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The Inefficiency Of Bank Modules As A Containment Response To Financial Contagion:

A Benchmark Result Derived Using A Partition Approach

by

Paul Youdell

A thesis submitted in fulfillment of the requirement for the degree of Doctor of Philosophy in Economics, University of Warwick, Department of Economics

19th November 2013

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0.1 Abbreviations

$\mathbf{A_l}$ $:= \{1, 2, 3.....l\}$

$\mathbb{N}$ $:= \{1, 2, 3.....\}$, the set of natural numbers starting with 1

$\mathbb{N}_k$ $:= \{k, k+1, k+2.....\}$, the set of natural numbers starting with k

CFG Characteristic Form Game

EBA Equilibrium Binding Agreement

EEBA Extended Equilibrium Binding Agreement

ICB Independent Commission on Banking

IMF International Monetary Fund

PFG Partition Form Game

P.O. Pareto Optimum

SME Small and Medium Enterprises

Acknowledgements

My supervisors Professors Sayantan Ghosal and Robert MacKay; Dr Ana Babus for the invite to attend the 2010 CFAP Financial Networks conference at Cambridge and the matching approach Babus (2010); Professor Peter Hammond who provided feedback on the mathematical methodology; I benefited from conversations with a number of visiting academics, in chronological order: Asher Wolinsky, Philip Reny, Francis Bloch, Guillermo Owen, Lord Robert May; attendees of the 2010 Warwick Economics PhD Poster session, including Dr Michael McMahon and Dr Michael Pitt, who both at an early stage seemed to think I was onto something; Andrés Carvajal for being generous with his time and energy in support of all the Warwick PhD economics students, and in particular for appointing me a research fellow of CRETA; Professors Chuck Blackorby and Kim Scharf for supporting my acceptance to the Warwick PhD Economics programme; many other Warwick Economics faculty including Professor Marcus Miller, Dr Lei Zhang, Dr Daniel Sgroi, and Professor Herakles Polemarchakis; Angela Rojas Rivera who saw both an early and a late version of the thesis; members of the Warwick PhD Reading Group who saw an even earlier version of this work; Guillaume Sublet for being a friend in the finest Proustian tradition: generous with his encouragement; fair in his criticism; Darina Dintcheva-Bis for talking with me about partition form games; all my contacts at the Warwick Complexity Centre, especially those on the 2007-2008 Complexity Msc course and those who listened to me present, in particular James Porter and Anthony Woolcock; Peng Sui for our conversations about financial networks; everyone from a financial regulation background who took an interest in my work, including Nicholas Beale at Sciteb and Peter Edmonds at the FSA; everyone I taught, including a special mention to all the 2010-2011 PhD Microeconomics students for keeping me honest in the classes, and giving me many wonderful experiences and memories: Shubhasish Barua, Moustafa Chatzouz, Lucio D'aguanno, Rocco D'este, Fei Ding, Monica Giovanniello, Jardine Husman, Christoph Koenig, Tony Liu, Giulia Lotti, Rigissa Megalokonomou, Andreas Menzel, Dimitri Migrow, Choong Nam, Quang Phan, Sukaina Riaz, David Ronayne, Mario Sanclemente Villegas, Cagri Savci, Andrew Siu, Xinxi Song, Simone Tonin, Giulio Trigilia, and Vaclav Zdarek; Erez Yerushalmi for introducing me to LyX; everyone who attended my 2012 RES Job Market presentation; everyone who attended one of the Warwick lunchtime presentations I gave; Amrita Sen for being my best friend, encouraging me, supporting me, feeding me, and listening to me; my parents, Betty and John Youdell, for always believing in me, especially when they did not understand; and finally, the investment banker friend who described himself as a financial estate agent. Thanks to the ESRC for PhD Fellowship funding 2007-10, and to the Warwick Economics department for a Teaching Fellowship 2010-2011.

Declaration

I hereby declare that this thesis is my own work, and none of the material contained in it has been used before in other research project or publication. Likewise, I confirm that this thesis has not been submitted for a degree at another university.

Abstract

Following the recent international financial crisis, a number of policy proposals have been made: one of which is the partitioning of banks into modules (groups), to contain financial shocks. The firewalls, which surround modules, prevent financial contagion: when a shock hits a bank it spreads to other banks in the same module, but not to banks in other modules. Conditional on bank modules avoiding shocks, businesses can achieve their latent business opportunities. The optimal banking system has a cost-benefit trade off: increased module size allows for more lucrative business opportunities, but increases systemic banking risk. This thesis, using a theoretical approach, assesses the importance of the distribution of business opportunities when using modules. When the distribution is uniform, the optimal structure of the banking industry is fully characterised: it surprisingly takes only two forms, either one all-encompassing module (containing all the banks), or atomistic modules (each module contains only one bank). The intuition behind this sharp characterisation is the increasing marginal returns that modules have on social welfare. A counter-example is constructed where, with a non-uniform matching of business opportunities, conversely, the efficient solution does have multiple modules each containing multiple banks. The model's policy recommendation is that the banking system needs to be designed in accordance with the financial requirements of businesses.

Keywords: Financial Regulation, Financial Stability, Network, Markov Chain, Contagion, Bank Run

JEL Classification: E44, D85, G01, G20, G21

Chapter 1

The Standard Model

1.1 Summary

This chapter starts by considering financial contagion: what it is, and how it has been modelled, both pre-crisis (customer liquidity shocks, see Allen and Gale (2000)), and post crisis (random networks of banks, where an exogenous negative shock disables a single bank, and the shock propagation is then calculated computationally, see Gai and Kapadia (2010)). It then considers the Haldane (2010) policy response: the partitioning of bank networks into modules (separate groups). It argues that banking modules need to be formed ex-ante, not ex-post (before shocks arrive, not afterwards), unlike for example Stiglitz (2010b).

My model uses cost-benefit analysis in the form of a welfare function to consider the optimality of bank modularisation. A Markov process is formed from the stochastic disabling banking shocks, and re-enabling module recoveries. As in Gintis (2012), such systems generically converge rapidly to a unique stationary distribution. This limiting state of the system gives the asymptotic probability that each module is enabled. These probabilities then micro-found an ex-ante welfare specification for every partition.

This chapter formulates a standard model under the assumption of a uniform distribution of business opportunities. The main result is that the use of modules is rejected, for all parametrisations. Specifically, any proper partition, (with both multiple modules and multiple banks per module), is inefficient. Hence, the efficient partition is either the grand coalition, (one big module containing all the banks), or the atomistic partition (each bank in its own separate singleton module).

The four step outline of the proof, which rejects not just symmetric interior partitions but also non-symmetric interior partitions, is as follows. First the bank one utility maximisation programme is considered where, without loss of generality, bank one is in module one. Second it is proven that the model has negative externalities: a merger between modules two and three always makes bank one worse off. This implies that the bank one argmax is of the form $\{x_1, 1, 1, 1, ..1\}$. Third it is proven that the bank one utility function, $v_1[x_1]$, has quasi-convexity, and hence that the bank one optimal partition is on the boundary. Fourth, by symmetry, this boundary partition is also welfare maximising.

This chapter then considers partition formation, and it shows that if agents are farsighted, then the Extended Equilibrium Binding Agreement (EEBA), from Diamantoudi and Xue (2007), results in the efficient partition being formed (Chapter 2 considers other partition formation solution concepts in Section 2.3.3). Finally, for realistic parametrisations, Section 1.9 argues that it is the atom partition that is efficient rather than the grand coalition.

1.2 Introduction

The 2007-09 financial crisis initially manifested itself in one place; one market: the US sub-prime mortgage market. However, the shock was contagious, and spread to affect banks world wide. This resulted in a number of major banks either collapsing, being bailed out or being nationalised. This included US investment banks, such as Lehman Brothers and Bear Stearns, and UK retail banks, such as Northern Rock, Royal Bank of Scotland (RBS), and Halifax Bank of Scotland (HBOS). However, banks worldwide were affected, and they dramatically reduced their lending: due to a lack of confidence and in order to rebuild capital buffers. The resulting credit crunch meant that businesses could no longer borrow, and hence there was a large opportunity cost from wasted business opportunities. Due to the large social cost of the crisis there is a strong political sense that, 'there must be a better way', and hence there is a search for policy responses: for example, in the UK by the Independent Commission on Banking (ICB). In this introduction, I will examine firstly, contagion (both non-financial and financial), and then secondly the policy response of grouping banks into modules.

The concept of contagion comes from epidemiology: the study of disease transmission. If Albert has a disease, and he has contact with Bill, then Bill in turn is infected.[1] This process is repeated, and there is a cascade effect, so that the disease is spread across the whole population. Similarly, financial contagion occurs when a negative shock hits one bank in a network, and that shock then spreads system wide.[2] During the crisis there are many factors that enabled the financial contagion to occur, these include: high levels of bank leverage, the complexity of the derivative market, and banker misbehaviour. However, one critical factor is high bank inter-connectedness: this thesis focuses on that factor.

[1]This assumes the disease is completely contagious. In a more complex model, the diffusion occurs probabilistically.

[2]The term contagion has been used in multiple contexts in economics. For example co-movements in stock markets. Here I am using the term restricted to banks in-line with the original definition (see Bagehot (1873)). See Moser (2003) for a survey of the use of contagion in economics.

Pre-crisis the main model of financial contagion was the "early-late" consumer model of Allen and Gale (2000). However, their model is focused on explaining how financial contagion can happen, rather than assessing policy responses; has shocks to consumer liquidity demand rather than banks assets; and is restricted only to very small networks: those with four banks. Further, the crisis has cast doubt on a number of the model's conclusions: for example, that the complete network (where every bank lends to every other bank), is the most stable.

Post-crisis a second stream of literature has emerged, for example Gai and Kapadia (2007), that has focused on the *mechanisms* which propagate shocks between banks. The criticism of the mechanism approach is that these are positive, rather than normative, models: the only agents are banks; there are no businesses (or consumers). So, it studies negative externalities between banks, but not of banks on businesses, and hence it does not model the effect of bank failures on the real economy. In contrast, this thesis develops a model containing businesses and derives a micro-founded welfare function which leads to the ability to assess efficiency.

Unlike Allen and Gale (2000), whose "central aim .. is to provide some microeconomic foundations for financial contagion", my aim is to take things to the next stage: take the existence of financial contagion as given, and assess a potential policy response. There are two main types of policy response: the first is to reduce the probability of an initial bank failure, for example through higher capital requirements, or ring fencing (separating retail and investment divisions of banks); the second is to reduce the probability of propagation between banks. My model assesses such a containment proposal from Haldane (2009, 2010): the use of modules to partition banks, so that banks within the same module are connected, but banks in different modules are not connected.

So, a banking module is a group of banks that are close together: in good states of the world, banks in the same module can do business at low cost; but in bad states, one financial shock disables all the banks in a module. An historical example of banking modules comes from the USA. In the USA, pre World War One, "the US system was based on unit banking - geographically isolated single-office banks" (Calomiris, 2010). Post World War One state banks, (which are licensed under state law), faced increasing competition from national banks, (which are licensed under federal law). However, the 1927 McFadden law, partitioned the USA into 49 modules: it confirmed that national banks were allowed branches, but restricted them to operating only in the state of their headquarters.[3] In 1956 this law was replaced by the Douglas Amendment to the 1956 Bank Holding Company (HC) Act, which let individual states decide whether to allow out of state banks to operate in their state. However, "until 1978, every state in the union barred banks from other states, so instead of one national banking system, we had more like 50 little banking systems, one per state", but "By 1992, all states but Hawaii had passed reciprocal entry laws of some sort." (Morgan, Rime and Strahan, 2004).[4] Further, the financial system became more internationalised through an interconnected network of bank lending (for example, EU banks holding sub-prime debt issued by American banks), and so, by the time of the 2007 crisis, the banking system was one all-encompassing module. However, despite this, my thesis is that the risk of financial contagion is not a sufficient argument for partitioning banks into (proper) modules.

In different contexts modules can be formed at different times: *ex-post* (after a shock hits), or *ex-ante* (only before a shock hits). The idea of modules in disease control is that we separate uninfected people from those who are infected. This is the concept of quarantine: we isolate infected countries, for example,

[3]There are currently 50 states plus Washington, District of Columbia; however, Alaska and Hawaii did not achieve statehood until 1959.

[4]"In 1978 Maine passed a law allowing entry by bank holding companies from any state that allowed entry by Maine banks." (Morgan, Rime and Strahan, 2004)

by cancelling transport links ex-post. However, in contrast, modules in some other environments need to be created ex-ante: there are firebreaks in forests to stop the spread of fire; and bulkheads in ships to form separate watertight compartments, so that one breach of the hull does not sink the ship. But these modules need to be ex-ante: we cannot wait until there is a fire before forming a firebreak (bulldozers move slowly; fire spreads fast); and we cannot install bulkheads in a ship that is sinking.[5]

Similarly, it is my contention that banking is such an environment, where modules need to be created ex-ante, not ex-post. In contrast, it is arguably feasible for *countries* to employ "circuit breakers": state contingent capital controls (see Stiglitz 2010b).[6] This is because governments determine their own laws, and there is a divide between national and international investors.[7] So, after such a policy change, there can remain a functioning internal economy. Whilst with banks, taking a topical example, suppose the regulator lets European banks hold American bonds as collateral, but then gets a signal that American debt is going bad. So the regulator decides to partition European from American banks, and orders them to get a new asset base. In order to buy new assets, European banks will need to sell off their American bonds. However, the price of American bonds will already have been lowered by the negative signal, and will be further decreased by the rush of European banks to sell. This fire sale means they will be unable to afford new assets, and thus will need re-financing: the same requirement as without the ex-post partitioning. Hence, I will be modelling modules

[5]The Internal Examiner has noted that "Actually, one way to fight a large forest fire is precisely to bulldoze firebreaks after the fire has broken out, but far enough away and soon enough so that they will indeed stop the fire spreading beyond them. One also 'fights with fire' by deliberately lighting minor back fires intended to remove fuel that would otherwise burn in the major fire.". This spoils the analogy somewhat!

[6]Stiglitz(2010b) does not have productive inter-country links. This may affect his results.

[7]The clarity of this dividing line may be unclear, for example in the case of the EU. But that is argument for ex-ante country modules; rather than an argument for ex-post banking modules.

in banking as being ex-ante, and further I will be considering ex-ante efficiency rather than interim efficiency or ex-post efficiency. Specifically, I am additionally assuming that, for reasons either of feasibility or preference, after the arrival of a disabling shock there is no change in the partitioning of the banks that remain enabled: once modules are in place they are not altered conditional on shocks, not just for one or two periods, but over an infinite time horizon.

Andrew Haldane, director of financial stability at the Bank of England, in 2009, 2010 policy papers, both recognises the potential for financial contagion, and via a watchmaker analogy from Simon (1962) advocates the use of modules. The analogy shows that, a watchmaker in an environment with stochastic shocks, has a much lower expected completion time, when he uses modules.[8] Hence he concludes: "What is second nature to the watch-maker needs to become second nature to the watchdog". In response, my model assesses the usefulness of this analogy: I use the welfare function to consider the optimality of bank modularisation.

My work here makes two main contributions to our understanding of the use of modules as a policy response to the risk of financial contagion. Firstly, the first chapter shows that, under the standard assumption of a uniform distribution of business opportunities, the use of modules is rejected, for all parametrisations. Specifically, any *proper* partition (both multiple modules and multiple banks per module), is inefficient. Hence, the efficient partition is either the *grand coalition* (one big module containing all the banks), or the *atomistic partition* (each bank in its own separate singleton module).

Secondly, the second chapter shows that, varying the structure of business op-

[8]Note this is not the same argument as advocated in Smith (1776), which gives the famous example of a pin factory where each worker specialises in one task and a more than 240 fold increase in productivity is achieved (see I.1.3). Instead with Simon (1962), modules reduce the amount of work in progress lost when an adverse shock hits the manufacturing process.

portunities can vary the optimal policy choice. For example, section 2.2.1.1 considers a variant model where matches are no longer distributed uniformly: instead banks are arranged in a circle, and matches are always between immediate neighbours. Under this model, generally the efficient partition is proper: there will be multiple modules, and each module will have multiple member banks. This shows the importance of understanding the financial requirements of businesses when designing bank networks.

1.3 Literature Review

There are two parts to this literature review. First, I will overview the literature on the policy of *containing* financial shocks. Within this part, I consider hierarchies, Haldane (2009), Simon (1962); modularisation, Haldane (2010), Stiglitz (2010b), Leitner (2005); neighbourhoods, Castiglionesi and Navarro (2007); the differential potential roles of banks (Allen and Carletti (2009)), and the requirement to consider the financial needs of businesses when designing the banking system (Mayer (2011)); and the ICB report (Independent Commission on Banking (2011). Then, in the second part, I consider the modelling of financial contagion, both using the traditional consumer liquidity model, Allen and Gale (2000); and the post crisis propagation mechanism literature which considers: different possible shock transmission pathways, Nier et al (2008); the robust yet fragile network characterisation, Gai and Kapadia (2010); the effect of multiple asset classes, May and Arinaminpathy (2010); and the effect of different network designs, Gai, Haldane and Kapadia (2011) and Georg (2011).

Haldane in two policy papers, Haldane (2009) and (2010), advocates splitting the banking system into components. Haldane (2009) argues for a hierarchical approach: he uses an analogy from Simon (1962), which compares two watchmakers. Tempus and Hora are each making identical watches from the same 1000 elements: it is just that their production processes are organised differently. Tempus, simply has one task of assembling all the elements together into a watch. By contrast, Hora has a hierarchy of tasks. In particular, Hora manufactures *recursively*: at each level the same number (10) of lower level components are used. So, Hora has 100 sub-assembly formation tasks, where each sub-assembly is formed from 10 elements; then Hora has 10 tasks of forming assemblies, where each assembly consists of 10 sub-assembles; and then one final task of forming the watch from the 10 assemblies.

Is it better to make watches with, or without hierarchies? This is in a stochastic environment, where disabling shocks can hit the watchmaker. If a shock hits, then previously completed tasks are unaffected; but all progress on the current task is lost. The watchmakers want to minimise expected completion time. The result of the model is that, Hora through using a hierarchical approach, is over 1000 times faster than Tempus. Haldane (2009) advocates the applicability of this example to banking, but is watchmaking an appropriate comparison? As Simon (1962) states: "Metaphor and analogy can be helpful, or they can be misleading. All depends on whether the similarities the metaphor captures are significant or superficial".

Haldane (2010) talks about the benefits of *modules* (non-hierarchical structures where the whole system is broken down into modules, but modules are not further broken down), as a policy response to potential shocks, for example using firebreaks to protect forests against fire. He again suggests this applies to banking: "(banking) has many of the same basic ingredients as other network industries, in particular the potential for viral spread and periodic systemic collapse."

Stiglitz (2010b) has a number of models containing modules; these are principally between countries, but are claimed to also apply to banks. Here is a stylised version of his modelling. There are a number of countries and a single consumption good: each country i has constant absolute risk aversion preferences (represented by utility function $u(x) = 1 - \exp[\alpha x]$, where x is consumption), and an initial endowment C.[9] This risk aversion is a difference from both Haldane (2010) and my model. The timings are as follows. Firstly, the social planner assigns the countries into modules. Secondly, within a module, countries agree state contingent goods transfers. Thirdly, each country then receives a shock ϵ_i which is either small (mean 0, variance σ^2) or big (negative infinity):[10]

$$\epsilon_i \overset{iid}{\sim} \begin{cases} \mathrm{D}(0, \sigma^2) & \text{probability } (1-p) \\ -\infty & \text{probability } p \end{cases}$$

The big shock case represents contagion: the big shock wipes out all countries in a module, and they all get a utility of 0. If all countries in a module only receive small shocks, then the state contingent transfers take place, and consumption occurs.

The program can be solved through backwards induction as follows. At the second stage, as the countries are ex-ante identical (in preferences, endowments and their shock distribution), and risk averse, there will be complete consumption smoothing within each module. At the first stage, module size x is chosen to maximise the expected utility of a sample country:

$$EU = (1-p)^x E[1 - \exp(\alpha[C + \frac{1}{x}\sum_{i=1}^{x} \epsilon_i])]$$

[9] Note that the coefficient of absolute risk aversion is $-\frac{u''[x]}{u'[x]} = -\frac{-\alpha^2 \exp[\alpha x]}{-\alpha \exp[\alpha x]} = -\alpha$, and so the requirement for agents to be risk averse (rather than risk loving) means that $\alpha < 0$.

[10] The $\mathrm{D}(0, \sigma^2)$ notation for the small shock case represents some probability distribution with mean 0 and variance σ^2.

Taking a 2nd order Taylor approximation gives:[11]

$$EU = (1-p)^x \left(1 - (1 + \frac{0.5}{x}\sigma^2\alpha^2)\exp[\alpha C]\right)$$

Log-linearising gives:

$$\ln[EU] = x\ln(1-p) - (1 + \frac{0.5}{x}\sigma^2\alpha^2)\exp[\alpha C]$$

and the first order condition for a maximum then gives

$$x^2 = -\frac{0.5\sigma^2\alpha^2\exp[\alpha C]}{\ln[1-p]}$$

There is a trade off in module size between risk aversion and contagion avoidance: the risk aversion creates a desire for consumption smoothing, and hence large modules; whilst the risk of contagion creates a demand for small modules. Note, that there are decreasing returns from larger modules: expected consumption is already close to the optimal level (in the good state of no contagion). There is only one class of goods, so ex-post, every transfer makes one country better off, but another country worse off. This contrasts with both normal trade, where there are gains for both the buyer and the seller (the buyer values the good at higher than the transaction price; the seller at lower than the transaction price), and my model, which has Pareto improving investment opportunities. Hence, the Stiglitz model has a lower desire for large modules, and so in general has an interior solution.

Leitner (2005), like this thesis, directly assesses the idea of bank modules. It models how the threat of contagion, may induce interbank support, when there is

[11] using $f(a+h) \approx f(a) + hf'(a) + 0.5h^2 f''(a)$ and so $E[f(a+h)] \approx f(a) + E[h]f'(a) + 0.5E[h^2]f''(a)$ where there is uncertainty in h but not in a.

neither pre-commitment nor a repeated game. However, it assumes that, illiquid banks (hit by shocks) can be saved by liquid banks (which avoided shocks): "To allow for some benefits from mutual insurance, I also assume that by pooling resources, the liquid banks can come up with the extra funds required for helping the illiquid bank, so that in the first best, no bank, whether liquid or illiquid, goes bankrupt". This is an environment with *private bail-ins* by other banks; rather than *public bailouts* by the government. So, Leitner (2005) is considering less severe shocks than in this thesis: when a bank is hit by a bad shock, it is possible for contagion to be prevented. In reality, when the illiquid bank gets very big, bail-ins become much less feasible: the private bail-in of LTCM (Long Term Capital Management) cost $3.6 Billion, and the positions formerly held by LTCM were successfully liquidated by their rescuers for a small profit (Partnoy, 2003); whilst, when Lehman Brothers applied for Chapter 11 bankruptcy it cited debt of $613 billion, but currently it is estimated that creditors will be paid $65 Billion, (New York Times 30th August 2011)).

Leitner (2005) has a multi-stage model, where the network for the n banks has to be chosen before the endowments are known: stage 1) social planner decides the bank network; stage 2) bank endowments are resolved; stage 3) banks make transfers to other banks; stage 4) banks invest (if they have enough money); stage 5) if *all* banks in a module invested then they each get return R. Leitner (2005) does not get completely conclusive results; it constructs an example where a non-trivial solution is best: the optimal network is neither empty nor fully connected. But the intuition is that: if the expected bank endowment is large, then we want fully connected bank networks to smooth out stochastic shocks that leave a few banks illiquid; conversely, if the expected endowment is small, then we want an empty network so the rare liquid banks can invest, without being stopped by the many illiquid banks. Leitner proves that the fully connected network is ex-ante strictly preferred to the empty network, if and only if,[12]

$$\text{Prob}(\sum_{i=1}^{n} \min(e_i, R) \geq n) > \text{Prob}(e_1 \geq 1)$$

Castiglionesi and Navarro (2007) is a financial fragility network model, not a financial contagion partition model, and includes both the potential for bank moral hazard and liquidity shocks. The setup has two types of agents, (depositors and shareholders), and a network of inter-bank links between (some) of the banks. Each bank can then invest in either a safe asset or a risky asset. Safe assets always generate enough returns to pay back the bank's depositors. In contrast, the risky assets may fail, in which case the depositor loses their deposit; or may succeed, in which the shareholders make extra profits, which gives the model its moral hazard feature. Banks that invest in risky assets are called gamblers, and the setup of the model is such that banks only have an incentive to gamble if their capital endowment is below a critical level.

There is a stochastic environment consisting of both negative liquidity shocks and failures of risky projects. The network of inter-bank links has both benefits (in providing resilience to liquidity shocks), and costs (when projects fail). If a bank receives a liquidity shock then that can be smoothed out by a linked liquid bank, (similar to the Leitner (2005) bail-in mechanism). If a risky project fails then the bank that gambled on that project fails, along with all its immediate neighbours (those with direct links to it), irrespective of their own investment choices and capital endowments. However, the shock can never propagate further: specifically, it does not take out any banks that are not the hit bank's neighbours, but are only neighbours of its neighbours or even more remote. This limited propagation means that this is a financial fragility model, not a financial

[12] This preference condition reflects an additional complication: as the highest return a bank can get is R, the highest they are prepared to invest (directly or in transfers), is R.

contagion model: a single shock cannot have systemic global effect; it can only have local effects. And hence this is a model where the outcome depends on the network of neighbours that each bank has; rather than on how the banks are partitioned into modules.

The form of the efficient network depends on the parametrisation, but the format is always a core and a periphery. If there is high aggregate bank capital then there is a complete network, and every bank is in the core for maximum resilience to liquidity shocks. Further, each bank is allocated enough capital to be incentivised to invest safely. Alternatively, if there is low aggregate bank capital, then the social planner finances as many banks as possible to the level where they will invest safely. Completed links are formed between these safe banks to form a core. The residual capital is split amongst the remaining banks, who will then gamble and form the periphery. Each peripheral bank will be connected to some core banks and to some peripheral banks. [13]

Haldane and May (2011) models the mechanics of how shocks move around system of banks, and so how financial contagion can occur; however, there are no businesses and no modules. Further, the model does not assign a social value to banks: it does not say how they matter. So, it is silent on questions of the type: "is it better to have some banks enabled all the time, or all the banks enabled some of the time?". Despite these difficulties, the conclusion of Haldane and May (2011) is pro-module:

> "modularity protects the systemic resilience of both natural and constructed networks. The same principles apply in banking."

Johnson (2011), in a direct response to Haldane and May (2011), cautions that: "Policy-makers may never fully appreciate a model's limitations... (so without

[13] What links the gambling banks form depends on the parameter values.

careful application) ... we simply increase risk, rather than reduce it". The financial crisis, was in part caused by complex financial derivative products created using physics models. So it would be ironic (and potentially tragic), if central bankers through inappropriate use of physics models of financial contagion, made policy mistakes in response. In using physics, there are two particular methodological issues. Firstly, economists do like to see themselves as scientists (studying how the world works), but economics has another aspect: policy work. And policy work is more engineering (problem solving), than science.[14] As, Keynes (1931) opined, "If economists could manage to get themselves thought of as humble, competent people on a level with dentists, that would be splendid.", and whilst there is a scientific aspect to dentistry, its primary role is treatment. Secondly, there is a difference in falsifiability between economic-science and physics: physicists can run laboratory experiments to test their models; whilst economics is an observational science like astronomy, so economists rely on real world data with all the ethical and practical issues that implies.[15]

The Allen and Carletti (2009) survey paper identifies four potential explanations for the existence of banks (or other financial intermediaries), in addition to, or instead of, a more decentralised financial market. The first is the traditional intertemporal *risk-smoothing* pathway: banks issue long term loans to business and householders, whilst offering accounts with short notice periods to savers. Because of the maturity mismatch between their assets and liabilities, however, banks are subject to the possibility of runs and systemic risk.

The other three explanations are *borrower monitoring*, (banks have the informa-

[14]See Mankiw (2006) for a discussion on the twin roles of an economist: scientist and engineer. Interestingly, writing *before* the recent financial crisis, he uses the example of the Great Depression to argue that the policy economist's toolkit has not moved forward much in the last 50 years.

[15]There has been a recent growth in laboratory economics, but that is new and there are issues, for example of external validity.

tion about their client business to ensure that they are trustworthy; in contrast market-based financiers face a free rider problem with respect to monitoring business effort); *economic growth*, (in a bank-based system there are close relationships between banks and businesses resulting in high growth; in contrast in a market-based system there are distant relationships between financiers and businesses resulting in low growth); and *corporate governance* (in a bank based system there are long-term relationship between a bank and its client firm, the holding of both debt and equity by the bank, and the active intervention of the bank should its client become financially distressed). My work includes the risk of systemic bank failure, (without the microfoundations of risk-smoothing, or the mechanics of liquidity shocks); requires banks to fund businesses in order for them to grow; and the inability for businesses to move banks can motivated in terms of borrower monitoring.

Mishkin (2007), p181, identifies that "A healthy and vibrant economy requires a financial system that moves funds from people who save to people who have productive investment opportunities.", and establishes, from Hackethal and Schmidt (2004), eight stylised facts: 1) stocks provide only small amount of business finance (about 10%); 2) bonds provide only small amount of business finance (about 10-15%); 3) *indirect finance* (where there is an intermediary between saver and borrower), is more important than *direct finance* (no intermediary between saver and borrower); 4) financial intermediaries (especially banks) provide most Finance; 5) there is a lot of financial regulation; 6) only big businesses use stock or bond markets; 7) collateral is important; and 8) debt contracts are complicated. My model is inline with the first six of these facts: 1) has no stock market; 2) has no bond market; 3) has indirect finance; 4) uses banks to provide finance; 5) considers another possible form of regulation: *modularisation*; and 6) focuses on businesses too small to use the stock market.

Mayer (2011) claims that: "One of the best-established associations in economics is between financial development and growth – countries with well-developed financial systems grow faster".[16] He argues that businesses need finance to grow, and that banks are especially important for financing medium sized businesses: small businesses are family funded; whilst large businesses use the stock market. He emphasises the importance of local banking; for example, in financing the industrial revolution during 19th century Britain. The strength of local banks is that they offer long term relationships to Small and Medium Enterprises (SMEs).[17] However, local banks are smaller and hence riskier.[18] In response to this riskiness, regulatory changes have made banks larger and hence safer; but also more distant from their business customers: resulting in a lack of funding for SMEs. He concludes, post crisis, "that the focus is ... on the immediate issue of avoiding another failure of the banks", whilst it should be on ensuring "that British banks provide sufficient financing for SMEs". This motivates, firstly my model's use of cost benefit analysis to assess the effect that different structures of the financial sector have on the business sector, and thus on the overall welfare of society; secondly my model's use of banks to finance businesses; and thirdly the inability of businesses in my model to change which bank they use.

It could be argued that this story about the dependence of businesses on banks can be undermined using Modigliani and Miller (1958), whose main result is the Modigliani-Miller Theorem, "a firm's valuation is the same whether financed by equity or debt". However, Modigliani and Miller (1958) themselves caution that "Misinterpretation .... can be avoided by remembering that this Proposition tells

[16] Colin Mayer is the Peter Moores dean and professor of management studies at Saïd Business School, University of Oxford.

[17] The internal examiner Professor Peter Hammond raised the interesting question of the effect of competition amongst local banks. The competition effect would reduce business lending costs, but the effect of firms changing banks to reduce costs would be to erode long term bank-business relationships.

[18] In the sense of higher chance of individual bank failure due to high exposure to local shocks rather than in the systematic sense of financial contagion.

us only that the type of instrument used to finance an investment is irrelevant to the question of whether or not the investment is worth while. This does not mean that the owners (or the managers) have no grounds whatever for preferring one financing plan to another; or that there are no other policy or technical issues in finance at the level of the firm."

Further, their results rely on strong assumptions such as, the absence of borrowing constraints, symmetric information, the absence of (distorting) taxes, the absence of agency costs, and complete markets. The complete market assumption is particularly onerous as it needs to apply over uncertainty, and that further requires state of world verification. As Mayer (1988) says, Modigliani and Miller (1958) require that "all contingencies must be appropriately contracted; there must be a complete set of (Arrow–Debreu State) contingent contracts." Further, Freixas and Rochet (1997) demonstrate "the discouraging fact that banks are useless in the Arrow–Debreu world". They show, that in a general equilibrium environment with complete market, banks make zero profits and "the size of bank's balance sheets have no impact on other economic agents", and thus conclude that "the Arrow–Debreu paradigm leads to a world in which banks are redundant institutions".[19] This shows the particular importance of the complete markets assumption in the Modigliani–Miller Theorem.

One of the primary concerns of the ICB was to prevent shocks jumping from investment banking to retail banking. Hence, their main proposal is ring-fencing: this is the internal separation of retail and investment divisions of universal

[19]Within the general equilibrium framework, the internal examiner, Peter Hammond, pointed out the potential relevancy of Green (1974), which has bankruptcy contagion in a temporary equilibrium model. Green (1974) is a continuation of Green (1973), which considered temporary equilibria in a multiple period environment. Green (1974) considers what happens if today it is not feasible for an agent to implement a transaction he agreed yesterday because today's endowment is found to be too small, and hence has to declare bankruptcy. Within this environment each agent can be a debtor to some agents but a creditor some other agents, leading to the potential for contagion.

banks; not the external separating of banks by firewalls.[20] As Kay (2011) said, in his evidence at the Treasury Select Committee hearings into the final version of the ICB report, "I think the ICB was asking what is the minimal change to the structure industry, that they could propose, that would achieve the objective of eliminating or reducing the taxpayer's subsidy to investment banking". Similarly, Hahn (2011) said to the committee, of the ICB report, "this is tinkering rather than stepping back (and making more wholesale changes)". For example they propose that inside the ring fence banks would be allowed to lend within the EEA (European Economic Area), but not outside: "(they would) not be allowed to engage in trading or other investment banking activities, provide services to financial companies, or services to customers outside the EEA", (ICB) 2011. Their justification for this proposed modularisation is that, "The UK's international treaty obligations make the appropriate geographic scope the EEA rather than the UK." Of course, the UK should respect its treaty obligations, however, it is feasible for treaties to be re-negotiated, and it should be assessed whether there are net benefits from such changes.

The Independent Commission on Banking (ICB), rather than choosing an optimal design for the banking system, were trying to reduce the size and frequency of governmental bail-outs of banks: "The package of banking reforms .. is designed to reduce the probability and impact of financial crises in the future", ICB (2011). Their approach was based on empirical judgement rather than mathematical modelling. They give three explanations for not using models. The first is "because no model exists which can both reliably account for the frequency and incidence of financial crises and encompass the effects of reform recommendations.", ICB (2011). And, clearly there are difficulties in modelling financial systems and the effectiveness of possible policy reforms. However, I feel

[20] Given this significant difference I would not consider the modularisation of banks as being an example of ring-fencing.

this is too pessimistic, particularly when considering policies to reduce the likelihood of financial contagion, rather than prevent an initial bank failure: with such policies we can take the initial financial shock as exogenous. The second explanation implicitly assumes that models are sensitive to parameter values: "Even if such a model did exist, sufficient empirical historical data about the relation between the recommendations and the frequency or impact of financial crises is not available to populate it.", ICB (2011). However, with my model, as the efficient partition always takes one of two forms, the policy implication is not generally critically dependent on parameter values. The third explanation is more philosophical, "attempts to quantify these effects are inherently limited, because ... future risks which will certainly arise but whose precise scale and nature is fundamentally unknowable.", ICB (2011). Again, in this aspect my model is resilient, for example the results are robust to the inclusion of common shocks that effect all banks directly.

I am now going to consider contagion, which consists of three parts: the first is *initiation*, the shock starts at only one node of a system; the second is *propagation*, the shock spreads node by node in an iterative process, (it is not just the initial node that can infect other nodes; every node, once infected, can spread the shock onto it's neighbours); and the third is the *result*, all of, or a significant proportion of, a network is affected. The term financial contagion is used when this concept is applied to a banking network: the initiation consists of an initial negative liquidity shock to one bank; and the shock is propagated through the inter-bank links.[21]

Pre-crisis Allen and Gale (2000) used the variable consumer liquidity demand

[21] The term contagion has been used in multiple contexts in economics. For example comovements in stock markets. Here I am restricting the term to banks in-line with the original definition (see Bagehot (1873)). See Moser (2003) for a survey of the use of contagion in economics.

story of Diamond and Dybvig (1983), both to motivate the existence of a network of inter-bank lending, and explain how that network can result in contagion. In their model inter-bank lending normally adds value, as it facilitates consumption smoothing. However, in the rare bad states of the world, when there is a large liquidity shock, this is propagated from one bank to the whole banking network; resulting in all consumers getting adversely affected. This is the cost of inter-bank lending: without it, the shock would have been contained to just the customers of a single bank, and there would have been no financial contagion. Notwithstanding this, the conclusion of the model is that if the state of the world where contagion occurs is rare enough, then inter-bank lending is ex-ante welfare enhancing. In the context of the present crisis, however, their model has a number of issues. Firstly, it is focused on explaining how financial contagion can happen, rather than assessing policy responses; for example, financial contagion in their model is a zero probability outcome.[22] Secondly, their model has shocks to consumer liquidity demand rather than banks' assets. Thirdly, the crisis has cast doubt on a number of the model's conclusions: for example, that the complete network (where every bank lends to every other bank), is the most stable. Fourthly, their analysis is restricted only to very small networks: specifically, those with four banks.[23]

The aim of Allen and Gale (2000) is to provide micro foundations for financial contagion, rather than to asses policy responses. The set-up has banks which smooth consumer consumption, but do not enable business investment. So, there are customers, but no non-financial businesses. Each bank has specified explicit

[22]Diamond and Dybvig (1983), that Allen and Gale (2000) uses for its model of bank runs, has *multiple* equilibrium; whilst Goldstein and Pauzner (2005) provides a bank run model where customers receive noisy signals that has a *unique* equilibrium (they use a method related to Carlsson and van Damme (1993) and Morris and Shin (1998)). Dasgupta (2004) uses this noisy signal approach to extend Allen and Gale (2000), and assess the probability of contagion.

[23]An extension to the paper claims that the arguments extend to the case of many banks; but few details are given.

inter-bank and bank-consumer lending. The inter-bank lending can take different network structures. However, due to the number of banks being restricted to four, only three different network structures can be analysed.[24] In increasing order of robustness to financial shocks, these are shown to be: circular, bilateral and complete. There are strong foundations to these results: the welfare function is derived from the preferences of consumers and all results are found using analytical methods.

In economics, the first contagion paper that theoretically considers the inter-bank propagation mechanism is Iori, Jafarey and Padilla (2006), which examines the potential for the interbank market to propagate liquidity crises.[25] Their model is behavioural, and they assess different networks using numerical simulations. Their results are that with homogeneous banks, "no evidence of the potential for contagion is found" and their conclusion for this case is that "an interbank market unambiguously stabilizes the system". Contrastingly, with heterogenous banks, the network has the characteristic later described by Gai and Kapadia (2010) as "robust but fragile": robustness as "interbank lending contributes to a lower incidence of bank failures through the mutual insurance role"; fragility as "at the same time, (inter-bank lending) does create the tendency for the system to display avalanches (episodes of multiple bank collapse)". The authors identify that shock transmission occurs through both direct ("the failure of one bank has

[24]The consumer liquidity shocks are perfectly negatively correlated between the odd numbered banks $(1, 3)$ and the even numbered banks $(2, 4)$: in state s_1 the proportion of consumers demanding liquidity are (w_h, w_l, w_h, w_l), whilst with state s_2 the proportions are (w_l, w_h, w_l, w_h). So firstly efficiency excludes the possibility of a disconnected bank. Secondly, risk is traded on the open market using state contingent Arrow-Debreu bonds. Hence, trading relationships may or may not be symmetric. Let $x \rightarrow y$ represent the situation where bank x buys a state contingent bond from bank y which pays out when bank x receives a negative liquidity shock. If the relationship is mutual then this represented by $x \leftrightarrow y$. And so we need to consider the following directed networks: i) bilateral network ($1 \leftrightarrow 2$, $3 \leftrightarrow 4$), ii) complete network ($1 \leftrightarrow 2$, $1 \leftrightarrow 3$, $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, $2 \leftrightarrow 4$, $3 \leftrightarrow 4$), and iii) circular network ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$).

[25]This paper is an expansion of the earlier Iori and Jafarey (2001) which was published in an econophysics journal. Further see Furfine (2003) for an early empirical contagion paper that considers US banks in February-March 1998.

knock-on effects on its creditors"), and indirect ("bank failures tend to weaken the system and drive it to an unstable state in which it becomes susceptible to further simultaneous failures") pathways. There are two main results on avalanches. The first is that "increasing inter-bank connectivity leads to increased probability of large avalanche sizes".[26] The second is that avalanche behaviour has a power law relationship: the log-log plot of the statistical distribution of avalanche size is linear (see the below figure from Iori and Jafarey (2001)). The existence of this power law relationship is consistent with the predictions of Self Ordered Criticality (SOC) theory, (see Bak, Chen, Scheinkman and Woodford, 1993).[27]

[26] An avalanche of size k occurs when an exogenous shock is applied that causes one bank to fail, results in a total of $k-1$ further banks failing after the full shock transmission effects are considered. For example, suppose an exogenous shock disables bank 1, that the knock on effects from that disable banks 2 and 3, and that the knock on effects from banks 2 and 3 disabling in turn disable banks 4 and 5. If there are no further knock-on effects then $k = 1 + 2 + 2 = 5$.

[27] Note that under SOC power-law relationships apply at the limiting state of the system. However, the Iori *et al.* results are not from the limiting state of the system: with the their model defaulting banks are never re-enabled, in the limiting state all banks are in default, and there can be no avalanches.

Figure 1.1: Iori and Jafarey (2001) Log-Log plot of avalanche size

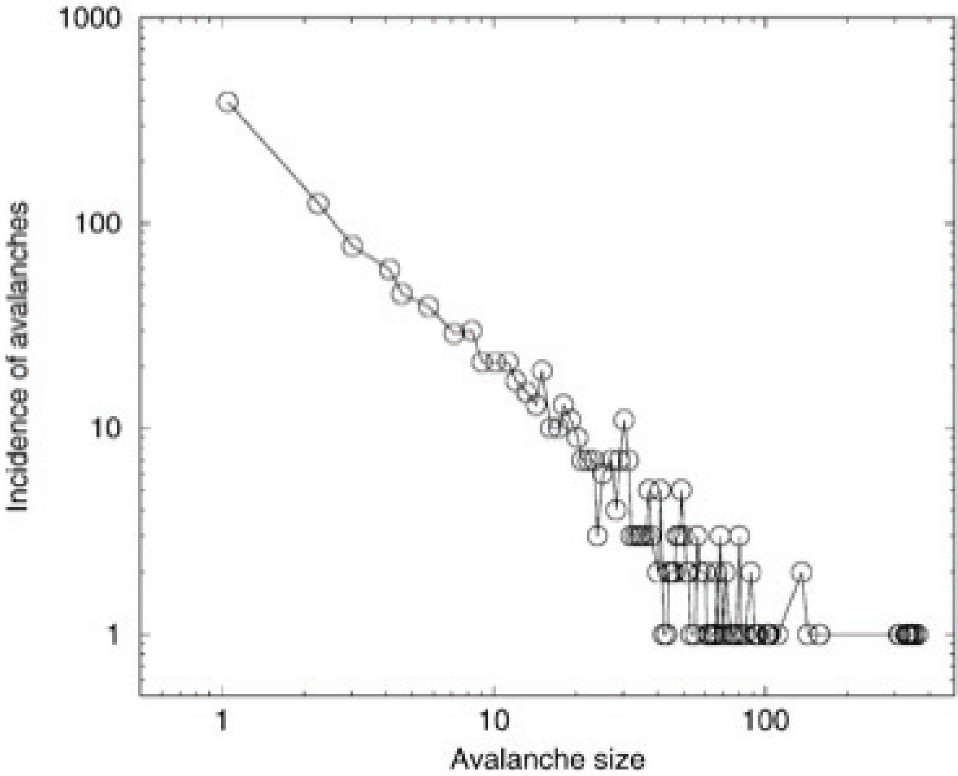


Fig. 4. Log–log plot of the statistical distribution of avalanche size.

However, it is not until post-crisis, that the propagation mechanism literature has fully emerged: this focuses on considering the *mechanics* of how the 2007 financial crisis could have been propagated worldwide from the US sub-prime market. In this literature there a number of major differences compared with Allen and Gale (2000): firstly, there is an arbitrary banking network, (there are an arbitrary number of banks, and the network of inter-bank lending is exogenous and irrational: the lending is not microfounded, as the preferences of banks are not specified); secondly, for each bank there is a balance sheet, (which specifies assets (loans to other banks, and external asset holdings), and liabilities (customer deposits, and loans from other banks)); thirdly, the initial shock affects bank assets rather than bank liquidity, (rather than a bank's customers withdrawing deposits, a bank's investments have failed). Fourthly, this literature instead of considering social welfare, specifies a propagation rule, and then constructs impulse response functions (the cascade resulting from shocking one bank). This is typically found using numerical simulations, although additionally Gai and Kapadia (2010), includes a probability generating function approach, and May and Arinaminpathy (2010), includes a mean field approximation.[28] In contrast, my model includes bank-business externalities and assesses efficiency.

Mechanism papers often consider different inter-bank propagation channels, other than just the direct inter-bank lending channel. For example, Nier, Yang, Yorulmazer, and Alentorn (2007) identifies four possible contagion channels: 1) inter-bank lending, 2) asset fire-sales, 3) asymmetric information (imperfect information about bank exposures, leading to a general drop in confidence), and 4) a common source of risk. In terms of my model it is helpful to consider the possible pathways that contagion can take, in order to understand the implications for what modularisation means in practical terms. The definition of a module is that it contains shocks within itself: so, for modularisation to be a feasible policy response to the first three pathways, the following are needed respectively: 1) no inter-module bank lending, 2) banks in separate modules hold separate classes of assets, and 3) common knowledge on bank exposures (inter-bank lending and bank assets).[29] In terms of the fourth pathway, common risk, modules do not

[28] The concept of a mean field approximation is that we ignore the distribution of a random variable, X. So $E[f(X)] \approx f(E[x])$. Hence, here the mean field approximation means that every bank has the same number of both credit and debtor banks. The idea of using mean field approximations is to ignore stochastic variation when making *predictions;* this contrasts with "First order equivalence", Malinvaud (1969), where stochastic variation is ignored when making *decisions*. For another economics example of a mean field approximation see Jackson (2008) $pp125 - 129$, which considers a poison network with growth. Jackson explains, "While we might prefer to calculate things like degree distributions analytically, this usually turns out to be intractable for all but the starkest of models."; whilst cautioning, "Analytically, we know distressingly little about when such approximation are good and when they are not."

[29] In terms of preventing asset fire-sales, we might think it sufficient to partition banks by what risky assets they have, but let all bank hold a common portfolio of safe assets. The

help.[30]

The stylised results of Gai and Kapadia (2010) are that, banking systems are *robust* (many shocks spread to few other banks), yet *fragile* (some shocks spread to many, or even all, banks). This fragility means the system has the capacity for financial contagion. For example, one model in Gai and Kapadia (2010) considers when propagation occurs through both the direct (inter-bank defaults), and indirect (asset price decreases due to fire sales), channels. Their results can be characterised in terms of z: the average degree.[31] In the region of $z = 1$ the extent of shock propagation jumps from approximately 0% of banks, to approximately 100% of banks.[32]

The main model of May and Arinaminpathy (2010) has multiple asset classes. Each bank holds n different assets: $n - c$ of which are idiosyncratic; whilst c are in a shared asset class. Each shared asset class has assets shared amongst $g + 1$ banks. The initial shock is assumed to cause one asset of one bank to lose all its value. As in Gai and Kapadia (2010), the shock is then propagated through both a direct inter-bank lending channel, and an indirect liquidity channel. However, here the liquidity channel takes 2 forms: strong (fire sales: experienced by banks holding other assets in the *same* class), and weak (confidence effect: experienced by banks holding other assets in *different* classes). With the inclusion of these weak liquidity effects, May and Arinaminpathy (2010) generically rejects intermediate propagations: "(with high initial bank net worth) very few simulations

problem here is that a shock to a certain type of risky asset, for example US sub-prime mortgages, will cause banks holding them to experience a liquidity shock, and hence tend to sell their liquid safe assets, for example US treasury bonds. This will decrease the value of US treasury bonds, and hence other banks holding those assets will experience a negative shock. So to prevent financial contagion spreading between modules via an asset fire-sales channel, requires complete separation in terms of what assets banks in different modules hold.

[30]Although it may be case that the common risk is not truly innate; so if we restrict bank lending by class then what was a common risk becomes a bank specific risk.

[31]The average degree is defined as follows: there is a directed network of n banks and a typical bank i has lent money to $j(i)$ other banks. So let $z := 1/n\sum_{i=1}^{n} j(i)$

[32]See figure 8 in Gai and Kapadia (2010)

have any failure other than the initiating one, but the few simulations that do show failures bring the entire system down, and thus have disproportionate effects on the average number of failures; quite small further decreases in g (initial bank net worth), result in a much higher proportion of all banks having failed (but still no intermediate propagations)."

So both, Gai and Kapadia (2010), and May and Arinaminpathy (2010), motivate the assumption in my model that financial contagion is full and immediate: a shock hits one bank, and is then immediately propagated to all the other banks connected to it (directly or indirectly). Similarly, in my model modularisation is also completely effective. Hence, when contagion occurs it always affects all banks in the affected module, but never spreads across module boundaries. This high cost-benefit ratio gives modularisation the most chance of being an effective policy.

Gai, Haldane and Kapadia (2011) is a recent mechanism paper which contains two innovations. Firstly it compares two different distributions for the random financial network: Poisson and geometric. In a Poisson network every inter-bank link exists with probability p and so the node degree has a Poisson distribution:

$$P(\text{node degree } k) = z^k * exp(-z)/k!$$

where $z = p * n$ is the average degree and n is the number of nodes.[33] An alternative random network is the geometric, or scale free network, which has fat tails: there are more nodes with high degree than in the Poisson network. One method of generating a geometric network is as follows: the initial network consists of a fully connected network of $m + 1$ nodes; the other $n - m$ nodes are then added sequentially. Each of those added nodes, i, forms links with

[33] See Newman et al (2001) page 4.

exactly m existing nodes, and the probability of connecting to existing node j is proportional to the existing degree of node j. So, if d_j is the existing degree of node j, then

$$P(i \text{ connect to } j{:}d_j{=}d) = d * P(i \text{ connect to } j{:}d_j{=}1)$$

The final node distribution after all the $n - m$ nodes have been added is given by

$$P(\text{node degree } k) = 2(m^2)k^{-3}$$

and this can be generalised to the geometric function

$$P(\text{node degree } k) = \gamma(m^\gamma)k^{-\gamma-1}$$

The following motivation of this generalisation is given by Jackson (2008) on p133. Suppose that at each time t, rather than a single node being introduced, that a group of new nodes are introduced. Suppose that a fraction α of links are from new nodes to existing nodes, and that a fraction $1 - \alpha$ of links are from new nodes to new nodes. This leads to the case $\gamma \equiv \frac{2}{\alpha}$.

The second innovation is it considers haircuts: if a bank has A^L liquid assets and can borrow up to $(1 - h)A^L$, then h is the haircut. Either a single bank may receive an idiosyncratic haircut shock or all banks may receive an aggregate haircut shock.

Gai, Haldane and Kapadia (2011) consider 6 different experiments and 4 different policy exercises. Each of these experiments and exercises specifies a calibration of the 20 parameters. The average node degree, z, is varied and for each choice of z, 1000 different realisations of both the random network and the node hit by the liquidity shock are selected. The extent of shock dissipation in each of the trials can then be measured. This enables the authors to estimate the average dissipation levels for different levels of connectivity within each experiment or exercise.

They often find that if a bank has more partners then a shock to any one partner has less effect, so there are *tipping points* z^*: if $z < z^*$ then $Prob(\text{contagion}) \approx 1$ and if $z > z^*$ then $Prob(\text{contagion}) \approx 0$. Experiment 1 considers a Poisson network and then "shocks a single bank into receiving a very large adverse idiosyncratic haircut shock which causes it to start hoarding liquidity". This gives a baseline result and the tipping point $z^* \approx 7.5$. In Experiment 2, the network is still Poisson but all banks face an increased haircut: 10% pre-shock to 20% post crisis. As expected an initial shock now has more effect and the tipping point moves to approximately 15.

The other four experiments consider geometric networks. In Experiment 3, shocks are randomly (uniformly) distributed. The network demonstrates resilience, as there is no tipping point: for all values of the average node degree, the $Prob(\text{contagion})$ is less than 1 and its maximum is approximately 0.5. In Experiment 4, the haircut shock is targeted on the highest degree node. The result is that there is now a tipping point and it has a very high value: approximately 60. Both these results are inline with the network literature: fat-tailed networks are robust to random shocks; but vulnerable to targeted shocks.[34] In Experiment 5, there is a return to random shocks but unsecured interbank liabilities are increased from 15% to 25% of the balance sheet. As in Experiment 3 there is no tipping point but the maximum probability of contagion increases from 0.5 to 0.7. Experiment 6 fixes the average connectivity, at 50, but varies the pre-shock aggregate haircut h; the post-shock aggregate haircut is fixed at 0.25. The result is that the contagion probability is 1 for h less than approximately

[34] See for example (Anderson and May, 1991; Albert et al., 2000)

0.06, and then the contagion probability decreases linearly for h up to 0.20.

Policy Exercise 1 considers the effect of tougher liquidity requirements. There is an increase in liquid asset holdings for all banks from 2% to 3.5%; resulting in increased resilience: the maximum $Prob$(contagion) drops to 0.2 from 0.5 compared with Experiment 3. Policy Exercise 2, considers systemic liquidity requirements, where banks that lend more are required to hold greater liquidity: banks are required to hold a minimum of 2% liquid assets, plus an amount equal to 10% of their total interbank assets. As average interbank assets are 15%, this implies average liquid asset holdings of 3.5%: the same as Policy Exercise 1. The result is that maximum $Prob$(contagion) drops to 0.1 from 0.2. The interpretation is that it is effective to focus liquidity requirements on high connectivity banks. Policy Exercise 3 considers counter-cyclical liquidity requirements to offset decreases in haircuts. There is a *tough* rule in which liquid assets are required to rise from 2% of total assets at an aggregate haircut of 0.25 to 4.5% when haircuts are zero, and an alternate *weak* rule, in which the liquid asset requirement only rises to 3.25%. Both rules show increased resilience compared with Experiment 6: the maximum contagion probability is below 0.1 under the tough rule, and below 0.5 under the weak rule. Policy Exercise 4, considers the effect of more restrained behaviour by shocked banks, for example because they have more confidence in the systemic integrity of the financial system due to increased network transparency. So shocked banks, instead of withdrawing all their liquidity from other banks, withdraw their liquidity needs plus 50% of the remainder. The result is that then the probability of contagion reduces to below 0.2 for all connectivities.

Ladley (2011) is a mechanism paper with firms as well as banks and households. Every agent has a utility function, however, agents' actions do not have rational microfoundations: firms and households follow behavioural rules; whilst the network of inter-bank lending is random. The paper considers the effectiveness of three different policy responses: increasing equity ratios, increasing reserve ratios, and constraining the maximum funds a bank may lend to a single counter-party; but not the effectiveness of modules.[35] The paper includes three robustness checks: firstly, parameter sensitivity; secondly, adding in a bank confidence channel, (during a crisis banks have less confidence that their loans will be repaid); and thirdly, adding in a creditworthiness channel, (riskier banks are charged a higher interest rate when they borrow). However, firstly, the paper has no efficiency assessment, and secondly, it relies on computational calculations providing no closed form solutions.

Bank regulators get blamed more for bank failures than they are praised for successful lending by banks to businesses. As Hahn (2011) said in his evidence to the ICB, "The regulatory regime is always going to be penalised for failures of banks; (but) it won't get a reward for the economy doing well. The Government gets the upside and the downside; the regulator only gets the downside". A number of the mechanism papers in this literature review have had contributions from Bank of England employees such as, Andrew Haldane, Sujit Kapadia, Erlend Nier, Jing Yang and Tanju Yorulmazer.[36] This may partly explain the propagation mechanism literature's modelling of inter-bank externalities, rather than bank-business externalities; lack of a business sector; lack of a welfare function; and its focus on bank capital losses, rather than the real economy.

The policy literature, the propagation mechanism literature and the recent empirical evidence, all motivate the assumption of the existence of financial contagion that this thesis makes. My aim in this thesis is to assess the validity

[35]Increasing equity ratios means requiring banks to hold more capital relative to their holdings of risky assets; increasing the reserve ratio means forcing banks to hold more of their assets in liquid form (for example as reserves with the central bank).

[36] The Bank of England is the British central bank and (partly) responsible for bank regulation.

of the pro-module hypothesis: it will explicitly compare bank networks with or without modules. A partition of modules can be thought of as a network with disconnected regions, where each region has full internal links.[37] In line with Allen and Gale (2008) p19, a cost benefit approach will be followed: "the costs of avoiding crises must be traded off against the costs of allowing crises". Whilst bank bail outs are large in absolute terms, they are small in size relative to the real economy costs; so in my model the cost of financial contagion is forgone investment opportunities, and hence is rooted in the real economy.[38] Therefore, this thesis abstracts from modelling bank balance sheets.

As in the propagation mechanism literature, I have an arbitrary number of banks. There are three key features in common with Allen and Gale (2000): firstly, the models have non-banks (consumers in Allen and Gale; non-financial businesses in this thesis); secondly, analytical methods are used; and thirdly, the results are interpreted using a welfare function micro-founded in stakeholder utility. Under the standard model, with a uniform distribution of business opportunities, the conclusion is contrary to Haldane and May (2011): for all parametrisations the use of modules is rejected as inefficient. In contrast, under an alternative model, where the matches are between neighbouring banks, the use of proper modules is efficient.

Allen and Gale (2008) p2 accepts that the financial system has "a basic function of allocating investment". A credit crunch has a much greater effect on businesses than householders. Householders are unable to get extra credit (for example to enter, or move up, the housing market), but they have got long term credit line availability, for example through 25 year mortgages. In contrast, businesses have floating credit lines on much shorter term lengths: this can result in them being

[37]Equivalently consider networks with transitive links where if x is connected to y and y is connected to z then x is connected to z.

[38]Further, bank bail outs are "mostly transfers" Allen and Gale (2008) p19.

unable to grow, or even carrying on trading. In the UK, the hearings into the ICB interim report (Treasury Select Committee 2011a, 2011b), demonstrated a similar focus as they asked many questions about the effect the credit crunch was having on businesses, but none on its effect on consumer credit. Therefore, my model focuses on businesses not consumers.

A third class of financial systemic risk models, exemplified by Garratt, Mahadeva and Svirydzenka (2011), rather than using particular experimental simulations, is more empirical in form: the aim is to summarise relevant features of the network without imposing too many assumptions. Their approach is to use lending data to infer contagion probabilities and hence formulate modules. The data they use is bilateral lending claims between banks in 21 countries, on a quarterly basis for the period 1985 Q1 to 2009 Q3. The data set was supplied by the Bank for International Settlements and data is aggregated at the country level, with each bank being associated with the country where their headquarters are located. The countries present are Austria, Australia, Belgium, Canada, the Cayman Islands, Switzerland, Germany, Greece, Denmark (excluding Faeroe Islands and Greenland), Spain, Finland, France (including Monaco), United Kingdom (excluding Guernsey, Isle of Man and Jersey), Ireland, Italy, Japan, Luxembourg, Netherlands, Portugal, Sweden, and the United States.

Garratt, Mahadeva and Svirydzenka (2011) model shocks as being transmitted through 2 channels: a credit channel (a bank defaulting on its loans transmits stress to its creditors) and a funding channel (when a bank is hit by a shock it reduces the funding it provides to banks it has previously lent to). In the network, each of the n countries is represented by 2 nodes a (C)redit node and a (F)unding node: country i has nodes F_i and C_i. So there are 4 contagion channels between each pair of countries: considering countries 1 and 2, i) if 1 defaults then 2 has reduced funding (contagion flow F_1 to C_2), ii) if 2 defaults then 1 has reduced assets (contagion flow from C_2 to F_1), iii) if 2 defaults then 1 has reduced funding (contagion flow from F_2 to C_1), and iv) if 1 defaults then 2 has reduced assets (contagion flow from C_1 to F_2).

Garratt, Mahadeva and Svirydzenka (2011) use an ad hoc model to translate funding and credit levels into contagion probabilities. Recalling that there are both credit and funding nodes: let x be a $2n*2n$ matrix representing the absolute amount of bilateral lending for each pair of nodes. Lending is always between funding and credit nodes so $x(F_i, F_j) = 0$ and $x(C_i, C_j) = 0$. Further, the amount lent from F_i to C_j is the same as the amount received by C_j from F_i. So $x(F_i, C_j) = x(C_j, F_i)$ represents the value of loans that banks in country j have made to banks in country i.

Except for the special case of within country contagion (contagion between funding and credit nodes of the same country), *contagion weight* v is the same as the bilateral lending. So between funding and credit nodes in different countries, contagion weight is given by the lending level. So $v(F_i, C_j) = x(F_i, C_j) = v(C_j, F_i) = x(C_j, F_i)$. There is no intra-funding or intra-credit node contagion: so $v(F_i, F_j) = 0$ and $v(C_i, C_j) = 0$. In the special case of intra-country contagion, the weight is given by the total level of outside funding that the country has: $v(F_i, C_i) = \sum_{j=1}^{n} x(F_i, C_j)$. The contagion weights v, are normalised to form the contagion probabilities π, $\pi(C_i, F_j) = \frac{v(C_i,F_j)}{\sum_{k=1}^{n} v(C_k,F_j)}$ and $\pi(F_i, C_j) = \frac{v(F_i,C_j)}{\sum_{k=1}^{n} v(F_k,C_j)}$.

The *prestige* p_{C_i} or p_{F_i} of a node is defined as the steady state probability of a shock being at that node. So the prestige vector p over all $2n$ nodes is a steady state solution of the π transition matrix, hence: $p = \pi p$. Similarly, p^m, the prestige of module m is given by the sum of the prestiges of all the nodes contained in module m. This prestige can be thought of as the frequency with which shocks visit the module.

Finally modules are formed from the contagion probabilities. The nodes are classified using 2 types of book: a single index book and (generically) multiple code books. In the index book, each module has a single entry. There is one code book per module, and each code book has 1 entry for each bank in that module. For intra-module travel, where a shock travels between 2 nodes in the same module, just the single code book is used. Whilst with inter-module travel, where a shock jumps to a new module, both the index book and the code book for the new module is used. It is more costly to use 2 books than 1, but it is also more costly to use nodes which have higher entry numbers in a code book. So the question is how can nodes be arranged into modules to minimise this cost?

Garratt, Mahadeva and Svirydzenka (2011) find the best partition using an entropy method. The entropy cost of a probability measure $\mathbf{z}$ is given by, $H(\mathbf{z}) = \sum z_i * Log[z_i]$. The entropy cost of a partition $\mathbb{M}$, consisting of M modules, is given by $L(\mathbb{M}) = q * H(Q) + \sum_{m=1}^{M} p^m * H(P^m)$, where q is the aggregate exit probability of all the modules, $H(Q)$ is the entropy cost of the index book, p^m is the prestige of module m and $H(P^m)$ is the entropy cost of the module m codebook.

Clustering is done at each date from 1985 Q1 to 2009 Q3. The results suggest a great deal of instability in the modularisation: there are 25 different partitions over the 99 quarters:

Table 1.1: Garratt, Mahadeva and Svirydzenka (2011) Number of Modules by quarter

Starting Quarter	Number of Quarters Lifetime	Number of Modules
1985 Q1	6	19
1986 Q3	4	18
1987 Q3	7	17
1989 Q2	1	18
1989 Q3	1	17
1989 Q4	9	19
1992 Q1	1	19
1992 Q2	1	19
1992 Q3	5	19
1993 Q4	13	18
1997 Q1	1	16
1997 Q2	8	18
1999 Q2	5	17
2000 Q3	1	16
2000 Q4	1	17
2001 Q1	4	16
2002 Q1	4	17
2003 Q1	6	16
2004 Q3	2	17
2005 Q1	1	16
2005 Q2	10	17
2007 Q4	4	17
2008 Q4	1	18
2009 Q1	1	17
2009 Q2	2	18

Given that there are 21 countries, in every period the estimated partition is close to the atomistic partition. We can simplify the picture by abstracting away from some of the non-key modules: these mainly consist of geographic relationships or of a small country's banks being interlinked with those of a big neighbour. Firstly, the Cayman Islands are in the same module as the US in every quarter. Garratt, Mahadeva and Svirydzenka (2011) acknowledge this as, "reflecting the fact that the Cayman Islands is an offshore centre for US banking. The IMF (International Monetary Fund) recently estimated that 57% of the assets of the

Cayman Islands banking system are overnight sweep accounts in branches of US banks.". Secondly, Luxembourg is in a two member module with Germany for 89 of the 99 quarters: in one of the other quarters it shares a module with Switzerland and in the other 9 it is in a singleton module. Thirdly, for one quarter Canada shares a module with the US: the rest of the time Canada is in a singleton Module. Fourthly, for 5 quarters the Netherlands and Belgium are in a 2 member module: the rest of the time they are in singleton modules. Fifthly, since 1999 q2, in every quarter Finland and Sweden have been in the same module, and since 2001 q1 for every quarter they have been joined by Denmark. Ignoring these links leaves the following *significant* non-singleton modules:

Table 1.2: Garratt, Mahadeva and Svirydzenka (2011)
Significant non-singleton Modules by quarter

Period	Period Lifetime	Significant non-Singleton Modules
1985 Q1	6	
1986 Q3	4	UK, Japan
1987 Q3	7	UK, Japan, US
1989 Q2	1	UK, Japan
1989 Q3	1	UK, Japan, US
1989 Q4	9	
1992 Q1	1	UK, Switzerland
1992 Q2	1	
1992 Q3	38	UK, Switzerland
2002 Q1	4	
2003 Q1	6	UK, Switzerland
2004 Q3	21	

This table shows that there are 2 significant deviations from the atomistic partition. The authors note the first of these: the series of modules in the 3 year period from 1986 Q3 to 1989 Q3, involving Japan, the United Kingdom, the United States and the Cayman Islands: "In the late 1980s four important financial centres formed one large supercluster". The second is the UK Swiss relationship covering 45 of the 50 of the quarters starting with 1992 Q1.

One of the robustness checks that Garratt, Mahadeva and Svirydzenka (2011) carried out was to, "(employ) the map equation on the data set before splitting into funding and credit arms to see if that could generate some interesting modular structure without splitting out these two channels that allow for intrabank claims." However, when this was done, their algorithm always reported that there is only one module for every period. So in this case they have found the other boundary solution of the grand coalition.

My model is parsimonious, and assesses a specific policy proposal in a believable, internally consistent scenario; it is not an attempt to do a multi-channel calibrated model of, for example the 2007–2009 financial crisis. This gives the standard model tractability (through a quasiconvexity argument); the variations considered in the second chapter are harder to do algebraically, but computational methods still give clear results.

1.4 Model Overview

This section starts with a visualisation story, and then continues with an overview of the model, the solution methodology, and its results. Subsequent sections then formally define the model and derive the results.

Imagine a small town consisting of a series of two storey buildings, each occupied by a single married couple. The wife works downstairs as a banker, and the husband upstairs as a businessman. The husband creates an invention which is clever and interesting; but not necessarily useful. The wife takes the invention and searches for another wife, whose husband has produced a compatible invention, which it can be matched with, in order to produce a completed sellable finished product.

The buildings are arranged in rows and each row forms a street.[39] Wives have close social connections to women in their own street, but are more distant from women in other streets. So, if a matching invention is present in the same street, then the match is certain to be successfully completed, but if the matching invention is in a different street, then the match may fail to occur.

The buildings can receive negative shocks in the form of lightning strikes: every period, each building has an independent and identical probability of being hit by a lightning strike. The effect of a strike is communal within streets (a lightning strike to one building burns down all buildings in a street), but idiosyncratic between streets (streets are separated enough so that fire cannot spread inter-street). Couples from a fire-damaged building cannot operate commercially until their house is repaired.

Fire-damaged streets require planning permission before they can be rebuilt, and each period each fire-damaged street has an independent and identical probability of getting planning permission. When the repairs occur, all houses in a street are repaired at once. Since fire and repairs both affect all the houses in a street simultaneously, at any time all the buildings in a street will be in the same state: either all operating normally, or all fire-damaged and non-operational.

The cost of fire and the benefits of business matches generate a trade off: if a street has a lot of buildings, then fire has a large cost; conversely, if a street contains only a few buildings, then it is harder to complete matches. So the question facing a town planner in this environment is: "What is the best town plan?". The model constructed in this chapter, answers that, the town planner always rejects *proper* partitions: street plans where there are both more than one street, and more than one house per street. My model argues, specifically, that for all parameterisations every interior partition is strictly dominated by at

[39] We allow the extreme case of a street consisting of a single building: a row of length one.

least one of the boundary partitions, so the efficient partition is always either the grand coalition (one street with all the buildings in it), or the atomistic partition (one building per street).[40] This choice between these 2 partitions depends on the parameter values.[41]

Now, I will overview the banking model, which is formed by a simple equivalence from the town planning model in terms of: stakeholders (husbands become businesses, wives become banks, and the town planner becomes the social planner); business opportunities (go from needing both of a matched pair of husbands to needing both of a matched pair of business); negative shocks initiation (exogenous stochastic lightning strikes become exogenous stochastic disabling financial shocks); shock propagation (fire transferring between adjoining buildings becomes shocks passing between connected banks); the negative effect of shocks (couples unable to take advantage of their invention become businesses unable to take advantage of their business opportunities); and the recovery process (the stochastic planning permission granting process becomes a stochastic bank recovery process).

The banking model in this thesis has two classes of stakeholders: banks and (non-financial) businesses. In the standard model, banks and businesses have aligned interests, so mathematically, it would be possible to construct the model with only one class. However, it is more natural to use two classes as there are two sectors to the economy (financial and business), and it makes interpretation cleaner as the investment opportunities are then clearly rooted in the business sector. Further, it lets us more easily consider extensions, where banks and businesses have different interests.

The role of businesses is to produce socially beneficial products; the role of banks

[40]This result holds for all shock probabilities, all recovery probabilities, all ratios of inter-street to within-street match values, and all population sizes.

[41]For a critical curve of parameters both the boundary partitions are efficient

is to facilitate investment between businesses. The stakeholders are risk neutral, both for tractability, and in line with the Haldane-Simon watchmaker analogy. There is no bank–business strategic interaction: each bank has a representative client business, (section 2.2.2.2 shows that there are still boundary solutions when we allow multiple businesses per bank.); businesses do not move between banks, (a motivation for this would be that, when a module is enabled there is no incentive for them to move, and when a module is disabled it is not feasible); and there is an exogenous split of investment returns between banks and businesses, (the efficiency results are robust to the addition of negotiation between banks and businesses over the distribution of investment returns, as long as the scenario where business opportunities are lost because negotiations break down does not occur).

The businesses are using banks to facilitate business investment. So, it is more natural to think of businesses which are medium sized; rather than small (which are family funded, for example by personal mortgages), or large (which have access to the stock market). The business opportunities come in the form of matches that requires two firms: for example, a finished product that requires both a manufacturing company and a service company.[42] The value of matches depends on whether the matches are *inside* (banks in same module), or *outside* (banks in different modules). Inside matches are of greater value than outside matches, because for example outside matches have higher transaction or search costs. The distribution of matches between businesses is independent of which banks the businesses use. Each match is identically distributed, and so we can just consider the value of a single sample business transaction.

Banks are either *enabled* or *disabled*: enabled banks can facilitate; disabled

[42]In the Fire-Invention analogy, the wives actively match inventions; whilst in the bank-business model, banks are required to provide banking services to businesses, in order for their exogenous matches to be productive.

banks cannot. Disabled banks need not be insolvent: they may be rebuilding their capital buffers and hence be unable to lend. Both businesses in the match need banking services, so in order for a match to be *productive,* both banks need to be enabled. So if either one or both banks are disabled, then the match is *unproductive,* and of no value.[43] When productive, the social benefit of an inside match is 2, whilst that of an outside match is $2\theta < 2$.

The model assumes risk neutrality and comparable utility, so the micro-allocation of returns is not relevant for welfare. However, for notational simplicity, assume that returns are split equally into four quarters: each bank and each business in the match gets a quarter of the total. So with an inside match each bank or business gets 0.5, and with an outside match each gets 0.5θ.

The financial economy is at risk of being hit by negative disabling financial shocks: these cause enabled banks to become disabled. Similarly, there is a stochastic recovery process that repairs disabled banks, and thus re-enables them. These disabling and enabling shocks combine to form a Markov process.[44] Each module is modelled as being at this Markov process's stationary distribution. The intertemporal model in section 2.4.1 shows that this assumption is robust.

The financial shocks will be formulated in discrete time, before considering the model in continuous time. In the discrete version, each period each enabled bank is independently with probability q hit by a shock. If a shock arrives, it then spreads to all other banks in the same module immediately, and with certainty.

[43] One motivation for a business not being able to borrow from an outside bank would be the use of relationship lending, (banks making lending decisions using information collected both historically and at the time of the lending decision); rather than transaction lending, (banks making lending decisions using information collected only at the time of the lending decision), (see Berger (2010) and DeYoung (2010)).

[44] The idea of a Markov process is that the current state, but not past state(s), matter in determining future states. The formal definition of Markov chains (the type of Markov process used in this thesis), is given in appendix B.

Each period a disabled module gets re-enabled with probability ρ. So, all banks in the same module are in the same state. Taking a continuous time limit of this process has two gains: it is more realistic and more tractable. The Markov process only has two states, so to align the discrete and continuous time versions we just need to equate the leaving times of each state.[45]

There is a trade off in module size: the benefit of large modules is that more of the matches are inside one module, and hence (when productive), are of higher value; the cost of large modules is that shocks have larger effects, so large modules are less likely to be enabled, and hence matches are less likely to be productive.

The decision maker, whether the social planner or the banker, makes this choice under an ex-ante basis: they know the parameter values and the probability distribution of both shocks and business opportunities; but do not know the actual future realisations of either the shocks, or the business opportunities. The standard model assumes that the system is already at the limiting state of the Markov process of shocks. So when the economy is centralised, the social planner makes the choice of bank partition using a farsighted utilitarian welfare function; in contrast, when the economy is decentralised, modules are formed by banks playing one of the games described in sections 1.8 and 2.3.3.

This stochastic formulation can be motivated in terms of modelling methodology, stakeholders' preferences, and the feasibility constraints that bank regulators face. There are two methodological reasons. Firstly, Appendix C shows that the Markov process converges exponentially fast, which provides re-assurance that errors in early periods from the system being far from its limiting state are not likely to be significant. Secondly, the Markov process formulation means that after a disabled module is re-enabled, the social planner has no reason to change their original choice of partition. This is because the social planner

[45] See Appendix C for the details.

has gained no extra information: no learning has occurred as they started with perfect information about the system's parameters. Hence, they are in the same position as when they made their initial choice.

There are two preference arguments. Firstly, choosing the structure of banking systems is a long run decision, and the model reflects this: it is reasonable to focus on the welfare of future generations. Secondly, Section 2.4.1 shows that the results are robust to considering an intertemporal model where welfare is summed over all time periods, rather than being evaluated just at the limiting state.

There are three feasibility explanations: firstly, in the form of what information the social planner receives, and how they can use it; secondly, there are lags in the effectiveness of module changes; and thirdly, the evidence of the response to the 2007–2009 financial crisis. In terms of information, firstly, the social planner does not receive the signals required to manipulate the economy for short term advantage: in the financial sector, they cannot predict which banks will go bust; and in the business sector, they cannot predict what the future business matches will be. Secondly, it is not feasible to re-structure inter-bank links continually: so even if they had short run information then they would not be able to use it.

There are lagged effects both when modules merge and when they split. After a merger, it takes time for banks that were previously in different modules to move closer, gain trust in each other, or communicate better. Hence, the full business benefits of being in the same module do not appear immediately in the form of lower transaction costs, or more effective matching. Similarly, after a split, it takes time for shock propagation channels between banks to disappear. And so the full gains that module separation offers in preventing financial shock propagation do not appear immediately.

The response to the crisis has shown four relevant points. Firstly, during a crisis the regulator is fire fighting, not fire preventing: they are focused on fixing bad banks, not changing relationships between the remaining good banks. Secondly, re-organising banks is a slow process: it is already 5 years since the financial crisis started and not much change has occurred; for example 4 years had passed before the ICB report was produced. Thirdly, the changes that are currently happening are explained by learning (regulators were previously too optimistic about light touch banking regulation), rather than by regulators having a pre-crisis planned strategy of state conditional actions. Fourthly, the currently planned changes to banking regulation are not conditional on the occurrence of bank failures. For example, there are three different sets of possible changes to bank reserve requirements: 1) higher reserve requirements for all banks all the time; 2) reserve requirements made pro business cycle, (higher when there is lots of business lending, lower when there is less), to smooth out lending; and 3) higher reserve requirements for banks with riskier lending. However, there is no requirement that one bank being in crisis means that another has a different reserve level.

The solution methodology developed in this work is as follows. There are many possible different partitions of n banks, and it is hard to systematically assess them all. However, we know that a partition that maximises a utilitarian welfare function must be Pareto optimal. So, we start by looking for Pareto optimal partitions, and in particular for Pareto optimal partitions that maximise the expected utility of bank 1. If the partition that maximises the expected utility of bank 1 is symmetric, then we know that it gives the same expected utility to all the other banks.[46] Hence it is the only Pareto optimal partition, and hence the solution to the welfare maximisation program.

[46]Utility is assumed to be non-transferable. If instead utility is transferable, then the social planner ex-ante will still want to maximise the total expected value of all business matches, and ex-post will give all the surplus to bank 1. Hence the social planner will still choose the same (trivial) partition.

Without loss of generality, we assume that bank 1 is in module 1. We have negative externalities: if two other modules merge then this makes bank 1 worse off.[47] So, bank 1 will want the banks not in module 1 to be in singleton modules, and so will choose a partition of form $\{x_1, 1, 1, ...1, 1, 1\}$, where $x_1 \in A_n := \{1, 2, 3, 4.....n\}$. Let $v_1[x_1]$ be the expected utility function for a member of module 1 (hereafter the utility function), and define $x_1^* := \arg\max_{x_1 \in A_n} v_1[x_1]$. We can now find x_1^* using standard single variable calculus methods.

The solution can, potentially, either be on the boundary ($x_1^* = 1$ or $x_1^* = n$), or in the interior ($1 < x_1^* < n$). If $x_1^* = 1$ then the *atomistic partition of singletons* $\{1, 1, ...1, 1\}$ maximises v_1, and by symmetry this partition gives all banks the same utility. So the atomistic partition is then the only Pareto optimal partition, and so maximises the utilitarian welfare function. Similarly, if $x_1^* = n$ then the *grand coalition* $\{n\}$ maximises v_1, is the only Pareto optimal partition, and so maximises the utilitarian welfare function. Either the atomistic partition of singletons or the grand coalition, will be called *trivial partitions* or *boundary partitions*. Any partition which is neither the atomistic partition, nor the grand coalition is a *non-trivial partition*, and also will be called an *interior partition*.

Conversely, if it was the case that $1 < x_1^* < n$ then there would be a non-trivial Pareto optimal partition, and there might be a non-trivial partition that maximised the welfare function. However, under the standard model, such non-trivial partitions are rejected for all parameterisations. Specifically, fix a parametrisation (number of banks (n), shock probability (q), recovery probability (ρ), and value of outside matches (θ)); then every feasible interior partition is strictly dominated by at least one of the boundary partitions. And hence the efficient

[47] Without loss of generality, consider a merger between modules 2 and 3, where bank 2 is a member of module 2. Bank 2 is now less likely to be enabled due to the exposure to financial contagion from module 3, but bank 2 is still an outsider to bank 1. So ex-ante a match between banks 1 and bank 2 has less expected value. Hence bank 1 is worse off.

configuration is proved to be a *trivial partition*: it is either the grand coalition $\{n\}$ or the atomistic partition $\{1, 1....1, 1\}$.[48] Generically, the solution set will consist of just one of the two boundary partitions; although, on a critical curve of parameters the solution set consists of both boundary partitions.

The intuition behind these results is that there are increasing marginal returns from more modules. If there is only one module, then $P(\text{outside match}) = 0$, and $P(\text{a match is productive : one module is disabled}) = 0$. Whilst if the number of modules is increased to two, then $P(\text{outside match}) = {}^1/_2$, and $P(\text{a match is productive:one module is disabled}) = {}^1/_4$: productivity requires that both the matched businesses are in the enabled module. More generally, suppose there are k equally sized modules, then $P(\text{outside match}) = (k{-}1)/k$ and $P(\text{a match is productive:one module is disabled}) = ((k{-}1)/k)^2$. This leads to one way of understanding the rejection of interior partitions. The "cost" of modules comes from outside matches; whilst the "benefit" comes from the resilience the system has to shocks. So these two probabilities can be used to estimate the cost and benefit of modules. Hence the "benefit" $((k{-}1)/k)^2$ is the square of the "cost" $(k{-}1)/k$: so there are increasing marginal returns. This explanation abstracts from a number of factors: for example, how module enablement probability changes with the number of modules, and the possibility of multiple modules being disabled. These factors are all included in the mathematical model given in the next section and the proofs given in subsequent sections.

The choice between the two trivial partitions can be interpreted as follows: *ceteris paribus*, increasing n past a critical value causes a switch in the efficient solution from the grand coalition, to the atomistic partition of singletons. The explanation is that as the number of banks increases, the probability of the one

[48] Efficient in terms either of Pareto optimality or utilitarian welfare maximisation.

all-encompassing module being enabled tends to zero.

A second interpretation is to consider the three cases of *necessity*, *co-ordination* and *protection*. When outside matches are of low value, $\theta \leq \frac{1}{n}$, we need the grand coalition to get a worthwhile return from the business opportunities. This is the *necessity* story and it holds irrespective of the level of shocks. When there are low levels of shocks, $\gamma < \frac{1-\theta}{(\theta n-1)}$, the efficient solution is to *co-ordinate* all the banks: keep them in sync through the grand coalition. Only if both the outside matches are of high value, $\theta > \frac{1}{n}$, and there are high levels of shocks, $\gamma > \frac{1-\theta}{(\theta n-1)}$, do we need the *protection* of singleton modules.[49]

A third interpretation is that it depends on the relative values of outside and inside matches ($\theta = (\theta/1)$), and the relative enablement probabilities ($\frac{P[n]}{P[1]}$). If $\theta < \frac{P[n]}{P[1]}$, then the grand coalition is efficient. If $\theta = \frac{P[n]}{P[1]}$, then both the trivial partitions are efficient. If $\theta > \frac{P[n]}{P[1]}$, then the atomistic partition is efficient.

If the matching process is changed from uniform to circular (imagine a circle of banks, where a business opportunity is always between immediate neighbours), then for module sizes above two the welfare function is now *quasi-concave*, and hence the efficient solution is generally interior. This case is covered in section 2.2.1.1.

One key stability concept used is Extended Equilibrium Binding Agreement (EEBA), from Diamantoudi and Xue (2007), which is fully defined in section 1.8 below. The EEBA is an extension of Equilibrium Binding Agreement (EBA), from Ray and Vohra (1997).[50] Under the EBA modules can only split; whilst under the EEBA there is complete flexibility: not only can existing modules split and merge, but it is feasible for any coalition to deviate, and form a new module or modules. In both concepts, they choose to do so if it makes sense

[49]For completeness, with $\gamma = \frac{1-\theta}{(\theta n-1)}$ there is indifference between the two trivial partitions.

[50]If the EBA is used instead similar results are obtained, but with the EEBA interpretation is cleaner.

on a *farsighted* basis: they accurately predict how other banks will respond to their initial deviation, and deviate if they prefer the final state to the current state. A partition is *stable* if there are no feasible deviations from it which are farsightedly rational. Under the standard model, the trivial partitions are stable if and only if they are efficient.[51] If we consider a variant model, where there are short run banks (zero lifetimes), then for all parametrisations the grand coalition is always stable, and no other partition is stable (see section 2.3.2.1).[52]

In the rest of this chapter: section 1.5 describes in detail the standard model; section 1.7 finds its efficient solutions; section 1.8 investigates their stability; section 1.9 calculates the efficient partition under different parametrisations; and finally section 1.10 considers the policy implications.

1.5 The Standard Model

This section formally defines the *standard model*. As motivated in the Model Overview pages 58-60, we will be evaluating the model at its asymptotic steady state. There are n identical risk neutral banks; each bank has a risk neutral business as a client. Banks facilitate investment between 'real' economy businesses. Each bank is either enabled or disabled. A pair of businesses each has half of a business opportunity. Each business needs their bank to be enabled in order for the match to be *productive*.[53]

Unproductive matches are of value 0. Productive matches are valued differently depending on whether the matched banks are in the same module (*inside*

[51] Further, under the generic case of there being a unique efficient outcome, no non-trivial partition is stable.

[52] This assumes that all the banks start enabled. If any are disabled then the same analysis applies just with the disabled banks excluded.

[53] As the bank-business relationships are fixed we can, without risk of confusion, talk about matches between banks rather than between businesses.

matches), or in different modules (*outside* matches):

Table 1.3: Standard Model Productive Matches: Distribution of Investment returns

Match	Business 1	Bank 1	Bank 2	Business 2	Total
Inside	0.5	0.5	0.5	0.5	2
Outside	0.5θ	0.5θ	0.5θ	0.5θ	$2\theta < 2$

The timings of the model are as follows:

1. Nature determines the system's parameters (n, θ, γ), where $n \in \mathbb{N}$ is the number of banks, $\theta \in (0, 1)$ is the relative value of outside matches and $\gamma \in (0, \infty)$ is the shock parameter.[54] These values are common knowledge.

2. The decision maker (social planner or any of the banks), determines a partition $(x_i \in \mathbb{N}_1)_{i=1}^{k}$ of $N = \{1, 2, 3....n\}$, the set of banks, such that $\sum_{i=1}^{k} x_i = n$, and $(x_i \geq 1)_{i=1}^{k}$.[55] The first constraint means that banks can be grouped, but not created or destroyed; the second requires a minimum module size of one. Let $\mathbb{P}$ be the feasible set of partitions that satisfy both these constraints.

3. The system is evaluated at its steady state. So Nature determines the state of each module independently using the stationary distribution, meaning that[56]

$$P(\text{Module } i \text{ enabled}) = \frac{1}{1 + \gamma x_i}$$

[54] Firstly, I exclude boundary parameter values for simplicity, but with them the key results remain. Secondly, Appendix C shows that the continuous time model that equates to the discrete time model has $\gamma = \frac{-Log[1-q]}{-Log[1-\rho]}$, where q is the bank disabling shock probability and ρ is the module re-enabling probability.

[55] Banks are ex-ante identical so we can identify a partition just in terms of the number of banks in each module.

[56] The derivation of this form for the module enablement probability is given in Appendix C

4. Nature allocates a random business opportunity, which is uniformly distributed on N^2. So $(P(b_1,\ b_2) = 1/n^2)_{b_1,b_2=1}^{n}$.[57]

5. Return is stochastically distributed:

Table 1.4: Standard Model Social Return Distribution

Match	Social Return	Probability
Productive:- Inside	2	$\sum_{i=1}^{k}\left(\frac{x_i^2}{n^2}P[x_i]\right)$
Productive:- Outside	2θ	$\sum_{i=1}^{k}\left(\sum_{j\neq i}\frac{x_i x_j}{n^2}P[x_i]P[x_j]\right)$
Unproductive	0	$1 - P(\text{Return is 2}) - P(\text{Return is }2\theta)$

Consider a partition $x = (x_i)_{i=1}^{k}$. Let $V_i[x]$ be the ex-ante expected total return to all banks in module i.[58] Let $v_i[x]{:=}\frac{1}{x_i}V_i[x]$ be the ex-ante unit expected return: the expected return to a single bank in module i. Let $W[x] := \sum_{i=1}^{k} V_i[x]$ be the social planner's ex-ante welfare function.

For a (generally asymmetric) partition, x:

- Bank Payoff: $v_i[(x_j)_{j=1}^{k}] := \frac{x_i}{n^2}P[x_i] + \sum_{j\neq i}\theta\frac{x_j}{n^2}P[x_i]P[x_j]$
- Module Worth $V_i[(x_j)_{j=1}^{k}] := \frac{x_i^2}{n^2}P[x_i] + \sum_{j\neq i}\theta\frac{x_i x_j}{n^2}P[x_i]P[x_j]$
- Welfare: $W[(x_i)_{i=1}^{k}] := \sum_{i=1}^{k}\left(\frac{x_i^2}{n^2}P[x_i] + \sum_{j\neq i}\theta\frac{x_i x_j}{n^2}P[x_i]P[x_j]\right)$

For the particular case of a symmetric partition, consisting of k modules each with d members, then, with a slight abuse of notation, these evaluate to:

- Bank Payoff $v_i[d] := \frac{d}{n^2}P[d] + \theta\frac{(n-d)}{n^2}P^2[d]$
- Module Worth $V_i[d] := \frac{d^2}{n^2}P[d] + \theta\frac{d(n-d)}{n^2}P^2[d]$
- Welfare $W[d] := \frac{d}{n}P[d] + \theta\frac{(n-d)}{n}P^2[d]$

[57] Allowing the self match $(b_1,\ b_1)$ makes the results cleaner. Without them we still always reject non-trivial partitions: just that the parameter condition for switching between the two trivial partitions is more complicated.

[58] As the banks and the businesses get the same return we can just consider the bank return.

So far we have been considering the return of a bank in module i, rather than the preferences of a generic bank j. These are linked by the straightforward condition, that j prefers the partition where its module gets the highest per bank return. So $x \succ_j x^*$ if and only if $v_i[x] > v_{i^*}[x^*]$: where j is in modules i and i^* for partitions x and x^* respectively. This completes our definition of what Diamantoudi and Xue (2007) call a simple coalition formation game with externalities. It is also a symmetric game in the sense of Yi (2003).

1.6 Partition Form Games

In co-operative game theory the main focus is on coalition formation rather than on the question of what coalition structure is the most efficient. And it is often assumed that the grand coalition will form; as Maskin (2003), in his Presidential address to the Econometric Society, says "Perhaps one reason that cooperative theory has not been more influential on the mainstream is that its two most important solution concepts for games of three or more players, the core and the Shapley value, presume that the grand coalition—the coalition of all players—*always* forms.". The formation, and efficiency, of the grand coalition is often driven by the assumption of *superadditivity*: "A game is superadditive if the union of two disjoint coalitions can obtain at least the sum of the payoffs of the two separate coalitions", (Maskin, 2008). A second criticism is that: "most cooperative theory ignores externalities, the possibility that a coalition can be affected by the actions of those not in the coalition", (Maskin, 2008). In contrast to such *characteristic form games* (CFGs), with *partition form games* (PFGs), there are externalities: how outside agents are grouped *does* affect you.[59] This same difference exists between PFGs and Club Goods models: see Appendix J

[59] See Definition 7 in Section 1.7.

for more details.

This section reviews the partition form game literature and explains how my standard model from Section 1.5 fits into that literature. Next, section 1.7 establishes properties for a general partition model under which the efficient partition is either the grand coalition or the atomistic partition of singletons, and shows that Section 1.5 constructs such a model. Then Section 1.8 starts by considering the extra difficulties in considering stability within PFGs; before applying the Extended Equilibrium Binding Agreement (EEBA) to the general partition model. The standard model is solved using four further solution concepts in Section 2.3.3: the three concepts used by Yi (1997) for symmetric games (The Simultaneous-Move Open Membership Game, The Unanimity Game and Equilibrium Binding Agreement (EBA)); as well as a PFG developed version of the Jackson Wolinsky (1996) network solution concept of bilateral stability.

In this section, first I consider the literature on partition formation, and start with attempts to extend the two main characteristic form solution concepts, the Shapley value and the core, to partition form games. Then I consider the two main approaches developed specifically to solve partition form games: blocking and bargaining. Second I consider efficiency conditions for partition form games; I start with superadditivity and how that condition needs to be altered in PFGs in order to ensure grand coalition efficiency. Some of the applications of these approaches to my standard model are included in the subsequent Efficiency and Stability sections, which focus on *hedonic* games.

Maskin (2003) develops an extension of the Shapley value for superadditive partition form games.[60] Maskin assumes four axioms for the partition formed and the payoff allocation.[61] The first axiom requires that coalitional worth not be

[60]See Shapley (1953).

[61]The axiom descriptions are adapted from Ray(2007), as well as of course Maskin (2003).

wasted: the payoff allocation must be such that the sum of payoffs allocated to members of each coalition equals the worth of that coalition. The second and third axioms consider players' marginal contributions: the second axiom requires that each player be assigned to a coalition where their marginal contribution is highest; the third requires that she indeed be allocated this marginal contribution. The fourth axiom requires inter-stage consistency: the final payoffs and partition that result if the partition of the first $j-1$ players has already formed, should be the same as those that result if player j then joins the coalition to which his gross marginal contribution is greatest.

Maskin (2003) argues that when there are significant positive externalities, due to the free rider effects, it is unreasonable to assume that the grand coalition will form; whilst conversely with negative externalities "the likelihood of the grand coalition forming would only be strengthened". He states three theorems. Theorem 1 is an existence result: for any superadditive partition form game, and any order of players there is a solution satisfying his axioms. Theorem 2 is a grand coalition formation result: if the PFG is superadditive and has negative externalities then the grand coalition will form. Theorem 3 is that his axioms are indeed an extension of the Shapley Value CFG concept: if the PFG is superadditive and there are no externalities, then taking the average over all the $n!$ orderings generates exactly the Shapley value. He provides detailed proofs for the 3 player case and claims, "The extension to $n > 3$ (more than 3 players), uses exactly the same methods.". However, counter examples with 4 players are provided in Hafalir (2007), and Cao and Yang (2011).[62] This motivates why the later papers considered below have considered stronger and different conditions from *bilateral superadditivity*, such as *supermodularity*, *multilateral superadditivity* and *grand coalition superadditivity*.

[62] Cao and Yang (2011) provide a counter example for Theorem 1; whilst Hafalir (2007) provides counter-examples for Theorems 2 and 3.

The second approach used to solve characteristic form games is the *core*: the set of feasible allocations $v = (v_j)_{j=1}^n$ that cannot be *blocked* by any deviating coalition D.[63] And we say that D *blocks* v, if coalition D can deviate and make themselves all better off. Formally, $\emptyset \subsetneq D$ and $\left(u_j^{cf}[D] > v_j\right)_{j \in D}$. Note that here the group D of deviators don't need to consider how the residual players $N \setminus D$ respond, as it is a CFG their payoff is independent of how $N \setminus D$ partition themselves into coalitions.

However, in contrast, in partition form games how the residual agents structure themselves *does* matter. This is a well known issue and, hence with PFGs there are different definitions of the core depending on how we model the other agents structuring themselves in respond to the deviation. For example, Hafalir (2007) provides 4 different definitions: the first, the *s-core*, is the core with singleton expectations: the deviators D expect $N \setminus D$ to partition themselves into singletons; the second, the *c-core*, is the core with cautious (pessimistic; punishing) expectations: the deviators D expect $N \setminus D$ to partition themselves into the partition that minimises the value of the module D; the third, the *m-core*, is the core with merging expectations: the deviators D expect the members of $N \setminus D$ to merge and form a single module in response; and the fourth, the *r-core*, is the core with rational expectations: the deviators D expect $N \setminus D$ to partition themselves into the partition that maximises the value of $N \setminus D$. In models with negative externalities, such as those considered here, the punishment response and the merging response are the same and hence the *c-core* and the *m-core* are the same. Further, within my standard model deviating from the efficient partition makes both the deviators and the non-deviators (weakly) worse off; hence for all 4 definitions of the core, the core contains the efficient partitions payoffs.

[63]Feasibility means that for some partition π, the payoff of π is greater than or equal to $\sum_{j=1}^n v_j$. It often assumed that this partition is the grand coalition.

A more subtle issue is the question of what allocations are feasible. If the Grand Coalition is efficient then the set of feasible allocations simply consists of the $v = (v_j)_{j=1}^n$ such that $\sum_{j \in N} v_j \leq V[N, \{N\}]$, where $V[N, \{N\}]$ is the worth of the grand coalition. With PFGs there is often an implicit assumption that the grand coalition is efficient: Hafalir (2007) explicitly assumes *supermodularity* which make the grand coalition efficient, but then is implicit about the feasible allocations when defining the core; whilst McQuillin (2009), when describing an extension and generalisation of Shapley approach to PFGs, acknowledges "the implicit supposition in the (his) Efficiency axiom that" the grand coalition is efficient. However, if this assumption no longer holds, and it is not (necessarily) the case that the grand coalition is efficient, then we need to define what are the feasible allocations. One way is as follows, and for each partition to allow transferable utility *within* modules, but not *between* modules. So the feasible allocations allowed by partition $\pi = (\pi_i)_{i=1}^k$ are $v^\pi = (v_j^\pi)_{j=1}^n$ such that $\sum_{j \in \pi_i} v_j^\pi \leq V[\pi_i, \pi]$, where $V[\pi_i, \pi]$ is the worth of module π_i within partition π. And an allocation is feasible in a PFG if it is feasible for some partition π.

One solution technique developed specifically for PFGs in de Clippel and Serrano (2008A) is that of *strict dominance*. The concept is first applied to coalitions: coalition S is a *strictly dominant coalition* if each member of S ranks S strictly as the best coalition, whatever the other players do.[64] Specifically, if π is a partition that contains S, π' is a partition that does not contain S, and j is a member of S, then require $v_j[\pi] > v_j[\pi']$. The concept is then applied iteratively to potentially form a *strictly dominant partition* as follows. The initial step is that any strictly dominant coalitions S form. The iterative step is that, given the existing coalitions, there maybe a new strictly dominant coalition T, which then forms. The iterative step is repeated as often as possible. If every member

[64] This equates to the members of S, as in the c-core, having pessimistic expectations.

ends up in a coalition then the resulting partition is called strictly dominant. de Clippel and Serrano (2008A) shows the usefulness of strictly dominant partitions: If a strictly dominant partition exists then it is both the only member of the pessimistic core and the equilibrium outcome of the sequential unanimity game they develop. Applying strict dominance to my standard model gives the following results: if the grand coalition is efficient then the grand coalition is a strictly dominant partition; however, if the atomic partition of singletons is efficient, then there are no strictly dominant coalitions.[65]

Yi (1997) considers *symmetric* games with ex-ante identical players, so it is the number of players in each module, rather than their identities, that determines payoffs. He surveys a number of different formation games, such as the Open Membership Game (Yi and Shin, 1995), the Infinite-Horizon Coalition Unanimity Game (Bloch, 1996), and the Equilibrium Binding Agreements from Ray and Vohra (1997). These approaches to partition formation are applied to my standard model in chapter 2 of this thesis.

In the second part of this section, we now move on from partition formation to partition efficiency. Characteristic function games (CFGs) are often assumed to be *superadditive*: this requires that if S and T are disjoint coalitions then $V[S \cup T] \geq V[S] + V[T]$. With CFGs there are no externalities onto other modules, so superadditivity implies that the grand coalition is efficient. This definition of *bilateral superadditivity* can be extended to PFGs:

[65] In the standard model, due to symmetry considerations, when the grand coalition is efficient it maximises not just the total welfare but also utility of every individual member, (see Proposition 14 in specific and Efficiency section 1.7 in general). When the atomic partition is efficient, the result follows from negative externalities: if a singleton module forms then the pessimistic response is a single module containing $n-1$ members; if a 2 member module forms then the optimistic response is a $n-2$ singleton modules. And $v_1[\{1, n-1\}] > v_1[\{2, 1, 1, 1....1\}]$ requires $\frac{1}{\theta} + n < 3$ and $n > 1$, which is impossible.

Definition 1. if $\pi = (\pi_i)_{i=1}^k$ is a partition then

$$V[\pi_1 \cup \pi_2, \{\pi_1 \cup \pi_2, \pi_{2+}\}] \geq V[\pi_1, \pi] + V[\pi_2, \pi]$$

where $\pi_{2+} := (\pi_i)_{i=3}^k$.[66]

However, with PFGs bilateral superadditivity does *not* imply grand coalition efficiency: if a PFG has negative externalities then although merging modules always benefit, outside modules may be worse off and the total payoff in the grand coalition can be less then the total payoff from some other partition.[67] Conversely, if a PFG has positive externalities, then super-additivity does indeed imply the efficiency of the grand coalition: merging modules always benefit, as do outside modules.

In order to produce sufficient conditions for grand coalition efficiency in PFGs, various strengthenings of bilateral superadditivity have been proposed. Ray (2007) proposes *grand coalition superadditivity*: directly postulating the efficiency of the grand coalition. Clippel and Serrano (2008), propose *multilateral superadditivity:*

Definition 2. Multilateral superadditivity requires that if $\pi = (\pi_i)_{i=1}^k$ is a partition and $1 \leq j \leq k$ then

$$V[\bigcup_{1 \leq i \leq j} \pi_i, \{\bigcup_{1 \leq i \leq j} \pi_i, \pi_{j+}\}] \geq \sum_{i=1}^{j} V[\pi_i, \pi]$$

where $\pi_{j+} := (\pi_i)_{i=j+1}^k$.

Note that for CFGs, by a simple iterative argument, bilateral superadditivity implies multilateral superadditivity. However, with PFGs this is no longer the

[66]Theorem 28 shows that with my standard model, the grand coalition is efficient when it is bilaterally superadditive.

[67]See Example 1 in Hafalir (2007).

case, as the list of outside modules changes. Taking $j = k$ shows that multilateral superadditivity is a sufficient condition for grand coalition efficiency, and taking $j = 2$ shows that multilateral superadditivity is a sufficient condition for bilateral superadditivity.

A third approach is *supermodularity*:

Definition 3. Suppose that $\pi = (\pi_i)_{i=1}^k$ is a partition and S, T are *any* coalitions such that $S \bigcup T = \pi_1 \bigcup \pi_2$: specifically we allow S and T *not* to be disjoint, allowing $S \bigcap T \neq \emptyset$. Without loss of generality, $c := |S \cap T|$, $t := |T \setminus S|$ and $s := |S \setminus T|$. Then supermodularity requires that

$$V[S \cup T, \{S \cup T, \pi_{2+}\}] + V[S \cap T, \{S \cap T,\ S \setminus T,\ T \setminus S,\ \pi_{2+}\}] \geq$$
$$V[S, \{S, T \setminus S, \pi_{2+}\}] + V[T, \{T,\ S \setminus T,\ \pi_{2+}\}]$$

where $\pi_{2+} := (\pi_i)_{i=3}^k$.[68]

Supermodularity implies both bilateral superadditivity (consider the case where $S = \pi_1$ and $T = \pi_2$), and multilateral superadditivity (see Proposition 1 in Hafalir (2007), which proves the result using induction). The supermodularity condition can be re-arranged into a 'convexity' increasing increases condition:

$$V[S \cup T, \{S \cup T, \pi_{2+}\}] - V[S, \{S, T \setminus S, \pi_{2+}\}] \geq$$
$$V[T, \{T,\ S \setminus T,\ \pi_{2+}\}] - V[S \cap T, \{S \cap T,\ S \setminus T,\ T \setminus S,\ \pi_{2+}\}]$$

This says that the increase in module worth when t members transfer from $T \setminus S$ to S forming $S \cup T$, is greater than the increase in module worth when t members transfer from $T \setminus S$ to $S \cap T$ forming T. With my standard model, Theorem

[68] Note that this is supermodularity of the module worth function V with respect to the embedded coalition lattice, rather than of the welfare function W with respect to the partition lattice. This prevents the use of the Topkis (1978, 1998) results. See Appendix L for details.

30 shows that the grand coalition is efficient if and only if it is supermodular. However, this thesis introduces a novel quasi-convexity condition, and in general supermodularity is a stronger condition than the quasi-convexity condition in two aspects: firstly, supermodularity is a condition on all partitions, whilst my quasi-convexity approach is a condition only on partitions with at most one non-singleton module; secondly, it is a convexity condition rather than a quasi-convexity condition.

Finally, within the PFG literature there is a lack of results on the efficiency of the atomic partition. Here my thesis makes a particular contribution: it gives efficiency results for the atomic partition in terms of quasi-convexity and subadditivity (Theorems 14 and 32). And for the standard model it shows that the subadditivity condition is strictly stronger than that required for efficiency of the atomic partition (Theorem 33).

1.7 Efficiency

In this section, I will first abstract from the standard model and consider general bank utility functions on the set of partitions. Then, I will consider the solution concept of efficiency. Next, I will show that if the preferences satisfy the three properties of *anonymity* (it is only the size of modules that count, and not their indices), *negative externalities* (if modules 2 and 3 merge, then members of module 1 are always worse off), and *strict quasi-convexity* (note, this is required to hold only on the set of partitions of with at most one non-singleton module, $\{x_1, 1, 1, 1....1\}$), then any efficient partition must exist on the boundary. I then show that the standard model satisfies these three properties generically, and so has boundary solutions.

I will start with a few mathematical preliminaries:

Definition 4. A *partition* $(X_i)_{i=1}^k$ of a set X is a pairwise disjoint covering so, $\bigcup\limits_i X_i = X$ and $(X_i \bigcap X_j = \emptyset)_{i \neq j}$. In mathematics, partitions are important special examples of lattices.[69] In lattice theory a member X_i of a partition is often called a block. However, here, in line with the motivation given above, the term *module* will be used. In contrast, the term *coalition* will refer to a general subset Z of X; coalitions may or may not be modules. Generally, let A_l be the first l strictly positive natural numbers, where l is itself a natural number: so $A_l := \{1, 2, 3.....l\}$. We restrict to consideration to *symmetric* models: we have n ex-ante identical banks, and so we just specify the number x_i of banks in each module i, this gives a general partition $(x_i)_{i=1}^k$. Note that generally i is an index on modules, not banks. Define the *grand coalition* to be $\{n\}$. Define the *atomistic partition of singletons* to be $\{1, 1, 1, 1....1\}$. These will be called the two *trivial partitions*; they will also be referred to as *boundary solutions.* Other partitions are *non-trivial,* and have *proper* or *real modularisations.* For a specific, $n \in \mathbb{N}$, let $\mathbb{P}^n$ be the set of all partitions of A_n, and let $\mathbb{P}_i^n$ be the subset of $\mathbb{P}^n$ where the module i is non-empty. Let $\mathbb{I}^n = \mathbb{P}^n \setminus \{\{n\}, \{1, 1, 1, 1...1\}$, be the proper partitions of A_n. Finally, taking the union over possible values of n: $\mathbb{P} := \bigcup\limits_{n \in \mathbb{N}} \mathbb{P}^n$ and $\mathbb{P}_i := \bigcup\limits_{n \in \mathbb{N}} \mathbb{P}_i^n$.

Next, we define when one partition can be formed from another partition, by merging modules:

Definition 5. Partition x is *strictly coarser* than y if, x and y are distinct, and each module in x is either: a module in y, or can be formed as a *merger* of modules in y. Specifically, if x has k modules and y has l modules then require $k < l$ and the existence of a mapping $f : A_l \to A_k$ *s.t.* $\left(x_j = \sum_{f(i)=j} y_i\right)_{j=1}^k$, where $A_l = \{1, 2, 3, 4.....l\}$ and $A_k = \{1, 2, 3, 4.....k\}$. Conversely, partition x is *strictly finer* than y, if and only if, partition y is *strictly coarser* than x.

[69]See Appendix L which includes a presentation of the partition lattice.

Next, we define the utility and welfare functions for this general model:

Definition 6. Consider a general partition $x = (x_i)_{i=1}^{k} \in \mathbb{P}$. Let the *module utility* function $V_i^g : \mathbb{P}_{\mathrm{i}} \to \mathbb{R}$, where $V_i^g[x]$ is the total return to all members of module i from partition x. Let the *bank utility* function $v_i^g : \mathbb{P}_{\mathrm{i}} \to \mathbb{R}$ such that $v_i^g[x]{:=}\frac{1}{x_i}V_i^g[x]$ is the return per member of module i. Let *welfare* function $W^g : \mathbb{P}_{\mathrm{i}} \to \mathbb{R}$ and $W^g[x] := \sum_{i=1}^{k} V_i^g[x]$ be the social planner's utilitarian welfare function. These three functions together form a general *partition model.* So far we have been considering the return of a bank in module i, rather than the preferences of a generic bank b. As in the standard model, these are linked by the straightforward condition, that bank b prefers the partition where its module gets the highest per bank return. So $x \succ_b^g x^*$ if and only if $v_i^g[x] > v_{i^*}^g[x^*]$: where b is in modules i and i^* for partitions x and x^* respectively.

The definition of a Partition Form Game originates in Lucas (1963). The modern notation, as in for example Hafalir (2007), is:

Definition 7. In a Partition Form Game any coalition $S \subseteq N$ generates a value $V[S, \pi]$ where π is a partition of N and $S \in \pi$.

Next, I define the three conditions which are required for the boundary solutions result to hold. The first, *anonymity,* intuitively means that agents do not care about their indices. This needs to apply both in terms of your own index and those of other modules. Anonymity means that the model forms a symmetric PFG. Formally:

Anonymity requires that both the following two properties hold for all *x:*

1. if $x_i \neq 0$ then $V_i^g[x] = V_1^g[x^*]$, where $x_1^* := x_i$, $x_i^* := x_1$, and $x_j^* = x_j$ for all other j.

2. if x^* is a permutation of x such that $x_i^* = x_i$ then $V_i^g[x] = V_i^g[x^*]$.[70]

The second condition is that the model satisfies *negative externalities* in the N1 sense of Yi (1997): intuitively, if two modules a and b merge, then banks in a third module c are always worse off. The formal definition is that:

Definition 8. The general partition model satisfies *negative externalities*, if when modules merge to form a larger module, outside modules not involved in the merger are strictly worse off. Specifically, if x is strictly coarser than y, and $x_i = y_i$, then that implies that $v_i[x] < v_i[y]$. Conversely, the general partition model satisfies *positive externalities*, if when modules merge to form a larger module, outside modules not involved in the merger are strictly better off. Specifically, if x is strictly coarser than y, and $x_i = y_i$, then that implies that $v_i[x] > v_i[y]$.

The third property is that partitions where at most one module is non-singleton, must possess *weak quasi-convexity.* So we now define the bank utility $v_{1,1}^g[x_1]$ for partitions where apart from a first module of size x_1, the other modules are singletons. This is formed as a restriction of the bank payoff function $v_1^g[x]$:

Definition 9. Bank utility function $v_{1,1}^g$ is defined by $v_{1,1}^g : \mathrm{A}_n \rightarrow \mathbb{R}$ s.t. $v_{1,1}^g[x_1] = v_1^g[x]$, where $x := \{x_1, 1, 1, 1, 1...1\}$.

Consider the general definition of strict and weak quasiconvexity:

Definition 10. Suppose $f : S \rightarrow \mathbb{R}$ where S is a convex subset of $\mathbb{R}^l$. Then f *is strictly quasiconvex* if and only if $f[\lambda x + (1-\lambda)x^0] < \max\{f[x], f[x^0]\}$ for all $\lambda \in (0,1)$ and $(x, x^0) \in S^2$ where $x \neq x^0$.[71] Then f *is weakly quasiconvex* if and only if $f[\lambda x + (1-\lambda)x^0] \leq \max\{f[x], f[x^0]\}$ for all $\lambda \in (0,1)$ and $(x, x^0) \in S^2$ where $x \neq x^0$.

[70] $x^* = (x_i^*)_{i=1}^k \in \mathbb{P}$ is a permutation of $x = (x_i)_{i=1}^k \in \mathbb{P}$ iff $\exists$ bijection $f : A_k \rightarrow A_k$ s.t. $(x_i^* = x_{f(i)})_{i=1}^k$

[71] See Definition 103 in Appendix F.

And applied to partitions with at most one non-singleton module, where the size of module 1 is a natural number:

Definition 11. Bank utility function with at most one non-singleton module is *strictly quasi-convex* if $\left(v^g_{1,1}[x_1] < \max\{v^g_{1,1}[a], v^g_{1,1}[b]\}\right)_{x_1=a+1}^{b-1}$ where $a < b$. Bank utility function with at most one non-singleton module is *weakly quasi-convex* if $\left(v^g_{1,1}[x_1] \leq \max\{v^g_{1,1}[a], v^g_{1,1}[b]\}\right)_{x_1=a+1}^{b-1}$.

Now we define the solution concept of strict efficiency:

Definition 12. x^* is *strictly efficient* if it maximises $W^g[x]$, the utilitarian welfare function.

We now consider the effect of these assumptions on a general model. First, if a model has negative externalities then a bank wants the other modules to be of minimal size:

Proposition 13. *If the partition model satisfies negative externalities and* $x^* \in \arg\max_{x \in \mathbb{P}^n} v_1[x]$ *then* $x^* = \{x^*_1, 1, 1, 1, 1....1\}$ *for some* x^*_1.

Proof. Proof by contradiction. Suppose partition x^* is not of the required form so there exists a second non-singleton module. Form partition y by splitting that module into singletons. However, as the model satisfies negative externalities, and as x^* is strictly coarser than y, and $x^*_1 = y_1$, then that implies that $v^g_1[x^*] < v^g_1[y]$. □

The following proposition shows that if there is *strict* quasi-convexity, then everyone is worse off at an inefficient partition:[72]

Proposition 14. *If the partition model has anonymity, negative externalities and* $v^g_{1,1}[x_1]$ *is* strictly *quasi-convex then*

[72]This proposition is used in the stability section 1.8.

1. *if* $v^g_{1,1}[1] > v^g_{1,1}[n]$ *then* $x \neq \{1,1,1...1\} \Rightarrow \left(x \prec^g_j \{1,1,1....1\}\right)^n_{j=1}$

2. *if* $v^g_{1,1}[n] > v^g_{1,1}[1]$ *then* $x \neq \{n\} \Rightarrow \left(x \prec^g_j \{n\}\right)^n_{j=1}$

3. *if* $v^g_{1,1}[1] = v^g_{1,1}[n]$ *then* $x \in \mathbb{I}^n \Rightarrow \left(x \prec^g_j \{n\} \text{ and } x \prec^g_j \{1,1,1...1\}\right)^n_{j=1}$.

Proof. Let us start by searching for the partition, or partitions, x^* that maximise(s) the utility of bank 1. By anonymity we can assume, without loss of generality, that bank 1 is in module 1, and so we want to maximise $v^g_1[x]$. The model has negative externalities, so by Proposition *13,* $x^* = \{x^*_1, 1, 1, 1, 1....1\}$ for some x^*_1. Further, as $v^g_{1,1}[x_1]$ is strictly quasi-convex, by Definition *11,* $x^* = \{1, 1, 1, , , 1\}$ *or* $x^* = \{n\}$. 1) follows, as by anonymity for all i, $v^g_i[\{1, 1....1\}] = v^g_1[\{1, 1....1\}]$, so if $v^g_{1,1}[1] > v^g_{1,1}[n]$ then $x \neq \{1,1,1....1\} \Rightarrow \left(x \prec^g_j \{1,1,1....1\}\right)^n_{j=1}$. 2) follows as partition $\{n\}$ only has one module, so it gives every bank the same utility, hence *if* $v^g_{1,1}[n] > v^g_{1,1}[1]$ *then* $x \neq \{n\} \Rightarrow \left(x \prec^g_j \{n\}\right)^n_{j=1}$. Similarly 3) follows. □

This is the main result, a general model with our three properties will always reject interior partitions as inefficient:

Theorem 15. *If the partition model has anonymity, negative externalities and* $v^g_{1,1}[x_1]$ *is weakly quasi-convex then the only possible candidate partitions to be strictly efficient are* $\{1, , , , , 1\}$ *and* $\{n\}$*.*

Proof. If $v^g_{1,1}[x_1]$ is not only weakly quasi-convex, but also strictly quasi-convex then, by Proposition 14, $\text{argmax}_{x\in\mathbb{P}} W^g[x] = \text{argmax}_{x\in\mathbb{P}} v^g_1[x] \subseteq \{\{1,1,1,1....1\}, \{n\}\}$. Now instead suppose that $v^g_{1,1}[x_1]$ is weakly quasi-convex, but not strictly quasi-convex. Then the previous arguments about $\{1, , , , , 1\}$ and $\{n\}$, still hold. So at least one of those 2 partitions is strictly efficient. However, suppose some interior x^*_1 argmax of $v^g_{1,1}[x_1]$ generates a further strictly efficient partition $x^* = \{x_1, 1, 1, ...1\}$. Suppose $v^g_1[\{1, 1111\}] \geq v^g_1[\{n\}]$ then x^* cannot be strictly efficient, as banks not in module 1 strictly prefer $\{1, 1, , , , 1\}$ to x^*, (due to negative externalities), and banks in module 1 are indifferent, (as both x^* and $\{1, 1, 11....1\}$ maximise $v_1^g[x]$).

Conversely, suppose that $v_1^g[\{n\}] > v_1^g[\{1, 1, 1...1\}]$, and so $\{n\}$ is both strictly efficient and an argmax for $v_1^g[x]$. We need $\left(x^* \preceq_j^g \{n\}\right)_{j=1}^n$, (as else we could put bank 1 in j's module and strictly increase their utility). And so we further need $\left(x^* \sim_j^g \{n\}\right)_{j=1}^n$, (as else x^* cannot be strictly efficient). So considering module 2 as a sample outside module: $v_2^g[x^*] = v_1^g[\{n\}]$. However, $v_2^g[x^*] < v_2^g[\{1, 1, ...1\}]$, as banks not in module 1 experience negative externalities from module 1. This now gives a contradiction: $v_1^g[\{1, 1, 1...1\}] < v_1^g[\{n\}] = v_2^g[x^*] < v_2^g[\{1, 1....1\}] = v_1^g[\{1, 1....1\}]$. $\square$

Next we derive a number of results about the inefficiency of the boundary partitions in terms of the elasticity of the module function, $V_{1,1}^g[x_1]$, rather than the bank function $v_1^g[x_1]$. Informally, $V_1^g[x_1]$, is the total worth of module 1, when all other modules are singletons. Formally

Definition 16. Module utility function with at most one non-singleton module $V_{1,1}^g : \mathrm{A}_n \rightarrow \mathbb{R}$ s.t. $V_{1,1}^g[x_1] = x_1 * v_{1,1}^g[x_1]$ or equivalently $V_{1,1}^g[x_1] = V_{1,1}^g[x]$, where $x := \{x_1, 1, 1, 1, 1...1\}$.

We now prove two proposition linking $V_{1,1}^g[x_1]$ and $v_{1,1}^g[x_1]$. With a slight abuse of notation, we consider $V_{1,1}^g[x_1]$ with domain extended from $\mathrm{A}_n := \{1, 2, 3, 4...n\}$ to the reals. This extension is required in order for point elasticities to be well defined.

Proposition 17. *Suppose $V_{1,1}^g[x_1]$ is differentiable and $V_{1,1}^g[x_1] \geq 0$. Then $v_1^{g'}[x_1] > 0$ if and only if $V_{1,1}^g$ is elastic at x_1.*

Proof. $\frac{dv^g_{1,1}[x_1]}{dx_1} = \frac{d\frac{V^g_{1,1}[x_1]}{x_1}}{dx_1} = \frac{x_1 V^{g'}_{1,1}[x_1] - V^g_{1,1}[x_1]}{x_1^2}$, where $V^{g'}_{1,1}[x_1] = \frac{dV^g_{1,1}[x_1]}{dx_1}$. So $\frac{dv^g_{1,1}[x_1]}{dx_1} > 0$ if and only if $x_1 V^{g'}_{1,1}[x_1] - V^g_{1,1}[x_1] > 0$ if and only if $\frac{x_1 V^{g'}_{1,1}[x_1]}{V^g_{1,1}[x_1]} > 1$ if and only if $\epsilon_{V^g_{1,1},x_1} > 1$, where $\epsilon_{V^g_{1,1},x_1} := \frac{x_1 V^{g'}_{1,1}[x_1]}{V^g_{1,1}[x_1]}$ is the elasticity of $V^g_{1,1}$ at x_1. □

Proposition 18. *Suppose $V^g_{1,1}[x_1]$ is differentiable and $V^g_{1,1}[x_1] \geq 0$. Then $v^{g'}_1[x_1] < 0$ if and only if $V^g_{1,1}$ is inelastic at x_1.*

Proof. See the proof of Proposition 17. □

Corollary 19. *If the partition model has anonymity, negative externalities and $V^g_{1,1}$ is elastic for all x_1 then the only strictly efficient partition is $\{n\}$.*

Proof. This follows directly from Theorem 15 and Proposition 17. □

Corollary 20. *If the partition model has anonymity, negative externalities and $V^g_{1,1}$ is inelastic for all x_1 then the only strictly efficient partition is $\{1, 1, 1...1\}$.*

Proof. This follows directly from Theorem 15 and Proposition 18. □

Corollary 21. *If the partition model has anonymity, negative externalities, $V^g_{1,1}[x_1]$ is convex and $V^g_{1,1}[0] = 0$ then the only strictly efficient partition is $\{n\}$.*

Proof. First note that the assumption $V^g_{1,1}[0] = 0$ is without loss of generality as $V^g_{1,1}$ is an extension of an original function defined on domain $\{1, 2, 3...n\}$. Second we use the quasi-monotone conditions for a convex function: from Theorem 21.2 in Simon and Blume (1994), if $V^g_{1,1}$is convex then $\frac{V^g_{1,1}[y_1] - V^g_{1,1}[x_1]}{y_1 - x_1} \leq V^{g'}_{1,1}[x_1]$, when $y_1 < x_1$. Taking $y_1 = 0$ gives that $V^g_{1,1}$ is elastic at x_1, and the result then follows directly from Corollary 19. □

Corollary 22. *If the partition model has anonymity, negative externalities, $V^g_{1,1}[x_1]$ is concave and $V^g_{1,1}[0] = 0$ then the only strictly efficient partition is $\{1, 1....1\}$.*

Proof. A similar methodology to that used in Corollary 21 shows that if $V_{1,1}^{g}[x_1]$ is concave then $V_{1,1}^{g}$ is inelastic for all x_1, and the result then follows directly from Corollary 20. □

Finally, we show that the standard model satisfies each of the required three conditions, and hence its efficient partition is always either the grand coalition or the atomistic partition of singletons.

Proposition 23. *The standard model satisfies anonymity for all parameter values.*

Proof. We demonstrate each of the two required conditions hold for all x:
if $x_i \neq 0$ then let $x_1^* := x_i$, $x_i^* := x_1$, and $x_j^* := x_j$ for all other j. Then:

$V_1[x^*] = \frac{x_1^{*2}}{n^2} P[x_1^*] + \sum_{j \neq 1} \theta \frac{x_1^* x_j^*}{n^2} P[x_1^*] P[x_j^*] = \frac{x_i^2}{n^2} P[x_i] + \sum_{j \neq 1} \theta \frac{x_i x_j^*}{n^2} P[x_i] P[x_j^*]$

$\Rightarrow V_1[x^*] = \frac{x_i^2}{n^2} P[x_i] + \sum_{j \neq 1, j \neq i} \theta \frac{x_i x_j}{n^2} P[x_i] P[x_j] + \sum_{j=i} \theta \frac{x_i x_1}{n^2} P[x_i] P[x_1]$

$\Rightarrow V_1[x^*] = \frac{x_i^2}{n^2} P[x_i] + \sum_{j \neq 1, j \neq i} \theta \frac{x_i x_j}{n^2} P[x_i] P[x_j] + \sum_{j=1} \theta \frac{x_i x_1}{n^2} P[x_i] P[x_1]$

$\Rightarrow V_1[x^*] = \frac{x_i^2}{n^2} P[x_i] + \sum_{j \neq i} \theta \frac{x_i x_j}{n^2} P[x_i] P[x_j] = V_i[x]$ as required

and

Suppose x^* is a permutation of x such that $x_i^* = x_i$ then $\exists$ bijection $f : A_k \to A_k$ s.t. $(x_i^* = x_{f(i)})_{i=1}^{k}$ and $f(i) = i$. So:

$V_i[x^*] = \frac{x_i^{*2}}{n^2} P[x_i^*] + \sum_{j \neq i} \theta \frac{x_i^* x_j^*}{n^2} P[x_i^*] P[x_j^*]$

$\Rightarrow V_i[x^*] = \frac{x_i^2}{n^2} P[x_i] + \sum_{j \neq i} \theta \frac{x_i x_j^*}{n^2} P[x_i] P[x_j^*]$

$\Rightarrow V_i[x^*] = \frac{x_i^2}{n^2} P[x_i] + \theta \frac{x_i}{n^2} P[x_i] \sum_{j \neq i} x_j^* P[x_j^*]$

$\Rightarrow V_i[x^*] = \frac{x_i^2}{n^2} P[x_i] + \theta \frac{x_i}{n^2} P[x_i] \sum_{j \neq i} x_j P[x_j] = V_i[x]$ as required. □

Intuitively, there are negative externalities in the standard model, as a merger between modules j and k has no benefits to module i, but does have costs as matches between module i and j, (or i and k), are now less likely to be productive. The formal proof is as follows:

Proposition 24. *The standard model satisfies negative externalities for all parameter values.*

Proof. By Proposition 23, the standard model has anonymity. So if we can prove that modules 2 and 3 merging makes a bank in module 1 worse off, then we are done. Let x be the original partition and y be the partition where modules 2 and 3 have merged, so $y = (x_1, x_2+x_3, x_4, ...x_k)$. Then $\frac{v_1[x]n^2}{P[x_1]}v_1[x] = x_1+\theta\sum_{j\geq 2} x_j P[x_j]$ and $\frac{v_1[y]n^2}{P[x_1]} = x_1 + \theta\left((x_2+x_3)P[(x_2+x_3] + \sum_{j>3} x_j P[x_j]\right)$. As $\gamma > 0$, $P[d]$ is strictly decreasing in d. So $x_2P[x_2] > x_2P[x_2+x_3]$ and $x_3P[x_3] > x_3P[x_2+x_3]$, and hence $x_2P[x_2]+x_3P[x_3] > (x_2+x_3)P[x_2+x_3]$. The result then follows. □

Proposition 25. *The standard model satisfies weak quasi-convexity for all parametrisations; is monotone for all parametrisations; and satisfies strict quasi-convexity when $P[n] \neq \theta P[1]$.*

Proof. We need to show that $(v_{1,1}[x_1] \leq max\{v_{1,1}[a], v_{1,1}[b]\})_{x_1=a+1}^{b-1}$. Recall that $v_1[(x_j)_{j=1}^k] := \frac{x_1}{n^2}P[x_1]+\sum_{j\neq 1}\theta\frac{x_j}{n^2}P[x_1]P[x_j]$, and so $v_{1,1}[x_1] := \frac{x_1}{n^2}P[x_1]+\theta\frac{n-x_1}{n^2}P[x_1]P[1]$. Hence $\frac{\partial v_{1,1}[x_1]}{\partial x_1} = (\gamma-\theta-\gamma\theta n+1)P[1]P[x1]$. This has the same sign for all x_1: specifically, $\frac{\partial v_{1,1}[x]}{\partial x_1} \geq 0$ if and only if $P[n] \geq \theta P[1]$. So not only is $v_{1,1}[x_1]$ weakly quasi-convex, it is monotonic. □

Corollary 26. *Under the standard model the strictly efficient partitions are as follows: $\{n\}$ is strictly efficient when $P[n] > \theta P[1]$; whilst $\{1,1,1....1\}$ is strictly efficient when $P[n] < \theta P[1]$; and both $\{n\}$ and $\{1,1,1....1\}$ are strictly efficient when $P[n] \neq \theta P[1]$. No other partition is ever strictly efficient.*

Proof. By Propositions 23, 24 and 25 the standard model has anonymity, negative externalities, and weak quasi-convexity. So by applying Theorem 15, partitions other than $\{n\}$ and $\{1, 1, , 1...1\}$ are rejected as inefficient. Then, by comparing $W[\{1, 1...1\}]$ and $W[\{n\}]$, the results on the choice between $\{1, 1, 1..1\}$ and $\{n\}$ follow. □

A second proof of this result for the standard model uses a continuity argument, and is given in Appendix F.

Finally we come to the concept of superadditivity. *Bilateral superadditivity* for anonymous symmetric PFGs of the type considered in this thesis, requires simply that the worth of a super-module, formed by merging 2 modules, is greater than the aggregate worth of the 2 original modules, where it is assumed that the other modules are unaltered. Specifically:

Definition 27. An anonymous symmetric PFG being *bilaterally superadditive* requires that, if $x = (x_i)_{i=1}^k$ is any partition then $V_1^g[x'] \geq V_1^g[x] + V_2^g[x]$, where $x' = \big(x_1 + x_2, (x_i)_{i=3}^k\big)$.

Theorem 28. *The standard model is bilaterally superadditive if and only if the grand coalition is a strictly efficient partition.*

Proof. Considering a general partition x and defining x', as in the Definition 27. The condition trivially holds if either $x_1 = 0$ or $x_2 = 0$. So we assume $x_1 \geq 1$ and $x_2 \geq 1$. Then, $\frac{V_1[x'] - (V_1[x] + V_2[x])}{(x_1 * P[x_1] * x_2 * P[x_2] * P[x_1 + x_2])} = (-2\theta - 2\gamma\theta t + \gamma t + 2) - R\,(\gamma^2\theta t + 2\gamma\theta)$, where $t := x_1 + x_2$ and $R := \sum_{i>2} x_i P[x_i]$. The coefficient on R is negative and hence it is hardest for this expression to be positive when R is maximised. As the model has negative externalities this occurs with $R = (n - x_1 - x_2) * P[1]$. Hence bilateral superadditivity holds if and only if $\theta < \theta_c[t] := \frac{2\gamma + \gamma^2 t + \gamma t + 2}{2\gamma + 2\gamma n + \gamma^2 (n-t)t + 2\gamma^2 t + 2}$ for all t such that $2 \leq t \leq n$. As $\frac{d\theta_c[t]}{dt} = \frac{\gamma(\gamma+1)\left(\gamma^2 t^2 + \gamma(4t-2) + 2\right)}{(2\gamma(n+1) + \gamma^2 t(n-t+2) + 2)^2} > 0$, the binding constraint occurs with $t = 2$. This gives the grand coalition efficiency condition, $\theta * P[1] \leq P[n]$, as required. $\square$

Definition 29. An anonymous symmetric PFG being *supermodular* requires that, if $x = (x_i)_{i=1}^k$ is any partition then $\mathrm{V}_1[x_1 + x_2 + x_3, 0, 0, x_{3+}] + \mathrm{V}_1[x_1, x_2, x_3, x_{3+}] \geq \mathrm{V}_1[x_1 + x_2, x_3, 0, x_{3+}] + \mathrm{V}_1[x_1 + x_3, x_2, 0, x_{3+}]$, where $x_{3+} = (x_i)_{i=4}^k$.[73]

Theorem 30. *The standard model satisfies supermodularity if and only if* $\theta P[1] \leq P[n]$.

Proof. Consider any partition $(x_i)_{i=1}^k$. Then let $e[x] := \mathrm{V}_1[x_1+x_2+x_3, 0, 0, x_{3+}] + \mathrm{V}_1[x_1, x_2, x_3, x_{3+}] - (\mathrm{V}_1[x_1 + x_2, x_3, 0, x_{3+}] + \mathrm{V}_1[x_1 + x_3, x_2, 0, x_{3+}])$. Trivially $e[x] = 0$ if either $x_2 = 0$ or $x_3 = 0$. So we can restrict ourselves to partitions where $x_2, x_3 \geq 1$ and consider $f[x] := \frac{e[x]}{P[x_1+x_2]P[x_1+x_3]P[x_1]P[x_1+x_2+x_3]x_3P[x_3]x_2P[x_2]}$, where $R := \sum_{i=4}^k x_i P[x_i]$. We want to know for which parametrisations (n, θ, γ), is $f[x] \geq 0$ for all partitions x, where $x_2, x_3 \geq 1$. Hence $f[x] \equiv C_0[x] - \theta C_1[x]$, where $C_0[x] := (\gamma x_2 + 1)(\gamma x_3 + 1)(2x_1\gamma + \gamma x_2 + \gamma x_3 + 2)$

and

$C_1[x] := x_1^2\gamma^2(\gamma x_2 + \gamma x_3 + 2) + (\gamma x_2 + 1)(\gamma x_3 + 1)(\gamma R(\gamma x_2 + \gamma x_3 + 2) + 2(\gamma x_2 + \gamma x_3 + 1)) + x_1\gamma\left(2\gamma R(\gamma x_2 + 1)(\gamma x_3 + 1) + \gamma^2 x_2^2 + \gamma x_2(4\gamma x_3 + 5) + \gamma^2 x_3^2 + 5\gamma x_3 + 4\right)$. The θ coefficient is negative, and so $f[x] > 0$ iff $\theta < \theta_{max}[x] := -\frac{C_0[x]}{C_1[x]}$. We now let $x_{2,3} := x_2 + x_3$, and $x_{123} := x_1 + x_2 + x_3$. Holding x_1, $x_{2,3}$ and x_{3+}constant we find the choice of x_3 that minimises $f[x]$. Differentiation gives that, $\frac{d\theta_{max}[x_1, x_{2,3} - x_3, x_3, x_{3+}]}{dx_3} = (x_{2,3} - 2x_3)\frac{g[x]}{h^2[x]}$, where

$g[x] := \gamma^3 \mathrm{x}_1(\gamma x_{2,3} + 2)\left(2\gamma^2 \mathrm{x}_1^2 + \gamma \mathrm{x}_1(3\gamma x_{2,3} + 4) + \gamma^2 x_{2,3} + 3\gamma x_{2,3} + 2\right) > 0$

[73]As in Section 1.6 above, this is supermodularity of the module worth function V with respect to embedded coalitions; rather than of welfare W with respect to partitions. See Appendix L for an explanation.

and

$h[x] := \gamma x_{3+}(\gamma x_3+1)(2\gamma x_1+\gamma x_{2,3}+2)(\gamma x_2+1)+\gamma x_1\left(\gamma^2 x_{2,3}^2+\gamma x_{2,3}(2\gamma x_3+5)-2\gamma^2 x_3^2+4\right)+$ $\gamma^2 x_1^2(\gamma x_{2,3}+2)+2(\gamma x_{2,3}+1)(\gamma x_3+1)(\gamma x_{2,3}-\gamma x_3+1)$. So minima occur with $x_3 = 1$ and with $x_3 = x_{2,3}-1$.[74] So without loss of generality $x_3 = x_{2,3}-1$ and hence $x_2 = 1$. Similarly $\frac{d\theta_{max}[x_1,1,x_{1,2,3}-x_1,x_{3+}]}{dx_1} = -\frac{l[x]}{m^2[x]}$ where:

$l[x] := \gamma(\gamma+1)\left(\gamma^2(x_{1,2,3}-1)x_{1,2,3}+\gamma(2x_{1,2,3}-1)+1\right)\left(\gamma^2(2x_1+1)+2\gamma x_{1,2,3}+2+\gamma^2(x_{1,2,3}-x_1-1)^2\right) > 0$ and $m[x] := \gamma(\gamma+1)R\left(\gamma^2\left(x_1^2+x_1-x_{1,2,3}^2+x_{1,2,3}\right)+\gamma(x_1-3x_{1,2,3}+2)-2\right)-(\gamma x_{1,2,3}+1)\left(\gamma^2\left(-\left(x_1^2+2x_1+2\right)\right)+\gamma x_{1,2,3}(\gamma(x_1+2)+2)+2\right)$. So we want to maximise x_1 bearing in mind the constraint that $x_3 \geq 1$. So $x_1 = x_{123}-2$, $x_2 = 1$ and $x_3 = 1$. Hence $\theta_{max}[x] = \theta_{max}[x_{1,2,3},1,1,x_{3+}] = \frac{\gamma+1}{(\gamma+1)\gamma R+\gamma x_{1,2,3}+1}$. So we want to maximise R: this requires $x_{3+} = \{1,1,1..1\}$ and hence gives $R = (n-x_{1,2,3}) * P[1,\gamma]$. Hence $\theta_{max}[x] = \theta_{max}[x_{1,2,3},1,1,\{1,1...1\}] = \frac{\gamma+1}{(\gamma n+1)}$.

□

Definition 31. An anonymous symmetric PFG being *bilaterally subadditive* requires that, if $x = (x_i)_{i=1}^k$ is any partition then $V_1^g[x'] \leq V_1^g[x]+V_2^g[x]$, where $x' = \left(x_1+x_2,(x_i)_{i=3}^k\right)$.

Theorem 32. *The atomic partition is strictly efficient for any symmetric bilaterally subadditive PFG with negative externalities.*

Proof. Suppose $x = (x_i)_{i=1}^k$ is a partition and module k is non-singleton. Then let $x' := ((x_i)_{i=1}^{k-1}, x_k-1, 1)$. Due to negative externalities the worth of each of the first $k-1$ modules is no lower; and due to subadditivity the members from x_k are better off as well. So by induction the atomic partition is efficient. □

Theorem 33. *The standard model is bilaterally subadditive if and only if* $\theta P[0.5n] \geq P[n]$.

[74] By symmetry both $x_3 = 1$ and $x_3 = x_{2,3}-1$ give the same value for $f[x]$.

Proof. Consider any partition $(x_i)_{i=1}^k$. Then let $e[x] := \mathrm{V}_1[x_1 + x_2, x_{2+}] - (\mathrm{V}_1[x_1, x_2, x_{2+}] + \mathrm{V}_1[x_2, x_1, x_{2+}])$. Trivially $e[x] = 0$ if either $x_2 = 0$ or $x_3 = 0$. So we assume $x_1 \geq 1$ and $x_2 \geq 1$. Let $f[x] := e[x]/(x_1 P[x_1] x_2 P[x_2] P[x_1 + x_2])$. Then $f[x] \equiv -2(\theta + \gamma\theta R - 1) - \gamma x_{1,2}(\theta(\gamma R + 2) - 1)$, where $R := \sum_{i=3}^k x_i P[x_i]$ and $x_{1,2} := x_1 + x_2 \geq 2$. Then,

$f[x] \equiv (\gamma x_{1,2} + 2) - \theta\,(2\gamma R + \gamma^2 R x_{1,2} + 2\gamma x_{1,2} + 2)$. The θ coefficient is negative so $f[x] \leq 0$ if and only if $\theta \geq \theta_{min}[x] := \frac{(\gamma x_{1,2}+2)}{2\gamma R + \gamma^2 R x_{1,2} + 2\gamma x_{1,2} + 2}$. As $\theta_{min}[x]$ is maximised when R is minimised, we set $R = (n - x_{1,2})P[n - x_{1,2}]$, and so $\theta_{min}[x_{1,2}] = \frac{\gamma x_{1,2}+2}{\frac{\gamma(n-x_{1,2})(\gamma x_{1,2}+2)}{\gamma n - \gamma x_{1,2}+1} + 2\gamma x_{1,2} + 2}$. Hence $\frac{d\theta_{min}[x_{1,2}]}{dx_{1,2}} = \frac{\gamma * g[x]}{h^2[x]}$, where $g[x] := (-2\gamma^2 n^2 - 4\gamma n + 2) + x_{1,2}\,(8\gamma + \gamma^2(4n - x_{1,2}))$ and $h[x] := \gamma n(3\gamma x_{1,2} + 4) - 3\gamma^2 x_{1,2}^2 - 2\gamma x_{1,2} + 2$. As $x_{1,2} \leq n$, $g[x]$ is quasi-convex and hence $\theta_{min}[x]$ is maximised either with $x_{1,2} = 2$ or with $x_{1,2} = n$. $\theta_{min}[x_{1,2} = n] - \theta_{min}[x_{1,2} = 2] = \frac{\gamma(n-2)\left(4\gamma + \gamma^2 n + 1\right)}{2(\gamma n+1)(3\gamma^2(n-2) + 2\gamma(n-1)+1)} \geq 0$, where $n \geq 2$, as $x_{12} \geq 2$. As $\theta_{min}[x_{1,2} = n] = \frac{P[n]}{P[0.5n]}$, the result now follows. $\square$

1.8 Stability

This section assesses the stability of the standard model using the EEBA (Extended Equilibrium Binding Agreement), building on the ideas, techniques and definitions which were introduced in Section 1.6. It does this by first describing the *hedonic* approach before reviewing the distinction between characteristic form games and partition form games. It then introduces the modelling of expectations within partition form games.

In a general *non-hedonic* co-operative game, there are two stages: agents firstly form modules, and then secondly play a strategic form game. Contrastingly, in a hedonic game there is no second stage: the partition is sufficient to define the payoffs each agent gets. Bogomolnaia, and Jackson (2002) introduced the idea of *hedonic coalition* games where, each agent's preferences over partitions are completely characterized by his preferences over the coalitions that he belongs to in each partition. Mathematically, $\pi \succeq_j \pi' \iff C_\pi(j) \succeq_j C_{\pi'}(j)$, where $C_\pi(j)$ is the coalition in π that j belongs to.[75] Diamantoudi and Xue (2007) extend the concept to form *hedonic partition* games, where for every agent there exists a complete, reflexive, and transitive binary relation on partitions. The banking models used in this thesis are all *hedonic*, and having a hedonic game simplifies some of the definitions used below, which show how partition form games differ from characteristic form games.

Typically in co-operative game theory there are no externalities: how outside agents organise themselves does not affect you. Your utility in such a *characteristic form* game is determined by the members of your coalition, but is independent of how the other outside agents are grouped. Formally, if N is the set of all agents then, for each nonempty coalition $S \subseteq N$, a *characteristic function* U^{cf} determines an $|S|$ dimensional vector, $U^{cf}[S] = \left(u_i^{cf}[S]\right)_{i \in S}$. In such characteristic form games, a central equilibrium notion is the *core*: the set of all *unblocked* coalitions. And we say that D *blocks* S if a coalition D of agents can deviate away from S, and make themselves all better off. Formally, $\emptyset \neq D \subseteq S$ and $\left(u_i^{cf}[D] > u_i^{cf}[S]\right)_{i \in D}$.

In contrast, with *partition form* games, there are externalities: how outside agents are grouped *does* affect you. So, in partition form games, utility is determined not only by the members of your module, but also by the modules formed by agents not in your module. Formally, there is a *partition function* U^{pf}, that for every partition π assigns a payoff to every agent, $U^{pf}[\pi] = \left(u_i^{pf}[\pi]\right)_{i \in N}$.[76]

[75] This description is adapted from Burani and Zwicker (2003).

[76] Recall from Definition 4 that a partition $\pi = (\pi_i)_{i=1}^{k}$ of a set N is a pairwise disjoint

With such games it is not trivial to define what the core is. Why? Suppose we start with partition π, from which a coalition S is considering deviating to form a new partition π^*, at which S gets higher payoffs: assuming the other modules do not react.[77] However, at π^* there may be a new coalition S^* that does want to react to form π^{**}; and at π^{**} it may be that S is worse off than they were at π. Further, this process is iterative: coalitions like S^* that might want to deviate in response to the S deviation, need to think how their deviation may provoke further deviations in turn. Hence, with partition form games, we need to include expectations in our solution concept.[78]

The EEBA (Extended Equilibrium Binding Agreement) concept, is a *farsighted* solution concept: "when contemplating a deviation, a coalition takes into consideration that further deviations may occur and that other deviating coalitions also apply similar reasoning. For farsighted agents, it is the final agreement their deviations lead to that matters.", Diamantoudi and Xue (2007). Note that farsightedness is a two part concept: such agents have not just *farsighted expectations* (the knowledge and ability to work out the end state of the system), but also *farsighted preferences* (it is *only* the end state they care about and not any of the intermediate states). In contrast, section 2.3.3 considers a number of solution concepts that are *myopic* in expectations or preferences: The Simultaneous-Move Open Membership Game, The Unanimity Game and Bilateral Stability.[79]

Now, I will define the preference relation. Then I will explain how the EEBA extends the original Equilibrium Binding Agreement (EBA), of Ray and Vohra (1997), through increasing the feasible set of allowable deviations. Then I will

covering so, $\bigcup_i \pi_i = N$ and $(\pi_i \bigcap \pi_j = \emptyset)_{i \neq j}$.

[77]Later in this section it is formally defined what new partition(s) a deviating coalition can form.

[78]For a detailed description of partition form games, including expectations, see Ray (2007); for a brief summary of a number of different extensions of the core to PFGs see section 1.6.

[79] Section 2.3.3 also considers the Equilibrium Binding Agreement (EBA) which like the EEBA is farsighted in both expectations and preferences.

define the indirect dominance relation. Then I will define a stable set, and then finally show that generically the unique EEBA solution of the standard model is the efficient partition.

We define a *coalition preference,* by requiring the individual preference to hold for all members of the coalition. Formally:

Definition 34. Let N be the set of agents, and let P and Q be partitions *of* N. *Coalition* S *strictly prefers* P *to* Q (notation $P \succ_S^{pf} Q$), if each member of coalition S strictly prefers partition P to Q. Specifically, require that $\left(P \succ_i^{pf} Q\right)_{i \in S}$, or in utility formation, $u_i^{pf}[P] > u_i^{pf}[Q]$ for all $i \in S$. Note, that S can be any subset of N: there is no requirement that S be a member of either of the partitions.

The idea of a *coalitional deviation* is as follows. Initially, there is a partition P consisting of modules $\{S_1, S_2....S_k\}$. A coalition T then deviates from partition P to form new modules $\{T_1, ..., T_l\}$. Each S_j module simply loses the members that have joined T. Formally:

Definition 35. We write $P \xrightarrow{T} P'$ to denote a *coalitional deviation* where the following conditions on P, T and P' hold. $P = \{S_1, ..., S_k\}$ is a partition of N. A coalition T can partition itself to form new modules, $(T_l)_{l=1}^L$. The resulting partition structure of N, before any further regrouping and restructuring, is P' such that:

1. $\left(T_l \in P'\right)_{l=1}^L$; that is the partitioning of T is itself included in the new partition structure.
2. $\forall j = 1, ..., k,\ S_j \bigcap T \neq \emptyset \Rightarrow S_j \setminus T \in P'$; that is, the residuals of all modules affected by the deviation of T are modules in the new partition structure.

3. $\forall j = 1, ..., k,\ S_j \bigcap T = \emptyset \Rightarrow S_j \in P\,'$; that is, all those modules that were unaffected by the deviation of T remain modules in the new partition structure.

The EEBA allows any sequence of coalitional deviations, so deviating banks can re-deviate: specifically, this allows $P_i \xrightarrow{T_i} P_{i+1}$, $P_j \xrightarrow{T_j} P_{j+1}$, $b \in T_i \cap T_j$, and $i \neq j$. In contrast, the EBA only allows internal deviations (splits within a module), and each agent can only deviate once. The benefit of EEBA is the added flexibility in allowing these more general deviations to take place; but the cost is that the solution set may not exist, or that there may be multiple solutions. However for a partition model with anonymity, negative externalities, and strict quasi-convexity, the solution always exists, and is unique except for a set of measure zero (when both the trivial partitions are efficient). So in this section the EEBA is used; section 2.3.3.4 shows the robustness of these results to using the EBA. These general partition model results apply directly to generic parametrisations of the standard model; except in the case where there is indifference between the two boundary partitions. In this special case of the standard model, there is only weak quasi-convexity and there are extra EEBA.

As explained above, with partition form games we need to consider expectations. Under the EEBA, this is done using the *indirect domination* relation: $P\,'$ indirectly dominates P if there exists a sequence of partitions $P = P_1, P_2$, ..., $P_k = P'$, and a sequence of coalitions $T_1, T_2, ..., T_{k-1}$ such that, at each stage there is both feasibility (the deviators T_j define a coalitional deviation $P_j \xrightarrow{T_j} P_{j+1}$), and farsighted individual rationality (each deviator T_j strictly prefers the end state P' to the current state P_j). More formally:

Definition 36. $P\,'$ *indirectly dominates* P (denoted by $P' \gg^{pf} P$), if there exists a sequence of partitions $P_1, P_2, ..., P_k$, where $P_1 = P$ and $P_k = P'$ and a sequence of coalitions $T_1, T_2, ..., T_{k-1}$ such that $\left(P_j \overset{T_j}{\rightarrow} P_{j+1} \, and \, P_j \prec^{pf}_{T_j} P\acute{} \right)_{j=1}^{k-1}$.

For its solution concept, the EEBA uses the *stable set* of von Neumann and Morgenstein (1944), applied to partitions under the indirect domination relation. Intuitively, the stable set requires that; firstly, no solution can be preferred to any other solution, and secondly, every non-solution must be inferior to some solution. Formally:

Definition 37. Consider a set X and some binary partial ordering $>$ on X. Then R where, $\emptyset \neq R \subseteq X$, is a *vN–M stable set* for $(X, >)$, if it is both *internally* and *externally stable*:

- R is *vN–M internally stable* for $(X, >)$, if there do not exist $P, P\acute{} \in R$ such that $P\acute{} > P$
- R is *vN–M externally stable* for $(X, >)$, if for any $P \in X \setminus R$, there exists some $P\acute{} \in R$ such that $P\acute{} > P$

Finally, we have the definition of an EEBA:

Definition 38. P is an *EEBA* (extended EBA), if there exists R s.t. $P \in R$ and R is a vN–M stable set of $(\mathbb{P}, \gg^{pf})$, where $\mathbb{P}$ is the set of partitions.

We are now ready to apply the EEBA concept, first to the general model, and then to the standard model.

Proposition 39. *In the general model, if there is anonymity, negative externalities, $v^g_{1,1}[x_1]$ is* strictly *quasi-convex, and if $\{n\}$ is the unique strictly efficient partition then $P' \neq \{n\} \Rightarrow \{n\} \gg^g P'$.*

Proof. We need to find a path from P' to $\{n\}$, that is both feasible (possible via a sequence of coalition deviations), and individually rational on a farsighted basis (for the deviators at every stage). Here that transition can occur directly in one stage. Firstly, $P' \overset{\{n\}}{\to} \{n\}$ is feasible. Secondly, by Proposition 14.2, $(\{n\} \succ_i^g P')_{i=1}^n$. Hence $\{n\} \gg^g P'$ as required. □

Theorem 40. *In the general model, if there is anonymity, negative externalities, $v_{1,1}^g[x_1]$ is strictly quasi-convex, and if $\{n\}$ is the unique strictly efficient partition, then $\{n\}$ is the unique EEBA.*

Proof. Existence. Consider $R = \{\{n\}\}$, the set containing the grand coalition partition. The set R only has one member, and so trivially it is internally stable. Suppose it is not externally stable and so there exists partition P' s.t. $\{n\} \not\gg^g P^{'}$. However, by Proposition 39 $\{n\} \gg^g P'$. This is a contradiction. Hence R is stable, and so $\{n\}$ is an EEBA.

Uniqueness. Suppose $R' \neq \{\{n\}\}$ is stable. If $\{n\} \notin R'$ then it will not be externally stable. So $\{n\} \in R'$. Suppose also that $P' \in R'$ and $P' \neq \{n\}$. But then it will not be internally stable as $\{n\} \gg^g P'$. Hence no such R' exists. □

There are similar results in the reverse case where $P[n] < \theta P[1]$, and hence the atomistic partition $\{1, 1, 1...1\}$ is the unique efficient partition:

Proposition 41. *In the general model, if there is anonymity, negative externalities, $v_{1,1}^g[x_1]$ is* strictly *quasi-convex, and if $\{1, 1, 1...1\}$ is the unique strictly efficient partition then $P' \neq \{1, 1, 1...1\} \Rightarrow \{1, 1, 1....1\} \gg^g P'$*

Proof. We need to show a path from P' to $\{1, 1, 1...1\}$ that is both feasible (possible via a sequence of coalition deviations), and individually rational on a farsighted basis (for the deviators at every stage). Here that transition can occur directly in one stage. Firstly, $P' \overset{\{n\}}{\to} \{1, 1, 1...1\}$ is feasible. Secondly, by Proposition 14.1, $(\{1, 1, 1..1\} \succ_i^g P')_{i=1}^n$. Hence $\{1, 1, 1....1\} \gg^g P'$ as required.

□

Theorem 42. *In the general model, if there is anonymity, negative externalities,* $v^g_{1,1}[x_1]$ *is* strictly *quasi-convex, and if* $\{1,1,1...1\}$ *is the unique strictly efficient partition then* $\{1,1,1....1\}$ *is the unique EEBA.*

Proof. Existence. Let $R := \{\{1,1,1...1\}\}$, be the set containing the atomistic partition of singletons. R only has one member, so trivially it is internally stable. Suppose it is not externally stable and so there exists partition P' s.t. $\{1,1,1..1\} \not\gg^g P'$. However, by Proposition 41, $\{1,1,1..1\} \gg^g P'$, so this is a contradiction. Hence R is stable, and so $\{1,1,1..1\}$ is an EEBA.

Uniqueness. Suppose $R' \neq \{\{1,1,1...1\}\}$ is stable. If $\{1,1,1...1\} \notin R'$ then it will not be externally stable. So $\{1,1,1...1\} \in R'$. Suppose also that $P' \in R'$ and $P' \neq \{1,1,1...1\}$. However, then it will not be internally stable as $\{1,1,1...1\} \gg^g P'$. Hence $R' = \{\{1,1,1...1\}\}$ and so $\{1,1,....1\}$ is the unique EEBA. $\square$

Proposition 43. *In the general model, if there is anonymity, negative externalities,* $v^g_{1,1}[x_1]$ *is* strictly *quasi-convex, and if* $\{1,1,1...1\}$ *and* $\{n\}$ *are the only strictly efficient partitions then* $P' \neq \{1,1,1...1\}$ *and* $P' \neq \{n\} \Rightarrow \{1,1,1....1\} \gg^g P'$ *and* $\{n\} \gg^g P'$

Proof. We need to show a path from P' to $\{1,1,1...1\}$ that is both feasible (possible via a sequence of coalition deviations), and individually rational on a farsighted basis (for the deviators at every stage). Here that transition can occur directly in one stage. Firstly, $P' \overset{\{n\}}{\rightarrow} \{1,1,1...1\}$ is feasible. Secondly, by Proposition 14.3, $(\{1,1,1..1\} \succ^g_i P')^n_{i=1}$. Hence $\{1,1,1....1\} \gg^g P'$ as required.

Similarly, we need to find a path from P' to $\{n\}$, that is both feasible (possible via a sequence of coalition deviations), and individually rational on a farsighted basis (for the deviators at every stage). Here that transition can occur directly in one stage. Firstly, $P' \overset{\{n\}}{\rightarrow} \{n\}$ is feasible. Secondly, by Proposition 14.3, $(\{n\} \succ_i^g P')_{i=1}^n$. Hence $\{n\} \gg^g P'$ as required. $\square$

Theorem 44. *In the general model, if there is anonymity, negative externalities, $v_{1,1}^g[x_1]$ is* strictly *quasi-convex, and if $\{1, 1, ...1\}$ and $\{n\}$ are both strictly efficient, then $\{1, 1, ...1\}$ and $\{n\}$ are both EEBA.*

Proof. Let $R = \{\{1, 1, 1...1\}, \{n\}\}$, be the set containing both the trivial partitions.

Internal Stability. Consider a path that starts at one of the trivial partitions and ends at the other. All banks are indifferent between $\{1, 1, 1....1\}$ and $\{n\}$. So no initial group of deviators can be strictly better off at the end state compared with the start state. Hence R is internally stable.

External Stability. Suppose it is not externally stable so there exists some partition P' s.t. $\{1, 1, 1...1\} \not\gg^g P'$ and $\{n\} \not\gg^g P'$. However, by Proposition 43 $\{1, 1, 1..1\} \gg^g P'$ and $\{n\} \gg^g P'$. This is a contradiction. Hence R is stable, and so both $\{1, 1, ...1\}$ and {n} are EEBA. $\square$

Now, with the standard model, we in turn consider the three cases, $P[n] > \theta P[1]$, $P[n] < \theta P[1]$ and $P[n] = \theta P[1]$.

Theorem 45. *In the standard model, if $P[n] > \theta P[1]$ then $\{n\}$ is the unique EEBA.*

Proof. The result follows from a direct application of Theorem 40: the model is strictly quasi-convex by Proposition 25; has anonymity by Proposition 23; has negative externalities by Proposition 24; and the partition $\{n\}$ is the unique efficient partition by Corollary 26. $\square$

Theorem 46. *In the standard model, if* $P[n] < \theta P[1]$ *then* $\{1, 1, ...1\}$ *is the unique EEBA.*

Proof. The result follows from a direct application of Theorem 42: the model is strictly quasi-convex by Proposition 25; has anonymity by Proposition 23; has negative externalities by Proposition 24; and the partition $\{n\}$ is the unique efficient partition by Corollary 26. □

With the critical parameter case, $P[n] = \theta P[1]$, we have weak, but not strict, quasi-convexity in $v_1[x_1]$; and hence we cannot use Theorem 44. Hence it is necessary to use an intermediate proposition, which for the special case of $P[n] = \theta P[1]$, generalises Proposition 13 to the ith module:

Proposition 47. *If* $P[n] = \theta P[1]$ *then* $\text{argmax } v_i[x] = \{\{(1)_{j=1}^{i-1}, x_i, (1)_{j=i+1}^{n+1-x_i}\} : x_i \in A_{n+1-i}\}$

Proof. Let us start by considering the partition(s) x that maximise $v_1[x]$, the utility of a bank in the first module. As there are negative externalities all the modules after the first one must be singletons. So $x = \{x_1, 1, 1, 1....1\}$: there must be one module of size x_1 and $n - x_1$ modules of size 1. So $v_1[x] = \frac{x_1}{n^2}P[x_1] + \theta\frac{(n-x_1)}{n^2}P[x_1]P[1]$. Hence $\frac{\partial v_1[x]}{\partial x_1} = (\gamma - \theta - \gamma\theta n + 1)P[1]P[x_1]$. So as $P[n] = \theta P[1]$, $\frac{\partial v_1[x]}{\partial x_1} \equiv 0$. This proves the result for the first module. The same argument applies for the ith module as appropriate, (with the caveat that we need there *to be* an ith module, and so we require the first $i - 1$ modules to be singleton modules). □

Theorem 48. *In the standard model, if* $P[n] = \theta P[1]$, *then* $\{1, 1, ...1\}$ *and* $\{n\}$ *are both EEBA. Further all other partitions with precisely one non-singleton module are also EEBA.*

Proof. Let R be the set containing all partitions with either 0 or 1 non-singleton modules. So it contains all elements such as $\{1, 1....1\}$, $\{x_1, 1, 1..1\}$, $\{1, x_2, 1, 1..1\}$,$\{1, 1, x_3, 1, 1..1\}$, $\{1, 1, 1, ..., x_i, 1, 1, 1, 1\}$, and $\{n\}$.

Internal Stability. We need to prove that there do not exist $P, P' \in R$ such that $P' \gg P$. Proof is by contradiction: suppose such a path from $P_1 := P$ to $P_t := P'$ exists, and that the length t is the minimal length. Then there is a set T_1 of deviators from P_1 to some P_2. Without loss of generality, assume that the potential non-singleton module within P, is module 1. Suppose T_1 contains no members of module 1 of P. Then we must have $P_2 = P$, or that P_2 is a permutation of P; however, in either case the path is now not of minimal length. Instead suppose T_1 contains some, or all, the members of module 1 of P. However by Proposition 47, $P \in \arg\max\ v_1[x]$. And so such members of module 1 of P that are deviating, cannot be strictly better off at the end partition P'. So there is a contradiction in this case as well. Hence R is internally stable.

External Stability. Suppose $P' \notin R$. And so that P' must have at least 2 non-singleton modules, so $P \notin \arg\max\ v_i[x]$, by Proposition 47. I will now show that there is a path from P' to $\{n\}$, that is both feasible (possible via a sequence of coalition deviations), and individually rational on a farsighted basis (for the deviators at every stage). The transition can occur directly in one stage. Firstly, $P' \overset{\{n\}}{\to} \{n\}$ is feasible. Secondly, the $\arg\max$ of $v_1[x]$ includes $\{n\}$. So $\{n\} \succ_i P'$ for all i. Hence $\{n\} \gg P'$ as required. $\square$

The internal examiner, Professor Peter Hammond, notes that in these proofs the transitions take place in one stage, and asked what this meant for the requirement of farsightedness. In general the EEBA definition includes multiple deviations because it allows situation such as $P = P_1,\ P_2,\ P_3 = P'$ where the deviators are T_1 and T_2 such that T_1 prefers P' to P and T_2 prefers P' to P_2 but does not prefer

P' to P: with each EEBA sequence, deviating agents prefer the end partition to the current partition at the point they deviate, but there is no requirement that they prefer the end partition to other partitions in the sequence.

However, here following Proposition 14 as the partition model has anonymity, negative externalities and $v^g_{1,1}[x_1]$ is strictly quasi-convex, there is generically a unique efficient partition such that at other partitions *everyone* is worse, and so the transition can be completed in one stage. One interpretation of this would be that the issues of farsightedness are non-critical. However, if we want to imagine that in practice that at each stage simple deviations are possible (a single module splitting; or 2 modules merging) but compound deviations are not feasible (multiple modules splitting or more than 2 modules merging), then the farsightedness requirement would again be non-trivial.

1.9 Parameterisations

So far we have considered the model in terms of three parameters: n, the number of banks; θ, the value of outside matches; and γ, the shock parameter. This section considers some sensible ranges of these parameters: the number of banks, n, is one of $\{10, 20, 30, 40, 50, 100, 1000\}$; the value of outside matches, θ, has a minimum of 0, a maximum of 1.0 and has an increment of 0.1; the shock parameter, γ, is one of $\{0\} \cup 0.01\mathbb{N}_9 \cup 0.1\mathbb{N}_{30}$, where $\mathbb{N}_n := \{1, 2...n\}$.[80] So, the n parameter is one of 7 values, the θ parameter is one of 11 values, and the γ parameter is one of 40 values. This gives a total of $7 * 11 * 40 = 3080$ different parametrisations. For each of these cases we consider which out of the atomistic partition of singletons, *Atom*, and the grand coalition, *GC*, is preferred.

[80]Firstly, the net effect is that the γ range is $\{0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0\}$. Secondly, Appendix E, shows that 3 is a reasonable upper limit for γ.

For each parametrisation, the welfare of the atomistic partition is given by W[1]; the welfare of the grand coalition is given by W[n]; and g, the percentage gain from choosing the atomistic partition over the grand coalition, is given by $100 * (W[1] - W[n])/W[n]$.[81] In 2444 cases the atomistic partition is strictly preferred; in 11 cases there is indifference; and in 625 cases the grand coalition is strictly preferred. This argues that for realistic parameter values the atomistic partition should be chosen.

How the distribution of the gain varies can be displayed for each parameter in a separate table. Each line of a table fixes one parameter value, but lets the other parameters vary. So, for example, the top line of the n table includes the parametrisations where, n is fixed at 10, θ varies between 0 and 1, and γ varies between 0 and 3. These $11 * 40 = 440$ cases are considered to see what the distribution of the gains is.

The grand coalition is the unique solution, if and only if $P[n] > \theta P[1]$. So the n table and the θ table show that as either n or θ increases, there is an increase, both in the number of cases with gain from atomisation, and in the gains in those cases. Re-arrangement gives that $GC \succ Atom$ if and only if ($\gamma < (1-\theta)/(\theta n - 1)$ or θn ≤ 1). The γ table shows that an increase in the shock parameter, γ, generally leads to a switch from the grand coalition to the Atomistic partition of singletons: however, this does not occur if $\theta n \leq 1$, for example when $\theta \in \{0, 0.1\}$ and $n = 10$.

[81] Here we are allowing modules to have fractional sizes. So the feasible set of partition is $\{(x_i)_{i=1}^{k} : x_i \in \mathbb{R},\, x_i \geq 1,\, k \in \mathbb{N} \, and \, \sum_{i=1}^{k} x_i = n\}$

Table 1.5: Standard Model n Table

n	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,25000)
	percentage gain from atomisation broken down by n											
10	151	169	115	5	0	0	0	0	0	0	0	0
20	106	116	75	66	58	19	0	0	0	0	0	0
30	91	87	63	45	50	40	37	26	1	0	0	0
40	83	68	51	47	34	32	33	33	30	19	10	0
50	78	63	44	37	31	27	30	26	23	25	25	31
100	67	23	39	21	22	26	15	22	13	11	18	163
1000	49	2	3	2	3	3	3	4	4	2	3	362
All	625	528	390	223	198	147	118	111	71	57	56	556

Table 1.6: Standard Model θ Table

θ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,25000)
	percentage gain from atomisation broken down by θ											
0.0	280	0	0	0	0	0	0	0	0	0	0	0
0.1	108	106	25	6	1	1	1	2	1	0	0	29
0.2	66	82	60	7	12	17	0	1	0	1	1	33
0.3	49	64	46	41	14	1	9	17	3	0	0	36
0.4	37	54	37	39	30	17	0	1	9	9	10	37
0.5	30	55	31	25	29	25	18	1	0	1	8	57
0.6	22	46	26	30	29	18	24	16	3	0	0	66
0.7	15	33	39	30	16	24	15	22	12	7	1	66
0.8	10	30	43	19	20	16	22	13	21	9	10	67
0.9	8	27	44	10	29	10	17	18	12	19	7	79
1.0	0	31	39	16	18	18	12	20	10	11	19	86
All	625	528	390	223	198	147	118	111	71	57	56	556

Table 1.7: Standard Model γ Table

	percentage gain from atomisation broken down by γ											
γ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,25000)
0.00	70	7	0	0	0	0	0	0	0	0	0	0
0.01	49	19	1	1	1	1	1	1	1	1	1	0
0.02	41	22	5	0	1	0	1	0	1	0	1	5
0.03	35	24	6	3	0	1	0	0	1	0	0	7
0.04	31	24	8	3	2	0	0	1	0	0	0	8
0.05	28	23	10	3	3	1	0	0	0	1	0	8
0.06	26	22	10	5	1	3	1	0	0	0	1	8
0.07	25	19	12	6	2	1	3	0	0	0	0	9
0.08	25	17	12	7	2	1	2	2	0	0	0	9
0.09	21	21	10	7	3	2	1	2	1	0	0	9
0.1	21	17	13	7	4	1	1	1	2	1	0	9
0.2	15	14	9	10	5	4	3	2	1	0	1	13
0.3	12	12	9	9	6	5	2	4	1	2	0	15
0.4	11	11	11	5	7	3	5	3	3	1	2	15
0.5	10	10	10	7	6	4	3	4	2	3	1	17
0.6	10	10	10	6	4	5	4	4	3	3	0	18
0.7	10	10	10	4	6	5	3	5	2	1	3	18
0.8	9	10	10	5	6	5	3	5	1	2	3	18
0.9	8	10	10	5	6	6	3	5	1	2	3	18
1.0	8	9	9	7	5	7	3	2	4	2	3	18
1.1	8	9	9	7	5	7	3	3	3	2	3	18
1.2	8	9	9	7	7	5	3	5	1	2	3	18
1.3	8	9	10	7	6	5	3	5	1	2	3	18
1.4	8	9	11	6	6	5	3	5	1	2	3	18
1.5	8	9	13	4	6	5	3	5	2	1	3	18
1.6	8	9	13	4	6	5	4	4	2	1	3	18
1.7	8	9	13	4	6	5	4	4	3	3	0	18
1.8	8	10	12	4	6	5	4	5	2	3	1	17
1.9	8	11	11	4	6	5	5	4	2	3	1	17
2.0	8	12	10	6	4	7	3	4	2	3	1	17
2.1	8	12	10	6	6	5	3	4	2	3	1	17
2.2	8	12	10	6	8	3	3	4	2	3	1	17
2.3	8	12	10	7	7	4	4	2	2	3	1	17
2.4	8	12	10	8	7	3	5	2	4	1	1	16
2.5	8	12	10	8	7	3	5	3	3	1	2	15
2.6	8	12	10	8	7	3	5	3	3	1	2	15
2.7	8	12	10	8	7	4	5	2	3	1	2	15
2.8	8	12	10	8	7	4	5	2	3	1	2	15
2.9	8	12	12	6	7	4	5	2	3	1	2	15
3.0	8	13	12	5	7	5	4	2	3	1	2	15
All	625	528	390	223	198	147	118	111	71	57	56	556

1.10 Interpretation

This chapter has constructed and considered the efficiency of a standard model, and concluded that, depending on the parametrisation, we should either: choose the grand coalition and accept boom and bust; or choose the atomistic partition and accept low income. This section considers the regulatory implications of this module rejection result: firstly, at what level of government should bank regulation take place; secondly, do inter-bank relationships need regulating, and thirdly, what are the implications of the 2007–2009 financial crisis.

The efficiency of the boundary partitions advocates the homogenous regulation of inter-bank relationships, as in both cases regulation of the inter-bank relationship is the same for every pair of banks (i, j): with the grand coalition it is low for all (i, j); whilst with the atomistic partition it is high for all (i, j). This is unlike a partition with real modularisation where regulation is low between banks in the same module, but high between banks in different modules. So, the grand coalition solution implies that banking regulation should be international; whilst the atomistic partition implies it should be local. Neither suggests it should be at an intermediate or regional level (for example by the EU).

The EEBA solution concept has farsighted agents, and Section 1.8 has showed that for the standard model, with this concept only the efficient modularisation is stable. This result argues against the need for regulation of banking networks. However, in contrast, chapter 2 will show that in formulations with myopic agents: the atomistic partition may be unstable, even when it is efficient; and the grand coalition stable, even when it is inefficient. So with myopic banks the conclusion is that there is a pro grand coalition bias.

Finally, it provides this two-part conflicting interpretation of the policy of highly interconnected banks that allowed the 2007–2009 financial crisis to become global. If the shock parameter is low enough, or the cost of high regulation large enough, then low regulation is optimal despite boom and bust. So, the occurrence of a single crisis is not sufficient to argue for a policy change. Conversely, however, as the number of banks increases, there is a tipping point at which the efficient solution jumps from the grand coalition to the atomistic partition, and indeed the parameterisations considered in section 1.9 suggest that separation into minimal sized banks is likely to be the optimal policy.

My model does not allow for growth in the banking sector, but it does suggest that in such an environment the efficient policy is to switch from low to high regulation once the number of banks reaches a certain critical level. And it is possible that the banking sector has grown past that critical level, and thus the pre crisis structure was sub-optimal.

Chapter 2

Criticality and Robustness

2.1 Summary

This chapter considers the criticality and robustness of the standard model in four aspects: the business sector, the financial sector, the social planner's preferences and partition formation. There is criticality with respect to an aspect of the standard model, if altering that aspect can result in the optimality of proper partitions, (where there are multiple modules, and each module will have multiple member banks). A method often used to demonstrate criticality is to computationally evaluate the altered model's welfare function for interior symmetric partitions. The method is valid as it leads to a lower bound for the welfare gain when interior non-symmetric partitions are additionally allowed. Conversely, there is robustness with respect to an aspect of the standard model, if on altering that aspect the standard model's rejection of proper partitions still remains. A method often used to demonstrate this is to computationally find the interior partition that maximises the utility of bank 1. As Appendix A shows, this leads to an upper bound for the welfare gain from interior partitions.

The business sector demonstrates criticality in aspects such as circular matching, (businesses are arranged in a circle, and matches are always between immediate neighbours); hypercube models, (either 4 businesses arranged in a square, or 8 businesses arranged in a cube); and non-uniform matching (with 3 or 4 businesses). Contrastingly, the financial sector demonstrates both robustness (where the probability of a bank receiving an initial shock increases with module size), and criticality (if banks receive biased incentives, for example in terms of their time horizons). The standard model is robust to altering the preferences of the social planner (either in terms of risk aversion or including intertemporal utility).

Further, this chapter applies different partition formation concepts to the standard model. The efficient partition is again always formed under either the Equilibrium Binding Agreement (EBA), of Ray and Vohra (1997), or the Unanimity Game of Bloch (1996). However, inefficient partitions can be formed under bilateral stability from Jackson and Wolinsky (1996), or the Open Membership game from Yi and Shin (2000).

2.2 Business

2.2.1 Circular Business Networks

2.2.1.1 Circular Matching

The standard model has uniform matching: this represents a 'global' world where business matches are as likely to be between businesses that are on opposite sides of the world, as they are to be between businesses that are close neighbours of each other. In contrast, here with the circular model we consider an environment where all matches are between business that are local to each other. The use of a circle of agents, rather than a line segment, is done partly to represent the circular (spherical) nature of the earth and partly to avoid introducing special matching behaviour at the end points.

This section considers a variant model where there is a circle of banks and will show that the efficient partition generally has *proper* modules: multiple modules each with multiple member banks.[1] We are attempting to show the inefficiency of the boundary partitions, so it is sufficient to check only the symmetric partitions: if there is a symmetric partition that beats both the trivial partitions, then there must be a partition in the bigger set of all partitions that beats both the trivial partitions. Hence, for simplicity, in this section we will consider only symmetric partitions, which partition the circle into k modules of length d, $\{[id,(i+1)d)\}_{i=0}^{k-1}$, where $kd = n$ and $d \geq 1$. We assume infinite divisibility of banks, and so the matching process is represented by a probability density function (pdf), rather than a probability function.[2] All matches are between neighbouring banks one unit distance apart: so the matching process has probability density function $p(i, i+1) = 1/n$ where $i \in [0, n]$.[3]

As in the standard model, matches are either *inside* (banks in same module), or *outside* (banks in different modules), however, the probability of an inside match is different:

Theorem 49. *The inside match probability is given by:* $P(\text{Inside}) = \begin{cases} \frac{d-1}{d} & d < n \\ 1 & d = n \end{cases}.$

Proof. Outside matching occurs when matches 'stretch' over a module boundary, so as matches are of length one, $P(inside)$ is the number of inter-module

[1] The next two sections 2.2.1.2 and 2.2.1.3 consider respectively circular models where the minimum module size is parametrised and where the match size is variable.

[2] Infinite divisibility is assumed to ensure that symmetric partitions are valid even when they have modules which contain a fractional number of banks.

[3] Where in a slight abuse of notation, addition is modulus n and so $n+1 = 1$. Hence, more generally, $p(z, z+1-n) = \frac{1}{n}$ when $z \in [n-1, n]$

boundaries divided by n. With k modules the number of boundaries is, 0 if $k = 1$ and k if $k > 1$. This discontinuity generates the two-part formula: when there are multiple modules, $(d < n)$, the match is inside one module, unless the first bank in the match is within one unit of its module's upper bound, so as each module is of length d, this gives the $\frac{d-1}{d}$ term; when there is only a single module $(d = n)$, the match is always inside. □

For symmetric sized modules the welfare function is given by:

$W_c[d]= P(inside)P[d]+\theta P(outside)P^2[d]$, where $P[d] =\frac{1}{1+\gamma d}$ is the standard module enablement probability. As with the matching probability, the welfare function has two parts: $W_c[1 \leq d \leq 0.5n]=\left(1-\frac{1}{d}\right)P[d]+\theta\left(\frac{1}{d}\right)P^2[d]$ and $W_c[d = n]=P[n]$.[4] This gives:

$$\frac{\partial W_c[d]}{\partial d}\,|_{d<n}= -\frac{(6\gamma\theta - 2\gamma + \theta - 1) + (d-2)\left(4\gamma^2 + 3\gamma\theta + \gamma\right) + \left(4\gamma^2 + \gamma\right)(d-2)^2 + \gamma^2(d-2)^3}{d^2(\gamma d + 1)^3}$$

The $(d - 2)$, $(d - 2)^2$ and $(d - 2)^3$ coefficients are all negative (note the leading minus sign). So, in the region where $d > 2$ the function is *quasi-concave* not *quasi-convex*: once the slope is negative it stays negative. So in general, with circular matching the efficient partition is non-trivial: it has real modules. So when the matching is circular this is a pro-module argument.

Next, for a range of different parametrisations we compute the partition that maximises $W_c[d]$. The number of banks, n, is one of $\{10, 20, 30, 40, 50, 100, 1000\}$. The value of outside matches, θ, has a minimum of 0.1, a maximum of 0.9 and has an increment of 0.1. The shock parameter, γ, is one of $\{0\} \cup 0.01\mathbb{N}_9 \cup 0.1\mathbb{N}_{30}$, where $\mathbb{N}_n := \{1, 2...n\}$.[5] So, the n parameter is one of 7 values, the θ parameter

[4]For general asymmetric partitions, the welfare function is $W_c[(x_i)_{i=1}^k] = \begin{cases} \sum_{i=1}^k \left(\frac{(x_i-1)}{n}P[x_i] + \frac{\theta}{n}P[x_i]P[x_{i+1}]\right) & k > 1 \\ P[n] & k = 1 \end{cases}$

[5]Firstly, the net effect is that the γ range is $\{0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0\}$. Secondly, Appendix E, shows that 3 is a reasonable upper

is one of 9 values, and the γ parameter is one of 40 values. This gives a total of $7 * 9 * 40 = 2520$ parametrisations. For each of these cases we consider which out of the atomistic partition of singletons, *Atom*, and the grand coalition, *GC*, is preferred.

For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_c[1], \mathrm{W}_c[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_c[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$.[6] The large gains from interior partitions, represented in Tables 2.1, 2.2 and 2.3 below, mean that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a circular distribution. If the gain had been calculated over all partitions rather than just symmetric partitions then each gain would have been weakly higher and so this criticality would only be stronger.

How the distribution of the gain varies can be displayed for each parameter in a separate table. The n table shows, firstly that there are nearly always gains from interior partitions, and secondly, as n increases that there are occasionally very high percentage gains from interior partitions. The θ table shows that there is a clear correlation between low θ and high gains from the interior partitions: this suggests the best boundary solution is the atomistic partition, and that when θ gets big that the atomistic partition becomes an increasingly valued option. The γ table shows that as γ increases, the gains from including interior partitions go up: there is an increased *co-ordination* value of getting both banks enabled at the same time.

limit for γ.

[6] Here we are allowing modules to have fractional sizes. So the feasible set of partition is $\{(x_i)_{i=1}^k : x_i \in \mathbb{R},\, x_i \geq 1,\, k \in \mathbb{N}\, and\, \sum_{i=1}^k x_i = n\}$

Table 2.1: Circular Matching n table

	Overall gain percentage distribution broken down by n											
n	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
10	36	236	88	0	0	0	0	0	0	0	0	0
20	14	229	51	33	33	0	0	0	0	0	0	0
30	13	222	53	28	13	16	15	0	0	0	0	0
40	13	214	61	27	10	10	4	7	14	0	0	0
50	13	209	62	30	10	9	6	5	5	4	7	0
100	13	201	60	35	11	10	9	5	5	4	4	3
1000	13	197	58	34	11	8	12	10	6	4	4	3
All	115	1508	433	187	88	53	46	27	30	12	15	6

Table 2.2: Circular Matching θ table

	Overall gain percentage distribution broken down by θ											
θ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
0.1	13	42	35	17	24	13	46	27	30	12	15	6
0.2	12	41	37	86	64	40	0	0	0	0	0	0
0.3	11	39	152	78	0	0	0	0	0	0	0	0
0.4	10	139	125	6	0	0	0	0	0	0	0	0
0.5	9	201	70	0	0	0	0	0	0	0	0	0
0.6	9	257	14	0	0	0	0	0	0	0	0	0
0.7	8	272	0	0	0	0	0	0	0	0	0	0
0.8	8	272	0	0	0	0	0	0	0	0	0	0
0.9	35	245	0	0	0	0	0	0	0	0	0	0
All	115	1508	433	187	88	53	46	27	30	12	15	6

Table 2.3: Circular Matching γ table

γ	Overall gain percentage distribution broken down by γ											
	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
0	63	0	0	0	0	0	0	0	0	0	0	0
0.01	9	50	2	0	1	0	0	0	1	0	0	0
0.02	6	49	6	0	1	0	0	1	0	0	0	0
0.03	4	51	6	1	0	0	0	1	0	0	0	0
0.04	3	51	5	3	0	0	0	1	0	0	0	0
0.05	2	51	6	3	0	0	0	1	0	0	0	0
0.06	0	51	8	2	1	0	0	1	0	0	0	0
0.07	0	51	8	2	1	0	1	0	0	0	0	0
0.08	0	49	10	2	0	1	1	0	0	0	0	0
0.09	0	48	9	4	0	1	1	0	0	0	0	0
0.1	0	48	9	4	0	0	2	0	0	0	0	0
0.2	7	38	8	7	1	0	2	0	0	0	0	0
0.3	7	38	8	6	1	1	2	0	0	0	0	0
0.4	7	38	8	5	1	1	3	0	0	0	0	0
0.5	7	38	6	7	1	1	3	0	0	0	0	0
0.6	0	45	6	7	1	0	4	0	0	0	0	0
0.7	0	45	6	7	1	0	4	0	0	0	0	0
0.8	0	45	6	7	0	1	4	0	0	0	0	0
0.9	0	45	6	7	0	1	4	0	0	0	0	0
1.0	0	44	7	7	0	1	0	4	0	0	0	0
1.1	0	44	7	7	0	1	0	4	0	0	0	0
1.2	0	43	8	7	0	1	0	4	0	0	0	0
1.3	0	35	16	7	0	1	0	4	0	0	0	0
1.4	0	35	16	1	6	1	0	4	0	0	0	0
1.5	0	35	16	0	7	1	0	1	3	0	0	0
1.6	0	35	16	0	7	0	1	1	3	0	0	0
1.7	0	35	16	0	7	0	1	0	4	0	0	0
1.8	0	35	10	6	7	0	1	0	4	0	0	0
1.9	0	35	10	6	7	0	1	0	4	0	0	0
2.0	0	35	10	6	7	0	1	0	1	3	0	0
2.1	0	28	17	6	7	0	1	0	1	3	0	0
2.2	0	28	17	6	7	0	1	0	1	3	0	0
2.3	0	28	17	6	2	5	1	0	1	3	0	0
2.4	0	28	17	6	2	5	1	0	1	0	3	0
2.5	0	28	17	6	2	5	1	0	1	0	3	0
2.6	0	28	17	6	2	5	1	0	1	0	3	0
2.7	0	28	17	6	2	5	1	0	1	0	3	0
2.8	0	28	17	6	2	5	1	0	1	0	1	2
2.9	0	21	24	6	2	5	1	0	1	0	1	2
3.0	0	21	18	12	2	5	1	0	1	0	1	2
All	115	1508	433	187	88	53	46	27	30	12	15	6

2.2.1.2 Circular Matching: variable minimum module size

In the previous section there are two normalisations of size 1: firstly, the minimum banks module size is 1, and secondly the match length is of size 1. And we cannot, in a mathematical model, make *two* such normalisation without losing generality.[7] This section tests the robustness of the circular model where the minimum module size is no longer 1: previously the maximum number of modules was n; now it is $n * s$. If $s = 1$ then the maximum number of modules is n, and the minimum module size is 1, as in the original circular model. If $s < 1$ then the maximum number of modules is less than n, and the minimum module size is greater than 1. If $s > 1$ then the maximum number of modules is greater than n, and the minimum module size is less than 1.

In this section the rest of the model stays the same. Again, we will consider only symmetric partitions, which partition the circle into k modules of length d, $\{[id,(i+1)d)\}_{i=0}^{k-1}$, where $kd = n$, $1 \leq k \leq ns$ and $k \in \mathbb{N}$. We continue to assume infinite divisibility of banks, and so the matching process is represented by a probability density function (pdf), rather than a probability function. All matches are still between neighbouring banks one unit distance apart: so the matching process has the same probability density function $p(i, i+1) = 1/n$ where $i \in [0, n]$. And hence the same welfare function, $W_c[d]$, still applies.

Next, for a range of different parametrisations we compute the partition that maximises $W_c[d]$. The s parameter has a minimum of 0.1, a maximum of 2 and an increment of 0.1. The other parameters have the same ranges as before: the number of banks, n, is one of $\{10, 20, 30, 40, 50, 100, 1000\}$; the value of outside matches, θ, has a minimum of 0.1, a maximum of 0.9 and has an increment of

[7] Although we might justify on practical terms why the normalisations could be the same. For example suppose there a number of small towns existing on a circle and each match is between business in neighbouring towns, and it is excluded on efficiency grounds for there to be multiple banks per town.

0.1; the shock parameter, γ, has a minimum of 0.1, a maximum of 3.0 and an increment of 0.1.[8] So, the s parameter is one of 20 values, the n parameter is one of 7 values, the θ parameter is one of 9 values, and the γ parameter is one of 30 values. This gives a total of $20 * 7 * 9 * 30 = 37800$ parametrisations.

For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_c[1/ns], \mathrm{W}_c[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_c[n/k]\}_{k=2}^{ns-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The large gains from interior partitions, represented in Tables 2.4, 2.5, 2.6 and 2.7 below, mean that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a circular distribution with variable minimum module size. If the gain had been calculated over all partitions rather than just symmetric partitions then each gain would have been weakly higher and so this criticality would only be stronger.

How the gain varies can be displayed for each parameter in a separate table. The s table shows the significant differences between this model and the previous model. If s is small (0.1 to 0.4) then modules in the atomistic partition are much bigger than the size of the matches: so we want smaller modules not bigger modules; hence the interior partitions are further from the ideal module size and there are very few gains from interior partitions. As s increases, there are more cases with gains from interior partitions, and once s is above 0.5 then in the majority of cases there are gain from interior partitions. With $s = 1$, we get the original circle model as a special case: minimum module size and match length are both 1, and this case represents a high point both for the frequency of gains from interior partitions, and for the size of gains from interior partitions. Once $s > 1$, the atomistic partition has modules of size less than 1, and hence

[8]This choice of n and s ensures that ns is an integer.

the atomistic partition always has outside matches. As s increases further, the atomistic partition has increasingly small modules, and hence the enablement probability of each atomic module increases, but the probability of an inside match is always 0 and so does not decrease further. Hence, once $s > 1$, the welfare of the atomistic partition is strictly increasing as s increases further, and hence the opportunities for gains from interior partitions are reduced.

Table 2.4: Circular Matching: variable minimum module size s table

	Overall gain percentage distribution broken down by s											
s	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
0.1	1890	0	0	0	0	0	0	0	0	0	0	0
0.2	1890	0	0	0	0	0	0	0	0	0	0	0
0.3	1872	18	0	0	0	0	0	0	0	0	0	0
0.4	1796	94	0	0	0	0	0	0	0	0	0	0
0.5	1276	614	0	0	0	0	0	0	0	0	0	0
0.6	434	1456	0	0	0	0	0	0	0	0	0	0
0.7	147	1743	0	0	0	0	0	0	0	0	0	0
0.8	65	1814	11	0	0	0	0	0	0	0	0	0
0.9	40	1455	387	8	0	0	0	0	0	0	0	0
1.0	28	1057	373	170	84	51	43	22	29	12	15	6
1.1	168	1001	343	146	84	37	43	24	26	15	3	0
1.2	301	974	301	112	64	41	49	24	24	0	0	0
1.3	462	882	267	113	28	61	47	30	0	0	0	0
1.4	604	795	242	89	22	81	49	8	0	0	0	0
1.5	715	719	237	59	63	65	32	0	0	0	0	0
1.6	770	706	231	23	88	65	7	0	0	0	0	0
1.7	875	615	223	17	116	42	2	0	0	0	0	0
1.8	934	556	223	17	141	17	2	0	0	0	0	0
1.9	987	503	223	17	156	2	2	0	0	0	0	0
2.0	1001	491	221	59	114	2	2	0	0	0	0	0
All	16255	15493	3282	830	960	464	278	108	79	27	18	6

Table 2.5: Circular Matching: variable minimum module size n table

n	Overall gain percentage distribution broken down by n											
	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
10	2423	2373	604	0	0	0	0	0	0	0	0	0
20	2345	2177	461	206	211	0	0	0	0	0	0	0
30	2321	2165	457	131	159	89	78	0	0	0	0	0
40	2307	2179	458	119	150	86	42	28	31	0	0	0
50	2297	2189	434	132	154	91	44	26	16	9	8	0
100	2284	2202	434	121	143	99	57	27	16	9	5	3
1000	2278	2208	434	121	143	99	57	27	16	9	5	3
All	16255	15493	3282	830	960	464	278	108	79	27	18	6

Table 2.6: Circular Matching: variable minimum module size θ table

θ	Overall gain percentage distribution broken down by θ											
	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
0.1	817	1008	453	204	788	414	278	108	79	27	18	6
0.2	840	986	1658	494	172	50	0	0	0	0	0	0
0.3	891	2412	771	126	0	0	0	0	0	0	0	0
0.4	979	2927	288	6	0	0	0	0	0	0	0	0
0.5	1540	2562	98	0	0	0	0	0	0	0	0	0
0.6	2124	2062	14	0	0	0	0	0	0	0	0	0
0.7	2618	1582	0	0	0	0	0	0	0	0	0	0
0.8	3039	1161	0	0	0	0	0	0	0	0	0	0
0.9	3407	793	0	0	0	0	0	0	0	0	0	0
All	16255	15493	3282	830	960	464	278	108	79	27	18	6

Table 2.7: Circular Matching: variable minimum module size γ table

	Overall gain percentage distribution broken down by γ											
γ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,900)	[900,1000)	[1000,1100)
0.1	351	729	111	47	0	0	22	0	0	0	0	0
0.2	479	607	99	42	11	18	4	0	0	0	0	0
0.3	545	559	92	20	23	19	2	0	0	0	0	0
0.4	576	528	96	5	35	17	3	0	0	0	0	0
0.5	601	503	84	17	35	17	3	0	0	0	0	0
0.6	610	494	78	23	35	16	4	0	0	0	0	0
0.7	611	493	78	23	35	16	4	0	0	0	0	0
0.8	614	490	78	23	30	17	8	0	0	0	0	0
0.9	611	487	84	28	25	17	8	0	0	0	0	0
1.0	597	498	81	34	25	17	4	4	0	0	0	0
1.1	589	506	81	34	25	13	8	4	0	0	0	0
1.2	582	502	92	34	25	13	8	4	0	0	0	0
1.3	581	477	118	34	20	18	8	4	0	0	0	0
1.4	576	481	113	34	26	18	4	8	0	0	0	0
1.5	574	483	113	24	36	18	4	5	3	0	0	0
1.6	560	497	113	24	36	10	12	5	3	0	0	0
1.7	545	505	120	18	42	10	12	4	4	0	0	0
1.8	545	498	121	24	37	15	12	4	4	0	0	0
1.9	536	507	115	24	43	15	8	8	4	0	0	0
2.0	536	507	115	24	43	10	13	4	5	3	0	0
2.1	535	501	122	24	43	10	13	4	5	3	0	0
2.2	531	505	122	24	43	10	13	4	5	3	0	0
2.3	514	508	136	24	33	20	11	6	5	3	0	0
2.4	507	515	124	30	39	20	9	8	5	0	3	0
2.5	499	523	124	30	39	15	14	4	6	3	3	0
2.6	498	524	124	30	39	15	14	4	6	3	3	0
2.7	496	519	131	30	39	15	14	4	6	3	3	0
2.8	495	520	131	30	34	20	14	4	6	3	1	2
2.9	481	513	152	30	29	25	10	8	6	3	1	2
3.0	480	514	134	42	35	20	15	8	6	0	4	2
All	16255	15493	3282	830	960	464	278	108	79	27	18	6

2.2.1.3 Circular Matching: variable match lengths

Consider an extension of the circle model where the *gap* z between matched banks is not fixed at 1, but instead is uniformly distributed between 0 and M. Specifically, consider a circle of circumference n, so that matched banks (x_1,

$x_2) \in [0,n)^2$ and $z := \mid x_2 - x_1 \mid \sim U[0,M]$. The gap has probability density function, $f(z) = \begin{cases} \frac{1}{M} & z \leq M \\ 0 & z > M \end{cases}$. We can now formulate the welfare function for the symmetric case of k modules of length d:

Theorem 50. *The Circular Model with variable match lengths has this welfare function for symmetric partitions:*

$$W_{gc}[d] = \begin{cases} \frac{d}{2M}P[d] + \theta\frac{(2M-d)}{2M}P[d]^2 & d \leq M \\ \left(1 - \frac{M}{2d}\right)P[d] + \theta\frac{M}{2d}P[d]^2 & M \leq d \leq 0.5n \\ P[n] & d = n \end{cases}$$

Proof. If the gap is z then in order for the match to be *inside* (both banks are in the same module), we require the first bank to be in the first $d - z$ units of the module. If the gap, z, is bigger than the size of the module, d, then the match is always *outside* (banks are in different modules). Also in the special case of 1 module, the match is always inside. Hence:

$$P(Inside:\ z) = \begin{cases} 1 & k = 1 \equiv d = n \\ \frac{d-z}{d} & z \leq d \\ 0 & z \geq d \end{cases}$$

So by integrating over z, we can calculate the inside match probability for the case of $k > 1$ symmetric modules: $P(Inside : k \in \mathbb{N}_2) = \int_{z=0}^{M} P(Inside : z)f(z)dz = \int_{z=0}^{Min\{d,M\}} \frac{d-z}{d}\frac{1}{M}dz = \frac{1}{dM}\int_{z=0}^{Min\{d,M\}}(d-z)dz$. This splits naturally into 2 cases: $P(Inside : d < M) = \frac{d}{2M}$ and $P(Inside : d < M) = 1 - \frac{M}{2d}$. The welfare function is given by $W_{gc}[d] = P(Inside)P[d] + \theta P(Outside)P[d]^2$, where $P[d] = \frac{1}{1+\gamma d}$ is the module enablement probability from the standard model. Including in the special case of 1 module, gives us the full 3-piece welfare function:

$$W_{gc}[d] = \begin{cases} \frac{d}{2M} P[d] + \theta \frac{(2M-d)}{2M} P[d]^2 & d \leq M \\ \left(1 - \frac{M}{2d}\right) P[d] + \theta \frac{M}{2d} P[d]^2 & M \leq d \leq 0.5n \\ P[n] & d = n \end{cases}$$

□

We now consider the argmax in each of the first two pieces, d_1^* and d_2^* respectively, and d^* the argmax of the whole function.

Theorem 51. *The first piece of $W_{gc}[d]$, where $d \leq M$, is quasi-convex and hence $d_1^* = 1$ or $d_1^* = M$.*

Proof. The first piece is the standard model with parametrisation $(n = 2M, \theta, \gamma)$. That model is quasi-convex and hence $d_1^* = 1$ or $d_1^* = M$. □

Corollary 52. *Hence if $M = 0.5n$ then the whole function consists of just the first piece and $d^* = 1$ or $d^* = M$.*

Theorem 53. *The 2nd piece of $W_{gc}[d]$ is quasi-concave and hence*

$$d_2^* = \begin{cases} M & \theta \geq \frac{1+\gamma M}{1+3\gamma M} \\ FOC & \frac{(\gamma n+2)\left(2M(\gamma n+1)-\gamma n^2\right)}{2M(3\gamma n+2)} < \theta < \frac{1+\gamma M}{1+3\gamma M} \\ 0.5n & \theta \leq \frac{(\gamma n+2)\left(2M(\gamma n+1)-\gamma n^2\right)}{2M(3\gamma n+2)} \end{cases}$$

Proof. With piece two, $W'_{gc}[d : M \leq d \leq 0.5n] = \frac{M\left(2\gamma^2 d^2 - 3\gamma d(\theta-1) - \theta + 1\right) - 2\gamma d^2(\gamma d+1)}{2d^2(\gamma d+1)^3}$ and so $\frac{2d^2 W'_{gc}[d]}{P[d]^3} = M\left(2\gamma^2 d^2 - 3\gamma d(\theta - 1) - \theta + 1\right) - 2\gamma d^2(\gamma d + 1)$. Considering this as a cubic expansion around $d = M$, gives that $W'_{gc}[d : M \leq d \leq 0.5n] > 0$ if and only if:

$$\left(-3\gamma\theta M^2+\gamma M^2-\theta M+M\right)-\left(2\gamma^2M^2+3\gamma\theta M+\gamma M\right)(d-M)-\left(2\gamma+4\gamma^2M\right)(d-M)^2-2\gamma^2(d-M)^3>0$$

The $(d-M)$, $(d-M)^2$, and $(d-M)^3$ coefficients are all negative, so the 2nd piece of $W_{gc}[d]$ is *quasi-concave* and hence:

$$d_2^* = \begin{cases} M & W_{gc}^{'}[d=M]\leq 0 \\ FOC & W_{gc}^{'}[d=M]>0 \text{ and } W_{gc}^{'}[d=0.5n]<0 \\ 0.5n & W_{gc}^{'}[d=0.5n]\geq 0 \end{cases}$$

Algebraic re-arrangement gives:

$$W_{gc}^{'}[d=M] \begin{cases} \leq 0 & 3\theta>1 \text{ and } M\geq \frac{1-\theta}{\gamma(3\theta-1)}<0 \\ >0 & 3\theta\leq 1 \text{ or } M<\frac{1-\theta}{\gamma(3\theta-1)} \end{cases}$$

$W_{gc}^{'}[d=0.5n]\geq 0\, if\ and\ only\ if\ M\geq \frac{\gamma n^2(\gamma n+2)}{(2\gamma^2n^2+6\gamma(1-\theta)n+4(1-\theta))}$.

The result then follows. □

Corollary 54. *If $M=1$ then the whole function is the middle piece.*

We can do further analysis by considering specific parametrisations: the number of banks, n, is from $\{10,\ 20,\ 30,\ 40,\ 50,\ 100,\ 1000\}$; θ, the value of outside matches, has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1; the shock parameter, γ, has a minimum of 0.1, a maximum of 3.0 and an increment of 0.1; M, the maximum match distance is set to be $M=m*n$, where m has a minimum of 0.01, a maximum of 0.5 and an increment of 0.01.[9]

We can consider the potential gains from interior partitions by using a separate table for each parameter. For each parametrisation, the welfare of the best

[9]Forming M from m in this way ensures that $M\leq 0.5n$ for all choices of n.

trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{gc}[1], \mathrm{W}_{gc}[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_{gc}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The low frequency of large gains from interior partitions, represented in Tables 2.8, 2.9, 2.10 and 2.11 below, suggests that the standard model rejection of interior partitions is robust to replacing a uniform distribution of business opportunities with a circular distribution with variable match length. However, if the gain had been calculated over all partitions rather than just symmetric partitions then each gain would have been weakly higher and so this may not then hold.

The quasi-concave region occurs with $mn \leq d \leq 0.5n$. As m increases this region gets smaller, and hence the potential for interior solutions is less. So in the m table, for larger m, both the frequency of interior solutions and the gains from interior solutions are lower. In the θ table, increases in θ results in, a reduction in the frequency of, and gains from, interior partitions: the atomistic partition has more outside matches than any interior partition; so as θ increases, there are greater benefits to the atomistic partition than to interior partitions. Similarly, increases in either n or γ, result in the same shift away from interior solutions.

Table 2.8: Circular Matching: variable match lengths m table

	Overall gain percentage distribution broken down by m											
m	(-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,100)	[100,140)
0.01	1724	149	9	3	3	0	0	1	0	0	0	1
0.02	1612	221	34	11	5	2	2	2	0	1	0	0
0.03	1511	282	55	20	10	6	1	2	1	1	0	1
0.04	1504	255	76	24	13	6	5	2	2	2	0	1
0.05	1461	285	77	29	16	9	2	6	1	2	0	2
0.06	1505	234	77	33	15	10	5	3	2	3	0	3
0.07	1552	188	76	31	13	11	6	5	1	3	1	3
0.08	1575	164	78	28	14	11	6	3	5	2	0	4
0.09	1578	171	69	28	15	9	5	7	2	2	1	3
0.10	1557	198	64	26	17	8	4	7	2	3	1	3
0.11	1553	207	59	29	14	8	5	5	3	4	0	3
0.12	1574	183	64	30	11	8	7	4	3	3	1	2
0.13	1602	165	57	26	13	8	6	4	3	3	1	2
0.14	1618	150	58	26	11	9	5	5	1	5	1	1
0.15	1641	128	58	26	12	8	5	5	3	2	2	0
0.16	1659	115	55	24	12	10	4	4	3	4	0	0
0.17	1675	107	50	23	13	8	5	4	1	4	0	0
0.18	1685	101	48	22	14	6	5	5	3	1	0	0
0.19	1694	94	50	18	14	8	5	4	3	0	0	0
0.20	1710	84	46	19	12	8	5	4	2	0	0	0
0.21	1720	81	41	16	13	9	6	3	1	0	0	0
0.22	1728	75	40	19	12	8	4	4	0	0	0	0
0.23	1737	67	40	21	11	6	7	1	0	0	0	0
0.24	1739	70	40	17	11	7	5	1	0	0	0	0
0.25	1745	69	37	17	13	8	1	0	0	0	0	0
0.26	1756	66	33	16	10	8	1	0	0	0	0	0
0.27	1765	60	31	15	12	7	0	0	0	0	0	0
0.28	1771	56	32	16	11	4	0	0	0	0	0	0
0.29	1778	51	32	17	9	3	0	0	0	0	0	0
0.30	1784	48	31	17	9	1	0	0	0	0	0	0
0.31	1788	46	32	17	7	0	0	0	0	0	0	0
0.32	1795	46	27	17	5	0	0	0	0	0	0	0
0.33	1800	46	28	15	1	0	0	0	0	0	0	0
0.34	1803	46	29	12	0	0	0	0	0	0	0	0
0.35	1812	41	28	9	0	0	0	0	0	0	0	0
0.36	1816	41	30	3	0	0	0	0	0	0	0	0
0.37	1820	44	26	0	0	0	0	0	0	0	0	0
0.38	1828	38	24	0	0	0	0	0	0	0	0	0
0.39	1829	42	19	0	0	0	0	0	0	0	0	0
0.40	1835	42	13	0	0	0	0	0	0	0	0	0
0.41	1837	46	7	0	0	0	0	0	0	0	0	0
0.42	1842	46	2	0	0	0	0	0	0	0	0	0
0.43	1848	42	0	0	0	0	0	0	0	0	0	0
0.44	1851	39	0	0	0	0	0	0	0	0	0	0
0.45	1854	36	0	0	0	0	0	0	0	0	0	0
0.46	1863	27	0	0	0	0	0	0	0	0	0	0
0.47	1870	20	0	0	0	0	0	0	0	0	0	0
0.48	1879	11	0	0	0	0	0	0	0	0	0	0
0.49	1890	0	0	0	0	0	0	0	0	0	0	0
0.50	1890	0	0	0	0	0	0	0	0	0	0	0
All	86263	4823	1782	720	371	214	112	91	42	45	8	29

Table 2.9: Circular Matching: variable match lengths θ table

Overall gain percentage distribution broken down by θ												
θ	(-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,100)	[100,140)
0.1	6236	1958	1181	461	251	144	88	71	28	45	8	29
0.2	8420	1387	342	154	88	51	24	20	14	0	0	0
0.3	9439	795	145	70	32	19	0	0	0	0	0	0
0.4	10033	359	73	35	0	0	0	0	0	0	0	0
0.5	10286	173	41	0	0	0	0	0	0	0	0	0
0.6	10402	98	0	0	0	0	0	0	0	0	0	0
0.7	10458	42	0	0	0	0	0	0	0	0	0	0
0.8	10489	11	0	0	0	0	0	0	0	0	0	0
0.9	10500	0	0	0	0	0	0	0	0	0	0	0
All	86263	4823	1782	720	371	214	112	91	42	45	8	29

Table 2.10: Circular Matching: variable match lengths n table

Overall gain percentage distribution broken down by n												
n	(-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,100)	[100,140)
10	10958	1690	572	194	68	18	0	0	0	0	0	0
20	11652	1075	405	165	92	57	31	16	6	1	0	0
30	12144	742	292	128	73	49	24	22	11	11	3	1
40	12444	549	229	103	58	39	24	21	12	13	2	6
50	12589	487	185	85	49	34	21	19	8	12	1	10
100	12990	276	96	44	28	17	12	11	5	8	2	11
10^3	13486	4	3	1	3	0	0	2	0	0	0	1
All	86263	4823	1782	720	371	214	112	91	42	45	8	29

Table 2.11: Circular Matching: variable match lengths γ table

Overall gain percentage distribution broken down by γ												
γ	(-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,100)	[100,140)
0.1	2249	342	175	113	74	54	39	34	23	10	8	29
0.2	2484	245	126	93	58	55	17	18	19	35	0	0
0.3	2614	211	116	58	57	29	26	39	0	0	0	0
0.4	2702	180	88	79	32	39	30	0	0	0	0	0
0.5	2759	172	89	46	47	37	0	0	0	0	0	0
0.6	2797	159	92	40	62	0	0	0	0	0	0	0
0.7	2826	156	81	46	41	0	0	0	0	0	0	0
0.8	2848	154	74	74	0	0	0	0	0	0	0	0
0.9	2873	143	71	63	0	0	0	0	0	0	0	0
1.0	2889	146	61	54	0	0	0	0	0	0	0	0
1.1	2906	148	55	41	0	0	0	0	0	0	0	0
1.2	2916	145	76	13	0	0	0	0	0	0	0	0
1.3	2923	144	83	0	0	0	0	0	0	0	0	0
1.4	2931	139	80	0	0	0	0	0	0	0	0	0
1.5	2938	134	78	0	0	0	0	0	0	0	0	0
1.6	2947	130	73	0	0	0	0	0	0	0	0	0
1.7	2954	130	66	0	0	0	0	0	0	0	0	0
1.8	2961	127	62	0	0	0	0	0	0	0	0	0
1.9	2964	131	55	0	0	0	0	0	0	0	0	0
2.0	2965	137	48	0	0	0	0	0	0	0	0	0
2.1	2967	140	43	0	0	0	0	0	0	0	0	0
2.2	2973	139	38	0	0	0	0	0	0	0	0	0
2.3	2975	146	29	0	0	0	0	0	0	0	0	0
2.4	2976	157	17	0	0	0	0	0	0	0	0	0
2.5	2982	162	6	0	0	0	0	0	0	0	0	0
2.6	2983	167	0	0	0	0	0	0	0	0	0	0
2.7	2986	164	0	0	0	0	0	0	0	0	0	0
2.8	2991	159	0	0	0	0	0	0	0	0	0	0
2.9	2992	158	0	0	0	0	0	0	0	0	0	0
3.0	2992	158	0	0	0	0	0	0	0	0	0	0
All	86263	4823	1782	720	371	214	112	91	42	45	8	7

2.2.2 Other Business Networks

2.2.2.1 Increased probability of self matching

The standard model assumes that the probability of a self match (b_1, b_1), a match where both businesses have the same bank, is $1/n^2$; the same probability as when the banks are different: $P[(b_1, b_2)] = 1/n^2$ for all (b_1, b_2). This section shows the robustness of the results of the standard model to varying that assumption.

In this section there are 2 classes of matches: self-match matches (class 1) and standard matches (class 2). If the class is 1 then the match is certain to be between businesses with the same bank. So $P[(b_1, b_1)|\, class1] = 1/n$. If the class is 2 then the match is distributed in the same way as in the standard model: self-matches can occur but are no more probable than any other match, so $P[(b_1, b_2)|\, class2] = 1/n^2$. The match is class 1 with probability s and is class 2 with probability $1 - s$. This means that: when $s = 0$, we have the standard model; when $s = {}^{-1}/_{(n-1)}$, we have a model where the probability of a self match is 0; and if $s = 1$ then the probability of a self match is 1.

This generates for a (generally asymmetric) partition, $x = (x_j)_{j=1}^k$:

- Bank Payoff $v_{si}[x] := s\frac{1}{n}P[x_i] + (1-s)\left(\frac{x_i}{n^2}P[x_i] + \sum\limits_{j \neq i} \theta \frac{x_j}{n^2} P[x_i]P[x_j]\right)$
- Module Worth $V_{si}[x] := s\frac{x_i}{n}P[x_i] + (1-s)\left(\frac{x_i^2}{n^2}P[x_i] + \sum\limits_{j \neq i} \theta \frac{x_i x_j}{n^2} P[x_i]P[x_j]\right)$
- Welfare $W_s[x] := s\sum\limits_{i=1}^{n} \frac{x_i}{n}P[x_i] + (1-s)\sum\limits_{i=1}^{n}\left(\frac{x_i^2}{n^2}P[x_i] + \sum\limits_{j \neq i} \theta \frac{x_i x_j}{n^2} P[x_i]P[x_j]\right)$

Theorem 55. *This model has boundary solutions*

Proof. The optimality of the trivial partitions can be shown using the same methodology as with the standard model. There are again negative externalities, and so the argmax for $v_{s1}[x]$ will be of the form $\{x_1, 1, 1, , , , 1\}$ for some x_1. The optimal x_1 can be found as follows. The derivative $v'_{s1}[x_1] := \frac{\partial v_{s1}[\{x_1,1,1,,,,1\}]}{\partial x_1} = -\frac{\theta + \gamma(\theta n - 1) + s(\gamma - \theta + 1)(\gamma n + 1) - 1}{(\gamma+1)(\gamma n x1 + n)^2}$ and so $v'_{s1}[x_1] > 0$ if and only if $s < \frac{\gamma - \theta + \gamma\theta(-n) + 1}{\gamma - \theta + \gamma^2 n - \gamma\theta n + \gamma n + 1}$; but this condition is independent of x_1, and so $v_{s1}[x_1]$ is monotonic. So the best partition for bank 1 is always either the grand coalition $\{n\}$, or the atomistic partition $\{1, 1, 1...1\}$. Hence, as in the standard model, only the trivial partitions can be strictly efficient. The preference between the two trivial partitions is given by the condition that, the grand coalition is weakly preferred to the atomistic partition if and only if $P[n] \geq (1-s)\theta P[1] + s$. $\square$

2.2.2.2 Multiple Businesses per Bank

We get similar results when we have multiple (f) businesses per bank, and matches are always between distinct businesses. This gives that:

$$v_{f1}[x_1] := \frac{1}{fn}\frac{(fx_1 - 1)}{fn-1}P[x_1] + \theta\frac{1}{fn}\frac{f(n-x_1)}{fn-1}P[x_1]P[1]$$

Hence $v'_{f1}[x_1] = \frac{\gamma(\gamma+1)+f(\gamma-\theta+\gamma\theta(-n)+1)}{(\gamma+1)(\gamma x_1+1)^2}$, and so the efficient solution is always a trivial partition.

2.2.2.3 Star Business Network

Consider a setup where there is 1 big firm and n small firms. Suppose the business match is always between the big firm and one of the small firms. This can be visualised as a star network: the big firm is the hub; each small firm is a spoke. Each firm is the single client of a bank. So there is 1 big bank and n small banks. Formulating financial shocks in discrete time, each small bank is hit independently by a disabling shock with probability q, whilst the big bank is hit by a disabling shock with probability Q. As in the standard model, $P[d] = \frac{1}{1+\gamma d}$, is the enablement probability for a module with d small banks (but not the big bank). Similar analysis to that used for the standard model gives that the enablement probability for a module containing d small banks plus the 1 big bank is $P_b[d] = \frac{1}{1+\Gamma+\gamma d}$, where $\gamma = \frac{Log[1-q]}{Log[1-\rho]}$ and $\Gamma = \frac{Log[1-Q]}{Log[1-\rho]}$.

Due to the negative externalities we know that the efficient partition will be of the form, $\{(1\ big,\ d\ small),\ 1small,\ 1small, ...1small\}$: it is inefficient for a module to contain multiple small firms, unless that module also contains the big firm. So we only need to consider the welfare function for partitions of that format, $W_{star}[d] = \frac{d}{n}P_b[d] + \theta\frac{(n-d)}{n}P_b[d]P[1]$. This represents that the matched small bank is in the same module as the big bank with probability $\frac{d}{n}$ and in a different module with probability $\frac{n-d}{n}$. When the banks are in different modules we have standard θ parameter to represent transaction costs.

Differentiation with respect to d gives that $W'_{star}[d] = \frac{(1+\Gamma)(\gamma-\theta+1)-\gamma\theta n}{(\gamma+1)(\gamma d+1+\Gamma)^2}$. So $\frac{W'_{star}[d]}{P[1]P_b[d]^2} = (1+\Gamma)(\gamma-\theta+1)-\gamma\theta n$. Hence $W_{star}[d]$, is monotonic and so the maximisation program has boundary solutions.

2.2.2.4 Trilateral Business Matches

In the standard model, business opportunities always involve two businesses. Here we consider the variant model where matches involve three businesses.[10] We first form $v_{tri,i}[x]$ the return for a sample member of module i. These are then aggregated to form the welfare function $W_{tri}[x]$.

Each match can be put into one of four different categories, from the perspective of module i. The first is when all three businesses are in module i. The second is when two businesses are in module i, but the third is in a distinct module j. The third is when one business is in module i, but the other two are in a distinct module j. The fourth is when the first business is in module i, and the other two businesses are in distinct modules j and k. In cases two and three where the businesses are split over two modules, the value of a productive match is θ_1. In case four where the businesses are split over three modules, the value of a productive match is θ_2. As there are increasing transaction costs from matches involving multiple modules, $0 < \theta_2 < \theta_1 < 1$. Keeping the same behaviour of financial shocks as in the standard model,

[10] Thanks to Professor Herakles Polemarchakis for suggesting the investigation of this aspect.

gives us: $v_{tri,i}[x] = \frac{x_i^2}{n^3}P[x_i] + \frac{2x_i}{n^2}P[x_i]\theta_1 \sum_{j\neq i} \frac{x_j}{n}P[x_j] + \frac{1}{n}\theta_1 P[x_i] \sum_{j\neq i} \frac{x_j^2}{n^2}P[x_j] + \frac{1}{n}\theta_2 P[x_i] \sum\limits_{j\neq i,\, k\neq i,\, j\neq k} \left(\frac{x_j x_k}{n^2}\right) P[x_j]P[x_k]$.

As $V_{tri,i}[x] = x_i * v_{tri,i}[x]$ and $W_{tri}[x] = \sum_{i=1}^{k} V_{tri,i}[x]$, aggregation gives: $W_{tri}[x] = \sum_{i=1}^{k} \left[\frac{x_i^2}{n^3}E_i + 2\frac{x_i}{n^2}E_i\theta_1 \sum_{j\neq i} \frac{1}{n}E_j + \frac{1}{n}E_i\theta_1 \sum_{j\neq i} \frac{x_j^2}{n^2}P[x_j] + \frac{1}{n}\theta_2 E_i \sum\limits_{j\neq i,\, k\neq i,\, j\neq k} \frac{1}{n^2}E_jE_k\right]$, where $E_l := x_l P[x_l]$ is the expected number of enabled banks in module l.

and for the symmetric case this simplifies to:

$W_{tri}[d] = \frac{1}{n^2}\left(d^2P[d] + 3d(n-d)P^2[d]\theta_1 + \theta_2((n-d)((n-2d)P^3[d])\right)$

Proposition 56. *If $\gamma \geq 0.5$ then the trilateral model has negative externalities.*

Proof. We now consider $v_{tri,1}[x]$, to see when it has negative externalities. Consider $x = (x_1, T - x_3, x_3, x_4, x_5.....x_k)$. This gives: $\frac{dv_{tri,1}[(x_1,T-x_3,x_3,x_4,x_5.....x_k)]}{dx_3} = P[x_1]P[x_3]^2P[T-x_3]^2(T-2x_3)\left\{\theta_2\left[\gamma R(\gamma T+2) + 2\gamma T + 2\right] + \theta_1(2\gamma x_1 - 1)(\gamma T+2)\right\}$, where $R := \sum_{j>3} x_j P[x_j]$. So, if $x_3 < 0.5T$ then $\frac{dv_{tri,1}[(x_1,T-x_3,x_3,x_4,x_5.....x_k)]}{dx_3} > 0$ if and only if $\theta_2(\gamma R(\gamma T+2) + 2\gamma T + 2) + \theta_1(2\gamma x_1 - 1)(\gamma T+2) > 0$. So $\gamma \geq 0.5$ is sufficient for negative externalities. □

Proposition 57. *If $\gamma < 0.5$, then there exist values of the other parameters (θ_1, θ_2, n) such that the trilateral model does not have negative externalities.*

Proof. From the proof of Proposition 56, in order for the model that to have negative externalities we need that $\theta_2\left[\gamma R(\gamma T+2) + 2\gamma T + 2\right] + \theta_1(2\gamma x_1 - 1)(\gamma T+2) > 0$ for all partitions and resulting values of R, T, and x_1. It is hardest for this condition to be positive when: R is minimised, which happens when all the members not in the first three modules are in one big module, resulting in $R = P[n-T-x_1]$; when T is maximised, which requires $T = n - x_1$; and that x_1 is minimised which requires $x_1 = 1$.[11] This combines to give $R = 0$,

[11] The result $R = P[n-T-x_1]$ only applies in the case where $n-T-x_1 > 0$. If $n-T-x_1 = 0$ then $R = 0$

$T = n-1$ and $x_1 = 1$. Hence the model has negative externalities if and only if $(2\gamma - 1)\left[\gamma(n-1)+2\right] + 2r\left[\gamma(n-1)+1\right] > 0$. where $r := {}^{\theta_2}\!/_{\theta_1}$. The second term is positive, but as $\gamma < 0.5$, the first term is negative. Hence for sufficiently small values of r, the whole expression is negative. $\square$

Theorem 58. *The v_{tri} program, where outside modules are restricted to being singletons, has boundary solutions.*

Proof. Consider the partitions of form $\left(x_1, (1)_{j=2}^{n+1-x_1}\right)$ where modules apart from module 1 are singletons. Then:

$$v_{tri,1}[\left(x_1,(1)_{j=2}^{n+1-x_1}\right)] = 2x_1(n-1)P[1]P[x_1]\theta_1 + x_1^2 P[x_1] + \theta_1(n-1)P[1]P[x_1] + \theta_2(n-1)(n-2)P[1]^2 P[x_1]$$

This gives:

$$\frac{1}{P[1]^2 P[x_1]^2}\frac{dv_{tri,1}[(x_1,(1)_{j=2}^{n+1-x_1}]}{dx_1} = -(n-1)\left[(\gamma^2-\gamma-2)\theta_1 + \gamma\theta_2(n-2)\right] + \left(2\gamma^2+4\gamma+2\right)x_1 + \left(\gamma^3+2\gamma^2+\gamma\right)x_1^2$$

The x_1 and x_1^2 coefficients are both positive. So once the slope is positive it stays positive. Hence for partitions of this form $v_{tri,1}$ is quasi-convex. $\square$

Corollary 59. *So if non-singleton outside modules can be rejected, for example due to $\gamma > 0.5$, then the efficient partition is a boundary solution.*

This corollary motivates why no cases were found with interior solutions to the welfare maximisation program. This rejection of interior partitions for every parametrisation argues that the standard model rejection of interior partitions is robust to replacing a bilateral distribution of business opportunities with a trilateral distribution.

For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{tri}[1], \mathrm{W}_{tri}[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_{tri}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The number of banks, n, is one of $\{10, 20, 30, 40, 50, 100, 1000\}$. The value of outside matches spread over 2 modules, θ_1, has a minimum of 0.1, a maximum of 0.9 and has an increment of 0.1. The value of outside matches spread over 3 modules, θ_2, is $\theta_1 * r$, and r has a minimum of 0, a maximum of 1 and has an increment of 0.1. We know there are no interior solutions, with $\gamma \geq 0.5$: so the shock parameter, γ, has a minimum of 0.1, a maximum of 0.4 and an increment of 0.1. So, the n parameter is one of 7 values, the θ_1 parameter is one of 9 values, the r parameter one of 11 values, and the γ parameter is one of 4 values. This gives a total of $7 * 9 * 11 * 4 = 2772$ parametrisations.

In every case considering symmetric partitions, the best boundary partition has a higher welfare than the best interior partition. This leads to the following conjecture:

Conjecture 60. *The trilateral program has boundary solutions for a large range of parameter values.*

The θ_1, r_2, n and γ tables, each show that when their parameter is increased then, whilst the best interior partition is still worse than the best boundary partition, the scale of the loss from choosing an interior partition is reduced:

Table 2.12: Trilateral Matching θ_1 Table

	Overall gain percentage distribution broken down by θ_1					
θ_1	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)
0.1	225	11	10	8	3	51
0.2	53	119	21	10	11	94
0.3	0	131	14	12	16	135
0.4	0	67	45	19	11	166
0.5	0	24	60	17	16	191
0.6	0	14	36	30	14	214
0.7	0	10	16	33	17	232
0.8	0	0	17	22	24	245
0.9	0	0	8	15	24	261
All	278	376	227	166	136	1589

Table 2.13: Trilateral Matching r_2 Table

	Overall gain percentage distribution broken down by r_2					
r_2	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)
0.0	33	117	60	33	8	1
0.1	29	67	40	29	26	61
0.2	29	48	28	22	22	103
0.3	29	37	19	19	17	131
0.4	26	30	16	14	14	152
0.5	26	20	17	10	12	167
0.6	25	16	11	11	9	180
0.7	23	13	11	8	8	189
0.8	21	11	10	7	6	197
0.9	19	10	7	6	10	200
1.0	18	7	8	7	4	208
All	278	376	227	166	136	1589

Table 2.14: Trilateral Matching n Table

	Overall gain percentage distribution broken down by n					
n	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)
10	77	68	62	52	44	93
20	55	70	48	35	31	157
30	53	58	36	29	20	200
40	38	62	28	21	19	228
50	33	55	28	18	13	249
100	18	39	17	10	9	303
10^3	4	24	8	1	0	359
All	278	376	227	166	136	1589

Table 2.15: Trilateral Matching γ Table

	Overall gain percentage distribution broken down by γ					
γ	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)
0.1	93	85	75	55	46	339
0.2	67	96	56	43	36	395
0.3	65	93	51	35	27	422
0.4	53	102	45	33	27	433
All	278	376	227	166	136	1589

2.2.2.5 Temporal distribution of business opportunities

We now show the resilience of the standard model to different *temporal* distributions of the business opportunities, when: the initial module enablement probabilities are given by the stationary distribution of financial shocks, and business opportunities remain independent of financial shocks. Let $s = (s_t)_{t\in\mathbb{R}_+}$ be a generic realisation of business opportunities, with $s_t \in \mathbb{R}$ opportunities at time t. Let $f(s)$ be the probability density function (pdf) of s and let $f_t(s_t)$ be the induced pdf of s_t. Let $W_{bis}[x]$ be the aggregate expected welfare given our distribution of s. Then by the law of total expectation $W_{bis}[x] = \int_s W_{bis}[x|s]f(s)ds$. The conditional expectation is given by $W_{bis}[x|s] = \int_{t=0}^{\infty} \delta^t s_t W[x]dt$, where δ is the intertemporal discount factor and $W[x]$ is the welfare function of the standard model. Hence, $W_{bis}[x] = \int_s \int_{t=0}^{\infty} \delta^t s_t W[x]dt\, f(s)ds$. As $W[x]$ is independent of both s and t, this gives: $W_{bis}[x] = W[x] \int_s \int_{t=0}^{\infty} \delta^t s_t dt\, f(s)ds$. So the optimal choice of x is the same as in the standard model.

2.2.3 Small number of banks

2.2.3.1 Regional trade

Mundell (1961) considers optimal currency areas and how they vary depending on the distribution of industry. He considers the example of a world consisting of just the USA and Canada, where eastern region of each country makes cars, and the western region makes wood products. He then argues in face of sectoral shocks that it may be better to have an eastern currency and a western currency; than a Canadian currency and an American currency. We can consider a similar simple example to consider financial stability.[12]

Imagine a world with 2 countries (North and South) and that each has a manufacturing sector in the west and a service sector in the east. So this gives 4 regions: $\{NW, NE, SW, SE\}$. Business opportunities are national (NW to NE or SW to SE) with probability $1-\phi$, or sectoral (NW to SW or NE to SE) with probability ϕ. The question is what partition of modules best enables business to operate. There are 4 symmetric partitions to choose from: 1) the grand coalition $G := \{\{NW, NE, SW, SE\}\}$; 2) the national partition $N := \{\{NW, NE\}, \{SW, SE\}\}$; 3) the sectoral partition $S := \{\{NW, SW\}, \{NE, SE\}\}$; and 4) the atomistic partition $A := \{\{NW\}, \{NE\}, \{SW\}, \{SE\}\}$.[13] Using the same shock dynamics as in the standard model, where θ is the value of inter-module matches and $P[d] = \frac{1}{1+\gamma d}$ is the module enablement probability, gives the following welfare function: $W_M[G] := P[4]$, $W_M[N] := (1-\phi)P[2] + \phi\theta P^2[2]$, $W_M[S] := \phi P[2] + (1-\phi)\theta P^2[2]$, and $W_M[A] := \theta P^2[1]$.

The choice between the national and sectoral partitions is determined by whether national or sectoral matches is more likely: $S \succ N \iff \phi > \frac{1}{2}$. Given the symmetry of the example, without loss of generality we assume $\phi > \frac{1}{2}$. The preference conditions comparing the sectoral partition with the boundary partitions are as follows: $S \succ A \iff \phi > \frac{\theta(P^2[1]-P^2[2])}{P[2]-\theta P^2[2]}$ and $S \succ G \iff \phi > \frac{(P[4]-\theta P^2[2])}{P[2]-\theta P^2[1]}$. Re-arrangement gives that S is the sole argmax partition if and only if, $\phi >$

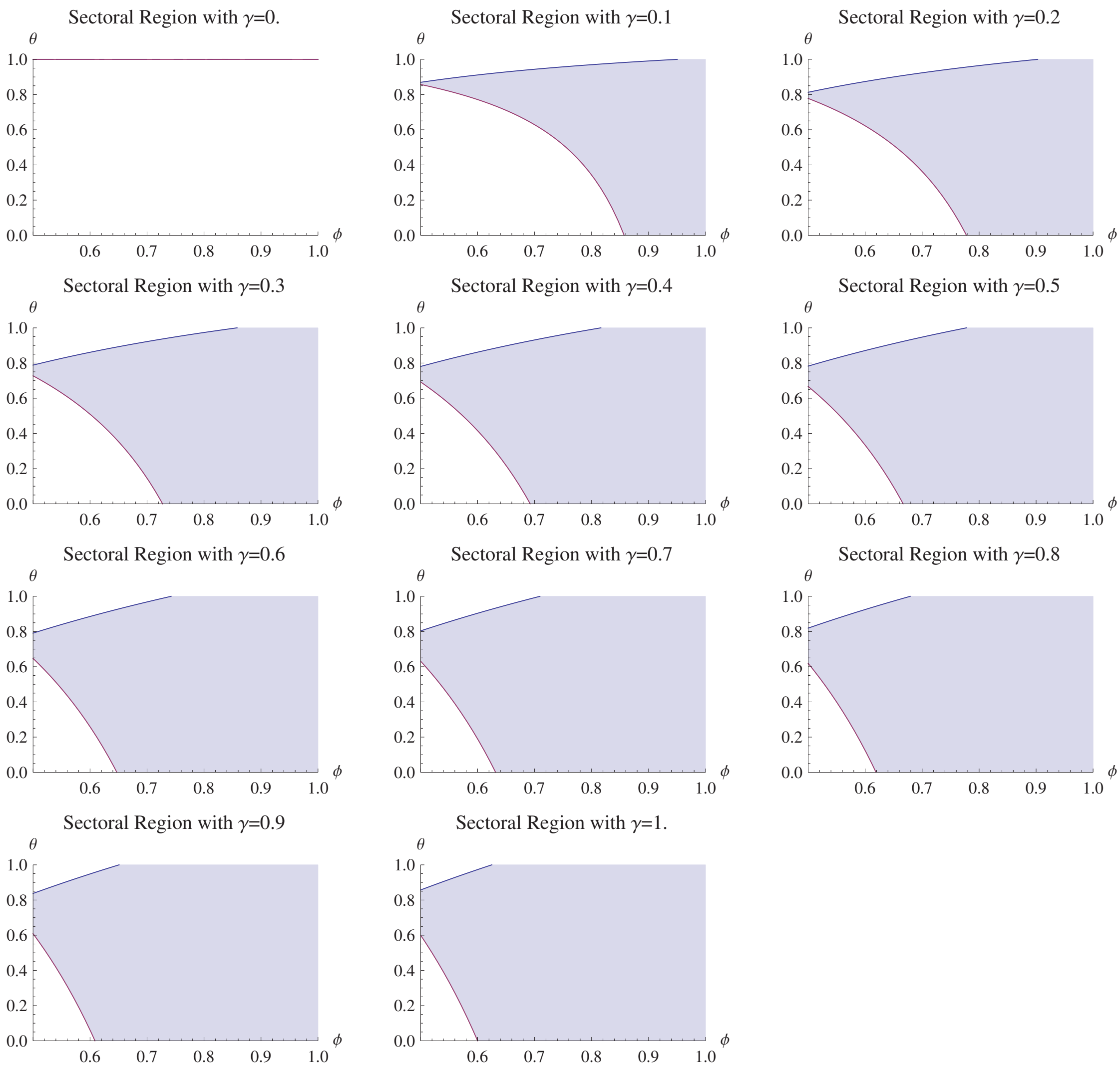
[12]Guillaume Sublet suggested investigating such an example.

[13]Allowing asymmetric partitions like $\{\{NW, NE, SW\}, \{SE\}\}$ makes the maths intractable. However, when only symmetric partition are allowed with most parametrisations there is an interior solution, and increasing the feasible set of interior solutions can only make interior solutions occur more often.

0.5 and $\theta > \theta_{min}$ and ($\theta < \theta_{max}$ or $\gamma \geq \varphi$) where, $\varphi := \frac{1+\sqrt{5}}{2}$ (the golden ratio), $\theta_{min} := \frac{(2\gamma+1)(2\gamma-\phi 4\gamma+1-\phi)}{(1-\phi)(4\gamma+1)}$, and $\theta_{max} := \frac{(\gamma+1)^2(2\gamma+1)\phi}{\gamma^2(\phi+3)+2\gamma(\phi+1)+\phi}$.

The following graphs plot this region in purple in a series of graphs with ϕ on the x-axis and θ on the y-axis. There is one graph for each value of γ, the shock parameter, starting at 0, incrementing by 0.1 and ending at 1:

Figure 2.1: Regional Trade Model Interior Sectoral Solution Plot



We can see that this region is large for all non-zero values of γ used. To the top left of the purple region is the region where the atomistic partition is strictly preferred to the sectoral partition: once γ reaches the golden ratio (approximately

1.61803), this region is empty. To the bottom left is the region where the grand coalition is strictly preferred to the sectoral partition. Each of the graphs plots ϕ between 0.5 and 1: If we consider what happens with ϕ between 0 and 0.5, then there is a symmetric relation for the preference of the national partition over the atomistic partition and the grand coalition.

For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := Max[W_M[A], W_M[G]]$; the welfare of the best interior partition (regional or sectoral partition) is given by $\mathrm{W}^i := Max[W_M[N], W_M[S]]$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. This gain was found for a range of different parametrisations: the shock parameter, γ had a minimum of 0.1, a maximum of 3 and an increment of 0.1; the sectoral match probability, ϕ had a minimum of 0.1, a maximum of 1 and an increment of 0.1 and outside match parameter, θ similarly had a minimum of 0.1, a maximum of 1 and an increment of 0.1. In general, the large gains from interior partitions, represented in Tables 2.16, 2.17 and 2.18 below, mean that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a four country orthogonal distribution. However, in the special case of uniform matching, where $\phi = 0.5$, there is both a low frequency of parametrisations (35%) where there are gains from an interior partition and small gains in those parametrisations (below 10% in all cases). In this case the robustness appears to remain.

The distribution of the gain percentages can be displayed for each of the 3 choice parameters. The γ table shows, as expected from the indifference graphs above, that for all values of γ, the frequency of interior solutions is low and that frequency decreases as γ decreases. The ϕ table has vertical symmetry given the symmetric choice between the sectoral partition in response to ϕ and the national partition the sectoral partition in response to $1 - \phi$. Only when the matching parameter, ϕ, is close to 0.5 are there no gains, or only small gains, from the interior partitions. In the θ table, there is not much of a correspondence between θ and the gain, except to observe that there are especially few cases of boundary solutions when θ is 0.6, 0.7 or 0.8.

Table 2.16: Regional Trade γ Table

Overall gain percentage distribution broken down by γ										
γ	[-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)
0.1	50	36	24	0	0	0	0	0	0	0
0.2	37	25	24	24	0	0	0	0	0	0
0.3	30	22	22	20	16	0	0	0	0	0
0.4	25	23	16	16	16	14	0	0	0	0
0.5	21	19	18	18	16	4	14	0	0	0
0.6	19	15	20	18	10	14	14	0	0	0
0.7	14	16	18	16	14	16	16	0	0	0
0.8	12	16	16	16	18	16	2	14	0	0
0.9	10	14	16	18	18	10	8	16	0	0
1.0	10	14	14	16	16	8	16	16	0	0
1.1	9	15	12	16	14	10	18	16	0	0
1.2	6	12	14	18	12	14	18	0	16	0
1.3	6	12	14	16	12	14	18	2	16	0
1.4	6	12	14	16	6	18	20	2	16	0
1.5	6	10	14	12	10	20	12	8	18	0
1.6	6	8	14	14	10	20	6	14	18	0
1.7	5	9	14	12	10	20	4	18	18	0
1.8	5	7	16	10	12	20	2	18	20	0
1.9	5	7	16	10	12	18	2	20	20	0
2.0	5	7	14	10	14	16	4	20	0	20
2.1	5	7	14	10	14	14	6	20	0	20
2.2	5	5	16	8	16	14	6	20	0	20
2.3	5	5	16	8	16	12	8	20	0	20
2.4	5	5	16	8	16	12	8	20	0	20
2.5	5	5	16	6	18	10	10	20	0	20
2.6	5	5	16	6	18	8	12	20	0	20
2.7	5	5	16	6	18	8	12	20	0	20
2.8	5	5	16	4	20	6	14	20	0	20
2.9	5	5	16	4	20	6	14	20	0	20
3.0	5	5	16	4	20	4	16	20	0	20
All	337	351	488	360	412	346	280	364	142	220

Table 2.17: Regional Trade ϕ Table

Overall gain percentage distribution broken down by ϕ										
ϕ	[-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)
0.0	0	4	13	14	13	13	27	35	71	110
0.1	2	9	14	19	20	32	57	147	0	0
0.2	7	16	19	28	46	128	56	0	0	0
0.3	17	26	37	93	127	0	0	0	0	0
0.4	45	68	161	26	0	0	0	0	0	0
0.5	195	105	0	0	0	0	0	0	0	0
0.6	45	68	161	26	0	0	0	0	0	0
0.7	17	26	37	93	127	0	0	0	0	0
0.8	7	16	19	28	46	128	56	0	0	0
0.9	2	9	14	19	20	32	57	147	0	0
1.0	0	4	13	14	13	13	27	35	71	110
All	337	351	488	360	412	346	280	364	142	220

Table 2.18: Regional Trade θ Table

Overall gain percentage distribution broken down by θ										
θ	[-100,0)	[0,10)	[10,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)
0.1	54	34	38	42	28	40	22	34	16	22
0.2	48	32	42	40	30	44	20	36	16	22
0.3	46	26	46	36	36	44	22	36	16	22
0.4	40	22	54	34	38	40	26	38	16	22
0.5	38	18	56	30	44	38	30	38	16	22
0.6	16	32	60	30	46	38	30	40	16	22
0.7	5	39	62	26	50	36	34	40	16	22
0.8	5	41	66	20	56	32	34	38	16	22
0.9	21	55	34	56	46	18	32	36	10	22
1.0	64	52	30	46	38	16	30	28	4	22
All	337	351	488	360	412	346	280	364	142	220

2.2.3.2 Orthogonal

The results in the previous section consider a world with 4 banks arranged in a square. This section generalises the setup to consider a D dimension orthogonal model: there are $n = 2^D$ banks and they are arranged at the vertices of a D dimensional hypercube. Bank positions can be represented in co-ordinate form as being the members of the set $Y := \{(y_i)_{i=1}^{D} : (y_i \in \{0,1\})_{i=1}^{D}\}$.

The business matches are distributed such that informally, each business trades with a business a 'distance' of 1 unit away. More precisely, matched businesses have the same co-ordinates in $D-1$ dimensions and different co-ordinates in exactly 1 dimension. So a match $(y, y^{'})$ between the banks in positions y and y' means that there exists dimension j s.t. $(y_i = y_i^{'})_{i \neq j}$ and $y_j \neq y_j^{'}$. The probabilistic distribution of matches is as follows. Each bank has a probability 2^{-D} of being the first bank picked. Without loss of generality, we order the dimensions such that the probability, $\phi_i \in (0, 1)$, of the second bank in the match being in dimension i is weakly decreasing as i increases: so $(\phi_i \geq \phi_{i+1})_{i=1}^{D-1}$. There are no self matches and normalisation requires $\sum_{i=1}^{D} \phi_i = 1$. For future use, we define the accumulative probability that the match is in one of the first j dimensions, $\Phi_j := \sum_{i=1}^{j} \phi_i$. This implies that, $\Phi_D = 1$, and we further define $\Phi_0 := 0$. Financial shocks operate as in the standard model: shocks are fully transmitted between banks in same module and there is no transmission between banks in different modules.

We now define a set of feasible partitions. Informally, partition X_j has 'breaks' in j dimensions and 'joins' in $D-j$ dimensions. So it has $k_j := 2^j$ modules each containing $d_j := 2^{D-j}$ banks. The gain from extra joins is that there are more high value inside matches, and fewer low value outside matches. The cost from extra joins is that each initial financial shock spreads further, and does more damage. So optimally, the $D-j$ join dimensions will be the first $D-j$ dimensions, as these have the highest matching probabilities, and the breaks will be in the last j dimensions, as these have the lowest matching probabilities. Formally, $\left(X_j := \{\{y \in Y : (y_i = m_i)_{i=D+1-j}^{D}\}_{m \in M_j}\}\right)_{j=0}^{D}$ where $M_j := \{((0)_{i=1}^{D-j}, (y_i)_{i=D+1-j}^{D}) : (y_i \in \{0, 1\})_{i=D+1-j}^{D}\}$. As special cases we have the boundary partitions, X_0 is the grand coalition and X_D is the atomistic partition.

The welfare function is

$$W_{\text{orth}}[X_i] = \Phi_{D-i} P[2^{D-i}] + \theta(1 - \Phi_{D-i}) P^2[2^{D-i}]$$

This can be interpreted in the usual way: the first term comes from inside matches; the second term from outside matches. From the standard model we use: θ, the value of outside matches; $P[d] = \frac{1}{1+\gamma d}$, the module enablement probability for a module of size d; and γ the shock parameter. The grand coalition and the atomistic partition have the expected welfares:

$$W_{\text{orth}}[X_0] = P[2^D]$$

and

$$W_{\text{orth}}[X_D] = \theta P^2[1]$$

We now now consider when X_{j+1} is preferred to X_j.

$$W_{\text{orth}}[X_j] = \Phi_{D-j} P[2^{D-j}] + \theta(1 - \Phi_{D-j}) P^2[2^{D-j}]$$

and

$$W_{\text{orth}}[X_{j+1}] = \Phi_{D-j-1} P[2^{D-j-1}] + \theta(1 - \Phi_{D-j-1}) P^2[2^{D-j-1}]$$

The trade off is formed as follows. Recall that Φ_i is an increasing function, so that partition X_j has the higher probability of an inside match: Φ_{D-j} compared with Φ_{D-j-1}. However, it also has the lower module enablement probability: $P[2^{D-j}]$ compared with $P[2^{D-j-1}]$. Re-arrangement gives that $X_{j+1} \succ X_j$ if and only if

$$\Phi_{D-j} < \frac{\gamma\theta n k_j \left(3\gamma n + 2k_{j+1}\right) + 2\Phi_{D-j-1}\left(\gamma n + 4\right)^2 \left(\gamma n + (1-\theta)k_{j+1}\right)}{\left(\gamma n + k_{j+1}\right)^2 \left(\gamma n + (1-\theta)k_j\right)}$$

In order for there to be an interior solution we need a partition X_j such that $X_j \succ X_0$ *and* $X_j \succ X_{Dim}$. The conditions for this are as follows:

Table 2.19: Hypercube Preference Conditions

	$X_j \succ X_0$	$X_j \succ X_{Dim}$
θ terms	$\frac{(\gamma d_j+1)\left(\frac{\left(1+\gamma d_j\right)}{(\gamma n+1)^2}-\Phi_{Dim-j}\right)}{(1-\Phi_{Dim-j})} < \theta$	$\theta < \frac{(\gamma+1)^2\ \Phi_{Dim-j}(\gamma d_j+1)}{(\gamma d_j+1)^2-\left((1+\gamma)^2(1-\Phi_{Dim-j})\right)}$
Φ_{Dim-j}	$\frac{(\gamma d_j+1)^2-\theta(\gamma n+1)^2}{(\gamma n+1)^2((1-\theta)+\gamma d_j)} < \Phi_{Dim-j}$	$\frac{\gamma\theta(d_j-1)((\gamma+2)+\gamma d_j)}{(\gamma+1)^2((1-\theta)+\gamma d_j)} < \Phi_{Dim-j}$

Here, $n = 2^D$ is the total number of banks, $k_j = 2^j$, is the number of modules in partition X_j, and $d_j = \frac{n}{k_j} = 2^{D-j}$ is the number of banks in each of those modules.

2.2.3.3 3-Dimension case

We now focus on the three-dimensional case, where there are $2^3 = 8$ banks, and four symmetric partitions: 1) X_0, the grand coalition, 1 module of 8 banks; 2) X_1, 2 squares, 2 modules of 4 banks; 3) X_2 4 lines, 4 modules of 2 banks; and 4) X_3 atomistic partition, 8 modules of 1 banks.[14] The welfare valuations are

$$W_{\text{orth}}[X_0] = \frac{1}{8\gamma + 1}$$

$$W_{\text{orth}}[X_1] = \frac{\theta(1-\Phi_2)}{(4\gamma+1)^2} + \frac{\Phi_2}{4\gamma+1}$$

[14] As in section 2.2.3.1, allowing asymmetric partitions makes the maths intractable. However, when only symmetric partition are allowed with most parametrisations there is an interior solution, and increasing the feasible set of interior solutions can only make interior solutions occur more often.

$$W_{\text{orth}}[X_2] = \frac{\theta(1 - \Phi_1)}{(2\gamma + 1)^2} + \frac{\Phi_1}{2\gamma + 1}$$

$$W_{\text{orth}}[X_3] = \frac{\theta}{(\gamma + 1)^2}$$

The parameters from the standard model have their standard ranges: $\theta \in (0, 1)$ and $\gamma \in (0, \infty)$. The parameters specific to this orthogonal model satisfy, $1/3 \leq \Phi_1 < \Phi_2 < 1$ and $\Phi_2 \leq 2 * \Phi_1$ and $\Phi_2 - \Phi_1 \geq 1 - \Phi_2$. In the case of *uniform* matching (no bias towards any dimension) then $\Phi_1 = \frac{1}{3}$ and $\Phi_2 = \frac{2}{3}$. In the case of *non-uniform* matching, the inequalities are all strict: $1/3 < \Phi_1 < \Phi_2 < 1$ and $\Phi_2 < 2 * \Phi_1$ and $\Phi_2 - \Phi_1 > 1 - \Phi_2$. The following conditions can be derived for when the interior partitions are strictly preferred to the boundary partitions:

Table 2.20: 3D Hypercube Preferences General Matching

strict preference conditions for an interior partition over a boundary partition

	X_0 (Grand Coalition) X_3 (Singletons)
X_1 (2 Squares)	$c_{10} := \frac{(4\gamma+1)\left(64\gamma^2\Phi_2+4\gamma(4\Phi_2-1)+\Phi_2-1\right)}{(8\gamma+1)^2(\Phi_2-1)} < \theta \;\; \theta < \frac{(\gamma+1)^2(4\gamma+1)\Phi_2}{\gamma^2(\Phi_2+15)+2\gamma(\Phi_2+3)+\Phi_2} =: c_{13}$
X_2 (4 lines)	$c_{20} := \frac{(2\gamma+1)\left(64\gamma^2\Phi_1+2\gamma(8\Phi_1-1)+\Phi_1-1\right)}{(8\gamma+1)^2(\Phi_1-1)} < \theta \;\; \theta < \frac{(\gamma+1)^2(2\gamma+1)\Phi_1}{\gamma^2(\Phi_1+3)+2\gamma(\Phi_1+1)+\Phi_1} =: c_{23}$

With Uniform Matching, this becomes:

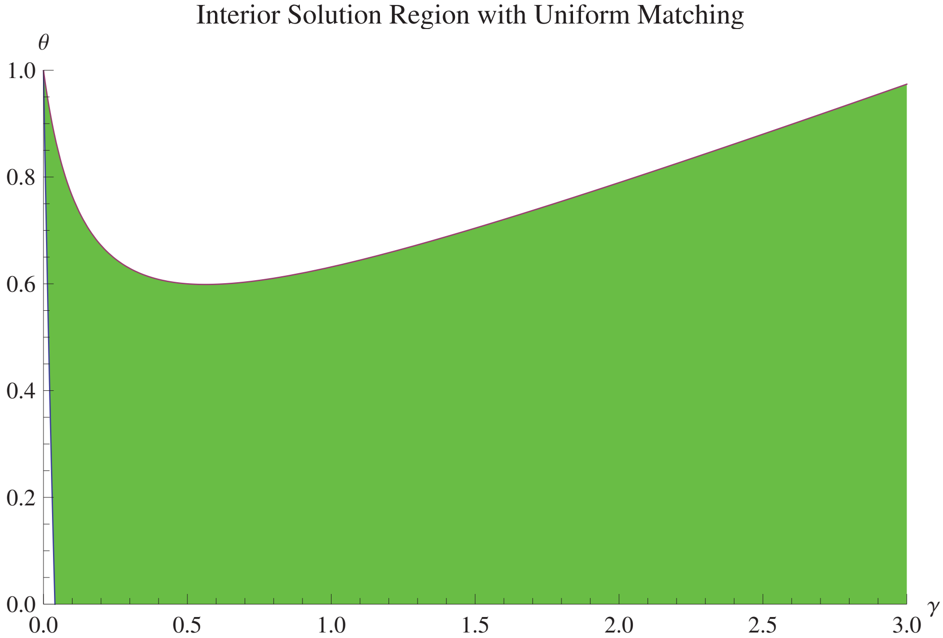
Table 2.21: 3D Hypercube Preferences Uniform Matching

strict preference conditions for an interior partition over a boundary partition

	X_0 (Grand Coalition) X_3 (Singletons)
X_1 (2 Squares)	$-\frac{(4\gamma+1)\left(128\gamma^2+20\gamma-1\right)}{(8\gamma+1)^2} < \theta \;\; \theta < \frac{2(\gamma+1)^2(4\gamma+1)}{47\gamma^2+22\gamma+2}$
X_2 (4 lines)	$-\frac{(2\gamma+1)\left(32\gamma^2+5\gamma-1\right)}{(8\gamma+1)^2} < \theta \;\; \theta < \frac{(\gamma+1)^2(2\gamma+1)}{10\gamma^2+8\gamma+1}$

To get an interior solution requires either, ($X_1 \succ X_0$ and $X_1 \succ X_3$) or ($X_2 \succ X_0$ and $X_2 \succ X_3$). With uniform matching this simplifies to $X_1 \succ X_0$and$X_2 \succ X_3$. Graphically this becomes:

Figure 2.2: 3D Uniform Matching Model Interior Solution Plot



The large green region shows that for most values of γ and θ, that there is an interior solution. Above this region there is an area where the atomistic partition is preferred: Once $\gamma \geq \frac{1}{4}\left(\sqrt{57}+5\right) = 3.14\,(3S.F.)$, this region becomes empty. On the far left there is a very small region where the grand coalition is preferred: Once $\gamma \geq \frac{1}{64}\left(\sqrt{57}-5\right) = 0.0398\,(3S.F.)$, this region also becomes empty.

For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := Max[(W_{\mathrm{orth}}[X_j : \Phi_1 = 1/3, \Phi_2 = 2/3])_{j=0,3}]$; the welfare of the best interior partition (regional or sectoral partition) is given by $\mathrm{W}^i := Max[(W_{\mathrm{orth}}[X_j : \Phi_1 = 1/3, \Phi_2 = 2/3])_{j=1}^{2}]$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The large gains from interior partitions, represented in Tables 2.22 and 2.23 below, mean that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a 3D Orthogonal Uniform Matching distribution.

This gain was found for a table of parametrisations: the shock parameter, γ had a minimum of 0.1, a maximum of 3 and an increment of 0.1; and θ had a minimum of 0.1, a maximum of 0.9 and an increment of 0.1. As expected from the graph above: as θ increases there are both fewer parametrisations with interior solutions and less gains from the parametrisations that do have interior solutions; the γ table shows fewest interior partitions with $\gamma = 0.6$.

Table 2.22: 3D Uniform Matching θ Table

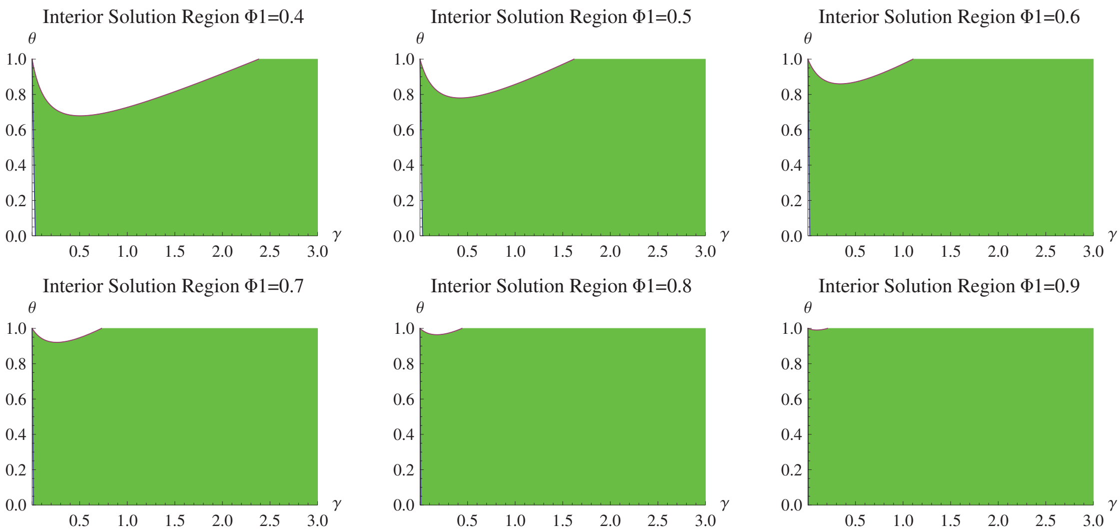
	3D Uniform Matching: percentage gain distribution broken down by θ								
θ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)
0.1	0	1	1	1	1	11	6	7	2
0.2	0	1	13	14	2	0	0	0	0
0.3	0	14	16	0	0	0	0	0	0
0.4	0	26	4	0	0	0	0	0	0
0.5	0	30	0	0	0	0	0	0	0
0.6	1	29	0	0	0	0	0	0	0
0.7	13	17	0	0	0	0	0	0	0
0.8	20	10	0	0	0	0	0	0	0
0.9	26	4	0	0	0	0	0	0	0
Total	60	132	34	15	3	11	6	7	2

Table 2.23: 3D Uniform Matching γ Table

	3D Uniform Matching: percentage gain distribution broken down by γ								
γ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)
0.1	2	7	0	0	0	0	0	0	0
0.2	3	4	2	0	0	0	0	0	0
0.3	3	4	1	1	0	0	0	0	0
0.4	3	4	1	0	1	0	0	0	0
0.5	3	4	1	0	0	1	0	0	0
0.6	4	3	1	0	0	1	0	0	0
0.7	3	4	1	0	0	1	0	0	0
0.8	3	4	1	0	0	1	0	0	0
0.9	3	4	1	0	0	1	0	0	0
1.0	3	4	1	0	0	1	0	0	0
1.1	3	4	1	0	0	1	0	0	0
1.2	3	4	1	0	0	1	0	0	0
1.3	3	4	1	0	0	1	0	0	0
1.4	3	4	1	0	0	1	0	0	0
1.5	2	4	1	1	0	1	0	0	0
1.6	2	4	1	1	0	0	1	0	0
1.7	2	4	1	1	0	0	1	0	0
1.8	2	4	1	1	0	0	1	0	0
1.9	2	4	1	1	0	0	1	0	0
2.0	2	4	1	1	0	0	1	0	0
2.1	1	5	1	1	0	0	1	0	0
2.2	1	5	1	1	0	0	0	1	0
2.3	1	5	1	1	0	0	0	1	0
2.4	1	5	1	1	0	0	0	1	0
2.5	1	5	1	1	0	0	0	1	0
2.6	1	5	1	1	0	0	0	1	0
2.7	0	5	2	1	0	0	0	1	0
2.8	0	5	2	1	0	0	0	1	0
2.9	0	5	2	0	1	0	0	0	1
3.0	0	5	2	0	1	0	0	0	1
Total	60	132	34	15	3	11	6	7	2

Considering the case of non-uniform matching in 3 dimensions, it is the case for all (γ, Φ_1, Φ_2) parametrisations that $c_{10} < c_{20} < c_{13} < c_{23}$. Hence there is an interior solution if and only if $c_{10} < \theta < c_{23}$. The interior solution is plotted below for a range of values of Φ_1:

Figure 2.3: 3D Non-Uniform Matching Model Interior Solution Plot



In each case Φ_2 has the minimal value $(\frac{\Phi_1+1}{2})$: if it is increased further then the region on the far left, which is barely visible, where the grand coalition is preferred becomes yet *smaller*; whilst the region at the top where the atomistic partition is preferred stays the same size. As Φ_1 increases across the plots, we see that the interior solution region becomes bigger.

Further, as with other models, we can consider a range of different parametrisations, and see what the percentage gain is from choosing the best interior partition compared with the best boundary partition: the welfare of the best trivial partition is given by $\mathrm{W}^b := \max[(W_{\mathrm{orth}}[X_j])_{j=0,3}]$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \max\left[(W_{\mathrm{orth}}[X_j])_{j=1}^{2}\right]$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The large gains from interior partitions, represented in Tables 2.24, 2.25, 2.26 and 2.27 below, mean that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a 3D Orthogonal Non-Uniform Matching distribution.

The θ and γ parameters have their usual ranges: θ has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1; γ has a minimum of 0.1, a maximum of 3.0 and an increment of 0.1. Next we calculate appropriate choices for the Φ_1 and Φ_2 ranges. The three required conditions are as follows: 1) $1/3 < \Phi_1 < \Phi_2 < 1$, (as Φ_j is an increasing function, and one of the dimensions must be chosen with a probability of at least 1/3); 2) $\Phi_2 < 2\Phi_1$, (as $\phi_2 < \phi_1$); and 3) $\Phi_2 - \Phi_1 > 1 - \Phi_2$, (as $\phi_2 > \phi_3$). These requirements combine to give the following conditional range for Φ_2 given $\Phi_1 : (1 + \Phi_1)/2 < \Phi_2|\Phi_1 < \min[2\Phi_1, 1]$. The implementation for Φ_1 and Φ_2 is as follows: Φ_1 has a minimum of 1/3, an increment of 0.1, and a maximum of $0.6 + \frac{1}{3}$; Φ_2 is one of 11 equally spaced out values, with a minimum of $(1 + \Phi_1)/2$, a maximum of $\min[2\Phi_1, 1]$, and an increment of $0.1\,(\min[2\Phi_1, 1] - (1 + \Phi_1)/2)$. The choice of Φ_2 is achieved using an intermediate variable f, so $\Phi_2 = \Phi_1 + f{*}0.1\,(\min[2\Phi_1, 1] - (1 + \Phi_1)/2)$.

The results are presented below in separate tables for f, Φ_1, θ and γ. In the f table the choice of f has very little effect: this is simply because given Φ_1, the choice of Φ_2 has very little effect. The top line of the Φ_1 table equates to the uniform case, and as Φ_1 increases, the welfare from X_2 increases, but the welfare from both the boundary partitions is unaltered. Hence there is a shift towards interior partitions, and higher gains are achieved. In the θ table, as θ increases, the welfare of the atomistic partition increases, and hence the gains from interior partitions are less. In the γ table, as γ increases somewhat surprisingly we see a shift towards interior solutions; however, this can be explained as follows. Recall that in this model there are no self matches: so, with the atomistic partition, we need two independent modules to be enabled in order for a match to be productive, an unlikely event when γ is large; however, with X_2 which has 1 dimensional modules, with probability Φ_1 a match is contained inside a single module.

Table 2.24: 3D Non-Uniform Matching: f Table

3D Non-Uniform Matching: percentage gain distribution broken down by f

f	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,2100)
0.0	100	768	398	188	110	76	54	39	31	126
0.1	100	767	397	190	109	77	54	39	31	126
0.2	100	765	399	190	109	77	54	39	31	126
0.3	100	763	400	190	110	76	55	39	31	126
0.4	100	760	401	192	110	75	56	39	31	126
0.5	100	757	399	197	110	72	58	40	31	126
0.6	100	756	400	197	110	71	57	42	31	126
0.7	100	756	400	197	110	71	54	44	32	126
0.8	100	755	401	196	109	73	52	45	32	127
0.9	100	754	402	195	109	74	52	44	32	128
1.0	100	750	405	193	110	76	51	42	33	130
Total	1100	8351	4402	2125	1206	818	597	452	346	1393

Table 2.25: 3D Non-Uniform Matching:Φ_1 Table

3D Non-Uniform Matching: percentage gain distribution broken down by Φ_1

Φ_1	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,2100)
10/30	660	1452	374	165	33	121	66	77	22	0
13/30	286	1528	465	250	120	58	79	55	57	72
16/30	110	1467	548	302	132	97	56	56	47	155
19/30	44	1286	661	308	195	80	77	66	44	209
22/30	0	1067	759	319	220	143	66	77	66	253
25/30	0	869	781	374	253	143	110	55	66	319
28/30	0	682	814	407	253	176	143	66	44	385
Total	1100	8351	4402	2125	1206	818	597	452	346	1393

Table 2.26: 3D Non-Uniform Matching: θ Table

3D Non-Uniform Matching: percentage gain distribution broken down by θ

θ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,2100)
0.1	0	28	66	52	58	176	212	232	214	1272
0.2	0	26	226	499	455	367	275	209	132	121
0.3	0	182	763	606	407	231	110	11	0	0
0.4	0	613	927	484	242	44	0	0	0	0
0.5	0	1100	836	330	44	0	0	0	0	0
0.6	11	1496	671	132	0	0	0	0	0	0
0.7	143	1661	484	22	0	0	0	0	0	0
0.8	330	1683	297	0	0	0	0	0	0	0
0.9	616	1562	132	0	0	0	0	0	0	0
Total	1100	8351	4402	2125	1206	818	597	452	346	1393

Table 2.27: 3D Non-Uniform Matching: γ Table

	3D Non-Uniform Matching: percentage gain distribution broken down by γ									
γ	(-100,0)	[0,100)	[100,200)	[200,300)	[300,400)	[400,500)	[500,600)	[600,700)	[700,800)	[800,2100)
0.1	44	451	198	0	0	0	0	0	0	0
0.2	77	361	123	88	44	0	0	0	0	0
0.3	77	354	100	66	57	17	22	0	0	0
0.4	77	356	100	50	34	26	17	11	11	11
0.5	77	357	99	50	22	28	13	14	11	22
0.6	77	335	121	50	22	27	13	15	11	22
0.7	66	341	115	61	11	38	13	15	11	22
0.8	66	341	114	51	22	37	13	5	11	33
0.9	66	319	121	55	33	25	19	11	11	33
1.0	66	318	122	54	23	33	22	10	12	33
1.1	55	319	121	65	23	33	21	11	12	33
1.2	44	319	121	64	35	22	30	12	2	44
1.3	44	308	132	55	44	22	29	4	11	44
1.4	44	297	143	55	44	22	17	16	11	44
1.5	33	286	154	66	33	33	16	17	11	44
1.6	33	286	121	88	33	22	33	22	10	45
1.7	22	275	143	87	34	22	33	21	1	55
1.8	22	264	154	74	47	22	22	30	3	55
1.9	22	253	165	66	55	22	22	18	15	55
2.0	22	242	165	77	43	34	22	11	22	55
2.1	11	242	165	77	44	44	22	11	22	55
2.2	11	231	175	67	55	33	11	33	21	56
2.3	11	220	165	88	55	22	22	22	20	68
2.4	11	220	154	99	55	22	22	22	19	69
2.5	11	220	154	99	44	33	22	22	11	77
2.6	11	176	198	99	44	33	22	22	11	77
2.7	0	165	198	99	54	45	22	22	11	77
2.8	0	165	198	88	64	35	22	22	11	88
2.9	0	165	187	88	66	33	33	11	22	88
3.0	0	165	176	99	66	33	22	22	22	88
Total	1100	8351	4402	2125	1206	818	597	452	346	1393

2.2.3.4 3-node case

Consider a 3-node model where, unlike in the standard model, there is non-uniform matching: the nodes are numbered $\{0, 1, 2\}$, and the matching probability between nodes i and j is q_{ij}.[15] There is no self matching and so there are 3 matching probabilities, which sum to one: $q_{01} + q_{02} + q_{12} = 1$. The 3 nodes have 5 possible partitions:

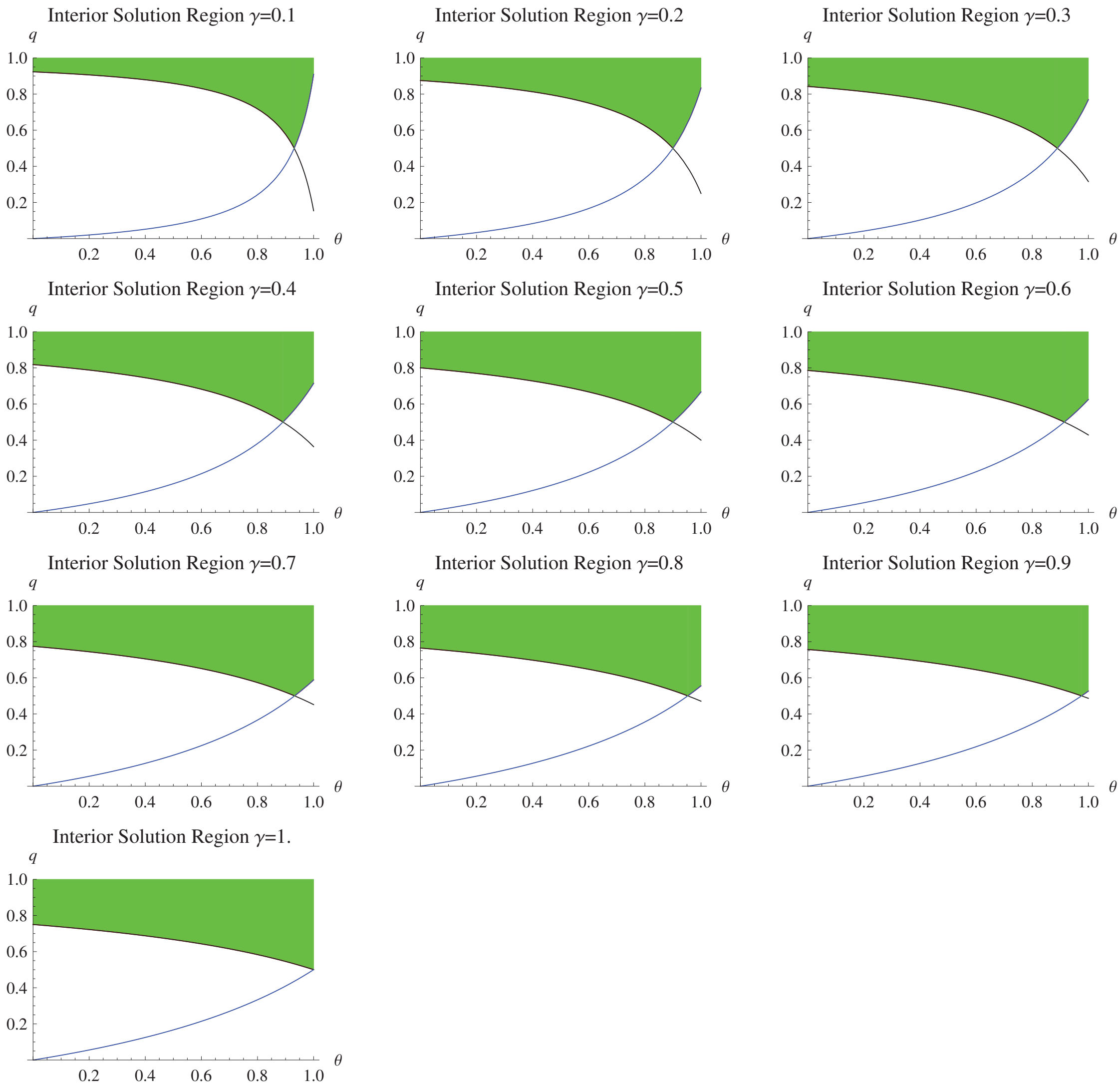
Table 2.28: 3 Nodes Model Partitioning Table

Partition	Modules	W_3
GC: Grand Coalition	$\{\{0,1,2\}\}$	$W_3[GC] = P[3]$
S_0: Separate node 0	$\{\{0\},\{1,2\}\}$	$W_3[S_0] = \theta(q_{01} + q_{02})P[1]P[2] + q_{12}P[2]$
S_1: Separate node 1	$\{\{1\},\{0,2\}\}$	$W_3[S_1] = \theta(q_{01} + q_{12})P[1]P[2] + q_{02}P[2]$
S_2: Separate node 2	$\{\{2\},\{0,1\}\}$	$W_3[S_2] = \theta(q_{02} + q_{12})P[1]P[2] + q_{01}P[2]$
$Atom$: Atomistic	$\{\{0\},\{1\},\{2\}\}$	$W_3[Atom] = \theta P^2[1]$

The best of the three S partitions comes from when the highest probability match is inside the 2 node module. Without loss of generality, we assume that $q := q_{01} \geq q_{02} \geq q_{12}$, (this also means that $q \geq 1/3$), and thus that $W_3[S] = W_3[S_2] \geq W_3[S_1] \geq W_3[S_0]$, where $S := S_2$. Hence the strict interior solutions are given when $W_3[S] > W_3[GC]$ and $W_3[S] > W_3[Atom]$. The preference conditions are $S \succ GC \Leftrightarrow q > q_{GC}[\gamma, \theta] := 1 - \frac{\gamma(\gamma+1)}{(3\gamma+1)(\gamma-\theta+1)}$ and $S \succ Atom \Leftrightarrow q > q_{Atom}[\gamma, \theta] := \frac{\gamma\theta}{(\gamma+1)\ (\gamma-\theta+1)}$. The green region shows the interior solution, where $q > \max[1 - \frac{\gamma(\gamma+1)}{(3\gamma+1)(\gamma-\theta+1)}, \frac{\gamma\theta}{(\gamma+1)\ (\gamma-\theta+1)}]$:

[15] One particular application of the 3 node model is to have 3 geographic blocks: Asia, USA, and EU. Each business opportunity comes from a product made in Asia and sold in either the USA or the EU; so each match is between Asia and one of the US and EU. Letting Asia be node 0, USA be node 1 and the EU node 2; the match distribution is $P[0,1] = q_{01}$ and $P[0,2] = 1 - q_{01}$.

Figure 2.4: 3 Node Model Interior Solution Plot



This leads to the following result:

Proposition 61. *In the 3-node model with no self matching a necessary condition for an interior solution is that* $q > 0.5$.

Proof. Recall for an interior solution we need both $q > q_{GC}[\gamma, \theta] := 1 - \frac{\gamma(\gamma+1)}{(3\gamma+1)(\gamma-\theta+1)}$ and $q > q_{Atom}[\gamma, \theta] := \frac{\gamma\theta}{(\gamma+1)\ (\gamma-\theta+1)}$. Fix an arbitrary value of γ, and note that, $\frac{\partial q_{GC}}{\partial \theta} = -\frac{\gamma(\gamma+1)}{(3\gamma+1)(\gamma-\theta+1)^2} < 0$, $\frac{\partial q_{Atom}}{\partial \theta} = \frac{\gamma}{(\gamma-\theta+1)^2} > 0$, $q_{GC}[\gamma, 0] = 1 - \frac{\gamma}{(3\gamma+1)} > 2/3$ and $q_{Atom}[\gamma, 0] = 0$. For $\theta = 0$, the q_{GC} condition is binding, but as θ increases, the q_{GC} condition becomes weaker, and the q_{Atom} condition becomes stronger, so for some values of γ, the q_{Atom} condition becomes binding. If we have such a γ then at the crossing point, $q_{GC}[\gamma,\theta] = q_{Atom}[\gamma,\theta]$ and $\theta = \theta^*[\gamma] := \frac{(\gamma+1)^2}{3\gamma+1}$. This gives that $q_{GC}[\gamma,\theta^*[\gamma]] = q_{Atom}[\gamma,\theta^*[\gamma]] = \frac{1}{2}$. If $\theta < \theta^*$ then the q_{GC} condition is binding and $\theta < \theta^* \Rightarrow q_{GC}[\gamma,\theta] > \frac{1}{2}$. Conversely, if $\theta > \theta^*$ then the q_{Atom} condition is binding and $\theta > \theta^* \Rightarrow q_{Atom}[\gamma,\theta] > \frac{1}{2}$. If γ is such that there is no crossing point, then the $q_{GC}[\gamma,\theta]$ is binding for all θ and $1 < \theta^*[\gamma] := \frac{(\gamma+1)^2}{3\gamma+1}$. So $q_{GC}[\gamma,\theta] > \frac{1}{2}$. $\square$

We can further analyse this model with our usual table approach. For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_3[Atom], \mathrm{W}_3[GC]\}$; the welfare of the best interior partition is given by $\mathrm{W}^i := \mathrm{W}_3[S]$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The high frequency (69%) of positive gains from interior partitions, represented in Tables 2.29, 2.30 and 2.31 below, suggests that the standard model rejection of interior partitions is critical to replacing a uniform distribution of business opportunities with a 3 Node distribution.

For an interior solution we need $q > 0.5$, so q has a minimum of 0.55, a maximum of 0.95 and an increment of 0.05. The θ and γ parameters have their usual ranges: θ has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1; γ has a minimum of 0.1, a maximum of 3.0 and an increment of 0.1. The q table has the clearest pattern: as expected, increasing q leads to greater gains from the interior partition. There are also increase in the gains from the interior partition when the θ and γ parameters increase, however, these are less dramatic.

Table 2.29: 3 Node Model q Table

	3 Node Model: percentage gain distribution broken down by q								
q	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)
0.55	261	9	0	0	0	0	0	0	0
0.60	234	34	2	0	0	0	0	0	0
0.65	161	81	28	0	0	0	0	0	0
0.70	65	89	94	22	0	0	0	0	0
0.75	25	33	90	101	21	0	0	0	0
0.80	13	16	27	75	123	16	0	0	0
0.85	6	9	15	27	57	134	22	0	0
0.90	2	7	10	15	24	50	126	36	0
0.95	0	4	7	11	16	24	41	97	70
Total	767	282	273	251	241	224	189	133	70

Table 2.30: 3 Node Model θ Table

	3 Node Model: percentage gain distribution broken down by θ								
θ	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)
0.1	127	22	24	24	17	20	19	12	5
0.2	117	26	27	22	19	22	20	12	5
0.3	108	29	30	18	23	23	21	12	6
0.4	103	32	21	24	25	25	21	12	7
0.5	98	22	26	29	27	27	19	14	8
0.6	72	34	33	31	31	27	18	15	9
0.7	65	35	34	33	31	26	20	17	9
0.8	49	39	43	34	31	21	24	19	10
0.9	28	43	35	36	37	33	27	20	11
Total	767	282	273	251	241	224	189	133	70

Table 2.31: 3 Node Model γ Table

	3 Node Model: percentage gain distribution broken down by γ								
γ	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)
0.1	53	20	8	0	0	0	0	0	0
0.2	39	18	14	10	0	0	0	0	0
0.3	34	13	15	14	5	0	0	0	0
0.4	30	13	11	14	13	0	0	0	0
0.5	27	11	11	12	12	8	0	0	0
0.6	27	8	13	10	11	12	0	0	0
0.7	26	10	9	11	10	12	3	0	0
0.8	24	10	10	11	10	9	7	0	0
0.9	23	10	11	9	9	9	10	0	0
1.0	23	9	10	9	9	10	11	0	0
1.1	24	8	9	8	11	9	10	2	0
1.2	23	9	9	8	10	9	8	5	0
1.3	23	8	9	9	9	9	7	7	0
1.4	23	9	8	8	10	8	6	9	0
1.5	23	9	7	9	9	8	7	9	0
1.6	24	8	7	9	8	8	8	9	0
1.7	23	8	8	8	8	8	9	9	0
1.8	23	8	8	8	8	8	9	9	0
1.9	23	8	8	8	7	8	10	9	0
2.0	23	7	9	8	7	8	9	9	1
2.1	23	7	9	7	7	9	9	8	2
2.2	23	7	9	7	7	9	8	7	4
2.3	22	8	8	8	7	8	9	6	5
2.4	23	8	7	8	7	8	8	6	6
2.5	23	8	7	8	7	8	8	5	7
2.6	23	8	7	7	8	8	7	4	9
2.7	23	8	8	6	8	8	7	4	9
2.8	23	8	8	6	8	8	6	5	9
2.9	23	8	8	6	8	7	7	5	9
3.0	23	8	8	5	8	8	6	6	9
Total	767	282	273	251	241	224	189	133	70

2.2.3.5 2 rich 2 poor

Imagine that there are 4 countries: 2 rich countries (R_1, R_2) and 2 poor countries (P_1, P_2). Each business opportunity is between a random rich country and a random poor country. So the distribution of matches is $P[R_1, P_1] = 0.25$, $P[R_1,$

$P_2] = 0.25$, $P[R_2, P_1] = 0.25$ and $P[R_2, P_2] = 0.25$. Countries are partitioned into modules and, as in the standard model, financial shocks are fully transmitted within modules; financial shocks are never transmitted between modules; and the module enablement probability is $P[d] = \frac{1}{1+\gamma d}$.

The possible partitions can be classified into five different types: firstly there is the *grand coalition* $\{\{RRPP\}\}$; secondly there is the *one singleton* partition that separates off one poor or rich country into a singleton module, $\{\{P\}, \{RRP\}\}$ or $\{\{R\}, \{RPP\}\}$; thirdly there is the heterogenous *pairs* partition that has each module consisting of one rich country and one poor country, $\{\{RP\}, \{RP\}\}$; fourthly there is the *split* partition that has 3 modules, one rich-poor module and two singletons $\{\{RP\}, \{R\}, \{P\}\}$; and fifthly there is the *atomistic partition* of singletons, $\{\{R\}, \{R\}, \{P\}, \{P\}\}$.[16] The welfares of the partitions are as follows:

Table 2.32: 2 Rich 2 Poor Model Welfare Table

Partition X	Description	Welfare $W_{rrpp}[X]$
$\{\{RRPP\}\}$	Grand Coalition	$P[4]$
$\{\{P\}, \{RRP\}\}$ or $\{\{R\}, \{RPP\}\}$	One Singleton	$(1/2)P[3] + (1/2)\theta P[3]P[1]$
$\{\{RP\}, \{RP\}\}$	Pairs	$(1/2)P[2] + (1/2)\theta P[2]^2$
$\{\{RP\}, \{R\}, \{P\}\}$	Split	$(1/4)P[2] + (1/4)\theta P[1]^2 + (1/2)\theta P[2]P[1]$
$\{\{R\}, \{R\}, \{P\}, \{P\}\}$	Atomistic Partition	$\theta P[1]^2$

Proposition 62. *If there is a strict interior solution then it is the pairs partition,* $I := \{\{RP\}, \{RP\}\}$

Proof. There are 3 interior partitions: i) One Singleton, ii) Pairs and iii) Split. First, the pairs partition strictly dominates the one singleton partition: $W_{rrpp}[\{\{RP\}, \{RP\}\}] - W_{rrpp}[\{\{R\}, \{RPP\}\}] = \frac{\gamma\left(2\gamma^2+\gamma(3-\theta)+1\right)}{2(\gamma+1)(2\gamma+1)^2(3\gamma+1)} > 0$. Second, if there is a solution at the split partition, then we need the split partition to be weakly preferred to the pairs partition, and strongly preferred to the atomistic partition:

[16] Firstly due to the symmetry we do not need to specify the specific rich or poor country in order to calculate the welfare. Secondly we have excluded the partitions {{RR},{PP}}, {{RR},{P},{P}}, {{R},{R},{PP}} as they have homogenous modules which increases contagion but have no business value.

$W_{rrpp}[\{\{R\},\{RPP\}\}] \geq W_{rrpp}[\{\{RP\},\{RP\}\}]$ and, $W_{rrpp}[\{\{R\},\{RPP\}\}] > W_{rrpp}[\{\{R\},\{R\},\{P\},\{P\}\}]$. This requires $\frac{(\gamma+1)^2(2\gamma+1)}{6\gamma^2+6\gamma+1} \leq \theta < \frac{(\gamma+1)^2}{4\gamma+1}$ and hence $8\gamma^2 + 6\gamma + 1 < 6\gamma^2 + 6\gamma + 1$, a contradiction. □

This leads to the condition for an interior solution:

Theorem 63. *There is a strict interior solution if and only if* $\frac{2\gamma+1}{4\gamma+1} < \theta < \frac{(\gamma+1)^2(2\gamma+1)}{7\gamma^2+6\gamma+1}$.

Proof. From Proposition 62 the pairs partition $I := \{\{RP\},\{RP\}$ is the only candidate interior maximum. The result then comes from comparing the pairs partition with the grand coalition $GC := \{RRPP\}$ and the atomistic partition $Atom := \{\{R\},\{R\},\{P\},\{P\}\}$: $I \succ Atom \Leftrightarrow \theta < \theta_{Atom}[\gamma] := \frac{(\gamma+1)^2(2\gamma+1)}{7\gamma^2+6\gamma+1}$ and $I \succ GC \Leftrightarrow \theta > \frac{2\gamma+1}{4\gamma+1}$. □

Corollary 64. *A necessary condition for an interior solution is that* $\theta > 0.5$

Proof. An interior solution requires $I \succ GC \Leftrightarrow \theta > \frac{2\gamma+1}{4\gamma+1} \Leftrightarrow \theta > \frac{2\gamma+0.5+0.5}{4\gamma+1} \Leftrightarrow \theta > \frac{1}{2} + \frac{0.5}{4\gamma+1} > 0.5$ □

Corollary 65. *A sufficient condition for the grand coalition to be optimal is that* $\theta \leq 0.5$

Proof. Follows from Corollary 64. □

Corollary 66. *A necessary condition for a strict atomistic solution is that* θ *is greater than the only real root of* $f[\theta] := 49\theta^3 - 43\theta^2 + 5\theta - 1$. *The root is given by*

$$\theta^* := \frac{\left(16+\sqrt[3]{379-21\sqrt{105}}+\sqrt[3]{379+21\sqrt{105}}\right)^2\left(11+2\sqrt[3]{379-21\sqrt{105}}+2\sqrt[3]{379+21\sqrt{105}}\right)}{147\left(90+8\sqrt[3]{379-21\sqrt{105}}+\left(379-21\sqrt{105}\right)^{2/3}+8\sqrt[3]{379+21\sqrt{105}}+\left(379+21\sqrt{105}\right)^{2/3}\right)} = 0.780(3d.p.).$$

Proof. $\frac{\partial \theta_{Atom}[\gamma]}{\partial \gamma} = \frac{2\left(7\gamma^4+12\gamma^3+4\gamma^2-2\gamma-1\right)}{(7\gamma^2+6\gamma+1)^2}$. This has only one first order point, a local minimum where $\gamma^* = \frac{1}{21}\left(-5+\left(379-21\sqrt{105}\right)^{1/3}+\left(379+21\sqrt{105}\right)^{1/3}\right)$. Evaluating $\theta_{Atom}[\gamma^*]$ gives the required result. □

The interior region is plotted in green below:

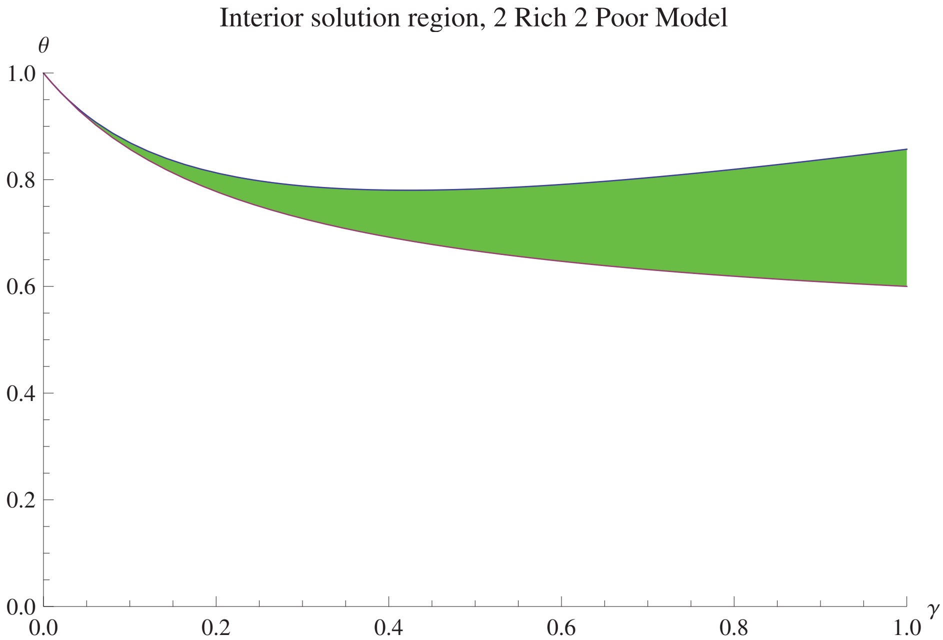


Figure 2.5: 2 Rich 2 Poor Model Interior Solution Plot

The percentage gain from welfare is shown in table 2.33 as a function of θ and γ, and the distribution of the gain is given in single variate θ and γ tables. For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{rrpp}[Atom], \mathrm{W}_{rrpp}[GC]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{W}_{rrpp}[I]$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. The following tables, 2.33, 2.34 and 2.35 show that the percentage gain is small (always below 10%), for each of the considered parametrisations. This argues for that the standard model rejection of interior partitions is robust to replacing a uniform distribution of business opportunities with a four country 2 Rich 2 Poor distribution.

The tables show that there can still be interior solutions when $\theta = 1$ and outside matches have the same value as inside matches. This occurs for large values of γ, specifically when $\gamma > \frac{1}{2}\left(1+\sqrt{5}\right) = 1.62$ $(2d.p.)$. Remember, that with the atomistic partition it is necessary for 2 independent modules to be enabled in order for the match to be productive; whilst with the pairs partition half the time we only need one module to be enabled in order for the match to be productive: for big enough γ the probability of 2 separate singleton modules being enabled is small enough that the pairs partition is preferred. The θ table shows firstly, that with intermediate values of θ there are most likely to be interior solutions, (this is consistent with the above graph), and secondly that to get the highest gains requires a large θ. The γ table shows that the highest gains occur with $\gamma = 2$; whilst the frequency of interior solutions increases as γ increases: specifically, once $\gamma > 2.2$ all solutions are interior.

Table 2.33: 2 Rich 2 Poor Percentage Gain Table

	Percentage Welfare Gain from the interior partition									
$\gamma\backslash\theta$	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.0
0.0	-23.0	-20.0	-18.0	-15.0	-13.0	-10.0	-7.5	-5.0	-2.5	0.0
0.1	-15.0	-12.0	-10.0	-7.6	-5.2	-2.8	-0.3	-2.0	-4.9	-7.6
0.2	-10.0	-8.2	-5.9	-3.6	-1.3	1.0	-2.8	-6.1	-9.1	-12.0
0.3	-7.6	-5.5	-3.3	-1.2	1.0	-1.0	-4.9	-8.3	-11.0	-14.0
0.4	-5.7	-3.7	-1.7	0.3	2.3	-1.7	-5.7	-9.3	-12.0	-15.0
0.5	-4.4	-2.5	-0.6	1.3	3.1	-1.6	-5.7	-9.4	-13.0	-16.0
0.6	-3.4	-1.7	0.1	1.9	3.6	-0.8	-5.1	-8.9	-12.0	-15.0
0.7	-2.7	-1.0	0.6	2.3	3.9	0.3	-4.1	-8.0	-12.0	-15.0
0.8	-2.1	-0.6	1.0	2.5	4.1	1.8	-2.7	-6.8	-10.0	-14.0
0.9	-1.7	-0.3	1.2	2.7	4.1	3.6	-1.1	-5.4	-9.1	-13.0
1.0	-1.4	0.0	1.4	2.8	4.2	5.6	0.7	-3.7	-7.6	-11.0
1.1	-1.1	0.2	1.5	2.8	4.2	5.5	2.6	-1.9	-5.9	-9.6
1.2	-0.9	0.3	1.6	2.9	4.1	5.4	4.7	0.0	-4.1	-7.9
1.3	-0.7	0.5	1.7	2.9	4.1	5.2	6.4	2.0	-2.3	-6.1
1.4	-0.6	0.6	1.7	2.8	4.0	5.1	6.3	4.2	-0.3	-4.3
1.5	-0.5	0.6	1.7	2.8	3.9	5.0	6.1	6.3	1.8	-2.3
1.6	-0.4	0.7	1.7	2.8	3.8	4.9	5.9	7.0	3.9	-0.4
1.7	-0.3	0.7	1.7	2.7	3.7	4.8	5.8	6.8	6.0	1.7
1.8	-0.2	0.8	1.7	2.7	3.7	4.6	5.6	6.6	7.5	3.7
1.9	-0.2	0.8	1.7	2.6	3.6	4.5	5.4	6.4	7.3	5.9
2.0	-0.1	0.8	1.7	2.6	3.5	4.4	5.3	6.2	7.1	8.0
2.1	-0.1	0.8	1.7	2.6	3.4	4.3	5.2	6.0	6.9	7.8
2.2	-0.0	0.8	1.7	2.5	3.3	4.2	5.0	5.9	6.7	7.5
2.3	0.0	0.8	1.6	2.5	3.3	4.1	4.9	5.7	6.5	7.3
2.4	0.0	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.3	7.1
2.5	0.1	0.8	1.6	2.4	3.1	3.9	4.7	5.4	6.2	6.9
2.6	0.1	0.8	1.6	2.3	3.1	3.8	4.5	5.3	6.0	6.8
2.7	0.1	0.8	1.6	2.3	3.0	3.7	4.4	5.2	5.9	6.6
2.8	0.1	0.8	1.5	2.2	2.9	3.6	4.3	5.0	5.7	6.4
2.9	0.1	0.8	1.5	2.2	2.9	3.5	4.2	4.9	5.6	6.3
3.0	0.2	0.8	1.5	2.1	2.8	3.5	4.1	4.8	5.5	6.1

Table 2.34: 2 Rich 2 Poor θ Table

	Percentage gain broken down by θ									
θ	(-100,0)	[0,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,7)	[7,8)	[8,9)
0.55	23	8	0	0	0	0	0	0	0	0
0.60	10	21	0	0	0	0	0	0	0	0
0.65	6	3	22	0	0	0	0	0	0	0
0.70	4	1	2	24	0	0	0	0	0	0
0.75	3	1	0	5	16	6	0	0	0	0
0.80	6	1	2	0	8	8	6	0	0	0
0.85	10	1	0	1	0	9	7	3	0	0
0.90	12	1	0	1	0	3	7	7	0	0
0.95	15	0	1	0	1	0	4	7	3	0
1.00	16	1	1	0	1	0	1	6	4	1
All	105	38	28	31	26	26	25	23	7	1

Table 2.35: 2 Rich 2 Poor γ Table

	Percentage gain broken down by γ									
γ	(-100,0)	[0,1)	[1,2)	[2,3)	[3,4)	[4,5)	[5,6)	[6,7)	[7,8)	[8,9)
0.0	9	1	0	0	0	0	0	0	0	0
0.1	10	0	0	0	0	0	0	0	0	0
0.2	9	0	1	0	0	0	0	0	0	0
0.3	9	1	0	0	0	0	0	0	0	0
0.4	8	1	0	1	0	0	0	0	0	0
0.5	8	0	1	0	1	0	0	0	0	0
0.6	7	1	1	0	1	0	0	0	0	0
0.7	6	2	0	1	1	0	0	0	0	0
0.8	6	1	1	1	0	1	0	0	0	0
0.9	6	0	1	1	1	1	0	0	0	0
1.0	4	2	1	1	0	1	1	0	0	0
1.1	4	1	1	2	0	1	1	0	0	0
1.2	3	2	1	1	0	2	1	0	0	0
1.3	3	1	1	2	0	1	1	1	0	0
1.4	3	1	1	1	1	1	1	1	0	0
1.5	2	1	2	1	1	0	1	2	0	0
1.6	2	1	1	1	2	1	1	1	0	0
1.7	1	1	2	1	1	1	1	2	0	0
1.8	1	1	1	1	2	1	1	1	1	0
1.9	1	1	1	1	1	1	2	1	1	0
2.0	1	1	1	1	1	1	1	1	1	1
2.1	1	1	1	1	1	1	1	2	1	0
2.2	1	1	1	1	1	1	2	1	1	0
2.3	0	2	1	1	1	2	1	1	1	0
2.4	0	2	1	1	2	1	1	1	1	0
2.5	0	2	1	1	2	1	1	2	0	0
2.6	0	2	1	1	2	1	1	2	0	0
2.7	0	2	1	2	1	1	2	1	0	0
2.8	0	2	1	2	1	1	2	1	0	0
2.9	0	2	1	2	1	2	1	1	0	0
3.0	0	2	1	2	1	2	1	1	0	0
All	105	38	28	31	26	26	25	23	7	1

2.3 Financial

2.3.1 Variable Shock Initialisation Probability

In the standard model, the module enablement probability is $P[d] = \frac{1}{1+\gamma d}$, where d is the module size variable and γ is the shock parameter. The shock parameter does not vary with module size variable. In this section we consider the variant α *model* where the module enablement probability is given by $P_\alpha[d] = \frac{1}{1+\gamma d^\alpha}$, and α is an extra parameter. This can be interpreted as follows: consider a module enablement probability $P_\Gamma[d] = \frac{1}{1+\Gamma(d)d}$, where $\Gamma(d)$ is some completely general shock parameter function $\Gamma : \mathbb{R} \to \mathbb{R}^+$; then the α model equates to the case where ϵ, the elasticity of Γ with respect to d, is constant and equals $\alpha - 1$.

In the standard model *independent* of the module size d, banks in enabled modules receive disabling shocks at rate $Log[1-q]$, and disabled modules receive enabling shocks at rate $Log[1-\rho]$. Let us now consider the α model. If $\alpha < 1$ then an increase in d decreases the module enablement probability by less than it would in the standard model: this equates to the case where banks in bigger modules receive fewer shocks and/or recover faster. If $\alpha = 1$ then we have the standard model. If $\alpha > 1$ then an increase in d decreases the module enablement probability by more than it would in the standard model: this equates to the case where banks in bigger modules receive more shocks and/or recover slower.

We now follow the solution methodology as followed with the standard model. This consists of finding the Pareto optimal allocations for bank 1. The welfare function for a general allocation $x = (x_i)_{i=1}^k$ is:

$$W_\alpha[(x_i)_{i=1}^k] = \sum_{i=1}^k (\frac{x_i}{n})^2 P_\alpha[x_i] + \sum_{i=1}^k \sum_{j \neq i} \theta(\frac{x_i}{n})(\frac{x_j}{n}) P_\alpha[x_i] P_\alpha[x_j]$$

Similarly the expected return for bank 1, assumed without loss of generality to be in module 1, is:

$$v_{1\alpha}[(x_i)_{i=1}^k] = (\frac{1}{n})(\frac{x_1}{n})P_\alpha[x_1] + \sum_{i=2}^{k} \theta(\frac{1}{n})(\frac{x_i}{n})P_\alpha[x_1]P_\alpha[x_i]$$

Lemma 67. *If $\alpha > 0$ then $v_{1\alpha}[(x_i)_{i=1}^k]$ has negative externalities. If $\alpha < 0$ then $v_{1\alpha}[(x_i)_{i=1}^k]$ has positive externalities.*

Proof. Consider $\Delta v_{1\alpha}$ the change in utility to bank 1 from merging two outside modules 2 and 3. So $\Delta v_{1\alpha} := v_{1\alpha}[x_1, x_2 + x_3, (x_i)_{i=4}^k] - v_{1\alpha}[x_1, x_2, x_3, (x_i)_{i=4}^k]$. Hence $\frac{n^2 \Delta v_{1\alpha}}{\theta P_\alpha[x_1]} = x_{2+3}P_\alpha[x_{2+3}] - x_2 P_\alpha[x_2] - x_3 P_\alpha[x_3]$. Re-arrangement gives that $\Delta v_{1\alpha} > 0$ if and only if

$$x_3 \frac{({x_3}^\alpha - (x_2 + x_3)^\alpha)}{P_\alpha[x_2]} + x_2 \frac{(x_2^\alpha - (x_2 + x_3)^\alpha)}{P_\alpha[x_3]} > 0$$

When $\alpha > 0$, each of the terms is always negative and so $\Delta v_{1\alpha} \leq 0$. Hence, $v_{1\alpha}[(x_i)_{i=1}^k]$ when $\alpha > 0$ has negative externalities.

Conversely, when $\alpha < 0$, each of the terms is always positive and so $\Delta v_{1\alpha} > 0$. Hence, $v_{1\alpha}[(x_i)_{i=1}^k]$ when $\alpha > 0$ has positive externalities. $\square$

Lemma 68. *If $\alpha < 0$ then the grand coalition is the solution to the α model.*

Proof. If $\alpha < 0$ then from Lemma 67 the partition that maximises the utility of bank 1 in module 1 will be of the format $(x_1, n - x_1)$ due to the presence of the positive externalities. Hence, let $v_{1\alpha p}[x_1] := v_{1\alpha}[(x_1, n - x_1)]$.

Further, let $\Delta v[x_1] := \frac{v_{1\alpha p}[n] - v_{1\alpha p}[x_1]}{\gamma P_\alpha[n] P_\alpha[x_1] P_\alpha[n - x_1]}$, which is positive if and only if there is a gain to bank 1 from being in the grand coalition (n) compared with being in the partition $(x_1, n - x_1)$. Letting $a = -\alpha$ and re-arranging gives:

$$\Delta v[x_1] = \gamma n^{-a} x_1^{-a}(n^{1+a} - x_1^{1+a})(n - x_1)^{-a} + n^{1-a} x_1^{-a}(n^a - x_1^a) + (n - x_1)^{1-a}$$

So $\Delta v[x_1] > 0$ for all x_1 and hence the grand coalition is the unique Pareto optimal partition. $\square$

Lemma 69. *If $\alpha = 0$ then the grand coalition is the solution to the α model.*

Proof. If $\alpha = 0$ and $x_1 > 0$ then $P_\alpha[x_1] = P[1]$. Define the normalised welfare function:

$$W_n[(x_i)_{i=1}^k] := \frac{n^2 W_{\alpha=0}[(x_i)_{i=1}^k]}{P[1]}$$

So:

$$W_n[(x_i)_{i=1}^k] = \sum_{i=1}^k x_i^2 + \sum_{i=1}^k \sum_{j \neq i} \theta x_i x_j P[1]$$

whilst:

$$W_n[x_1+x_2, (x_i)_{i=3}^k] = (x_1+x_2)^2 + \sum_{i=3}^k x_i^2 + (x_1+x_2) \sum_{j=3}^k \theta x_j P[1] + \sum_{i=3}^k \theta x_i \sum_{j \neq i} x_j P[1]$$

So

$$W_n[x_1+x_2, (x_i)_{i=3}^k] - W_n[(x_i)_{i=1}^k] = 2x_1x_2 + \theta P[1] \left((x_1 + x_2) \sum_{j=3}^k x_j + \sum_{i=3}^k x_i \sum_{j \neq i} x_j - \sum_{i=1}^k \sum_{j \neq i} x_i x_j \right)$$

Hence

$$W_n[x_1 + x_2, (x_i)_{i=3}^k] - W_n[(x_i)_{i=1}^k] = 2x_1x_2(1 - \theta P[1])$$

So merging modules always increases welfare and hence the grand coalition maximises welfare. $\square$

We now consider only partitions of the form $\{x_1, 1, 1, ...1\}$, where all modules apart from the first module are singletons and define

$$v_{1\alpha}[x_1] := (\frac{1}{n})(\frac{x_1}{n})P_\alpha[x_1] + \theta(\frac{1}{n})(\frac{n - x_1}{n})P_\alpha[x_1]P_\alpha[1]$$

Then $v'_{1\alpha}[x_1] = v'_{1\alpha n}[x_1](\gamma - \theta + 1)n^{-2}P_\alpha^2[x_1]P_\alpha[1]x_1^{-1}$ where

$$v'_{1\alpha n}[x_1] := -Nx_1^{\alpha-1} - (\alpha - 1)\gamma \mathrm{x}_1^\alpha + 1$$

and $N := \frac{n\alpha\gamma\theta}{\gamma-\theta+1} > 0$.

So $v'_{1\alpha}[x_1] > 0$ if and only if the normalised function $v'_{1\alpha n}[x_1] > 0$. We can now characterise $v_{1\alpha}[x_1]$:

Theorem 70. *If $0 < \alpha \leq 1$ then $v_{1\alpha}[x_1]$ is quasi-convex. Conversely if $\alpha \geq 1$ then $v_{1\alpha}[x_1]$ is quasi-concave.*

Proof. If $\alpha = 1$ then this comes from the standard model results, as $v'_{1\alpha}[x_1 : \alpha = 1] = \frac{\gamma-\theta+\gamma\theta(-n)+1}{(\gamma+1)(\gamma \mathrm{x}1+1)^2}$

For $\alpha \neq 1$ then this will proved using $v'_{1\alpha n}[x_1]$. If $\alpha < 1$ then $\alpha - 1 < 0$. Suppose $y_1 > x_1$ then $-Ny_1^{\alpha-1} > -Nx_1^{\alpha-1}$ and $\gamma(1 - \alpha)y_1^\alpha > \gamma(1 - \alpha)x_1^\alpha$. Hence if $v'_{1\alpha}[x_1] > 0$ and $y_1 > x_1$ then $v'_{1\alpha}[y_1] > 0$. Hence if $0 < \alpha \leq 1$ then $v_{1\alpha}[x_1]$ is quasi-convex.

If $\alpha > 1$ then $\alpha - 1 > 0$. A similar argument as in the above case then shows that if $v'_{1\alpha}[x_1] < 0$ and $y_1 > x_1$ then $v'_{1\alpha}[y_1] < 0$. Hence if $\alpha \geq 1$ then $v_{1\alpha}[x_1]$ is quasi-concave. □

Corollary 71. *If* $0 < \alpha < 1$ *then the model has boundary solutions.*

Proof. This follows from applying Theorem 15: the α model clearly possesses anonymity; Lemma 67 shows that the model has negative externalities; and Theorem 70 shows that there is quasi-convexity. □

So the standard model, where $\alpha = 1$, is at a tipping point between the quasi-convex and quasi-concave regions. However, when $\alpha > 1$ despite $v_{1\alpha}[x_1]$ being quasi-concave it may be monotonic and hence not have interior solutions. Considering $v'_{1\alpha n}[1]$ and $v'_{1\alpha n}[n]$ leads to 9 different cases:

Table 2.36: α Model Characterisation Table

Characterisation of Argmax $v_{1\alpha}[x_1]$ when $\alpha > 1$		
$v'_{1\alpha n}[1]$	$v'_{1\alpha n}[n]$	x_1^*
< 0	< 0	1
< 0	$= 0$	impossible due to quasi-concavity
< 0	> 0	impossible due to quasi-concavity
$= 0$	< 0	1
$= 0$	$= 0$	impossible due to quasi-concavity
$= 0$	> 0	impossible due to quasi-concavity
> 0	< 0	interior
> 0	$= 0$	n
> 0	> 0	n

This leads to the following characterisation theorem:

Theorem 72. *When* $\alpha > 1$: *if* $v'_{1\alpha n}[1] \leq 0$ *then the atomistic partition is the only Pareto optimal partition; if* $v'_{1\alpha n}[n] \geq 0$ *then the grand coalition is the only Pareto optimal partition; if* $v'_{1\alpha n}[1] > 0$ *and* $v'_{1\alpha n}[n] < 0$ *then there is an interior Pareto optimal partition.*

Proof. The results follow directly from the table above. $\square$

Corollary 73. *If the atomistic partition is the solution of the standard model then it is also the sole solution to the* α *model when* $\alpha > 1$.

Proof. Re-arrangement gives firstly that

$\frac{v'_{1\alpha n}[1]}{P^3[1]} = -(\alpha - 1)\gamma^2 - \theta - \gamma(-\alpha\theta + \alpha + \theta + \alpha\theta n - 2) + 1$, and secondly that $v'_{1\alpha n}[1] \leq 0$ if and only if $\alpha \geq \frac{\gamma^2 - \gamma\theta + 2\gamma - \theta + 1}{\gamma^2 - \gamma\theta + \gamma + \gamma\theta n}$. If the standard model has the atomistic solution then $P[n] < \theta P[1]$ and hence $\frac{\gamma^2 - \gamma\theta + 2\gamma - \theta + 1}{\gamma^2 - \gamma\theta + \gamma + \gamma\theta n} < 1 < \alpha$.

A second argument is as follows. Differentiation gives that $\frac{\partial v_{1\alpha}[x_1]}{\partial \alpha} = -\gamma x_1^\alpha \log(x_1)(\theta n + x_1(\gamma - \theta + 1))P[1]P_\alpha^2[x_1]$. As $\log[1] = 0$, $\frac{\partial v_{1\alpha}[1]}{\partial \alpha} = 0$. The $\frac{\partial v_{1\alpha}[x_1]}{\partial \alpha}$ expression has a leading negative and so when $x_1 > 0$ the other factors are always positive. So $x_1 > 0 \Rightarrow \frac{\partial v_{1\alpha}[x_1]}{\partial \alpha} < 0$. Hence $v_{1\alpha}[1] = v_1[1]$ and $(v_{1\alpha}[x_1] < v_1[x_1])_{x_1 > 0}$. So if $1 = \text{argmax}_{1 \leq x_1 \leq n} v_1[x_1]$, then $1 = \text{argmax}_{1 \leq x_1 \leq n} v_{1\alpha}[x_1]$. $\square$

Corollary 74. *In order for the* α *model to have an interior solution we require that* $\alpha > 1$ *and that the grand coalition be the solution of the standard model.*

Proof. When $\alpha \leq 0$ then by Lemmas 68 and 69 there is no interior solution. When $0 < \alpha \leq 1$ the non-existence of interior solutions follows from the quasi-convexity shown in Theorem 70 and the negative externalities shown in Lemma 67. When $\alpha > 1$ the result follows from Corollary 73 $\square$

Corollary 75. *If* $\alpha > 1$ *and the grand coalition is the solution of the* α *model then the grand coalition is also the solution of the corresponding standard model.*

Proof. This again follows from Corollary 73. But also directly: $v'_{1\alpha n}[n] \geq 0$ if and only if $n^\alpha \leq \frac{\gamma - \theta + 1}{\alpha\gamma^2 + \alpha\gamma - \gamma^2 + \gamma\theta - \gamma}$ if and only if $\alpha \leq 1 + \frac{\gamma - \theta - \gamma\theta n^\alpha + 1}{\gamma^2 n^\alpha + \gamma n^\alpha}$. This requires $\gamma - \theta - \gamma\theta n^\alpha + 1 > 0$ and hence that $P[n^\alpha] > \theta P[1]$. As $\alpha > 1$ it follows that

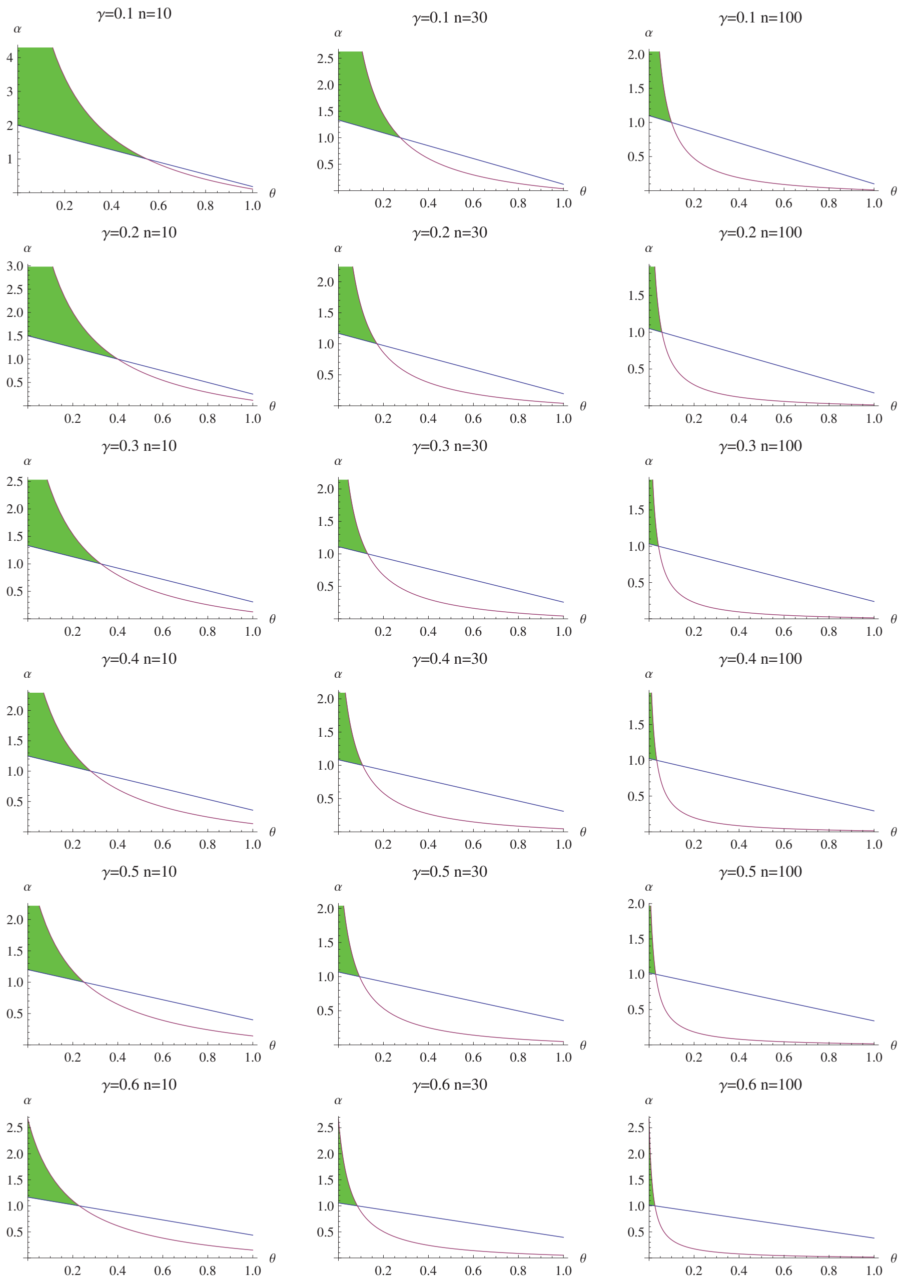
$n < n^{\alpha}$, and so $P[n] > \theta P[1]$ and hence that the grand coalition is the solution to the standard model. $\square$

Corollary 76. *The α model has an interior Pareto Optimal partition if and only if* $Max\{1, \frac{\gamma-\theta+\gamma^2 n^{\alpha}-\gamma\theta n^{\alpha}+\gamma n^{\alpha}+1}{\gamma^2 n^{\alpha}+\gamma n^{\alpha}}\} < \alpha < \frac{\gamma^2-\gamma\theta+2\gamma-\theta+1}{\gamma^2-\gamma\theta+\gamma+\gamma\theta n}$.

Proof. The need for $\alpha > 1$ follows from Lemmas 68, 69 and Corollary 71. Theorem 72 gives that if $\alpha > 1$ then there is a interior PO partition if and only if both $v'_{1\alpha n}[1] > 0$ and $v'_{1\alpha n}[n] < 0$. The definitions of $v'_{1\alpha n}[1]$ and $v'_{1\alpha n}[n]$, gives that $v'_{1\alpha n}[n] < 0$ iff $\alpha > \alpha_{lower}[n, \theta, \gamma, \alpha] := \frac{\gamma-\theta+\gamma^2 n^{\alpha}-\gamma\theta n^{\alpha}+\gamma n^{\alpha}+1}{\gamma^2 n^{\alpha}+\gamma n^{\alpha}}$ and that $v'_{1\alpha n}[1] > 0$ iff $\alpha < \alpha_{upper}[n, \theta, \gamma] := \frac{\gamma^2-\gamma\theta+2\gamma-\theta+1}{\gamma^2-\gamma\theta+\gamma+\gamma\theta n}$. $\square$

This is a graphical representation of the interior solution region:

Figure 2.6: α Model Interior Solution Plot



Theorem 77. *If the standard model has the grand coalition as a solution then there is a value of $\alpha > 1$ with an interior Pareto Optimum of the α model.*

Proof. From Corollary 76 the α model has an interior P.O. if both $\frac{\frac{\gamma-\theta+1}{n^{\alpha}}+\gamma^2-\gamma\theta+\gamma}{\gamma^2+\gamma} < \alpha < \frac{\gamma^2-\gamma\theta+2\gamma-\theta+1}{\gamma^2-\gamma\theta+\gamma+\gamma\theta n}$ and $\alpha > 1$. The two conditions will be checked in turn.

As $n^\alpha > n$, a sufficient condition for condition 1 is $\frac{\frac{\gamma-\theta+1}{n}+\gamma^2-\gamma\theta+\gamma}{\gamma^2+\gamma} < \alpha < \frac{\gamma^2-\gamma\theta+2\gamma-\theta+1}{\gamma^2-\gamma\theta+\gamma+\gamma\theta n}$, which equates to $\frac{(\gamma-\theta+1)(\gamma n+1)}{\gamma(\gamma+1)n} < \alpha < \frac{(\gamma+1)(\gamma-\theta+1)}{\gamma(\gamma+\theta(n-1)+1)}$. The interval is non-empty if and only if $\theta \mathrm{P}[1] < P[n]$: the condition for the grand coalition to be the solution of the standard model.

We now need to check that the intersection of this range with $(1, \infty)$ is non-empty. This requires $\frac{(\gamma+1)(\gamma-\theta+1)}{\gamma(\gamma+\theta(n-1)+1)} > 1$, which occurs when $\theta \mathrm{P}[1] < P[n]$: the condition for the grand coalition to be the solution of the standard model. □

We now consider a range of different parametrisations to investigate the potential that $v_{1\alpha}[x_1]$ has for interior solutions. The number of banks, n, is one of $\{10, 20, 30, 40, 50, 100, 1000\}$. The value of outside matches, θ, has a minimum of 0.1, a maximum of 0.9, and has an increment of 0.1. The shock parameter γ has a minimum of 0.1, a maximum of 3.0, and an increment of 0.1. The α parameter, has a minimum of 1.0, a maximum of 3.0 and an increment of 0.1. The n parameter is one of 7 values, the θ parameter is one of 9 values, the γ parameter is one of 30 values and the α parameter is one of 21 values. This gives a total of $7 * 9 * 21 * 30 = 39690$ parameterisations.

In each of these cases, assuming x_1 to be integer, it is firstly assessed whether there is a boundary solution or an interior solution. Secondly, the percentage gain from the best interior solution over the best boundary solution is computed. For each parametrisation, the bank 1 utility of the best trivial partition is given by $\mathrm{v}^b := \mathrm{Max}\{v_{1\alpha}[1], v_{1\alpha}[n]\}$; the bank 1 utility of the best symmetric interior partition is given by $\mathrm{v}^i := \mathrm{Max}\{v_{1\alpha}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (v^i - v^b)/v^b$. In 99.65% of these cases the $v_{1\alpha}$ program has a boundary solution. In cases where it does not, the gain has a maximum of 35%. The high frequency of parametrisations with boundary solutions, and the absence of large gains when there are interior solutions, leads to the conclusion that the standard model rejection of interior partitions is robust to replacing a distribution of financial shocks which is independent of module size, with a distribution where the incidence of financial shocks is higher in bigger modules.

The gain percentage results can be presented in a summary table for each of the different dimensions $(n, \theta, \gamma, \alpha)$ of the parameters. In the α table, for every value of α, the number of cases with interior solutions is low: the highest proportion (1.59%) occurs with $\alpha = 1.10$, whilst with $\alpha > 2.7$ there are no cases with interior solutions. When $\alpha = 1$, we have the standard model and so there are no gains from interior partitions. With α slightly bigger than 1, there are a few examples where, in the standard model the grand coalition is optimal, but in the α model, due to the decreased enablement probability of large modules, an interior partition with smaller modules is now optimal. Once α becomes large, the solution is always the atomistic partition. The n table shows that, when n is small, interior solutions are more frequent, and are of greater benefit. Once n becomes large then, in the standard model, the atomistic partition is always preferred, and so this is also the case in the α model. Specifically, if n is chosen to be 100 or 1000 then there are no gains from interior partitions for all choices of the other parameters. Similarly, in the standard model, if θ or γ increases then generically the solution converges to the atomistic partition, and so the θ table and the γ tables show the same result for the α model.[17]

[17] As θ tends to 1 the result holds for all parametrisations. As γ tends to ∞ the result holds unless $\theta n \leq 1$.

Table 2.37: α Model α Table

	Overall gain percentage broken down by α							
α	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)
1.0	1890	0	0	0	0	0	0	0
1.1	1860	28	2	0	0	0	0	0
1.2	1866	14	9	0	1	0	0	0
1.3	1871	9	6	3	0	1	0	0
1.4	1876	5	5	3	0	1	0	0
1.5	1877	7	3	0	1	2	0	0
1.6	1881	3	2	2	2	0	0	0
1.7	1883	3	1	2	0	0	0	1
1.8	1885	1	3	0	0	0	1	0
1.9	1885	3	1	0	0	1	0	0
2.0	1886	3	0	0	1	0	0	0
2.1	1887	2	0	0	1	0	0	0
2.2	1888	1	0	1	0	0	0	0
2.3	1889	0	0	1	0	0	0	0
2.4	1889	0	1	0	0	0	0	0
2.5	1889	0	1	0	0	0	0	0
2.6	1889	1	0	0	0	0	0	0
2.7	1889	1	0	0	0	0	0	0
2.8	1890	0	0	0	0	0	0	0
2.9	1890	0	0	0	0	0	0	0
3.0	1890	0	0	0	0	0	0	0
All	39550	81	34	12	6	5	1	1

Table 2.38: α Model n Table

	Overall gain percentage broken down by n							
n	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)
10	5585	51	17	9	4	2	1	1
20	5641	15	9	2	1	2	0	0
30	5656	7	5	1	0	1	0	0
40	5663	4	2	0	1	0	0	0
50	5665	4	1	0	0	0	0	0
100	5670	0	0	0	0	0	0	0
1000	5670	0	0	0	0	0	0	0
All	39550	81	34	12	6	5	1	1

Table 2.39: α Model θ Table

	Overall gain percentage broken down by θ							
θ	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)
0.1	4308	54	25	11	5	5	1	1
0.2	4384	17	7	1	1	0	0	0
0.3	4401	7	2	0	0	0	0	0
0.4	4408	2	0	0	0	0	0	0
0.5	4409	1	0	0	0	0	0	0
0.6	4410	0	0	0	0	0	0	0
0.7	4410	0	0	0	0	0	0	0
0.8	4410	0	0	0	0	0	0	0
0.9	4410	0	0	0	0	0	0	0
All	39550	81	34	12	6	5	1	1

Table 2.40: α Model γ Table

	Overall gain percentage broken down by γ							
γ	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)
0.1	1254	32	20	5	6	4	1	1
0.2	1298	15	5	4	0	1	0	0
0.3	1310	8	3	2	0	0	0	0
0.4	1315	5	2	1	0	0	0	0
0.5	1317	4	2	0	0	0	0	0
0.6	1320	2	1	0	0	0	0	0
0.7	1320	2	1	0	0	0	0	0
0.8	1321	2	0	0	0	0	0	0
0.9	1321	2	0	0	0	0	0	0
1.0	1321	2	0	0	0	0	0	0
1.1	1322	1	0	0	0	0	0	0
1.2	1322	1	0	0	0	0	0	0
1.3	1322	1	0	0	0	0	0	0
1.4	1322	1	0	0	0	0	0	0
1.5	1322	1	0	0	0	0	0	0
1.6	1322	1	0	0	0	0	0	0
1.7	1322	1	0	0	0	0	0	0
1.8	1323	0	0	0	0	0	0	0
1.9	1323	0	0	0	0	0	0	0
2.0	1323	0	0	0	0	0	0	0
2.1	1323	0	0	0	0	0	0	0
2.2	1323	0	0	0	0	0	0	0
2.3	1323	0	0	0	0	0	0	0
2.4	1323	0	0	0	0	0	0	0
2.5	1323	0	0	0	0	0	0	0
2.6	1323	0	0	0	0	0	0	0
2.7	1323	0	0	0	0	0	0	0
2.8	1323	0	0	0	0	0	0	0
2.9	1323	0	0	0	0	0	0	0
3.0	1323	0	0	0	0	0	0	0
All	39550	81	34	12	6	5	1	1

2.3.2 Biased Incentives

2.3.2.1 Zero Lifetime Bankers

This section considers bankers who are short lived. In particular it will assume the limiting case of them having zero lifetime.[18] Imagine bankers who firstly, know that there is a last round of business opportunities before they retire, and secondly, are able to reorganise the modules before the business opportunities come through. For the sake of simplicity assume that initially all the modules are enabled.[19]

The per bank utility function has the same format as in the standard model, but with an altered module enablement probability:

$$v_{i0}[(x_j)_{j=1}^n] := \frac{x_i}{n^2}P_0[x_i] + \sum_{j\neq i}\theta\frac{x_j}{n^2}P_0[x_i]P_0[x_j]$$

where $P_0[d]$ is the enablement probability of a module of size d. There is no time for a shock to hit in zero time, so every module stays enabled, and hence $P_0[d] = 1$ for all d. Therefore:

$$v_{i0}[(x_j)_{j=1}^n] = \frac{x_i}{n^2} + \sum_{j\neq i}\theta\frac{x_j}{n^2} = \frac{x_i + \theta(n - x_i)}{n^2} = \frac{\theta n + (1-\theta)x_i}{n^2}$$

However, recall from the standard model that $P[d] = \frac{1}{1+\gamma d}$ with $\gamma = \frac{-Log[1-q]}{-Log[1-\rho]}$ where γ is the shock parameter and so $P_0[d] = P[d : \gamma = 0]$. So the EEBA analysis above for the standard model still applies: just that the bankers have the wrong γ. $P[n : \gamma = 0] = 1$ and $\theta P[1 : \gamma = 0] = \theta$. So as $\theta < 1$ the grand coalition is the only stable partition. Hence when $P[n] < \theta P[1]$ the market

[18]An extension in the next section shows that the results extend to the case of short lived bankers with non-zero life times.

[19]If some modules are initially disabled then the result is that all enabled modules merge and the disabled modules remain separate.

outcome is inefficient.

2.3.2.2 Short Run Bankers

Section 2.3.2.1 considers the case of zero lifetime bankers and concludes that such bankers will, for all parameterisations, choose the grand coalition. This section extends the results to the case where the business opportunity is at time $t > 0$ in the future, after which the bankers retire. Assuming that initially all banks are enabled, then as the Markov process converges exponentially fast, we get the time t module enablement probability, $P_t[d] = P[d] + ((1-q)^d(1-\rho))^t(1-P[d])$, where the terms have their standard meanings: $P[d]$ is the module enablement probability of the standard model, d is the module size, q is the probability that a bank is hit by a disabling shock, and ρ is the re-enablement probability. In this section I will consider for what size of t the short run bankers inefficiently prefer the grand coalition over the atomistic partition of singletons.

The welfare function for short run bankers with symmetric partitions is given by, $\mathrm{W}_t[\mathrm{d}] := {}^{d}/_{n}\mathrm{P}_t[d] + \theta\,{}^{(n-d)}/_{n}\mathrm{P}_t[d]^2$. We calculate the percentage gain, g, the social planner has from his preferred choice, compared with the choice of the period t short run banker. This is done on the basis that: firstly, the social planner remains concerned solely about welfare in the limiting state; and secondly, that the banker has to choose a boundary partition. Precisely: $W_{sp} = \max\{W[1], W[n]\}$; $d_{bank} = \arg\max_{d\in\{1,n\}} W_t[d]$; and $g = 100 * (W_{sp} - W[d_{bank}])/W[d_{bank}]$.

Using a similar approach as with other variant models, we include the time t as one dimension of the parametrisations: the time, t, has a minimum of 0, a maximum of 20 and an increment of 1; the number of banks, n, is one of 10, 20, 30, 40, 50, 100, 1000; the value of outside matches, θ, has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1; the disabling shock probability, q, comes from the range, $0.001N_9 \cup 0.01N_9 \cup \{0.1\}$; and the enabling probability, ρ, comes from the range $0.001N_9 \cup 0.01N_9 \cup \{0.1,\ 0.2,\ 0.3\}$, where $N_9 := \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So, the t parameter is one of 21 values, the n parameter is one of 7 values, the θ parameter is one of 9 values, the q parameter is one of 19 values, and the ρ parameter is one of 21 values. This gives a total of $21*7*9*19*21 = 527877$ parametrisations.

The next five tables record the percentage gain to a social planner from deciding the partition themselves, rather than leaving it to the market: they assess how biased a market of period t bankers is; and the final sixth, bias length table, records directly the number of biased cases at each period. The t table shows that as time increases, the banker's choice converges to the social planner's choice. The n table shows that if there are lots of banks, then it is less likely that bankers want the grand coalition even in the short run; but when they do choose the wrong partition, the social costs are very high. As θ increases, there are lower costs from outside matches; less gains to the bankers from moving away from the atomistic partition; and hence less social costs from the market solution. As the enablement probability q increases, the time before a disabling shock arrive reduces, and hence the banker's choice is less biased. The re-enablement probability, ρ, does not have much effect on the banker's choice: they are short run and there is unlikely to be enough time for the system to both be disabled and then re-enabled again before time t. The bias length table shows that: firstly, once t reaches a couple of years, that there is a large decrease in the number of biased choices by bankers; but secondly, also that there is a substantial number of residual cases where the banker's choice is biased even for large t (up to 20 years).

Table 2.41: Short Run Bankers t Table

Overall gain percentage distribution broken down by t													
t	[0,0]	(0,10]	(10,20]	(20,30]	(30,40]	(40,50]	(50,60]	(60,70]	(70,80]	(80,90]	(90,100]	(100,1000]	(1000,∞]
0	2955	800	858	812	768	630	758	511	610	453	482	11293	4207
1	16045	680	626	548	504	387	483	296	355	244	270	4543	156
2	18852	523	457	403	367	287	364	222	274	183	206	2962	37
3	20188	504	396	354	316	245	313	181	227	136	153	2111	13
4	21076	479	357	324	284	204	279	137	200	103	133	1556	5
5	21715	459	343	292	247	165	240	111	172	88	118	1182	5
6	22193	443	322	268	216	140	214	91	144	72	98	934	2
7	22585	445	299	231	189	121	196	75	123	63	85	724	1
8	22883	432	272	189	171	107	176	66	101	59	71	610	0
9	23136	422	244	171	150	95	153	58	92	56	63	497	0
10	23360	404	225	154	135	86	131	54	77	50	53	408	0
11	23514	412	201	136	123	78	111	49	71	46	47	349	0
12	23649	393	186	123	111	75	96	47	63	42	40	312	0
13	23814	378	171	115	98	62	85	42	56	39	34	243	0
14	23922	367	161	100	90	58	80	40	48	35	31	205	0
15	24025	350	146	92	82	53	70	36	43	31	27	182	0
16	24096	341	136	87	70	50	68	32	34	30	25	168	0
17	24170	332	122	81	64	48	57	30	33	29	23	148	0
18	24234	317	112	78	58	43	56	27	33	27	22	130	0
19	24293	307	105	72	53	42	55	27	30	26	20	107	0
20	24359	291	97	68	51	41	50	26	25	23	18	88	0
All	455064	9079	5836	4698	4147	3017	4035	2158	2811	1835	2019	28752	4426

Table 2.42: Short Run Bankers n Table

Overall gain percentage distribution broken down by n													
n	[0,0]	(0,10]	(10,20]	(20,30]	(30,40]	(40,50]	(50,60]	(60,70]	(70,80]	(80,90]	(90,100]	(100,1000]	(1000,∞]
10	58003	4434	2379	1939	1514	1024	1434	558	1144	419	488	2075	0
20	61750	2018	1439	808	663	741	1112	457	377	470	598	4978	0
30	63734	1036	893	914	903	337	327	332	371	402	436	5726	0
40	65148	686	550	488	562	496	722	239	230	231	228	5831	0
50	66091	600	376	381	367	288	359	388	599	135	182	5550	95
100	68622	296	192	163	136	124	79	179	89	175	79	4175	1102
1000	71716	9	7	5	2	7	2	5	1	3	8	417	3229
All	455064	9079	5836	4698	4147	3017	4035	2158	2811	1835	2019	28752	4426

Table 2.43: Short Run Bankers θ Table

Overall gain percentage distribution broken down by θ													
θ	[0,0]	(0,10]	(10,20]	(20,30]	(30,40]	(40,50]	(50,60]	(60,70]	(70,80]	(80,90]	(90,100]	(100,1000]	(1000,∞]
0.1	48847	3616	1451	1014	787	450	612	244	484	55	38	791	264
0.2	48618	1753	1337	538	463	637	886	214	277	349	414	2836	331
0.3	48482	811	801	1140	968	256	343	397	254	346	450	4045	360
0.4	48950	664	556	553	635	625	1018	226	175	197	197	4494	363
0.5	49722	607	509	407	417	346	453	402	918	168	133	4012	559
0.6	50635	572	466	352	331	246	272	266	280	333	404	3892	604
0.7	51820	471	321	353	256	202	195	197	215	179	211	3626	607
0.8	53236	370	261	211	192	169	149	123	131	144	101	2945	621
0.9	54754	215	134	130	98	86	107	89	77	64	71	2111	717
All	455064	9079	5836	4698	4147	3017	4035	2158	2811	1835	2019	28752	4426

Table 2.44: Short Run Bankers q Table

Overall gain percentage distribution broken down by q													
q	[0,0]	(0,10]	(10,20]	(20,30]	(30,40]	(40,50]	(50,60]	(60,70]	(70,80]	(80,90]	(90,100]	(100,1000]	(1000,∞]
0.001	18936	949	694	618	460	341	411	353	268	342	209	3973	229
0.002	20027	721	612	484	446	298	384	329	316	229	314	3385	238
0.003	20988	641	561	419	451	350	370	206	306	222	179	2855	235
0.004	21826	576	443	436	373	274	427	169	315	143	174	2396	231
0.005	22470	472	447	369	389	240	344	188	238	143	166	2076	241
0.006	22955	502	396	297	331	258	331	121	199	145	147	1863	238
0.007	23425	421	341	374	200	247	311	103	211	65	174	1678	233
0.008	23695	453	337	261	262	177	299	98	195	80	122	1570	234
0.009	23901	464	301	240	267	166	267	113	172	68	110	1477	237
0.01	24193	365	312	228	256	113	273	99	148	84	84	1389	239
0.02	25229	366	233	162	130	103	118	75	88	63	74	897	245
0.03	25575	386	192	156	81	94	72	62	51	50	42	779	243
0.04	25791	388	153	115	89	67	75	48	48	36	37	699	237
0.05	25876	379	172	100	88	53	70	33	45	35	38	659	235
0.06	25972	404	130	85	77	56	52	32	44	31	33	637	230
0.07	25975	422	134	97	59	52	59	31	50	23	32	624	225
0.08	26053	402	115	93	55	49	58	31	40	27	27	611	222
0.09	26076	382	132	86	64	42	60	33	40	21	31	599	217
0.1	26101	386	131	78	69	37	54	34	37	28	26	585	217
All	455064	9079	5836	4698	4147	3017	4035	2158	2811	1835	2019	28752	4426

Table 2.45: Short Run Bankers ρ Table

Overall gain percentage distribution broken down by ρ													
ρ	[0,0]	(0,10]	(10,20]	(20,30]	(30,40]	(40,50]	(50,60]	(60,70]	(70,80]	(80,90]	(90,100]	(100,1000]	(1000,∞]
0.001	20324	576	385	367	277	251	270	135	199	131	146	1907	169
0.002	20349	355	395	278	253	176	332	113	239	94	120	2207	226
0.003	20413	345	331	208	246	183	322	103	196	135	153	2263	239
0.004	20475	373	275	204	269	136	340	97	219	112	166	2231	240
0.005	20556	358	289	172	248	142	345	138	199	87	162	2202	239
0.006	20582	423	193	287	186	236	269	106	191	155	121	2146	242
0.007	20665	444	181	224	285	158	267	123	211	117	142	2084	236
0.008	20743	406	215	291	190	213	273	105	176	129	190	1970	236
0.009	20836	367	213	295	161	275	169	216	153	169	119	1928	236
0.01	20903	378	263	208	245	190	249	124	220	111	147	1862	237
0.02	21414	408	333	225	336	105	202	181	143	88	80	1391	231
0.03	21717	498	349	263	266	110	194	122	124	89	75	1102	228
0.04	22041	450	430	219	234	120	152	100	125	46	79	923	218
0.05	22288	488	367	237	188	130	134	73	90	60	69	801	212
0.06	22460	512	338	262	158	99	118	88	65	66	37	727	207
0.07	22650	548	269	232	141	116	83	83	58	55	49	655	198
0.08	22839	462	308	196	133	91	97	68	52	57	43	596	195
0.09	22902	562	228	203	102	107	79	69	56	41	42	556	190
0.1	23025	529	230	173	116	92	71	61	48	44	45	516	187
0.2	23771	352	140	92	71	47	42	33	27	25	22	370	145
0.3	24111	245	104	62	42	40	27	20	20	24	12	315	115
All	455064	9079	5836	4698	4147	3017	4035	2158	2811	1835	2019	28752	4426

Table 2.46: Short Run Bankers Bias Table

Time t	Frequency of cases where at time t the grand coalition is preferred by the banker but the social planner prefers the atomistic partition.
0.0	22182
0.5	12107
1.0	9090
1.5	7375
2.0	6256
2.5	5480
3.0	4849
3.5	4374
4.0	3920
4.5	3585
5.0	3259
5.5	3018
6.0	2768
6.5	2530
7.0	2361
7.5	2230
8.0	2062
8.5	1906
9.0	1808
9.5	1676
10.0	1581
10.5	1506
11.0	1417
11.5	1359
12.0	1286
12.5	1194
13.0	1120
13.5	1063
14.0	1009
14.5	954
15.0	912
15.5	876
16.0	843
16.5	801
17.0	771
17.5	739
18.0	708
18.5	677
19.0	648
19.5	612
20.0	583
Total	527877

2.3.2.3 Banks and Businesses have different transaction costs

We now consider how the returns from matches may be split up between banks and businesses. The proportions may be different for inside and outside matches, and this can affect the banks' choice of partition. As in the standard model, assume that inside and outside matches have total values 1 and θ respectively. The standard model assumes that the bank and the business receive equal shares. Now let us generalise and assume that the proportion of the return received by the bank is α and β in the two cases:

Table 2.47: Bank-Business Distribution Table

	Inside	Outside
Bank	α	$\beta\theta$
Business	$1-\alpha$	$(1-\beta)\theta$
Total	1	θ

The social planner has the standard θ program and hence has welfare function: $W_{SP}[(x_i)_{i=1}^k] := W[(x_i)_{i=1}^k : \Theta = \theta] = \sum_{i=1}^n \left(\frac{x_i^2}{n^2}P[x_i] + \sum_{j\neq i} \theta \frac{x_i x_j}{n^2} P[x_i]P[x_j]\right)$. The bank has inside and outside returns of α and $\beta\theta$ respectively. So the bank welfare function is: $W_{Bank}[(x_i)_{i=1}^k] := \sum_{i=1}^n \left(\alpha\frac{x_i^2}{n^2}P[x_i] + \sum_{j\neq i} \beta\theta \frac{x_i x_j}{n^2} P[x_i]P[x_j]\right)$. This can be re-normalised to the standard welfare program with $\Theta = \frac{\beta\theta}{\alpha}$, as $W_{Bank}[(x_i)_{i=1}^k] = \alpha * W[(x_i)_{i=1}^k : \Theta = \frac{\beta\theta}{\alpha}]$. However it is possible that $\frac{\beta\theta}{\alpha} > 1$, and so the banker prefers outside matches to inside matches. We need to check that in this case, where $\theta > 1$, the standard model still has boundary solutions:

Proposition 78. *The standard model with $\theta > 1$, and all other parameters still in their standard ranges, has the atomistic boundary solutions to the welfare program.*

Proof. First note the model still has negative externalities: if modules 2 and 3 merge to form a 'super module' then module 1 is worse off as banks in the super module are less likely to be enabled. So the partition that maximises the utility of bank 1 will be of form $\{x_1, 1, 1, 1, 1, 1..1\}$. As in the standard model the bank 1 utility function is given by $v_1[x_1] = \frac{x_1}{n^2}P[x_1] + \theta\frac{(n-x_1)}{n^2}P[x_1]P[1]$. Hence $\frac{\partial v_1[x_1]}{\partial x_1} = (1 - \theta + \gamma(1 - \theta n))\, P[1]P[x_1]^2$. As $\theta n > 1$ and $1 - \theta < 0$ we get, $\frac{\partial v_1[x_1]}{\partial x_1} < 0$ and hence the atomistic partition of singletons is the solution to both the $v_1[x_1]$ program and the general $W[x]$ program. □

So, the bank will still want a boundary solution, but not necessarily the same boundary as the social planner. Secondly, the form of the bias depends on only the relative level of the 2 shares $\lambda := \beta/\alpha$, but not on the absolute levels α and β: from the standard model, $\{n\} \succ_{SP} \{1, 1, 1....1\} \iff P[n] > \theta P[1]$ and $\{n\} \succ_B \{1, 1, 1....1\} \iff P[n] > \lambda\theta P[1]$, where $\succ_{SP}$ and $\succ_B$ represent the preferences of the social planner and the banker respectively. Similarly let x^*_{SP} and x^*_B represent the argmax of the social planner and the bank. So if $\beta < \alpha$ then the banker is biased towards the grand coalition, conversely if $\beta > \alpha$ then the banker is biased towards the atomistic partition of singletons. The condition for the banker to inefficiently choose the grand coalition ($x^*_B = \{n\}$ and $x^*_{SP} = \{1, 1, 1, 1...1\}$) is $\lambda\theta < \frac{P[n]}{P[1]} < \theta$. A necessary condition for this to happen is $\lambda < 1$, which equates to $\beta < \alpha$. Conversely, the condition for the banker to inefficiently choose the atomistic partition ($x^*_B = \{1, 1, 1, 1...1\}$ and $x^*_{SP} = \{n\}$) is $\theta < \frac{P[n]}{P[1]} < \lambda\theta$. A necessary condition for this to happen is $\lambda > 1$, which equates to $\alpha < \beta$.

The percentage loss calculation is as follows. Firstly, the welfare of the social planners choice is calculated using the unbiased match value θ, and is just $W_{SP} = \max[W[1 : \theta], W[n : \theta]$. Secondly, the banker choice, d_{bank}, is calculated using, $\lambda\theta$, the biased match value: $d_{bank} = \text{argmax}_{1,n} W[d : \lambda\theta]$. The loss, l, is then calculated as the percentage loss they experience by delegating the decision to the banks, rather than making the decision themselves. Hence, $l = 100 * (W[d_{bank} :$

$\theta] - W_{SP})/W_{SP}$.

Using the same approach as with other variant models, we consider a range of parametrisations. The number of banks, n, is one of 10, 20, 30, 40, 50, 100, 1000. The value of outside matches, θ, has a minimum of 0, a maximum of 1.0 and an increment of 0.1. The shock parameter, γ, has a minimum of 0, a maximum of 3.0, and an increment of 0.1. The λ parameter is one of $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. So, the n parameter is one of 7 values, the θ parameter is one of 11 values, the γ parameter is one of 31 values, and the λ parameter is one of 20 values. This gives a total of $7 * 11 * 31 * 20 = 47740$ parametrisations.

The λ table below shows, that for very small values of λ, the banker is biased towards the grand coalition: in particular when $\lambda = 0$, the banker always chooses the grand coalition, resulting in many cases where the bankers's outcome is inefficient. However, once λ reaches 0.2, whilst there can be large costs from the banker's bias, these are rare: in 77.5% (1791) of cases the banker's choice of boundary partition is the same as the social planner. Once λ equals 1 the banker is unbiased, and so always makes the same choice as the social planner. Once λ is above 1, the direct of bias now switches: the banker is biased towards the atomistic partition. However, in 2033 of the 2310 cases the social planner prefers the atomistic partition already; so the banker's bias has little effect. So with $\lambda > 1$, there are few cases of losses due to the banker bias, and when they occur they are generally small in size.

The n table shows that as n increases, firstly, there are fewer case with losses, but secondly, when losses do occur they are larger in magnitude. This is because, with large n, the social planner more strongly wants the atomistic partition: so even with their λ bias the bank is likely to still want the atomistic partition; however, when the λ bias is strong enough to make the bank choose the grand coalition, the loss to the social planner is large. In the θ table, when $\theta = 0$, both the social planner and the banker always choose the grand coalition. For large values of θ, the same pattern emerges, and for the same reason, as in the n table: there are fewer case with losses; but when losses occur they are large in magnitude. In the γ table, when $\gamma = 0$: the social planner always prefers the grand coalition (strictly if $\theta < 1$; weakly if $\theta = 1$); however, the bank chooses the atomistic partition if $\theta\lambda > 1$. Apart from when γ is small, for example 0.1 or 0.2, there are few defections as, even with a potential λ bias, there needs to be a low γ for the bank to select the grand coalition.

Table 2.48: Bank-Business Distribution Model λ Table

	Percentage loss broken down by λ										
λ	(-100,-90)	[-90,-80)	[-80,-70)	[-70,-60)	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	0
0	545	473	304	230	150	115	88	61	41	25	355
0.1	0	68	164	229	150	115	88	61	41	25	1446
0.2	0	0	23	82	85	114	88	61	41	25	1868
0.3	0	0	0	14	40	54	86	61	41	25	2066
0.4	0	0	0	0	12	26	41	61	41	25	2181
0.5	0	0	0	0	0	8	21	37	41	25	2255
0.6	0	0	0	0	0	0	5	19	41	25	2297
0.7	0	0	0	0	0	0	0	4	23	25	2335
0.8	0	0	0	0	0	0	0	0	8	19	2360
0.9	0	0	0	0	0	0	0	0	0	11	2376
1	0	0	0	0	0	0	0	0	0	0	2387
2	0	0	0	0	0	1	11	14	23	28	2310
3	0	0	0	0	9	11	14	16	23	28	2286
4	0	0	0	8	9	12	14	16	23	28	2277
5	0	0	0	8	10	12	14	16	23	28	2276
6	0	0	7	9	10	12	14	16	23	28	2268
7	0	0	7	9	10	12	14	16	23	28	2268
8	0	0	7	9	10	12	14	16	23	28	2268
9	0	0	7	9	10	12	14	16	23	28	2268
10	0	0	7	9	10	12	14	16	23	28	2268
All	545	541	526	616	515	528	540	507	525	482	42415

Table 2.49: Bank-Business Distribution Model n Table

	Percentage loss broken down by n										
n	(-100,-90)	[-90,-80)	[-80,-70)	[-70,-60)	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	0
10	0	0	5	130	206	200	185	224	261	253	5356
20	0	25	169	145	111	90	156	46	151	127	5800
30	0	118	130	111	77	77	30	149	59	38	6031
40	9	165	105	74	70	33	112	54	18	18	6162
50	56	146	83	78	26	106	33	16	18	28	6230
100	181	85	29	71	17	14	15	9	9	9	6381
1000	299	2	5	7	8	8	9	9	9	9	6455
All	545	541	526	616	515	528	540	507	525	482	42415

Table 2.50: Bank-Business Distribution Model θ Table

	Percentage loss broken down by θ										
θ	(-100,-90)	[-90,-80)	[-80,-70)	[-70,-60)	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	0
0.0	0	0	0	0	0	0	0	0	0	0	4340
0.1	29	2	0	69	23	123	160	197	238	234	3265
0.2	30	20	51	78	94	81	125	52	139	127	3543
0.3	30	30	72	133	60	63	35	129	59	29	3700
0.4	40	41	84	65	117	11	113	37	9	10	3813
0.5	57	52	69	75	9	151	18	15	9	9	3876
0.6	58	67	65	48	36	64	76	7	0	10	3909
0.7	59	87	66	12	74	18	6	63	8	0	3947
0.8	69	85	53	23	62	6	0	7	63	0	3972
0.9	78	79	42	51	27	5	7	0	0	63	3988
1.0	95	78	24	62	13	6	0	0	0	0	4062
All	545	541	526	616	515	528	540	507	525	482	42415

Table 2.51: Bank-Business Distribution Model γ Table

	Percentage loss broken down by γ										
γ	(-100,-90)	[-90,-80)	[-80,-70)	[-70,-60)	[-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	0
0.0	0	0	35	49	56	56	63	63	63	63	1092
0.1	9	12	26	43	49	47	39	48	17	58	1192
0.2	14	19	25	31	30	30	42	23	45	9	1272
0.3	15	25	28	20	31	28	22	23	27	19	1302
0.4	17	24	28	16	24	23	18	24	26	19	1321
0.5	18	23	20	24	20	15	20	22	27	10	1341
0.6	18	27	14	27	15	19	14	20	26	9	1351
0.7	21	21	20	18	13	17	11	20	16	18	1365
0.8	21	21	20	18	12	22	18	8	15	9	1376
0.9	21	20	20	20	13	17	17	16	7	10	1379
1.0	21	20	16	23	15	8	15	23	0	20	1379
1.1	21	20	16	22	14	8	15	13	9	19	1383
1.2	21	16	18	21	12	8	15	12	9	18	1390
1.3	21	16	18	21	12	8	15	12	9	18	1390
1.4	21	16	18	18	15	8	15	12	9	18	1390
1.5	21	16	18	18	14	8	14	12	9	17	1393
1.6	20	17	17	12	19	8	13	12	9	16	1397
1.7	18	19	16	12	15	11	12	12	9	16	1400
1.8	18	19	16	12	15	10	12	11	17	8	1402
1.9	18	19	12	16	9	16	12	11	16	9	1402
2.0	18	19	8	18	9	15	12	10	16	9	1406
2.1	18	17	10	18	9	15	12	10	16	9	1406
2.2	18	15	12	18	8	15	12	10	15	9	1408
2.3	18	15	12	18	8	15	12	10	15	9	1408
2.4	17	15	12	16	10	15	12	10	14	9	1410
2.5	17	15	12	16	10	15	12	10	14	9	1410
2.6	17	15	12	14	12	15	12	10	14	9	1410
2.7	17	15	12	14	12	15	12	10	14	9	1410
2.8	17	15	12	14	12	15	12	10	14	9	1410
2.9	17	15	12	14	12	12	15	10	14	9	1410
3.0	17	15	11	15	10	14	15	10	14	9	1410
All	545	541	526	616	515	528	540	507	525	482	42415

2.3.3 Partition Formation

In section 1.8 we considered the EEBA game where agents are farsighted in both expectations and preferences. Ideally, we want expectations and preferences aligned in terms of their timeframe. Recall that, the preferences in the standard model rely on the Markov process being at its limiting state; all the stability games in this section, like the standard model, have farsighted preferences. However, the first two formations, (Bilateral Stability and the Open Membership Game), here have agents with myopic expectations, who only consider the direct effect of their actions. These games for their validity rely on the assumption that after the first deviation, it is assumed that subsequent deviations are not feasible (see subsections 2.3.3.1 and 2.3.3.2).[20] The final two games in this section, the Unanimity Game and the EBA (Equilibrium Binding Agreements), like the EEBA game, have full farsightedness in both preferences and expectations. The following stability table considers these four new games, along with the earlier EEBA and the Zero Lifetime Bankers:

Table 2.52: The Stability Bias Table

		Atomistic Partition Stability Level				
		Never	Less	Correct	Too	Always
Grand Coalition Stability Level	Never					
	Less					
	Correct			EEBA Unanimity Game	EBA	
	Too			Bilateral Stability		
	Always	Zero Lifetime Bankers		Simultaneous-Move		

In table 2.52, the results from the six approaches are summarised: when is each trivial partition stable? Is it *correctly stable* (stable if and only if it is efficient)? Or is it biased: either *too stable* (stable for some parameter values at which it is inefficient), or *less stable* than ideal (unstable for some parameter values at

[20] An alternative motivation for myopic expectations would be agents who only have short run preferences (see sections 2.3.2.1 and 2.3.2.2), and for example, will be retired before any response occurs.

which it is efficient)? Further, the pro or anti-stability bias may be extreme: the partition may be either, *always stable* (stable for all parameter values), or *never stable* (unstable for all parameter values).

The EEBA and Unanimity Games give us 'goldilocks' results: each of the trivial partitions is stable if and only if it is efficient. However, in contrast three of the other solution concepts all suggest that there is a bias towards the grand coalition: with bilateral stability, the atomistic partition is correctly stable, but the grand coalition is too stable; with the Simultaneous-Move game, the atomistic partition is correctly stable, but the grand coalition is always stable; and with Zero Lifetime Bankers, the atomistic partition is never stable, but the grand coalition is always stable.[21]

2.3.3.1 Bilateral Stability

The concept of *bilateral stability*, from Jackson and Wolinsky (1996), is often used in the network literature. In network theory, with bilateral stability, the feasible set of changes is that individual agents can unilaterally remove links but it takes a pair of agents to bilaterally form a new link. So, if no single agent gains from removing any of their links, and no pair of agents gains from creating a link between them, then the network is *bilaterally stable*.

Extending this concept to partition form games, means considering an environment where there are random opportunities for individual modules to split, and pairs of modules to merge. Bilateral stability requires that no module wants to split, and no pair of modules want to merge. Bilateral stability ignores the potential for subsequent deviations, and so requires myopic agents: either in

[21] With the EBA modules can split, but not merge, resulting in a structural bias towards the atomistic partition. In particular, for all payoff functions by definition the atomic partition is always an EBA. However, the convention is to pick the coarsest EBA as the solution. And under this convention for the standard model the solution EBA is always efficient: see Corollary 96 and Proposition 97.

expectations (agents do not believe that future deviations are possible), or in preferences (agents have short run preferences and do not care about the long run outcome). In the standard model, the agents' preferences are farsighted: the payoffs in the model are evaluated using asymptotic probabilities. Therefore, to justify the use of this solution concept, we need a story that makes subsequent deviations unfeasible. For example, imagine that, a banking commission is re-designing the banking network, whilst structural changes (in the form of bank mergers and bank splits), are on-going. After the commission announces their new design for the bank network, there is time for one final structural change in response, before the network is finalised. So, the timing of such a story would be:

1. The social planner assigns a partition x.
2. At random, either a single module x_i or, a pair of modules (x_i, x_j), is given a chance to deviate. The single module x_i can split; the pair of modules x_i and x_j can merge. The resulting partition $x^*(x_i)$ or $x^*(x_i, x_j)$ is now fixed forever.[22]

Partition x is *bilaterally stable* if, for any possible x_i or (x_i, x_j), there is no feasible profitable deviation, and so x^* always equals x. For my banking model, when are the trivial partitions bilaterally stable? Starting at the atomistic partition, it is not feasible for a singleton module to split, but it is feasible for two modules (without loss of generality these are assumed to be modules 1 and 2), to merge. The initial partition gives both banks 1 and 2 an expected utility of:

$$v_1[\{1, 1, 1....1\}] = \frac{1}{n^2}P[1] + \frac{1}{n^2}\theta(n-1)P[1]P[1]$$

[22] In June 2010 the UK government setup the Independent Commission on Banking (ICB) to make recommendation on how to reform the banking industry. But it knew that as soon as it issued its recommendations that it would be dissolved: this happened in September 2011. This kind of setup motivates why the commission has to make a static choice, and why they can not respond and make further alterations to the banking network.

whilst merging gives them each:

$$v_1[\{2, 1, 1....1\}] = \frac{2}{n^2}P[2] + \frac{1}{n^2}\theta(n-2)P[1]P[1]$$

Re-arranging gives that $\{1, 1, 1....1\}$ is bilaterally stable if and only if $P[n] \leq \theta P[1]$: the condition for the atomistic partition to be efficient.

Now considering the grand coalition $\{n\}$, the alternative is to split to form $\{0.5n, 0.5n\}$.[23] So the initial partition gives each bank an expected utility of $v_1[\{n\}] = \frac{1}{n}P[n]$; whilst deviating gives each bank an expected utility of $v_1[\{0.5n, 0.5n\}] = \frac{1}{2n}P[0.5n] + \frac{1}{2n}\theta P[0.5n]P[0.5n]$. Re-arranging gives that $\{n\}$ is bilaterally stable, if and only if $n \leq \frac{n}{2\theta} - \frac{(1-\theta)}{\theta\gamma}$. This is weaker than the condition for $\{n\}$ to be efficient. So, there exist parameter values for which $\{n\}$ is bilaterally stable, but not efficient.[24]

2.3.3.2 The Simultaneous-Move Open Membership Game

The simultaneous-move open membership game comes from Yi and Shin (2000). It defines a game where there is an exogenous list of *groups*, and every agent simultaneously picks one group to be in. Every agent gets to be in the group they asked to be in; each group becomes a module, and thus a partition is formed. This partition becomes permanent. The solution concept is that the agents strategies need to form a Nash equilibrium of the game.

The permanency of the partition motivates the use of asymptotic payoffs. We look to see when each of the trivial partitions can be formed as a Nash equilibrium of this game. Recall $v_i[(x_j)_{j=1}^n] = \frac{x_i}{n^2}P[x_i] + \sum_{j\neq i}\theta\frac{x_j}{n^2}P[x_i]P[x_j]$. Starting with

[23] If there are gains to splitting, then the partition with the highest total utility over the 2 modules is the symmetric partition $\{0.5n, 0.5n\}$.

[24] See Appendix J for a discussion of how models of club goods differ from partition form games.

the atomistic partition, the equilibrium strategy gives bank 1 a utility of $v_1[\{1, 1, 1....1\}] = \frac{1}{n^2}P[1] + \frac{1}{n^2}\theta(n-1)P[1]P[1]$; whilst deviating gives $v_1[\{2, 1, 1....1\}] = \frac{2}{n^2}P[2] + \frac{1}{n^2}\theta(n-2)P[1]P[1]$.[25] Re-arranging gives that $\{1, 1, 1....1\}$ is a Nash equilibrium, if and only if, $P[n] \leq \theta P[1]$: the condition for the atomistic partition to be efficient. [26]

Considering the grand coalition, the equilibrium strategy has a payoff of $v_1[\{n\}] = \frac{1}{n}P[n]$; whilst deviating gives $v_1[\{1, n-1\}] = \frac{1}{n^2}P[1] + \frac{\theta(n-1)}{n^2}P[1]P[n-1]$. Rearrangement shows that for all parameter values $\{n\}$ is a Nash equilibrium. The intuition behind why a single bank never wants to deviate is that $\{1, n-1\}$ gives bank 1 most of the cost of a big module (low enablement probability for partner banks that bank 1 could be matched with), but few of the benefits (low rather than high match values as it is in a different module).

2.3.3.3 The Unanimity Game

The Yi (1997) paper considers the Unanimity game of Bloch (1996). In the Unanimity game a module forms if and only if all proposed members agree to form the module. Suppose the N players are labelled $\{P_1, P_2, P_3...P_N\}$. First, P_1 makes a proposal for a module, e.g., $\{P_1, P_3, P_4, P_7\}$. Then each of the other proposed module members, starting with the smallest index (here it is P_3), accepts or rejects the proposal. If P_3 accepts, then it is P_4's turn to accept or reject the proposal, and so on. Module formation needs unanimity, so if any of the other proposed members rejects P_1's proposal, then the current proposal is completely thrown out: firstly, there is *no* module formation among the players who agreed to the original proposal, and secondly the player who first rejected

[25] Many different strategies have an outcome which is a trivial partition. But, the simplest for the atomistic partition is each bank i picking group i, and every bank picking group 1 for the grand coalition.

[26] This is the same derivation as with bilateral stability.

the proposal starts over by proposing another module. If, instead, all proposed members accepted P_1's proposal, then they form a module and the remaining players continue the module formation game, starting with the player with the smallest index making a proposal. Notice that once a module forms, it cannot break apart, admit new members, or merge with other modules, regardless of how the rest of the players form modules.

Bloch (1996) shows that the Unanimity game yields the same stationary subgame perfect equilibrium structure as the following "Size Announcement" game: player P_1 first announces the size of his module s_1, and the first s_1 players form a size s_1 module, and then player P_{s_1+1} proposes s_2, and the next s_2 players form a size s_2 module, and so on until P_N is reached. Intuitively, the equivalence between the two games holds because of the identical players in the Unanimity game: the *proposer* has reason to care about his module size but no reason to care about the labels of the other player he selects; the *receiver* has no reason to reject any proposal as what is best for the proposer is best for the receiver.

Under the standard model, the formation results for the Unanimity Game are as follows:

Theorem 79. *Under the standard model if $P[n] > \theta P[1]$ then proposing the grand coalition $\{n\}$ is the unique SPE solution of the Unanimity game.*

Proof. Firstly we know that the Unanimity game is equivalent to the size announcement game, so we can assume that if player 1 rationally proposes $\{n\}$ then it will be accepted by the other players. Secondly if $P[n] > \theta P[1]$, then $\{n\}$ is the unique partition that maximises $v_1[x]$, so any other partition gives player 1 a worse payoff. □

Theorem 80. *Under the standard model if $P[n] < \theta P[1]$ then each player proposing a singleton module is an SPE solution of the Unanimity game.*

Proof. As $P[n] < \theta P[1]$, then $\{1, 1, 1...1\}$ is the partition that maximises the utility of each and every player. So consider the strategy where each player proposes a singleton module. Then no deviation can lead to a higher payoff. $\square$

2.3.3.4 Equilibrium Binding Agreement (EBA)

Chapter One in Section 1.8 considered the Extended Equilibrium Binding Agreement (EEBA), from Diamantoudi and Xue (2007). This section will now consider the original Equilibrium Binding Agreement (EBA), of Ray and Vohra (1997), which has a smaller feasible set of allowable deviations but has the same *coalition preference* relation, formed by requiring the individual preference to hold for all members of the coalition. Formally:

Definition 81. Let N be the set of agents, and let P and Q be partitions *of* N. Coalition S strictly prefers P to Q (notation $P \succ_S Q$), if each member of coalition S strictly prefers partition P to Q. Specifically, require that $(P \succ_i Q)_{i \in S}$, or in utility formation, $u_i(P) > u_i(Q)$ for all $i \in S$. Note, that S can be any subset of N: there is no requirement that S be a member of either of the partitions.

With the EEBA *coalitional deviations* are allowed; whilst with the EBA only *internal coalitional deviations* are allowed: with the EEBA any members can form a module and deviate; whilst with the EBA all the deviating members need to come from the *same* module. Formally:

Definition 82. We write $P \xmapsto{T} P'$ to denote an *internal coalitional deviation* where the following conditions on P, T and P' hold:

1. P is a partition of N
2. $T \subsetneq S \in P$; that is T is a strict subset of *one* of the modules in P.

3. $P' = P \setminus S \cup \{S \setminus T, T\}$; that is P' has one more module than P, and all those modules that were unaffected by the deviation of T remain modules in the new partition structure.

Specifically, the EEBA allows any sequence of coalitional deviations, so deviating agents can re-deviate: this allows $P_i \stackrel{T_i}{\rightarrow} P_{i+1}$, $P_j \stackrel{T_j}{\rightarrow} P_{j+1}$, $b \in T_i \cap T_j$ and $i \neq j$. In contrast, the EBA only allows internal deviations (splits within a module), and further each agent can only deviate once: $P_i \stackrel{T_i}{\mapsto} P_{i+1}$, $P_j \stackrel{T_j}{\mapsto} P_{j+1}$, $b \in T_i \; and \; i \neq j \Rightarrow b \notin T_j$. This single deviation property is captured within the definition of *RV-reachable*, which allows a sequence of deviations, but at each stage forces newly formed modules to be members of the *final* partition: no member can be in more than one module in the final partition, and so can only be in (at most) one deviating coalition.

Definition 83. *P´ is RV-reachable* from P if there exists a sequence of partitions $P_1, P_2, ..., P_k$, where:

1. $P_1 = P$ and $P_k = P'$; the sequence of partitions needs to start with P and end with $P^{'}$
2. and a sequence of modules $(T_j \in P')_{j=1}^{k-1}$ such that $\left(P_j \stackrel{T_j}{\mapsto} P_{j+1}\right)_{j=1}^{k-1}$; the deviating coalition at each stage is a module in the *final* partition P'

There are a number of different, but equivalent, definitions of RV dominance, the dominance relation used by the EBA. The original Ray and Vohra (1997) definition is explicitly recursive. However, Diamantoudi and Xue (2007) prove that there are two other equivalent formulations, and we use a definition based on the second of these.[27]

[27]See Diamantoudi and Xue (2007) Proposition 1 and Corollary 1, and Bloch and Datta (2010) definition 15.

Definition 84. *P' RV dominates* P (denoted by $P' \gg^{RV} P$) if there exists a sequence of partitions $P_1, P_2, ..., P_k$ and a sequence of modules $T_1, T_2, ..., T_{k-1}$ such that:

1. *P' is RV-reachable* from P through the sequence of partitions and modules.
2. $P \prec_{T_1} P'$; the initial deviators prefer the end partition P' to the starting partition P.
3. If $Q = P_2$ or Q is RV-reachable from P_2 via a strict subset $R \subset \{T_2, T_2, ..., T_{k-1}\}$ then there exists $S \in \{T_2, T_2, ..., T_{k-1}\} \setminus R$ s.t. $Q \prec_S P'$; Suppose T_1 deviates because they prefer P' to P, and there is a sequence of deviators $(T_j)_{j=1}^{k-1}$ that leads from P to P' (in the RV-reachable sense). However, perhaps not all the other potential deviators deviate: for RV-dominance we require that whatever subset of the $(T_j)_{j=1}^{k-1}$ do not deviate there is at least one T_j who has not deviated yet, that prefers P' to their current partition Q.

For its solution concept, the EBA uses the same *stable set* of von Neumann and Morgenstein (1944), as the EEBA does; but it uses RV domination instead of indirect domination. Recall, that intuitively, the stable set requires that: firstly, no solution can be preferred to any other solution; and secondly, every non-solution must be inferior to some solution. Formally:

Definition 85. Consider a set X and some binary ordering $>$ on X. Then R, where$\emptyset \neq R \subseteq X$, is a *vN–M stable set* for $(X, >)$, if it is both *internally* and *externally stable*:

R is *vN–M internally stable* for $(X, >)$, if there do not exist $P, P' \in R$ such that $P' > P$

R is *vN–M externally stable* for $(X, >)$, if for any $P \in X \setminus R$, there exists some $P' \in R$ such that $P' > P$

Finally, we have the definition of an EBA:

Definition 86. P is an *EBA*, if there exists R s.t. $P \in R$ and R is a vN–M stable set of $(\mathbb{P}, \gg^{RV})$, where $\mathbb{P}$ is the set of partitions.

The Yi (1997) paper gives a number of useful results for partition form games under negative externalities, including four different conditions. The first and most important, $(N1)$ we met in chapter 1:

Definition 87. $(N1)$ The game satisfies *negative externalities*, if when modules merge to form a larger module, outside modules not involved in the merger are strictly worse off. Specifically, if x is strictly coarser than y, and $x_i = y_i$, then that implies that $v_i[x] < v_i[y]$.

And recall:

Definition 5. Partition x is *strictly coarser* than y if each module in x can be formed as a merger of modules in y. Specifically, if x has k modules and y has l modules then require $k < l$ and the existence of a mapping $f : A_l \to A_k$ *s.t.* $\left(x_j = \sum_{f(i)=j} y_i\right)_{j=1}^{k}$ where $A_l = \{1, 2, 3, 4.....l\}$ and $A_k = \{1, 2, 3, 4.....k\}$. Conversely, partition x is *strictly finer* than y, if and only if, partition y is *strictly coarser* than x.

We now introduce the next 2 definitions: which are about the internal effects of changes in the partition structure (that is the effects on players who are members of modules which change in size). With $(N2)$, a member of a module becomes better off if his module merges with larger or equal-sized module:

Definition 88. $(N2)$ If $x = (x_i)_{i=1}^{k}$ and $x' = (\sum_{i=1}^{j} x_i, (x_i)_{i=j+1}^{k})$ are partitions where $(x_1 \leq x_i)_{i=1}^{j}$ then $v_1[x'] > v_1[x]$.

With $(N3)$, a member of a non-singleton module becomes better off if he leaves his module to join another module of equal or larger size:

Definition 89. $(N3)$ If $x = (x_i)_{i=1}^k$ and $x' = ((x_i)_{i=1}^{j-1}, x_j - 1, (x_i)_{i=j+1}^{l-1}, x_l + 1, (x_i)_{i=l+1}^n)$ are partitions and $x_l \geq x_j \geq 2$ then $v_j[x] < v_l[x']$.

Here I introduce the stronger variant condition $(N3^*)$ which removes the requirement for the departing and receiving modules to be non-singletons:

Definition 90. $(N3^*)$ If $x = (x_i)_{i=1}^k$ and $x' = ((x_i)_{i=1}^{j-1}, x_j - 1, (x_i)_{i=j+1}^{l-1}, x_l + 1, (x_i)_{i=l+1}^n)$ are partitions and $x_l \geq x_j \geq 1$ then $v_j[x] < v_l[x']$.

Theorem 91. *Under the standard model,* $(N3^*)$ *holds if and only if* $P[n] > P[1]\theta$.

Proof. Without loss of generality assume $j = 1$ and $l = 2$. Recall

$$v_1[x] = \frac{x_1}{n^2}P[x_1] + \sum_{j \neq i} \theta \frac{x_j}{n^2} P[x_1]P[x_j]$$

and similarly

$$v_2[x^{'}] = \frac{x_2+1}{n^2}P[x_2+1] + \theta\frac{x_1-1}{n^2}P[x_2+1]P[x_1-1] + \sum_{j>2}\theta\frac{x_j}{n^2}P[x_2+1]P[x_j]$$

If $v_2[x^{'}] > v_1[x]$ then $D[x, x'] > 0$ where

$$D[x, x'] := \frac{n^2(v_2[x^{'}] - v_1[x])}{(1 - x_1 + x_2)P[x_1]P[x_1-1]P[x_2]P[x_2+1]}$$

The term $D[x, x']$ can be re-arranged to give

$$D[x,x'] = 1-\gamma-\theta+\gamma(1-\theta)(x_1+x_2)+\gamma^2(1-\theta)(x_1-1)x_2 \\ -R\gamma\theta\left\{1+\gamma(x_1+x_2-1)+\gamma^2(x_1-1)x_2\right\}$$

where $R := \sum_{j>2} x_j P[x_j]$. The R coefficient is negative, so it is hardest for $D[x, x']$ to be positive when R is maximised: this occurs when the other $n-x_1-x_2$ members are arranged in singleton modules and $R = (n-x_1-x_2)P[1]$. So $x = (x_1, x_2, 1, 1, 1....1)$ and $x' = (x_1-1, x_2+1, 1, 1, 1...1)$. This equates to requiring that the two conditions

$$\frac{\theta\gamma}{1-\gamma+\gamma(x_1+x_2)+\gamma^2(x_1-1)x_2}+\frac{\gamma\theta\,(n-x_1-x_2)}{\gamma+1}\leq 1-\theta$$

and

$$\frac{\gamma}{1+\gamma(x_1+x_2)+\gamma^2(x_1-1)x_2}<1-\theta$$

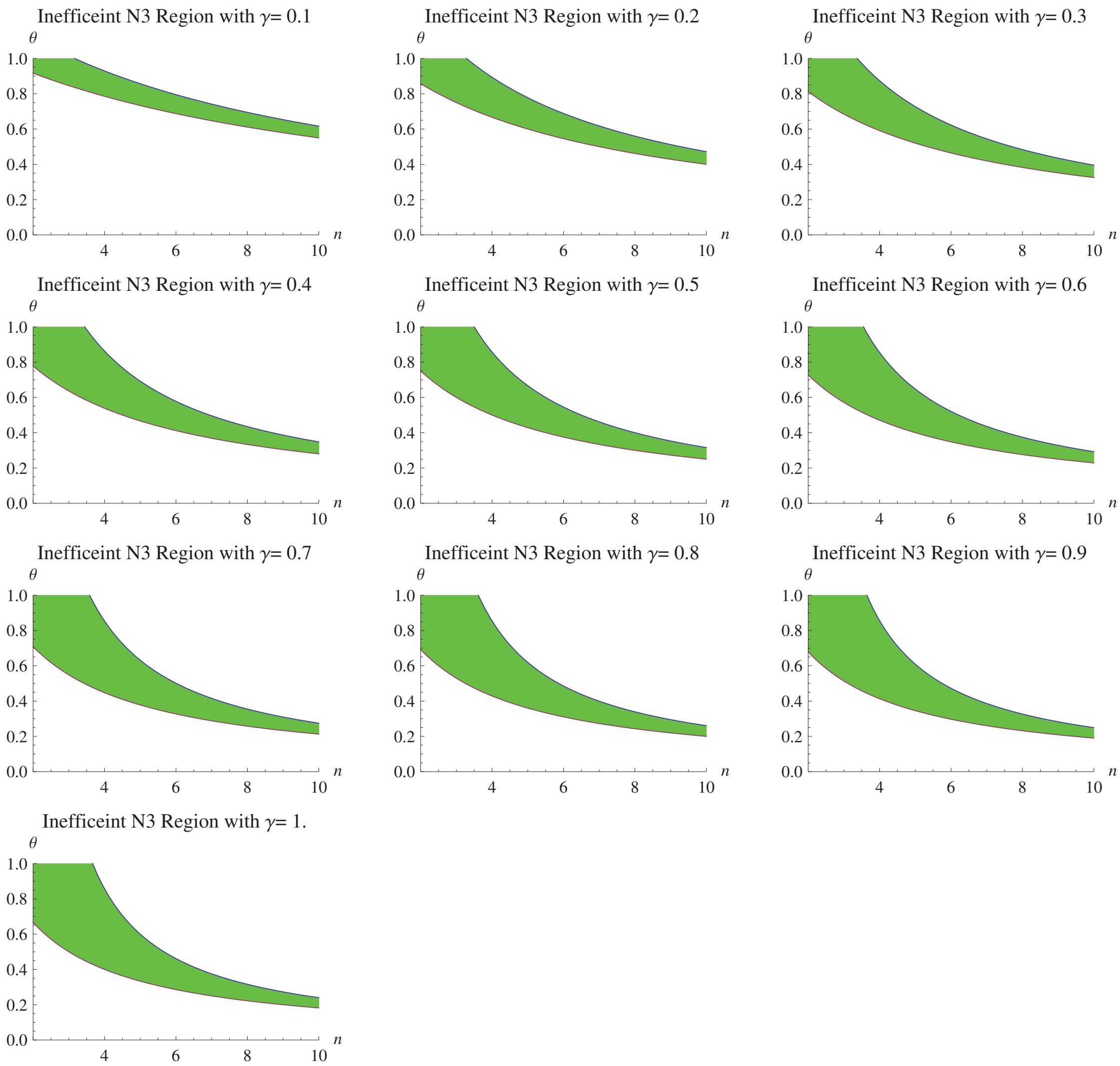
both hold for all suitable x_1 and x_2. However, these conditions are strictly harder to solve for smaller x_1 and x_2. So we need only to consider the case with $x_1 = x_2 = 1$, which gives $P[n] > P[1]\theta$: the grand coalition efficiency condition. □

Corollary 92. *Under the standard model if $P[n] > P[1]\theta$ then $(N3)$ holds.*

Proof. $(N3)$ is weaker then $(N3^*)$, so this follows directly from Theorem 91. Further, following the same argument as in that theorem's proof and considering the case $x_1 = x_2 = 2$, gives the following necessary and sufficient condition for $(N3)$ to hold: $\gamma^2(2-2\theta(n-3))+\gamma(3-\theta n)+1-\theta>0$. □

The region where the $(N3)$ condition holds but the grand coalition is inefficient, is plotted below in green, with n on the x-axis and θ on the y-axis, with plot for each value of the shock parameter γ between 0.1 and 1.0 with an increment of 0.1:

Figure 2.7: $(N3)$ holds but Grand Coalition Inefficient Plots



The final condition concerns the effect of a merger between a non-singleton and a singleton module on the non-singleton module. Formally: [28]

Definition 93. d^* is the largest integer which satisfies $(2 \leq x_i \leq d^*$ and $x = (x_j)_{j=1}^k$ a partition$) \Rightarrow v_i[x] \geq v_i[x']$ where $x' := ((x_j)_{j=1}^{i-1}, x_i - 1, 1, (x_j)_{j=i+1}^k)$

[28] Yi (1997) has different notation and uses k_0 for what I have called d^*.

So intuitively, if there is a module of size smaller than or equal to d^* and we split off a singleton module, then the members of the remaining module can be no better off, *whatever* the initial structure of the other modules.[29]

Theorem 94. *For the standard model,* $d^* = \begin{cases} n & \theta P[1] \leq P[n] \\ 1 & \theta P[1] > P[n] \end{cases}$

Proof. Recall from the standard model that

$$v_i[x] := \frac{x_i}{n^2}P[x_i] + \sum_{j \neq i} \theta \frac{x_j}{n^2} P[x_i]P[x_j]$$

and

$$v_i[x'] = \frac{x_i - 1}{n^2}P[x_i - 1] + \sum_{j \neq i} \theta \frac{x_j}{n^2} P[x_i - 1]P[x_j] + \theta \frac{1}{n^2} P[x_i - 1]P[1]$$

Hence

$$n^2(v_i[x] - v_i[x']) = (x_i + \theta R)P[x_i] - (x_i + \theta R - 1 + \theta P[1])\, P[x_i - 1]$$

where $R := \sum_{j \neq i} x_j P[x_j]$. Without loss of generality we let $i = 1$ and consider module 1. Factorising gives:

$$n^2(v_1[x] - v_1[x']) = (\gamma - \theta - \gamma^2\theta R - \gamma\theta R - \gamma\theta x_1 + 1)P[1]P[x_1]P[x_1 - 1]$$

So $v_1[x] \geq v_1[x']$ if and only if $(\gamma - \theta - \gamma^2\theta R - \gamma\theta R - \gamma\theta x_1 + 1) \geq 0$. We are looking for x_1 where this condition holds for all possible specifications $(x_j)_{j>1}$ of the rest of the partition and hence we want to consider the partition that maximises R.

[29] Yi (1997) defines this concept as definition 4.1 in section 4.1 However different notation and terminology is used: k_0 is the maximum module size, C represents a partition (although the terminology partition is not used), and $\pi(k; C)$ considers a module of size k and represents the utility per member of that module.

This occurs when the other modules are all singletons and $R = (n - x_1)P[1]$. This gives $v_1[x] \geq v_1[x']$ if and only if $\gamma - \theta + \gamma\theta(-n) + 1 \geq 0$. This equates to $v_1[x] \geq v_1[x']$ if and only if $\theta\mathrm{P}[1] \leq P[n]$ □

The condition on d^* is the same as the condition as whether the grand coalition or the atomistic partition of singletons is efficient. So if the grand coalition is efficient, then starting from any partition any module that merges with a singleton module weakly gains. Conversely, if the partition of singletons is efficient then for any other partition there is at least one module that would strictly gain from a split where it lost one member.

Now we formulate the Equilibrium Binding Agreement (EBA) solutions. Consider part 1 of Proposition (4.3) from Yi (1997):

Proposition 95. *Under (N1) if partition $x = (x_i)_{i=1}^k$ is such that $(x_i \leq d^*)_{i=1}^k$ then x is a stable coalition structure under the Equilibrium Binding Agreements rule.*

Corollary 96. *If $\theta\mathrm{P}[1] \leq P[n]$ then every partition x is a stable coalition structure under the Equilibrium Binding Agreements rule.*

Proof. By Theorem 94, if $\theta\mathrm{P}[1] \leq P[n]$ then $d^* = n$. Hence every partition satisfies the $(x_i \leq d^*)_{i=1}^k$ condition of Proposition 95 and the result follows. □

This multiplicity of EBAs is not a surprise: the EBA allows for splits of modules but does not allow mergers between modules. So when the grand coalition is efficient and we start at a general partition x, then the only feasible change is for a module to split, and that takes the system further from the efficient partition. In contrast, when the atomistic partition is efficient, we can derive the uniqueness of the atomistic EBA from first principles:

Proposition 97. *If* $\theta \mathrm{P}[1] > P[n]$ *then only the atomistic partition* $\{1, 1, 1...1\}$ *is a stable coalition structure under the Equilibrium Binding Agreements rule.*

Proof. The atomistic partition is stable as it allows no internal coalitional deviations.[30] To show uniqueness, consider some other partition x, which must have at least one non-singleton module. Through a choice of internal coalitional deviations $\{1, 1, 1...1\}$ is RV-reachable from x. And as $\{1, 1, 1...1\}$ is the unique Pareto efficient partition it RV-dominates x. Hence, due to the internal stability property of stable sets, x cannot be a stable coalition structure. $\square$

With the EBA modules can split, but not merge, resulting in a structural bias towards the atomistic partition. So, in particular, for all payoff functions by definition the atomic partition is always an EBA. However, firstly the convention is to pick the coarsest EBA as the solution. See both Bloch and Dutta (2010), "The 'solution' of the game is the set of coarsest EBAs.", and Ray (2007), "Typically, many coalition structures admit EBAs. Which of these should be considered as *the* set of EBAs for the game? The answer to this question depends on what we consider to be the 'initial' coalition structure under which negotiations commence. In keeping with the spirit of our exercise, which is to understand the outcomes of free and unconstrained negotiation, we take it that the initial structure is the grand coalition itself. Under this supposition, it is natural to focus on the set of equilibrium binding agreements for the grand coalition, or, if this set is empty, on the next level of refinement for which the set of EBAs is nonempty." With this convention, for the standard model, under the generic case of a unique efficient partition, then the solution EBA is the efficient partition. Secondly if we had an approach, which was the reverse of the EBA, (where the initial partition

[30]Definition 86 defines EBAs in terms of stable sets. In an earlier equivalent formulation, Ray and Vohra (1997), the atomistic partition is directly defined to be stable.

is the atomistic partition; whilst, mergers, but not splits, are allowed), then my conjecture is that we would get the opposite result: a pro-Grand Coalition bias.

2.4 Social Planner Preferences

2.4.1 Intertemporal Model

The standard model assumed that the system was already at the limiting state of the Markov process of shocks. This section shows that the results are robust to considering a model with a uniform inter-temporal distribution of business opportunities. It considers a model, where welfare is summed over all time periods, rather than being evaluated just at the limiting state: this is the only difference the intertemporal model has compared with the standard model.

Appendix I derives the form of vc_1, the intertemporal expected utility function for a member of module 1; shows it to be of the *ratio quadratic* form defined in appendix H; and hence proves that for all parametrisations, vc_1 is either quasi-convex, or quasi-concave (or monotonic and hence both quasi-convex and quasi-concave). This leads to the following characterisation:

Table 2.53: Intertemporal Model Characterisation Table

condition on vc_1		Stationary Points	vc_1 description	x_1^*
$vc_1[1]<vc_1[2]$	$vc_1[n-1]<vc_1[n]$	none	increasing	n
$vc_1[1]<vc_1[2]$	$vc_1[n-1]=vc_1[n]$	1 local max in range $[n-1,n]$	quasi-concave	$n-1$ and n
$vc_1[1]<vc_1[2]$	$vc_1[n-1]>vc_1[n]$	1 local max	quasi-concave	interior[31]
$vc_1[1]=vc_1[2]$	$vc_1[n-1]<vc_1[n]$	1 local min in range $[1,2]$	quasi-convex	n
$vc_1[1]=vc_1[2]$	$vc_1[n-1]=vc_1[n]$		impossible	
$vc_1[1]=vc_1[2]$	$vc_1[n-1]>vc_1[n]$	1 local max in range $[1,2]$	quasi-concave	1 and 2
$vc_1[1]>vc_1[2]$	$vc_1[n-1]<vc_1[n]$	1 local min in range $[2,n-1]$	quasi-convex	1 or n or both
$vc_1[1]>vc_1[2]$	$vc_1[n-1]=vc_1[n]$	1 local min in range $[n-1,n]$	quasi-convex	1
$vc_1[1]>vc_1[2]$	$vc_1[n-1]>vc_1[n]$	none	decreasing	1

[31] Either the floor or the ceiling of the stationary point

Now, we evaluate vc_1, bank 1's utility, for a range of different parametrisations, $(\beta, n, \theta, q, \rho)$. There does not appear to be a simple closed form solution for the vc_1 maximisation program, so the behaviour needs to be investigated computationally; however, there are four reasons why this is not onerous. Firstly, here we do have closed forms for the expected utility function vc_1, and so we can directly assess the value of each policy option *without* using a Monte Carlo approach; conversely, in the propagation mechanism literature, there is no closed form for the expected utility function, and so it needs to be sampled from the data generating process using a Monte Carlo approach. Secondly, table 2.53 shows that $x_1^* := \text{argmax}_{x_1 \in N} vc_1[x_1]$ is strictly interior, if and only if, both $vc_1[1] < vc_1[2]$ and $vc_1[n-1] > vc_1[n]$: we do not need to evaluate vc_1 for all possible partitions in order to select parametrisations with interior solutions. Thirdly, there are only $n-2$ interior cases to evaluate, so the computational search cost is linear in parameters.[32] Fourthly, when vc_1 has an interior solution, that solution is also the *unique* local maximum: so once a local maximum is found the search can be terminated.[33]

For each parametrisation, the utility of the best trivial partition is given by $\text{v}^B := \text{Max}\{vc_1[1], vc_1[\text{n}]\}$; the utility of the best interior partition is given by $vc^I := \text{Max}\{\text{vc}_1[x_1]\}_{x_1=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $g := 100 * (v^I - v^B)/v^B$. Without loss of generality, we can consider one period to be a year, and we assume that the discount factor β is 0.97, 0.98 or 0.99.[34] The number of banks, n, is one of {10, 20, 30, 40, 50, 100, 1000}. The value of outside matches, θ, has a minimum of 0.1, a maximum

[32] Certain well known computer science problems, for example the traveling salesman problem, are believed to only have solutions that are *exponential* in parameters: making it impractical to find their solutions for large examples.

[33] This is because in order for vc_1 to have an interior maximum, it is necessary that it is quasi-concave.

[34] This is without loss of generality as the argmax is homogenous of degree 0 with respect to jointly $Log[\beta]$, $Log[1-q]$ and $Log[1-\rho]$.

of 0.9 and an increment of 0.1. The disabling shock probability, q, comes from this list of values, $\{0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$; and the enabling probability, ρ, independently comes from this list, $\{0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.3\}$. The β parameter is one of 3 values, the n parameter is one of 7 values, the θ parameter is one of 9 values, the q parameter is one of 19 values, and the ρ parameter is one of 21 values. This gives a total of $3 * 7 * 9 * 19 * 21 = 75411$ parametrisations.

In nearly all these parametrisations (74601 out of 75411), vc_1 has a boundary solution and, in such cases, as argued earlier by symmetry, this solution also then solves the welfare maximisation program. In contrast, only 810 (1.7%) of these cases have interior solutions. Further, there is only ever a small gain to bank 1 from choosing an interior solution: the highest gain in the table is 3.6%.[35] Due to the existence of negative externalities, we know that at interior solutions, banks not in module 1 lose out: they prefer $\{1, 1, \ldots, 1, 1, 1, 1\}$ to $\{x_1^*, 1, 1, \ldots, 1, 1\}$, and hence prefer $\text{argmax}_{x_1 \in \{1,n\}} vc_1[x_1]$ to x_1^*. So, when comparing interior and boundary partitions, the bank 1 utility percentage gain, is an upper bound to the welfare percentage gain, and it is possible that even when vc_1 does have an interior solution that Wc does not: the costs of the losers may dominate the benefits of the winners. In summary, with the intertemporal model, the welfare program rarely has interior solutions, and when they do occur they only give a small increase in welfare. Hence, the conclusion is that the standard model rejection of interior partitions is robust to replacing asymptotic preferences with intertemporal preferences.

The results can be further broken down in term of each of the five choice parameters. These show that interior solutions are most likely to occur for: low

[35] This occurs with ($\beta = 0.97$,$n = 20$, $\theta = 0.1$, $q = 0.02$, $\rho = 0.001$)

number of banks, n; low levels of θ, the value of outside matches; low levels of ρ, the re-enablement probability; low levels of β, the discount factor; and for intermediate values of q, the disabling shock probability.

Table 2.54: Intertemporal Model n Table

	Overall gain percentage distribution broken down by n								
n	(-100,0)	[0,0.5)	[0.5,1)	[1,1.5)	[1.5,2)	[2,2.5)	[2.5,3)	[3,3.5)	[3.5,4)
10	10463	160	62	35	26	20	7	0	0
20	10581	105	38	20	15	6	5	2	1
30	10655	69	22	13	7	3	3	1	0
40	10680	52	18	8	7	4	4	0	0
50	10704	41	14	5	6	2	1	0	0
100	10745	21	6	1	0	0	0	0	0
1000	10773	0	0	0	0	0	0	0	0
All	74601	448	160	82	61	35	20	3	1

Table 2.55: Intertemporal Model θ Table

	Overall gain percentage distribution broken down by θ								
θ	(-100,0)	[0,0.5)	[0.5,1)	[1,1.5)	[1.5,2)	[2,2.5)	[2.5,3)	[3,3.5)	[3.5,4)
0.1	7994	161	84	48	44	25	19	3	1
0.2	8179	113	45	18	15	8	1	0	0
0.3	8272	73	17	13	2	2	0	0	0
0.4	8316	50	10	3	0	0	0	0	0
0.5	8347	28	4	0	0	0	0	0	0
0.6	8363	16	0	0	0	0	0	0	0
0.7	8374	5	0	0	0	0	0	0	0
0.8	8377	2	0	0	0	0	0	0	0
0.9	8379	0	0	0	0	0	0	0	0
All	74601	448	160	82	61	35	20	3	1

Table 2.56: Intertemporal Model q Table

	Overall gain percentage distribution broken down by q								
q	(-100,0)	[0,0.5)	[0.5,1)	[1,1.5)	[1.5,2)	[2,2.5)	[2.5,3)	[3,3.5)	[3.5,4)
0.001	3947	21	1	0	0	0	0	0	0
0.002	3931	33	5	0	0	0	0	0	0
0.003	3914	38	13	2	2	0	0	0	0
0.004	3919	30	12	7	0	1	0	0	0
0.005	3916	30	11	8	2	1	1	0	0
0.006	3915	30	12	5	6	0	1	0	0
0.007	3905	36	11	6	6	3	2	0	0
0.008	3899	35	14	8	5	7	0	1	0
0.009	3900	33	14	9	5	4	2	2	0
0.01	3899	34	13	8	9	2	4	0	0
0.02	3917	23	6	6	8	3	5	0	1
0.03	3927	23	9	3	4	3	0	0	0
0.04	3933	14	9	6	4	2	1	0	0
0.05	3936	12	8	3	3	5	2	0	0
0.06	3939	14	5	4	3	2	2	0	0
0.07	3945	14	4	3	1	2	0	0	0
0.08	3948	12	5	3	1	0	0	0	0
0.09	3953	10	4	1	1	0	0	0	0
0.1	3958	6	4	0	1	0	0	0	0
All	74601	448	160	82	61	35	20	3	1

Table 2.57: Intertemporal Model ρ Table

	Overall gain percentage distribution broken down by ρ								
ρ	(-100,0)	[0,0.5)	[0.5,1)	[1,1.5)	[1.5,2)	[2,2.5)	[2.5,3)	[3,3.5)	[3.5,4)
0.001	3530	21	15	9	4	8	3	0	1
0.002	3531	24	10	7	8	3	6	2	0
0.003	3531	22	14	9	8	4	2	1	0
0.004	3525	33	14	6	4	6	3	0	0
0.005	3529	31	11	7	8	3	2	0	0
0.006	3534	25	16	7	6	2	1	0	0
0.007	3538	28	11	6	5	2	1	0	0
0.008	3539	26	11	10	1	3	1	0	0
0.009	3541	29	9	4	5	2	1	0	0
0.01	3540	27	10	6	7	1	0	0	0
0.02	3547	23	12	5	3	1	0	0	0
0.03	3557	25	6	2	1	0	0	0	0
0.04	3567	18	4	1	1	0	0	0	0
0.05	3568	19	3	1	0	0	0	0	0
0.06	3567	20	3	1	0	0	0	0	0
0.07	3569	18	3	1	0	0	0	0	0
0.08	3572	16	3	0	0	0	0	0	0
0.09	3570	18	3	0	0	0	0	0	0
0.1	3575	14	2	0	0	0	0	0	0
0.2	3582	9	0	0	0	0	0	0	0
0.3	3589	2	0	0	0	0	0	0	0
All	74601	448	160	82	61	35	20	3	1

Table 2.58: Intertemporal Model β Table

	Overall gain percentage distribution broken down by β								
β	(-100,0)	[0,0.5)	[0.5,1)	[1,1.5)	[1.5,2)	[2,2.5)	[2.5,3)	[3,3.5)	[3.5,4)
0.97	24807	179	66	36	24	14	10	0	1
0.98	24862	155	52	28	18	15	6	1	0
0.99	24932	114	42	18	19	6	4	2	0
All	74601	448	160	82	61	35	20	3	1

2.4.2 Risk Aversion

2.4.2.1 Risk Averse Social Planner

Suppose we now have a risk averse social planner, but we keep the same distribution for $b[x]$, the business returns as a function of the partition x, as in the standard model:

$$b[\theta] \sim \begin{cases} 0 & 1 - P_I[x] - P_O[x] \\ \theta & P_O[x] \\ 1 & P_I[x] \end{cases}$$

Here $P_I[x]$ and $P_O[x]$ are the probabilities of enabled inside and outside matches: as in the standard model, they are given by $P_I[x] = \sum_{i=1}^{k}(\frac{x_i}{n})^2 P[x_i]$ and $P_O[x] = \sum_{i=1}^{k}\sum_{j\neq i}(\frac{x_i}{n})(\frac{x_j}{n})P[x_i]P[x_j]$; note specifically, that these functions are independent of θ. We model the social planner as penalising high variances in business returns, and having a quadratic utility function, $W_{RA}[x] := E[b[\theta]] - \lambda \text{Var}[b[\theta]]$, where $1 > \lambda > 0$ is a fixed parameter.

Theorem 98. *A necessary condition for this program to have an interior solution is that* $\theta < \frac{P[n]}{P[1]} < \frac{\theta - \lambda\theta^2}{1-\lambda}$

Proof. As $\text{Var}[b] = E[b[\theta]^2] - E^2[b[\theta]]$, this means that $W_{RA}[x] = E[b[\theta]] - \lambda E[b^2[\theta]] + \lambda E^2[b[\theta]]$. Further, $b^2[\theta]$ has the same distribution as $b[\theta^2]$, and hence $W_{RA}[x] = E[b[\theta]] - \lambda E[b[\theta^2]] + \lambda E^2[b[\theta]]$. Simplifying using and letting $W[x,\theta]$ be the welfare for the standard model with x as the partition and θ as the value of outside matches, gives: $W_{RA}[x] = (1-\lambda)W[x, \frac{\theta-\lambda\theta^2}{1-\lambda}] + \lambda(W[x,\theta])^2$.[36]

As $\lambda < 1$, we have an average of standard programs each with positive weights. We know that standard programs have boundary solutions. So if both the $\frac{\theta-\lambda\theta^2}{1-\lambda}$

[36]The key step is that $E[b[\theta]] = \sum_{i=1}^{k}(\frac{x_i}{n})^2 P[x_i] + \theta\sum_{i=1}^{k}\sum_{j\neq i}(\frac{x_i}{n})(\frac{x_j}{n})P[x_i]P[x_j]$ and $E[b[\theta^2]] = \sum_{i=1}^{k}(\frac{x_i}{n})^2 P[x_i] + \theta^2\sum_{i=1}^{k}\sum_{j\neq i}(\frac{x_i}{n})(\frac{x_j}{n})P[x_i]P[x_j]$. So $E[b[\theta]] - \lambda E[b[\theta^2]] = (1-\lambda)\sum_{i=1}^{k}(\frac{x_i}{n})^2 P[x_i] + \theta(1-\theta\lambda)\sum_{i=1}^{k}\sum_{j\neq i}(\frac{x_i}{n})(\frac{x_j}{n})P[x_i]P[x_j]$

and θ standard model programs have the *same* boundary solution, then the risk averse $W_{RA}[x]$ will also have that boundary solution. Conversely, in order for the risk averse program to have an interior solution it is necessary for the 2 standard programs to have different boundary solutions. Recall that in the standard model, the grand coalition is strictly preferred if and only if $P[n] > \theta P[1]$. Further as $\lambda < 1$, we have $\frac{\theta - \lambda\theta^2}{1-\lambda} > \theta$. So in order for the risk averse program to have an interior solution, it is necessary that the standard model θ program has a grand coalition solution, and that the θ program has an atomistic solution. This equates to the required condition, $\theta < \frac{P[n]}{P[1]} < \frac{\theta - \lambda\theta^2}{1-\lambda}$.

□

The risk aversion model was assessed for a range of parameterisations. As the welfare function is specified directly at the social planner level rather than at the individual bank level, the welfare function is used rather than the per bank utility function. The number of banks n, has a minimum of 3, a maximum of 100 and an increment of 1. The value of outside matches, θ, has a minimum of 0, a maximum of 1 and an increment of 0.1. The shock parameter γ has a minimum of 0, a maximum of 3 and an increment of 0.1. The risk aversion parameter, λ, has a minimum of 0, a maximum of 1.0 and an increment of 0.1. This gives a total of $98 * 11 * 31 * 11 = 367598$ different parametrisations. Only 7925 of these cases satisfy the $\theta < \frac{P[n]}{P[1]} < \frac{\theta - \lambda\theta^2}{1-\lambda}$ condition. In each case the best interior symmetric partition was compared to the best boundary partition. In every case the best interior partition had lower welfare than the best boundary partition. Hence, the conclusion is that the standard model rejection of interior partitions is robust to changing the social planner from being risk neutral to having quadratic risk aversion. Further, the absence of interior solutions leads to the following conjecture:

Conjecture 99. *The risk averse social planner program has boundary solutions for all parameter values.*

The following projection tables are produced using the same parameter ranges as above, except that n is restricted to the standard range, $(10, 20, 30, 40, 50, 100, 1000)$. For each parametrisation, the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{RA}[1], \mathrm{W}_{RA}[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_{RA}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100 * (W^i - W^b)/W^b$. When either $\lambda = 0$ or $\lambda = 1$ the social planner has preferences equal to those given in the θ parametrisation of the standard model. However, with $\lambda = 1$ the welfare function is the square of the standard model's utility function, and so the losses from an interior partition are larger. This motivates why in the λ table, when λ is larger, the losses are larger. In the n table, as n increases the loss from choosing an interior partition decreases. This is because for nearly all parametrisations the best boundary partition is the atomistic partition, and the nearest interior partition occurs with $k = n - 1$ modules; so as n increases the nearest interior partition gets closer to the atom partition. In the γ table, when $\gamma = 0$, the efficient partition is the grand coalition and the closest interior partition has 2 modules each with $0.5n$ members: this has outside matches half the time and so there is a big drop in welfare. For small, but strictly positive γ, the efficient partition rapidly switches to become the atomistic partition, and the $n-1$ module interior partition can get close to it, resulting in smaller losses. However, as γ increases, the cost of larger modules increases, and so the loss from interior partitions increases.

Table 2.59: Risk Aversion Model λ Table

	Overall gain percentage distribution broken down by λ							
λ	[-75,-50)	[-20,-10)	[-10,-5)	[-5,-4)	[-4,-3)	[-3,-2)	[-2,-1)	[-1,0]
0.0	49	54	308	162	289	430	511	584
0.1	58	56	327	146	291	435	506	568
0.2	58	66	329	158	286	444	492	554
0.3	58	70	340	168	291	442	477	541
0.4	58	84	350	176	276	464	449	530
0.5	58	105	356	177	287	452	433	519
0.6	58	132	343	191	312	429	414	508
0.7	61	166	331	221	326	384	401	497
0.8	61	199	367	218	324	337	397	484
0.9	63	230	407	215	326	285	387	474
1.0	67	289	462	199	321	269	338	442
All	649	1451	3920	2031	3329	4371	4805	5701

Table 2.60: Risk Aversion Model n Table

	Overall gain percentage distribution broken down by n							
n	[-75,-50)	[-20,-10)	[-10,-5)	[-5,-4)	[-4,-3)	[-3,-2)	[-2,-1)	[-1,0]
10	149	1168	1725	250	251	142	53	13
20	103	92	1780	630	454	362	268	62
30	87	56	272	990	1086	645	477	138
40	80	48	62	128	1253	1245	673	262
50	78	40	47	22	273	1934	963	394
100	76	25	23	10	12	43	2367	1195
10^3	76	22	11	1	0	0	4	3637
All	649	1451	3920	2031	3329	4371	4805	5701

Table 2.61: Risk Aversion Model θ Table

	Overall gain percentage distribution broken down by θ							
θ	[-75,-50)	[-20,-10)	[-10,-5)	[-5,-4)	[-4,-3)	[-3,-2)	[-2,-1)	[-1,0]
0.0	150	149	283	135	204	260	500	706
0.1	110	83	158	105	187	450	710	584
0.2	88	62	209	158	339	535	463	533
0.3	77	69	305	207	311	482	418	518
0.4	77	77	358	195	320	447	412	501
0.5	77	82	406	180	334	418	401	489
0.6	70	109	423	185	334	397	390	479
0.7	0	201	426	201	328	376	389	466
0.8	0	215	434	216	327	350	388	457
0.9	0	175	495	227	334	332	376	448
1.0	0	229	423	222	311	324	358	520
All	649	1451	3920	2031	3329	4371	4805	5701

Table 2.62: Risk Aversion Model γ Table

	Overall gain percentage distribution broken down by γ							
γ	[-75,-50)	[-20,-10)	[-10,-5)	[-5,-4)	[-4,-3)	[-3,-2)	[-2,-1)	[-1,0]
0.0	532	154	77	7	0	0	0	77
0.1	71	65	17	1	5	27	91	570
0.2	27	52	30	25	34	55	201	423
0.3	11	37	69	31	47	95	248	309
0.4	4	31	93	38	56	127	247	251
0.5	2	23	108	43	81	136	222	232
0.6	1	23	115	49	94	147	197	221
0.7	1	23	118	61	91	173	172	208
0.8	0	28	119	64	98	187	155	196
0.9	0	27	131	66	103	190	150	180
1.0	0	30	131	68	114	184	152	168
1.1	0	32	136	72	107	199	150	151
1.2	0	34	140	66	118	198	148	143
1.3	0	38	143	64	120	190	153	139
1.4	0	41	145	64	124	186	151	136
1.5	0	44	146	66	125	182	148	136
1.6	0	45	149	66	127	177	148	135
1.7	0	48	148	68	124	170	155	134
1.8	0	49	150	70	129	159	156	134
1.9	0	49	145	78	134	152	156	133
2.0	0	51	143	85	132	147	148	141
2.1	0	51	145	90	128	144	146	143
2.2	0	52	146	89	131	141	145	143
2.3	0	53	145	92	130	138	146	143
2.4	0	53	144	88	136	137	146	143
2.5	0	53	145	89	136	128	146	150
2.6	0	53	146	87	138	125	146	152
2.7	0	53	148	86	140	122	145	153
2.8	0	53	149	87	139	121	145	153
2.9	0	53	149	86	143	118	146	152
3.0	0	53	150	85	145	116	146	152
All	649	1451	3920	2031	3329	4371	4805	5701

2.4.2.2 General Equilibrium Approach

This section shows that a general equilibrium model, under the assumption of complete markets, reduces to a standard model, and thus has boundary solutions. The setup is as follows: firstly, suppose there are n banks, each with T

businesses and that each business has a risk averse utility function $u[.]$. Secondly, compared with the risk neutral model we need 2 parameters, not 1, to model outside matches: let p be the probability that a match can be made between matched businesses in 2 different modules and let v be the value of a successfully completed outside match. This leads to the following welfare function for symmetric matches: $W_{GE}[d] = \frac{d}{n}P[d]nT * u[\frac{1}{nT}] + \frac{(n-d)}{n}P^2[d]pnT * u[\frac{v}{nT}]$.

This can be derived as follows: as the businesses are risk averse and there are complete markets the result will be complete consumption smoothing. So when there is a successful inside match each of the nT businesses gets $\frac{1}{nT}$ units of consumption, whilst when there is a successful outside match each business gets $\frac{v}{nT}$. Dividing by nT and without loss of generality setting $u[\frac{1}{nT}] = 1$, gives: $\frac{W_{GE}[d]}{nT} = \frac{d}{n}P[d] + \frac{(n-d)}{n}P^2[d]pu[\frac{v}{nT}]$. So this program is the same as the standard model with $\theta = pu[\frac{v}{nT}]$, and hence has boundary solutions.[37]

2.4.2.3 Within Module Consumption Smoothing

Suppose that businesses are risk averse and can smooth consumption with other businesses in the same module, but not with businesses in other modules. As in the standard model, n is the number of banks, and γ is the shock parameter. And as in the general equilibrium case in section 2.4.2.2 above, p is the probability that two businesses in distinct enabled modules can match, v is the value of that match once made and $u[.]$ is the utility function, where we assume $u[0] = 0$. Considering symmetric partitions: in the case of an inside match, the return of 1 is shared equally amongst d businesses and so each has utility $u[\frac{1}{d}]$; in the case of an outside match, the return of v is shared equally amongst the $2d$ businesses in the 2 modules and so each has utility $u[\frac{v}{2d}]$. This leads to the following welfare

[37] p is a probability so $0 < p < 1$ and v is the value of an outside match so $0 < v < 1$. We have normalised $u(0) = 0$ and $u(\frac{1}{nT}) = 1$, so $0 < u(\frac{v}{nT}) < 1$. Hence $0 < pu(\frac{v}{nT}) < 1$. So this is a valid value for θ.

function:

$$W_{cs}[d]{:=}(d/n)P[d]d * u[1/d] + p((n-d)/n)P[d]^2 * 2d * u[v/2d]$$

Risk averse preferences are represented by a utility function of form $u[c] = c^{\omega}$, with $0 < \omega < 1$, so there is constant relative risk aversion with a coefficient of relative risk aversion of $1 - \omega$.[38]

This model satisfies negative externalities, and so we consider $v_{1,CS}[x_1]$, the utility that bank 1 gets when it is in a module of size x_1 and all other modules are singletons.[39] The utility function is as follows:

$$\mathrm{v}_{1,cs}[x_1]{:=}(x_1/n)^2 P[x_1](1/x_1)^{\omega} + p(x_1/n)((n-x_1)/n)2P[x_1]P[1](0.5v/x_1)^{\omega}$$

The first term represents inside matches: there is an inside match involving module one with probability $(x_1/n)^2$, and then each of the module 1 businesses gets a utility of $(1/x_1)^{\omega}$. The second term represents outside matches: this requires ones of the selected modules to be module 1, a x_1/n probability event, and one module is not module 1, a $(n-x_1)/n$ probability event. The 2 is present because module 1 could be either the first or the second module selected. If the match is completed, then the value v is shared equally between the 2 modules, and then each of the module 1 businesses gets a utility of $(0.5v/x_1)^{\omega}$.

By considering $\frac{\partial v_{1,cs}[x_1]}{\partial x_1}$, we can use our usual technique to see if $\mathrm{v}_{1,cs}[x_1]$ is quasi-

[38] If the coefficient of relative risk aversion is greater than or equal to 1 then we can no longer have $u[0] = 0$, indeed we require $u[0] = -\infty$. This causes difficulties as even in good states of the world when the match is successfully completed, businesses in other modules get no return.

[39] Similarly we know that the model has negative externalities: if an outside module has more than one member then it is less likely to be enabled and bank 1 is worse off.

convex. Re-arrangement gives:

$$\frac{\partial v_{1,cs}[x_1]}{\partial x_1} = \frac{dv_{1,cs}[x_1]P[1]P[x_1]^2}{n^2(x_1^{\omega})}$$

where

$$dv_{1,cs}[x_1] := ax_1^2 + bx_1 + c$$

$$a := (1-\omega)\gamma\,(\gamma + 1 - 2p\upsilon)$$

$$b := (-\gamma\omega + 2\gamma - 2\gamma np\upsilon\omega + 2p\upsilon\omega - 4p\upsilon - \omega + 2)$$

$$c := 2np\upsilon(1-\omega)$$

$$\upsilon := (0.5v)^{\omega}$$

Theorem 100. *If $a \neq 0$ then $v_{1,cs}[x_1]$ either has boundary solutions or an interior solution at $x_1 = \frac{-b-\sqrt{(b^2-4ac)}}{2a}$.*

Proof. As $\frac{\partial v_{1,cs}[x_1]}{\partial x_1} = 0$ if and only if $dv_{1,cs}[x_1] = 0$, the roots of $dv_{1,cs}[x_1]$ equate to the first order points of $v_{1,cs}[x_1]$. So if the quadratic function $dv_{1,cs}[x_1]$ does not have any roots in the range $[1, n]$ then $\mathrm{v}_{1,cs}[x_1]$ is monotonic and has boundary solutions. The function $dv_{1,cs}[x_1]$ has an x_1^2 term whose coefficient can either be positive or negative. If $a > 0$ then $x_{1-} := \frac{-b-\sqrt{(b^2-4ac)}}{2a}$ is a local maximum and $x_{1+} := \frac{-b+\sqrt{(b^2-4ac)}}{2a}$ is a local minimum: as $x_1 < x_{1-} \Rightarrow dv_{1,cs}[x_1] > 0$, $x_{1-} < x_1 < x_{1+} \Rightarrow dv_{1,cs}[x_1] < 0$ and $x_1 > x_{1+} \Rightarrow dv_{1,cs}[x_1] > 0$. So comparing $dv_{1,cs}[x_{1-}]$ with $dv_{1,cs}[1]$ and $dv_{1,cs}[n]$ shows if the $v_{1,cs}$ program has an interior solution or not.[40] If $a < 0$ then $x_{1-} := \frac{-b-\sqrt{(b^2-4ac)}}{2a}$ is a local maximum and $x_{1+} := \frac{-b+\sqrt{(b^2-4ac)}}{2a}$ is a local minimum: note that as $a < 0$ that $x_{1+} < x_{1-}$, so $x_1 < x_{1+} \Rightarrow dv_{1,cs}[x_1] < 0$, $x_{1+} < x_1 < x_{1-} \Rightarrow dv_{1,cs}[x_1] > 0$ and $x_1 > x_{1-} \Rightarrow$

[40] If x_{1-} is complex or outside the range $[1, n]$ then we already know that there is no interior solution.

$dv_{1,cs}[x_1] < 0$. Again comparing $dv_{1,cs}[x_{1-}]$ with $dv_{1,cs}[1]$ and $dv_{1,cs}[n]$ shows if the $v_{1,cs}$ program has an interior solution or not. $\square$

Theorem 101. *If $a = 0$ then $v_{1,cs}[x_1]$ either has boundary solutions or an interior solution where $\omega = P[x_1]$.*

Proof. If $a = 0$ then $(\gamma + 1 - 2pv) = 0$, so $b = -\gamma n(1+\gamma)\omega < 0$, and $c = n(1+\gamma)(1-\omega) > 0$. Hence $dv_{1,cs}[x_1] < 0$ if and only if $bx_1 + c < 0$ if and only if $-\gamma\omega x_1 + (1-\omega) < 0$. Once this condition holds, it stays true for larger x_1, so $v_{1,cs}[x_1]$ is quasi-concave (which is expected given the signs of b and c). Hence $v_{1,cs}[x_1]$ has a maximum when $x_1 = \frac{(1-\omega)}{\gamma\omega}$, or equivalently when $\omega = P[x_1]$. So an interior solution requires $1 < \frac{(1-\omega)}{\gamma\omega} < n$. $\square$

We now consider the within module smoothing model for a range of different parametrisations. The number of banks n, is either 10, 20, 30, 40, 50, 100 or 1000. The shock parameter γ has a minimum of 0.1, a maximum of 3 and an increment of 0.1. The outside match completion probability, p, has a minimum of 0, a maximum of 1.0 and an increment of 0.1. The outside match completion value, v, has a minimum of 0, a maximum of 1.0 and an increment of 0.1. The risk aversion parameter, ω, has a minimum of 0.1, a maximum of 1.0 and an increment of 0.1. This gives a total of $7 * 30 * 11 * 11 * 10 = 254100$ different parametrisations.

As noted in Theorem 100 above, we only need to check one value of x_1 for a potential interior global maximum of the $v_{1,cs}$ program. If the $v_{1,cs}$ program has an interior solution, then the $W_{cs}[d]$ program is checked at each of its interior partitions. In each of these cases the percentage *gain* from the best interior symmetric partition over the best boundary partition was computed: the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{cs}[1], \mathrm{W}_{cs}[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_{cs}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100*(W^i - W^b)/W^b$. In total 20445 cases were found where the $v_{1,cs}$ program has an interior solution; this total reduced to 2341 (0.92%) cases where the $W_{1,cs}[d]$ program has an interior solution. This rareness of interior solutions leads to the conclusion that the standard model rejection of interior partitions is robust to changing businesses from being risk neutral, to having constant relative risk aversion and being able to consumption smooth within their own module.

These cases are represented below in five tables that show how the gain varies with the value of each of the five parameters. In the ω table, there are no gains from interior partitions with either $\omega = 0.1$ or with $\omega = 0.9$. The explanation is that the coefficient of relative risk aversion is $1 - \omega$; so with $\omega = 0.9$, the level of risk aversion is low and we are close to the risk neutral model ($\omega = 1.0$) with its boundary solutions; whilst with $\omega = 0.1$, there is a high level of risk aversion which makes the businesses highly value something over nothing, and creates a preference for the grand coalition, as then the business is certain to be part of the module that receives the business opportunity.[41] In the γ table, with the shock parameter γ in the range 0.7 to 3.0 no examples of interior welfare solutions are found. This can be explained as being due to the low enablement probability of of non-singleton modules when the shock parameter is high. The other tables show that conversely increases in the number of banks n, the outside match completion probability,p, and the outside match completion value v, each lead to more interior solutions:

[41] With $\omega = 0.1$ the grand coalition is always preferred to the partition of singletons.

Table 2.63: Consumption Smoothing Model ω Table

ω	Overall gain percentage distribution broken down by ω									
	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,70)
0.1	25410	0	0	0	0	0	0	0	0	0
0.2	25272	79	31	9	15	1	2	1	0	0
0.3	25075	199	68	7	4	39	2	2	3	11
0.4	24939	319	16	87	3	4	0	2	4	36
0.5	24918	353	52	6	5	68	8	0	0	0
0.6	25017	253	8	132	0	0	0	0	0	0
0.7	25163	221	26	0	0	0	0	0	0	0
0.8	25145	265	0	0	0	0	0	0	0	0
0.9	25410	0	0	0	0	0	0	0	0	0
1.0	25410	0	0	0	0	0	0	0	0	0
All	251759	1689	201	241	27	112	12	5	7	47

Table 2.64: Consumption Smoothing Model γ Table

	Overall gain percentage distribution broken down by γ									
γ	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,70)
0.1	7631	487	51	147	10	76	10	4	7	47
0.2	7851	420	65	90	5	36	2	1	0	0
0.3	8026	366	62	4	12	0	0	0	0	0
0.4	8194	253	23	0	0	0	0	0	0	0
0.5	8341	129	0	0	0	0	0	0	0	0
0.6	8436	34	0	0	0	0	0	0	0	0
0.7	8470	0	0	0	0	0	0	0	0	0
0.8	8470	0	0	0	0	0	0	0	0	0
0.9	8470	0	0	0	0	0	0	0	0	0
1.0	8470	0	0	0	0	0	0	0	0	0
1.1	8470	0	0	0	0	0	0	0	0	0
1.2	8470	0	0	0	0	0	0	0	0	0
1.3	8470	0	0	0	0	0	0	0	0	0
1.4	8470	0	0	0	0	0	0	0	0	0
1.5	8470	0	0	0	0	0	0	0	0	0
1.6	8470	0	0	0	0	0	0	0	0	0
1.7	8470	0	0	0	0	0	0	0	0	0
1.8	8470	0	0	0	0	0	0	0	0	0
1.9	8470	0	0	0	0	0	0	0	0	0
2.0	8470	0	0	0	0	0	0	0	0	0
2.1	8470	0	0	0	0	0	0	0	0	0
2.2	8470	0	0	0	0	0	0	0	0	0
2.3	8470	0	0	0	0	0	0	0	0	0
2.4	8470	0	0	0	0	0	0	0	0	0
2.5	8470	0	0	0	0	0	0	0	0	0
2.6	8470	0	0	0	0	0	0	0	0	0
2.7	8470	0	0	0	0	0	0	0	0	0
2.8	8470	0	0	0	0	0	0	0	0	0
2.9	8470	0	0	0	0	0	0	0	0	0
3.0	8470	0	0	0	0	0	0	0	0	0
All	251759	1689	201	241	27	112	12	5	7	47

Table 2.65: Consumption Smoothing Model n Table

	Overall gain percentage distribution broken down by n									
n	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,70)
10	36287	13	0	0	0	0	0	0	0	0
20	36233	64	3	0	0	0	0	0	0	0
30	36170	113	13	4	0	0	0	0	0	0
40	36111	161	17	11	0	0	0	0	0	0
50	36057	199	25	17	1	1	0	0	0	0
100	35854	335	48	49	2	10	2	0	0	0
1000	35047	804	95	160	24	101	10	5	7	47
All	251759	1689	201	241	27	112	12	5	7	47

Table 2.66: Consumption Smoothing Model p Table

	Overall gain percentage distribution broken down by p									
p	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,70)
0	23100	0	0	0	0	0	0	0	0	0
0.1	23086	14	0	0	0	0	0	0	0	0
0.2	23057	38	0	5	0	0	0	0	0	0
0.3	23022	63	2	9	1	1	2	0	0	0
0.4	22980	94	3	16	1	5	1	0	0	0
0.5	22931	127	12	17	1	8	1	1	1	1
0.6	22881	156	20	23	2	11	1	1	1	4
0.7	22810	207	30	26	2	17	1	0	1	6
0.8	22725	266	39	34	3	20	2	1	1	9
0.9	22629	333	44	48	7	22	2	1	2	12
1.0	22538	391	51	63	10	28	2	1	1	15
All	251759	1689	201	241	27	112	12	5	7	47

Table 2.67: Consumption Smoothing Model v Table

	Overall gain percentage distribution broken down by v									
v	(-100,0)	[0,5)	[5,10)	[10,15)	[15,20)	[20,25)	[25,30)	[30,35)	[35,40)	[40,70)
0	23100	0	0	0	0	0	0	0	0	0
0.1	23029	56	2	10	1	1	1	0	0	0
0.2	22994	76	7	13	0	7	1	0	1	1
0.3	22953	105	12	16	0	10	1	0	0	3
0.4	22917	132	16	16	4	10	0	0	2	3
0.5	22878	163	17	22	2	10	1	2	0	5
0.6	22843	185	23	25	4	13	0	0	2	5
0.7	22809	207	30	27	4	15	1	0	0	7
0.8	22773	238	27	33	4	13	3	2	0	7
0.9	22742	258	33	35	5	16	2	0	2	7
1.0	22721	269	34	44	3	17	2	1	0	9
All	251759	1689	201	241	27	112	12	5	7	47

We have only been able to consider cases where the coefficient of relative risk aversion is less than 1. So for robustness reasons the constant absolute risk aversion case is now considered. With absolute risk aversion $u[c] = 1 - \exp(-\alpha c)$. Note that the coefficient of absolute risk aversion is $-\frac{u''[c]}{u'[c]} = -\frac{-\alpha^2 exp[-\alpha c]}{-\alpha exp[-\alpha c]} = -\alpha$, and so the requirement for agents to be risk averse is that $\alpha > 0$. Outside matches are modelled using 2 parameters p and v: p is the probability that an outside match is successfully completed and v is the gross value of the outside match. With an inside match the value of 1 is spread equally amongst the x_i members; whilst with an outside match the value of $0.5v$ is shared amongst the members of each the matched modules x_i and x_j. This leads to the following welfare function:

$$W_{\text{cara}}[x] = \sum_{i=1}^{k} \frac{x_i^2}{n^2}(1 - e^{-\frac{\alpha}{x_i}})P[x_i] + \sum_{i=1}^{k} \frac{x_i}{n} \sum_{j \neq i} p \frac{x_j}{n} \left(x_i(1 - e^{-\frac{\alpha v}{2x_i}}) + x_j(1 - e^{-\frac{\alpha v}{2x_j}}) \right) P[x_i]P[x_j]$$

and for symmetric cases this becomes

$$W_{\text{cara}}[d] = \frac{d}{n} d \left(1 - \exp\left(-\frac{\alpha}{d}\right)\right) P[d] + p\frac{(n-d)}{n} (2d) \left(1 - \exp\left(-\frac{\alpha v}{2d}\right)\right) P^2[d]$$

We consider the absolute risk aversion within module smoothing model for a range of different parametrisations. The number of banks n, is either 10, 20, 30, 40, 50, 100 or 1000. The shock parameter γ has a minimum of 0.1, a maximum of 3 and an increment of 0.1. The outside match completion probability, p, has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1. The outside match completion value, v, has a minimum of 0.1, a maximum of 0.9 and an increment of 0.1. The risk aversion parameter, α, has a minimum of 0.1, a maximum of 2.0 and an increment of 0.1. This gives a total of $7 * 40 * 9 * 9 * 20 = 453600$ different parametrisations.

In each of these cases the percentage *gain* from the best interior symmetric partition over the best boundary partition was computed: the welfare of the best trivial partition is given by $\mathrm{W}^b := \mathrm{Max}\{\mathrm{W}_{\text{cara}}[1], \mathrm{W}_{\text{cara}}[n]\}$; the welfare of the best symmetric interior partition is given by $\mathrm{W}^i := \mathrm{Max}\{\mathrm{W}_{\text{cara}}[n/k]\}_{k=2}^{n-1}$; and g, the percentage gain from choosing an interior partition, is given by $100*(W^i - W^b)/W^b$. In total 21776 (4.8%) cases were found where the $W_{\text{cara}}[d]$ program has an interior solution, and the highest percentage gain from an internal solution was 31%. This rareness of interior solutions leads to the conclusion that the standard model rejection of interior partitions is robust to changing businesses from being risk neutral, to having constant absolute risk aversion and being able to consumption smooth within their own module.

Table 2.68: Consumption Smoothing Model n Table

	Overall gain percentage distribution broken down by n									
	(-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	[0,10)	[10,20)	[20,30)	[30,40)
10	212	3585	7044	5859	15573	32383	144	0	0	0
20	65	1683	4127	6707	5927	45341	938	12	0	0
30	33	1252	2669	5843	5680	47535	1737	51	0	0
40	20	1022	2006	4307	6405	48523	2407	110	0	0
50	10	869	1703	3358	6616	49129	2959	152	4	0
100	0	840	815	1578	4476	52138	4591	334	28	0
1000	0	840	440	220	100	54891	7552	653	103	1
All	340	10091	18804	27872	44777	329940	20328	1312	135	1

Table 2.69: Consumption Smoothing Model γ Table

	Overall gain percentage distribution broken down by γ									
γ	(-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	[0,10)	[10,20)	[20,30)	[30,40)
0.00	306	5776	2963	1507	717	71	0	0	0	0
0.01	34	1987	3676	2426	1135	757	924	302	98	1
0.02	0	945	3116	2986	1638	811	1476	333	35	0
0.03	0	561	2138	3131	2364	1015	1821	308	2	0
0.04	0	398	1550	3243	2551	1290	2105	203	0	0
0.05	0	247	1250	3018	2828	1621	2261	115	0	0
0.06	0	120	1093	2557	3139	2028	2363	40	0	0
0.07	0	45	901	2228	3271	2526	2359	10	0	0
0.08	0	12	771	1949	3184	3133	2290	1	0	0
0.09	0	0	701	1720	3237	3508	2174	0	0	0
0.1	0	0	625	1491	3357	3841	2026	0	0	0
0.2	0	0	20	867	2414	7539	500	0	0	0
0.3	0	0	0	500	1701	9110	29	0	0	0
0.4	0	0	0	199	1436	9705	0	0	0	0
0.5	0	0	0	49	1254	10037	0	0	0	0
0.6	0	0	0	1	1076	10263	0	0	0	0
0.7	0	0	0	0	923	10417	0	0	0	0
0.8	0	0	0	0	803	10537	0	0	0	0
0.9	0	0	0	0	714	10626	0	0	0	0
1.0	0	0	0	0	641	10699	0	0	0	0
1.1	0	0	0	0	581	10759	0	0	0	0
1.2	0	0	0	0	533	10807	0	0	0	0
1.3	0	0	0	0	490	10850	0	0	0	0
1.4	0	0	0	0	454	10886	0	0	0	0
1.5	0	0	0	0	423	10917	0	0	0	0
1.6	0	0	0	0	391	10949	0	0	0	0
1.7	0	0	0	0	363	10977	0	0	0	0
1.8	0	0	0	0	345	10995	0	0	0	0
1.9	0	0	0	0	317	11023	0	0	0	0
2.0	0	0	0	0	299	11041	0	0	0	0
2.1	0	0	0	0	284	11056	0	0	0	0
2.2	0	0	0	0	264	11076	0	0	0	0
2.3	0	0	0	0	249	11091	0	0	0	0
2.4	0	0	0	0	235	11105	0	0	0	0
2.5	0	0	0	0	222	11118	0	0	0	0
2.6	0	0	0	0	210	11130	0	0	0	0
2.7	0	0	0	0	199	11141	0	0	0	0
2.8	0	0	0	0	188	11152	0	0	0	0
2.9	0	0	0	0	178	11162	0	0	0	0
3.0	0	0	0	0	169	11171	0	0	0	0
All	340	10091	18804	27872	44777	329940	20328	1312	135	1

Table 2.70: Consumption Smoothing Model p Table

	Overall gain percentage distribution broken down by p									
p	(-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	[0,10)	[10,20)	[20,30)	[30,40)
0.1	195	2534	3020	3690	6034	34137	731	54	5	0
0.2	72	2103	2875	3661	6098	34491	1007	77	15	1
0.3	34	1472	2814	3593	5951	35097	1334	91	14	0
0.4	18	1105	2510	3453	5476	35941	1777	107	13	0
0.5	9	859	2089	3327	4920	36795	2256	132	13	0
0.6	7	655	1703	3105	4554	37490	2718	154	14	0
0.7	4	529	1433	2745	4239	38106	3133	195	16	0
0.8	1	435	1306	2264	3968	38662	3509	236	19	0
0.9	0	399	1054	2034	3537	39221	3863	266	26	0
All	340	10091	18804	27872	44777	329940	20328	1312	135	1

Table 2.71: Consumption Smoothing Model v Table

	Overall gain percentage distribution broken down by v									
v	(-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	[0,10)	[10,20)	[20,30)	[30,40)
0.1	189	2525	3030	3684	6038	34886	48	0	0	0
0.2	72	2087	2877	3662	6054	35281	367	0	0	0
0.3	35	1456	2802	3600	5739	35946	822	0	0	0
0.4	19	1103	2489	3455	5211	36603	1516	4	0	0
0.5	10	861	2082	3311	4814	37030	2259	33	0	0
0.6	7	664	1701	3094	4580	37311	2945	98	0	0
0.7	5	538	1443	2732	4379	37504	3577	211	11	0
0.8	3	446	1311	2281	4161	37624	4162	376	36	0
0.9	0	411	1069	2053	3801	37755	4632	590	88	1
All	340	10091	18804	27872	44777	329940	20328	1312	135	1

Table 2.72: Consumption Smoothing Model α Table

	Overall gain percentage distribution broken down by α									
α	(-60,-50)	[-50,-40)	[-40,-30)	[-30,-20)	[-20,-10)	[-10,0)	[0,10)	[10,20)	[20,30)	[30,40)
0.1	0	427	895	1312	1681	18327	38	0	0	0
0.2	0	441	898	1315	1712	18171	143	0	0	0
0.3	1	445	901	1329	1746	17987	271	0	0	0
0.4	1	456	908	1341	1771	17772	431	0	0	0
0.5	4	461	915	1345	1809	17551	595	0	0	0
0.6	4	473	921	1359	1835	17318	770	0	0	0
0.7	7	480	921	1369	1878	17085	940	0	0	0
0.8	7	481	931	1380	1921	16857	1103	0	0	0
0.9	13	489	935	1381	1981	16671	1205	5	0	0
1.0	13	501	939	1383	2038	16506	1285	15	0	0
1.1	15	516	942	1388	2106	16356	1328	29	0	0
1.2	17	519	950	1401	2177	16211	1356	49	0	0
1.3	21	526	952	1416	2268	16055	1379	63	0	0
1.4	25	531	959	1420	2393	15894	1370	88	0	0
1.5	27	535	960	1435	2531	15712	1368	107	5	0
1.6	28	541	970	1439	2715	15471	1372	133	11	0
1.7	33	557	971	1448	2881	15243	1368	164	15	0
1.8	37	565	972	1462	3022	15055	1352	191	24	0
1.9	42	569	980	1467	3124	14908	1331	223	36	0
2.0	45	578	984	1482	3188	14790	1323	245	44	1
All	340	10091	18804	27872	44777	329940	20328	1312	135	1

2.5 Interpretation

The efficiency table 2.73 below assesses the role for interior partitions, and each row represents a different model. Models are split depending on which aspect they differ from the standard model: the business sector, the financial sector or the social planner. Business models are further broken down into 4 sections: circular business networks, hypercube business networks, networks with small numbers of banks, and robust networks. Within each section, models are in order of decreasing criticality: earlier models differ most from the standard model in having a large role for interior partitions; later models have less role for interior partitions.

Criticality is assessed using the percentage gain distribution, which measures the role for interior partitions, and there are three different types of percentage gain calculation represented in the table. In the first type, the $W[x]$ programme, the gain is calculated by assessing the social welfare over all partitions. In the second type, the $W[d]$ programme, the gain is calculated by assessing the social welfare over all symmetric partitions. As the $W[d]$ programme has a smaller feasible set the $W[d]$ gain is weakly less than the $W[x]$ gain, but is definitely a valid measure when it shows criticality. In the third type, the $v_1[x_1]$ programme, the gain is calculated by assessing the utility of bank 1, assessed over partitions of format $\{x_1, 1, 1, 1, 1\}$, where outside modules are all singletons. Due to the presence of negative externalities, bank 1 is weakly better off than banks not in module 1. So the $v_1[x_1]$ programme gain is weakly higher than the $W[d]$ programme gain, but is definitely a valid measure when it shows robustness.

For each model, the accumulative distribution of gains is assessed: firstly, the $(-100, 0]$ column gives the proportion of cases where the optimal partition is not an interior partition; secondly, the $(-100, 10]$ column gives the proportion of cases where either the optimal partition is not an interior partition, or the gain from the interior solution is less than or equal to 10%; and thirdly, the $(-100, 100]$ column gives the proportion of cases where either the optimal partition is not an interior partition, or the gain from the interior solution is less than or equal to 100%. The final column of the table indicates which type of programme ($W[x]$, $W[d]$ or $v_1[x_1]$), the gain was assessed using.

In the original circle model, the size of a singleton module is the same as the match length, and there are many interior solutions (99%). The difficulties with the two boundary partitions are as follows: the atomistic partition has zero probability of an inside match; whilst the grand coalition has all encompassing connections, many of which have no business value as there are no distant business matches, but they do have a financial cost as banks are exposed to contagion. The variable minimum module size model shows that the original model has a minimum module size which results in the largest role for interior partitions. Any change in the size of singleton modules results in the welfare of the atomistic partition strictly increasing: when singleton modules are smaller, then the probability of an inside match is still zero, but exposure to financial shocks is reduced; when singleton modules are bigger, then the atomistic partition moves towards the optimal partition of the original program. With the variable match length model, some of the matches will be of length less than 1 and so will be inside a singleton module, this increases the welfare of the atomistic partition, causes the welfare function to be quasi-convex (rather than quasi-concave) for small module sizes, and makes it harder for interior partitions to deliver improvements.

Four different hypercube models are considered: firstly, there are either 4 banks arranged in a 2-dimensional square, or 8 banks arranged in a 3-dimensional cube; secondly, the matching is either uniform (the match is equally likely to be in each dimension), or non-uniform (the match is more likely to be in certain dimensions). The result is that there are more gains from interior partitions, either when the hypercube is of higher dimension, or when matching is non-uniform: with more dimensions, there are more interior partitions to compare against; with non-uniform matching, the interior partition can contain the dimensions where the business match is likely to be made inside the module, and have the dimensions less likely to to offer business matches outside the module.

With a small number of banks, and non-uniform matching, it is often the case that an interior partition is best. Conversely, the robust business networks section shows five variations to the standard model which result in models where the total rejection of interior partitions remains. For four of the models, the rejection is proven: Increased Probability of Self Matching, Multiple Businesses per Bank, and the Star Business Network and Temporal distribution of business opportunities. For the fifth, Trilateral Business Matches, the result holds for all the sampled cases, but is not yet proven.

Concerning the financial sector, letting larger modules have an increased probability of an initial banking shock, does not result in significant gains for interior partitions, this is because: firstly, the welfare of the atomistic partition is unaltered; secondly, the welfare of all other partitions is reduced; and thirdly the atomistic partition is preferred to the grand coalition, for nearly all parametrisations of the standard model. Considering the social planner, the variant models show robustness to the rejection result, both when intertemporal utility is considered and when risk aversion is included.

A significant correlation is between model symmetry and the existence of interior solutions. Here symmetry can be interpreted as occurring when the unconditional and conditional distributions of the second business in a match are the same: $P(m_2|m_1) = P(m_2)$ for all choices of m_1 and m_2 the first and second members of the match. This is consistent with Appendix L which proves that if the Welfare function is both symmetric and quasi super-modular then the solutions will be on the boundary.[42]

[42] Appendix L uses a different definition of symmetry but the only model that changes status is 2.2.2.3 the Star Business Network.

Table 2.73: Banking Models Efficiency Table

Interior Partition Gain Case Proportion: none; up to small; up to large.						
Model			Interior Partition Percentage Gain Distribution			
			$(-100, 0]$	$(-100, 10]$	$(-100, 100]$	type
Section	Description	Symmetric	none	none,small	none,large	
1.5	Standard Model	Y	100%	100%	100%	$W[x]$
Business						
	Circular Business Networks					
2.2.1.1	Original Circular Model	N	1%		57%	$W[d]$
2.2.1.2	Variable Minimum Module Size	N	43%		84%	$W[d]$
2.2.1.3	Variable Match Lengths	N	91%		96%	$W[d]$
	Hypercube Business Networks					
2.2.3.2	3D Non-Uniform Matching	N	5%		46%	$W[d]$
2.2.3.1	2D Non-Uniform Matching	N	10%	21%	100%	$W[d]$
2.2.3.2	3D Uniform Matching	N	22%		71%	$W[d]$
2.2.3.1	2D Uniform Matching	N	65%	100%	100%	$W[d]$
	Small number of banks					
2.2.3.4	3 node case	N	31%	54%	100%	$W[x]$
2.2.3.5	2 rich 2 poor	N	33%	100%		$W[x]$
	Robust Business Networks					
2.2.2.4	Trilateral Business Matches	Y	100%		100%	$W[d]$
2.2.2.1	Increased Self Matching	Y	100%		100%	$W[x]$
2.2.2.2	Multiple Businesses per Bank	Y	100%		100%	$W[x]$
2.2.2.3	Star Business Network	Y	100%		100%	$W[x]$
2.2.2.5	Temporal Business Opportunities	Y	100%		100%	$W[x]$
Financial						
2.3.1	Variable Shock Initialisation	Y	99.6%	99.99%	100%	$v_1[x]$
Social Planner						
2.4.1	Intertemporal Model	Y	99%	100%	100%	$v_1[x]$
2.4.2.3	Consumption Smoothing	Y	99.1%	99.8%	100%	$W[d]$
2.4.2.1	Risk Aversion	Y	100%			$W[d]$
2.4.2.2	General Equilibrium	Y	100%			$W[x]$

The below stability table considers the stability of each of the boundary partitions under each of the different partition formation concepts. The first model, EBA (Equilibrium Binding Agreement), does not allow modules to merge and hence the atomistic partition is always stable.[43] In contrast, with the EEBA (Extended Equilibrium Binding Agreement), or the Unanimity Game, each boundary partition is stable only for parametrisations of the standard model where it is efficient. This is because firstly, generically the standard model only has one Pareto optimal partition, and secondly, there is enough freedom either in terms of coalition deviations (EEBA), or bargaining (the Unanimity Game), to reach that efficient partition. So with each of these three concepts, the solution is efficient.

With the other three concepts, the solutions include inefficient partitions. With Bilateral Stability, the grand coalition is too stable: an initial deviation from the grand coalition would give the alternative partition $\{0.5n, 0.5n\}$, and it may be the case that the grand coalition is inefficient, $W[\{n\}] < W[\{1, 1, 1, 1, 1, 1\}]$ but still beats the alternative, $W[\{n\}] > W[\{0.5, 0.5n\}]$. Whilst with Open Membership the solution is the Nash Equilibrium of a simultaneous move game, resulting in the grand coalition being stable, for all parametrisations: if every other bank is in a single big module, no individual bank wants to deviate to $\{1, n-1\}$, as the member of the singleton module, for a successful business match, except in the unlikely case of a self-match, still needs the large module to be enabled in order to be productive, but then, as an outsider, they only get a low value θ match. Finally, with Zero Lifetime Bankers, the atomistic partition is never stable: the zero lifetime means that bankers ignore financial risk; they just want to maximise the value of business matches; and so they always choose the grand coalition. The Short Run Bankers model, in Section 2.3.2.2, shows that this bias can last for a few years.

[43] Although with the EBA it is normally accepted that the 'solution' of the game is the partition with the largest modules (Bloch and Dutta (2010)).

Table 2.74: Banking Models Stability Table

Boundary Partition Stability with each of the different market games					
Game		Payoffs	Expectations	atomistic Partition Stability	Grand Coalition Stability
2.3.3.4	EBA	Farsighted	Farsighted	Always	Correct
1.8	EEBA	Farsighted	Farsighted	Correct	Correct
2.3.3.3	Unanimity Game	Farsighted	Myopic	Correct	Correct
2.3.3.1	Bilateral Stability	Farsighted	Myopic	Correct	Too
2.3.3.2	Open Membership	Farsighted	Myopic	Correct	Always
2.3.2.1	Zero Lifetime Bankers	Myopic	Myopic	Never	Always

2.6 Conclusion

The policy contribution of the first chapter was that it mathematically assessed the policy proposal of containing financial shocks, by partitioning banks into modules separated by firewalls: previous authors having relied on analogy or intuition. The methodological contribution of the chapter was the use of a theoretical economic model containing a micro-founded welfare function leading to good interpretability: previous contributions having used behavioural and computational approaches. In the standard model, where business opportunities are uniformly distributed, a strong result was derived: the full characterisation of the optimal structure of the banking industry. Surprisingly, for any choice of the parameters, the optimal partition took one of two forms, either it was the grand coalition (one module containing all the banks), or it was the atomistic partition (each module contains only one bank): the intuition behind this sharp characterisation was that modules have increasing marginal returns. Section 1.9 went further and argued that for realistic parametrisations, it is the atomistic partition that will be optimal; whilst Section 1.7 proved that the efficiency of boundary partitions holds for a more general class of models; and finally, Section 1.8 considered partition formation, and it showed that if agents are farsighted, then the Extended Equilibrium Binding Agreement (EEBA), from Diamantoudi and Xue (2007), results in the efficient partition being formed.

Chapter 2 considered the criticality and robustness of the standard model in four areas: the business sector, the financial sector, the social planner's preferences and partition formation. Section 2.2 showed that the efficiency characterisation result is robust in a number of aspects of the business sector including, the temporal distribution of the business opportunities; the elasticity of the shock parameter; the number of businesses per bank; and the number of businesses linked in a match. However, it also showed that there is criticality with respect to the structure of the business sector in a number of other aspects: varying the distribution of business opportunities can result in the optimality of proper partitions, (where there are multiple modules, and each module will have multiple member banks). Examples constructed that demonstrate this criticality include circular matching, (businesses are arranged in a circle, and matches are always between immediate neighbours); hypercube models, (either 4 businesses arranged in a square, or 8 businesses arranged in a cube); and non-uniform matching (with 3 or 4 businesses).

Next Sections 2.3 and 2.4 further showed that there is robustness both in the financial sector (where the probability of a bank receiving an initial shock increases with module size), and in the preferences of the social planner (when the planner is either risk averse or has intertemporal utility): with these variations the results of boundary solutions remain. Further, Section 2.3.3 applied different partition formation concepts to the standard model. This showed that the efficient partition is again always formed under either the Equilibrium Binding Agreement (EBA), of Ray and Vohra (1997), or the Unanimity Game of Bloch (1996). However, inefficient partitions can be formed under bilateral stability from Jackson and Wolinsky (1996), or the Open Membership game from Yi and

Shin (2000).

The robustness and the criticality of the current work could be assessed in a number of other areas, and these forms the basis of potential future work. Firstly, the current models have a static number of banks and businesses; so one extension would be to develop a dynamic model where the number of banks and businesses grows over time. Secondly, in this thesis financial shocks never cross from one module to another, and never infect some but not all banks in a module. So a model with probabilistic contagion and probabilistic firewalls would be another extension. Thirdly in the current models banks cannot make bad investments; so a model with strategic bank investment behaviour would be another extension. Fourthly, the current models consider ex-ante efficiency rather than interim efficiency or ex-post efficiency. Specifically, they assume that after the arrival of a disabling shock the social planner does not alter the partitioning of banks that remain enabled: once modules are in place, they are not altered conditional on shocks, not just for one or two periods but over an infinite time horizon. So an extension would be to allow the social planner to make such a state conditional choice of partition, and this could be modelled using a Markov Decision Process.

Appendix A

An Upper Limit to the Welfare gain from interior partitions

This section will consider the gain in choosing an interior partition over a boundary partition. This section will show that the gain in the v_1 program, which considers only the utility of banks in module 1, provides an upper bound for the gain in the full W program. The argument relies on there being negative externalities.

The setup is as follows. As in all variants of model, consider a partition $x = (x_i)_{i=1}^k$. Let $V_i[x]$ be the total utility to all banks in module i. Let $v_i[x]{:=}\frac{1}{x_i}V_i[x]$ be the utility of a single bank in module i. Let $W[x] := \sum_{i=1}^k V_i[x]$ be the social planner's welfare function.

The W program feasibility set is as follows. The decision maker chooses a partition $(x_i \in \mathbb{N}_1)_{i=1}^k$ of $N = \{1, 2, 3....n\}$, the set of banks, such that $\sum_{i=1}^k x_i = n$, and $(x_i \geq 1)_{i=1}^k$. The first constraint means that banks can be grouped, but not created or destroyed; the second requires a minimum module size of one. Hence the feasible set of partitions is

$$\mathbb{P} = \{(x_i \in \mathbb{N}_1)_{i=1}^k : \sum_{i=1}^{k} x_i = n \text{ and } (x_i \geq 1)_{i=1}^k \text{ and } k \in \mathbb{N}_1\}$$

The maximum of the v_1 program can be found in a reduced set: as there are negative externalities, a module always looses from mergers between other modules; hence $x^* = \arg\max_{x \in \mathbb{P}} v_1[x] \Rightarrow x^* = (x_1, 1......1)$ for some $x_1 \in \mathbb{N}_1$. Extended the definition of $v_1[x_1]$, and let $v_1[x_1] := v_1[(x_1, 1, 1......1)]$, where $x_1 \in \mathbb{N}_1$. Define, $x_1^* := \arg\max_{x_1 \in A_n} v_1[x_1]$.

We now consider the percentage gain from choosing an interior partition over a boundary.[1] With the W program, the welfare of the best boundary partition is given by $W^b := \text{Max}\{W[(1,1....1)], W[(\text{n})]\}$; the welfare of the best interior partition is given by $W^i := \text{Max}\{W[x] : x \in \mathbb{P} \setminus \{(1,,,,,1),(n)\}\}$; and g_W, the percentage gain in welfare from choosing an interior partition, is given by $g_W := 100*(W^i - W^b)/W^b$. With the v_1 program, the utility of the best boundary partition is given by $v_1^b := \text{Max}\{v_1[1], v_1[\text{n}]\}$; the utility of the best interior partition is given by $v_1^i := \text{Max}\{\text{v}_1[x_1]\}_{x_1=2}^{n-1}$; and g_1, the utility percentage gain from choosing an interior partition, is given by $g_1 := 100 * (v_1^i - v_1^b)/v_1^b$.

Theorem 102. $g_W \leq g_1$

Proof. By contradiction. Suppose not then, $g_W > g_1$. By re-arrangement, $g_W = 100 * (W^i/W^b) - 100$ and $g_1 := 100 * (v_1^i/v_1^b) - 100$. So $g_W > g_1$ if and only if $(W^i/W^b) > (v_1^i/v_1^b)$. Recall $W^b := \text{Max}\{W[(1,1....1)], W[(\text{n})]\}$ and $v_1^b := \text{Max}\{v_1[1], v_1[\text{n}]\}$. But by symmetry $W[(1,1....1)] = nv_1[1]$ and $W[(\text{n})] = nv_1[\text{n}]$. So $W^b = nv_1^b$. Hence $g_W > g_1$ if and only if $(W^i/n) > (v_1^i)$. But this is a contradiction: it would mean that the highest utility is less than the average

[1]Of course, my main thesis is that the standard model, and many variations of it, the optimal partition is a boundary one. In such cases the gain is negative.

utility. □

Appendix B

Markov Processes

The disabling, and re-enabling, shocks to the bank network will form a *Markov process.*[1] In a *Markov process* the current state, but not the past states, matter in determining the future state: $P(s_{t+1}|s_t, s_{t-1}......s_1, s_0) = P(s_{t+1}|s_t)$, where $s_t \in S$ is the state at time t. A *Markov chain* is a time-homogenous Markov process: $P(s_{t+1} = j|s_t = i) = P(s_1 = j|s_0 = i)$, for all times t, and all states i and j. A Markov chain is *ergodic* if there exists a stationary probability distribution P^* which is the limiting probability distribution irrespective of the initial probability distribution. The t period transition probability is represented by $P(i, j, t) :=P(s_t = j : s_0 = i)$, and the 1 period transition probability by $P_{i,j} := P(i, j, 1)$. A Markov chain is *aperiodic,* if for all states i $\exists\, T\ s.t.\, t > T \Rightarrow P(i, i, t) > 0$. A Markov chain is *irreducible,* if all states *communicate*; that is for all states i, j $\exists t\ s.t.\ P(i, j, t) > 0$. In order for a finite state discrete time Markov chain to be ergodic it is sufficient that it is aperiodic and irreducible. For a continuous time, finite state-space, Markov chain to be ergodic it is sufficient just that it is irreducible. For a continuous time, infinite state space, Markov chain to be

[1]For full textbook treatments of Markov processes see Norris (1999), Grimmett and Stirzaker (2001) chapter 6, or Ross (2010) chapter 4. For a brief description of Markov Processes, as well as an another application of them in an economic environment, see Gintis (2012).

ergodic it is sufficient that it is *positive recurrent*. *Positive recurrent* means that for all states i, $P(T_i < \infty) = 1$ and $E[T_i]$ is finite, where T_i is the *return time* for state i: the first time the process returns to state i after having left it.

Appendix C

Continuous Time Markov Process Formulation

The first section of this appendix derives the unique stationary distribution of the stochastic process of enabling and disabling shocks. The second section shows that this stationary distribution is a global attractor: whatever the initial probability of module enablement is then the asymptotic limit is that the probability of module enablement is the stationary probability. The third and final section shows that this convergence occurs exponentially fast.

C.1 Stationary distribution

Consider $X(t)$ a continuous time Markov process. Let S be the set of states, and i, j be typical distinct states. The transitions are specified by the leaving rates q_{ij} from state i to state j. Then

$$P(X(t+h)=j|X(t)=i)=q_{ij}h+o(h) \text{ where } i \neq j$$

and by symmetry the *staying rate* is given by

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

This gives an r period *staying* probability given by,

$$P(X(\tau) = i \, \forall \tau \in (t, t+r) | X(t) = i) = e^{q_{ii} r}$$

Considering a continuous time formulation of the banking shocks process, there are two states, (E)nabled and (D)isabled, and four transition rates q_{EE}, q_{DD}, q_{ED} and q_{DE}. By symmetry $q_{ED} = -q_{EE}$ and $q_{DE} = -q_{DD}$. So a general continuous time shock process can be specified just by the staying rates q_{EE} and q_{DD}.

A continuous time Markov process represents the same system as a discrete time Markov process, if for every state the two processes have the same probability distributions on both destinations and staying (equivalently leaving) times. Here both versions have only two states, and so both must have the same trivial probability distribution on destination states: when the process leaves a state i, it must be to go the other state j. So equivalence requires us only to equate staying times. With the discrete time version we know the one period staying probabilities:

$$P_{EE} = (1 - q)^d$$

and

$$P_{DD} = 1 - \rho$$

So for the continuous time version this requires

$$q_{EE} = d \log[1 - q]$$

$$q_{DD} = \log[1-\rho]$$

and hence

$$q_{ED} = -d\log[1-q]$$

$$q_{DE} = -\log[1-\rho]$$

For the stationary distribution we need the *master balance* condition to hold:

$$P_E q_{ED} = P_D q_{DE}$$

this requires that the flow rate from state E to state D is the same as the flow rate from D to E. Further, as there are only 2 states:

$$P_D = 1 - P_E$$

Hence

$$P_E = \frac{q_{DE}}{(q_{DE} + q_{ED})} = \frac{-\log[1-\rho]}{-\log[1-\rho] - d\log[1-q]}$$

This gives the following module enablement probability:

$$P(d) := P_E = \frac{1}{1+\gamma d}$$

where the shock parameter $\gamma := \frac{-Log[1-q]}{-Log[1-\rho]}$ is the ratio of disablement and re-enablement rates.

C.2 Convergence

Next we consider the convergence of the Markov process. Let $p(t)$ be the matrix of transition probabilities from time 0 up to time t. So $p(t)$ has entry (i, j) given by:

$$p_{ij}(t) := P\left(X_t = j | X_0 = i\right)$$

Hence $p(t = 0) = I$ where I is the identity matrix. Next we consider the rate of change of $p(t)$. At time $t = 0$, define $Q := p'(t = 0) = \lim_{h\to 0} \frac{p(h)-p(0)}{h}$. So $Q = \lim_{h\to 0} \frac{p(h)-I}{h}$. As $P(X(t + h) = j|X(t) = i) = q_{ij}h + o(h)$ where $i \neq j$, this gives $Q_{ij} = q_{ij}$. As this is a Markov Chain, for general time t the derivative alters only due to differences in $p[t]$. So $p'[t] = Q\, p[t]$. This differential equation can be solved to get:

$$p[t] = exp[tQ]$$

where the exponential of a matrix has its expected definition: $exp[tQ] = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k$

Let $\pi[t]$ be the unconditional distribution of $X[t]$ at time t in row vector form, so

$$\pi[t] = \pi[0]p[t]$$

If Q has a 'nice' eigenvalue-eigenvector expansion $Q = U\Lambda U^{-1}$, where U is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvectors then $Q^k = U\Lambda^k U^{-1}$ and so

$$\pi[t] = \pi[0] \sum_{k=0}^{\infty} \frac{t^k}{k!} U\Lambda^k U^{-1} = \pi[0]U \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k \right) U^{-1} = \pi[0]Uexp[t\Lambda]U^{-1}$$

Listing the enabled state first and the disabled state second gives $Q = \begin{pmatrix} +a & -a \\ -b & +b \end{pmatrix}$, where $a := d\log[1-q]$ and $b := \log[1-\rho]$. So the eigenvalues are 0 and $a+b$, which have corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} +a \\ -b \end{pmatrix}$ respectively.[1] So $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & a+b \end{pmatrix}$ and $U = \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix}$. Then, as required:

$$U\Lambda U^{-1} = \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a+b \end{pmatrix} \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix}^{-1}$$

$$U\Lambda U^{-1} = \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a+b \end{pmatrix} \frac{1}{a+b} \begin{pmatrix} b & +a \\ 1 & -1 \end{pmatrix}$$

$$U\Lambda U^{-1} = \frac{1}{a+b} \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a+b & -a-b \end{pmatrix}$$

$$U\Lambda U^{-1} = \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} +a & -a \\ -b & +b \end{pmatrix} = Q$$

So using $\pi[t] = \pi[0] U exp[t\Lambda] U^{-1}$

$$\pi[t] = \pi[0] \begin{pmatrix} 1 & +a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & exp[(a+b)t] \end{pmatrix} \frac{1}{a+b} \begin{pmatrix} b & +a \\ 1 & -1 \end{pmatrix}$$

$$\pi[t] = \frac{\pi[0]}{a+b} \begin{pmatrix} 1 & +a * exp[(a+b)t] \\ 1 & -b * exp[(a+b)t] \end{pmatrix} \begin{pmatrix} b & +a \\ 1 & -1 \end{pmatrix}$$

[1] The Internal Examiner has pointed out that as the eigenvalues have negative real parts that the system will converge exponentially to the stable solution.

$$\pi[t] = \frac{\pi[0]}{a+b} \begin{pmatrix} b + a * exp[(a+b)t] & a - a * exp[(a+b)t] \\ b - b * exp[(a+b)t] & a + b * exp[(a+b)t] \end{pmatrix}$$

Using $a = d\log[1-q] = \log[(1-q)^d]$ and $b = \log[1-\rho]$ gives $exp[(a+b)t] = (1-q)^{dt}.(1-\rho)^t = \left((1-q)^d.(1-\rho)\right)^t$. As $(1-q) < 1$ and $(1-\rho) < 1$ we get $\lim_{t\to\infty} exp[(a+b)t] = 0$. So

$$\pi^* := \lim_{t\to\infty} \pi[t] = \frac{\begin{pmatrix} \pi_E[0] & \pi_D[0] \end{pmatrix}}{a+b} \begin{pmatrix} b & a \\ b & a \end{pmatrix}$$

$$\pi^* = \frac{\begin{pmatrix} b & a \end{pmatrix}}{a+b} = \begin{pmatrix} P[d] & 1 - P[d] \end{pmatrix}$$

Hence irrespective of the initial distribution the limiting state is the stationary distribution.

C.3 Rate of Convergence

Further:

$$\pi[t] - \pi^* = exp[(a+b)t] \frac{\pi[0]}{a+b} \begin{pmatrix} +a & -a \\ -b & +b \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t] \frac{\begin{pmatrix} \pi_E[0] & 1 - \pi_E[0] \end{pmatrix}}{a+b} \begin{pmatrix} +a & -a \\ -b & +b \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t] \begin{pmatrix} -\frac{b}{a+b} + \pi_E[0] & \frac{b}{a+b} - \pi_E[0] \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t] \begin{pmatrix} -\pi_E^* + \pi_E[0] & \pi_E^* - \pi_E[0] \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t] \begin{pmatrix} -\pi_E^* + \pi_E[0] & \pi_E^* - 1 + 1 - \pi_E[0] \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t] \begin{pmatrix} -\pi_E^* + \pi_E[0] & -\pi_D^* + \pi_D[0] \end{pmatrix}$$

$$\pi[t] - \pi^* = exp[(a+b)t](\pi[0] - \pi^*)$$

So the process converges exponentially fast. This can be re-stated as follows: if $P_0(d)$ is the initial enablement probability, $P_t(d)$ is the probability of enablement at a general time t and $P_\infty(d) = P(d)$ is the limiting distribution then:

$$P_t(d) - P_\infty(d) = \left[(1-q)^d(1-\rho)\right]^t [P_0(d) - P_\infty(d)]$$

Appendix D

Distribution of Re-enablements

The standard model has a simple representation of the re-enablement process, and assumes that re-enablements are distributed *memorylessly*: in each unit of time there is the same conditional probability of a re-enablement. So in discrete time formulation this means,

$$P[\text{re-enablement at time } t+1 \mid \text{no re-enablement by time } t] = P[\text{re-enablement at time } 1]$$

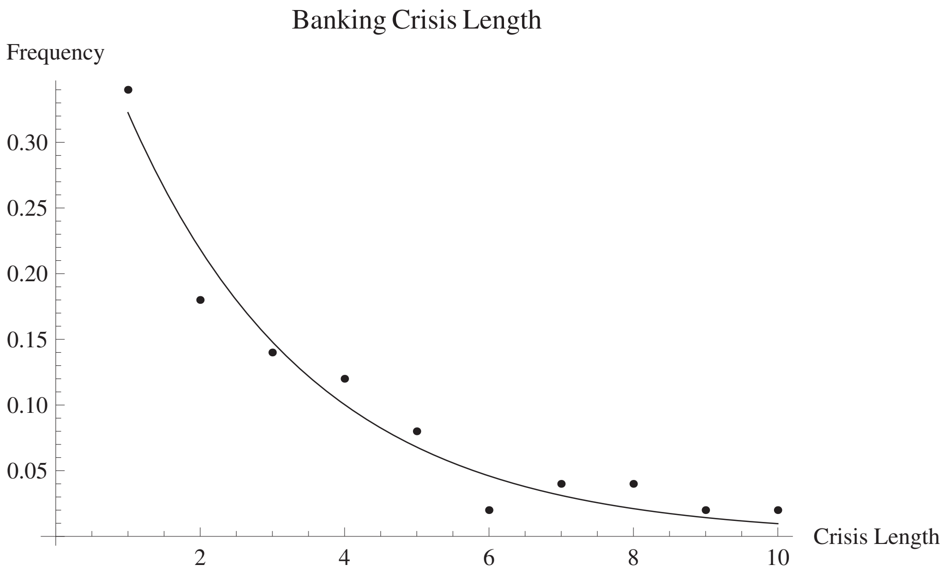
This implies that the crisis lengths should be distributed geometrically. The empirical data in table 6 of Frydl (1999), extracted by from Caprio and Klingebiel (1996), has the following distribution:

Table D.1: Frydl (1999) Crisis Length Table

Crisis length (t)	Observations ($x(t)$)	Geometric probability
1	17	ρ
2	9	$\rho(1-\rho)$
3	7	$\rho(1-\rho)^2$
4	6	$\rho(1-\rho)^3$
5	4	$\rho(1-\rho)^4$
6	1	$\rho(1-\rho)^5$
7	2	$\rho(1-\rho)^6$
8	2	$\rho(1-\rho)^7$
9	1	$\rho(1-\rho)^8$
10	1	$\rho(1-\rho)^9$

The mean crisis length is 3.1. With a geometric distribution, the Maximum Likelihood Estimator is the reciprocal of the mean and so $\hat{\rho} = 0.32\,(2.d.p.)$. The resulting probability function is a good fit for the empirical data:

Figure D.1: Frydl (1999) Banking Crisis Length Plot



Appendix E

Typical Shock Parameter Values

From Appendix C, the probability of a module remaining disabled for at least r periods is $e^{-rLog[1-\rho]}$, where ρ is the module enablement probability. So if T_ρ is the median time for a module to remain disabled, then $0.5 = e^{-T\rho Log[1-\rho]}$, and hence $T_\rho = \frac{\text{Log}[0.5]}{\text{Log}[1-\rho]}$. Similarly the median time we expect a module of d banks to remain enabled is $T_q[d] = \frac{\text{Log}[0.5]}{\text{d*Log}[1-q]}$, where q is the bank disablement probability.

The table below plots three things: firstly, for each value of ρ, it gives the median recovery time T_ρ; secondly, for each value of q, it gives the median enabled time $T_q[1]$ for a singleton module; and finally, for each combination of q and ρ it gives the shock parameter $\gamma := \frac{-Log[1-q]}{-Log[1-\rho]}$.[1] We would expect the median recovery time to be less than 20 years, and the median enablement time of a singleton module to be greater than 10 years.[2] In the table when $\rho = 0.03$, (where $T_\rho = 22.8$), and $q = 0.07$, (where $T_q = 9.6$), we get that $\gamma = 2.38$. This motivates the levels of γ studied in chapters 1 and 2, where γ is in the range 0 to 3.

[1]The median enablement time for a module of size d is $\frac{T_q}{d}$.

[2]For example, using historical US data, there often periods of 10 years when there are *no* bank failures leading to financial contagion: 1857, Ohio Life Insurance and Trust Company; 1873, Northern Pacific Railroad and Jay Cooke and Company, +16 years; 1884, Grant and Ward, +11 years; 1893, National Cordage Company, +9 years; 1907, Knickerboker Trust Company, +14; 1930, Bank of United States, +23 (data from both Mishkin (2007) and Calomiris (2010)).

Table E.1: Shock Parameter Table

		How the shock parameter γ depends on q and ρ											
	ρ	0.01	0.02	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.1**	**0.2**	**0.3**
	Median Recovery T_ρ	69.0	34.3	**22.8**	**17.0**	**13.5**	**11.2**	**9.6**	**8.3**	**7.4**	**6.6**	**3.1**	**1.9**
q	Median Enablement T_q												
0.001	**692.8**	0.10	0.05	**0.03**	**0.02**	**0.02**	**0.02**	**0.01**	**0.01**	**0.01**	**0.01**	**0.00**	**0.00**
0.002	**346.2**	0.20	0.10	**0.07**	**0.05**	**0.04**	**0.03**	**0.03**	**0.02**	**0.02**	**0.02**	**0.01**	**0.01**
0.003	**230.7**	0.30	0.15	**0.10**	**0.07**	**0.06**	**0.05**	**0.04**	**0.04**	**0.03**	**0.03**	**0.01**	**0.01**
0.004	**172.9**	0.40	0.20	**0.13**	**0.10**	**0.08**	**0.06**	**0.06**	**0.05**	**0.04**	**0.04**	**0.02**	**0.01**
0.005	**138.3**	0.50	0.25	**0.16**	**0.12**	**0.10**	**0.08**	**0.07**	**0.06**	**0.05**	**0.05**	**0.02**	**0.01**
0.006	**115.2**	0.60	0.30	**0.20**	**0.15**	**0.12**	**0.10**	**0.08**	**0.07**	**0.06**	**0.06**	**0.03**	**0.02**
0.007	**98.7**	0.70	0.35	**0.23**	**0.17**	**0.14**	**0.11**	**0.10**	**0.08**	**0.07**	**0.07**	**0.03**	**0.02**
0.008	**86.3**	0.80	0.40	**0.26**	**0.20**	**0.16**	**0.13**	**0.11**	**0.10**	**0.09**	**0.08**	**0.04**	**0.02**
0.009	**76.7**	0.90	0.45	**0.30**	**0.22**	**0.18**	**0.15**	**0.12**	**0.11**	**0.10**	**0.09**	**0.04**	**0.03**
0.01	**69.0**	1.00	0.50	**0.33**	**0.25**	**0.20**	**0.16**	**0.14**	**0.12**	**0.11**	**0.10**	**0.05**	**0.03**
0.02	**34.3**	2.01	1.00	**0.66**	**0.49**	**0.39**	**0.33**	**0.28**	**0.24**	**0.21**	**0.19**	**0.09**	**0.06**
0.03	**22.8**	3.03	1.51	**1.00**	**0.75**	**0.59**	**0.49**	**0.42**	**0.37**	**0.32**	**0.29**	**0.14**	**0.09**
0.04	**17.0**	4.06	2.02	**1.34**	**1.00**	**0.80**	**0.66**	**0.56**	**0.49**	**0.43**	**0.39**	**0.18**	**0.11**
0.05	**13.5**	5.10	2.54	**1.68**	**1.26**	**1.00**	**0.83**	**0.71**	**0.62**	**0.54**	**0.49**	**0.23**	**0.14**
0.06	**11.2**	6.16	3.06	**2.03**	**1.52**	**1.21**	**1.00**	**0.85**	**0.74**	**0.66**	**0.59**	**0.28**	**0.17**
0.07	**9.6**	7.22	3.59	**2.38**	**1.78**	**1.41**	**1.17**	**1.00**	**0.87**	**0.77**	**0.69**	**0.33**	**0.20**
0.08	8.3	8.30	4.13	2.74	2.04	1.63	1.35	1.15	1.00	0.88	0.79	0.37	0.23
0.09	7.3	9.38	4.67	3.10	2.31	1.84	1.52	1.30	1.13	1.00	0.90	0.42	0.26
0.1	6.6	10.48	5.22	3.46	2.58	2.05	1.70	1.45	1.26	1.12	1.00	0.47	0.30

Appendix F

Quasiconvexity Results

Definition 103. Suppose $f : S \to \mathbb{R}$ where S is a convex subset of $\mathbb{R}^l$. Then, f is *weakly quasiconvex* if and only if the lower level set $P^a = \{x : f(x) \leq a\}$ is convex for each real number a . Similarly, f is *weakly quasiconcave* if and only if the upper level set $P^a = \{x : f(x) \geq a\}$ is convex for each number a. Equivalently f is *weakly quasiconcave* if and only if $f(\lambda x + (1 - \lambda)x^0) \geq \min\{f(x), f(x^0)\}$ for all $\lambda \in (0, 1)$ and $(x, x^0) \in S^2$

f is *strictly quasiconcave* if $f(\lambda x + (1 - \lambda)x^0) > \min\{f(x), f(x^0)\}$ for all $\lambda \in (0, 1)$ and $(x, x^0) \in S^2$ where $x \neq x^0$. f is *strictly quasiconvex if and only if* $-f$ *is strictly quasiconcave. So* f is *strictly quasiconvex if and only if* $f(\lambda x + (1 - \lambda)x^0) < \max\{f(x), f(x^0)\}$ for all $\lambda \in (0, 1)$ and $(x, x^0) \in S^2$ where $x \neq x^0$.

Proposition 104. *Consider* $f : [1, n] \to \mathbb{R}$. *If* f *is weakly quasiconvex then* f *does not have a strict interior local maximum.*

Proof. Proof by contradiction. Suppose f has some strict local interior maximum x^* and $f(x^*) = a$. It is a strict maximum, so there exists some $\delta > 0$ such that if $x \in [x^* - \delta, x^* + \delta]$ and $x \neq x^*$ then $f(x) < f(x^*)$. The set $P := \{x : f(x) \leq$

$\max\{f(x^*-\delta), f(x^*+\delta)\}\}$ contains both $x^*-\delta$ and $x^*+\delta$ but does not contain x^*. Hence it is not convex. □

Corollary 105. *Consider* $f : [1, n] \to \mathbb{R}$. *If* f *is weakly quasiconvex then* $\operatorname{argmax} f$ *includes a boundary solution.*

Proof. By Proposition 104 no interior point can be a strict local maximum. And if it is not a strict local maximum then it is not a strict global maximum. So the argmax includes at least one of 1 and n. □

Theorem 106. *Suppose* f *is* C^1. *Then in the following conditions,* (1) $\Rightarrow$(2) $\Rightarrow$(3):

(1) $f'(x) > 0$ and $y > x \Rightarrow f'(y) > 0$.

(2) f *weakly quasi-convex.*

(3) $f'(x) > 0$ and $y > x \Rightarrow f'(y) \geq 0$

Proof. (1) $\Rightarrow$ (2)

Proof by contradiction. Suppose not: then there exists level a s.t. $P^a = \{x : f(x) \leq a\}$ is not convex and so $\exists\, x_0, x_1, x_2\; s.t.\; x_0 < x_1 < x_2$, $f(x_0) \leq a$, $f(x_1) > a$ and $f(x_2) \leq a$. So the average gradient between x_0 and x_1 is positive and the average gradient between x_1 and x_2 is negative. Hence $\exists x_{01} s.t.\, x_0 \leq x_{01} \leq x_1 and f'(x_{01}) > 0$. Similarly $\exists x_{12} s.t.\, x_1 \leq x_{12} \leq x_2 and f'(x_{12}) < 0$. So $f'(x_{01}) > 0$ and $x_{12} > x_{01}$ but $f'(x_{12}) < 0$. This contradicts (1).

(2) $\Rightarrow$ (3)

Proof by contradiction. Suppose not: then there exist $y > x s.t. f'(x) > 0 > f'(y)$. Then, as f is C^1, $\exists x^* s.t.\, x < x^* \leq y\; and\; f'(x^*) = 0$ and x^* is a local maximum. This contradicts Proposition 104. □

Theorem 107. *Suppose $f : \mathbb{R} \to \mathbb{R}$. Then:*

i) if f is a monotonic function, then it is both weakly quasiconvex and quasiconcave.[1]

ii) if f increases monotonically to a global maximum and then falls monotonically, then f is weakly quasiconcave.

ii) if f decreases monotonically to a global minimum and then increases monotonically, then f is weakly quasiconvex.

Proof. i) See Simon and Blume (1994), example 21.11 p524. ii) See Simon and Blume (1994), example 21.12 p524. iii) consider $-f$ and apply result ii). $\square$

Theorem 108. *$W[d]$, the welfare function from the standard model evaluated at symmetric partitions, has boundary solutions.*

Proof. By differentiation, $W'[d] = \frac{(-\theta-2\gamma\theta n+1)+d(\gamma\theta+\gamma)}{(1+\gamma d)^3}$. So $W'[d] = t[d]/l[d]$ where: $t[d]$ $=(-\theta - 2\gamma\theta n + 1) + d(\gamma\theta + \gamma)$ and $l[d] = (1 + \gamma d)^3$. As $l[d] > 0\,\forall d$, by Theorem 106 we get that W is weakly quasi-convex if $t[d] > 0$ and $e > d \Rightarrow t[e] > 0$. This is true as $(\gamma\theta + \gamma) > 0$. Hence $W[d]$ is weakly quasiconvex and has a boundary solution as required. $\square$

Corollary 109. *The general welfare function $W[x]$ has boundary solutions*

Proof. From Theorem 108, symmetric welfare $W[d]$ has boundary solutions. From Theorem 115 in appendix G, the symmetric and general programs have the same maximal value. Hence, the general program also has a boundary solution. $\square$

[1] Monotonic means a function that is either: weakly decreasing or weakly increasing.

Appendix G

Asymmetric Partitions

This section proves that for the standard model, the symmetric partitions maximise welfare: no asymmetric partition has higher welfare than the best symmetric partition.

Recall, that the welfare for symmetric partitions (where $d \in \mathbb{R}_{++}$ is the module size), is given by, $W_{sym}[d] := \frac{d}{n}P[d] + \theta\frac{n-d}{n}P[d]^2$ and $P[d] := \frac{1}{1+\gamma d}$. Let $N := \{1, 2, 3, ...n\}$. The number of (non-empty) modules, k will be in N and we need $n = kd$. So the welfare maximising program over symmetric partitions is: $\max_{k \in N} W_{sym}[\frac{n}{k}]$

The welfare for (generally) asymmetric partitions is given by, $W[(x_i)_{i=1}^n] := \sum_{i=1}^n \frac{x_i^2}{n^2}P[x_i] + \theta \sum_{i=1}^n \frac{x_i}{n}P[x_i] \sum_{j\neq i}^n \frac{x_j}{n}P[x_j]$. In this section, we force there always to be exactly n modules, but we allow empty modules: so module i is of size $x_i \in \{0\} \bigcup [1, n]$. Further, we need to only specify the size of the first $n-1$ modules, as the total has to be n. So, $\sum_{i=1}^n x_i = n$. So, the welfare maximisation program for general partitions is $\arg\max_{x \in A} W[x]$ where $A := \{(x_i \in \mathbb{R}_+)_{i=1}^{n-1} : x_i \in \{0\} \bigcup [1, n] \textit{ and } \big(\sum_{i=1}^{n-1} x_i \leq n-1 \textit{ or } \sum_{i=1}^{n-1} x_i = n\big)\}$.

The proof of the main theorem uses the Extreme-Value Theorem:

Proposition 110. *Let $f : C \to \mathbb{R}$, be a continuous function and $C \subseteq \mathbb{R}^{n-1}$ be a closed bounded set. Then f has a maximum point in C.*

Proof. See Sydsæter, K., Hammond, P., Seierstad, A, and Strøm, A. (2005) Theorem 3.1.3. □

Definition 111. For A, use the metric based on the L^1 norm: so $d : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}$ s.t. $d(x, y) = \sum_{i=1}^{n} \mid x_i - y_i \mid$ where $x_n := n - \sum_{i=1}^{n-1} x_i$ and $y_n := n - \sum_{i=1}^{n-1} y_i$.

Lemma 112. *A is compact: closed and bounded*

Proof. A is *bounded* as it is contained within the ball of radius $n + 1$ centred on the origin.

Let $A_f := \{(x_i \in \mathbb{R}_+)_{i=1}^{n-1} : x_i \in [f(i), nf(i)]\text{and} \left(\sum_{i=1}^{n-1} x_i \leq n-1 \text{ or } \sum_{i=1}^{n-1} x_i = n\right)\}$, where $f \in F = \{f : \{1, 2, 3...n-1\} \to \{0, 1\}\}$. So if $f[i] = 0$ then $x_i = 0$ and if $f[i] = 1$ then $x_i \in [1, n]$. Hence $A = \bigcup_{f \in F} A_f$ (a finite union as the function f can only take 2^{n-1} different forms). Each A_f is closed as it is defined by inequalities satisfied by continuous functions. Hence A is also closed. (See Theorem 12.10 in Simon and Blume (1994) p268). □

Lemma 113. *W is continuous (with the absolute metric on C)*

Proof. $W[(x_i)_{i=1}^n] = \sum_{i=1}^n V_i[x]$ where $V_i[(x_i)_{i=1}^n] := x_i^2 P[x_i] + \theta x_i P[x_i] \sum_{j \neq i}^n x_j P[x_j]$. So W is the sum of continuous functions, and hence is continuous itself. □

Theorem 114. *The supremum of the general programs exists and is achieved so: $Max_{x \in A} W[x]$ exists, $Sup_{x \in A} W[x]$ exists and $Max_{x \in A} W[x] = Sup_{x \in A} W[x]$*

Proof. 1 is an upper bound in 1 for $W[x]$. So $Sup_{x \in A} W[x]$ exists. By lemma 112 A is a compact set, and by lemma 113 W is a continuous function, so by

Proposition 110 W must achieve its maximum. So $Max_{x\in A}W[x] = Sup_{x\in A}W[x]$.

$\square$

Theorem 115. *The symmetric and general programs have the same maximum:* $\max_{x\in A}W[x] = \max_{k\in N} W_{sym}[\frac{n}{k}]$

Proof. From Theorem 114 $\sup_{x\in A}W[x]$ exists and $\max_{x\in A}W[x] = \sup_{x\in A}W[x]$. Further, $W[(d)_{i=1}^{k},(0)_{i=k+1}^{n-1}] = W_{sym}[d]$, and so $\sup_{x\in A}W[x] \geq \max_{k\in N} W_{sym}[\frac{n}{k}]$.

Suppose $W^* := \sup_{x\in A} W[x] > \max_{k\in N} W_{sym}[\frac{n}{k}] =: W^*_{sym}$. Then define the set of asymmetric partitions that strictly dominate the best symmetric partition, $S_0 := \{x : W[x] > W^*_{sym}\} \neq \emptyset$. And by Theorem 114, there is a non-empty set of partitions that maximise W, $S_1 := \{x :\ x \in A\, and\, W[x] = W^*\} \neq \emptyset$. Define $k(x)$ to be the number of non-empty modules in x, and define k^* to be the least number of modules in a partition that achieves W^*. So $k^* := min\{k(x) : x \in S_1\}$. Now consider the set of partitions that maximise W, with this minimal number of non-empty modules, $S_2 := \{x :\ x \in S_1\, and\, k[x] = k^*\} \neq \emptyset$.

Take $x \in S_2$. Then there must exists i, j s.t. $0 \neq x_i \neq x_j \neq 0$. WLOG assume $i = 1$ and $j = 2$, so: $0 \neq x_1 \neq x_2 \neq 0$. We now consider 2 alternative partitions that re-assign the members of the first 2 modules: the merged partition x^m and the averaged out partition x^a. Partition x^m *merges* the banks from the first 2 modules into a single module, so $x^m := (0, x_1 + x_2, (x_i)_{i=3}^{n-1})$. $k(x^m) = k(x^m) - 1 < k^*$ so $x^m \notin S_2$ and hence $W[x] > W[x^m]$. Conversely partition x^a *averages* out the differences between the size of the first 2 modules, so $x^a := (0.5(x_1 + x_2), 0.5(x_1 + x_2), (x_i)_{i=3}^{n-1})$. $x \in S_2 \Rightarrow x \in S_1$, so $W[x] \geq W[x^a]$. Algebraic manipulation gives that:

$W[x] > W[x^m] \Leftrightarrow \frac{P[0.5t]}{P[t]} > -2\gamma R + \frac{1}{\theta}$

and

$W[x] \geq W[x^a] \Leftrightarrow \frac{P[0.5t]}{P[t]} \leq -2\gamma R + \frac{1}{\theta}$

where $t := x_1 + x_2$ is the total number of banks in the first two modules, and $R := \sum_{i=3}^{n} x_i P[x_i]$ is the total expected number of enabled banks in the other modules.

This is a contradiction, hence $Max_{x \in A} W[x] = \max_{k \in N} W_{sym}[\frac{n}{k}]$ as required. $\square$

So there cannot be an asymmetric sized partition that has a strictly higher welfare than the best symmetrically sized partition. [1]

[1] Corollary 26 in section 1.7 goes further and shows that, for no parameterisations, can there be a partition with asymmetrically sized modules that has the *same* welfare as the best symmetrically sized partition.

Appendix H

Ratio Quadratic Welfare

This appendix considers a general class of welfare functions: the *ratio quadratic* where both the numerator and the denominator are quadratic functions of the policy variable x:[1]

Definition 116. Define the class of *ratio quadratic* functions to be $h : [1, n] \to \mathbb{R}$ such that $h[x] = \frac{ax^2+bx+c}{ex^2+fx+g}$ where $(a, b, c, e, f, g) \in \mathbb{R}^6$ *and* $e > 0$.

A first example of this class occurs when we have continuous time welfare, see section 2.4.1. A second occurs when we take the standard model, but then alter the probability of inside and outside matches.

Theorem 117. *If h_0 is ratio quadratic and the denominator has negative roots, then h_0 is at least one of quasi-convex and quasi-concave. Further it has either 0 or 1 stationary points.*

The proof of this theorem relies of simply transforming the objective function into partial fraction format giving $h[x] = \sigma + \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2}$. However, despite this the proof is still quite lengthy: firstly, because of the need to consider a range of

[1]Hopefully, it is not too confusing that in this appendix that, x is a *single* number, whilst in the main text it is a *partition*.

special cases, such as double roots, and secondly because of the need to consider a wide range of different parameter values.

Proof. This proof relies on the fact that quasi-convexity and quasi-concavity properties only rely on the sign of the h' derivative, and not the level of the derivative. Hence, we will be able to take a sequence of normalisations (h_1, h_2, h_3, h_4 and h_5), that each leave the sign of the derivative unaltered.

Start with $h_0[x] = \frac{a_0x^2+b_0x+c_0}{e_0x^2+f_0x+g_0} = \frac{a_0}{e_0} + \frac{b_1x+c_1}{e_0x^2+f_0x+g_0}$, where $b_1 = b_0 - \frac{a_0f_0}{e_0}$ and $c_1 = c_0 - \frac{a_0g_0}{e_0}$. Now let $h_1[x] := \frac{b_1x+c_1}{e_0x^2+f_0x+g_0}$. As $e_0 \neq 0$, $h_1[x] = \frac{b_2x+c_2}{x^2+f_2x+g_2}$, where $b_2 = \frac{b_1}{e_0}$, $c_2 = \frac{c_1}{e_0}$, $f_2 = \frac{f_0}{e_0}$ and $g_2 = \frac{g_0}{e_0}$. So let $h_2[x] := \frac{b_2x+c_2}{x^2+f_2x+g_2}$. Factoring the denominator gives: $h_3[x] := \frac{b_2x+c_2}{(x-\alpha_1)(x-\alpha_2)}$, where we know that $\alpha_1 < 0$ and $\alpha_2 < 0$.

Starting with the special case of a double root ($\alpha_1 = \alpha_2$): $h_3'[x|\alpha = \alpha_1 = \alpha_2] = \frac{-b_2x-\alpha b_2-2c_2}{(x-\alpha)^3}$. Note that $x_1 > 0$ and $\alpha < 0$, so $x - \alpha > 0$. So, if $b_2 < 0$ then h is quasi-convex; if $b_2 > 0$ then h is quasi-concave; and finally if $b_2 = 0$ then h is both quasi-convex and quasi-concave.

In the general case of distinct roots ($\alpha_1 \neq \alpha_2$), then without (further) loss of generality, $\alpha_2 > \alpha_1$, and $h_3[x] = \frac{1}{(\alpha_2-\alpha_1)}\left[\frac{r}{(x-\alpha_1)} + \frac{s}{(x-\alpha_2)}\right]$, where $r = -\alpha_1b_2 - c_2$ and $s = \alpha_2b_2 + c_2$. Let $h_4[x] := \frac{r}{(x-\alpha_1)} + \frac{s}{(x-\alpha_2)}$. So $h_4'[x] = \frac{-r}{(x-\alpha_1)^2} - \frac{s}{(x-\alpha_2)^2}$, and hence: $h_4'[x] > 0 \Leftrightarrow -r(x-\alpha_2)^2 > s(x-\alpha_1)^2$.

Therefore, $h_4[x]$ has 4 parameters: r, s, α_1 and α_2. One can be removed by a change in the variable: let $h_5[y] := \frac{r}{(y+\delta)} + \frac{s}{y}$, where $y := x - \alpha_2$, $\delta := \alpha_2 - \alpha_1 > 0$, $h_5 : [L, U] \to \mathbb{R}$, $L := 1 - \alpha_2 > 1 > 0$, and $U := n - \alpha_2 > n > 1 > 0$. Then $h_5'[y] > 0 \Leftrightarrow h_n'[y] := -(r+s)y^2 - 2\delta sy - s\delta^2 > 0$.

The next 6 results covering certain special cases, come from evaluating the sign of the $[y^2]$ and $[y]$ coefficients. Firstly, if $r + s > 0$ and $s > 0$, then once $h_5'[y]$ becomes negative it stays negative: if $z > y$ and $h_5'[y] < 0$, then $h_5'[z] < 0$. So

h is quasi-concave. Secondly and conversely, if $r + s < 0$ and $s < 0$, then $h'_n[y]$ is monotonic increasing. So h is quasi-convex. Thirdly, if $r + s = 0$ and $s \geq 0$, then h is quasi-concave. Fourthly, if $r + s = 0$ and $s \leq 0$ then h is quasi-convex. Fifthly, if $r = 0$ then $r + s = s$, and so $r + s$ and s will have the same sign: this is covered by the previous 4 cases. Sixthly, if $s = 0$ then $h'_n[y] = -ry^2$ and h is monotonic.

More generally, we need to consider the roots of $h'_n[y]$. The (real) roots of $h'_n[y]$ are the only points at which the sign of h' can change. As $h'_n[y]$ is quadratic, there are at most 2 such points. If there are no real roots in the range $[L, U]$, then h is monotonic, and hence both quasi-concave and quasi-convex. If, there is one (single) root in the range $[L, U]$, and increasing y causes $h'_n[y]$ to go from *negative to positive,* then h is *quasi-convex.* If there is one (single) root in the range $[L, U]$, but conversely increasing y causes $h'_n[y]$ to go from *positive to negative,* then h is *quasi-concave.* If there is a double root in the range $[L, U]$, then h has a point of inflection and so h is (weakly) monotonic. Only if there are 2 roots in the range $[L, U]$, will h be neither quasi-concave nor quasi-convex: with 2 roots there will be both a local maximum and a local minimum. We show that this is not possible by considering the first order conditions.

We dealt with the cases where $s = 0$ or $r = 0$ above, so we now assume that $s \neq 0$ and $r \neq 0$, and consider the cases where r and s have the same, or different, *signs* ($Sign[a] = +$ if $a > 0$; $Sign[a] = -$ if $a < 0$). Note that, $h'_5[y] = -\frac{r}{(y+\delta)^2} - \frac{s}{y^2}$ and $h''_5[y] = \frac{2r}{(y+\delta)^3} + \frac{2s}{y^3}$. So for $\hat{y}$ to be a first order point of h_5, requires that $-\frac{s}{\hat{y}^2} = \frac{r}{(\hat{y}+\delta)^2}$. If $Sign[r] = Sign[s]$, then this has no solutions, and h is monotonic. As $s \neq 0$ and $r \neq 0$, the first order condition becomes $-\frac{\hat{y}^2}{s} = \frac{(\hat{y}+\delta)^2}{r}$. So $h''_5[\hat{y}] = \frac{2r}{(\hat{y}+\delta)^2}\left(\frac{-\delta}{(\hat{y}+\delta)\hat{y}}\right)$, and $Sign[h''_5[\hat{y}]] = Sign[-r] = Sign[s]$. If $Sign[-r] = Sign[s] = +$, then any first order point is a local minimum. Conversely, if $Sign[-r] = Sign[s] = -$, then any first order point is a local maximum. So all the first order points of h are of the same type. So h_5 cannot have both local minima and local maximum, and hence it has either 0 or 1 first order condition points. Hence h_0 is either quasi-convex, quasi-concave or both. $\square$

So when $h : [1, n] \to \mathbb{R}$, there is a strict interior solution if and only if $h'[1] > 0$ and $h'[n] < 0$.[2]

If x is now restricted to being an integer, so that $h : A_n = \{1, 2, 3, 4.....n\} \to \mathbb{R}$, then there is a strict interior solution if and only if both $h[1] < h[2]$ and $h[n-1] > h[n]$.

The decision maker decides the partition of banks into modules; the partition is then regarded as fixed. One justification for this assumption would be that, when any module is disabled it is not feasible to alter the partition, and when all modules are enabled there is no incentive to alter the partition.

[2] A strict interior solution just means that the argmax excludes both the boundary solutions: there is an interior solution that is *strictly* preferred to both boundary solutions.

Appendix I

Intertemporal Model

The intertemporal model has the same stochastic process as the standard model: there are financial shocks (disabling shocks hitting enabled *banks* at rate $\log[1-q]$; enabling shocks hitting disabled *modules* at rate $log[1-\rho]$), and business opportunities (business matches are distributed uniformly, so $(P(b_1,\ b_2) = 1/n^2)_{b_1,b_2=1}^{n}$.

The enablement probability converges exponentially, meaning that $(P_t[d] - P_\infty[d]) = \left((1-q)^d(1-\rho)\right)^t (P_0[d] - P_\infty[d])$ where: d is the module size, $P_0[d]$ is the initial enablement probability, $P_t[d]$ is the probability of enablement at a general time t, and $P_\infty[d]$ is the limiting distribution.[1] This intertemporal model, considers a system that starts with fully enabled banks, so $P_0[d] = 1$, and has the same Markov process of shocks as the standard model, and so has the same limiting probability: $P_\infty[d] = P[d] = \frac{1}{1+\gamma d}$. Hence, for general time t, $P_t[d] = P[d] + \left((1-q)^d(1-\rho)\right)^t (1 - P[d])$.

When the partition is of form $\{x_1, 1, 1, ...1, 1, 1\}$, the expected utility function for bank 1 at time t is: $vt_1[x_1] := \beta^t(1/n)\left(({}^{x_1}/_n)P_t[x_1] + ({}^{(n-x_1)}/_n)\theta P_t[x_1]P_t[1]\right)$.[2]

[1] See Appendix Cfor more details.

[2] Similarly, the social welfare at time t for a general partition $x = (x_i)_{i=1}^k$, is given by $Wt[x] := \beta^t \left(\sum_{i=1}^k ({}^{x_i}/_n)^2 P_t[x_i] + \theta \sum_{i=1}^k \sum_{j\neq i} ({}^{x_i}/_n)({}^{x_j}/_n) P_t[x_i]P_t[x_j]\right)$.

The explanation for this specification is as follows. The leading β^t term gives the time t discount factor. Without loss of generality, we assume that all the utility goes to the first bank in a match: this is bank 1 with probability $1/n$. The match is inside with probability x_1/n; it is then productive if and only if module 1 is enabled: this occurs with probability $P[x_1]$. The match is outside with probability $(n-x_1)/n$, and it is then of value θ. Outside matches are productive if and only if both modules are enabled: the two modules are independently enabled with probabilities $P[x_1]$ and $P[1]$.

Assuming that business matches occur at all times with the same intensity, we integrate $vt_1[x_1]$ from time 0 to time infinity to get $vc_1[x_1]$.[3] This can be arranged into *ratio quadratic* format, so $vc_1[x_1] = \frac{ax_1^2+bx_1+c}{e(x_1-\alpha_1)(x_1-\alpha_2)}$ where the denominator coefficients are: $e = \text{Log}[\beta]\text{Log}[\beta(q-1)(\rho-1)]Log^2[1-q]n^2$, $\alpha_1 = -1-(\text{Log}[\beta]+2\text{Log}[1-\rho])/\text{Log}[1-q]$, and $\alpha_2 = -(\text{Log}[1-\rho]+\text{Log}[\beta])/\text{Log}[1-q]$.[4] We see that $e > 0$ and $\alpha_1 < \alpha_2 < 0$, so Theorem 117 from appendix H holds. This gives the results given in table 2.53 in section 2.4.1, including that vc_1 is either quasi-convex, or quasi-concave (or both).

[3] Similarly $Wc[x]$ is the integral of $Wt[x]$, so $Wc[x] := \int_{t=0}^{t=\infty} Wt[x]dt$

[4] For completeness, the numerator coefficients are:
$a = 2\theta\log(\beta)\log(1-\rho)\log(1-q) + (\theta-1)\log(\beta)\log(1-q)\log(\beta-\beta q) - 2\log(\beta)\log(1-\rho)\log(1-q) + \theta\log^2(1-\rho)\log(1-q) - \log(1-\rho)\log^2(1-q) - \log^2(1-\rho)\log(1-q)$,
$b = 5(\theta-1)\log(\beta)\log^2(1-\rho) + 4\theta\log^2(\beta)\log(1-\rho) - 4\log^2(\beta)\log(1-\rho) + 2(\theta-1)\log^3(1-\rho) - 2\theta n\log(\beta)\log(1-q)\log(1-\rho) - \theta n\log(\beta)\log(1-q)\log(\beta-\beta q) - \theta n\log(1-q)\log^2(1-\rho) + 2\theta\log(\beta)\log(1-q)\log(1-\rho) + \theta\log^2(\beta)\log(\beta-\beta q) - 5\log(\beta)\log(1-q)\log(1-\rho) - \log(\beta)\log^2(\beta-\beta q) + \theta\log(1-q)\log^2(1-\rho) - 3\log(1-q)\log^2(1-\rho) - \log^2(1-q)\log(1-\rho)$,
and $c = -5\theta n\log(\beta)\log^2(1-\rho) - 4\theta n\log^2(\beta)\log(1-\rho) - 2\theta n\log^3(1-\rho) - 2\theta n\log(\beta)\log(1-q)\log(1-\rho) - \theta n\log^2(\beta)\log(\beta-\beta q) - \theta n\log(1-q)\log^2(1-\rho)$

Appendix J

Club Goods

Buchanan (1965) introduces the economic analysis of joint consumption via club membership. He considers a setup where for such club goods, the utility of agent i depends on the level of good acquired by i's club and the number of members of that club. This can be modelled mathematically as follows.[1] The setup is that there is a set S of agents (index i or k) and n different goods (index j) that each agent can consume. Consumption can be private (individual i consumes quantity x_j^i separate from the rest of society); public (individual i consumes quantity x_j jointly with the rest of society and so we need $x_j^i = x_j^k$, for each pair of agents i and k), or within a club (clubs are formed endogenously and consumption takes place within them). We can capture the preferences of agent i using a utility function of the form

$$U^i[(x_j^i, m_j^i)_{j=1}^n]$$

where $x_j^i \in \mathbb{R}$ is an amount of good j that person i has access to shared amongst the club m_j^i. With a society that allows such club consumption we need to both update the clearing conditions from private goods and to introduce consistency

[1] This is a generalisation of the original Buchanan (1965) setup in that I allow agent i to care about the identity of his fellow club members. This makes it easier to specify the conditions for both consistency of choice and clearing.

of choice criteria:

- Consistency of choice

  1. You have to belong to your own club. So $i \in m^i_j$. Note that, someone who might describe themselves as not being in a club can be described as being in their own singleton club $\{i\}$.
  2. Everyone is in one and only one club. So the m^i_j collectively form a partition of S. All agents need to agree on the implementation of the same partition: there exists $p_j \in P_j$ such that $\left(i \in m^i_j \in p_j \in P_j\right)_{i \in S}$.
  3. Here P_j is a list of allowable partitions (of the agents S) for good j. Two special cases are as follows:
     - If P_j consists just of the grand coalition then good j is *public* and this forces that $m^i_j = S$ for all i.
     - If P_j consists just of the atomic coalition then good j is *private* and this forces that $m^i_j = \{i\}$ for all i.
  4. Everyone within a club has the same access to a pool of resources: so $m^i_j = m^k_j \Rightarrow x^i_j = x^k_j$
     - In a slight abuse of notation let x^m_j represent this common quantity, the amount of good j consumed jointly by the members of club m. Precisely, $m = m^i_j \Rightarrow x^m_j := x^i_j$, whilst $m \neq m^i_j \; \forall i \Rightarrow x^m_j := 0$

- Clearing

  - Suppose the gross endowment of good j is ω_j
  - Aggregate consumption of good j is $\sum_{m \in p_j} x^m_j$: we only need to count the consumption by each club once.

– So the clearing conditions are $\left(\omega_j = \sum_{m \in p_j} x_j^m\right)_{j=1}^n$

But note this is different from the partition function model. With the club model agent i's utility depends only on the members of the clubs he is in $\left(m_j^i\right)_{j=1}^n$ and how much resource his clubs has $\left(x_j^i\right)_{j=1}^n$. It is independent of the resource levels of other clubs $\left(\left(x_j^k\right)_{i \notin m_j^k}\right)_{j=1}^n$, and the membership of those clubs$\left(\left(m_j^k\right)_{i \notin m_j^k}\right)_{j=1}^n$. So a club model could capture the notion that the probability that a module is enabled depends on the membership list of that module; but it could not capture that the utility of bank i depends not only on the membership of its own module but also of the membership of the other modules. Although later papers allow an agent to be a member of multiple clubs for a single good (see for example Ellickson, Grodal, Scotchmer and Zame (1999, 2001)); they do not remove the restriction that externalities are contained within a club.

Appendix K

Mathematica Appendix

Mathematica was used to support this work in a number of ways. Firstly it was used to sketch graphically the welfare function for symmetric partitions. Secondly it was used to do algebraic manipulation. Thirdly it was used to derive the parametrisation tables that show for each model how the percentage gain (or loss) from choosing the best interior partition over the best boundary partition. The code for forming every gain table is not given as the code is very similar in the different models. As examples, the explanation of how this was down for the standard model, the original circular model and the variable shock initialisation model are given below:

K.1 Standard Model

First the Welfare and module enable probability functions are defined:

$$W[\texttt{d\_}, \texttt{n\_}, \theta\texttt{\_}, \gamma\texttt{\_}] := d * P[d, \gamma] + \theta * (n - d) * P[d, \gamma]^2$$
$$P[\texttt{d\_}, \gamma\texttt{\_}] := (1 + \gamma * d)^{-1}$$

Then a data matrix is formed:

$$\texttt{StdModelData} = \texttt{Flatten}[\texttt{Table}[\{n, \theta, \gamma, P[n, \gamma]/P[1, \gamma], W[1, n, \theta, \gamma], W[n, n, \theta, \gamma], 100 * (W[1, n, \theta, \gamma] - W[n, n, \theta, \gamma])/W[n, n, \theta, \gamma]\}, \{n, \{10, 20, 30, 40, 50, 100, 1000\}\}, \{\theta, 0.1, 1, 0.1\}, \{\gamma, 0.1, 3.0, 0.1\}], 2];$$

Each row is a different parametrisation. The columns are 1) n, 2) θ, 3) γ, 4) $P[n]/P[1]$, 5) $W[1]$, 6) $W[n]$, 7) $100 * (W[1] - W[n])/W[n]$.

Next a function is defined that for each row categories the percentage gain into one of 12 bins:

$$\texttt{StdCounts}[\texttt{data\_}]\texttt{:=}\texttt{BinCounts}[\texttt{Flatten}[\texttt{Drop}[\texttt{data}, \texttt{None}, 6]], \{\{-100, 0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 25000\}\}]$$

Finally each output table is produced:

$$\texttt{StdnOutput} = \texttt{Map}[\texttt{StdCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{StdModelData}, \#[[1]]\&], \#1[[1,1]] < \#2[[1,1]]\&]];\text{(* this forms table 1.5 *)}$$
$$\texttt{Std}\theta\texttt{Output} = \texttt{Map}[\texttt{StdCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{StdModelData}, \#[[2]]\&], \#1[[1,2]] < \#2[[1,2]]\&]];\text{(* this forms table 1.6 *)}$$
$$\texttt{Std}\gamma\texttt{Output} = \texttt{Map}[\texttt{StdCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{StdModelData}, \#[[3]]\&], \#1[[1,3]] < \#2[[1,3]]\&]];\text{(* this forms table 1.7 *)}$$

Taking the n table as an example the output is produced as follows. Starting from the inside and working out, The $GatherBy$ command forms separate $n10$, $n20$, $n30$, $n40, ...$ matrices from the data matrix: one matrix for each value of n. So for example the $n10$ matrix has one row for each parametrisation where $n = 10$ and the same columns as the data matrix. Next the $Sort$ command ensures that the n matrices are in numerical order. Finally the Map command evaluates the $StdCounts$ function for each n matrix: each n matrix producing a row of output. The θ and γ tables are produced similarly.

K.2 Original Circular Model

The circular welfare function is defined as:

```
WC[n_,n_,θ_,γ_]:=P[n,
γ] (* include special rule for single module case with no boundaries*)
WC[d_,n_,θ_,γ_]:=(1 − d^-1)P[d,γ] + θd^-1 P[d,γ]^2 (* for the general case, there is an inside match with probability (d−1)/d; an outside match with probability 1/d *)
```

where $P[d,\gamma]$has the same definition as in the standard model.

The percentage gain from an interior partition is defined as follows:

```
WCperc[n_,θ_,γ_]:=Module[{Wi,Wb},
  Wi=Max[Table[WC[n/k,n,θ,γ],{k,2,n-1}]];
   (*best interior option*)[1]
  Wb=Max[WC[1,n,θ,γ],WC[n,n,θ,γ]];
   (*best boundary option*)
  100*(Wi − Wb)/Wb
   (*what % the best interior option is higher than best boundary option*)
]
```

The data matrix is formed as follows:

```
Circledata = Flatten[Table[{n,θ,γ,WCperc[n,θ,γ]},{n,{10,20,
30,40,50,100,1000}},{θ,0.1,0.9,0.1},{γ,0.1,3,0.1}],2];
```

[1]Note that with this model it is necessary to check all the interior partitions. However, with certain models, such as the variable shock initialisation model below, it is possible to run a pre-process that excludes parametrisations where it is certain the best interior partition has lower welfare than the best boundary partition.

The bin categorisation function used is:

$$\texttt{CircleCounts}[\texttt{data\_}]\texttt{:=BinCounts}[\texttt{Flatten}[\texttt{Drop}[\texttt{data}, \texttt{None}, 3]],$$
$$\{\{-100, 0, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100\}\}]$$

Then finally the three output tables are formed:

$$\texttt{CirclenOutput} = \texttt{Map}[\texttt{CircleCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{Circledata},$$
$$\#[[1]]\&], \#\texttt{1}[[1, 1]] < \#\texttt{2}[[1, 1]]\&]];$$
$$\texttt{Circle}\theta\texttt{Output} = \texttt{Map}[\texttt{CircleCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{Circledata},$$
$$\#[[2]]\&], \#\texttt{1}[[1, 2]] < \#\texttt{2}[[1, 2]]\&]];$$
$$\texttt{Circle}\gamma\texttt{Output} = \texttt{Map}[\texttt{CircleCounts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{Circledata},$$
$$\#[[3]]\&], \#\texttt{1}[[1, 3]] < \#\texttt{2}[[1, 3]]\&]];$$

K.3 Variable Shock Initialisation Model

The payoff per member of module 1 when the other modules are all singletons, $v_{1\alpha}[x_1]$, is given by the mathematica code:

$$\texttt{v1}\alpha[\texttt{x1\_}, \texttt{n\_}, \theta\texttt{\_}, \alpha\texttt{\_}, \gamma\texttt{\_}]\texttt{:=x1}P[\texttt{x1}, \alpha, \gamma] + \theta P[\texttt{x1}, \alpha, \gamma](n - \texttt{x1})P[1, \alpha, \gamma]$$

and the module enablement probability is given by:

$$P[\texttt{d\_}, \alpha\texttt{\_}, \gamma\texttt{\_}]\texttt{:=}(1 + \gamma d^{\alpha})^{-1}$$

The percentage gain in $v_{1\alpha}[x_1]$ from the best interior partition over the best boundary partition is given by

```
v1αperc[n_, θ_, α_, γ_]:=Module[{vi, vb, Perc, x1},
   vi = Max[Table[v1α[x1, n, θ, α, γ], {x1, 2, n − 1}]];
   (*best interior option*)
   vb = Max[v1α[1, n, θ, α, γ], v1α[n, n, θ, α, γ]];
   (*best boundary option*)
   100 ∗ (vi − vb)/vb
   (*what % best interior option is higher than best boundary option*)
]
```

The data matrix is formed as follows:

```
v1αdata = Map[Append[#, v1αperc[#[[1]], #[[2]], #[[3]], #[[4]]]]&,
Select[Select[Flatten[Table[{n, θ, α, γ, v1α[2, n, θ, α, γ] − v1α[1,
n, θ, α, γ], v1α[n, n, θ, α, γ] − v1α[n − 1, n, θ, α, γ]}, {n, {10, 20, 30,
40, 50, 100, 1000}}, {θ, 0, 0.9, 0.1}, {α, 1.0, 30, 0.1}, {γ, 0.1, 3, 0.1}], 3],
#[[5]] > 0&], #[[6]] < 0&]];
```

Note, that from Theorem 72 the α model is quasi-concave when $\alpha \geq 1$. So in order to get an interior solution we need $v_{1\alpha}[n-1] > v_{1\alpha}[n-1]$ and $v_{1\alpha}[1] < v_{1\alpha}[2]$. The v1$\alpha$data code therefore discards parametrisations from either of these conditions does not hold: this makes the code more efficient as it does not waste time looking check for interior gains when this is not feasible.

The bin categorisation function is given by:

```
v1αCounts[data_]:=BinCounts[Flatten[Drop[data, None, 6]],
{{−100, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 250}}]
```

The four output matrices are given by:

$\texttt{v1}\alpha\texttt{nOutput} = \texttt{Map}[\texttt{v1}\alpha\texttt{Counts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{v1}\alpha\texttt{data}, \#[[1]]\&],$

$\#\texttt{1}[[1,1]] < \#\texttt{2}[[1,1]]\&]];$

$\texttt{v1}\alpha\theta\texttt{Output} = \texttt{Map}[\texttt{v1}\alpha\texttt{Counts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{v1}\alpha\texttt{data}, \#[[2]]\&],$

$\#\texttt{1}[[1,2]] < \#\texttt{2}[[1,2]]\&]];$

$\texttt{v1}\alpha\alpha\texttt{Output} = \texttt{Map}[\texttt{v1}\alpha\texttt{Counts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{v1}\alpha\texttt{data}, \#[[3]]\&],$

$\#\texttt{1}[[1,3]] < \#\texttt{2}[[1,3]]\&]];$

$\texttt{v1}\alpha\gamma\texttt{Output} = \texttt{Map}[\texttt{v1}\alpha\texttt{Counts}, \texttt{Sort}[\texttt{GatherBy}[\texttt{v1}\alpha\texttt{data}, \#[[4]]\&],$

$\#\texttt{1}[[1,4]] < \#\texttt{2}[[1,4]]\&]];$

At this stage it is necessary to reform correctly the ≤ 0 column in the output table by adding back in the deleted cases:

$\texttt{TableForm}[\texttt{Map}[\texttt{Prepend}[\#,$

$(611100/7) - \#[[1]] - \#[[2]] - \#[[3]] - \#[[4]] - \#[[5]] - \#[[6]] - \#[[7]]]\&,$

$\texttt{Drop}[\texttt{v1}\alpha\texttt{nOutput}, \texttt{None}, 1]], \texttt{TableHeadings} \to \{\texttt{Table}[n, \{n, \{10,$

$20, 30, 40, 50, 100, 1000\}\}], \{-100, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90,$

$100, 250\}\}]$

$\texttt{TableForm}[\texttt{Map}[\texttt{Prepend}[\#,$

$(611100/9) - \#[[1]] - \#[[2]] - \#[[3]] - \#[[4]] - \#[[5]] - \#[[6]] - \#[[7]]]\&,$

$\texttt{Drop}[\texttt{v1}\alpha\theta\texttt{Output}, \texttt{None}, 1]], \texttt{TableHeadings} \to \{\texttt{Table}[\theta, \{\theta, 0.0, 0.5,$

$0.1\}], \{-100, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 250\}\}]$

$\texttt{TableForm}[\texttt{Map}[\texttt{Prepend}[\#,$

$(611100/21) - \#[[1]] - \#[[2]] - \#[[3]] - \#[[4]] - \#[[5]] - \#[[6]] - \#[[7]]]\&,$

$\texttt{Drop}[\texttt{v1}\alpha\alpha\texttt{Output}, \texttt{None}, 1]], \texttt{TableHeadings} \to \{\texttt{Table}[\alpha, \{\alpha, 1.1,$

$30, 0.1\}], \{-100, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 250\}\}]$

$\texttt{TableForm}[\texttt{Map}[\texttt{Prepend}[\#,$

$(611100/30) - \#[[1]] - \#[[2]] - \#[[3]] - \#[[4]] - \#[[5]] - \#[[6]] - \#[[7]]]\&,$

$\texttt{Drop}[\texttt{v1}\alpha\gamma\texttt{Output}, \texttt{None}, 1]], \texttt{TableHeadings} \to \{\texttt{Table}[\gamma, \{\gamma, 0.1, 3,$

$0.1\}], \{-100, 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 250\}\}]$

Appendix L

Lattice Appendix

This appendix first defines what a *lattice* is. It then compares two different relevant lattice formations: the *partition lattice* and the *embedded coalition lattice*. The implications for the use of the Topkis (1978) supermodularity approach are then drawn out.

A *partial ordering* $\leq$ is a binary relation that satisfies the following conditions:

- (reflexivity) $a \leq a$
- (antisymmetry) if $a \leq b$ and $b \leq a$ then $a = b$.
- (transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$.

A *poset* consists of a set L, together with such a partial ordering $\leq$. If two elements, a and b, of a poset have a (unique) least upper bound (or supremum), denoted, $a \vee b$, it is their *join*. If two elements, a and b, of a poset have a (unique) greatest lower bound (or infimum), denoted $a \wedge b$, it is their *meet*. A poset $(L, \leq)$ is a *lattice* if it satisfies the following two closure axioms:

- Existence of binary joins: for any two elements a and b of L, the set L contains the join $a \vee b$.

- Existence of binary meets: for any two elements a and b of L, the set L contains the meet $a \wedge b$.

One condition that a lattice function $f : L \to \mathbb{R}$ may satisfy is *supermodularity*: for any two elements a and b of L, $f[a \wedge b] + f[a \vee b] \geq f[a] + f[b]$.

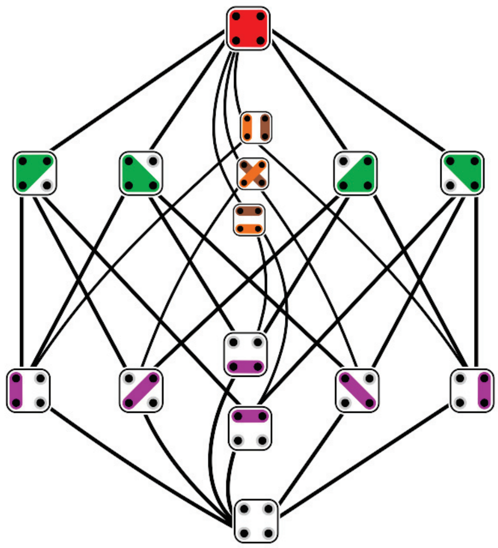
The partition lattice is a well known and important example of lattices: see for example Gratzer (2005) pp250–263 and Roman (2008) pp110–120. The presentation given here in terms of equivalence relations is based on section 4 of Nation (1991). An equivalence relation on a set Y is a binary relation E satisfying, for all $x, y, z \in Y$:

1. (reflexivity) xEx
2. (symmetry) xEy implies yEx
3. (transitivity) if xEy and yEz then xEz.

We think of an equivalence relation as partitioning the set Y into blocks of E-related elements, called equivalence classes: x and y are in the same equivalence class if and only if xEy. Conversely, any partition of Y into a disjoint union of blocks induces an equivalence relation on Y: xEy if and only if x and y are in the same block. So, in a slight abuse of notation, we write xEy and $(x, y) \in E$ interchangeably. We will be defining a lattice on X, the set of all partitions of Y, (or alternatively let X be the set of all equivalence relations on Y so $X = EqY$).

The next stage is to define an ordering on X. The ordering used is the *refinement* ordering: $E_1 \leq E_2$ if and only if xE_1y implies xE_2y. So E_1 is a refinement of E_2 if and only if each E_1 equivalence class is contained entirely within a single E_2 equivalence class. The greatest element of X is the universal relation Y^2: $xE_{Y^2}y$ iff (x, y) in Y^2; so the only equivalence class is the grand coalition Y. The least element of X is the equality relation $=$, so: $xE_=y$ iff $x = y$; this gives atomic equivalence classes each containing a single member of Y. The *meet* operation on X is set intersection, which means that $x(E_1 \wedge E_2)y$ if and only if xE_1y and xE_2y.[1] The *join* operation is given by transitive closure, which means that $x(E_1 \vee E_2)y$ if and only if there exists a finite sequence from x to y using other members of Y as intermediate points and equating using either E_1 or E_2 at each stage; so formally, there exists a finite sequence $(x_j \in Y)_{j=0}^{k}$ such that $x_0 = x$, $\left(x_{j-1}E_{i_j}x_j \text{ for some } i_j \in \{1,2\}\right)_{j=1}^{k}$, and $x_k = y$.[2] Finally, here is a *Hasse diagram* for the case where Y has 4 elements.

Figure L.1: Partition Lattice with 4 elements ordered by refinement



from http://en.wikipedia.org/wiki/File:Set_partitions_4;_Hasse;_circles.svg

A related but distinct lattice of *embedded coalitions* is formed in section 3 of Grabisch (2010). Let N be the set of n players, let $\Pi(N)$ be the set of partitions of the n players. Two specific partitions are the grand coalition partition $\pi^{GC} := \{N\}$, and the atomistic partition of singletons $\pi^{Atom} := \{\{i\} : i \in N\}$. An

[1]See Nation (1991) page 35
[2]Again, see Nation (1991) page 35

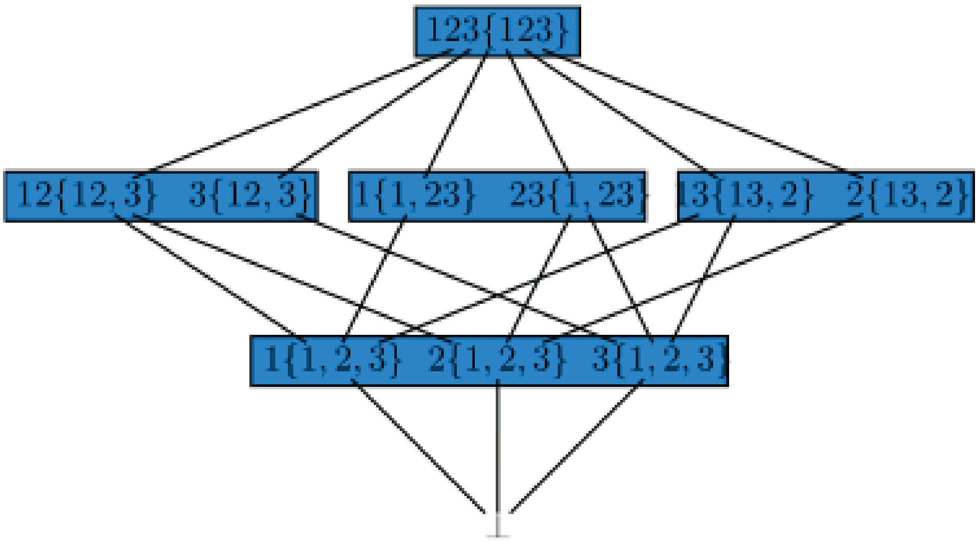
embedded coalition consists of a partition and an equivalence class from that partition. Formally, it is a pair (S, π), where $\pi \in \Pi(N)$ and $S \in \pi$. Let $C(N)$ denote all such pairs and thus be the set of embedded coalitions. A naturally ordering on $C(N)$ is the *product ordering* which requires the ordering to apply both to the subset S and the partition π: $(S, \pi) \sqsubseteq (S', \pi')$ iff $S \subseteq S'$ and $\pi \leq \pi'$, where $\leq$ is the refinement ordering on partitions specified above.[3] With this ordering the top element of $C(N)$ is (N, π^{GC}). However, partitions do not include empty classes and so $C(N)$ has no least element, since all elements of the form $(\{i\}, \pi^{Atom})$ are minimal. So for mathematical convenience, we introduce an artificial bottom element $b := (\emptyset, \pi^{Atom})$, and define $C^b(N) = C(N) \cup \{b\}$. Proposition 2 of Grabisch (2010) shows that, for any $n > 2$, $\left(C^b(N), \sqsubseteq\right)$ is a lattice where the meet and the join are given by:

- $(S, \pi) \wedge (S', \pi') = \begin{cases} (S \cap S', \pi \wedge \pi') & \text{if } S \cap S' \neq \emptyset \\ b & \text{if } S \cap S' = \emptyset \end{cases}$
- $(S, \pi) \vee (S', \pi') = (T \cup T', \rho)$, where T, T' are the blocks of $\pi \vee \pi'$ containing respectively S and S', and ρ is the partition obtained by merging T and T' in $\pi \vee \pi'$.

This is a Hasse diagram for the 3 player case:

[3] Similarly, $\pi \wedge \pi'$ is the meet given by the refinement ordering on partitions specified above and, $\pi \vee \pi'$ is the join given by the refinement ordering on partitions specified above.

Figure L.2: Embedded Coalition Lattice with 3 elements



This is Figure 1 from Grabisch (2010) and elements with the same partition are framed in blue.

Note that Topkis (1978, 1998) gives results on the maximisation of a supermodular function specified on a lattice; specifically how the argmax alters with parameter variation. However, whilst there are clean conditions for the module worth V_i to be supermodular with respect to the embedded coalition lattice (see Definition 29 and Theorem 30); there are not clean results for the welfare function W to be supermodular with respect to the partition lattice (the supermodularity conditions need to be solved separately for each value of n). Partition Lattice Supermodularity for the Welfare function W requires that for all partitions S and T that:

$$\mathrm{W[S \wedge T]} + \mathrm{W}[S \vee T] \geq W[S] + W[T]$$

If $n = 3$ then the only potentially non-trivial case is when S and T are distinct interior partitions for example $S = \{12, 3\}$ and $T = \{1, 23\}$. This leads to the requirement $W[1, 2, 3] + W[123] \geq W[12, 3] + W[1, 23]$, which holds for all parametrisations.

If $n = 4$ then the four potentially non-trivial cases are

$$W[123, 4] + W[1, 2, 3, 4] \geq W[12, 3, 4] + W[13, 2, 4]$$

$$W[1234] + W[1, 2, 3, 4] \geq W[12, 3, 4] + W[234, 1]$$

$$W[1234] + W[1, 2, 3, 4] \geq W[12, 3, 4] + W[14, 23]$$

$$W[1234] + W[1, 4, 23] \geq W[123, 4] + W[14, 23]$$

In order for all 4 conditions to hold need that either $\theta \leq \frac{\gamma+1}{-6\gamma^2+3\gamma+1}$ or $3\gamma \geq 1$.

Hence the partition supermodularity approach was not used in this thesis. However, Milgrom and Shannon (1994) introduced the weaker condition of *quasi-supermodularity*:

Definition 118. a lattice function $f : L \to \mathbb{R}$ is *quasi-supermodular* if for any two elements a and b of L, $f[a] - f[a \wedge b] \geq 0 \Rightarrow f[a \vee b] - f[b] \geq 0$ and $f[a] - f[a \wedge b] > 0 \Rightarrow f[a \vee b] - f[b] > 0$

and proved the following characterisation theorem:

Theorem 119. *Milgrom and Shannon (1994) Theorem 4 Let $f : L \times T \to R$, where L is a lattice, T is a partially ordered set and $S \subseteq L$. Then* $\operatorname{argmax} f(l, t)_{l \in S}$ *is monotone non-decreasing in (t, S) iff both f is quasi-supermodular in L and satisfies the single crossing property*

which uses two additional definitions. Firstly,

Definition 120. Let L be a lattice (choice set), let T a partially ordered set (of parameters), and $f : L \times T \to \mathbb{R}$. Then f satisfies the *single crossing property* iff if $l' > l''$ and $t' > t''$ then $f(l', t'') > f(l'', t'') \Rightarrow f(l', t') > f(l'', t')$ and $f(l', t'') \geq f(l'', t'') \Rightarrow f(l', t') \geq f(l'', t')$

and secondly,

Definition 121. Suppose that $M : T \to 2^L$ and T is a lattice. Then the set function M is *monotone nondecreasing* if $t' \geq t$ implies that $M(t') \geq_S M(t)$ with respect to the set ordering. if $m \in M(t)$ and $m' \in M(t')$ then $m \wedge m' \in M(t)$ and $m \vee m' \in M(t')$

In the case of symmetric functions, quasi-supermodularity with respect to the partition lattice is sufficient to give boundary solutions. A *symmetric function* is one where the value depends only on the number of members in each module and does not depend on the sequencing of modules:[4]

Definition 122. Suppose $f : P^n \to \mathbb{R}$ where P^n is the lattice of partitions of n objects. Then f is symmetric iff there exists$f_X : \mathbb{N}^{\mathrm{n}} \to \mathbb{R}$ such that

1. $f(l) = f_X(x)$ where $x = (x_i)_{i=1}^k$ and $(x_i = |l_i|)_{i=1}^k$, where k is the number of modules in partition l and $|l_i|$ represents the number of elements in module l_i.
2. If $x' = \tau(x)$ and τ is a permutation then $f_X(x') = f_X(x)$.

This leads to the following characterisation theorem:

Theorem 123. *If $f : P^n \to \mathbb{R}$ where P^n is the lattice of partitions, f is quasi-supermodular with respect to the partition lattice and f is symmetric then the argmax is a subset of the 2 boundary partitions (the grand coalition and the atomic partition of singletons).*

[4]The use of the term *symmetric functions* is from the mathematics literature, see for example MacDonald (2005). In economics equivalently a welfare function would be described as having *anonymity,* which is the term I used in Section 1.7.

Proof. The proof is by contradiction. Suppose $p \in \mathrm{argmax}_{p \in P^n} f(p)$ and $p = (p_i)_{i=1}^k$ is an interior partition. Let $(x_i = |p_i|)_{i=1}^k$.Without loss of generality we can assume that the modules in p are arranged in non-increasing size so $x_i \geq x_{i+1}$. And as there are only a finite number of partitions in P^n we can assume that p has the (weakly) largest module. So $q \in \mathrm{argmax}_{p \in P^n} f(p)$ and $q_j \in q$ implies $|q_j| \leq x_1$. Without loss of generality we can assume the n objects are assigned to modules in increasing order. So $p_1 = \{1, 2, 3, ...x_1 - 1, x_1\}$,$p_2 = \{x_1 + 1, x_1 + 2, x_1 + 3, ...x_1 + x_2\}$ and in general $p_i = \{1 + \sum_{r=1}^{i-1} x_r, 2 + \sum_{r=1}^{i-1} x_r, ...x_i + \sum_{r=1}^{i-1} x_r\}$.

Now consider the partition p^m where a single member is switched around between the first and mth modules. So, $p_1^m = \{1, 2, 3, ...x_1 - 2, x_1 - 1, \sum\limits_{r=1}^{m} x_r\}$ and $p_m^m = \{1 + \sum_{r=1}^{m-1} x_r, 2 + \sum_{r=1}^{m-1} x_r, ...x_m - 2 + \sum_{r=1}^{m-1} x_r, x_m - 1 + \sum_{r=1}^{m-1} x_r, x_1\}$. Whilst a general element is unaltered, so $p_i^m = p_i = \{1 + \sum_{r=1}^{i-1} x_r, 2 + \sum_{r=1}^{i-1} x_r, ...x_i + \sum_{r=1}^{i-1} x_r\}$ when $i \neq 1$ and $i \neq m$.

So $|p \wedge p^m| = (x_1 - 1, 1, x_2, x_3..., x_m - 1, 1, x_{m+1}, ...x_k)$ and $|p \vee p^m| = (x_1 + x_m, x_2, x_3..., x_{m-1}, x_{m+1}, ...x_k)$.

As $p \in \mathrm{argmax}_{p \in P^n} f(p)$, $f[p] \geq f[p \wedge p^m]$. So by quasi-supermodularity $f[p \vee p^m] \geq f[p^m]$ and as f symmetric, $f_X[|p \vee p^m|] \geq f_X[|p^m|]$. But as p^m is a permutation of p, $f_X[|p^m|] = f_X[|p|]$. So as $p \in \mathrm{argmax}_{p \in P^n} f(p)$, we require $p \vee p^m \in \mathrm{argmax}_{p \in P^n} f(p)$, but $p \vee p^m$ has a module of size $x_1 + x_m > x_1$ which is a contradiction. $\square$

However in practice it is difficult to show that a partition lattice function is quasi-supermodular as the number of partitions of n objects is given by the nth Bell number and this sequence grows very quickly: $1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975$,... An alternative approach that involves considering less cases is to use the $f_X(.)$function directly. Suppose that for all x:

$f_X[x_1 + x_2, x_3, x_{-3}] \geq f_X[x_1, x_2, x_3, x_{-3}] \Rightarrow f_X[x_1 + x_2 + x_3, x_{-3}] \geq f_X[x_1 + x_2, x_3, x_{-3}]$

where $x_{-3} := (x_i)_{i=4}^k$

So if merging the first 2 modules of x increases f then repeating the process increases f further. This quasi-supermodularity type condition is similarly sufficient for boundary solutions. Let $g[r] := f[x_1+x_2+rx_3, 0, x_3-rx_3, x_{-3}]-f[x_1+rx_2, x_2-rx_2, x_3, x_{-3}]$. Then the condition becomes $g[0] \geq 0 \Rightarrow g[1] \geq 0$. For the standard model:

$$g[0] \geq 0 \Leftrightarrow \theta \leq \theta_0^c$$

where

$$\theta_0^c := \frac{(\gamma 0.5s+1)(\gamma x_3+1)}{2\gamma R+\gamma^2 Rs+\gamma^3 Rsx_3+2\gamma^2 Rx_3+\gamma s+2\gamma^2 sx_3+3\gamma x_3+1}$$

which uses the substitutions $R := \sum_{i=5}^k x_i P[x_i]$ and $s := x_1+x_2$. Similarly:

$$g[1] \geq 0 \Leftrightarrow \theta \leq \theta_1^c := \frac{\gamma 0.5t+1}{2\gamma R+\gamma^2 Rt+\gamma t+1}$$

where $t := x_1+x_2+x_3$.

Comparison of θ_0^c and θ_1^c gives that:

$$\theta_1^c-\theta_0^c = \frac{\gamma(\gamma s+1)x_3(\gamma t+3)}{2(\gamma R(\gamma t+2)+\gamma t+1)(\gamma R(\gamma s+2)(\gamma x_3+1)+2\gamma^2 sx_3+2\gamma x_3+2\gamma t+1)} > 0$$

So this provides an alternative proof that the standard model has boundary solutions. Unfortunately attempts to apply the same method to some of the other models were unsuccessful.

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