption. Households may also undertake temporary disinvestment decisions on domestic capital to improve their financial balances. The representative agent seeks to maximize an intertemporal utility consumption function, $U(c)$, and discounts future consumption exponentially at a constant rate $\rho \in \Re^+$. To achieve this objective, agents solve an infinite horizon consumption, $c(t) \in \Re^+$, and investment, $i(t) \in \Re$, dynamic optimization problem *a la Merton*, taking into account the evolution of their budget constraint, $b(t) \in \Re$, and their domestic capital accumulation, $k(t) \in \Re^+$. The objective of each agent is to maximize the flow of discounted consumption outcomes,

$$U(c) = \int_0^T u(c(t)) e^{-\rho t} dt, \text{ with } u(c(t)) = c(t)^\gamma, \tag{2.26}$$

where γ defines the intertemporal substitution elasticity in consumption, measuring the willingness to substitute consumption between different periods. A smaller γ means that the household is more willing to substitute consumption over time. We impose the usual constraint on the intertemporal substitution parameter, $0 < \gamma < 1$, such that $u'(c(t)) > 0$. This specification for utility belongs to the family of constant relative risk aversion (CRRA) utility functions and is widely used in optimization setups, where savings behaviour is crucial, such as economic growth problems. This setup also guarantees the concavity of the utility function, $u''(c(t)) < 0$. This is a necessary condition to obtain optimal solutions to our dynamic optimization problem as an initial value problem [14].

[14]We follow closely the seminal results of Arrow and Kurtz [1970], Chapter 2- Methods of optimization over time, which guarantee the *Pontryagin* first order conditions are sufficient for determining an optimum solution in infinite horizon dynamic optimization problems with constant dis-

Following Barro and Sala-i-Martin [2004], setups with infinitely lived households have the following interpretation. Each household contains more than one adult, defining the current generation population. In making plans, these adults take account of the welfare and resources of their prospective descendants. We model this intergenerational interaction by imagining that the current generation maximizes utility and incorporates a budget constraint over an infinite horizon. That is, although individuals have finite lives, we consider an immortal extended family.

This economy has N identical firms, owned by each household and producing an homogeneous good, $y(t) \in \Re^{+}$, that requires just capital inputs, $k(t)$. We assume this simplification for mathematical reasons, nevertheless, the domestic capital stock can be considered as a broad measure of available capital in the economy used in the production of goods. The technology level of firms is identical and given exogenously by parameter A. We do not consider the possibility of technological progress in this economy. The flow of output produced by each firm, at a given period, is given by a AK production function, expressed by equation (2.27), following the simple Romer [1986] endogenous growth framework with marginal and average product constant at the level $A \in \Re^{+}$,

$$y(t) = Ak(t). \tag{2.27}$$

As usual in open economy frameworks, we assume that agents and firms have full access to international capital markets. Households can accumulate foreign debt/assets, $b(t)$, for which they pay/receive an exogenous interest, expressed in terms of the real international interest rate, r, plus a risk premium defined by the evolution of their real financial balances ratio, $b(t)/k(t)$. We assume that foreign debt payments, $b(t) > 0$, and returns on foreign assets, $b(t) < 0$, follow a convex specification, $\Xi(t)$, where, $\Xi(t)'_{b(t)} > 0$ and $\Xi(t)''_{b(t)} > 0$, for $b(t) > 0$. This specification follows closely the original proposal by Bardhan [1967][15]. Interest payments are defined by

$$\Xi(t) = rb(t)\left(1 + \frac{d}{2}\frac{b(t)}{k(t)}\right), \tag{2.28}$$

where parameter $d \in \Re$ stands for the exogenous institutional risk premium, resulting from international capital markets sentiments on the quality of the debt bonds issued by the economy. This assumption is justified by bias arising from historical

count, provided that the objective function is concave and the transversality conditions are fulfilled. Turnovsky [1995], chapter 3- Intertemporal optimization, discusses the definition of transversality conditions for two sector models, such as ours.

[15] Stiglitz and Weiss [1981] have shown that even in cases of individual borrowing, because of informational asymmetries or problems associated with moral hazard, risk premium or credit constraints, or both, are known to exist.

and psychological perceptions. We assume that macroeconomic factors are priced in the risk premium valuation through the net foreign assets to domestic capital ratio. A higher value of d means that holding the country debt bonds yields a higher risk for international investors, but investment by nationals on foreign assets pays a greater premium. A smaller value of d means that holding the country debt bonds yields a small risk for international investors, but investment by nationals on foreign assets pays a smaller premium. This setup can be interpreted in terms of the degree of development and international financial integration of a given economy. International investors' sentiment towards a mature economy is less severe, as a consequence of the higher degree of international trade and financial integration. This phenomenon can be explained by historical, political and economic factors, which bias international investors sentiments towards *successful* economies, while disregarding real economic information. It can also result from information costs, which deter international investors from acquiring relevant information on the state of a specific economy and increases investors reliance on individual or collective market beliefs. A smaller d represents also a smaller premium for residents investing in foreign assets. This can be interpreted as a result of the higher degree of international financial integration in mature economies. Residents of developed economies require smaller premiums on their foreign investments due to smaller transaction and information costs of investing abroad, arising from financial innovation in developed economies banking systems. Therefore, d can be ultimately interpreted as a measure of the degree of openness and maturity of an economy. We also consider the hypothesis of an economy facing negative institutional risk premium, $d < 0$. We consider that strong market sentiment may drive institutional risk premium to be negative, when certain institutional macroeconomic scenarios arising from international liquidity bias, strong domestic bias towards home assets and specific international institutional frameworks are fulfilled for a given economy. We provide a detailed discussion on this matter in the context of existence of investment adjustment costs and detail four possible dynamic setups with a relevant economic interpretation.

Agents take investment decisions on domestic assets and face convex investment adjustment costs on these decisions, given by function $\Omega(t)$, following the famous Hayashi [1982] proposal:

$$\Omega(t) = i(t)\left(1 + \frac{h}{2}\frac{i(t)}{k(t)}\right). \tag{2.29}$$

In a closed economy framework convex investment adjustment costs are usually interpreted in the context of installation costs. In an open economy framework, the installation cost parameter, $h \in \Re$, has the following interpretation: if (i) $h < 0$, institutional conditions impose bias on investment in domestic assets, if (ii) $h > 0$, institutional conditions impose bias on investment in foreign assets. This mechanism is linked to the previous discussion on the degree of openness and maturity of an economy. Empirical evidence suggests the predominance of bias towards investment in domestic assets. This is known as the equity home bias puzzle. Evidence on this phenomena was first brought forward by French and Poterba [1991]. Since mature developed economies offer smaller costs on investment in international assets, following our assumption on the higher degree of sophistication of its financial sector industry, we can assume that these economies face institutional conditions that promote smaller bias towards investment internationalization. The opposite is expected in less developed economies, where institutional conditions impose higher costs on investment in foreign assets. On the other hand, it is widely known that economies facing dire financial conditions, due to severe balance of payment imbalances leading to currency crises, increase the incentives for households to substitute domestic assets by foreign assets. Capital flights in this fashion are a consequence of domestic asset devaluation arising from currency value collapse and the consequent inflationary dynamics, which drive down the value of domestic assets against foreign assets. Although we don't consider currency in our model, we can consider that such extreme situations impose extraordinary institutional conditions, which lead to bias on investment in foreign assets.

We conclude the presentation of our economy with the definition of domestic and net foreign capital dynamics. Agents receive capital returns, $r_k \in \Re^+$, on domestic assets equal to the marginal productivity of firms, following the usual neoclassical assumption on market clearing conditions for perfectly competitive domestic capital markets. The marginal returns on domestic capital are given by the exogenous technology rate of firms,

$$r_k = \frac{\partial y(t)}{\partial k(t)} = A. \tag{2.30}$$

We can now write the intertemporal budget constraint for the representative agent, in terms of foreign debt/assets accumulation. This constraint is given by the following differential equation,

$$\dot{b}(t) = c(t) + i(t)\left(1 + \frac{h}{2}\frac{i(t)}{k(t)}\right) + rb(t)\left(1 + \frac{d}{2}\frac{b(t)}{k(t)}\right) - r_k k(t). \tag{2.31}$$

Firms accumulate capital following agents' investment decisions and face a depreciation rate of their capital stock equal to $\delta \in \Re^+$, following the usual linear differential specification for capital dynamics,

$$\dot{k}(t) = i(t) - \delta k(t). \tag{2.32}$$

In our setup, we assume that agents can have temporary disinvestment decisions, in order to improve their foreign net assets balances or increase their consumption levels. In the long run, we assume that the following asymptotic condition is fulfilled:

$$\liminf_{t\to\infty} \frac{i(t)}{k(t)} > \delta. \tag{2.33}$$

We can now put forward the dynamic optimization problem faced by the aggregate economy. The problem we propose to analyse is an aggregate version of the original problem, which yields an identical solution to the solutions obtained from the aggregate representative agent and central planner dynamic optimization problems. Recall that aggregation in a representative agent framework is given by assuming $X_i(t) = Nx_i(t)$, where $i \in \{1, ..., 4\}$, and $X_i(t)$, $x_i(t)$ correspond to each of our aggregate and individual variables, consumption, $c(t)$, net foreign assets, $b(t)$, domestic capital, $k(t)$, and investment, $i(t)$, respectively. The representative agent problem is defined in the following fashion. First we solve the dynamic optimization problem by maximizing (2.26) subject to (2.31) and (2.32). The aggregate dynamics of this economy are then obtained after substituting in first order conditions the market clearing condition (2.30) and aggregating variables, following the rule $x_i(t) = X_i(t)/N$. The central planner problem, on the other hand, is obtained by substituting first market clearing conditions and aggregating variables in (2.26), (2.31) and (2.32), and then solving the respective intertemporal maximization problem. We propose to analyse an aggregate version of this problem, which is given by the maximization of the aggregate objective function given in (2.34), below, subject to the central planner aggregate state conditions. This option is justified because this intertemporal optimization problem represents a simplified version of both the representative agent and central planner problems. Although the optimality condition for aggregate consumption still depends on the size of the economy, when we solve the representative agent or central planner problem, this parameter disappears when we define the *Keynes-Ramsey* consumption equations that are obtained from first order conditions. It is easy to observe that the central planner state conditions do not depend on N. Therefore, we can simplify the dynamic optimization problem of this economy by assuming the following general aggregate representation, [16],

$$
\begin{aligned}
&\underset{C(t),I(t)}{MAX} \int_0^\infty e^{-\rho t} C(t)^\gamma \, dt \\
&\text{subject to the solution of:} \\
&\dot{B}(t) = C(t) + I(t)\left(1 + \tfrac{h}{2}\tfrac{I(t)}{K(t)}\right) + rB(t)\left(1 + \tfrac{d}{2}\tfrac{B(t)}{K(t)}\right) - r_k K(t); \\
&\dot{K}(t) = I(t) - \delta K(t);
\end{aligned}
\tag{2.34}
$$

given some initial conditions, $B(0)$ and $K(0)$, and satisfying the transversality conditions (A.8) and (A.9) defined in appendix section A.1, guaranteeing that solutions to (2.34) do not grow too fast.

We finish this section with a discussion on the economic interpretation of different institutional parameter scenarios that arise when investment adjustment costs and risk premium are considered together in a growth model of an open economy. Four dynamic scenarios with relevant economic interpretation can be considered, when we take into account the interactions between these two institutional parameters. As discussed in the previous paragraphs, the expected scenario, according to economic theory, is given by an economy facing positive risk premium and bias towards domestic assets, $d > 0$ and $h < 0$. However, there are empirical and theoretical grounds to assume that an economy may benefit from both negative risk premium and bias towards home assets, $d < 0$ and $h < 0$. We consider two institutional frameworks that may produce macroeconomic outcomes consistent with this specific scenario: (i) Flight to liquidity driven by reserve currency status; and (ii) Excess liquidity arising from international capital flows. The first scenario arises in economies with currencies that function as strategic reserve assets in international capital markets. Historically, this status has been held by the UK pound during the gold standard period and afterwards by the US dollar following the second world war and the Bretton Woods agreement. For historical, economic and geostrategic reasons, these two economies benefited from international financial bias, which resulted in higher demand and increased liquidity in both foreign exchange and sovereign bond markets. Longstaff [2004] provides evidence that during liquidity flights arising from international financial crisis, investors are willing to pay a premium to hold US bonds. This strong liquidity effect may lead to negative risk premium scenarios. According to Ludvigson and Ng [2009] sovereign bond markets are strongly driven

[16]This dynamic problem yields the same solution as the central planner and representative agent problems and, therefore, represents a slight simplification of the economic growth problem described in the previous paragraphs. This result can be easily confirmed following our definitions in section A.1. We discard the demonstration of this result in order to contain our already long presentation.

by market sentiment, which leads to an acyclical behavior of risk premium. The authors give evidence of acyclical and negative risk premium dynamics in the US sovereign bond market and attribute this behaviour to investor decisions driven by market sentiments and macroeconomic factors. The authors link this outcome to theories that sustain that investors demand compensation for increased risks during economic downturns, which drives risk premium higher, and relax these demands during expansions, where risks are considered to be smaller. Other currencies have also benefited from reserve currency status and have been accepted by international investors as substitutes to the US dollar in recent decades. Besides the UK pound, we can include in this set of currencies the Deutsche Mark, now replaced by the Euro, the Japanese Yen and the Swiss Franc. The Japanese case is of special interest to our discussion, since it is linked to a strong home bias on domestic assets and a liquidity trap environment driven by historically low interest rates. Goyal and McKinnon [2003] provides empirical evidence on Japan's consistent negative risk premium on sovereign bonds and links this outcome to the strong home bias on domestic assets effect mixed with the low interest rates environment in a context of an ineffective monetary policy. The common economic factors shared by the above mentioned economies are long run growth, export capacity, credit worthiness and creditor protection, strong property rights and historical low to moderate inflation. The second scenario arises in economies that benefit from strong international liquidity flows, which were driven by international low interest rates and resulted in a distortion of domestic bond markets, due to lower perceived default risk and improved creditworthiness. Agenor [1998] provides an insightful theoretical discussion on this issue and maintains that this was the main cause driving the boom and bust of Asian economies during the nineties. The European periphery countries experienced the same environment with the introduction of the Euro and the period of low interest rates that followed. Again excess liquidity drove risk premium to low, and most likely negative levels, due to a perceived increased creditworthiness and lower default risk. This effect was a result of market perceptions about European institutional developments, which fuelled the belief that exchange rate risk between EU nations had vanished. During several years, European periphery countries yields on sovereign bonds were historically low and even negative, when compared to benchmark German sovereign bonds. This link was broken in the aftermath of the 2008 financial crisis, and since then economic factors have dominated international investors' decisions and bond market outcomes, leading to a return to positive spreads relative to German bonds. Finally, some open economies benefit from the status of commodity currencies, due to their strategic importance for world commodity markets. In recent decades the Canadian, Australian and New Zealand dollar, as well as the Norwegian Krone have benefited from this specific status. These economies usually experience excess liquidity driven by international financial flows during strong expansion periods. Rising demand for strategic commodities in world markets, leads to a rising demand on commodity currency assets, which results in currency appreciation and increased liquidity in domestic bond markets. Excess liquidity of foreign reserves may lead to severe distortions on risk premium in the absence of appropriate institutions. In recent decades, several commodity exporting economies followed the Norwegian institutional framework and constituted sovereign wealth funds, with the objective of investing commodity based revenues in foreign assets, to avoid distortions in domestic markets arising from excess foreign reserve liquidity.

We conclude this discussion on institutional scenarios with a description of macroeconomic nightmare scenarios arising in economies facing positive risk premium and bias towards foreign assets, $d, h > 0$. Recall that we discussed previously that scenarios consistent with $h > 0$, can be related to balance of payment crisis and expectations of currency crisis and debt default scenarios. In this critical environment, international investors price in this risk assuming strong probability of losses and demand a higher premium to hold the stressed economy sovereign bonds. Before the default scenario becomes inevitable, countries seek to lower the premium demanded for their bonds by guaranteeing debt roll over through bilateral agreements. This is usually arranged through IMF intervention and the implementation of structural adjustment programs. Loans are guaranteed by IMF stockholders and the soundness of the institutional arrangement is monitored by IMF economists. This institutional arrangement seeks to roll over debt repayments, until market risk premium on domestic bonds returns to affordable levels and there is no longer bias towards foreign assets. This institutional arrangement seeks to avoid macroeconomic nightmare scenarios, by guaranteeing a temporary debt subsidy at a negative real premium, for an economy facing bias towards foreign assets, $d < 0$ and $h > 0$. In recent years IMF interventions have come under severe criticism because of its consistent inability to achieve the desired goals and leaving economies worst off. Some authors suggest that institutionally imposed negative risk premiums scenarios creates moral hazard incentives for both the creditor and debtor. These authors argue that such arrangements promote the delay of economic adjustment by the debtor and reduce negotiation willingness of creditors. This non-cooperative situation delays the achievement of a permanent solution to the unsustainable debt problem and usually results in higher costs for both debtors and creditors. Miller and Zhang [2000] and

Corsetti et al. [2006] discuss this problem in detail and propose a standstill solution, or debt repayment freeze, during the economic adjustment program period, with the purpose of reducing the moral hazard consequences of negative imposed institutional risk premium. Since our model cannot predict such outcomes, it is of far more importance in this context to understand how international investors systematically fail to forecast unsustainable debt dynamics and price in the increasing risk on demanded premium, before the situation becomes irreversible. We believe that Ludvigson and Ng [2009] hypothesis of strong market sentiment driving acyclical risk premium dynamics is a consistent explanation of this phenomenon. During good times investors fail to scrutinize correctly real risk premium and allow economies to accumulate excessive debt. The negative risk valuation of an economy debt dynamics functions as an incentive to continue to accumulate excessive debt, because it allows for short run economic and political gains. As soon as the situation deteriorates, investors penalize this behaviour and demand higher risk premium on the country bonds. Rising risk premium leads to devaluation and increased inflation expectations by domestic investors, who eventually bias their investments towards foreign assets. At this point IMF interventions provide temporary liquidity through debt subsidies and again guarantee negative risk premium, but now in an environment with bias towards foreign assets. Whatever the outcome of the adjustment program, sustainable long run growth dynamics are only achieved when bias towards domestic assets is restored. At this point investors will still be vigilant of a country's debt dynamics and demand a positive risk premium on bonds. Market sentiment now penalizes this economy. Eventually international investors' memory fades and this risk premium cycle can potentially restart. Although our proposal does not account for risk premium dynamics, we propose to study this phenomena assuming all these four scenarios separately.

2.4 Stationary dynamics for the aggregate economy

To derive the relevant dynamical system describing the optimal solution to (2.34), we first derive in section A.1 of the appendix the *Pontryagin* necessary first order conditions, which are given in (A.2) to (A.7). These conditions are sufficient if they fulfil admissibility conditions, given in (A.10), and transversality conditions, (A.8) and (A.9), following the seminal result by Arrow and Kurtz [1970]. We start the derivation of this optimal control problem by taking the time derivatives of the optimality conditions, given by (A.2) and (A.3). We obtain the following expressions:

$$\dot{\lambda}(t) = -\gamma(\gamma-1)\,C(t)^{\gamma-2}\,\dot{C}(t); \tag{2.35}$$

$$\dot{q}(t) = -\dot{\lambda}(t)\left(1+\frac{hI(t)}{K(t)}\right) - \frac{h\lambda(t)}{K(t)}\dot{I}(t) + \frac{h\lambda(t)\,I(t)}{K(t)^2}\dot{K}(t). \tag{2.36}$$

We then substitute these expressions and the optimality conditions, in the co-state conditions (A.6) and (A.7), and obtain the two possible *Keynes-Ramsey* optimal consumption rules for this economy[17], $\dot{C}_B(t)$ and $\dot{C}_K(t)$[18], arising from consumption driven either by net foreign assets accumulation or domestic capital accumulation,

$$\dot{C}_B(t) = \frac{C(t)}{\gamma-1}\left(\rho - r - rd\frac{B(t)}{K(t)}\right), \tag{2.37}$$

$$\begin{aligned}\dot{C}_K(t) = \frac{C(t)}{(\gamma-1)\left(1+\frac{hI(t)}{K(t)}\right)}\Bigg[&\frac{hI(t)\,\dot{K}(t)}{K(t)^2} - \frac{h\dot{I}(t)}{K(t)} + \left(1+\frac{hI(t)}{K(t)}\right)(\rho+\delta)\\ &-\frac{rdB(t)^2}{2K(t)^2} - \frac{hI(t)^2}{2K(t)^2} - r_k\Bigg].\end{aligned} \tag{2.38}$$

Optimal investment decisions in this economy are given by imposing $\dot{C}_B(t) = \dot{C}_K(t)$. After some fair amount of calculus and the substitution of the state condition for capital accumulation, (A.7), we obtain the differential equation driving investment activities[19],

$$\dot{I}(t) = \tfrac{I(t)^2}{2K(t)} + \left(r + \tfrac{rdB(t)}{K(t)}\right)I(t) + \left(r + \tfrac{rdB(t)}{K(t)} + \delta - r_k\right)\tfrac{K(t)}{h} - \tfrac{rdB(t)^2}{2hK(t)}. \tag{2.39}$$

[17]By *Keynes-Ramsey* consumption rules, we mean the intertemporal dynamic consumption decisions that are obtained for this control variable in an optimal control problem with a constant intertemporal discount rate. In macroeconomics literature these dynamic equations are known by *Keynes-Ramsey* consumption rules, following the work by the two famous Cambridge scholars, that related intertemporal consumption decisions with the discounted value of expected future incomes and optimal savings for capital accumulation. It is our opinion that in open economy optimization problems with two state variables, this rule is not unique, since state defined income accumulation can vary in its source. Therefore it is reasonable to impose two possible consumption paths that satisfy the optimal investment condition. In this model, optimal investment decisions impose an indifference rule on the intertemporal marginal adjustment between different assets, in order to allow for distinct capital accumulation decisions. This mechanism has the following interpretation, investors will always choose to accumulate assets that adjust faster to optimum outcomes rather than invest in assets that yield longer adjustment rates. In economics jargon the co-state variables represent the shadow price (or marginal value) of a specific asset.

[18]To obtain the second *Keynes-Ramsey* consumption rule, $\dot{C}_K(t)$, it is convenient to start by substituting the optimality condition for consumption, (A.2), and its time derivative, (2.35), in the optimality condition for investment, (A.3), and in its time derivative, (2.36).

[19]We would like to stress that this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining investment dynamics is obtained when substituting directly (2.37) while deriving (2.39).

This economy is thus defined by the dynamical system given by the differential equations of the controls in consumption and investment, (2.37) and (2.39), and the state conditions for net foreign financial assets and domestic capital accumulation, defined by (A.6) and (A.7). We define a stationary dynamical system by taking advantage of the scaled invariance of the dynamics, and redefine the variables, $X_i(t)$, in terms of domestic capital units:

$$Z_i(t) = \frac{X_i(t)}{K(t)} \Longrightarrow \dot{Z}_i(t) = \frac{\dot{X}_i(t)}{K(t)} - \frac{X_i(t)}{K(t)}\frac{\dot{K}(t)}{K(t)}, \tag{2.40}$$

where i reduces to $i \in \{1,2,4\}$ and $Z_i(t)$ defines scaled consumption, net foreign assets and investment, respectively. Following this rule, we redefine the system in terms of the scaled controls and scaled state equations:

$$\dot{Z}_1(t) = Z_1(t)\left(\frac{\rho - r(1 + dZ_2(t)) + (\delta - Z_4(t))(\gamma - 1)}{\gamma - 1}\right); \tag{2.41}$$

$$\dot{Z}_2(t) = Z_4(t)\left[1 + \tfrac{hZ_4(t)}{2}\right] + Z_2(t)\left[r + \tfrac{rdZ_2(t)}{2} + \delta - Z_4(t)\right] + Z_1(t) - r_k; \tag{2.42}$$

$$\dot{Z}_4(t) = -\tfrac{Z_4(t)^2}{2} + (r + rdZ_2(t) + \delta)Z_4(t) - \tfrac{rdZ_2(t)^2}{2h} + \tfrac{\delta + r + rdZ_2(t) - r_k}{h}. \tag{2.43}$$

Since the system is now independent of domestic capital dynamics, which only depends endogenously on investment outcomes, we have reduced the dynamics of this economy to three dimensions[20]. Domestic capital is given by the following expression,

$$K(t) = K(0)\, e^{\int_0^t (Z_4(s) - \delta) ds}, \tag{2.44}$$

following the result in (2.33). We now introduce the notion of acceptable solutions to the system given by (2.41), (2.42) and (2.43). We assume that such solutions can be described as an ergodic invariant set[21] with a well defined invariant probability measure, whose expectation operator we denote by $\langle\rangle$, such that $\lim_{t\to\infty} t^{-1}\int_0^t Z_i(t) \to \langle Z_i \rangle$. Following this definition, we can redefine domestic capital dynamics in the long run by taking the asymptotic limit of expression (2.44).

[20]Recall that $Z_3(t) = 1$, so scaled domestic capital dynamics scales out of the system. To avoid confusion with the scaled variable notation, we decided to maintain the original indexes throughout this chapter.

[21]By invariant set we refer to solutions of the scaled dynamical system that can be defined as distributions obtained from bounded trajectories of $\lim_{t\to\infty} Z_i(t)$ in the phase plane. We follow the definition of an invariant set composed by asymptotic limit sets of points given in Guckenheimer and Holmes [1983]. Let ϕ_t be a flow such that the α limit set of x for ϕ_t is the set of accumulation points of $\phi_t(x), t \to -\infty$. The ω limit set of x for ϕ_t is the set of accumulation points of $\phi_t(x), t \to \infty$. The α and ω limits of x are its asymptotic limit sets.

We obtain

$$\lim_{t\to\infty} t^{-1}\log K(t) = \langle Z_4\rangle - \delta. \tag{2.45}$$

Following our definition of solutions given by invariant sets, we can now define the constraint on scaled investment activities that imposes the existence of long run growth dynamics, $\lim_{t\to\infty} K(t) = \infty$, exponentially, $\langle Z_4\rangle > \delta$. In order to guarantee the existence of an optimum solution arising from the *Pontryagin* maximum conditions, we need to check under which circumstances the transversality conditions are fulfilled. For that purpose we rearrange expressions (A.8) and (A.9) in terms of scaled variables and substitute the co-state variables from the optimality conditions (A.2) and (A.3). The transversality conditions are now given by:

$$\lim_{t\to\infty} -\gamma\left(Z_1(t)\,K(t)\right)^{\gamma-1} Z_2(t)\,K(t)\,e^{-\rho t} = 0; \tag{2.46}$$

$$\lim_{t\to\infty} \gamma\left(Z_1(t)\,K(t)\right)^{\gamma-1}\left(1+hZ_4(t)\right)K(t)\,e^{-\rho t} = 0. \tag{2.47}$$

Recall that we defined domestic capital dynamics, $K(t)$ in (2.44), as a function of scaled investment dynamics, $Z_4(t)$. Assuming that we only accept solutions for the dynamical system defined in (2.41) to (2.43), given by invariant sets, following the result in (2.45), we can rearrange the transversality conditions given in (2.46) and (2.47) in a intuitive fashion by taking the scaled limit of the logarithm of (2.46) and (2.47), and solving the transversality constraints as an asymptotic inequality[22]. The transversality conditions are now given by:

$$\lim_{t\to\infty} t^{-1}\log\left[-\gamma\left\langle Z_1\right\rangle^{\gamma-1} K(0)^{\gamma}\left\langle Z_2\right\rangle e^{[(\gamma-1)(\langle Z_4\rangle-\delta)+\langle Z_4\rangle-\delta-\rho]t}\right] < 0; \tag{2.48}$$

$$\lim_{t\to\infty} t^{-1}\log\left[\gamma\left\langle Z_1\right\rangle^{\gamma-1} K(0)^{\gamma}\left(1+h\left\langle Z_4\right\rangle\right) e^{[(\gamma-1)(\langle Z_4\rangle-\delta)+\langle Z_4\rangle-\delta-\rho]t}\right] < 0. \tag{2.49}$$

From (2.48) or (2.49) it is straightforward to obtain the transversality constraint for the existence of an optimal solution as a function of the invariant probability measure describing scaled investment trajectories. Transversality conditions impose the following asymptotic condition $\limsup_{t\to\infty} Z_4(t) < \delta + \rho/\gamma$. Given the long run growth restriction, (2.45), requiring that the following asymptotic condition is fulfilled, $\liminf_{t\to\infty} Z_4(t) > \delta$, the optimal growth constraint for the problem defined in (2.34) is the interval,

$$\delta < \langle Z_4\rangle < \delta + \frac{\rho}{\gamma}. \tag{2.50}$$

[22]Recall that a dynamic process that scales exponentially, $w(t) \sim e^{\psi t}$, can be defined asymptotically in the following fashion, $\lim_{t\to\infty} t^{-1}\log w(t) = \psi$. If $\psi > 0 \Rightarrow w(t) \to \infty$. If $\psi < 0 \Rightarrow w(t) \to 0$.

The constraint on scaled investment dynamics defined in (2.50), guarantees that stationary solutions to the system described by (2.41) to (2.43) fulfilling this constraint are optimal growing solutions to the dynamic optimization problem defined in (2.34). The intuition for this outcome is straightforward. As the initial value solution is defined in terms of capital units, stationary solutions to the dynamical system consistent with (2.50), guarantee that the evolution of both controls and state variables are consistent with the growth rate of capital defined in (2.45), which depends solely on scaled investment dynamics. Thus a constraint on investment trajectories is sufficient to bound solutions and guarantee that these are optimal.

2.5 Steady states, linearized dynamics and local stability conditions

We now turn our attention to the study of steady states and local qualitative dynamics. The dynamical system described by (2.41), (2.42) and (2.43) has two sets of steady states with specific economic meaning. We define the complete set of steady states as $\widetilde{\widetilde{Z}} = \left\{\widetilde{Z}^{*}, \widetilde{Z}^{**}\right\}$, where the first set of steady states, $\widetilde{Z}^{*}$, is obtained by setting $Z_1^{*} = 0$. This set of steady states violates the non zero constraint for consumption. We shall refer to this set as non feasible steady states throughout this chapter. The second set of steady states, $\widetilde{Z}^{**}$, is obtained assuming $Z_1^{**} \neq 0$ and is consistent with an economic meaningful solution when $Z_1^{**} > 0$.

The first set of steady states is given by the intersection of two quadratic curves defined by the system, $\dot{Z}_2(t) \wedge \dot{Z}_4(t) = 0$. The derivation of $\widetilde{Z}^{*}$ appears to require the solution of a fourth order equation. It can be solved using a numerical polynomial solver routine. In section A.2, we provide the detailed description of an efficient and accurate algorithm to perform this computation[23]. Alternatively, one can note that the two quadratics, (Z_2^{*}, Z_4^{*}) happen to have the same center, $\left(Z_{2,0}^{*}, Z_{4,0}^{*}\right) = \left(1 + h\left(rd + \delta + r\right)\left(1 - hrd\right)^{-1}, \left(rd + \delta + r\right)\left(1 - hrd\right)^{-1}\right)$. Assuming the transformation $z_2^{*} = Z_2^{*} - Z_{2,0}^{*}$ and $z_4^{*} = Z_4^{*} - Z_{4,0}^{*}$, the system, $\dot{Z}_2(t) \wedge \dot{Z}_4(t) = 0$ reduces to:

$$\frac{rd}{2}\left(z_2^{*}\right)^2 + \frac{h}{2}\left(z_4^{*}\right)^2 - z_2^{*} z_4^{*} - C_2 = 0; \tag{2.51}$$

$$-\frac{rd}{2h}\left(z_2^{*}\right)^2 - \frac{\left(z_4^{*}\right)^2}{2} + rd z_2^{*} z_4^{*} - C_4 = 0; \tag{2.52}$$

[23]We put forward this algorithm because it provides an efficient and accurate method to compute the intersection of two conic sections and because the numerical computations described in section 2.7 are still based on this methodology.

where C_2 and C_4 are given by the following parameter expressions:

$$C_2 = Z_{4,0}^* \left(1 + \frac{h}{2} Z_{4,0}^*\right) + \left(r + \frac{rd}{2} Z_{2,0}^* + \delta - Z_{4,0}^*\right) - r_k; \tag{2.53}$$

$$C_4 = Z_{4,0}^* \left(r + rdZ_{2,0}^* + \delta - \frac{Z_{4,0}^*}{2}\right) - \frac{rd}{2h} Z_{2,0}^* + \frac{r + rdZ_{2,0}^* + \delta - rk}{h}. \tag{2.54}$$

Multiplying (2.52) by h, the solution to z_2^* in terms of is z_4^* obtained after adding expressions (2.51) and (2.52),

$$z_2^* = \frac{-\left(C_2 + hC_4\right)}{1 - rdh} \frac{1}{z_4^*}. \tag{2.55}$$

Substituting back (2.55) in (2.52), solutions to z_4^* are given by the resulting bi-quadratic equation

$$\frac{\left(rdh - 1\right)^2}{2} \left(z_4^*\right)^4 + \left(rdC_2 + C_4\right)\left(rdh - 1\right)\left(z_4^*\right)^2 + \frac{rd}{2h}\left(C_2 + hC_4\right) = 0, \tag{2.56}$$

whence,

$$z_4^* = \pm \frac{\sqrt{-\left(rdC_2 + C_4\right) \pm \sqrt{rdh^{-1}\left(rdh - 1\right) C_2^2 + \left(1 - rdh\right) C_4^2}}}{rdh - 1}. \tag{2.57}$$

In the case of the economic feasible steady states, $\widetilde{Z}^{**}$, the solution can be obtained analytically by solving the quadratic equation given by $\dot{Z}_4\left(t\right) = 0$, after substituting by the solution of $\dot{Z}_1\left(t\right) = 0 \wedge Z_1^{**} \neq 0$. This operation yields the following quadratic equation for Z_4^{**},

$$\begin{gathered}\left[-\tfrac{(\gamma-1)^2}{2hrd} - \gamma + \tfrac{1}{2}\right]\left(Z_4^{**}\right)^2 + \left[\rho + \delta\gamma + \tfrac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd} - \tfrac{\gamma-1}{h}\right] Z_4^{**} \\ +\rho + \delta\gamma - \tfrac{(\rho-r+\delta(\gamma-1))^2}{2rd} - r_k = 0.\end{gathered} \tag{2.58}$$

The solution to (2.58), defining the economic feasible steady state for Z_4^{**} is thus given by

$$\begin{gathered}Z_4^{**} = \frac{-\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right)}{-\frac{(\gamma-1)^2}{hrd}-2\gamma+1} \\ \pm\frac{\sqrt{\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right)^2-4\left[\left(-\frac{(\gamma-1)^2}{2hrd}-\gamma+\frac{1}{2}\right)\left(-\frac{(\rho-r+\delta(\gamma-1))^2}{2rd}+\rho+\delta\gamma-r_k\right)\frac{1}{h}\right]}}{-\frac{(\gamma-1)^2}{hrd}-2\gamma+1}.\end{gathered} \tag{2.59}$$

Scaled consumption and net financial asset equilibrium expressions can be computed in terms of Z_4^{**} solutions. These conditions are given in (2.60) and (2.61), below:

$$Z_2^{**} = \frac{\rho - r + \delta\left(\gamma - 1\right)}{rd} - Z_4^{**}\frac{\left(\gamma - 1\right)}{rd}; \tag{2.60}$$

$$Z_1^{**} = r_k - Z_4^{**}\left(1 + \frac{h}{2}Z_4^{**}\right) - \left(r + rdZ_2^{**} + \delta - Z_4^{**}\right)Z_2^{**}. \tag{2.61}$$

We continue the analytical discussion of our dynamical system with the definition of general *Jacobian* matrix for this system, evaluated in the vicinity of a given fixed point, $\bar{Z}_i \in \widetilde{Z}$. The general *Jacobian* is defined by

$$J = \begin{bmatrix} \frac{\rho - r\left(1 + d\bar{Z}_2\right) + \left(\delta - \bar{Z}_4\right)\left(\gamma - 1\right)}{\gamma - 1} & -\frac{rd}{\gamma - 1}\bar{Z}_1 & -\bar{Z}_1 \\ 1 & r + rd\bar{Z}_2 + \delta - \bar{Z}_4 & 1 + h\bar{Z}_4 - \bar{Z}_2 \\ 0 & rd\bar{Z}_4 - \frac{rd}{h}\left(\bar{Z}_2 - 1\right) & -\bar{Z}_4 + r + rd\bar{Z}_2 + \delta \end{bmatrix}. \tag{2.62}$$

The generalized characteristic equation, $\det\left(J - \Lambda I\right) = 0$, for this *Jacobian* matrix, using the simplifications, $J_{2,2} = J_{3,3}$ and $J_{3,2} = J_{2,3}rd/h$, comes to

$$\left(J_{1,1} - \Lambda\right)\left(J_{2,2} - \Lambda\right)^2 + \frac{rdJ_{1,3}J_{2,3}}{h} - \frac{rd\left(J_{2,3}\right)^2}{h}\left(J_{1,1} - \Lambda\right) - \left(J_{2,2} - \Lambda\right)J_{1,2} = 0, \tag{2.63}$$

where Λ is the eigenvalue solution to (2.63) and I the identity matrix. The remaining simplifications can be assumed for each set of fixed points: (i) $J_{1,2}, J_{1,3} = 0$, for the set of fixed points given by $\widetilde{Z}_i^*$; and (ii) $J_{1,1} = 0$ and $J_{1,2} = J_{1,3}rd/\left(\gamma - 1\right)$, for the set of fixed points given by $\widetilde{Z}_i^{**}$. In section A.3 we provide a general description of linearized dynamics and define hyperbolicity conditions in the vicinity of each specific set of steady states for this system.

To put forward sufficient conditions guaranteeing local stability of economic feasible solutions, $\widetilde{Z}_i^{**}$, we resort to the *Routh-Hurwitz Criterion*, following the seminal paper by Hurwitz [1964]. The *Routh-Hurwitz Criterion* guarantees that all solutions to a polynomial of degree n have a negative real part. The advantage of following this approach is that it allows us to impose local stability conditions without having to compute the eigenvalues of (A.31). To determine the signs of the solutions of a cubic polynomial, we start by defining generically (A.32) as,

$$a_0\left(\Lambda^{**}\right)^3 + a_1\left(\Lambda^{**}\right)^2 + a_2\Lambda^{**} + a_3 = 0. \tag{2.64}$$

The *Hurwitz* matrices for a cubic polynomial are generically given by:

$$H_0 = [a_0]\,, H_1 = [a_1]\,, H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}. \tag{2.65}$$

The *Routh-Hurwitz Criterion* guarantees that solutions of the polynomial defined in (2.64) have a negative real part, $Re\left(\Lambda^{**}\right) < 0$, if the determinants of the *Hurwitz* matrices are positive, $|H_0|, |H_1|, |H_2|, |H_3| > 0$. Given this definition, sufficient conditions for local stability are given by:

$$\begin{gathered} |H_0| = a_0 = 1 > 0; \\ |H_1| = a_1 = -2J^{**}_{2,2} > 0 \Longrightarrow J^{**}_{2,2} < 0; \\ |H_2| = a_1 a_2 - a_3 = \\ = 2J^{**}_{2,2}\left(\tfrac{rd}{\gamma-1}J^{**}_{1,3} - \left(J^{**}_{2,2}\right)^2 + \tfrac{rd}{h}\left(J^{**}_{2,3}\right)^2\right) + rdJ^{**}_{1,3}\left(\tfrac{1}{h}J^{**}_{2,3} - \tfrac{1}{\gamma-1}J^{**}_{2,2}\right) > 0; \\ |H_3| = a_3\,|H_2| > 0 \Longrightarrow a_3 = -rdJ^{**}_{1,3}\left(\tfrac{1}{h}J^{**}_{2,3} - \tfrac{1}{\gamma-1}J^{**}_{2,2}\right) > 0. \end{gathered} \tag{2.66}$$

We now focus on the evaluation of local qualitative dynamics in the context of a broad parameter space. We start with the definition of an algorithm for the exploration of a parameter space defined by a vector of parameters, $\widetilde{\mu}$, for a given parameter space with j parameters, where each μ_j domain is described as a bounded set of real numbers, $\mu_j \in \Re \wedge \mu_j = \left[\mu_j^{min} \;\; \mu_j^{max}\right]$. A robust algorithm that maps all possible outcomes is given by a grid search of the complete parameter space, such that the grid search has a small enough step size. According to the linearization theorem, local qualitative dynamics are robust to small parameter changes in hyperbolic autonomous dynamical systems. Following this definition, the size of the individual parameter domain can be described as the discrete sum of all its partitions, $\sum_{m=1}^{p} \mu_{m,j}$, where m is the index of each equal partition, $\mu_{m,j}$, of the original parameter domain, $\mu_j = \left[\mu_j^{min} \;\; \mu_j^{max}\right]$. Assuming that we choose a large enough number of partitions, p, such that each partition of the individual parameter space is small enough and therefore robust under the linearization theorem. Then the maximum number of parameter combinations, $\prod_j \sum_m \mu_{m,j}$, corresponds to the iterations required to perform a grid search on the entire parameter space, $\widetilde{\mu}$. Now, for example, if we consider that a grid search with a step size equal to 10^{-2} is consistent with the previous definition, the total parameter space to explore assuming $\rho, \gamma, \delta, r, r_k \in [0\ 1]$ and $d, h \in [-10\ 10]$, requires a grid search procedure that performs $4 \cdot 10^{16}$ iterations, in order to cover all possible parameter combinations. Given that this is not a feasible computational task, we propose to explore this vast parameter space assuming a stochastic variation of the grid search procedure described. Instead of grid searching each possible combination, we propose to draw parameter combinations stochastically, assuming a uniform distribution of the parameter space, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$ and $d, h \sim U(-10, 10)$. If we draw large enough samples of uniformly random distributed numbers for a given parameter space, then the total parameter space covered by the samples will asymptotically approach the original parameter space. Therefore, we can define an accurate probability measure of a given event, by computing the sample averages of parameter combinations consistent with these events.

The first conclusion drawn from the application of the stochastic search routine, for the parameter space defined in the previous paragraph, is the absence of local stable solutions. Several samples[24] of size 10^9 were computed and not a single outcome satisfied both the optimal growth condition and the *Routh-Hurwitz criterion*[25]. The qualitative dynamic outcomes of stable steady-states solutions in the saddle point sense, consistent with the optimal growth constraint on long run scaled investment dynamics (2.50), can be characterized[26] in terms of the risk premium parameter: (i) if $d > 0$, there are only saddle solutions with stable dimension equal to one and these correspond to the positive root of (2.59); and (ii) if $d < 0$, we can have saddle solutions with stable dimension equal to two for the positive root of (2.59). This last outcome is more likely to occur when there is a small bias towards home assets. Further, when $d < 0$, there are parameter combinations where the negative root of (2.59) is a saddle solution of stable dimension equal to one consistent with (2.50). These two saddle solutions may coexist for specific institutional scenarios. We discuss this result further in section 2.8. In the next sections, we provide several examples of the application of this stochastic method and, when convenient, portray some of the sample results obtained.

[24]We compute samples using a C routine compiled with the GSL scientific library.

[25]This result is confirmed by numerical computation of eigenvalues.

[26]To determine the qualitative dynamics of the fixed points we assumed that: (i) saddles with stable dimension equal to one are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) < 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) < 0$, (ii) saddles with stable dimension equal to two are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) > 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) < 0$, and (iii) divergent solutions are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) > 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) > 0$.

2.6 Local singularities: Saddle-node, Hopf and Fold-Hopf bifurcations

We now turn our attention to the description of bifurcations arising from the set of economic feasible fixed points. We start by describing the conditions required for the existence of saddle-node bifurcations also known as folds. A saddle-node bifurcation is a co-dimension one singularity that imposes dramatic qualitative changes in the system behaviour. It occurs when two fixed points collide and disappear. This bifurcation is associated with dramatic dynamic phenomena, such as hysteresis or catastrophe. In the vicinity of this bifurcation, small parameter perturbations may provoke changes in the phase-space organization and give rise to path dependence and global nonlinear phenomena, such as heteroclinic and homoclinic orbits. A recent example of this bifurcation in a continuous time model of endogenous growth is found in Chen and Guo [2008]. Saddle-node bifurcations arise in regions where an equilibrium is at a branching point, one of the eigenvalues is equal to zero and the remaining eigenvalues are real. Following our steady state formulae for economic feasible outcomes, given in (2.59) to (2.61), an optimal candidate for a saddle-node bifurcation is the parameter constraint that guarantees the square root term in (2.59) is equal to zero. For convenience, we choose to vary r_k. At the branching point of Z_4^{**}, the bifurcation parameter, r_k^{**}, is equal to,

$$r_k^{**} = \frac{-h\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right)^2}{-\frac{2(\gamma-1)^2}{hrd}-4\gamma+2} + \rho + \delta\gamma - \frac{(\rho-r+\delta(\gamma-1))^2}{2rd}. \tag{2.67}$$

Recall that equilibrium for Z_4^{**} is now given by:

$$Z_4^{**}\left(r_k^{**}\right) = \frac{-\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right)}{-\frac{(\gamma-1)^2}{hrd}-2\gamma+1}. \tag{2.68}$$

In section A.4, we describe sufficient conditions for the existence of a saddle-node bifurcation. First, we recall a necessary condition that has to be fulfilled at the critical equilibrium point, $\det\left(J^{**}\right) = 0 \Rightarrow \Lambda_1^{**} = 0$. This condition is described in (A.38) and (A.39). In (A.40), we confirm that this condition is met at the branching point, defined in (2.68), which confirms the result in (2.67). To guarantee that the remaining condition for the existence of a saddle-node bifurcation is fulfilled, we solve the characteristic polynomial, (A.41), in the vicinity of (2.67), and obtain the remaining eigenvalues, Λ_2^{**} and Λ_3^{**}, defined in (A.42). This condition is given in (A.43). Substituting (A.43) with the *Jacobian* terms, and rearranging, the existence of a saddle-node bifurcation requires that: (i) when $d > 0$ we have $Z_1^{**}(r_k^{**}) > h\left[r + rdZ_2^{**}(r_k^{**}) + \delta - Z_4^{**}(r_k^{**})\right]^2$; and (ii) when $d < 0$ we have $0 < Z_1^{**}(r_k^{**}) < h\left(r + rdZ_2^{**}\left[r_k^{**}\right) + \delta - Z_4^{**}(r_k^{**})\right]^2$ and $h > 0$.

To confirm the existence of saddle-node bifurcations for this economy, we performed a numerical evaluation of possible outcomes, assuming that growth and optimality conditions are fulfilled for solutions in the feasible economic space. We computed samples following the stochastic sampling method discussed in the previous section[27]. The outcomes obtained suggest that saddle-node bifurcations are a common outcome for a broad range of parameter combinations and are more likely to occur in institutional scenarios where there is bias toward home assets, $h < 0$, and positive risk premium, $d > 0$. Numerical results suggest that when $d > 0$, we have $h > 0$ and $\Lambda_2^{**}\Lambda_3^{**} < 0$. When $d < 0$, numerical results suggest that $\Lambda_2^{**}, \Lambda_3^{**} > 0$.

We continue the discussion on local bifurcations with a description of analytical conditions for the existence of general and attracting *Hopf* bifurcations. The attracting *Hopf* bifurcation is usually related to the existence of limit cycles that can be observed physically. The existence of *Hopf* bifurcations in models of endogenous growth implies the coexistence of optimal growth and cycles. This literature has established itself in growth theory during the last two decades, in what has been established as optimal growth and cycle models. Some recent papers on this subject that follow the same base modeling assumptions of our proposal are the proposals by Slobodyan [2007], Nishimura and Shigoka [2006] and Wirl [2002]. Examples of earlier literature on this subject can be found in the papers by Lordon [1995], Greiner and Semmler [1996], Greiner [1996], Drugeon [1998], Benhabib and Nishimura [1998] and Asada et al. [1998].

General *Hopf* bifurcations require that the following set of eigenvalue conditions is fulfilled: (i) $\mathrm{Re}\left(\Lambda_{2,3}^{**}\right) = 0$; (ii) $\mathrm{Im}\left(\Lambda_{2,3}^{**}\right) \neq 0$. The existence of an attracting *Hopf* bifurcation requires additionally that $\mathrm{Tr}\left(J^{**}\right) < 0 \Rightarrow \mathrm{Re}\left(\Lambda_1^{**}\right) < 0$. For reasons of convenience, we define these conditions using the *Hurwitz Determinants* described in section 2.5. Following Liu [1994], an attracting *Hopf* bifurcation for vector fields in $\Re^3$, occurs if the following generic conditions are fulfilled: (i) $\left|H_0\left(\mu^{**}\right)\right|, \left|H_1\left(\mu^{**}\right)\right| > 0$; (ii) $\left|H_2\left(\mu^{**}\right)\right| = 0$; (iii) $a_3 > 0$; and (iv) $\partial\left|H_2\left(\mu^{**}\right)\right|/\partial\mu \neq 0$. Where, μ is the bifurcation parameter and μ^{**} is the bifurcation parameter at the *Hopf* equilibrium point, which is obtained from the solution of the second condition. The last condition guarantees that the eigenvalues cross the imaginary axis with non-zero speed. A general *Hopf* bifurcation does not require

[27]The results were obtained from a sample with 10^9 random draws of uniform distributed numbers, $\rho, \gamma, \delta, r \sim U(0,1)$ and $d, h \sim U(-10, 10)$. We restricted the outcomes of this sample to solutions where $0 < r_k^{**} < 1$.

that $|H_1(\mu^{**})| > 0$ is fulfilled. Given this set of general conditions, the existence of attracting *Hopf* bifurcations in our economy, requires that the following conditions are fulfilled:

$$\begin{gathered} -rdJ_{1,3}^{**}\left(\tfrac{1}{h}J_{2,3}^{**} - \tfrac{1}{\gamma-1}J_{2,2}^{**}\right) > 0; \\ 2J_{2,2}^{**}\left(\tfrac{rd}{\gamma-1}J_{1,3}^{**} - \left(J_{2,2}^{**}\right)^2 + \tfrac{rd}{h}\left(J_{2,3}^{**}\right)^2\right) + rdJ_{1,3}^{**}\left(\tfrac{1}{h}J_{2,3}^{**} - \tfrac{1}{\gamma-1}J_{2,2}^{**}\right) = 0; \qquad (2.69) \\ J_{2,2}^{**} < 0. \end{gathered}$$

To test the coexistence of optimal growth and cycles, we explore numerically solutions consistent with $Z_1^{**} > 0$, (2.50) and (2.69), assuming r_k as bifurcation parameter. For this purpose, we adapted our routine, to explore a bifurcation interval, $0 < r_k < 1$, for a given stochastic combination of parameters[28]. As expected, the modified stochastic search routine was not able to detect the existence of parameter combinations consistent with optimal growth dynamics undergoing attracting *Hopf* bifurcations[29]. Given this outcome, we focused our efforts on the detection of general *Hopf* bifurcations. The samples obtained show that only the positive root of Z_4^{**} is consistent with the existence of general *Hopf* bifurcations. This bifurcation scenario is more likely to occur when $d, h < 0$, but may also occur when $d < 0$ and $h > 0$. Finally, our results show that $0 < \Lambda_1^{**} < 1$ and in most of the cases small.

We finish this section with the description of necessary conditions required for the existence of a codimension two fold-hopf bifurcation in this system[30]. This bifurcation is born from the merging of the two previously discussed instabilities. When the saddle-node and *Hopf* bifurcation curves are tangential in the parameter space a fold-hopf bifurcation is born. This singularity is characterized in vector fields in $\Re^3$ by: (i) $\det(J^{**}) = 0 \Rightarrow \Lambda_1^{**} = 0$; (ii) $\mathrm{Tr}(J^{**}) = 0 \Rightarrow \mathrm{Re}\left(\Lambda_{2,3}^{**}\right) = 0$; and (ii) $\mathrm{Im}\left(\Lambda_{2,3}^{**}\right) \neq 0$. The presence of this bifurcation shows that there is a path

[28] We explore supercritical *Hopf* bifurcations by varying parameter r_k along the interval $0 < r_k < 1$, assuming a step size iteration of magnitude 10^{-2}. The outcomes discussed in this section were obtained from a sample with 10^7 random draws, assuming that the remaining parameters are uniformly distributed as in previous experiments, $\rho, \gamma, \delta, r, \sim U(0,1)$ and $d, h \sim U(-10, 10)$. In this experiment we consider that the supercritical *Hopf* bifurcation point is given by the average value of the crossing interval, when there is a change in sign consistent with (2.69). Steady state outcomes take into account this adjustment.

[29] The easiest path to detect a supercritical *Hopf* bifurcation in hyperbolic autonomous dynamical systems is to continue equilibrium from a spiral attractor. The numerical results described in section 2.5 suggest that this is not a likely outcome. The results obtained in this section confirm our conclusions regarding steady-state stability.

[30] This bifurcation is also called Gavrilov-Guckenheimer, saddle-node *Hopf*, zero *Hopf* and zero pair bifurcation. A detailed technical discussion of this bifurcation scenario can be found in Guckenheimer and Holmes [1983] and Kuznetsov [1998], chapter 7 section 4 and chapter 8 section 5, respectively.

towards complex dynamics in this system. Fold-Hopf points are associated with several nonlinear phenomena. The influence of this bifurcation is not limited to parameter regimes in its close vicinity, it stretches far beyond the specific bifurcation point and may give rise to a cascade of complex dynamic transitions, including the local birth of chaos. In this framework, policy analysis has to take into account the increased complexity of possible model outcomes.

Although the unfolding of a fold-hopf bifurcation scenario is not fully known and in some sense impossible to describe in all detail, four transition scenarios can be considered for flows in $\Re^3$. The first two scenarios imply subcritical transitions and no torus formation. The unfolding of the most simple of these scenarios may not be associated with global nonlinear phenomena, but at least one limit cycle is known to exist[31]. The remaining scenarios may arise with subcritical and supercritical transitions and give rise to torus bifurcations and complex global dynamics. These transitions may create saddle node bifurcations of periodic orbits on the invariant torus, torus breakdown and chaos, heteroclinic orbits on a sphere (heteroclinic cycles), bursting and Sil'nikov bifurcations leading to chaos. A formal definition of the exact unfolding scenarios for this system requires the computation of the normal form coefficients using numerical continuation software.

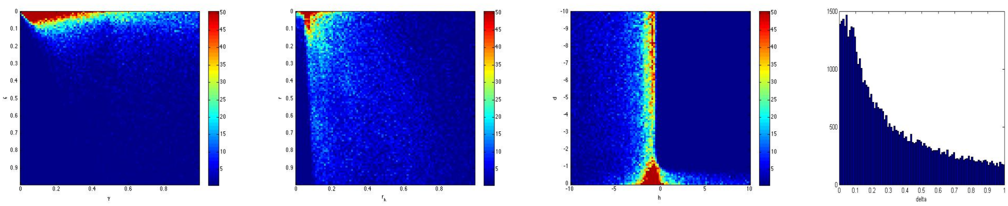
This bifurcation scenario has been gaining a greater attention in other fields of applied mathematics[32]. This has not been the case in the field of economic dynamics. We were only able to find one article where this topic is discussed in economic literature. Brito [1999] proves the existence of fold-hopf bifurcations generally for optimal control problems with one control and three state variables, that have solutions given by flows in $\Re^3$.

We finish this discussion with a description of the sufficient conditions for the existence of the fold-hopf bifurcation. The easiest path to obtain these conditions is to continue equilibrium from the saddle-node bifurcation point. A necessary condition for the existence of a fold-hopf bifurcation is given by setting $J^{**}_{2,2}\left(r^{**}_k\right), J^{**}_{2,3}\left(r^{**}_k\right) = 0$, following the saddle-node condition (A.39). We have a fold-hopf bifurcation, given in terms of parameters r^{**}_k and ρ^{**}, when condition (A.45) is fulfilled, and provided that there is negative risk premium, $d < 0$, following the eigenvalue solution in (A.44). If $d > 0$ we have a neutral saddle with $\Lambda^{**}_1 = 0$ and $\sum \Lambda^{**}_{2,3} = 0$. The second parameter constraint, ρ^{**}, is given in (A.46). Again we re-

[31]See Kuznetsov [1998] page 339.

[32]Crommelin et al. [2004] discusses heteroclinic cycles driving atmospheric regimes, Hilker and Malchow [2006] portrays strange periodic attractors in population dynamics, and Izhikevich [2001] shows bursting in neurocience dynamic systems in the vicinity of fold-hopf bifurcations.

sort to our stochastic routine to map the the parameter space for this bifurcation[33]. Below in Figure 2.2 we portray the sampling outcomes obtained for meaningful optimal economic solutions consistent with fold-hopf bifurcations of equilibrium. The graphics below portray the parameter frequencies, depicted as heat maps for specific parameter combinations, consistent with computed fold-hopf bifurcations of equilibrium. This sample shows that fold-hopf bifurcations are more likely to occur when there is negative risk premium and bias towards home assets. We may also have scenarios, where $h > 0$ and $d < 0$, but d takes a small negative value. The vast majority of fold-hopf points occur when $h < 0$ and small. To get further insight in the possible transitions arising from the fold-hopf point, we perturbed each of the system parameters, $\tilde{\mu}$, by $\Delta\mu = \pm 0.05$, for the entire computed sample. For transitions where both solutions are consistent with $Z_1^{**} > 0$ and (2.50), we may have saddle-repellor and saddle-saddle scenarios. The first scenario suggests that unfoldings are simple for this case and relevant nonlinear phenomena is not a likely outcome. However, the saddle-saddle unfolding scenario may lead to complex dynamic phenomena, as a result of the complex organization of the saddle's invariant manifolds. The existence of a general *Hopf* bifurcations in the vicinity of these transitions, as previously discussed, may also play a role on the complexity of both unfolding scenarios. Still, the absence of attracting *Hopf* bifurcations in the vicinity of the unfolding, limits the range of nonlinear phenomena that may arise in this system. In section 2.8, we discuss with more detail some of the complex dynamics that may arise from this bifurcation, and put forward some conjectures and examples with meaningful economic interpretation.



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Histogram δ

Figure 2.2: Parameter density distributions for Fold-Hopf bifurcation

[33]The parameter distributions for the fold-hopf bifurcation were computed numerically for a sample with 10^9 random draws, assuming parameters distributed uniformly, $\gamma, \delta, r \sim U(0,1)$, $h \sim U(-10,10)$ and $d \sim U(-10,0)$. We considered fold-hopf co-dimension two outcomes consistent with $0 < r_k^{**}, \rho^{**} < 1$, $Z_1^{**}(r_k^{**}, \rho^{**}) > 0$ and (2.50).

2.7 Economic space boundary dynamics

The existence of non feasible steady states in economic models has not shared the same amount of attention in literature, when compared to its economic counterparts. From a mathematical modeling point of view, the unjustified existence of these fixed points is sufficient ground to question the quality of a proposal. Since many economic models share this specific feature, we link the absence of a consistent discussion on this topic, to the lack of explanatory power of linear and quasi-linear proposals. In this section, we discuss the role of non feasible fixed points in a economic setting and derive policy rules that guarantee that orbits starting within the economic space, stay in this region. This concept is crucial, since it solves the modeling issue described, and introduces a novel policy objective. In Figure 2.3, we describe this mechanism schematically. Flows in the vicinity of the nullcline plane, $\dot{Z}_1(t) = 0$ for $Z_1(t) = 0$, are locally repelling for $Z_1(t) > 0$, when the growth rate of scaled consumption, $\dot{Z}_1(t)$, is positive in the vicinity of this separatrix plane. Economic recovery can be achieved by an endogenous dynamic mechanism if the necessary institutional framework is in place.

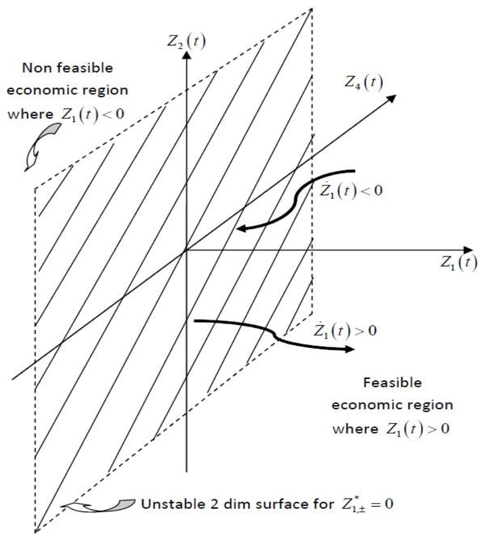


Figure 2.3: Separatrix planes for feasible economic regions

The mechanism described in Figure 2.3 has the following mathematical interpretation. Any trajectory, $\Delta(t)$, which starts or is in the vicinity of the region described by $Z_1(t) \approx 0 \wedge Z_1(t) > 0$, will stay in the space described by $Z_1(t) > 0$, if the following invariant condition is fulfilled, $\dot{Z}_1(t) = Z_1(t) f(Z_1(t), Z_2(t), Z_4(t)) > 0 \implies f(Z_1(t), Z_2(t), Z_4(t)) > 0$. This condition guarantees that the invariant plane, $\dot{Z}_1(t) = 0$ when $Z_1(t) = 0$, is locally repelling. Since the scaled consumption equation for this economy, (2.41), is already in the functional form of the invariant condition, a sufficient condition for the scenario described in Figure 2.3 is given by:

$$\frac{\rho - r\left(1 + dZ_2\left(t\right)\right) + \left(\delta - Z_4\left(t\right)\right)\left(\gamma - 1\right)}{\gamma - 1} > 0. \tag{2.70}$$

However, the invariant condition given in (2.70), does not add much to our knowledge of the system. It just guarantees that trajectories starting in the economic space will stay there. In order to have a greater insight on the dynamics in the vicinity of this plane, we have to consider the qualitative dynamic properties of the nullcline surface dominating this region. For presentation purposes, we shall assume for now that a necessary condition for the existence of a repelling frontier, requires that the local dynamics of $\tilde{Z}_i^*$, have all at least an unstable dimension equal to two. Since in section A.3, we are able to derive the general expressions in (A.37), for the eigenvalues describing local dynamics in the vicinity of $\tilde{Z}_i^*$. It is straightforward to define a set of rules that guarantee this outcome is fulfilled. Two scenarios can be considered, when we take into account the interaction between the institutional parameters, d and h. When a country faces an institutional framework consistent with $dh < 0$, the boundary has a unstable dimension equal at least to two when $J_{2,2}^* > 0$,

$$r + rdZ_2^* > Z_4^* - \delta. \tag{2.71}$$

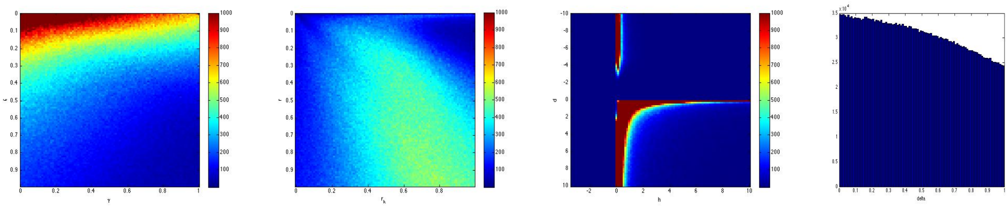
This rule has two possible interpretations, when we to take into account the net financial status of an economy towards the rest of the world. In economies facing contraction and accumulation of foreign debt, $Z_2\left(t\right) > 0$, the growth rate of debt should be bigger than the growth rate of domestic capital. This rule allows for growth of domestic assets, as long as it occurs at a smaller rate than the interest growth on foreign bonds. If a country faces negative risk premium it might impose disinvestment on domestic assets. Recall that condition (2.71) only guarantees that local dynamics in this plane have an unstable dimension equal to two. To guarantee that solutions are repelled in all dimensions, the growth rate of consumption has also to be positive in the vicinity of the economic frontier, $J_{1,1}^* > 0$, following the result in (A.37). Since in the long run both foreign debt and domestic capital have to follow a balanced growth path, two hypothesis can be considered, in order to guarantee that the gap between growth paths does not widen in the long run. The first solution implies a contraction of the consumption growth rate and of the domestic capital accumulation rate, in order to reduce the level of debt. This hypothesis implies reduced investment and slower growth dynamics. The second hypothesis involves offsetting domestic capital returns to allow for investment and consumption growth to catch up with the faster debt growth rate. This scenario is consistent with challenges posed to economies with current balance deficits and facing contraction due to excess debt service. In economies facing contraction and accumulation of foreign assets, $Z_2(t) < 0$, the growth rate of investments abroad might impose disinvestment in domestic assets. In broader terms, this rule implies that policy has to guarantee that the rate of growth of revenues on foreign asset from domestic investment abroad, cannot be used exclusively on domestic capital accumulation. These foreign capital influxes have to finance domestic consumption. When an economy faces positive risk premium it might impose disinvestment on domestic capital. This rule provides an indirect instrument guaranteeing that foreign based capital revenues are used in domestic consumption activities during contractions. In the long run, growth stability has to achieved by an increased growth rate of consumption and domestic capital. This scenario is consistent with challenges posed to economies with current balance surplus facing contractions due to reduced world demand for their goods and services. In economies facing an institutional framework consistent with, $dh > 0$, two policy rules have to be considered:

$$J^*_{2,2} > 0 \wedge J^*_{2,2} > J^*_{2,3}\sqrt{\frac{rd}{h}}; \tag{2.72}$$

$$J^*_{2,2} < 0 \wedge J^*_{2,2} < J^*_{2,3}\sqrt{\frac{rd}{h}} \Longrightarrow J^*_{1,1} > 0. \tag{2.73}$$

The rule described in (2.72) has the same interpretation as the rule described in (2.71). Local dynamics in the boundary surface are only repelling when $J^*_{1,1}, J^*_{2,2} > 0$ holds. The interpretation for creditor and debtor economies given in the previous paragraph, still holds for this case. When the constraint, $J^*_{2,2} > 0$, does not hold, the only policy solution available is to guarantee that the growth rate of consumption is always positive, as described in (2.73). This is a last resort option. The policy-maker has to guarantee consumption growth in the event of severe institutional environment, $d, h > 0$, as observed in economies facing balance of payment crisis leading to exchange rate crisis. In such cases, only direct intervention to curb consumption dynamics guarantees that local dynamics in the vicinity of $\tilde{Z}^*_i$ have at least two unstable dimensions. In this scenario, it is not guaranteed that trajectories are repelled in all dimensions near the economic frontier, as non feasible fixed points have at least one stable dimension.

We finish this section with the sampling results obtained for local dynamics satisfying the rules described in (2.71) to (2.73). In Figure 2.4, below, we portray the parameter distributions consistent with these rules. [34]. A quick inspection shows that the existence of local dynamics with an unstable dimension equal to two is more likely to occur when $h, d > 0$. It is also a likely outcome for institutional scenarios described by $d < 0$ and $h > 0$. In section 2.3, we related these institutional scenarios with economies facing dire economic conditions. In our opinion, this result has the following interpretation. This nonlinear setup is capable of capturing the existence of a dynamic mechanism that avoids ever declining economic trajectories for countries facing severe institutional and financial conditions. Economies with favourable institutional frameworks do not require the existence of this dynamic mechanism.

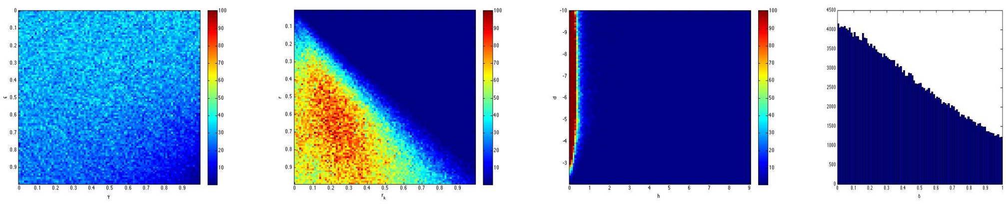


(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

Figure 2.4: Parameter distributions for co-dimension two repelling frontier regions

When we consider the sample described in Figure 2.4 and assume that $\mathrm{Re}\left(\Lambda^*_{1,2,3}\right) > 0$, is fulfilled for all $\tilde{Z}^*_i$, two patterns arise. First, local repelling dynamics in the economic boundary are only consistent with $d < 0$ and $h > 0$. In section 2.3, we related this institutional scenario to foreign interventions guaranteeing a temporary negative risk premium environment, for countries facing capital flights due to dire domestic financial conditions. The results in Figure 2.5 seem to support foreign policy interventions that guarantee a temporary debt subsidy to distressed nations. Second, sampling results suggest that returns on domestic assets have to be smaller than the international interest rate, $r_k < r$. We conclude that a dynamic recovery path may exist for countries facing productivity problems, as long as they are able to access foreign capital at a sustainable level. Finally, recall that this set of rules only guarantees that flows are repelled when they approach the economic frontier. Convergence to a balanced growth regime depends on additional factors. If these are not met, there is a risk that boundary interactions result in explosive debt dynamics and create an unsustainable economic environment.

[34] The results portrayed in this section were obtained from a sample with 10^9 random draws of uniform distributed parameter, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$, $d, h \sim U(-10, 10)$. The numerical computation of steady-states followed the definitions given in section A.2. Only computed steady-states consistent with a maximum absolute error smaller than 10^{-5} were considered.



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

Figure 2.5: Parameter density distributions for repelling frontier regions

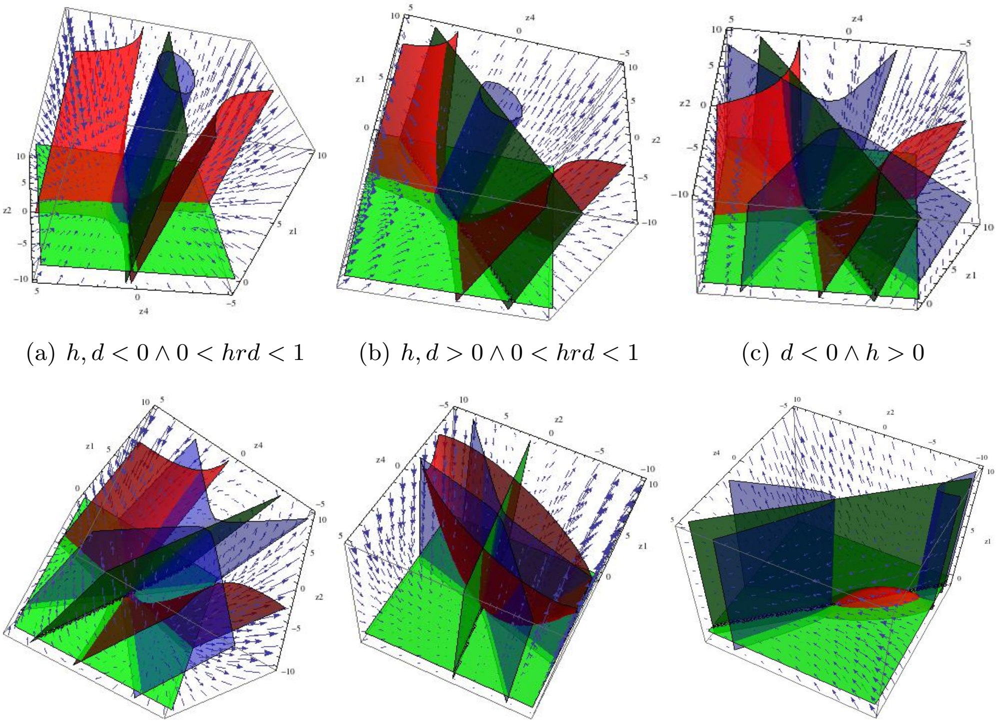
2.8 Global dynamics: Conjectures, examples and policy implications

We conclude this presentation with a discussion on the global dynamics[35] of this system. We start with a description of the phase-space organization for $Z_1(t) > 0$. In the previous section, we discussed the local dynamics of the nullcline plane $\dot{Z}_1(t) = 0$ for $Z_1(t) = 0$. When $Z_1(t) > 0$, the $\dot{Z}_1(t) = 0$ nullcline defines another plane that intersects the boundary plane given by $Z_1(t) = 0$. The intersection of these planes is given by the line described by the steady-state expression (2.60). The remaining nullclines are described by two quadrics. In section A.5, we classify geometrically these quadratic surfaces and show that the phase space organization depends on institutional scenarios for h and d combinations. At $hrd = 1$ the phase-space simplifies dramatically. For the remaining cases we distinguish between six relevant scenarios. The main phase space scenarios and the corresponding vector fields are portrayed in Figure 2.6 below, assuming reasonable numerical values for the model parameters.

The scenarios depicted in(2.6) portray the challenge posed to the policy-maker in a nonlinear environment. Different institutional setups impose drastic changes in the phase space organization. Further, the absence of local stable solutions, implies that the existence of asymptotic orbital stable solutions, for a given institutional scenario, requires the existence of an attractive set[36]. Alternatively, we can assume that the policy-maker challenge is a boundary value problem, more specifically a *Turnpike* control problem, where the policy-maker role is to impose discontinuous jumps in the control variable, in order to guarantee that the distance travelled from the initial point to the terminal value is maximizes the policy objec-

[35] A detailed mathematical introduction to the topic of global dynamics for flows in $\Re^3$ can be found in Wiggins [1988] and Wiggins [2003].

[36] By attractive set we refer to the broad definition of an attractor, where flows starting in the neighbourhood of the attractive set, called the attractor basin, asymptotically evolve towards an invariant closed subset of the phase space. This invariant set is the attractive set.



(a) $h, d < 0 \wedge 0 < hrd < 1$ (b) $h, d > 0 \wedge 0 < hrd < 1$ (c) $d < 0 \wedge h > 0$

(d) $d > 0 \wedge h < 0$ (e) $h, d < 0 \wedge hrd < 0 \wedge hrd > 1$ (f) $h, d > 0 \wedge hrd < 0 \wedge hrd > 1$

Figure 2.6: Phase space organization for different institutional scenarios

tive over time. *Turnpike* theory has its roots in modern growth theory. For optimal growth models, McKenzie [1976] frames the policy problem[37], as one of finding the fastest route to the desired solution, when the departure point is far from the final long run solution[38]. In our setup, the *Turnpike* problem reduces to a problem of placing orbits on the stable manifold of a saddle solution that fulfils $\max\left(Z_1^{**}\right)$ and (2.50). Initial values for the two controls variables, $Z_1(0)$ and $Z_4(0)$, can be chosen for this purpose. Discontinuous jumps of the control variables for $t > 0$ can also be considered. However, several issues arise with this approach in a nonlinear environment with multiple equilibria. First, it is unlikely that the stable manifolds of optimal saddle solutions can be computed exactly. This is particularly true when $d > 0$, since feasible solutions have a stable dimension equal to one. A realistic option is to shoot trajectories towards the boundary saddle value solution and take into account system dynamics in the vicinity of the saddle stable manifold. This is a technically feasible task but of difficult application for nonlinear vector fields in

[37]For some recent developments and open problems in *Turnpike* theory and optimal growth see McKenzie [1998].

[38]If the origin is in the vicinity of the final solution and the *Turnpike* far away, then the best policy option may not involve the *Turnpike*.

$\Re^3$. The literature on this subject suggests orbital control at a value loss along the stable eigenvector (or eigenplane) obtained from linearisation of equilibrium in the vicinity of the saddle solution. In discounted problems the quality of this method worsens as the turnpike distance increases. This is a result of the value loss boundaries widening as $t \to \infty$. Second, in the event of a small parameter perturbation for $t > 0$ and $Z_i(t) = Z_i^{**}$, the *Turnpike* solution may no longer be the best and/or an optimal boundary solution to the control problem. In this context, a *Turnpike* control policy: (i) imposes a discontinuous jump towards the stable manifold of the new best optimal saddle solution, and we have a *Turnpike* heteroclinic connection of equilibria; or (ii) the parameter perturbation leads to the disappearance of optimal saddle solutions and the sole policy option available relies on the existence of a *Turnpike* path towards an attractor solution asymptotically consistent with (2.50). Thus, *Turnpike* control in this context may impose global nonlinear phenomena, in the form of heteroclinic connections, and saddle path global solutions only exist for a given parameter set and may be not robust to perturbations of the model parameters. This second hypothesis stresses the need to evaluate the existence of attractive sets driven by global interactions in nonlinear multi-equilibria growth models. As flows bounded by hyperbolic compact sets are likely to arise in the vicinity of phase space singularities, the analysis of bifurcations is crucial to understand the specific nonlinear global phenomena that may arise in this system. We conclude that the existence of a *Turnpike* control policy is not sufficient to guarantee the existence of optimal solution path's for a given constrained parameter set. In such context, the study of both local and global nonlinear phenomena is required to increase the policy options available to the policy-maker.

To demonstrate our arguments, we start by considering two conjectures. We evaluate the conditions for the existence of heteroclinic and homoclinic dynamics consistent with the definition of attractive sets. In this setup, heteroclinic orbits[39] correspond to flows connecting long run growth regimes, while homoclinic orbits[40] can be linked to temporary structural change dynamics. Recall that in section 2.1, we introduced the concept of endogenous structural change, as a valid hypothesis to explain the structural breaks observed in macroeconomic data, and portrayed

[39] An orbit, $\Delta(t)$, is said to be heteroclinic if it connects two fixed points, Z_a^{**} and Z_b^{**}, such that: (i) $\Delta(t) \longrightarrow Z_b^{**}$ as $t \longrightarrow +\infty$, and (ii) $\Delta(t) \longrightarrow Z_a^{**}$ as $t \longrightarrow -\infty$. Where Z_b^{**} is a stable feasible solution that fulfils (2.50).

[40] An orbit, $\Delta(t)$, is said to be homoclinic if $\Delta(t) \longrightarrow Z_a^{**}$ as $t \longrightarrow \pm\infty$. Where Z_a^{**} is a economic meaningful equilibrium of our system that fulfils (2.50). Homoclinic orbits are characterized by having a infinite period but finite length. In layman's terms, this means that diverging flight trajectories eventually converge to the stable manifold of the saddle equilibrium, where they will stay longer and longer, before starting another flight. The invariant set describing homoclinic flows can thus be asymptotically approximated to Z_a^{**}.

this empirical phenomena in Figure 2.1, for US and UK $\log(GDP)$ time series. We related this hypothesis to the existence of heteroclinic and homoclinic phenomena. However, heteroclinic and homoclinic orbits are not usually consistent with the strict definition of structurally stable solutions[41] [42]. In general terms, the strict structural stability criterion imposes that the qualitative features of a system are robust under C^1 small perturbations. In layman terms, structural stability implies that the qualitative properties of a system reaming unchanged with small parameter perturbations[43]. This criterion imposes severe limitations to the study of nonlinear global phenomena, since solutions defined by attractive sets may be asymptotically, but not structurally, stable under small perturbations. As Guckenheimer and Holmes [1983] puts it: *"This principle was embodied in a stability dogma, in which structurally unstable systems were regarded as somehow suspect. This dogma stated that, due to measurement uncertainties, etc., a model of a physical system was valuable only if its qualitative properties did not change with perturbations."*. The authors suggest that the constraints imposed by the structural stability paradigm[44] may not be relevant in all settings, and in some setups, it might be more reasonable to consider a stability definition that takes into account the complexity of global nonlinear phenomena.

[41] Structural stability is a fundamental concept of dynamic systems theory. It was introduced by the russian mathematicians Aleksandr Andronov and Lev Pontryagin. A formal proof of *Andronov-Pontryagin* structural stability criterion exists only for vector fields in $\Re^2$. The proof is given by Peixoto [1959a] and Peixoto [1959b].

[42] The existence of homoclinic and heteroclinic solutions consistent with the definition of structural stability has been identified in the mathematical literature for vector fields in $\Re^3$ and higher dimensions. These recent results in dynamical systems theory show the limitations of evaluating the structural stability of solutions, based on the strict assumptions of the *Andronov-Pontryagin* criterion, for higher order systems. For example, Guckenheimer and Holmes [1988] give evidence of the existence of structurally stable heteroclinic cycles in vector fields in $\Re^3$.

[43] There are many extensions of the strict mathematical definition of structural stability in applied nonlinear science. It is common to find proposals that define structural stability following the *Andronov-Pontryagin* strict criterion, but take into account model specificities and theoretical implications of different structural stability definitions, to provide a consistent measure of the structural stability of a given system. Some examples of this approach can be found in economic literature. Fuchs [1975] discusses the implications for economics of the notion of structural stability. Anderlini and Canning [2001] propose bounded rationality as a criterion of structural stability in dynamic games with fully rational players. Zhang [2002] emphasizes the need for a structural dynamic approach in economics and discusses possible implications of complexity theory for the study of economic processes.

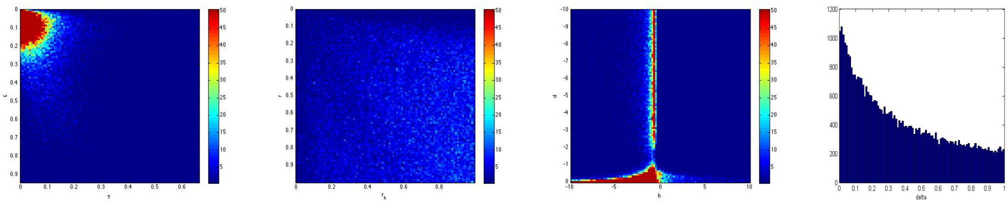
[44] The structural stability dogma also played a role on the development of economic theory. The neoclassical critic on the evolutionary economic paradigm has been based on the grounds that evolutionary economic models are inherently structurally unstable. In a recent article, Veneziani and Mohun [2006], reviews the neoclassical critic of Goodwin's growth cycle model and evolutionary dynamics approach. The author suggests that the neoclassical dismissal of evolutionary dynamic proposals on these grounds, is not in accordance with the modern mathematical concept of structural stability. In a recent working paper, Matteo [2009] reviews early discussions on structural stability and economic dynamics by Morishima and discusses its implications for neoclassical growth theory, following *Solow's* seminal proposal on economic growth.

"Thus the stability dogma might be reformulated to state that the only properties of a dynamical system which are physically relevant are those which are preserved under perturbations of the system. The definition of physical relevance will clearly depend upon the specific problem.". Taking into account this broader definition, we argue that a reasonable criterion for relevant solutions is, asymptotic orbital stability under small perturbations consistent with optimal growth dynamics. Even if these perturbations lead to qualitative changes and transitions between attractors. In a policy framework our argument has the following interpretation. The policy-maker should acknowledge the complexity of interactions driving the short run economic process and decisions should be restricted to policies that promote a long run stable growth environment. Even if this *laissez faire* approach results in the economy undergoing structural changes in the short run.

A conjecture consistent with the above criterion is the heteroclinic cycle scenario. This hypothesis has an interesting economic interpretation and introduces novel challenges to macroeconomic policy definition. In our setup, the interaction between saddles, with different stable dimensions, may be consistent with the existence of heteroclinic cycles. In section 2.6, we referred how the unfolding of a fold-hopf bifurcation might be consistent with this phenomena[45]. Heterocilinic cycles in this context arise from homoclinic bifurcations and are preceded by chaotic parameter regimes. Long run growth dynamics driven by heteroclinic cycles are characterized by long lasting fast growing regimes that undergo increasing, and then decreasing periods of volatility, before a crisis event drives the economy abruptly to the slow growth regime of the past. Evolutionary growth theories suggest that severe crisis, or the downturn of the long wave cycle, is preceded by fast growth regimes with low volatility. In a recent article on the *Great Moderation*, the 2007-2008 financial crisis and the resulting strong economic contraction, Bean [2010] suggests that: *"The longer the low volatility period lasts, the more reasonable it is to assume that it is permanent. But as tail events are necessarily rarely observed, there is always going to be a danger of underestimating risks"*. According to the author, the forecasting problem faced by the decision maker is exacerbated by the lack of information that is required to learn the higher moments of economic distributions. In other words, the complexity of nonlinear phenomena poses dramatic challenges to the management of macroeconomic risks and the lack of knowledge about the true dynamics driving the economic process may produce policies with dire consequences for long run growth.

[45]The interaction of the stable and unstable dimensions of the two saddles creates a compact set with a sphere geometry, which results in dense orbits connecting the two equilibrium. See Crommelin et al. [2004] for a clear geometric description and discussion of this phenomena in the vicinity of fold-hopf bifurcations.

As heteroclinic cycle scenarios have the potential to unleash a similar cascade of dynamic events, we propose to evaluate this conjecture and its implications throughout this chapter. We start by evaluating the feasibility of this conjecture in our setup by considering a conservative scenario, where Z_a^{**} and Z_b^{**} both fulfil (2.50) and $Z_1^{**} > 0$. This conjecture requires the co-existence of economic meaningful multiple saddle equilibrium solutions with different stable dimensions that are consistent with optimal growth constraints. Below, in Figure 2.7, we portray the parameter distributions consistent with this conjecture[46].



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

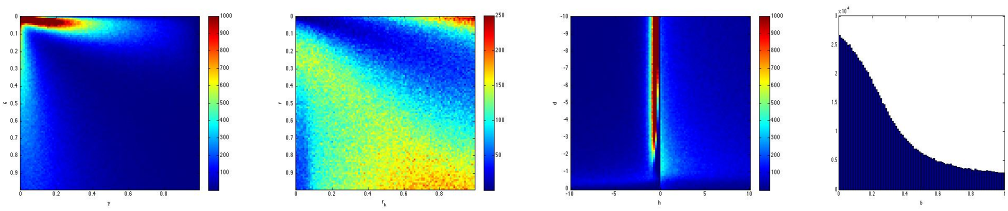
Figure 2.7: Parameter density distributions for saddle-saddle optimal scenarios

Saddle-saddle interactions are more likely to occur when $h, d < 0$, but may also occur when $h > 0$, $d < 0$, for small values of h and d. We conclude that meaningful heteroclinic cycle scenarios are only likely to occur in the vicinity of fold-hopf bifurcations, following the results portrayed in Figure 2.2.

The second conjecture proposed, is related to the structural change phenomena observed in Figure 2.1, for US $\log(GDP)$ data. In this example, the estimation procedure computed two breaks for the second quarter of 1978 and the last quarter of 1983. The difference between the estimated slope shifts is approximately zero. We extrapolate that a temporary medium run crisis led the economy out of its long run growth path, into a diverging and then converging flight, which eventually rested in the initial long run growth equilibrium. This dynamic event is consistent with the mathematical definition of a homoclinic orbit. In this scenario, an economy will stay long periods in the vicinity of long run equilibrium, but will undergo crisis or hysteria for short periods of time. Homoclinic phenomena has been gaining attention in recent growth literature. Benhabib et al. [2008] and Mattana et al. [2009], for example, evaluate homoclinic bifurcations in continuous time endogenous growth models. However, homoclinic orbits are most likely not robust to small perturbations of the model parameters.. To overcome this issue, we focus on a conjecture consistent with the existence of *Sil'nikov* homoclinic phenomena. We had already

[46] The results were obtained using our stochastic sampling method, from a sample with 10^9 random draws, assuming parameters distributed uniformly, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$, $d, h \sim U(-10, 10)$.

mentioned in section 2.6 that this scenario may occur in the vicinity of fold-hopf bifurcations. The original *Sil'nikov* scenario is a basic criteria for system complexity, where strange attractors are born from transitions from a homoclinic bifurcation [47]. The *Sil'nikov* bifurcation can be described in the following fashion. If the leading eigenvalue condition is fulfilled in the vicinity of a saddle focus homoclinic bifurcation, then trajectories diverge faster along the one dimensional outset than the convergent trajectories along the two dimensional inset. In the vicinity of this parameter region there is a transition where orbits generated by the system become increasingly more complex homoclinic loops and by definition represent dense solutions to the system[48]. We evaluate under what conditions saddle focus solutions fulfil the leading eigenvalue condition. The parameter distributions consistent with this conjecture are given below in Figure 2.8[49].



(a) Density ρ, γ (b) Density r, r_k (c) Density d,h (d) Prob. δ

Figure 2.8: Parameter density distributions for *Sil'nikov* saddle focus scenario

As in the previous example, sample results suggest that this scenario is more likely to occur in the vicinity of fold-hopf bifurcations, when $d < h < 0$, and may also occur when $d < 0$ and $h > 0$. A closer look at the h, d density plots in Figure 2.8 and Figure 2.7 reveals a pattern consistent with the joint distribution described in Figure 2.2 for the fold-hopf sample. To confirm the existence of *Sil'nikov* homoclinic scenarios, the application of numerical continuation methods and the simulation of

[47] Other *Sil'nikov* scenarios can be considered for this system. Piscitellia and Sportelli [2004] demonstrate the existence of inverted *Sil'nikov* homoclinic phenomena in a nonlinear continuous version of the inventory cycle model. This scenario involves divergence in the two dimensional outset and convergence in the one dimensional inset of a saddle focus solution. The leading eigenvalue condition is given by $|\Lambda_1^{**}| < Re\left(\Lambda_{2,3}^{**}\right)$. This bifurcation scenario was originally proposed by Tresser [1984]. The coexistence of saddle focus solutions of different stable dimensions also suggests the possibility of multi-circuit *Sil'nikov* homoclinic dynamics, following the proposal by Gaspard [1983]. See Piscitellia and Sportelli [2004] for a demonstration of this phenomenon in an economic setup.

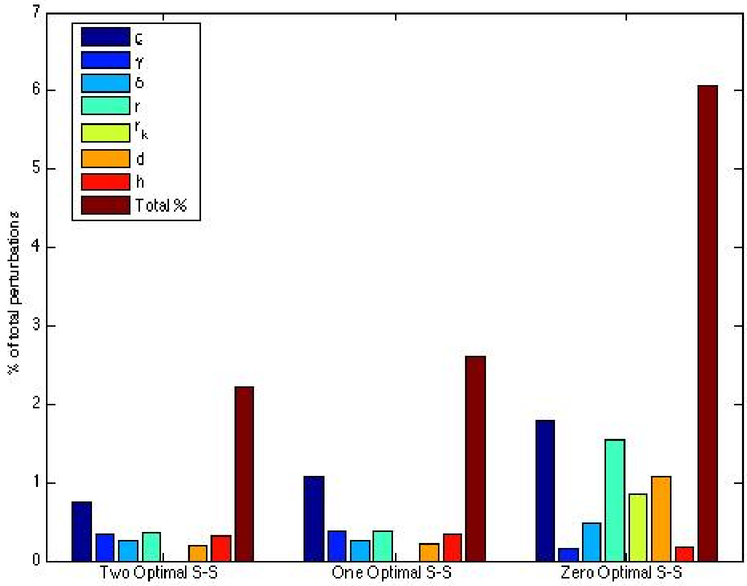
[48] Sufficient conditions for the existence of saddle focus homoclinic bifurcations require non-degeneracy, inclination and orbit properties to be fulfilled. The leading eigenvalue condition is given by, $\Lambda_1^{**} > \left|Re\left(\Lambda_{2,3}^{**}\right)\right|$. A recent survey and detailed mathematical discussion on this topic can be found in Homburg and Sandstede [2010].

[49] The results portrayed follow the same specifications of the previous example. We check for saddle focus solutions consistent with (2.50), by computing first the discriminant of J^{**}, Γ, and then checking if necessary conditions, $\Gamma < 0$, $Re\left(\Lambda_{2,3}^{**}\right) < 0$ and $\Lambda_1^{**} > \left|Re\left(\Lambda_{2,3}^{**}\right)\right|$, are fulfilled.

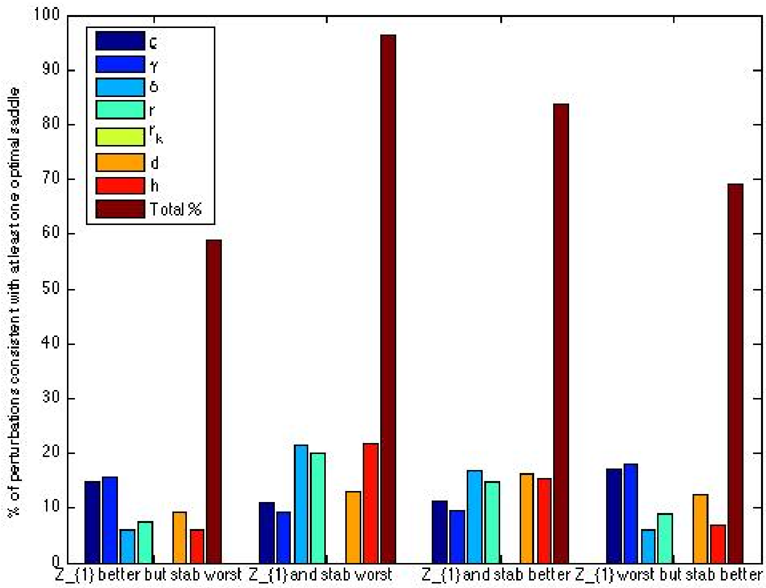
orbits using normal forms, is required. The computed fold-hopf bifurcation sample provides an interesting starting point. However, one word of advice, performing such task is comparable to searching for a needle in the haystack. Fold-hopf points may undergo transitions consistent with this scenario, but other transitions are possible. Given the amount of bifurcation points computed and the vast parameter space, the choice of an optimal candidate for numerical bifurcation analysis is not an obvious decision.

Recall that in the beginning of this section, we referred to the implications of considering *Turnpike* control solutions in a multiple equilibria setup, when small parameter perturbations are considered. In Figure 2.7 we portrayed the parameter samples describing the co-existence of optimal saddle solutions. To test the hypothesis of heteroclinic connections of equilibria arising from *Turnpike* control dynamics, we evaluated the saddle-saddle sample and checked for qualitative and quantitative changes induced by small parameter perturbations. Equation 2.52, below, portrays the results obtained. For simplicity reasons, we now assume that Z_a^{**} and Z_b^{**} are the initial saddle solutions and $Z_a^{**} > Z_b^{**}$, while $Z_a^{**}(\Delta\mu)$ and $Z_b^{**}(\Delta\mu)$, are the resulting steady-states assuming a parameter perturbation equal to $\Delta\mu = \pm 0.01$. The first figure portrays the quantitative sensitivity of Z_a^{**} and Z_b^{**} to parameter perturbations as a percentage of the total perturbations. We considered three cases of interest: (i) on the left we have $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}(\Delta\mu) > 0$ and $\delta < Z_{4,a}^{**}(\Delta\mu), Z_{4,b}^{**}(\Delta\mu) < \delta + \rho/\gamma$; (ii) on the center we have $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}(\Delta\mu)$, $Z_{1,b}^{**}(\Delta\mu) > 0$ and $\delta < Z_{4,b}^{**}(\Delta\mu) < \delta + \rho/\gamma$, while $Z_{1,a}^{**}(\Delta\mu) < 0$ and/or $Z_{4,a}^{**}(\Delta\mu)$ does not fulfil (2.50); and (iii) on the right $Z_{1,a}^{**}(\Delta\mu), Z_{1,b}^{**}(\Delta\mu) < 0$ and/or $Z_{4,a}^{**}(\Delta\mu), Z_{4,b}^{**}(\Delta\mu)$ does not fulfil (2.50). We then focused on the first two cases and checked their qualitative outcomes. The bottom figure describes the results obtained. From left to right, the four cases of interest are described by: (i) $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with one stable dimension; (ii) $Z_{1,b}^{**}(\Delta\mu) < Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with one stable dimension; (iii) $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with two stable dimensions; and (iv) $Z_{1,b}^{**}(\Delta\mu) < Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with two stable dimensions.

The results portrayed in Figure 2.9 show that there are parameter regimes where complex outcomes may also arise from *Turnpike* control dynamics. If *Turnpike* dynamics are not able to impose heteroclinic connection paths of equilibria, then a small parameter perturbation may throw the economy into a low growth regime, or worst, into a non-optimal growth regime. Moreover, there are parameter regimes, where small perturbations lead to a phase-space organization, where orbits on the stable manifold of a saddle are no longer consistent with the notion of optimal growth



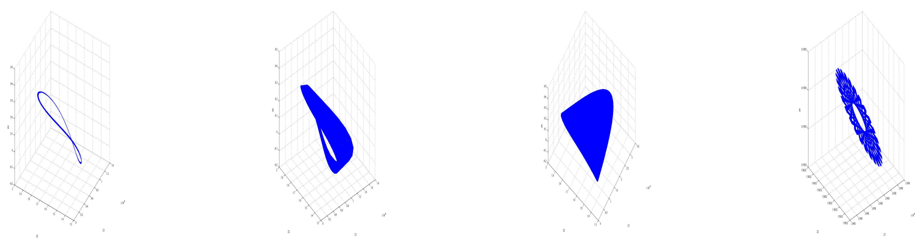
(a) Parameter sensitivity



(b) Best saddle outcomes

Figure 2.9: Saddle-saddle scenario sensitivity to parameter perturbations

dynamics. In this specific case, optimal dynamics may only be feasible in the vicinity of attractors arising from the complex organization of the invariant manifolds. This result stresses the importance of analysing global dynamics in economic systems. To illustrate this argument, we finish this discussion with an example of a parameter regime consistent with complex global phenomena. For this purpose, we consider a non orthodox parameter set, in the vicinity of a fold-hopf point, where $d < 0$ and $h > 0$. The simulated orbit is portrayed below, in the left picture of Figure 2.10 which is labelled as optimal dynamics, given that the central moment of the distribution described by limit cycle flow fulfils the optimal growth constraint defined in (2.50)[50]. For this combination of parameters, there is only a feasible, but non optimal, steady-state solution. We conclude that in this phase space region, orbits are driven by the complex organization of the feasible and boundary fixed points manifolds. This solution is not structurally stable. When a small perturbation is imposed, $\Delta\gamma = 0.0001$, trajectories are attracted to another region of the phase space. This dramatic transition is portrayed by the center pictures of Figure 2.10, which we labelled as initial transition and full transition, respectively. The last figure, labelled asymptotic dynamics, portrays the asymptotic behaviour of the transition flow.



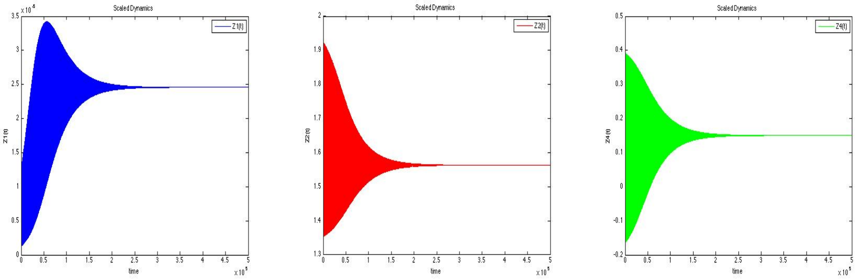
(a) Optimal dynamics (b) Initial transition (c) Full transition (d) Asympt. dynamics

Figure 2.10: Optimal dynamics and transition of a structurally unstable solution

The asymptotic behaviour of this flow is characterized by small amplitude limit cycle dynamics. This behaviour suggests that both repelling and attracting forces are at work in this region, as the observed oscillations are greater than expected error amplitudes produced by the numerical integrator at this error tolerance. We are not able to confirm if this orbit fulfils (2.50) asymptotically, since $\langle Z_4 \rangle = 0.1502 \approx \delta + \rho/\gamma = 0.1502$. This attractive set is robust to a wide range of perturbations,

[50]We integrate all the orbits using a Runge-Kutta of the $8^{th} - 7^{th}$ order and set the relative and absolute error tolerance to 10^{-10}. This orbit is obtained for the parameter set: $\rho = 0.099704$, $\gamma = 0.731579$, $\delta = 0.013929$, $r = 0.892695$, $r_k = 0.747145$, $d = -0.542038$, $h = 0.506959$, given initial conditions: $Z_1(0) = 0.000007$, $Z_2(0) = 1.921564$, $Z_4(0) = 0.150233$, where $\langle Z_4 \rangle = 0.133 < \delta + \rho/\gamma = 0.1502$, consistent with (2.50). The fold-hopf point is: $\rho = 0.099704$, $\gamma = 0.731479$, $\delta = 0.013929$, $r = 0.893695$, $r_k = 0.947145$, $d = -0.441038$, $h = 0.6812$.

but sensitive to the sign of the parameter variation. For example, another transition to a different attractor occurs for small perturbations of d or h, if $\Delta d, \Delta h < 0$. We portray this dynamic transition in our next example. Given that numerical integration routines are not able to correctly capture system dynamics near complex singularities, we cannot rule out the possibility of further nonlinear phenomena, such as bursting, which is known to arise in the vicinity of fold-hopf bifurcations. The time series describing the variables transition from the initial flow, portrayed in the picture labelled full transition in Figure 2.10, are given below in Figure 2.11.

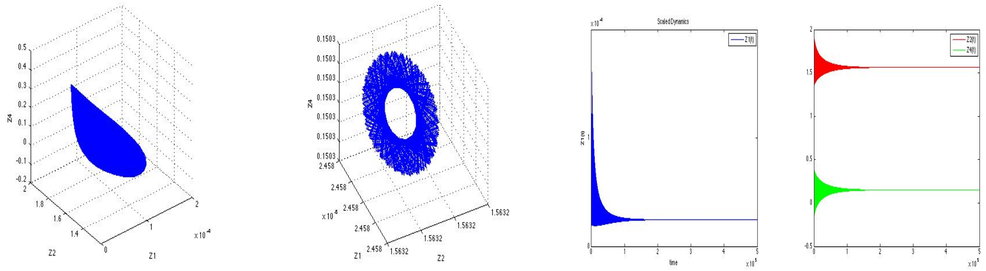


(a) $Z_1(t)$ (b) $Z_2(t)$ (c) $Z_4(t)$

Figure 2.11: Scaled dynamics during transition

The dynamics portrayed in Figure 2.11 show an over-indebted economy undergoing structural change, where initial investment dynamics are extremely volatile and domestic capital growth rates alternate between expansion and contraction periods. This behaviour is a result of the initial high level of debt. Investment expansions depend on foreign capital flows, which further increases the debt load until it becomes unsustainable and investment has to contract. As the economy accumulates productive capital, investment volatility decreases and the economy settles in a long run regime with small amplitude cycles. The structural change occurring in this economy is portrayed by the dynamics of consumption. Consumption volatility increases during the transition period, before decreasing and settling in the small amplitude long run cycle growth regime.

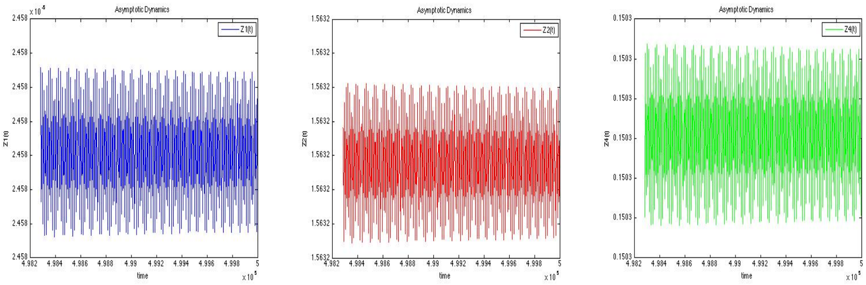
Finally, we evaluated the basin of attraction for this attractor and confirmed that it holds for a broad range of $Z_2(0)$ and $Z_4(0)$ values. However, it is sensitive to small perturbations of $Z_1(0)$. We had already mentioned that the behaviour of this attractor is sensitive to the sign of parameter perturbations and that orbits may converge to another attractive set. We demonstrate this transition by imposing a small variation on initial consumption, $\Delta Z_1(0) = 0.0001$. The dynamics of this transition are portrayed below in Figure 2.12. The two figures on the left portray the phase space transition and asymptotic dynamics, respectively. The two figures on the right show the time series obtained for this transition.



(a) Full transition (b) Asymptotic dynamics (c) $Z_1\,(t)$ (d) $Z_2\,(t)\,, Z_4\,(t)$

Figure 2.12: Transition to second attractor due to sensitivity in initial conditions

Although the convergence process is somewhat similar for both attractive sets. The slightly different departure point sends the economy to a region with a lower consumption level, as investment volatility decreases. Asymptotic dynamics for this case are consistent with (2.50), $\langle Z_4 \rangle = 0.1495 < \delta + \rho/\gamma = 0.1502$. The phenomena portrayed by these two transitions, illustrates the challenges faced by policy-makers in economies facing dire institutional and financial conditions. We describe this challenge in the following fashion. There is a path towards expansion and stronger long run growth dynamics. However, the probability of the economy converging to this growth regime is low. If the ideal conditions are not met, it is more likely that the economy spirals down to the low growth regime. Finally, we cannot exclude further nonlinear phenomena arising from this transition, for the same reasons described previously. Asymptotic dynamics, portrayed in Figure 2.13, are now consistent with small amplitude quasi-periodic motion. It is also possible that these two attractive sets correspond to a sole attractor, or flows cycle between the two attracting regions. A more in depth analysis of this phase space region is required to be able to shed some light in these hypotheses.



(a) $Z_1\left(t\right)$ (b) $Z_1\left(t\right)$ (c) $Z_1\left(t\right)$

Figure 2.13: Asymptotic scaled dynamics

2.9 Conclusions and further research

In this chapter we proposed a simple endogenous growth model, where asymptotic orbital stable solutions are only feasible for attractive sets arising from global interactions of multiple equilibria. We also described how the assumption of dynamic solutions defined in a *Turnpike* control setting is not sufficient to accommodate the complexities that may arise from this setup. We show that these outcomes are the natural consequence of considering multiple nonlinear mechanisms. This conclusion has several implications for policy in a dynamic setting. First, the analysis of local bifurcations in multiple equilibria setups, is crucial for the definition of policy rules in nonlinear environments. Second, the evaluation of global conjectures allows for a broader perspective of the challenges faced by a particular economy. Using a stochastic sampling method, we were able to map effectively the parameter space describing the institutional conditions for the existence of specific local phenomena and relevant global conjectures. We emphasize two main results from this analysis. When economies face a positive risk premium, the existence of optimal growth outcomes is limited to a saddle solution with only one stable dimension. This result suggests, as expected, that risk premium on sovereign debt plays a crucial role on the long run financial stability of an economy. This outcome could also be interpreted in another fashion. The capacity of the policy-maker to impose financial repression policies, consistent with real negative interest rates, is crucial to guarantee a stable financial and long run growth environment. In a recent working paper, Reinhart and Sbrancia [2011] suggests an important role of financial repression and moderate inflation policies on the post war deleveraging period and subsequent decades of financial stability in developed western economies. Drelichman and Voth [2008] gives historical evidence on this phenomenon and suggests that the British Empire capacity to overspend its rivals during the eighteenth century expansion period was linked to financial repression policies. The author compares this result with the decline of the Spanish Empire during the sixteenth century, which resulted in a series of defaults on Spanish sovereign debt. Our analysis of the phase space boundaries suggests that in the event of capital flights, due to severe economic conditions, policies capable of maintaining a negative risk premium, such as bilateral debt subsidies, are crucial for the existence of a dynamic recovery path. This result is in accordance with both economic theory and the modern policy paradigm. However, our model suggests that negative risk premium is only a necessary condition for recovery. The existence of a recovery consistent with convergence to a long run growth path requires that further institutional conditions are met. Finally, sampling results suggest that asymptotic orbitally stable solutions driven by complex global dynamics, and consistent with the definition of an attractor, are only likely to occur for institutional scenarios in the vicinity of fold-hopf bifurcations. This result has several implications for policy, as the unfolding of fold-hopf bifurcations has the potential to unleash a cascade of complex global dynamic events, and the full bifurcation scenario is still not fully understood. We give an example of the complex outcomes that may arise in this system, for an economy facing dire financial and institutional conditions, and describe the challenges posed to policy in this environment. To shed some light on this and other possible complex nonlinear phenomena, a thorough analysis based on modern numerical bifurcation analysis techniques is still required.

Finally, it is possible to scale this system and reduce the parameter space by assuming a translation to the center of the quadratic nullclines. The analysis of the resulting reduced system might provide important clues about the global organization of the phase space. We are aware of this hypothesis but leave this exercise to a future discussion.

Chapter 3

Optimal Beliefs and Self-Confirming Equilibrium for a Class of Games with Economic Applications

3.1 Introduction

In this chapter we introduce a conjecture regarding the existence of a class of open loop nonlinear multi-player general sum differential games with optimal subjective beliefs. Our framework departs from the hypothesis first put forward by Dockner et al. [1985], describing the conditions for the existence of qualitative and explicit solutions in a class of two person general sum differential games with state-separability properties. Dockner et al. [1985] suggested that under some conditions it might be possible to solve qualitatively differential games that have state-separability properties, if optimal control conditions are consistent with non interacting dynamics with respect to state and controls. According to Dockner et al. [1985], *"differential games that possess these properties will be termed state-separable games, since the determination of Nash optimal controls can be done separately from the determination of the state variables"*. The literature on differential games with state-separability properties is a vast growing field in economics literature. Research in this field is not limited to Dockner et al. [1985] results. Caputo [1997], for example, discusses further state-separability properties, for a class of discounted infinite horizon optimal control models similar to the one discussed in this chapter. The papers by Ling and Caputo [2011] and Bacchiega et al. [2010] are recent examples of the literature dealing with Caputo [1997] state-separability hypothesis and its applications in economics theory. We extend Dockner et al. [1985] definitions on differential games with the state-separability property, and propose a setup, where player controls depend on beliefs about the state of the game that are derived from optimality conditions. In the specific class of games discussed in this chapter, optimal control conditions impose individual beliefs that guarantee that controls are independent from the evolution of the state of the game. Optimal game solutions in this setup have to be consistent with individual state-separable solutions, therefore consistent with the broad definition by Dockner et al. [1985] on state-separable differential games. In this context, it is our opinion that the optimal solution to the game has to be consistent with a Self-Confirming Equilibrium (SCE) that satisfies the state-separability property of the optimal individual solution. The notion of strategic SCE in non-cooperative incomplete information games was first proposed by Fudenberg and Levine [1993a] for repeated games where players observe their opponents actions[1]. According to Fudenberg and Levine [1993a] the existence of stable belief based solutions in games, implies that player decisions under uncertainty are rational and equilibrium is self-fulfilling for a given history of observed information sets about the state of the nature. SCE is a thus a weak notion of equilibrium for games where players settle in an outcome of nature that is not contradicted by evidence[2]. Fudenberg and Levine [1993a] describe generally the SCE concept for repeated incomplete information games in the following fashion:

> *"The concept of self-confirming equilibrium is motivated by the idea that noncooperative equilibria should be interpreted as the outcome of a learning process, in which players revise their beliefs using their observations of previous play. Suppose that each time the game is played, the players observe the actions observed by their opponents, but that players do not observe the actions their opponents would have played at the information sets that were not reached along the path of play. Then, if a self-confirming equilibrium occurs repeatedly, no player ever observes play that contradicts his beliefs, so the equilibrium*

[1]The original concept was introduced in the working paper by Fudenberg and Kreps [1988]. See also the paper by Fudenberg and Levine [1993b] on steady-state learning and *Nash* equilibrium. A broad overview on the topic of learning in incomplete information non-cooperative games can be found in the book by Fudenberg and Levine [1998]. Recently, Battigalli and Siniscalchi [2003] proposed an extension of the SCE notion to signalling games. Finally, Kamada [2010] proposes a definition for strongly consistent SCE.

[2]The original proposal on SCE deals with solutions to extensive form games, where players have incomplete information and subjective beliefs. See Fudenberg and Kreps [1995] for a discussion on this topic and on the topic of learning in extensive form games.

is "self-confirming" in the weak sense of not being inconsistent with the evidence."

Following the seminal work by economist Thomas Sargent on the causes of American inflation, the topic of SCE has been gaining ground in the broader field of macroeconomic dynamics literature as an interesting hypothesis for evaluating macroeconomic policy models. A detailed account of the macroeconomic dynamics literature relating inflationary periods to large deviations from SCE can be found in Sargent [1999]. Our definition of SCE follows closely the economic dynamics approach and defines equilibrium in a dynamic game setup as an asymptotic property of the system. According to Hansen and Sargent [2001], Fudenberg and Levine [1993a] and Sargent [1999], *"...advocate the concept of self-confirming equilibrium partly because it is the natural limit point of a set of adaptive learning schemes"*. Further, Hansen and Sargent [2001] goes on and describes *"...the concept of SCE, a type of rational expectations that seems natural for macroeconomics."*. In a recent discussion on this topic, Cho and Sargent [2008] define SCE has a rational expectation equilibrium outcome. However, the authors stress that a rational expectation equilibrium does not necessarily correspond to a SCE. More recently, there has been an increasing interest on the properties of SCE to model economic phenomena. Fershtman and Pakes [2009], for example, propose the adoption of SCE, given that it is *"...an equilibrium notion for dynamic games with asymmetric information which does not require a specification for players' beliefs about their opponents' types. This enables us to determine equilibrium conditions which, at least in principle, are testable and can be computed using a simple reinforcement learning algorithm."*. Fudenberg and Levine [2009] relates the concept of SCE with the macroeconomics paradigm known as the *Lucas Critique*[3], which states that economic policy is ineffective when policy forecasts rely on past aggregate data and not on microeconomic fundamentals, such as individual preferences and beliefs.

In the setups proposed in chapter 3 and chapter 4, only SCE solutions can qualify as optimal solutions asymptotically. The class of games discussed extends the concept of SCE to differential games where agents only have information about their own pay-offs. This information hypothesis was initially forwarded in Fudenberg and Levine [1993a]. The authors speculate that in this class of incomplete information games the key feature is to determine how much information the individual pay-offs convey. It is our opinion that the class of games we propose provides a unique framework to study the existence and stability of SCE solutions in nonlinear

[3]Following the famous proposal by Lucas [1976].

multi-player dynamic games. Our intuition on this matter is straightforward. Non-cooperative differential games[4], as defined in the seminal paper of Isaacs [1954], are a natural generalization of optimal control theory. Therefore, open loop no feedback differential games with the state-separability property, as defined by Dockner et al. [1985], have optimal solutions given by the set of independent individual solutions. These solutions are defined independently from the state evolution of the game, by each player optimal control dynamics. On the other hand, state-separability in our class of games arises from an optimum belief condition regarding the evolution of the state of the game. In this framework, we consider that the optimal solution to this class of differential games has to be simultaneously consistent with self-fulfilling beliefs and with the optimal solution to the individual state-separable problem. In chapter 4 we extend this framework to a class of games where agents have subjective beliefs about the evolution of state of the game and optimal asymptotic solutions are also described by a SCE. In the last game proposed, we demonstrate that Fudenberg and Levine [1993a] information conjecture, regarding the existence of a SCE in games where agents only observe their individual pay-offs, is a crucial hypothesis for the description of asymptotic dynamics in this class of open loop differential games. In this setup, we propose to evaluate asymptotic game dynamics as the limiting outcome of an adaptive learning scheme between interacting learning agents that extrapolate past moments of nature from available pay-off information. We show that a statistical analysis of different learning schemes and decision criteria which mimics game dynamics in the equilibrium frontier, can be performed as a stochastic repeated game, where players are chosen randomly and their actions performed sequentially.

The initial motivation for this proposal departs from the recent focus on the study of nonlinear economies as multi-player games. In economics literature this topic is framed by the seminal proposal of Grandmont [1998], on the stability analysis of equilibrium in large socio-economic systems. In this paper, the author puts forward an extensive discussion on the vast implications and mathematical challenges of undertaking stability analysis in large systems with decentralized decision dynamics under incomplete information. As in Grandmont [1998] proposal, we discuss in this chapter and the next *"how adaptive learning may or may not lead to stability and convergence to self-fulfilling expectations in large socioeconomic systems where no agent, or collection of agents, can act to manipulate macroeconomic outcomes"*. In this chapter, as previously referred, our focus is on the conditions for the ex-

[4] The book by Başar and Olsder [1995] provides a modern detailed overview on the topic of dynamic non-cooperative game theory.

istence of SCE solutions when agents have optimal beliefs. We show that under some simplifying assumptions it is possible to determine and evaluate qualitatively the dynamics of a self-fulfilling equilibrium asymptotically. In the next chapter, we introduce the concept of subjective belief based SCE solutions for a class of differential games. We use this setup to evaluate Grandmont's main hypothesis, which states that if adaptive learning agents seeking convergence to a SCE are influenced by the nearby dynamics and rather uncertain, then a self-fulfilling equilibrium will not be stable. Grandmont defined this phenomenon as the *uncertainty principle*. In chapter 4, we show that when agents are able to reduce the uncertainty about future outcomes there is convergence to a SCE. On the other hand, when an environment with persistent uncertainty is considered, agents are no longer able to concur on a self-fulfilling equilibrium, thus confirming Grandmont's main hypothesis.

To demonstrate our state-separability conjecture for differential games where optimality conditions impose beliefs consistent with non interacting dynamics between player's controls and the state of the game evolution, we propose two differential games that are set up as growing economies driven by agents' financial strategic decisions. The two games proposed are extensions of the general intertemporal maximization setup discussed in chapter 2. We now consider an endogenous growth environment populated by a discrete set of asymmetric players that take consumption and investment decisions under incomplete information. In this chapter we assume that players continue to face risk premium but there is no investment adjustment costs. We reintroduce investment bias in the games discussed in chapter 4. Linear productive capital dynamics continue to be the growth engine of the economy. The papers by Clemhout and Wan [1995], Vencatachellum [1998], Bethamann [2008] and Hori and Shibata [2010] are some examples of the literature dealing with growing economies defined as multi-player non-cooperative dynamic games. However, given that our specific setup departs from a well known theoretical financial framework, which has its roots in the Merton [1970] intertemporal consumption-investment problem, one can interpret our specific proposals as a simplified version of multi-player non-cooperative portfolio games[5]. One may also interpret these models as a foreign exchange game, where agents invest in domestic risk-free deposits, and may leverage their domestic investments by selling bonds to foreign investors, or invest part of their capital in foreign deposits and face exchange rate risk.

[5]We refer to Merton's thesis in the context of the authors earlier proposals on optimal control and economics dynamics that later influenced the modelling of open economies. Modern portfolio theory is based on the authors seminal articles, Merton [1969], Merton [1971] and Merton [1973a], which later led to the development of the famous Merton-Black-Scholes model, following Black and Scholes [1973] and Merton [1973b].

In both of the games proposed, we consider that players' decisions are coupled by the evolution of aggregate risk premium. The foreign bond market measures aggregate risk premium as the ratio of net aggregate financial assets to domestic capital. Optimal beliefs in this context are a condition imposed by first order *Pontryagin* maximum conditions. This outcomes guarantees the state-separability of the game in a perverse fashion. In the one hand, the existence of optimal solutions to these games must be consistent with individual optimal belief solution, given by the solution to the state-separable problem. On the other hand, optimal beliefs are only consistent with the true state of the game, when agents concur on a solution that fits their individual beliefs. Given these two contradictory results, we propose to answer the following relevant questions in this chapter. Are players with optimal naive beliefs able to converge to a SCE, where their expectations are self-fulfilling? If game dynamics allow the existence of a SCE, what is the qualitative nature of this solution? Is it a stable equilibrium? Is there history dependence resulting from multiple SCE solutions, thus making game solutions a conjectural SCE[6]? The answer to the first question is yes, but it will depend on the degree of asymmetry among players and/or the degree of nonlinearity considered. In the former case, it is possible to evaluate qualitatively the dynamics in the vicinity of a SCE, but existence and stability of solutions involves considering a smaller or higher degree of asymmetry among players. In the latter case, players are required to learn SCE. We can confirm the existence of optimal solutions by geometrically defining SCE solutions as intersections between feasible conjectural solutions and actual outcomes in a bounded interval. However, dynamics in the vicinity of SCE solutions now depend on the gradients of the individual learning functions, which are correlated and depend on higher order moments. In the likely scenario that learning dynamics lead to non smooth strategic dynamics, then it is not possible to use standard qualitative techniques to evaluate SCE. Both these issues are discussed in detail in Grandmont [1998]. Finally, when multiple SCE solutions and learning dynamics coexist, then it is not possible to describe game dynamics in a well defined mathematical fashion. However, the individual player dynamics defined by the state-separate solution, provide some insight on what are the most likely outcomes.

To demonstrate our main hypotheses, we depart from a simple conjecture regarding the solution and analysis of dynamic games under incomplete information. We argue that if beliefs are consistent with the existence of asymptotic equilibrium solutions, then it is possible to evaluate strategic equilibrium outcomes. If the belief

[6]See Wellman et al. [1998] on the implications and characterization of conjectural equilibrium solutions in the field of multi-agent learning.

function is known and open loop Nash Controls fulfil *Lipchitz* continuity conditions, then it is possible to obtain a full qualitative description of the game dynamics and the stability of SCE can be evaluated at least locally. Equilibrium in this class of games can be fully described as a *Cauchy* boundary value problem, under this set of conditions, as long as we have knowledge of the value and gradient of the belief function, evaluated in the vicinity of the asymptotic equilibrium region. We portray this hypothesis in the game proposed in section 3.3. In this game player beliefs impose a unique equilibrium and under some simplifying assumptions it is possible to define a set of Nash strategic controls that are *Lipchitz* continuous. The existence of a SCE solution requires the existence of constraints on individual parameter distributions. If agents' strategies are a result of both beliefs and learning, then, in the likely case that player control's are no longer consistent with *Lipchitz* continuity, asymptotic game dynamics can always be evaluated numerically as a Hidden Markov Model (HMM)[7] of a static version of the game asymptotic SCE solution. A static version of the game should describe the most faithfully possible the game asymptotic solution as a multi-objective optimization problem under uncertainty. We discuss this hypothesis in the game proposed in section 3.4. In this setup, optimality conditions impose multiple belief equilibria and require that players learn the true state of the game, in order to pursue their optimal investment strategies.

We organize our presentation in the following fashion. In 3.2 we put forward the general framework for a class of exponentially discounted differential games with concave pay-offs, and introduce our main conjecture regarding the existence of state-separable games with optimal beliefs. We then put forward the general conditions for the existence of an optimal SCE solution for this class of games. In 3.3 and 3.4, we demonstrate our main conjecture and evaluate the implications of our hypotheses in two non-cooperative consumption/investment differential games, where agents seek to maximize consumption utility, choosing open loop consumption and investment strategies. In both games, players accumulate productive assets linearly and may choose to invest in bonds or leverage their productive asset portfolio. Player dynamics are coupled by risk premium dynamics, which is driven by the aggregate ratio of net financial assets to productive capital. In the first example, discussed in 3.3, we consider that players face an asymmetric institutional market risk premium measure and model this asymmetry as an individual model parameter. We show that the existence of an asymptotic SCE solution is not consistent with the existence of asymmetries in institutional risk premium valuations or, on the other hand, other parameter asymmetries have to be considered to define a SCE solution. In this

[7] A HMM is a *Markov* decision process where agents lack information about the state evolution.

section, we assume the existence of a perfect capital market with an efficient regulator, which guarantees that institutional risk premium is defined optimally. Given some further simplifications, we are able to perform an extended numerical qualitative analysis of SCE solutions for a wide range of institutional scenarios. In the second example, we drop the institutional risk premium assumption and consider that individual risk premium market valuations are now a function of the players' individual ratio of net assets to productive capital. We show that a SCE solution in this framework is only possible when complex learning dynamics are considered. The complexity of this solution is a direct outcome of considering additional non-linearities in our initial setup. We propose a static version of this game and discuss the implementation of a *Markov* switching regime model under incomplete information to evaluate learning outcomes. Given the *Bayesian* nature of this problem, we discard the numerical evaluation of the game asymptotic dynamics on the ground that the computational costs of implementing a HMM are not justified in this specific environment. We discuss the HMM approach with detail in chapter 4 for the game proposed in section 4.4. In this chapter we focus our analysis on a geometrical description of conditions guaranteeing the existence of SCE solutions, and evaluate the robustness of these outcomes for a reasonable range of state outcomes. We finish this discussion with a description of likely outcomes in the vicinity of SCE solutions for two distinct institutional scenarios, based on the qualitative analysis results of the solution to the individual fully state-separable problem.

3.2 General setup

In this section, we define the general setup for the class of differential games we wish to consider, and put forward the main conjectures for the existence of solutions driven by optimal beliefs. Consider the following general N-Player non-cooperative differential game[8] faced by player $i \in N$:

$$\begin{aligned}
&\underset{u_i(t)}{MAX} \int_0^T \beta_i(t)\, \pi_i(u_i(t))\, dt \\
&\text{subject to the solution of:} \\
&\dot{x}_i(t) = g_i(u_i(t), X(t))\,; \\
&x_i(0) = x_{i,0}.
\end{aligned} \tag{3.1}$$

where:

[8]This specific class of games is contained in the broad general framework for exponentially discounted differential games with concave utility discussed in Dragone et al. [2008].

- $N = \{1, \ldots, n\}$- Discrete set of players;
- $\beta_i(t)$- Discount function for player i;
- $u_i(t) = \{u_i^1, \ldots, u_i^k\} \in \Re^k$- Finite dimensional vector of player i controls;
- $\pi_i(u_i(t)) \in \Re^{k'}$- Instantaneous pay-off for player i, where $k' \leq k$;
- $x_i(t) = \{x_i^1, \ldots, x_i^w\} \in \Re^w$-Finite dimensional vector of player i states;
- $x_i(0) = \{x_i^1, \ldots, x_i^w\} \in \Re^w$- Finite dimensional vector of initial conditions on player i states
- $X(t) = [x_1(t), \ldots, x_n(t)] \in \Re^{w'}$- Finite dimensional vector of state variables, where $w' \leq w$.

We consider solutions to (3.1) consistent with players choosing open loop no feedback strategies, $\eta(t) = \{X(0)\}, \forall t \in [0\ T]$, where $\eta(t)$ is the information set available to players at period t. Players seek to maximize a concave pay-off function, $\pi_i(u_i(t))$. Pay-offs in this class of games are discounted at an individual constant exponential rate, $\beta_i(t) = e^{-\rho_i t}$, where ρ_i is player i discount rate. These conditions lead to solutions that can be defined as initial value problems, given by *Pontryagin* first order conditions. In this framework, *Pontryagin* maximum conditions are sufficient for the existence of an optimal solution to (3.1), provided that transversality conditions, following Arrow and Kurtz [1970], are fulfilled[9], thus guaranteeing that $x_i(t)$ does not grow too fast. We propose the following conjecture regarding the existence and form of Nash open loop controls for games belonging to the class of differential games described generally in (3.1), which are consistent with the existence state-separable solutions when players have optimal beliefs. We start by assuming the following hypothesis regarding player i optimal strategies. We consider games described generally by (3.1), where player i optimal Nash controls, obtained from first order *Pontryagin* conditions, can be defined generally in the following fashion,

$$\dot{u}_i(t) = f_i(u_i(t), X(t)). \tag{3.2}$$

Following the conjecture in (3.2), we consider that a game has solutions consistent with the existence of individual optimal subjective beliefs, $X_{opt}^{(i)}(t)$, if optimality conditions also impose the existence of an unique belief function, $X_{opt}^{(i)}(t) = v_i(t)$.

[9]For this class of control problems, Arrow and Kurtz [1970] have shown that first order conditions are sufficient, provided that transversality conditions, defined generally by $\lim_{t\to\infty} e^{-\rho_i t}\Gamma_i(t)\, x_i(t) = 0$, are fulfilled. $\Gamma_i(t)$ are the co-state variables of the optimal control problem defined in (3.1), describing the marginal adjustment of the players control to the individual state evolution, when no feedback strategies are considered.

We further assume that this belief function is consistent with a game with the state-separability property. Recall first that Dockner et al. [1985] defined state-separability for a class of differential games where players choose their strategies given the evolution of the state of the game, $\dot{X}(t)$, in the following fashion: (i) Open loop Nash controls are described generally as $\dot{u}_i(t) = v_i(\tilde{u}(t), t)$, where $\tilde{u} \in \Re^{k \times n}$ is a vector of player controls; (ii) which is consistent with the existence of a Nash equilibrium N-tupple defined generally by $\tilde{u}^* = \{u_1^*, \ldots, u_n^*\}$, where $\tilde{u}^* \in \Re^{k \times n}$. In the setup described in (3.1) players choose the controls given the evolution of their individual state variable, $\dot{x}_i(t)$. We propose that a necessary condition for the existence of state-separable solutions for this class of games requires that optimality conditions impose the existence of beliefs that are at most a function of the player controls and individual state variables[10]. Optimal beliefs for this class of games can thus be generally defined in the following fashion:

$$X_{opt}^{(i)}(t) = v_i(u_i(t), x_i(t)). \tag{3.3}$$

The assumption forwarded in (3.3) is crucial to our proposal. Our first intuition is that if the existence of optimal beliefs is consistent with a differential game with the state-separability property, then optimal solutions to (3.1) have to be consistent with solutions to the individual state-separable solution, that is obtained by substituting (3.3) in (3.2), and in the state condition of the general game defined in (3.1). The optimal control solution to (3.1) is thus defined by individual solutions to the state-separable system, after considering the optimal belief condition defined in(3.3):

$$\dot{u}_i(t) = f_i(u_i(t), v_i(t)); \tag{3.4}$$

$$\dot{x}_i(t) = g_i(u_i(t), v_i(t)). \tag{3.5}$$

On the other hand, differential game theory does not allow us to tamper with the state condition of the game. The intuition on this matter is straightforward. The state condition of a dynamic game defines the evolution of the state of the game, so, even if players have optimal beliefs, the game evolution takes into account the true state outcomes and not the ones resulting from belief based decisions. Thus,

[10]This assumption is rather general. In the game discussed in 3.3, optimal beliefs are a function of game parameters, while in 3.4, optimal beliefs are a function of the evolution of the player state variables. However, we maintain this general assumption throughout this section, because this specific conjecture is consistent with Dockner et al. [1985] state-separability definition.

by definition, the solution to the game defined in (3.1) should be given by a system, where players follow strategies that are in accordance with their optimal beliefs, as defined in (3.4), but where the game evolution, $X(t)$, is defined by the original state condition of the game described in (3.1). The game solution to (3.1) is thus correctly defined generally by the following system:

$$\dot{u}_i(t) = f_i(u_i(t), v_i(t)); \tag{3.6}$$

$$\dot{x}_i(t) = g_i(u_i(t), X(t)). \tag{3.7}$$

Before describing the necessary conditions for the existence of an optimal solution for the differential game described in (3.1) with optimal controls and beliefs defined by (3.2) and (3.3), respectively, it is convenient to introduce a general definition for SCE. To put forward a broad definition for SCE in a dynamic setup, we follow closely the general definition given in Sargent [2008] for SCE, as a limiting behaviour of adaptive multi agent learning schemes. The general definition that we put forward in this section is broad enough to accommodate the game analysis undertaken in this chapter, where we focus on the description of necessary conditions for the existence of SCE, consistent with optimal beliefs and a state separable solution to (3.1). In chapter 4, we extend this definition for games with subjective beliefs and describe SCE as an outcome arising from a process of adaptive learning in a complex co-evolutionary environment, where players use inference methods to determine the true state of the game. Let p denote a probability density and $\mathbf{x}^{\boldsymbol{\tau}}$ a history of the state of nature, $\mathbf{x}(\boldsymbol{\tau}), \mathbf{x}(\boldsymbol{\tau}-1), \ldots, \mathbf{x}(0)$, driven by players decisions. Partition $\mathbf{x}^{\boldsymbol{\tau}} = [\mathbf{y}^{\boldsymbol{\tau}}, \mathbf{u}^{\boldsymbol{\tau}}]$, where $\mathbf{u}^{\boldsymbol{\tau}}$ is a vector of decisions taken by agents and $\mathbf{y}^{\boldsymbol{\tau}}$ a vector of all other variables. Let $p(\mathbf{y}^{\infty}, \mathbf{u}^{\infty}|\boldsymbol{\alpha})$ be a joint density describing the true data generating process conditional on a parameter vector $\boldsymbol{\alpha} \in \boldsymbol{\Omega}_{\boldsymbol{\alpha}}$ and $p(\mathbf{y}^{\infty}, \mathbf{u}^{\infty}|\boldsymbol{\theta})$ an approximating model, where $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\boldsymbol{\theta}}$ is a parameter vector describing agents subjective density. A decision maker chooses a history dependent plan assuming a sequence $\mathbf{h}$ of functions

$$\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{h}(\mathbf{x}^{\boldsymbol{\tau}}|\boldsymbol{\theta}), \quad \boldsymbol{\tau} \geq 0 \tag{3.8}$$

to maximize a Pareto criterion that can be expressed as expected utility under density $p(\mathbf{x}^{\infty}|\boldsymbol{\theta})$, where the decision maker as preferences ordered by

$$\int \mathbf{U}(\mathbf{y}^{\infty}, \mathbf{u}^{\infty}) \, p(\mathbf{y}^{\infty}, \mathbf{u}^{\infty}|\boldsymbol{\theta}) \, d(\mathbf{y}^{\infty}, \mathbf{u}^{\infty}). \tag{3.9}$$

This gives rise to the sequence of decisions $\mathbf{u}(\mathbf{h}|\boldsymbol{\theta})^{\infty}$. A SCE is thus a parameter vector $\bar{\boldsymbol{\theta}}$ for the approximating model that satisfies the following data-matching conditions

$$p\left(\mathbf{y}^{\infty}, \mathbf{u}\left(h|\bar{\boldsymbol{\theta}}\right)^{\infty}|\bar{\boldsymbol{\theta}}\right) = p\left(\mathbf{y}^{\infty}, \mathbf{u}\left(h|\bar{\boldsymbol{\theta}}\right)^{\infty}|\boldsymbol{\alpha}\right). \tag{3.10}$$

We can now put forward the conditions for the existence of optimal solutions to this class of differential games. To show that only SCE solutions qualify as optimal solutions for this class of games, it is convenient to start by describing the necessary conditions for the existence of equilibrium solutions in each of the proposed systems, (3.4)-(3.5) and (3.6)-(3.7), separately. Recall that the existence of optimal beliefs, following (3.3), imposes optimal solutions described by the state-separable system given generally in (3.4) and (3.5). Therefore optimal solutions to (3.1) have to be consistent with the existence of equilibrium solutions to this system. Assuming that the asymptotic condition for individual optimal beliefs is fulfilled for the set of n players,

$$\lim_{t\to\infty} X_{opt}^{(i)}(t) = \bar{X}_{opt}^{(i)}, \tag{3.11}$$

given a set of player asymmetries and initial conditions. Where $\bar{X}_{opt}^{(i)}$ describes the set of optimal equilibrium beliefs. Then an optimal solution to (3.1), has to be consistent with the existence of an equilibrium solution for (3.4)-(3.5), $\bar{X}_{opt}^{(i)}$ guarantees that the asymptotic condition, $f_i(u_i(t), v_i(t)), g_i(u_i(t), v_i(t)) = 0$ as $t \to \infty$, is fulfilled. On the other hand, we require the existence of equilibrium solutions for the game defined in (3.6) and (3.7). Assuming that the result in (3.11) is consistent with the existence of an unique equilibrium solution, $\bar{X}$, for the true state of the game,

$$\lim_{t\to\infty} X(t) = \bar{X}, \tag{3.12}$$

such that (3.11) and (3.12) are consistent with the existence of an equilibrium solution for (3.4)-(3.5), and $f_i(u_i(t), v_i(t)), g_i(u_i(t), X(t)) = 0$ as $t \to \infty$, is also fulfilled. Then the game defined in (3.1) with dynamics described by the system (3.4)-(3.5), has solutions consistent with the existence of optimal beliefs. Our intuition from here on is straightforward. If (3.12) and (3.11) bound the state-space of the systems, (3.4)-(3.5) and (3.6)-(3.7), in equilibrium, then we argue that a game solution is optimal, when it coincides with the optimal control solution to the state-separable problem. This implies assuming the existence of SCE solutions, following the general definition in (3.10), where players optimal beliefs are self-fulfilling,

$$|X_{opt}^{(i)}\left(t\right) - X\left(t\right)| \rightarrow 0, \text{ as } t \rightarrow \infty. \tag{3.13}$$

In layman terms, we mean that solutions to the system defined in (3.6) and (3.7), are only consistent with the optimality condition defined in (3.3), if they are also solutions to the optimal control solution defined in (3.4) and (3.5). Thus only SCE solutions qualify as optimal solutions to the class of games with optimal beliefs defined in (3.1).

Following the result in (3.13), we argue that we can evaluate qualitatively SCE solutions, if the belief function ensures that the players' strategic Nash dynamic controls, $\dot{u}_i\left(t\right) = f_i\left(t\right)$, as defined generally by (3.2), are *Lipschitz* continuous in $X_{opt}^{i}\left(t\right)$ and continuous in t. This is a crucial assumption, following the *Cauchy-Lipschitz* theorem, to guarantee the existence and uniqueness of solutions to first order differential equations given a set of initial conditions. For games fulfilling this property, SCE outcomes can be at least qualitatively evaluated using standard analysis techniques for hyperbolic dynamical systems. Our intuition goes as follows. SCE solutions to incomplete information differential games can always be evaluated qualitatively, if asymptotic dynamics can be described as a *Cauchy* boundary problem and strategies are smooth when the individual beliefs of player i are taken into account. In games where players follow smooth strategies, described as initial value solutions, we require knowledge of the value and gradient of the belief function in the game asymptotic frontier, in order to be able to perform qualitative analysis of self-fulfilling solutions. If strategies are not smooth, which is likely to occur in games where players have to learn a SCE, but asymptotic solutions are still consistent with a game with a bounded state-space, then asymptotic dynamics can always be evaluated numerically as a HMM. In such scenarios, the qualitative analysis of the game asymptotic outcomes is limited to statistical analysis of different hypotheses for the learning process driving players decisions. This approach is computationally costly and analysis of equilibrium is limited in scope. The main difficulty faced when simulating and sampling a stochastic *Markov* process under uncertainty is related to the computational cost of performing inference in a large scale. To overcome this issue, we suggest that SCE solutions can be revealed geometrically and some considerations about game dynamics can be obtained through qualitative analysis of the state-separable solution. Recall that given the result in (3.12), state-separable dynamics are always consistent with a well defined *Cauchy* boundary problem, if (3.4)-(3.5) has an equilibrium solution.

In the next two sections, we illustrate the two hypotheses discussed above in two nonlinear multi-player consumption/investment differential games. In the first example, we show that SCE solutions can be obtained by imposing parameter constraints and that under simple assumptions, it is possible to describe fully the qualitative dynamics in equilibrium. In the second example, we show that the existence of SCE solutions is only achievable if we consider the existence of complex learning dynamics.

3.3 A consumption and investment game with coupled institutional risk premium

To demonstrate these conjectures, we consider two nonlinear extensions of the general investor problem and set it up as a non-cooperative differential game under incomplete information. In this section, we consider an economy populated by a discrete set of players, $N = \{1, \ldots, n\}$, that seeks to maximize their intertemporal pay-offs, given by a consumption utility function, $U_i(C_i)$, subject to the evolution of individual net financial assets, $B_i(t) \in \Re$, describing the budget constraint of each player, and productive capital accumulation, $K_i(t) \in \Re^+$, where $i \in N$. In order to pursue this objective, agents choose open loop, $\eta(t) = \{X(0)\}$, no feedback consumption, $C_i(t) \in \Re^+$ and investment strategies, $I_i(t) \in \Re^+$, and discount future consumption exponentially at a constant rate $\rho_i \in \Re^+$, in a game of infinite duration, $\forall t \in [0\,, T]$ and $T = \infty$. Player decisions are coupled by a risk premium mechanism that depends on the overall evolution of the state of the game, defined by $X(t) = \{B(t), K(t)\}$, where $B(t) = \sum_{i \in N} B_i(t)$ and $K(t) = \sum_{i \in N} K_i(t)$. The objective of each player is to maximize the flow of discounted consumption pay-offs,

$$U_i(C_i) = \int_0^T u_i(C_i(t))\, e^{-\rho t} dt, \;\; \text{with} \;\; u_i(C_i(t)) = C_i(t)^{\gamma_i}\,, \tag{3.14}$$

where γ_i is again the intertemporal substitution elasticity between consumption in any two periods, measuring the willingness to substitute consumption between different periods. We impose the usual constraint on the intertemporal substitution parameter, $0 < \gamma_i < 1$, such that $u_i'(C_i(t)) > 0$, and the concavity of the utility function is guaranteed, $u_i''(C_i(t)) < 0$. This is a necessary condition to define optimal solutions to our open loop differential game as an initial value problem.

Each player faces a budget constraint describing the evolution of net financial assets, $\dot{B}_i(t)$. We again consider that players are bond buyers when $B_i(t) < 0$ and bond sellers when $B_i(t) > 0$. Each player uses their financial resources to finance consumption and investment activities, and to repay interest on their outstanding bonds or reinvest in financial assets. Players have revenues arising from productive capital investments, $r_k K_i(t)$, where $r_k \in \Re^+$ is the marginal revenue of investment in productive capital, and receive interest payments on holdings of financial assets, if they are bond investors. Interest payments follow a convex specification defined by, $rB_i(t)[1 + d_i B(t)/K(t)]$, where $r \in \Re^+$ stands as usual for the real market interest rate and $d_i \in \Re$ is an institutional measure of risk premium, resulting from capital markets' sentiments on the quality of the bonds issued by a specific player. A higher value of d_i means that holding player i debt bonds yields a higher risk for other investors, but investment by player i in financial assets pays a greater premium. A smaller value of d_i means that holding player i debt bonds yields a smaller risk for investors but investment by player i in financial assets pays a smaller premium. Such outcomes are reinforced if agents' financial situations match the aggregate financial situation of the economy. However, bond contract holders are rewarded with smaller interest premiums when the aggregate economy is a net seller of bond contracts, and agents selling debt contracts benefit from smaller interest premiums when the aggregate economy is a net buyer of bonds. Finally, players accumulate productive capital exponentially following their strategic investment decisions, $\dot{K}_i(t) = I_i(t) - \delta K_i(t)$, and productive capital accumulation is subject to depreciation, which is defined by the common capital depreciation rate, $\delta \in \Re^+$. We assume that players playing open loop strategies do not commit to a common investment strategy, but their decisions will be such that they always fulfil growth, $\liminf_{t\to\infty} I_i(t) K_i(t)^{-1} > \delta$, and optimality conditions.

Following the description of the decision problem faced by each member of this economy, the non-cooperative game faced by player $i \in N$, is defined by the following dynamic optimization problem:

$$\begin{aligned}
&\underset{C_i(t),I_i(t)}{MAX} \int_0^\infty e^{-\rho_i t} C_i(t)^{\gamma_i}\, dt \\
&\text{subject to the solution of:} \\
&\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + d_i \tfrac{B(t)}{K(t)}\right) - r_k K_i(t)\,; \\
&\dot{K}_i(t) = I_i(t) - \delta K_i(t)\,;
\end{aligned} \tag{3.15}$$

satisfying the transversality conditions, (B.8) and (B.9), guaranteeing that solutions to (3.15) do not grow too fast.

Due to its simplicity, the framework proposed in (3.15) can have different interpretations. These interpretations depend on the economic context we choose to consider[11]. This game can be interpreted as an economy populated by investors

[11]The simplicity of these proposals have both advantages and drawbacks. The main advantages of the simplified framework proposed in this section and the next is, in our opinion, its mathematical tractability and the flexibility it allows in terms of economic interpretation. On the other hand, the

that seek to finance their intertemporal consumption, given the returns of their portfolios. Investors may finance consumption by investing their initial capital in a portfolio composed of a risk-free asset and a risky asset that is linked to other players investment decisions and market institutional conditions. This type of investor chooses to diversify their portfolios to finance present and future consumption. Otherwise, investors may choose to hold only risk-free assets in their portfolio and finance their investment and consumption decisions through the accumulation of financial debt. In this particular case, investors use risk-free asset returns to roll on existing debt contracts. Present and future accumulation of risk-free assets guarantee the sustainability of their leveraged position. Another interesting interpretation is given by an economy populated by exchange rate speculators that can deposit their capital domestically and borrow capital from abroad, or invest in foreign currency and face exchange risk in foreign currency deposits. Throughout the remainder of this chapter and the next, we shall follow a twofold approach regarding the interpretation of solutions to the games discussed in section 3.3, section 3.4, section 4.3 and section 4.4. When convenient, we shall consider that these setups define growing economies populated by agents that invest in productive/domestic assets and choose to be either net borrowers or net lenders. However, the alternative interpretation as a portfolio investor game is also reasonable and provides interesting insight on some of the results arising from these specific proposals.

Following this introduction, we now focus on the the solution to the open loop case, described by the *Pontryagin* maximum conditions given in B.1.1. We start by deriving the optimal *Keynes-Ramsey* consumption strategies. We follow the same procedure described in section 2.4 for the general problem discussed in chapter 2. The first optimal consumption strategy is defined in the following fashion. We start by taking the time derivative from the optimality condition in consumption given by (B.2), which is given by,

$$\dot{\lambda}_i(t) = -(\gamma_i - 1)\gamma_i C_i(t)^{\gamma_i - 2}\dot{C}_i(t) \Longleftrightarrow \dot{\lambda}_i(t) = \lambda_i(\gamma_i - 1) C_i(t)^{-1}\dot{C}_i(t). \quad (3.16)$$

Substituting (3.16) and (B.2), in the co-state condition (B.4), we obtain the optimal

extreme simplicity of these proposals does not take into account the diversity and complexity of real economic phenomena. Moreover, these setups are loosely related to neoclassical economic theory, but they do not not take into account all theoretical assumptions usually required in mainstream economics. We consider that this trade-off eventually arises when one considers nonlinear economies in a differential game framework. The economic modeller must take into account this natural trade-off, and seek a reasonable balance between the mathematical tractability of the proposed problem, and its economic interpretation, in the context of related theoretical fundamentals.

Keynes-Ramsey consumption open loop strategy, describing optimal consumption strategies driven by net financial assets accumulation:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(\rho_i - r - rd_i \frac{B(t)}{K(t)}\right). \tag{3.17}$$

In order to obtain the second *Keynes-Ramsey* consumption strategy, we follow the same procedure as above. First we take the time derivative of the optimality condition for investment decisions, given by equation (B.3). We obtain:

$$\dot{q}_i(t) = -\dot{\lambda}_i(t) \Longleftrightarrow \dot{q}_i(t) = \lambda_i(\gamma_i - 1)\, C_i(t)^{-1}\, \dot{C}_i(t). \tag{3.18}$$

Substituting (3.18) and (B.2), after considering the result in (B.3), in the co-state condition describing the shadow price of productive capital, (B.5), the second optimal consumption strategy[12] is given by:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}(\rho_i + \delta - r_k). \tag{3.19}$$

Now we need to impose the optimal accumulation rule that guarantees indifference between consumption strategies[13] for player i. Setting (3.17) equal to (3.19), and substituting the capital accumulation equation, (B.7), we obtain the following result defined in terms of the aggregate net financial balances ratio, $B(t)/K(t)$,

$$\frac{B(t)}{K(t)} = \frac{r + \delta - r_k}{rd_i}. \tag{3.20}$$

Given that in an open loop setup, agents do not have information on the evolution of the state of the system, this outcome can be interpreted as an individual belief regarding the true outcome of aggregate market risk premium. In the absence of information on the evolution of the state of the system, players base their decisions on individual beliefs. Since beliefs depend on d_i, we shall have asymmetric beliefs arising from asymmetries on market determined institutional risk premium. We deal with this feature of the game later on and focus now on the description of asymptotic conditions guaranteeing the existence of SCE solutions consistent with (3.20). The open loop solution to this game is defined by consumption, (3.17), net financial assets, (B.6), and productive capital dynamics, (B.7), assuming the

[12]Condition (3.19) defines optimal consumption paths assuming income arising from the accumulation of productive capital while condition (3.17) defines consumption financed by financial net assets accumulation.

[13]We would like to stress that this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining optimal beliefs is obtained when substituting directly (3.17) while deriving (3.20).

existence of optimal beliefs, as defined by (3.20), in (3.17). This system defines a solution described by a set of non-stationary variables. It is necessary to define a scaling rule consistent with the existence of stationary dynamics. We follow the same procedure described in chapter 2 and define a stationary dynamical system by taking advantage of the scaled invariance of the dynamics. We redefine the variables, now described by $X_{m,i}(t)$, again in terms of domestic capital units:

$$Z_{m,i}(t) = \frac{X_{m,i}(t)}{K_i(t)} \Rightarrow \dot{Z}_{m,i}(t) = \frac{\dot{X}_{m,i}(t)}{K_i(t)} - \frac{X_{m,i}(t)}{K_i(t)}\frac{\dot{K}_i(t)}{K_i(t)}, \tag{3.21}$$

where $m \in \{1,2,4\}$ and $X_{m,i}(t)$ defines consumption, $C_i(t)$, net foreign assets, $B_i(t)$, investment, $I_i(t)$, and $Z_{m,i}(t)$ each corresponding scaled variable for each player $i \in N$, respectively. Following the rule given in (3.21) the stationary solution to (3.15) comes out as:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i + \delta - r_k - (Z_{4,i}(t) - \delta)(\gamma_i - 1);}{\gamma_i - 1}\right]; \tag{3.22}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t) + Z_{2,i}(t)\left[r + rd_i Z_2(t) - Z_{4,i}(t) + \delta\right] - r_k; \tag{3.23}$$

where:

$$K_i(t) = K_i(0)\, e^{\int_0^t (Z_{4,i}(s)-\delta)ds} \Rightarrow K(t) = \sum_{i\in N} K_i(0)\, e^{\int_0^t (Z_{4,i}(s)-\delta)ds}; \tag{3.24}$$

$$Z_2(t) = \frac{\sum_{i\in N} Z_{2,i}(t)\, K_i(t)}{\sum_{i\in N} K_i(t)}. \tag{3.25}$$

The proposed solution to (3.15) does not provide any information regarding scaled investment strategies. We solve this issue by assuming that in the long run agents commit to linear strategies on investment per unit of capital that are consistent with the existence of equilibrium, $\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} \Rightarrow \lim_{t\to\infty} \dot{Z}_{1,i}(t) = 0$. These strategies are optimal if they solve the stationary differential system, given in (3.22) and (3.23), and the solution is consistent with the tranversality conditions given in (B.8) and (B.9). If such strategies are asymptotically consistent with players' optimal beliefs, $Z_2^{b,i}(t)$, following condition (3.20), we consider that equilibrium is a SCE, when $\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2$. We start by defining scaled investment strategy equilibrium, and assume these strategies guarantee asymptotic convergence to a feasible scaled consumption strategic equilibrium, $Z_1(t) > 0, \forall t \in T$. Setting

$\dot{Z}_{1,i}(t) = 0$ we obtain:

$$\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \frac{\rho_i + \delta - r_k}{\gamma_i - 1} + \delta. \tag{3.26}$$

Following (3.26), long run capital dynamics can be defined in the following fashion:

$$\lim_{t\to\infty} t^{-1} \log K_i(t) = \bar{Z}_{4,i} - \delta > 0. \tag{3.27}$$

Given (3.26), we can redefine consumption dynamics as a function of scaled investment strategies:

$$Z_{1,i}(t) = Z_{1,i}(0)\, e^{\int_0^t \left(\bar{Z}_{4,i} - Z_{4,i}(s)\right) ds} \tag{3.28}$$

Given the result in (3.28), we know that it is optimal for agents to undergo transitions that improve their initial scaled consumption strategies, such that in equilibrium $\bar{Z}_{1,i} > Z_{1,i}(0)$[14] . This result is sufficient to study the qualitative dynamics of this game SCE, when we assume that (3.26) is a globally stable solution to $Z_{4,i}(t)$ dynamics. We can now derive the conditions for the existence of game outcomes consistent with an asymptotic optimal SCE. We start by defining aggregate state dynamics, as given in (3.25), asymptotically. First, we assume that there is an unique equilibrium solution for individual state dynamics, $\bar{Z}_{2,i}$, obtained by solving $\dot{Z}_{2,i} = 0$. The long run outcome of $\bar{Z}_2$ is given by the asymptotic limit of (3.25), following the result in (3.26) for productive capital dynamics in the long run. Aggregate risk premium in the long run is thus given by:

$$\lim_{t\to\infty} Z_2(t) = \bar{Z}_2 = \frac{\sum_{j\in L} \bar{Z}_{2,j} K_j(0)}{\sum_{j\in L} K_j(0)}, \tag{3.29}$$

where player $j \in L$ corresponds to the subset of players that have scaled investment strategies consistent with $\bar{Z}_{4,j} = \max\left(\bar{Z}_{4,i}\right)$[15]. This result has a straightforward interpretation. Aggregate risk premium dynamics are driven by the game investment leaders in the long run. The long run risk premium condition given in (3.29) defines a relative measure that takes into account the financial situation of the ensemble of leaders in this economy and weights it against their initial productive capital endowments. In the long run, market forces price aggregate risk premium following the financial outcomes of the players choosing more aggressive, and therefore riskier investment strategies. Leverage based aggressive investment strategies raise the game

[14] if $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$ then $Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$.

[15] The asymptotic limit of a sum ratio of equal exponential terms with different coefficients is given by the sum ratio of the coefficients of the fastest growing exponential terms.

bond premium, while aggressive investment strategies from players with a diversified portfolio lower the game bond premium. This long run market risk premium measure can be justified by the existence of information costs that deter investors from acquiring relevant information. Under these circumstances it is a reasonable decision to price aggregate risk based on a sample of aggressive investors and their portfolio decisions. Recall now that we assumed that the existence of a SCE requires optimal beliefs, (3.20), to be fulfilled at least asymptotically. Substituting (3.29) in (3.20) and solving in terms of d_i, we obtain the condition guaranteeing beliefs are consistent with a SCE asymptotically:

$$d_i = \frac{r + \delta - r_k}{r\bar{Z}_2}. \tag{3.30}$$

The existence of a SCE solution is not consistent with asymmetric individual institutional risk premium measures. This game is only consistent with the existence of a SCE if the asymmetry assumption regarding individual institutional risk premium measures is dropped[16] and we consider that d_i is endogenously determined as a function of $\bar{Z}_2$. We can interpret this measure in the context of a market regulator with perfect information about the state of the game, which sets institutional risk premium in accordance with the economy aggregate outcomes. Following this result we can define the equilibrium solution, $\bar{Z}_{2,i}$, by setting $\dot{Z}_{2,i} = 0$, and substituting the asymptotic condition for optimal beliefs, (3.30). Individual state dynamics in the long run are given by:

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i}}{r + rd_i\bar{Z}_2 - \bar{Z}_{4,i} + \delta}. \tag{3.31}$$

The individual and aggregate state outcomes given in (3.31) and (3.29) are not unique solutions for investment leaders. To solve this issue we substitute the constraint for SCE, (3.30), in (3.31), and obtain:

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i}}{2r + 2\delta - r_k - \bar{Z}_{4,i}}. \tag{3.32}$$

The result in (3.32) confirms that a unique SCE exists in this game, when individual institutional risk premium valuations are the same for all players. This solution is optimal if $\bar{Z}_1 > 0$ and the transversality constraints are fulfilled. In order to define conditions for transversality in this differential game, we follow the same procedure described in section 2.4. First, it is convenient to redefine (B.8) and (B.9)

[16] One could also consider that SCE with asymmetric risk premium requires the existence of asymmetric productive capital returns. We do not explore this hypothesis because it is of no particular relevance to the qualitative description of equilibrium.

in terms of the scaled variables. Substituting the co-state variables by the optimality conditions, (B.2) and (B.3), and imposing the scaling rule, (3.21), the transversality conditions come out as:

$$\lim_{t\to\infty} \left(-\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i - 1} Z_{2,i}(t) K_i(t) e^{-\rho_i t} = 0; \tag{3.33}$$

$$\lim_{t\to\infty} \left(\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i - 1} K_i(t) e^{-\rho_i t} = 0. \tag{3.34}$$

These expressions can be rearranged by taking the scaled limit of the logarithm of (3.33) and (3.34) and solving the transversality constraints as an asymptotic inequality. The constraints (3.33) and (3.34) are now given by:

$$\lim_{t\to\infty} t^{-1} \log\left[\left(-\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i - 1} Z_{2,i}(t) K_i(t) e^{-\rho_i t}\right] < 0; \tag{3.35}$$

$$\lim_{t\to\infty} t^{-1} \log\left[\left(\gamma_i Z_{1,i}(t) K_i(t)\right)^{\gamma_i - 1} K_i(t) e^{-\rho_i t}\right] < 0. \tag{3.36}$$

From (3.35) or (3.36) it is straightforward to obtain the transversality constraint for the existence of an optimal solution as a function of long run scaled investment decisions. Assuming that capital dynamics grows asymptotically at a constant rate, following the result in (3.27), and that the scaled variables are consistent with balanced long run growth dynamics $\lim_{t\to\infty} Z_{m,i}(t) \to \bar{Z}_{m,i}$, where $\bar{Z}_{1,i}, \bar{Z}_{4,i} \in \Re^+$ and $\bar{Z}_{2,i} \in \Re$, the final condition for existence of asymptotic optimal open loop investment strategies guaranteeing long run productive capital growth is given by:

$$\delta < \bar{Z}_{4,i} < \delta + \frac{\rho_i}{\gamma_i}. \tag{3.37}$$

Having described the conditions for the existence of SCE solutions consistent with player's optimal beliefs, for the non-cooperative game given in (3.15), we now focus on the qualitative description of this solution. We base our approach on a weak argument for asymptotic stability. This argument is based on the results described in (3.26) to (3.32), which guarantee that a self-fulfilling equilibrium is always achieved asymptotically and independent of other players decisions, when institutional risk premium is unique and a function of the state of game. Since in the long run there are no longer transitions driven by $Z_{1,i}(t)$ and $Z_2(t)$ dynamics, when we assume $Z_{4,i}(t)$ dynamics always converges to the equilibrium defined in (3.26), we can evaluate qualitatively the local stability of the SCE strategies by testing the stability of the system describing scaled net assets dynamics, $\left\{\dot{Z}_{2,1}(t), \ldots, \dot{Z}_{2,n}(t)\right\}$. We start by defining the n by n *Jacobian* matrix describing individual state dynamics in the vicinity of SCE,

$$J = \begin{bmatrix} \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,1}(t)}{\partial Z_{2,n}(t)} \\ \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,2}(t)}{\partial Z_{2,n}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,1}(t)} & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,2}(t)} & \cdots & \frac{\partial \dot{Z}_{2,n}(t)}{\partial Z_{2,n}(t)} \end{bmatrix}_{Z_{2,i}(t)=\bar{Z}_{2,i}}, \tag{3.38}$$

where the partial derivatives of this *Jacobian* are given generally by the following expressions:

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} = r - Z_{4,i}(t) + \delta + rd_i \left[Z_2(t) + Z_{2,i}(t) \frac{K_i(t)}{\sum_{i \in N} K_i(t)} \right]; \tag{3.39}$$

$$\frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)} = rd_i Z_{2,i}(t) \frac{K_w(t)}{\sum_{i \in N} K_i(t)}, w \neq i \wedge w, i \in N. \tag{3.40}$$

To evaluate these derivatives in equilibrium, we have to distinguish between investment and non investment leaders. If players $i, w \in N$ are investment leaders, $i, w \in L$, then (3.39) and (3.40) evaluated in equilibrium come out as,

$$\left. \frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} \right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + rd_i \left[\bar{Z}_2 + \bar{Z}_{2,i} \frac{K_i(0)}{\sum_{j \in L} K_j(0)} \right], i \in L, \tag{3.41}$$

$$\left. \frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,w}(t)} \right|_{Z_i(t)=\bar{Z}_i} = rd_i \bar{Z}_{2,i} \frac{K_w(0)}{\sum_{j \in l} K_j(0)}, w \neq i \wedge w \in L. \tag{3.42}$$

If players $i, w \in N$ are not investment leaders, $i, w \notin L$, then (3.40) vanishes and (3.39) reduces to,

$$\left. \frac{\partial \dot{Z}_{2,i}(t)}{\partial Z_{2,i}(t)} \right|_{Z_i(t)=\bar{Z}_i} = r - \bar{Z}_{4,i} + \delta + rd_i \bar{Z}_2. \tag{3.43}$$

The local stability of SCE solutions for the game defined in (3.15) can be easily evaluated numerically. If all the eigenvalue solutions of (3.38) have a negative real part, then we can state that SCE solutions are at least locally weakly asymptotically stable. A robust argument for local asymptotic stability would have to take into account transitions to equilibrium arising from $Z_{4,i}(t)$ decisions and $Z_2(t)$ non-autonomous dynamic transitions[17].

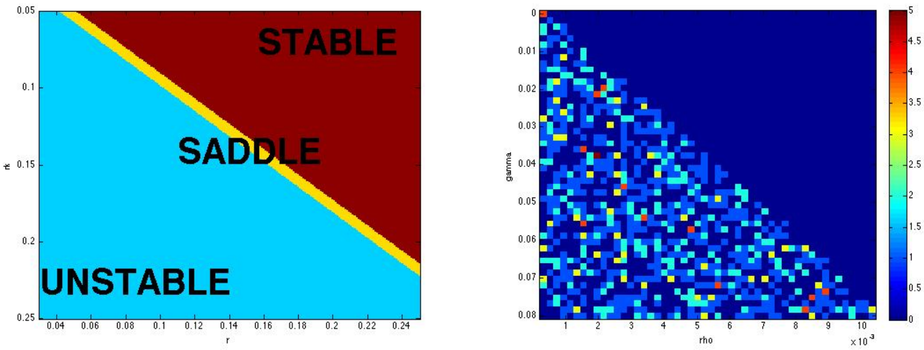
To demonstrate our conjecture, we require a final assumption. In this section, we analyse the qualitative dynamics assuming that players commit to initial investment equilibrium strategies, $Z_{4,i}(t) = \bar{Z}_{4,i} \Rightarrow Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$. This simplifying hypothesis has the advantage of not requiring any assumption regarding individual investment transitions to equilibrium, thus reducing greatly the burden required to perform a vast qualitative analysis. It allows for a systematic evaluation of the local stability of SCE solutions for large populations, given different probabilistic assumptions regarding initial conditions and parameter distributions. This assumption has the following economic interpretation. Players choose beliefs such that their consumption outcome relative to their wealth, measured by productive assets accumulation, is stable throughout the duration of the game. This result is in accordance with the *Life Cycle* hypothesis for intertemporal consumption[18]. In the end of this section, we evaluate the dynamics of this game assuming two different hypotheses for the initial consumption endowments. We shall consider that $Z_{1,i}(0)$ is given by random outcomes distributed according to $Z_{1,i}(0) \sim U(0,1)$ and $Z_{1,i}(0) \sim \exp(1)$. In these experiments, we consider the existence of a robust population, with initial productive/riskless asset endowments randomly given by an exponential distribution, $K_i(0) \sim \exp(1)$. By robust population, we mean a discrete set of $n = 1000$ agents with uniform randomly drawn individual characteristics, $\rho_i, \gamma_i \sim U(0,1)$, such that $\bar{Z}_{4,i}$ outcomes, defined by (3.26), fulfil the optimal growth constraint, (3.37), for the range institutional scenarios, $r_k \in [0.05\ 0.25]$ and $r \in [0.03\ 0.25]$, where $\delta = 0.03$. For simplification reasons, we consider that the state of the game is driven by a fixed pool of investors, which is defined by a fixed share of the population. We set this share at 30%, and consider that the aggregate risk faced by investors, (3.29), is obtained from the share of aggressive players with higher rates of investment per unit of capital. This assumption is consistent with risk setting in real markets. The LIBOR spread for example, is defined by a similar institutional mechanism, where only the average interest rate on credit transactions

[17] If we consider that the fixed point defined by $\bar{Z}_{1,i} = 0$ is always a repelling solution, following the discussion on $Z_{4,i}(t)$ dynamics, then we can assume that SCE is a globally stable solution if it fulfils the *Routh-Hurwitz Stability Criterion*. If we additionally assume that $Z_{4,i}(t)$ dynamics are *Lipschitz* continuous, then it is possible to give a full description of this game dynamics.

[18] The *Life Cycle* consumption hypothesis was forwarded by Brumberg and Modigliani [1954] and Friedman [1957]. This theory suggests that individuals make saving and consumption decisions, in order to maintain a stable consumption pattern throughout their lives. Evidence suggests that the *Life Cycle* hypothesis is not consistent with saving and consumption patterns observed in older members of the population. Older generations show patterns of precautionary saving that can be explained by: (i) intergenerational altruism; (ii) increased caution in spending; and (iii) poor retirement planning based on optimistic assumptions about life expectancy.

faced by a pool of the large financial institutions is considered.

Given the set of assumptions described in the previous paragraph, we start the discussion on the qualitative dynamic outcomes of this game with the analysis of a population, where initial consumption decisions are distributed following $Z_{1,i}(0) \sim U(0,1)$. In Figure 3.1, below, we portray, in the picture on the left, the numerical qualitative outcomes for the a given range of institutional scenarios, $r_k \in [0.05\; 0.25]$ and $r \in [0.03\; 0.25]$. These results were obtained by numerically computing the eigenvalues of the *Jacobian* defined in (3.38), in the vicinity of SCE solutions described by conditions (3.26) to (3.32). As previously discussed, we defined a robust population by randomly drawing individuals asymmetric parameters ρ_i and γ_i. The population characteristics are depicted by the density plot on the right. In this setup, robust populations are characterized by patient investors with a low intertemporal elasticity. The stability diagram is dominated by stable solutions, all eigenvalues have negative real parts, and unstable solutions[19], all eigenvalues have positive real parts. This diagram suggests that a stable SCE solution requires institutional scenarios ordered in the following fashion, $r_k \leq r$. There is also a transition region, described in yellow, that is consistent with saddle type solutions for this game. In this region we have eigenvalues with positive and negative real parts. Bifurcations of equilibria can also arise as a result of a degenerate *Jacobian*, $\det(J) = 0$[20]. In the vicinity of these singularities, higher-dimensional nonlinear phenomena may occur.



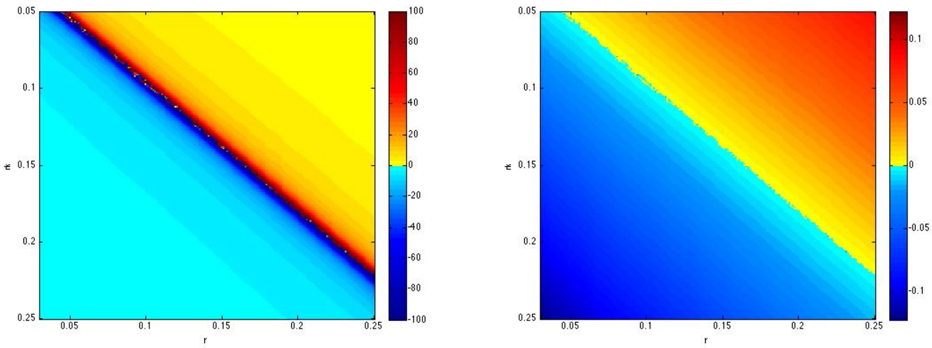
(a) Stability diagram (b) Distribution of preferences

Figure 3.1: Stability diagram and distribution of preferences for $Z_{1,i}(0) \sim U(0,1)$

[19]By unstable solutions we mean that the SCE is a completely unstable solution, which implies that the SCE solution is time-reverse stable.

[20]In these simulations we only evaluate the real part of the two leading eigenvalues, $\min\{\mathrm{Re}(\Lambda)\}$ and $\max\{\mathrm{Re}(\Lambda)\}$, where Λ stands for the set of eigenvalues solving the characteristic polynomial of (3.38) in the vicinity of SCE.

To better understand the qualitative results portrayed in Figure 3.1, it is convenient to illustrate the distributions for the endogenous variables, $\bar{Z}_2$ and d, defined in expressions (3.32) and (3.30), respectively. In Figure 3.2, we observe that stability requires that the investment leaders are bond sellers, $\bar{Z}_2 > 0$, and institutional risk premium, d, is positive but small. On the other hand, when leaders diversify their portfolios, $\bar{Z}_2 < 0$, and the regulator sets risk premium at a small but negative rate, SCE solutions are unstable. The critical transition region is characterized by leaders accumulating vast amounts of financial/foreign assets or vast amounts of debt. The regulator in this region sets risk premium close to zero.



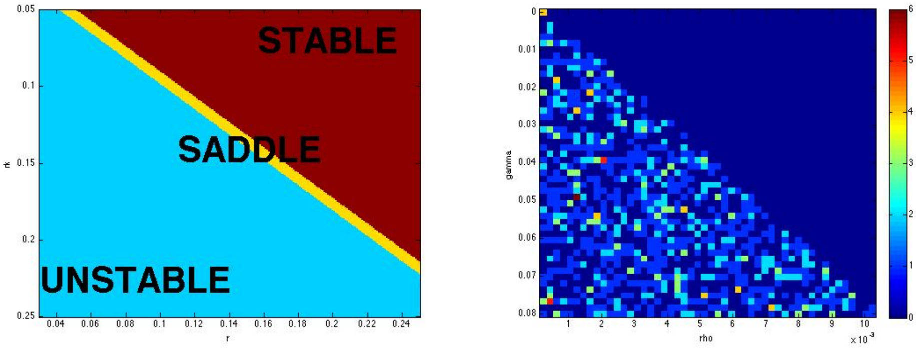
(a) $\bar{Z}_2$ outcomes

(b) d outcomes

Figure 3.2: Game and institutional risk premium outcomes for $Z_{1,i}(0) \sim U(0,1)$

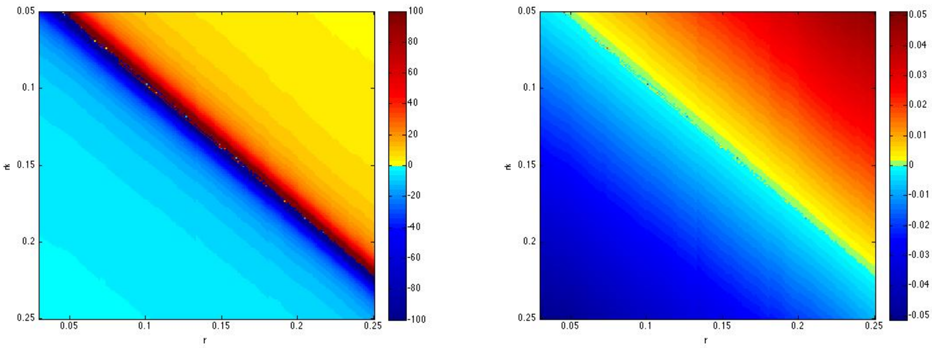
We then evaluated the qualitative dynamics of this solution with the analysis of a game where initial consumption decisions are now distributed according to $Z_{1,i}(0) \sim \exp(1)$. We followed the same procedure as in the previous experiment and start by determining robust populations for the same range of institutional scenarios. Finally, we computed numerically the eigenvalues of (3.38) in the vicinity of the corresponding SCE solutions. The results portrayed in Figure 3.3 and Figure 3.4 do not show any major difference from the results discussed above in Figure 3.1 and Figure 3.2. This outcome suggests that scaled consumption values do not seem to interfere much with the qualitative dynamics of this game. To have a better insight in this matter, we have to consider sampling results larger for populations sets, assuming different distribution hypotheses for initial consumption and the population share of investment leaders[21].

[21]Our numerical simulations suggest that there are no significant differences between these two hypotheses for games with a share population of leaders ranging from 10% to 90%. Given the constraints imposed for the existence of robust populations, it is likely that this result holds for a broad range of scaled consumption values in large populations.



(a) Stability diagram (b) Distribution of preferences

Figure 3.3: Stability diagram and distribution of preferences for $Z_{1,i}(0) \sim exp(1)$



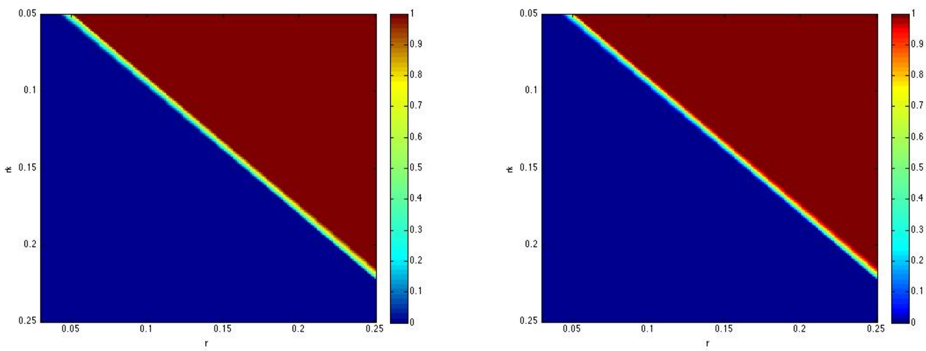
(a) $\bar{Z}_2$ outcomes (b) d outcomes

Figure 3.4: Game and institutional risk premium outcomes for $Z_{1,i}(0) \sim exp(1)$

We finish this presentation with a discussion on the qualitative dynamics of the state-separable problem in the vicinity of SCE solutions. The solution and qualitative analysis of the state-separable system, assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given in B.2.1. The condition describing qualitative dynamics in the vicinity of equilibrium is given generally by (B.26). Rearranging this expression, we obtain a condition for stability in terms of the marginal revenue on domestic capital for the state-separable game in the vicinity of the SCE solution,

$$r_k > \frac{2r(\gamma_i - 1)}{\gamma_i - 2} + \delta - \frac{\rho_i}{\gamma_i - 2}. \tag{3.44}$$

Following (3.44), we can evaluate numerically the individual qualitative dynamics obtained from the state-separable solution described in B.2.1, in the vicinity of the game SCE solutions for the two population scenarios described previously. In this analysis, we portray the percentage of players with state-separable attracting dynamics in the vicinity of the SCE. These results are given in Figure 3.5, below, for each hypothesis of initial consumption decisions, $Z_{1,i}(0)$, considered previously. Given that consumption does not play a role in the state-separable solution, the differences observed are related only to player asymmetries. A quick glance at the figures below, shows that the game stable and unstable regions, portrayed in Figure 3.1 and Figure 3.3, coincide with regions, where state-separable dynamics are consistent with 100% of players having attracting or repelling solutions, respectively. The parameter range describing the saddle region for the game SCE solution is now characterized in the state-separable system, by a region where only a share of the population has qualitative dynamics consistent with an attracting solution. We can draw a simple conclusion from this outcome. If we consider the game solution has a sum of its individual parts, we might be tempted to consider that this region is not consistent with stable SCE solutions, as a percentage of players does not agree with this SCE. However, the game solution suggests the existence of a saddle equilibrium and bifurcation phenomena. Thus we conclude that there are institutional regimes, where game dynamics in the vicinity of a SCE cannot be described by just the sum of its individual parts, as defined by the state-separable outcomes, and weak emergence phenomena might occur in this economy, as a result of aggregate interactions consistent with higher-dimensional nonlinear phenomena.



(a) $Z_{1,i}(0) \sim U(0,1)$ (b) $Z_{1,i}(0) \sim exp(1)$

Figure 3.5: Percentage of players with stable dynamics

In this section, we showed that when beliefs are unique, strategies are *Lipshitz* continuous and the game solution is bounded asymptotically, it is possible to perform a qualitative analysis of SCE solutions. This setup allowed us to show that weak emergence phenomena may arise for some parameter regions. In the next section, we show that the introduction of further nonlinearities leads to solutions where a SCE is only achievable when learning dynamics are considered. We show that feasible SCE solutions can be identified geometrically and some conjectures about possible game outcomes can be put forward, based on the qualitative analysis of the state-separable problem.

3.4 A consumption and investment game with coupled endogenous risk premium

The second game we propose is largely based on the setup discussed in the previous section with a simple exception. We now drop the institutional risk premium hypothesis and consider that individual risk premium depends on the ratio of net financial assets to productive capital. Interest payments/revenues are now given by $rB_i(t)[1 + (B_i(t)/K_i(t))(B(t)/K(t))]$. In this setup, institutional conditions driving risk premium no longer depend on market driven beliefs, but on information regarding the player financial balances. The inclusion of an additional nonlinearity in the risk premium mechanism allows for the introduction of several novel features in this economy. Bond buyers are now rewarded when the aggregate economy is a net buyer of bond contracts, and penalized when the economy is a net issuer of bonds. Bond issuers benefit from smaller interest premiums, when the aggregate economy is net buyer of bonds and are penalized if the aggregate economy is a net issuer of bonds. Given this very brief introduction, the non-cooperative differential game faced by player i is given by the following dynamic optimization problem:

$$\begin{array}{l} \underset{C_i(t),I_i(t)}{MAX} \int_0^\infty e^{-\rho_i t} C_i(t)^{\gamma_i}\, dt \\ \text{subject to the solution of:} \\ \dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + \frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right) - r_k K_i(t); \\ \dot{K}_i(t) = I_i(t) - \delta K_i(t); \end{array} \tag{3.45}$$

satisfying the transversality conditions, (B.18) and (B.19), guaranteeing that solutions to (3.45) do not grow too fast. The optimal *Keynes-Ramsey* consumption strategies for (3.45), following the procedure described in 3.3, and given the maximum conditions in B.1.2, are now defined by:

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(\rho_i - r - 2r\frac{B(t)\,B_i(t)}{K(t)\,K_i(t)}\right); \tag{3.46}$$

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(\rho_i + \delta - r\frac{B(t)\,B_i(t)^2}{K(t)\,K_i(t)^2} - r_k\right). \tag{3.47}$$

The optimality condition for indifference in capital accumulation is obtained by setting (3.46) equal to (3.47). Following the discussion in 3.3, we solve this equality in terms of the aggregate coupled risk premium mechanism, $B(t)/K(t)$. The optimal belief for this game is,

$$\frac{B(t)}{K(t)} = (\delta + r - r_k)\left[r\frac{B_i(t)}{K_i(t)}\left(\frac{B_i(t)}{K_i(t)} - 2\right)\right]^{-1}. \tag{3.48}$$

Substituting the optimal belief, (3.48), in the optimal consumption strategy, (3.46), and then scaling(3.46) and (B.16), following the rule given in (3.21), we obtain the stationary dynamical system defining the general open loop dynamic solution for the non-cooperative game given in (3.45), in terms of player i dynamics:

$$\dot{Z}_{1,i}(t) = \frac{Z_{1,i}(t)}{\gamma_i - 1}\left[\rho_i - r - \frac{2r(r + \delta - r_k)}{(Z_{2,i}(t) - 2)} - (\gamma_i - 1)(Z_{4,i}(t) - \delta)\right]; \tag{3.49}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t) + Z_{2,i}(t)\left[r + rZ_{2,i}(t)\,Z_2(t) - Z_{4,i}(t) + \delta\right] - r_k; \tag{3.50}$$

where capital dynamics, $K_i(t)$, and state dynamics, $Z_2(t)$, are given by (3.24) and (3.25). Again, there is no information regarding investment strategies. We start by defining the equilibrium condition for state dynamics, $\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i}$, assuming the existence of an asymptotic equilibrium solution for investment strategies, $\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i}$, consistent with $\dot{Z}_{1,i}(t) = 0$, and the growth and transversality condition defined in (3.37). Assuming solutions are consistent with a feasible economic outcome, $Z_1(t) > 0, \forall t \in T$, the individual state equilibrium solution is given by:

$$\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i} = \frac{2r(r + \delta - r_k)}{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)} + 2. \tag{3.51}$$

Following these assumptions on the long run dynamics of $Z_{2,i}(t)$ and $Z_{4,i}(t)$ and the result in (3.51), aggregate state dynamics is again defined asymptotically by (3.29), following the discussion in 3.3. This set of assumptions is again sufficient to define the conditions for the existence of SCE solutions consistent with the player's optimal beliefs. We start by redefining player beliefs asymptotically as a function of the scaled variables. Optimal beliefs, $Z_2^{b,i}(t)$, in the long run are now given by:

$$\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} = (r+\delta-r_k)\left[r\bar{Z}_{2,i}\left(\bar{Z}_{2,i}-2\right)\right]^{-1}. \tag{3.52}$$

Rearranging (3.52) and solving in terms of $\bar{Z}_{2,i}$, we obtain:

$$\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i} = 1 \pm \sqrt{1+\frac{r+\delta-r_k}{r}\left(\bar{Z}_2^{b,i}\right)^{-1}}. \tag{3.53}$$

Recall now that the existence of a SCE requires that $\bar{Z}_2 = \bar{Z}_2^{b,i}$ is fulfilled. From the result in (3.53), and following the equilibrium condition for individual state dynamics, (3.51), it becomes clear that agents have to pursue scaled investment strategies that involve learning dynamics and are consistent with the existence of a self confirming equilibrium asymptotically, $\lim_{t\to\infty} Z_{4,i}\{E[Z_2(t)]\} \to \bar{Z}_{4,i}\left(\bar{Z}_2\right)$[22]. This feature of the game becomes clear when we substitute (3.51) in (3.53), and obtain the asymptotic condition for scaled investment strategies that fulfils the optimal belief condition and the existence of a SCE. Investment strategies have to be consistent with

$$\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \delta + \frac{\rho_i - r}{\gamma_i - 1} - \frac{2r(r+\delta-r_k)}{(\gamma_i-1)\left(-1\pm\sqrt{1+\frac{r+\delta-r_k}{r}\bar{Z}_2^{-1}}\right)}, \tag{3.54}$$

where we already assumed $\bar{Z}_2 = \bar{Z}_2^{b,i}$. This solution is an asymptotic optimal solution to (3.45) if tranversality conditions, given by (B.18) and (B.19), are fulfilled. Following the discussion in the previous section and the results described in (3.33) to (3.36), transversality and growth conditions impose that long run scaled investment strategies are consistent with (3.37). We can finish the description of this game equilibrium with the definition of scaled consumption strategies, $\bar{Z}_{1,i}$. We define long run scaled consumption strategies by setting $\dot{Z}_{2,i} = 0$ and assuming that a SCE solution, as defined in (3.52) to (3.54), is achieved:

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i} - \bar{Z}_{2,i}\left(r + r\bar{Z}_{2,i}\bar{Z}_2 - \bar{Z}_{4,i} + \delta\right). \tag{3.55}$$

From (3.55) it is obvious that that the best strategy for player i consists in the maximization of the expected value of scaled consumption strategies, $E\left(\bar{Z}_{1,i}\right)$, given some learning process driving scaled investment strategies to a feasible SCE.

[22] For reasons of simplicity, we considered that the learning process takes only into account the expected value of $Z_2(t)$. Given the specific nature of this learning process it might be reasonable to consider other decision criteria that takes into account the uncertainty faced by players.

A static version of this game, describing the individual asymptotic outcomes, given the equilibrium solution defined in (3.51) to (3.55), is given by the following multi-objective maximization problem under uncertainty:

$$\begin{aligned}
&\underset{\bar{Z}_{4,i}}{MAX}\, E\left(\bar{Z}_{1,i}\right) \\
&\text{Subject to the solution of:} \\
&\bar{Z}_{2,i} = 2r\left(r+\delta-r_k\right)\left[\rho_i - r - \left(\bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right]-\delta\right)\left(\gamma_i-1\right)\right]^{-1}+2,
\end{aligned} \tag{3.56}$$

such that $\bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right]$ fulfils (3.37), where $E\left(\bar{Z}_2^i\right)$ is the individual expectation about the evolution of the asymptotic state of the game, $\bar{Z}_2$. The players objective, $E\left(\bar{Z}_{1,i}\right)$, is defined by:

$$E\left(\bar{Z}_{1,i}\right) = r_k - \bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right] - \bar{Z}_{2,i}\left(r + r\bar{Z}_{2,i}E\left(\bar{Z}_2^i\right) - \bar{Z}_{4,i}\left[E\left(\bar{Z}_2^i\right)\right]+\delta\right), \tag{3.57}$$

and the *N-tuple* of strategies, Π_i, for player i are given by the set of feasible investment strategic actions, following the result in (3.54):

$$\Pi_i = \left\{ \begin{array}{c} \delta + \dfrac{\rho_i - r}{\gamma_i - 1} - \dfrac{2r\left(r+\delta-r_k\right)}{\left(\gamma_i-1\right)\left(-1-\sqrt{1+\frac{r+\delta-r_k}{r}E\left(\bar{Z}_2^i\right)^{-1}}\right)}, \\ \delta + \dfrac{\rho_i - r}{\gamma_i - 1} - \dfrac{2r\left(r+\delta-r_k\right)}{\left(\gamma_i-1\right)\left(-1+\sqrt{1+\frac{r+\delta-r_k}{r}E\left(\bar{Z}_2^i\right)^{-1}}\right)} \end{array} \right\}. \tag{3.58}$$

The asymptotic dynamics of this game can thus be examined as a HMM, more concretely by a *Markov* switching regime model that mimics the co-evolutionary learning process faced by players. Recall that in this setup players have to learn a SCE, otherwise solutions are not optimal. Since in an open loop setup, agents have no information about the evolution of the state of the game, it is not clear how players can learn a SCE if their decisions can only rely on their individual beliefs. We suggest that players are able to learn the past moments of the evolution of the state of the game by simple extrapolation. Starting with an initial guess, $E\left[\bar{Z}_2^i\right]$, players are able to determine the last outcome of $\bar{Z}_2$, by measuring their individual forecasting errors. At any given moment of the game, the agent observed forecast error, ϵ_i, is given by:

$$\bar{Z}_{1,i} - E\left[\bar{Z}_{1,i}\right] = \bar{Z}_{1,i}\left[\bar{Z}_{4,i}, \bar{Z}_{2,i}, \bar{Z}_2\right] - \bar{Z}_{1,i}\left[E\left(\bar{Z}_{4,i}\right), E\left(\bar{Z}_{2,i}\right), E\left(\bar{Z}_2^i\right)\right] = \epsilon_i. \quad (3.59)$$

To extrapolate the past moments of $\bar{Z}_2$, we just have to substitute expressions $\bar{Z}_{1,i}$ and $E\left[\bar{Z}_{1,i}\right]$ by the equivalent steady-state expressions, following the result in (3.55), and after rearranging we obtain:

$$\bar{Z}_2 = -\frac{\bar{Z}_{4,i} - E\left(\bar{Z}_{4,i}\right) + \bar{Z}_{2,i}\left(r - \bar{Z}_{4,i} - \delta\right) - E\left(\bar{Z}_{2,i}\right)\left[r + rE\left(\bar{Z}_{2,i}\right)E\left(\bar{Z}_2^i\right) - E\left(\bar{Z}_{4,i}\right) + \delta\right] + \epsilon_i}{r\bar{Z}_{2,i}^2}. \quad (3.60)$$

The result in (3.60) guarantees that players have access to a distribution of past moments of the evolution of the state of the game. Given the *Bayesian* nature of this decision process, portrayed by the existence of individual optimal beliefs, we suggest that learning has to take into account these beliefs as a form of individual bias. We argue that any inference process considered should be a *Bayesian* inference process that takes into account the existence of a posterior distribution of the past moments of the game, the evolution of players forecast errors and the players optimal beliefs, as a prior assumption on future outcomes. This hypothesis has its roots in modern economic reasoning. Morris [1995], for example, puts forward a strong argument regarding the importance of considering individual priors as opposed to the common prior assumption usually found in orthodox economics literature.

> *"Perhaps the most compelling argument against the common prior assumption is the following reductio argument. If individuals had common prior beliefs then it would be impossible for them to publicly disagree with each other about anything, even if they started out with asymmetric information. Since such public 'agreeing to disagree' among apparently rational individuals seems to be common, in economic environments as elsewhere, an assumption which rules it out is surely going to fail to explain important features of the world."*

We do not put forward any specific proposal regarding learning dynamics in this game and discard the numerical evaluation of SCE solutions has a HMM solution to (3.56). The reasons for this decision are the following. First, the evaluation of learning outcomes as to rely on the numerical simulation and sampling of outcomes for the proposed HMM, following the *Markov Chain Monte Carlo* method. We already referred that this method is computationally costly. Second, given the uncertain nature of this decision process, it is not clear how a robust population can be defined in such a way that player strategies are in accordance with (3.37), and more important that player consumption outcomes are economically feasible, $\bar{Z}_{1,i} > 0$. Recall that in a co-evolutionary learning framework, such as this one, as players learn and choose their best strategies, they change the environment faced by other players in a dramatic fashion. Later in this section, we show that this phenomena is likely to occur in this setup, even when a robust population is considered. Finally, following Grandmont [1998], the local stability in complex multi-agent environments with adaptive learning dynamics depends on the degree of confidence that an agent has regarding the local stability of the system. Grandmont [1998] defined this property as the *uncertainty principle* faced by learning agents in complex environments. In other words, equilibrium may not be stable when complex learning strategies are considered. Since the evaluation of SCE solutions has to rely on complex learning strategies, and these are decided *a priori* by the modeller, the qualitative dynamics of this system will be always a function of the specific co-evolutionary process considered. We acknowledge that the best approach to this game asymptotic solution, should be based on the evaluation of (3.56) as a HMM. By discarding this approach, we cannot guarantee that a SCE solution is achievable and we cannot put forward any results regarding the stability of solutions. To overcome this issue and still provide some intuition on possible solutions to the problem defined in (3.56), we propose a method to evaluate the existence and economic feasibility of SCE solutions, based on geometrical evaluation of conjectural solutions when players commit to an initial investment strategy. We discuss the implications of feasible solutions based on the analysis of the state-separable system described in B.2.2.

The method we propose to evaluate the existence of SCE solutions to the problem defined in (3.56), relies in a straightforward geometrical approach that can be easily applied. To determine the existence of consistent SCE solutions, we start by evaluating the actual outcomes obtained, when players share a common conjecture about the state of the game. Finally a SCE solution, consistent with the general definition in (3.13), is defined by the intersection of the curve describing the actual observable outcomes, given a common conjectural expectation on $\bar{Z}_2$, and the 45 degree line that crosses the origin. We exemplify our approach schematically in Figure 3.6 below.

To exemplify our method, we start by testing institutional scenarios that are consistent with the existence of a robust population of $n = 1000$ players[23] in a reasonable interval of common conjectural outcomes of the state of the game, $\bar{Z}_2 \subset [-2.501 \;\; 2.501]$, where at least one of the strategies defined in (3.58) is con-

[23] Again we consider the same conditions described in the previous section.

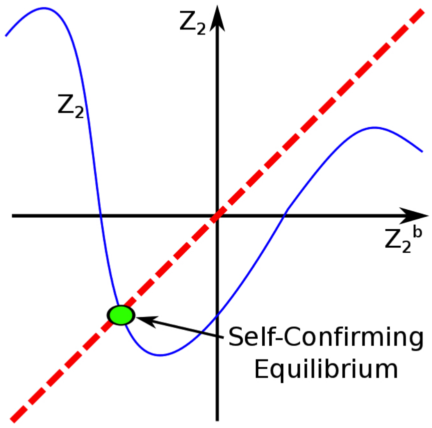


Figure 3.6: Conjectural and actual SCE solutions.

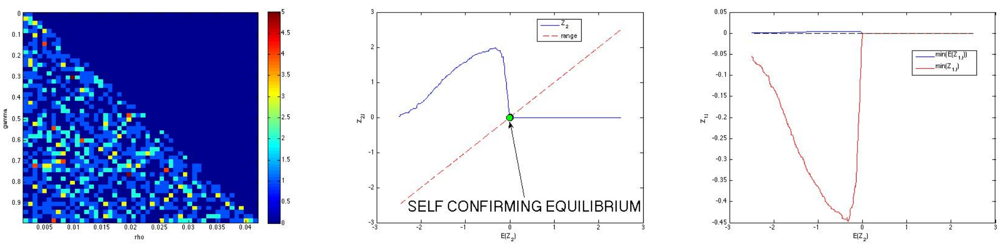
sistent with $\bar{Z}_{1,i} > 0$[24]. Numerical simulations suggest that institutional scenarios consistent with the existence of a robust population require that, $r_k \approx r + \delta$ and $r_k \neq r + \delta$[25]. We evaluate two scenarios for r_k, assuming $r = 0.05$ and $\delta = 0.03$ fixed. The results for the first scenario, $r_k = 0.07999$, are described below in Figure 3.7. Robust populations are again described by a set of patient agents, but now consumption elasticities are evenly distributed. The figure in the center shows that there is a unique SCE solution in this institutional scenario. However, the figure on the right, describing the worst player actual scaled consumption outcome, suggests that strategic interactions in this framework might not be consistent with feasible economic solutions for the entire set of agents in this economy[26]. The reason behind this dramatic outcome can be explained by wrong conjectures regarding the actual outcome of the game. The figure in the center shows that when player's conjectures are consistent with $\bar{Z}_2^b < 0$, these conjectures are systematically flawed, as actual outcomes are consistent with $\bar{Z}_2 > 0$. This result suggests that the numerical simulation of this institutional scenario as a HMM, might not be consistent with the economic constraints imposed by our model. Finally, the qualitative analysis of the state-separable solution, given in B.2.2, suggests that individual solutions, when $r_k < r + \delta$, may be either stable or unstable. The description of qualitative equilibrium solutions for the state-separable problem is provided in subsection B.2.2. These results suggest that the stability of solutions in this institutional scenario depends

[24]When both strategies are consistent with $\bar{Z}_{1,i} > 0$, we consider that players choose the strategy that yields the best consumption outcome.

[25]Recall that $r_k - \delta$ can be interpreted as the net marginal revenue of domestic capital. This result suggests that feasible solutions can only be considered for economies where the net marginal revenue of domestic assets is close to the international interest rate.

[26]Our numerical results suggest that this is a common outcome among the population.

on the relation between individual strategic investment outcomes and the individual state outcomes. Given that this outcomes depend endogenously on the the state of the game, it is unlikely that the state-separable stability condition is fulfilled for the overall set of players. Further, as we showed in the previous section, the game qualitative dynamics in the vicinity of a SCE, cannot be fully explained by the individual dynamics of the state-separable problem. Given that co-evolutionary strategic interactions, arising from learning dynamics, play a role on the possible outcomes of this game. It is reasonable to assume the existence of emergence phenomena for this institutional setup.



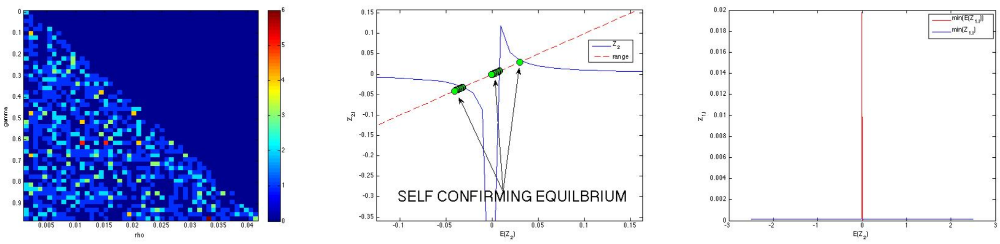
(a) Distribution of preferences (b) SCE (c) Worst player

Figure 3.7: Robust population distributions and SCE outcomes for $r_k = 0.07999$.

We finish this presentation with a discussion of an institutional scenario consistent with $r_k > r + \delta$. We now consider $r_k = 0.0801$. The results for this scenario, depicted below in Figure 3.8, suggest that the feasibility problems arising from wrong conjectures about aggregate risk premium outcomes are no longer an issue. The computed population shares the same broad characteristics described in the previous example. However, we no longer have a unique SCE solution. Our numerical routine detects three distinct SCE solutions[27]. The first consequence of this result is that if players are able to learn and concur in a specific SCE solution, the final outcome is always conjectural. Hu and Wellman [1996], for example, show that solutions to competitive multi-agent models with learning dynamics are highly sensitive to initial conditions. Another hypothesis is that players do not concur on an unique solution and wonder between different equilibrium. This conjecture is suggested by the qualitative analysis of the state-separable solution. As discussed in the end of subsection B.2.2, institutional scenarios where $r_k > r + \delta$, are always consistent with saddle solutions for the individual problem. Again, we stress that we can only speculate about the possible outcomes that may arise when complex

[27]The repetition of intersection dots is a consequence of our computational procedure. This routine controls for intersections to the left and to the right of the 45 degree line. Although we considered very small error tolerances, we were not able to eliminate this problem fully.

learning strategies are considered. However, this final example suggests that the analysis of this game should be constrained to institutional scenarios consistent with $r_k > r + \delta$, where SCE solutions are of a conjectural nature and players' conjectures robust to strategic interactions under uncertainty. An alternative approach would be the analyses of the open loop feedback *Nash* solution to the games of leaders and followers. An example of this approach for a class of multi-player general sum differential games with leaders and followers is given in Bacchiega et al. [2010]. If there are feedback *Nash* equilibrium solutions consistent with a SCE, then we can evaluate under what conditions is the Dynamic Programming problem consistent with an optimal solution to (3.45). However, given the coupled nature of the state of this game, it is unlikely that an analytical solution to the uncoupled leader/follower open loop feedback game can be derived.



(a) Distribution of preferences (b) SCE (c) Worst player

Figure 3.8: Robust population distributions and SCE outcomes for $r_k = 0.0801$.

3.5 Conclusions and further research

In this chapter, we proposed the existence of optimal beliefs for a class of differential games consistent with a specific state-separability property and related the existence of optimal solutions with the concept of SCE in a non-cooperative incomplete information game setup. In the first example discussed, we showed that a SCE outcome is feasible, when asymmetries between players are either limited or else further asymmetries are considered. Grandmont [1998] had already suggested this relation as crucial for the existence and stability of self-fulfilling outcomes in large socio economic systems. We also showed that a qualitative analysis of equilibrium is feasible using standard dynamical system techniques and portrayed the existence of weak emergence phenomena, by comparing the game outcomes with the individual solution obtained from the analysis of the state-separable system. In the second example discussed, we showed that the introduction of further nonlinearities has dramatic implications. In this setup, we have to consider that agents are able to coordinate asymptotically their belief based decisions and learn a SCE. We suggested two possible paths to evaluate SCE solutions, based on a static version of the game equilibrium solution. First, we proposed that the asymptotic outcomes of this game can be evaluated as a multi-objective maximization problem. We then proposed a simple geometric approach to determine the existence of solutions consistent with a SCE and test the robustness of individual decisions under uncertainty. This approach allows for a description of feasible optimal solutions and a discussion of possible outcomes based on the analysis of the state-separable solution. Despite its simplicity, this approach allowed us to determine that only institutional scenarios where the net marginal revenue of domestic capital is slightly greater than the international interest rate, are robust for a co-evolving environment with strategic interactions. This result paves the way for a future evaluation of this game as a HMM, where different hypotheses regarding complex learning dynamics can be tested, with the objective of determining under what conditions a SCE solution can be achieved. We discuss this methodology in the next chapter for games where players have subjective beliefs about the state of the game. An alternative approach would be the evaluation of open loop feedback *Nash* solutions to (3.45) consistent with a SCE for the uncoupled games of leaders and followers. This is an interesting option to explore in the future that may provide some insight on the qualitative dynamics in the vicinity of SCE equilibria.

Chapter 4

Subjective Beliefs and Unintended Consequences for a Class of Games with Economic Applications

4.1 Introduction

In this chapter we discuss the implications of introducing further nonlinearities in the incomplete information non-cooperative differential game setup discussed in chapter 3. Our specific proposal departs from a conjecture regarding the existence of optimal solutions to a class of exponentially discounted games that lack the state-separability property. In this framework, players lack the relevant information to pursue their optimal strategies and have to base their decisions on subjective beliefs[1]. We argue that solutions to this class of games are optimal if decisions based on subjective beliefs are consistent with the existence of a stable SCE.

We propose to answer the following question. Are players able to concur on a SCE, where their expectations are self-fulfilling and their beliefs are subjective? To frame this conjecture, we describe in section 4.2 a general setup for the class of non-cooperative differential games discussed in section 3.2, which have solutions

[1]This outcome can be linked to the paradox of relevance. The paradox of relevance states that either by ignorance or choice, individuals lack the relevant information to pursue their actions. In a dynamic game context the relevance paradox can be described in the following fashion. When deciding their strategies, players seek only the information that they perceive as relevant. However, this information might not be sufficient to allow players to pursue their optimal strategies. This problem can arise because players cannot access the relevant information, because they did not consider important information as crucial or cost effective. When this phenomenon dominates the decision making process, the player is trapped in a paradox. When decision makers are not aware of essential information to pursue their decisions in the best possible way, then strategic choices will lead to unintended consequences. Such consequences can be either positive or negative.

consistent with coupled open loop strategies, defined as an initial value problem[2]. Solutions in this class of games are optimal if player beliefs converge asymptotically to a SCE. To demonstrate the validity and implications of this conjecture, we now consider the existence of individual investment adjustment costs in the differential games discussed in section 3.3 and section 3.4.

To demonstrate our main hypotheses, we extend the conjectures discussed in chapter 3, regarding the existence and qualitative analysis of solutions for dynamic games under incomplete information. In the game proposed in section 4.3 we show that a full qualitative analysis is possible when we consider that players have naive beliefs consistent with the existence of an asymptotic SCE solution. In this game, player beliefs impose a unique equilibrium. The existence of a SCE solution requires the existence of constraints on individual parameter distributions. Given a simplified hypothesis for the evolution of beliefs, assuming state independent control dynamics, we are able to give a full description of strategic dynamics and a qualitative analysis of state-space dynamics in the vicinity of the SCE solution, following the same procedure discussed in section 3.3. Numerical results suggest that SCE are not consistent with local stability. In the second game proposed, we show that the game solution can only be described as a multi-objective optimization problem under uncertainty. We evaluate this solution numerically as a multi-criteria HMM, and show that under a linear learning regime, players converge to a SCE. We also show the existence of unintended consequences and strong emergence phenomena, as a result of persistent uncertainty.

When beliefs are no longer consistent with the existence of an unique solution, standard dynamical systems techniques cannot be applied. However, if beliefs guarantee that the state-space of the game is bounded asymptotically, then long run strategic dynamics, as shown previously, can be described as a multi-objective optimization problem under uncertainty. We portray this hypothesis in section 4.4. In this section, we show that the game solution is defined by an individual equilibrium region in $\Re^2$, bounded by growth and transversality conditions. This is a complex geometric problem that may or may not have solutions consistent with a SCE. To check the existence of a SCE in this framework, we propose that this optimization problem can be evaluated numerically as a HMM, where players learn a SCE in

[2]This option of analysing a specific class of games and discuss specific applications as a long tradition in differential game theory. In a review of the first 25 years of economic applications in differential games, following the pioneering work of Isaacs [1954], Clemhout and Wan [1979] already suggests that the analysis of general sum multi-player differential games is limited to the study of specific applications. As the authors put it: *"A frontal assault can be promising only when a general theory of partial differential equations becomes available. So far, progress is made by solving special classes of differential games..."*.

a co-evolving environment. This approach has several drawbacks. First, in incomplete information learning environments, it is unlikely that players will follow smooth strategies. As a consequence, the introduction of learning dynamics most probably does not allow for a rigorous mathematical description of the game dynamics and equilibrium solution. Başar and Olsder [1995] describe this crucial dilemma in the following fashion.

> *"Hence, a relaxation of the Lipschitz-continuity condition on the permissible strategies could make an optimal control problem quite ill-defined. In such a problem, the single player may be satisfied with smooth (but sub-optimal) strategies. In differential games, however, it is unlikely that players are willing to restrict themselves to smooth strategies voluntarily. If one player would restrict his strategy to be Lipschitz, the other player(s) may be able to exploit this. ... In conclusion, non-Lipschitz strategies cannot easily be put into a rigorous mathematical framework. On the other hand, in many games, we do not want the strategy spaces to comprise only smooth mappings."*[3]

Further, player strategies in an uncertain learning environment are not in accordance with the notion of dynamic consistency. This is a result of decision making in an uncertain environment, where player belief based decisions for some future period might not be optimal when that future period arrives. SCE outcomes in this context are subgame imperfect equilibrium solutions since they violate the *Bellman Principle of optimality.* It is well known fact that in incomplete information environments backward induction cannot be applied to non singleton information sets. This co-evolutionary phenomenon is later on illustrated by the HMM setup proposed to analyse the hypersurface resulting from the equilibrium solution obtained from first order maximum conditions. In this context, even if for a given set of parameters and learning strategies a SCE is unique and achievable solution, the dynamic decision paths followed by players, given the same set of initial conditions, are most likely different for each simulation. In our setup, this phenomenon is exacerbated because the equilibrium hypersurface is evaluated as a repeated equilibrium game, where players are chosen randomly and their moves depend on individual subjective beliefs and learning strategies. Second, as a consequence of the methodology proposed, a consistent evaluation of the asymptotic game dynamics as a HMM requires the sampling and statistical analysis of numerical solutions, assuming different hypotheses. As we referred in chapter 3, the analysis of solutions using this method

[3] Part II, chapter 5, section 5.3.

poses several limitations. It is a computationally expensive method and its outcomes are a result of subjective modelling decisions regarding players' learning dynamics. On the other hand, given that the existence of optimal solutions requires players to learn a SCE, all learning mechanisms considered have to fulfil this specific *Bayesian Incentive*, and solutions can be ordered according to the *Bayesian Efficiency* SCE criterion for optimality. This approach merges concepts from the fields of evolutionary multi-objective optimization, machine learning and game theory. We discuss this topic in detail in subsection 4.4.1, when describing the specific HMM proposed. In subsection 4.4.2 and subsection 4.4.3, we evaluate this game assuming two different learning mechanisms and portray the existence of strong emergence phenomena. In the first example, we show that agents are able to self-organize in an asymptotically robust SCE. In the second example, we show that belief-based decisions result in a stationary co-evolutionary dynamic process driven by permanent strategic interactions. The SCE attractor is now described by an invariant set that can only be evaluated statistically. We consider this outcome as a natural consequence of decision under subjective beliefs in nonlinear co-evolving environments. Individual belief-based decisions lead to unintended consequences that effectively change the environment faced by others, and as a consequence, their beliefs. This feedback loop drives the complex dynamics observed. Although this solution does not represent an optimal solution to the game proposed, we believe that this result provides a crucial link between the social sciences and decision theory paradigm of unintended consequences[4] and the mathematics and statistics paradigm of subjective probability[5].

[4]The law of unintended consequences was a concept introduced by the sociologist Robert K. Merton, in his seminal essay Merton [1936]. In very broad terms, Robert K. Merton defined the possible causes of the unintended consequences of purposive social action as ignorance, error, conventions and self or institutional interest. The author defines the process of social evolution as a consequence of social decision making based on subjective assumptions. Subjective decisions play a crucial role on the development of the complex inter-relations which organize society, and inevitably lead to individual, collective or institutional decisions with unintended consequences. The author describes this process in the following fashion.*"The empirical observation is incontestable: activities oriented toward certain values release processes which so react as to change the very scale of values which precipitated them. This process may in part be due to the fact that when a system of basic values enjoins certain specific actions, adherents are not concerned with the objective consequences of these actions but only with the subjective satisfaction of duty well performed. Or, action in accordance with a dominant set of values tends to be focused upon that particular value-area. But with the complex interaction which constitutes society, action ramifies, its consequences are not restricted to the specific area in which they were initially intended to center, they occur in interrelated fields explicitly ignored at the time of action. Yet it is because these fields are in fact interrelated that the further consequences in adjacent areas tend to react upon the fundamental value-system."*.

[5]The paradigm of subjectivity, as a key concept in modern probability theory, is framed by the proposals of *Henri Poincaré*, *Frank Ramsey* and *Bruno De Finneti*. Although this discussion has older roots in philosophy and scientific thought, these three famous authors are fundamental for the development of key modern concepts in probability theory. For example, the formal definition of

When considered in a nonlinear co-evolutionary context, these two forces reveal, in our opinion, the evolutionary nature of decentralized competitive economic dynamics. The paradigm of subjectivity, as the driver of social and economic decisions, is a crucial concept to understand how unintended consequences arise and lead to complex evolutionary dynamics in a non-cooperative differential game framework.

4.2 General setup

Consider again the general N-Player non-cooperative differential game framework discussed in section 3.2 and described by (3.1). We now assume the following conjecture regarding the existence of state-dependent optimal open loop strategies, for the game given in (3.1). The conjecture is that player i optimal control solutions to (3.1) are defined generally in the following fashion,

$$\dot{u}_i(t) = f_i(u_i(t), X(t)). \tag{4.1}$$

When optimality conditions impose strategic solutions that are consistent with (4.1), we have to acknowledge that players lack the relevant information to achieve their optimal goals. To pursue their optimal strategies in this environment, players require full knowledge of the state of the game, $\eta_i = \{X(t)\}$, but when choosing the relevant information to pursue their strategies, either by ignorance or choice, players' knowledge about the state of the game is incomplete. As players lack crucial information to pursue their goals, they have to rely on subjective beliefs. In a open loop setup, we shall consider that belief dynamics are defined by a

the general *Bayes* theorem, was first introduced by *Pierre-Simon Laplace*. This is usually referred to as the inductive reasoning theorem. However, it is *Henry Poincaré* that later forwarded a crucial justification for the introduction of the inverse probability reasoning concept in the field of calculus of probability, when he introduced the doctrine of conventionalism, and subjective probability theory. The crucial modern hypothesis of subjectivity can be attributed to *Bruno De Finetti* main paradigm. Probability as a measure of an event departing from an objective perspective simply does not exist. Therefore, all probabilities are subjective. *De Finetti* argues that there is always an inherent degree of uncertainty driving one's beliefs about a given phenomenon. *Ramsey* is one of the first proponents of subjective probability, but describes this paradigm in terms of the degree of confidence or specific beliefs, one might have regarding the probability of an event. This concept is closer to *Poincaré's* doctrine of conventionalism, since it allows for conventions to be determined and improved, assuming methods that increase the degree of confidence regarding the probability of a specific phenomenon. According to *Poincaré*, this could be achieved by proving the validity of conventions empirically. However, as *Ramsey* put it, the degree of confidence that one may have regarding the probability of a given phenomenon is also impaired by subjectivity. Therefore, there is always a degree of belief regarding any given probability measure.Gower [1997] gives a detailed review, survey and discussion on this topic. The author focus the discussion on *Poincaré's* vision and provides a detailed review of *Poincaré's* proposals on conventions and subjectivity, which include date back to his work on non euclidean geometry and dynamical systems. Regarding *Ramsey* and *De Finetti*, their main contributions are found in Ramsey [1926] and de Finetti [1937], respectively.

function, $X_b^{(i)}$, that depends on the information available to the player. We characterize beliefs as naive if there is no incentive to learn the true state of the game. Given that in the setup defined by (3.1), players only have information about the evolution of their individual state and their strategies, we define subjective beliefs in the following fashion: (i) player beliefs are independent of other player decisions, $X_b^{(i)}(t) = v_i(u_i(t))$; (ii) player decisions are coupled but beliefs depend solely on individual outcomes observed, $X_b^{(i)}(t) = v_i(u_i(t), x_i(t))$; and finally (iii) players can extrapolate past moments of the state of the game from individual outcomes, $X_b^{(i)}(t) = v_i(u_i(t), x_i(t), X(t-\Delta t))$[6]. Following this set of assumptions, the problem faced by player i under subjective beliefs, is generally defined by the following dynamical system:

$$\dot{u}_i(t) = f_i\left(u_i(t), X_b^{(i)}(t)\right); \tag{4.2}$$

$$\dot{x}_i(t) = g_i(u_i(t), X(t)). \tag{4.3}$$

To evaluate the game defined in general by (4.2) and (4.3), we require the existence of solutions consistent with asymptotic convergence to a equilibrium solution. A necessary condition for the existence of subjective belief solutions to (3.1), is that the following set of conditions is fulfilled,

$$\lim_{t\to\infty} X_b^{(i)}(t) = \bar{X}_b^{(i)} \wedge \lim_{t\to\infty} X = \bar{X} \wedge f_i\left(\bar{u}_i, \bar{X}_b^{(i)}\right), g_i\left(\bar{u}_i, \bar{X}\right) = 0, \tag{4.4}$$

where $\bar{X}_b^{(i)}$, $\bar{X}$ and $\bar{u}_i$, define an unique equilibrium solution that bounds the state-space of the game asymptotically. Solutions to (4.2) and (4.3) are optimal solutions to (3.1), if the transversality conditions are fulfilled for a equilibrium satisfying (4.4), and beliefs match the true state outcomes in the long run,

$$|X_b^{(i)}(t) - X(t)| \to 0 \text{ and } f_i\left(\bar{u}_i, \bar{X}\right), g_i\left(\bar{u}_i, \bar{X}\right) = 0 \text{ as } t \to \infty. \tag{4.5}$$

Condition (4.5) implies that optimal open loop solutions to (3.1), given control solutions defined by (4.1), and strategic decisions driven by beliefs, following (4.2), require the existence of a SCE solution. Given that the game solution does

[6]In subsection 4.4.1 we show that this is a valid hypothesis. In this environment players observe forecasting errors and use their available information to extrapolate past outcomes of the state of the game.

not provide any information regarding the evolution of beliefs, a qualitative evaluation of the SCE using standard methods requires specific assumptions about belief dynamics. If the belief function guarantees that a SCE is achieved asymptotically and (4.2) is smooth, then it is possible to perform a full qualitative analysis of the game defined in (3.1). In the next section, we portray this hypothesis assuming a naive specification for the evolution of beliefs, in a game where conditions impose a unique equilibrium solution. In section 4.4, beliefs are no longer consistent with a unique solution, but equilibrium, growth and transversality conditions bound the state-space of the game asymptotically. In this framework, the existence of a SCE can only be evaluated as a learning outcome in a co-evolutionary environment. Beliefs are no longer consistent with the existence of smooth strategies. We evaluate the existence of SCE solutions numerically, as a HMM, for two different learning mechanisms. To undertake this analysis, we propose to evaluate the asymptotic dynamics of the game equilibrium region. The equilibrium region is obtained as previously described. First we impose the existence of equilibrium solutions for beliefs and the state of the game. Then we solve the dynamical system obtained from first order condition for equilibrium, such that condition (4.4) is fulfilled. In the setup we propose, the equilibrium region is described by a bounded nullcline hypersurface when players have finite beliefs about the true state of the game. We propose that dynamics in this region can be analysed as a HMM, where agents seek to learn the true state of the game and SCE is a limiting outcome of an adaptive learning scheme. This approach seeks to mimic the co-evolutionary nature of the game by setting the asymptotic solution as a stochastic repeated game, where agents decisions are taken under uncertainty and players moves are sequential and random, as opposed to the simultaneous strategic environment defined by first order conditions. Applying the *Monte Carlo* method and simulating equilibrium dynamics as a *Markov* game, a statistical description of the differential game solution can be obtained through sampling, following the *Law of Large Numbers.* It is thus convenient to extend the general SCE definition, described in section 3.2, to accommodate the existence of adaptive learning dynamics, as proposed in Sargent [2008]. Suppose that each player begins with an initial estimate $\hat{\boldsymbol{\theta}}(0)$ at $\boldsymbol{\tau} = 0$ and uses the following recursive learning algorithm,

$$\hat{\boldsymbol{\theta}}(\boldsymbol{\tau}+1) - \mathbf{E}[\boldsymbol{\alpha}^{\boldsymbol{\tau}}] = \mathbf{e}\left[\hat{\boldsymbol{\theta}}(\boldsymbol{\tau})\right], \tag{4.6}$$

where $\mathbf{E}[\boldsymbol{\alpha}^{\boldsymbol{\tau}}]$ is an average of past observed moments of the true state of the game, which can be extrapolated from players observed pay-offs. $\mathbf{e}\left[\hat{\boldsymbol{\theta}}(\boldsymbol{\tau})\right] \sim \mathbf{e}[0, \boldsymbol{\Phi}]$ an error distribution with a zero mean and second moment determined as a function of the last forecast, $\mathbf{\Phi}\left[\hat{\boldsymbol{\theta}}\left(\boldsymbol{\tau}\right)\right]$. Then a adaptive learning outcome is a SCE when player forecasts converge,

$$\hat{\boldsymbol{\theta}}\left(\boldsymbol{\tau}\right) \to \bar{\boldsymbol{\theta}}, \tag{4.7}$$

and a limiting outcome when uncertainty vanishes, $\mathbf{\Phi}\left[\hat{\boldsymbol{\theta}}\left(\boldsymbol{\tau}\right)\right] \to 0$, such that future forecasts are no longer taken under uncertainty, $\mathbf{\Phi}\left[\bar{\boldsymbol{\theta}}\right] = 0$, when deviations from SCE are no longer observed.

4.3 A consumption and investment game with investment bias and coupled institutional risk premium

To demonstrate these conjectures, we consider two extensions of the non-cooperative differential games under incomplete information proposed in chapter 3. In this section we extend the game framework described in section 3.3 and assume that players now face convex investment costs in their budget constraint. In this non-cooperative differential game framework, investment costs are again defined by the convex mechanism described in chapter 2, $I_i\left(t\right)\left(1 + h_i I_i\left(t\right)/K_i\left(t\right)\right)$, where parameter $h_i \in \Re$ has the following interpretation in a game environment: if (i) $h_i < 0$, institutional conditions and/or individual characteristics impose bias on investment in productive/domestic assets, if (ii) $h_i > 0$, institutional conditions and/or individual characteristics impose bias on investment in financial/foreign assets.

Following this brief description of the decision problem faced by each member of this economy, the non-cooperative game faced by player $i \in N$, is defined by the following dynamic optimization problem:

$$\begin{aligned}
&\underset{C_i(t),I_i(t)}{MAX} \int_0^{\infty} e^{-\rho_i t} C_i\left(t\right)^{\gamma_i} dt \\
&\text{subject to the solution of:} \\
&\dot{B}_i\left(t\right) = C_i\left(t\right) + I_i\left(t\right)\left(1 + h_i \tfrac{I_i(t)}{K_i(t)}\right) + rB_i\left(t\right)\left(1 + d_i \tfrac{B(t)}{K(t)}\right) - r_k K_i\left(t\right); \\
&\dot{K}_i\left(t\right) = I_i\left(t\right) - \delta K_i\left(t\right);
\end{aligned} \tag{4.8}$$

satisfying the transversality conditions, (C.8) and (C.9), guaranteeing that solutions to (4.8) do not grow too fast.

We now put forward the solution to the open loop case, following the necessary and sufficient first order maximum conditions given in C.1.1. We start by deriving the optimal *Keynes-Ramsey* consumption strategies. The first *Keynes-*

Ramsey consumption strategy is obtained as usual by taking the time derivative of (C.2) and substituting both these expressions in the co-state condition (C.4). After some manipulations we obtain,

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(\rho_i - r - rd_i\frac{B(t)}{K(t)}\right). \tag{4.9}$$

To obtain the second *Keynes-Ramsey* consumption strategy, we first have to take the time derivative of the optimality condition on investment decisions (C.3). We obtain the following differential equation,

$$\dot{q}_i(t) = -\dot{\lambda}_i(t)\left(1 + 2h_i\frac{I_i(t)}{K_i(t)}\right) - \lambda_i(t)\,2h_i\frac{\dot{I}_i(t)}{K_i(t)} + \lambda_i(t)\,2h_i\frac{I_i(t)}{K_i(t)}\frac{\dot{K}_i(t)}{K_i(t)}. \tag{4.10}$$

Substituting (4.10) and optimality conditions (C.2) and (C.3) in the co-state condition defined in (C.5), where $\dot{\lambda}(t)$ is again given by the time derivative of (C.2), we obtain the second *Keynes-Ramsey* consumption strategy:

$$\begin{aligned}\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(1 + 2h_i\frac{I_i(t)}{K_i(t)}\right)^{-1}\Bigg[(\rho_i + \delta)\left(1 + 2h_i\frac{I_i(t)}{K_i(t)}\right) - h_i\frac{I_i(t)^2}{K_i(t)^2} \\ -r_k - 2h_i\frac{\dot{I}_i(t)}{K_i(t)} + 2h_i\frac{I_i(t)}{K_i(t)}\frac{\dot{K}_i(t)}{K_i(t)}\Bigg].\end{aligned} \tag{4.11}$$

Now we need to impose the optimal accumulation rule that guarantees indifference between consumption strategies for player i. Setting (4.9) equal to (4.11), and substituting the capital accumulation equation, (C.7), this rule defines strategic investment decisions[7], which are given by,

$$\dot{I}_i(t) = \frac{I_i(t)^2}{2K_i(t)} + \left(r + rd_i\frac{B(t)}{K(t)}\right)I_i(t) + \left(r + rd_i\frac{B(t)}{K(t)} + \delta - r_k\right)\frac{K_i(t)}{2h_i}. \tag{4.12}$$

The optimal open loop solution to the game defined in (4.8) is thus given by consumption, (4.9), investment, (4.12), net financial assets, (C.5), and productive capital dynamics, (C.6). This system defines a solution described by a set of non-stationary variables. Following the rule given in (3.21), we scale our variables in terms of domestic capital units and obtain the stationary dynamical system describing the

[7] As discussed in chapter 2 this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining strategic investment dynamics is obtained when substituting directly (4.9) while deriving (4.11).

general dynamic solution for the non-cooperative game given in (4.8), in terms of player i dynamics,

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i - r - rd_i Z_2(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1}\right], \tag{4.13}$$

$$\begin{aligned}\dot{Z}_{2,i}(t) = Z_{4,i}(t)\left[1 + h_i Z_{4,i}(t)\right] + Z_{2,i}(t)\left[r + rd_i Z_2(t) - Z_{4,i}(t) + \delta\right] \\ + Z_{1,i}(t) - r_k,\end{aligned} \tag{4.14}$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + \left[r + rd_i Z_2(t) + \delta\right] Z_{4,i}(t) + \frac{r + rd_i Z_2(t) + \delta - r_k}{2h_i}, \tag{4.15}$$

where $K_i(t)$, $K(t)$ and $Z_2(t)$ are again given by (3.24) and (3.25).

Since players follow open loop strategies, we have to assume that individual solutions to this game can only be correctly described by strategic decisions based on subjective beliefs about the state of the game. Following the general discussion in section 4.2, and the assumption in (4.2), we define individual beliefs about the evolution of $Z_2(t)$ as $Z_2^{b,i}(t)$. The intuition is straightforward. When choosing their strategies, players discard the use of relevant information about the state of the economy. To pursue these strategies, players have to base their decisions on beliefs. Beliefs can be of a static, dynamic and/or stochastic nature, as long as they are consistent with the existence of a strategic equilibrium that bounds the state-space of this game asymptotically. We now consider that individual strategic dynamics are given by (4.16) and (4.17), below, instead of (4.13) and (4.15), respectively. The individual state dynamics continue to be defined by (4.14), while strategic dynamics for player i are now given by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i - r - rd_i Z_2^{b,i}(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1}\right]; \tag{4.16}$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + \left[r + rd_i Z_2^{b,i}(t) + \delta\right] Z_{4,i}(t) + \frac{r + rd_i Z_2^{b,i}(t) + \delta - r_k}{2h_i}. \tag{4.17}$$

We now focus on the description of asymptotic solutions to this game. Following the conjecture on the existence of solutions to this class of games, defined by condition (4.4), beliefs have to be consistent with the existence of an asymptotic strategic equilibrium solution. In this specific setup, the evolution of beliefs has to be consistent with $\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2{}^{b,i} \wedge \dot{Z}_{1,i}\left(\bar{Z}_2{}^{b,i}\right) = 0$. Setting $\dot{Z}_{1,i}(t) = 0$, the existence of individual equilibrium beliefs consistent with feasible asymptotic consumption outcomes, $\bar{Z}_{1,i} \in \Re^+$, requires that

$$\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} = \frac{\rho_i - r - (\gamma_i - 1)\left(\bar{Z}_{4,i} - \delta\right)}{rd_i}, \tag{4.18}$$

is fulfilled. Condition (4.18) defines a unique belief solution about the long run evolution of the state of the game that depends on individual investment decisions and parameter asymmetries. This result has both advantages and disadvantages. If we consider that belief dynamics are given by a *Lipschitz* continuous function, consistent with $\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i}$, then, given some simplifying assumptions, it is possible to perform qualitative analysis using standard dynamical systems techniques. On the other hand, the existence of an asymptotic SCE solution, as defined by (4.5), consistent with an optimal solution to (4.8), requires that we impose further constraints on this game solution, such that the existence of a unique solution fulfilling $\lim_{t\to\infty} Z_2(t) = \bar{Z}_2^{b,1}, \ldots \bar{Z}_2^{b,n}$ is guaranteed. This set of conditions requires that individual parameter asymmetries are distributed in a unique fashion. To demonstrate this result, it is convenient to define first the remaining necessary conditions for the existence of equilibrium solution consistent with a bounded state-space for this game dynamic solution. We start by defining scaled investment equilibrium, $\bar{Z}_{4,i}$. Setting $\dot{Z}_{4,i}(t) = 0$ and substituting $\bar{Z}_2^{b,i}$ by the result in (4.18), we obtain:

$$\begin{aligned}\lim_{t\to\infty} Z_{4,i}(t) = \bar{Z}_{4,i} = \frac{-\left[\rho_i + \delta\gamma_i - (\gamma_i - 1)(2h_i)^{-1}\right]}{1 - 2\gamma_i} \\ \frac{\pm\sqrt{\left[\rho_i + \delta\gamma_i - (\gamma_i - 1)(2h_i)^{-1}\right]^2 - (2 - 4\gamma_i)(\rho_i + \delta\gamma_i - r_k)(2h_i)^{-1}}}{1 - 2\gamma_i}.\end{aligned} \tag{4.19}$$

Now, it is convenient to redefine domestic/productive capital accumulation in the long run, as a function of long run investment decisions. Taking the asymptotic limit of expression (3.24), long run productive capital dynamics can be expressed as a function of (4.19). Following the result in section 3.3, $K_i(t)$ when $t \to \infty$ is again given by (3.27).

Recall now that strategic consumption dynamics are defined endogenously by $Z_{4,i}(t)$ and $Z_2^{b,i}(t)$ dynamic transitions to equilibrium. Specific assumptions regarding belief dynamics have to be taken into account in this context. We discuss this topic with further detail later in this section and focus now on the definition of state equilibrium. State dynamics are defined asymptotically in the same fashion as in section 3.3. First, we assume that there is a unique equilibrium solution for individual state dynamics, $\bar{Z}_{2,i}$, obtained from solving $\dot{Z}_{2,i} = 0$. We then assume that aggregate state dynamics, $\bar{Z}_2$, is given by the asymptotic limit of (3.25), given the result in (3.27) for asymptotic productive capital dynamics. In the long run the state of the game, $\lim_{t\to\infty} Z_2(t) = \bar{Z}_2$, is again defined by (3.29), following the discussion in section 3.3 on aggregate risk premium driven by the game investment leaders. Again we shall consider that player $j \in L$ corresponds to the subset of players that have scaled investment strategies consistent with $\bar{Z}_{4,j} = \max\left(\bar{Z}_{4,i}\right)$. Following this last definition, we define individual state equilibrium, $\bar{Z}_{2,i}$, as the solution to $\dot{Z}_{2,i}(t) = 0$. Individual state dynamics in the long run are given by:

$$\lim_{t\to\infty} Z_{2,i}(t) = \bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{1,i} - \bar{Z}_{4,i}\left(1 + h_i \bar{Z}_{4,i}\right)}{r + r d_i \bar{Z}_2 - \bar{Z}_{4,i} + \delta}, \tag{4.20}$$

where we assume that $\lim_{t\to\infty} Z_{1,i}\left(Z_{4,i}(t), Z_2^{b,i}(t)\right) = \bar{Z}_1$ is a unique asymptotic outcome arising from belief and investment transitions to equilibrium. The result in (4.20) confirms that the existence of a unique equilibrium solution for $Z_{2,i}(t)$ requires the existence of a SCE, guaranteeing the existence of a unique solution to (3.29). To guarantee the existence of a SCE, we have to impose a specific distribution for the individual institutional risk premium parameter, d_i, such that $\lim_{t\to\infty} Z_2(t) = \bar{Z}_2^{b,1}, \ldots \bar{Z}_2^{b,n}$, is fulfilled. This constraint can be considered, because $\bar{Z}_{4,i}$ does not depend on d_i, following the result in (4.19). Substituting $\bar{Z}_2^{b,i}$ by $\bar{Z}_2$ in (4.18), and then solving in terms of d_i, we obtain the parameter condition guaranteeing beliefs are consistent with a unique SCE asymptotically. Individual risk premium distributions consistent with a SCE are now defined by:

$$d_i = \frac{\rho_i - r - (\gamma_i - 1)\left(\bar{Z}_{4,i} - \delta\right)}{r \bar{Z}_2}. \tag{4.21}$$

From the result in (4.21), it is straightforward to confirm that (4.20) no longer depends on $\bar{Z}_2$ outcomes. Since investment equilibrium in the long run is not a function of d_i, institutional risk premium can be determined as a function of (4.19) and (4.20). The result in (4.21) can be interpreted as the optimal price of risk in a market with perfect information about the optimal state of the game, which sets individual institutional risk premium in accordance with the economy aggregate outcomes, player specific preferences and long run investment decisions. We are now able to define the final set of conditions for the existence of a SCE to (4.8). Before putting forward this set of conditions, it is convenient to determine under what conditions are the transversality constraints fulfilled, in order to guarantee that an optimal SCE solution, arising from the *Pontryagin* maximum conditions, exists. For that purpose we rearrange expressions (C.8) and (C.9) in terms of scaled variables and substitute the co-state variables by the optimality conditions, (C.2) and (C.3). The transversality conditions are now given by:

$$\lim_{t\to\infty} -\gamma_i \left(Z_{1,i}(t) K_i(t)\right)^{\gamma_i-1} Z_{2,i}(t) K_i(t) e^{-\rho_i t} = 0; \tag{4.22}$$

$$\lim_{t\to\infty} \gamma_i \left(Z_{1,i}(t) K_i(t)\right)^{\gamma_i-1} \left(1 + 2h_i Z_{4,i}(t)\right) K_i(t) e^{-\rho_i t} = 0. \tag{4.23}$$

Again recall that we defined long run capital dynamics in (3.27) as a function of $\bar{Z}_{4,i}$. Assuming that we only accept equilibrium solutions for the dynamical system defined by (4.14), (4.16) and (4.17), following the results in (4.18), (4.19), (4.20) and (4.21), we can rearrange the transversality conditions given in (4.22) and (4.23) as an asymptotic inequality. Conditions (4.22) and (4.23) are now given by the following inequalities:

$$\lim_{t\to\infty} t^{-1} \log \left[-\gamma_i \bar{Z}_{1,i}^{\gamma_i-1} K_i(0)^{\gamma_i} \bar{Z}_{2,i} e^{\left[(\gamma_i-1)\left(\bar{Z}_{4,i}-\delta\right)+\bar{Z}_{4,i}-\delta-\rho_i\right]t}\right] < 0; \tag{4.24}$$

$$\lim_{t\to\infty} t^{-1} \log \left[\gamma_i \bar{Z}_{1,i}^{\gamma_i-1} K_i(0)^{\gamma_i} \left(1 + 2h_i \bar{Z}_{4,i}\right) e^{\left[(\gamma_i-1)\left(\bar{Z}_{4,i}-\delta\right)+\bar{Z}_{4,i}-\delta-\rho_i\right]t}\right] < 0. \tag{4.25}$$

From (4.24) or (4.25) the growth and transversality constraint for the existence of an optimal solution for the problem defined in (4.8) is again given by:

$$\delta < \bar{Z}_4 < \delta + \frac{\rho_i}{\gamma_i}. \tag{4.26}$$

We now focus on the description of strategic dynamics, assuming that beliefs do not depend on the state evolution. To simplify our approach, we assume that beliefs are a function of investment strategies, $Z_2^{b,i}(t) = S_i\left(Z_{4,i}(t)\right)$, and fulfil the equilibrium condition defined in (4.18), such that $S_i\left(\bar{Z}_{4,i}\right) = \bar{Z}_2^{b,i}$. This naive hypothesis regarding players' beliefs has both advantages and disadvantages. The main advantage is that it allows for a great deal of simplification and a full description of strategic dynamics. This approach paves the way for a full qualitative analysis of this game in the vicinity of a SCE. However, this analysis is based on a weak asymptotic argument. We can only guarantee the validity of qualitative outcomes when we assume that $Z_2(t)$ non-autonomous transitions, arising from $Z_{4,i}(t)$ dynamics, stay in the vicinity of the SCE solution. We start by redefining consumption dynamics.

Given the general hypothesis on the evolution of beliefs, we can redefine $Z_{1,i}(t)$ as a function of $Z_{4,i}(t)$:

$$Z_{1,i}(t) = Z_{1,i}(0)\, e^{\int_0^t \left(S_i\left(\bar{Z}_{4,i}\right) - S_i\left(Z_{4,i}(s)\right)\right)ds}. \tag{4.27}$$

Now recall that the existence of a SCE solution, requires the existence of a stable equilibrium for $Z_{4,i}(t)$. Substituting $Z_2^{b,i}(t)$ by $S_i\left(Z_{4,i}(t)\right)$ in (4.17), the general condition for stability of $Z_{4,i}(t)$ solutions can be defined as,

$$\bar{Z}_{4,1} > \frac{\rho_i + \delta\gamma_i}{\gamma_i - rd_i S_i'\left(\bar{Z}_{4,i}\right)} + \frac{rd_i S_i'\left(\bar{Z}_{4,i}\right)}{2h_i\left[\gamma_i - rd_i S_i'\left(\bar{Z}_{4,i}\right)\right]}. \tag{4.28}$$

The result in (4.28) has several implications. When we consider $S_i'\left(\bar{Z}_{4,i}(t)\right) = 0$, the stability condition reduces to $\bar{Z}_{4,1} > \delta + \rho_i/\gamma_i$. This result is not consistent with the constraint required for the existence of an optimal solution. Beliefs cannot be static in this setup. We consider then a simple hypothesis and define belief dynamics as,

$$Z_2^{b,i}(t) = \frac{\rho_i - r - (\gamma_i - 1)\left(Z_{4,i}(t) - \delta\right)}{rd_i}. \tag{4.29}$$

Following (4.29), player consumption dynamics reduce to $Z_{1,i}(t) = Z_{1,i}(0), \forall t \in T$. This assumption can also be interpreted in terms of the *Life Cycle* hypothesis for intertemporal consumption, following the description in section 3.3. Given the uncertainty regarding state outcomes, players rather choose consumption profiles that are not distorted by investment transitions to equilibrium. Substituting (4.29) in (4.17) we can redefine investment decisions as,

$$\dot{Z}_{4,i}(t) = Z_{4,i}(t)^2 + \frac{b}{a} Z_{4,i}(t) + \frac{c}{a}, \tag{4.30}$$

where $a = 1/2 - \gamma_i$, $b = \rho_i + \delta\gamma_i - (\gamma_i - 1)(2h_i)^{-1}$ and $c = (\rho_i + \delta\gamma_i - r_k)(2h_i)^{-1}$. Equation (4.30) defines a *Ricatti* equation that has an explicit solution. Before describing the solution to the above differential equation, it is convenient to investigate the properties of its coefficients, given the optimality and stability constraints defined in (4.26) and(4.28). First, recall that stable optimal solutions to (4.30) require that $\delta < \bar{Z}_{4,i} < -b(2a)^{-1} < \delta + \rho_i/\gamma_i$. This condition implies that $sign(b) \neq sign(a)$, for solutions consistent with $\bar{Z}_{4,i} > 0$. Second, feasible solutions to (4.30), $\bar{Z}_{4,i} \in \Re^+$, require that $b^2 > 4ac$. Given these definitions, the solution to (4.30), is given by (4.31), below, following the solution to the general *Ricatti* equation described in C.2. Investment strategic dynamics are defined explicitly as,

$$Z_{4,i}(t) = -\frac{\sqrt{b^2-4ac}\tanh\left\{\frac{\sqrt{b^2-4ac}}{2a}\left[t+\frac{2a}{\sqrt{b^2-4ac}}\operatorname{arctanh}\left(-\frac{2aZ_{4,i}(0)+b}{\sqrt{b^2-4ac}}\right)\right]\right\}+b}{2a}. \tag{4.31}$$

We now conclude the description of the conditions for the existence of a SCE solution to (4.8), with the definition of equilibrium for $Z_{2,i}(t)$. Substituting (4.21) and $\bar{Z}_{1,i} = Z_{1,i}(0)$ in (4.20), $\bar{Z}_{2,i}$ is given by the following expression,

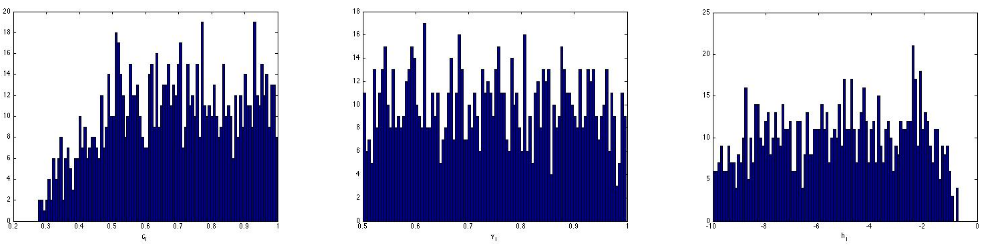
$$\bar{Z}_{2,i} = \frac{r_k - Z_{1,i}(0) - \bar{Z}_{4,i}\left(1 + h_i\bar{Z}_{4,i}\right)}{\rho_i - \gamma_i\left(\bar{Z}_{4,i} - \delta\right)}. \tag{4.32}$$

Having described the conditions for the existence of SCE solutions consistent with a stable strategic equilibrium, for the non-cooperative game given in (4.8). We now focus on the qualitative description of this solution. We base our approach on the weak argument for asymptotic stability discussed in section 3.2. This argument is based on the results described in (4.18) to (4.32), which guarantee that a SCE is always achieved asymptotically and independent of other players decisions, when institutional risk premium is defined by a unique distribution that depends on the asymptotic outcome of the state of the game. Since in the long run there are no longer transitions driven by $Z_{1,i}(t)$ and $Z_2(t)$ dynamics, when we assume $Z_{4,i}(t)$ dynamics always converges to the equilibrium defined in (4.19), we can evaluate qualitatively the local stability of the SCE strategies by testing the stability of the system describing scaled net assets dynamics, $\left\{\dot{Z}_{2,1}(t), \ldots, \dot{Z}_{2,n}(t)\right\}$. The n by n *Jacobian* matrix describing individual state dynamics in the vicinity of a SCE is again given by (3.38) and its partial derivatives defined by conditions (3.39) to (3.43).

The local stability of SCE solutions for the game defined in (4.8) can be easily evaluated numerically. If all the eigenvalue solutions of (3.38) have negative real part, then we can state that SCE solutions are at least locally weakly asymptotically stable. A robust argument for local asymptotic stability would have to take into account transitions to equilibrium arising from $Z_{4,i}(t)$ decisions and $Z_2(t)$ non-autonomous dynamics. The result in (4.31) can be used to test the robustness of qualitative numerical results. We discard this analysis in this chapter and focus on the evaluation of (3.38). To evaluate the dynamics in the vicinity of equilibrium for this game, we again assume that initial consumption and productive/riskless asset endowments are given by random outcomes distributed according to $Z_{1,i}(0) \sim U(0,1)$ and $K_i(0) \sim \exp(1)$, respectively. To test the qualitative dynamics in the vicinity of a SCE, we again have to consider the existence of a robust population.

By robust population, we now mean a discrete set of $n = 1000$ agents with uniform randomly drawn individual characteristics, $\rho_i, \gamma_i \sim U(0,1)$ and $h_i \sim U(-10,10)$, such that $\bar{Z}_{4,i}$ outcomes, defined by (4.19), fulfil the optimal growth constraint, (4.26), for the range institutional scenarios, $r_k \in [0.01\ 0.25]$ and $r \in [0.03\ 0.5]$, where $\delta = 0.03$[8]. For simplification reasons, we again consider that the state of the game is driven by a fixed pool of investors, which is defined by a fixed share of the population, following the discussion in section 3.3. We set this share at 30%, and consider that the aggregate risk faced by investors, (3.29), is again obtained from the share of aggressive players with higher rates of investment per unit of capital.

In Figure 4.1, below, we portray the parameter distributions describing the computed robust population. In this setup, robust populations are characterized by a large set of impatient investors with high intertemporal elasticity and bias towards domestic/productive assets. This distribution of characteristics is consistent with the existence of an unique stable equilibrium for $\bar{Z}_{4,i}$, given by the positive root of (4.19). Further numerical analyses, assuming different hypothesis, suggests that the distributions portrayed, provide a good picture of a robust population, given the set of conditions required for the existence of stable strategic solutions and a SCE.



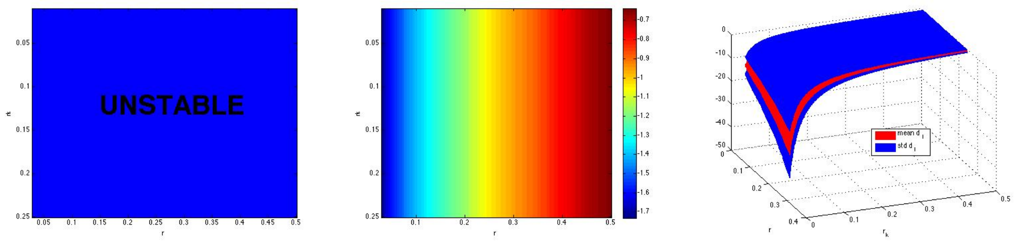
(a) Distribution of ρ_i (b) Distribution of γ_i (c) Distribution of h_i

Figure 4.1: Robust population distributions.

Figure 4.2, below, portrays the qualitative results for the computed SCE solution, the sample density of $\bar{Z}_2$ and the mean and standard deviation for d_i distribution. The stability diagram shows that SCE solutions are repelling for all the range of institutional scenarios considered. Our numerical analysis suggests the existence of saddle solutions, when different scenarios are considered. However, these solutions have very few stable dimensions. This outcome is robust to different distributions of $Z_{1,i}(0)$ and $K_i(0)$, and also to different hypothesis regarding the

[8] Given that there are two possible solutions to $\bar{Z}_{4,i}$, following (4.19), the routine tests the robustness of each solution. When both equilibrium are robust, eigenvalue solutions to (3.38) are computed by choosing randomly one of the solutions. Our extensive numerical analysis suggests that for this random set of parameters only one of the roots of (4.19) is consistent with (4.26).

share size of the population of leaders. We conclude that the optimal SCE solution proposed, based on the naive evolution of beliefs following (4.29), is inherently an unstable[9]. game solution for (4.8).



(a) Stability Diagram (b) Distribution of $\bar{Z}_2$ (c) d_i sample moments

Figure 4.2: Stability diagram and risk premium distributions.

We attribute this dramatic result to the existence of SCE solutions that impose a negative institutional risk premium on the majority of players. This outcome implies that investors in financial assets get a negative real yield on their investment, since the game premium is always negative[10]. Moreover, leveraged players do not benefit from this arrangement either, since this setup imposes a positive premium on their debt. This SCE arrangement does not benefit any type of investor. In our opinion, this result is a consequence of players' belief rigidity and rationality. An optimal unique SCE solution to (4.8) can only be imposed when we consider the existence of a credit market with complete information that is able to price individual risk optimally by taking into account player's asymmetries. This is a result of players' belief rigidity. However, players do not concur on this solution, because they believe they will be better off pursuing other objectives, even though only the SCE solution described is consistent with an optimal solution. For agents facing uncertainty, this decision can be considered as a rational one. Finally, other hypotheses regarding the evolution of beliefs could be considered, such as naive beliefs that at least allow for consumption transitions. Given that our numerical results suggest that SCE solutions are unstable for a wide range of consumption distributions, a qualitative analysis based on the two step approach proposed would not be able to take into account the impact of this transitions on the stability of a SCE. Therefore, we conclude that the existence of stable SCE solutions for (4.8) requires

[9]In these simulations we again evaluate the real part of the two leading eigenvalues, $\min\{\mathrm{Re}(\Lambda)\}$ and $\max\{\mathrm{Re}(\Lambda)\}$, where Λ stands for the set of eigenvalues solving the characteristic polynomial of (3.38) in the vicinity of SCE. By unstable solutions we again mean that the SCE is a completely unstable solution, which implies that the SCE solution is time-reverse stable.

[10]It is possible that some bond investors get a negative nominal yield, which means that they pay interest on their investments.

that we take into account the role of the controls and beliefs gradients evaluated in the vicinity of a SCE. An interesting exercise would involve the description of the class of smooth belief functions that is consistent with the existence of stable SCE institutional scenarios. Alternatively, we could drop all naive assumptions regarding the evolution of beliefs and evaluate the existence of learning mechanisms consistent a SCE. We discuss this approach in the next section for a game where players' risk premium dynamics are given by the evolution of his state conditions, following the proposal in section 3.4. This outcome illustrates the implications of introducing further nonlinearities in competitive economic setups, when we take into account the results obtained for the game described in section 3.3. The inclusion of further nonlinearities not only introduces belief subjectivity in this class of games, but also is not consistent with the existence of stable optimal SCE solutions for games where players have simple naive beliefs.

4.4 A consumption and investment game with investment bias and coupled endogenous risk premium

To introduce strategic interactions in our framework, we propose an extension to the game discussed in the previous section. We now drop the institutional risk premium hypothesis and consider that individual risk premium depends on the ratio of net financial assets to productive capital. Interest payments/revenues are now given by $rB_i(t)\left[1+\left(B_i(t)/K_i(t)\right)\left(B(t)/K(t)\right)\right]$, following the proposal in (3.4). Given this brief introduction, the non-cooperative differential game faced by player i is now given by the following dynamic optimization problem:

$$\begin{array}{l}
\underset{C_i(t),I_i(t)}{MAX}\int_0^\infty e^{-\rho_i t}C_i(t)^{\gamma_i}\,dt \\
\text{subject to the solution of:} \\
\dot{B}_i(t)=C_i(t)+I_i(t)\left(1+h_i\frac{I_i(t)}{K_i(t)}\right)+rB_i(t)\left(1+\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right)-r_kK_i(t)\,; \\
\dot{K}_i(t)=I_i(t)-\delta K_i(t)\,;
\end{array}\qquad(4.33)$$

satisfying the transversality conditions, (C.18) and (C.19), guaranteeing that solutions to (4.33) do not grow too fast. The optimal *Keynes-Ramsey* consumption strategies for (4.33), following the procedure described in section 4.3, and given the maximum conditions in C.1.2, are now defined by:

$$\dot{C}_i(t)=\frac{C_i(t)}{\gamma_i-1}\left(\rho_i-r-2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right);\qquad(4.34)$$

$$\dot{C}_i(t) = \frac{C_i(t)}{\gamma_i - 1}\left(1 + 2h_i\frac{I_i(t)}{K_i(t)}\right)^{-1}\left[(\rho_i + \delta)\left(1 + 2h_i\frac{I_i(t)}{K_i(t)}\right) - h_i\frac{I_i(t)^2}{K_i(t)^2}\right.$$
$$\left. - r_k - 2h_i\frac{\dot{I}_i(t)}{K_i(t)} + 2h_i\frac{I_i(t)}{K_i(t)}\frac{\dot{K}_i(t)}{K_i(t)} - r\frac{B_i(t)^2}{K_i(t)^2}\frac{B(t)}{K(t)}\right]. \tag{4.35}$$

We obtain the optimal rule that guarantees indifference between consumption strategies for player i, by imposing equality between consumption strategies, (4.34) and (4.35), and substituting by the capital accumulation equation (C.17). Again, this rule is given by the dynamics of strategic investment decisions,

$$\dot{I}_i(t) = \frac{I_i(t)^2}{2K_i(t)} + \left(r + 2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right)I_i(t) - \frac{r}{2h_i}\frac{B_i(t)^2}{K_i(t)}\frac{B(t)}{K(t)}$$
$$+ \left(r + 2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)} + \delta - r_k\right)\frac{K_i(t)}{2h_i}. \tag{4.36}$$

Following the scaling rule given in (3.21), we obtain the stationary dynamical system defining the general solution for the non-cooperative game given in (4.33), in terms of player i dynamics:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i - r - 2rZ_{2,i}(t)Z_2(t) - (\gamma_i - 1)(Z_{4,i}(t) - \delta)}{\gamma_i - 1}\right]; \tag{4.37}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + Z_{4,i}(t)\left[1 + h_i Z_{4,i}(t)\right]$$
$$+ Z_{2,i}(t)\left[r + rZ_{2,i}(t)Z_2(t) - Z_{4,i}(t) + \delta\right] - r_k; \tag{4.38}$$

$$\dot{Z}_{4,i}(t) = -\frac{Z_{4,i}(t)^2}{2} + \left[r + 2rZ_{2,i}(t)Z_2(t) + \delta\right]Z_{4,i}(t) - \frac{r}{2h_i}Z_{2,i}(t)^2 Z_2(t)$$
$$+ \left[r + 2rZ_{2,i}(t)Z_2(t) + \delta - r_k\right]\frac{1}{2h_i}; \tag{4.39}$$

where productive capital dynamics, $K_i(t)$, and aggregate state dynamics, $Z_2(t)$, are given by (3.24) and (3.25), respectively. A quick glance at the strategic controls of player i, defined by (4.37) and (4.39), shows that players lack the relevant information to pursue their optimal strategies. We follow the same set of assumptions regarding the existence of feasible strategic solutions based on beliefs, given in section 4.3.

This involves assuming the existence of individual beliefs, $Z_2^{b,i}(t)$, as previously. Substituting $Z_2^{b,i}(t)$ in (4.37) and (4.39), we obtain the belief-based controls for player i. The dynamical solution to (4.33) is now given by the evolution of individual state dynamics, (4.38), while strategic dynamics are defined by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i - r - 2rZ_{2,i}(t)\,Z_2^{b,i}(t) - (\gamma_i - 1)\,(Z_{4,i}(t) - \delta)}{\gamma_i - 1}\right]; \tag{4.40}$$

$$\begin{aligned}\dot{Z}_{4,i}(t) &= -\frac{Z_{4,i}(t)^2}{2} + \left[r + 2rZ_{2,i}(t)\,Z_2^{b,i}(t) + \delta\right] Z_{4,i}(t)\\ -\frac{r}{2h_i}Z_{2,i}(t)^2\,Z_2^{b,i}(t) &+ \left[r + 2rZ_{2,i}(t)\,Z_2^{b,i}(t) + \delta - r_k\right]\frac{1}{2h_i}.\end{aligned} \tag{4.41}$$

Following the discussion in section 4.2, we have to guarantee that the dynamic solution to (4.33) is consistent with the existence of belief solutions that bound the state-space of the game asymptotically. First, we check under what conditions beliefs are consistent with the existence of asymptotic strategic equilibrium. This assumption requires that the set of conditions, $\lim_{t\to\infty} Z_2^{b,i}(t) = \bar{Z}_2^{b,i} \wedge \dot{Z}_{1,i}(t), \dot{Z}_{4,i}(t) = 0$, is fulfilled. From (4.40) and (4.41) we obtain:

$$\bar{Z}_2^{b,i} = \frac{\rho_i - r - (\gamma_i - 1)\left(\bar{Z}_{4,i} - \delta\right)}{2r\bar{Z}_{2,i}}; \tag{4.42}$$

$$\bar{Z}_2^{b,i} = \frac{\frac{\bar{Z}_{4,i}^2}{2} - (r+\delta)\,\bar{Z}_{4,i} - (r + \delta - r_k)\,(2h_i)^{-1}}{2r\bar{Z}_{2,i}\bar{Z}_{4,i} - r\,(2h_i)^{-1}\,\bar{Z}_{2,i}^2 + r\,(h_i)^{-1}\,\bar{Z}_{2,i}}. \tag{4.43}$$

From the results in (4.42) and (4.43) we will show that belief solutions consistent with strategic equilibrium are no longer unique, but given by a continuous curve of equilibrium solutions. This result has dramatic implications. However, as we demonstrate later on, it has a straightforward interpretation in the context of a portfolio game. To justify our claim we start by eliminating $\bar{Z}_2^{b,i}$ from (4.42) and (4.43). After rearranging we obtain,

$$\begin{aligned}\left(\frac{1}{2} - \gamma_i\right)\bar{Z}_{4,i}^2 + \left(\frac{\gamma_i - 1}{4h_i}\right)\bar{Z}_{2,i}\bar{Z}_{4,i} + \left[(\gamma_i - 1)\left(\delta - \frac{1}{2h_i}\right) + \rho_i + \delta\right]\bar{Z}_{4,i}\\ + \left(\frac{r - \rho_i - (\gamma_i - 1)\,\delta}{4h_i}\right)\bar{Z}_{2,i} + \frac{\gamma_i\delta + \rho_i - r_k}{2h_i} = 0.\end{aligned} \tag{4.44}$$

Equilibrium condition (4.44) describes a quadratic curve relating individual investment decisions and financial outcomes. It defines an asymptotic frontier solution, where player beliefs are always fulfilled. This frontier is always defined by a hyperbola[11], since $[(\gamma_i - 1) / (4h_i)]^2 > 0$. The frontier defined in (4.44) is defined by the right hand side of a hyperbola for feasible values of investment strategies, $\bar{Z}_{4,i} \in \Re^+$. We show this result numerically later on. First, it is convenient to rearrange (4.44) as an equilibrium solution for $\bar{Z}_{2,i}$ in terms of $\bar{Z}_{4,i}$. Rearranging (4.44) we obtain,

$$\begin{aligned}\bar{Z}_{2,i} = -4h_i \frac{\left(\frac{1}{2} - \gamma_i\right)\left(\bar{Z}_{4,i}\right)^2 + \left[(\gamma_i - 1)\left(\delta - \frac{1}{2h_i}\right) + \rho_i + \delta\right]\bar{Z}_{4,i}}{(\gamma_i - 1)\bar{Z}_{4,i} + r - \rho_i - (\gamma_i - 1)\delta} \\ -4h_i \frac{(\gamma_i \delta + \rho_i - r_k)(2h_i)^{-1}}{(\gamma_i - 1)\bar{Z}_{4,i} + r - \rho_i - (\gamma_i - 1)\delta}.\end{aligned} \tag{4.45}$$

In this setup, beliefs provide only information about the game asymptotic frontier. We do not have any information about the value of beliefs in equilibrium. Under these circumstances, the definition of naive assumptions regarding the evolution of beliefs is no longer an option. The analysis of equilibrium and the evaluation of SCE solutions in this framework is a complex geometric problem. We propose to analyse this solution as a stochastic decision process under uncertainty. To illustrate our approach, we start by defining equilibrium for consumption strategies, $\bar{Z}_{1,i}$. Setting $\dot{Z}_{2,i} = 0$ and solving for $\bar{Z}_{1,i}$, we obtain,

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i}\left(1 + h_i \bar{Z}_{4,i}\right) - \bar{Z}_{2,i}\left(r + r\bar{Z}_{2,i}\bar{Z}_2 - \bar{Z}_{4,i} + \delta\right), \tag{4.46}$$

where $\bar{Z}_2$ is again defined by the investment leaders' actions, following the result in (3.29). From (4.46), it is clear that $\bar{Z}_{1,i}$ outcomes depend solely on investment decisions, given the result in (4.45). Since the state of the game, $\bar{Z}_2$ is also a function of $\bar{Z}_{4,i}$ outcomes, we have to check under what conditions $\bar{Z}_{4,i}$ is bounded, to bound the state-space of the game asymptotically. In section 4.3 we showed that investment decisions are bounded by growth and transversality conditions. The interval defined in (4.26) is again a necessary condition for the existence of optimal solutions to the game defined in (4.33). However, as solutions might not be unique, it is convenient to redefine the notion of acceptable solutions to the system given by (4.38), (4.40) and (4.41), following the definitions given in chapter 2. We assume

[11]This outcome can be related to Merton [1972] proposal on the efficient portfolio frontier, which has been one of the main workhorses of modern financial and portfolio decision theory. The efficient portfolio theory suggests that the relationship between the expected value and the standard deviation of a given portfolio is correctly described by the right hand side of an hyperbola.

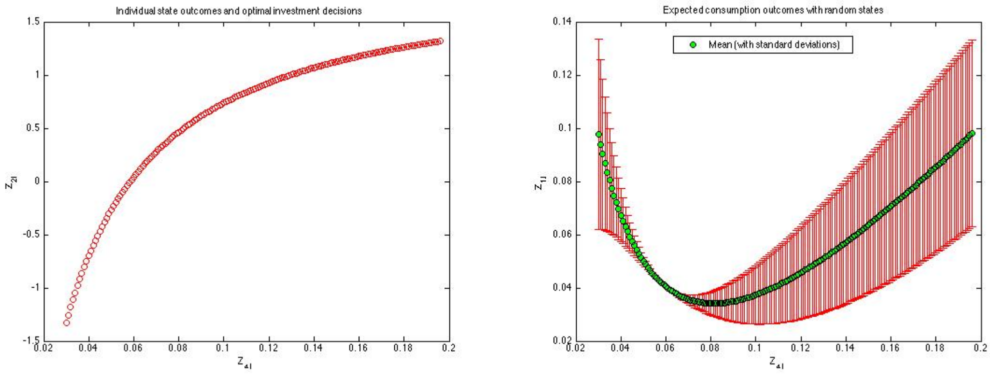
now, that solutions can be described as an ergodic invariant set with a well defined invariant probability measure, whose expectation operator we denote by $\langle\rangle$, such that $\lim_{t\to\infty} t^{-1}\int_0^t Z_{m,i}(t) \to \langle Z_{m,i}\rangle$. If this invariant set is unique, $\langle Z_{m,i}\rangle = \bar{Z}_{m,i}$, then solutions are consistent with a SCE when players concur in a unique outcome for $\bar{Z}_2$. Following this definition, we can redefine $K_i(t)$ in the long run, by taking the asymptotic limit of (3.24). Long run productive/domestic capital dynamics are defined by

$$\lim_{t\to\infty} t^{-1}\log K_i(t) \to \langle Z_{4,i}\rangle - \delta. \tag{4.47}$$

Following the result in (4.47), the constraint on scaled investment activities that imposes the existence of long run dynamics consistent with exponential growth, $\lim_{t\to\infty} K_i(t) \to \infty$, is given by $\langle Z_{4,i}\rangle > \delta$. To obtain the transversality constraint, we follow the same procedure described in section 4.3, given in (4.22) to (4.25), but taking into account solutions described by the invariant probability measure of a ergodic invariant set, $\langle Z_{m,i}\rangle$, with well defined lower and upper bounds given by $\liminf_{t\to\infty} t^{-1}\int_0^t Z_{m,i}(t) \geq \min\left(\bar{Z}_{m,i}\right)$ and $\limsup_{t\to\infty} t^{-1}\int_0^t Z_{m,i}(t) \leq \max\left(\bar{Z}_{m,i}\right)$, respectively. The growth and transversality constraint is now given in terms of the invariant probability measure for investment decisions,

$$\delta < \langle Z_{4,i}\rangle < \delta + \frac{\rho_i}{\gamma_i}, \text{ where } \min\left(\bar{Z}_{4,i}\right) > \delta \text{ and } \max\left(\bar{Z}_{4,i}\right) < \delta + \frac{\rho}{\gamma_i}. \tag{4.48}$$

The solution to (4.33), assuming the existence of subjective beliefs, is given by an asymptotic frontier bounded by growth and transversality conditions. It is no longer possible to evaluate this game using qualitative dynamical methods, since equilibrium for any given period t is now a function of individual subjective beliefs and players investment decisions. It is possible to provide a geometric description of the possible individual game outcomes and put forward some conjectures regarding feasible solution scenarios, if we assume that individual beliefs, $Z_2^{b,i}$, follow a distribution with a finite second moment. To portray this feature, we compute the equilibrium outcomes for a feasible set of investment strategies, as defined in (4.48), given a set of parameters values, $\{r, r_k, \delta, \rho_i, \gamma_i, h_i\} = \{0.02, 0.1, 0.03, 0.05, 0.3, -0.01\}$, and assuming a random sample for state outcomes, $\bar{Z}_2 \sim N(\alpha, 1)$, where $\alpha \sim N(0, 0.1)$ is also a random number. Results are given in Figure 4.3 below. The picture on the left portrays individual state outcomes, $\bar{Z}_{2,i}\left(\bar{Z}_{4,i}\right)$, for a given bounded vector of optimal investment strategies described by the growth and transversality constraint in (4.48). Individual state outcomes are given by the right hand side of the hyperbola describing equilibrium condition (4.44). The picture on the right portrays the expected values and standard deviations for consumption outcomes for the same range of individual optimal investment decisions, obtained from a $\bar{Z}_2$ sample with one thousand random draws.



(a) Individual state outcomes (b) Expected consumption under a random state

Figure 4.3: Individual state and consumption outcomes assuming a random state

Figure 4.3 portrays the indecision faced by players in an environment with incomplete information. The sampled distribution of scaled consumption outcomes suggests that this player may be undecided between following conservative or aggressive investment strategies, when seeking to maximize his expected long run consumption pay-off. The explanation for this result is straightforward. Equilibrium condition (4.46) defines a 4^{th} order polynomial for $\bar{Z}_{1,i}$ in terms of $\bar{Z}_{4,i}$, that depends on state outcomes, $\bar{Z}_2$, which are a result of other players investment decisions and initial capital endowments, following the result in (3.29). Given different parameter combinations and assuming normally distributed state outcomes, $\bar{Z}_2 \sim N(\alpha, 1)$, with $\alpha \in [-2 \ldots 2]$, numerical results for different values of α suggest that we may have one global maximum, interior or on the edges, or two local maxima, most likely on the edges. The geometric definition of a maximum for player i depends on his beliefs about the state of the game, which in turn depends on other players' beliefs. Further, if we consider that players beliefs evolve, as they try to learn state outcomes, their actions lead to changes in the environment faced by other players. Under these circumstances a geometric approach is not an option. Game asymptotic solutions are, in our opinion, best described as a multi-objective optimization problem, where the existence of multiple decision criteria in a co-evolutionary framework is taken into account. We are interested in evaluating two possible game outcomes. If game dynamics, given a set of rules describing an adaptive learning and multi-criteria decision scheme, are consistent with strategic coordination, leading to an unique asymptotic equilibrium solution, as defined in (4.6) and (4.7), then we argue that this solution is consistent with a SCE . The outcome of this portfolio game between asymmetric investors has thus stable solutions consistent with the *return to the dollar* property[12]. Given the existence of strategic interactions and uncertainty driving the evolution of consumption outcomes, it is also likely that under certain conditions this property is not fulfilled and game solutions can only be described statistically when the asymptotic dynamics are consistent with an ergodic distribution. In this specific case we can argue that players have a positive probability of *returning to the dollar* for every reachable state of the game but do not settle in any of the visited states. To test these hypotheses, we propose to evaluate the game asymptotic solution as a stochastic repeated game, where agents seek to learn a SCE and maximize their equilibrium consumption outcomes under uncertainty. The choice of a stochastic game setup to investigate asymptotic dynamics in nonlinear differential games has several advantages. First, this is a setup that mimics the co-evolutionary nature of the original game and allows the introduction of both adaptive learning and multiple criteria decision in a coherent mathematical fashion. Second, a continuous time version of the asymptotic solution can be simulated as a HMM. This implies that player moves are chosen randomly and strategies are determined sequentially, as opposed to the simultaneous play arising in differential games. As previously discussed, equilibrium dynamics can be simulated as a continuous time Markov game and a statistical description of outcomes can be obtained through sampling, following the *Markov Chain Monte Carlo* method. This is a particularly interesting approach computationally since one our main objectives is to investigate limiting outcomes described by a SCE. For adaptive schemes that are not consistent with a SCE when $t \to \infty$, HMM setups may impose undesirable limits to the number of sampling trials, range and population size, thus limiting the quality of samples and our knowledge of the solution long run dynamics.

4.4.1 Asymptotic dynamics under uncertainty

Following the discussion in the previous section, we now focus on the description of a stochastic continuous time game for the equilibrium solution of the non-cooperative game defined in (4.33), as a multi criteria Markov decision process with incomplete information, which we define here generally as a multi-objective maximization ex-

[12] This definition was introduced in the famous paper by Georgescu-Roegen [1951], to describe the robustness of equilibrium in many agent systems with decentralized strategic decisions. The author coined this expression to describe the asymptotic dynamics on the production-possibility frontier, which he characterized in some cases as lacking the *return to the dollar* property or of being of the saddle type.

pectation problem. Given the individual asymptotic equilibrium solution defined by equations (4.45) to (4.48), player i's objective in the asymptotic equilibrium frontier is to

$$\underset{\bar{Z}_{4,i}}{MAX}\, E\left[\bar{Z}_{1,i}\right], \tag{4.49}$$

subject to (4.45) and (4.48). In this framework, players seek to maximize the expected consumption outcomes under uncertainty, $E\left[\bar{Z}_{1,i}\right] = \bar{Z}_{1,i}\left[\bar{Z}_{2,i}\left(\bar{Z}_{4,i}\right), \bar{Z}_2^i\right]$, describing player i asymptotic pay-off function as defined by equilibrium condition (4.46), given a choice of optimal $\bar{Z}_{4,i}$ strategic actions in a finite interval fulfilling constraint (4.48), describing player i optimal strategic space. Player i pursues this objective assuming individual beliefs about state outcomes, $\bar{Z}_2^i$, describing player i finite set of belief states. We shall assume that $\bar{Z}_2^i$ is a set defined by the first two moments of a belief distribution, S_i^b, with a finite second moment, such that $\bar{Z}_2^i \sim S_i^b\left(E\left(S_i^{past}\right), \Phi_i^2\right)$, where $S_i^{past} \subset S$ is the player i set of available past information about the set of states of the game, S, and $\Phi_i \in \Re$ a function of player i observed forecast errors, ϵ_i, which fulfils $\Phi_i = 0$ for $E\left(\bar{Z}_2^i\right) = Z_2 \Rightarrow \epsilon_i = 0$, following the general conditions described in (4.6) and (4.7) for the existence of SCE solutions in an adaptive learning scheme.

To evaluate the decision problem defined in (4.49), we propose to model it as a Markov process under incomplete information. For that purpose we shall consider a multi-criteria decision process, where players co-evolve in a stochastic learning environment. The application of stochastic processes to evaluate multi-objective optimization problems, is an inter-disciplinary approach that merges concepts of evolutionary optimization, machine learning and non-cooperative game theory. This approach focuses on the development of evolutionary algorithms capable of evaluating outcomes in complex interacting environments[13]. In a review on modern evolutionary multi-objective optimization methods, Zitzler et al. [2003] summarizes this

[13]This approach is gaining ground in economics literature. In a recent review on these topics, Castillo and Coello [2007] discuss the potential applications of these inter-disciplinary methods in the fields of economic and finance. This approach are already widely used in modern portfolio optimization theory. Steuer et al. [2008] gives an overview of the inter-disciplinary aspects of multi-criteria portfolio optimization. The author suggests that the inclusion of further decision criteria, can improve existing models of expected portfolio maximization under uncertainty. Hens and Schenk-Hoppe [2005] shows that mean-variance portfolio strategies in incomplete markets are not evolutionary stable, whereas diversified portfolio strategies consistent with the CAPM rule are evolutionary stable. Elliott et al. [2010] proposes a regime switching HMM for mean-variance portfolio selection and provides a recent survey on the application of stochastic processes in finance. Finally, Ahmed and Hegazi [2006] provides a discussion on three inter-disciplinary aspects of portfolio optimization: (i) multi-objective optimization; (ii) dynamical re-balancing; and (iii) evolutionary game theory.

approach in the following fashion. *"The term evolutionary algorithm (EA) stands for a class of stochastic optimization methods that simulate the process of natural evolution."*

We now focus on the description of the specific HMM setup. First, we define the strategic decision space for player i, as a one dimensional lattice with reflective boundary conditions. Player i's strategic space is defined by a discrete bounded set, $\Lambda_i \in \Re^{\sigma_i}$, where $\min(\Lambda_i) > \delta$, $\max(\Lambda_i)) < \delta + \rho_i/\gamma_i$ and $\sigma_i = \#\Lambda_i$. Given a small number v[14], the set, Λ_i, describing player i strategic decision space can be defined as,

$$\Lambda_i = \left\{ \delta + v, \delta + 2v, \ldots, \delta + \frac{\rho_i}{\gamma_i} - v \right\}. \tag{4.50}$$

Given an investment strategy value, $\bar{Z}_{4,i} \in \Lambda_i$, player i's feasible set of strategic actions, $\Delta\tilde{Z}_{4,i}$, is defined by: (i) $\left\{ \Delta\bar{Z}^{-}_{4,i}, \Delta\bar{Z}^{0}_{4,i}, \Delta\bar{Z}^{+}_{4,i} \right\} = \{-v, 0, v\}$, if $\delta + v < \bar{Z}_{4,i} < \delta + \rho_i\gamma_i^{-1} - v$; (ii) $\left\{ \Delta\bar{Z}^{0}_{4,i}, \Delta\bar{Z}^{+}_{4,i} \right\} = \{0, v\}$, if $\bar{Z}_{4,i} = \delta + v$; (iii) $\left\{ \Delta\bar{Z}^{-}_{4,i}, \Delta\bar{Z}^{0}_{4,i} \right\} = \{-v, 0\}$, if $\bar{Z}_{4,i} = \delta + \rho_i\gamma_i^{-1} - v$. Figure 4.4, below, portrays the evolution of player i investment strategies, for a given $\bar{Z}_{4,i} \in \Lambda_i$,

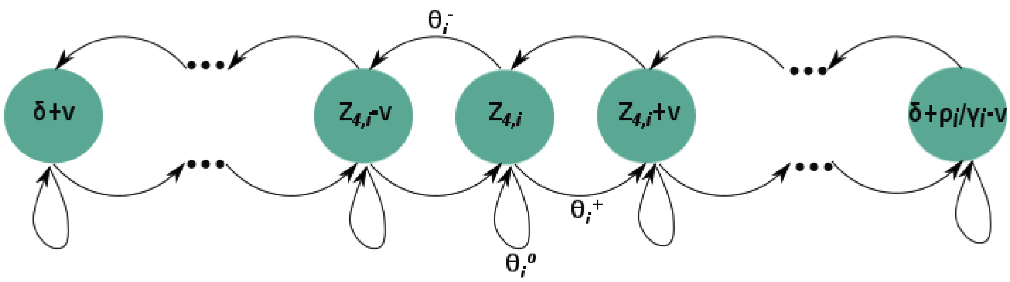


Figure 4.4: Player investment transitions in a lattice

where $\theta_i \in \Re^{+}$ are player i transition rates between reachable $\bar{Z}_{4,i}$ states. Transition rates fulfil the usual probability transition rules for a stochastic matrix describing a *Markov* chain over the finite state-space of strategies: (i) $0 \leq \theta_i \leq 1$; and (ii) $\theta_i^{-} + \theta_i^{0} + \theta_i^{+} = 1$. The total probability of decreasing, maintaining or increasing investment is given by, $P\left(\Delta\bar{Z}_4^{-}\right) = n^{-1}\sum_{i\in N}\theta_i^{-}$, $P\left(\Delta\bar{Z}_4^{0}\right) = n^{-1}\sum_{i\in N}\theta_i^{0}$ and $P\left(\Delta\bar{Z}_4^{+}\right) = n^{-1}\sum_{i\in N}\theta_i^{+}$, respectively. By setting strategic actions in this fashion, we seek that the HMM proposed is able to capture the co-evolutionary nature of decision under uncertainty suggested by our framework. Players follow investment paths and adjust these as the environment changes, instead of radically changing

[14] For simulation purposes, we assume $v = 10^{-3}$.

their investment strategies with every state innovation. This option is consistent with investment strategies in financial markets, where radical trading decisions might trigger market movements that lead to important portfolio losses. It also guarantees that an absorbing state, $\sum_{i \in N} \theta_i^0 = n \Rightarrow P\left(\Delta \bar{Z}_4^0\right) = 1$, consistent with a SCE solution, as defined in (4.5), exists.

As mentioned previously, we shall assume that individual transition rates, θ_i, are computed following a simple multi-criteria decision process that takes into account the uncertainty faced by players, given the available information about the state of the game. In the absence of information, modern machine learning theory suggests the use of variational methods for inference and forecasting purposes[15]. A popular approach to inference in HMM environments, is the *Expectation Maximization*(EM) algorithm, which requires some knowledge about the evolution of the state of the system. We discard the use of sophisticated machine learning methods, because of efficiency problems arising when performing inference in a large scale. To overcome this issue, we propose the use of simple *Bayesian* learning mechanisms. In subsection 4.4.1, we demonstrate the effectiveness of this approach to describe game outcomes consistent with the existence of SCE solutions. *Bayesian* inference methods require information about the evolution of the state of the game. In this setup, players can extrapolate relevant information by monitoring their forecast errors. The forecasting error, ϵ_i, observed by player i, is generally given by,

$$\bar{Z}_{1,i} - E\left[\bar{Z}_{1,i}\right] = \bar{Z}_{1,i}\left[\bar{Z}_{4,i}, \bar{Z}_{2,i}, \bar{Z}_2\right] - \bar{Z}_{1,i}\left[\bar{Z}_{4,i}, \bar{Z}_{2,i}, E\left(\bar{Z}_2^i\right)\right] = \epsilon_i. \tag{4.51}$$

Substituting $\bar{Z}_{1,i}$ and $E\left[\bar{Z}_{1,i}\right]$, by the equivalent steady-state expression, following (4.46), we obtain,

$$\bar{Z}_2(\tau) = E\left[\bar{Z}_2^i(\tau)\right] + \frac{\epsilon_i(\tau)}{r\bar{Z}_{2,i}^2(\tau)}, \tag{4.52}$$

where τ describes the time period of the continuous time HMM[16]. Finally, following (4.52), we define player i forecast rule, as a simple error correcting mechanism based on the last inferred outcome, $\bar{Z}_2(\tau)$, of the state of the game:

[15]For an introduction to this topic see Bishop [2006], chapter 10, or Barber [2012], chapter 21.

[16]As we are describing the asymptotic dynamics of the game defined in (4.33), we shall use this notation when referring to asymptotic time. In a continuous time *Markov* process, time evolution is defined as a random distributed exponential variable, following $\Delta\tau \sim \exp(n)$, where $E[\Delta\tau] = n^{-1}$, defines the average waiting time for player i.

$$E\left[\bar{Z}_2^i\left(\tau+\Delta\tau\right)\right]=E\left[\bar{Z}_2^i\left(\tau\right)\right]+\frac{\epsilon_i\left(\tau\right)}{r\bar{Z}_{2,i}^2\left(\tau\right)}\Rightarrow E\left[\bar{Z}_2^i\left(\tau+\Delta\tau\right)\right]=\bar{Z}_2\left(\tau\right). \tag{4.53}$$

The result given in (4.53) defines the simplest inference rule available to players[17], in this incomplete information setup. This result has several advantages. It allows for a simplification of the expected value computation process, thus greatly reducing the computational time required for the simulation of the HMM, and consequently, the cost of introducing other decision criteria. The existence of information about past moments of the game paves the way for the introduction and evaluation of different reinforcement learning mechanisms. The SCE optimality condition can then be used to rank these mechanisms according to their performance.[18]. We propose that players' beliefs about the evolution of the state of the game have to be consistent with *Bayesian* learning in a competitive environment, in order to preserve the co-evolutionary nature of this decision process. This assumption involves introducing as decision criteria, an individual measure of uncertainty, which we defined initially as the second moment of a belief distribution with finite variance, Φ_i. In this specific application, we assume for simplicity that Φ_i evolves according to some linear function of the observed forecasting errors, $\Phi_i\left(\epsilon_i\left(\tau\right)\right)$. Following this set of assumptions and the result in (4.53), we assume that individual beliefs, $\bar{Z}_2^i$, about future values of $\bar{Z}_2$ are normally distributed,

$$\begin{cases} \bar{Z}_2^i\left(\tau+\Delta\tau\right)\sim N\left(\bar{Z}_2\left(\tau\right),\Phi_i^2\right) & \text{if } \tau>0,\\ \bar{Z}_2^i\left(\tau+\Delta\tau\right)\sim N\left(0,1\right) & \text{if } \tau=0.\end{cases} \tag{4.54}$$

We are now able to describe the computation of individual transition rates. At a given period, τ, the players objective is to take investment decisions that are

[17] The result in (4.52) shows that players can learn the past moments of $\bar{Z}_2$. If expectations were driven solely by beliefs, then the learning rule would be a belief-based decision. See Feltovich [2000] for a discussion on this topic.

[18] In economic theory the reinforcement learning assumption has the following interpretation. It justifies how equilibrium may or may not arise when agents have bounded rationality. The consistent expectations equilibrium (CEE) hypothesis by Hommes and Sorger [1998], for example, suggests that linear forecasting decisions, based on past available information, represent simple rational adaptive learning rules for nonlinear incomplete information environments, that are consistent with the existence of self-fulfilling belief solutions As the authors put it, *"...agents form expectations about future variables in such a way that their beliefs are consistent with the observed realizations in a linear statistical sense. In other words, it is supposed that agents act like econometricians using linear statistical techniques and, in doing so, they do not make systematic forecasting errors..."*. An application of this hypothesis in a nonlinear dynamic optimization setup can be found in Hommes and Rosser [2001]. For a thorough review on the topic of multi-agent reinforcement learning see Tuyls and Nowé [2005].

consistent with the long run maximization of their expected outcomes under uncertainty. At a given $\bar{Z}_{4,i} \in \Lambda_i$, players decide what is the best action to undertake given the full range of achievable investment decisions to the left and right of $\bar{Z}_{4,i}$, which are defined by the following discrete sets, $\Lambda_i^L = \left\{\delta + v, \ldots, \bar{Z}_{4,i} - v\right\}$ and $\Lambda_i^R = \left\{\bar{Z}_{4,i} + v, \ldots, \delta + \rho_i/\gamma_i - v\right\}$, respectively. To evaluate the best path to follow, players compute the maximum expected consumption values, given (4.53), for all the range of their feasible strategic space, $E_{\tau+\Delta\tau}\left[\bar{Z}_{1,i}\left(\Lambda_i, \bar{Z}_2(\tau)\right)\right]$, and determine the following quantities, $Z_{1,i}^{L,\max} = \max\left\{E_{\tau+\Delta\tau}\left[\bar{Z}_{1,i}\left(\Lambda_i^L, \bar{Z}_2(\tau)\right)\right]\right\}$, $Z_{1,i}^{R,\max} = \max\left\{E_{\tau+\Delta\tau}\left[\bar{Z}_{1,i}\left(\Lambda_i^R, \bar{Z}_2(\tau)\right)\right]\right\}$ and $Z_{1,i}^{C} = E_{\tau+\Delta\tau}\left[Z_{1,i}\left(\bar{Z}_{4,i}, \bar{Z}_2(\tau)\right)\right]$. The best expected achievable outcome is defined generally by $Z_{1,i}^{best} = \max\left\{Z_{1,i}^{L,\max}, Z_{1,i}^{C}, Z_{1,i}^{L,\max}\right\}$, while the remaining outcomes are defined as $Z_{1,i}^{worst,1}$ and $Z_{1,i}^{worst,2}$. Finally, each agent determines the following quantities, taking into account the uncertainty regarding the best achievable outcome[19]:

$$\begin{cases} \Theta_i^{worst,1} = Z_{1,i}^{worst,1}\left(\bar{Z}_2(\tau) + \Phi_i\right) - Z_{1,i}^{best}\left(\bar{Z}_2(\tau) - \Phi_i\right) \\ \text{if} \\ Z_{1,i}^{worst,1}\left(\bar{Z}_2(\tau) + \Phi_i\right) > Z_{1,i}^{best}\left(\bar{Z}_2(\tau) - \Phi_i\right), \\ \Theta_i^{worst,1} = 0 \quad \text{otherwise.} \end{cases} \tag{4.55}$$

$$\begin{cases} \Theta_i^{worst,2} = Z_{1,i}^{worst,2}\left(\bar{Z}_2(\tau) + \Phi_i\right) - Z_{1,i}^{best}\left(\bar{Z}_2(\tau) - \Phi_i\right) \\ \text{if} \\ Z_{1,i}^{worst,2}\left(\bar{Z}_2(\tau) + \Phi_i\right) > Z_{1,i}^{best}\left(\bar{Z}_2(\tau) - \Phi_i\right), \\ \Theta_i^{worst,2} = 0 \quad \text{otherwise.} \end{cases} \tag{4.56}$$

$$\Theta_i^{best} = Z_{1,i}^{best}\left(\bar{Z}_2(\tau) + \Phi_i\right) - Z_{1,i}^{best}\left(\bar{Z}_2(\tau) - \Phi_i\right). \tag{4.57}$$

Finally, transition rates are given by a weighted average of all achievable outcomes that are greater than the lower confidence bound of $Z_{1,i}^{best}$ expected outcomes,

$$\begin{cases} \theta_i^{best} = \Theta_i^{best}\left(\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2}\right)^{-1} \\ \theta_i^{worst,1} = \Theta_i^{worst,1}\left(\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2}\right)^{-1} \\ \theta_i^{worst,2} = \Theta_i^{worst,2}\left(\Theta_i^{best} + \Theta_i^{worst,1} + \Theta_i^{worst,2}\right)^{-1} \end{cases}. \tag{4.58}$$

To describe the mechanics driving this decision process, we resort to the example portrayed in Figure 4.3. In this setup, the best expected outcome for agent i, is

[19] Where $Z_{1,i}^{worst,1}\left(\bar{Z}_2(\tau) \pm \Phi_i\right)$, $Z_{1,i}^{worst,2}\left(\bar{Z}_2(\tau) \pm \Phi_i\right)$ and $Z_{1,i}^{best}\left(\bar{Z}_2(\tau) \pm \Phi_i\right)$, describe $Z_{1,i}^{L,max}$, $Z_{1,i}^{C}$ and $Z_{1,i}^{R,max}$, evaluated at $\bar{Z}_2(\tau) \pm \Phi_i$.

in one of the extremes of Λ_i. It is not clear which one of the two extremes has the best expected value. However, by evaluating the landscape in this fashion, we know that the transition rates for a player with an interior strategy, $\delta+v < \bar{Z}_{4,i} < \delta+\rho_i\gamma_i^{-1}-v$, are ranked according to, $0 \leq \theta_i^0 < \theta_i^-, \theta_i^+$ and $\theta_i^- \approx \theta_i^+$. This ordering of rates is, in our opinion, a reasonable weighting of the uncertainty faced by player i. This process allows players to re-evaluate their investment priorities in a co-evolutionary environment, where one player's decision may lead to drastic changes in the landscape. To illustrate the HMM described in this section, we simulate it numerically as a *Markov* chain. Our main objective is to show the crucial role played by uncertainty. As in this framework, players' expected outcomes are homogeneous, uncertainty is the only distinctive criteria driving player decisions[20]. This feature of the HMM allows us to focus on the dramatic consequences of uncertainty in the quality of solutions that may arise from (4.49). Our approach is purely illustrative. The simple metric proposed serves mainly the purpose of avoiding the computational costs of performing large scale inference and paves the way for a future statistical analyses of this HMM outcomes based on the *Markov Chain Monte Carlo* method. A more thorough analysis will involve assuming a more realistic decision criteria based on a consistent probability measure, which takes into account the heterogeneity and evolutionary nature of the individual inference process. In the next two sections, we demonstrate the framework proposed for two different hypothesis describing the evolution of individual uncertainty. In these simulations, we consider an institutional framework defined by $\{r, \delta, r_k\} = \{0.05, 0.03, 0.11\}$. Each experiment draws a distinct robust population [21], with initial investment positions distributed randomly following, $\bar{Z}_{4,i}(\tau = 0) \sim U_d(\Lambda_i)$.

[20]The inclusion of individual memory, is a natural extension in this HMM. It allows the limited introduction of heterogeneity about expected state outcomes, without adding further complications to this setup. The general *Bayesian* forecast rule in an environment with individual memory, following the set of conditions described, would be given by $\bar{Z}_2^i(\tau+\Delta\tau) \sim N\left(\tilde{Z}_2(\tau), \tilde{\Phi}_i(\tilde{\epsilon}_i(\tau))^2\right)$, $\forall\, \tau > 0$, where $\tilde{Z}_2(\tau)$, is the sample mean of player i observed posterior distribution up to period τ, and $\tilde{\Phi}_i(\tilde{\epsilon}_i(\tau))^2$, is a function of the estimated sample variance given a prior belief that depends on the evolution of player i past forecast errors, $\tilde{\epsilon}_i(\tau)$.

[21]By robust population, we mean a discrete set of $n = 1000$ agents with randomly drawn distributed individual characteristics, $\rho_i, \gamma_i \sim U(0,1)$ and $h_i \sim U(-10,10)$, that are consistent with $\delta + \rho_i/\gamma_i < 1$, $\#\Lambda_i \geq 10$ and $\bar{Z}_{1,i} > 0, \forall \bar{Z}_{4,i} \in \Lambda_i$, given a reasonable range of state outcomes, $-1.5 < \bar{Z}_2 < 1.5$. Following the definitions in section 4.3, $K_i(0)$ is again given by random draws of an exponential distribution, $K_i(0) \sim \exp(1)$, and the evolution of the state of the game is driven by a 30% population share of aggressive investors.

4.4.2 Self-confirming equilibrium

Before putting forward the results of this experiment, it is convenient to frame the definition of SCE, given in (4.5), with the literature on evolutionary multi-objective optimization. In this field, the analysis of solutions relies on the definition of a *Pareto Dominance* criterion[22] or *Pareto Efficiency* frontier, to develop efficient algorithms and order feasible solutions. A strategy is *Pareto Dominant*, if it lies in the *Pareto Efficient* frontier. This approach cannot be extended to decision problems under uncertainty. In incomplete information environments, solutions are by definition conjectural equilibria outcomes[23], since equilibrium is a result of co-evolution under uncertainty. In evolutionary multi-objective optimization environments with uncertainty, a common approach is to order solutions in terms of some *Bayesian Efficiency* criterion. This involves considering a consistent *Bayesian* learning incentive. A reasonable definition of *Bayesian Efficiency* for the problem defined in (4.49), is given by the SCE condition. To be able to order solutions to (4.49), consistent with this notion of *Bayesian Efficiency*, we have to introduce an incentive compatible rule in the *Bayesian* learning mechanism of our HMM. Recall that in subsection 4.4.1, we mentioned that a SCE requires the existence of an absorbing state, $\sum_{i \in N} \theta_i^0 = n$. Following our definition in (4.54), a SCE is achieved for this HMM if $lim_{\tau \to \infty} \Phi_i\left(\epsilon\left(\tau\right)\right) = 0 \Rightarrow E\left[\bar{Z}_2^i\left(\tau + \Delta\tau\right)\right] = \bar{Z}_2\left(\tau\right), \forall i \in N$. To include a valid *Bayesian Incentive* in this setup, we shall consider that when $e_i\left(\tau\right) = 0$, agents no longer forecast future outcomes under uncertainty, $\Phi_i\left(0\right) = 0$. We demonstrate the existence of SCE solutions for the HMM proposed, assuming that uncertainty depends linearly on the last observed forecasting error. In this experiment, Φ_i evolves in the following fashion[24],

$$\Phi_i\left(\tau + \Delta\tau\right) = 10^8 |\epsilon_i\left(\tau\right)|. \tag{4.59}$$

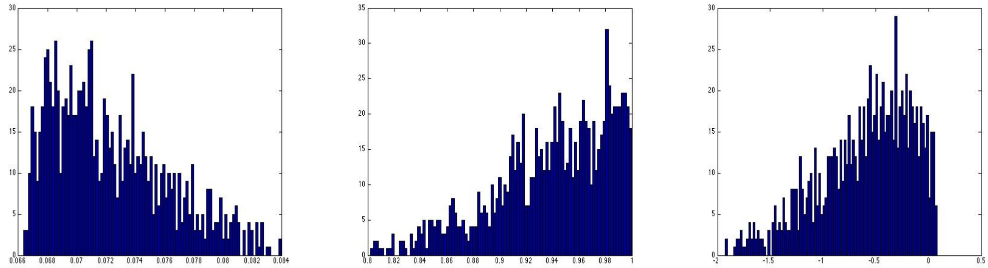
We start this presentation with the parameter distributions describing a random robust population. These are portrayed below, by the histograms of ρ_i, γ_i and h_i. In this institutional setup, robust populations are characterized by a group of patient players with a high rate intertemporal substitution of consumption, where the

[22]See Gajda et al. [2010] for a formal definition of *Pareto Dominance* and a mathematical discussion on multi-objective optimization problems.

[23]We consider that SCE solutions are a particular stronger case of conjectural equilibrium solutions. See Wellman et al. [1998] for a discussion on this topic in a continuous time pure exchange economy model. In economics literature a popular extension of this topic is the conjectural variations approach. Itaya and Shimomura [2001] provides a thorough discussion on this method for multi-player public goods games.

[24]The constant 10^8 in (4.59) is used to portray the robustness of this linear learning rule in guaranteeing convergence to a SCE for this HMM.

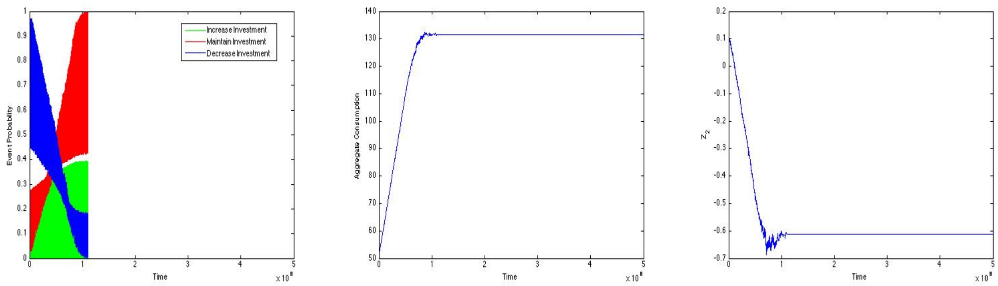
vast majority of players is biased towards investment in productive/riskless assets.



(a) Distribution of ρ_i (b) Distribution of γ_i (c) Distribution of h_i

Figure 4.5: Robust population distributions.

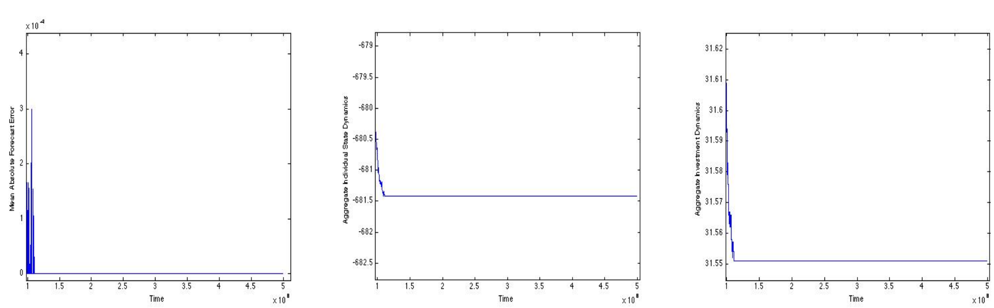
The next figure portrays the HMM convergence to a SCE. The dynamics $\bar{Z}_2(\tau)$, on the right, describe an initial fast transition to the equilibrium region, followed by overshooting phenomenon arising before players concur on a SCE solution. In the figure on the left, we observe that uncertainty about investment decisions decreases during the initial convergence moment and suddenly increases before settling in the SCE solution. This behaviour during transitions is, in our opinion, linked to dynamics of learning under uncertainty. In a first phase, there is a clear trend driving strategies towards a region where agents outcomes are close to optimum. At this point, higher aggregate outcomes are achievable, but they come at the expense of increased uncertainty among players. This eventually leads to a review of the long run strategies by an increasing share of the population. The reason for this phenomenon is straightforward. There is a set of players that is now worst off. As the co-evolution process develops, players eventually concur on a equilibrium that best fits their individual interests. This self-organization process materializes in conjectural SCE solution when uncertainty vanishes completely from the system.



(a) Event probability (b) $\sum_{i\in N}\bar{Z}_{1,i}(\tau)$ dynamics (c) $\bar{Z}_2(\tau)$ dynamics

Figure 4.6: Total probability, game and consumption dynamics.

Figure 4.7 describes the asymptotic outcomes for the mean absolute forecast errors, aggregate state and investment dynamics. The figure on the left shows that the learning rule described in (4.59) is consistent with the existence of optimal solutions to (4.33), as the result $lim_{\tau \to \infty} \Phi_i (\epsilon (\tau)) = 0$ guarantees that (4.5) is fulfilled. Additional simulations suggest that this specific setup is always consistent with convergence to a SCE solution.

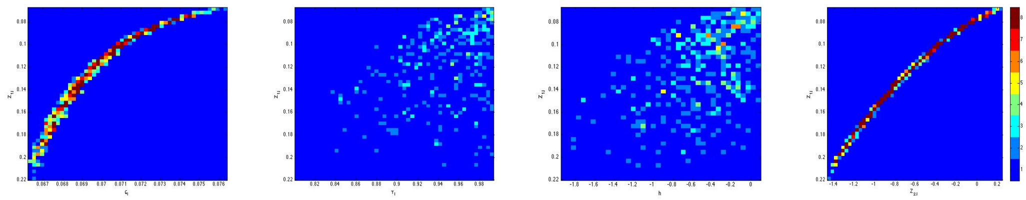


(a) $n^{-1} \sum_{i \in N} |\epsilon_i (\tau)|$ dynamics (b) $\sum_{i \in N} \bar{Z}_{2,i} (\tau)$ dynamics (c) $\sum_{i \in N} \bar{Z}_{4,i} (\tau)$ dynamics

Figure 4.7: Asymptotic mean absolute forecast errors and aggregate dynamics.

We conclude this discussion with an analysis of the effects of player asymmetries and portfolio decisions on consumption outcomes. These results are given below in Figure 4.8. The figure on the left shows that there is a linear/logarithmic relation between player patience and his consumption outcomes. This result is consistent with the fundamental law of growth theory, which states that agents with higher savings rate (smaller discount rates), are able to sustain higher consumption profiles in the future[25]. A similar relation is observed when players portfolios are compared against consumption outcomes. In the figure in the right, we observe that leveraged players have the smallest share of consumption to productive assets. These results suggest that players with diversified portfolios and higher propensity to save will perform better in this economy. The remaining figures portray the relation between intertemporal substitution of consumption and investment bias on the final consumption outcomes. The joint density plots show that the majority of players with higher γ_i and higher h_i have lower consumption profiles. However, there is no clear pattern arising in these two cases.

[25] This result holds for growing economies, where savings finance profitable investments in the present that increase capital in the future.



(a) Density $(\bar{Z}_{1,i}, \rho_i)$ (b) Density $(\bar{Z}_{1,i}, \gamma_i)$ (c) Density $(\bar{Z}_{1,i}, h_i)$ (d) Density $(\bar{Z}_{1,i}, \bar{Z}_{2,i})$

Figure 4.8: Joint densities of final outcomes versus player asymmetries.

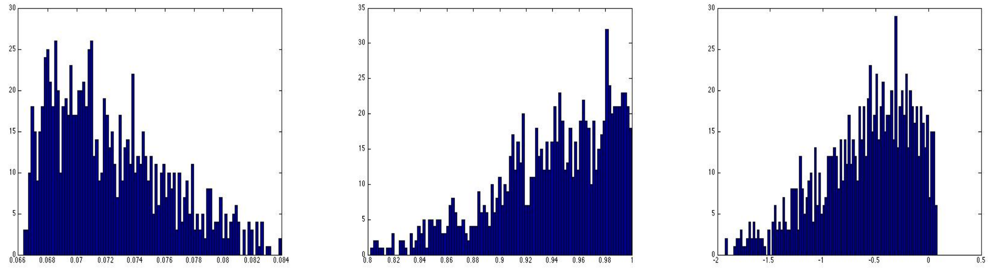
4.4.3 Unintended consequences

To portray game dynamics consistent with strategic interactions and unintended consequences driven by subjective beliefs as a possible outcome to (4.33), we now assume that agents have homogeneous beliefs about uncertainty, except when no forecast errors are observed. In this setup, uncertainty is no longer a linear function of $\epsilon_i(\tau)$, but given by a *Heaviside* step function. This specification allows us to show that above some threshold, uncertainty plays a crucial role on the quality of solutions. It also also guarantees the existence of a *Bayesian Incentive* mechanism that is compatible with the existence of an absorbing state for this HMM. This simplification has the sole purpose of showing the co-evolutionary nature of solutions to (4.49). Solutions in this setup are not unique and asymptotic dynamics are better defined as invariant sets. Although these solutions no longer fulfil the *Bayesian Efficiency* criterion and, therefore, are non-optimal, numerical results suggest that consumption outcomes are quasi-optimal in the long run, when compared to the previous example. Uncertainty is now defined as

$$\Phi_i(\tau + \Delta\tau) = \begin{cases} 1 & \text{if} \quad |e_i(\tau)| > 0 \\ 0 & \text{if} \quad |e_i(\tau)| = 0 \end{cases}. \tag{4.60}$$

We start this presentation with the description of player asymmetries. A quick glance at the histograms given in Figure 4.5 and Figure 4.9 shows that there is no significant differences between the two samples.

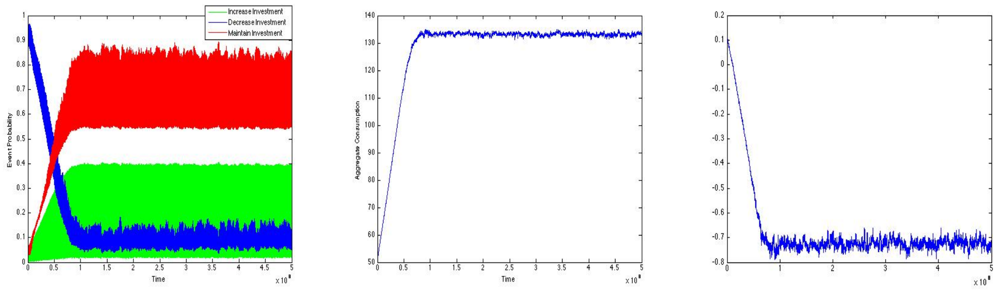
Figure 4.10, below, portrays the main results for this experiment. Again, we observe a fast convergence to long run dynamics. However, asymptotic dynamics are no longer described by a unique SCE solution. The stochastic process does not converge to an absorbing state. This solution is best defined as an invariant set, and asymptotic dynamics are described by a stationary distribution. A quick inspection reveals that aggregate consumption outcomes converge to values close to the SCE



(a) Distribution of ρ_i (b) Distribution of γ_i (c) Distribution of h_i

Figure 4.9: Robust population distributions.

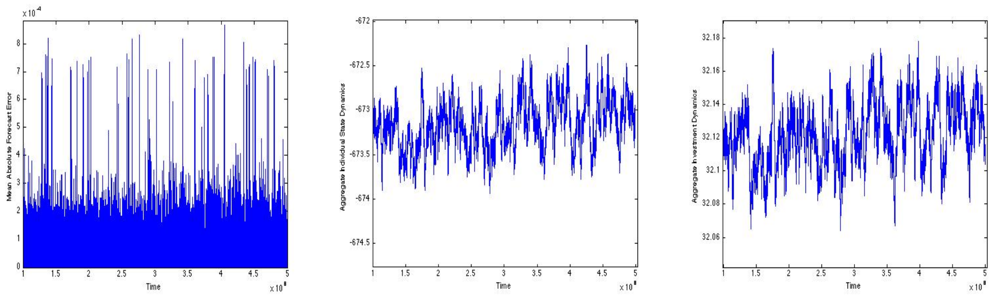
solution portrayed in Figure 4.6. Although these two experiments cannot be directly compared, since initial values and population characteristics are not identical, this result suggests that the invariant set describing this game asymptotic dynamics can be characterized as a quasi-optimal solution to (4.49). The level of uncertainty faced by the ensemble of players has dramatic consequences on the quality and complexity of solutions to (4.33). The picture on the left, describing the dynamics of total probability, portrays the evolutionary mechanism driving the complex macro dynamics observed. In the long run, a minority of players is undecided between choosing conservative or aggressive investment strategies, $0.1 < n^{-1}\sum_{i\in N}\theta_i^{+} + n^{-1}\sum_{i\in N}\theta_i^{-} < 0.5$. The indecision leads to unintended consequences, which in turn feeds strategic interactions and further changes in the environment. On the other hand, a majority of players maintains their long run investment strategies, $0.5 < n^{-1}\sum_{i\in N}\theta_i^{0} < 0.9$, despite the changes imposed by the undecided minority. In a continuous co-evolution process, undecided investors drive the market and reshape the economic landscape. The dynamics of this economy are thus dominated by the decisions of a minority.



(a) Event probability (b) $\sum_i \bar{Z}_{1,i}(\tau)$ dynamics (c) $\bar{Z}_2(\tau)$ dynamics

Figure 4.10: Total probability, consumption and game dynamics.

We finish this presentation with a figure portraying the asymptotic dynamics of $n^{-1}\sum_i |\epsilon_i(\tau)|$, $\sum_i \bar{Z}_{2,i}(\tau)$ and $\sum_i \bar{Z}_{4,i}(\tau)$. The dynamics of $n^{-1}\sum_i |\epsilon_i(\tau)|$ portray the consequences of persistently high uncertainty on the forecasting process. Errors in this setup alternate between stable and bursting periods with different amplitudes. The largest bursts identify the decisions that imposed greater changes on the state of the game. The consequences of these interactions are portrayed in the dynamics of $\sum_i \bar{Z}_{2,i}(\tau)$ and $\sum_i \bar{Z}_{4,i}(\tau)$. Investment and portfolio decisions are asymptotically stationary but consistently drift away from the mean. Further insight on this and other matters requires a thorough statistical analysis of this stochastic process.



(a) $n^{-1}\sum_i |\epsilon_i(\tau)|$ dynamics (b) $\sum_i \bar{Z}_{2,i}(\tau)$ dynamics (c) $\sum_i \bar{Z}_{4,i}(\tau)$ dynamics

Figure 4.11: Asymptotic mean absolute forecast errors and aggregate dynamics.

4.5 Conclusions and further research

In this chapter, we described a class of differential games, where players' optimal strategies require information that is not available to them. Decisions in this context, we argued, have to be based on subjective beliefs, and optimality requires the existence of a SCE solution. To demonstrate this conjecture, we proposed two extensions of the consumer-investor problem. In the first example discussed in section 4.3, beliefs impose a unique strategic solution and the existence of a SCE requires the existence of credit markets with complete information that are able to price risk optimally. Solutions can be evaluated qualitatively when beliefs are naive. Numerical results show that game solutions are not stable for a wide range of institutional scenarios. This result is not surprising, as these constraints impose a game environment where both creditors and debtors believe that they will be worse off. We concluded that stable SCE solutions require that the strict assumptions imposed on the evolution of beliefs are at least partially dropped. This result illustrates the dramatic consequences of introducing further nonlinear mechanisms in competitive economic models. In section 4.4, we showed that the introduction of additional nonlinearities in the initial game is sufficient to introduce multiple equilibria. Solutions are consistent with a game with a state-space bounded asymptotically, but standard qualitative methods cannot be employed in this context. We propose that game solutions can be evaluated as a multi-objective optimization problem under uncertainty. Given the specific nature of this game, we proposed to evaluate game solutions as a multi-criteria HMM, consistent with learning in a co-evolutionary environment, and focused on the role played by uncertainty. First, we showed the existence of SCE solutions, when uncertainty depends linearly on forecast errors. When uncertainty is persistent, belief-based decisions lead to unintended consequences and strategic interactions that have dramatic effects on the macro evolution of this economy. Solutions are now of a quasi-optimal nature. This phenomenon is driven by a minority of players, which suggests the existence of strong emergence phenomena. Further insight on these two hypotheses requires the computation of samples and a thorough statistical analysis. It is our opinion that the statistical analysis of quasi-optimal solutions to nonlinear dynamic games is bound to play an important role in future economics literature. Finally, the introduction of learning heterogeneity represents a natural extension to the simple framework proposed. It would also be interesting to investigate the implications of introducing evolutionary adaptation in the decision process discussed.

Appendix A

Appendix to chapter 1

A.1 Optimal control conditions

The current value *Hamiltonian* for the intertemporal maximization problem given in (2.34) is,

$$H^{*}\left[C\left(t\right),I\left(t\right),B\left(t\right),K\left(t\right),\lambda\left(t\right),q\left(t\right)\right]=C\left(t\right)^{\gamma}+\lambda\left(t\right)\dot{B}\left(t\right)+q\left(t\right)\dot{K}\left(t\right), \quad \text{(A.1)}$$

where $\dot{B}(t)$ and $\dot{K}(t)$ are given in (A.6) and (A.7). The *Pontryagin* necessary and sufficient conditions for the existence of an optimum solution for (2.34), are given by:

Optimality conditions

$$\frac{\partial H^{*}}{\partial C\left(t\right)}=0\Leftrightarrow\gamma C\left(t\right)^{\gamma-1}=-\lambda\left(t\right); \quad \text{(A.2)}$$

$$\frac{\partial H^{*}}{\partial I\left(t\right)}=0\Leftrightarrow q\left(t\right)=-\lambda\left(t\right)\left(1+h\frac{I\left(t\right);}{K\left(t\right)}\right) \quad \text{(A.3)}$$

Multiplier conditions

$$\frac{\partial\lambda\left(t\right)}{\partial t}=\rho\lambda\left(t\right)-\frac{\partial H^{*}}{\partial B\left(t\right)}\Leftrightarrow\dot{\lambda}\left(t\right)=\lambda\left(t\right)\left(\rho-r-rd\frac{B\left(t\right)}{K\left(t\right)}\right); \quad \text{(A.4)}$$

$$\frac{\partial q(t)}{\partial t}=\rho q\left(t\right)-\frac{\partial H^{*}}{\partial K(t)}\Leftrightarrow\dot{q}\left(t\right)=q\left(t\right)\left(\rho+\delta\right)+\lambda\left(t\right)\left(\frac{rdB(t)^{2}}{2K(t)^{2}}+\frac{hI(t)^{2}}{2K(t)^{2}}+r_{k}\right); \quad \text{(A.5)}$$

State conditions

$$\frac{\partial B(t)}{\partial t} = \frac{\partial H^*}{\partial \lambda(t)} \Leftrightarrow$$
$$\Leftrightarrow \dot{B}(t) = C(t) + I(t)\left(1 + \frac{hI(t)}{2K(t)}\right) + rB(t)\left(1 + \frac{dB(t)}{2K(t)}\right) - r_k K(t); \tag{A.6}$$

$$\frac{\partial K(t)}{\partial t} = \frac{\partial H^*}{\partial q(t)} \Leftrightarrow \dot{K}(t) = I(t) - \delta K(t); \tag{A.7}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda(t) B(t) e^{-\rho t} = 0; \tag{A.8}$$

$$\lim_{t\to\infty} q(t) K(t) e^{-\rho t} = 0; \tag{A.9}$$

Admissibility conditions

$$B_0 = B(0), K_0 = K(0). \tag{A.10}$$

A.2 Non-feasible steady states

Following the discussion in section 2.5, on $\widetilde{Z}^* \in \Re$, we show in this section under what conditions this set of steady states can be computed numerically. The general expressions describing the intersection of two quadratic curves are given by

$$\underbrace{\frac{rd}{2}(Z_2^*)^2}_{AX^2} - \underbrace{Z_2^* Z_4^*}_{BXY} + \underbrace{\frac{h}{2}(Z_4^*)^2}_{CY^2} + \underbrace{(r+\delta) Z_2^*}_{DX} + \underbrace{Z_4^*}_{EY} - \underbrace{r_k}_{F} = 0, \tag{A.11}$$

$$-\underbrace{\frac{rd}{2h}(Z_2^*)^2}_{GX^2} + \underbrace{rdZ_2^* Z_4^*}_{HXY} - \underbrace{\frac{1}{2}(Z_4^*)^2}_{IY^2} + \underbrace{\frac{rd}{h} Z_2^*}_{JX} + \underbrace{(r+\delta) Z_4^*}_{KY} + \underbrace{(r+\delta-r_k)\frac{1}{h}}_{L} = 0. \tag{A.12}$$

It is now convenient to characterize the geometry of the quadratic curves described by (A.11) and (A.12). We discard the case of infinitely many equilibrium and the hypothesis that these quadratic curves are represented by degenerate conic sections. Following these assumptions, we characterize each quadratic curve by determining the quantities:

$$B^2 - 4AC = 1 - rdh; \tag{A.13}$$

$$H^2 - 4GI = rd^2 - \frac{rd}{h} = rd\left(rd - \frac{1}{h}\right). \tag{A.14}$$

Following (A.13) the curve (A.11) is defined by: (i) a hyperbola when $hrd < 1$; (ii) a parabola when $hrd = 1$; and (iii) an ellipse when $hrd > 1$. Given (A.14) the curve (A.12) is: (i) a hyperbola when $hrd < 0 \wedge hrd > 1$; (ii) a parabola when $hrd = 1$; and (ii) and ellipse when $0 < rdh < 1$. The general solution to the system defined by (A.11) and (A.13) is given by a fourth order equation. We can solve this system analytically when the parabola constraint is considered, as it allows for a reduction of the fourth order equation to a second order one. To solve the intersection of (A.11) and (A.12), assuming $rdh = 1$, it is convenient to rearrange (A.11) in the following fashion[1],

$$Y^2 = -\frac{A}{C}X^2 - \frac{B}{C}XY - \frac{D}{C}X - \frac{E}{C}Y - \frac{F}{C}. \tag{A.15}$$

Substituting the Y^2 term in (A.12) by the expression given in (A.15) we obtain:

$$\left(G - \tfrac{AI}{C}\right)X^2 + \left(H - \tfrac{BI}{C}\right)XY + \left(J - \tfrac{DI}{C}\right)X + \left(K - \tfrac{EI}{C}\right)Y + L - \tfrac{FI}{C} = 0. \tag{A.16}$$

Recall now that the first two terms in (A.16) are given by the following expressions, $G - AIC^{-1} = -rd\,(2h)^{-1} + rd\,(2h)^{-1} = 0$ and $H - BIC^{-1} = rd - h^{-1}$. When $rdh = 1$, the second term vanishes and (A.16) reduces to,

$$X = -\frac{KC - EI}{JC - DI}Y - \frac{LC - FI}{JC - DI}. \tag{A.17}$$

Substituting now (A.17) in (A.11), we obtain,

$$\begin{aligned} &\left[\tfrac{KC-EI}{JC-DI}\left(A\tfrac{KC-EI}{JC-DI} - B\right) + C\right]Y^2 + \left[\tfrac{LC-FI}{JC-DI}\left(A\tfrac{LC-FI}{JC-DI} - D\right) + F\right] \\ &\quad + \left[2A\tfrac{(KC-EI)(LC-FI)}{(JC-DI)^2} - \tfrac{B(LC-FI)+D(KC-EI)}{JC-DI} + E\right]Y = 0. \end{aligned} \tag{A.18}$$

The non feasible steady states when $hrd = 1$ are given by

$$Z_2^* = -\frac{\left(r + \delta + \frac{1}{h}\right)h}{rd + r + \delta}Z_4^* - \frac{\delta + r - 2r_k}{rd + r + \delta} \quad \text{and} \quad Z_4^* = \frac{-\Phi \pm \sqrt{\Phi^2 - 4\Theta\Psi}}{2\Theta}, \tag{A.19}$$

where Θ, Φ and Ψ are defined by the following set of expressions:

$$\begin{aligned} &\Theta = \left[\tfrac{KC-EI}{JC-DI}\left(A\tfrac{KC-EI}{JC-DI} - B\right) + C\right], \quad \Psi = \left[\tfrac{LC-FI}{JC-DI}\left(A\tfrac{LC-FI}{JC-DI} - D\right) + F\right] \\ &\qquad \text{and} \quad \Phi = \left[2A\tfrac{(KC-EI)(LC-FI)}{(JC-DI)^2} - \tfrac{B(LC-FI)+D(KC-EI)}{JC-DI} + E\right]. \end{aligned} \tag{A.20}$$

[1] To allow for a clearer presentation, we shall use the general expressions of both quadratic curves throughout this section.

Substituting Θ, Φ and Ψ by system parameters we obtain:

$$\begin{aligned} \Theta = 2h,\ \ \Phi = \tfrac{2(r+\delta-2r_k)}{rd+r+\delta} - (r+\delta)\,h + 1 \ \text{ and} \\ \Psi = \tfrac{r+\delta-2r_k}{rd+r+\delta}\left[\tfrac{rd}{2}\left(\tfrac{r+\delta-2r_k}{rd+r+\delta}\right) - (r+\delta)\right] - r_k. \end{aligned} \tag{A.21}$$

We now describe a numerical algorithm for the computation of the general solution of the system defined in (A.11) and (A.12). We propose a two step solution to solve this problem in a robust and efficient fashion. First, we define a solution that is linear in terms of one of the coordinates solution of (A.11) and (A.12). We choose coordinate Y for this purpose. Multiplying (A.11) by I and (A.12) by C, and imposing equality between the resulting expressions, we obtain the solution for Y in terms of the coordinate solution of X,

$$Y = \frac{(GC - AI)\,X^2 + (JC - DI)\,X + (LC - FI)}{(EI - KC) + (BI - HC)\,X}. \tag{A.22}$$

We now have to determine the coordinate solution for X. This solution is given by a fourth order polynomial. First we rearrange the original expressions, (A.11) and (A.12), as quadratic polynomials in terms of Y coordinate. Then we set the resulting system in a *Sylvester* matrix form:

$$\begin{bmatrix} C & BX+E & AX^2+DX+F & 0 \\ 0 & C & BX+E & AX^2+DX+F \\ I & HX+K & GX^2+JX+L & 0 \\ 0 & I & HX+K & GX^2+JX+L \end{bmatrix} \begin{bmatrix} Y^3 \\ Y^2 \\ Y^1 \\ Y^0 \end{bmatrix} = 0. \tag{A.23}$$

In order to obtain the coordinate solution in terms of X, we follow *Bezout's* theorem, and determine the resultant of the two original polynomials. To obtain the resultant, we set the determinant of the *Sylvester* matrix defined in (A.23) equal to zero. This condition is given by:

$$\begin{aligned} \left[C\left(GX^2+JX+L\right)-I\left(AX^2+DX+F\right)\right]^2 \\ -[C(HX+K)-I(BX+E)]\left[(BX+E)\left(GX^2+JX+L\right)-(HX+K)\left(AX^2+DX+F\right)\right]=0. \end{aligned} \tag{A.24}$$

Solving the above expression, we obtain the fourth order polynomial in terms of X coordinates. After a fair amount of calculus we obtain the following equation,

$$b_0X^4 + b_1X^3 + b_2X^2 + b_3X + b_4 = 0, \tag{A.25}$$

where the coefficients of this polynomial are given by the following expressions:

$$b_0 = (CG - IA)^2 + (HA - BG)(CH - IB); \tag{A.26}$$

$$\begin{aligned} b_1 = {} & 2(CJ - ID)(CG - IA) - (KB + HE)(CG + IA) \\ & + (BJ - HD)(IB - CH) + 2(CHKA + IBEG); \end{aligned} \tag{A.27}$$

$$\begin{aligned} b_2 = {} & (CJ - ID)^2 + 2(CL - IF)(CG - IA) \\ & + (HF - BL)(CH - IB) - (KB + HE)(CJ + ID) \\ & + (KA - EG)(CK - IE) + 2(CHKD + IBEJ); \end{aligned} \tag{A.28}$$

$$\begin{aligned} b_3 = {} & 2(CL - IF)(CJ - ID) - (KB + HE)(CL + IF) \\ & + (KD - EJ)(CK - IE) + 2(CHKF + IBEL); \end{aligned} \tag{A.29}$$

$$b_4 = (CL - IF)^2 + (KF - EL)(CK - IE). \tag{A.30}$$

To finish this procedure, we must now employ a polynomial solver and obtain the coordinate solution in terms of X, given the solution defined in (A.25) to (A.30), and then substitute this solution in (A.22) to obtain the corresponding Y coordinate[2].

A.3 Linearized dynamics and non-degeneracy conditions

Recall that given the restrictions described in section 2.5, the *Jacobian* in the vicinity of the economic meaningful steady states, $\widetilde{Z}^{**}$, is given generically by:

$$J^{**} = \begin{bmatrix} 0 & \frac{rd}{\gamma-1} J^{**}_{1,3} & J^{**}_{1,3} \\ 1 & J^{**}_{2,2} & J^{**}_{2,3} \\ 0 & \frac{rd}{h} J^{**}_{2,3} & J^{**}_{2,2} \end{bmatrix}. \tag{A.31}$$

The characteristic equation for this *Jacobian* comes,

$$\begin{aligned} (\Lambda^{**})^3 - 2J^{**}_{2,2}(\Lambda^{**})^2 - \Lambda^{**}\left(\frac{rd}{\gamma-1}J^{**}_{1,3} - \left(J^{**}_{2,2}\right)^2 + \frac{rd}{h}\left(J^{**}_{2,3}\right)^2\right) \\ -rdJ^{**}_{1,3}\left(\frac{1}{h}J^{**}_{2,3} - \frac{1}{\gamma-1}J^{**}_{2,2}\right) = 0, \end{aligned} \tag{A.32}$$

[2] For this purpose we built a C routine and compiled our code with the GNU scientific library(GSL) polynomial solver, which is based on the *Horner's* method for stability. We then obtain absolute computation errors by substituting the numerical solution in the original system, (A.11) and (A.12), and test their accuracy for an error tolerance defined by $|\max_{error}\{X^*, Y^*\}| \leq 10^{-5}$. For this error tolerance , all computed solutions consistent with, $X^*, Y^* \in \Re$, were accepted. We confirmed this procedure by running a routine in MATLAB using the built-in polynomial solver function *roots* and no significant differences were found.

where Λ^{**} stands for the eigenvalues solving the characteristic polynomial in the vicinity of $\widetilde{Z}_i^{**}$. The condition guaranteeing the *Jacobian* defined in (A.31) is non-degenerate, $Det\left(J^{**}\right) \neq 0$, is given by:

$$rdJ_{1,3}^{**}\left(\frac{J_{2,3}^{**}}{h}-\frac{J_{2,2}^{**}}{\gamma-1}\right) \neq 0 \Longrightarrow rdJ_{1,3}^{**} \neq 0 \wedge J_{2,3}^{**} \neq \frac{h}{\gamma-1} J_{2,2}^{**}. \tag{A.33}$$

We now focus on the linearized dynamics in the vicinity of of the non feasible set of steady states. Given the restrictions described in section 2.5 the *Jacobian* in the vicinity of $\widetilde{Z}^{*}$, is given by:

$$J^{*}=\left[\begin{array}{ccc} J_{1,1}^{*} & 0 & 0 \\ 1 & J_{2,2}^{*} & J_{2,3}^{*} \\ 0 & \frac{rd}{h} J_{2,3}^{*} & J_{2,2}^{*} \end{array}\right]. \tag{A.34}$$

The characteristic equation for this *Jacobian* comes,

$$\left(J_{1,1}^{*}-\Lambda^{*}\right)\left[\left(J_{2,2}^{*}-\Lambda^{*}\right)^{2}-\frac{rd}{h}\left(J_{2,3}^{*}\right)^{2}\right]=0, \tag{A.35}$$

where Λ^{*} stands for the eigenvalues solving the characteristic polynomial in the vicinity of $\tilde{Z}_i^{*}$. Non-degeneracy condition, $Det\left(J^{*}\right) \neq 0$, impose the following restriction,

$$J_{1,1}^{*}\left[\left(J_{2,2}^{*}\right)^{2}-\frac{rd}{h}\left(J_{2,3}^{*}\right)^{2}\right] \neq 0 \Longrightarrow J_{1,1}^{*} \neq 0 \wedge J_{2,2}^{*} \neq \pm J_{2,3}^{*} \sqrt{\frac{rd}{h}}. \tag{A.36}$$

The solution of the characteristic equation defined in (A.35) is given by:

$$\Lambda^{*}=J_{1,1}^{*} \wedge \Lambda^{*}=J_{2,2}^{*} \pm J_{2,3}^{*} \sqrt{\frac{rd}{h}}. \tag{A.37}$$

A.4 Local bifurcation analysis

In this section, we provide the analytical conditions for the existence of saddle-node and fold-hopf bifurcations, following the discussion in section 2.6. In order to put forward the sufficient conditions for existence of saddle-node bifurcations in this system, we first start by proving that the bifurcation constraint, r_k^{**}, given in (2.67), is consistent with $Det\left(J^{**}\right)=0$. Recall that according to *Viète*'s theorem the product of eigenvalues is given by,

$$\prod_{j=1}^{3} \Lambda_i^{**}=rdJ_{1,3}^{**}\left(\frac{1}{h} J_{2,3}^{**}-\frac{1}{\gamma-1} J_{2,2}^{**}\right), \tag{A.38}$$

where j is the eigenvalue index. Since $rdJ_{1,3}^{**} \neq 0$, we require that the following condition is fulfilled,

$$\frac{1}{\gamma - 1}\left(r + rdZ_2^{**} + \delta - Z_4^{**}\right) = \left(1 + hZ_4^{**} - Z_2^{**}\right)\frac{1}{h}. \tag{A.39}$$

Substituting the equilibrium expression for Z_2^{**}, (2.60), and solving in terms of Z_4^{**}, we confirm that r_k^{**} is consistent with the existence of this singularity, $\Lambda_1^{**} = 0$, and equal to the equilibrium expression for $Z_4^{**}\left(r_k^{**}\right)$, given in (2.68),

$$Z_4^{**} = \frac{-\left(\rho + \delta\gamma + \frac{(\gamma-1)(\rho - r + \delta(\gamma-1))}{hrd} - \frac{\gamma-1}{h}\right)}{-\frac{(\gamma-1)^2}{hrd} - 2\gamma + 1} = Z_4^{**}\left(r_k^{**}\right). \tag{A.40}$$

In three dimensional systems a saddle-node bifurcation occurs if the remaining eigenvalues are of opposite signs, $\Lambda_2^{**} \cdot \Lambda_3^{**} < 0$. Following (A.39), the characteristic equation is now given by:

$$-\Lambda^{**}\left[\left(J_{2,2}^{**} - \Lambda^{**}\right)^2 - \frac{rdh}{\gamma - 1}\left(J_{2,2}^{**}\right)^2 - \frac{rd}{\gamma - 1}J_{1,3}^{**}\right] = 0. \tag{A.41}$$

The eigenvalues at the bifurcation point are thus given by,

$$\Lambda_1^{**} = 0 \wedge \Lambda_{2,3}^{**} = J_{2,2} \pm \sqrt{\frac{rdh}{\gamma - 1}\left(J_{2,2}^{**}\right)^2 + \frac{rd}{\gamma - 1}J_{1,3}^{**}}. \tag{A.42}$$

The existence of a saddle-node bifurcation can be put in terms of parameter d. We have a saddle-node if

$$J_{1,3}^{**} < -h\left(J_{2,2}^{**}\right)^2 \text{ and } d > 0; \text{ or if } J_{1,3}^{**} > -h\left(J_{2,2}^{**}\right)^2 \text{ and } d < 0. \tag{A.43}$$

To define analytically the fold-hopf bifurcation point, it is convenient to continue equilibrium from the saddle node bifurcation defined by r_k^{**}. Continuing equilibrium from this point a fold-hopf bifurcation is guaranteed to exist if $J_{2,2}^{**}\left(r_k^{**}\right) = 0 \wedge J_{2,3}^{**}\left(r_k^{**}\right) = 0$, and provided that $d < 0$. At this singular point we have a zero eigenvalue, $\Lambda_1^{**} = 0$, and the remaining eigenvalues, $\Lambda_{2,3}^{**}$, are given by a pure imaginary conjugate pair, following the result in (A.42). The expression for the non negative eigenvalues is given by,

$$\Lambda_{2,3}^{**} = \pm\sqrt{\frac{rd}{\gamma - 1}J_{1,3}^{**}}, \quad \frac{rd}{\gamma - 1}J_{1,3}^{**} < 0. \tag{A.44}$$

To obtain the parameter constraint required for the existence of a codimension two fold-hopf bifurcation, we have to solve the system given by $J_{2,2}^{**}\left(r_k^{**}\right)=0 \wedge J_{2,3}\left(r_k^{**}\right)=0$. Substituting we obtain,

$$Z_4^{**}\left(r_k^{**}\right)=\frac{r+\delta+rd}{1-hrd} \wedge Z_2^{**}\left(r_k^{**}\right)=1+hZ_4^{**}\left(r_k^{**}\right). \tag{A.45}$$

Substituting (A.45) in (A.40), we obtain the second parameter condition, in terms of parameter ρ. This condition is given by the following expression,

$$\begin{aligned}\rho^{**}=&-\left(\frac{hrd}{\gamma-1+hrd}\right)\left[\delta\gamma+\frac{(\gamma-1)\left(\delta\left(\gamma-1\right)-r\right)}{hrd}-\frac{\gamma-1}{h}\right.\\&\left.-\left(\frac{r+\delta+rd}{1-hrd}\right)\left(-\frac{(\gamma-1)^2}{hrd}-2\gamma+1\right)\right].\end{aligned} \tag{A.46}$$

A.5 Geometric analysis of the quadric nullcline surfaces

To classify the nullclines described by the quadrics, $\dot{Z}_2\left(t\right),\dot{Z}_4\left(t\right)=0$, it is first convenient to redefine these surfaces as a matrix product, $\chi^T\Sigma_i\chi^T$, where $\chi=\left[Z_2\left(t\right),Z_4\left(t\right),Z_1\left(t\right),1\right]$. The matrix Σ_2 for the nullcline $\dot{Z}_2\left(t\right)=0$ and its upper sub-matrix, $\Sigma_2^{'}$, are given in general terms, following the notation in A.2, by:

$$\Sigma_2=\begin{bmatrix} A & \frac{B}{2} & 0 & \frac{D}{2}\\ \frac{B}{2} & C & 0 & \frac{E}{2}\\ 0 & 0 & 0 & \frac{1}{2}\\ \frac{D}{2} & \frac{E}{2} & \frac{1}{2} & F\end{bmatrix} \text{ and } \Sigma_2^{'}=\begin{bmatrix} A & \frac{B}{2} & 0\\ \frac{B}{2} & C & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{A.47}$$

For the nullcline $\dot{Z}_4\left(t\right)=0$, Σ_4 and $\Sigma_4^{'}$, are given by,

$$\Sigma_4=\begin{bmatrix} G & \frac{H}{2} & 0 & \frac{J}{2}\\ \frac{H}{2} & I & 0 & \frac{K}{2}\\ 0 & 0 & 0 & 0\\ \frac{J}{2} & \frac{K}{2} & 0 & L\end{bmatrix} \text{ and } \Sigma_4^{'}=\begin{bmatrix} G & \frac{H}{2} & 0\\ \frac{H}{2} & I & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{A.48}$$

To define these geometric surfaces we need to define the following quantities: (i) $\det\left(\Sigma\right)$, $\det\left(\Sigma^{'}\right)$; (ii) $\operatorname{rank}\left(\Sigma\right)$, $\operatorname{rank}\left(\Sigma^{'}\right)$; and (iii) $\det\left(\Sigma^{'}-\pi I\right)=0$. Where π is the eigenvalue solution to the characteristic equation of $\Sigma^{'}$ and I the identity matrix. For the first quadric surface, $\dot{Z}_2\left(t\right)=0$, we obtain the following quantities. When $hrd\neq 1$, $\det\left(\Sigma_2\right)=\left(1-hrd\right)/16$ and the matrix has full rank, $\operatorname{rank}\left(\Sigma_2\right)=4$, while $\det\left(\Sigma_2^{'}\right)=0$ and $\operatorname{rank}\left(\Sigma_2^{'}\right)=2$. When $hrd=1$, $\det\left(\Sigma_2\right)=0$ and

$\text{rank}\left(\Sigma_2\right) = 2$, while $\text{rank}\left(\Sigma_2^{'}\right) = 1$. The surface is a parabolic cylinder when $hrd = 1$. Now we need to evaluate the signs of the non negative eigenvalue solutions, π_2, for $\Sigma_2^{'}$. These are given by,

$$\pi_2 = 0 \vee \pi_2 = \frac{-\frac{1}{2}\left(rd + h\right) \pm \sqrt{\frac{1}{4}\left(rd + h\right)^2 + 1 - rdh}}{-2}. \tag{A.49}$$

When $hrd > 1$, we have eigenvalues with the same signs. The quadric $\dot{Z}_2\left(t\right) = 0$ is: (i) an elliptic paraboloid when $hrd > 1$; and (ii) a hyperbolic paraboloid when $hrd < 1$. For the second quadric, $\dot{Z}_4\left(t\right) = 0$, we obtain the following quantities. When $hrd \neq 1$, $\det\left(\Sigma_4\right) = 0$ and $\text{rank}\left(\Sigma_4\right) = 3$, while $\det\left(\Sigma_4^{'}\right) = 0$ and $\text{rank}\left(\Sigma_4^{'}\right) = 2$. When $hrd = 1$, $\text{rank}\left(\Sigma_4\right) = 3$ and $\text{rank}\left(\Sigma_4^{'}\right) = 1$, the surface is a parabolic cylinder. Now we need to evaluate the eigenvalues, π_4, of $\Sigma_4^{'}$. These are given by the following expressions,

$$\pi_4 = 0 \vee \pi_4 = \frac{\frac{1}{2}\left(\frac{rd}{h} + 1\right) \pm \sqrt{\frac{1}{4}\left(\frac{rd}{h} + 1\right)^2 + rd\left(rd - \frac{1}{h}\right)}}{-2}. \tag{A.50}$$

When $0 < rdh < 1$, the quadric, $\dot{Z}_4\left(t\right) = 0$, is an elliptic cylinder[3]. When $hrd < 0 \wedge hrd > 1$, the quadric, $\dot{Z}_4\left(t\right) = 0$, is a hyperbolic cylinder.

[3]To determine if this elliptic cylinder is real or imaginary the eigenvalues of Σ_4 have to be checked. If the non negative eigenvalues have opposite signs then we have an real elliptic cylinder.

Appendix B

Appendix to chapter 2

B.1 Optimal control conditions

B.1.1 Optimal control conditions for the non-cooperative game with coupled institutional risk premium

The current value *Hamiltonian* for the non-cooperative game in (3.15) is:

$$\begin{aligned} &H\left[B_i\left(t\right), K_i\left(t\right), B\left(t\right), K\left(t\right), \lambda_i\left(t\right), q_i\left(t\right), C_i\left(t\right), I_i\left(t\right)\right]^* = \\ &= C_i\left(t\right)^{\gamma_i} + \lambda_i\left(t\right)\dot{B}_i\left(t\right) + q_i\left(t\right)\dot{K}_i\left(t\right), \end{aligned} \tag{B.1}$$

where $\dot{B}_i\left(t\right)$ and $\dot{K}_i\left(t\right)$ are given in (B.6) and (B.7). The general *Pontryagin* maximum conditions for the existence of optimal open loop solution are given by:

Optimality conditions

$$\gamma_i C_i\left(t\right)^{\gamma_i - 1} = -\lambda_i\left(t\right); \tag{B.2}$$

$$q_i\left(t\right) = -\lambda_i\left(t\right); \tag{B.3}$$

Multiplier conditions

$$\dot{\lambda}_i\left(t\right) = \lambda_i\left(t\right)\left(\rho_i - r - rd_i\frac{B\left(t\right)}{K\left(t\right)}\right); \tag{B.4}$$

$$\dot{q}_i\left(t\right) = q_i\left(t\right)\left(\rho_i + \delta\right) + \lambda_i\left(t\right)\left(r_k\right); \tag{B.5}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + d_i \frac{B(t)}{K(t)}\right) - r_k K_i(t)\,; \tag{B.6}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t)\,; \tag{B.7}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda_i(t)\, B_i(t)\, e^{-\rho_i t} = 0; \tag{B.8}$$

$$\lim_{t\to\infty} q_i(t)\, K_i(t)\, e^{-\rho_i t} = 0; \tag{B.9}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0)\,, K_{i,0}(t) = K_i(0)\,. \tag{B.10}$$

B.1.2 Optimal control conditions for the non-cooperative game with coupled endogenous risk premium

The current value *Hamiltonian* for the non-cooperative game in (3.45) is:

$$\begin{aligned} &H\left[B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)\right]^* = \\ &= C_i(t)^{\gamma_i} + \lambda_i(t)\,\dot{B}_i(t) + q_i(t)\,\dot{K}_i(t), \end{aligned} \tag{B.11}$$

where $\dot{B}_i(t)$ and $\dot{K}_i(t)$ are given in (B.16) and (B.17). The general *Pontryagin* maximum conditions for the existence of optimal open loop solution are given by:

Optimality conditions

$$\gamma_i C_i(t)^{\gamma_i - 1} = -\lambda_i(t)\,; \tag{B.12}$$

$$q_i(t) = -\lambda_i(t)\,; \tag{B.13}$$

Multiplier conditions

$$\dot{\lambda}_i(t) = \lambda_i(t)\left(\rho_i - r - 2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right); \tag{B.14}$$

$$\dot{q}_i(t) = q_i(t)(\rho_i + \delta) + \lambda_i(t)\left(r\frac{B_i(t)^2}{K_i(t)^2}\frac{B(t)}{K(t)} + r_k\right); \tag{B.15}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t) + rB_i(t)\left(1 + \frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right) - r_k K_i(t); \tag{B.16}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \tag{B.17}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \tag{B.18}$$

$$\lim_{t\to\infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \tag{B.19}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \tag{B.20}$$

B.2 Qualitative analysis of state-separable systems

B.2.1 Qualitative analysis for the non-cooperative game with coupled institutional risk premium

The state-separable solution to the game defined in (3.15), assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given by the system defined by scaled consumption, (3.22), scaled net financial assets, (3.23), and productive capital dynamics, (B.7), after substituting (3.20) in (3.22) and (3.23). The dynamics assuming state-separability are given by:

$$\dot{Z}_{1,i}(t) = Z_{1,i}(t)\left[\frac{\rho_i + \delta - r_k - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)}{\gamma_i - 1}\right]; \tag{B.21}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + \bar{Z}_{4,i} + Z_{2,i}(t)\left[2r + 2\delta - r_k - \bar{Z}_{4,i}\right] - r_k; \tag{B.22}$$

$$K_i(t) = K_i(0) e^{\left(\bar{Z}_{4,i} - \delta\right)t}. \tag{B.23}$$

The steady states consistent with a feasible solution, $\bar{Z}_1 > 0, \forall t \in T$, are defined by the following expressions:

$$\bar{Z}_{4,i} = \frac{\rho_i + \delta - r_k}{\gamma_i - 1} + \delta; \tag{B.24}$$

$$\bar{Z}_{2,i} = \frac{r_k - \bar{Z}_{4,i} - Z_{1,i}(0)}{2r + 2\delta - r_k - \bar{Z}_{4,i}}. \tag{B.25}$$

Given that agents commit to an initial investment strategy consistent with equilibrium for (B.22), the dynamics in the vicinity of (B.24) and (B.25), are described by the dynamics of net financial assets. Qualitative dynamics are thus defined by,

$$\frac{\partial Z_{2,i}(t)}{\partial Z2, i(t)} = 2r + 2\delta - r_k - \bar{Z}_{4,i}. \tag{B.26}$$

B.2.2 Qualitative analysis for the non-cooperative game with coupled endogenous risk premium

The solution to the state-separable optimal control problem defined in (3.45), assuming $Z_{4,i}(t) = \bar{Z}_{4,i}, \forall t \in T$, is given by the system defined by scaled consumption, (3.49), scaled net financial assets, (3.50), and productive capital dynamics, (B.17), after substituting (3.48) in (3.49) and (3.50). The dynamics assuming state-separability are given by:

$$\dot{Z}_{1,i}(t) = \frac{Z_{1,i}(t)}{\gamma_i - 1}\left[\rho_i - r - \frac{2r(r + \delta - r_k)}{(Z_{2,i}(t) - 2)} - (\gamma_i - 1)(Z_{4,i}(t) - \delta)\right]; \tag{B.27}$$

$$\dot{Z}_{2,i}(t) = Z_{1,i}(t) + \bar{Z}_{4,i} + Z_{2,i}(t)\left[r + \frac{(r + \delta - r_k)}{Z_{2,i}(t) - 2} - \bar{Z}_{4,i} + \delta\right] - r_k; \tag{B.28}$$

$$K_i(t) = K_i(0)\, e^{\left(\bar{Z}_{4,i} - \delta\right)t}. \tag{B.29}$$

The steady states consistent with a feasible solution, $Z_1(t) > 0, \forall t \in T$, are defined by the following expressions:

$$\bar{Z}_{1,i} = r_k - \bar{Z}_{4,i} - \left(\frac{2(r+\delta-r_k)}{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)} + 2\right)\left(\frac{r + \rho_i - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i + 1)}{2}\right); \tag{B.30}$$

$$\bar{Z}_{2,i} = \frac{2(r + \delta - r_k)}{\rho_i - r - \left(\bar{Z}_{4,i} - \delta\right)(\gamma_i - 1)} + 2. \tag{B.31}$$

Qualitative dynamics in the vicinity of (B.30) and (B.31), are defined by the following *Jacobian* matrix[1],

$$J = \begin{bmatrix} 0 & \frac{2\bar{Z}_{1,i}r(r+\delta-r_k)}{\left(\bar{Z}_{2,i}-2\right)^2(\gamma_i-1)} \\ 1 & r-\bar{Z}_{4,i}+\delta-2\frac{(r+\delta-r_k)}{\left(\bar{Z}_{2,i}-2\right)^2} \end{bmatrix}. \tag{B.32}$$

Following the usual conditions for qualitative dynamics in hyperbolic autonomous planar systems[2]. We define the main qualitative features of this solution based on the general eigenvalue solution, Λ, to the characteristic equation, $\det(J-\Lambda I)=0$[3], of the *Jacobian* defined in (B.32):

$$\Lambda = \frac{-tr\,(J) \pm \sqrt{tr\,(J)^2 - 4\det(J)}}{2}, \tag{B.33}$$

where $\det(J) = -J_{1,2}$ and $tr\,(J) = J_{2,2}$. Substituting the steady state expressions, (B.30) and (B.31), the main qualitative dynamic features of the state-separability in the vicinity of equilibrium are the following: (i) saddle point when $r_k > r+\delta$; (ii) node when $r_k < r+\delta \wedge \bar{Z}_{4,i} > \Omega$; (iii) repellor when $r_k < r+\delta \wedge \bar{Z}_{4,i} < \Omega$; and (iv) at $\bar{Z}_{4,i} = \Omega$ we have an *Hopf* bifurcation when $r_k < r+\delta$, where

$$\Omega = r+\delta-2\frac{(r+\delta-r_k)}{\left(\bar{Z}_{2,i}-2\right)^2}. \tag{B.34}$$

[1] The *Jacobian* is non-degenerate when $J_{1,2}\left(\bar{Z}_{m,i}\right) \neq 0$.

[2] Recall that in planar systems we have: (i) an attractor when $\det(J) > 0$ and $Tr\,(J) < 0$; (ii) a saddle point when $\det(J) < 0$; and (iii) a repelling solution when $\det(J) > 0$ and $Tr\,(J) > 0$.

[3] Where I is the identity matrix.

Appendix C

Appendix to chapter 3

C.1 Optimal control conditions

C.1.1 Optimal control conditions for the non-cooperative game with investment bias and coupled institutional risk premium

The current value *Hamiltonian* for the non-cooperative game in (4.8) is:

$$\begin{aligned} H\left[B_i\left(t\right), K_i\left(t\right), B\left(t\right), K\left(t\right), \lambda_i\left(t\right), q_i\left(t\right), C_i\left(t\right), I_i\left(t\right)\right]^* = \\ = C_i\left(t\right)^{\gamma_i} + \lambda_i\left(t\right)\dot{B}_i\left(t\right) + q_i\left(t\right)\dot{K}_i\left(t\right), \end{aligned} \tag{C.1}$$

where $\dot{B}_i\left(t\right)$ and $\dot{K}_i\left(t\right)$ are given in (C.6) and (C.7). The general *Pontryagin* maximum conditions for the existence of optimal open loop solutions are given by:

Optimality conditions

$$\gamma_i C_i\left(t\right)^{\gamma_i - 1} = -\lambda_i\left(t\right); \tag{C.2}$$

$$q_i\left(t\right) = -\lambda_i\left(t\right)\left(1 + 2h_i \frac{I_{i(t)}}{K_i\left(t\right)}\right); \tag{C.3}$$

Multiplier conditions

$$\dot{\lambda}_i\left(t\right) = \lambda_i\left(t\right)\left(\rho_i - r - rd_i \frac{B\left(t\right)}{K\left(t\right)}\right); \tag{C.4}$$

$$\dot{q}_i\left(t\right) = q_i\left(t\right)\left(\rho_i + \delta\right) + \lambda_i\left(t\right)\left(h_i \frac{I_i\left(t\right)^2}{K_i\left(t\right)^2} + r_k\right); \tag{C.5}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t)\left(1 + h_i \frac{I_i(t)}{K_i(t)}\right) + rB_i(t)\left(1 + d_i \frac{B(t)}{K(t)}\right) - r_k K_i(t); \quad \text{(C.6)}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \quad \text{(C.7)}$$

Transversality conditions

$$\lim_{t \to \infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \quad \text{(C.8)}$$

$$\lim_{t \to \infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \quad \text{(C.9)}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \quad \text{(C.10)}$$

C.1.2 Optimal control conditions for the non-cooperative game with investment bias and coupled endogenous risk premium

The current value *Hamiltonian* for the non-cooperative game in (4.33) is:

$$\begin{gathered} H\left[B_i(t), K_i(t), B(t), K(t), \lambda_i(t), q_i(t), C_i(t), I_i(t)\right]^* = \\ = C_i(t)^{\gamma_i} + \lambda_i(t)\dot{B}_i(t) + q_i(t)\dot{K}_i(t), \end{gathered} \quad \text{(C.11)}$$

where $\dot{B}_i(t)$ and $\dot{K}_i(t)$ are given in (C.16) and (C.17). The general *Pontryagin* maximum conditions for the existence of optimal open loop solutions are given by:

Optimality conditions

$$\gamma_i C_i(t)^{\gamma_i - 1} = -\lambda_i(t); \quad \text{(C.12)}$$

$$q_i(t) = -\lambda_i(t)\left(1 + 2h_i \frac{I_{i(t)}}{K_i(t)}\right); \quad \text{(C.13)}$$

Multiplier conditions

$$\dot{\lambda}_i(t) = \lambda_i(t)\left(\rho_i - r - 2r\frac{B_i(t)}{K_i(t)}\frac{B(t)}{K(t)}\right); \quad \text{(C.14)}$$

$$\dot{q}_i(t) = q_i(t)(\rho_i + \delta) + \lambda_i(t)\left(h_i \frac{I_i(t)^2}{K_i(t)^2} + r\frac{B_i(t)^2}{K_i(t)^2}\frac{B(t)}{K(t)} + r_k\right); \tag{C.15}$$

State conditions

$$\dot{B}_i(t) = C_i(t) + I_i(t)\left(1 + \tfrac{h_i I_i(t)}{K_i(t)}\right) + rB_i(t)\left(1 + \tfrac{B_i(t)B(t)}{K_i(t)K(t)}\right) - r_k K_i(t); \tag{C.16}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t); \tag{C.17}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) B_i(t) e^{-\rho_i t} = 0; \tag{C.18}$$

$$\lim_{t\to\infty} q_i(t) K_i(t) e^{-\rho_i t} = 0; \tag{C.19}$$

Admissibility conditions

$$B_{i,0}(t) = B_i(0), K_{i,0}(t) = K_i(0). \tag{C.20}$$

C.2 General *Ricatti* equation solution

The general *Ricatti* equation of interest is defined by the following first order ordinary differential equation,

$$\frac{\partial \chi(t)}{\partial t} = \chi(t)^2 + \frac{b}{a}\chi(t) + \frac{c}{a}, \tag{C.21}$$

where $a/b < 0$, $b^2 - 4ac > 0$. To solve the above *Ricatti* equation it is convenient to start by dividing everything by the right hand side expression and then take integrals with respect to time. Equation (C.21) is now given by:

$$\int_0^t \frac{\frac{\partial \chi(s)}{\partial s}}{\chi(s)^2 + ba^{-1}\chi(s) + ca^{-1}} ds = \int_0^t 1 ds. \tag{C.22}$$

The right hand side integral is given by $t + o_1$, where o_1 is a constant of integration. We can now focus on the solution of the left hand side integral. We start by simplifying this integral by assuming $u = \chi(s) \Rightarrow du = \frac{\partial \chi(s)}{\partial s} ds$,

$$\int_0^t \frac{1}{u^2 + ba^{-1}u + ca^{-1}} du. \tag{C.23}$$

It is possible to obtain a solvable integral expression to (C.23) by factoring out constants through substitution. First we complete the square in (C.23),

$$\int_0^t \frac{1}{\frac{1}{4a^2}\left(4ac - b^2\right) + \left(u + \frac{b}{2a}\right)^2} du. \tag{C.24}$$

The next substitution is straightforward. We consider, $p = u + \frac{b}{2a} \Rightarrow dp = du$, and (C.24) after rearranging comes,

$$\int_0^t \frac{1}{\frac{1}{4a^2}\left(4ac - b^2\right) + p^2} dp = \int_0^t \frac{-\frac{4a^2}{b^2-4ac}}{-\frac{4a^2}{b^2-4ac}p^2 + 1} dp. \tag{C.25}$$

Factoring out constants and then setting $y = -2a\left(b^2 - 4ac\right)^{-\frac{1}{2}} p$, where $dy = -2a\left(b^2 - 4ac\right)^{-\frac{1}{2}} dp$, we obtain,

$$\frac{2a}{\sqrt{b^2 - 4ac}} \int_0^t \frac{1}{1 - y^2} dy. \tag{C.26}$$

Since $\int 1/\left(1 - y^2\right) dy = \operatorname{arctanh}(y)$, we substitute everything back and obtain the solution to (C.22),

$$\frac{2a}{\sqrt{b^2 - 4ac}} \operatorname{arctanh}\left(-\frac{2a\chi(t) + b}{\sqrt{b^2 - 4ac}}\right) = t + o, \quad \text{where } o = 0_1 - o_2. \tag{C.27}$$

Taking tanh from both sides the general solution to (C.21) is given by,

$$\chi(t) = -\frac{\sqrt{b^2 - 4ac}\tanh\left(\frac{1}{2a}(t + o)\sqrt{b^2 - 4ac}\right) + b}{2a}. \tag{C.28}$$

Now we consider the existence of an initial value, $\chi(0)$. Setting $t = 0$ in (C.28), taking arctanh on both sides and solving in terms of o we obtain:

$$o = \frac{2a}{\sqrt{b^2 - 4ac}} \operatorname{arctanh}\left[-\frac{(2a\chi(0) + b)}{\sqrt{b^2 - 4ac}}\right]. \tag{C.29}$$

Substituting (C.29) in (C.28), we obtain the general explicit solution to (C.21),

$$\chi(t) = -\frac{\sqrt{b^2 - 4ac}\tanh\left\{\frac{\sqrt{b^2-4ac}}{2a}\left[t + \frac{2a}{\sqrt{b^2-4ac}}\operatorname{arctanh}\left(-\frac{2a\chi(0)+b}{\sqrt{b^2-4ac}}\right)\right]\right\} + b}{2a}. \tag{C.30}$$

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