WHITNEY STRATIFICATIONS : FAULTS AND DETECTORS

BY

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PREFACE

It is a pleasure to thank everyone who has provided me with help, encouragement or inspiration during the four years of research 1973-77 which have culminated in this thesis.

During the first of those four years Christopher Zeeman supervised me. I am grateful to him for having convinced me that it is both more essential and more rewarding to participate in mathematical research than to remain merely a well-informed spectator.

Before I went to Cambridge, when I was debating if I should concentrate on pure mathematics, it was an article by Christopher Zeeman describing the present time as the " golden age of pure mathematics " which persuaded me to do so. Another article of his, this time in " Manifold " , tempted me towards topology. I first heard of the wonders of Catastrophe Theory from him at an evening meeting of the Archimedeans Society in November 1970 , and interested by this talk, and drawn by the creative aura emanating from my copies of " Manifold " , I decided to come to Warwick.

In my M.Sc. year 1972-73 , I was lucky to have Clint McCrory as supervisor : he initiated me into the secrets of differential topology via the works of John Milnor, and by running a seminar on Whitney stratifications, helped to determine the future course of my research.

After my M.Sc. dissertation — a write-up of Zeeman's lectures on the proof of Thom's theorem classifying elementary catastrophes — I was looking for ways of using stratifications in singularity theory. On learning that Brieskorn was to give a survey talk on complex singularities at the 1974 B.M.C. held at Brighton, I devoured Milnor's " Singular points of complex hypersurfaces " in the week before the conference and was well rewarded by Brieskorn's stunning display of the lights and facets of the jewelled geometry of complex singularities. There was also a short talk by Jim Timourian describing a conjecture of Teissier that " Milnor number constant implies Whitney's condition (b) " ([30] , [31]). On discovering that this was part of a theory (equisingularity) which involved a fine study of Whitney regularity and both used and produced results about singularities, I decided to work on Teissier's conjecture.* About the same time Poenaru suggested I go to Orsay, and with Rolph Schwarzenberger's practical assistance as Chairman of the Department, I prepared to do so, in the mean time making contact with the research group at Liverpool, who were studying Whitney stratifications during 1974-75 as part of the proof of the topological stability theorem ([7]).

At Liverpool I was able to discuss with Chris Gibson and Eduard Looijenga, both of whom provided me with friendly encouragement. Moreover there I had the opportunity of being directed by Terry Wall, whose critical advice has been of great assistance to me throughout these past three years, especially in gauging the worth of my various ideas and results. I am pleased to be able to present here (see §3 and §5) proofs of the conjectures concerning geometric versions of Whitney regularity which were put forward by him in [43].

In Orsay I had the good fortune to be offered a teaching post, which although delaying the completion of my thesis by taking up time and energy, was interesting, gave me a taste of responsibility, and provided necessary financial support.

With the equisingularity team at the Ecole Polytechnique I have had many pleasant and profitable discussions : notably with Jean-Pierre Henry, Jean-Jacques Risler, and Lê Dũng Tràng, and especially with Bernard Teissier, whose unfailing enthusiasm and willing ear I have much appreciated.

I thank Tony Iarrobino for persuading me to give seminars ; Bob MacPherson for discussions about twenty-first century mathematics ; and of course René Thom, without whom the greater part of the work contained here, and much of the work of those mathematicians named above, would not as yet exist, and whose Monday seminars at the I.H.E.S. are a constant source of delight and inspiration.

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* Counterexamples were found by Briancon and Speder [2] in January 1975.

I am also indebted to my wife, Marie-Hélène, for her patient support, faith, and understanding.

Finally I acknowledge with thanks the Research Studentship provided by the Science Research Council while I was at Warwick, and the French Government Scholarship which enabled me to write up my results.

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CHAPTER 0. INTRODUCTION

This work deals with properties of Whitney (a)- and (b)-regularity. The regularity conditions prescribe the local behaviour of limits of tangent spaces to smooth manifolds, which are usually strata of a stratification. So, first, what is a stratification ?

A stratification Σ of a subset V of a C^1 manifold M is a partition of V into connected C^1 submanifolds, called the strata of Σ. Σ is locally finite if each point of V has a neighbourhood meeting only finitely many strata.

Example 0.1. V a connected C^1 submanifold of M. There is a trivial stratification of V with just one stratum.

Example 0.2. V the underlying space of a linearly embedded simplicial complex. There is a natural stratification whose strata are the interiors of the simplices of the complex.

Example 0.3. V an analytic variety in $\mathbb{R}^n$. Let $S(V)$ be the set of points where V is not a submanifold of maximal dimension. Write $S^2(V) = S(S(V))$, etc. Suppose r is the smallest integer such that $S^{r+1}(V) = \phi$. Let $\mathcal{G}(A)$ denote the set of connected components of a set A. Then

$$\mathcal{G}(V-S(V)) \amalg \mathcal{G}(S(V)-S^2(V)) \amalg \ldots \amalg \mathcal{G}(S^{r-1}(V)-S^r(V)) \amalg \mathcal{G}(S^r(V))$$

defines a locally finite stratification of V called the full partition by dimension (by Whitney in [46]).

The Whitney conditions

Let X, Y be disjoint C^1 submanifolds of a C^1 manifold M and let y be a point in $Y \cap \overline{X}$.

X is (a)-regular over Y at y if,

(a) Given a sequence of points $\{x_i\}$ in X tending to y, such that $T_{x_i}X$ tends to τ, then $T_yY \subset \tau$.

X is (b)-regular over Y at y if,

(b) Given sequences $\{x_i\}$ in X, $\{y_i\}$ in Y, both tending to y, such that $T_{x_i}X$ tends to τ, and the unit vector in the direction of $\overrightarrow{x_iy_i}$ tends to λ, then $\lambda \subset \tau$.

These conditions were first defined by Whitney in [45] and [46]. Accounts of them have been given by Thom in [35] and [36], by Mather in [21] and [22], by Wall in [43] and [44], and by Gibson and Wirthmüller in [7].

Following Thom, we say that X is (b')-regular over Y at y if, for some C^1 local retraction π associated to a C^1 tubular neighbourhood of Y near y (see §5),

(b') Given a sequence $\{x_i\}$ in X tending to y, such that $T_{x_i}X$ tends to τ and the unit vector in the direction of $\overrightarrow{x_i\pi(x_i)}$ tends to λ, then $\lambda \subset \tau$.

(b) clearly implies (b') for any π. Also (b) implies (a), since given any vector v in T_yY and any sequence $\{x_i\}$ in X we can choose $\{y_i\}$ in Y coming in to y in the direction of v so slowly that $\overrightarrow{x_iy_i}/|\overrightarrow{x_iy_i}|$ tends to v (see Mather [21]). Conversely, if (a) holds and (b') holds for some π, we arrive at (b) by decomposing the vector λ into the sum of two vectors, one in T_yY and one in $T_y(\pi^{-1}(y))$. (Compare Wall [43]) To sum up,

(0.4) $$\boxed{(b') + (a) \iff (b)}$$

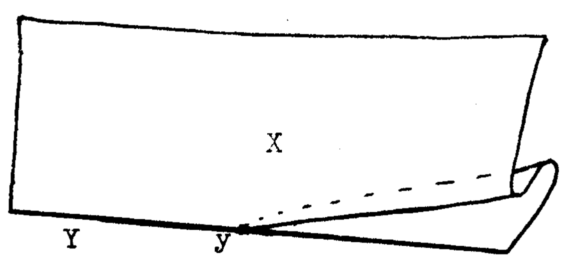
We shall make frequent use of this equivalence.

A stratification Σ is <u>(a)-regular</u> if, for each pair of strata X, Y and at every point $y \in Y \cap \overline{X}$, X is (a)-regular over Y at y. Similarly, we speak of (b)-regular stratifications. We call a locally finite (b)-regular stratification a <u>Whitney stratification</u>.

<u>Example 1</u>. (0.1) is trivially a Whitney stratification since there is only one stratum, and (a)- and (b)-regularity are conditions on a pair of strata.

<u>Example 2</u>. The stratification in (0.2) defined by a linearly embedded simplicial complex is a Whitney stratification by the next example.

<u>Example 3</u>. Let $\overline{X}$ be a C^1 submanifold-with-boundary of a C^1 manifold M, with interior X and boundary Y. Then X is (b)-regular over Y, since <u>(b)-regularity is invariant under C^1 diffeomorphism</u> (see Corollary 5.3), and $\mathbb{R}^p \times (0,\infty)^q \times 0^r$ is (b)-regular over $\mathbb{R}^p \times 0^{q+r}$ in $\mathbb{R}^{p+q+r}$. (b)-regularity is far from being a topological invariant.



Pictured is a topological manifold-with-boundary $\overline{X}$, with interior X a C^1 manifold and boundary Y a line, such that X is not (b)-regular over Y at y; we say the pair (X,Y) has a (b)-fault at y (see below).

Example 4. The stratification defined in (0.3) by the full partition by dimension of an analytic variety is not necessarily a Whitney stratification. We give the standard examples.

1) $V \equiv \{y^2 = t^2x^2 + x^3\} \subset \mathbb{R}^3$.

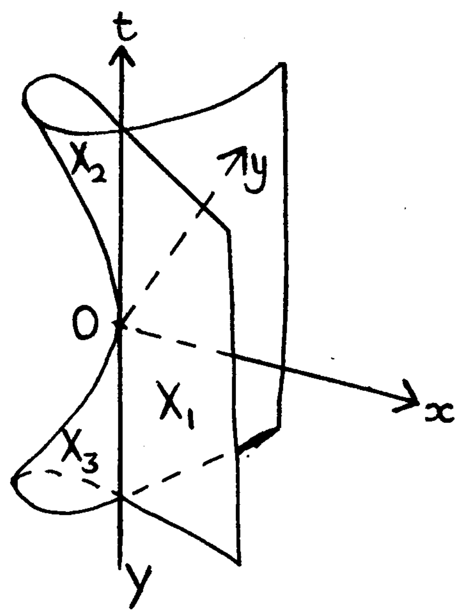
Let Y be the t-axis, and X be $V - Y$.

Then set $X_1 = X \cap \{x > 0\}$,

$X_2 = X \cap \{x < 0\} \cap \{t > 0\}$,

$X_3 = X \cap \{x < 0\} \cap \{t < 0\}$.

X_1 is (b)-regular over Y at 0, but X_2 and X_3 are not (b)-regular over Y at 0. However all three are (a)-regular over Y at 0. The reader may check that $X_1 \amalg X_2 \amalg X_3 \amalg Y$ is the full partition by dimension of V.



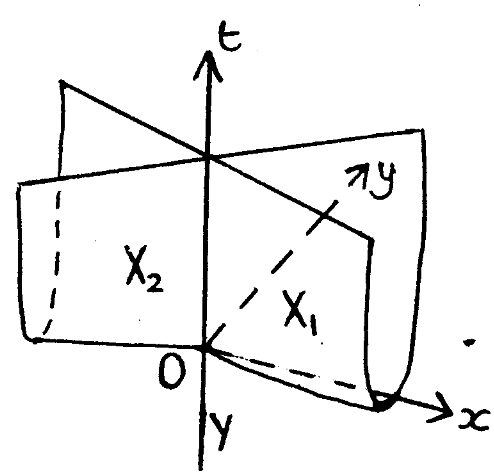
2) $V \equiv \{y^2 = tx^2\} \subset \mathbb{R}^3$.

Let Y be the t-axis, and X be $V - Y$.

Then set $X_1 = X \cap \{x > 0\}$, $X_2 = X \cap \{x < 0\}$.

X_1 and X_2 are neither (a)-regular over Y at 0, but are both (b')-regular over Y.

Again $X_1 \amalg X_2 \amalg Y$ is the full partition by dimension of V.



The fact that we do not get a Whitney stratification from the full partition by dimension of an analytic variety is only a minor handicap because of the following theorem.

Theorem(Whitney [45],[46]) : Every analytic variety admits an analytic Whitney stratification.

This is proved by showing that every locally finite analytic stratification (i.e. whose strata are locally analytic manifolds) admits an analytic Whitney stratification as a refinement : this is because (b)-regularity is <u>generic</u> — the set of points where (b) fails for a pair (X,Y) of analytic strata is contained in the complement of an open dense subset of Y .

The class of sets for which (b)-regularity is generic has been extended by Lojasiewicz [18] and Hironaka [12]. See also Hardt [10] and Gabrielov's thesis.

<u>Definition</u> : A subset of $\mathbb{R}^n$ which is globally (resp. locally at each point of $\mathbb{R}^n$) a finite union of subsets each of the form $\{ f_i = 0 , g_j > 0 \mid i=1,..,p; j=1,\ldots,q \}$ where the $\{f_i\}$, $\{g_j\}$ are polynomial (resp. analytic) functions on $\mathbb{R}^n$, is called <u>semialgebraic</u> (resp. <u>semianalytic</u>) .

<u>Theorem</u>(Lojasiewicz [18]) : <u>Every semianalytic set admits an analytic stratification, and every analytic stratification of a semianalytic set admits an analytic Whitney stratification as a refinement.</u>

A more accessible proof, for semialgebraic sets, was given by Wall [43].

<u>Definition</u> : A <u>subanalytic</u> set in $\mathbb{R}^n$ is the image of a semianalytic set in $\mathbb{R}^m$, some m , by a proper analytic map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

<u>Theorem</u>(Hironaka [12]) : <u>Every subanalytic set admits an analytic stratification, and every analytic stratification of a subanalytic set admits an analytic Whitney stratification as a refinement.</u>

So far we have discussed the existence of Whitney stratifications. Among the most important applications of Whitney regularity are the consequences of the following results.

Theorem A : Let Σ be a locally finite stratification of a closed subset of a C^1 manifold M . Σ is (a)-regular $\Longleftrightarrow$ the set of maps transverse to Σ is open in $C^1(N,M)$ for all C^1 manifolds N .

See §1 for a precise statement and proof of Theorem A .

Theorem B : A Whitney stratification is locally topologically trivial.

Theorem B was conjectured by Thom and proved by Mather [21] .

Neither Theorem A nor Theorem B makes use of analyticity. However in most of the work done either on the Whitney conditions themselves — as in Speder's thesis [29], and Teissier's study of the equisingularity of hypersurfaces [30] , [31], and the equimultiplicity theorem of Hironaka [11] — or using the Whitney conditions as tools — as in the proof of the topological stability theorem [7] , and the Lefschetz hyperplane theorems of Hamm and Lê [9] , and the extensions of characteristic class theory to singular varieties by MacPherson [19 , 20], and M.-H. Schwartz [26] — extensive use of the special properties of analytic varieties has been made. And it was for complex analytic hypersurfaces that Zariski demanded a theory of equisingularity [49,50] .

This thesis can be thought of as a study of aspects of the theory of equisingularity of smooth stratified sets, the plans of which were drawn in Thom's "Ensembles et morphismes stratifiés" [36] . When there are improvements in the case of subanalytic sets we give them; and we make special mention of any relations with complex hypersurfaces.

With Theorem B in mind, we make all our counterexamples topological manifolds-with-boundary, hence topologically trivial, whenever possible. This shows well the great difference in the nature of the results found here, and those obtained for complex hypersurfaces, for which topological triviality has fairly strong consequences, including (a)-regularity.

The basic local situation is as follows : let X and Y be C^1 submanifolds (and, when appropriate, subanalytic subsets) of $\mathbb{R}^n$, with $Y \subset \overline{X} - X$. Y is the <u>base stratum</u>, and X the <u>attaching stratum</u>. When X is (b)-regular over Y at 0 in Y, we will say that the pair (X,Y) is (b)-regular at 0, or that $(X,Y)_0$ is (b)-regular. When $(X,Y)_0$ is not (b)-regular, we say that $(X,Y)_0$ is a <u>(b)-fault</u> : we justify this term below.

<u>Faults and detectors</u> :

When some equisingularity condition E is not satisfied at a point of a stratification, it is natural to call the point an <u>E-fault</u> (so retaining the geological terminology). Many proofs showing that one equisingularity condition implies another are by <u>reductio ad absurdum</u> : we suppose that the second condition fails, and then we show that the first condition necessarily fails as well. When we can do this we say we have <u>detected the fault</u> (the point where the second condition fails). In the same way counterexamples to implications between equisingularity conditions tend to be faults which are not detectable in some given way. Most of the results given in this thesis consist of taking an equisingularity condition E and deciding whether possible detectors are effective or ineffective in detecting every E-fault. We hope that this will clarify and motivate the point of view taken throughout.

CHAPTER 1. WHITNEY (a)-REGULARITY

We begin by showing that (a)-regularity is precisely the condition to impose on a stratification in order that the maps transverse to the stratification form an open set, i.e. that transversality be stable, as well as being generic (the transverse maps always form a dense set). (a)-regularity was introduced by Whitney in [45] as a sufficient condition for this to be true ; at the time it was thought that (t)-regularity (defined in § 2) was the condition required, and that (a) was only useful in that it implied (t) (see the introduction to [45]). This is true in the analytic case, since then (t) and (a) are equivalent as proved in Theorem 2.5 below (and [37]), but we give examples (2.1 and 2.4) showing that (t) is in general weaker than (a). (a) is necessary and sufficient for openness : the sufficiency was proved in detail by E. A. Feldman in [5] and we prove necessity here in Theorem 1.1. The only difficulty in the proof is to find a transverse map with a given transverse 1-jet at a given point : for this we show that in a suitably chosen Baire subspace of the space of maps containing the given jet at the given point, transverse maps are dense.

Example 2.1 , showing (t) to be weaker than (a) in the smooth case, has (a) failing for a sequence on a curve (in the ambient space) tangent to the base stratum, thus defining an (a)-fault not detectable by transverse submanifolds. To show that the property that the (a)-fault be given by sequences tangent to the base stratum does not characterise those (a)-faults which are not detectable by transverse submanifolds, we give a second example (2.4) which uses a basic semialgebraic object called a "barrow", which is defined in 2.3 . We then prove, in Theorem 2.5 , that (t) is equivalent to (a) when curve selection is available, and obtain as a consequence in this case the conjecture of C. T. C. Wall [43] that (a)-regularity be equivalent to the condition that the fibres of a

C^1 retraction onto the base stratum be transverse to the attaching stratum for all retractions. We prove this conjecture in general as Theorem 3.3 after rephrasing the conjecture to read "do transverse C^1 foliations detect (a)-faults ? " Example 3.6 shows, using the barrows of 2.3 , that transverse C^2 foliations do not detect all (a)-faults.

To complete §2 we discuss results relating to a theorem of T.-C. Kuo , that (a)-regularity implies that transversals to the base stratum have germs at 0 of their intersection with the attaching stratum, of a single topological type, and we prove a partial converse to Kuo's theorem.

Finally in §4 we describe the analogues of the results proved here about (a)-regularity of stratified sets for the (a_f) condition on stratified morphisms.

1. (a)-regularity and stability of transverse maps

C^k topologies

First we briefly define the weak and strong C^k topologies on the space of C^k mappings between two C^k manifolds $(1 \leq k \leq \infty)$.

A thorough treatment of these topologies is given in Hirsch's book "Differential Topology" [13] . Other versions are given by Morlet [24] , Feldman [5] , and Golubitsky and Guillemin [8] .

Let N , P be C^k manifolds. $C^k(N,P)$ denotes the set of C^k mappings from N to P , $J^k(N,P)$ denotes the bundle of k-jets associated to such mappings, and $j^k : C^k(N,P) \longrightarrow C^0(N,J^k(N,P))$ is the associated jet map. The map $j^k f : N \longrightarrow J^k(N,P)$ is called the k-jet prolongation of f .

A basis for the weak C^k topology on $C^k(N,P)$ is given by taking all sets of the form $\{f \in C^k(N,P) : j^k f(K) \subset U\}$ where K is a compact subset of N , and U is an open subset of $J^k(N,P)$.

A basis for the <u>strong C^k topology</u> (also known as the Whitney C^k topology) on $C^k(N,P)$ is given by taking all sets of the form $\{f \in C^k(N,P) : j^k f(N) \subset U\}$ where U is an open subset of $J^k(N,P)$.

If N is compact these topologies are clearly the same.

Transversality

We shall use the notation $\pitchfork$ for "is transverse to".

If X, Y are C^1 submanifolds of a C^1 manifold M,

$$X \pitchfork Y \text{ at } m \iff T_m X + T_m Y = T_m M$$

$$X \pitchfork Y \iff X \pitchfork Y \text{ at } m, \ \forall\, m \in X \cap Y$$

If $f : N \longrightarrow M$ is a C^1 map,

$$f \pitchfork X \text{ at } n \iff T_{f(n)}X + (df)_n(T_n N) = T_{f(n)}M \quad \text{or} \quad f(n) \notin X$$

$$f \pitchfork X \iff f \pitchfork X \text{ at } n, \ \forall\, n \in f^{-1}(X)$$

If $z \in J^1(N,M)$ is a 1-jet, and $f \in C^1(N,M)$ is a map representing z (at $n \in N$

$$z \pitchfork X \iff f \pitchfork X \text{ at } n$$

We say X is transverse to a stratification Σ, and write $X \pitchfork \Sigma$, when $X \pitchfork S$ $\forall$ strata S of Σ.

We say X is transverse to a foliation $\mathcal{F}$ of M at x, and write $X \pitchfork \mathcal{F}$ at x, when X is transverse at x to the leaf of $\mathcal{F}$ through x.

We say a foliation $\mathcal{F}$ of a submanifold X is transverse at x to a foliation $\mathcal{G}$ of a submanifold Y, and write $\mathcal{F} \pitchfork \mathcal{G}$ at x, when the leaf of $\mathcal{F}$ through x is transverse at x to the leaf of $\mathcal{G}$ through x. (This requires that X be transverse to Y at x.)

Now we are in a position to state Theorem A of the introduction.

Theorem 1.1 Let Σ be a locally finite stratification of a closed subset V of a C^1 manifold M. Then the following conditions are equivalent :

(1) Σ is (a)-regular,

(2) for every C^1 manifold N, $\{z \in J^1(N,M) : z \pitchfork \Sigma\}$ is open in $J^1(N,M)$,

(3) for every C^1 manifold N, $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ is open in $C^1(N,M)$ with the strong C^1 topology,

(4) there is some integer r, $1 \leqslant r \leqslant \max(1, \min_{S \in \Sigma}(\dim S))$, and some C^1 manifold N with $\dim N = \dim M - r$, for which $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ is open in $C^1(N,M)$ with the strong C^1 topology.

Notes 1.2 (i) (1) $\Longleftrightarrow$ (2) is proved by Wall [44]. In fact he asserts that (2) implies that V is closed, which is not quite true. Consider the case where $V = M - \text{pt.}$, and Σ has a single stratum.

(ii) (1) $\Longrightarrow$ (3) is implicit in Thom [34] (1964) and explicit in [35,36], but see the discussion in §2 below. It was proved by Feldman [5], who describes Σ as cohesive if Σ is (a)-regular, and now appears as Exercise 15 at the end of Chapter 3 of Hirsch's "Differential Topology" [13]. Feldman's proof went unnoticed by several specialists in the theory to the extent that a very short false proof of (1) $\Longrightarrow$ (3) appeared several times (see the discussion and counterexample in §2), and in 1975, D. W. Bass [1] wrote "there seems to be no published proof of this". This was probably due to Feldman's use of the term "cohesive" before "(a)-regular" came into common usage ; also his proof appeared as a technical lemma in a paper on immersion theory rather than in a paper on stratification theory. Observe also that before the term "stratification" was accepted people talked of "submanifold complex" and "manifold collection".

(iii) We have the same theorem replacing C^1 everywhere by C^k $(1 \leqslant k \leqslant \infty)$, as the problem reduces to a study of 1-jets.

(iv) The set of C^k maps transverse to Σ $(1 \leqslant k \leqslant \infty)$ is dense in $C^k(N,M)$ with the strong C^k topology by applying Thom's transversality theorem countably often as in [8] or [13] , even without applying (a)-regularity. Thus if Σ is (a)-regular, the maps transverse to Σ in $C^k(N,M)$ form an open dense set in the strong C^k topology (C^1-open implies C^k-open).

(v) If each stratum is closed, then it follows from the result that for a closed submanifold W of M , $\{f \in C^k(N,M) : f \pitchfork W\}$ is open (see [8] or [13]), that $\{f \in C^k(N,M) : f \pitchfork \Sigma\}$ is open. But we do not assume the strata are closed (only that $V = |\Sigma|$ is closed) and in almost every situation of interest they will not be closed.

Proof of Theorem 1.1 : (2) implies (3) by definition of the strong topology. That (3) implies (4) is immediate. We shall prove that (1) implies (2), and that (4) implies (1), which will establish the equivalences.

(1) implies (2) :

Suppose (2) is not satisfied for some C^1 manifold N . Then there is a 1-jet $z \in J^1(N,M)$, with $z \pitchfork \Sigma$ and a sequence $\{z_n\} \in J^1(N,M)$ such that z_n tends to z as n tends to ∞, but for all n , z_n is not transverse to Σ . Let ν , μ denote the maps $J^1(N,M) \longrightarrow N$, $J^1(N,M) \longrightarrow M$, taking source and target respectively. Let $x = \nu(z)$, $x_n = \nu(z_n)$, $y = \mu(z)$, $y_n = \mu(z_n)$. Since $z \pitchfork \Sigma$ and $z_n \not\pitchfork \Sigma$, for all sufficiently large n we have that $y_n \neq y$. Also clearly $y_n \in V$ for all n . Since V is closed, and since $y_n \longrightarrow y$ $(n \to \infty)$ we have that $y \in V$. Let S be the stratum of Σ containing y . Since Σ is locally finite, we can suppose (by taking a subsequence) that for all n , y_n belongs to the same stratum S' . $S' \neq S$ since S is a C^1 submanifold. Thus $y \in S \cap (\bar{S}' - S')$ and S' is (a)-regular over S by the hypothesis (1).

Now by means of a chart for M at y we can identify all the tangent spaces (and their subspaces) to M at points near y, with $\mathbb{R}^m$ (and its subspaces) , where $m = \dim M$.

Let P_n (resp. P) denote the vector subspace of $\mathbb{R}^m$ determined by the jet z_n (resp. z) $\forall$ n . By choosing a further subsequence we can suppose that the dimension of P_n is constant for all n. (It is possible however that the dimension of P is less than that of P_n.) Because grassmannians are compact we may suppose by taking more subsequences that $\{P_n\}$ tends to a limit P_∞ and $\{T_{y_n}S'\}$ tends to a limit τ. Then $P \subseteq P_\infty$, and, since S' is (a)-regular over S, $T_yS \subseteq \tau$.

$z \pitchfork \Sigma$ means that $P \pitchfork T_yS$, and so $P_\infty \pitchfork \tau$. Then $\exists\, \varepsilon > 0$ such that if $d(P_\infty, Q) < \varepsilon$ $(Q \in G^m_{\dim P_\infty})$, and $d(\tau, T) < \varepsilon$ $(T \in G^m_{\dim S'})$, then $Q \pitchfork T$ (transversality is an open condition on vector subspaces). Now choose n_1 such that $\forall\, n \geq n_1$, $d(P_\infty, P_n) < \varepsilon$, and n_2 such that $\forall\, n \geq n_2$, $d(\tau, T_{y_n}S') < \varepsilon$. Then $\forall\, n \geq \max(n_1, n_2)$, $P_n \pitchfork T_{y_n}S'$, i.e. $z_n \pitchfork \Sigma$, contradicting the choice of $\{z_n\}$, and proving that (1) implies (2).

<u>(4) implies (1)</u>:

Suppose that Σ is not (a)-regular. Then there is a point y in V contained in a stratum Y of Σ $(\dim Y \geq 1)$, and a sequence of points $\{x_i\}$ of V in a stratum X of Σ such that $x_i \longrightarrow y$ as $i \longrightarrow \infty$, and $T_{x_i}X \longrightarrow \tau$ as $i \longrightarrow \infty$, and there is a vector $v \in T_yY$ such that $v \notin \tau$. Let E be the 1-dimensional subspace of T_yM spanned by v. Choose a basis for T_yM such that

$$T_yY = E \oplus W_1 \oplus T_1$$
$$\tau = T_1 \oplus T_2$$
$$T_yM = E \oplus W_1 \oplus W_2 \oplus T_1 \oplus T_2$$

where T_1, T_2, W_1, W_2 are vector subspaces of T_yM and T_1, W_1, W_2 are perhaps empty. Then find a subspace H of T_yM with $\dim H = m - r$ $(= \dim N)$, such that $T_2 \oplus W_2 \subseteq H \subseteq T_1 \oplus T_2 \oplus W_1 \oplus W_2$ (this is possible since $1 \leq r \leq \dim Y$). Then $H + T_yY = T_yM$, but $H + \tau \neq T_yM$. Let $p \in N$, and define

$$\mathcal{D}_H = \{ f \in C^1(N,M) : f(p) = y,\ (df)_p(T_pN) = H \}.$$

<u>Lemma 1.3</u>: $\exists\, g \in \mathcal{D}_H$ <u>such that</u> $g \pitchfork \Sigma$.

Choose a chart (W,ψ) for N at p such that $g|_W$ is an embedding (if $g\in\mathcal{D}_H$, $(dg)_p$ has maximal rank), and choose a chart (U,ϕ) for M at y such that $g(W)\subset U$. Then it is not hard, since we have reduced the problem to one for $C^1(\mathbb{R}^{m-r},\mathbb{R}^m)$, to construct, for each i such that $x_i\in U$, an f_i in $C^1(N,M)$ such that,

(i) $f_i|_{N-W} = g|_{N-W}$,

(ii) $f_i|_W$ is an embedding,

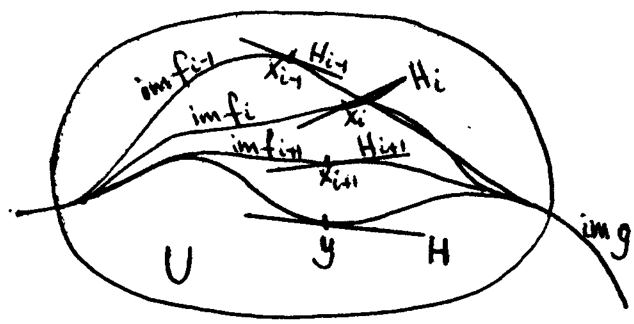
(iii) $f_i(W)\subset U$, $f_i(p)=x_i$,

(iv) $(df_i)_p(T_pN) = H_i \subseteq T_{x_i}X\oplus W_1\oplus W_2$,

for i sufficiently large, where we have considered W_1, W_2 as subspaces of $T_{x_i}M$.

(v) $H_i \longrightarrow H$ $(i\longrightarrow\infty)$,

(vi) $f_i\longrightarrow g$ $(i\longrightarrow\infty)$ in the strong C^1 topology.



Then for each sufficiently large i, f_i is not transverse to X at x_i, since $E\not\subset H_i + T_{x_i}X$, i.e. f_i is not transverse to Σ. But by the lemma, $\lim f_i = g$ is transverse to Σ, thus we have a contradiction to the hypothesis of (4) that the set of maps transverse to Σ is open in $C^1(N,M)$, completing the proof that (4) implies (1).

<u>Proof of lemma 1.3</u> : Choose charts (U,ϕ) for M at y, (W,ψ) for N at p, and a C^1 map $h: N\longrightarrow M$ such that $h(W)\subset U$, $h|_W$ is an embedding, $h(p)=y$, and $(dh)_p(T_pN)=H$. Let $W'\subset W$ be an open set containing p, with compact closure $\overline{W'}\subset W$. Then $\exists\ \delta>0$ such that if $f\in\mathcal{V}_{\delta,\overline{W'}}(h)$, which is $\{f\in C^1(N,M) : |j^1f(x) - j^1h(x)|<\delta\ \forall x\in\overline{W'}\}$, then $f|_{W'}$ is an embedding (see [13], Chapter 2, Lemma 1.3). Let $\overline{\mathcal{V}_{\delta,\overline{W'}}(h)}$ denote the weak C^1 closure of the weakly open set $\mathcal{V}_{\delta,\overline{W'}}(h)$, and let $\mathcal{E}_H = \mathcal{D}_H\cap\overline{\mathcal{V}_{\delta/2,\overline{W'}}(h)}$. Then $\mathcal{E}_H$ is weakly C^1 closed in $C^1(N,M)$. For, consider any limit point f_0 of a convergent sequence in $\mathcal{D}_H$ with the weak C^1 topology. Clearly $f_0(p)=y$

and $(df_0)_p(T_pN) \subseteq H$; however the inclusion can be strict : the rank of f can drop at p . But if $f_0 \in \overline{\mathcal{V}_{\delta/2,\overline{W}'}(h)} \subset \mathcal{V}_{\delta,\overline{W}'}(h)$, f_0 has maximal rank at p since $f_0|_{W'}$ is an embedding by choice of δ . Thus $(df_0)_p(T_pN) = H$, and $f_0 \in \mathcal{D}_H$. Hence $\mathcal{E}_H$ is weakly C^1 closed. Now we quote

<u>Theorem 1.4</u> : <u>Any weakly</u> C^k <u>closed subspace of</u> $C^k(N,M)$ <u>is a Baire space in the strong</u> C^k <u>topology</u> $(1 \leq k \leq \infty)$.

<u>Proof</u>. See [13] , Chapter 2, Theorem 4.4 , or [24] .

Using this result we can now apply the usual procedure of the Thom transversality theorem (as in [8] , or [13]) to prove that $\{f \in \mathcal{E}_H : f \pitchfork \Sigma\}$ is strongly dense in $\mathcal{E}_H$. Cover each stratum S of Σ by countably many compact coordinate discs $\{K^S_\alpha\}^{S\in\Sigma}_{\alpha\in A}$ such that if $y \in K^Y_{\alpha(y)}$ then no other K^Y_α contains y , and if $f \in \mathcal{E}_H$, then $f(\overline{W}') \cap K^Y_{\alpha(y)} = y$. Now verify that for each S and each α , $\{f \in \mathcal{E}_H : f \pitchfork S$ on $K^S_\alpha\}$ is open and dense in $\mathcal{E}_H$ with the strong C^1 topology. The proof of this is a local argument near K^S_α and goes through as for the standard proof in $C^1(N,M)$ by the choice of $K^Y_{\alpha(y)}$. (Given $f \in \mathcal{E}_H$, f not transverse to Y on $K^Y_{\alpha(y)}$, we can find an arbitrarily small perturbation of f to a map $g \in \mathcal{E}_H$ which is transverse to Y on $K^Y_{\alpha(y)}$, and such that $g|_{\overline{W}'} = f|_{\overline{W}'}$.) Because there are countably many strata (Σ being assumed locally finite) , and because $\mathcal{E}_H$ is a Bairé space in the strong C^1 topology (Theorem 1.4) , we deduce that

$$\{f \in \mathcal{E}_H : f \pitchfork S \text{ on } K^S_\alpha , \forall \alpha , \forall S\} = \{f \in \mathcal{E}_H : f \pitchfork \Sigma\}$$

is strongly dense in $\mathcal{E}_H$. Since $\mathcal{E}_H \neq \emptyset$, as $h \in \mathcal{E}_H$, we have shown the existence of some g in $\mathcal{E}_H$, and hence in $\mathcal{D}_H$, with $g \pitchfork \Sigma$. This completes the proof of Lemma 1.3 .

<u>Notes on the proof</u> : 1. It is not clear if $\mathcal{D}_H$ is a Baire space . This is the reason for introducing $\mathcal{E}_H$ in the proof of Lemma 1.3 . Certainly $\mathcal{D}_H$ is

not weakly closed, since the rank at p of a limit map may be less than the rank of the maps of a sequence in $\mathcal{D}_H$, convergent in $C^1(N,M)$.

2. The proof of (4) implies (1) shows that if there is a C^1 manifold N with $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ open, then Σ is (a)-regular over the strata of dimension $\geq \dim M - \dim N$.

2. (a)-regularity and transverse submanifolds

Consider the following condition on a pair of adjacent strata (X,Y) at a point $0 \in Y \cap (\overline{X} - X)$, with X, Y C^1 submanifolds of $\mathbb{R}^n$.

(t) Given a C^1 submanifold S of $\mathbb{R}^n$ transverse to Y at 0, there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that S is transverse to X in U.

If (t) is satisfied for $(X,Y)_0$ we say X is (t)-regular over Y at 0. If X is (t)-regular over Y for each point in $Y \cap (\overline{X} - X)$ we say X is (t)-regular over Y. If each pair of adjacent strata of a stratification are (t)-regular, then Σ is a (t)-regular stratification.

Since spanning is an open condition, it follows at once that (a) implies (t). The false argument referred to above to prove (1) implies (3) of Theorem 1.1 is

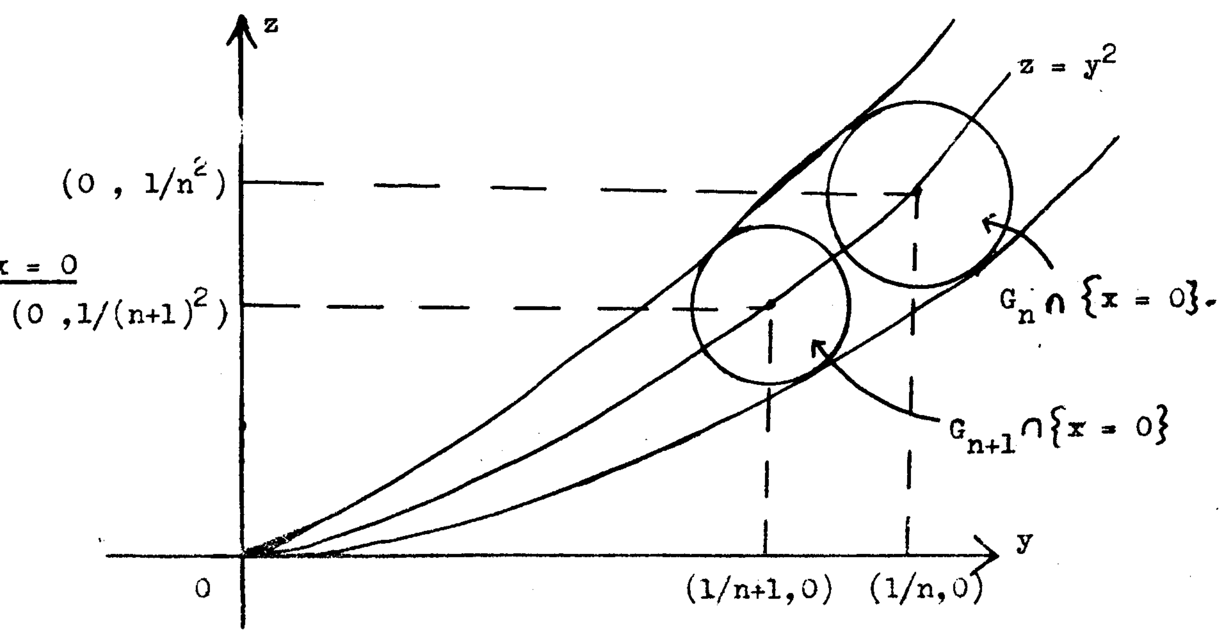
$$\Big[(a) \text{ implies } (t)\Big] \text{ implies } \Big[\underset{(1)}{\underset{|||}{(a)}} \text{ implies } \underbrace{\text{openness of transverse maps}}_{(3)}\Big]$$

This suggests that (t) implies the openness of transverse maps, which is false in general, although true in the case of subanalytic strata (or any situation where the curve selection lemma is available), as proved in Theorem 2.5 below. Thom, in [34] mentioned that (t) implied that the transverse maps formed an open set in the semialgebraic case. In [35] he used this to deduce that (a) implies that the transverse maps are open, again using analyticity. The mistake first occurs in [36] where he repeats the argument, but does not assume analyticity. The error was then copied by Wall [41], Trotman [37], and Chenciner [4]. Although [37] contains an example showing that (t) does not imply (a), I did not then realise that (a) was equivalent to the openness of transverse maps, and missed the fact that the example there was actually a counterexample to (t) implies openness. A fortuitous remark by E. Bierstone at Oslo in August 1976 led to the recognition of the counterexample which follows.

<u>Example 2.1</u>. <u>A</u> (t)-<u>regular stratification which is not</u> (a)-<u>regular</u> [39].

Let (x, y, z) be coordinates in $\mathbb{R}^3$. Take Y to be the y-axis, and let $X = \left(\bigcup_{n=1}^{\infty}\{f_n = 0,\ g_n \leq 0\}\right) \cup \left(\bigcap_{n=1}^{\infty}\{x = 0,\ g_n \geq 0,\ z > 0\}\right)$ where $\{g_n \leq 0\}$ defines the cylinder G_n of radius $1/3n(n+1)$ with axis the line $\{y = 1/n,\ z = 1/$ and where $\{f_n = 0\}$ defines the surface F_n obtained from $\{x = ((y^2 + z^2) - \frac{1}{2})^2\}$ by translating the origin to $(0, 1/n, 1/n^2)$ and reducing by a factor of $3n(n+1)/\sqrt{2}$ so that F_n intersects ∂G_n exactly where $\{x = 0\}$ is tangent to F_n.

<u>Figure: x = 0</u>



X is a C^1 submanifold and is semialgebraic on the complement of the origin. The normal vector to X at the point

$$x_n = \left(1/24\sqrt{2}n(n+1),\ (1/n) + 1/3\sqrt{2}n(n+1),\ 1/n^2\right)$$

is $(2 : 1 : 0)$ for all n. Hence the limit as n tends to ∞ is $(2 : 1 : 0)$

and (a) fails. (For (a) to hold, all limits of normals would have to be of the form $(c_1 : 0 : c_2)$, where c_1, c_2 are not both zero.)

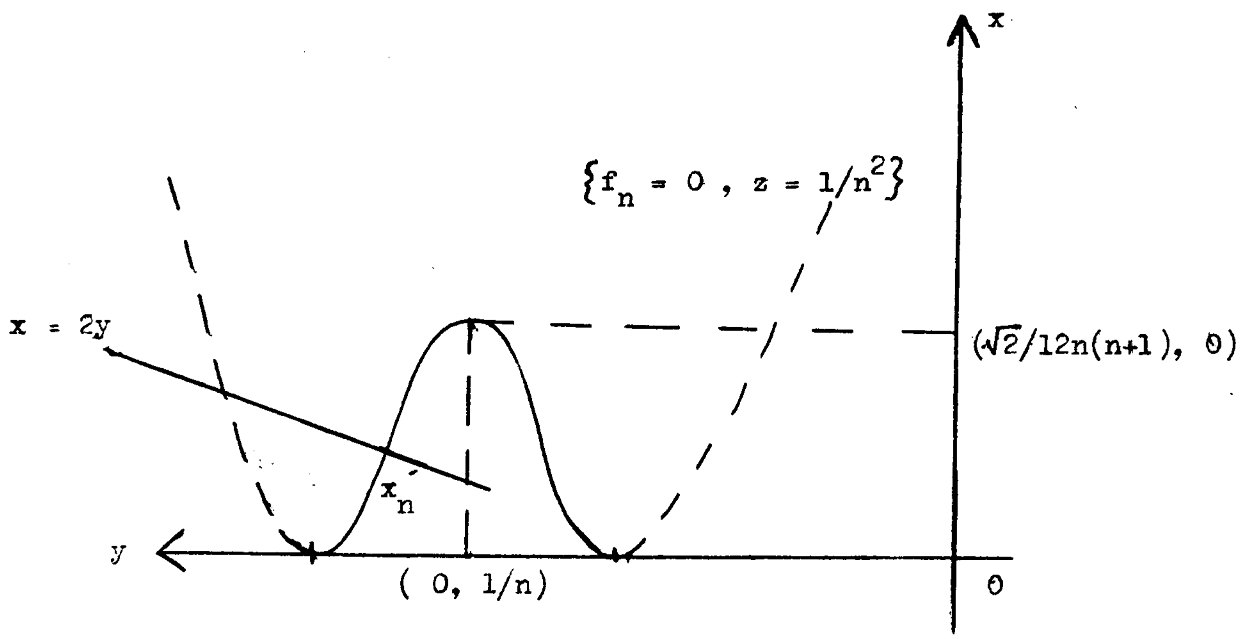


Figure: $z = 1/n^2$

(t) holds since any submanifold transverse to Y will intersect X near Y only at points near which X is defined by $\{x = 0\}$. Hence the stratification Σ of $\mathbb{R}^3$ defined by $\{Y, X, \mathbb{R}^3 - (X \cup Y)\}$ is (t)-regular.

Now we verify explicitly that the set of maps transverse to Σ is not open. The mapping h in $C^1(S^2, \mathbb{R}^3)$ defined by inclusion of the sphere of radius 1 and tangent $\{2x + y = 0\}$ at 0 and with centre at $(-1/\sqrt{5}, -2/\sqrt{5}, 0)$ is transverse to the stratification, but for each n the mapping h_n defined by inclusion of the unit sphere with tangent at x_n the plane

$$\{2x + y = (5 + 12\sqrt{2}(n+1))/(12\sqrt{2}n(n+1))\}$$

and with 0 in the bounded component of $\mathbb{R}^3 - h_n(S^2)$, is not transverse to X at x_n. Since $\{h_n\}$ tends to h in the weak C^1 topology, which is also the strong C^1 topology (since S^2 is compact), the set of mappings transverse to Σ is not open in $C^1(S^2, \mathbb{R}^3)$.

Thus (t) cannot replace (a) in the statement of Theorem 1.1.

Note that by smoothing near each circle $\{x = 0 , g_n = 0\}$, X can be made into a C^∞ submanifold of $\mathbb{R}^3$, with the normal vector to X at each x_n as before, for all n , thus producing a C^∞ counterexample.

Construction 2.2 (Hills, or Round Barrows)

The example above used a simple construction of a C^1 semialgebraic hill which will prove useful as a building block for both examples and proofs of theorems. Consider the curve $\{x = (y^2 - 1)^2\}$ in $\mathbb{R}^2$: it has tangent parallel to the y-axis for $y = \pm 1$.

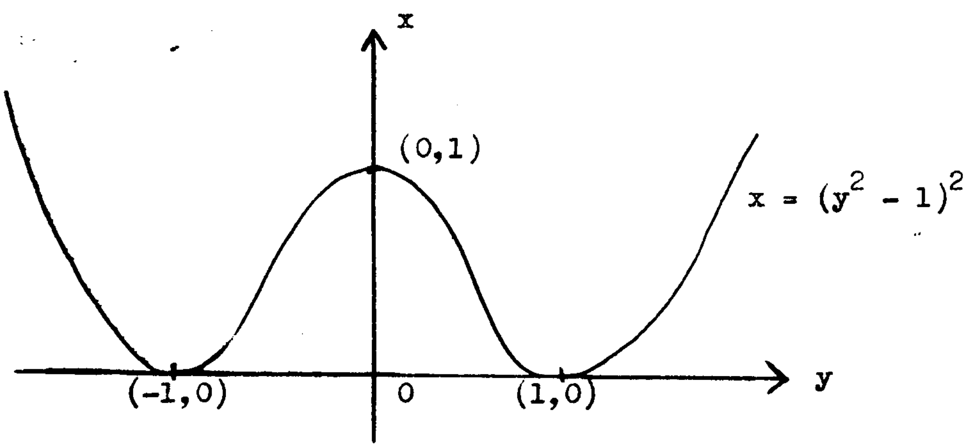


Figure : Hill of dimendion one

Rotating in $\mathbb{R}^3$ about the x-axis, and cutting around the circle $\{y^2 + z^2 = 1 , x = 0\}$ and then inserting in the plane $\{x = 0\}$ with the disc $\{y^2 + z^2 \leq 1 , x = 0\}$ removed, gives a C^1 semialgebraic manifold. The vital property of the curve $\{x = (y^2 - 1)^2\}$ which will be used again and again is that in the region $\{y^2 \leq 1\}$ the tangent to the curve is furthest from $\{x = 0\}$ when $y = \pm 1/\sqrt{3}$, and at the points $(4/9 , \pm 1/\sqrt{3})$ the normal is $(1 : \pm 8/3\sqrt{3})$.

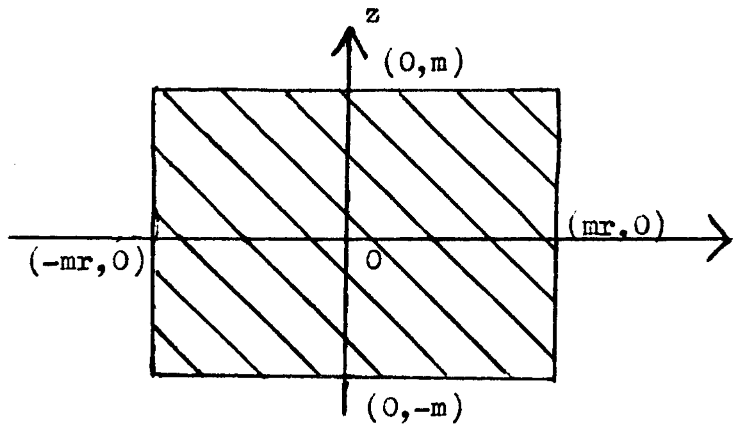
Construction 2.3 (Long Barrows)

Consider the surface in $\mathbb{R}^3$ with coordinates x , y , z ,

$$m^7 r^3 x = (m^2 - z^2)^2 (m^2 r^2 - y^2)^2$$

where $m , r \in [0 , \infty)$. The normal to the surface at (x , y , z) is

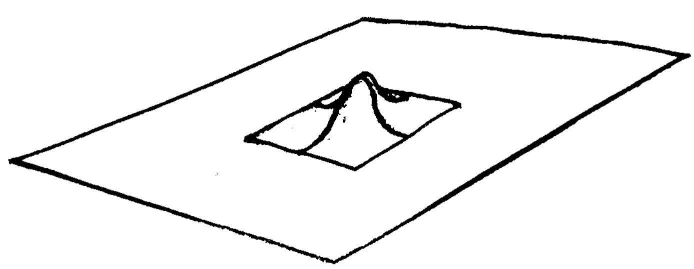
$$(m^7r^3 : 4(m^2 - z^2)^2(m^2r^2 - y^2)y : 4(m^2r^2 - y^2)^2(m^2 - z^2)z) .$$



On $\{z^2 = m^2, x = 0\}$ and $\{y^2 = m^2r^2, x = 0\}$ the normal is $(1 : 0 : 0)$, and thus we can cut along these lines to obtain the surface

$$B(m,r) \equiv \{m^7r^3x = (m^2 - z^2)^2(m^2r^2 - y^2)^2 , z^2 \leqslant m^2 , y^2 \leqslant m^2r^2\}$$

and we can insert $B(m,r)$ in the plane $\{x = 0\}$ with a rectangle $\{ x = 0 , z^2 \leqslant m^2 , y^2 \leqslant m^2r^2\}$ removed, to give a C^1 semialgebraic manifold.



At (mrx , mry , mz) for $z^2 \leqslant 1$, $y^2 \leqslant 1$, the normal is now $(1 : 4y(1 - z^2)^2(1 - y^2) : 4rz(1 - z^2)(1 - y^2)^2)$. Thus as m varies $B(m,r)$ varies in size, but the tangent structure (that is the set of points in $P^2(\mathbb{R})$ defined by the normals or tangents to the surface) remains the same. But as r varies the normals change, and as r tends to 0 the normals tend to lie in the arc of lines $\{ (1 : \frac{8\lambda}{3\sqrt{3}} : 0) : \lambda \in [- 1 , 1]\}$

We call this surface $B(m,r)$ a <u>(long) barrow of magnitude</u> m , <u>ratio</u> r <u>with axis</u> Oz , <u>and centre</u> O , <u>and base</u> yOz . The axis, centre, and base will always be specified. Calculation shows that for $r < \sqrt{3}/4$, the normal to the surface is furthest from $(1 : 0 : 0)$ when $y = \pm mr/\sqrt{3}$ and $z = 0$, and at these points , $(4mr/9, \pm mr/\sqrt{3}, 0)$, the normal is $(1 : \pm 8/3\sqrt{3} : 0)$. (Compare Construction 2.2)

<u>Linguistic Note</u> : The term <u>barrow</u> is used because of the resemblance of the surface to the ancient burial mounds called by that name in England, when r is small.

<u>Example 2.4</u> : This will show that the phenomenon that (t) be strictly weaker than (a) is not solely due to the possibility of (a)-faults given by sequences tangent to the base stratum as in Example 2.1 : that is, it is not true that (a) holds for those sequences on curves with limiting direction not tangent to the base stratum.

In $\mathbb{R}^3$ with coordinates (x, y, z) let Y be the y-axis, and let X be $\left(\bigcup_{n=1}^{\infty} \{f_n = 0 , g_n \leqslant 0\}\right) \cup \left(\bigcap_{n=1}^{\infty} \{x = 0 , g_n \geqslant 0 , z > 0\}\right)$ where $\{f_n = 0\}$ is the equation defining the barrow $B(m_n, r_n)$ with centre $(0, 1/n, 1/n)$ and axis $\{x = 0 , z + y = 2/n\}$, with base in the plane $\{x = 0\}$, and $\{g_n \leqslant 0\}$ defines the interior of the rectangular base of the barrow. X is a C^1 manifold, and is semialgebraic on the complement of the origin in $\mathbb{R}^3$. We choose $\{(m_n, r_n)\}_{n=1}^{\infty}$ such that,

(1) r_n tends to 0 as n tends to ∞,

(2) the barrows are pairwise disjoint (in particular m_n tends to 0),

(3) m_n tends to 0 fast enough so that the n^{th} barrow $B(m_n, r_n)$ is contained in the 2-sphere with centre $(0, 1/n, 1/n)$ and radius $1/2n^2$ (so $m_n = 1/4n^2$ will do).

By (1) the set of limiting normals is exactly $\{(1 : (4\sqrt{2}/3\sqrt{3})\lambda : (4\sqrt{2}/3\sqrt{3})\lambda) : 0 \leqslant |\lambda| \leqslant 1\}$. (Cf. Construction 2.3) Thus (a) fails, since for (a) to hold <u>all</u> limiting normals must be of the form $(c_1, 0, c_2)$.

By (3) the set of barrows is contained in the horn which is tangent to $\{z = y , x = 0\}$ and which intersects the plane $\{z + y = 2t\}$ in a circle of radius t^2 . Hence a C^1 submanifold S transverse to Y at 0 intersects infinitely many barrows only if $\{z = y\ ,\ x = 0\} \subset T_0S$. But then S will be transverse to all barrows in some neighbourhood of 0 . For, suppose S were nontransverse to infinitely many barrows ; then N_0S would be one of the limiting $(\ 1 : (4\sqrt{2}/3\sqrt{3})\lambda : (4\sqrt{2}/3\sqrt{3})\lambda\ )$. But $\{z = y\ ,\ x = 0\} \subset T_0S$, and S is transverse to $\{x = 0\ ,\ z = 0\}$ at 0 , thus N_0S is of the form $(\ \mu : \nu : -\nu\ )$ with $\nu \neq 0$, which is not a limiting normal to X .

Thus we have shown that (t) holds and that (a) fails along sequences which are not tangent to Y .

As in example 2.1 , by smoothing near the base of each barrow we obtain a C^∞ example.

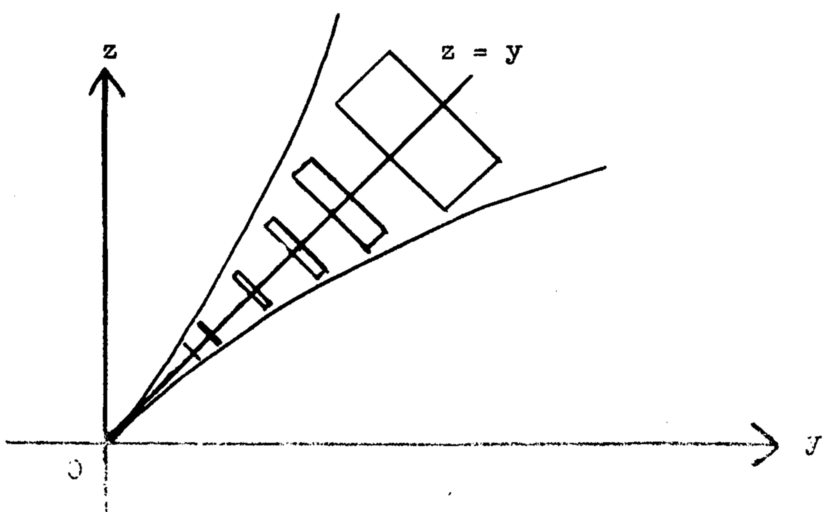


Figure : $x = 0$.

Now we shall prove that (t) and (a) are equivalent in the subanalytic case. Precisely, we have the following result.

Theorem 2.5 : Let X , Y be C^1 submanifolds of R^n and let $0 \in Y \cap (\overline{X} - X)$ and let X be a subanalytic set. Then X is (a)-regular over Y at 0 if and only if for every semianalytic C^1 submanifold S transverse to Y at 0 there is some neighbourhood U of 0 in which S is transverse to X .

The proof will depend upon two technical lemmas which we display for future reference.

Curve Selection Lemma 2.6 : Let U be a subanalytic subset of the analytic space A, and let $0 \in \overline{U}$. Then there is an analytic arc

$$\alpha : [0, 1] \longrightarrow A$$

such that $\alpha(0) = 0$, $\alpha(t) \in U$ if $t \neq 0$.

Proofs of Lemma 2.6 : (1) Subanalytic U : Hironaka [12, Proposition 3.9].

(2) Semianalytic U : Lojasiewicz [18, page 103].

(3) Semialgebraic U : Milnor [23, Chapter 2].

(Of course, (1) implies (2), and (2) implies (3).)

Lemma 2.7 : Let X^m be a C^1 submanifold of $\mathbb{R}^n$, and a subanalytic subset of $\mathbb{R}^n$. Then $\{(x, T_xX) : x \in X\}$ is a subanalytic subset of $\mathbb{R}^n \times G^n_m(\mathbb{R})$.

Proof : See Verdier [40, Lemma 1.6].

Lemma 2.7, with semianalytic replacing subanalytic each time, follows after partition into real analytic manifolds from the proof of Whitney [47] for complex analytic varieties. A short proof of Lemma 2.7, with semialgebraic replacing subanalytic each time, appears in Gibson [6, page 30].

Proof of Theorem 2.5 : Only if - this is immediate since spanning (and hence transversality) is an open condition.

If - Suppose (a) fails. Thus there is a unit vector $v \in T_0Y$, a sequence $\{x_i\} \in X$ such that x_i tends to 0, and $T_{x_i}X$ tends to a limit τ, and $v \not\subset \tau$.

Choose $\mathcal{E} > 0$ and $i_0 \in \mathbb{N}$ such that $\forall\, i \geq i_0$, $d(v, T_{x_i}X) > \mathcal{E}$, where $d(v, P)$ denotes the distance between $P \in G^n_m(\mathbb{R})$ and the endpoint of the unit vector v, both considered as subspaces of $\mathbb{R}^n$ at 0.

Define $V_1 = \mathbb{R}^n \times \{P \in G^n_m(\mathbb{R}) : d(v,P) > \mathcal{E}\} \subset \mathbb{R}^n \times G^n_m(\mathbb{R})$

$V_2 = \{(x, T_xX) : x \in X\} \subset \mathbb{R}^n \times G^n_m(\mathbb{R})$.

V_1 is semialgebraic, and V_2 is subanalytic by Lemma 2.7 , since X is assumed to be subanalytic. Semialgebraic sets are subanalytic, and the finite intersection of subanalytic sets is subanalytic (by Hironaka [12]). Hence $V_1 \cap V_2$ is subanalytic and $(0, \tau) \in \overline{V_1 \cap V_2}$ satisfies the hypotheses of the curve selection lemma 2.6. Thus there is an analytic arc in $\mathbb{R}^n \times G^n_m(\mathbb{R})$ (which is an analytic, even algebraic, manifold) ,

$$\alpha : [0, 1] \longrightarrow \mathbb{R}^n \times G^n_m(\mathbb{R})$$

with $\alpha(0) = (0, \tau)$ and $\alpha(t) \in V_1 \cap V_2$ if $t > 0$.

Write $\alpha_1(t)$ for the $\mathbb{R}^n$-component of $\alpha(t)$; the $G^n_m(\mathbb{R})$-component is $T_{\alpha_1(t)}X$. Let $N_t \in G^n_{n-1}(\mathbb{R})$ denote the normal space at $\alpha_1(t)$ to the C^1 manifold-with-boundary $\alpha_1([0, 1])$, and let the vector v_t be the projection of v into N_t spanning $\langle v_t \rangle \in G^n_1(\mathbb{R})$.

We shall define an analytic arc $\sigma : [0,1] \longrightarrow G^n_{n-2}(\mathbb{R})$ such that

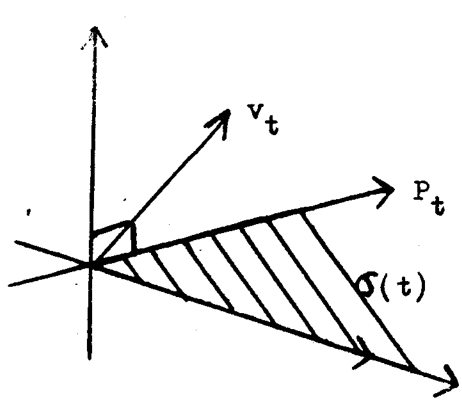
$$\sigma(t) \oplus \langle v_t \rangle = N_t \qquad (*)$$

Then the union of the $\{\sigma(t)\}$, considered as embedded $(n-2)$-planes in $\mathbb{R}^n$ passing through the points $\alpha_1(t)$ defines an analytic manifold-with-boundary S' of dimension $(n-1)$. Reflection in N_0 extends S' to a C^1 manifold S which is a semianalytic subset of $\mathbb{R}^n$, and which is transverse to Y at 0 by $(*)$. However we shall show that no neighbourhood U of 0 exists within which S is transverse to X .

<u>Construction of σ</u> :

Let $P_t = N_t \cap T_{\alpha_1(t)}X \in G^n_{m-1}(\mathbb{R})$. Then $0 \neq v_t \notin P_t$ by definition of $V_1 \cap V_2$. Let $\sigma(t) = P_t \oplus (P_t \oplus \langle v_t \rangle)^{\perp} \in G^n_{n-2}(\mathbb{R})$, where $(\ )^{\perp}$ denotes orthogonal complement in N_t .

<u>Figure</u> : N_t $(n=4, m=2)$.



σ satisfies (*) by construction, and so it only remains to show that S fails to be transverse to X in any given neighbourhood U of 0 . Now there exists some $t_0 \in (0,1]$ such that $U \cap \alpha_1(0,1] \supset \alpha_1(0,t_0]$. But S' (and hence S) is not transverse to X at any point of $\alpha_1(0,1]$. For, if A_t denotes the tangent space to the curve $\alpha_1(0,1]$ at $\alpha_1(t)$,

$$T_{\alpha_1(t)}X = P_t \oplus A_t \subset \sigma(t) \oplus A_t = T_{\alpha_1(t)}S .$$

This completes the proof of Theorem 2.5.

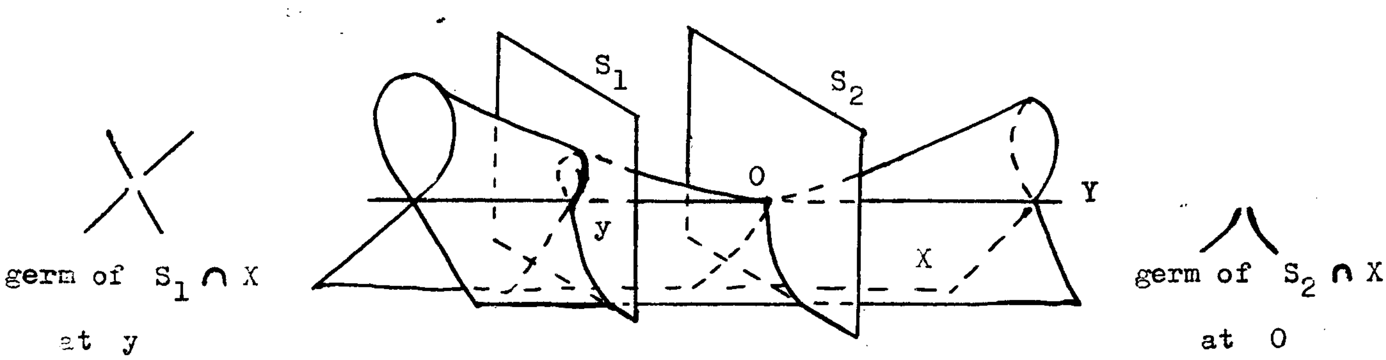
Note 2.8 : Even if X and Y are C^∞ submanifolds we cannot restrict to C^∞, or even C^2, semianalytic submanifolds S , since (a) may fail only near a cusp of type "$y^2 = x^3$" , each branch of which is a C^1 manifold-with-boundary, but not a C^2 manifold-with-boundary. The same type of example excludes restricting to analytic submanifolds S , although by the proof of 2.5 we can restrict to analytic submanifolds-with-boundary S , since the statement that S be transverse to Y at 0 still makes sense if $0 \in Y \cap \partial S$. The proof of 2.5 also shows that we can restrict to those S which are " ruled submanifolds " , that is a differentiable one-dimensional family of planes of codimension 2 in $\mathbb{R}^n$. Moreover it suffices to consider all submanifolds of some fixed dimension greater than or equal to the codimension of Y , by a small adjustment in the proof (choose $\sigma_1(t) \subset \sigma(t)$, where $\sigma_1(0) + T_0Y = N_0$, $\sigma_1(t) \in G^n_{c-1}(\mathbb{R})$, and $c \geqslant \operatorname{codim} Y$).

T.-C. Kuo has recently proved the following result, which is related to the questions already treated in this section.

Theorem 2.9 (Kuo) : Let X , Y be C^∞ submanifolds of $\mathbb{R}^n$, $Y = \overline{X} - X$ in some neighbourhood of Y . Suppose X is (a)-regular over Y at $0 \in Y$. Let S_1 , S_2 be C^∞ submanifolds transverse to Y at 0 , with $\dim S_i = n - \dim Y_i$ ($i = 1, 2$). Then the germs of $S_1 \cap X$ and $S_2 \cap X$ at 0 are homeomorphic.

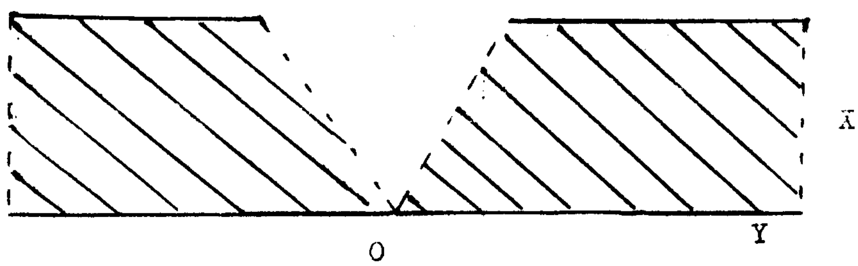
Proof : In [15] .

This is an attractive result since it parallels the Thom-Mather theorem (Theorem B of the introduction) that (b)-regularity implies topological triviality. Explicitly, if X is (b)-regular over Y and S_1 and S_2 are two submanifolds transverse to Y at points y_1 and y_2 in Y (with $y_1 \neq y_2$ allowed) , then the germs of $S_1 \cap X$ at y_1 and $S_2 \cap X$ at y_2 are homeomorphic. This follows from Corollary 10.6 of [21] . (a)-regularity is definitely insufficient for the latter property as shown by the figure below.



Conjecture 2.10 : Theorem 2.9 is true with the weaker hypothesis that X be (t)-regular over Y at 0 .

Observe that the hypothesis $Y = \overline{X} - X$ rather than $Y \subset \overline{X} - X$ is essential in 2.9 and 2.10 , as shown by the next figure.



We might also ask if the converse of Theorem 2.9 is true. However examples 2.1 and 2.4 show that this is not so. In both examples X is not (a)-regular over Y at 0 , but any C^1 submanifold transverse to Y (at 0) intersects

X in a topological open half-line near 0. We do though have a converse to 2.9 if we replace (a)-regularity by (t)-regularity as in Theorem 2.11 below.

Definition : Let X, Y be C^1 submanifolds of $\mathbb{R}^n$, and $0 \in Y \subset \overline{X} - X$. The pair (X, Y) is said to have homeomorphic C^k transversals of dimension s at 0 ($1 \leq k \leq \infty$, codim $Y \leq s \leq n$) if,

(h_s^k) Given a C^k submanifold S of dimension s transverse to Y at 0, the topological type of the germ of $S \cap X$ at 0 is independent of S.

Theorem 2.9 says that (a) implies $(h^\infty_{\text{cod } Y})$. From the proof of 2.9 [15], one sees that (a) implies $(h^2_{\text{cod } Y})$, but it is left in doubt whether (a) implies $(h^1_{\text{cod } Y})$ since the proof makes use of a (tangent) vector field in a blowing-up.

Write (t_s^k) for condition (t) restricted to those C^1 submanifolds S of class C^k ($1 \leq k \leq \infty$) and dimension s (codim $Y \leq s \leq n$). Then we have,

Theorem 2.11 : Let X, Y be disjoint C^k submanifolds of $\mathbb{R}^n$, and let $0 \in Y \cap \overline{X}$, with $1 \leq k \leq \infty$. Then

$$(h_s^k) \text{ implies } (t_s^k) \text{ if } \begin{cases} k = 1 \\ \text{or} \\ k > 1 \text{ and } s > n - \dim X . \end{cases}$$

(David Epstein has given a counterexample showing that the restriction on s when $k > 1$ is necessary.)

Proof : Suppose X is not (t_s^k)-regular over Y at 0. Then there is some C^k submanifold S of dimension s, transverse to Y at 0, and an infinite sequence of points x_i in X, tending to 0, such that S and X are not transverse at x_i, for all i.

We are working locally at 0, so we can suppose that S is the image of a C^k embedding $i_S : (\mathbb{R}^s, 0) \longrightarrow (S, 0) \subset (\mathbb{R}^n, 0)$.

Choose a sequence of pairwise disjoint balls B_i of radius r_i and centre x_i, which are contained in coordinate charts for X, such that $i_S^{-1}(S \cap B_i) = D_i$ is an open subset of $\mathbb{R}^s$, and diffeomorphic to $\mathbb{R}^s$. Let $s_i = i_S^{-1}(x_i)$.

We shall show the existence of a C^k embedding $g : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that,

(I) $g = i_S$ off $\bigcup_{i=1}^{\infty} D_i$,

(II) for all i, $g(\mathbb{R}^s) \cap X \cap B_i$ is not homeomorphic to a manifold of dimension $(s + \dim X - n)$, and is nonempty.

From (I) it follows that i_S and g have the same k-jet at 0, so that in particular $g(\mathbb{R}^s) = S'$ is transverse to Y at 0.

<u>Existence of g when $k = 1$</u>:

Finding such a g is particularly simple when $k = 1$.

Fix i, and let ϕ_i be a C^1 diffeomorphism of B_i, fixing x_i, so that $\phi_i(X \cap B_i)$ is affine. By an arbitrarily small C^1-perturbation of i_S near s_i we can change $i_S|_{D_i}$ to a C^1 embedding $g_i : (D_i, s_i) \longrightarrow (B_i, x_i)$, such that there are open neighbourhoods N_i and L_i of s_i in $\mathbb{R}^s$ with $N_i \subset \overline{N}_i \subset L_i \subset \overline{L}_i \subset D_i$, and $g_i|_{D_i - L_i} = i_S|_{D_i - L_i}$, and $\phi_i \circ g_i|_{N_i} = d(\phi_i \circ i_S)(s_i)|_{N_i}$. (We have pushed $\phi_i(S)$ onto its tangent space near x_i.)

Near x_i we now have two affine subspaces $\phi_i(X \cap B_i)$ and $(\phi_i \circ g_i)(N_i)$ which intersect at x_i, but are not transverse at x_i, and hence intersect in an affine subspace of dimension greater than $d = \max(-1, s + \dim X - n)$. Thus $\dim(\phi_i(X \cap B_i) \cap (\phi_i \circ g_i)(D_i))$ is greater than d, and hence

$$(*) \qquad \dim(X \cap g_i(D_i)) > d .$$

In particular $X \cap g_i(D_i)$ is nbnempty.

Now define $g : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^n, 0)$ by setting $g|_{D_i} = g_i$ for all i, and g equal to i_S elsewhere.

For g to be a C^1 embedding, it suffices to choose $\{g_i\}$ such that $|j^1(i_S)(s) - j^1(g_i)(s)| < r_i/2^i$ for all $s \in D_i$, for all i.

Then (I) is satisfied by construction, and (*) gives (II).

<u>Existence of g when $k > 1$, and $s > (n - \dim X)$</u> :

Fix i. We shall change $i_S|_{D_i}$ to a C^k embedding $g_i : (D_i, s_i) \longrightarrow (B_i, x_i)$ by an arbitrarily small C^k-perturbation (less than $r_i/2^i$, say) near s_i, so that there are open neighbourhoods N_i and L_i of s_i in R^s with $\overline{N}_i \subset L_i$, and $\overline{L}_i \subset D_i$ such that $g_i|_{D_i - L_i} = i_S|_{D_i - L_i}$, and such that $g_i^{-1}(X) \cap N_i$ is homeomorphic to a cone in $\mathbb{R}^s$, of the form

$$\sum_{j=1}^{s+\dim X-n+1} \varepsilon_j z_j^2 = 0 , \quad \text{where} \quad \varepsilon_j = \pm 1 ;$$

hence $g_i^{-1}(X) \cap N_i$ is not homeomorphic to a topological manifold of dimension $(s + \dim X - n)$, and is nonempty.

The existence of such a g_i follows from the Perturbation Lemma of May (Lemma 1A of his thesis [53] ; Damon has given a detailed proof of a more precise perturbation in Lemma 3.1 of [51]) applied to the C^k embedding i_S at 0, using the hypothesis $s > n - \dim X$. The Perturbation Lemma is stated for C^∞ maps and uses the C^∞ Morse Lemma, however the proof works for C^k maps $(k \geq 2)$, using the C^2 Morse Lemma due to Kuiper ([52] ; Ostrowski [55] and Takens [56] provide different proofs) . Note that the classical proof of the Morse Lemma is only valid for C^3 functions (see [13], Chapter 6, Section 1) .

(I) and (II) now follow for the C^k embedding g defined in terms of i_S and $\{g_i\}$, as in the case of $k = 1$. This completes our proof of the existence of g.

<u>Lemma 2.12</u> : <u>There is some</u> C^k <u>submanifold</u> S'' <u>of dimension</u> s, <u>with</u> $0 \in S''$, <u>transverse to</u> Y <u>at</u> 0 <u>and transverse to</u> X <u>near</u> 0.

<u>Proof</u> : This proof will be similar to that of Lemma 1.3 .

Let $\mathcal{E}_S = \{ f \in C^k(S , \mathbb{R}^n) : f(0) = 0 \}$. $\mathcal{E}_S$ is weakly closed in the C^k topology, and thus, by Theorem 1.4 , $\mathcal{E}_S$ is a Baire space in the strong

C^k topology. Now we apply the standard procedure of covering X by countably many coordinate discs $\{K_\alpha\}$, and proving that $\{f \in \mathcal{E}_S : f \pitchfork X \text{ on } K_\alpha\}$ is open and dense in $\mathcal{E}_S$ in the strong C^k topology, for each α, to deduce that $\{f \in \mathcal{E}_S : f \pitchfork X\}$ is dense in $\mathcal{E}_S$.

Choose a weak C^k neighbourhood $\mathcal{V}_{\delta,\bar{V}}(i_S)$ of the (C^k) mapping i_S defined by inclusion of S in $\mathbb{R}^n$, where δ is a positive real number, V is a neighbourhood of 0 in S, with compact closure $\bar{V}$, and if $f \in \mathcal{V}_{\delta,\bar{V}}(i_S)$, $f|_V$ is a C^k embedding transverse to Y at 0 (Lemma 1.3 in Chapter 2 of Hirsch [13] gives δ, V for such a C^1 neighbourhood, and the same δ, V provide an adequate C^k neighbourhood). Then the strong C^k neighbourhood $\mathcal{V}_{\delta,S}(i_S)$ has $\mathcal{V}_\pitchfork = \mathcal{V}_{\delta,S}(i_S) \cap \{f \in \mathcal{E}_S : f \pitchfork X\}$ as a strongly C^k dense subset. For any f in $\mathcal{V}_\pitchfork$, $S'' = f(V)$ satisfies the requirements of Lemma 2.12.

(Recall that $\mathcal{V}_{\delta,\bar{V}}(i_S) = \{f \in C^k(S, \mathbb{R}^n) : |j^k f(z) - j^k i_S(z)| < \delta, \forall z \in \bar{V}\}$.)

Let S'' be given by Lemma 2.12. Then $S'' \cap X$ is either empty in some neighbourhood of 0, or is a topological manifold of dimension $(s + \dim X - n)$. Let S' be given as the image of the embedding g constructed above. Then the germs at 0 of $S' \cap X$ and $S'' \cap X$ are of distinct topological types, by (II), and so (h^k_s) is not satisfied, thus proving Theorem 2.11.

<u>Corollary 2.13</u> : <u>If</u> X <u>is subanalytic and the pair</u> (X, Y) <u>has homeomorphic</u> C^1 <u>transversals of dimension</u> s <u>at</u> 0 <u>for some</u> s, $n - 1 \geq s \geq \operatorname{codim} Y$, <u>then</u> X <u>is</u> (a)-<u>regular over</u> Y <u>at</u> 0.

<u>Proof</u> : Combine Theorem 2.11 with Theorem 2.5, using the remark at the end of Note 2.8 that for any s, $n - 1 \geq s \geq \operatorname{codim} Y$, (t^1_s) implies (a).

<u>Remark</u> : Conjecture 2.10 and Theorem 2.11 are in accord with the general principle of Thom that instability of topological type corresponds to a lack of transversality.

One of the original motivations for this work was the hope of generalising the theorems about equisingularity of families of complex hypersurfaces achieved by Zariski and the French School (led by Teissier). We now explain how the results just described fit in with this idea.

Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k , 0 \times \mathbb{C}^k) \longrightarrow (\mathbb{C}, 0)$ be a complex analytic function such that $Y = 0 \times \mathbb{C}^k$ contains the singular set of F. Let $r : \mathbb{C}^{n+1} \times \mathbb{C}^k \longrightarrow Y$ be an analytic retraction. In [30] we find the following implications :-

(T.E) topological type of $F^{-1}(0) \cap r^{-1}(y)$ is constant as y varies in Y

$\Downarrow$

(μ) the Milnor number $\mu(F^{-1}(0) \cap r^{-1}(y))$ is constant as y varies in Y

$\Downarrow$

(a) $(F^{-1}(0) - Y)$ is (a)-regular over Y

(The first implication is (0.1.4) of [30], and is also sketched on page 68 of [23]. The second implication is (II.3.10) of [30]; a different proof appears in [16].)

In [31], Teissier denotes by (S.T.E) the condition that (T.E) hold for <u>all</u> such retractions r. Corollary 2.13 can now be thought of as a generalisation of the implication : (S.T.E) implies (a). Also Kuo's Theorem 2.9 has as a direct consequence that (T.E) implies (S.T.E), a result left unsettled in [31] .

The example given by Teissier in the post-script to [31] is instructive . Consider $V \equiv \{y^3 = tx^2 + x^5\}$ in $\mathbb{R}^3$ and let Y be the t-axis, and $X = V - Y$. Then X is topologically trivial over Y , and the topological type of the intersection of X with each plane $\{t = \text{constant}\}$ is constant, so that (T.E) holds for $r : \mathbb{R}^3 \longrightarrow Y$ defined by $(x, y, t) \longmapsto t$. However X is not (a)-regular over Y at 0 , and (X,Y) does not have homeomorphic C^1 transversals of dimension 2 at 0 as is seen from the figure.

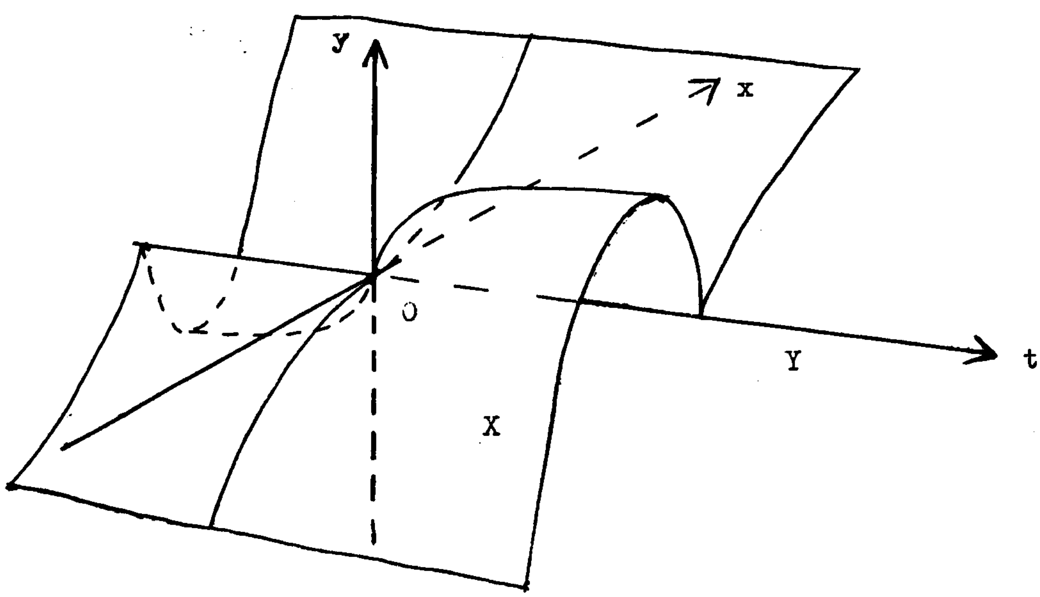


Figure : $y^3 = tx^2 + x^5$.

3. (a)-regularity and transverse foliations

In his paper " Regular Stratifications " [43] C. T. C. Wall noted that if a pair of adjacent strata (X,Y) in $\mathbb{R}^n$ are (a)-regular at 0 in Y then,

(a_s) Given a C^1 local retraction π onto Y defined near 0 , then there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that $\pi|_{X \cap U}$ is a submersion.

He suggested that the converse was also true, and this will be the main result of this section.

First note that $\pi|_{X \cap U}$ is a submersion if and only if the fibres of π are transverse to X in U. Then we see that (a_s) implies (t). For, given a C^1 submanifold S transverse to Y at 0 we can choose a chart at 0 in which S and Y become linear and then take a linear retraction π whose fibres lie in S. If the fibres of π are transverse to X, S will be transverse to X. Thus we obtain,

Corollary 3.1 : Let X', Y be C^1 submanifolds of $\mathbb{R}^n$ and let $0 \in Y \subset \overline{X} - X$ and let X be a subanalytic set. Then X is (a)-regular over Y at 0 if and only if X is (a_s)-regular over Y at 0.

Proof : As above, (a) implies (a_s), and (a_s) implies (t). Now apply Theorem 2.5.

Clearly if Y is an analytic manifold we can restrict to C^1 local retractions π whose fibres are semianalytic : further improvements on Corollary 3.1 may be culled from Note 2.8.

Remark 3.2 : In both examples 2.1 and 2.4 we can choose a (linear) retraction π whose fibres are translates (over Y) of a limiting tangent plane for which (a) fails, and these fibres fail to be transverse to X at each point of a sequence tending to 0.

Before we prove that (a_s) implies (a), we give a helpful reformulation of (a_s) suggested by Dennis Sullivan.

$(\mathcal{F}^k)$ Given a C^k foliation $\mathcal{F}$ transverse to Y at 0, there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that $\mathcal{F}$ is transverse to X in U.

It is clear that (a_s) is equivalent to $(\mathcal{F}^1)$. Given $(\mathcal{F}^1)$, (a_s) follows since the fibres of a C^1 local retraction define a foliation transverse to Y of codimension the dimension of Y. Given (a_s), $(\mathcal{F}^1)$ follows by choosing a retraction whose fibres are contained in the leaves of the foliation.

So the question of whether (a_s) implies (a) can be formulated as : <u>do transverse C^1 foliations detect (a)-faults</u> ?

<u>Theorem 3.3 (" Transverse C^1 foliations detect (a)-faults ")</u>

<u>Let X, Y be C^1 submanifolds of $\mathbb{R}^n$, and let $0 \in Y \subset \overline{X} - X$. Then X is (a)-regular over Y at 0 if and only if X is $(\mathcal{F}^1)$-regular over Y at 0</u>

<u>Proof</u> : We have already established that (a) implies $(\mathcal{F}^1)$. So suppose that there is an (a)-fault at 0 given by a sequence $\{x_i\} \in X$ tending to 0, with $\tau = \lim T_{x_i} X$, and $T_0 Y \not\subset \tau$.

We shall adjust a codimension 1 foliation by hyperplanes parallel to a hyperplane containing τ so as to be nontransverse to X at infinitely many x_i.

<u>Construction 3.4 (Ripples)</u>

Given a hyperplane $H \in G^n_{n-1}(\mathbb{R})$, a real number $s \in [0,\frac{1}{2}]$, and a real number $r > 0$, we construct a C^1 foliation $\mathcal{F}^s_H$ of codimension 1 of the ball B^n_r of radius r with centre 0 in $\mathbb{R}^n$ such that

(1) for all $x \in B^n_r - B^n_{\frac{1}{2}r}$, $T_x \mathcal{F}^s_H = H$,

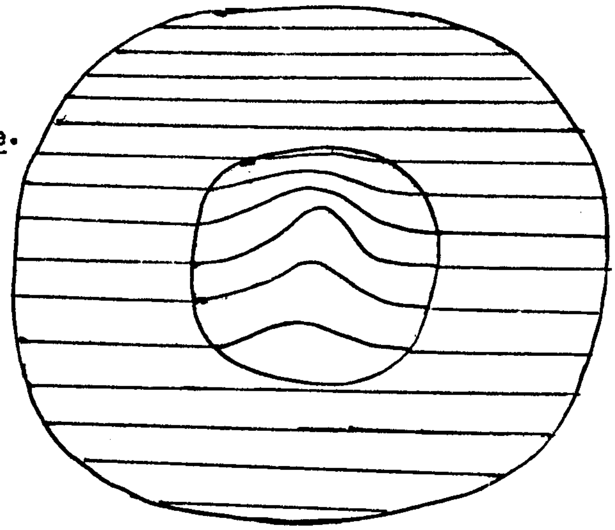
(2) for all $x \in B^n_{\frac{1}{2}r}$, $d(H, T_x \mathcal{F}^s_H) \leq s$,

(3) for all $K \in G^n_{n-1}(\mathbb{R})$ such that $d(K,H) = s$, there is a unique $x_K \in B^n_{\frac{1}{2}r}$ such that $T_{x_K} \mathcal{F}^s_H = K$,

(4) there is a C^1 diffeomorphism $\phi^s_H : B^n_r \longrightarrow B^n_r$ such that $\phi^s_H(\mathcal{F}^s_H)$ is the trivial foliation $\mathcal{F}^0_H$ by hyperplanes parallel to H, and such that $\phi^s_H|_{B^n_r - B^n_{\frac{1}{2}r}} = \mathrm{id}|_{B^n_r - B^n_{\frac{1}{2}r}}$, and $d\phi^s_H$ tends to the identity uniformly as s tends to 0, i.e. $\forall\ \varepsilon > 0$, $\exists\ s_\varepsilon > 0$ such that

$s < s_\varepsilon$ implies $|d\Phi_H^s(x) - I| < \varepsilon$ for all $x \in B_r^n$.

Figure : Foliation with a ripple.



(We shall postpone the verification of Construction 3.4 until after the proof of Theorem 3.3 . The reader may in any case prefer to admit the verification as geometrically evident.)

Choose a one-dimensional subspace $V \subset T_0 Y$ such that $V \not\subset \tau$. Define a hyperplane H by $\tau \oplus (\tau \oplus V)^\perp$, where $(\ )^\perp$ denotes orthogonal complement in $T_0 \mathbb{R}^n$.

Since $T_{x_i} X$ tends to τ as i tends to ∞ , there is some i_0 such that $i \geqslant i_0$ implies $V \not\subset T_{x_i} X$. Then for all $i \geqslant i_0$ define a hyperplane H_i by $T_{x_i} X \oplus (T_{x_i} X \oplus V)^\perp \subset T_{x_i} \mathbb{R}^n$. Then H_i tends to H as i tends to ∞. Pick $i_1 \geqslant i_0$ such that $|H_i - H| < \frac{1}{2}$ for $i \geqslant i_1$.

Now pick an infinite sequence of pairwise disjoint balls $B_{r_i}(x_i)$ with radius r_i and centre x_i . This is possible since 0 is the only accumulation point of $\{x_i\}_{i=1}^{\infty}$. Then for all i , $0 \notin B_{r_i}(x_i)$.

For all $i \geqslant i_1$, place inside $B_{r_i}(x_i)$ a " ripple " : a foliated ball $B_i = B_{\frac{1}{2}r_i}(y_i)$ with radius $\frac{1}{2}r_i$, centre y_i , and the foliation $\mathcal{F}_i = \mathcal{F}_H^{|H_i - H|}$ given by Construction 3.4 such that $x_i = x_{H_i}$, i.e. $T_{x_i}\mathcal{F}_i = H_i$. (There are two possible positions for the ripple.) Define a foliation $\mathcal{F}$ on $\mathbb{R}^n$ by the trivial foliation $\mathcal{F}_H$ by hyperplanes parallel to H on $\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)$, together with $\mathcal{F}_i$ on B_i for all $i \geq i_1$. $\mathcal{F}$ will be a C^1 foliation if we can define a C^1 diffeomorphism $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ taking $\mathcal{F}$ onto $\mathcal{F}_H$. Let $\phi|_{\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)}$ = identity, and $\phi|_{B_i} = \phi_H^{|H_i - H|}$ as defined in Construction 3.4. To check that ϕ is a C^1 diffeomorphism it is enough to check that $d\phi(x)$ is continuous at 0 and equal to the identity at 0.

Given $\varepsilon > 0$, (4) of Construction 3.4 gives us an $s_\varepsilon > 0$. Pick $i_2 \geq i_1$ such that $|H_i - H| < s_\varepsilon$ for all $i \geq i_2$. Let $\delta = \min\limits_{\substack{x \in \overline{B}_i \\ i_1 \leq i < i_2}} \{|x|\}$. δ is well-defined and nonzero since $0 \notin \bigcup_{i=i_1}^{i_2 - 1} \overline{B}_i \subset \bigcup_{i=i_1}^{i_2-1} B_{r_i}(x_i)$.

Then $|x| < \delta$ implies $x \notin \bigcup_{i=i_1}^{i_2-1} B_i$, so

$$
\begin{aligned}
|d\phi(x) - I| &\leq \max_{\substack{x' \in B_i \\ i \geq i_2}} \{|d\phi_H^{|H_i - H|}(x') - I|\} \\
&< \varepsilon \quad \text{by (4) of Construction 3.4,}
\end{aligned}
$$

and the choice of s_ε, i_2.

Thus $d\phi(x)$ is continuous near 0, and $d\phi(0) = I$ (the identity matrix). Hence $\mathcal{F}$ is a C^1 foliation and $T_0\mathcal{F} = H$, so that $\mathcal{F}$ is transverse to Y at 0 ($V \not\subset H$ by definition of H). But for all $i \geq i_1$, $T_{x_i}\mathcal{F} = T_{x_i}\mathcal{F}_i = H_i$ and $T_{x_i} X \subseteq H_i$, so that $\mathcal{F}$ is nontransverse to X at x_i. This shows that X is not $(\mathcal{F}^1)$-regular over Y at 0, proving Theorem 3.3.

<u>Verification of Construction 3.4</u>: It suffices to take $H = \mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n$ and $n = 2$. For $n > 2$ the calculations are similar.

Consider,
$$
\begin{cases}
y = \lambda + (1-\lambda^2)^2(x^2 - a^2)^2 & \lambda^2 \leq 1, \ x^2 \leq a^2 \\
y = \lambda & \lambda^2 \leq 1, \ a^2 \leq x^2 \leq 1,
\end{cases}
$$

with the constant a in $(0,1)$ to be chosen shortly.

We shall prove that this defines a C^1 foliation of $[-1,1]^2$ of codimension 1, with the leaves corresponding to fixed values of λ. (If $n > 2$, take $x_n = \lambda + (1-\lambda^2)^2 (\sum_{i=1}^{n-1} x_i^2 - a^2)^2$, et cetera.)

Multiplying by $r/4$ gives a foliation of $[-r/4, r/4]^2$ which fits into the ball $B_{\frac{1}{2}r}(0)$ and extends trivially to a foliation $\mathcal{F}_a$ of $B_r(0)$ which satisfies (1). The leaf with normal vector furthest from $(0:1)$ is clearly given by $\lambda = 0$, and this normal is $(1 : \mp(8a^3)/(3\sqrt{3}))$ at the points $((4/9)a^4, \pm a/\sqrt{3})$. (Compare Construction 2.2)

Write $\nu_a = (8a^3)/(3\sqrt{3})$. Then $|(1:\nu_a) - (1:0)| = (\nu_a)/(1+\nu_a^2)^{\frac{1}{2}}$.

So, given s, choose a such that

$$\frac{\nu_a^2}{1+\nu_a^2} = s^2,$$

$$\text{i.e.} \quad \nu_a^2 = \frac{s^2}{1-s^2}.$$

$$\text{Then} \quad a^6 = \frac{27\, s^2}{64(1-s^2)}.$$

With this choice of a, (2) and (3) of 3.4 are satisfied.

Note that for $s \in [0,\frac{1}{2}]$ we have : $a^6 \leq 9/64$ (*).

Define $\phi_a : [-1,1]^2 \longrightarrow [-1,1]^2$ by

$$\phi_a(x,y) = \begin{cases} (x,y) & a^2 \leq x^2 \leq 1 \\ (x,\ y + (1-y^2)^2(x^2-a^2)^2) & x^2 \leq a^2 \end{cases}.$$

ϕ_a is then a C^1 map. Elementary calculation using (*) shows that ϕ_a is injective. Now

$$d\phi_a(x,y) = \begin{pmatrix} 1 & 0 \\ 4x(x^2-a^2)(1-y^2)^2 & 1 - 4y(1-y^2)(x^2-a^2)^2 \end{pmatrix} \text{ if } x^2 \leq a^2 .$$

and $d\phi_a(x,y)$ is the identity matrix if $a^2 \leq x^2 \leq 1$.

Calculation using (*) shows that $d\phi_a(x,y)$ is always nonsingular. Thus ϕ_a is a C^1 diffeomorphism of $[-1,1]^2$, which after scalar multiplication by $r/4$

as described above may be extended by the identity to a C^1 diffeomorphism of $B_r(0)$ since $d\phi_a(x,\pm 1)$ is the identity matrix. It defines the foliation.

ϕ^s_H will be the inverse of the resulting diffeomorphism. It only remains to verify (4) of Construction 3.4 , i.e. to show that $d(\phi_a^{-1})$ tends uniformly to the identity matrix as a tends to 0 ; but this follows from the same result for $d\phi_a$, and this in turn follows from the expression above.

Thus we have verified conditions (1) - (4) of Construction 3.4 .

<u>Corollary 3.5</u> : (a)-<u>regularity is a</u> C^1 <u>diffeomorphism invariant</u> .

<u>Proof</u> : $(\mathcal{F}^1)$ is clearly C^1 diffeomorphism invariant .

Having shown that transverse C^1 foliations detect (a)-faults, we give an example of an (a)-fault which is not detectable by transverse C^2 foliations, showing that Theorem 3.3 is sharp. The details of this example were worked out with the help of Anne Kambouchner.

<u>Example 3.6</u> : <u>An</u> (a)-<u>fault not detectable by transverse</u> C^2 <u>foliations</u>.

In $\mathbb{R}^3$ let (x, y, z) be coordinates, and let Y be the y-axis, and let X be $(\bigcap_{n=1}^{\infty}\{x = 0 , g_n \geq 0 , z>0\}) \cup (\bigcup_{n=1}^{\infty}\{f_n = 0 , g_n \leq 0\})$, where g_n is a function of y and z and $\{g_n \leq 0\}$ intersects $\{x = 0\}$ in a rectangle of length m_n , width $m_n r_n$, and $\{f_n = 0\}$ defines the barrow B_n of magnitude m_n , ratio r_n , axis $\{x = 0 , y + \tan(\theta_n)z = (1/2n) + (\tan\theta_n)/2n\}$, and centre $p_n = (0, 1/2n , 1/2n)$ with base in the plane $\{x = 0\}$.(Cf. 2.3.)

First choose a monotonic **decreasing** sequence $\{m_n\}$ such that for any choice of θ_n , and any $r_n \leq 1$, the barrows are pairwise disjoint (and do not intersect Y). Now let δ_n be the radius of the largest 2-sphere $S^2_\delta(0)$ such that $S^2_\delta(0) \cap B_n \neq \emptyset$ when $r_n = 1$ and θ_n takes all values in $[-\pi/2 , \pi/2]$.

Then set $r_n = (3\sqrt{3}/8)\delta_n^{\frac{2}{3}}$ and $\theta_n = \sin^{-1}((3\sqrt{3}/8)(\delta_n^{\frac{1}{3}} + \delta_n^{\frac{2}{3}}))$, so defining

B_n completely, and hence specifying X.

(Note that $(3\sqrt{3}/4)\delta_n^{\frac{1}{3}} < 1$, i.e. $\delta_n < 64/81\sqrt{3}$, and so this choice of θ_n is possible for all $n \geqslant 1$, by the choice of the centre $p_1 = (0, \frac{1}{2}, \frac{1}{2})$ of B_1.)

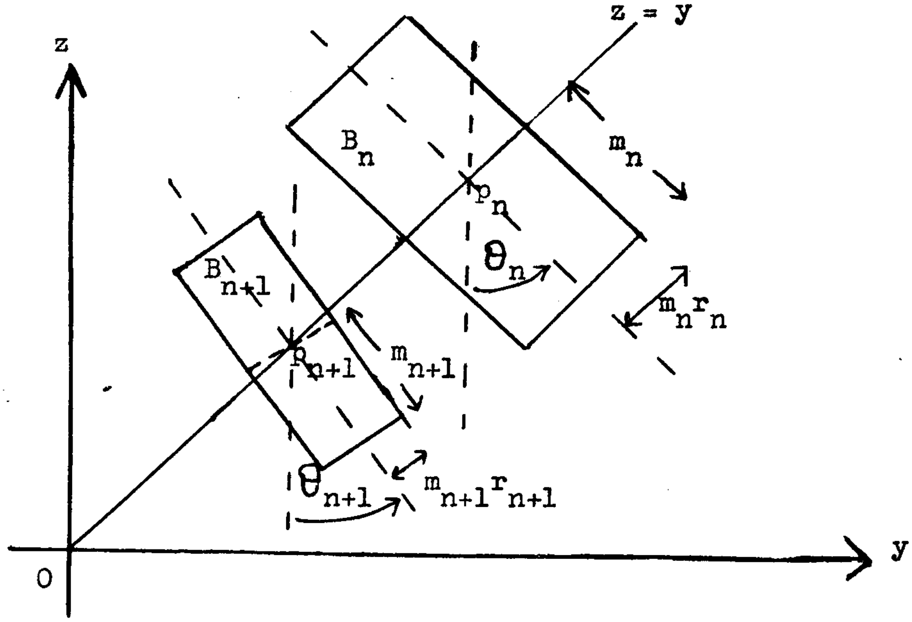


Figure : $x = 0$

Since $\{\delta_n\}$ is a monotonic decreasing sequence, tending to 0, both $\{r_n\}$ and $\{\theta_n\}$ are monotonie decreasing to 0. Thus (cf. Construction 2.3) the set of limiting normals to X at 0 is $\{(1 : \lambda : 0) : -8/3\sqrt{3} \leqslant \lambda \leqslant 8/3\sqrt{3}\}$ Hence (a) fails at 0 for the pair (X,Y).

Suppose $(\mathcal{F}^2)$ does not hold at 0 for (X,Y). Then there is a C^2 foliation $\mathcal{F}$ which is transverse to Y at 0 and which is not transverse to X in any neighbourhood of 0. Necessarily $\mathcal{F}$ is of codimension 1 and $T_0\mathcal{F}$ (the tangent at 0 to the leaf of $\mathcal{F}$ passing through 0) must be of the form $(1 : \alpha : 0)$ where $0 < |\alpha| \leqslant 8/3\sqrt{3}$.

We shall show that there is a constant $C > 0$ and an n_0 such that for all $n \geqslant n_0$ and for all $p \in B_n$,

$$|N_pX - (1 : \alpha : 0)| > C\delta_n^{\frac{1}{3}} \qquad (*)$$

($N_p X$ is the normal space to X at p.) The proof of $(*)$ will be given later.

Let $\phi : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$ denote the C^2 diffeomorphism defining

$\mathcal{F}$ so that the leaves of $\mathcal{F}$ are the images of $\{\mathbb{R}^2 \times w\}_{w \in \mathbb{R}}$. Then $d\phi(0)(\mathbb{R}^2 \times 0)$ is the plane with normal $(1 : \alpha : 0)$.

Since ϕ is C^2, the map $(\mathbb{R}^3, 0) \longrightarrow (GL_3(\mathbb{R}), d\phi(0))$ is C^1 and

$$p \longmapsto d\phi(\phi^{-1}(p))$$

thus there exist $\varepsilon > 0$ and $M > 0$ such that

$$\left| d\phi(\phi^{-1}(p)) - d\phi(0) \right| < M|p| \quad , \text{ for all } p \in B_\varepsilon(0) .$$

It follows at once that

$$\left| (d\phi(\phi^{-1}(p)) - d\phi(0))\big|_{\mathbb{R}^2 \times 0} \right| < M|p| \text{ , for all } p \in B_\varepsilon(0) \quad ,$$

or in other words that

$$\left| T_p\mathcal{F} - T_0\mathcal{F} \right| < M|p| \text{ , for all } p \in B_\varepsilon(0) .$$

Now, by hypothesis, $\mathcal{F}$ is nontransverse to X at some point of B_n, for infinitely many n, i.e. for infinitely many n, there exists $p \in B_n$ such that $T_p\mathcal{F} = T_pX$. Let $n_1 \geq n_0$ be such that for all $n \geq n_1$, if $p \in B_n$, then $|p| < \varepsilon$. Then for infinitely many $n \geq n_1$, there exists $p \in B_n$ such that $M|p| > |N_pX - (1 : \alpha : 0)|$. But assuming $(*)$ and using the choice of δ_n, we know that for all $n \geq n_0$, and for all $p \in B_n$, $|N_pX - (1 : \alpha : 0)| > C|p|^{\frac{1}{3}}$. These last two inequalities are absurd, since there is some n_2 such that for all $n \geq n_2$, and for all $p \in B_n$, $|p| < (C/M)^{3/2}$, i.e. $M|p| < C|p|^{\frac{1}{3}}$. Thus we obtain a contradiction, showing that $(\mathcal{F}^2)$ holds, and that transverse C^2 foliations cannot detect this (a)-fault.

Proof of $(*)$: A short calculation shows that for all n the set of normals to B_n (rotated back through θ_n), is contained in

$$\left\{ (1 : \lambda : \mu) : \lambda \in [-8/3\sqrt{3} , 8/3\sqrt{3}] , \mu \in [-8r_n/3\sqrt{3} , 8r_n/3\sqrt{3}] \right\} .$$

It will suffice to establish $(*)$ in the euclidean norm $| \ |_e$ in the usual chart for $P^2(\mathbb{R})$ centred at $(1:0:0)$ given by the homogeneous coordinates $(\nu : \lambda : \mu) \longmapsto (\lambda/\nu , \mu/\nu)$, since this norm is equivalent to the standard one. ($\left|(1:\lambda':\mu') - (1:\lambda'':\mu'')\right|_e = ((\lambda' - \lambda'')^2 + (\mu' - \mu'')^2)^{\frac{1}{2}}$.)

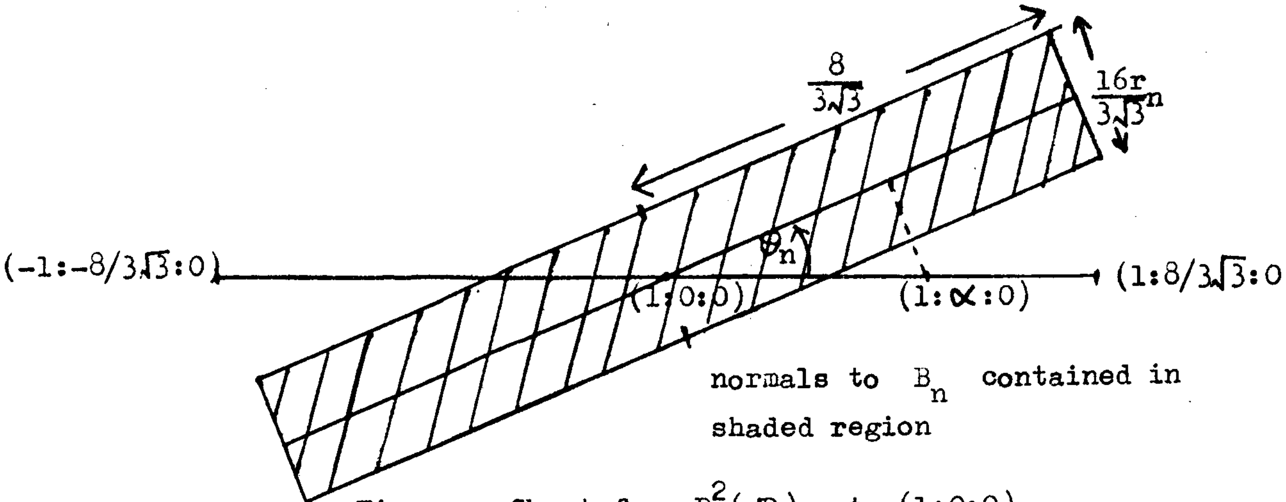


Figure : Chart for $P^2(\mathbb{R})$ at $(1:0:0)$.

It is evident from the figure above and the choice of r_n and θ_n that there exists n' such that for all $n \geqslant n'$, $(1:\alpha:0)$ is outside the shaded region which contains the normals to B_n . We calculate the minimal distance of $(1:\alpha:0)$ from a normal of B_n . This is clearly $(\alpha \sin\theta_n - 8r_n/3\sqrt{3})$. Thus for all $n \geqslant n'$ and all $p \in B_n$,

$$\begin{aligned} |N_pX - (1:\alpha:0)|_e &\geqslant \alpha\sin\theta_n - 8r_n/3\sqrt{3} \\ &= \alpha(3\sqrt{3}/8)(\delta_n^{\frac{1}{3}} + \delta_n^{\frac{2}{3}}) - \delta_n^{\frac{2}{3}} \\ &= \delta_n^{\frac{1}{3}}(\ (3\sqrt{3}\,\alpha/8) - \delta_n^{\frac{1}{3}}(1 - (3\sqrt{3}\,\alpha/8))\ ) . \end{aligned}$$

Since δ_n tends to 0 as n tends to ∞ , there exists $n_0 \geqslant n'$ such that for all $n \geqslant n_0$, and all $p \in B_n$,

$$|N_pX - (1:\alpha:0)|_e > (3\sqrt{3}\,\alpha/16)\,\delta_n^{\frac{1}{3}} .$$

Thus we obtain $(*)$.

Note 3.7 : We have in fact proved slightly more by the above example. Namely that a transverse foliation, with C^1 leaves, which is C^1 with a Lipschitz derivative in the direction transverse to the leaves, cannot detect this (a)-fault. If $(\mathcal{F}^{1,p})$ denotes the condition similar to $(\mathcal{F}^1)$ but restricting to foliations defined by a C^1 diffeomorphism C^1 along the leaves and C^p transverse to the leaves, then clearly $(\mathcal{F}^{1,p})$ implies $(\mathcal{F}^{1,q})$ if $p < q$ (and $(\mathcal{F}^{1,p})$ implies (t) for all $p \leqslant \infty$). Also it is (now) easy to construct examples showing $(\mathcal{F}^{1,q})$ does not imply $(\mathcal{F}^{1,p})$ when $p < q$. Simply set

$$\theta_n = \sin^{-1}(3\sqrt{3}(\delta_n^{p-\frac{2}{3}} + \delta_n^{2p-\frac{4}{3}})/8) ,$$
$$r_n = (3\sqrt{3}\,\delta_n^{2p-\frac{4}{3}})/8 ,$$

and repeat the argument of 3.6 .

4. Detecting Thom faults in stratified mappings.

Since the regularity condition imposed on a stratified morphism is formally very similar to (a)-regularity we note here the analogues of the results we have proved about (a)-regularity in §§1-3.

Following [6] , let $f : N \longrightarrow P$ be a C^1 map, between C^1 manifolds N and P , and let X and Y be C^1 submanifolds of N such that $f|_X$ and $f|_Y$ have constant rank, and let $0 \in Y \subset \overline{X} - X$. We say that X is (a_f)-regular over Y at 0 (in the terminology of Gibson [6], X is Thom regubar over Y at 0 relative to f) if,

(a_f) Given a sequence $\{x_i\}$ in X , such that x_i tends to 0 as i tends to ∞, and $\ker d_{x_i}(f|_X)$ converges to a plane τ , then $\ker d_0(f|_Y) \subseteq \tau$.

Since $f|_X$ is of constant rank, the fibres of $f|_X$ form the leaves of a foliation $\mathfrak{F}_X^f$ of X , and similarly for Y . Thus (a_f) may be stated,

(a_f) Given a sequence $\{x_i\}$ in X , such that x_i tends to 0 as i tends to ∞, and $T_{x_i}(\mathfrak{F}_X^f)$ converges to a plane τ , then $T_0(\mathfrak{F}_Y^f) \subseteq \tau$.

Here $T_0(\mathfrak{F}_Y^f)$ denotes the tangent space at 0 to the leaf of $\mathfrak{F}_Y^f$ passing through 0 .

The natural analogue of (t)-regularity is,

(t_f) Given a C^1 submanifold S such that S is transverse to $\mathcal{F}^f_Y$ at 0, there is a neighbourhood of 0 in which S is transverse to $\mathcal{F}^f_X$.

Similarly the analogue of $(\mathcal{F}^k)$ is,

$(\mathcal{F}^k_f)$ Given a C^k foliation $\mathcal{F}$ of N transverse to $\mathcal{F}^f_Y$ at 0, there is a neighbourhood of 0 in which $\mathcal{F}$ is transverse to $\mathcal{F}^f_X$.

Note 4.1 : (i) Another way to say that S is transverse to $\mathcal{F}^f_Y$ at 0 is to say that the rank of $f|_{S \cap Y}$ at 0 equals the rank of $f|_Y$.

(ii) If f has rank zero on X and Y then (a_f), (t_f), $(\mathcal{F}^k_f)$ become (a), (t), $(\mathcal{F}^k)$ respectively.

With these definitions all of the results proved in §2 and §3 have corresponding versions, with just some nuances.

Thus, $(a_f) \Longleftrightarrow (\mathcal{F}^1_f) \Longrightarrow (t_f)$ by merely mimicking the proofs that $(a) \Longleftrightarrow (\mathcal{F}^1) \Longrightarrow (t)$.

Example 4.2 : Take Example 2.1 and define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $(x, y, z) \longmapsto z$. (a_f) fails since the tangent to $\mathcal{F}^f_X$ at x_n will be the vector $(2, 1, 0)$ for all n. (t_f) holds since no submanifold transverse to Y intersects the horn containing the sequences on which (a_f) fails. ($\mathcal{F}^f_Y$ is the trivial foliation with one leaf.) Thus (t_f) does not imply (a_f).

Example 4.3 : If we define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $(x, y, z) \longmapsto z$ and examine Example 2.4 we find that X is not (t_f)-regular over Y at 0 : it is easy to find a C^1 submanifold, with tangent plane at the origin spanned by the lines $\{z = y, x = 0\}$ and $\{z = 0, y = x\}$, which is not transverse to $\mathcal{F}^f_X$ on a sequence of points in X tending to 0.

To obtain an example with (t_f) and not (a_f) we can either take f to be the constant map (see Note 4.1 (ii)), or add a fourth variable w, and consider $X_1 = X \times \mathbb{R}$, $Y_1 = Y \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$ and let $f : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ be projection $(x, y, z, w) \longmapsto w$. Then X_1 is (t_f)-regular over Y_1 at 0, but not (a_f)-regular.

Example 4.4 : As in Example 4.3 we take Example 3.6, let $X_1 = X \times \mathbb{R}$, $Y_1 = Y \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$, and take $f : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ to be projection $(x, y, z, w) \longmapsto w$. Then X_1 is $(\mathcal{F}_f^2)$-regular over Y_1 at 0, but not (a_f)-regular. (X is neither $(\mathcal{F}_f^2)$-regular nor (a_f)-regular over Y at 0.) Thus $(\mathcal{F}_f^2)$-regularity does not imply (a_f)-regularity.

The next result is an analogue of Theorem 2.5.

Theorem 4.5 : Let X, Y be C^1 submanifolds of $\mathbb{R}^n$, and let $0 \in Y \subset \overline{X} - X$, and let X be a subanalytic set. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a subanalytic map (i.e. the graph of f is subanalytic in $\mathbb{R}^n \times \mathbb{R}^p$), such that $f|_X$ and $f|_Y$ are of constant rank. Then X is (a_f)-regular over Y at 0 if and only if for every semianalytic C^1 submanifold S transverse to $\mathcal{F}_Y^f$ at 0, there is some neighbourhood of 0 in which S is transverse to $\mathcal{F}_X^f$.

Proof : The proof is similar to that of Theorem 2.5, save that instead of proving that $\{(x, T_xX) : x \in X\}$ is subanalytic, we must prove that $\{(x, T_x(\mathcal{F}_X^f)) : x \in X\}$ is subanalytic. But this reduces to proving that $\{(x, T_xX) : x \in X\}$ is subanalytic. For, $T_x(\mathcal{F}_X^f) = \ker d_x(f|_X) = \ker d_xf \cap T_xX$, and $\ker d_xf$ is a fixed subspace of $\mathbb{R}^n$ if we suppose (as we can) that f is a linear projection, since f is the composition of an embedding onto its graph followed by a linear projection (cf. page 30 of [6]). Theorem 4.5 follows.

Finally we consider a possible analogue of Theorem 1.1. Let $g : M \longrightarrow N$

and $f : N \longrightarrow P$ be C^1 maps between C^1 manifolds, and X a submanifold of N. Then,

$$g \pitchfork \ker d_x(f|_X) \text{ for all } x \in X \iff g \pitchfork \mathcal{F}^f_X$$
$$\iff g \pitchfork \text{fibres of } f|_X$$
$$\iff f|_X \circ g : M \longrightarrow f(X) \text{ is a submersion.}$$

Then the analogue of Theorem 1.1 is as follows, writing " $g \pitchfork \mathcal{F}^f_\Sigma$ " for " $g \pitchfork \mathcal{F}^f_X$ for all X in Σ ".

<u>Hypothesis 4.6</u> : <u>Let</u> Σ <u>be a locally finite stratification of a closed subset</u> V <u>of a</u> C^1 <u>manifold</u> M, <u>and let</u> $f : M \longrightarrow P$ <u>be a</u> C^1 <u>map</u>, P <u>a</u> C^1 <u>manifold, such that for each stratum</u> X <u>of</u> Σ, $f|_X$ <u>has constant rank. Then the following conditions are equivalent</u> :

(1) Σ <u>is</u> (a_f)-<u>regular</u>,

(2) <u>for every</u> C^1 <u>manifold</u> N, $\{z \in J^1(N,M) : z \pitchfork \mathcal{F}^f_\Sigma\}$ <u>is open in</u> $J^1(N,M)$,

(3) <u>for every</u> C^1 <u>manifold</u> N, $\{g \in C^1(N,M) : g \pitchfork \mathcal{F}^f_\Sigma\}$ <u>is open in</u> $C^1(N,M)$ <u>with the strong</u> C^1 <u>topology</u>,

(4) <u>there is some integer</u> r, $1 \leq r \leq \max(1, \min_{X \in \Sigma} (\operatorname{rank} f|_X))$, <u>and some</u> C^1 <u>manifold</u> N <u>with</u> $\dim N = \dim M - r$, <u>for which</u> $\{g \in C^1(N,M) : g \pitchfork \mathcal{F}^f_\Sigma\}$ <u>is open in</u> $C^1(N,M)$ <u>with the strong</u> C^1 <u>topology</u>.

One can prove $(1) \iff (2) \Longrightarrow (3) \Longrightarrow (4)$ without much difficulty, by copying the proof of Theorem 1.1 . To make Hypothesis 4.6 into a theorem we must prove (4) implies (1) . If we try to copy the proof that (4) implies (1) in Theorem 1.1 we arrive at,

<u>Question 4.7</u> : If X is a C^1 submanifold of $\mathbb{R}^m$, $0 \in \overline{X} - X$, and $f : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a C^1 map such that $f|_X$ has constant rank, then given a plane H and a C^1 manifold N with $\dim N = \dim H$, and $n \in N$, is $\{g \in C^1(N,\mathbb{R}^m) : g \pitchfork \mathcal{F}^f_X, \ g(n) = 0, \ d_n g(T_n N) = H\}$ nonempty ? ✱

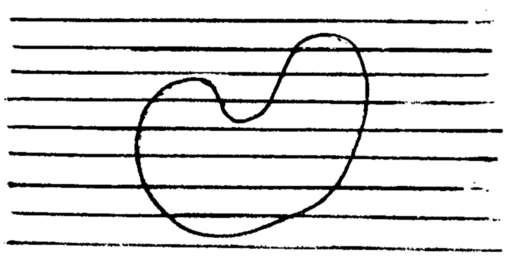
✱ See over.

A positive answer to Question 4.7 would suffice to prove Hypothesis 4.6. To prove that (3) implies (1) it suffices to answer Question 4.8 , which is *a priori* weaker than 4.7 .

Question 4.8 : Is there some C^1 manifold N for which Question 4.7 has a positive response ?*

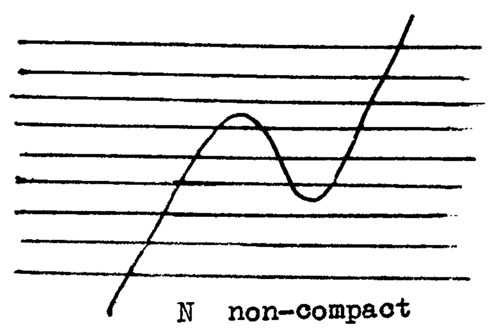
Note 4.9 : The proof of Lemma 1.3 made use of the local transversality lemma : the set of C^1 maps transverse to a submanifold on a compact coordinate disc is open and dense. The corresponding statement that C^1 maps transverse to the leaves of a foliation on a compact coordinate disc be dense is clearly false (although openness is easy). (Cf. page 193 of [42] .)

Consider



N compact

or



N non-compact

So another method of proof is required to attack (3) implies (1) of Hypothesis 4.6 .

Observe also that the figures above show that the set of C^1 maps transverse to $\mathcal{F}^f_\Sigma$ is not dense (cf. Note 1.2 (iv)).

Finally we remark that the results of §§1-3 could also be extended to the "generalised condition (a) for O-bundles" of M.-H. Schwartz in [27] .

* An example of David Epstein shows that the answer to Questions 4.7 and 4.8 is no. However Hypothesis 4.6 is still undecided : a finer study is needed.

CHAPTER 2. WHITNEY (b)-REGULARITY

In this chapter we consider various natural ways of detecting (b)-faults.

The most striking property of (b)-regularity in the theory of smooth stratified objects is that a (b)-regular stratification is locally topologically trivial, as proved by Mather in [21]. The proof shows <u>en route</u> that (b)-regularity implies a condition we have called (b_s) in [38], namely that for any C^1 tubular neighbourhood of the base stratum, associated to which are a retraction π and a distance function ρ, the fibres of $(\pi \times \rho)$ (which are embedded spheres) are transverse to the attaching stratum. This has an exact counterpart in the implication (a) implies (a_s) (see §3). In [43] C. T. C. Wall conjectured that (a_s) implied (a) and that (b_s) implied (b); we proved these implications in the semianalytic case in [37] and [38]. In Chapter 1 (Theorem 3.3) we have shown that (a_s) implies (a) in general, by perturbing a transverse foliation with an infinite sequence of ripples so as to detect a given (a)-fault. The same idea will be used in §5 to prove that (b_s) implies (b); this time we use the ripples (of 3.4) to perturb a foliation by spheres (the fibres of $\pi \times \rho$) of the complement of the base stratum, so as to detect a given (b)-fault.

In §6 we study how (b)-regularity behaves with respect to generic sections. We show that, if Y is linear, and if, for a generic set of linear spaces H containing Y, $(X \cap H, Y)_0$ is (b)-regular, then any (b)-fault of (X,Y) at 0 cannot be too "deep". Conversely, we show that if (X,Y) is (b)-regular at 0, then for generic such H, $(X \cap H, Y)$ is (b)-regular at 0.

Knowing that (b)-regularity is generic for subanalytic sets — see the introduction — it is natural to ask what are the strongest generic regularity conditions. In [40] J.-L. Verdier introduced (w)-regularity, proved that it implied (b)-regularity, and showed that it was generic (and also that it gave local trivialisations by integrating <u>continuous</u> vector fields tangent to the strata, whereas the vector fields resulting from (b)-regularity may theoretically be discontinuous). (w)-regularity is easily seen to imply Kuo's ratio test (r), and hence (r) too is generic. In §7 we give examples which show that even for semialgebraic strata, (b), (r) and (w) are distinct, and that (r) and (w) are not invariant under C^1 diffeomorphisms, although they are preserved by C^2 diffeomorphisms.

5. (b)-regularity and tubular neighbourhoods.

Following Mather in [22], we first define what is meant by a C^1 tubular neighbourhood.

<u>Definition 5.1</u> : Let X be a C^1 submanifold of a C^1 manifold M. A C^1 <u>tubular neighbourhood</u> T of X in M is a quadruple $(p, E, \varepsilon, \phi)$ where $p : E \longrightarrow X$ is an inner product bundle of class C^1, $\varepsilon : X \longrightarrow \mathbb{R}^+$ is a positive C^1 function on X, and ϕ is a C^1 diffeomorphism of $B_\varepsilon = \{e \in E : \|e\| < \varepsilon(\pi(e))\}$ onto an open subset of M which commutes with the zero section ζ of E:

$$\begin{tikzcd} B_\varepsilon \arrow[dr, "\phi"] & \\ X \arrow[u, "\zeta"] \arrow[r, hook] & M \end{tikzcd}$$

We set $|T| = \phi(B_\varepsilon)$. The map $\pi_T = p \circ \phi^{-1} : |T| \longrightarrow X$ will be called the C^1 <u>retraction</u> π_T <u>associated to</u> T, and the non-negative function $\rho_T = \rho_E \circ \phi^{-1} : |T| \longrightarrow \mathbb{R}$, where $\rho_E(e) = \|e\|^2$ for $e \in E$, will be called the C^1 <u>distance function</u> ρ_T <u>associated to</u> T.

(We have, similarly, C^r tubular neighbourhoods.)

It is clear that the map $(\pi_T, \rho_T) : |T| - X \longrightarrow X \times \mathbb{R}$ is a submersion.

As what follows will be entirely local, we can restrict to the situation of adjacent strata in $\mathbb{R}^n$.

Let X, Y be C^1 submanifolds of $\mathbb{R}^n$ and let $0 \in Y \subset \overline{X} - X$. We say that X is (b_s)-<u>regular over</u> Y if for all C^1 tubular neighbourhoods T of Y, there is a neighbourhood N of Y in $|T|$ such that $(\pi_T, \rho_T)|_{X \cap N}$ is a submersion.

Given a C^1 chart for Y at 0,

$$\phi : (U, U \cap Y, 0) \longrightarrow (\mathbb{R}^n, \mathbb{R}^m \times 0^{n-m}, 0),$$

the standard tubular neighbourhood of $\mathbb{R}^m \times 0^{n-m}$ in $\mathbb{R}^n$ provides a retraction $\pi_\phi = \phi^{-1} \circ \pi_m \circ \phi : U \longrightarrow Y \cap U$, where $\pi_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m \times 0^{n-m}$ is linear projection taking $(x_1, \dots, x_n)$ to $(x_1, \dots, x_m, 0, \dots, 0)$, and a distance function $\rho_\phi = \rho \circ \phi : U \longrightarrow \mathbb{R}^+$, where $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is the function $\rho(x_1, \dots, x_n) = \sum_{i=m+1}^{n} x_i^2$. We refer to the tubular neighbourhood T_ϕ of $U \cap Y$ in U.

We say X is (b_s)-<u>regular over</u> Y <u>at</u> 0 when,

(b_s) Given a C^1 chart (U, ϕ) at 0 for Y as a C^1 submanifold of $\mathbb{R}^n$, there is a neighbourhood U' of 0, $U' \subset U$, such that $(\pi_\phi, \rho_\phi)|_{X \cap U'}$ is a submersion.

The following lemma justifies our use of the term (b_s)-regularity in the local and global cases.

<u>Lemma 5.2</u> : <u>X is (b_s)-regular over Y if and only if X is (b_s)-regular over Y at y, for all $y \in Y$.</u>

<u>Proof</u> : " If " : Given a sequence of points on X tending to Y, at which $(\pi_T, \rho_T)|_X$ is not submersive, there must be some convergent subsequence with a limit y_0 in Y. The implication follows.

" Only if " : Given a point y_0 of Y and a C^1 tubular neighbourhood T_ϕ of a neighbourhood $U \cap Y$ of y_0 in Y defined by a C^1 chart (U, ϕ)

for Y at y_o, it will suffice to find a C^1 tubular neighbourhood T of Y and a neighbourhood U' of y_o, $U' \subset U$, such that $T|_{U' \cap Y} = T_\phi|_{U' \cap Y}$. This follows from the Tubular Neighbourhood Theorem of [22], which is proved in [21].

For a simpler proof, let ψ be a C^1 diffeomorphism of $\mathbb{R}^n$ which is the identity outside some neighbourhood of y_o, and such that there is a smaller neighbourhood W of y_o, $W \subset U$, such that the fibres of the retraction $\psi \circ \pi_\phi \circ \psi^{-1}$ intersect $\psi(W)$ in a C^1 field of planes transverse to $\psi(Y)$, and such that $\rho_\phi \circ \psi^{-1}$ is the square of the function measuring distance from $\psi(Y)$ in R^n. Extend this local C^1 field to a globally defined (over $\psi(Y)$) C^1 field of planes (whose dimension is the codimension of Y) transverse to $\psi(Y)$ In Theorem 4.5.1 of [13] Hirsch shows how to obtain a tubular neighbourhood of $\psi(Y)$, so that the transverse planes contain the fibres of the associated retraction. There is also a very careful proof of this fact by Munkres on page 51 of [54]. Pulling back by ψ^{-1} we have a tubular neighbourhood T of Y with the required properties. This completes the proof of Lemma 5.2.

In [43] C. T. C. Wall conjectured that (b_s)-regularity is a necessary and sufficient condition for (b)-regularity. Applying Lemma 5.2, together with the convention that X is (b)-regular over Y when X is (b)-regular over Y at y for all y in Y, we see that the local and global versions of the conjecture are equivalent. We now prove the local version.

<u>Theorem 5.3</u> : <u>Let</u> X, Y <u>be disjoint</u> C^1 <u>submanifolds of</u> $\mathbb{R}^n$, <u>and let</u> $0 \in Y$. <u>Then</u> X <u>is</u> (b)-<u>regular over</u> Y <u>at</u> 0 <u>if and only if</u> X <u>is</u> (b_s)-<u>regular over</u> Y <u>at</u> 0.

<u>Proof</u> : " Only if " was proved by Mather as Lemma 7.3 in [21], and in fact in 1964 by Thom on page 10 of [35]. For another published proof see Lemma 2.3 of [48].

It is left to prove " if ".

Suppose X is (b_s)-regular over Y at 0. It follows at once that X is (a_s)-regular over Y at 0 (see §3), so that we can apply Theorem 3.3 to show that (a) holds. Suppose (b) fails : we shall derive a contradiction.

By (0.4), (b') must fail for every C^1 retraction onto Y.

Let π_1 (resp. π_2) be the local linear retraction defined near 0 of $\mathbb{R}^n$ onto Y (resp. T_0Y) orthogonal to T_0Y. Then (b') fails for π_1, and there is a sequence $\{x_i\}$ in X tending to 0 such that $\lambda_i = \dfrac{x_i\pi_1(x_i)}{|x_i\pi_1(x_i)|}$ tends to a limit λ, and $T_{x_i}X$ tends to a limit τ, and $\lambda \not\subset \tau$.

The C^1 diffeomorphism defined near 0,

$$\begin{aligned} \alpha : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ p &\longmapsto p + (\pi_2(p) - \pi_1(p)) \end{aligned}$$

preserves $\{\lambda_i\}$, λ and τ, and sends Y onto T_0Y, hence we may identify Y with $\mathbb{R}^m \times 0^{n-m}$ in $\mathbb{R}^n$. Write $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m \times 0^{n-m}$ for the projection mapping $(x_1,\dots,x_n)$ to $(x_1,\dots,x_m,0,\dots,0)$. Then, continuing to write $\{x_i\}$ and X for their images by α, we have that $\lambda_i = \dfrac{x_i\pi(x_i)}{|x_i\pi(x_i)|}$ tends to λ, which is not contained in $\tau = \lim T_{x_i}X$.

Now let A be a linear automorphism of $0^m \times \mathbb{R}^{n-m}$ such that $A(\lambda)$ and $A(\tau \cap \mathbb{R}^{n-m})$ are orthogonal. By applying the linear change of coördinates $(I_m, A) : \mathbb{R}^m \times \mathbb{R}^{n-m} \circlearrowleft$ we may suppose that λ and τ are orthogonal. The function measuring distance from Y is $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, taking $(x_1,\dots,x_n)$ to $\sum_{i=m+1}^{n} x_i^2$. We shall construct a C^1 diffeomorphism ϕ of $\mathbb{R}^n$ with $\phi|_{\mathbb{R}^m \times 0^{n-m}} =$ identity, such that the tangent space to X is contained in the tangent space to the fibre of $\rho_\phi = \rho\circ\phi$ on an infinite subsequence of the sequence $\{x_i\}$, so that (b_s) fails for (X, Y) at 0.

As in the proof of Theorem 3.3, pick an infinite sequence of pairwise disjoint balls $B_{r_i}(x_i) = B_i$ with centre x_i and of radius r_i. Then $0 \notin B_i$ for all i. We shall obtain ϕ by perturbing the foliation of $\mathbb{R}^n - (\mathbb{R}^m \times 0^{n-m})$ by the level hypersurfaces of ρ, within each B_i.

Let $H = \lambda^{\perp} \in G^n_{n-1}(\mathbb{R})$, and note that $H = \tau \oplus (\tau \oplus \lambda)^{\perp}$ because τ and λ have been assumed orthogonal. Since $T_{x_i}X$ tends to τ, and λ_i tends to λ, as i tends to ∞, there is some i_o such that $i \geqslant i_o$ implies $\lambda_i \not\subset T_{x_i}X$. Then for all $i \geqslant i_o$ we define a hyperplane

$$H_i = T_{x_i}X \oplus (T_{x_i}X \oplus \lambda_i)^{\perp} \subset T_{x_i}\mathbb{R}^n .$$

H_i tends to H as i tends to ∞. Pick $i_1 \geqslant i_o$ such that $|H_i - H| < 1/4$ for $i \geqslant i_1$.

Let $\delta_i > 0$. Then it is clear that we can find a C^1 diffeomorphism $\psi_i : (B_i, x_i) \circlearrowleft$, equal to the identity near ∂B_i, such that $d\psi_i(x_i) = I_n$ (the identity matrix), $|j^1(\psi_i)(p) - j^1(\mathrm{id}_{\mathbb{R}^n})(p)| < \delta_i$ and $|j^1(\psi_i^{-1})(p) - j^1(\mathrm{id}_{\mathbb{R}^n})(p)| < \delta_i$ for all $p \in B_i$, and such that for some t_i, $0 < t_i < r_i$, the image by ψ_i of the foliation of $B_{t_i}(x_i)$ by the level hypersurfaces of ρ is the trivial foliation by hyperplanes parallel with $K_i = T_{x_i}(\rho^{-1}(\rho(x_i)))$. Now $K_i = \lambda_i^{\perp}$, by definition of λ_i, and so K_i tends to $H = \lambda^{\perp} = (\lim \lambda_i)^{\perp}$ as i tends to ∞. Pick $i_2 \geqslant i_1$ such that $|K_i - H| < 1/4$ for all $i \geqslant i_2$. Then $|K_i - H_i| \leqslant 1/2$ for $i \geqslant i_2$, by our choice of i_1 and i_2.

For all $i \geqslant i_2$ we now perturb the trivial foliation of $B_{t_i}(x_i)$ by planes parallel with K_i by placing inside $B_{t_i}(x_i)$ a "ripple" : a foliated ball $B_{\frac{1}{2}t_i}(y_i)$ of radius $\frac{1}{2}t_i$, centre y_i, with the foliation $\mathcal{F}^{|H_i-K_i|}_{K_i}$ given by Construction 3.4, such that $x_i = x_{H_i}$ (the tangent at x_i to the leaf of the foliation passing through x_i is H_i). In the notation of 3.4, $\phi^{|H_i-K_i|}_{K_i}$ is the C^1 diffeomorphism defining the resulting foliation of $B_{t_i}(x_i)$, and we may extend $\phi^{|H_i-K_i|}_{K_i}$ by the identity to the rest of B_i.

Set $\phi_i = \psi_i \circ \phi^{|H_i-K_i|}_{K_i} \circ \psi_i^{-1} : B_i \circlearrowleft$. ϕ_i is a C^1 diffeomorphism, and the tangent space at x_i to $(\rho \circ \phi_i)^{-1}(\rho(\phi_i(x_i)))$ is H_i which contains $T_{x_i}X$ by definition (we have used here for the second time that $d\psi_i(x_i) = I_n$). Compare the figure overleaf.

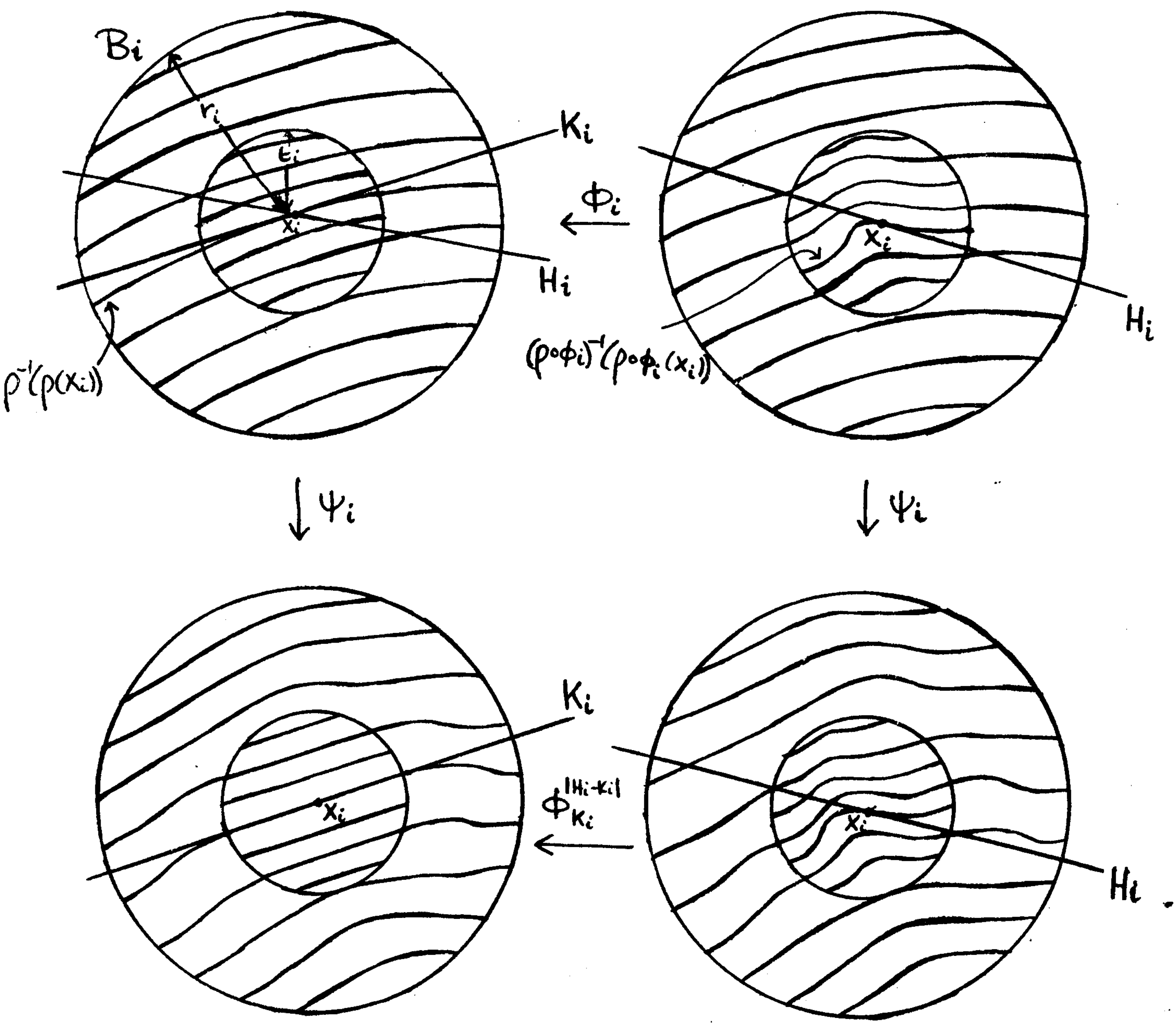


Figure : Construction of Φ_i .

We have yet to fix δ_i. It is easy to verify that $\sup_{p\in B_i} |d\phi_i(p) - I_n|$ may be set as near as we please to $\sup_{p\in B_i} |d\phi_{K_i}^{|H_i-K_i|}(p) - I_n|$, by choosing δ_i small.

Let δ_i be chosen such that,

$$\sup_{p\in B_i} |d\phi_i(p) - I_n| \leq 2 \sup_{p\in B_i} |d\phi_{K_i}^{|H_i-K_i|}(p) - I_n| . \qquad (*)$$

Define $\phi : \mathbb{R}^n \circlearrowleft$ by setting $\phi|_{\mathbb{R}^n - (\bigcup_{i\geq i_2} B_i)}$ = identity, and $\phi|_{B_i} = \phi_i$ for $i \geq i_2$. To verify that ϕ is a C^1 diffeomorphism it is enough to check that $d\phi(p)$ is continuous at 0, and that $d\phi(0) = I_n$.

Given $\varepsilon > 0$, (4) of Construction 3.4 gives an $s_{\frac{1}{2}\varepsilon} > 0$. Pick $i_3 \geq i_2$ such that $|H_i - H|$ and $|K_i - H|$ are each less than $\frac{1}{2}s_{\frac{1}{2}\varepsilon}$ for all $i \geq i_3$. Then $|H_i - K_i| < s_{\frac{1}{2}\varepsilon}$ for all $i \geq i_3$. Let $\delta = \min_{\substack{p\in B_i \\ i_2\leq i<i_3}} \{|p|\}$.

Then δ is well-defined and nonzero since $0 \notin \bigcup_{i=i_2}^{i_3-1} B_i$.

Let $p \in \mathbb{R}^n$ be such that $|p| < \delta$. Then $p \notin \bigcup_{i=i_2}^{i_3-1} B_i$, and thus

$$\begin{aligned}
|d\phi(p) - I_n| &\leq \max_{\substack{p'\in B_i \\ i\geq i_3}} \{|d\phi_i(p') - I_n|\} \\
&\leq 2 \max_{\substack{p'\in B_i \\ i\geq i_3}} \{|d\phi_{K_i}^{|H_i-K_i|}(p') - I_n|\} && \text{(by } (*)) \\
&\leq 2 . \tfrac{1}{2}\varepsilon && \text{(by choice of } i_3 \text{ and } s_{\frac{1}{2}\varepsilon} \text{ - see 3.4)} \\
&= \varepsilon .
\end{aligned}$$

Hence $d\phi(p)$ is continuous at 0, and $d\phi(0)$ is the identity matrix.

By construction, the fibre of $\rho_\phi = \rho\circ\phi$ is not transverse to X at x_i , and hence neither is the fibre of $(\pi_\phi , \rho_\phi) = (\pi\circ\phi , \rho\circ\phi)$, so that $(\pi_\phi , \rho_\phi)|_X$ is not a submersion near x_i . Hence we have shown that X fails to be (b_s)-regular over Y at 0 , using the hypothesis that X is not (b)-regular over Y at 0 .

This completes the proof of Theorem 5.3 .

<u>Corollary 5.4</u> : (b)-<u>regularity is a</u> C^1 <u>invariant</u>.

<u>Example 5.5</u> : Theorem 5.3 is sharp : C^2 tubular neighbourhoods do not detect all (b)-faults. Consider once again Example 3.6 . There we have a (b)-fault, since it is an (a)-fault. However for all C^1 distance functions ρ (associated to a C^1 tubular neighbourhood), the fibres of ρ are transverse to X near 0 . For, all limiting tangent planes to X at 0 contain the z-axis , and near 0 all points (x, y, z) on X have x/z small, and at such points the normal to the fibre of ρ will be close to $(0 : 0 : 1)$. (To see that near 0 , if (x, y, z) is on X , then x/z is small, notice that the x-coordinate of the points in each barrow B_n is bounded above by $m_n r_n$, while the z-coordinate is bounded below by m_n , and r_n tends to 0 as n tends to ∞ and we approach 0 .)

Since we have shown in 3.6 that all C^2 retractions have their fibres transverse to X near 0 , it follows that for all C^2 tubular neighbourhoods T of Y , the fibres of (π_T, ρ_T) are transverse to X near 0 .

<u>Note 5.6</u> : <u>A semianalytic version of 5.3</u> .

We refer to [38] for a proof that (b_s) implies (b) when X and Y

are semianalytic. A careful reading of the proof in [38] shows that semianalytic (b)-faults can be detected by C^1 semianalytic tubular neighbourhoods, i.e. we can suppose the maps in the definition of tubular neighbourhood to have semianalytic graphs.

Note 5.7 : On μ-constant implies topological triviality.

In [17] Lê Dũng Tràng and Ramanujam prove that for a family of complex hypersurfaces (with isolated singularity) defined by

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^k , 0 \times \mathbb{C}^k) \longrightarrow (\mathbb{C} , 0)$$

with $F(z, t) = F_t(z)$, that $\mu(F_t)$ constant implies that the topological type of $F_t^{-1}(0)$ is constant, provided $n \neq 2$. Timourian has proved further that the family is topologically trivial (see [33]).

If one could prove that $\mu(F_t)$ constant implied the existence of a C^1 tubular neighbourhood T of $0 \times \mathbb{C}^k$ with the fibres of (π_T, ρ_T) transverse to $F^{-1}(0)$ near 0, one could then apply the proof of Mather in [21] to give topological triviality, so removing the restriction $n \neq 2$. Applying Theorem 5.3, we know from the counterexamples of Briançon and Speder in [2] that $\mu(F_t)$ constant does not imply that $(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k)$ is (b)-regular, and hence does not imply (b_s), and indeed following the proof in [38] that (b_s) implies (b) it is easy to construct explicit semianalytic tubular neighbourhoods T with the fibres of (π_T, ρ_T) nontransverse to $F^{-1}(0)$ along the curve through 0 for which (b) fails. There are though <u>some</u> tubular neighbourhoods T for which the fibres of (π_T, ρ_T) are transverse to $F^{-1}(0)$ in their examples, since in each case $F(z, t)$ is weighted homogeneous in z, and so the <u>standard</u> spheres cut $F^{-1}(0)$ transversally. Thus, even though $n = 2$, we can derive topological triviality from [21].

A more promising way of removing the restriction that $n \neq 2$ looks to be a new theorem of Kuo (Theorem 2 in [15]) which may give topological triviality directly from the hypothesis that $\mu(F_t)$ be constant. This depends on whether $\mu(F_t)$ constant implies that there is some constant $C < 1$ and a neighbourhood U of 0 such that $(\partial F/\partial t\,(z,t))/|\mathrm{grad}\,F| < C\,|z|/|t|$ whenever $(z,t) \in U \cap F^{-1}(0)$. We shall leave this question for the present.

6. (b)-regularity and generic sections

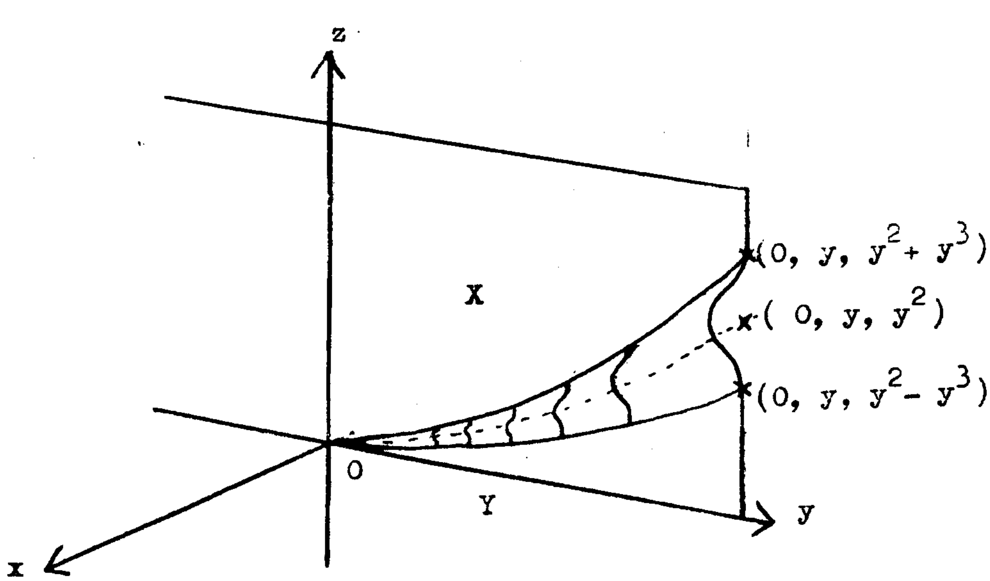
Part I . Detecting (b)-faults with generic sections.

The work in this section was motivated by the result of Teissier in [30] that " μ^*-constant " implies (b)-regularity for a family of complex hypersurfaces. Using the converse result (proved by Briançon and Speder in [3]) we find that if we have topological triviality, and (b) for generic hyperplane sections, then (b) follows. That this result does not generalise to real semialgebraic strata is shown by the next example.

Example 6.1 : In the open subset of $\mathbb{R}^3$ (with (x, y, z) as coordinates) where $y^2 < 1$, let Y be the y-axis, and let X be

$$\{x = 0\,,\ (z - y^2)^2 \geq y^6,\ z > 0\} \cup \{y^9 x = ((z - y^2)^2 - y^6)^2,\ (z - y^2)^2 \leq y^6,\ z > 0\}.$$

X is a C^1 manifold, and a semialgebraic set.



Then X is topologically trivial along Y and , since the non-linear part of X is contained in a horn tangent to Y , X is (a)-regular over Y .

But X is not (b)-regular over Y at 0 : on the curve

$$\gamma(t) = (9t^3/16 , t , t^2 + \tfrac{1}{2}t^3)$$

which lies in X , the normal tends to $(1, 0, 3/2)$, so that the limiting tangent space does not comtain $0z$, which is the limit of $\frac{x_i\pi(x_i)}{|x_i\pi(x_i)|}$ for all sequences $\{x_i\}$ on X tending to 0 , since the radius (y^3) of the horn tends to 0 faster than the height (y^2) above Y of the centre of the horn.

Also if $x = \alpha z$ defines the plane H_α , which contains Y , then H_α intersects X near 0 only if $\alpha = 0$. Thus $(X \cap H_\alpha , Y)$ is not a (b)-fault (by default) for generic sections H_α containing Y .

<u>Notation</u> : Let (X,Y) be a pair of adjacent strata, and let $0 \in Y \subset \overline{X} - X$. Suppose Y is a linear space, and that π is orthogonal projection onto Y . We let $\mathcal{K}_0(X,Y)$ (resp. $\Lambda_0(X,Y)$) denote the set of limit vectors for which (b) (resp. (b')) fails.

$$\mathcal{K}_0(X,Y) = \{ \lambda : \exists \{x_i\} \in X , \{y_i\} \in Y , \lambda = \lim \frac{x_iy_i}{|x_iy_i|} \not\subset \tau = \lim T_{x_i}X \}$$

$$\Lambda_0(X,Y) = \{ \lambda : \exists \{x_i\} \in X , \lambda = \lim \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|} \not\subset \tau = \lim T_{x_i}X \} .$$

In Example 6.1 , $\Lambda_0(X,Y) = \{(0: 0: 1)\}$, $\mathcal{K}_0(X,Y) = \{(0: a: b) : b \neq 0\}$.

It is easy to see that (when $\dim \Lambda_0(X,Y)$ is defined),

$$\dim \Lambda_0(X,Y) + \dim Y \leq \dim \mathcal{K}_0(X,Y)$$

if $\mathcal{K}_0(X,Y) \neq \emptyset$.

If X is (a)-regular over Y at 0 , then by the proof of (0.4) that (b) is equivalent to (a) + (b') ,

$$\mathcal{K}_0(X,Y) \subset \Lambda_0(X,Y) \oplus T_0Y ,$$

and hence,

$$\dim \mathcal{K}_0(X,Y) \leq \dim \Lambda_0(X,Y) + \dim Y .$$

Thus if (a)-regularity holds and $\Lambda_0(X,Y) \neq \emptyset$ (or, equivalently, $\mathcal{K}_0(X,Y) \neq \emptyset$),

$$\dim \mathcal{K}_0(X,Y) = \dim \Lambda_0(X,Y) + \dim Y .$$

That is, the dimension of $\Lambda_0(X,Y)$ determines the dimension of $\mathcal{K}_0(X,Y)$, so that we can restrict our attention to $\Lambda_0(X,Y)$.

We say that X <u>is</u> $(b_{\mathrm{cod}\ k})$-<u>regular over</u> Y <u>at</u> 0 for $0 \leq k \leq \mathrm{cod}\ Y - 1$, when Y is linear (as it will be throughout this first part of §6), if

$(b_{\mathrm{cod}\ k})$ There is an open dense subset $\mathcal{L}$ of the set of linear subspaces of codimension k containing Y, such that if $L \in \mathcal{L}$, $L \pitchfork X$ near 0, and $X \cap L$ is (b)-regular over Y at 0 in L.

We must suppose $L \pitchfork X$ to be able to talk of (b)-regularity of $X \cap L$ over Y. In the case where X is the nonsingular part of a family of complex analytic hypersurfaces with singular locus Y, there is a Zariski open dense subset of the set of linear subspaces of (complex) codimension k containing Y, consisting of subspaces transverse to X (moreover the topological type of their intersection with X is well-defined : see Chapter 1, §1 of [30]). It was this situation which motivated the work in this section : see Note 6.9.

The following theorem says that $(b_{\mathrm{cod}\ k})$ implies that $\dim \Lambda_0(X,Y) < k$. Here $\dim \Lambda_0(X,Y)$ is the maximal integer r, $-1 \leq r \leq \mathrm{cod}\ Y - 1$, for which $\Lambda_0(X,Y)$ has a point near which it is a differentiable submanifold of $G_1^{\mathrm{cod}\ Y}(\mathbb{R})$ of dimension r. This is the same as the usual dimension of $\Lambda_0(X,Y)$ when X is subanalytic, for then $\Lambda_0(X,Y)$ is the union of countably many compact manifolds-with-boundary of varying dimensions, the largest of which being the dimension of $\Lambda_0(X,Y)$; this will follow from the proof of the theorem.

We point out that a section of a pair (X,Y), as in the title of §6, is a linear subspace of $\mathbb{R}^n$ containing Y, which is assumed to be linear. Thus Theorem 6.2 describes the extent to which generic sections detect (b)-faults.

Theorem 6.2 : Let Y be a linear subspace of $\mathbb{R}^n$ containing 0, and let X be a C^2 submanifold (resp. and a subanalytic subset) of $\mathbb{R}^n$ such that $Y \subset \overline{X} - X$. Suppose there is an open dense (resp. dense) subset $\mathcal{L}_k'$ of the set $\mathcal{L}_k$ (of linear subspaces of codimension k in $\mathbb{R}^n$ which contain Y) such that $L \in \mathcal{L}_k'$ implies $L \pitchfork X$ near 0 and $X \cap L$ is (b)-regular over Y at 0.

Then $\dim \Lambda_0(X,Y) < k$.

Proof : We first state two assertions which we shall prove once we have shown how they give the theorem.

Assertion 6.3 : Let $Y \subset \mathbb{R}^n$ be linear, $0 \in Y$, and X a C^2 submanifold of $\mathbb{R}^n$, $Y \subset \overline{X} - X$, such that $\dim \Lambda_0(X,Y) = i \geqslant k$.

Then there is a dense subset $\mathcal{L}_k^d$ of a nonempty open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$, such that if $L \in \mathcal{L}_k^d$, there is a sequence $\{x_i\}$ in $X \cap L$ such that x_i tends to 0 as i tends to ∞, and $\lim \dfrac{x_i\pi(x_i)}{|x_i\pi(x_i)|} \not\subset \lim T_{x_i}X$.

Assertion 6.4 : In Assertion 6.3, if X is also a subanalytic subset of $\mathbb{R}^n$, we may take $\mathcal{L}_k^d = \mathcal{L}_k^o$.

(The conclusion of Assertion 6.4 is that there is a nonempty subset of $\mathcal{L}_k$ consisting of linear sections containing "bad" sequences, and that this subset may be taken to be open, not merely dense in some open set.)

Suppose that Theorem 6.2 is false.

Take Y and X which satisfy the hypotheses of Theorem 6.2, and yet

$$\dim \Lambda_0(X,Y) = i \geq k .$$

Assume for the moment that X is not subanalytic, and apply Assertion 6.3.

Assertion 6.3 gives $\mathcal{L}_k^d$, which is dense in the nonempty open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$, and hence meets the open dense subset $\mathcal{L}_k'$ of $\mathcal{L}_k$ described in the hypotheses of Theorem 6.2.

Take $L \in \mathcal{L}_k' \cap \mathcal{L}_k^d$. Then $L \pitchfork X$ near 0 and $(X \cap L, Y)_0$ is (b)-regular. Hence for all sequences $\{x_i\}$ in $X \cap L$ tending to 0,

$$\lim \frac{x_i \pi(x_i)}{x_i \pi(x_i)} \subset \lim T_{x_i}(X \cap L) .$$

But $T_{x_i}(X \cap L) \subset T_{x_i}X$ for all i, and so $\lim T_{x_i}(X \cap L) \subset \lim T_{x_i}X$.

Thus,

$$\lim \frac{x_i \pi(x_i)}{x_i \pi(x_i)} \subset \lim T_{x_i}X .$$

However this is not true for all $\{x_i\}$ in $X \cap L$ since $L \in \mathcal{L}_k^d$, by Assertion 6.3. Thus we find a contradiction, showing that Theorem 6.2 is valid when X is not subanalytic so long as Assertion 6.3 is true.

The argument for subanalytic X is similar : the dense subset $\mathcal{L}_k'$ of $\mathcal{L}_k$ must meet the open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$ given by Assertion 6.4.

We shall have to prove Assertions 6.3 and 6.4 separately, but we first set up the situation which is common to both.

Rotate the coordinate axes so that $Y = \mathbb{R}^{n-m} \times 0^m$. Let $\dim X = d$.

Define $\gamma : X \longrightarrow G_1^m \times G_d^n$ and let G denote the closure of the graph of

$$x \longmapsto \left(\frac{x\pi(x)}{|x\pi(x)|}, T_x X \right)$$

γ in $\mathbb{R}^n \times G_1^m \times G_d^n$ (we write G_1^m for $G_1^m(\mathbb{R})$,etc.). Since X is C^2 , γ is a C^1 map. Let p and q denote the projections from $\mathbb{R}^n \times G_1^m \times G_d^n$ onto $\mathbb{R}^n$ and G_1^m respectively. $p|_{\gamma(X)}$ is a C^1 diffeomorphism.

If ℓ is a line through 0 in $\mathbb{R}^m$, let $\hat{\ell}$ denote the line in $\mathbb{R}^n$ given by the inclusion $0^{n-m} \times \mathbb{R}^m \hookrightarrow \mathbb{R}^n$. Then $B = \{(\ell,\tau) \in G_1^m \times G_d^n : \hat{\ell} \not\subset \tau\}$ is an open subset of $G_1^m \times G_d^n$.

From now on we write Λ for $\Lambda_0(X,Y)$. Observe that

$$\Lambda = q(G \cap p^{-1}(0) \cap (\mathbb{R}^n \times B)) .$$

Given a subspace L in $\mathcal{L}_k$ we can write $L = Y \times \tilde{L}$ where $\tilde{L} \in G_{m-k}^m$. Given $A \in G_{m-k}^m$, write $A^* = \{\ell \in G_1^m : \ell \subset A\} \subset G_1^m$.

Let D_0 be a compact coordinate disc (of dimension $m-1$) for Λ as a C^1 submanifold of G_1^m of dimension i . D_0 exists by hypothesis on dim Λ .

<u>Proof of Assertion 6.3</u>:

<u>Lemma 6.5</u> : <u>There is a dense subset</u> $\mathcal{L}_k^d$ <u>of the open set</u>

$$\mathcal{L}_k^0 = \{L \in \mathcal{L}_k : (\tilde{L})^* \pitchfork \Lambda \text{ on } \Lambda \cap D_0\}$$

<u>such that for all</u> $L \in \mathcal{L}_k^d$, $L \pitchfork X$ <u>near</u> 0 <u>and there is an open ball</u> B_L <u>such that</u> (i) $\bar{B}_L \subset \mathbb{R}^n \times B$ <u>and</u> $q(B_L) \subset D_0$,

(ii) <u>if</u> $F_L = q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(X) \cap \{z \in G : G \pitchfork q^{-1}(\tilde{L})^* \text{ at } z\}$, <u>then</u> $q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(0)$ <u>has nonempty intersection with</u> $\bar{F}_L$.

Assuming Lemma 6.5 , let $L \in \mathcal{L}_k^d$, and let $\{z_i\}$ be a sequence of points in F_L tending to a limit z_0 in $q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(0)$. Let $x_i = p(z_i)$ for all i . Then $\{x_i\}$ is a sequence of points in X tending to $p(z_0) = 0$. Also for all i , $x_i \in L$ since $q(\gamma(x_i)) \in (\tilde{L})^*$. Finally

$\lim \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|} = \hat{\ell} \not\subset \tau = \lim T_{x_i}X$ since $(0, \ell, \tau) \in \bar{F}_L \subset \bar{B}_L \subset \mathbb{R}^n \times B$, by (i) and (ii) of Lemma 6.5, and so $(\ell, \tau) \in B$, i.e. $\hat{\ell} \not\subset \tau$. This completes the proof of Assertion 6.3.

<u>Proof of Lemma 6.5</u> :

<u>Sublemma 6.6</u> : <u>Given a</u> C^1 <u>retraction</u> $r : D_0 \longrightarrow \Lambda \cap D_0$, <u>there is a dense subset</u> W <u>of</u> $\Lambda \cap D_0$ <u>such that if</u> $\ell \in W$, $(r\circ q)^{-1}(\ell)$ <u>contains a sequence</u> $\{a_i\}$ <u>in</u> $p^{-1}(X) \cap G \cap (\mathbb{R}^n \times B)$ <u>tending to a point in</u> $p^{-1}(0) \cap q^{-1}(D_0)$ <u>such that</u> $(r\circ q)^{-1}(\ell)$ <u>is transverse to</u> G <u>at</u> a_i <u>for all</u> i.

<u>Proof</u> (after L. Siebenmann) :

Let $W_N = \{\ell \in \Lambda \cap D_0 : \exists\ a_\ell^N \in G \cap (r\circ q)^{-1}(\ell)$ with $(r\circ q)^{-1}(\ell) \pitchfork G$ at a_ℓ^N and $0 < |\pi(p(a_\ell^N))| < 1/N\}$, for N a positive integer. a_ℓ^N is inside a region R_N of radius $1/N$ around $p^{-1}(0)$. W_N is open since transversality is an open condition. W_N is dense (and hence nonempty) by Sard's theorem applied to the C^1 map $(r\circ q)|_{G \cap R_N \cap q^{-1}(D_0)}$. Note that $G \cap R_N \cap q^{-1}(D_0)$ is nonempty since,

$$(\Lambda \cap D_0) \subset q(p^{-1}(0) \cap \overline{(G \cap R_N \cap q^{-1}(D_0))}\ ).$$

Because $\Lambda \cap D_0$ is a C^1 manifold, it is locally compact and Hausdorff, and hence is a Baire space. Thus $W = \bigcap_{N=1}^{\infty} W_N$ is dense in $\Lambda \cap D_0$. Given $\ell \in W$, there is a limit point of $\{a_\ell^N\}$ in $p^{-1}(0)$ since $p^{-1}(0)$ is compact ($p^{-1}(0) \cong G_1^m \times G_d^n$). This limit point will be in $q^{-1}(D_0)$ since D_0 is closed. Then W satisfies the properties required for Sublemma 6.6.

Now we can prove Lemma 6.5.

Given L in $\mathcal{L}_k^o$ with $(\tilde{L})^* \pitchfork \Lambda$ at ℓ in $\Lambda \cap D_0 \cap (\tilde{L})^*$, there is a neighbourhood U of L in the k-dimensional family in $\mathcal{L}_k^o$ which is defined by the $(k+1)$-dimensional linear subspace orthogonal to L and containing the line ℓ, such that if $L' \in U$, $(\tilde{L}')^* \pitchfork \Lambda$ in $\Lambda \cap D_0$. $\{(\tilde{L}')^* : L' \in U\}$

defines a foliation of codimension k transverse to Λ near ℓ .

Choose a C^1 retraction $r : D_0 \longrightarrow \Lambda \cap D_0$ such that $r^{-1}(\ell) \subset (\tilde{L})^*$ and for all ℓ' in some neighbourhood of ℓ in $\Lambda \cap D_0$, $r^{-1}(\ell') \subset (\tilde{L'})^*$, where L' is the element of U such that $\ell' \in (\tilde{L'})^*$. By Sublemma 6.6 , arbitrarily near ℓ there is some $\ell' \in W$. Hence arbitrarily near L in $\mathcal{L}_k^o$ there is some L' (in U) with $(\tilde{L'})^* \pitchfork \Lambda$ in D_0 and such that $q^{-1}((\tilde{L'})^*)$ contains a sequence of points $\{a_i\}$ in $G \cap (\mathbb{R}^n \times B)$ tending to a limit a_0 in $G \cap p^{-1}(0) \cap q^{-1}((\tilde{L'})^*)$ such that for all i , $q^{-1}((\tilde{L'})^*)$ is transverse to G at a_i . Choose an open ball $B_{L'}$ around a_0 such that $q(B_{L'}) \subset D_0$ and $\bar{B}_{L'} \subset \mathbb{R}^n \times B$. Then (i) and (ii) of Lemma 6.5 are satisfied since $a_0 \in \bar{F}_{L'} \cap q^{-1}((\tilde{L'})^*) \cap G \cap B_{L'} \cap p^{-1}(0)$. This completes the proof of Lemma 6.5 .

<u>Proof of Assertion 6.4</u> :

<u>Lemma 6.7</u> : <u>There is a compact coordinate disc</u> D <u>for</u> Λ <u>as a submanifold of dimension</u> i <u>in</u> G_1^m , <u>with</u> $D \subset \mathrm{Int}\, D_0$, <u>such that if</u> T <u>is a</u> C^1 <u>submanifold of</u> G_1^m <u>of dimension</u> $(m-k-1)$ <u>which is transverse to</u> Λ <u>on</u> $D \cap \Lambda$, <u>then there is an open ball</u> $B_T \subset \mathbb{R}^n \times B$ <u>such that,</u>

(i) $\bar{B}_T \subset \mathbb{R}^n \times B$ <u>and</u> $q(B_T) \subset D$,

(ii) $F_T = q^{-1}(T) \cap G \cap p^{-1}(X) \cap B_T$ <u>is a</u> C^1 <u>submanifold of</u> G <u>of codimension</u> k .

(iii) $\emptyset \neq q^{-1}(T) \cap G \cap p^{-1}(0) \cap B_T \subset \bar{F}_T$.

We leave the proof of Lemma 6.7 for the moment.

Let $\mathcal{L}_k^o = \{L \in \mathcal{L}_k : (\tilde{L})^* \pitchfork \Lambda \text{ on } D\}$. Let $M_L = p(F_{(\tilde{L})^*})$ for L in $\mathcal{L}_k^o$. By Lemma 6.7(ii) and the fact that $p|_{\gamma(X)}$ is a C^1 diffeomorphism , M_L is a C^1 submanifold of X of codimension k , and $0 \in \bar{M}_L$ by (iii) . Also if $x \in M_L$ then $q(\gamma(x)) \in (\tilde{L})^*$ by definition of M_L , and hence

$q(\gamma(x)) \subset \tilde{L}$ by definition of $(\ )^*$, so that $x \in \pi(x) \times \tilde{L} \subset Y \times \tilde{L} = L$. Thus $M_L \subset L$ for $L \in \mathcal{L}_k^o$.

Let $\{x_i\}$ be a sequence in M_L tending to 0 as i tends to ∞. To complete the proof of Assertion 6.4 we must show that

$$\hat{\ell} = \lim \frac{x_i \pi(x_i)}{|x_i\, \pi(x_i)|} \not\subset \lim T_{x_i} X = \tau .$$

Now for all i $(x_i, \gamma(x_i)) \in F_{(\tilde{L})*}$, by (ii) of Lemma 6.7 and the definition of M_L. Hence $(0, \lim \gamma(x_i)) = (0, \ell, \tau) \in \overline{F}_{(\tilde{L})*} \subset \overline{B}_{(\tilde{L})*} \subset \mathbb{R}^n \times B$ using (i) and (ii) of Lemma 6.7. Thus $(\ell, \tau) \in B$, i.e. $\hat{\ell} \not\subset \tau$, by the definition of B. This completes the proof of Assertion 6.4.

<u>Proof of Lemma 6.7</u> : First, G is subanalytic in $\mathbb{R}^n \times G_1^m \times G_d^n$. For we can partition X into a locally finite set of real analytic submanifolds by [12] (See also [10] and [40]), then complexify each real analytic part, apply the argument of §17 in [46], take real parts, and finally take closures, using that the closure of a subanalytic set is subanalytic [12]. The closures match up since X is C^2.

Then apply Lemma 4.3.3 of [12] to G to give a (b)-regular stratification $\mathcal{G}$ of G such that $G \cap p^{-1}(Y)$ and $G \cap p^{-1}(0)$ are each the union of strata of $\mathcal{G}$. Since $\Lambda = q(G \cap p^{-1}(0) \cap (\mathbb{R}^n \times B))$ has dimension i there is some stratum S of $\mathcal{G}$ contained in $G \cap p^{-1}(0)$ such that $\dim (q(S) \cap D_0) = i$. By the implicit function theorem there is an open subset V of S contained in $\mathbb{R}^n \times B$ such that $q(V) \subset \Lambda \cap D_0$ is a C^1 submanifold of dimension i, and $q|_V$ has rank i. Let D be a compact coordinate disc for $D_0 \cap q(V)$ as a submanifold of dimension i.

Suppose T is a C^1 submanifold of dimension $(m-k-1)$ in G_1^m, transverse to Λ on $D \cap \Lambda$. Then $q^{-1}(T)$ is transverse to S on V since $q|_V$ has constant rank. Let $z \in V \cap q^{-1}(T)$. By (a)-regularity of $\mathcal{G}$ there is an open ball B_T in $\mathbb{R}^n \times B$ such that $z \in (B_T \cap S) \subset V$ and such that $q^{-1}(T)$ is transverse to every stratum of $\mathcal{G}$ within B_T. We may further suppose that $q(B_T) \subset D$, proving (i) of Lemma 6.7.

By definition of G, there is a stratum S_1 of $\mathcal{G}$, not meeting $p^{-1}(Y)$, such that $z \in \overline{S}_1$, i.e. $S \cap \overline{S}_1 \neq \emptyset$. Then by 10.4 of [21], $q^{-1}(T) \cap S \cap B_T \subset \overline{q^{-1}(T) \cap S_1 \cap B_T}$.

Repeating the argument given above for S for each stratum of $\mathcal{G}$ in $p^{-1}(0)$ adjacent to S we find that $q^{-1}(T) \cap G \cap p^{-1}(0) \cap B_T$ is nonempty and contained in $\overline{F}_T$, where $F_T = q^{-1}(T) \cap G \cap p^{-1}(X) \cap B_T$, and that F_T is a C^1 submanifold of G of codimension k. This proves (ii) and (iii) and completes the proof of Lemma 6.7.

We have now completed the proof of Theorem 6.2.

<u>Note 6.8</u> : (1) In the proof of Lemma 6.7 we cited the result of Mather (10.4 of [21]) that if X is (b)-regular over Y in $\mathbb{R}^n$ and S is a submanifold of $\mathbb{R}^n$ transverse to Y then $S \cap Y \subset \overline{S \cap X}$. It is amusing that for complex analytic X, Y, and S, this follows from (a)-regularity : see the appendix of [25].

(2) If X, Y are complex analytic in $\mathbb{C}^n$ we obtain the same theorem, but involving complex linear subspaces of complex codimension k, and with the conclusion that $\dim_{\mathbb{C}} \Lambda_0(X,Y) < k$.

<u>Note 6.9</u> : In the context of a family of complex hypersurfaces with isolated singularity, if one could prove that $\mu(F_t)$ constant implies that

$$\dim_{\mathbb{C}} \Lambda_0(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k) \neq 0,$$

then using Theorem 6.2 we would obtain an inductive proof of the result of Teissier that μ^*-constant implies (b)-regularity for the pair $(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k)$ ([30]).

In the only known examples of a μ-constant family which is not (b)-regular (due to Briançon and Speder), $\dim_{\mathbb{C}} \Lambda_0(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k) > 0$.

For example, consider $F(x, y, z, t) = x^3 + txy^3 + y^4z + z^9$ (due to Speder. Cf. [2]) . Analogous to the calculation in [2] we find that (b) fails on a curve $\gamma(u) = (\beta u^5, \alpha u^3, h(\alpha u^3)\alpha u^3, u)$ where $h : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies $h(0) = 1$ and $h(y)y^5 + (h(y)y)^9 = y^5$ (h exists by the implicit function theorem), and α , β are complex numbers defined by the equations $\beta^3 + \alpha^3\beta + \alpha^5 = 0$, $3\beta^2 + \alpha^3 = 0$. The limit of orthogonal secant vectors λ is $(0 : 1 : 1)$ and the limit of normal vectors ν is $(0 : 3\beta + 4\alpha^2 : \alpha^2)$. λ is not contained in the limiting tangent space orthogonal to ν since $3\beta + 5\alpha^2 \neq 0$.

Now consider the curve $\gamma_\theta(u) = (\beta_\theta u^5, \alpha_\theta u^5, h_\theta(\alpha_\theta u^3)\alpha_\theta u^3(1+\theta), u)$ where $\theta \in \mathbb{C}$, $|\theta| < \varepsilon$ for some positive $\varepsilon < 1$, and $h_\theta : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies $h_\theta(0) = 1$ and $h_\theta(y)y^5(1+\theta) + (h_\theta(y)y(1+\theta))^9 = y^5(1+\theta)$, and α_θ , β_θ are complex numbers defined by the equations $\beta_\theta^3 + \alpha_\theta^3\beta_\theta + (1+\theta)\alpha_\theta^5 = 0$, $3\beta_\theta^2 + \alpha_\theta^3 = 0$. Then $\lambda_\theta = (0 : 1 : 1+\theta)$ and $\nu_\theta = (0 : 3\beta_\theta + 4(1+\theta)\alpha_\theta^2 : \alpha_\theta^2)$. λ_θ is not contained in the limiting tangent space orthogonal to ν_θ since $3\beta_\theta + 5(1+\theta)\alpha_\theta^2 \neq 0$ for small θ , i.e. for ε sufficiently small. As θ varies we obtain a complex 1-dimensional subset of $\Lambda_0(X,Y)$ and thus $\dim_{\mathbb{C}} \Lambda_0(X,Y) \geq 1$. In fact $\dim_{\mathbb{C}} \Lambda_0(X,Y) = 1$ here since the family is equimultiple (with multiplicity 3), which is the same as saying that $(X \cap L, Y)$ is (b)-regular for generic complex linear subspaces L of codimension 2 containing Y , or again that $X \cap L = \emptyset$ for generic L . (Recall $X = F^{-1}(0) - (0 \times \mathbb{C})$, and $Y = 0 \times \mathbb{C}$, the t-axis)

Remark 6.10 : It would be interesting to have a converse to Theorem 6.2 , i.e. a proof that $\dim \Lambda < k$ implies $(b_{\text{cod } k})$, when generic linear subspaces of codimension k are transverse to X . We consider a weak form of such a converse in the second part of §6 .

Part II . Preservation of (b)-regularity under generic sections.

Let X , Y be C^1 submanifolds of $\mathbb{R}^n$, and $0 \in Y \subset \overline{X} - X$. We call a C^1 submanifold of dimension $(n-k)$ containing Y a section of codimension k (cod $Y < k \leqslant 0$).(This term was reserved for linear subspaces in Part I.) Denote the set of germs at 0 of sections of codimension k by $\mathcal{S}_k$. In the notation of Whitney [46] [47], the set of limits of tangent planes to X given by sequences on X tending to 0 is $\tau(X,0) \subset G^n_{\dim X}(\mathbb{R})$. Let $\mathcal{S}^*_k$ denote the subset of $\mathcal{S}_k$ consisting of germs at 0 of sections S of codimension k such that T_0S is transverse to every element of $\tau(X,0)$ in $T_0\mathbb{R}^n$. We give $\mathcal{S}_k$ the topology induced from the topology on $G^n_{n-k}(\mathbb{R})$ by the map $\sigma \longmapsto T_0\sigma$.

Theorem 6.11 : Let X be (b)-regular over Y at 0 , and let S be a representative of $\sigma \in \mathcal{S}^*_k$. Then $S \pitchfork X$ near 0 and $X \cap S$ is (b)-regular over Y at 0 .

Proof : It suffices to prove the result for $k = 1$, since we may consider a section of codimension k as the intersection of k sections of codimension 1 . Let $\sigma \in \mathcal{S}^*_1$, and let S be a representative of σ . It is clear that $S \pitchfork X$ near 0 , so that it makes sense to test for (b)-regularity.

Let $\{x_i\}$ and $\{y_i\}$ be sequences in $X \cap S$ and Y tending to 0 so that $\frac{x_iy_i}{|x_iy_i|}$ tends to λ , $T_{x_i}(X \cap S)$ tends to τ_S and $T_{x_i}X$ tends to τ .

$T_0S \pitchfork \tau$ since $S \in \mathcal{S}^*_1$, and clearly $\tau_S \subset \tau \cap T_0S$. Thus $\tau_S = \tau \cap T_0S$. Since X is (b)-regular over Y at 0 , $\lambda \subset \tau$. But S is a C^1 submanifold, and thus $\lambda \subset T_0S$, and $\lambda \subset \tau_S$, showing that $X \cap S$ is (b)-regular over Y at 0 , and completing the proof of the theorem.

If $\mathcal{S}^*_k$ were open and dense in $\mathcal{S}_k$ (in the topology given by the tangents at 0), we would have proved that (b) implies ($b_{\text{cod }k}$). Our next result describes sufficient conditions for this to be so.

Theorem 6.12 : Let X be (a)-regular over Y at 0 in $\mathbb{R}^n$, and let $\mathcal{T}(X,0)$ have a finite partition into C^1 submanifolds of dimension less than or equal to $\dim X - \dim Y - 1$. Then $\mathcal{S}_k^*$ is open and dense in $\mathcal{S}_k$.

Proof : It suffices to prove the theorem when $k = 1$.

Let $\operatorname{codim} Y = m$, $\dim X - \dim Y = p$. By definition of the topology on $\mathcal{S}_k$, it suffices to show that $\{P \in G^n_{n-1}(\mathbb{R}) : T_0Y \subset P,\ P \pitchfork T,\ \forall\ T \in \mathcal{T}(X,0)\}$ is open and dense in $\{P \in G^n_{n-1}(\mathbb{R}) : T_0Y \subset P\}$.

Lemma 6.13 : Let $\mathcal{K}$ be a compact set in $G^m_p(\mathbb{R})$ partitioned into a finite number of C^1 submanifolds of dimension $\leq (p-1)$. Then $\{Q \in G^m_{m-1}(\mathbb{R}) : Q \pitchfork K,\ \forall\ K \in \mathcal{K}\}$ is open and dense in $G^m_{m-1}(\mathbb{R})$.

Assuming Lemma 6.13 we obtain the required result since if $T \in \mathcal{T}(X,0)$ then $T_0Y \subset T$ by (a)-regularity of X over Y at 0, and since $\mathcal{T}(X,0)$ is compact, being a closed subset of a compact space.

Proof of Lemma 6.13 : We assert that if $\mathcal{K}_1$ is a C^1 submanifold of $G^m_p(\mathbb{R}) = G^m_p$ of dimension $\leq (p-1)$, and we are given $K \in \mathcal{K}_1$ and a compact coordinate neighbourhood N of K in $\mathcal{K}_1$ then $\{Q \in G^m_{m-1} : Q \pitchfork K',\ \forall K' \in N\}$ is open and dense in G^m_{m-1}. For $\{Q \in G^m_{m-1} : K < Q\}$ has dimension $(m-p-1)$, since it is isomorphic to G^{m-p}_1. Thus $\{Q \in G^m_{m-1} : \exists\ K' \in N \text{ with } K' < Q\}$ has dimension $(m-p-1) + \dim \mathcal{K}_1 \leq (m-p-1) + (p-1) = m - 2$, and is closed. Hence its complement, which is $\{Q \in G^m_{m-1} : Q \pitchfork K'\ \forall\ K' \in N\}$, is open and dense in G^m_{m-1}.

Now cover $\mathcal{K}$ by a countable number of compact coordinate discs for each submanifold of the finite partition. Since G^m_{m-1} is a Baire space we deduce that $\{Q \in G^m_{m-1} : Q \pitchfork K \ \forall K \in \mathcal{K}\}$ is dense. Since $\mathcal{K}$ is assumed to be compact it is also open. This completes the proof of Lemma 6.13 and hence of Theorem 6.12.

Note 6.14 : If X is subanalytic, $\tau(X,0)$ is also subanalytic. (Intersect the closure of $\{(x,T_xX) : x \in X\}$ in $\mathbb{R}^n \times G^n_{\dim X}$, which is subanalytic by Lemma 2.7, with $0 \times G^n_{\dim X}$.) Then $\tau(X,0)$ has a locally finite partition into C^1 submanifolds by [12], and the partition will be finite since $\tau(X,0)$, a closed subset of a compact space, is compact.

Examples 6.15 : In Example 6.1, $1 = \dim \tau(X,0) > \dim X - \dim Y - 1 = 0$. For an algebraic example consider Example 4 (1) on page 4 of the introduction. Again $1 = \dim \tau(X,0) > \dim X - \dim Y - 1 = 0$. In both cases $\mathcal{S}^*_1$ is not dense in $\mathcal{S}_1$. However $(b_{\text{cod } 1})$ does hold, so that the following result, which is a consequence of Theorems 6.11 and 6.12, is not sharp.

Corollary 6.16 *If* Y *is linear,* X *is* (b)-*regular over* Y *at* 0, *and* $\tau(X,0)$ *has a finite partition into* C^1 *submanifolds of dimension at most* $\dim X - \dim Y - 1$, *then* X *is* $(b_{\text{cod } k})$-*regular over* Y *at* 0.

Proof : Apply Theorems 6.11 and 6.12, and note that the topology on $\mathcal{S}_k$ was that induced from $G^n_{n-k}(\mathbb{R})$.

Remark 6.17 : In [3], Briançon and Speder prove that (b)-regularity implies μ^*-constant for a family of complex hypersurfaces with isolated singularity. They show that (b)-regularity implies that μ^{n+1} is constant, then, essentially, that $\mathcal{S}^*_k$ is open and dense in $\mathcal{S}_k$, so that applying Theorem 6.11, one obtains the constancy of the rest of the μ^i.

7. Stronger generic regularity.

Let X be a C^1 submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of R^n such that $0 \in Y \subset \overline{X} - X$.

According to Verdier [40], X is (w)-regular over Y at 0 if,

(w) There is a constant $C > 0$ and a neighbourhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$, $y \in U \cap Y$, then $d(T_x X, T_y Y) \leq C\|x - y\|$. *

Verdier proves that (w) implies (b). Here we give an example showing that (b) does not imply (w), even for algebraic strata.

Example 7.1 : In $\mathbb{R}^3$ with (x, y, t) as coordinates, let V be $\{y^4 = t^4 x + x^3\}$. Let Y be the t-axis, and X be $(V - Y)$.

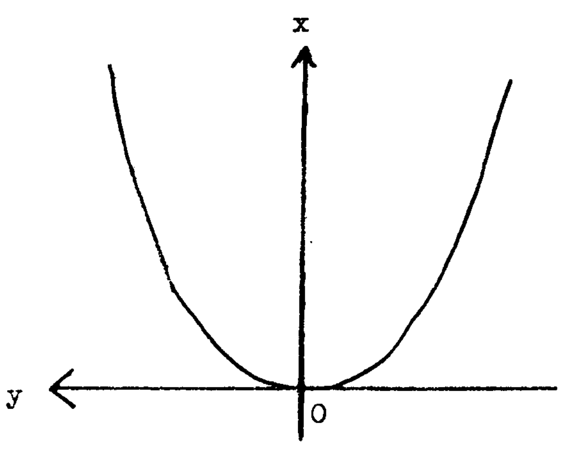


Figure : $t = 0$.

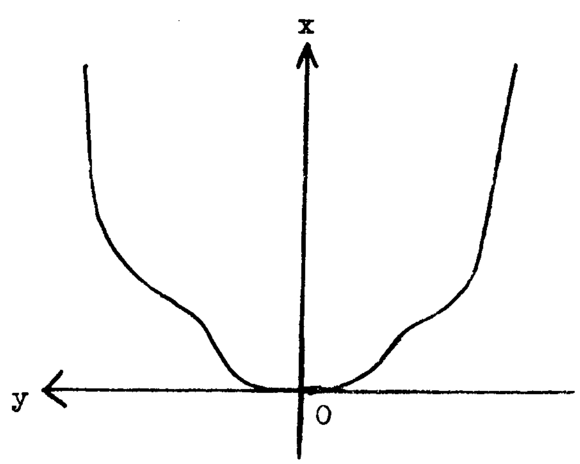


Figure : $t \neq 0$.

From the figures it is clear that V is a topological manifold near 0, and in particular that X is topologically trivial along Y. It will follow from the calculations of §8 that X is (b)-regular over Y at 0. In fact in this example X is C^1 trivial along Y : V is a C^1 submanifold. We show that at 0 there is a unique limiting tangent plane, with normal $(1 : 0 : 0)$ — a chart for V at 0 follows easily.

* See Addendum 7.13 for the definition of $d(\, , \,)$.

The normal to X at $(x, y, t) = (x, (t^4x + x^3)^{1/4}, t)$ is

$$(3x^2 + t^4 : -4(t^4x + x^3)^{3/4} : 4t^3x) \tag{7.2}$$

Since X is algebraic it suffices to consider curves on X through 0 defined by an analytic arc $\gamma(s) = (x(s),t(s))$, $s \in [0,1]$. If $|t(s)/x(s)|$ is bounded as s tends to 0 , the normal is

$$(3 + t^2(t/x)^2 : -4(t^{4/3}(t/x)^{8/3} + x^{1/3})^{3/4} : 4t^2(t/x))$$

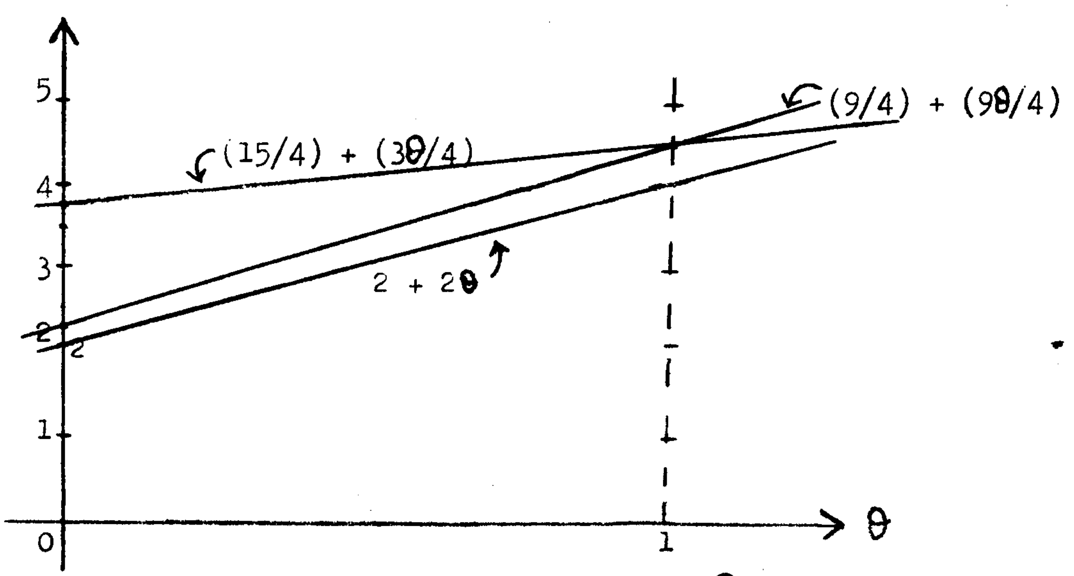
and tends to $(1 : 0 : 0)$. If $|t(s)/x(s)|$ is not bounded as s tends to 0 we set $x = ct^{1+\theta}$ + (higher terms in t), $\theta > 0$. The normal becomes

$$(3c^2t^{2+2\theta} + t^4 : -4(ct^{5+\theta} + c^3t^{3+3\theta})^{3/4} : 4ct^{4+\theta})$$

disregarding higher terms.

<u>$\theta \geq 1$</u> : $4 < 18/4 = \min((15/4) + (3\theta/4) , (9/4) + (9\theta/4)) < 5 \leq 4 + \theta$, hence the normal tends to $(1 : 0 : 0)$.

<u>$\theta < 1$</u> : $2 + 2\theta < (9/4) + (9\theta/4) < (15/4) + (3\theta/4)$, and so once again we find $(1 : 0 : 0)$.



<u>Figure</u> : <u>Justification of the inequalities when</u> $\theta < 1$.

<u>(w) fails</u> : Consider the curve $\gamma(s) = (s^2, (2s^6)^{1/4}, s)$ on X . From (7.2) we find that the normal to X at $\gamma(s)$ is $(4s^4 : -4(2s^6)^{3/4} : 4s^5)$ and hence that $d(T_{\gamma(s)}X, T_0Y) = 4s^5/((4s^4)^2 + \dots)^{\frac{1}{2}} \sim s$. Now $\|\gamma(s) - \pi_Y(\gamma(s))\| = \|(s^2, (2s^6)^{1/4}, 0)\| \sim s^{3/2}$. Hence X fails to be (w)-regular over Y at 0 .

As a consequence (w)-regularity is not a C^1 diffeomorphism invariant. However it is clear from the definition of (w) that it is a C^2 diffeomorphism invariant, or more precisely that it is invariant under a C^1 diffeomorphism with a Lipschitz derivative.

Note 7.3 : No example has been found so far of complex analytic strata for which (b) holds and (w) fails. In the special case of a family of complex hypersurfaces with isolated singularity parametrised by Y it is known that (b) and (w) are equivalent. This is because (w) is a trivial consequence of (c)-cosecance as defined by Teissier in [32] . It follows from [3] and [31] that (b) implies (c)-cosecance.

Now we suppose that Y is linear (apply a local analytic isomorphism at 0 to $\mathbb{R}^n$). Let π denote orthogonal projection onto Y .

We can reformulate (w) by saying that for x , y near 0 , $\dfrac{d(T_xX,T_yY)}{\|x - y\|}$ is bounded, and so in particular $\dfrac{d(T_xX,T_0Y)}{\|x - \pi(x)\|}$ is bounded for x near 0 .

Then it is clear that if X is (w)-regular over Y at 0 then $(X,Y)_0$ satisfies the ratio test (r) of Kuo (defined in [14]) :

(r) Given any vector $v \in T_0Y$, $\displaystyle\lim_{\substack{x \to 0 \\ x \in X}} \frac{|\pi_x(v)|.\|x\|}{\|x - \pi(x)\|} = 0$.

Here π_x denotes orthogonal projection onto the normal space to X at x , so that $|\pi_x(v)| = d(T_xX, v)$.

Kuo proved in [14] ,

Theorem 7.4 (Kuo) : (1) (r) implies (b) ,

(2) (b) implies (r) if Y is of dimension one.

Proof : In each case the proof in [14] uses the curve selection lemma with the assumption that X be a semianalytic set. Using Lemma 2.6 we can use the same proof when X is a subanalytic set.

Corollary 7.5 : (w) implies (b) .

Example 7.6 : For an example showing that (r) does not imply (w) apply Theorem 7.4 (2) to Example 7.1 .

Actually we can make more precise what was proved in [14] . It is shown there that (b) is equivalent to the conjunction of (a) and

(r') If $\gamma(t)$, $t \in [0,1]$, is an analytic arc on X with $\gamma(0) = 0$, then $\lim_{t\to 0} \frac{|\pi_t(v)| \, \|\gamma(t)\|}{\|\gamma(t) - \pi(\gamma(t))\|} = 0$, where v is the tangent at 0 to the arc in Y defined by $\pi \circ \gamma(t)$ (when nonzero) and π_t is projection onto the normal space to X at $\gamma(t)$.

It is obvious that (r) implies (a) + (r') , and that (a) + (r') implies (r) when Y is of dimension one. With this in mind we now give an example of a pair of semialgebraic strata, with Y of dimension two, X (b)-regular over Y , and where (r) fails to hold for a curve $\gamma(t)$ and a vector v spanning the orthogonal complement in $T_0 Y$ to the subspace spanned by the tangent at 0 to the curve in Y defined by $\pi \circ \gamma(t)$.

This example, discovered at Oslo in August 1976 (see [39]), gives the first (b)-regular pair of subanalytic strata which fail the ratio test (r) (introduced in 1970) . It is an open question whether real algebraic or complex analytic examples exist, although from the argument for (w) in Note 7.3 we see that (b) is equivalent to (r) when X is the nonsingular part of a complex hypersurface.

Example 7.7 : Let (x, y, z, w) be coordinates in $\mathbb{R}^4$, and let Y be the plane $\{z = w = 0\}$. Define the semialgebraic set,

$$X = \{w = 0 \,,\, 2(x^2 + (z - y^p)^2) \geq y^{2p} \,,\, z > 0\}$$
$$\cup \{y^q w = (x^2 + (z - y^p)^2 - y^{2p}/2)^2 \,,\, 2(x^2 + (z - y^p)^2) \leq y^{2p} \,,\, z > 0\}$$

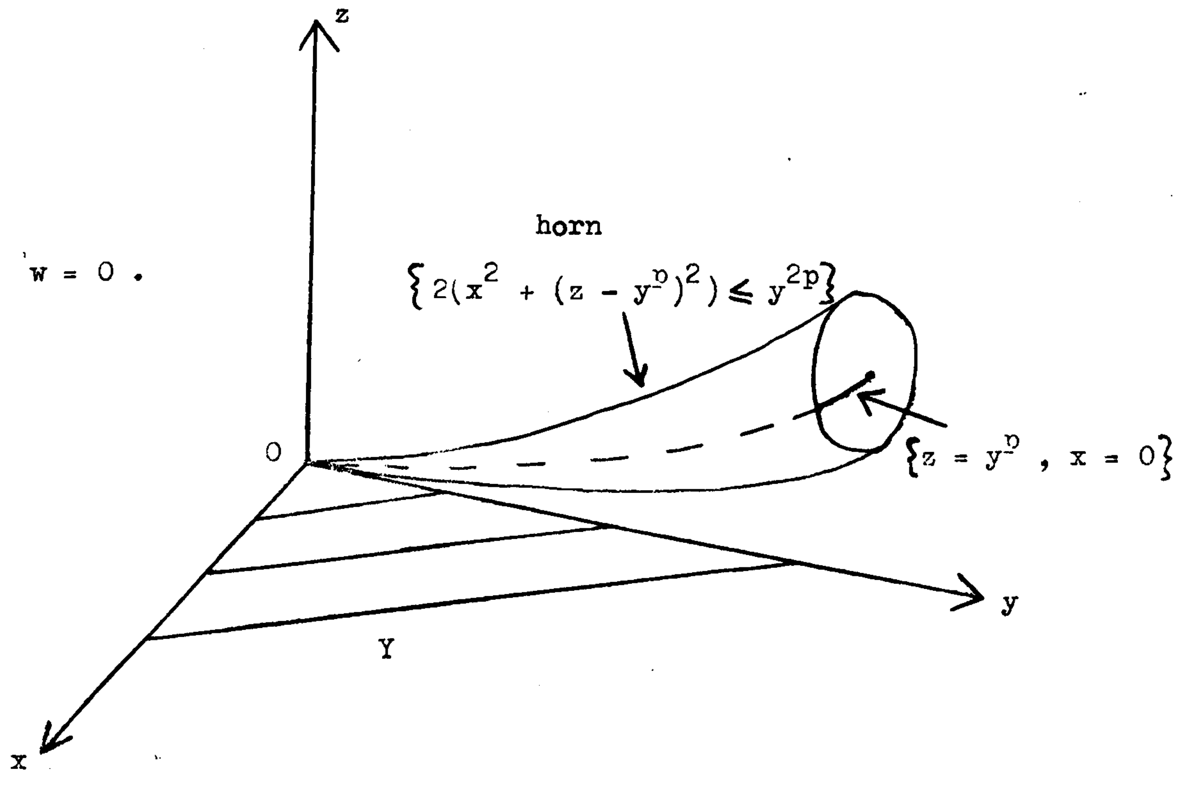
where p and q are positive integers satisfying

$$2p < q < 3p \ . \tag{7.8}$$

(For example let $p = 2$, $q = 5$.)

Observe that because the algebraic variety defined by the equality in the second part of the expression for X has $\{w = 0\}$ as tangent space at every point of its intersection with $\{2(x^2 + (z - y^p)^2) = y^{2p}\}$, X is a C^1 submanifold of $\mathbb{R}^4$ (compare Construction 2.2).

Figure : $w = 0$.



Assertion 7.9 : (b) holds.

Proof : We show that there is a single limiting tangent 3-plane for sequences on X tending to 0, namely $\{w = 0\}$. It suffices to consider the points on $\{y^q w = (x^2 - y^{2p}/2)^2\}$ (with y fixed) where $d^2w/dx^2 = 0$, since at these points the normal is furthest from the (w)-direction (cf. 2.2).

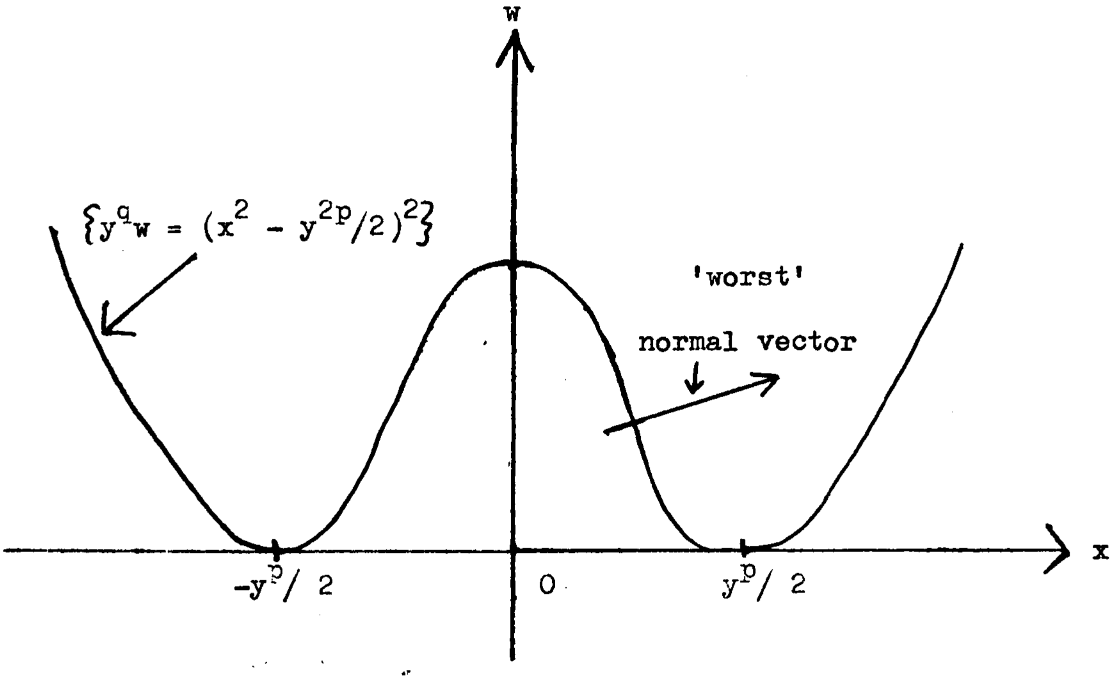


Figure : $z = y^p$, y fixed.

$d^2w/dx^2 = 0$ when $6x^2 = y^{2p}$, and the normal vector is $(\pm(4/3\sqrt{6})y^{3p} : -y^q)$ which tends to $(0 : 1)$ as y tends to 0 since $q < 3p$ by (7.8) . Hence $\{w = 0\}$ is the unique limiting tangent plane.

At the points on X where the secant vector defined by orthogonal projection onto Y is furthest from the z-direction , the secant vector is contained in the tangent space to X . Hence Oz is the unique limit of tangent vectors, and (b') holds. (a) holds (since $\{w = 0 , z = 0\} \subset \{w = 0\}$) , so we can apply the result that (a) + (b') is equivalent to (b) (0.4) to show that (b) holds, proving the assertion.

Assertion 7.10 : (r) fails to hold.

Proof : Consider the curve $\gamma(t) = (t^p/\sqrt{6} , t , t^p , t^{4p-q}/9)$ which lies on X . The normal vector to X at $\gamma(t)$ is,

$$((4/3\sqrt{6})t^{3p} : ((2p/3) - (q/9))t^{4p-1} : 0 : -t^q) .$$

Let π_t denote projection onto this normal space. Then

$$|\pi_t(Ox)| \sim \frac{t^{3p}}{\|(t^{3p} , t^{4p-1} , 0 , t^q)\|} \sim \frac{t^{3p}}{t^q} ,$$

since, by (7.8) , $q < 3p$.

$$\frac{\|\gamma(t)\|}{\|\gamma(t)-\pi(\gamma(t))\|} = \frac{\|(t^p/6\ ,\ t\ ,\ t^p\ ,\ t^{4p-q}/9)\|}{\|(\ 0\ ,\ 0\ ,\ t^p\ ,\ t^{4p-q}/9)\|} \sim \frac{t}{t^p} \ .$$

Hence the ratio (as in the definition of (r)) becomes t^{2p-q+1} , which does not tend to zero since $2p < q$ by (7.8) . This proves Assertion 7.10 .

Finally we check that (w) fails to hold.

$$d(T_{\gamma(t)}X,\ T_{\pi(\gamma(t))}Y) \sim t^{3p-q} \ ,$$

$$d(\gamma(t)\ ,\pi(\gamma(t))) \sim t^p \ ,$$

so that (w) fails exactly when $2p < q$.

<u>Note 7.11</u> : The proof of Assertion 7.9 gives in fact that $\overline{X}$ is a C^1 manifold-with-boundary. Basing the construction on $\{w = (x^{2k} - 1/2)^2\}$, $1 < k < \infty$, instead of $k = 1$ as here, we can build similar examples with X a C^k submanifold and semialgebraic subset of $\mathbb{R}^4$. However $\overline{X}$ will still be a submanifold-with-boundary of class C^1 , not C^2 . (r) , like (w) , is a C^2 diffeomorphism invariant, but not a C^1 diffeomorphism invariant. In this context note that there is no C^2 version of the lemma showing that wings are generically submanifolds-with-boundary of class C^1 (see [43]). Hence the proof in [43] that (b) is generic does not apply directly to (r) or (w) .

(As a counterexample to a C^2 version it suffices to take the product of $\mathbb{R}$ and a semi-cubical cusp in $\mathbb{R}^3$.)

<u>Note 7.12</u> : In [14] there is an example of Kuo showing that (r) does not imply (b) if X is merely smooth. Kuo has also an example where Y is 1-dimensional, (b) holds, and (r) fails, and of course X merely smooth (private communication). This is why we assumed subanalyticity of X from the beginning of §7 .

<u>Addendum 7.13.</u> If A, B are vector subspaces of $\mathbb{R}^n$, let

$$d(A, B) = \sup_{\substack{b \in B \\ |b|=1}} |b - \pi_A(b)|$$

where π_A is orthogonal projection onto A. This is not symmetric in A and B. Clearly $d(A, B) = 0$ if and only if $A \supseteq B$.

(Compare [14], [40], [46], [47] in all of which the order is the reverse of the above.)

CHAPTER 3. COMPUTATIONS

During a talk delivered at the Göttingen Catastrophe Theory Conference in October 1973, C. T. C. Wall suggested that it would be useful to determine Whitney regularity in the following case : $X \equiv \{y^a = t^b x^c + x^d\} - \{t\text{-axis}\}$ in $\mathbb{R}^3$ or $\mathbb{C}^3$, $Y \equiv \{t\text{-axis}\}$, with a , b , c , d positive integers.

We determine (a)- and (b)-regularity completely in the complex case and record this together wih the calculations for the real case that have been made. These calculations have proved useful in providing Example 7.1 (showing (b) to be strictly weaker than (w) even for algebraic strata), and in answering several questions posed by J.-J. Risler concerning algebraic stratifications not regular over $\mathbb{C}$, yet regular over $\mathbb{R}$.

8. $\underline{y^a = t^b x^c + x^d}$.

The tables below collect the results which are obtained.

Key : ✓ - regularity holds ; ✗ - there is a fault at 0 ; ? - undecided .

Table 8.1 : (a)-regularity over $\mathbb{C}$.

$a = 1$ ✓ (8.6)

$$a > 1 \begin{cases} d \leq c \quad \checkmark \quad (8.7) \\ c < d < b + c \begin{cases} a \leq b \quad \checkmark \quad (8.12) \\ a > b \begin{cases} d < ac/(a-b) \quad \checkmark \quad (8.12) \\ d \geq ac/(a-b) \quad \times \quad (8.12) \end{cases} \end{cases} \\ b + c \leq d \quad \times \quad (8.8) \end{cases}$$

Table 8.2 : (a)-regularity over $\mathbb{R}$.

$a = 1 \quad \checkmark \quad (8.6)$

$$a > 1 \begin{cases} d \leq c \quad \checkmark \quad (8.7) \\ c < d < b+c \begin{cases} a \leq b \quad \checkmark \quad (8.11, 8.12) \\ a > b \begin{cases} d < ac/(a-b) \quad \checkmark \quad (8.12) \\ d \geq ac/(a-b) \begin{cases} b \equiv 0 \ (2) \begin{cases} d \equiv c \ (2) \quad \checkmark \quad (8.14) \\ d \equiv c+1 \ (2) \quad \times \quad (8.13) \end{cases} \\ b \equiv 1 \ (2) \quad \times \quad (8.13) \end{cases} \end{cases} \end{cases} \\ b+c \leq d \begin{cases} b \equiv 0 \ (2) \begin{cases} d \equiv c+1 \ (2) \quad \times \quad (8.9) \\ d \equiv c \ (2) \begin{cases} a \leq b \quad \checkmark \quad (8.11) \\ b < a < b+c \begin{cases} d = b+c \quad \checkmark \quad (8.15) \\ b+c < d \quad ? \end{cases} \\ b+c \leq a \quad \times \quad (8.10) \end{cases} \end{cases} \\ b \equiv 1 \ (2) \quad \times \quad (8.9) \end{cases} \end{cases}$$

Table 8.3 : (b)-regularity over $\mathbb{C}$.

$a = 1 \quad \checkmark \quad (8.16)$

$$a > 1 \begin{cases} d \leq c \quad \checkmark \quad (8.17) \\ c < d \quad \times \quad (8.18) \end{cases}$$

Table 8.4 : (b')-regularity over $\mathbb{R}$. (Not (b))

$a = 1 \quad \checkmark \quad (8.16)$

$$a > 1 \begin{cases} d \leq c \quad \checkmark \quad (8.17) \\ c < d \begin{cases} b \equiv 0 \ (2) \begin{cases} d \equiv c \ (2) \begin{cases} d < a \quad \checkmark \quad (8.20) \\ a \leq d \quad ? \end{cases} \\ d \equiv c+1 \ (2) \quad \times \quad (8.19) \end{cases} \\ b \equiv 1 \ (2) \quad \times \quad (8.19) \end{cases} \end{cases}$$

Note 8.5 : It is easy to show that if (a) (resp (b'), resp. (b)) holds over $\mathbb{C}$, then (a) (resp. (b') , resp. (b)) holds over $\mathbb{R}$.

Write $f(x, y, t) = -y^a + t^b x^c + x^d$. Then (a) holds at 0 if and only if $\dfrac{\partial f/\partial t(x, y, t)}{|\text{grad} f(x, y, t)|}$ tends to 0 as (x, y, t) tends to 0 on X , i.e. if and only if at least one of $\dfrac{\partial f/\partial t}{\partial f/\partial x}$ and $\dfrac{\partial f/\partial t}{\partial f/\partial y}$ tend to 0 . We have that

$$\begin{aligned} \text{grad } f &= (\partial f/\partial x, \partial f/\partial y, \partial f/\partial t) \\ &= (dx^{d-1} + cx^{c-1}t^b, -ay^{a-1}, bt^{b-1}x^c) . \end{aligned}$$

<u>8.6</u>: (a) <u>holds if</u> $a = 1$.

$$\frac{\partial f/\partial t}{\partial f/\partial y} = \frac{bt^{b-1}x^c}{-1} \longrightarrow 0 \text{ as } x \longrightarrow 0 .$$

<u>8.7</u> : (a) <u>holds if</u> $d \leq c$.

We may suppose $\partial f/\partial x \not\equiv 0$, for $\partial f/\partial x$ is identically zero only on $\{dx^{d-c} + ct^b\} = 0$, and since $d \leq c$, this surface intersects X in an isolated point at 0 . Then $\left|\dfrac{\partial f/\partial t}{\partial f/\partial x}\right| \sim \dfrac{t^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} = \dfrac{t^{b-1}x^{c-d+1}}{d + cx^{c-d}t^b} \longrightarrow 0$ as x tends to 0 if $d \leq c$.

<u>8.8</u> : (a) <u>fails over</u> $\mathbb{C}$ <u>if</u> $d \geq b + c$ <u>and</u> $a > 1$.

Consider the curve on which $\partial f/\partial y \equiv 0$, i.e. $y = 0 = x^c(t^b + x^{d-c})$. Let $t^b = -x^{d-c}$. Then

$$\frac{\partial f/\partial t}{\partial f/\partial x} = \frac{bt^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} \sim \frac{x^{c+(d-c)(b-1)/b}}{dx^{d-1} - cx^{d-1}}$$

$$\sim x^{c-d+1+(d-c)(b-1)/b}$$

$$= x^{(b+c-d)/b} \not\longrightarrow 0$$

if $d \geq b + c$. Hence if $d \geq b + c$, (a) fails on $\{y = 0 = t^b + x^{d-c}\}$.

<u>8.9</u> : (a) <u>fails over</u> $\mathbb{R}$ <u>if</u> $d \geq b+c$, $a > 1$ <u>and either</u> $b \equiv 1 \pmod 2$ <u>or</u> $(d-c) \equiv 1 \pmod 2$, <u>or both</u> .

As in 8.8 $\{t^b = -x^{d-c}\} \cap X$ has a branch through 0 .

8.10 : (a) fails over R if $b+c \leq a$, $b+c \leq d$, $b \equiv 0 \pmod 2$ and $d \equiv c \pmod 2$.

$\partial f/\partial y \not\equiv 0$ since $\{t^b = -x^{d-c}\} \cap X$ has no branches near 0. Let $x = \lambda t$, $\lambda \neq 0$. Then

$$\frac{\partial f/\partial t}{\partial f/\partial y} \sim \frac{x^{b+c-1}}{(x^{b+c} + x^d)^{(a-1)/a}} \sim x^{b+c-1-(b+c)(a-1)/a} = x^{(b+c-a)/a}.$$

Thus $\dfrac{\partial f/\partial t}{\partial f/\partial y} \to 0$ along $\{x = \lambda t\}$ if $a > b+c$.

Also

$$\frac{\partial f/\partial t}{\partial f/\partial x} \sim \frac{x^{b+c-1}}{dx^{d-1} + cx^{b+c-1}} \quad \text{on } \{x = \lambda t\}$$

$$\sim \frac{x^{b+c-1}}{cx^{b+c-1}} \quad \text{since } d \geq b+c$$

$$\not\to 0.$$

Hence (a) fails along $\{x = \lambda t\}$.

8.11 : (a) holds over R if $a \leq b$, $b \equiv 0 \pmod 2$, and $d \equiv c \pmod 2$.

$\partial f/\partial y \not\equiv 0$ since $\{t^b x^c + x^d\} \neq 0$ except at 0 if b and $(d-c)$ are even.

$$\frac{\partial f/\partial t}{\partial f/\partial y} \sim \frac{t^{b-1}x^c}{(t^b x^c + x^d)^{1-1/a}} \leq \frac{t^{b-1}x^c}{(t^b x^c)^{1-1/a}} = t^{b-1-b(a-1)/a}x^{c/a} = t^{(b-a)/a}\, x^{c/a} \to 0 \text{ if } a \leq b.$$

8.12 : Let $c < d < b+c$. Then (a) holds over $\mathbb{C}$ if and only if either $a \leq b$ or, $a > b$ and $d < ac/(a-b)$.

After curve selection (2.6) we can reduce to the case of curves along which $|t/x|$ is bounded or unbounded as x and t tend to 0.

(i) $|t/x|$ is bounded. Then $\partial f/\partial x \not\equiv 0$ and

$$\frac{\partial f/\partial t}{\partial f/\partial x} \sim \frac{t^{b+c-d}(t/x)^{d-c-1}}{d + ct^{b+c-d-1}(t/x)^{d-c}} \to 0, \text{ if } c < d < b+c.$$

(ii) $|x/t|$ tends to 0.

$$\frac{\partial f/\partial t}{\partial f/\partial x} \sim \frac{(x/t)}{c + dx^{d-c}/t^b} \to 0 \text{ if } dx^{d-c}/ct^b \not\to -1.$$

Let $dx^{d-c}/ct^b \to -1$.

Then $\dfrac{\partial f/\partial t}{\partial f/\partial y} \sim \dfrac{t^{b-1}x^c}{(t^bx^c + x^d)^{1-1/a}} \sim \dfrac{x^{c+(d-c)(b-1)/b}}{((1-d/c)x^d)^{1-1/a}}$

$\sim x^{c+(d-c)(b-1)/b - d(a-1)/a}$ since $d > c$

$\sim x^{(ac-d(a-b))/ab}$ which

$\to 0$ if $d(a-b) < ac$

$\not\to 0$ if $d(a-b) \geqslant ac$, when (a) fails

along $dx^{d-c} + ct^b = 0$.

<u>8.13</u> : (a) <u>fails over</u> $\mathbb{R}$ <u>if</u> $c < d < b+c$, $a > b$, $d \geqslant ac/(a-b)$ <u>and either</u> $b \equiv 1 \pmod 2$ <u>or</u> $d \equiv c+1 \pmod 2$, <u>or both</u>.

As in 8.12, (a) fails along $dx^{d-c} + ct^b = 0$.

<u>8.14</u> : (a) <u>holds over</u> $\mathbb{R}$ <u>if</u> $c < d < b+c$, $b \equiv 0 \pmod 2$, $d \equiv c \pmod 2$.

8.12 shows that (a) fails only for curves on which $dx^{d-c}/ct^b \to -1$, and these curves have no points on X near 0 if b and $d-c$ are even.

<u>8.15</u> : (a) <u>holds over</u> $\mathbb{R}$ <u>if</u> $b \leqslant a < b+c = d$, $b \equiv 0 \pmod 2$, $d \equiv c \pmod 2$.

(i) $|x/t|$ bounded near 0.

$$\frac{\partial f/\partial t}{\partial f/\partial y} \sim t^{b-1-b(a-1)/a}\, x^{c-c(a-1)/a} = t^{(b-a)/a}\, x^{c/a}$$

$$= x/t^{\,(a-b)/a}\, x^{(b+c-a)/a}$$

$$\to 0 \text{ if } b \leqslant a < b+c .$$

(ii) $|t/x|$ tends to 0.

Suppose t tends to x^θ, $\theta > 1$.

$$\frac{\partial f/\partial t}{\partial f/\partial x} = \frac{bt^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} \sim x^{c+b\theta-\theta-d+1} = x^{(b-1)(\theta-1)} \quad \text{if } d = b+c .$$

$$\to 0 .$$

This completes our calculations of (a)-regularity — the inquisitive reader can work out for himself the remaining cases of (a)-regularity over $\mathbb{R}$: when $b < a \leqslant b+c < d$ and $b \equiv 0 \pmod 2$, $d \equiv c \pmod 2$.

(b') holds at 0 if and only if $\dfrac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y, \partial f/\partial t)|}$ tends to 0 as (x, y, t) tends to $(0, 0, 0)$.

<u>8.16</u> : (b) <u>holds if</u> $a = 1$.

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y,\partial f/\partial t)|} = \frac{(d-1)x^d + (c-1)t^b x^c}{|(x,y)|.|(\partial f/\partial x,\ 1\ ,\partial f/\partial t)|}$$

$$= \frac{(d-1)x^{d-1} + (c-1)t^b x^{c-1}}{|(1,y/x)|.|(\partial f/\partial x,1,\partial f/\partial t)|}$$

$$\to 0 .$$

Now use (8.6) and (0.4).

<u>8.17</u> : (b) <u>holds if</u> $d \leq c$.

Since by (8.7) (a) holds, by (0.4) it is enough to show that $\dfrac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y)|}$ tends to 0, i.e. $\dfrac{(c-a)t^b x^c + (d-a)x^d}{|\cdot\cdot|.|\ \cdots\cdots\ |}$ tends to 0. Since $d \leq c$, it is enough to show that $\dfrac{x^d}{|\cdot|.|\cdot\cdot|}$ tends to 0 when $d \neq a$, and $\dfrac{t^b x^c}{|\cdot|.|\cdot\cdot\cdot|}$ tends to 0 when $d = a$.

(i) <u>$d > a$</u>. $$\frac{x^d}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y)|} = \frac{x^{d-1}}{|(1,y/x)|.|(cx^{c-1}t^b+dx^{d-1},-a(t^b x^c+x^d)^{1-1/a})|}$$

$$\sim \frac{x^{(d/a)-1}}{|(1,y/x)|.|(\ldots, -a(t^b x^{c-d} + 1)^{1-1/a})|} .$$

$\to$ 0 as $d > a$, unless $t^b x^{c-d} + 1$ tends to 0, but there are no such points near 0 as $d \leq c$.

(ii) <u>$d < a$</u>. $$\frac{x^d}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y)|} = \frac{x}{|(x,-(t^b x^c+x^d)^{1/a})|.|(d+cx^{c-d}t^b,\ldots)|}$$

$$= \frac{x^{1-d/a}}{|(x,-(t^b x^{c-d}+1)^{1/a})|.|(d+cx^{c-d}t^b,\ldots)|}$$

$\to$ 0 since $d < a$, and $d \leq c$.

(iii) <u>$d = a$</u>. $$\frac{t^b x^c}{|(x,y)|.|(cx^{c-1}t^b+dx^{d-1},\partial f/\partial y)|} = \frac{t^b x^{c-d}}{|(1,y/x)|\ |(ct^b + d,\ldots)|}$$

$\to$ 0 since $d \leq c$.

<u>8.18</u> : (b') <u>fails over</u> $\mathbb{C}$ <u>if</u> $c < d$ <u>and</u> $a > 1$, <u>and</u> (a) <u>holds</u>.

$y = 0$ and $\partial f/\partial y \equiv 0$ on $t^b x^c + x^d = 0$. Then

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y)|} = \frac{(d-c)x^d}{|(x,0)|.|(ct^b x^{c-1} + dx^{d-1},\ 0)|}$$

$$= \frac{(d-c)x^{d-1}}{|(1,0)|.|((d-c)x^{d-1},\ 0)|}$$

$\longrightarrow 0$, so (b') fails, and hence (b) fails.

<u>8.19</u> : (b') <u>fails over</u> $\mathbb{R}$ <u>if</u> $a > 1$, $c < d$ <u>and either</u> $b \equiv 1$ (<u>mod</u> 2) <u>or</u> $d \equiv c+1$ (<u>mod</u> 2) <u>or both</u>.

$X \cap \{t^b x^c + x^d = 0\}$ has real branches through 0 if b or $(d-c)$ is odd.

<u>8.20</u> : (b') <u>holds over</u> $\mathbb{R}$ <u>if</u> $d < a$, $b \equiv 0$ (mod 2) <u>and</u> $d \equiv c$ (mod 2) .

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y, \partial f/\partial t)|} = \frac{(d-a)x^d + (c-a)t^b x^c}{|(x,(t^b x^c + x^d)^{1/a})|.|(ct^b x^{c-1} + dx^{d-1},\ldots,\ldots)|}$$

$$= \frac{(d-a)x^{1-d/a}}{|(x^{1-d/a},(t^b x^{c-d} + 1)^{1/a})|.|(ct^b x^{c-d} + d,\ldots,\ldots)|}$$

$$+ \frac{(c-a)x^{1-d/a}}{|(x^{1-d/a},(t^b x^{c-d} + 1)^{1/a})|.|(c + dx^{d-c}t^{-b},\ldots,\ldots)|}$$

$\longrightarrow 0$ if $d < a$.

This completes our calculations of (b')- and (b)-regularity save for the case $1 < a \leq d$, $c < d$, $b \equiv 0$ (mod 2) , $d \equiv c$ (mod 2) , over $\mathbb{R}$.

<u>Example 8.21</u> : J. J. Risler asked for an example which was (a)-regular over $\mathbb{R}$, but not over $\mathbb{C}$. By 8.11 and 8.8 it suffices that $a \leq b \leq d-c$, $b \equiv 0$ (mod 2) and $d \equiv c$ (mod 2) . For example $\{y^2 = t^2 x + x^3\}$.

<u>Example 8.22</u> : For an example which is (b)-regular over $\mathbb{R}$ but not over $\mathbb{C}$, 8.12 , 8.13 , and 8.20 give $c < d < a \leq b$ (or $c < d < a$, $b < a$, $d < ac/(a-b)$) $b \equiv 0$ (mod 2) , $d \equiv c$ (mod 2) . For example $\{y^4 = t^4 x + x^3\}$ or $\{y^5 = t^4 x + x^3\}$.

Example 8.23 : If an equimultiple example is demanded, satisfying the requirements of 8.22 , consider $\{y^2 = t^2x^2 + x^4\}$. By 8.8 (a) fails over $\mathbb{C}$, and by 8.11 (a) holds over $\mathbb{R}$. It remains to check that (b') holds over $\mathbb{R}$, using (0.4) .

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|\,|(\partial f/\partial x, \partial f/\partial y)|} = \frac{2x^4}{|(x,(t^2x^2+x^4)^{\frac{1}{2}})|\,.\,|(4x^3+2t^2x,\ -2(x^4+t^2x^2)^{\frac{1}{2}})|}$$

$$= \frac{2\ x}{|(1,(t^2+x^2)^{\frac{1}{2}})|\,.\,|(4x + 2t^2/x,\ -2(1+(t/x)^2)^{\frac{1}{2}})|}$$

$$\longrightarrow 0 \quad \text{as} \quad (x,t) \text{ tends to } 0 \text{ since } X \cap \{t^2 + x^2 = 0\}$$

has no branches passing through 0 . Hence (b) holds over $\mathbb{R}$.

Note 8.24 : Table 8.3 corresponds with the known fact that for families of plane curves, " μ -constant " is equivalent to (b)-regularity ([30]) .

BIBLIOGRAPHY

1. Bass, D. W. Topological classification of endomorphisms of line bundles, Warwick University preprint, 1975 .

2. Briançon, J. & Speder, J.-P. La trivialité topologique n'implique pas les conditions de Whitney, C. R. Acad. Sci. Paris, tome 280, p. 365-367, 1975 .

3. Briançon, J. & Speder, J.-P. Les conditions de Whitney impliquent μ^*-constant, Annales de l'Institut Fourier de Grenoble, 26(2), p. 153-163, 1976 .

4. Chenciner, A. Hanoi notes, 1975 .

5. Feldman, E. A. The geometry of immersions. I , Trans. Amer. Math. Soc. 120, p. 185-224, 1965 .

6. Gibson, C. G. Construction of canonical stratifications, in Topological stability of smooth mappings, by C. G. Gibson et al., Springer lecture notes 552, Berlin, p. 8-34, 1976 .

7. Gibson, C. G., Wirthmüller, K., du Plessis, A. A., Looijenga, E. J. N., Topological stability of smooth mappings, Springer lecture notes 552, Berlin, 1976 .

8. Golubitsky, M. & Guillemin, V., Stable mappings and their singularities,

Springer-Verlag, New York, 1974 .

9. Hamm, H. A. & Lê Dũng Tràng, Un théoreme de Zariski du type de Lefschetz, Ann. Sci. Ec. Norm. Sup. 4ème série, t.6, p. 317-366, 1973 .

10. Hardt, R., Stratification of real analytic mappings and images, Inventiones 28, p. 193-208, 1975 .

11. Hironaka, H., Normal cones in analytic Whitney stratifications, Publ. Math. I. H. E. S. 36, 1969 .

12. Hironaka, H. Subanalytic sets, in Number Theory, Algebraic Geometry and Commutative Algebra, volume in honour of Y. Akizuki, Kinokuniya, Tokyo, p. 453-493, 1973 .

13. Hirsch, M. Differential topology, Springer-Verlag, New York, 1976 .

14. Kuo, T.-C. The ratio test for analytic Whitney stratifications, Liverpool singularities symposium I. Springer lecture notes 192, Berlin, p. 141-149, 1971 .

15. Kuo, T.-C. On Thom-Whitney stratification theory, preprint, 1976 .

16. Lê Dũng Tràng & Saito, K. La constance du nombre de Milnor donne des bonnes stratifications, C. R. Acad. Sci. Paris, t. 277 (22 Oct. 1973), p. 793-795, 1973 .

17. Lê Dũng Tràng & Ramanujam, C. P. The invariance of Milnor's number implies the invariance of topological type, Amer. J. Math. 98, p. 67-78, 1976 .

18. Lojasiewicz, S. Ensembles semi-analytiques, I. H. E. S. notes, 1965 .

19. MacPherson, R. D. Chern classes for singular algebraic varieties, Ann. of Math. 100 , p. 423-432, 1974 .

20. MacPherson, R. Characteristic classes for singular varieties, preprint, 1976 .

21. Mather, J. Notes on topological stability, Harvard University, 1970 .

22. Mather, J. Stratifications and mappings, in Dynamical Systems (ed. M. M. Peixoto), Academic Press, p. 195-223, 1971 .

23. Milnor, J. Singular points of complex hypersurfaces, Ann. of Math. Studies 61, Princeton, 1968 .

24. Morlet, C. Le lemme de Thom et les théorèmes de plongement de Whitney I : les topologies des espaces d'applications, Séminaire Henri Cartan (Topologie différentielle), 4, 1961-62 .

25. Pham, F. & Teissier, B. Fractions lipschitziennes d'une algebre analytique complexe et saturation de Zariski, Ecole Polytechnique notes, 1969 .

26. Schwartz, M.-H. Classes caractéristiques définies par une stratification d'une variété analytique complexe, C. R. Acad. Sci. Paris, t. 260, p. 3262-3264, 3535-3537, 1965 .

27. Schwartz, M.-H. Sections holomorphes des espaces lineaires et autres espaces, Lille I preprint, 1976 .

28. Speder, J.-P. Equisingularité et conditions de Whitney, Amer. J. Math. 97(3), p. 571-588, 1975 .

29. Speder, J.-P. Equisingularité et conditions de Whitney, Thèse d'Etat, Université de Nice, 1976 .

30. Teissier, B. Cycles evanescents, sections planes, et conditions de Whitney, in Singularités à Cargèse, Astérisque 7-8, Société Mathématique de France, p. 285-362, 1973 .

31. Teissier, B. Introduction to equisingularity problems, A. M. S. Algebraic geometry symposium, Arcata 1974, Providence, Rhode Island, p. 593-632, 1975 .

32. Teissier, B. Variétés polaires I : invariants polaires des singularités d'hypersurfaces, Inventiones Math. 40, p. 267-292, 1977 .

33. Timourian, J. The invariance of Milnor's number implies topological triviality, Amer. J. Math. 99, p. 437-446, 1977 .

34. Thom, R. Local topological properties of differentiable mappings, in Differential analysis, Oxford University Press, London, p. 191-202, 1964 .

35. Thom, R. Propriétés différentielles locales des ensembles analytiques, Séminaire Bourbaki no. 281, 1964-65 .

36. Thom, R. Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. 75 , p. 240-284, 1969 .

37. Trotman, D. J. A. A transversality property weaker than Whitney (a)-regularity, Bull. London Math. Soc., 8, p. 225-228, 1976 .

38. Trotman, D. J. A. Geometric versions of Whitney regularity, Math. Proc. Camb. Phil. Soc. 80, p. 99-101, 1976 .

39. Trotman, D. J. A. Counterexamples in stratification theory : two discordant horns, in Singularities of real and complex mappings, Proc. Nordic Summer School in Math., Oslo 1976, to appear, 1977 .

40. Verdier, J.-L. Stratifications de Whitney et théorème de Bertini-Sard, Inventiones Math. 36, p. 295-312, 1976 .

41. Wall, C. T. C. Stratified sets : a survey, Liverpool Singularities Symposium I, Springer lecture notes 192, Berlin, p. 133-140, 1971

42. Wall, C. T. C. Lectures on C^{∞} stability and classification, Liverpool Singularities Symposium I, Springer lecture notes 192, Berlin, p. 178-206, 1971 .

43. Wall, C. T. C. Regular stratifications, Dynamical Systems - Warwick 1974, Springer lecture notes 468, p. 332-344, 1975 .

44. Wall, C. T. C. Geometric properties of generic differentiable manifolds, preprint, 1977 .

45. Whitney, H. Local properties of analytic varieties, Diff. and comb. topology (ed. S. Cairns), Princeton, p. 205-244, 1965 .

46. Whitney, H. Tangents to an analytic variety, Ann. of Math. 81, p. 496-549, 1965 .

47. Whitney, H. Complex analytic varieties, Addison-Wesley, Reading, Mass., 1972

48. Wirthmüller, K. Stratifications and flows, in Topological stability of smooth mappings, C. G. Gibson et al., Springer lecture notes 552, Berlin, p. 35-63, 1976 .

49. Zariski, O. Contributions to the problem of equisingularity, C. I. M. E. Varenna 1969. Edizioni Cremonese Roma, 1970 .

50. Zariski, O. Some open questions in the theory of singularities, Bull. Amer. Math. Soc. 77(4), p. 481-491, 1971 .

51. Damon, J. Topological stability in the nice dimensions, preprint, 1977 .

52. Kuiper, N.H. C^1-equivalence of functions near isolated critical points, Symposium on infinite-dimensional topology, Baton Rouge 1967. edited by R. D. Anderson, Annals Study 69, Princeton, 1972 .

53. May, R. D. Transversality properties of topologically stable mappings, Harvard doctoral thesis, 1973 .

54. Munkres, J. R. Elementary differential topology, Annals Study 54, Princeton, 1968 .

55. Ostrowski, A. On the Morse-Kuiper theorem, Aequationes Math. 1, p. 66-76, 1968

56. Takens, F. A note on sufficiency of jets, Inventiones Math. 13, p.225-231, 19

SUPPLEMENTS

1. Reference [37].
2. Reference [38].
3. Reference [39].

A TRANSVERSALITY PROPERTY WEAKER THAN WHITNEY (A)-REGULARITY

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Let X and Y be C^∞ manifolds embedded in $\mathbb{R}^n$, and let $x \in X \subset \bar{Y}$. The (a)-regularity condition due to Whitney is,

(a) Given $\{y_i\} \in Y$ such that $y_i \to x$ and $T_{y_i} Y \to \tau$ as $i \to \infty$, then $T_x X \subseteq \tau$.

Since spanning is an open property, (a) implies,

(t) Given a C^1 submanifold S of $\mathbb{R}^n$ meeting X transversely at x, $\exists$ a neighbourhood U of x such that S is transverse to Y in $Y \cap U$.

Conversely, (t) implies (a) if Y is semianalytic. This we prove using the curve selection lemma, and we give an example where Y is a C^∞ manifold and (t) holds at a point x where (a) fails.

The importance of (t) follows from,

THEOREM. *Let N, P be C^∞ manifolds, with P partitioned into finitely many submanifolds $\mathscr{P}$, such that*

(i) *if $X, Y \in \mathscr{P}$, and $X \cap \bar{Y} \neq \varnothing$, then $X \subset \bar{Y}$, (frontier property)*

(ii) *if $X, Y \in \mathscr{P}$, and $x \in X \subset \bar{Y}$,* (t) *is satisfied at x.*

Then the set of C^∞ mappings $f : N \to P$ which are transverse to the members of $\mathscr{P}$ is open and dense in $C^\infty(N, P)$ with the Whitney C^∞ topology.

The remark above that (a) implies (t) enables us to restate the theorem with (a) replacing (t). See for example [2, 3].

1. *The semianalytic case*

PROPOSITION. (t) *implies* (a) *if Y is a semianalytic manifold.*

Proof. Suppose (a) fails at $x \in X \subset \bar{Y}$.

Choose a unit vector $v \in T_x X$ and a sequence $\{y_i\} \in Y$ such that $y_i \to x$ and $T_{y_i} Y \to \tau$ as $i \to \infty$, and $v \notin \tau$. Then $\exists\, \varepsilon > 0$ and $i_0 \in \mathbb{N}$ such that,

$$\forall i \geqslant i_0, \quad d(v, T_{y_i} Y) > \varepsilon,$$

where $d(v, T_{y_i} Y)$ denotes the distance between $T_{y_i} Y$ and the endpoint of the translation of v from x to y_i. Suppose $\dim Y = m$, and let

$$V_1 = \mathbb{R}^n \times \{P \in G_{m, n-m} : d(v, P) > \varepsilon\} \subset \mathbb{R}^n \times G_{m, n-m},$$

$$V_2 = \bigcup_{y \in Y} (y, T_y Y) \subset \mathbb{R}^n \times G_{m, n-m}.$$

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Here $G_{m,n-m}$ denotes the grassmann manifold of m-places in n-space. V_1 is semialgebraic, and V_2 is semianalytic (since Y is assumed semianalytic); hence $V_1 \cap V_2$ is semianalytic and $(x, \tau) \in \overline{V_1 \cap V_2}$ satisfies the hypotheses of the curve selection lemma. See [**1**; p. 103].

Thus $\exists$ an analytic arc in $\mathbb{R}^n \times G_{m,n-m}$, $\alpha : [0, 1] \to \overline{V_1 \cap V_2}$ with $\alpha(0) = (x, \tau)$ and $\alpha(t) \in V_1 \cap V_2$ if $t \neq 0$. Write $\alpha_1(t)$ for the $\mathbb{R}^n$-component of $\alpha(t)$; the $G_{m,n-m}$-component is $T_{\alpha_1(t)}Y$. Let $N_t \in G_{n-1,1}$ be the normal space at $\alpha_1(t)$ to the C^1 manifold-with-boundary $\alpha_1([0, 1])$, and let the vector v_t be the projection of v into N_t spanning $\langle v_t \rangle \in G_{1,n-1}$.

We shall define a C^1 arc $\sigma : [0, 1] \to G_{n-2,2}$ such that

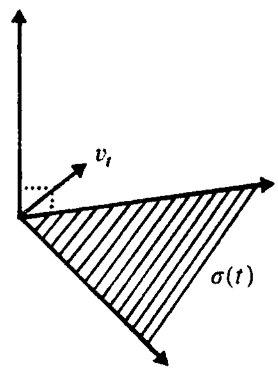
$$\sigma(t) \oplus \langle v_t \rangle = N_t. \tag{1}$$

Then the union of the $\sigma(t)$, considered as embedded $(n-2)$-planes in $\mathbb{R}^n$ passing through the points $\alpha_1(t)$, defines a C^1 manifold-with-boundary S' of dimension $n-1$. Reflection in N_0 extends S' to a C^1 $(n-1)$-manifold S, which is transverse to X at $x \in \operatorname{Int} S$ by (1). However, we shall show that no neighbourhood U of x exists within which S is transverse to Y; so (t) fails as required.

Construction of σ:

Let $P_t = N_t \cap T_{\alpha_1(t)}Y \in G_{m-1,n-m+1}$. Then $0 \neq v_t \notin P_t$ (definition of $V_1 \cap V_2$).
Let $\sigma(t) = P_t \oplus (P_t \oplus \langle v_t \rangle)^\perp \in G_{n-2,2}$,

where $(\ )^\perp$ denotes the orthogonal complement in N_t.



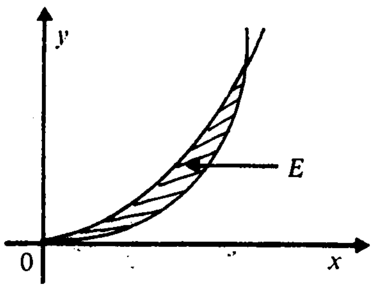
Picture of N_t in the case $n = 4$, $m = 2$.

σ satisfies the required properties and so it remains to show S fails to be transverse to Y at some point in any given neighbourhood U of x. Given $U \exists$ some $s \in (0, 1]$ such that $U \cap \alpha_1(0, 1] \supset \alpha_1(0, s]$; but S' (and hence S) is not transverse to Y at any point of $\alpha_1(0, 1]$. The proof is complete.

2. *Counterexample in the non-semianalytic case*

We construct a pair of C^∞ manifolds X and Y, $X \subset \bar{Y}$ such that at a point $x \in X$ (t) is satisfied, yet (a) is not.

Let x, y be co-ordinates for $\mathbb{R}^2$ and let X be the x-axis. Y will be the union of a countable sequence of C^∞ curves $\{Y_n\}_{n=1}^\infty$ which tend to X as $n \to \infty$. Let $E = \{(x, y) \in R^2 : x \geqslant 0, y \geqslant 0, x^4 \leqslant y \leqslant x^2\}$.

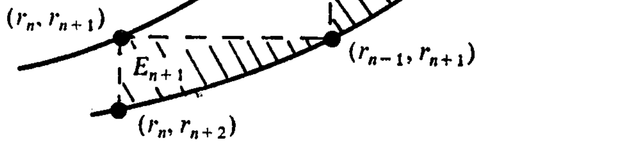


We shall define Y so that (i) the tangents to Y outside E are parallel to X.

Assertion. If (i) is true then (t) holds at $(0, 0)$.

Proof. Let S be a C^1 submanifold of $\mathbb{R}^2$ transverse to the x-axis at 0. We may suppose S is 1-dimensional. Then in a neighbourhood U of the origin, S does not intersect E, and so $S \cap U$ meets $Y \cap U$ only at points p of Y where $T_p Y$ is parallel to X. By continuity and transversality $\exists$ a neighbourhood V of 0 in which the tangent to S has gradient strictly nonzero. Hence S is transverse to Y in $V \cap U$.

The sequence for which (a) fails will lie inside E.
Let $0 < a < 1$, and let $r_n = a^{2^n}$, so that $r_{n+1} = r_n^2 \; \forall n \geqslant 0$.
If $E_n = \{(x, y) \in E : y \leqslant r_n, x \geqslant r_{n-1}\}$, then $E_n \cap E_{n+1}$ is the single point (r_{n-1}, r_{n+1}).



Let (s_n, s_n^4) be the point of intersection in E_{n+1} of $y = x^4$ and $y+x = r_n+r_{n+1}$, so in particular $s_n > r_n$. Let Y_n be the graph of a smooth decreasing function of x such that $y = r_{n+1}$ if $x \leqslant r_n$, $y = s_n^4$ if $x \geqslant s_n$, and Y_n includes a segment with gradient -2 and mid-point m_n half way between (r_n, r_n^2) and (s_n, s_n^4). Then (i) holds.
Clearly $m_n \to (0, 0)$ as $n \to \infty$. And $T_{m_n} Y = T_{m_n} Y_n$ is a line of gradient -2 $\forall n$, and so tends to $y = -2x$ as $n \to \infty$; thus (a) fails.

Further examples. A counterexample with Y 2-dimensional is obtained at once by rotating about X in $\mathbb{R}^3$.
With a little more effort we can produce an example of a 2-dimensional connected Y so that the triple $(\mathbb{R}^3, \bar{Y}, X)$ is homeomorphic to $(\mathbb{R}^3, \mathbb{R}^2, 0 \times \mathbb{R})$ and $\bar{Y}$ is the

plane $z = 0$ outside a 3-dimensional "dart" which intersects $z = 0$ in the E given above. Inside the dart Y contains a decreasing sequence of hemispheres so that we also have a counterexample to the implication corresponding to (t) $\Rightarrow$ (a) for (b)-regularity. Details of this and its semi-analytic case will appear in [4].

3. *Further properties*

Consider, for $x \in X \subset \bar{Y}$,

(t′) Given an r-plane P meeting X transversely at x, $\exists$ a neighbourhood U of x in which P is transverse to Y.

(t) implies (t′), but a counterexample to the converse is obtained by defining X and Y as in §2, except this time keeping the "bad" points of Y in between $x = y^2$ and $x = y^4$. A counterexample in the algebraic case is given by

$$V \equiv \{(x, y, z) : y^5 = z^3 x + x^3\} \text{ in } \mathbb{R}^3$$

with X the z-axis, $Y = V - \{z\text{-axis}\}$, and x the origin. A sequence of points along a branch of $V \cap \{3x^2 + z^3 = 0\}$ contradicts (a) (and hence (t) by our proposition) but (t′) is satisfied.

Consider also for $x \in X \subset \bar{Y}$, with X and Y embedded in $\mathbb{R}^n$,

(a_s) Given a smooth local retraction $\pi_X : \mathbb{R}^n \to X$, x has a neighbourhood U such that $\pi_X|(Y \cap U)$ is a submersion.

(a_s) implies (t) since we can choose a chart at x in which X and S are both linear, and use a linear retraction. C. T. C. Wall has conjectured in [5] that (a_s) implies (a); our proposition shows this to be so if Y is semi-analytic. (A result also obtained by C. G. Gibson and E. Looijenga.) Note that for the counterexamples in §2 it is easy to find smooth retractions π_X for which (a_s) fails, so the conjecture stands.

References

1. S. Lojasiewicz, *Ensembles semi-analytiques*, I.H.E.S. Lecture-notes, 1965.
2. R. Thom, "Ensembles et morphismes stratifiés", *Bull. Amer. Math. Soc.*, 75 (1969), 240–284.
3. D. J. A. Trotman and E. C. Zeeman, "Classification of elementary catastrophes of codimension $\leqslant 5$", *Warwick Lecture Notes*, 1974.
4. D. J. A. Trotman, *Ph.D. Thesis* (Warwick University).
5. C. T. C. Wall, "Regular stratifications", *Dynamical Systems-Warwick* 1974, Springer Lecture Notes 468 (1975), 332–344.

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Geometric versions of Whitney regularity

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Let X^m and Y^n be C^1 manifolds embedded in $\mathbb{R}^p$, $m < n < p$, and let $x \in X \subset \bar{Y} - Y$. In (4) C. T. C. Wall considered the following conditions:

(a_s) For any local C^1 retraction at x, π: $\mathbb{R}^p \to X$, x has a neighbourhood U such that $\pi|_{Y\cap U}$ is a submersion.

(b_s) For any local C^1 tubular neighbourhood of X at x, given by π: $\mathbb{R}^p \to X$ and ρ: $\mathbb{R}^p \to \mathbb{R}_+ \cup \{0\}$, where $\rho^{-1}(0) = X$, x has a neighbourhood U such that $(\pi, \rho)|_{Y\cap U}$ is a submersion.

Wall conjectured that (a_s) and (b_s) are respectively equivalent to Whitney's conditions (a) and (b):

(a) Given $y_i \in Y$ so that, as $i \to \infty$, $y_i \to x$ and $T_{y_i} Y \to \tau$, then $T_x X \subset \tau$.

(b) Given $y_i \in Y$ and $x_i \in X$ so that, as $i \to \infty$, $y_i \to x$, $x_i \to x$, $T_{y_i} Y \to \tau$ and $y_i - x_i/|y_i - x_i| = \lambda_i \to \lambda$, then $\lambda \subset \tau$.

It is not difficult to show that (a) implies (a_s). See (2), p. 35, for a proof that (b) implies (b_s); this enabled Mather to show that if X is a stratum of a (b)-regular stratification Σ, then Σ is locally topologically trivial over X. In (3), § 3, it is proved that (a_s) implies (a) if Y is semianalytic. Here we prove the following,

THEOREM. (b_s) *implies* (b) *if* X *and* Y *are semianalytic*. (C. G. Gibson has also obtained this result.)

Note. The conjectured equivalences have been verified in exactly the cases where the curve selection lemma is applicable. It would be interesting to know if they are true in the general, i.e. non-semianalytic, case, so as to have geometric versions of the regularity conditions available, avoiding sequences.

Proof of the theorem. Suppose (b) fails; we shall show that (b_s) fails.

We have sequences $x_i \in X$, $y_i \in Y$ tending to x, $T_{y_i} Y \to \tau$, and $y_i - x_i/|y_i - x_i| = \lambda_i \to \lambda$. Since $\lambda \not\subset \tau$ we may suppose that $d(\lambda, \tau) > \epsilon > 0$ for some ϵ, with distance $d(\ ,\ )$ defined appropriately. Then, for some i_0, $d(\lambda_i, T_{y_i} Y) > \epsilon$ when $i \geq i_0$.

Let G^r_s denote the Grassmannian of s-planes in $\mathbb{R}^r$, a compact analytic manifold. Set

$$V_1 = \{(v, P) \in G^p_1 \times G^p_n : d(v, P) > \epsilon\}$$

and

$$V_2 = \{(x, y, y - x/|y - x|, T_y Y) : x \in X, y \in Y\}.$$

Then V_1 is semialgebraic, and V_2 is semianalytic since both X and Y are semianalytic by hypothesis. Hence

$$V = (X \times Y \times V_1) \cap V_2$$

is a semianalytic subset of $\mathbb{R}^p \times \mathbb{R}^p \times G_1^p \times G_n^p$, and $(x, x, \lambda, \tau) \in \overline{V}$ satisfies the hypotheses of the curve selection lemma. See (1), p. 103. This provides an analytic curve

$$\alpha\colon [0,1] \to X \times \overline{Y} \times G_1^p \times G_n^p,$$
$$t \mapsto (x_t, y_t, \lambda_t, T_{y_t} Y),$$

where $\lambda_t = y_t - x_t / |y_t - x_t|$, $y_t \in Y$ if $t \neq 0$, and $d(\lambda_t, T_{y_t} Y) > \epsilon$.

Write η for the C^1 manifold-with-boundary $\bigcup_t y_t$, and ξ for $\bigcup_t x_t$, contracting the domain of α if necessary.

Since we are trying to show that (b_s) fails, and (b_s) implies (a_s), we may assume that (a_s) holds. Then by (3), § 3, since Y is semianalytic, (a) holds. This implies that

$$T_x \eta = T_x \xi. \tag{*}$$

For, suppose not. Then

$$\begin{aligned} \lambda &\subset T_x \xi \oplus T_x \eta \\ &\subset T_x X + T_x \eta \\ &\subset \tau \end{aligned}$$

using (a). But $\lambda \not\subset \tau$ by hypothesis, giving (*).

Notation. Given distinct lines λ, λ' in the plane meeting at a point q, and a point q' on λ' at unit distance from q, consider the circles with tangent λ at q which contain q' in their interior. If $\epsilon = d(\lambda, \lambda')$ let r_ϵ denote the lower limit of the radii of these circles.

LEMMA. *There exists a local C^1 retraction defined on a neighbourhood U of x in $\mathbb{R}^p$, $\pi\colon U \to X$, such that for each t, $\pi^{-1}(x_t)$*

(i) *is the intersection with U of a $(p-m)$-plane containing λ_t,*

(ii) *is transverse to Y in U,*

(iii) *contains a $(p-m)$-disc D_t of radius $r_\epsilon |y_t - x_t|$ with $y_t \in \partial D_t$, $x_t \in \operatorname{Int} D_t$, and*

$$T_{y_t}(Y \cap \pi^{-1}(x_t)) \subset T_{y_t}(\partial D_t),$$

(iv) *intersects η only at y_t.*

Proof. Because (b) fails and (a) holds, $\lambda \not\subset T_x X$. Thus there exists a $(p-m)$-plane transverse to X at x, and containing λ. Using (*) and the analytic dependence of y_t, λ_t, and $T_{y_t} Y$ upon t, we can find an analytic, and hence a C^1, fibre bundle over ξ, restricting α if necessary, so that the fibre over x_t is a $(p-m)$-plane containing λ_t. Choose a C^1 diffeomorphism ϕ of an open neighbourhood U of x in $\mathbb{R}^p$, so that $\phi(X \cap U)$ is affine and $\phi(\xi \cap U)$ is a line. Extend the fibration over $\phi(\xi)$ to the rest of $\phi(X \cap U)$ by parallel translation, and pull back by ϕ^{-1} to give a C^1 retraction $\pi\colon U \to X$ with each fibre C^1 diffeomorphic to $\mathbb{R}^{p-m}$, and which satisfies (i).

For (ii) use (a_s), shrinking U if necessary, and observe that $\pi|_Y$ is a submersion at y if and only if $\pi^{-1}(\pi(y))$ is transverse to Y at y. (ii) tells us that $Y \cap \pi^{-1}(x_t)$ is a C^1 $(n-m)$-manifold.

Let D_t be a disc of radius $r_\epsilon |y_t - x_t|$ in the $(p-m)$-plane of (i), with y_t on its boundary and so that

$$T_{y_t}(Y \cap \pi^{-1}(x_t)) \subset T_{y_t}(\partial D_t).$$

Because $d(\lambda_t, T_{y_t} Y) > \epsilon$ and r_ϵ is a decreasing function of ϵ, x_t belongs to the interior of D_t. For sufficiently small t, $D_t \subset \pi^{-1}(x_t)$, giving (iii).

Finally use (*), restricting α if necessary, to ensure that $\eta \cap \pi^{-1}(x_t) = y_t$. This proves (iv) and completes the proof of the lemma.

Project λ_t onto $N_{y_t}(\partial D_t)$ to give $\mu_t \in G_1^p$. By (iii) each μ_t is non-zero and

$$\mu_t \subset N_{y_t}(Y \cap \pi^{-1}(x_t)).$$

Now we construct a tubular function ρ so that $\rho(y_t) = t$ and

$$\mu_t \subset N_{y_t}((\pi, \rho)^{-1}(x_t, t)).$$

This will show that Y is not transverse to the fibre of (π, ρ) at y_t, for each t, which is the same as saying that $(\pi, \rho)|_Y$ is not a submersion at y_t, for each t, so that (b_s) fails.

It suffices then to find ρ so that

$$\partial D_t = (\pi, \rho)^{-1}(x_t, t)$$

for each t. Let ϕ be as in the proof of the lemma, and for each $t > 0$ let P_t be obtained by first translating $\phi(\partial D_t)$ along $\phi(\xi)$, using (iv), and then over $\phi(X \cap U)$ orthogonal to $\phi(\xi)$. Shrink U so that

$$\bigcup_{t>0} \phi^{-1}(P_t) = U \backslash (X \cap U).$$

Then we have a C^1 fibration

$$\rho: U \backslash (X \cap U) \to (0, 1],$$

with $\rho^{-1}(t) = \phi^{-1}(P_t)$ a C^1 manifold C^1 diffeomorphic to $S^{p-m-1} \times \mathbb{R}^m$. Setting $\rho|_{X \cap U} \equiv 0$ extends ρ to be C^1 on U, and ρ is the required tubular function. This completes the proof of the theorem.

REFERENCES

(1) Lojasiewicz, S. *Ensembles semianalytiques* (I.H.E.S. notes, 1965).

(2) Mather, J. N. *Notes on topological stability* (Harvard notes, 1970).

(3) Trotman, D. J. A. A transversality property weaker than Whitney (*a*)-regularity. *Bull. London Math. Soc.* (to appear).

(4) Wall, C. T. C. Regular stratifications. *Dynamical Systems—Warwick 1974* (Springer Lecture Notes 468 (1975), Berlin), 332–344.

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COUNTEREXAMPLES IN STRATIFICATION THEORY: TWO DISCORDANT HORNS*

D. J. A. Trotman

One of the useful properties of Whitney's (a)-regularity condition (as defined in [13]) is that the set of mappings transverse to the strata of an (a)-regular stratification is open and dense. That this set is open has often been justified by remarking that (a)-regularity implies that a submanifold transverse to a stratum at a given point is transverse to all other strata in some neighborhood of the point, a condition I have called (t)-regularity in [10]. Our first example shows that this reasoning is wrong: transversality to a (t)-regular stratification need not be open. However we verify directly that transversality to an (a)-regular stratification is open.

Our second example is that of a pair of real semialgebraic strata which are (b)-regular (as defined in [13]) but which fail Kuo's ratio test ([4], where Kuo proved that no such example exists when the smaller stratum has dimension one), and hence do not satisfy the property (w) used by Verdier in [12], where it was remarked that such an example was not known.

1. (a)-regularity and transversality

Let X, Y be C^1 submanifolds of $\mathbb{R}^n$ and let $0 \in Y \subset \bar{X} - X$. Consider the following regularity conditions for the pair (X, Y) at 0.

(a) Given x_i in X tending to 0, if $T_{x_i}X$ tends to τ, then $T_0 Y \subset \tau$.

(t) Given a C^1 submanifold S meeting Y transversely at 0, then there is a neighborhood U of 0 in $\mathbb{R}^n$ such that S is transverse to X within U.

Call a stratification *(a)-regular* if each pair of strata (X, Y) satisfies (a) at each point of Y. Similarly for a *(t)-regular* stratification.

NOTE 1.1. That (a) implies (t) is immediate.

NOTE 1.2. It is *not* a consequence of 1.1 that mappings transverse to each of the strata of an (a)-regular stratification form an open set, as suggested for example in [8], [9], [10], [11]. It is in fact a direct consequence of (a)-regularity.

* The title was suggested by Tony Iarrobino.

PROPOSITION 1.3. *Let N, P be C^∞ manifolds. Let P contain a closed subset Q partitioned into a locally finite union of submanifolds forming an* (a)*-regular stratification $\mathcal{S}$, i.e. if X, Y are strata of $\mathcal{S}$, then at each point of $Y \cap \bar{X}$, condition* (a) *is satisfied. Then $T_{\mathcal{S}} = \{f \in C^\infty(N, P) : f$ is transverse to each stratum of $\mathcal{S}\}$ is open in $C^\infty(N, P)$ with the Whitney C^1 topology (and hence with the Whitney C^∞ topology).*

PROOF*. Suppose that $T_{\mathcal{S}}$ is not open, so that there exists f in $T_{\mathcal{S}}$, a sequence $\{g_i\}$ tending to f in $C^\infty(N, P)$ with $g_i \notin T_{\mathcal{S}}$, a stratum X, and a sequence $\{a_i\}$ tending to a_0 in N such that g_i is not transverse to X at a_i. It is clear that $f(a_0) \notin X$, since X is a smooth submanifold. So let Y be the stratum containing $f(a_0)$. $(df)_{a_0}(T_{a_0}N)$ and $T_{f(a_0)}Y$ span $T_{f(a_0)}P$, and so for i sufficiently large $(dg_i)_{a_i}(T_{a_i}N)$ and $T_{g_i(a_i)}X$ span $T_{g_i(a_i)}P$, by (a) and the assumption that g_i tends to f. This gives a contradiction, proving the proposition.

NOTE 1.4. (i) When W is a submanifold of P, it is a corollary of Thom's Transversality Theorem that $T_W = \{f \in C^\infty(N, P) : f$ is transverse to $W\}$ is dense in $C^\infty(N, P)$ with the Whitney C^∞ topology. (See for example [2].) Hence $T_{\mathcal{S}}$ is both open and dense in $C^\infty(N, P)$ with the Whitney C^∞ topology.

(ii) If W is closed, T_W is open, as proved in [2], but here the strata of $\mathcal{S}$ are not assumed to be closed.

(iii) It is easily verified that $\mathcal{S}$ is (a)-regular if and only if the set of jets transverse to $\mathcal{S}$ is open. This observation is due to C. T. C. Wall.

In [10] the curve selection lemma is used to prove that (t) implies (a) if X is semianalytic. It is equally true if X is subanalytic for Hironaka proved a curve selection lemma for subanalytic sets in [3] (proposition 3.9. See [5] for a proof for semialgebraic sets). Hence if the strata are subanalytic the transversal mappings to a (t)-regular stratification do form an open set.

In the next section we shall give an example of a finite (t)-regular stratification for which the set of transversal mappings is not open, and so in particular it is not (a)-regular. This is an explicit version of an example mentioned in [10].

I stress this point at length because I had mistakenly thought that proposition 1.3 was true with (a) replaced by (t). Thus in [11] (t) is used in the definition of stratification given in chapter 8. There the strata are semialgebraic (corollary 3.6 of [11]), so we could use the result of [10] mentioned above to give (a), and then apply proposition 1.3. Alternatively

* A detailed proof appears as Proposition 3.6 in E. A. Feldman, The geometry of immersions, I, Trans. Amer. Math. Soc. 120 (1965), 185–224.

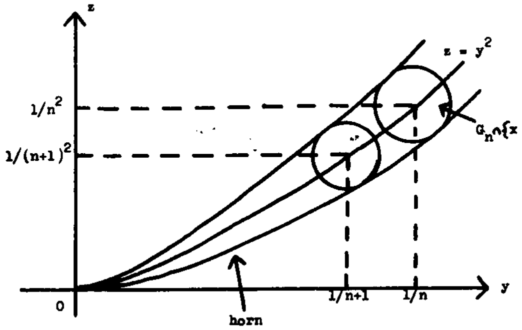

Figure 1. $x=0$.

one can use the following elementary formulation suggested by E. C. Zeeman.

PROPOSITION 1.5. *Let X, Y be C^1 submanifolds of a C^1 manifold P, and suppose that $\phi: M \to C^1(N, P)$ is continuous, M is a topological space, N is a C^1 manifold, $Y=\phi(m)(N)$ for some $m \in M$, and for all open sets $U \subset M$ containing m, there is an $m' \in U$ such that $\phi(m')(N) \subset X$. Then the pair (X, Y) satisfies* (a) *at each point of Y.*

The proof is left as an exercise.

2. The first horn

Let (x, y, z) be coordinates in $\mathbb{R}^3$. Take Y to be the y-axis, and let $X=(\bigcup_{n=1}^{\infty}\{f_n=0, g_n \leqq 0\}) \cup (\bigcap_{n=1}^{\infty}\{x=0, g_n \geqq 0\})$, where $g_n \leqq 0$ defines the cylinder G_n of radius $1/3n(n+1)$ with axis the line $y=1/n$, $z=1/n^2$, and where $f_n=0$ defines the surface F_n obtained from $x=(y^2+z^2)^2-(y^2+z^2)+\frac{1}{4}$ by translating the origin to $(0, 1/n, 1/n^2)$ and reducing by a factor of $3n(n+1)/\sqrt{2}$ so that F_n intersects ∂G_n exactly where $x=0$ is tangent to F_n. See figures 1 and 2.

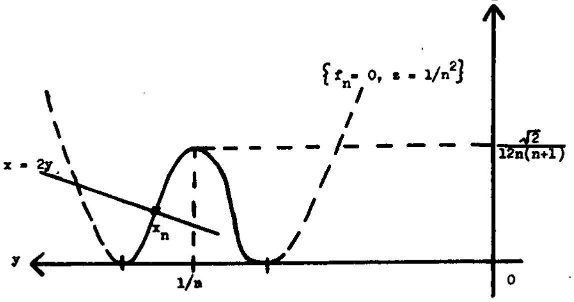

Figure 2. $z=1/n^2$.

X is a C^1 submanifold and is semialgebraic on the complement of the origin.

The normal vector to X at the point

$$x_n = (1/24\sqrt{2}n(n+1),\ (1/n)+1/3\sqrt{2}n(n+1),\ 1/n^2)$$

is $(2, 1, 0)$ for all n. Hence the limit as n tends to ∞ is $(2, 1, 0)$ and (a) fails. (t) holds since any submanifold transverse to Y will intersect X near Y only at points near which X is defined by $x = 0$. Hence the stratification of $\mathbb{R}^3$ defined by $\{Y, X, \mathbb{R}^3 - (X \cup Y)\}$ is (t)-regular. Now the mapping h in $C^\infty(\mathbb{R}^2, \mathbb{R}^3)$, defined by inclusion of the plane $2x + y = 0$, is transverse to the stratification, but for each n the mapping h_n defined by inclusion of the plane

$$2x + y = (5 + 12\sqrt{2}(n+1))/(12\sqrt{2}n(n+1))$$

is not transverse to X at x_n. Since h_n tends to h as n tends to ∞, mappings transverse to the stratification are not open in $C^\infty(\mathbb{R}^2, \mathbb{R}^3)$.

Note that by smoothing near each circle $\{x = 0, g_n = 0\}$, X can be made into a C^∞ submanifold of $\mathbb{R}^3$, with normal vector at x_n as before, for each n. Hence proposition 1.3 with (t) replacing (a) is false.

3. (b)-regularity and the ratio test

Let X be a C^1 submanifold and a semianalytic (or subanalytic) set in $\mathbb{R}^n$. Let $Y \subset \bar{X} - X$ be an analytic submanifold of $\mathbb{R}^n$. The pair (X, Y) are *(b)-regular* at $0 \in Y$ if,

(b) Given x_i in X and y_i in Y tending to 0, if $T_{x_i}X$ tends to τ, and the unit vector in the direction $x_i y_i$ tends to λ, then $\lambda \subset \tau$.

Apply a local analytic isomorphism at 0 to $\mathbb{R}^n$ so that, near 0, Y becomes affine. Let π denote orthogonal projection onto Y and define,

(b′) Given x_i in X tending to 0, if $T_{x_i}X$ tends to τ, and the unit vector in the direction $x_i\pi(x_i)$ tends to λ, then $\lambda \subset \tau$.

LEMMA 3.1. (a) + (b′) $\Leftrightarrow$ (b).

In [4] T.-C. Kuo introduced the following condition, which he called the *ratio test.*

(r) Given x_i in X tending to 0, and any vector $v \in T_0 Y$,

$$\lim_{i\to\infty} \frac{|\pi_i(v)| \cdot |x_i|}{|x_i - \pi(x_i)|} = 0.$$

Here π_i denotes orthogonal projection onto the normal space to X at x_i. Kuo proved two theorems in [4]:

THEOREM 3.2. (r)$\Rightarrow$(b).

THEOREM 3.3. (b)$\Rightarrow$(r) *if* Y *is* 1-*dimensional.*

In each case the proof uses the curve selection lemma with the assumption that X is a semianalytic set. As remarked in §1, by [3] we know that the same proof can be used if X is a subanalytic set.

In the next section we give an example with Y 2-dimensional where (b) holds and (r) fails to hold. X will be a semialgebraic C^1 submanifold of dimension 3 in $\mathbb{R}^4$. I do not know of such an example where X is the smooth part of an algebraic variety. In the special case of a family of complex hypersurfaces with isolated singularity parametrized by Y it is known that (b) and (r) are equivalent, for Y of arbitrary dimension. This is because (r) is a trivial consequence of (c)-cosécance as defined by Teissier in [7] and discussed by him in this volume. It follows from [1] and [6] that (b) implies (c)-cosécance.

Verdier has introduced the following condition in [12],

(w) There is a constant $C>0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that if $x\in U\cap X$, and $y\in U\cap Y$, $d(T_xX, T_yY)\leqq Cd(x, y)$.

This is just (c)-cosécance restricted to X, so that it makes sense when X is not a variety. (w) trivially implies (r), hence (b) does not imply (w), when the dimension of Y is greater than 1, by the example in the next section. Even when Y is 1-dimensional, (b) can hold and yet (w) fail: in $\mathbb{R}^3$ let X be $\{x=0, z>0, z^2\leqq y^2\}\cup\{z^5x^2=(y^2-z^2)^4, x\geqq 0, z>0, z^2\geqq y^2\}$, let Y be $\{x=z=0\}$, and consider the curve $X\cap\{z^2=3y^2\}$. Thus (w) is strictly stronger than (r) by theorem 3.3.

4. The second horn

Let (x, y, z, w) be coordinates in $\mathbb{R}^4$, and let Y be the plane $z=w=0$. Define the semialgebraic set,

$$X=\{w=0, 2(x^2+(z-y^p)^2)\geqq y^{2p}, z>0\}$$

$$\cup\{y^qw=(x^2+(z-y^p)^2)^2-y^{2p}(x^2+(z-y^p)^2)+y^{4p}/4,$$

$$2(x^2+(z-y^p)^2)\leqq y^{2p}, z>0\}$$

where p and q are positive integers satisfying,

$$2p<q<3p. \tag{4.1}$$

For example let $p=2$, $q=5$.

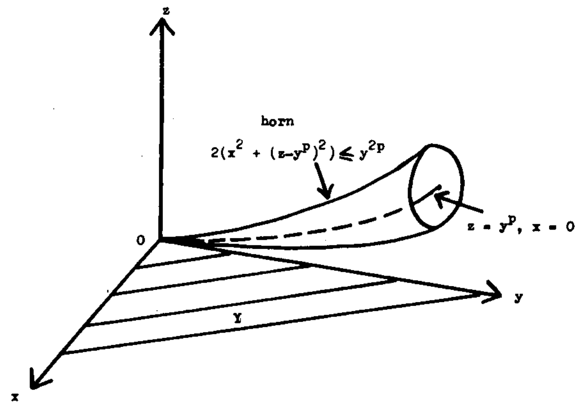

Figure 3. $w=0$.

Observe that because the algebraic variety defined by the equality in the second part of the expression for X has $w=0$ as tangent space at every point of its intersection with $2(x^2+(z-y^p)^2)=y^{2p}$, X is a C^1 submanifold of $\mathbb{R}^4$.

ASSERTION 4.2. (b) holds.

PROOF. We show that there is a single limiting tangent 3-plane for sequences on X tending to 0, namely $w=0$. It suffices to consider the points on $y^q w = x^4 - y^{2p}x^2 + y^{4p}/4$ (with y fixed) where $d^2w/dx^2=0$, since at these points the normal is furthest from the w-direction.

$d^2w/dx^2=0$ when $6x^2=y^{2p}$, and the normal vector is $(\pm(\frac{4}{3}\sqrt{6})y^{3p}, -y^q)$ which tends to $(0, 1)$ as y tends to 0 since $q<3p$ by (4.1). Hence $w=0$ is the unique limiting tangent plane.

At the points on X where the sécant vector defined by orthogonal projection to Y is furthest from the z-direction, the sécant vector is contained in the tangent space to X. Hence $0z$ is the unique limit of sécant

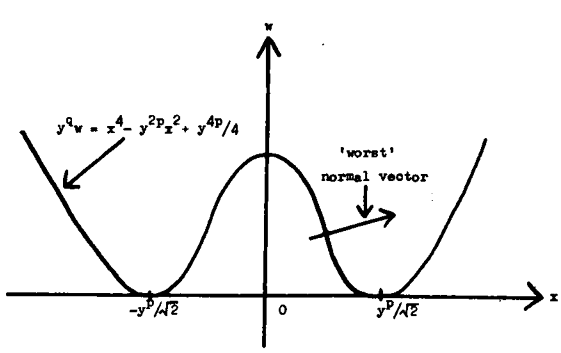

Figure 4. $z=y^p$, y fixed.

vectors, and (b′) holds. (a) holds (since $\{w=0, z=0\} \subset \{w=0\}$), so we can apply lemma 3.1 to show that (b) holds, proving the assertion.

ASSERTION 4.3. (r) fails to hold.

PROOF. Consider the curve $\gamma(t) = (t^p/6, t, t^p, t^{4p-q}/9)$ which lies on X. The normal vector to X at $\gamma(t)$ is,

$$((\tfrac{4}{3}\sqrt{6})t^{3p}, ((2p/3)-(q/9))t^{4p-1}, 0, -t^q).$$

Let π_t denote projection onto this normal space. Then,

$$|\pi_t(0x)| \sim \frac{t^{3p}}{|(t^{3p}, t^{4p-1}, 0, t^q)|} \sim \frac{t^{3p}}{t^q},$$

since by (4.1) $q<3p$.

$$\frac{|\gamma(t)|}{|\gamma(t)-\pi(\gamma(t))|} = \frac{|(t^p/\sqrt{6}, t, t^p, t^{4p-q}/9)|}{|(0, 0, t^p, t^{4p-q}/9)|} \sim \frac{t}{t^p}.$$

Hence the ratio (as in the definition of (r)) becomes t^{2p-q+1}, which does not tend to zero since $2p<q$ by (4.1). This proves assertion 4.3.

Finally we check that Verdier's condition (w) fails to hold.

$$d(T_{\gamma(t)}X, T_{\pi(\gamma(t))}Y) \sim t^{3p-q},$$
$$d(\gamma(t), \pi(\gamma(t))) \sim t^p,$$

hence (w) fails exactly when $2p<q$.

NOTE 4.4. Basing the construction on $w = x^{4k} - x^{2k} + \frac{1}{4}$, $1<k<\infty$ (instead of $k=1$ as here), we can build similar examples with X a C^k submanifold.

REFERENCES

[1] J. BRIANÇON and J.-P. SPEDER, Les conditions de Whitney impliquent μ^*-constant, *Annales de l'Institut Fourier de Grenoble 26*(2) (1976), 153–163.

[2] M. GOLUBITSKY and V. GUILLEMIN, *Stable mappings and their singularities*, Springer-Verlag, New York, 1974.

[3] H. HIRONAKA, Subanalytic sets, *Number Theory, Algebraic Geometry and Commutative Algebra, volume in honour of Y. Akizuki*, Kinokuniya, Tokyo, 1973, 453–493.

[4] T.-C. KUO, The ratio test for analytic Whitney stratifications, *Liverpool Singularities Symposium* I, Springer Lecture Notes 192, Berlin (1971), 141–149.

[5] J. MILNOR, *Singular points of complex hypersurfaces*, Ann. of Math. Studies No. 61, Princeton (1968).

[6] B. TEISSIER, Introduction to equisingularity problems, *A.M.S. Algebraic Geometry Symposium, Arcata 1974*, Providence (1975), 593–632.

[7] B. TEISSIER, Variétés polaires I: Invariants polaires des singularités d'hypersurfaces, Publication M242.0376 du *Centre de Math. de l'Ecole Polytechnique*, 1976.

[8] R. THOM, Propriétés differentielles locales des ensembles analytiques, *Séminaire Bourbaki 1964–65*, exposé no. 281.

[9] R. THOM, Ensembles et morphismes stratifiés, *Bull. Am. Math. Soc. 75* (1969), 240–284.

[10] D. J. A. TROTMAN, A transversality property weaker than Whitney (A)-regularity, *Bull. Lond. Math. Soc. 8* (1976), 225–228.

[11] D. J. A. TROTMAN and E. C. ZEEMAN, Classification of elementary catastrophes of codimension $\leqq 5$, *Structural stability, the theory of catastrophes, and applications in the sciences*, Springer Lecture Notes No. 525 (1976), 263–327.

[12] J.-L. VERDIER, Stratifications de Whitney et théorème de Bertini-Sard, *Inventiones Math.* (1976), *36*, 295–312.

[13] H. WHITNEY, Local properties of analytic varieties, *Diff. and Comb. Topology*, edited by S. Cairns, Princeton (1965), 205–244.

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