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Gröbner Bases over fields with Valuations and Tropical Curves by Coordinate Projections

by

Andrew John Chan

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself, except where stated otherwise, and has not been submitted in any previous application for any degree or award.

The work presented was carried out by the author, except for the work of Chapter 3 which was carried out jointly with my supervisor Diane Maclagan [Chan and Maclagan, 2013]. A preprint of our paper "Gröbner bases over fields with valuations" can be found on the ArXiv:1303.0729.

Abstract

In the emerging field of tropical geometry, algebraic varieties are replaced by polyhedral objects called tropical varieties. The algebraic and tropical variety share many invariants, but due to its polyhedral structure the tropical variety is often easier to work with. In this thesis, we look at two problems related to constructing tropical varieties.

In the first, we extend the theory of Gröbner bases to the case where we are looking over a field with a valuation. The motivation is that we can use these Gröbner bases in order to compute tropical varieties over fields with valuations. We discuss some complexity and implementation issues and present a family of ideals whose Gröbner basis with respect to the p-adic valuation is small, but all of whose standard Gröbner bases are large.

In the second, we investigate finding tropical curves over fields with the trivial valuation from their two-dimensional coordinate projections. A tropical curve has the support of a one-dimensional fan, and we use its coordinate projections to reconstruct the rays of this fan. We discuss some implementation issues and we see examples of tropical curves which can be computed using our projection techniques which cannot be computed with existing techniques.

Chapter 1

Introduction

In the emerging field of tropical geometry, we replace a subvariety X of the n-dimensional algebraic torus with a polyhedral object $\text{trop}(X)$ called a tropical variety. A fundamental question is to determine the structure of this polyhedral object. This is because the algebraic variety X and tropical variety $\text{trop}(X)$ share many of the same invariants, such as their degree and dimension. This means that understanding $\text{trop}(X)$ would help us to understand X. Due to its polyhedral nature, the tropical variety is often easier to understand as combinatorics can be used to study its polyhedral structure. In this thesis we will investigate two different constructions of tropical varieties. The first, which we will see in Chapter 3, uses the theory of Gröbner bases, and the second, which we will see in Chapter 4, uses coordinate projections.

Let K be an algebraically closed field equipped with a valuation, which is a function $K^* \to \mathbb{R}$ where K^* denotes the non-zero elements of K. The Fundamental Theorem of Tropical Geometry (Theorem 2.1.5) will give us a way of constructing tropical varieties over a field with a valuation using a variant of Gröbner theory that takes the valuation of coefficients into account. Currently, computational work is focused on the case of $K = \mathbb{Q}$ with the trivial valuation (which is the valuation for which all non-zero elements have valuation zero) as this can be analysed using standard Gröbner techniques where the valuations of the coefficients do not play a role.

Buchberger [1965] introduced an algorithm which could compute these standard Gröbner bases. In Chapter 3, joint work with my supervisor Diane Maclagan [Chan and Maclagan, 2013], we will extend this Gröbner theory to our situation where we take the valuations of coefficients into account.

In Section 3.6 we introduce `GroebnerValuations`, which is a computer package to compute Gröbner bases over $\mathbb{Q}$ with the p-adic valuation. As is common for standard Gröbner bases over $\mathbb{Q}$, the coefficients can grow very large. However, we show (Proposition 3.3.4) that valuations of coefficients occurring in a Gröbner basis over a field with a valuation can be bounded in terms of the valuations and absolute values of the coefficients of the original generators. This motivates working over $\mathbb{Z}/p^m\mathbb{Z}$ for some suitably large $m \in \mathbb{N}$. We see this, and other implementation issues in Section 3.4.

We end this part by seeing a family of examples of ideals whose Gröbner basis with respect to the 2-adic valuation has size two but any of whose standard Gröbner bases with respect to any standard term ordering has size at least linear in the degree. This gives a usefulness to this work outside of tropical geometry as it provides an example where an ideal has a small p-adic Gröbner basis but all of whose standard Gröbner bases are large.

In the second part of this document, we concentrate on constructing tropical varieties over a field with the trivial valuation. The current methods to construct these tropical varieties comes from the work of Bogart, Jensen, Speyer, Sturmfels, and Thomas [2007]. A key step in their algorithms is the construction of a tropical curve. Thus in Chapter 4 we concentrate on tropical curves as any improvement in the tropical curve algorithm would provide improvement to the tropical variety algorithm. Further, we see in Section 4.3 an example of a tropical curve which cannot be computed efficiently using the existing methods, but which can be computed using the methods described in Chapter 4.

Let C be a one-dimensional subvariety of the n-dimensional algebraic torus $(K^*)^n$. Then the Structure Theorem of Tropical Varieties (Theorem 2.2.5) will tell us that the tropical curve $\text{trop}(C)$ has the support of a one-dimensional fan in $\mathbb{R}^n$. We look to use a set of projections to find $\text{trop}(C)$. We will do this by recovering the points that are in the pre-image of all of the projections in this set. This will be a superset of the rays of $\text{trop}(C)$. In Section 2.5 we will see that Bieri and Groves [1984] and Hept and Theobald [2009] show that we can always choose these projections sufficiently generically so that in this way, we will recover only the rays of $\text{trop}(C)$. However, it is often difficult to determine whether projections are sufficiently generic. Additionally, generic projections can be difficult to compute as, for example, the degrees of generators can grow very large.

The key idea is that we will restrict our attention to coordinate projections. In general, these are not generic enough to recover only the rays of the tropical curve, but they are usually easier to compute. In Section 4.2 we provide algorithms to reconstruct tropical curves from a set of coordinate projections in two main steps.

The first is recovering a set of rays which are in the pre-image of all the coordinate projections in our set. This is a superset of the rays in trop(C), and in general will contain extra rays not in trop(C) as our coordinate projections will not be sufficiently generic. So, the second step will be to determine which of the rays in this superset are rays of trop(C).

In Section 4.5, we introduce `TropicalCurves` which is a computer package to compute tropical curves from its coordinate projections using these algorithms. We also discuss various implementation issues.

This document is structured as follows. It is comprised of three chapters:

- **Chapter 2** introduces the basics of tropical geometry. The Fundamental Theorem 2.1.5 gives three equivalent formulations of tropical varieties and the Structure Theorem 2.2.5 asserts that it has the support of a rational polyhedral complex. We then look at the existing methods of Bogart, Jensen, Speyer, Sturmfels, and Thomas [2007] to compute tropical varieties. We outline the work of Bieri and Groves [1984] and Hept and Theobald [2009] on tropical varieties from regular projections and the tropical elimination theory of Sturmfels and Tevelev [2008]. We end by looking at how to compute the degree of a tropical curve from tropical intersection theory.

- **Chapter 3** studies Gröbner bases over fields with valuations. We see how the algorithms from standard Gröbner theory need to be altered when considering valuations of coefficients. We discuss complexity and implementation issues and end with an example of a Gröbner basis over $\mathbb{Q}$ whose p-adic Gröbner basis has size two, but any of whose standard Gröbner bases have size at least linear in the degree.

- **Chapter 4** looks at tropical curves over fields with the trivial valuation. The fan structure of a tropical curve is reconstructed from its two-dimensional coordinate projections. We see some implementation issues, and an example of a tropical curve which cannot be computed using existing techniques, but which can be computed using these coordinate projection methods.

Chapter 2

An Introduction to Tropical Geometry

Tropical geometry can be thought of as a piecewise linear approximation to algebraic geometry where an algebraic variety is replaced by a rational polyhedral complex called a tropical variety. The adjective *tropical* was given to this area of research by a group of French mathematicians including Jean-Eric Pin, Dominique Perrin and Christian Choffrut to honour their Brazilian friend and colleague Imre Simon [1988] who pioneered the use of the tropical semi-ring. This semi-ring, also known as the min-plus semi-ring, originally had important applications to Optimisation Theory and Theoretical Computer Science [Perrin, 1990].

In this chapter, we set out the Tropical Geometry background which we need. Full details can be found in the draft book *"Introduction to Tropical Geometry"* [Maclagan and Sturmfels, 2013]. We begin by defining the tropicalisation $\text{trop}(X)$ of an algebraic variety X in terms of the intersection of tropical hypersurfaces. The Fundamental Theorem 2.1.5 gives us two different, but equivalent, formulations of $\text{trop}(X)$ and the Structure Theorem 2.2.5 tells us that it has the support of a weighted balanced rational polyhedral complex. We then discuss the methods of Bogart, Jensen, Speyer, Sturmfels, and Thomas [2007] to compute $\text{trop}(X)$ and see that the construction of a tropical curve is a key step in their algorithms. We end with some technical material which we will need, including an outline of some tropical elimination theory, and the tropical intersection theory used to find the degree of a tropical curve combinatorially.

2.1 Tropical Varieties and the Fundamental Theorem

The *tropical semi-ring* is $(\mathbb{R} \cup \{\infty\}, \otimes, \oplus)$ where *tropical multiplication* $\otimes$ is the usual addition and *tropical addition* $\oplus$ is the usual minimum. These operations satisfy the familiar axioms of arithmetic; for example they are both commutative and the distributive, 0 is the identity element for tropical multiplication and ∞ is the identity element for tropical addition. In fact all of the ring axioms are satisfied except for the existence of an additive inverse as there is no well defined tropical subtraction. Thus $(\mathbb{R} \cup \{\infty\}, \otimes, \oplus)$ has the structure of a semi-ring.

Let $x_1, \ldots, x_n$ be variables which represent elements in the tropical semi-ring $(\mathbb{R} \cup \{\infty\}, \otimes, \oplus)$. A *tropical monomial* is any product of the variables. By evaluating these tropical monomials with classical arithmetic, the tropical monomial $x_1^{a_1} \ldots x_n^{a_n}$ for some $a_i \in \mathbb{N}$ can be thought of as representing the ordinary linear form $\Sigma_{i=1}^{n} a_i x_i$. As a shorthand, we let $x^a := x_1^{a_1} \ldots x_n^{a_n}$ for $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. A *tropical polynomial* is simply a finite linear combination of tropical monomials: $f(x_1, \ldots, x_n) = \bigoplus_{i=1}^{s} a \otimes x_1^{i_1} \ldots x_n^{i_n}$. This represents a piecewise linear function $\mathbb{R}^n \to \mathbb{R}$.

Let K be an algebraically closed field and by K^* denote the non-zero elements of K. A *real valuation* on K is a function $\mathrm{val}\colon K^* \to \mathbb{R}$ such that the following axioms are satisfied:

1. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$;
2. $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in K^*$.

The image of the valuation map, denoted Γ, is called the *value group*. After scaling if necessary, we can assume that Γ contains 1. We assume that there always exists a group homomorphism $\phi\colon \Gamma \to K^*$ with $\mathrm{val}(\phi(w)) = w$ which is denoted by $\phi(w) = t^w$. This always exists if K is algebraically closed [see, for example, Maclagan and Sturmfels, 2013, Lemma 2.1.15]. The *valuation ring* of K is

$$R = \{a \in K : \mathrm{val}(a) \geq 0\},$$

which consists of all elements of K which have non-negative valuation. It is a local ring with unique maximal ideal

$$\mathfrak{m} = \{a \in K : \mathrm{val}(a) > 0\}.$$

The quotient ring $\Bbbk = R/\mathfrak{m}$ is called the *residue field* of K. For $a \in R$ we denote by $\overline{a}$ the image of a in the residue field $\Bbbk$.

Example 2.1.1. Let $K = \mathbb{Q}$ be the field of rational numbers and fix a prime $p \in \mathbb{Z}$. Any rational number $n \in \mathbb{Q}$ can be written in the form $n = p^m \frac{a}{b}$ for some $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ such that p does not divide a or b. Then the *p-adic valuation* is the map $\mathrm{val}_p \colon \mathbb{Q}^* \to \mathbb{Z}$ where $\mathrm{val}_p(n) = m$. The valuation ring is $\{a \in \mathbb{Q} : \mathrm{val}_p(a) \geq 0\} = \mathbb{Z}_{(p)}$ which has unique maximal ideal $\{a \in \mathbb{Q} : \mathrm{val}_p(a) > 0\} = \langle p \rangle$ and residue field $\mathbb{Z}/p\mathbb{Z}$. The image of val_p is $\Gamma = \mathbb{Z}$ and we can take ϕ to be $\phi(w) = p^w$. ◇

Example 2.1.2. An important example of a field with a valuation is the field of *Puiseux series* over $\mathbb{C}$ denoted by $\mathbb{C}\{\{t\}\}$. An element of $\mathbb{C}\{\{t\}\}$ is of the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \dots$ where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < a_3 < \dots$ are rational numbers bounded below by a_1 and with a common denominator. We can write $\mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}(\!(t^{1/n})\!)$, the union of Laurent series in the formal variable $t^{1/n}$. The field of Puiseux series over $\mathbb{C}$ is algebraically closed [see, for example, Markwig, 2010, Theorem 6]. We define a valuation $\mathrm{val} \colon \mathbb{C}\{\{t\}\} \to \mathbb{R}$ which sends a Puiseux series to its lowest exponent of t. That is, $\mathrm{val}(c(t)) = a_1$. The valuation ring consists of power series with rational exponents with common denominator, and the maximal ideal consists of those Puiseux power series whose constant term is zero. The residue field is $\mathbb{C}$ and the image of val is $\Gamma = \mathbb{Q}$. We can take ϕ to be $\phi(w) = t^w$.

In the Puiseux series definition, the field $\mathbb{C}$ can be replaced by any field K and $K\{\{t\}\}$, the field of Puiseux series over K, can be defined in an analogous way. Similarly to over $\mathbb{C}$, the field of Puiseux series over K is algebraically closed if K is an algebraically closed field of characteristic zero [see, for example, Maclagan and Sturmfels, 2013, Theorem 2.1.5]. In fact, in this case, the field of Puiseux series $K\{\{t\}\}$ is the algebraic closure of the field of Laurent series $K(\!(t)\!)$ [Ribenboim, 1999, 7.1.$A(\beta)$, p.186]. ◇

Example 2.1.3. Let K be an algebraically closed field. Then the *trivial valuation* is the valuation val such that $\mathrm{val}(a) = 0$ for all $a \neq 0$. These are important to consider as many objects we will consider, for example Gröbner bases and the Gröbner complex, originate from the case where we do not consider the valuations of coefficients. Unfortunately, several of the theorems and results in tropical geometry require that the valuation be non-trivial. If K is a field of characteristic zero then we can consider the field K as a subfield of the field of the Puiseux series field $K\{\{t\}\}$. The trivial valuation on K can be thought of as the restriction of the valuation on the field of Puiseux series over K from Example 2.1.2 to K, noting that for the Puiseux series valuation, we have that $\mathrm{val}(a) = 0$ for all $a \in K \subseteq K\{\{t\}\}$. If K has positive characteristic, then as $K\{\{t\}\}$ may not be algebraically closed we need to consider the generalised power series ring $K(\!(\Gamma)\!)$, where Γ is the image of the valuation map val, for val the usual valuation on the Puiseux series. The generalised power series ring $K(\!(\Gamma)\!)$ consists of elements of the form $\sum_{g \in \Gamma} \alpha_g t^g$ where $\alpha_g \in K$ and the set of supports $\{g : \alpha_g \neq 0\}$ is a well-ordered set. Then Poonen [1993, Corollary 5] showed that $K\{\{t\}\}$ is isomorphic to a subfield of the field of generalised power series $K(\!(\Gamma)\!)$ which is algebraically closed [Poonen, 1993, Corollary 4]. We can then think of K with the trivial valuation as the restriction of the valuation on $K(\!(\Gamma)\!)$ to K. $\diamond$

Let $\mathbb{P}^n$ be the n-dimensional projective space over K. Let T^n be the n-dimensional algebraic torus over K. The homogeneous coordinate ring of $\mathbb{P}^n$ is the polynomial ring $K[x_0, x_1, \ldots, x_n]$ and the coordinate ring of T^n is the Laurent polynomial ring $K[x_1^{\pm}, \ldots, x_n^{\pm}]$.

Remark 2.1.4. In tropical geometry we are interested in varieties X which are contained in the algebraic torus T^n. By the inclusion $i \colon T^n \hookrightarrow \mathbb{P}^n$ we can think of X as a projective variety in the following way. The map i is given by the projective closure in $\mathbb{P}^n$ of the map $x \mapsto [1 : x]$. Algebraically, if $X \subseteq T^n$ is given by an ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then first set $J = I \cap K[x_1, \ldots, x_n]$. The homogenisation of J is the Zariski closure of $i(X)$ in $\mathbb{P}^n$ [Maclagan and Sturmfels, 2013, Proposition 2.2.6]. In this way, we can think of a variety of the algebraic torus T^n given by an ideal in $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ as a variety of $\mathbb{P}^n$ given by a homogeneous ideal in $K[x_0, x_1, \ldots, x_n]$.

Conversely, if X is a subvariety of $\mathbb{P}^n$, then we will consider $X^0 = X \cap T^n$ as this is then a subvariety of T^n. $\diamond$

Let $f = \Sigma_{u \in \mathbb{Z}^n} c_u x^u$ be a Laurent polynomial in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We define the map trop, which takes a polynomial to its *tropicalisation*, by sending coefficients to their valuations, the usual addition $+$ to tropical addition $\oplus$ and the usual multiplication $\times$ to tropical multiplication $\otimes$. If $f = \Sigma_{u \in \mathbb{Z}^n} c_u x^u$ is a usual polynomial then its tropicalisation is $\mathrm{trop}(f) = \min\{\mathrm{val}(c_u) + \Sigma_{i=1}^n u_i x_i\}$.

The *tropical hypersurface* $\mathrm{trop}(V(f))$ defined by the polynomial f is the set of points of $\mathbb{R}^n$ where the minimum in $\mathrm{trop}(f)$ is achieved at least twice. The *tropical pre-variety* of a finite set $\{f_1, \ldots, f_s\}$ is the intersection of the tropical hypersurfaces $\mathrm{trop}(V(f_1)), \ldots, \mathrm{trop}(V(f_s))$. Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $X = V(I)$. Then the tropical variety of X is

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)).$$

A finite set $\{f_1, \ldots, f_s\} \subseteq I$ is called a *tropical basis* for I if its tropical pre-variety equals the tropical variety $\mathrm{trop}(X)$. That is:

$$\mathrm{trop}(X) = \bigcap_{i=1}^{s} \mathrm{trop}(V(f_i)).$$

We shall see in the Fundamental Theorem 2.1.5 another more algebraic formulation of the tropical variety. For this, we shall need the theory of initial ideals.

Fix a *weight vector* $w \in \Gamma^n$ and set $W := \operatorname{trop}(f)(w) = \min\{\operatorname{val}(c_u) + w.u : c_u \neq 0\}$. The *initial form* of f with respect to w is defined as

$$\operatorname{in}_w(f) := \sum_{u \in \mathbb{Z}^n : \operatorname{val}(c_u)+w.u=W} \overline{c_u t^{-\operatorname{val}(c_u)}} x^u,$$

which is a polynomial in the Laurent polynomial ring $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over the residue field. When considering initial terms with respect to a monomial term ordering $\prec$ (see [Cox, Little, and O'Shea, 2007, Section 2.2] for more information on monomial term orderings) $\operatorname{in}_{\prec}(I)$ is always a monomial term. However, this is not always the case for our initial terms as we are considering initial ideals with respect to a weight vector. For example, consider $x+3y+12z \in \mathbb{Q}[x, y, z]$ where $\mathbb{Q}$ is equipped with the 2-adic valuation. Then the initial form with respect to the weight vector $(1, 1, 1)$ is $\operatorname{in}_{(1,1,1)}(x + 3y + 12z) = x + y$.

The *initial ideal* of I with respect to w is the ideal generated by the initial forms of all polynomials in I:

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) : f \in I \rangle .$$

The initial ideal is an ideal in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Theorem 2.1.5 (The Fundamental Theorem of Tropical Algebraic Geometry)**.** *Let K be an algebraically closed field with non-trivial valuation* val*. Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $X = V(I)$ its variety in the algebraic torus T^n. Then the following three subsets of $\mathbb{R}^n$ coincide.*

1. *The tropical variety* $\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}(V(f))$*;*
2. *The closure of* $\{w \in \Gamma^n : \operatorname{in}_w(I) \neq \langle 1 \rangle\}$ *in* $\mathbb{R}^n$*;*
3. *The closure of* $\{(\operatorname{val}(u_1), \ldots, \operatorname{val}(u_n)) : (u_1, \ldots, u_n) \in X\}$ *in* $\mathbb{R}^n$*.*

The Fundamental Theorem gives us three equivalent formulations of tropical varieties. The first in terms of the intersection of tropical hypersurfaces which we have already seen, the second in terms of initial ideals, and the third as the image of the valuation map. This third formulation allows us to think of the tropical variety as being a shadow of its algebraic counterpart. The second formulation in terms of initial ideals gives an algebraic way to check if points are in a tropical variety. This is because initial ideals over the polynomial ring can be finitely computed using

Gröbner bases as we shall see in Chapter 3. By Remark 2.1.4, we can think of a variety of the algebraic torus T^n given by an ideal in $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ as a variety $X \subseteq \mathbb{P}^n$ with defining homogeneous ideal $I \subseteq K[x_0, x_1, \ldots, x_n]$. In this situation, fix a weight vector $w \in \Gamma^{n+1}$. Then $\{g_1, \ldots, g_s\}$ is called a *Gröbner basis* for I with respect to w if the initial ideal $\text{in}_w(I)$ is generated by $\{\text{in}_w(g_1), \ldots, \text{in}_w(g_s)\}$. In this case, the second set of the Fundamental Theorem 2.1.5 becomes the closure in $\mathbb{R}^{n+1}$ of the set

$$\{w \in \Gamma^{n+1} : \text{in}_w(I) \text{ does not contain a monomial}\}.$$

We omit the proof of the Fundamental Theorem 2.1.5 (details can be found in [Maclagan and Sturmfels, 2013, Theorem 3.2.5]) and instead we provide an example.

Example 2.1.6. Consider $K = \mathbb{C}$ with the trivial valuation and consider the variety $X = V(x + y + 1) \subseteq K[x^{\pm 1}, y^{\pm 1}]$. Theorem 2.1.5 requires our valuation to be non-trivial; recall from Example 2.1.3 that we can consider $\mathbb{C}$ as a subfield of the field of Puiseux series $\mathbb{C}\{\{t\}\}$. We construct $\text{trop}(X)$ in the three ways as described in Theorem 2.1.5 and demonstrate that they are all equal.

Firstly, the definition of $\text{trop}(X) = \text{trop}(V(x + y + 1))$ is the set of all $w = (w_1, w_2) \in \mathbb{R}^2$ such that the minimum in $\text{trop}(x + y + 1)(w)$ is achieved at least twice. By definition, $\text{trop}(x + y + 1)(w) = \min\{w_1, w_2, 0\}$. This minimum is achieved twice at

1. w_2 and 0 when $w = (\alpha, 0)$ for all $\alpha > 0$;
2. w_1 and 0 when $w = (0, \alpha)$ for all $\alpha > 0$;
3. w_1 and w_2 when $w = (-\alpha, -\alpha)$ for all $\alpha > 0$.

This minimum is achieved three times when $w = (0, 0)$. Thus the tropical variety $\text{trop}(X)$ equals the three half lines spanned by positive multiples of $(1, 0)$, $(0, 1)$, $(-1, -1)$ and the origin as shown in Figure 2.1.

Secondly, $\text{trop}(X)$ is seen as the closure in $\mathbb{R}^2$ of the set of all $w \in \Gamma^2$ such that $\text{in}_w(I) \neq \langle 1 \rangle$. As I is generated by a single polynomial $x + y + 1$, this is the same as finding all $w \in \Gamma^2$ such that $\text{in}_w(x + y + 1)$ is not a monomial. The initial form $\text{in}_w(x + y + 1)$ is not a monomial at

1. $w = (\alpha, 0)$ for all $\alpha > 0$ where $\text{in}_w(x + y + 1) = y + 1$;
2. $w = (0, \alpha)$ for all $\alpha > 0$ where $\text{in}_w(x + y + 1) = x + 1$;
3. $w = (-\alpha, -\alpha)$ for all $\alpha > 0$ where $\text{in}_w(x + y + 1) = x + y$;

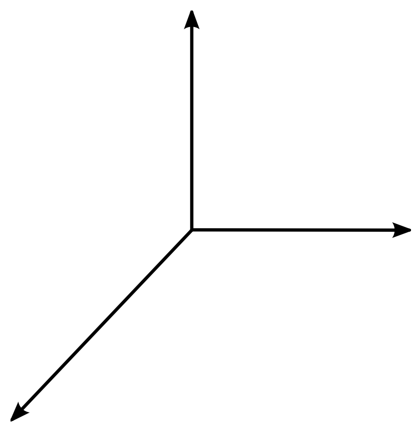

Figure 2.1: A tropical line in the plane

4. $w = (0, 0)$ where $\text{in}_w(x + y + 1) = x + y + 1$.

Again, we see that $\text{trop}(X)$ is as in Figure 2.1.

Finally, consider $\text{trop}(X)$ as the closure of the image of the algebraic variety under the valuation map. The variety is $V(x + y + 1) = \{(a, -1 - a) : a \in \mathbb{C}\{\{t\}\}, a \neq 0, 1\}$ and so consider $(\text{val}(a), \text{val}(-1 - a))$. If $\text{val}(a) > 0$ then as $\text{val}(a + b) = \min\{\text{val}(a), \text{val}(b)\}$ if $\text{val}(a) \neq \text{val}(b)$ we have that $\text{val}(-1 - a) = \min\{\text{val}(-1), \text{val}(-a)\} = \min\{0, \text{val}(a)\} = 0$. Then $(\text{val}(a), \text{val}(-1-a)) = (\text{val}(a), 0)$. If $\text{val}(a) < 0$ then by similar arguments, we have that $(\text{val}(a), \text{val}(-1 - a)) = (\text{val}(a), \text{val}(a))$. If $\text{val}(a) = 0$ and $a = b - 1$ for some b with positive valuation, then $(\text{val}(a), \text{val}(-1 - a)) = (0, \text{val}(b))$; otherwise, $(\text{val}(a), \text{val}(-1 - a)) = (0, 0)$.

Again, we conclude that $\text{trop}(X)$ is as in Figure 2.1. $\diamond$

2.2 Polyhedra and the Structure Theorem

We shall see that tropical varieties have a polyhedral structure for which we shall need some background in polyhedral geometry. For full details, see for example Ziegler [1995, Chapter 1].

A *polyhedron* P in $\mathbb{R}^n$ is the intersection of finitely many closed half spaces. That is, for some $A \in \text{Mat}(m \times n, \mathbb{R})$ and $z \in \mathbb{R}^m$, it can be presented in the form

$$P = P(A, z) := \{x \in \mathbb{R}^n : Ax \leq z\} \tag{2.1}$$

where by $Ax \leq z$ we mean that if $a_1, \ldots a_m$ are the rows of A and $z = (z_1, \ldots, z_m)$, then we have inequalities $a_i \cdot x \leq z_i$ for all $1 \leq i \leq m$.

A bounded polyhedron in $\mathbb{R}^n$ is called a *polytope*. That is, it can be described as the bounded intersection of finitely many closed half-spaces. We call a polytope given with this description an *H-polytope* as it is given by an intersection of *half-spaces*. Equivalently [Ziegler, 1995, Theorem 1.1], a polytope can be described as the convex hull of a finite set of points. That is, if $V = \{v_1, \dots, v_k\}$ is a finite set in $\mathbb{R}^n$ then P can be presented in the form

$$P = \operatorname{conv}(V) := \left\{ \lambda_1 v_1 + ... + \lambda_k v_k : \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

We call a polytope given with this description a *V-polytope* as it is given by its set of *vertices*.

Using Fourier-Motzkin Elimination [Ziegler, 1995, Theorem 1.4] we can turn a V-polytope into an H-polytope and vice versa. We shall use this in Section 2.3.1 when we are computing tropical hypersurfaces. We shall also need the *lattice length* of an edge of a polygon, which is defined to be the number of interior lattice points plus one.

Let P be a polyhedron in $\mathbb{R}^n$. The *face* of P minimising some $w \in \mathbb{R}^n$ is the set:

$$\operatorname{face}_w(P) = \{y \in P : w \cdot y \leq w \cdot x \text{ for all } x \in P\}.$$

For some subgroup $\Gamma \subseteq \mathbb{R}^n$, P is called *Γ-rational* if $A \in \operatorname{Mat}_{m\times n}(\mathbb{Q})$ and $z \in \Gamma^m$ in (2.1). For the case where $\mathbb{Q} \subseteq \Gamma$, this is equivalent to P having rational facet normals and vertices in Γ^n. The *affine span*, $\operatorname{aff}(P)$, of a polyhedron P is the affine subspace $u + \operatorname{span}\{v - u : v \in P\}$, for some $u \in P$, the dimension of which is the *dimension of P*. The zero-dimensional faces are called *vertices* and the one-dimensional faces are called *edges*. Faces which are not contained in any larger proper face are called *facets*. The *relative interior* of P is its interior in its affine span. The *lineality space* of P is the largest affine subspace contained in P. That is, if V is a subspace of $\mathbb{R}^n$ for which $x + v \in P$ for all $x \in P$ and $v \in V$, then it is the lineality space of P.

A *polyhedral complex* Σ is a collection of polyhedra for which if the intersection of any two polyhedra is non-empty, then it is a common face of each. It is called *Γ-rational* if every polyhedron in this collection is itself Γ-rational. The *lineality space* of a polyhedral complex is the intersection of the lineality spaces of all the polyhedra in the complex. The *support* $|\Sigma|$ of Σ is the set of all points which are contained in some polyhedron in Σ:

$$|\Sigma| = \{x \in \mathbb{R}^n : x \in P \text{ for some } P \in \Sigma\}.$$

A polyhedral complex Σ is *pure of dimension d* if every polyhedron in Σ which is maximal with respect to inclusion has dimension d. If Σ is pure of dimension d, then we say it is *connected through codimension one* if for every two d-dimensional polyhedra P and Q in Σ, there is a chain

$$P = P_0, P_1, \ldots, P_t = Q$$

for which P_i and P_{i+1} intersect in a unique codimension-one polyhedron for every $0 \le i \le s-1$.

Proposition 2.2.1. *Let X be an irreducible d-dimensional subvariety of T^n. Then the tropical variety* $\operatorname{trop}(X)$ *is the support of a pure d-dimensional Γ-rational polyhedral complex which is connected through codimension one.*

Let X be an irreducible d-dimensional subvariety of $\mathbb{P}^n$ given by an ideal $I \subseteq K[x_0, x_1, \ldots, x_n]$. Let $X^0 = X \cap T^n$ with tropicalisation $\operatorname{trop}(X^0)$, which is the support the Γ-rational polyhedral complex Σ in $\mathbb{R}^n$. The support $|\Sigma|$ of this polyhedral complex is determined by the defining ideal I and so is a fixed invariant of the ideal, but the polyhedral complex structure applied to it may vary. For example, trivially, one face can be subdivided into two, but there are also non-trivial examples [see, for example, Maclagan and Sturmfels, 2013, Example 3.2.9]. We thus fix a polyhedral complex structure on $|\Sigma|$. For example, it could inherit a polyhedral complex structure from the Gröbner complex $\Sigma(I)$, which we now define.

For a homogeneous ideal $I \subseteq K[x_0, x_1, \ldots, x_n]$, the Gröbner complex $\Sigma(I)$ of I is the polyhedral complex in $\mathbb{R}^{n+1}$ whose $(n+1)$-dimensional open cells are in bijection with the distinct initial ideals of I. Given $w \in \Gamma^{n+1}$, the *Gröbner cell* $C_w(I)$ is the closure in $\mathbb{R}^{n+1}$ of

$$\{w' \in \Gamma^{n+1} : \operatorname{in}_{w'}(I) = \operatorname{in}_w(I)\}.$$

This is a Γ-rational polyhedron [Maclagan and Sturmfels, 2013, Proposition 2.5.2] whose lineality space contains $(1, \ldots, 1)$ as the ideal I is homogeneous. Thus after we quotient out by this lineality space, it is a polyhedron in $\mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1)$. An ideal has only finitely many different initial ideals [Maclagan and Sturmfels, 2013, Lemma 2.5.4], and so has only finitely many different Gröbner cells. The *Gröbner complex* $\Sigma(I)$ is the finite collection of Gröbner cells $C_w(I)$ for all $w \in \Gamma^{n+1}$. It is a Γ-rational polyhedral complex in $\mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1)$ [Maclagan and Sturmfels, 2013, Theorem 2.5.3] where every Gröbner cell is a face of some n-dimensional Gröbner cell. In the case where we are considering the trivial valuation, the Gröbner complex is the Gröbner fan of I. Gröbner fans were introduced much earlier than Gröbner complexes by Mora and Robbiano [1988], and are consequently a much more well-studied object. The Gröbner complex is thus a generalisation of the Gröbner fan to the case where we are looking at fields with valuations.

By the Fundamental Theorem 2.1.5, it follows that $\operatorname{trop}(X)$ consists of the Gröbner cells $C_w(I)$ for which $\operatorname{in}_w(I)$ does not contain a monomial. This endows $\operatorname{trop}(X)$ with the structure of a polyhedral complex. This also gives a naïve algorithm for computing the tropical variety. We could first compute the Gröbner complex, and then $\operatorname{trop}(X)$ is the subcomplex consisting of those cells $C_w(I)$ for which $\operatorname{in}_w(I)$ does not contain a monomial. However, this turns out to be an inefficient method, as we explain in the following example [Bogart, Jensen, Speyer, Sturmfels, and Thomas, 2007, Example 6.1].

Example 2.2.2. Consider the homogeneous ideal $I = \langle x_5^3 - 2x_4x_5x_6 + x_3x_6^2 + x_4^2x_7 - x_3x_5x_7, x_4x_5^2 - x_4^2x_6 - x_3x_5x_6 + x_2x_6^2 + x_3x_4x_7 - x_2x_5x_7, x_3x_5^2 - x_3x_4x_6 - x_2x_5x_6 + x_2x_4x_7, x_4^2x_5 - 2x_3x_4x_6 + x_3^2x_7, x_3x_4x_5 - x_2x_5^2 - x_3^2x_6 + x_2x_3x_7, x_3^2x_5 - x_2x_4x_5 - x_2x_3x_6 + x_2^2x_7, x_4^3 - x_2x_5^2 - x_3^2x_6 - x_2x_4x_6 + 2x_2x_3x_7, x_3x_4^2 - 2x_2x_4x_5 + x_2^2x_7, x_3^2x_4 - x_2x_4^2 - x_2x_3x_5 + x_2^2x_6, x_3^3 - 2x_2x_3x_4 + x_2^2x_5\rangle \subseteq \mathbb{C}[x_1, \dots, x_7]$. Then $V(I)$ defines a curve times a three-dimensional torus, and so the tropicalisation $\operatorname{trop}(V(I))$ is a four-dimensional fan with a three-dimensional lineality space. Thus after quotienting out by the lineality space it defines a tropical curve. We compute, using gfan [Jensen], that $\operatorname{trop}(V(I))$ has five rays spanned by $(0, 5, -4, -13, -22, 74, -40)$, $(0, 4, 1, -2, -5, -8, 10)$, $(0, -5, 11, -8, 8, -11, 5)$, $(0, -5, -17, 76, -41, -53, 40)$ and $(0, -10, -13, -16, 86, -22, -25)$ with three dimensional lineality space spanned by $(-1, 0, 0, 0, 0, 0, 0)$, $(0, -1, 0, 1, 2, 3, 4)$ and $(0, 0, -1, -2, -3, -4, -5)$. However, $\Sigma(I)$ is full dimensional and so has many more cones than $\operatorname{trop}(V(I))$. In fact, $\Sigma(I)$ has 7167 rays whereas $\operatorname{trop}(V(J))$ has only five. $\diamond$

Having fixed a polyhedral complex structure on $\operatorname{trop}(X)$, we now explain how to define multiplicities on the d-dimensional polyhedra of Σ. To do this, we first need the definition of the multiplicity of a minimal associated prime of an ideal (see [Eisenbud, 1995, Chapter 3] for full details).

An ideal Q of $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is called *primary* if $fg \in Q$ implies that either $f \in Q$ or $g^m \in Q$ for some integer $m > 0$. We can write our ideal I as the intersection of primary ideals $I = \bigcap_{i=1}^s Q_i$ where each Q_i is a primary ideal whose radical is the prime ideal P_i. If the P_i are all unique and no Q_i is redundant in this expression then it is called a *primary decomposition* of I. Such an expression is not unique, but it turns out that the collection of primes $\{P_i\}$ is independent of the choice of primary decomposition [Eisenbud, 1995, Proposition 3.13(a)]. These P_i are called the *minimal associated primes* of I. The *multiplicity* of a minimal prime P_i of I is defined to be the length of $(K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/Q_i)_{P_i}$ as an $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$-module:

$$\text{mult}(P_i, I) = \text{length}((K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/Q_i)_{P_i}).$$

The multiplicity of a minimal associated prime of I is used to define the multiplicity of a d-dimensional polyhedron σ of the d-dimensional polyhedral complex $\Sigma = \text{trop}(V(I))$. Fix some w in the relative interior of σ and define the *multiplicity* of σ to be

$$\text{mult}(\sigma) = \sum_{P \text{ a minimal associated prime of } \text{in}_w(I)} \text{mult}(P, \text{in}_w(I)).$$

Consider any other w' in the relative interior of σ. We have $\text{in}_{w'}(I) = \text{in}_w(I)$ as they are both contained in the same Gröbner cone of I. Thus this definition of multiplicity is independent of the choice of w in the relative interior of σ.

We now explain how this multiplicity can be effectively computed (see [Maclagan and Sturmfels, 2013, Lemma 3.4.6] for full details and a proof). For this we need a multiplicative change of coordinates. This is given by an automorphism $\phi\colon (K^*)^n \to (K^*)^n$. It has an induced map on rings $\phi^*\colon K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by $\phi^*(x_i) = x^{a_i}$ for some $a_i \in \mathbb{Z}^n$. As ϕ is an automorphism, ϕ^* is an isomorphism. Let A be the matrix whose columns are $a_1, \ldots, a_n$. Notice that as ϕ is an isomorphism, A is invertible and also that $A \in \text{GL}(n, \mathbb{Z})$. We apply a multiplicative change of coordinates so that the polyhedron σ in Σ has affine span $\text{span}(e_1, \ldots, e_d)$. Let w be a relative interior point of σ, $\Bbbk[x_{d+1}, \ldots, x_n]$ the polynomial ring in variables $x_{d+1}, \ldots, x_n$ over the residue field $\Bbbk$ and $J = \text{in}_w(I) \cap \Bbbk[x_{d+1}, \ldots, x_n]$. Then $V(J)$ is a finite set of points, the number of which when counted with multiplicity is the *multiplicity* of σ:

$$\text{mult}(\sigma) = \dim_{\Bbbk}(\Bbbk[x_{d+1}, \ldots, x_n]/J).$$

By assigning multiplicities to all maximal dimensional polyhedra in this way, we can endow Σ with the structure of a *weighted polyhedral complex* where the d-dimensional polyhedron σ in Σ has weight given by $\text{mult}(\sigma)$.

Example 2.2.3. Consider the ideal $I = \langle 1 + x^2 + x^2y + xy^2 + y^2 \rangle \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ where $\mathbb{C}$ has the trivial valuation. Then $\text{trop}(V(I))$ has five rays generated by $u_1 = (1, 0)$, $u_2 = (0, 1)$, $u_3 = (-1, 0)$, $u_4 = (0, -1)$ and $u_5 = (-1, -1)$. We determine the multiplicities. For the ray generated by u_1, the initial ideal is $\text{in}_{u_1}(I) = \langle y^2 + 1 \rangle = \langle y + i \rangle \cap \langle y - i \rangle$, and so the ray has multiplicity two. For the ray generated by u_2, the initial ideal is $\text{in}_{u_2}(I) = \langle x^2+1\rangle = \langle x+i\rangle \cap \langle x-i\rangle$, and so the ray has multiplicity two. For the ray generated by u_3, the initial ideal is $\text{in}_{u_3}(I) = \langle x^2+x^2y\rangle = \langle 1+y\rangle$, and so the ray has multiplicity one. For the ray generated by u_4, the initial ideal is $\text{in}_{u_4}(I) = \langle xy^2+y^2\rangle = \langle x+1\rangle$, and so the ray has multiplicity one. For the ray generated by u_5, the initial ideal is $\text{in}_{u_5}(I) = \langle x^2y+xy^2\rangle = \langle x+y\rangle$, and so the ray has multiplicity one. This is shown in Figure 2.2 where all rays have multiplicity one except those indicated with a 2. ◇

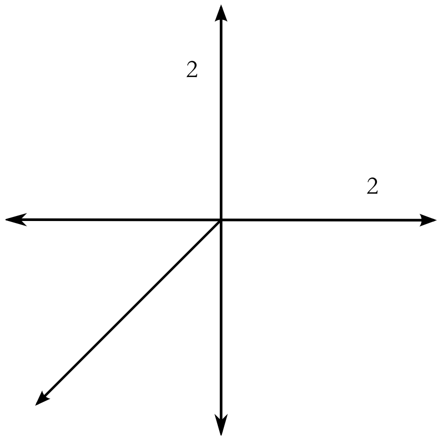

Figure 2.2: A weighted fan in $\mathbb{R}^2$

We define what it means for a weighted polyhedral complex to be balanced by first considering the case of a one-dimensional fan. Let Σ be a one-dimensional weighted Γ-rational polyhedral fan in $\mathbb{R}^n$. Denote by $u_i \in \mathbb{Z}^n$ the first lattice point of the i-th ray of Σ and suppose it has multiplicity m_i. The one-dimensional fan Σ is said to be *balanced* if

$$\sum_i m_i u_i = 0.$$

Example 2.2.4. Returning to Example 2.2.3, the fan for $\text{trop}(V(I))$ is balanced as

$$2\cdot\begin{pmatrix}1\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\end{pmatrix}+1\cdot\begin{pmatrix}-1\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\-1\end{pmatrix}+1\cdot\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \quad \diamond$$

Let Σ be a weighted Γ-rational polyhedral complex in $\mathbb{R}^n$. To see that Σ is balanced we reduce to the case of a one-dimensional fan. Let $P \in \Sigma$ be a polyhedron, then the *star* of P, $\text{star}_\Sigma(P)$ is a fan in $\mathbb{R}^n$ with cones indexed by those $Q \in \Sigma$ which have P as a face. Fix $w \in P$, then the cone of $\text{star}_\Sigma(P)$ indexed by face Q is

$$\{v \in \mathbb{R}^n : \exists \varepsilon > 0 \text{ with } w + \varepsilon v \in Q\} + \text{aff}(P) - w$$

which is independent of the choice of w. For example, the star of the ray generated by $(1, 0)$ in the tropical curve of Example 2.2.3 is the horizontal axis, and the star of the origin is the whole fan of Figure 2.2. If Σ is a pure weighted d-dimensional, Γ-rational polyhedral complex with P a $(d-1)$-dimensional polyhedron in Σ, then we say that Σ is *balanced at* P if the one dimensional fan

$$\text{star}_\Sigma(P)/\{\text{aff}(P) - w\}_{w\in P}$$

is balanced after inheriting the weights from Σ. The complex Σ is said to be *balanced* if it is balanced at all $(d-1)$-dimensional polyhedra in Σ.

The *Structure Theorem of Tropical Varieties* [see Maclagan and Sturmfels, 2013, Theorem 3.3.5] asserts that tropical varieties have this additional structure.

Theorem 2.2.5. *(Structure Theorem for Tropical Varieties) Let X be an irreducible d dimensional subvariety of the torus T^n. Then* $\text{trop}(X)$ *is the support of a balanced weighted Γ-rational polyhedral complex pure of dimension d. If K has characteristic zero then this complex is connected through codimension one.*

2.3 Computing Tropical Varieties

A natural question to ask is how can $\text{trop}(X)$ be efficiently computed. Recall that the Fundamental Theorem 2.1.5 says that for $X \subseteq T^n$ with defining ideal $I \subseteq K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$, the tropical variety $\text{trop}(X)$ is the closure in $\mathbb{R}^n$ of the set $\{w \in \Gamma^n : \text{in}_w(I) \neq \langle 1\rangle\}$. In Chapter 3 we explain how over a field with a valuation we can compute a Gröbner basis which can be used to compute $\text{in}_w(I)$. We examine existing methods to compute tropical varieties over fields with the trivial valuation. This allows us to use the standard theory of Gröbner bases where we do not take the valuations into account. We shall see firstly that tropical hypersurfaces, which are defined by a single polynomial f can be constructed from the Newton polygon $\text{Newt}(f)$. We then shall see how tropical curves can be constructed before seeing how an arbitrary dimensional tropical variety can be constructed using the methods of Bogart, Jensen, Speyer, Sturmfels, and Thomas [2007].

2.3.1 Tropicalising Hypersurfaces in $\mathbb{P}^2$

In this section, we consider tropicalising hypersurfaces which are also curves; that is, hypersurfaces in $\mathbb{P}^2$ defined by a single homogeneous equation in $K[x_0, x_1, x_2]$. We concentrate on the case of tropicalising hyperplanes in $\mathbb{P}^2$ as we shall explicitly require these algorithms for our constructions in Chapter 4. In that chapter, we will be looking at a different methods of computing tropical curves from projections to coordinate planes. The work of this section will allow us to compute the tropicalisations of the coordinate projection of these tropical curves as they are tropical hypersurfaces. Similar methods can be used to construct tropical hypersurfaces in $\mathbb{P}^n$. In particular, the tropical hypersurface $\operatorname{trop}(V(f))$ for some polynomial $f \in K[x_0, x_1, \ldots, x_n]$ can be constructed from its Newton polygon $\operatorname{Newt}(f)$.

As the defining equation $f \in K[x_0, x_1, x_2]$ is homogeneous the tropical variety has $(1,1,1)$ in its lineality space so we consider the tropical variety after first quotienting out by $(1,1,1)$. This corresponds to first dehomogenising the defining equation which we can then consider as a polynomial f in $K[x_1^{\pm 1}, x_2^{\pm 1}]$. We discuss how to construct tropical hypersurfaces in $\mathbb{R}^2$ defined by a single polynomial f from its Newton polygon and how this can be computed using Fourier-Motzkin elimination. We shall also see how the multiplicities can be recovered from the lengths of corresponding edges of the Newton polygon.

We start with some polyhedral geometry. Let P be a convex polygon in $\mathbb{R}^2$, F a face of P and set $\mathcal{N}_F(P) := \{w \in \mathbb{R}^2 : w \cdot y \leq w \cdot x \text{ for all } x \in P, y \in F\}$. That is, $\mathcal{N}_F(P)$ is the collection of all $w \in \mathbb{R}^2$ for which $\operatorname{face}_w(P) \subseteq F$. Consider $w = (0,0)$. As $(0,0) \cdot y \leq (0,0) \cdot x$ for all $x, y \in P$ it follows that for all faces F of P we have that $(0,0) \in \mathcal{N}_F(P)$. Suppose that $w, w' \in \mathcal{N}_F(P)$ then $w \cdot y \leq w \cdot x$ and $w' \cdot y \leq w' \cdot x$ for all $x \in P$ and $y \in F$ and so $(\alpha w + \beta w') \cdot y \leq (\alpha w + \beta w') \cdot x$ for all $\alpha, \beta \geq 0$ from which it follows that $\alpha w + \beta w' \in \mathcal{N}_F(P)$. We conclude that $\mathcal{N}_F(P)$ is a polyhedral cone in $\mathbb{R}^2$ and is called the *normal cone* of P at F.

The *normal fan* of P is the set of all normal cones at all faces of P:

$$\mathcal{N}(P) := \{\mathcal{N}_F(P) : F \text{ is a face of } P\}.$$

By $\mathcal{N}_0(P)$ we denote the subfan of $\mathcal{N}(P)$ containing only the cones of $\mathcal{N}(P)$ which are not maximal dimensional.

Lemma 2.3.1. *Suppose E and F are faces of a convex polygon P with $E \subsetneqq F$, then $\mathcal{N}_E(P) \supsetneqq \mathcal{N}_F(P)$.*

Proof. Let $w \in \mathcal{N}_F(P)$. Then for all $x \in P$ and $y \in F$ we have that $w \cdot y \leq w \cdot x$. As this holds for all $y \in F$ it must hold for all $y \in E \subseteq F$ and so $w \in \mathcal{N}_E(P)$.

Now suppose that $\mathcal{N}_E(P) = \mathcal{N}_F(P)$ and let $w \in \mathcal{N}_E(P)$. Then by definition $\operatorname{face}_w(P) \supseteq E$ and so for all $x \in E$ and $y \in P$ it follows that $w \cdot x \leq w \cdot y$. But $w \in \mathcal{N}_E(P)$ so $w \in \mathcal{N}_F(P)$ and in particular $w \cdot x' \leq w \cdot y$ for all $x' \in F$ and $y \in P$ meaning that $x' \in \operatorname{face}_w(P)$ for all $x' \in F$ and so $F \subseteq E$. By assumption $E \subseteq F$ and so $E = F$ contradicting the assumption that $E \subsetneqq F$. □

Let $P \subseteq \mathbb{R}^2$ be a two-dimensional polytope. If F is a face of P then by, for example [Cox, Little, and Schenck, 2011, Proposition 2.3.8(a)], we have that

$$\dim(F) + \dim(\mathcal{N}_F(P)) = 2.$$

Combining this with Lemma 2.3.1, we see that there is a bijective dimension-reversing inclusion between the faces of the convex polygon P and the cones in its normal fan $\mathcal{N}(P)$.

Returning to constructing the fan which is the support of a tropical curve in $\mathbb{P}^2$, let C be a curve defined by polynomial $f = \sum c_u x^u$ in $K[x_1^{\pm 1}, x_2^{\pm 1}]$. The *Newton polygon* of f is defined as the convex hull of the exponents of x which have non-zero coefficients:

$$\mathrm{Newt}(f) := \mathrm{conv}\{u : c_u \neq 0\}.$$

We can recover the fan that is the support of $\mathrm{trop}(C)$ from the Newton polygon of f as the following result explains.

Lemma 2.3.2. *Let $f \in K[x_1^{\pm 1}, x_2^{\pm 1}]$ be a polynomial and let Σ be the polyhedral complex which consists of the non-maximal cones of the Newton polygon of f. Then the tropicalisation of $V(f)$ is the support of the complex Σ. That is,*

$$\mathrm{trop}(V(f)) = \mathcal{N}_0(\mathrm{Newt}(f)).$$

Further, the multiplicity of the ray of $\mathrm{trop}(X)$ *which is normal to the edge E of* $\mathrm{Newt}(f)$ *is the lattice length of E.*

Proof. Let $P = \mathrm{Newt}(f)$. If f is a monomial, then P is a single point. Setting $P = \{x\}$, then for all $w \in \mathbb{R}^2$ we have that $w \cdot x \leq w \cdot x$ and it follows that $\mathcal{N}_P(P) = \mathbb{R}^2$ is the only cone, and so $\mathcal{N}_0(P) = \emptyset$. Clearly $\mathrm{trop}(V(f))$ is empty as $\mathrm{in}_w(f)$ is always a monomial, and so the two sets coincide as required.

Now suppose that f is not a monomial. Let $u_1, \ldots, u_s$ be the exponents of the monomials in f which have non-zero coefficients. Now, $w \in \mathrm{trop}(V(f))$ means that $\mathrm{in}_w(f)$ is not a monomial and so the minimum in $\min_i\{w \cdot u_i\}$ is achieved at least twice, which after relabelling, we assume occurs for $i = 1, \ldots, r$ for some $r \leq s$. So $w \cdot u_1 = \cdots = w \cdot u_r \leq w \cdot u_i$ for all $1 \leq i \leq s$ which implies that $w \cdot (\sum_{i=1}^r \alpha_i u_i) \leq w \cdot (\sum_{j=1}^s \beta_j u_j)$ for all $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^r \alpha_i = 1$ and $\sum_{j=1}^s \beta_j = 1$. Notice that $\sum_{i=1}^r \alpha_i u_i \in \mathrm{conv}\{u_1, \ldots, u_r\}$ and $\sum_{j=1}^s \beta_j u_j \in \mathrm{conv}\{u_1, \ldots, u_s\} = P$ and so by definition, we have that $w \in \mathcal{N}_F(P)$ for $F = \mathrm{conv}\{u_1, \ldots, u_r\}$. If F is one-dimensional, then by the dimension reversing correspondence between faces of P and cones of $\mathcal{N}(P)$, it follows that $\mathcal{N}_F(P)$ is also one-dimensional and so as $\mathcal{N}(P)$ is two-dimensional we have that $w \in \mathcal{N}_0(P)$ as required. If F is two-dimensional, then $\mathcal{N}_F(P)$ is zero-dimensional, and so again $w \in \mathcal{N}_0(P)$ as required.

For the reverse inclusion, notice that $w \in \mathcal{N}_0(P)$ means that $w \in \mathcal{N}_F(P)$ for some face F such that $\mathcal{N}_F(P)$ is zero-dimensional or one-dimensional. If $\mathcal{N}_F(P)$ is one-dimensional this corresponds to F also being one-dimensional and so after relabelling if required, we can write $F = \text{conv}\{u_1, u_2\}$. By the definition of $\mathcal{N}_F(P)$ it follows that $w \cdot u_1 \leq w \cdot u_2$ and $w \cdot u_1 \geq w \cdot u_2$ and so it must follow that we have equality. Further we have that $w \cdot u_1, w \cdot u_2 \leq w \cdot u_i$ for all $1 \leq i \leq s$ by the definition of $\mathcal{N}_F(P)$. Thus $w \cdot u_1 = w \cdot u_2 \leq w \cdot u_i$ for all $1 \leq i \leq s$ and so $w \in \text{trop}(V(f))$. If $\mathcal{N}_F(P)$ is zero-dimensional, it must be the origin and so F is the whole of P. Again we must have that $w \cdot u_i = w \cdot u_j$ for all $1 \leq i, j \leq s$ and so $w \in \text{trop}(V(f))$.

Let $\rho \in \text{trop}(V(f))$ be a one-dimensional cone in the tropical variety which by the first part of this Lemma, ρ is a one dimensional cone of the normal fan to $\text{Newt}(f)$. By the dimension reversing inclusion of the normal, ρ corresponds to an edge E of $\text{Newt}(f)$. By a monomial change of coordinates, we can transform $\text{Newt}(f)$ such that E lies horizontally. As we are working in the Laurent polynomial ring, we can translate the Newton polygon as we wish as this corresponds to multiplication by monomials. We move $\text{Newt}(f)$ so that the transformed Newton polygon $\text{Newt}(f')$ lies in the upper half plane and the chosen transformed edge E' has vertices $(0,0)$ and $(k,0)$ for some $k > 0$. This means that $\mathcal{N}_{E'}(\text{Newt}(f'))$ extends in the direction $(0,1)$.

Let $\rho' = (0,1)$. Then as $\text{Newt}(f')$ lies in the upper half plane, f' has only non-negative powers of x_2. Further, all terms in f' which involve x_1 only have non-negative powers of x_1 due to the position of $\text{Newt}(f')$. Thus $\text{in}_{\rho'}(f')$ is a polynomial in x_1 only. Let $\text{in}_{\rho'}(f') = \sum_{i=0}^{k} c_i x_1^i$ where $c_0, c_k \neq 0$. Recall that ρ' has multiplicity corresponding to the number of points in $V(\text{in}_{\rho'}(V(f')))$. As K is algebraically closed and f' is of degree k with no monomial factors, it has k roots in K and so the ray ρ' has multiplicity k. Also, E' has $k-1$ interior lattice points. By the definition of multiplicity of ρ, we see that it equals the multiplicity of ρ' of our tropical curve under the coordinate change.

We claim that the original edge E in $\text{Newt}(f)$ and the modified edge E' of the translated Newton polygon $\text{Newt}(f')$ both have the same number of interior lattice points. Let $\phi\colon (K^*)^2 \to (K^*)^2$ be the monomial change of coordinates with induced map on rings $\phi^*\colon K[x_1^{\pm 1}, x_2^{\pm 1}] \to K[x_1^{\pm 1}, x_2^{\pm 1}]$ given by matrix $A \in \text{GL}(2, \mathbb{Z})$. As A has all integer entries, lattice points in E are sent to lattice points in E'. Then as $A \in \text{GL}(2, \mathbb{Z})$ it has a inverse in $\text{GL}(2, \mathbb{Z})$ and so by analogous arguments, all lattice points in E' are sent to lattice points in E. Thus they must have the same number of lattice points as A is an isomorphism. $\square$

Using Fourier-Motzkin Elimination, we can turn the Newton polytope from a V-polytope $\text{conv}\{u : c_u \neq 0\}$ to an H-polytope $\{x \in \mathbb{R}^2 : Ax \leq z\}$ for some $A \in \text{Mat}(m \times 2, \mathbb{Z})$ and $z \in \mathbb{R}^m$. We claim that if $a_1, \dots, a_m$ are the rows of A, then the one-dimensional cones of the normal fan to $\text{Newt}(f)$ are generated by $a_1, \dots, a_m$. This allows us to easily compute the one-dimensional skeleton of the normal fan to $\text{Newt}(f)$, which is the tropical variety $\text{trop}(V(f))$. To see this, we firstly show that a_i is contained in some normal cone of P at some face F. Faces of P are given by $F_j = \{x \in P : a_j \cdot x = z_j\}$ for all $1 \leq j \leq m$. We show that a_i is contained in $\mathcal{N}_{F_i}(P)$ by showing that $\text{face}_{a_i}(P) = F_i$:

$$\begin{aligned}\text{face}_{a_i}(P) &= \{y \in P : a_i \cdot y \leq a_i \cdot x \quad \forall x \in P\} \\ &= \{y \in P : a_i \cdot y = z_i \quad \forall x \in P\} \\ &= F_i.\end{aligned}$$

The second equality follows as $a_i \cdot y$ is a constant for all $y \in \text{face}_{a_i}(P)$ as $a_i \cdot y \leq a_i \cdot y'$ for all $y, y' \in \text{face}_{a_i}(P)$. So $a_i \cdot y' \leq a_i \cdot y$ and then $a_i \cdot y = a_i \cdot y' =: z_i$. Now we show that if a_i is in the normal fan of P then $a_i \cdot x \leq z_i$ is a defining inequality for P as an H-polytope. Let $y \in F_i$ then $a_i \cdot y = z_i$, but we know that $\text{face}_{a_i}(P) = F_i$ so for all $x \in P$ we have that $a_i \cdot x \leq a_i \cdot y = z_i$.

We summarise the results of this section in the following algorithm. It computes the tropicalisation of a hypersurface in $\mathbb{P}^2$ from the Newton polygon of its defining equation, with the multiplicities of the edges being the lattice length of the corresponding edge.

Algorithm 2.3.3. Input: Polynomial $f \in K[x_0, x_1, x_2]$.

Output: A one-dimensional balanced weighted fan which is the support of $\text{trop}(V(f))$.

1. Write f in the form $\sum_{u=(u_0,u_1,u_2)\in\mathbb{N}^3} c_u x^u$. Define a finite set of points $V = \{(u_1, u_2) : c_u \neq 0, u = (u_0, u_1, u_2) \text{ in the expression } f = \sum_{u\in\mathbb{N}^3} c_u x^u\}$ in $\mathbb{N}^2$. Let $P = \text{conv}(V)$ be the V-polytope defined by V.
2. Define $Q = \{(x, t) \in \mathbb{R}^{2+m} : x = Vt \quad t \geq 0 \text{ and } (1, \dots, 1) \cdot t = 1\}$.
3. Use Fourier-Motzkin elimination to project away the t variables and write P as H-polytope $P = \{x \in \mathbb{R}^2 : Ax \leq z\}$ for some $A \in \text{Mat}(m \times 2, \mathbb{Z})$ and $z \in \mathbb{R}^m$.

4. Let m be the lattice lengths of the edges of P.

5. Let Σ be the one-dimensional fan which has rays spanned by the rows of A.

Output Σ, m.

2.3.2 Tropical Curves

In this section, we focus on how to compute tropical curves. Let X be an irreducible one-dimensional variety in $\mathbb{P}^n$ with defining ideal I and tropicalisation $\mathrm{trop}(X)$. Recall that $\{g_1, \ldots, g_s\}$ is a tropical basis for I if its tropical pre-variety equals $\mathrm{trop}(X)$. So if we can construct a tropical basis then we would be able to construct the tropical variety $\mathrm{trop}(X)$ as the intersection of those finitely many tropical hypersurfaces.

Let $\mathcal{B} = \{f_1, \ldots, f_r\}$ be a generating set for I and construct the tropical pre-variety for $\mathcal{B}$. This is the finite intersection of tropical hypersurfaces each of which is a polyhedral fan by the Structure Theorem 2.2.5. Thus the intersection $\bigcap_{f \in \mathcal{B}} \mathrm{trop}(V(f))$ is also a polyhedral fan Σ, which may not be pure dimensional.

Consider a cone σ in Σ whose dimension is greater than one. Then as $\mathrm{trop}(X)$ is a tropical curve, by the Structure Theorem 2.2.5 it is a polyhedral complex of dimension one and so the whole of σ cannot be in $\mathrm{trop}(X)$. As $\mathrm{trop}(X)$ is one-dimensional and the cone is two-dimensional, we can find a generic relative interior point $w \in \sigma$ such that $\mathrm{in}_w(I)$ contains a monomial x^u. Let $\mathcal{G}$ be a reduced Gröbner basis for I with respect to w and let r be the normal form on division of x^u with respect to $\mathcal{G}$. By properties of Gröbner bases [see, for example, Cox, Little, and O'Shea, 2007, Section 2.6, Proposition 1(ii)] this means that there is some $f \in I$ such that $x^u = f - r$ with the property that $x^u = \mathrm{in}_w(f)$. Additionally, as r is obtained on division by $\mathcal{G}$, it depends only on the reduced Gröbner basis $\mathcal{G}$ and not on the choice of $C_w(I)$. This means that for $f = x^u + r$, if we choose any $w' \in C_w(I)$, then as $\mathrm{in}_{w'}(I) = \mathrm{in}_w(I)$, $\mathcal{G}$ is a Gröbner basis for I with respect to w' and we would have that $\mathrm{in}_{w'}(f) = x^u$. So f is a *witness* for $C_w(I)$ not being in the tropical variety. We add f to $\mathcal{B}$ and this excludes the Gröbner cone $C_w(I)$ from being in the tropical variety.

Now, suppose that σ is zero or one-dimensional, and let $w \in \sigma$ be a relative interior point. If $\mathrm{in}_w(I)$ contains a monomial, then σ does not live in the tropical variety and so we need to add a polynomial which excludes this cone. Proceed as above to find a witness to add to $\mathcal{B}$. Suppose now that $\mathrm{in}_w(I)$ does not contain a monomial. Then as σ is one-dimensional, any other relative interior point of σ is of the form $w' = \alpha w$ for some $\alpha > 0$. Thus $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$ and we would have that $w' \in \mathrm{trop}(X)$. Thus $\sigma \in \mathrm{trop}(X)$.

As I has only finitely many initial ideals and as at each step we exclude at least one, we only need to add a finite number of polynomials in order to recover $\mathrm{trop}(X)$ in this way, and so we are left with a tropical basis for I and a way to find the tropical variety $\mathrm{trop}(X)$.

We then repeat this process on the enlarged set $\mathcal{B}$ until all the cones are certified to be in the tropical variety. Thus we have a tropical basis and can compute the tropical curve by the intersection $\mathrm{trop}(X) = \bigcap_{f \in \mathcal{B}} \mathrm{trop}(V(f))$.

2.3.3 Computing Other Tropical Varieties

In this section, we consider how to construct the tropicalisation $\mathrm{trop}(X)$ for an irreducible d-dimensional variety $X \subseteq \mathbb{P}^n$ contained in the torus T^n and with defining ideal I. We use that $\mathrm{trop}(X)$ is connected through codimension one to pass from one maximal dimensional cone to another through a common facet. This is known as a *Gröbner walk* and is demonstrated for the case of tropical surfaces in Figure 2.3. We start at the red shaded face in the first diagram, then walk to the connecting edge coloured red in the second diagram. We then walk to the connecting red shaded face in the final diagram. This walk is performed by computing some tropical curve which has a ray for each neighbouring maximal dimensional cone. Continuing over all facets of all maximal dimensional cones, we recover the entire tropical variety. In this subsection, we outline how this works.

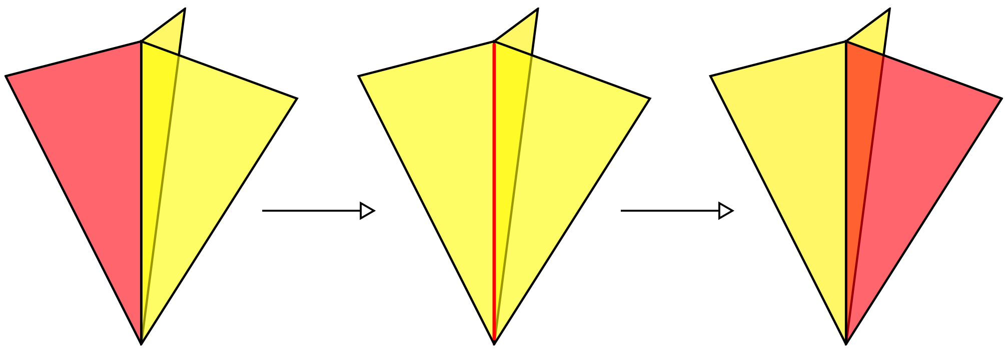

Figure 2.3: Walking around a tropical surface

Suppose that we have a maximal dimensional cone of the tropical variety $\mathrm{trop}(X)$. This corresponds to finding a Gröbner basis of I with respect to w such that $\mathrm{in}_w(I)$ does not contain a monomial for which the Gröbner cone $C_w(I)$ is d-dimensional. Let F be a facet of $C_w(I)$ and u a relative interior point of F. Consider the initial ideal $\mathrm{in}_u(I)$. As u is a relative interior point of F, $\mathrm{in}_u(I)$ is homogeneous with respect to the span of F. Thus as F is $(d-1)$-dimensional, $\mathrm{in}_u(I)$ has a $(d-1)$-

dimensional lineality space. So $V(\text{in}_u(I)) = C \times T^{d-1}$ and after we quotient out by the lineality space, the tropicalisation is a curve $\text{trop}(C)$. By Section 2.3.2, we can compute the tropicalisation $\text{trop}(C)$ which is a one-dimensional polyhedral fan, and so is a collection of rays and the origin. This is shown for the case of a tropical surface in Figure 2.4. The maximal two-dimensional cone that we are looking at is shaded red, and the point u in green. Then the tropical curve $\text{trop}(V(\text{in}_u(I)))$, after we quotient out by the torus, is drawn in blue. Observe that this tropical curve has a ray pointing in the direction of each neighbouring two-dimensional cone.

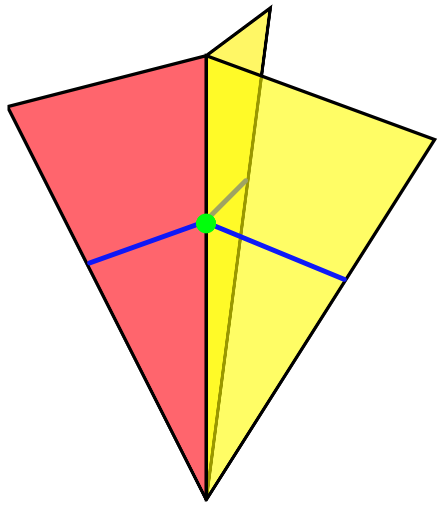

Figure 2.4: A tropical curve with a ray pointing in the drection of each two-dimensional cone of the tropical surface

By Kalkbrener and Sturmfels [1995, Theorem 2], for I a prime ideal, the initial complex associated to the initial ideal $\text{in}_w(I)$ is strongly connected. Then by Kalkbrener and Sturmfels [1995, proof of Theorem 1], $V(\text{in}_w(I))$ is also connected. Thus in Chapter 4 where we are looking at computing tropical curves from coordinate projections, we will assume that the input curve is connected.

Let v be a primitive ray generator of $\text{trop}(V(\text{in}_u(I)))$. This means that $v \in \text{trop}(V(\text{in}_u(I)))$ and so by definition $\text{in}_v(\text{in}_u(I))$ does not contain a monomial. For sufficiently small $\varepsilon > 0$, there exists $w' = u + \varepsilon v$ such that $\text{in}_v(\text{in}_u(I)) = \text{in}_{u+\varepsilon v}(I) = \text{in}_{w'}(I)$ [Maclagan and Sturmfels, 2013, Lemma 2.4.5]. Then w' is a point in the interior of a neighbouring cone to F. Repeating for all rays of $\text{trop}(V(\text{in}_u(I)))$ we obtain the neighbouring cones of the facet F and then repeating for all the facets of $C_w(I)$ we obtain a collection of cones which are adjacent to the selected cone $C_w(I)$.

We repeat this procedure for all of the new cones that we have found. For each of them, we find the facets, then a relative interior point u for the facet then compute the tropical curve $\text{trop}(V(\text{in}_u(I)))$, each ray of which corresponds to a neighbouring cone.

By continuing this process until we add no new cones, we have then recovered the tropical variety.

It remains to show that we can find some maximal dimensional cone of the tropical variety as a starting point for this procedure. One possibility is to compute the entire Gröbner fan. However, this is not very efficient as the Gröbner fan may be much larger than the tropical variety, as shown in Example 2.2.2. Instead, currently probabilistic heuristics are used to find a starting cone [Bogart, Jensen, Speyer, Sturmfels, and Thomas, 2007, Algorithm 4.12].

2.4 Elimination Theory of Tropical Varieties

In Chapter 4 we will use Elimination Theory and Tropical Elimination Theory in our reconstruction of tropical curves from coordinate projections. We use Elimination Theory [see, for example, Cox, Little, and O'Shea, 2007, Chapter 3, Section 1] to find equations of the projection of a curve to coordinate planes. Then, by Tropical Elimination Theory [Sturmfels and Tevelev, 2008], the tropicalisation of this projection is the projection of the tropicalisation of the original curve. We outline these results here.

Let I be an ideal in $K[x_1, \ldots, x_n]$. Then the *l-th elimination ideal* of I is an ideal in $K[x_{l+1}, \ldots, x_n]$ which is defined by

$$I_l = I \cap K[x_{l+1}, \ldots, x_n].$$

The following Elimination Theorem [see, for example, Cox, Little, and O'Shea, 2007, Chapter 3, Theorem 2] tells us that a basis for this elimination ideal can be obtained from a Gröbner basis for I with respect to the lexicographic term order.

Theorem 2.4.1. *Let I be an ideal in $K[x_1, \ldots, x_n]$ where $\mathcal{G}$ is a Gröbner basis for I with respect to the lexicographic ordering with $x_1 > x_2 > \cdots > x_n$. Then for all $1 \leq l \leq n$, $\mathcal{G}_l = \mathcal{G} \cap K[x_{l+1}, \ldots, x_n]$ is a Gröbner basis for the l-th elimination ideal I_l.*

The Elimination Theorem 2.4.1 tells us that from the Gröbner basis with respect to the lexicographic term order, we can recover the l-th elimination ideal for all $1 \leq l \leq n$. For our purposes, we shall only require a single elimination ideal and so it does not make sense to compute a Gröbner basis with respect to the lexicographic term order. This is especially true as the lexicographic term order can lead to some very large Gröbner bases [see Bayer and Mumford, 1993, pp.11-12]. Instead we use an *elimination term order* where any monomial involving one of the $x_1, \ldots, x_l$ is greater than any of the monomials in $K[x_{l+1}, \ldots, x_n]$ and if $\mathcal{G}$ is a Gröbner basis with respect to this term ordering, then $\mathcal{G}_l = \mathcal{G} \cap K[x_{l+1}, \ldots, x_n]$ is a Gröbner basis for the l-th elimination ideal I_l.

In Tropical Elimination Theory [Sturmfels and Tevelev, 2008], varieties are replaced by their tropicalisations. Let $X \subseteq T^n$ and $Y \subseteq T^m$ be a subvarieties of the same dimension. Denote by N_n the dual lattice to the lattice of characters of $\mathbb{P}^n$ and similarly for N_m. Let $\operatorname{trop}(X) \subseteq \mathbb{R}^n$ and $\operatorname{trop}(Y) \subseteq \mathbb{R}^m$ be the tropicalisations of X and Y respectively.

Suppose that $f\colon X \to Y$ is a dominant map which is generically finite of degree δ and α is the homomorphism of tori specified by the $\mathbb{Z}$-linear map $A\colon (N_n)_{\mathbb{Q}} \to (N_m)_{\mathbb{Q}}$. Sturmfels and Tevelev [2008, Theorem 1.1] tell us that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\text{trop}} & \operatorname{trop}(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle A} \\ Y & \xrightarrow{\text{trop}} & \operatorname{trop}(Y) \end{array}$$

and so $A(\operatorname{trop}(X)) = \operatorname{trop}(Y)$. If σ is a maximal dimensional cone of $\operatorname{trop}(Y)$ then its multiplicity is given by

$$m_\sigma = \frac{1}{\delta} \sum_{\gamma \in \operatorname{trop}(X)\colon A(\gamma) \supseteq \sigma} m_\gamma \operatorname{index}(\gamma, \sigma) \tag{2.2}$$

where $\operatorname{index}(\gamma, \sigma)$ denotes the index of the sublattice of N_m generated by $A(\gamma \cap N_n)$ inside of the sublattice generated by $\sigma \cap N_m$.

Example 2.4.2. Let $I = \langle xy + 10y^2 - 23yz - 4y + 64z - 48, y^2 - 4yz + 4z^2 + 2y - 3z, 23y^2 + 4xz - 52yz - 18y + 171z - 128\rangle \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be a one-dimensional ideal which defines a subvariety X of $(\mathbb{C}^*)^3$. Let $\mathbb{C}$ be equipped with the trivial valuation. Let f be the projection map onto the first two coordinates. This map has degree one, and so $\delta = 1$. We verify the results of Sturmfels and Tevelev [2008] here showing that projection and tropicalisation commute.

First we tropicalise then take the projection. Using the methods of Section 2.3.2 we see that $\operatorname{trop}(X)$ is a one-dimensional fan in $\mathbb{R}^3$ with five rays generated by

$$\begin{aligned}
\rho_1 &= (1,0,0)\\
\rho_2 &= (0,1,0)\\
\rho_3 &= (2,0,1)\\
\rho_4 &= (-1,-1,-1)\\
\rho_5 &= (-1,1,1)
\end{aligned}$$

with multiplicities $m_{\rho_1} = 1$, $m_{\rho_2} = 1$, $m_{\rho_3} = 1$, $m_{\rho_4} = 2$ and $m_{\rho_5} = 1$. Then projecting onto the first two coordinates, we have four rays

$$\sigma_1 = (1,0), \sigma_2 = (0,1), \sigma_3 = (-1,-1), \sigma_4 = (-1,1).$$

We use (2.2) to find their multiplicities. To find m_{σ_1} we see that both ρ_1 and ρ_3 project to σ_1. We need to now compute the lattice indices. For ρ_1, $\text{index}(\sigma_1,\rho_1)$ is the index of the sublattice of $\mathbb{Z}^2$ generated by $f((1,0,0)) = (1,0)$ inside the sublattice generated by $(1,0)$. This has index one. For ρ_3, $\text{index}(\sigma_1,\rho_3)$ is the index of the sublattice of $\mathbb{Z}^2$ generated by $f((2,0,1)) = (2,0)$ inside the sublattice generated by $(1,0)$. This has index two. So then

$$m_{\sigma_1} = \frac{1}{1}\left[1\cdot 1 + 1\cdot 2\right] = 3.$$

We similarly compute the other multiplicities using (2.2) as follows:

$$\begin{aligned}
m_{\sigma_2} &= \frac{1}{1}\left[1\cdot 1\right] = 1\\
m_{\sigma_3} &= \frac{1}{1}\left[2\cdot 1\right] = 2\\
m_{\sigma_4} &= \frac{1}{1}\left[1\cdot 1\right] = 1.
\end{aligned}$$

Now, suppose we project and then tropicalise the result. By Elimination Theory, we find that the projection is generated by polynomial $x^2y - 12xy^2 + 9y^3 + 155xy + 32y^2 - 192x + 20y - 16$. Using Algorithm 2.3.3 we see that the tropicalisation has rays $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ with multiplicities $m_{\sigma_1}, m_{\sigma_2}, m_{\sigma_3}, m_{\sigma_4}$ as required. ◇

2.5 Tropical Varieties by Regular Projections

Let X be an irreducible d-dimensional subvariety of T^n with defining ideal I. In proving that the tropical variety $\text{trop}(X)$ has the structure of a polyhedral complex, Bieri and Groves [1984] showed that a tropical variety can be obtained by considering the intersection of the pre-images of sufficiently general projections. Hept and Theobald [2009] then showed that you can always find these projections such that the tropical variety is then given by an intersection of tropical hypersurfaces. In this section, we outline this material as it forms the background to Chapter 4 where we reconstruct a tropical curve from its coordinate projections.

Let X be a subvariety of T^n with tropicalisation $\operatorname{trop}(X)$. Then $\operatorname{trop}(X)$ is the support of a polyhedral complex Σ in $\mathbb{R}^n$. A projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ is called *geometrically regular* with respect to Σ [Hept and Theobald, 2009, Definition 3.6] if:

1. if σ is a k-dimensional face of Σ then $\dim(\pi(\sigma)) = k$;
2. if $\pi(\sigma) \subseteq \pi(\tau)$ for some $\sigma, \tau \in \Sigma$ then $\sigma \subseteq \tau$.

That is, if it respects dimensions and inclusion of faces. Bieri and Groves [1984, Section 4.2] also considered these geometrically regular projections but they simply called them *regular projections*. However Hept and Theobald [2009] gave them the name geometrically regular to avoid confusion with another class of regular projections which they also defined, algebraically regular projection. Thus we shall stick to the name geometrically regular of Hept and Theobald [2009].

For $\pi\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ a rational projection, $\pi^{-1}\pi \operatorname{trop}(V(I))$ is a tropical variety. If π is additionally geometrically regular, then $\dim(\operatorname{trop}(V(I))) = d$ and then $\pi^{-1}\pi \operatorname{trop}(X)$ is a tropical hypersurface. Thus it is defined by a single equation. We can find this equation by first applying a coordinate change so that π is a coordinate projection and finding this equation by the Elimination Theory of Section 2.4.

In proving that the tropical variety $\operatorname{trop}(X)$ has the structure of a polyhedral complex, Bieri and Groves [1984, Proof of Theorem 4.4] also showed that there exists $n-d+1$ geometrically regular projections $\pi_0, \dots, \pi_{n-d}\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ in some dense open set in the space of all projections such that

$$\operatorname{trop}(X) = \bigcap_{i=0}^{n-d} \pi_i^{-1}\pi_i(\operatorname{trop}(X)).$$

From above, for all $0 \le i \le n-d$ we have that $\pi_i^{-1}\pi_i(\operatorname{trop}(X)) = \operatorname{trop}(V(g_i))$ for some $g_i \in I$. Thus we can write $\operatorname{trop}(X)$ as the finite intersection of tropical hypersurfaces. This means that we can find $g_0, g_1, \dots, g_{n-d} \in I$ such that

$$\operatorname{trop}(X) = \bigcap_{i=0}^{n-d} \operatorname{trop}(V(g_i)).$$

We will use these projection ideas in order to find new ways to construct tropical curves in Chapter 4 where we will restrict ourselves to coordinate projections.

2.6 The Degree of Tropical Curves

Let C be a curve in T^n with tropicalisation $\text{trop}(C)$. We shall see in (2.3) how the degree of a tropical curve can be defined. The following theorem asserts that both C and $\text{trop}(C)$ have the same degree.

Theorem 2.6.1. *[Maclagan and Sturmfels, 2013] Let C be a curve in T^n. Then $\deg(C) = \deg(\text{trop}(C))$.*

In Chapter 4 we shall use the fact that we know the degree of the algebraic curve C as then by Theorem 2.6.1 we know the degree of the tropical curve $\text{trop}(C)$. We outline the combinatorial calculation of the degree of a tropical curve here. It requires some tropical intersection theory. Let Σ_1 and Σ_2 be two weighted balanced Γ-rational polyhedral fans in $\mathbb{R}^n$ which are the support of two tropical varieties. Suppose that Σ_1 is one-dimensional and Σ_2 is $(n-1)$-dimensional. We define the following intersection product [Katz, 2012]:

$$\Sigma_1 \cdot \Sigma_2 = \sum_{(\sigma_1,\sigma_2)\in\Sigma_1\times\Sigma_2} \mu_{\sigma_1\sigma_2} m_{\sigma_1} m_{\sigma_2} [\sigma_1 \cap \sigma_2],$$

where m_{σ_1} is the multiplicity of σ_1 in Σ_1, m_{σ_2} is the multiplicity of σ_2 in Σ_2 and

$$\mu_{\sigma_1\sigma_2} = \begin{cases} [\mathbb{Z}^n : \mathbb{Z}\langle\sigma_1, \sigma_2\rangle] & \text{if } \sigma_1 \cap (\sigma_2 + u) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$$

for some generic $u \in \mathbb{Z}^n$ such that $\sigma_1 \cap (\sigma_2 + u)$ is a finite set of points. We define the *degree* of this intersection:

$$\deg(\Sigma_1 \cdot \Sigma_2) = \sum_{(\sigma_1,\sigma_2)\in\Sigma_1\times\Sigma_2} \mu_{\sigma_1\sigma_2} m_{\sigma_1} m_{\sigma_2}.$$

If $\Sigma_2(u)$ represents the fan Σ_2 translated by u, then we can think of $\Sigma_1 \cdot \Sigma_2$ as being the set of intersection points of Σ_1 and $\Sigma_2(u)$. As u is chosen to be generic, it follows that this intersection is a finite number of points each point being the intersection of a cone $\sigma_1 \in \Sigma_1$ and $\sigma_2 + u \in \Sigma_2(u)$ with multiplicity $\mu_{\sigma_1\sigma_2}$. That the degree does not depend on the choice of generic u follows from Allermann and Rau [2010, Lemma 9.14].

Example 2.6.2. Let Σ_1 and Σ_2 be two one-dimensional fans in $\mathbb{R}^2$. Suppose that Σ_1 has rays generated by $\rho_1 = (1, 0)$, $\rho_2 = (0, -1)$ and $\rho_3 = (-2, 1)$ with multiplicities 2, 1 and 1 respectively, and that Σ_2 has rays generated by $\sigma_1 = (1, 0)$, $\sigma_2 = (0, 1)$ and $\sigma_3 = (-1, -1)$ all with multiplicities one. let $u = (-1, -1)$ then the intersection $\Sigma_1 \cdot \Sigma_2$ as shown in Figure 2.5 and consists of the points $(-1, 1)$ being the intersection

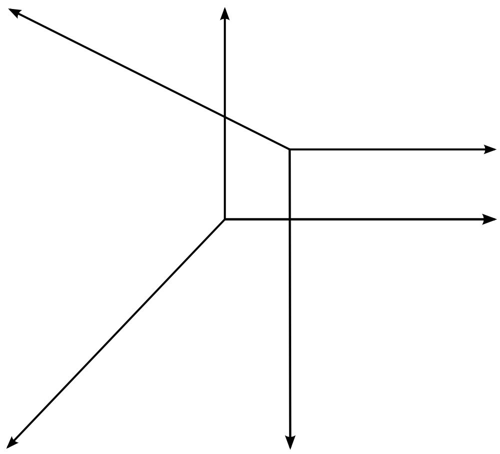

Figure 2.5: The intersection of Σ_1 and Σ_2 of Example 2.6.2

of ρ_3 with σ_2 and the point $(0, -1)$ that is the intersection of ρ_2 with σ_1. Then

$$\mu_{\rho_3\sigma_2} = [\mathbb{Z}^2 : \mathbb{Z}\langle(-2, 1), (0, 1)\rangle] = \left|\det \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}\right| = 2;$$

$$\mu_{\rho_2\sigma_1} = [\mathbb{Z}^2 : \mathbb{Z}\langle(0, -1), (1, 0)\rangle] = \left|\det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right| = 1;$$

and so $\deg(\Sigma_1 \cdot \Sigma_2) = 2 + 1 = 3$. Notice that if we had chosen another generic u, say $u = (1, 0)$, then we would obtain the same final answer $\deg(\Sigma_1 \cdot \Sigma_2) = 3$. ◇

We now explain how to use this tropical intersection theory to find the degree of a tropical curve combinatorially. Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$ and $e_0 = -e_1 - \dots - e_n$. Let $\mathrm{trop}(C)$ be a tropical curve which is the support of a one-dimensional weighted balanced fan Σ in $\mathbb{R}^n$. If ρ is the first lattice point on a ray of Σ then we can decompose it into a sum of $e_0, e_1, \dots, e_n$ as

$$\rho = \sum_{i=0}^{n} a_i e_i$$

for some $a_i \in \mathbb{N}$. We say a decomposition is *minimal* if there does not exist another decomposition with smaller a_i. This occurs if and only if at least one a_i equals zero.

Such a minimal decomposition is unique. The *support* of ρ is $\operatorname{supp}(\rho) = \{e_i : a_i \neq 0$ in its minimal decomposition$\}$.

Let L be the weighted balanced fan which is the tropicalisation of a hypersurface defined by a linear polynomial of full support. That is L is the $(n-1)$-dimensional fan consisting of all cones generated by all $(n-1)$-element subsets of $\{e_0, e_1, \ldots, e_n\}$ which all occur with multiplicity one. Following Allermann and Rau [2010], we define the degree of $\operatorname{trop}(C)$ as follows

$$\deg(\operatorname{trop}(C)) = \deg(\Sigma \cdot L). \tag{2.3}$$

Example 2.6.3. Consider the ideal $I = \langle xy^2 + y^2 + 1\rangle \subseteq K[x^{\pm 1}, y^{\pm 1}]$ and let $C = V(I)$. Then by Section 2.3.1 we see that $\operatorname{trop}(C)$ is the fan Σ_1 of Example 2.6.2 and see that $\deg(\operatorname{trop}(C)) = 3$ as we would expect as the generating polynomial of the hypersurface has degree 3. ◇

Fix some $j \in \{0, 1 \ldots, n\}$. In the definition of tropical intersection, we translate one of the tropical varieties by a generic $u \in \mathbb{Z}^n$. Choose u to be generic and such that Σ and $L(u)$ intersect only in rays ρ of Σ for which $e_j \in \operatorname{supp}(\rho)$ and cones σ of L for which e_j is not a generator. We do this by choosing u to be $ce_j + \varepsilon$ which means that we are moving L in the e_j direction. The intersection condition for each ray of Σ gives us a closed set of points of u which are not suitable. As Σ has only finitely may rays, again we have a closed set of unsuitable choices for u. The set of generic u is open and Zariski dense in $\mathbb{Z}^n$ [Fulton and Sturmfels, 1997], and so we can always find a suitably generic u. With this choice of u, the ray ρ of Σ intersects $L(u)$ in the cone σ where σ is generated by $A = \{e_i : 0 \leq i \leq n, \quad i \neq j, k\}$ for some $0 \leq k \leq n$ where $\operatorname{supp}(\rho) \subseteq A \cup \{e_j\}$.

This intersection contributes $\mu_{\rho\sigma} m_\rho$ to the degree equation as $m_\sigma = 1$ for all cones of L. The factor $\mu_{\rho\sigma}$ is the lattice index $[\mathbb{Z}^n : \mathbb{Z}\langle \rho, \sigma\rangle]$ which can be seen as the absolute value of the determinant of the $n \times n$ matrix M whose rows are given by ρ and the e_i for which $e_i \in A$. If the minimal decomposition of ρ is $\sum_{i=0}^n a_i e_i$ then as $\operatorname{supp}(\rho) \subseteq A \cup \{e_j\}$ we can perform row operations on M which do not affect its determinant to make the row ρ become $a_j e_j$. This matrix then has determinant equal to $\pm a_j$ and so does M.

It then follows that if Σ has rays $\rho_1, \ldots, \rho_s$ where ρ_i has minimal decomposition

$$\rho_i = \sum_{j=0}^{n} a_{ij} e_j,$$

for $1 \leq i \leq s$ then for all $0 \leq j \leq n$

$$\deg(\text{trop}(C)) = \sum_{i=1}^{s} m_i a_{ij}. \tag{2.4}$$

This gives a combinatorial way to compute the degree of a tropical curve without having to find a generic u.

Example 2.6.4. Returning to Example 2.6.3, we find the degree of $\text{trop}(C)$ using this combinatorial rule. We first find minimal decompositions for the rays of $\text{trop}(C)$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis vectors of $\mathbb{Z}^2$ and $e_0 = -e_1 - e_2 = (-1, -1)$. Then the minimal decompositions are:

$$\begin{aligned} \rho_1 &= (1, 0) = e_1; \\ \rho_2 &= (0, -1) = e_0 + e_1; \\ \rho_3 &= (-2, 1) = 2e_0 + 3e_2, \end{aligned}$$

where ρ_1 has multiplicity two and ρ_2 and ρ_3 both have multiplicity one. Then the description above tells us that the degree of $\text{trop}(C)$ is the total number of rays in the direction e_0, e_1, e_2 counted with multiplicity. Counting in the direction e_0 we get

$$\deg(\text{trop}(C)) = 2 \cdot 0 + 1 \cdot 1 + 1 \cdot 2 = 3.$$

Notice that counting in the directions e_1 or e_2 would give us the same answers

$$\begin{aligned} \deg(\text{trop}(C)) &= 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 3 \\ \deg(\text{trop}(C)) &= 2 \cdot 0 + 1 \cdot 0 + 1 \cdot 3 = 3. \end{aligned}$$ ◇

Chapter 3

Gröbner Bases Over Fields with Valuations

3.1 Introduction

Let X be a subvariety of $\mathbb{P}^{n-1}$ and let $X^0 = X \cap T^n$. Then by the Fundamental Theorem 2.1.5, we see that the tropical variety $\text{trop}(X^0)$ is the closure in $\mathbb{R}^n$ of those $w \in \Gamma^n$ for which the initial ideal $\text{in}_w(I) \neq \langle 1 \rangle$. We can use the theory of Gröbner bases in order to compute these initial ideals, and so construct the tropical variety.

We saw in the Section 2.3.3 that most prior computational work in tropical geometry has concerned ideals with coefficients in $\mathbb{Q}$ with the trivial valuation as this can be treated using standard Gröbner techniques. In this case (without valuations), an algorithm for computing Gröbner bases was developed by Bruno Buchberger [1965] in his PhD thesis. For more general valued fields, such as $K = \mathbb{Q}$ with the p-adic valuation val_p, the standard Gröbner algorithms need to be modified. This is explained in Section 3.2. The main issue is that the standard normal form algorithm need not terminate. The solution is to replace it by a modification of Mora's tangent cone algorithm.

Unlike the standard basis case, we get a strong normal form; see Remark 3.2.7. In Sections 3.3 and 3.4 we discuss complexity and implementation issues. Degree bounds are as for usual Gröbner bases (Theorem 3.3.1). While the valuations of coefficients in a reduced Gröbner basis cannot be bounded by the valuations of the original generators (Example 3.3.3), for the case $K = \mathbb{Q}$ with the p-adic valuation, we can bound the valuations of coefficients in a reduced Gröbner basis using the valuations and absolute values of coefficients of the generators; see Proposition 3.3.4.

A theoretical consequence of these results is that the tropical variety of an ideal only depends on the field defined by the coefficients of the generators; see Corollary 3.2.13. We expect these algorithms to also have applications outside tropical geometry. In particular, they can lead to smaller Gröbner bases. In Section 3.5 we give a family of ideals in $\mathbb{Q}[x_1, x_2, x_3]$ for which the size of the p-adic Gröbner basis is constant but the smallest size of a traditional Gröbner basis grows unboundedly.

The algorithms described in this Chapter have been implemented in the computational algebraic geometry software `Macaulay2` [Grayson and Stillman] in the package `GroebnerValuations` [Chan, 2013a].

The material in this Chapter is joint work with my supervisor Diane Maclagan [Chan and Maclagan, 2013].

3.2 Gröbner Theory

Let S be the polynomial ring $K[x_1, \ldots, x_n]$, and fix a weight vector $w \in \Gamma^n$. Note that in this Chapter, our homogeneous polynomial ring S is in the n variables $x_1, \ldots, x_n$ instead for the $n+1$ variables $x_0, x_1, \ldots, x_n$ of Chapter 2 for ease of notation. Recall that for homogeneous ideal $I \subset S$, a finite set $\mathcal{G} = \{g_1, \ldots, g_r\} \subset S$ is called a *Gröbner basis* for I with respect to w if $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_r) \rangle$. We require the ideal to be homogeneous for it to exhibit the expected properties of Gröbner bases; see Remark 3.2.12.

In the standard case where we are not considering the valuations, or where we have the trivial valuation, Buchberger [1965] introduced an algorithm to compute Gröbner bases. Let $\mathcal{B}$ be a generating set for the ideal I. Then for all $g, g' \in \mathcal{B}$, the S-polynomial $S(g, g')$ is the sum of g and g' which cancels their leading terms. Buchberger showed that $\mathcal{B}$ is a Gröbner basis for I if and only if all S-polynomials have zero remainder on division by $\mathcal{B}$. We can then compute a Gröbner basis by adding the non-zero remainders of S-polynomials back to $\mathcal{B}$ until all S-polynomials have zero remainder. We shall see Buchberger's Algorithm in Algorithm 3.2.9.

Lemma 3.2.1. *Let I be an ideal in S with $\prec$ an arbitrary term order and $w \in \Gamma^n$. If $\mathcal{G} = \{g_1, \ldots, g_r\}$ is a generating set for I such that $\{\mathrm{in}_w(g) : g \in \mathcal{G}\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to $\prec$, then it is a Gröbner basis for I with respect to w.*

Proof. As $\{\mathrm{in}_w(g) : g \in \mathcal{G}\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to $\prec$, in particular it is a generating set. Thus $\langle \mathrm{in}_w(g) \colon g \in \mathcal{G} \rangle = \mathrm{in}_w(I)$, which by definition tells us that $\mathcal{G}$ is a Gröbner basis for I with respect to w. □

This result tells us that if we can compute a basis $\mathcal{G}$ for an ideal I such that $\{\mathrm{in}_\prec(\mathrm{in}_w(g)) : g \in \mathcal{G}\}$ generates $\mathrm{in}_\prec(\mathrm{in}_w(I))$ then it is automatically a Gröbner basis for I with respect to w.

3.2.1 The Modified Normal Form Algorithm

A Gröbner basis for an ideal I over a field with a valuation can be computed by a modification of the standard Buchberger algorithm as we explain in Section 3.2.2. The main difference is in the normal form algorithm which computes the remainder of a polynomial on division by a set of other polynomials. The difficulty is that a naïve implementation of the normal form algorithm need not terminate, as the following example shows.

Example 3.2.2. Let $K = \mathbb{Q}$ with the 2-adic valuation. Consider the standard normal form algorithm, where the term to be cancelled at each stage is taken to be the term whose coefficient has the lowest valuation. Using this to compute the remainder of $x \in \mathbb{Q}[x, y, z]$ on division by $\{x - 2y, y - 2z, z - 2x\}$, we reduce x by $x - 2y$ to get $2y$. This is then reduced by $y - 2z$ to get $4z$, which in turn is reduced by $z - 2x$ to get $8x$. This reduction continues indefinitely. $\diamond$

This problem also arises in the theory of standard bases [see Cox, Little, and O'Shea, 2005, Section 4.3]. The solution in that setting, Mora's tangent cone algorithm, is to allow division by previous partial quotients. Termination is assured by a descending nonnegative integer invariant called the écart which measures the difference in degrees between two possible initial terms of a polynomial. A difficulty in generalizing this function to Gröbner bases with valuations is that this difference must take the valuations of the coefficients into account, so would naturally lie in the not-necessarily-well-ordered group Γ. Even for the valuation val_p on $\mathbb{Q}$, where $\Gamma = \mathbb{Z}$, the standard écart function does not work directly.

The following algorithm modifies Mora's algorithm to take into account the valuations of the coefficients. It uses a function $E(f, g)$, which takes two homogeneous polynomials and returns a nonnegative integer. In Lemma 3.2.6 we give one option for this function which ensures termination. We present the algorithm with the function E unspecified as more efficient functions E may exist.

As in all normal form algorithms this is a generalisation of long division, which works by cancelling the "leading term" of the polynomial f. An added complication is that we do not assume that the weight vector w is generic, so the leading term $\text{in}_w(f)$ is not necessarily a monomial. For this reason we also fix an arbitrary monomial term order $\prec$ (in the sense of usual Gröbner theory) to determine which term of $\text{in}_w(f)$ to cancel. If w is sufficiently generic with respect to the input polynomials $\prec$ will play no role. For $f \in K[x_1, \ldots, x_n]$, $\text{in}_\prec(\text{in}_w(f)) = \alpha x^u$ denotes the leading term, including the coefficient. We denote by $\operatorname{lm}(f)$ the monomial x^u occurring in $\operatorname{in}_\prec(\operatorname{in}_w(f))$ and by $\operatorname{lc}(f)$ the coefficient of x^u in f. Note that $\operatorname{lc}(f) \in K$, not $\Bbbk$, and that $\operatorname{lc}(f)$ and $\operatorname{lm}(f)$ depend on both w and $\prec$.

We also use the following partial order on polynomials, which plays the role of comparing initial monomials in usual Gröbner bases.

Definition 3.2.3. Fix homogeneous polynomials $f, g \in K[x_1, \ldots, x_n]$, $w \in \Gamma^n$, and a term order $\prec$. Write $\operatorname{lm}(f) = x^u$, $\operatorname{lm}(g) = x^v$, $\operatorname{lc}(f) = a$, and $\operatorname{lc}(g) = b$. Then $f < g$ if $\operatorname{val}(a) + w \cdot u < \operatorname{val}(b) + w \cdot v$ or $\operatorname{val}(a) + w \cdot u = \operatorname{val}(b) + w \cdot v$ and $x^u \succ x^v$. In addition $f < 0$ for all nonzero f.

For example, if $\mathbb{Q}$ has the 2-adic valuation, $w = (1, 2)$ and $\prec$ is the lexicographic term order with $x_1 > x_2$, then $x_1^2 < x_2^2 < x_1^5 < 2x_2^2$. Note that if $f \geq h$ and $g \geq h$ then $f \pm g \geq h$.

Algorithm 3.2.4. Input: Homogeneous polynomials $\{g_1, \ldots, g_s\}$, f in $S = K[x_1, \ldots, x_n]$, a weight vector $w \in \Gamma^n$, and a term order $\prec$.
Output: Homogeneous polynomials $h_1, \ldots, h_s, r \in S$ satisfying

$$f = \sum_{i=1}^{s} h_i g_i + r,$$

where $h_i g_i \geq f$ for $1 \leq i \leq s$, and $r \geq f$. Write $r = \sum b_v x^v$ with $b_v \in K$. Then in addition $b_v \neq 0$ implies x^v is not divisible by any $\operatorname{lm}(g_i)$.

We call r a *remainder*, or *normal form*, of dividing f by $\{g_1, \ldots, g_s\}$.

1. **Initialise:** Set $T = \{g_1, \ldots, g_s\}, h_{10} = \cdots = h_{s0} = 0, q_0 = f, r_0 = 0$. Set $j = 0$.

2. **Loop:** While $q_j \neq 0$ do:

   (a) **Move to remainder:** If there is no $g \in T$ with $\operatorname{lm}(g)$ dividing $\operatorname{lm}(q_j)$, then set $r_{j+1} = r_j + \operatorname{lc}(q_j) \operatorname{lm}(q_j)$, $q_{j+1} = q_j - \operatorname{lc}(q_j) \operatorname{lm}(q_j)$, and $h_{ij+1} = h_{ij}$ for all i. Set $T = T \cup \{q_j\}$.

   (b) **Divide:** Otherwise:

      i. Choose $g \in T$ such that $\operatorname{lm}(g)$ divides $\operatorname{lm}(q_j)$ with $E(q_j, g)$ minimal among all such choices.

      ii. If $E(q_j, g) > 0$ then set $T = T \cup \{q_j\}$.

      iii. Since $\operatorname{lm}(g)$ divides $\operatorname{lm}(q_j)$ there is a monomial x^v with $\operatorname{lm}(x^v g) = \operatorname{lm}(q_j)$. Set $c_v = \operatorname{lc}(q_j)/\operatorname{lc}(x^v g) \in K$. Let $p = q_j - c_v x^v g$.

iv. If $g = g_l$ for some $1 \le l \le s$, then set $q_{j+1} = p$, $h_{lj+1} = h_{lj} + c_v x^v$, $h_{ij+1} = h_{ij}$ for $i \ne l$, and $r_{j+1} = r_j$.

v. If g was added to T at some previous iteration of the algorithm, so $g = q_m$ for some $m < j$, then set $q_{j+1} = 1/(1-c_v)p$, $h_{ij+1} = 1/(1-c_v)(h_{ij} - c_v h_{im})$, and $r_{j+1} = 1/(1-c_v)(r_j - c_v r_m)$ for all i.

(c) $j = j + 1$.

3. **Output:** Output $h_i = h_{ij}$ for $1 \le i \le s$, and $r = r_j$.

Example 3.2.5. Let $f = x^2 + y^2 + z^2 \in \mathbb{Q}[x, y, z]$ where $\mathbb{Q}$ has the 2-adic valuation, and let $g_1 = y + 16z$. Fix $w = (3, 2, 1)$, and let $\prec$ be the lexicographic order with $x \prec y \prec z$. For clarity we underline the term of a polynomial f containing $\mathrm{lm}(f)$. We do not specify the function $E(f, g)$, assuming that it is always positive. Then the algorithm proceeds as follows.

1. $T = \{\underline{y} + 16z\}$, $h_{10} = 0$, $q_0 = x^2 + y^2 + \underline{z^2}$, $r_0 = 0$, $j = 0$.

2. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2\}$, $h_{11} = 0$, $q_1 = x^2 + \underline{y}^2$, $r_1 = z^2$, $j = 1$.

3. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2, x^2 + \underline{y}^2\}$, $h_{12} = y$, $q_2 = \underline{x}^2 - 16yz$, $r_2 = z^2$, $j = 2$.

4. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2, x^2 + \underline{y}^2, \underline{x}^2 - 16yz\}$, $h_{13} = y$, $q_3 = -16yz$, $r_3 = x^2 + z^2$, $j = 3$.

5. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2, x^2 + \underline{y}^2, \underline{x}^2 - 16yz, -16yz\}$, $h_{14} = y - 16z$, $q_4 = 256z^2$, $r_4 = x^2 + z^2$, $j = 4$.

6. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2, x^2 + \underline{y}^2, \underline{x}^2 - 16yz, -16yz, 256z^2\}$. In this case we divide by $g = x^2 + y^2 + z^2 = q_0$, so $c_v = 256$. Thus $h_{15} = -1/255(y - 16z)$, $q_5 = 1/255(256x^2 + \underline{256y}^2)$, $r_5 = -1/255(x^2 + z^2)$, and $j = 5$.

7. $T = \{\underline{y} + 16z, x^2 + y^2 + \underline{z}^2, x^2 + \underline{y}^2, \underline{x}^2 - 16yz, -16yz, -16z^2, 256/255x^2 + \underline{255/256y}^2\}$. Then $g = x^2 + y^2 = q_1$, so $c_v = 256/255$. Thus $h_{16} = 255(1/255(y - 16z)) = y - 16z$, $q_6 = 0$, $r_6 = -255(-1/255(x^2 + z^2) - 256/255z^2) = x^2 + 257z^2$, and $j = 6$.

8. Output $h_1 = y - 16z$ and $r = x^2 + 257z^2$.

Note that $x^2 + y^2 + z^2 = (y - 16z)(y + 16z) + x^2 + 257z^2$ and no term of $x^2 + 257z^2$ is divisible by $\mathrm{lm}(y + 16z) = y$. $\diamond$

Proof of correctness. We show correctness assuming termination.

We show that the following properties hold at each stage of the algorithm:

1. $f = q_j + \sum_{i=1}^{s} h_{ij} g_i + r_j$;

2. $h_{ij} g_i \geq f$;

3. $r_j \geq f$;

4. No term of r_j is divisible by any $\text{lm}(g_i)$;

5. $q_j \geq f$;

6. If $q_{j+1} \neq 0$ then $q_{j+1} > q_j$.

These properties all hold at the initialization step by construction. We now show they continue to hold after each of the three types of iteration step. We also show that in step 2(b)v of the algorithm we have $1 - c_v \neq 0$. In all cases, write $\text{lc}(q_j)\,\text{lm}(q_j) = c_j x^{\alpha_j}$. There are three possibilities for the division step, which we consider separately.

Case 1: *Move to remainder.* Suppose there is no $g \in T$ with $\text{lm}(g)$ dividing $\text{lm}(q_j)$. Then the only values that change are q_j and r_j, but we have $q_j + r_j = q_{j+1} + r_{j+1}$ by construction, so the equality 1 holds. Condition 2 holds at stage $j+1$ since it held at stage j. Since properties 3 and 5 hold for j, property 3 holds for $j+1$. The term that is added to r_{j+1} is not divisible by any $\text{lm}(g_i)$, so property 4 still holds. The term $c_{j+1} x^{\alpha_{j+1}}$ is a nonleading term of q_j, so property 6 follows, which also implies property 5.

Case 2: *Divide, with $g = g_m$.* Suppose the chosen g with $\text{lm}(g)$ dividing $\text{lm}(q_j)$ is g_m for some $1 \leq m \leq s$. Since $q_j + h_{mj} g_m = q_{j+1} + h_{mj+1} g_m$ by construction, the equality 1 holds in this case as well. Since $h_{mj} g_m \geq f$, and $q_j \geq f$, we have $h_{mj+1} g_m \geq f$. As the remainder term does not change, properties 3 and 4 still hold. Since $q_{j+1} = q_j - c_v x^v g_l$, we cancel the leading term of q_j, so all terms of q_{j+1} are the sum of a nonleading term of q_j and a term of $c_v x^v q_l$ that is larger than $c_j x^{\alpha_j}$. This implies that $q_j \prec q_{j+1}$ (property 6), which implies property 5 for $j+1$ as above.

Case 3: *Divide, with $g = q_m$.* Finally, we consider the case that the chosen g with $\text{lm}(g)$ dividing $\text{lm}(q_j)$ is q_m for some $m < j$. Since all q_i are homogeneous of the same degree, $x^v = 1$ in this setting and $c_v = c_j / c_m$. Since property 6 holds for all smaller values, we have $\text{val}(c_j) + w \cdot \alpha_j > \text{val}(c_m) + w \cdot \alpha_m$. Thus $x^{\alpha_m} = x^{\alpha_j}$ implies $\text{val}(c_v) > 0$, so $1 - c_v \neq 0$.

Now $f = q_m + \sum_{i=1}^{s} h_{im} g_i + r_m$, so $q_{j+1} = 1/(1 - c_v)(q_j - c_v q_m)$, which equals $1/(1-c_v)((f - \sum_{i=1}^{s} h_{ij} g_i - r_j) - c_v(f - \sum_{i=1}^{s} h_{im} g_i - r_m))$. Thus $f = q_{j+1} + \sum_{i=1}^{s} 1/(1-c_v)(h_{ij} - c_v h_{im}) g_i + 1/(1-c_v)(r_j - c_v r_m) = q_{j+1} + \sum_{i=1}^{s} h_{ij+1} g_i + r_{j+1}$. This is equality 1.

Since $\text{val}(1-c_v) = 0$, we have $\text{val}(1/(1-c_v)) = 0$. Note the following property of the order $<$ of Definition 3.2.3: if $p_1 \geq p_2$ and $c \in K$ satisfies $\text{val}(c) \geq 0$ then $cp_1 \geq p_2$. Then properties 2 and 3 for $j+1$ follow from the analogous properties for j and m. No term in either r_j or r_m is divisible by any $\text{lm}(g_i)$, so the same is true for r_{j+1}. Finally $p > q_j$ by construction, so $q_{j+1} = 1/(1-c_v)p > q_j$ as above, so properties 5 and 6 also hold. □

Lemma 3.2.6. *For homogeneous polynomials $f, g \in S$ with $f = \sum c_u x^u$ and $g = \sum b_u x^u$, set $E(f,g) := |\{u : b_u \neq 0, c_u = 0\}|$. Algorithm 3.2.4 terminates for this choice of function E.*

Proof. There are only a finite number of possible supports $\text{supp}(q_j) = \{u : c_u \neq 0\}$ of the polynomials $q_j = \sum c_u x^u$, as they all have the same degree. Thus after some step j no new support will occur, so there will be $q_m \in T_j$ with $\text{supp}(q_m) \subseteq \text{supp}(q_j)$, and so $E(q_j, q_m) = 0$. Since we remove the leading term of q_j at the jth step, either by moving it to the remainder, or by cancelling it, when $\text{supp}(q_m) \subseteq \text{supp}(q_j)$ we have $\text{supp}(q_{j+1}) \subsetneq \text{supp}(q_j)$. Since the size of the support cannot decrease indefinitely, the algorithm must terminate. □

Remark 3.2.7. Note that Algorithm 3.2.4 gives a strong normal form (no term of the remainder is divisible by any of the monomials $\{\text{lm}(g_i) : 1 \leq i \leq s\}$), as opposed to the weak normal form that occurs in the standard basis case. See Greuel and Pfister [2008, Section 1.6] for details of normal forms in the standard basis case. ◇

Remark 3.2.8. Algorithm 3.2.4 also holds, with the same proof in the following modified setting. Let $K = \mathbb{Q}$ with the p-adic valuation. The valuation val_p restricts to a function, which we also denote by val_p, from $\mathbb{Z}/p^m\mathbb{Z}$ to the semigroup $\{0, 1, \ldots, m-1\} \cup \infty$, where ∞ acts as an absorbing element. Note that $\text{val}_p(ab) = \text{val}_p(a) + \text{val}_p(b)$ and $\text{val}_p(a+b) \geq \min(\text{val}_p(a), \text{val}_p(b))$ for $a, b \in \mathbb{Z}/p^m\mathbb{Z}$. We can then define the partial order $<$ on polynomials in $\mathbb{Z}/p^m\mathbb{Z}[x_1, \ldots, x_n]$ in the same way as in Definition 3.2.3. Also note that in step 2(b)v of the algorithm, since $1 - c_v$ has valuation zero (as shown in the proof), it is not divisible by p, so is a unit in $\mathbb{Z}/p^m\mathbb{Z}$. This means that the algorithm and its proof go through in this setting. This variant is used in Section 3.4.2. ◇

3.2.2 Buchberger's Algorithm

As in standard Gröbner theory, we can use the normal form algorithm to compute a Gröbner basis using Buchberger's algorithm. Let f, g be two polynomials in

$K[x_1, \ldots, x_n]$. We define the S-polynomial of f and g to be

$$S(f,g) := \mathrm{lc}(g)\frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(f)}f - \mathrm{lc}(f)\frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(g)}g.$$

Algorithm 3.2.9. Input: A list $\{f_1, \ldots, f_l\}$ of homogeneous polynomials in S, a weight-vector $w \in \Gamma^n$, and a term order $\prec$.
Output: A list $\{g_1, \ldots, g_s\}$ of homogeneous polynomials in S such that $\{\mathrm{in}_\prec(\mathrm{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\mathrm{in}_\prec(\mathrm{in}_w(I))$ for $I = \langle f_1, \ldots, f_l \rangle$.

1. Set $\mathcal{G} = \{f_1, \ldots, f_l\}$. Set $\mathcal{P} = \{(g, g') : g, g' \in \mathcal{G}\}$.
2. While $\mathcal{P} \neq \emptyset$:
   (a) Pick any $(g, g') \in \mathcal{P}$. Set $\mathcal{P} = \mathcal{P} \setminus \{(g, g')\}$.
   (b) Let r be the normal form on dividing $S(g, g')$ by $\mathcal{G}$. If $r \neq 0$ then set $\mathcal{G} = \mathcal{G} \cup \{r\}$, and $\mathcal{P} = \mathcal{P} \cup \{(r, g) : g \in \mathcal{G}\}$.
3. Return $\mathcal{G}$.

The proof of the finiteness and correctness of this algorithm is almost the same as the proof for standard Gröbner bases [see, for example, Cox, Little, and O'Shea, 2007, Chapter 2]. We will prove it by using a generalisation of the normal form.

Let $f \in S$ be a homogeneous polynomial and $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a finite subset of S. We say that f has a *standard representation* with respect to $\mathcal{G}$ if it can be written in the form

$$f = \sum_{i=1}^{s} a_i g_i$$

such that $f \leq a_i g_i$ for all i where $a_i \neq 0$.

Remark 3.2.10. Note that if f has zero normal form with respect to $\mathcal{G}$ then f also has a standard representation with respect to $\mathcal{G}$. This follows from Algorithm 3.2.4. However in general, the converse may be false. This is because Algorithm 3.2.4 depends on the ordering of the polynomials in $\mathcal{G}$. For example, let $f = xy^2 - xz^2$ and $\mathcal{G} = \{g_1 = xy + z^2, g_2 = y^2 - z^2\}$ and $w = (1, 10, 100)$. Then Algorithm 3.2.4 tells us that

$$f = y \cdot g_1 + 0 \cdot g_2 + (-x - y)z^2$$

and so f has normal form $-x - y$ with respect to $\mathcal{G}$, which in particular is non-zero.

However, notice that we can write f as

$$f = 0 \cdot g_1 + x \cdot g_2$$

which is a standard representation for f with respect to $\mathcal{G}$. $\diamond$

The following Proposition allows us to prove the correctness and finiteness of Buchberger's Algorithm. We do not prove this Proposition here, as it will be a Corollary of Proposition 3.4.5 in Section 3.4.

Proposition 3.2.11. *Let $\mathcal{G} = \{g_1, \dots, g_s\}$ be a subset of S. If every S-polynomial $S(g_i, g_j)$ of elements of $\mathcal{G}$ has a standard representation with respect to $\mathcal{G}$ then $\{\mathrm{in}_\prec(\mathrm{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\mathrm{in}_\prec(\mathrm{in}_w(I))$ for $I = \langle g_1, \dots, g_s \rangle$.*

In standard Gröbner theory where we are not considering valuations, this proposition is known as Buchberger's Theorem and was proved in his thesis [Buchberger, 1965]. It allows us to easily check if a generating set is a Gröbner basis by seeing if all S-polynomials have a standard representation. Note that by Remark 3.2.10, it is sufficient to show that all S-polynomials have zero normal form as this implies that they have a standard representation. With Proposition 3.2.11, we can now prove the correctness and finiteness of Algorithm 3.2.9.

Proof of Algorithm 3.2.9. We first show finiteness. At each pass through the main while loop, we denote by $\mathcal{G}'$ the updated $\mathcal{G}$. That is, $\mathcal{G}' = \mathcal{G} \cup \{r\}$, the union of $\mathcal{G}$ with some possibly non-zero remainder r of an S-polynomial of $\mathcal{G}$. Since $\mathcal{G} \subseteq \mathcal{G}'$ we have that $\langle \mathrm{in}_\prec(\mathrm{in}_w(g)) : g \in \mathcal{G} \rangle \subseteq \langle \mathrm{in}_\prec(\mathrm{in}_w(g')) : g' \in \mathcal{G}' \rangle$ and in particular if $\mathcal{G} \neq \mathcal{G}'$ then the inclusion is strict. Thus the $\langle \mathrm{in}_\prec(\mathrm{in}_w(g)) : g \in \mathcal{G} \rangle$ form an ascending chain of ideals which must stabilise as $K[x_1, \dots, x_n]$ is Noetherian. So after finitely many steps we must have $\langle \mathrm{in}_\prec(\mathrm{in}_w(g')) : g' \in \mathcal{G}' \rangle = \langle \mathrm{in}_\prec(\mathrm{in}_w(g)) : g \in \mathcal{G} \rangle$ and so $\mathcal{G}' = \mathcal{G}$ and the algorithm terminates.

For correctness, we first show that at every step of the algorithm $\mathcal{G} \subseteq I$. Initially this is true as $\mathcal{G}$ is a generating set for I so $\mathcal{G} \subseteq I$. As the normal form of an S-polynomial is also in I it follows that for the updated $\mathcal{G}$, $\mathcal{G} \subseteq I$ holds. Secondly, we show that at every step of the algorithm the S-polynomial $S(g_i, g_j)$ has a standard representation for all $g_i, g_j \in \mathcal{G}$ with $(g_i, g_j) \notin \mathcal{P}$. This means that we are checking that all S-polynomials $S(g_i, g_j)$ which have been encountered at some previous stage of the algorithm have a standard representation. As $S(g_i, g_j)$ was encountered at some previous stage of the algorithm, it either has zero normal form and so by Remark 3.2.10 it has a standard representation, or it has a non-zero normal form which is added to $\mathcal{G}$. So for $\mathcal{G} = \{g_1, \dots, g_s\}$, $S(g_i, g_j)$ has a representation $\sum_{i=1}^{s} h_i g_i + r$ where $S(g_i, g_j) \leq h_i g_i$ and $S(g_i, g_j) \leq r$ as it is a normal form. This is a standard representation with respect to $\mathcal{G} \cup \{r\}$.

On termination, $\mathcal{P} = \emptyset$ and so $S(g_i, g_j)$ has a standard representation for all $g_i, g_j \in \mathcal{G}$. Thus by Proposition 3.2.11, if $\mathcal{G} = \{g_1, \ldots, g_s\}$ then $\{\text{in}_{\prec}(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_{\prec}(\text{in}_w(I))$. □

After applying Algorithm 3.2.9 we have found a set $\{g_1, \ldots, g_s\} \subset I$ such that $\{\text{in}_{\prec}(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_{\prec}(\text{in}_w(I))$. This means that $\{\text{in}_w(g_i) : 1 \leq i \leq s\}$ is a standard Gröbner basis for $\text{in}_w(I)$ with respect to $\prec$, so in particular this set generates $\text{in}_w(I)$. By Lemma 3.2.1 we thus conclude that the set $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I with respect to w. This proof also holds in the variation discussed in Remark 3.2.8.

Note that Algorithm 3.2.9 is potentially rather complex. The normal form r of every S-polynomial that is added to the set $\mathcal{G}$ enlarges the set of S-polynomials to consider by $S(r, g)$ for all $g \in \mathcal{G}$. We discuss issues of complexity in Section 3.3 and discuss in Section 3.4 how the growth of S-polynomials can be managed by eliminating unnecessary pairs with no further computation.

This Gröbner theory shares many of the properties of standard Gröbner bases:

1. The Gröbner basis $\{g_1, \ldots, g_s\}$ generates I. The proof here is the standard one: if $f \in I$ then the normal form r of f with respect to $\{g_1, \ldots, g_s\}$ lies in I, but $\text{in}_{\prec}(\text{in}_w(r)) \notin \text{in}_{\prec}(\text{in}_w(I))$ unless $r = 0$.

2. For any homogeneous ideal I, $w \in \Gamma^n$, and monomial term order $\prec$ there is a unique reduced Gröbner basis. This is a Gröbner basis $\{g_1, \ldots, g_s\}$ with the property that the $\text{in}_{\prec}(\text{in}_w(g_i))$ minimally generate $\text{in}_{\prec}(\text{in}_w(I))$, and no monomial in g_i except $\text{lm}(g_i)$ is divisible by any $\text{lm}(g_j)$. This follows, as in the standard case, from the existence of a strong normal form. Specifically, if $\text{in}_{\prec}(\text{in}_w(I)) = \langle x^{u_1}, \ldots, x^{u_s} \rangle$, then let r_i be the remainder on dividing x^{u_i} by any Gröbner basis for I with respect to w and $\prec$. Set $g_i = x^{u_i} - r_i$.

3. The Hilbert function of the two ideals I and $\text{in}_w(I)$ (which live in different polynomial rings) agree. While this follows, as in the standard case, from the existence of a strong normal form, there are other proofs; see, for example, [Speyer, 2005, Chapter 2] or [Maclagan and Sturmfels, 2013, Corollary 2.4.7].

Remark 3.2.12. We remark that the assumption that the ideal I, and the Gröbner basis $\{g_1, \ldots, g_s\}$, are homogeneous is necessary for many of these properties of

Gröbner bases. For example, a finite set $\{g_1, \ldots, g_s\} \subset I$ with the property $\text{in}_w(I) = \langle \text{in}_w(g_1), \ldots, \text{in}_w(g_s) \rangle$ need not generate I if it is not homogeneous. A simple example is given by $I = \langle x \rangle \subseteq \mathbb{Q}[x]$ with the 2-adic valuation: for $w = 0$ the set $\{g_1 = x+2x^2\}$ satisfies $\text{in}_w(I) = \langle x \rangle = \langle \text{in}_w(g_1) \rangle$, but $\langle x \rangle \neq \langle x + 2x^2 \rangle$. ◇

This algorithmic approach to these initial ideals also has the following consequence for tropical geometry.

Corollary 3.2.13. *Let K be a field with a valuation* val *for which there is a homomorphism $\phi : \Gamma \to K^*$ with* $\text{val}(\phi(w)) = w$*, and for which Γ is a dense subgroup of $\mathbb{R}$. Let L be an extension field of K with a valuation that restricts to* val *on K. Let $Y \subseteq (K^*)^n$, and let $Y_L = Y \times_{\text{Spec}(K)} \text{Spec}(L)$. Then* $\text{trop}(Y) = \text{trop}(Y_L)$.

Proof. Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideal of $Y \subset (K^*)^n$. Then the ideal of Y_L is given by $I_L = IL[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let J be the homogenisation of $I \cap K[x_1, \ldots, x_n]$ in $K[x_0, x_1, \ldots, x_n]$ and J_L the homogenisation of $I_L \cap L[x_1, \ldots, x_n]$ in $L[x_0, x_1, \ldots, x_n]$. This intersection can be calculated by a (standard) Gröbner computation, so the ideals J and J_L have the same generators: $J_L = JL[x_1, \ldots, x_n]$. The definition of the initial ideal of an ideal taking the valuation of the coefficients into account extends naturally to the Laurent polynomial ring. By the Fundamental Theorem 2.1.5 $w \in \Gamma^n$ lies in $\text{trop}(Y)$ if and only if $\text{in}_w(I) \neq \langle 1 \rangle$, and thus if and only if $\text{in}_w(J)$ does not contain a monomial. Since J and J_L have the same generators, Algorithm 3.2.9 implies that regarding the elements of a Gröbner basis for J with respect to w as living in $L[x_1, \ldots, x_n]$ gives a Gröbner basis for J_L with respect to w. The residue field $\mathbb{L}$ of L is an extension field of $\mathbb{k}$, so this means that $\text{in}_w(J_L) = \text{in}_w(J)\mathbb{L}[x_1, \ldots, x_n]$. An ideal contains a monomial if and only if the saturation by the product of all the variables is the unit ideal. Since this can be decided by a (standard) Gröbner basis computation, this means that $\text{in}_w(J_L)$ contains a monomial if and only if $\text{in}_w(J)$ does. Since Γ is dense in $\mathbb{R}$, this implies that $\text{trop}(Y) = \text{trop}(Y_L)$. □

3.3 Complexity

Given a bound on the degrees of generators for I, it is useful to have a bound on the degrees of elements in a reduced Gröbner basis. The degree bounds in this context are the same as for usual Gröbner bases [Möller and Mora, 1984; Dubé, 1990], as we show below. We also give a bound on the valuations of coefficients occurring in a reduced Gröbner basis when working over $\mathbb{Q}$ with the p-adic valuation. For the degree bounds we use the formulation of Dubé [1990].

Theorem 3.3.1. *Let $I = \langle f_1, \ldots, f_l \rangle \subset K[x_1, \ldots, x_n]$ be a homogeneous ideal, with $\deg(f_i) \leq d$ for $1 \leq i \leq l$. Fix $w \in \Gamma^n$. Then there is a Gröbner basis $\{g_1, \ldots, g_s\}$ for I with respect to w with $\deg(g_i) \leq 2(d^2/2 + d)^{2^{n-2}}$.*

Proof. In Dubé [1990] it is shown that if $\deg(f_i) \leq d$ for $1 \leq i \leq l$, and $\{g'_1, \ldots, g'_s\}$ is a standard homogeneous Gröbner basis with respect to some term order $\prec$, then the degree of each g'_i is bounded by $2(d^2/2+d)^{2^{n-2}}$. The proof given actually shows more: if M is any monomial ideal whose Hilbert function agrees with that of I, then M is generated in degrees at most $2(d^2/2 + d)^{2^{n-2}}$. Denote by S_K the polynomial ring $K[x_1, \ldots, x_n]$ and by $S_{\Bbbk}$ the polynomial ring $\Bbbk[x_1, \ldots, x_n]$. By Maclagan and Sturmfels [2013, Corollary 2.4.7] we have $\dim_{\Bbbk}(S_{\Bbbk}/\operatorname{in}_w(I))_\delta = \dim_K(S_K/I)_\delta$ for all degrees δ. Since the initial ideal $\operatorname{in}_w(I)$ is again a homogeneous ideal, all of its monomial initial ideals have the same Hilbert function, so we have

$$\dim_{\Bbbk}(S_{\Bbbk}/\operatorname{in}_\prec(\operatorname{in}_w(I)))_\delta = \dim_{\Bbbk}(S_{\Bbbk}/\operatorname{in}_w(I))_\delta = \dim_K(S_K/I)_\delta.$$

Let M be the monomial ideal in S_K with the same generators as $\operatorname{in}_\prec(\operatorname{in}_w(I)) \subset S_{\Bbbk}$. As the Hilbert function of a monomial ideal does not depend on the coefficient field, M has the same Hilbert function as I, so by Dubé [1990] M is generated in degrees at most $2(d^2/2 + d)^{2^{n-2}}$. Choose homogeneous polynomials $\{g_1, \ldots, g_s\} \subset I$ such that $\{\operatorname{in}_\prec(\operatorname{in}_w(g_1)), \ldots, \operatorname{in}_\prec(\operatorname{in}_w(g_s))\}$ is a minimal generating set for $\operatorname{in}_\prec(\operatorname{in}_w(I))$. Then $\operatorname{in}_w(I) = \langle \operatorname{in}_w(g_1), \ldots, \operatorname{in}_w(g_s) \rangle$ so $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I with respect to w. Since we have $\deg(\operatorname{in}_\prec(\operatorname{in}_w(g_i))) \leq 2(d^2/2 + d)^{2^{n-2}}$ by above, we deduce that $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I with respect to w with $\deg(g_i) \leq 2(d^2/2+d)^{2^{n-2}}$ for $1 \leq i \leq s$ as required. □

Remark 3.3.2. For $w \in \Gamma^n$, let $J_w = \operatorname{in}_w(I)$, and let J'_w be standard initial ideal of I with respect to the weight vector w. For $\ell \gg 0$, and generic w, we have $J_{\ell w} = J'_{-\ell w}$; the minus sign is because the initial ideal taking the valuation into account uses min instead of max. This means that any usual initial ideal, and thus any usual Gröbner basis, occurs in this setting, so any improvement to Theorem 3.3.1 would also have to improve the bounds of Möller and Mora [1984] and Dubé [1990]. ◇

Since the valuations of coefficients also play an important role in computing these Gröbner bases, it is also useful to bound the valuations that may occur. This is not possible in full generality, as the following example shows.

Example 3.3.3. Let $K = \mathbb{Q}(t)$ with the valuation of a rational function given by taking the lowest exponent occurring in a Taylor series for the function. Fix an integer $a \gg 0$ and weight vector $w = (1, a, 2a)$. Let I be the ideal in $K[x, y, z]$

generated by the two polynomials $f = x+z$ and $g = x^2+(1+t^a)xz+xy$. We compute a Gröbner basis by looking at the S-polynomial $S(f,g) = xf - g = -xy - t^a xz$. Computing the remainder on division by $\{f, g\}$ we obtain $yz+t^a z^2$ which is a nonzero polynomial with initial term yz. It is added to the Gröbner basis at this stage by Buchberger's Algorithm (Algorithm 3.2.9). Notice that we started with polynomials where the valuations of all the coefficients were zero and we have an element of the reduced Gröbner basis which has a coefficient with valuation a showing that unbounded valuations may potentially occur when computing Gröbner bases. The field $K = \mathbb{Q}(t)$ is only chosen for concreteness; such an example exists for any nontrivially-valued field. $\diamond$

When $K = \mathbb{Q}$ with the p-adic valuation the valuation of coefficients that can occur in a reduced Gröbner basis can be bounded in terms of the absolute values of the original coefficients.

Let $I = \langle f_1, \dots, f_l \rangle$ be a homogeneous ideal in $\mathbb{Q}[x_1, \dots, x_n]$ with $\deg(f_i) \leq \delta$ for $1 \leq i \leq l$. Fix val to be the p-adic valuation on $\mathbb{Q}$. Write $f_i = \sum c_{u,i} x^u$ where we assume (by clearing denominators or dividing by a common factor) that $c_{u,i} \in \mathbb{Z}$ and that for each i we have $\min_u \operatorname{val}(c_{u,i}) = 0$.

Proposition 3.3.4. *Let $I = \langle f_1, \dots, f_l \rangle$ be a homogeneous ideal in $\mathbb{Q}[x_1, \dots, x_n]$ with assumptions as above. Let $C = \max_{u,i} |c_{u,i}|$. Fix $w \in \Gamma^n$. Then there is a Gröbner basis $\{g_1, \dots, g_s\}$ for I with respect to w with $g_i = \sum_u b_{u,i} x^u$ with*

$$\operatorname{val}(b_{u,i}) \leq A/2 \log_p(C^2 A),$$

where $A = \dim_{\mathbb{Q}}(I_D)$ for $D = 2(\delta^2/2 + \delta)^{2^{n-2}}$.

Proof. As the Hilbert functions of I and $\operatorname{in}_w(I)$ agree [Maclagan and Sturmfels, 2013, Corollary 2.4.7] we have that $\dim_{\mathbb{Q}} I_d = \dim_{\mathbb{Z}/p\mathbb{Z}}(\operatorname{in}_w(I)_d)$ for all d. Fix a term order $\prec$ on $\mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_n]$. Let $H(d) = \dim_{\mathbb{Q}}(I_d)$.

For $d \leq D$, form an $H(d) \times \binom{n+d-1}{d}$ matrix A_d with columns indexed by the monomials of degree d ordered so that those in $\operatorname{in}_\prec(\operatorname{in}_w(I))_d$ come first. The rows of A_d correspond to a $\mathbb{Q}$-basis for I_d; we may take these to be monomial multiples of the generators f_i, so all entries of A_d have absolute value at most C.

Let the submatrix of A_d indexed by the first $H(d)$ columns be denoted by M_d. Note that M_d has full rank; if not since A_d has rank $H(d)$, there would be a vector in the row-space of A_d with its first $H(d)$ entries zero, and thus there would be a non-zero polynomial f in I_d for which $\operatorname{in}_\prec(\operatorname{in}_w(f))$ does not lie in $\operatorname{in}_\prec(\operatorname{in}_w(I))$, which is a contradiction.

Set $B_d = M_d^{-1} A_d$. Note that the first $H(d)$ columns of B_d are an identity matrix, so the minor $\det((B_d)_J)$ of B_d indexed by the set $J := (\{1, \ldots, H(d)\} \cup \{j\}) \backslash \{i\}$ equals $(-1)^{H(d)-i}(B_d)_{ij}$. Since $(B_d)_J = M_d^{-1}(A_d)_J$,

$$\begin{aligned}
\mathrm{val}_p((B_d)_{ij}) &= \mathrm{val}_p(\det((B_d)_J)) \\
&= \mathrm{val}_p(\det(M_d^{-1}(A_d)_J)) \\
&= -\mathrm{val}_p(\det(M_d)) + \mathrm{val}_p(\det((A_d)_J)).
\end{aligned}$$

Hadamard's inequality (see for example [Garling, 2007, Corollary 14.2.1]) states that if M is an $N \times N$ matrix with the absolute value of the entries bounded by C, then $|\det(M)| \leq C^N N^{N/2}$. Thus $|\det((A_d)_J)| \leq C^{H(d)} H(d)^{H(d)/2}$. Since $\det(A_d)_J$ is an integer, $\mathrm{val}_p(\det((A_d)_J) \leq \log_p(\det((A_d)_J))$. By construction all entries of M_d have nonnegative valuation, so $\mathrm{val}_p(\det(M_d)) \geq 0$. Thus

$$\mathrm{val}_p((B_d)_{ij})) \leq \log_p(C^{H(d)} H(d)^{H(d)/2}) = H(d)/2 \log_p(C^2 H(d)).$$

By Theorem 3.3.1 there is a Gröbner basis $\{g_1, \ldots, g_s\}$ for I with respect to w with $\deg(g_i) \leq D$, which can be chosen so $\{\mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_s)\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to $\prec$. By construction of the matrix B_d if g_i has degree d then the coefficients of g_i form a row of the matrix B_d. Thus the valuation of the coefficients of g_i is bounded as above. Since $H(d)$ is an increasing function of d, the bound is largest when $d = D$, so $H(d) = A$, from which we see that the valuations of any of the coefficients of any g_i is bounded by $A/2 \log_p(C^2 A)$ as required. □

3.4 Implementation Issues

Whilst we have proved that Algorithm 3.2.9 terminates correctly in finite time, we have said nothing about its efficiency. Adding polynomials to $\mathcal{G}$ during the algorithm increases the complexity as we then have many more S-polynomials to consider before termination. Also the order in which the S-polynomials are selected can affect whether it can be reduced to zero by $\mathcal{G}$ and hence whether its normal form is added to $\mathcal{G}$ or not. Such a selection strategy can drastically alter the complexity of the algorithm and even for standard Gröbner bases without valuations, no optimal strategy is known. However, Buchberger [1979] provided criteria in the standard case for when it is known that we do not need to consider certain S-polynomials in Buchberger's Algorithm 3.2.9 as they will a priori have a standard representation. We investigate these for our Gröbner bases.

Another issue that is common for Gröbner algorithms with coefficients in

$\mathbb{Q}$ is coefficient blow-up. However, we show that for the case $K = \mathbb{Q}$ with the p-adic valuation, we can perform computations over the finite field $\mathbb{Z}/p^m\mathbb{Z}$ for some suitably large $m \in \mathbb{N}$ and then lift the results to $\mathbb{Q}$. This helps avoid unboundedly large coefficients occurring.

The two main ways which we will look at to improve the efficiency of the Algorithm 3.2.9 are:

1. Using criteria to decide a priori that we do not need to consider certain S-polynomials;

2. When $K = \mathbb{Q}$ with the p-adic valuation, we work over $\mathbb{Z}/p^m\mathbb{Z}$ for some suitably large $m \in \mathbb{N}$.

3.4.1 Choice of S-Polynomials - Buchberger's Criteria

For standard Gröbner bases, Buchberger [1979] introduced criteria for when it is known a priori that we do not need to consider certain S-polynomials in Algorithm 3.2.9. We see that these criteria hold true for our Gröbner bases over fields with valuations.

Suppose we are at some intermediate stage of Buchberger's Algorithm whereby we have a set $\mathcal{P}$ of critical pairs still to consider and we are about to compute the S-polynomial of the pair (f_i, f_j). Then

B1 holds if $\operatorname{lcm}(\operatorname{lm}(f_i), \operatorname{lm}(f_j)) = \operatorname{lm}(f_i)\operatorname{lm}(f_j)$;

B2 holds if there exists some $k \neq i, j$ such that the pairs (f_i, f_k) and (f_j, f_k) are not in $\mathcal{P}$ and $\operatorname{lm}(f_k)$ divides $\operatorname{lcm}(\operatorname{lm}(f_i), \operatorname{lm}(f_j))$.

For standard Gröbner bases, Buchberger [1979] showed that if either of these conditions hold then we know a priori that the S-polynomial has a standard representation. The proof in the standard case can be found for example in Cox, Little, and O'Shea [2007, Section 2.9]: the proof for B1 is Proposition 4, and the proof for B2 is Proposition 10. The criterion B1 is known as Buchberger's first criterion and B2 is known as Buchberger's second criterion. Before proving that these hold for Gröbner bases over fields with valuations, we provide an example of its usefulness.

Example 3.4.1. Let $K = \mathbb{Q}$ with the 2-adic valuation and let S be the polynomial ring $\mathbb{Q}[x_1, \ldots, x_9]$. Let I be the ideal generated by polynomials $\{-3x_1x_4 + 6x_3x_4 + 3x_1x_5 + 92x_2x_5 + 2x_3x_5 - 23x_2x_6 - 2x_3x_6, x_1x_8 + 7x_2x_8 - 4x_3x_8 - 6x_1x_9 - 3x_2x_9, x_4x_8 + 3x_5x_8 - 3x_6x_8 - 24x_5x_9 - 3x_6x_9, -x_2x_4 - 4x_3x_4 + x_2x_5 + 4x_3x_5 +$

$23x_2x_6+2x_3x_6, -13x_1x_7-4x_3x_7+7x_2x_8+28x_3x_8-65x_1x_9-3x_2x_9-32x_3x_9, x_4x_7+27x_5x_7-9x_6x_8+5x_4x_9+135x_5x_9-9x_6x_9, -4x_2x_5-16x_3x_5+3x_1x_6+x_2x_6-2x_3x_6, 13x_2x_7-8x_3x_7+x_2x_8+4x_3x_8+59x_2x_9-64x_3x_9, 8x_5x_7+x_6x_7-3x_6x_8+40x_5x_9+5x_6x_9, 4x_2x_5x_8+16x_3x_5x_8+20x_2x_6x_8-10x_3x_6x_8-24x_2x_5x_9-96x_3x_5x_9-3x_2x_6x_9-12x_3x_6x_9\}$. This is the general fibre of a Mustafin variety in the sense of Cartwright, Häbich, Sturmfels, and Werner [2011]. Its special fibre is the initial ideal with respect to $w = (1, \dots, 1)$.

At some intermediate step of Buchberger's Algorithm 3.2.9 we compute the normal form of the S-polynomial $6x_3x_4x_6x_7+3x_1x_5x_6x_7+24x_1x_4x_5x_7+92x_2x_5x_6x_7+2x_3x_5x_6x_7-23x_2x_6^2x_7-2x_3x_6^2x_7-9x_1x_4x_6x_8+120x_1x_4x_5x_9+15x_1x_4x_6x_9$ of the polynomials $-3x_1x_4+6x_3x_4+3x_1x_5+92x_2x_5+2x_3x_5-23x_2x_6-2x_3x_6$ and $x_6x_7+8x_5x_7-3x_6x_8+40x_5x_9+5x_6x_9$. Notice that the condition B1 holds, so we know a priori that this S-polynomial will have a standard representation, however when we try to compute the normal form, after a few divisions we obtain a leading coefficient of $1.02624\cdots\times 10^{37{,}746}$ and after a few more divisions we have exceeded the memory capabilities of the computer.

By implementing Buchberger's Criterion, the algorithm no longer considers this critical pair and we compute the Gröbner basis to be $\{3x_1x_4-6x_3x_4-3x_1x_5-92x_2x_5-2x_3x_5+23x_2x_6+2x_3x_6, x_1x_8+7x_2x_8-4x_3x_8-6x_1x_9-3x_2x_9, x_4x_8+3x_5x_8-3x_6x_8-24x_5x_9-3x_6x_9, x_2x_4+4x_3x_4-x_2x_5-4x_3x_5-23x_2x_6-2x_3x_6, 13x_1x_7+4x_3x_7-7x_2x_8-28x_3x_8+65x_1x_9+3x_2x_9+32x_3x_9, x_4x_7+27x_5x_7-9x_6x_8+5x_4x_9+135x_5x_9-9x_6x_9, -4x_2x_5-16x_3x_5+3x_1x_6+x_2x_6-2x_3x_6, 13x_2x_7-8x_3x_7+x_2x_8+4x_3x_8+59x_2x_9-64x_3x_9, 8x_5x_7-3x_6x_8+40x_5x_9+5x_6x_9+x_6x_7, -4x_2x_5x_8-16x_3x_5x_8-20x_2x_6x_8+10x_3x_6x_8+24x_2x_5x_9+3x_2x_6x_9+96x_3x_5x_9+12x_3x_6x_9\}$. ◇

The following Proposition is the criterion B1, Buchberger's first criterion for Gröbner bases over fields with valuations.

Proposition 3.4.2. *Suppose that f and g are distinct polynomials in S such that* $\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) = \operatorname{lm}(f)\operatorname{lm}(g)$. *Then $S(f,g)$ has a standard representation with respect to $\{f,g\}$.*

Proof. We can write $f = \sum_{i=1}^{k} a_i x^{u_i}$ and $g = \sum_{j=1}^{l} b_j x^{v_j}$ for $a_i, b_j \in K$ and $u_i, v_j \in \mathbb{N}^n$. We assume that $a_i x^{u_i} \leq a_{i+1}x^{u_{i+1}}$ and $b_j x^{v_j} \leq b_{j+1}x^{v_{j+1}}$ for all $1 \leq i \leq k-1$ and $1 \leq j \leq l-1$ and that the u_i are distinct, and the v_j also distinct. Then $\operatorname{lm}(f) = x^{u_1}$ and $\operatorname{lm}(g) = x^{v_1}$. By hypothesis, $\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) = \operatorname{lm}(f)\operatorname{lm}(g) = x^{u_1}x^{v_1}$, so we can write the S-polynomial $S(f,g)$ as

$$S(f,g) = b_1 x^{v_1} f - a_1 x^{u_1} g = b_1 x^{v_1} \sum_{i=2}^{k} a_i x^{u_i} - a_1 x^{u_1} \sum_{j=2}^{l} b_j x^{v_j}. \tag{3.1}$$

We first claim that these two sums have no terms in common. For a contradiction, suppose that for some $2 \leq i \leq k$ and $2 \leq j \leq l$ we have that $x^{v_1} x^{u_i} = x^{u_1} x^{v_j}$. Then both x^{u_1} and x^{v_1} divide $x^{v_1} x^{u_i} = x^{u_1} x^{v_j}$ and so it follows that $\operatorname{lcm}(\operatorname{lm}(x^{u_1}), \operatorname{lm}(x^{v_1})) = x^{u_1} x^{v_1}$ divides $x^{v_1} x^{u_i}$. This means that x^{u_1} divides x^{u_i}. As f is homogeneous, x^{u_1} and x^{u_i} have the same degree and so $x^{u_1} = x^{u_i}$. This contradicts the assumption that the u_i are distinct.

By the ordering on the $a_i x^{u_i}$ and $b_j x^{v_j}$, the smallest term of $S(f,g)$ is either $a_2 b_1 x^{u_2} x^{v_1}$ or $a_1 b_2 x^{u_1} x^{v_2}$. We assume that $a_1 b_2 x^{u_1} x^{v_2}$ is the smaller and so is the lead term of $S(f,g)$. Notice that we can write $b_1 x^{v_1} = g - \sum_{j=2}^{l} b_j x^{v_j}$ and $a_1 x^{u_1} = f - \sum_{i=2}^{k} a_i x^{u_i}$. Then

$$\begin{aligned}
S(f,g) &= b_1 x^{v_1} f - a_1 x^{u_1} g \\
&= \left(g - \sum_{j=2}^{l} b_j x^{v_j} \right) f - \left(f - \sum_{i=2}^{k} a_i x^{u_i} \right) g \\
&= gf - \left(\sum_{j=2}^{l} b_j x^{v_j} \right) f - fg + \left(\sum_{i=2}^{k} a_i x^{u_i} \right) g \\
&= -\sum_{j=2}^{l} b_j x^{v_j} f + \sum_{i=2}^{k} a_i x^{u_i} g.
\end{aligned} \tag{3.2}$$

We claim that (3.2) is a standard representation of $S(f,g)$ with respect to $\{f,g\}$. For this, we need to show that $S(f,g) \leq b_j x^{v_j} f$ for all $2 \leq j \leq l$ and that $S(f,g) \leq a_i x^{u_i} g$ for all $2 \leq i \leq k$. By the ordering of the $b_j x^{v_j}$ the smallest term of $b_j x^{v_j} f$ for all $2 \leq j \leq l$ is $a_1 b_2 x^{u_1} x^{v_2}$. As the sums in (3.1) have no terms in common, it follows that $x^{v_2} x^{u_1}$ and $x^{u_2} x^{v_1}$ are distinct and so do not cancel. As $a_1 b_2 x^{u_1} x^{v_2} \leq b_j x^{v_j} g$ for $2 \leq j \leq l$ and $a_1 b_2 x^{u_1} x^{v_2} \leq a_i x^{u_i} f$ for $2 \leq i \leq k$, it follows that $S(f,g) = -\left(\sum_{j=2}^{l} b_j x^{v_j} \right) f + \left(\sum_{i=2}^{k} a_i x^{u_i} \right) g$ is a standard representation of $S(f,g)$ with respect to $\{f,g\}$. □

In order to prove criterion B2, we need to introduce the notion of a T-representation. Let $f \in S$ be a homogeneous polynomial and let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a finite subset of S. Fix some $T \in S$. We say that f has a *T-representation* with respect to $\mathcal{G}$ if it can be written in the form

$$f = \sum_{i=1}^{s} a_i g_i$$

such that $T \leq a_i g_i$ for all i where $a_i \neq 0$. A T-representation is a measure of how far the representation is from being a standard representation. Due to the definition of the order $\leq$ of Definition 3.2.3, we may take T to be a monomial term should we require. Notice that if f has an f-representation, or an $\operatorname{lc}(f)\operatorname{lm}(f)$-representation, then it has a standard representation.

Proposition 3.4.3. *Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a finite set such that there exist distinct elements $g_1, g_2, f \in \mathcal{G}$ where*

1. $\operatorname{lm}(f)$ *divides* $\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$*;*
2. *For some $T_1 \in S$ such that $T_1 > \operatorname{lc}(g_1)\operatorname{lc}(f)\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f))$, $S(g_1, f)$ has a T_1-representation with respect to $\mathcal{G}$;*
3. *For some $T_2 \in S$ such that $T_2 > \operatorname{lc}(g_2)\operatorname{lc}(f)\operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f))$, $S(g_2, f)$ has a T_2-representation with respect to $\mathcal{G}$.*

Then for some $T \in S$ such that $T > \operatorname{lc}(g_1)\operatorname{lc}(g_2)\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$, $S(g_1, g_2)$ has a T-representation with respect to $\mathcal{G}$.

Proof. As K is a field, we can assume for simplicity that $\operatorname{lc}(f) = \operatorname{lc}(g_1) = \operatorname{lc}(g_2) = 1$. By assumption, for some $T_1 > \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f))$, $S(g_1, f)$ has a T_1-representation:

$$S(g_1, f) = \sum_{i=1}^{s} h_{i1} g_i$$

with respect to $\mathcal{G}$ where $T_1 \leq h_{i1} g_i$ for all $1 \leq i \leq s$. Similarly, for some $T_2 > \operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f))$, $S(g_2, f)$ has a T_2-representation:

$$S(g_2, f) = \sum_{i=1}^{s} h_{i2} g_i$$

with respect to $\mathcal{G}$ where $T_2 \leq h_{i2} g_i$ for all $1 \leq i \leq s$.

By assumption we have that $\operatorname{lm}(f)$ divides $\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$. By the definition of lcm, both $\operatorname{lm}(g_1)$ and $\operatorname{lm}(g_2)$ also divide $\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$. It follows that $\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f))$ and $\operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f))$ both divide $\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$. This means that we can find monomials $s_1, s_2 \in S$ such that

$$s_1 \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f)) = \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2));$$
$$s_2 \operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f)) = \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2)),$$

and so we also can find monomials $u_1, u_2, v_1, v_2 \in S$ such that

$$\operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f)) = u_1 \operatorname{lm}(g_1) = v_1 \operatorname{lm}(f);$$
$$\operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f)) = u_2 \operatorname{lm}(g_2) = v_2 \operatorname{lm}(f).$$

Then

$$\begin{aligned} s_1 S(g_1, f) - s_2 S(g_2, f) =& s_1(u_1 g_1 - v_1 f) + s_2(v_2 f - u_2 g_2) \\ =& s_1 u_1 g_1 - s_2 u_2 g_2 \\ =& S(g_1, g_2) \end{aligned}$$

where the second equality holds as $s_1 v_1 = s_2 v_2$. Then we can write $S(g_1, g_2)$ as

$$S(g_1, g_2) = s_1 \sum_{i=1}^{s} h_{i1} g_i + s_2 \sum_{i=1}^{s} h_{i2} g_i \tag{3.3}$$

which we claim is a T-representation for $S(g_1, g_2)$ for some $T > \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$. By the choice of the representation for $S(g_1, f)$ we have

$$s_1 h_{i1} g_i \geq s_1 T_1 > s_1 \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(f)) = \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$$

for all $1 \leq i \leq s$ where $h_{ij} \neq 0$ and by the choice of the representation for $S(g_1, f)$ we have

$$s_2 h_{i2} g_i \geq s_2 T_2 > s_2 \operatorname{lcm}(\operatorname{lm}(g_2), \operatorname{lm}(f)) = \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$$

Thus choosing T to be the minimum over all $s_1 h_{i1} g_i$ for $1 \leq i \leq s$ where $h_{i1} \neq 0$ and all $s_2 h_{i2} g_i$ for $1 \leq i \leq s$ where $h_{i2} \neq 0$, we see that (3.3) is a T-representation for $S(g_1, g_2)$ for some $T > \operatorname{lcm}(\operatorname{lm}(g_1), \operatorname{lm}(g_2))$. $\square$

In order to prove that we can use Buchberger's Criteria B1 and B2 in algorithms to compute Gröbner bases over a field with valuations, we need to show that if for all $g_i, g_j \in \mathcal{G}$, the S-polynomials $S(g_i, g_j)$ have a T-representation for some $T > \operatorname{lc}(g_i) \operatorname{lc}(g_j) \operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))$ then $\mathcal{G}$ is a Gröbner basis for I. To show this, we require the following preparatory lemma.

Lemma 3.4.4. *Fix* $v_1, \dots, v_m \in K^n$ *and* $\omega_1, \dots, \omega_m \in \mathbb{R}$. *For* $\lambda \in K^m$, *write* $s(\lambda) = \min(\mathrm{val}(\lambda_i) + \omega_i)$. *Then for fixed* $v \in span(v_1, \dots, v_m)$ *there is a choice of* $\lambda \in K^m$ *with* $\sum \lambda_i v_i = v$ *that maximizes* $s(\lambda)$ *among all such choices.*

Proof. We first show that for any λ with $\sum \lambda_i v_i = v$ there is a λ' with $\sum \lambda'_i v_i = v$, $\{v_i : \lambda'_i \neq 0\}$ linearly independent, and $s(\lambda') \geq s(\lambda)$. Indeed, if $\{v_i : \lambda_i \neq 0\}$ is linearly dependent, then there is $c \neq 0$ with $\sum c_i v_i = 0$ and $c_i \neq 0$ only when $\lambda_i \neq 0$. After relabelling and rescaling we may assume that $\mathrm{val}(c_1) + \omega_1 = \min(\mathrm{val}(c_i) + \omega_i)$, and $c_1 = \lambda_1$. Let $\lambda' = \lambda - c$. Then for every i

$$\begin{aligned}
\mathrm{val}(\lambda'_i) + \omega_i &= \mathrm{val}(\lambda_i - c_i) + \omega_i \\
&\geq \min(\mathrm{val}(\lambda_i), \mathrm{val}(c_i)) + \omega_i \\
&= \min(\mathrm{val}(\lambda_i) + \omega_i, \mathrm{val}(c_i) + \omega_i) \\
&\geq \min(\mathrm{val}(\lambda_i) + \omega_i, \mathrm{val}(\lambda_1) + \omega_1) \\
&\geq s(\lambda),
\end{aligned}$$

so $s(\lambda') \geq s(\lambda)$. Since $\{i : \lambda'_i \neq 0\} \subseteq \{i : \lambda_i \neq 0\}$, after iterating a finite number of times $\{v_i : \lambda'_i \neq 0\}$ is linearly independent. The lemma then follows from the observation that if $\{v_i : \lambda_i \neq 0\}$ is linearly independent, then the λ_i are determined, so the maximum $s(\lambda)$ is achieved at one of these finitely many choices. □

Proposition 3.4.5. *Let* $\mathcal{G} = \{g_1, \dots, g_s\}$ *be a subset of* S. *If every* S*-polynomial* $S(g_i, g_j)$ *of elements of* $\mathcal{G}$ *has a* T*-representation with respect to* $\mathcal{G}$ *for some* $T > \mathrm{lc}(g_i)\,\mathrm{lc}(g_j)\,\mathrm{lcm}(\mathrm{lm}(g_i), \mathrm{lm}(g_j))$ *then* $\{\mathrm{in}_\prec(\mathrm{in}_w(g_i)) : 1 \leq i \leq s\}$ *generates* $\mathrm{in}_\prec(\mathrm{in}_w(I))$ *for* $I = \langle g_1, \dots, g_s \rangle$.

Proof. Let $f \in I$ be a non-zero polynomial for which $\mathrm{in}_\prec(\mathrm{in}_w(f)) \notin \langle \mathrm{in}_\prec(\mathrm{in}_w(g_i)) : 1 \leq i \leq s \rangle$. We aim for a contradiction. As $f \in I$, we can write $f = \sum_{i=1}^s h_i g_i$ for some homogeneous polynomials $h_i \in S$. Write $\mathrm{lm}(h_i g_i) = x^{u_i}$. We may assume that $\min(\mathrm{val}(\mathrm{lc}(h_i g_i)) + w \cdot u_i)$ is maximal over all choices of description $f = \sum_{i=1}^s h_i g_i$. That a maximum exists follows from Lemma 3.4.4 applied to the vector space $S_{\deg(f)}$, with the v_i all polynomials of the form $x^u g_j$ where x^u is a monomial of degree $\deg(f) - \deg(g_j)$, and $\omega_i = w \cdot u'$ for $\mathrm{lm}(x^u g_j) = x^{u'}$.

After renumbering if necessary, we may assume that $\min(\mathrm{val}(\mathrm{lc}(h_i g_i)) + w \cdot u_i) = \mathrm{val}(\mathrm{lc}(h_j g_j)) + w \cdot u_j$ for $1 \leq j \leq d$, and that in addition $x^{u_1} = x^{u_i}$ for $1 \leq i \leq d' \leq d$ with x^{u_1} the largest x^{u_i} among those $i \leq d$. We may further assume that d' is as small as possible among descriptions achieving the maximum.

Since $\text{in}_\prec(\text{in}_w(h_i g_i)) = \text{in}_\prec(\text{in}_w(h_i))\,\text{in}_\prec(\text{in}_w(g_i)) \in \langle \text{in}_\prec(\text{in}_w(g_1)), \ldots, \text{in}_\prec(\text{in}_w(g_s))\rangle$, $x^{u_1} \neq \text{lm}(f)$. This means that $\text{lm}(\sum_{i=1}^{d'} h_i g_i) \neq \text{lm}(f)$, so $\text{val}(\sum_{i=1}^{d'} \text{lc}(h_i g_i)) > \min(\text{val}(\text{lc}(h_i g_i)))$, and so in particular $d' \geq 2$. By hypothesis the S-polynomial $S(g_1, g_2)$ has a T-representation for some $T > \text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))$ and so we can write $S(g_1, g_2) = \sum_{i=1}^{s} h'_i g_i$ where $h'_i g_i \geq T > \text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))$. Then

$$
\begin{aligned}
f &= \sum_{i=1}^{s} h_i g_i \\
&= \sum_{i=1}^{s} h_i g_i - \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))} \left(S(g_1, g_2) - \sum_{i=1}^{s} h'_i g_i \right) \\
&= \left(h_1 - \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_1)\,\text{lm}(g_1)} + \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))} h'_1 \right) g_1 \\
&\quad + \left(h_2 - \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_2)\,\text{lm}(g_2)} + \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))} h'_2 \right) g_2 \\
&\quad + \sum_{i=3}^{s} \left(h_i + \frac{\text{lc}(h_1 g_1) x^{u_1}}{\text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))} h'_i \right) g_i \\
&= \sum_{i=1}^{s} \tilde{h}_i g_i,
\end{aligned}
$$

where $\tilde{h}_i$ is defined to be the polynomial multiplying g_i in the previous line. By construction $\tilde{h}_1 > h_1$ and $\tilde{h}_i \geq h_i$ for all $i \geq 2$. Write $x^{\tilde{u}_i}$ for $\text{lm}(\tilde{h}_i g_i)$. Thus we have a new expression for f with either $\min(\text{val}(\text{lc}(\tilde{h}_i g_i)) + w \cdot \tilde{u}_i)$ larger or this minimum the same and d' smaller, which contradicts our assumptions on the respective maximality and minimality of these quantities. We thus conclude that f does not exist and so $\text{in}_\prec(\text{in}_w(I)) = \langle \text{in}_\prec(\text{in}_w(g_1)), \ldots, \text{in}_\prec(\text{in}_w(g_s))\rangle$ as required. □

Recall that the key result in order to prove Algorithm 3.2.9 was Proposition 3.2.11 which said that for $\mathcal{G} = \{g_1, \ldots, g_s\}$, if all the S-polynomials $S(g_i, g_j)$ have a standard representation with respect to $\mathcal{G}$, then $\{\text{in}_\prec(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_\prec(\text{in}_w(I))$ for $I = \langle g_1, \ldots, g_s\rangle$. We can now prove this result as it is a corollary of the previous proposition.

Proof of Proposition 3.2.11. By assumption, each S-polynomial $S(g_i, g_j)$ has a standard representation with respect to $\mathcal{G}$. As $S(g_i, g_j) > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$, if we set $T = S(g_i, g_j)$ then this standard representation is a T-representation for some $T > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$. By Proposition 3.4.5, $\{\text{in}_\prec(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_\prec(\text{in}_w(I))$ for $I = \langle g_1, \ldots, g_s\rangle$, and the result follows. □

We now incorporate Buchberger's first criterion B1 of Proposition 3.4.2 and Buchberger's second criterion B2 of Proposition 3.4.3 into Algorithm 3.2.9

Algorithm 3.4.6. Input: A list $\{f_1, \ldots, f_l\}$ of homogeneous polynomials in S, a weight-vector $w \in \Gamma^n$, and a term order $\prec$.
Output: A list $\{g_1, \ldots, g_s\}$ of homogeneous polynomials in S such that the set $\{\text{in}_\prec(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_\prec(\text{in}_w(I))$ for $I = \langle f_1, \ldots, f_l \rangle$.

1. Set $\mathcal{G} = \{f_1, \ldots, f_l\}$. Set $\mathcal{P} = \{(g, g') : g, g' \in \mathcal{G}\}$.
2. While $\mathcal{P} \neq \emptyset$:
   (a) Pick $(g, g') \in \mathcal{P}$. Set $\mathcal{P} = \mathcal{P} \setminus \{(g, g')\}$.
   (b) If $\text{lcm}(\text{lm}(g), \text{lm}(g')) \neq \text{lm}(g)\,\text{lm}(g')$ and CriterionB2$(g, g', \mathcal{P})$ is false then let r be the normal form on dividing $S(g, g')$ by $\mathcal{G}$. If $r \neq 0$ then set $\mathcal{G} = \mathcal{G} \cup \{r\}$, and $\mathcal{P} = \mathcal{P} \cup \{(r, g) : g \in \mathcal{G}\}$.
3. Return $\mathcal{G}$.

where CriterionB2$(g, g', \mathcal{P})$ is Buchberger's second criterion of Proposition 3.4.3 and so it is true if there exists $p \neq f_i, f_j$ in $\mathcal{G}$ such that $\text{lm}(p)$ divides $\text{lcm}(\text{lm}(f_i), \text{lm}(f_j))$ and where pairs $(f_i, p), (f_j, p)$ are not in $\mathcal{P}$.

Proof. The termination of Algorithm 3.4.6 follows from the termination of Algorithm 3.2.9 as if Algorithm 3.4.6 had an infinite loop, then so would Algorithm 3.2.9.

For correctness, note that as in Algorithm 3.2.9, at every step of the algorithm $\mathcal{G} \subseteq I$. By Proposition 3.4.5, to show that the output $\mathcal{G} = \{g_1, \ldots, g_s\}$ has the property that $\{\text{in}_\prec(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_\prec(\text{in}_w(I))$ we need to verify that every S-polynomial $S(g_i, g_j)$ has a T-representation for some $T > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$. We check that at every step of the algorithm, the S-polynomial $S(g_i, g_j)$ for $g_i, g_j \in \mathcal{G}$, $(g_i, g_j) \notin \mathcal{P}$ has a T-representation for some $T > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$.

By Remark 3.2.10 if $S(g_i, g_j)$ has zero normal form with respect to $\mathcal{G}$, then it has a standard representation with respect to $\mathcal{G}$. Further, as $S(g_i, g_j)$ has a standard representation and $S(g_i, g_j) > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$, then this standard representation is a T-representation for some $T > \text{lc}(g_i)\,\text{lc}(g_j)\,\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$. Thus, if $S(g_i, g_j)$ has a zero normal form, then it has a suitable T-representation. Similarly, if $S(g_i, g_j)$ has a non-zero normal form with respect to $\mathcal{G}$ then it can be written as $\sum_{i=1}^{s} h_i g_i + r$ which as in the proof of Algorithm 3.2.9 we saw was a standard representation with respect to $\mathcal{G} \cup \{r\}$ and so is a suitable T-representation. If

$\text{lcm}(\text{lm}(g_i), \text{lm}(g_j)) = \text{lm}(g_i)\,\text{lm}(g_j)$, then by Proposition 3.4.2, $S(g_i, g_j)$ has a standard representation with respect to $\{g_i, g_j\}$ and so it has a suitable T-representation.

We can then deduce that (g_i, g_j) satisfies Buchberger's second criterion B2 and so $S(g_i, g_j)$ was tested out in step 2(b) of the Algorithm 3.4.6. This means that there exists $p \neq g_i, g_j$ in $\mathcal{G}$ such that $\text{lm}(p)$ divides $\text{lcm}(\text{lm}(g_i), \text{lm}(g_j))$ and where pairs (f_i, p) and (f_j, p) are not in $\mathcal{P}$ at that step of the algorithm. Thus, for some $T_1 \in S$ such that $T_1 > \text{lc}(g_1)\,\text{lc}(f)\,\text{lcm}(\text{lm}(g_1), \text{lm}(f))$, $S(g_1, f)$ has a T_1-representation with respect to $\mathcal{G}$, and for some $T_2 \in S$ such that $T_2 > \text{lc}(g_2)\,\text{lc}(f)\,\text{lcm}(\text{lm}(g_2), \text{lm}(f))$, $S(g_2, f)$ has a T_2-representation with respect to $\mathcal{G}$. Then, by Proposition 3.4.3, there is a $T \in S$ such that $T > \text{lc}(g_1)\,\text{lc}(g_2)\,\text{lcm}(\text{lm}(g_1), \text{lm}(g_2))$ and $S(g_1, g_2)$ has a T-representation with respect to $\mathcal{G}$.

At termination, $\mathcal{P} = \emptyset$ and so every S-polynomial $S(g_i, g_j)$ has a suitable T-representation, and so by Proposition 3.4.5 we have that $\{\text{in}_\prec(\text{in}_w(g_i)) : 1 \leq i \leq s\}$ generates $\text{in}_\prec(\text{in}_w(I))$ as required. □

3.4.2 Working over $\mathbb{Z}/p^m\mathbb{Z}$

While it is sometimes unavoidable to get large coefficients when computing a Gröbner basis over $\mathbb{Q}$, these coefficients do not always have large p-adic valuation. This motivates working in $\mathbb{Z}/p^m\mathbb{Z}$ via the method suggested in Remark 3.2.8.

This requires the following subroutine, which details how to compute a Gröbner basis for I given generators for $\text{in}_\prec(\text{in}_w(I))$.

Algorithm 3.4.7. Input: Homogeneous generators $\{f_1, \ldots, f_l\}$ for an ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$. A weight vector $w \in \mathbb{Z}^n$ and a term order $\prec$. Generators $\mathcal{I} = \{x^{u_1}, \ldots, x^{u_s}\}$ for $\text{in}_\prec(\text{in}_w(I))$.

Output: A reduced Gröbner basis for I with respect to w and $\prec$.

1. $\mathcal{G} = \emptyset$.
2. For each degree d of a monomial $x^{u_i} \in \mathcal{I}$ do:
   (a) Let $h = \dim_{\mathbb{Q}} I_d$. Form the $h \times \binom{n+d-1}{d}$ matrix A_d whose rows are the coefficients of a $\mathbb{Q}$-basis for I_d. The columns of A_d are indexed by the monomials of degree d, and we assume that the monomials in $\text{in}_\prec(\text{in}_w(I))_d$ come first in the ordering. The rows can be taken to be monomial multiples of the f_i.
   (b) Let B_d be the result of multiplying A_d by the inverse of the first $h \times h$ submatrix of A_d. This submatrix is invertible by the argument of the proof of Proposition 3.3.4.

(c) For each $x^{u_i} \in \mathcal{I}$ of degree d, let g_i be the polynomial corresponding to the row of B_d that contains a 1 in the column corresponding to x^{u_i}. Add g_i to $\mathcal{G}$.

3. Output $\mathcal{G}$.

Proof of correctness of algorithm 3.4.7. The chosen polynomials have the property that no monomial other than x^{u_i} lies in $\mathrm{in}_\prec(\mathrm{in}_w(I))$, so $\mathrm{in}_\prec(\mathrm{in}_w(g_i)) = x^{u_i}$. Thus the initial ideal $\mathrm{in}_\prec(\mathrm{in}_w(I))$ equals $\langle \mathrm{in}_\prec(\mathrm{in}_w(g_1)), \dots, \mathrm{in}_\prec(\mathrm{in}_w(g_r))\rangle$, so the output is a reduced Gröbner basis as required. □

We incorporate this into the following algorithm, which computes a Gröbner basis modulo p^m for large m.

Algorithm 3.4.8. Input: A list $\{f_1, \dots, f_l\}$ of homogeneous polynomials in $\mathbb{Q}[x_1, \dots, x_n]$, a prime p, a weight-vector $w \in \Gamma^n$, and a term order $\prec$.
Output: A Gröbner basis for $\langle f_1, \dots, f_l\rangle$.

1. Let $I = \langle f_1, \dots, f_l\rangle$. Let $f_i^w = f_i(p^{w_1}x_1, \dots, p^{w_n}x_n)$ for $1 \leq i \leq l$. Clear denominators in the f_i^w, and saturate the resulting ideal in $\mathbb{Z}[x_1, \dots, x_n]$ by $\langle p\rangle$. Let $\tilde{I}_w$ be the image of this ideal in $\mathbb{Z}/p^m\mathbb{Z}[x_1, \dots, x_n]$.

2. Compute $\mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w))$ using Algorithm 3.2.9.

3. Lift the resulting initial ideal to a Gröbner basis for I using Algorithm 3.4.7.

Note that the fact Algorithm 3.2.9 does compute $\mathrm{in}_\prec(\mathrm{in}_w(\tilde{I}))$ follows from Remark 3.2.8. The following lemma shows that for m sufficiently large this initial ideal equals $\mathrm{in}_\prec(\mathrm{in}_w(I))$, so Algorithm 3.4.7 will terminate with the correct answer.

Lemma 3.4.9. *For $m \gg 0$ Algorithm 3.4.8 terminates with the correct answer.*

Proof. We first show that for $m \gg 0$ we have $\mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w)) = \mathrm{in}_\prec(\mathrm{in}_w(I))$. Note that if $f = \sum c_u x^u$ with $c_u \in \mathbb{Z}$ with $\mathrm{val}(c_u) < m$, then the image $\tilde{f}$ of f in $\mathbb{Z}/p^m\mathbb{Z}[x_1, \dots, x_n]$ satisfies $\mathrm{in}_\prec(\mathrm{in}_0(\tilde{f})) = \mathrm{in}_\prec(\mathrm{in}_0(f))$. Let $I_w = \langle f_i^w\rangle \subseteq \mathbb{Q}[x_1, \dots, x_n]$, so $\mathrm{in}_w(I) = \mathrm{in}_0(I_w)$. By Proposition 3.3.4 there is a bound in terms of the absolute value of the coefficients of the generators of I on the maximum valuation that occurs in a reduced Gröbner basis. For m larger than this bound we have $\mathrm{in}_\prec(\mathrm{in}_w(I)) \subseteq \mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w))$.

For the reverse inclusion, fix $x^u \in \mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w))$. Choose $f \in \tilde{I}_w$ with $\mathrm{in}_\prec(\mathrm{in}_0(f)) = x^u$. By the definition of $\tilde{I}_w$ there is $g \in I_w$ with $f = \tilde{g}$. By construction $\mathrm{in}_0(g) = \mathrm{in}_0(f)$, so $x^u = \mathrm{in}_\prec(\mathrm{in}_0(g)) \in \mathrm{in}_\prec(\mathrm{in}_0(I_w)) = \mathrm{in}_\prec(\mathrm{in}_w(I))$.

In the first step of the algorithm, note that generators of the ideal obtained by clearing denominators and saturating by $\langle p\rangle$ generate $I_w \cap \mathbb{Z}_{\langle p\rangle}[x_1, \ldots, x_n]$. Since the image of an ideal $J \subset \mathbb{Z}[x_1, \ldots, x_n]$ in $\mathbb{Z}/p^m[x_1, \ldots, x_n]$ equals the ideal obtained by first taking the image of J in $\mathbb{Z}_{\langle p\rangle}[x_1, \ldots, x_n]$ and then taking the image in $\mathbb{Z}/p^m\mathbb{Z}[x_1, \ldots, x_n]$ (using that $\mathbb{Z}_{\langle p\rangle}/\langle p^m\rangle \cong \mathbb{Z}/p^m\mathbb{Z}$), $\tilde{I}_w$ is the image of $I_w \cap \mathbb{Z}[x_1, \ldots, x_n]$ in $\mathbb{Z}/p^m\mathbb{Z}[x_1, \ldots, x_n]$. The second step computes $\mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w))$ by Remark 3.2.8. The equality $\mathrm{in}_\prec(\mathrm{in}_0(\tilde{I}_w)) = \mathrm{in}_\prec(\mathrm{in}_w(I))$ then guarantees that we have the correct input for Algorithm 3.4.7, so the algorithm terminates correctly. □

The bound on m to guarantee that we are in the situation given in Proposition 3.3.4, may be ridiculously large, and not tight. If instead one uses an ad hoc choice for m, step 3 of Algorithm 3.4.8 will fail if the bound chosen was too low. We can thus iterate, repeating the computation with a larger value of m. This is often the best choice in practice.

3.5 Cardinality

In this section we give an example which shows that a p-adic Gröbner basis may be significantly smaller than any standard Gröbner basis. This gives another motivation to study such Gröbner bases.

Recall that a monomial ideal M is strongly stable, or Borel fixed, if for all $x^u \in M$ with $u_j > 0$ and $i < j$ we have $x_i/x_j x^u \in M$. Our construction requires a special case of the following elementary lemma.

Lemma 3.5.1. *Fix degrees $d_1, \ldots, d_l$, and let $\mathbb{P} = \prod_{i=1}^{l} \mathbb{P}^{\binom{d_i+n-1}{d_i}-1}$ be the parameter space for sequences of homogeneous polynomials $f_1, \ldots, f_l \subset K[x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_l$, where K has characteristic zero. Then there is a Zariski-open set $U \subseteq \mathbb{P}$ for which if $p \in U$ then the ideal $I = \langle f_1, \ldots, f_l\rangle$ generated by the polynomials corresponding to p has the property that $\mathrm{in}_\prec(I)$ is strongly stable for all term orders $\prec$. There are points in U with any prescribed valuations.*

Proof. Fix a term order $\prec$. Note that $G = \mathrm{PGL}(n, K)$ acts on $\mathbb{P}$ by change of coordinates on each factor. There is a nonempty open set $V \subset G \times \mathbb{P}$ for which $\mathrm{in}_\prec(gI)$ is constant for all $(g, p) \in V$. Denote this initial ideal by $M_\prec$. The existence of this open set V follows from the theory of comprehensive Gröbner bases [Weispfenning, 2006]. For a fixed $p \in \mathbb{P}$, there is an open set $V' \subset G$ for which the initial ideal $\mathrm{in}_\prec(gI)$ equals the generic initial ideal $\mathrm{gin}_\prec(I)$, which is strongly stable [see Eisenbud, 1995, Theorem 15.23]. By considering any $p \in \mathbb{P}$ for which there is some $g \in G$ with $(g, p) \in V$, we see that the initial ideal $M_\prec$ is strongly stable.

Since V is open in $G \times \mathbb{P}$, the set $U_\prec = \{p \in \mathbb{P} : (\mathrm{id}, p) \in V\}$ is open in $\mathbb{P}$, and $\mathrm{in}_\prec(I) = M_\prec$ for all $p \in U_\prec$. The group G acts on $G \times \mathbb{P}$ by $h \cdot (g, p) = (gh^{-1}, hp)$. Note that the set $V \subset G \times \mathbb{P}$ is invariant under this action. This means that the set $U_\prec$ is nonempty, as given any $(g, p) \in V$, we also have $(\mathrm{id}, g^{-1}p) \in V$. If $M_\prec = M_{\prec'}$ for two different term orders $\prec, \prec'$, then we can take $U_\prec = U_{\prec'}$, as the two term orders agree on the initial terms of a reduced Gröbner basis of any $I = I(p)$ with $p \in U_\prec$. The first part of the lemma then follows from the observation that the Hilbert functions of all initial ideals $M_\prec$ agree and there are only a finite number of strongly stable ideals with a given Hilbert function, so there are only a finite number of open sets $U_\prec$ to intersect to obtain an open set $U \subset \mathbb{P}$ with $\mathrm{in}_\prec(I)$ strongly stable for any $p \in U$ and any term order $\prec$.

Since $U \subset \mathbb{P}$ is open, so is its intersection with an affine chart $\mathbb{A}^{\sum_{i=1}^{l} \binom{d_i+n-1}{d_i} - l}$. This contains the complement of a hypersurface $V(f)$ where $f \in K[x_1, \ldots, x_N]$ for $N = \sum_{i=1}^{l} \binom{d_i+n-1}{d_i} - l$. We now show by induction on N that the valuations of a point outside $V(f)$ can be prescribed. When $N = 1$, $V(f)$ is a finite set, so the base case follows from the fact that there are infinitely many elements of K with a given valuation. Now assume that the claim is true for smaller N, and write $f = gx_1^m+$ lower order terms, where $g \in K[x_2, \ldots, x_N]$. Then by induction there is $x' = (x_2, \ldots, x_N)$ with $g(x') \neq 0$ and with $\mathrm{val}(x')$ prescribed. By the base case there is x_1 with prescribed valuation for which the univariate polynomial $f(x_1, x')$ is nonzero. Then $(x_1, x') \in U$ is the desired point. □

The other ingredient needed for the construction is the notion of a Stanley decomposition for a monomial ideal $M \subseteq K[x_1, \ldots, x_n]$. For $\sigma \subseteq \{1, \ldots, n\}$ and a monomial x^u we denote by (x^u, σ) the set of monomials $\{x^{u+v} : v_i = 0 \text{ for } i \notin \sigma\}$. A Stanley decomposition for M is a union $\{(x^{u_i}, \sigma_i) : 1 \leq i \leq s\}$ such that every monomial in M lies in a unique set (x^{u_i}, σ_i). The key fact about Stanley decompositions is that the Hilbert function $\dim_K I_t$ of I is the sum $\sum_{i=1}^{s} \binom{t-|u_i|+|\sigma_i|-1}{|\sigma_i|-1}$.

Theorem 3.5.2. *Fix an even integer $d = 2e$. Let $I = \langle f, g\rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$ be two generic polynomials of degree d where every coefficient of f except x_1^d and every coefficient of g except $x_2^e x_3^e$ has positive 2-adic valuation, and the remaining two coefficients have valuation zero. Then $\mathrm{in}_0(I) = \langle x_1^d, x_2^e x_3^e\rangle$ with the 2-adic valuation, but any standard initial ideal $\mathrm{in}_\prec(I)$ has at least $1/2(d + 3)$ generators.*

Proof. Note first that the existence of f, g satisfying these conditions follows from Lemma 3.5.1, from which it also follows that every standard initial ideal $\mathrm{in}_\prec(I)$ is Borel-fixed. That $\{f, g\}$ is a 2-adic Gröbner basis for I with respect to $w = 0$ follows from Buchberger's criterion B1.

Fix a term order $\prec$, and let $\text{in}_\prec(I) = \langle x^{u_1}, \ldots, x^{u_s}\rangle$. Write $\{1,2,3\} = \{i_1, i_2, i_3\}$ so that $x_{i_1} \succ x_{i_2} \succ x_{i_3}$. For $u \in \mathbb{N}^3$, denote by $m(u)$ the index $m(u) = \max(j : u_{i_j} \neq 0) \in \{1,2,3\}$. Then since $\text{in}_\prec(I)$ is Borel-fixed, the decomposition $\{(x^{u_i}, \{i_{m(u_i)}, \ldots, i_3\}) : 1 \leq i \leq s\}$ is a Stanley decomposition for $\text{in}_\prec(I)$. This means that $\dim_{\mathbb{Q}}(\text{in}_\prec(I)_t) = \sum_{i=1}^s \binom{t-|u_i|+3-m(u_i)}{3-m(u_i)}$. Without loss of generality we may assume that $x^{u_1} = x_{i_1}^d$, and $m(u_i) \geq 2$ for $i \geq 2$. Since I is generated in degree d, $|u_i| \geq d$ for all i. Since the Hilbert function of I and any initial ideal (standard or 2-adic) agree, the fact that the 2-adic initial ideal of I is $\langle x_1^d, x_2^e x_3^e\rangle$ implies that $\dim_{\mathbb{Q}}(I_t) = 2\binom{t-d+2}{2}$ for $d \leq t < 2d$. Thus for $d \leq t < 2d$ we have

$$\begin{aligned} 2\binom{t-d+2}{2} &= \sum_{i=1}^{s} \binom{t-|u_i|+3-m(u_i)}{3-m(u_i)} \\ &\leq \binom{t-d+2}{2} + (s-1)\binom{t-d+1}{1} \end{aligned}$$

so

$$1/2(t-d+2)(t-d+1) \leq (s-1)(t-d+1).$$

Then setting $t = 2d-1$ we see that $s \geq 1/2(d+3)$, as required. □

3.6 A Macaulay2 package to Compute Gröbner bases over fields with valuations

Consider $K = \mathbb{Q}$ with the p-adic valuation for some prime p. The algorithms in this Chapter are implemented in the package `GroebnerValuations` [Chan, 2013a] for the computational algebraic geometry system Macaulay2 [Grayson and Stillman] to compute these Gröbner bases with coefficients in the rational numbers with the p-adic valuation. The package `GroebnerValuations` has been submitted to "The Journal of Software for Algebra and Geometry". `GroebnerValuations` allows computation of:

1. the Gröbner basis of an ideal with `groebnerVal`;
2. the initial ideal of an ideal with `leadForm`;
3. the normal form of a polynomial with respect to a set of polynomials with `normalForm`.

It was in using this package that we encountered problems with the blow-up of coefficients that we needed to utilise the improvements and alterations discussed in the Section 3.4.

We demonstrate usage in by first installing the package then specifying the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4]$ and a polynomial $f = 2x_1 + x_2 + 8x_3 - 2x_4$:

```
i1 : installPackage "GroebnerValuations"

i2 : R = QQ[x_1..x_4];

i3 : f = 2*x_1+x_2+8*x_3-2*x_4
o3 = 2x  + x  + 8x  - 2x
       1    2     3     4
o3 : R
```

Considering the 2-adic valuation, we compute the initial form of f with respsect to the weight vectors $(1, 1, 1, 1)$ and $(1, 3, 7, 1)$:

```
i4 : leadForm(f,{1,1,1,1})
o4 = x
      2
     ZZ
o4 : --[x , x , x , x ]
      2  1   2   3   4

i5 : leadForm(f,{1,3,7,1})
o5 = x  + x
      1    4
     ZZ
o5 : --[x , x , x , x ]
      2  1   2   3   4
```

So with respect to the 2-adic valuation, $\text{in}_{(1,1,1,1)}(f) = x_2$ and $\text{in}_{(1,3,7,1)}(f) = x_1 + x_4$. Consider the ideal $I = \langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4\rangle$ in S and compute the initial ideal with respect to 2-adic valuation for weight vector $(1, 1, 1, 1)$ and with respect to the 3-adic valuation for weight vector $(1, 11, 3, 19)$:

```
i6 : I=ideal(2*x_1^2+3*x_1*x_2+24*x_3*x_4,8*x_1^3+x_2*x_3*x_4+18*x_3^2*x_4)
                2                       3                 2
o6 = ideal (2x   + 3x x   + 24x x , 8x   + x x x   + 18x x )
                1     1 2      3 4    1    2 3 4       3 4
o6 : Ideal of R

i7 : leadForm(I,{1,1,1,1})
```

```
o7 = | x_1x_2 x_2x_3x_4 x_1^2x_3x_4+x_1x_3^2x_4 |

             ZZ                  1       ZZ                  3
o7 : Matrix (--[x , x , x , x ])  <--- (--[x , x , x , x ])
              2  1   2   3   4            2  1   2   3   4

i8 : leadForm(I,{1,11,3,19},Prime=>3)

o8 = | x_1^2 x_1x_3x_4 x_1x_2^2x_3 x_1x_2^4 x_3^4x_4^2 |

             ZZ                  1       ZZ                  5
o8 : Matrix (--[x , x , x , x ])  <--- (--[x , x , x , x ])
              3  1   2   3   4            3  1   2   3   4
```

We have computed that $\text{in}_{(1,1,1,1)}(I) = \langle x_1x_2, x_2x_3x_4, x_1^2x_3x_4\rangle$ with respect to the 2-adic valuation and $\text{in}_{(1,11,3,19)}(I) = \langle x_1^2, x_1x_3x_4, x_1x_2^2, x_3^4x_4^2\rangle$ with respect to the 3-adic valuation. We now compute the Gröbner basis with respect to the 2-adic valuation and the 3-adic valuations for weight vector $(1,1,1,1)$:

```
i9 : groebnerVal(I,{1,1,1,1})

o9 = | 2/3x_1^2+x_1x_2+8x_3x_4 8x_1^3+x_2x_3x_4+18x_3^2x_4
     ------------------------------------------------------------------------
     -12x_1^4+x_1^2x_3x_4-27x_1x_3^2x_4+12x_3^2x_4^2 |

             1       3
o9 : Matrix R  <--- R

i10 : groebnerVal(I,{1,1,1,1},Prime=>3)

o10 = | x_1^2+3/2x_1x_2+12x_3x_4
      ------------------------------------------------------------------------
      18/145x_1x_2^2-96/145x_1x_3x_4+x_2x_3x_4+18/145x_3^2x_4 |

              1       2
o10 : Matrix R  <--- R
```

We see that $\{2/3x_1^2+x_1x_2+8x_3x_4, 8x_1^3+x_2x_3x_4+18x_3^2x_4, -12x_1^4+x_1^2x_3x_4-27x_1x_3^2x_4+12x_3^2x_4^2\}$ is a Gröbner basis for I with respect to the 2-adic valuation and $\{x_1^2+3/2x_1x_2+12x_3x_4, 18/145x_1x_2^2-96/145x_1x_3x_4+x_2x_3x_4+18/145x_3^2x_4\}$ is a Gröbner basis for I with respect to the 3-adic valuation. Finally we compute the normal form of $x_1^3+x_2^3+x_3^3+x_4^3$ with respect to polynomials $\{x_1+12x_4, x_2-8x_1, x_3-128x_1\}$ and weight vector $(1,3,2,1)$:

```
i11 : g = x_1^3+x_2^3+x_3^3+x_4^3;

i12 : normalForm(g,{x_1+12*x_4,x_2-8*x_1,x_3-128*x_1},{1,3,2,1})

                2     2       2                                             2
o12 = (1, {x  + 8x  + 128x  - 12x x  - 768x x  - 196608x x  + 302063760x ,
            1     2       3      1 4       2 4          3 4              4
      ------------------------------------------------------------------------
        2                  2   2                          2                3
```

```
      x  - 96x x  + 9216x , x  - 1536x x  + 2359296x }, -3624765119x )
       2      2 4        4   3        3 4            4                4
o12 : Sequence
```

We see that $x_1^3 + x_2^3 + x_3^3 + x_4^3 = (x_1^2 + 8x_2^2 + 128x_3^2 - 12x_1x_4 - 768x_2x_4 - 196608x_3x_4 + 302063760x_4^2)(x_1 + 12x_4) + (x_2^2 - 96x_2x_4 + 9216x_4^2)(x_2 - 8x_1) + (x_3^2 - 1536x_3x_4 + 2359296x_4^2)(x_3 - 128x_1) - 3624765119x_4^3$.

Recall that the two main ways to improve the speed and efficiency of the algorithms are

1. Using criteria to decide a priori that certain S-polynomials reduce to zero;
2. By working over $\mathbb{Z}/p^m\mathbb{Z}$ for some suitably large $m \in \mathbb{N}$.

These two improvements are implemented in the package, with the second as an option. We demonstrate how this works by computing the special fibre of a Mustafin variety [Cartwright, Häbich, Sturmfels, and Werner, 2011, Definition 1.1] in the case where $p = 2$. The following code computes the Mustafin variety as in Example 2.2 of Cartwright, Häbich, Sturmfels, and Werner [2011]. In our example we replace the matrices g_1, g_2, g_2 from Example 2.2 with the matrices A, B, C below.

```
i13 : R = QQ[x_1..x_9];
i14 : y1 = matrix{{x_1},{x_2},{x_3}};
i15 : y2 = matrix{{x_4},{x_5},{x_6}};
i16 : y3 = matrix{{x_7},{x_8},{x_9}};
i17 : A = matrix{{1,8,16},{2,1,32},{4,1,4}};
i18 : B = matrix{{19,3,7},{5,8,1},{2,64,3}};
i19 : C = matrix{{1,12,8},{11,1,6},{1,9,5}};
i20 : A1 = flatten entries(A*y1);
i21 : B1 = flatten entries(B*y2);
i22 : C1 = flatten entries(C*y3);
i23 : M = matrix{A1,B1,C1};
i24 : J = minors(2,M);
```

The special fibre of the Mustafin variety is the initial ideal with respect to the weight vector $(1, \ldots, 1)$. When computing this example, the coefficients grow very large and we are unable to complete the computation using the memory space of the computer and so we need to work over $\mathbb{Z}/p^m\mathbb{Z}$:

```
i25 : leadForm(J,{1,1,1,1,1,1,1,1,1},ModPn=>true)
o25 = | x_4x_8+x_5x_9 x_1x_7+x_2x_7+x_1x_8 x_5x_7+x_5x_8+x_6x_8+x_5x_9
      --------------------------------------------------------------------
      x_4x_7+x_4x_9+x_5x_9+x_6x_9 x_2x_9 x_1x_9 x_1x_4 x_2x_4
      --------------------------------------------------------------------
      x_2x_5+x_1x_6+x_2x_6 |
```

In Theorem 3.5.2 we saw a family of ideals which have small 2-adic Gröbner basis but where all other standard Gröbner bases are large and grow linearly with the degrees of the generators. Consider the case of degree 30 polynomials. We start by finding two random polynomials f and g which satisfy the hypotheses and forming the ideal $\langle f, g\rangle$:

```
i23 : S = ZZ[x_1,x_2,x_3];
i24 : T1 = S/((x_1)^30);
i25 : use S;
i26 : T2 = S/((x_2)^15*(x_3)^15);
i27 : use S;
i28 : f = x_1^30+2*(lift(random(30,T1),S));
i29 : g = x_2^15*x_3^15+2*(lift(random(30,T2),S));
i30 : U = QQ(monoid S);
i31 : h = map(U,S);
o31 : RingMap U <--- S
i32 : K = h ideal(f,g);
o32 : Ideal of U
```

We now compute the $\text{in}_{(1,1,1)}(\langle f, g\rangle)$ with respect to the 2-adic valuation to show that it has only 2 elements, and as a demonstration show that the Gröbner basis with respect to the graded reverse lexicographic ordering has 61 elements.

```
i33 : leadForm(K,{1,1,1})
o33 = | x_1^30 x_2^15x_3^15 |
               ZZ              1       ZZ              2
o33 : Matrix (--[x , x , x ])  <--- (--[x , x , x ])
                2  1   2   3           2  1   2   3
i34 : #(flatten entries gens gb K)
o34 = 61
```

Chapter 4

Tropical Curves from Coordinate Projections

4.1 Introduction

Let K be an algebraically closed field equipped with the trivial valuation. Denote by $\mathbb{P}^n$ the n-dimensional projective space over the field K with n-dimensional algebraic torus T^n and $S = K[x_0, x_1, \dots, x_n]$ the homogeneous coordinate ring of $\mathbb{P}^n$. Let X be an irreducible m-dimensional subvariety of T^n with defining ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. As we saw in Chapter 2, the tropicalisation of X is defined to be $\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f))$ and by the Structure Theorem 2.2.5 it has the support of a balanced weighted rational m-dimensional polyhedral complex Σ that is connected through codimension one. A fundamental question in tropical geometry is how to find this polyhedral complex Σ.

In Section 2.3.3, we saw that a first answer was given by Bogart, Jensen, Speyer, Sturmfels, and Thomas [2007] in the paper "Computing Tropical Varieties". They provided algorithms to compute tropical varieties which have been implemented in the computer software package `gfan` [Jensen]. These algorithms use the fact that $\mathrm{trop}(X)$ is connected through codimension one. The idea is to 'walk' from one maximal dimensional cone to another by passing through a facet. We find these neighbouring maximal dimensional cones by computing a tropical curve which has a ray passing in the direction of each neighbouring maximal dimensional cone. If u is a generic relative interior point of a facet of some maximal dimensional cone then, we saw in Section 2.3.3 that $V(\mathrm{in}_u(I)) = C \times (K^*)^{\dim I - 1}$ for some curve C. As the initial complex associated to the initial ideal is connected [Kalkbrener and Sturmfels, 1995, Theorem 2], this tells us that C is connected. The tropicalisation trop(C) defines a tropical curve which has a ray pointing in the direction of each neighbouring maximal dimensional cone. So the construction of tropical curves is a key step of these algorithms to construct tropical varieties. Any improvements in the algorithm for computing a tropical curve would result in improvements to the algorithms for constructing arbitrary dimensional tropical varieties. In particular, this is a bottleneck in the algorithms and we present examples in Section 4.3 of a curve for which we cannot compute its tropicalisation using `gfan`.

Let $C \subseteq T^n$ be a curve with defining ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We saw in Section 2.5 that Bieri and Groves [1984] showed that there is a dense open set in the set of all projections such that for n of these projections $\pi_1, \ldots, \pi_n \colon \mathbb{R}^n \to \mathbb{R}^2$ we have that

$$\operatorname{trop}(C) = \bigcap_{i=1}^{n} \pi_i^{-1}\pi_i(\operatorname{trop}(C)). \tag{4.1}$$

In fact, $\pi^{-1}\pi(\operatorname{trop}(C))$ is a tropical hypersurface and we can compute the equation for this hypersurface by changing coordinates so that the projection is a coordinate projection before using elimination theory, as described in Section 2.4, to find an equation for the projection. Combining these two results, we see that we can find suitably generic geometrically regular projections, such that (4.1) holds and $\pi_i^{-1}\pi_i(\operatorname{trop}(C)) = \operatorname{trop}(V(g_i))$ for some $g_i \in I$. Thus $\{g_1, \ldots, g_n\}$ is a tropical basis for I and we find trop(C) by intersecting all of the tropical hypersurfaces trop($V(g_i)$) as we saw in Section 2.3.2. We demonstrate how we can recover the support of a one-dimensional tropical variety from suitably generic projections with an example.

Example 4.1.1. Let $I = \langle xz + 4yz - z^2 + 3xw - 12yw + 5zw, xy - 4y^2 + yz + xw + 2yw - zw, x^2 - 16y^2 + 8yz - z^2 + 14xw - 8yw + 2zw\rangle$ be a homogeneous ideal in $\mathbb{C}[x, y, z, w]$ where $\mathbb{C}$ is equipped with the trivial valuation. Then I defines a curve in $\mathbb{P}^3$. Let $C = V(I) \cap T^n$. Then trop(C) is a one-dimensional tropical curve in $\mathbb{R}^4/\mathbb{R}(1, 1, 1, 1)$. We consider the tropicalisations as fans living in $\mathbb{R}^3$ after we quotient out by the lineality space $(1, 1, 1, 1)$ so that the w coordinate is zero. We see that trop(C) has four rays spanned by

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).$$

In Figure 4.1 we see trop(C) as the rays passing through alternate vertices of a cube in $\mathbb{R}^3$ with vertices at $(\pm 1, \pm 1, \pm 1)$.

By the discussion above, we can find three geometrically regular projections which recover the tropical curve. We consider projections onto the planes defined by $y = 0$, $x = 0$ and $x + 2z = 0$ as shown in Figure 4.2.

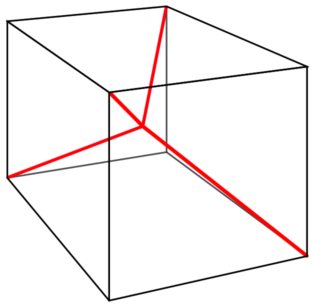

Figure 4.1: A tropical curve as the alternate vertices of a cube

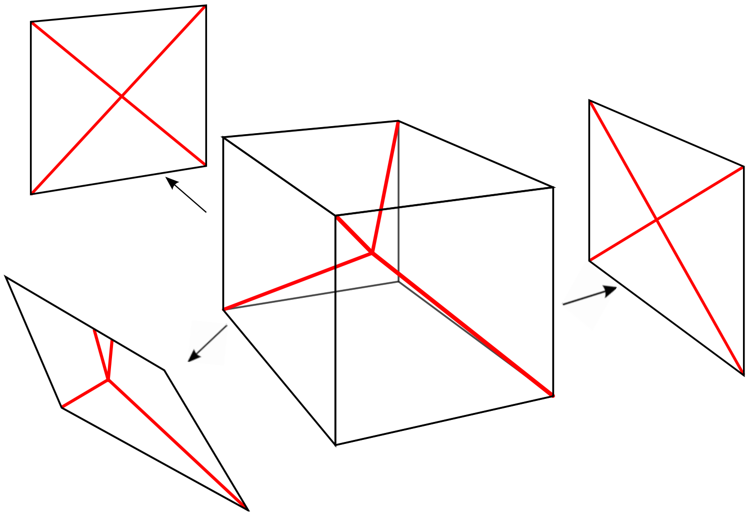

Figure 4.2: A tropical curve in $\mathbb{R}^3$ with three generic projections

We recover the rays of $\text{trop}(C)$ from these three projections starting with the projection to $y = 0$. This has image spanned by positive multiples of four rays generated by $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ in the xz-plane. Thus we see that the points in $\mathbb{R}^3$ that project to this projection can have any y-coordinate with the x and z coordinates positive multiples of the rays of the projected curve. That is, the points of $\mathbb{R}^3$ which project to the $y = 0$ projection are of the form (a, b, a), $(a, b, -a)$, $(-a, b, a)$ and $(-a, b, -a)$ for $a \geq 0$ and any $b \in \mathbb{R}$. This is shown in Figure 4.3.

Then we look to see which of those points project to the $x = 0$ projection. We now recover eight rays which can be seen as those passing through all vertices of the cube. This is shown in Figure 4.4.

Finally, we see that of these eight rays, only four project on our third projection and we have recovered the tropical curve $\text{trop}(C)$. This is shown in Figure 4.5 $\diamond$

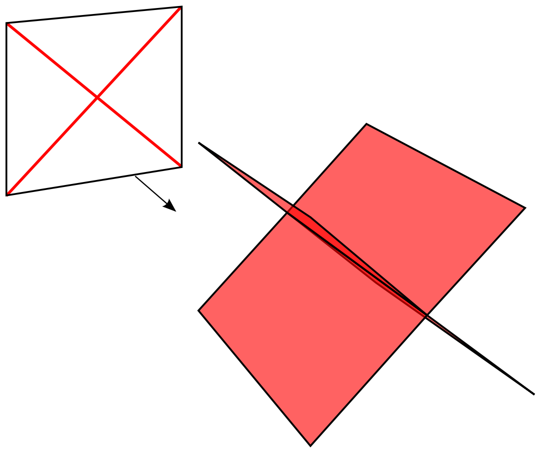

Figure 4.3: The points in $\mathbb{R}^3$ which project to the $y = 0$ projection

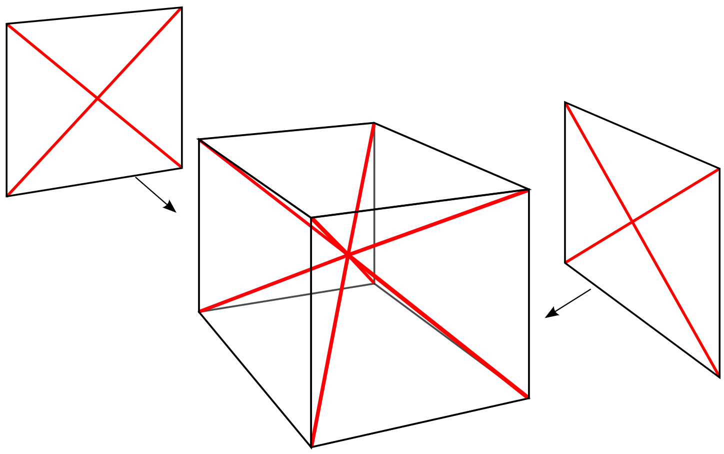

Figure 4.4: The points in $\mathbb{R}^3$ which project to the $y = 0$ and $x = 0$ projection

In order to use these techniques to computationally find $\text{trop}(C)$, we need to be able to find suitably generic projections. However, it is difficult to check if a set of projections is generic or not. Additionally, projections other than coordinate projections are often difficult to compute and the degree of polynomials involved may grow extremely large. Thus the key idea in this Chapter is that we shall restrict our attention to coordinate projections. These are usually easier to compute, but in general they will not be generic enough for the results of Bieri and Groves [1984] and Hept and Theobald [2009]. So even if we consider all coordinate projections, we may not have a tropical basis with which we can recover $\text{trop}(C)$. We see one such example in Example 4.2.11.

In this Chapter, we explain methods to be able to reconstruct $\text{trop}(C)$ from its coordinate projections. The reconstruction has three main steps:

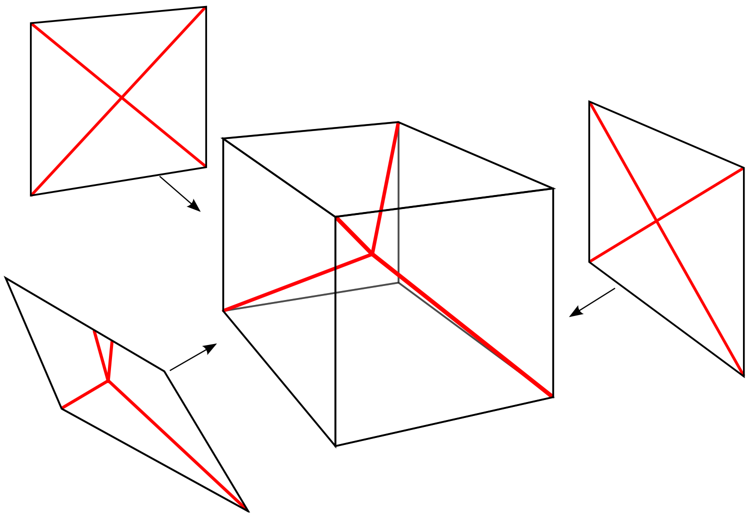

Figure 4.5: A tropical curve in $\mathbb{R}^3$ recovered from three generic projections

1. Find the image of the tropical curve under two-dimensional coordinate projections. In general these coordinate projections may not be geometrically regular but are easier to compute by elimination theory. This is discussed in Section 4.2.1.

2. Find the set of rays which project to the projected tropical curves found in Step 1. In general, this will form a finite superset of the rays in the tropical curve $\text{trop}(C)$. We provide algorithms to determine this superset of rays in Section 4.2.2.

3. Determine which of the rays from the superset found in Step 2 are rays of the tropical curve $\text{trop}(C)$ and find their multiplicities in $\text{trop}(C)$. The algorithms we provide for this in Section 4.2.3 use the multiplicity equation (2.2) from Tropical Elimination Theory, equations from the degree of the curve C which we saw in Section 2.6, and equations which come from the balancing condition. This limits the number of additional initial ideals that we need to compute.

In Section 4.3 we examine an example of a tropical curve which cannot be computed using `gfan` but which can be computed using these coordinate projection techniques. In Section 4.4 we see some implementation issues and ways in which our algorithms can be optimised. Finally, we introduce the `Macaulay2` [Grayson and Stillman] package `TropicalCurves` [Chan, 2013b] in Section 4.5 which implements these algorithms to compute tropical curves from coordinate projections.

4.2 Reconstructing Tropical Curves from Coordinate Projections

Let $I \subseteq S$ be a homogeneous prime ideal defining an irreducible curve in $\mathbb{P}^n$. Let $C = V(I) \cap T^n$. Recall from Section 2.3.3 that a key step in the construction of a tropical variety is to construct the tropicalisation of a connected curve. Thus, we will assume here that C is connected. In this section, we explain how to reconstruct the tropicalisation of C from its projections to coordinate planes. This procedure has three main steps. In the first we find equations for the projections and then tropicalise them. In the second we find a superset of the rays in $\mathrm{trop}(C)$, and in the third we determine which rays in the superset are rays of the tropical curve.

By the Structure Theorem 2.2.5, $\mathrm{trop}(C)$ has the support of a weighted balanced Γ-rational one-dimensional fan in $\mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$. This can be thought of as a fan in $\mathbb{R}^n$ after we identify $\mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ with $\mathbb{R}^n$. Algebraically, this corresponds to looking at a coordinate slice of the fan. We look at the slice of the fan where $x_0 = 0$ which corresponds to dehomogenising with respect to x_0. If ρ is a ray of $\mathrm{trop}(C)$, then the first lattice point of ρ is called the *primitive generator* of ρ. By abuse of notation, we shall let ρ denote a ray of $\mathrm{trop}(C)$ as well its primitive generator.

4.2.1 Finding the Projections

Let $\pi_{ij} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be the projection map to coordinates x_0, x_i, x_j. As $C \subseteq \mathbb{P}^n$ is one-dimensional, its image $C_{ij} := \pi_{ij}(C) \subseteq \mathbb{P}^2$ is then either one-dimensional or zero-dimensional. Suppose that the image is zero-dimensional. This means that C_{ij} is a finite set of points. As we are assuming that C is connected, the image under projection is also connected and so is a single point. Thus C is contained in a hyperplane. When this happens, the defining ideal I of C will contain a linear form in x_0, x_i, x_j. We require that C and its image C_{ij} under the projection π_{ij} to have the same dimension, so we will thus assume that I does not contain such a linear form and that C_{ij} is then one-dimensional so that it is a curve in $\mathbb{P}^2$. It is then given by a single homogeneous polynomial in $K[x_0, x_i, x_j]$.

Fix a monomial ordering $\succ$ where $x_k \succ x_0, x_i, x_j$ for all $k \neq i, j$, $1 \leq k \leq n$, and let $\mathcal{G}$ be a Gröbner basis for I with respect to $\succ$. Then, in Section 2.4 we saw that by Elimination Theory, $\mathcal{G} \cap K[x_0, x_i, x_j]$ is a Gröbner basis for $I \cap K[x_0, x_i, x_j]$. Geometrically $I \cap K[x_0, x_i, x_j]$ is the ideal defining the Zariski closure of the image C_{ij} in $\mathbb{P}^2$.

As the curve C is connected, irreducible and is not contained in any linear hyperplane of the form $ax_0 + bx_i + cx_j$, each fibre of the projection map cannot be a component of the curve. Thus fibres cannot be one-dimensional and so must be a finite set of zero-dimensional points. Thus the projection map $\pi_{ij}\colon C \to C_{ij}$ is a surjective map generically finite of some degree δ. We denote the tropicalisations of C and C_{ij} by trop(C) and trop(C_{ij}) respectively. We choose identifications of $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which send the first coordinates to zero. Then abusing notation, we let $x_1,\ldots,x_n$ be the coordinates of $\mathbb{R}^n$ and x_i, x_j the coordinates of $\mathbb{R}^2$. Thus, trop(C) is a one-dimensional fan in $\mathbb{R}^n$ and trop(C_{ij}) is a one-dimensional fan in $\mathbb{R}^2$. We also denote by $\pi_{ij}\colon \mathbb{R}^n \to \mathbb{R}^2$ the projection map onto the coordinates x_i, x_j induced from $\pi_{ij}\colon C \to C_{ij}$ and so sends trop(C) to trop(C_{ij}). By Tropical Elimination Theory [Sturmfels and Tevelev, 2008], which we saw in Section 2.4, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\text{trop}} & \text{trop}(C) \\ \Big\downarrow{\scriptstyle \pi_{ij}} & & \Big\downarrow{\scriptstyle \pi_{ij}} \\ C_{ij} & \xrightarrow{\text{trop}} & \text{trop}(C_{ij}). \end{array}$$

Additionally, if σ is a ray of trop(C_{ij}) then its multiplicity is given by

$$m_\sigma = \frac{1}{\delta} \sum_{\rho \in \text{trop}(C) : \sigma \subseteq \pi_{ij}(\rho)} m_\rho \cdot \text{index}(\rho, \sigma)$$

where this sum is over all rays $\rho \in \text{trop}(C)$ which project to σ. Here index(ρ, σ) denotes the index of the lattice generated by $\pi_{ij}(\rho)$ inside the lattice generated by σ. The index of a sublattice L' in L is given by the determinant of the matrix which sends the generators of the sublattice L' to the generators of the lattice L. The image of ρ is $\pi_{ij}(\rho) = (\rho_i, \rho_j)$ where ρ_i is the x_i coordinate of ρ, and ρ_j is the x_j coordinate of ρ. As the image of ρ under π_{ij} is $\sigma = (\sigma_i, \sigma_j)$ we must have that $(\rho_i, \rho_j) = a \cdot (\sigma_i, \sigma_j)$, where $\gcd(\sigma_i, \sigma_j) = 1$ as it is a primitive ray generator, and so it follows that $a = \gcd(\rho_i, \rho_j)$. Thus where we are considering coordinate projections, we can write the multiplicity formula as:

$$m_\sigma = \frac{1}{\delta} \sum_{\rho \in \text{trop}(C) : \sigma \subseteq \pi_{ij}(\rho)} m_\rho \cdot \gcd(\rho_i, \rho_j). \tag{4.2}$$

Remark 4.2.1. It follows from (4.2) that for any $\rho \in \text{trop}(C)$ such that $\sigma \subseteq \pi_{ij}(\rho)$

the multiplicity m_ρ can be bounded above by

$$m_\rho \leq \frac{\delta \cdot m_\sigma}{\gcd(\rho_i, \rho_j)} \tag{4.3}$$

which means that the maximum possible multiplicity of ρ in $\text{trop}(C)$ such that it still projects to σ is

$$\left\lfloor \frac{\delta \cdot m_\sigma}{\gcd(\rho_i, \rho_j)} \right\rfloor$$

as it must be a positive integer. $\diamond$

Let $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ be a coordinate projection. If C is a curve in $\mathbb{P}^n$ and L a hyperplane in $\mathbb{P}^n$ such that C is not contained in L then $C \cap L$ is zero-dimensional. We denote by m the number of points, counted with multiplicity, in $C \cap L$ that do not map to $\pi(C)$.

Lemma 4.2.2. *Let $C \subseteq \mathbb{P}^n$ be an irreducible connected curve of degree d_1 and let $\pi\colon C \subseteq \mathbb{P}^n \dashrightarrow \pi(C) \subseteq \mathbb{P}^2$ be a coordinate projection to x_0, x_1, x_2 which is generically finite of degree δ. Suppose that C is not contained in any hyperplane of x_0, x_1, x_2 and that $\pi(C)$ is a curve in $\mathbb{P}^2$. Let L be a generic hyperplane in $\mathbb{P}^n$ defined by a polynomial $l = ax_0 + bx_1 + cx_2$ which projects to a line $\pi(L)$ in $\mathbb{P}^2$. Suppose that $\pi(C \cap L)$ has degree d_2 and let m be defined as above. Then $d_1 = \delta \cdot d_2 + m$.*

Proof. We first note that as C is not contained in any hyperplane, it is not contained in L. Thus $C \cap L$ is zero-dimensional and so is a finite set of points. We can also choose L generically so that it does not intersect $\overline{\pi(C)} \backslash \pi(C)$. This is because there are only finitely many points in $\overline{\pi(C)} \backslash \pi(C)$.

We count the points in $C \cap L$ in two different ways. Firstly the degree of C can be defined as the number of points in $C \cap C'$ for a complementary dimensional subspace C'. As C is a curve, this means considering a hyperplane so we can take $C' = L$ as C is not contained in L. Thus the number of points in $C \cap L$ counted with multiplicity is d_1, the degree of C.

Secondly, these points are either points which do not map to points in $\mathbb{P}^2$ under the projection π, or they are points in the pre-image of π. By the hypothesis, there are m points, counted with multiplicity, in $C \cap L$ that do not map to $\pi(C)$. For the points in the pre-image of π, we count the points in $\pi(C \cap L)$. Then as π is of degree δ, this will correspond to δ times this number of points. As $\pi(C \cap L)$ is a zero-dimensional ideal, the number of points counted with multiplicity will generically equal the degree d_2. Thus there are $\delta \cdot d_2$ points which map to $\pi(C)$, and this point count gives $\delta \cdot d_2 + m$ points in $C \cap L$.

Combining these two ways of counting the points of $C \cap L$ we have

$$d_1 = \delta \cdot d_2 + m$$

as required. □

Remark 4.2.3. In Lemma 4.2.2, it is possible to choose the line $l = ax_0 + bx_1 + cx_2$ generically so that $\pi(C \cap L) = \pi(C) \cap \pi(L)$. In fact, the inclusion $\pi(C) \cap \pi(L) \supseteq \pi(C \cap L)$ always holds. Let I be the defining ideal of C and $J = \langle l \rangle$ the defining ideal of L. Then the condition $\pi(C \cap L) = \pi(C) \cap \pi(L)$ is equivalent to showing that $(I + J) \cap K[x_0, x_i, x_j] = (I \cap K[x_0, x_i, x_j]) + (J \cap K[x_0, x_i, x_j])$. Let $\{f_1, \ldots, f_s\}$ be an elimination Gröbner basis for I with respect to a monomial term order where $x_0, x_i, x_k > x_l$ for all $1 \leq l \leq n$ where $l \neq i, j$, and let f_s be the defining equation for $I \cap K[x_0, x_i, x_j]$. The ideal $J \cap K[x_0, x_i, x_j]$ is defined by l so that $(I \cap K[x_0, x_i, x_j]) + (J \cap K[x_0, x_i, x_j]) = \langle f_s, l \rangle \subseteq \langle f_1, \ldots, f_s, l \rangle \subseteq (I + J) \cap K[x_0, x_i, x_j]$. Thus $\pi(C) \cap \pi(L) \supseteq \pi(C \cap L)$ always holds.

For the reverse inclusion, we need the linear polynomial l to be chosen such that $(I + J) \cap K[x_0, x_i, x_j] \subseteq (I \cap K[x_0, x_i, x_j]) + (J \cap K[x_0, x_i, x_j])$. Suppose that I is generated in degree d, then we need to show that $(I + J)_d \cap K[x_0, x_i, x_j] \subseteq (I_d \cap K[x_0, x_i, x_j]) + (J_d \cap K[x_0, x_i, x_j])$. That is, we cannot have any polynomials in $(I + J)_d \cap K[x_0, x_i, x_j]$ which are not contained in $\langle f_s, l \rangle$. This is an open condition on the coefficients of l.

In the case where $\pi(C \cap L) = \pi(C) \cap \pi(L)$, it follows that $\deg(\pi(C \cap L)) = \deg(\pi(C))$ which is the degree of the polynomial defining the hypersurface $\pi(C)$. In practice, it appears that finding such a generic line is computationally time consuming and that it is in fact easier to simply compute the degree of $\pi(C \cap L)$. ◇

Algorithm 4.2.4. Input: An ideal $I \subseteq S$ defining a connected irreducible curve C in $\mathbb{P}^n$ and a projection map $\pi \colon C \dashrightarrow \pi(C)$ to $\mathbb{P}^2$.

Output: The degree δ of the projection map π.

1. Let $d_1 = \deg(C)$.
2. For $J = \langle ax_0 + bx_1 + cx_2 \rangle \subseteq S$ defining a hyperplane $L = V(J) \subseteq \mathbb{P}^n$, set $d_2 = \deg(\pi(C \cap L))$.
3. Let $P = \langle x_0, x_1, x_2 \rangle$ then define $M_1 = (I + J) : P^\infty$ and $M_2 = (I + J) : M_1^\infty$.
4. Let $m = \deg M_2$.

Output $\delta = (d_1 - m)/d_2$.

Proof of Algorithm. The ideal M_2 is the saturation of $I+J$ with respect to the ideal M_1 and so M_2 defines the variety of points in $V(I+J)$ which are not in $V(M_1)$. The ideal M_1 is the saturation of $I+J$ with respect to P thus M_1 defines the variety of points in $V(I+J)$ which are not in P. Then, the ideal M_2 is the ideal of points in $V(I+J)$ which are also in $V(P)$, and thus corresponds to the points in $C \cap L$ which do not map to $\mathbb{P}^2$. As M_2 is a zero-dimensional ideal, its degree counts these points with multiplicity. By Lemma 4.2.2, $d_1 = \delta d_2 + m$ and so the result follows. □

For a given homogeneous ideal I in S, the following algorithm computes the tropical curves which are the projections of $\operatorname{trop}(V(I))$ to two dimensional coordinate hyperplanes. It first finds equations for the projection of I to those coordinates, then constructs the tropical curve from the normal fan of its Newton polygon. We equip each projected tropical curve with a positive integer which is the degree of the algebraic projection map as this will be required to determine multiplicities using (4.2) in future steps of the reconstruction process.

Algorithm 4.2.5. Input: A homogeneous ideal $I \subseteq S$ defining an irreducible connected curve $C \subseteq \mathbb{P}^n$.

Output: A set $P = \{\mathcal{C}_{ij}\}$ of tropical curves where $\mathcal{C}_{ij} \subseteq \mathbb{R}^2$ is the projection of $\operatorname{trop}(C) \subseteq \mathbb{R}^n$ to the coordinate plane with coordinates x_i, x_j, and a set $D = \{\delta_{ij}\}$ where δ_{ij} is the degree of the projection map $\pi_{ij}\colon C \subseteq \mathbb{P}^n \dashrightarrow \mathbb{P}^2$.

Initialisation: $P = \emptyset$, $D = \emptyset$, $T = \{(x_i, x_j) : 1 \leq i < j \leq n\}$

For all (x_i, x_j) in T do:

1. Compute the elimination ideal I_{ij} which eliminates variables x_k for all $k \neq 0, i, j$.
2. Compute the degree δ_{ij} of the projection map $\pi\colon \mathbb{P}^n \to \mathbb{P}^2$ by Algorithm 4.2.4.
3. If I does not contain a linear form in x_0, x_i, x_j then $\pi_{ij}(C)$ is one-dimensional. Let $I_{ij} = \langle f_{ij} \rangle$ be the defining ideal of $\pi_{ij}(C)$ and compute $\mathcal{C}_{ij} = \operatorname{trop}(V(f_{ij}))$ by Algorithm 2.3.3.
4. $P = P \cup \{\mathcal{C}_{ij}\}$, $D = D \cup \{\delta_{ij}\}$.

Output P, D.

4.2.2 Reconstructing the pre-image rays

Consider the tropical curve $\operatorname{trop}(C)$ in $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1) \cong \mathbb{R}^n$ and let $\pi_{ij}\colon \operatorname{trop}(C) \subseteq \mathbb{R}^n \to \mathbb{R}^2$ be the coordinate projection to the coordinates x_i, x_j. The image of the tropical curve $\operatorname{trop}(C)$ under this projection is denoted by $\mathcal{C}_{ij} := \pi_{ij}(\operatorname{trop}(C))$.

In this section, we explain how to find the points in $\mathbb{R}^n$ which under coordinate projections project to our collection of tropical plane curves $\{\mathcal{C}_{ij}\}_{(i,j)\in P}$ where the set P indexes our projections. A primitive vector $\rho \in \mathbb{R}^n$ with associated multiplicity bound m_ρ is called an (i,j)-*candidate* for $\mathrm{trop}(C)$ if either $\pi_{ij}(\rho)$ spans a ray $\sigma \in \mathcal{C}_{ij}$ with multiplicity m_σ where by Remark 4.2.1 we require that $m_\rho \cdot \gcd(\rho_i, \rho_j) \leq \delta_{ij} \cdot m_\sigma$, or $\pi_{ij}(\rho) = (0,0)$. It is called a *candidate* for $\mathrm{trop}(C)$ if it is an (i,j)-candidate for all $(i,j) \in P$. That is ρ is an (i,j)-candidate for $\mathrm{trop}(C)$ if it projects to $\mathcal{C}_{ij}$ and is a candidate for $\mathrm{trop}(C)$ if it projects to $\mathcal{C}_{ij}$ for all $(i,j) \in P$. In this language, the aim of this section is to find a set of all candidates for $\mathrm{trop}(C)$.

We partially reconstruct candidates for $\mathrm{trop}(C)$ by building up rays considering one new projection at a time. Suppose we have partially reconstructed a candidate ρ and we are considering a projected curve $\mathcal{C}_{ij}$. Then we are looking to find the points which agree with ρ on the coordinates already reconstructed and which are additionally (i,j)-candidates for $\mathrm{trop}(C)$.

Example 4.2.6. Suppose that we have partially reconstructed a candidate $\rho = (1,0,3,*,*,*) \in \mathbb{R}^6$ which has been reconstructed to coordinates x_1, x_2, x_3 where the $*$s in position x_4, x_5, x_6 indicate that these coordinates have yet to be reconstructed. Suppose that we are considering the projection π_{45} to coordinates x_4, x_5 with $\mathcal{C}_{45}$ having rays $(1,0), (0,1), (-1,-1)$. Then $(1,0,3,1,0,*)$, $(1,0,3,0,1,*)$ and $(1,0,3,-1,-1,*)$ all agree with ρ on coordinates x_1, x_2, x_3 and are additionally $(4,5)$-candidates for the tropical curve. ◇

The multiplicity bound (4.3) also gives a bound on the different rays which we can reconstruct. However, as we see in the following example, where the projection map has large degree, or the ray in the projected tropical curve has large multiplicity, we can combine these rays multiple times.

Example 4.2.7. As in Example 4.2.6, suppose that we have a partially reconstructed a candidate $\rho = (1,0,3,*,*,*) \in \mathbb{R}^6$ which has been reconstructed to coordinates x_1, x_2, x_3 and that we are considering the projection π_{45} to coordinates x_4, x_5. Suppose that ray $\sigma = (-1,-1)$ is in $\mathcal{C}_{45}$ which we are attempting to combine with ρ. If σ has a high multiplicity, then we can combine it with ρ in multiple ways. For example, suppose that $m_\sigma = 5$, then we can combine ρ with σ to form the partially reconstructed candidate $(1,0,3,-1,-1,*)$, ρ with 2σ to form the partially reconstructed candidate $(1,0,3,-2,-2,*)$, ρ with 3σ to form the partially reconstructed candidate $(1,0,3,-3,-3,*)$, ρ with 4σ to form the partially reconstructed candidate $(1,0,3,-4,-4,*)$ and ρ with 5σ to form the partially reconstructed candidate $(1,0,3,-5,-5,*)$. These are all valid $(4,5)$-candidates which extend ρ on the coordinates already reconstructed. ◇

For a fixed partially reconstructed candidate ρ, consider the projected tropical curve $\mathcal{C}_{ij}$ so that we are looking for those rays which agree with ρ on the coordinates already reconstructed and which are additionally (i,j)-candidates for $\mathrm{trop}(C)$. How we extend ρ with the rays of $\mathcal{C}_{ij}$ depends on i and j, and on whether ρ has been partially reconstructed to x_i or x_j or both or neither. There are three cases that we need to consider.

Projections of Type 0. In this case, x_i and x_j have already been reconstructed in the partially reconstructed candidate ρ. We then have to check to see if ρ is an (i,j)-candidate for $\mathrm{trop}(C)$. If not, then it is discarded as it not possible to extend to a candidate for $\mathrm{trop}(C)$ whilst also agreeing on the coordinates already reconstructed.

Projections of Type 1. In this case, only one of the x_i and x_j have been reconstructed in the partially reconstructed candidate ρ. After relabelling coordinates if necessary, we can assume that the x_i-coordinate has been reconstructed in the partially reconstructed candidate ρ and that the x_j-coordinate has not. Let ρ_i be the x_i-coordinate of ρ. This case splits into two subcases.

Projections of Type 1(a). In this subcase, ρ_i is non-zero. Here, we extend ρ by combining with the rays of $\mathcal{C}_{ij}$ whose x_i-coordinate has the same sign as ρ_i. For example, consider again the partially reconstructed candidate $\rho = (1,0,3,*,*,*)$ from Example 4.2.6 which has been partially reconstructed to coordinates x_1, x_2, x_3 and suppose that this time we are considering the projected tropical curve $\mathcal{C}_{14}$ to coordinates x_1, x_4 which has rays $(1,0), (0,1), (-1,-1)$. Then we can only combine ρ with the ray $(1,0)$ to form $(1,0,3,0,*,*)$ which is also a $(1,4)$-candidate which extends ρ.

Projections of Type 1(b). In this subcase, ρ_i is zero. Here, we extend ρ by combining with the rays of $\mathcal{C}_{ij}$ whose x_i-coordinate is also zero. For example, consider again the partially reconstructed candidate $\rho = (1,0,3,*,*,*)$ from Example 4.2.6 which has been partially reconstructed to coordinates x_1, x_2, x_3 and suppose that this time we are considering the projected tropical curve $\mathcal{C}_{24}$ to coordinates x_2, x_4 which has rays $(1,0), (0,1), (-1,-1)$ this time all with multiplicity two. Then we can combine ρ only with the ray $(0,1)$. Notice that the ray $(0,1)$ has multiplicity two which means that we can think of it as the ray $(0,1)$ with multiplicity two or as the ray $(0,2)$ with multiplicity one. In this way ρ can be combined with $(0,1)$ to form two new partially reconstructed candidates $(1,0,3,1,*,*)$ and $(1,0,3,2,*,*)$ which are both $(2,4)$-candidates which extend ρ.

Projections of Type 2. In this case, both x_i and x_j have not been reconstructed in the partially reconstructed candidate ρ. Thus we can extend ρ with each ray of $\mathcal{C}_{ij}$. We saw this case in Example 4.2.6.

The numbering of these projections refers to the number of new coordinates that they would add to the partially reconstructed candidate. That is, a projection of Type 0 would not add any new coordinates, a projection of Type 1 would add one new coordinate and a projection of Type 2 would add two new coordinates.

We now explain the algorithm which allows the reconstruction of all candidates given a set of two-dimensional coordinate projections. We first set up some notation.

Consider a partially reconstructed candidate ρ which has been partially reconstructed to coordinates $x_1, \dots, x_r$ where we have considered projections indexed by a set P. To each such ρ we associate a multiplicity bound m_ρ. This encodes the greatest multiplicity with which the partially reconstructed candidate ρ can occur in order to still be an (i,j)-candidate for all $(i,j) \in P$. So if the partially reconstructed candidate $\rho = (1,0,3,*,*,*)$ from Example 4.2.6 occurred with multiplicity bound $m_\rho = 2$ then $2\rho = (2,0,6,*,*,*)$ is an equally valid partially reconstructed candidate that we need to consider. Suppose as in Example 4.2.6 that we are combining ρ with the ray $(1,0)$ with multiplicity 1 of the projected tropical curve $\mathcal{C}_{45}$. Then ρ can be extended to form $(1,0,3,1,0,*)$ and $(2,0,3,1,0,*)$ both with multiplicity bound 1 which both agree with ρ on the coordinates already reconstructed and are additionally $(4,5)$-candidates for $\text{trop}(C)$.

We also encode each partially reconstructed candidate ρ with an index set Q_ρ which indexes the projections that are still remaining to consider in the reconstruction of the partially reconstructed candidate, and index set Y_ρ which indexes the coordinate variables which have already been reconstructed in ρ.

The following algorithm reconstructs all candidates for $\text{trop}(C)$ from a set of projected tropical curves indexed by elements of the set P. The 'if' loop takes a partially reconstructed candidate and extends it by combining it with the rays of a projected tropical curve which we have not yet considered. How the partially reconstructed candidate is extended depends on the type of the projection we are adding. After considering all projected tropical curves in P, we have then recovered a candidate for $\text{trop}(C)$. As another piece of notation, we let $* + a = a$ for all a.

Algorithm 4.2.8. Input: An index set P which indexes the projected tropical curves $\{\mathcal{C}_{ij} : (i,j) \in P\}$ where each $\mathcal{C}_{ij}$ comes with a positive integer δ_{ij}.

Output: The set T of all candidates for $\text{trop}(C)$.

Initialisation: $T = \emptyset$, $\mathcal{S} = \{\rho = (*, \dots, *)\}$ where $m_\rho = \infty$, $Q_\rho = P$ and $Y_\rho = \emptyset$.

While $\mathcal{S} \neq \emptyset$ do:

I. Choose any $\rho \in S$. Set $\mathcal{S} = \mathcal{S} - \{\rho\}$

II. If $Q_\rho = \emptyset$ and $\rho \neq (0, \ldots, 0)$ then $T = T \cup \{\rho\}$. If $Q_\rho = \emptyset$ and $\rho = (0, \ldots, 0)$ then $T = T$. Otherwise do:

(a) Choose any $(i, j) \in Q_\rho$.

(b) Updating $\mathcal{S}$. This depends on $\{i, j\} \cap Y_\rho$.

1. **Case 0: If $|\{i, j\} \cap Y_\rho| = 2$ then π_{ij} is a projection of Type 0.**

   (i) If $(\rho_i / \gcd(\rho_i, \rho_j), \rho_j / \gcd(\rho_i, \rho_j))$ is equal to a ray σ of $\mathcal{C}_{ij}$ where in addition $\gcd(\rho_i, \rho_j) \leq \delta_{ij} m_\sigma$, then we define $\theta = \rho$. Set $m_\theta = \min(m_\rho, \lfloor \delta_{ij} m_\sigma / \gcd(\rho_i, \rho_j) \rfloor)$, $Q_\theta = Q_\rho - \{(i, j)\}$ and $Y_\theta = Y_\rho$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$

   (ii) If $(\rho_i, \rho_j) = (0, 0)$ then set $\mathcal{S} = \mathcal{S} \cup \{\rho\}$ where $m_\rho = m_\rho$, $Q_\rho = Q_\rho - \{(i, j)\}$ and $Y_\rho = Y_\rho$.

2. **Case 1(a): If $|\{i, j\} \cap Y_\rho| = 1$ and $\rho_i \neq 0$, where after relabelling if necessary, we assume that $\{i, j\} \cap Y_\rho = \{i\}$, then π_{ij} is a projection of Type 1(a).**

   This splits into two subcases depending on the sign of ρ_i.

   (i) If $\rho_i > 0$ then set $U = \{\sigma : \sigma \text{ is a ray of } \mathcal{C}_{ij} \text{ and } \sigma_i > 0\}$. For all $\sigma \in U$ with multiplicity m_σ define $a := \operatorname{lcm}(\rho_i, \sigma_i)/\rho_i$ and $b := \operatorname{lcm}(\rho_i, \sigma_i)/\sigma_i$. Then when $a \leq m_\rho$ and $b \leq \delta_{ij} m_\sigma$, define $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = a\rho_l$ for $l \neq j$ and $\theta_j = b\sigma_j$. Set $m_\theta = \lfloor \min(m_\rho/a, \delta_{ij} m_\sigma / b) \rfloor$, $Q_\theta = Q_\rho - \{(i, j)\}$ and $Y_\theta = Y_\rho \cup \{j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$

   (ii) If $\rho_i < 0$ then set $U = \{\sigma : \sigma \text{ is a ray of } \mathcal{C}_{ij} \text{ and } \sigma_i < 0\}$. For all $\sigma \in U$ with multiplicity m_σ define $a := \operatorname{lcm}(\rho_i, \sigma_i)/|\rho_i|$ and $b := \operatorname{lcm}(\rho_i, \sigma_i)/|\sigma_i|$. Then when $a \leq m_\rho$ and $b \leq \delta_{ij} m_\sigma$, let $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = a\rho_l$ for $l \neq j$ and $\theta_j = b\sigma_j$. Set $m_\theta = \lfloor \min(m_\rho/a, \delta_{ij} m_\sigma / b) \rfloor$, $Q_\theta = Q_\rho - \{(i, j)\}$ and $Y_\theta = Y_\rho \cup \{j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$.

3. **Case 1(b): If $|\{i, j\} \cap Y_\rho| = 1$ and $\rho_i = 0$, where after relabelling if necessary, we assume that $\{i, j\} \cap Y_\rho = \{i\}$, then π_{ij} is a projection of Type 1(b).**

   (i) Set $U = \{\sigma : \sigma \text{ a ray of } \mathcal{C}_{ij} \text{ and } \sigma_i = 0\}$. For all $\sigma \in U$ with multiplicity m_σ let $J = \{(u, v) : 1 \leq u \leq m_\rho, 1 \leq v \leq \delta_{ij} m_\sigma, \gcd(u, v) = 1\}$. Then for all $(u, v) \in J$ let $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = u\rho_l$ for

$l \neq j$ and $\theta_j = v\sigma_j$. Set $m_\theta = \lfloor \min(m_\rho/u, \delta_{ij} m_\sigma/v) \rfloor$, $Q_\theta = Q_\rho - \{(i,j)\}$ and $Y_\rho = Y_\rho \cup \{j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$.

(ii) Let $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = \rho_l$ for $l \neq j$ and $\theta_j = 0$. Set $m_\theta = m_\rho$, $Q_\theta = Q_\rho - \{(i,j)\}$ and $Y_\theta = Y_\rho \cup \{j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$.

4. **Case 2: If** $|\{i,j\} \cap Y_\rho| = 0$ **then** π_{ij} **is a projection of Type 2.**

(i) Let U be the set of rays of $\mathcal{C}_{ij}$. For all $\sigma \in U$ with corresponding multiplicity m_σ let $J = \{(u,v) : 1 \leq u \leq m_\rho, 1 \leq v \leq \delta_{ij} m_\sigma, \gcd(u,v) = 1\}$. Then for all $(u,v) \in J$ let $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = u\rho_l$ for $l \neq i,j$ and $\theta_i = v\sigma_i$ and $\theta_j = v\sigma_j$. Set $m_\theta = \lfloor \min(m_\rho/u, \delta_{ij} m_\sigma/v) \rfloor$, $Q_\theta = Q_\rho - \{(i,j)\}$ and $Y_\theta = Y_\rho \cup \{i,j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$

(ii) Let $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = \rho_l$ for $l \neq i,j$ and $\theta_i = \theta_j = 0$. Set $m_\theta = m_\rho$, $Q_\theta = Q_\rho - \{(i,j)\}$ and $Y_\theta = Y_\rho \cup \{i,j\}$. Let $\mathcal{S} = \mathcal{S} \cup \{\theta\}$.

Return T.

The steps 1(ii), 3(ii) and 4(ii) of Algorithm 4.2.8 are the situations where the partially reconstructed candidate ρ does not project to a one-dimensional ray of $\mathcal{C}_{ij}$, but it projects to the origin. For example, consider the partially reconstructed candidate $\rho = (1, 0, 3, *, *, *)$ from Example 4.2.6 which has been partially reconstructed to coordinates x_1, x_2, x_3 and suppose that we are considering the projected tropical curve $\mathcal{C}_{45}$ to coordinates x_4, x_5. Then Step 4(i) deals with combining ρ with the rays of $\mathcal{C}_{45}$. However, we need to deal with the case where ρ projects to the point $(0,0)$ in $\mathcal{C}_{45}$. Thus, we form the partially reconstructed candidate $(1, 0, 3, 0, 0, *)$ which is a $(4,5)$-candidate (as it projects to the point $(0,0) \in \mathcal{C}_{45}$) which extends ρ.

Before proving that this algorithm does give the desired result, we first consider an example of constructing a one-dimensional fan in $\mathbb{R}^3$ from coordinate projections.

Example 4.2.9. Suppose that we have a projected tropical curve to coordinates x_1, x_2 with rays $\{(1,0), (0,1), (1,2), (-2,-3)\}$ all with multiplicity 1, to coordinates x_1, x_3 with rays $\{(1,3), (1,-1), (0,-1), (-1,0)\}$ with multiplicities $1, 1, 2, 2$ respectively, and to coordinates x_2, x_3 with rays $\{(0,1), (-1,0), (2,-1), (1,-2)\}$ with multiplicities $3, 3, 1, 1$ respectively. In all cases, the degree of the projection map is 1. In the language of Algorithm 4.2.8 we have $P = \{(1,2), (1,3), (2,3)\}$ and $\delta_{12} = \delta_{13} = \delta_{23} = 1$. Initialising, we have $T = \emptyset$, $\mathcal{S} = \{\rho\}$ where $\rho = (*, *, *)$, $m_\rho = \infty$ and $Q_\rho = P$.

1. Choose $\rho = (*, *, *)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = P$, we choose $(1,2) \in Q_\rho$ where π_{12} is a projection of Type 2 for ρ so we are in Case 2. We can combine

ρ with every ray of $\mathcal{C}_{12}$ and so $S = S \cup \{(1,0,*), (0,1,*), (1,2,*), (-2,-3,*)\}$ with multiplicity bound $1 = \lfloor \min(\infty, 1 \cdot 1/1) \rfloor$ and $Q = \{(1,3),(2,3)\}$ for each of these new partially reconstructed candidates added to $\mathcal{S}$. Adding the rays which extend ρ and project to the point $(0,0)$ in $\mathcal{C}_{12}$, we set $\mathcal{S} = \mathcal{S} \cup \{(0,0,*)\}$ with multiplicity bound ∞ and $Q = \{(1,3),(2,3)\}$ for the new partially reconstructed candidate added to $\mathcal{S}$

2. Choose $\rho = (1,0,*)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1,3),(2,3)\}$, we choose $(1,3) \in Q_\rho$ where π_{13} is a projection of Type 1(a) for ρ so we are in Case 1(a). We can combine ρ with the rays $(1,3)$ and $(1,-1)$ of $\mathcal{C}_{13}$ and so $\mathcal{S} = \mathcal{S} \cup \{(1,0,3),(1,0,-1)\}$ with multiplicity bound $1 = \lfloor \min(1/1, 1 \cdot 1/1) \rfloor$ and $Q = \{(2,3)\}$ for each of these new partially reconstructed candidates added to $\mathcal{S}$.

3. Choose $\rho = (1,0,3)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(2,3)\}$, we choose $(2,3) \in Q_\rho$ where π_{23} is a projection of Type 0 for ρ so we are in Case 0. Then as $\pi_{23}(\rho) \in \mathcal{C}_{23}$ and $Q_\rho = \emptyset$, we set $T = T \cup \{(1,0,3)\}$.

4. Choose $\rho = (1,0,-1)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(2,3)\}$, we choose $(2,3) \in Q_\rho$ where π_{23} is a projection of Type 0 for ρ so we are in Case 0. But as $\pi_{23}(\rho) \notin \mathcal{C}_{23}$ we are done.

5. Choose $\rho = (0,1,*)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1,3),(2,3)\}$, we decide to choose $(2,3) \in Q_\rho$ where π_{23} is a projection of Type 1(a) for ρ so we are in Case 1(a). We explain why it is a sensible choice to choose $(2,3)$ instead of $(1,3)$, as we did in previous steps, in Section 4.4. We can combine ρ with the ray $(1,-2)$ of $\mathcal{C}_{23}$ and so $\mathcal{S} = \mathcal{S} \cup \{(0,1,-2)\}$ with multiplicity bound $1 = \lfloor \min(1/1, 1 \cdot 1/1) \rfloor$ and $Q = \{(2,3)\}$ for the new partially reconstructed candidates added to $\mathcal{S}$.

6. Choose $\rho = (0,1,-2)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1,3)\}$, we choose $(1,3) \in Q_\rho$ where π_{13} is a projection of Type 0 for ρ so we are in Case 0. Then as $\pi_{13}(\rho) \in \mathcal{C}_{13}$ and $Q_\rho = \emptyset$, we set $T = T \cup \{(0,1,-2)\}$.

7. Choose $\rho = (1,2,*)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1,3),(2,3)\}$, we choose $(2,3) \in Q_\rho$ where π_{23} is a projection of Type 1(a) for ρ so we are in Case 1(a). We can combine ρ with the ray $(2,-1)$ of $\mathcal{C}_{23}$ and so $\mathcal{S} = \mathcal{S} \cup \{(1,2,-1)\}$ with multiplicity bound $1 = \lfloor \min(1/1, 1 \cdot 1/1) \rfloor$ and $Q = \{(1,3)\}$ for the new partially reconstructed candidates added to $\mathcal{S}$. Note that we cannot combine ρ with the ray $(1,-2)$ of the projected tropical curve $\mathcal{C}_{24}$ as $b = \text{lcm}(1,2)/1 =$

$2 < 1 = m_{(1,-2)}$. We would be able to perform such a combination if $(1,-2)$ has multiplicity greater than one.

8. Choose $\rho = (1, 2, -1)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1, 3)\}$, we choose $(1, 3) \in Q_\rho$ where π_{13} is a projection of Type 0 for ρ so we are in Case 0. Then as $\pi_{13}(\rho) \in \mathcal{C}_{13}$ and $Q_\rho = \emptyset$, we set $T = T \cup \{(1, 2, -1)\}$.

9. Choose $\rho = (-2, -3, *)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1, 3), (2, 3)\}$, we choose $(1, 3) \in Q_\rho$ where π_{13} is a projection of Type 1(a) for ρ so we are in Case 1(a). We can combine ρ with the ray $(-1, 0)$ of $\mathcal{C}_{13}$ as it occurs with multiplicity 2, and so $\mathcal{S} = \mathcal{S} \cup \{(-2, -3, 0)\}$ with multiplicity bound $1 = \lfloor \min(1/1, 1 \cdot 2/2) \rfloor$ and $Q = \{(2, 3)\}$ for the new partially reconstructed candidates added to $\mathcal{S}$.

10. Choose $\rho = (-2, -3, 0)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(2, 3)\}$, we choose $(2, 3) \in Q_\rho$ where π_{23} is a projection of Type 0 for ρ so we are in Case 0. Then as $\pi_{23}(\rho) \in \mathcal{C}_{23}$ and $Q_\rho = \emptyset$, we set $T = T \cup \{(-2, -3, 0)\}$.

11. Choose $\rho = (0, 0, *)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(1, 3), (2, 3)\}$, we choose $(1, 3) \in Q_\rho$ where π_{13} is a projection of Type 1(b) for ρ so we are in Case 1(b). We can combine ρ with the ray $(0, -1)$ of $\mathcal{C}_{13}$ and so $\mathcal{S} = \mathcal{S} \cup \{(0, 0, -1)\}$ with multiplicity bound $2 = \lfloor \min(\infty, 2 \cdot 1/1) \rfloor$ and $Q = \{(2, 3)\}$ for the new partially reconstructed candidate added to $\mathcal{S}$. We also have to add the rays which extend ρ and project to the point $(0, 0)$ in $\mathcal{C}_{13}$. We set $\mathcal{S} = \mathcal{S} \cup \{(0, 0, 0)\}$ with multiplicity bound ∞ and $Q = \{(2, 3)\}$ for the new partially reconstructed candidate added to $\mathcal{S}$.

12. Choose $\rho = (0, 0, -1)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(2, 3)\}$, we choose $(2, 3) \in Q_\rho$ where π_{23} is a projection of Type 0 for ρ so we are in Case 0. But as $\pi_{23}(\rho) \notin \mathcal{C}_{23}$ we are done.

13. Choose $\rho = (0, 0, 0)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \{(2, 3)\}$, we choose $(2, 3) \in Q_\rho$ where π_{23} is a projection of Type 0 for ρ so we are in Case 0. But as $\pi_{23}(\rho) = (0, 0)$ so we set $S = S \cup \{(0, 0, 0)\}$ with multiplicity bound ∞ and $Q = \emptyset$ for the new partially reconstructed candidate added to $\mathcal{S}$.

14. Choose $\rho = (0, 0, 0)$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$. As $Q_\rho = \emptyset$ we are done.

Now as $\mathcal{S} = \emptyset$, we output $T = \{(1, 0, 3), (0, 1, -2), (1, 2, -1), (-2, -3, 0)\}$. ◇

Proof of Algorithm 4.2.8. At each pass through the algorithm, we replace a partially reconstructed candidate ρ with a finite collection of partially reconstructed candidates $\theta_1, \ldots, \theta_s$. Thus, at each stage the set $\mathcal{S}$ is finite. As there are only finitely many projections in P, we only have finitely many projections of Type 0. If we consider a projection of Type 0, then $|Y_{\theta_i}| = |Y_\rho|$ and the set of projections of Type 0 cannot increase. Thus at some point we will have to add a projection of Type 1(a), 1(b) or 2. If we consider a projection of Type 1(a), 1(b) or 2, then $|Y_{\theta_i}| > |Y_\rho|$. As $|Y_\rho|$ cannot increase indefinitely (it is bounded above by the number of variables n), at some point of the algorithm we will only have projections of Type 0 left to add. For each remaining candidate $\rho \in \mathcal{S}$, it now remains to check those finite number of projections in $(i, j) \in Q_\rho$ to see if ρ is an (i, j)-candidate for trop(C). We have a finite number of projections to check for a finite number of rays and termination follows.

For correctness, we need to show that each ray in T is indeed a candidate for trop(C). To do this, we show that after considering projected tropical curve $\mathcal{C}_{ij}$ the newly added rays are (i, j)-candidates for trop(C) as well as still being (k, l)-candidates for all the tropical curves $\mathcal{C}_{kl}$ already considered. We consider each case separately, corresponding to the different types of projections that we are adding. In each case, we need to check that the rays added to $\mathcal{S}$ are (i, j)-candidates for trop(C) and that they remain (k, l)-candidates for trop(C) for all projected tropical curves $\mathcal{C}_{kl}$ which we have already considered at some previous stage.

In Case 0, we are considering projections of Type 0 where both x_i and x_j are already reconstructed in the partially reconstructed candidate ρ. When ρ is not deleted from S, we verify first that ρ is an (i, j)-candidate for trop(C). As $(\rho_i/\gcd(\rho_i, \rho_j), \rho_j/\gcd(\rho_i, \rho_j))$ is equal to a ray σ of $\mathcal{C}_{ij}$ it follows that $\pi_{ij}(\rho)$ generates a ray of $\mathcal{C}_{ij}$. From Remark 4.2.1, it follows that the multiplicity bound has maximal possible value $\left\lfloor \frac{\delta_{ij} \cdot m_\sigma}{\gcd(\rho_i, \rho_j)} \right\rfloor$ in order to still be an (i, j)-candidate. In order to remain a (k, l)-candidate for all previously considered $(k, l) \in P$, we require the multiplicity bound also to be smaller than m_ρ.

In Case 1(a), we are considering projections of Type 1(a) where only one of x_i and x_j are reconstructed in the partially reconstructed candidate ρ. We assume that after relabelling if necessary that x_i has been reconstructed and that x_j has not been reconstructed. In Case 1(a), we additionally have that ρ_i is non-zero. We consider only the case where $\rho_i > 0$ as the other case is analogous. In order to be able to extend ρ using σ, we require that their x_i coordinates are equal. For $a = \mathrm{lcm}(\rho_i, \sigma_i)/\rho_i$ and $b = \mathrm{lcm}(\rho_i, \sigma_i)/\sigma_i$ we have that $(a\rho)_i = (b\sigma)_i$ and so we can combine ρ and σ to get new ray $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = a\rho_l$ for $l \neq j$ and $\theta_j = b\sigma_j$. The factors of a and b serve to increase the corresponding lattice indices, and so from Remark 4.2.1, the multiplicity bound has maximal possible value $\left\lfloor \frac{\delta_{ij} \cdot m_\sigma}{b} \right\rfloor$ in order to still be an (i, j)-candidate. In order to remain a (k, l)-candidate for all previously considered $(k, l) \in P$, again from Remark 4.2.1 the multiplicity bound has maximum possible value $\lfloor \frac{m_\rho}{a} \rfloor$.

In Case 1(b), we are considering projections of Type 1(b) where only one of x_i and x_j are reconstructed in the partially reconstructed candidate ρ. We assume that after relabelling if necessary that x_i has been reconstructed and that x_j has not been reconstructed. In Case 1(b), we additionally have that ρ_i is equal to zero. In order to be able to extend ρ using σ, we require that σ_i is also zero. In such a case, for all $1 \leq u \leq m_\rho$ and $1 \leq v \leq \delta_{ij} m_\sigma$ we can form $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = u\rho_l$ for $l \neq j$ and $\theta_j = v\sigma_j$. The factors of u and v serve to increase the corresponding lattice indices, and so from Remark 4.2.1, the multiplicity bound has maximal possible value $\left\lfloor \frac{\delta_{ij} \cdot m_\sigma}{v} \right\rfloor$ in order to still be an (i, j)-candidate. In order to remain a (k, l)-candidate for all previously considered $(k, l) \in P$, again from Remark 4.2.1, the multiplicity bound has maximum possible value $\lfloor \frac{m_\rho}{v} \rfloor$.

In Case 2, we are considering projections of Type 2 where both of x_i and x_j have not been reconstructed in the partially reconstructed candidate ρ. For any $\sigma \in \mathcal{C}_{ij}$ for all $1 \leq u \leq m_\rho$ and $1 \leq v \leq \delta_{ij} m_\sigma$ we can form $\theta = (\theta_1, \ldots, \theta_n)$ where $\theta_l = u\rho_l$ for $l \neq i, j$ and $\theta_i = v\sigma_i$ and $\theta_j = v\sigma_j$. The factors of u and v serve to increase the corresponding lattice indices, and so from Remark 4.2.1, the multiplicity bound has maximal possible value $\left\lfloor \frac{\delta_{ij} \cdot m_\sigma}{v} \right\rfloor$ in order to still be an (i, j)-candidate. In order to remain a (k, l)-candidate for all previously considered $(k, l) \in P$, again from Remark 4.2.1 the multiplicity bound has maximum possible value $\lfloor \frac{m_\rho}{v} \rfloor$.

It remains to show that the algorithm recovers a superset of the rays of trop(C). That is, all the rays of trop(C) are also candidates for trop(C) and so will be recovered by the algorithm. Suppose not, and let σ be a non-zero ray of trop(C) which is not a candidate for trop(C) and so is not reconstructed by the algorithm. This means that there is some $(i, j) \in P$ such that $\pi_{ij}(\sigma)$ is not in the tropical curve $\mathcal{C}_{ij}$. However, this contradicts that $\mathcal{C}_{ij}$ is the image of trop(C) under the projection map π_{ij} and so no such ray σ can exist. □

Lemma 4.2.10. *Let P be a set indexing coordinate projections $\pi_{ij} \colon \mathbb{R}^n_{x_1,\ldots x_n} \to \mathbb{R}^2_{x_i,x_j}$ whose image is the tropical curve $\mathcal{C}_{ij}$ and $P' = \{i : \exists j$ such that $(i, j) \in P$ or $(j, i) \in P\}$. Then the set of points in the pre-image of these projections forms a one-dimensional fan in $\mathbb{R}^n$ if and only if $P' = \{1, \ldots, n\}$.*

Proof. Suppose that $P' = \{1, \ldots, n\}$. Thus for each coordinate x_i for $1 \leq i \leq n$ we can find a projection in P projecting to x_i. Using the set P and the associated projected tropical curves as input to Algorithm 4.2.8, we output a set of candidates which all project to the $\mathcal{C}_{ij}$ for all $(i, j) \in P$. Thus they are all contained in the pre-images of all these projections. Further, as we have projections in P which projects to each coordinate $x_1, \ldots, x_n$, each candidate has been reconstructed in all coordinates and so spans a one-dimensional ray in $\mathbb{R}^n$. Thus they form a one-dimensional fan in $\mathbb{R}^n$.

Now suppose that P indexes projections where $P' \neq \{1, \ldots, n\}$. After relabelling the variables if necessary, we can assume that we do not have any projections to x_n. Using the set P and the associated projected tropical curves as input to Algorithm 4.2.8, we output a set S of candidates which project to $\mathcal{C}_{ij}$ for all $(i, j) \in P$. By the first part of the lemma, this is a one-dimensional fan in the coordinates in P'. However, as there is no projection in P to x_n, this coordinate is not reconstructed and so we can set the x_n-coordinate to be any value in $\mathbb{R}$ and the ray would still project to all projected tropical curves indexed by P. In particular, this ray is not one-dimensional. □

4.2.3 Finding the tropical curve

In the previous section we saw how given a set of coordinate projections of a tropical curve $\text{trop}(C)$ onto all variables $x_1, \ldots, x_n$, we can find a set T containing all one-dimensional candidates for $\text{trop}(C)$. In this section we determine how to find which of these candidates are actual rays of $\text{trop}(C)$. Recall from the Fundamental Theorem 2.1.5 that $w \in \mathbb{R}^n$ is in the tropical curve $\text{trop}(C)$ if and only if $\text{in}_w(I) \neq \langle 1 \rangle$. A naïve algorithm would be to compute $\text{in}_\rho(I)$ for all $\rho \in T$ and note that ρ is in $\text{trop}(C)$ if and only if $\text{in}_\rho(I) \neq \langle 1 \rangle$. However, even though this gives us a solution to the problem of finding the rays of $\text{trop}(C)$, in practice we may have to compute many Gröbner bases. This turns out not to be ideal as many of these Gröbner bases can be difficult to compute. Instead, we try to determine the rays of $\text{trop}(C)$ from this list of candidates by computing as few Gröbner bases as possible.

A natural question to ask is: If we use all the coordinate projections, are all the candidates we have recovered actually rays in the tropical curve $\text{trop}(C)$? This would mean that in this case, this step of the reconstruction procedure would be trivial. It turns out that this is unfortunately not true. The following example gives two different tropical curves both of which have the same coordinate projections to two-dimensional planes. This example demonstrates how coordinate projections are in general not generic enough to apply the results of Bieri and Groves [1984] and Hept and Theobald [2009].

Example 4.2.11. Recall from Example 4.1.1 that the tropicalisation of the curve defined by $I = \langle xz + 4yz - z^2 + 3xw - 12yw + 5zw, xy - 4y^2 + yz + xw + 2yw -$

$zw, x^2 - 16y^2 + 8yz - z^2 + 14xw - 8yw + 2zw\rangle \subseteq \mathbb{C}[x, y, z, w]$ is a one-dimensional fan in $\mathbb{R}^3$ with four rays spanned by

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).$$

Consider now the curve defined by $J = \langle xy - 3xz + 3yz - w^2, 3xz^2 - 12yz^2 + xzw + 4yzw + 5zw^2 - w^3, 4y^2z - 9yz^2 + 2yzw - yw^2 + 4zw^2, x^2z - 36yz^2 + 11xzw + 12yzw - xw^2 + 16zw^2 - 3w^3\rangle \subseteq \mathbb{C}[x, y, z, w]$. This defines a tropical curve $\text{trop}(V(J))$ with four rays spanned by

$$(1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1).$$

Then as shown in Figure 4.6, $\text{trop}(V(I))$ and $\text{trop}(V(J))$ have the same coordinate projections to all three planes given by $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. We see that the rays of $\text{trop}(V(I))$ and $\text{trop}(V(J))$ correspond to rays passing through the alternate vertices of the cube with vertices at $\{(\pm 1, \pm 1, \pm 1)\}$. The fan for $\text{trop}(V(I))$ is given by the red rays and the fan for $\text{trop}(V(J))$ given by the blue rays. This means that given these projections as input into Algorithm 4.2.8 we would recover both the rays for $\text{trop}(V(I))$ and $\text{trop}(V(J))$ as candidates. $\diamond$

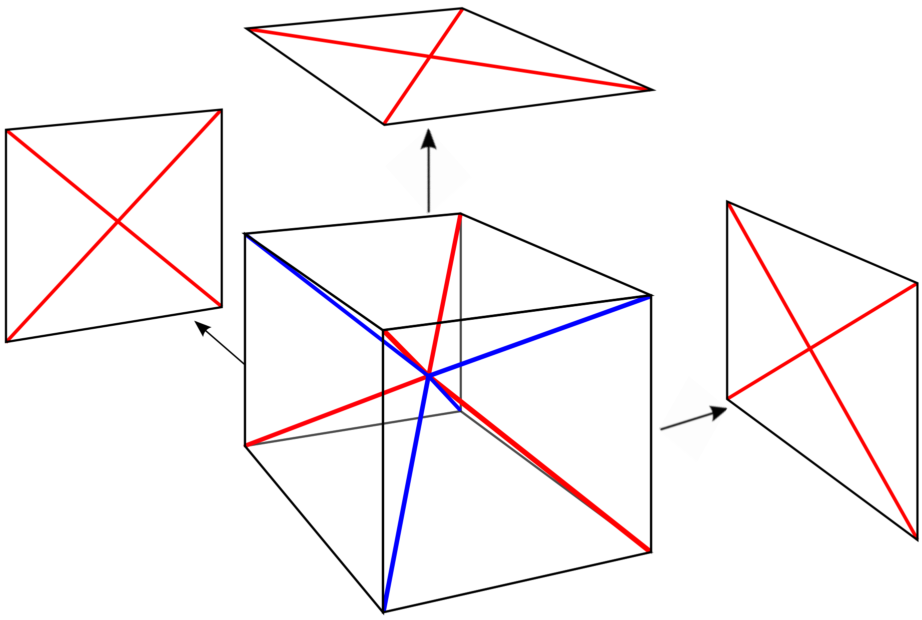

Figure 4.6: Two tropical curves in $\mathbb{R}^3$ with the same coordinate projections

Suppose that after passing through Algorithm 4.2.8 we have a set $T = \{\rho_1, \dots, \rho_s\}$ of candidates for $\text{trop}(C)$. We want to find non-negative integers

$m_1, \dots, m_s$ such that ρ_i appears in trop(C) with multiplicity m_i and if ρ_i does not appear in trop(C) then m_i equals zero. The idea is to use conditions coming from the projection multiplicity equation (4.2) and balancing. We illustrate this by continuing Example 4.2.11.

Example 4.2.12. Continuing Example 4.2.11, suppose that we are attempting to reconstruct trop$(V(I))$ and so after passing through Algorithm 4.2.8 we have a set of candidates $T = \{\rho_1 = (1,1,1), \rho_2 = (-1,1,1), \rho_3 = (1,-1,1), \rho_4 = (1,1,-1), \rho_5 = (-1,-1,1), \rho_6 = (-1,1,-1), \rho_7 = (1,-1,-1), \rho_8 = (-1,-1,-1)\}$, We are trying to find the non-negative multiplicities m_i with which they occur in trop$(V(I))$. We show that we can find these multiplicities by computing only one additional Gröbner basis. We compute a Gröbner basis for $\rho_3 = (1,-1,1)$ and see that $\text{in}_{\rho_3}(I)$ contains a monomial and so ρ_3 cannot be in our tropical curve. Consider the projection to $\{x = 0\}$ and we see that now the only ray projecting onto $(-1,1)$ is $\rho_5 = (-1,-1,1)$ so this must be a ray of the tropical variety with multiplicity 1, equal to the multiplicity of the projected ray as the projection map here has degree 1. Using similar arguments on the projections to $\{y = 0\}$ and $\{z = 0\}$ we see that $\rho_1 = (1,1,1)$ and $\rho_7 = (1,-1,-1)$ must both live in the tropical curve. Looking again at the projection to $\{x = 0\}$ we see that rays $\rho_1 = (1,1,1)$ and $\rho_2 = (-1,1,1)$ both project to the ray $(1,1)$. The multiplicity equation 4.2 tells us that $m_1+m_2 = 1$. But as we already know that $m_1 = 1$, we deduce that $m_2 = 0$. We can similarly exclude the rays ρ_4 and ρ_8. $\diamond$

The idea in general is to find and solve a system of equations in the unknowns m_i, from the projection multiplicity equation (4.2), the balancing condition, and from the degree of the tropical curve. We see the equations we get in general by first considering the balancing condition. If ρ_i is a candidate for trop(C), then let $\rho_{i,j}$ denote the x_j-coordinate of ρ_i. Then, the balancing condition says that for all $1 \leq j \leq n$ we have

$$\sum_{k=1}^{s} m_k \rho_{k,j} = 0 \tag{4.4}$$

which gives n equations in the variables $m_1, \dots, m_s$. In the case of Example 4.2.11 this gives us three equations one for each coordinate of $\mathbb{R}^3$:

$$\begin{aligned} m_1 - m_2 + m_3 + m_4 - m_5 - m_6 + m_7 - m_8 = 0; \\ m_1 + m_2 - m_3 + m_4 - m_5 + m_6 - m_7 - m_8 = 0; \\ m_1 + m_2 + m_3 - m_4 + m_5 - m_6 - m_7 - m_8 = 0, \end{aligned} \tag{4.5}$$

for which there are clearly still infinitely many solutions.

So we secondly consider the equations from the projection multiplicity equation 4.2. Consider the projection π_{ij} to coordinates x_i, x_j where $\mathcal{C}_{ij}$ is the projected tropical curve in these coordinates and let σ be a primitive ray generator of a ray of $\mathcal{C}_{ij}$. Then its multiplicity is given by

$$m_\sigma = \frac{1}{\delta} \sum_{\rho_k \in \mathcal{S} : \sigma \subseteq \pi_{ij}(\rho_k)} m_k \cdot \gcd(\rho_{k,i}, \rho_{k,j}). \tag{4.6}$$

This gives one equation in terms of the variables $m_1, \dots, m_s$ for each ray of each projected tropical curve. Note however that these equations will in general not be linearly independent. Returning again to Example 4.2.11, we recover equations

$$\begin{aligned} &m_1 + m_2 = 1, m_3 + m_5 = 1, m_4 + m_6 = 1, m_7 + m_8 = 1; \\ &m_1 + m_3 = 1, m_2 + m_5 = 1, m_4 + m_7 = 1, m_6 + m_8 = 1; \\ &m_1 + m_4 = 1, m_2 + m_6 = 1, m_3 + m_7 = 1, m_5 + m_8 = 1, \end{aligned} \tag{4.7}$$

from the projection to $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ respectively.

Finally, we consider equations from the degree of the tropical curve. We saw in Theorem 2.6.1 that the algebraic curve and its tropicalisation have the same degree. Thus, as we know the degree of the algebraic curve, for example from its Hilbert polynomial, we also know the degree of the tropical curve. Further, we saw in Section 2.6 how we can compute the degree of the tropical curve combinatorially from tropical intersection theory. Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{Z}^n$ and $e_0 = -e_1 - \cdots - e_n$. If $\rho_1, \dots, \rho_r$ are the rays of $\operatorname{trop}(C)$ with multiplicity $m_1, \dots, m_r$ and for all $1 \leq i \leq r$ we have minimal decomposition

$$\rho_i = \sum_{j=0}^{n} a_{ij} e_j,$$

then for all $0 \leq j \leq n$, if $\operatorname{trop}(C)$ is of degree D, by (2.4) we have that

$$\sum_{i=1}^{r} a_{ij} m_i = D.$$

So if $\rho_1, \dots, \rho_s$ is our collection of candidates and we have minimal decompositions $\rho_i = \sum_{j=0}^{n} a_{ij} e_j$, then we have $n + 1$ equations for the degree

$$\sum_{i=1}^{s} a_{ij} m_i = D \tag{4.8}$$

one equations for each $0 \leq j \leq n$, in the variables $m_1, \ldots, m_s$. In the case of Example 4.2.11, for $e_0 = (-1,-1,-1)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ we have minimal decompositions:

$$\begin{aligned}
\rho_1 &= e_1 + e_2 + e_3;\\
\rho_2 &= e_0 + 2e_1 + 2e_3;\\
\rho_3 &= e_0 + 2e_1 + 2e_3;\\
\rho_4 &= e_0 + 2e_1 + 2e_2;\\
\rho_5 &= e_0 + 2e_3;\\
\rho_6 &= e_0 + 2e_2;\\
\rho_7 &= e_0 + 2e_1;\\
\rho_8 &= e_0.
\end{aligned}$$

As $V(I)$ has degree 3 in $\mathbb{P}^2$, we get the following 4 equations from computing the degree:

$$\begin{aligned}
m_2 + m_3 + m_4 + m_5 + m_6 + m_7 &= 3;\\
m_1 + 2m_3 + 2m_4 + 2m_7 &= 3;\\
m_1 + 2m_2 + 2m_4 + 2m_6 &= 3;\\
m_1 + 2m_2 + 2m_3 + 2m_5 &= 3.
\end{aligned} \tag{4.9}$$

Thus we have three linear systems of equations (4.4), (4.6) and (4.8) coming from the balancing condition, multiplicity equations and degree calculations respectively. Combining these sets of equations we then have a system of linear equations in the m_i to solve which can be written in matrix form $Am = b$ for some matrix A where we are solving for the unknowns $m = (m_1, \ldots, m_s)$. Returning again to Example 4.2.11, we combine the three linear system of equations (4.5), (4.7) and (4.9) and written in matrix form becomes

$$\begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}.$$

Reducing this by row operations, we get the equivalent system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

And so using the multiplicity equation, balancing condition and equations from the degree of the curve, we have found a unique solution

$$m = (1, 0, 0, 0, 1, 1, 1, 0),$$

which was obtained without having to compute any further initial ideals. We thus conclude that the tropical curve consists of one-dimensional rays $\rho_1 = (1,1,1)$, $\rho_5 = (-1,-1,1)$, $\rho_6 = (-1,1,-1)$ and $\rho_7 = (1,-1,-1)$ each of which appears with multiplicity one.

Remark 4.2.13. We saw in Example 4.2.12 that the balancing and multiplicity conditions alone do not distinguish these two fans which have the same projections to two-dimensional coordinate planes. However, they can be distinguished by considering degrees. This is because $V(I)$ has degree 3 but $V(J)$ has degree 4. In the first case, we add equations (4.9) to our matrix system and in the second case we add the equations:

$$\begin{aligned} m_2 + m_3 + m_4 + m_5 + m_6 + m_7 &= 4; \\ m_1 + 2m_3 + 2m_4 + 2m_7 &= 4; \\ m_1 + 2m_2 + 2m_4 + 2m_6 &= 4; \\ m_1 + 2m_2 + 2m_3 + 2m_5 &= 4, \end{aligned}$$

to our matrix system. In each case, these extra equations differentiate the two tropical curves.

However, there are further examples where the cube configuration is not centred on the origin for which both tropical curves have the same projections to two-dimensional coordinate planes and both curves have the same degree. For example, consider the two fans Σ_1, Σ_2 in $\mathbb{R}^3$ where Σ_1 has rays generated by rays

$$(1,1,1), (1,0,0), (0,1,0), (0,0,1), (-1,-1,-1)$$

with multiplicities 2, 2, 2, 2 and 4 respectively and Σ_2 has rays generated by

$$(1,1,0), (1,0,1), (0,1,1), (-1,-1,-1)$$

with multiplicities 2, 2, 2 and 4 respectively. Then Σ_1 and Σ_2 have the same projection to two-dimensional coordinate planes and they are both the support of tropical curves of degree four. Thus they cannot be differentiated using the equations from the degree and so we would need to compute some initial ideals in order to differentiate between the two possible tropical curves. The rays $(1,1,1), (1,0,0), (0,1,0), (0,0,1)$ and $(1,1,0), (1,0,1), (0,1,1)$ are the vertices of the cube from Example 4.2.11 translated by $(1,1,1)$ so that the cube is not longer centred on the origin. We then need to include the ray $(-1,-1,-1)$ in order to ensure that the fans are balanced. ◇

The following algorithm outlines how to compute the rays of $\operatorname{trop}(C)$ given a set of candidates. It uses equations (4.4), (4.6) and (4.8) to limit the number of further initial ideal computations that we need to perform.

Algorithm 4.2.14. Input: A set $T = \{\rho_1, \ldots, \rho_s\}$ of candidates for $\operatorname{trop}(C)$ containing $\operatorname{trop}(C)$ where $C = V(I)$, and an index set P which indexes the projected tropical curves $\{\mathcal{C}_{ij} : (i,j) \in P\}$ where each $\mathcal{C}_{ij}$ come with a positive integer δ_{ij}.

Output: A set $\mathcal{R}$ of rays $\operatorname{trop}(C)$ where each ray $\rho \in \mathcal{R}$ comes with a positive integer m_ρ.

1. To each ρ_k in T we associate variable m_k. Let $\rho_{k,i}$ denote the i-th coordinate of ρ_k.

2. To each $(i,j) \in P$ and for each ray σ in $\mathcal{C}_{ij}$ with multiplicity m_σ we have the multiplicity equation $\delta_{ij} m_\sigma = \sum_{k:\pi_{ij}(\rho_k)\supseteq\sigma} m_k \gcd(\rho_{k,i}, \rho_{k,j})$ in the unknowns m_k.

3. For all $1 \leq j \leq n$ we have an equation from balancing $\sum_{k=1}^{s} m_k \rho_{k,j}$.

4. (a) Let $e_1, \ldots, e_n$ be the standard basis vectors for $\mathbb{R}^n$ and $e_0 = -e_1 - \cdots - e_n$.
   (b) For all $1 \leq k \leq s$, write $\rho_k = \sum_{l=0}^{n} a_{kl} e_l$ where $a_{kl} \geq 0$ for all $0 \leq l \leq n$ and where at least one a_{kl} is zero.
   (c) Add $n+1$ equations of the form $\sum_{k=1}^{s} a_{kl} m_k = D$ for all $0 \leq l \leq n$.

5. Write the equations in row reduced matrix form $Am = b$ by performing integer Gaussian elimination.

6. While A has fewer than s non-zero rows, do:
   (a) Choose $1 \leq \alpha \leq |\operatorname{rows}(A)|$ such that row α of A has more than one non-zero entry. Suppose that the l-th entry of the row α is non-zero.
   (b) Let $J = \operatorname{in}_{\rho_l}(I)$. If $(J : (x_1 \ldots x_n)^\infty) = \langle 1 \rangle$ then add equation $m_l = 0$ to the system of equations, otherwise, add the equation $m_l = \dim(S/\operatorname{in}_{\rho_l}(I))$.
   (c) Let $Am = b$ be this new system of equations after reducing to row reduced form.

7. Return $\mathcal{R} = \{\rho_k : b_k \neq 0\}$ where the ray ρ_i has multiplicity b_i.

Proof. We first show termination by showing that in the 'while' loop, at worst, we have to add s equations to the matrix system $Am = b$. If a row of A has more than one non-zero entry, then as the matrix system is in row reduced form, the multiplicities of the rays corresponding to the non-zero entries have not yet been determined. Thus, we add an equation $m_l = b_l$ which determines one of them. Then after reducing the new system to row reduced form, the only equation involving m_l is the one we have added. So A is of the form

$$\begin{pmatrix} & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$

and in particular, the only row whose l-th entry is non-zero is the one we have just added. Then at future steps, we will never add any new equations involving m_l. As we start with s candidates for $\mathrm{trop}(C)$ and at each step we either exclude one from being in $\mathrm{trop}(C)$ or determine its multiplicity, then as we are always analysing new candidates, after at most s steps, the first s rows of A is the $s \times s$ identity matrix, and and in particular A has exactly s non-zero rows.

To show correctness we show that all equations we add to the system are satisfied. Firstly, the balancing and multiplicity equations hold from the balancing condition of the Structure Theorem 2.2.5 and the multiplicity equation 4.2 of Sturmfels and Tevelev [2008]. The equations from the degree of the curve hold from the discussion in Section 2.6. Consider the equations we add in Step 6 of the algorithm. If $(J : (x_1 \dots x_n)^\infty) = \langle 1 \rangle$ then the initial ideal J contains a monomial and so ρ_α is not contained in the tropical curve. We set $m_\alpha = 0$. If not, then we set m_α to be the multiplicity of ρ_α in $\mathrm{trop}(C)$. □

4.3 Application and Examples

The key application of computing tropical curves by coordinate projections is as a potential replacement for the subroutine in `gfan` [Jensen] which computes tropical curves. This is because, as we saw in Section 2.3.3 the construction of tropical curves plays a key role in their algorithms and is often a bottleneck in computations. This means that any improvements in the tropical curves algorithm would result in potential improvements in the algorithms used in `gfan`. In this section, we look at an example which cannot be computed using `gfan` but which can be computed using the coordinate projection reconstruction methods as described in this chapter.

Example 4.3.1. Let $R = \mathbb{C}[x_0, x_1, \dots, x_{11}]$ be the polynomial ring in 12 variables where $\mathbb{C}$ is equipped with the trivial valuation. Let $I = \langle x_1^4 - x_0^3 x_{11} + 2x_0^2 x_1 x_{11} -$

$x_0x_1^2x_{11}, x_2^4 - x_0^3x_{11} + 2x_0^2x_2x_{11} - x_0x_2^2x_{11}, x_3^4 - x_0^3x_{11} + 2x_0^2x_3x_{11} - x_0x_3^2x_{11}, x_4^4 - x_0^3x_{11} + 2x_0^2x_4x_{11} - x_0x_4^2x_{11}, x_1 - x_2 + x_5, x_1 - x_3 + x_6, x_1 - x_4 + x_7, x_2 - x_3 + x_8, x_2 - x_4 + x_9, x_3 - x_4 + x_{10}\rangle$ be a homogeneous ideal defined by 10 equations. Then I defines a curve in $\mathbb{P}^{11}$ whose tropicalisation is $\text{trop}(V(I))$. As $\text{trop}(V(I))$ is a tropical curve in $\mathbb{R}^{11} \cong \mathbb{R}^{12}/\mathbb{R}(1,\ldots,1)$, by the Structure Theorem 2.2.5 it has the support of a weighted balanced one-dimensional fan in $\mathbb{R}^{11}$. Thus it can be given as a finite collection of weighted rays in $\mathbb{R}^{11}$. Computations were carried out on a MacBook Air with an Intel i5 processor and 8Gb of RAM. When input into `gfan` to try and compute these rays, the computations do not complete despite being left to run overnight. When using the algorithms from Section 4.2 which computes $\text{trop}(V(I))$ from its coordinate projections, the computations terminate after one minute. We see that $\text{trop}(V(I))$ has the support of a one-dimensional fan in $\mathbb{R}^{11}$ with ray generators the columns of the matrix

$$M = \begin{pmatrix} -4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 2 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with multiplicities given by $m = (6, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$ where the ray generated by the ith column of M has multiplicity m_i. ◇

4.4 Implementation Issues

The reconstruction discussed in the previous section has three main steps for reconstructing the tropical curve $\text{trop}(C)$. In the first, we recover the projections of $\text{trop}(C)$ to two-dimensional coordinate planes by finding an equation for the projected algebraic curve by elimination theory. The tropical curve is then the tropical hypersurface defined by that equation. This is Algorithm 4.2.5. The second step is to find a set of candidate rays for $\text{trop}(C)$ which project to the projections determined in the first step. We do this by building up coordinates to the partially reconstructed candidate one by one. This is in Algorithm 4.2.8. The final step is to determine which of the candidate rays from the previous step are rays of the tropical curve. In Algorithm 4.2.14 we use the multiplicity equation of Sturmfels and Tevelev [2008], the balancing condition and equations from the degree of the curve to minimise the number of initial ideals we need to compute.

In this section, we look at ways in which we can implement these algorithms for computational efficiency. We start by determining that we need not have all projections to two-dimensional coordinate planes as input to Algorithm 4.2.8 in order to have a finite collection of one-dimensional candidates as output. This allows a more efficient use of Algorithm 4.2.5 as we know that we do not need all of these projections and so we can abort those which are computationally expensive.

We then look at ways in which Algorithm 4.2.8 can be implemented more efficiently. The main way in which we do this is in the choice of projection to consider at each step of the algorithm. Choosing a different selection can lead to many more partially reconstructed candidates at each step, although the final output will always be the same. We discuss one strategy in Section 4.4.2, and another less efficient strategy in Section 4.4.3. Finally, in the work so far, we have only considered projections to x_0, x_i, x_j for all $1 \leq i < j \leq n$ as we are considering the tropical curve after we we quotient by the lineality space so that $x_0 = 0$. In Section 4.4.4 we consider how we can adapt what we have done so that we use all projections to x_i, x_j, x_k for all $0 \leq i < j < k \leq n$.

4.4.1 Number of Projections needed

In an ideal world, we would have all projections to coordinate planes as input to Algorithm 4.2.8. However, the projections are constructed by computing elimination Gröbner bases which in practice may be time consuming [see Bayer and Mumford, 1993, pp.11-12]. However, it follows from Lemma 4.2.10 that we only require a minimum of $\lceil \frac{n}{2} \rceil$ projections, so that we have a projection to each x_i for all $1 \leq i \leq n$, in order to recover finitely many one-dimensional candidates for $\text{trop}(C)$. Explicitly, if n is even we can choose projections to

$$x_1x_2, x_3x_4, \ldots, x_{n-1}x_n,$$

and if n is odd then we can choose projections to

$$x_1x_2, x_3x_4, \ldots, x_{n-2}x_{n-1}, x_{n-1}x_n.$$

On the other hand, considering extra projections, for example the projection to x_1x_3, would eliminate those candidates which are not in the pre-image of the extra projections. Further, they provide additional multiplicity equations for use in Algorithm 4.2.14 which restricts the number of initial ideal computations required in Step 6. Thus, we need a compromise between the number of projections we have whilst weighing up the fact that we will not require all of them for our reconstruction techniques to work. After ensuring that we have sufficient projections for the hypotheses of Lemma 4.2.10, we can time out any subsequent elimination Gröbner basis computations which take too long, say longer than one second. We will then know that Algorithm 4.2.8 will return a finite set of one-dimensional candidates which can be checked with Algorithm 4.2.14 to determine $\mathrm{trop}(C)$.

4.4.2 Choosing the Projection to Add

The choice of the projected tropical curve to add at each stage of the algorithm makes a considerable difference to the complexity of the algorithm. We examine this by considering how the type of projection that we are adding affects the set $\mathcal{S}$ of partially reconstructed candidates. Suppose that we have some partially reconstructed candidate $\rho \in \mathcal{S}$ and we are combining this with the rays of some projected tropical curve $\mathcal{C}_{ij}$.

If π_{ij} is a projection of **Type 0**, then this corresponds to the case where x_i and x_j have already been reconstructed in ρ. If $\pi_{ij}(\rho)$ spans a ray of $\mathcal{C}_{ij}$, or if $\pi_{ij}(\rho) = (0,0)$, then it is retained as it is an (i,j)-candidate for $\mathrm{trop}(C)$. If not, then it is discarded. Thus in this case, we simply remove the rays from $\mathcal{S}$ which are not (i,j)-candidates for $\mathrm{trop}(C)$ and so the size of $\mathcal{S}$ does not grow.

If π_{ij} is a projection of **Type 1(a)**, then this corresponds to the case where only one of x_i and x_j are reconstructed in the partially reconstructed candidate ρ. We assume that after relabelling if necessary, that x_i has been reconstructed and that x_j has not been reconstructed. In Case 1(a), we additionally have that ρ_i is non-zero. In order to combine ρ with a ray σ of $\mathcal{C}_{ij}$ we saw that we need to find multiples of ρ and σ such that their x_i coordinates to agree. This happens when they have the same sign and when $\mathrm{lcm}(\rho_i, \sigma_i)/\rho_i \leq m_\rho$ and $\mathrm{lcm}(\rho_i, \sigma_i)/\sigma_i \leq \delta_{ij} m_\sigma$. In this case, the rays ρ and σ can be combined uniquely. We thus only add one ray to $\mathcal{S}$ for each compatible ray of $\mathcal{C}_{ij}$.

If π_{ij} is a projection of **Type 1(b)**, then this corresponds to the case where only one of x_i and x_j are reconstructed in the partially reconstructed candidate ρ. We assume that after relabelling if necessary, that x_i has been reconstructed and that x_j has not been reconstructed. In Case 1(b), we additionally have that ρ_i equals zero. In order to combine ρ with a ray σ of $\mathcal{C}_{ij}$ we require that σ_i is also zero.

Thus only rays $(0, 1)$ and $(0, -1)$ can be combined with ρ, if they are also rays of $\mathcal{C}_{ij}$. However, they can be both combined multiple times according to the multiplicity bounds.

If π_{ij} is a projection of **Type 2**, then this corresponds to the case where both of x_i and x_j have not been reconstructed in the partially reconstructed candidate ρ. Here, we can combine ρ with every ray σ of $\mathcal{C}_{ij}$ multiple times according to multiplicity bounds.

So in order to reduce the number of rays we need to consider, we should consider projections in the following order:

1. Projections of Type 0;
2. Projections of Type 1(a);
3. Projections of Type 1(b);
4. Projections of Type 2.

This motivates the following variant of Algorithm 4.2.8 where we always choose projections in that ordering.

Algorithm 4.4.1. Input: An index set P which indexes the projected tropical curves $\{\mathcal{C}_{ij} : (i, j) \in P\}$.

Output: A set T of candidates for $\text{trop}(C)$.

Initialisation: $T = \emptyset$, $X = \{1, \dots, n\}$, $\mathcal{S} = \{\rho = (*, \dots, *)\}$ where $m_\rho = \infty$, $Q_\rho = P$ and $Y_\rho = \emptyset$.
While $\mathcal{S} \neq \emptyset$ do:

1. Choose any $\rho \in \mathcal{S}$. Let $Z_\rho = X - Y_\rho$ and set $\mathcal{S} = \mathcal{S} - \{\rho\}$.
2. Choose any $j \in Z_\rho$. Set $W = \emptyset$.
   (a) If there exists $i \in Y_\rho$ such that $(i, j) \in P$ and $\rho_i \neq 0$ then π_{ij} is a projection of Type 1(a) and follow **Case 1(a)** of Algorithm 4.2.8. Let U denote the set of rays added to $\mathcal{S}$ and set $W = \{i\}$.
   (b) If not, then if there exists $i \in Y_\rho$ such that $(i, j) \in P$ and $\rho_i = 0$ then π_{ij} is a projection of Type 1(a) and follow **Case 1(b)** of Algorithm 4.2.8. Let U denote the set of rays added to $\mathcal{S}$ and set $W = \{i\}$.
   (c) If not then there are no $i \in Y_\rho$ such that $(i, j) \in P$ and so choose any $j \neq i \in Z_\rho$ and then π_{ij} is a projection of Type 2 and follow **Case 2** of Algorithm 4.2.8. Let U denote the set of rays added to $\mathcal{S}$ and set $W = \{i, j\}$.

3. For all $\tau \in U$ and all $(w, k) \in P$ such that $w \in W$ and $k \in Y_\rho$ then π_{wk} is a projection of Type 0 and follow **Case 0** of Algorithm 4.2.8.

Return T.

Proof. Correctness and termination of this algorithm is the same as Algorithm 4.2.8 as here we have simply changed the order with which we are looking at projections. After adding coordinates to a partially reconstructed candidate in Step 3, we then consider all Type 0 projections which involve those added coordinates. □

4.4.3 Alternative Solutions which were not Improvements

In our algorithms, we reconstruct all candidates for $\text{trop}(C)$. We do this by building up the coordinates of a partially reconstructed candidate with compatible rays of a projected tropical curve until all coordinates have been reconstructed. We select a partially reconstructed candidate and then combine it with projected tropical curves until it is fully reconstructed. An alternative would be to select a projected tropical curve and reconstruct all partially reconstructed candidates in the set $\mathcal{S}$ with this same projected tropical curve. One advantage of this strategy is that as we are adding the same projection simultaneously to each of the partially reconstructed candidates. Thus these computations can be done more efficiently as we are doing them to all partially reconstructed candidates at once. However, we have seen in Section 4.4.2 that the choice of projection make a difference to the number of partially reconstructed candidates in $\mathcal{S}$ which we have to consider. We wish to consider projections in the order Type 0 then Type 1(a) then Type 1(b) then Type 2. However, when we are combining each partially reconstructed candidate in $\mathcal{S}$ with the same projection, its type will depend on the individual partially reconstructed candidate. So even it if is of Type 1(a) for one partially reconstructed candidate, then it may be of Type 1(b) for another. Then we would add many more partially reconstructed candidates to $\mathcal{S}$ than if we had chosen a different projection.

Another improvement is to use the polynomials encountered at all steps of the algorithm in order to eliminate candidates which we know cannot be in the tropical curve. For example when computing the elimination Gröbner bases in Algorithm 4.2.5 to find the equation of the projected curve, we computed many polynomials which live in our ideal. We can use them to check whether candidates are rays of the tropical curve or not. That is, for candidate ρ, we can compute $\text{in}_\rho(f)$ for all polynomials in I that we have encountered. If any $\text{in}_\rho(f)$ is a monomial then $\text{in}_\rho(I)$ would contain a monomial and so ρ cannot be in the tropical variety by the Fundamental Theorem 2.1.5.

For example, in Example 4.3.1 we have 1196 different polynomials which are in the ideal that we can use to successfully cut down the number of rays in the superset from 177 to 18. In this case, we now have a unique solution to the equations determining the multiplicity meaning no further Gröbner basis calculations are necessary as we do not need to compute any initial ideals. However, in practice, this is not an efficient solution as it is actually quicker to simply check if each of the rays live in the tropical curve or not by the computation of initial ideals.

4.4.4 Other Two-Dimensional Coordinate Projections

In our algorithms, we recover the rays of the tropical curve together with their multiplicities. As C is a curve in $\mathbb{P}^n$, the tropical curve $\text{trop}(C)$ lives in $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$. We can consider its lift to $\mathbb{R}^{n+1}$ where $\text{trop}(C)$ is a two-dimensional fan with a one-dimensional lineality space spanned by $(1,\ldots,1)$. We consider the projection $\pi_{ij}\colon \mathbb{R}^{n+1} \to \mathbb{R}^3$ to coordinates x_0, x_i, x_j. The tropical curves $\text{trop}(C)$ and $\mathcal{C}_{ij} = \pi_{ij}(\text{trop}(C))$ have one-dimensional lineality spaces spanned by $(1,\ldots,1)$ and $(1,1,1)$ respectively. After quotienting out by this lineality space, we think of them as living in $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ respectively. So far in this Chapter, we have chosen identifications which send the x_0-coordinate to zero, and crucially this means that the map $\pi_{ij}\colon \mathbb{R}^n \to \mathbb{R}^2$ is still a projection.

The key point that we need for the map $\pi_{ij}\colon \mathbb{R}^n \to \mathbb{R}^2$ to still be a projection, is that the identifications $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ are the same. For the work thus far in this Chapter, this has always been the case as we have chosen the identification which sends x_0 to zero. This allows us to easily combine rays in Algorithm 4.2.8 as we are looking at the same slice of the tropical curve. However, this choice was arbitrary and we could have chosen identifications sending any x_i to zero instead. Algebraically, this would correspond to dehomogenising the equations with respect to x_i. We can now consider the projections to x_i, x_j, x_k for $0 \leq i < j < k \leq n$ and then choose an identification which sends x_k to zero.

Let $\pi_{ijk}\colon \text{trop}(C) \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^3$ be a coordinate projection where $\mathbb{R}^3$ has coordinates x_i, x_j, x_k for some $0 \leq i < j < k \leq n$. The image of $\text{trop}(C)$ under the projection map π_{ijk} is the tropical curve denoted by $\mathcal{C}_{ijk}$.

Recall that in Section 4.2, we reconstructed tropical curves in three steps. We explain the main changes to these steps in order to accommodate these extra two-dimensional coordinate projections.

1. Algorithm 4.2.5 found the projected tropical curves $\mathcal{C}_{ij}$ which are the coordinate projections of $\text{trop}(C)$ to x_0, x_i, x_j for $1 \leq i < j \leq n$. The modified

algorithm would now find the projected tropical curves $\mathcal{C}_{ijk}$ to x_i, x_j, x_k for $0 \leq i < j < k \leq n$.

2. Algorithm 4.2.8 recovered a superset of the rays in $\mathrm{trop}(C)$ by reconstructing candidates. Now, in order to combine a partially reconstructed candidate ρ in $\mathbb{R}^{n+1}$ with a ray of the projected tropical curve σ in $\mathbb{R}^3$, we first need to choose the same identification for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which sets the x_k-coordinate to zero. (In Algorithm 4.2.8, we always set the x_0-coordinate to zero as all of our projections project to x_0.) This ensures that the induced map $\mathbb{R}^n \to \mathbb{R}^2$ is still a projection. We see these changes and how they affect the results in Algorithm 4.4.3.

3. Algorithm 4.2.14 determined which of the candidates in the superset are rays of the tropical curve. This used notions of multiplicity, balancing and degree of the tropical curve. For this, we need all rays to have the same identification. Thus at the end of the new Algorithm 4.4.3, we choose the identification which sends the x_0-coordinate to zero for all candidates and Algorithm 4.2.14 follows through to find the rays of $\mathrm{trop}(C)$.

As we are now looking at projections to coordinates x_i, x_j, x_k for all $0 \leq i < j < k \leq n$ we can use these extra projections to differentiate between the two tropical curves in Example 4.2.11. However, there are new examples of two different tropical curves which have the same projections to all x_i, x_j, x_k.

Example 4.4.2. Consider the two-dimensional fans in $\mathbb{R}^4$ which both have a one-dimensional lineality space spanned by $(1,1,1,1)$. They have rays spanned by

$$(1,1,-1,-1),(1,-1,-1,1),(1,-1,1,-1),$$
$$(-1,-1,1,1),(-1,1,-1,1),(-1,1,1,-1),$$

and

$$(1,1,1,-1),(1,1,-1,1),(1,-1,1,1),(-1,1,1,1),$$
$$(-1,-1,-1,1),(-1,-1,1,-1),(-1,1,-1,-1),(1,-1,-1,-1),$$

respectively, each ray with multiplicity one. Let $\mathbb{R}^4$ have coordinates x_0, x_1, x_2, x_3, and consider the coordinate projection to x_i, x_j, x_k for $0 \leq i < j < k \leq 3$. We then see that both fans above have the same image under this coordinate projections after we have quotiented out by the lineality space spanned by $(1,1,1)$. Notice that as for Example 4.2.11 this corresponds to the alternative vertices of a hypercube in $\mathbb{R}^4$ centred on the origin with vertices at $(\pm 1, \pm 1, \pm 1, \pm 1)$.

This example can be generalised to $\mathbb{R}^n$ in the following way. Consider the hypercube in $\mathbb{R}^n$ which is centred on the origin and has vertices at $(\pm 1, \ldots, \pm 1)$ and let Σ_0 and Σ_1 be one-dimensional fans with rays passing through alternate vertices. That is, Σ_0 consists of rays passing through vertices the product of whose entries is equal to 1, and Σ_1 consists of rays passing through vertices the product of whose entries is equal to -1. Then Σ_0 and Σ_1 agree on coordinate projections. Notice that the tropical curves which have support Σ_0 and Σ_1 respectively have different degrees and so can be differentiated using the methods of Section 4.2.3. However, as in Remark 4.2.13 there are also examples of tropical curves with the same degree which have the same projections to coordinate planes, for example by translating one of the cube configurations away from the origin. As an example, consider two fans in $\mathbb{R}^4$, the first with rays generated by

$$(1,1,1,1), (1,1,0,0), (1,0,0,1), (1,0,1,0),$$
$$(0,0,1,1), (0,1,0,1), (0,1,1,0), (-1,-1,-1,-1),$$

where $(-1,-1,-1,-1)$ has multiplicity eight and all other rays have multiplicity two, and the second with rays generated by

$$(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1),$$
$$(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0), (-1,-1,-1,-1)$$

where $(-1,-1,-1,-1)$ has multiplicity eight and all other rays have multiplicity two. Both of these tropical curves have degree eight and so cannot be differentiated using the equations from the degree. This corresponds to translating the hypercube with vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ from the beginning of this example by $(1,1,1,1)$, where we include the rays $(-1,-1,-1,-1)$ for balancing. $\diamond$

A further advantage of using these extra projections is that we have more equations in the multiplicities of the candidates which helps limit the number of initial ideal computations in Step 6 of Algorithm 4.2.14.

We first review some notation. Let ρ be a partially reconstructed candidate, with associated multiplicity bound m_ρ. Suppose that we are trying to reconstruct ρ to variables $x_0, x_1, \ldots, x_n$. In this setting the definition of a candidate naturally extends. We call ρ an (i,j,k)-*candidate* for $\operatorname{trop}(C)$ if, after choosing identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which sets the x_k-coordinate to zero, either $\pi_{ijk}(\rho)$ spans a ray $\sigma \in \mathcal{C}_{ij}$ with multiplicity m_σ and $m_\rho \leq m_\sigma$, or $\pi_{ijk}(\rho) = (0,0)$. It is called a *candidate* for $\text{trop}(C)$ if it is an (i,j,k)-candidate for all (i,j,k).

Suppose that we are trying to extend ρ with the rays of the projected tropical curve $\mathcal{C}_{ijk}$. Recall from Section 4.2.2 that $\mathcal{C}_{ijk}$ can be of Type 0, Type 1(a) or 1(b), or Type 2, which affects how it interacts with the partially reconstructed candidate ρ. These follow through to similar cases as we explain below. Before we can combine a partially reconstructed ray with the rays of a projected tropical curve, we need to choose identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$. Recall that we need to choose the same identification for both. This equates to adding multiples of the lineality space to rays so that some chosen x_i-coordinate equals zero. There is also now an additional type of projection which we shall encounter which will be a projection of Type 3.

Projections of Type 0. In this case, x_i, x_j and x_k have already been reconstructed in the partially reconstructed candidate ρ. Choose identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which sets the x_k-coordinate to zero. We then have to check to see if ρ is an (i,j,k)-candidate for $\text{trop}(C)$.

Projections of Type 1. In this case, two of the x_i, x_j and x_k have been reconstructed in the partially reconstructed candidate ρ. After relabelling coordinates if necessary, we can assume that the x_i and x_k coordinates have been reconstructed in the partially reconstructed candidate ρ and that x_j has not. Choose identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which sets the x_k-coordinate to zero. Then analogously to the projection of Type 1 from Section 4.2.2 this splits into projections of Type 1(a) and 1(b) depending on the value of ρ_i.

Projections of Type 2. In this case, only one of x_i, x_j and x_k have been reconstructed in the partially reconstructed candidate ρ. After relabelling coordinates if necessary, we can assume that the x_k-coordinate has been reconstructed in the partially reconstructed candidate ρ. Choose identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\dots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ which sets the x_k-coordinate to zero. We can then extend ρ with each ray of $\mathcal{C}_{ijk}$ analogously to projections of Type 2 from Section 4.2.2.

Projections of Type 3. In this case, none of x_i, x_j and x_k have been reconstructed in the partially reconstructed candidate ρ and so theoretically we can combine ρ with each ray of $\mathcal{C}_{ijk}$ multiple times as for projections on Type 2. However, as both the partially reconstructed candidate ρ and the rays of $\mathcal{C}_{ijk}$ can have any multiple of the lineality space added to them, there are infinitely many partially reconstructed candidates which are (i,j,k)-candidates and which agree with ρ on coordinates already reconstructed.

The following algorithm reconstructs all candidates from a set $\{\mathcal{C}_{ijk}\}$ of projected tropical curves. As when considering a projection of Type 3 there are infinitely many suitable reconstructions, such projections are avoided. We give conditions in Proposition 4.4.4 for when we have enough projections so that we never have to add projections of Type 3. To partially reconstructed candidate ρ we assign a set Y_ρ which indexes the variables already reconstructed in ρ. We then only extend the partially reconstructed candidate ρ with a projected tropical curve $\mathcal{C}_{ijk}$ where $\{i,j,k\} \cap Y_\rho \neq \emptyset$, which ensures that $\mathcal{C}_{ijk}$ is not a projection of Type 3.

Algorithm 4.4.3. Input: An index set P which indexes the projected tropical curves $\{\mathcal{C}_{ijk} : (i,j,k) \in P\}$. The set P satisfies the hypotheses of Proposition 4.4.4 so that we do not have to consider Projections of Type 3.

Output: The set T of all candidates for $\text{trop}(C)$.

Initialisation: Set $T = \emptyset$ and $\mathcal{S} = \emptyset$. Choose any $(i,j,k) \in P$. Let U be the set consisting of rays of $\mathcal{C}_{ijk}$ and $\rho = (0,0,0)$ where $m_\rho = \infty$. For all $\sigma \in U$ with corresponding multiplicity m_σ let $\theta = (\theta_0, \theta_1, \ldots, \theta_n)$ where $\theta_l = *$ for $l \neq i,j,k$, $\theta_i = \sigma_i$, $\theta_j = \sigma_j$ and $\theta_k = \sigma_k$. Set $m_\theta = m_\sigma$, $Q_\theta = P - \{(i,j,k)\}$ and $Y_\theta = \{i,j,k\}$. Set $\mathcal{S} = \mathcal{S} \cup \{\theta\}$.
While $\mathcal{S} \neq \emptyset$ do:

I. Choose any $\rho \in \mathcal{S}$. Set $\mathcal{S} = \mathcal{S} - \{\rho\}$.

II. If $Q_\rho = \emptyset$ and $\rho \neq (0,\ldots,0)$ then $T = T \cup \{\rho\}$. Otherwise, while $Q_\rho \neq \emptyset$ do:

   (a) Choose any $(i,j,k) \in Q_\rho$ such that $\{i,j,k\} \cap Y_\rho \neq \emptyset$. After relabelling if necessary, we can assume that $k \in \{i,j,k\} \cap Y_\rho$. Choose identifications for $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ and $\mathbb{R}^2 \cong \mathbb{R}^3/\mathbb{R}(1,1,1)$ so that ρ and the rays of $\mathcal{C}_{ijk}$ have x_k-coordinate equal to zero. We do this by adding multiples of the lineality space $(1,\ldots,1)$.

   (b) Updating $\mathcal{S}$. This depends on $\{i,j,k\} \cap Y_\rho$.

      1. **Case 0: If** $|\{i,j,k\} \cap Y_\rho| = 3$ **then** π_{ijk} **is a projection of Type 0.**
      Follow Case 0 of Algorithm 4.2.8.

      2. **Case 1(a): If** $|\{i,j,k\} \cap Y_\rho| = 2$ **and** $\rho_i \neq 0$**, where after relabelling if necessary we assume that** $\{i,j,k\} \cap Y_\rho = \{i,k\}$**, then** π_{ijk} **is a projection of Type 1(a).**
      Follow Case 1(a) of Algorithm 4.2.8.

      3. **Case 1(b): If** $|\{i,j,k\} \cap Y_\rho| = 2$ **and** $\rho_i = 0$**, where after relabelling if necessary we assume that** $\{i,j,k\} \cap Y_\rho = \{i,k\}$**, then** π_{ijk} **is a**

projection of Type 1(b).

Follow Case 1(b) of Algorithm 4.2.8.

4. **Case 2: If** $|\{i,j,k\} \cap Y_\rho| = 1$ **then** π_{ijk} **is a projection of Type 2.**
   Follow Case 2 of Algorithm 4.2.8.

Scale all rays in T with $(1,\ldots,1)$ so that the x_0-coordinate is zero.
Return T.

Proof. The proof of correctness and termination follows directly from the corresponding proof of Algorithm 4.2.8. □

We define a hypergraph $\mathcal{G}_P$ associated to a set P which indexes the projections which we use as input to Algorithm 4.4.3. We call this the *graph of projections* for P. The set of vertices of $\mathcal{G}_P$ is $\{v_0, v_1, \ldots, v_n\}$ where the vertex v_i indexes the x_i-coordinate. We have an edge $\{v_i, v_j, v_k\}$ for every triple (i,j,k) in P.

The idea of the algorithm from this graph point of view is that we start at a vertex and walk to all other vertices by passing along edges. For example, we could start at v_0 and use edge $\{v_0, v_1, v_2\}$ to walk to vertices v_1, v_2 then by edge $\{v_1, v_2, v_3\}$ walk to vertex v_3, and so on. We use conditions on the hypergraph $\mathcal{G}_P$ to give conditions on the set of projections P to ensure that we will have finitely many one-dimensional candidates for $\operatorname{trop}(C)$.

Proposition 4.4.4. *Let P be a set indexing coordinate projections $\pi_{ijk}\colon \mathbb{R}^{n+1}_{x_0,x_1,\ldots x_n} \to \mathbb{R}^3_{x_i,x_j,x_k}$ whose image is the tropical curve $\mathcal{C}_{ijk}$ which has a one-dimensional lineality space spanned by $(1,1,1)$. Let $\mathcal{G}_P$ be the graph of projections for P and T a set of rays which are (i,j,k)-candidates for* $\operatorname{trop}(C)$ *for all $(i,j,k) \in P$. Then $\mathcal{G}_P$ is connected if and only if T is a finite set of two-dimensional rays in $\mathbb{R}^{n+1}$ each with a one-dimensional lineality space spanned by $(1,\ldots,1)$.*

Proof. Suppose that $\mathcal{G}_P$ is connected. Consider reconstructing candidates by Algorithm 4.4.3. As $\mathcal{G}_P$ is connected, at every step we can choose a projection in P such that P is not of Type 3 until we have recovered all vertices of $\mathcal{G}_P$. By the proof of Algorithm 4.2.8, it follows that we have recovered a finite collection of two-dimensional candidates each with one-dimensional lineality space.

Conversely suppose that $\mathcal{G}_P$ is not connected. Using P as input to Algorithm 4.4.3 notice that when considering projections of Type 0, 1 or 2 we share at least one coordinate with those already reconstructed. This means that the graph $\mathcal{G}_P$ is connected. As $\mathcal{G}_P$ is not connected, at some point, we must have to consider a projection of Type 3. From the discussion before Algorithm 4.4.3, we see that adding a projection of Type 3 gives infinitely many partially reconstructed candidates. □

Remark 4.4.5. The implementation issues from previous subsections can also be implemented in this reworking of the reconstruction algorithm. For example, when we have a choice of projection to consider we do so in preference order: Type 0, as they remove partially reconstructed candidates from $\mathcal{S}$, then Type 1(a), as they add a unique partially reconstructed candidate to $\mathcal{S}$ for each compatible ray of the projected tropical curve, then Type 1(b), as we can only combine with the rays $(0, 1)$ and $(0, -1)$ if they occur in the projected tropical curve but multiple times according to multiplicity bounds, then finally Type 2, as we add multiple partially reconstructed candidates to $\mathcal{S}$ for each ray of the projected tropical curve. ◇

Remark 4.4.6. When we have decided which is the optimal projection to add as explained in Section 4.4.2 we may have still have a choice of which of these projections of a certain type to add. It appears heuristically that we should add the projection which adds coordinate x_i for which the corresponding vertex in the graph of projection has the highest valency. This would mean that there is a greater choice of projections to add at the next step and so a greater likelihood of finding one of Type 0 then one of Type 1 and then one of Type 2. ◇

4.5 A Macaulay2 package to Compute Tropical Curves from Coordinate Projections

Let $K = \mathbb{Q}$ with the trivial valuation. In this case, the algorithms in this Chapter are implemented in the package `TropicalCurves` [Chan, 2013b] for the computer algebraic geometry system Macaulay2 [Grayson and Stillman]. The package `TropicalCurves` allows the computation of tropical curves from coordinate projections. The main function of this package is `tropicalCurve` which takes a homogeneous ideal I defining a curve in $\mathbb{P}^n$ and outputs the rays and multiplicities of the one-dimensional fan in $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1)$ whose support is the tropical curve $\text{trop}(V(I))$.

We demonstrate usage by first installing the package then specifying the polynomial ring $\mathbb{Q}[x, y, z]$ and ideal $I = \langle x + y + z \rangle$

```
i1 : installPackage "TropicalCurves";
i2 : QQ[x,y,z];
i3 : I = ideal(x+y+z);
```

Then using `tropicalCurves` we find the rays and multiplicities of $\text{trop}(V(I))$

```
i4 : tropicalCurve I
```

```
o4 = (| 1 0 -1 |, {1, 1, 1})
      | 0 1 -1 |
o4 : Sequence
```

The output comes in the form (A, m) for some matrix A with integer entries and a list m with positive integer entries. The columns of A are minimal ray generators of $\text{trop}(V(I))$ and the ray spanned by the i-th column of A has multiplicity the i-th entry in the list m. In this example, we see that $\text{trop}(V(I))$ consists of three rays spanned by $(1, 0)$, $(0, 1)$ and $(-1, -1)$ each with multiplicity one, which agrees with the calculations in Example 2.1.6.

We next compute Example 4.2.11 which is the example where there are two tropical curves with the same projections to coordinate planes. Here we have an ideal $I = \langle xz + 4yz - z^2 + 3xw - 12yw + 5zw, xy - 4y^2 + yz + xw + 2yw - zw, x^2 - 16y^2 + 8yz - z^2 + 14xw - 8yw + 2zw\rangle$ in the ring $\mathbb{Q}[x, y, z, w]$ and we find the rays of the tropical curve using `tropicalCurve`.

```
i5 : f_1 = x*z+4*y*z-z^2+3*x*w-12*y*w+5*z*w;
i6 : f_2 = x*y-4*y^2+y*z+x*w+2*y*w-z*w;
i7 : f_3 = x^2-16*y^2+8*y*z-z^2+14*x*w-8*y*w+2*z*w;

i8 : I = ideal(f_1,f_2,f_3);

i9 : tropicalCurve I

o9 = (| -1 -1 1  1 |, {1, 1, 1, 1})
      | -1 1  -1 1 |
      | 1  -1 -1 1 |
o9 : Sequence
```

As expected, we see the tropical curve has four rays spanned by $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$ and $(1, 1, 1)$ each with multiplicity one. This agree with earlier calculations. Recall that the reconstruction step recovers eight rays spanned and using the equations from the multiplicity condition, the balancing condition, and the degree of the curve, we can recover the rays in the tropical curve.

Consider the ideal $I = \langle a^2 + 2bc + ad + e^2, ab + bc + cd + de, ac + bd + ce\rangle$ in $\mathbb{Q}[a, b, c, d, e]$. Then using `tropicalCurve` we compute the tropical curve $\text{trop}(V(I))$:

```
i10 : QQ[a,b,c,d,e];
i11 : I = ideal(a^2+2*b*c+a*d+e^2,a*b+b*c+c*d+d*e,a*c+b*d+c*e);

i12 : tropicalCurve I

o12 = (| 1 0 0  0 -2 1  -1 1  0 |, {3, 1, 1, 2, 1, 1, 3, 1, 1})
```

```
        | 0 1 1  1 -1 -1 -1 1  0 |
        | 0 0 -1 2 -1 1  -1 0  0 |
        | 0 0 1  1 -2 2  -1 -1 1 |
o12 : Sequence
```

We end with the example in Section 4.3 which was the example which could not be computed using `gfan`. We have ideal $I = \langle x_1^4 - x_0^3x_{11} + 2x_0^2x_1x_{11} - x_0x_1^2x_{11}, x_2^4 - x_0^3x_{11} + 2x_0^2x_2x_{11} - x_0x_2^2x_{11}, x_3^4 - x_0^3x_{11} + 2x_0^2x_3x_{11} - x_0x_3^2x_{11}, x_4^4 - x_0^3x_{11} + 2x_0^2x_4x_{11} - x_0x_4^2x_{11}, x_1 - x_2 + x_5, x_1 - x_3 + x_6, x_1 - x_4 + x_7, x_2 - x_3 + x_8, x_2 - x_4 + x_9, x_3 - x_4 + x_{10}\rangle$ in $\mathbb{Q}[x_0, \ldots, x_{11}]$.

```
i13 : QQ[X_0..X_11];
i14 : g_1 = X_1^4-X_0^3*X_11+2*X_0^2*X_1*X_11-X_0*X_1^2*X_11;
i15 : g_2 = X_2^4-X_0^3*X_11+2*X_0^2*X_2*X_11-X_0*X_2^2*X_11;
i16 : g_3 = X_3^4-X_0^3*X_11+2*X_0^2*X_3*X_11-X_0*X_3^2*X_11;
i17 : g_4 = X_4^4-X_0^3*X_11+2*X_0^2*X_4*X_11-X_0*X_4^2*X_11;
i18 : g_5 = X_1-X_2+X_5;
i19 : g_6 = X_1-X_3+X_6;
i20 : g_7 = X_1-X_4+X_7;
i21 : g_8 = X_2-X_3+X_8;
i22 : g_9 = X_2-X_4+X_9;
i23 : g_10 = X_3-X_4+X_10;
i24 : I = ideal(g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8,g_9,g_10);
```

and we compute the tropical curve with `tropicalCurve`.

```
i25 : C = tropicalCurve I;

i26 : A = C#1

o26 = | -4 2 2 2 2 2 2 0 0 0 0 0 0 |
      | -3 1 1 1 2 2 2 0 0 0 0 0 0 |
      | -3 2 2 1 1 2 1 0 0 0 0 0 0 |
      | -3 1 2 2 1 1 2 0 0 0 0 0 0 |
      | -3 2 1 2 2 1 1 0 0 0 0 0 0 |
      | -3 1 1 1 1 3 1 0 0 1 0 0 0 |
      | -3 1 1 1 1 1 3 1 0 0 0 0 0 |
      | -3 1 1 1 3 1 1 0 0 0 0 1 0 |
      | -3 1 3 1 1 1 1 0 0 0 0 0 1 |
      | -3 3 1 1 1 1 1 0 0 0 1 0 0 |
      | -3 1 1 3 1 1 1 0 1 0 0 0 0 |
                11         13
o26 : Matrix ZZ   <--- ZZ
```

```
i27 : m = C#2

o27 = {6, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}
o27 : List
```

The output tells us that $\text{trop}(V(I))$ consists of 13 one-dimensional rays in $\mathbb{R}^{11}$ spanned by the columns of the matrix A where the multiplicity of the ray spanned by the ith column of A has multiplicity m_i.

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