Categories for fixpoint semantics

by

Daniel J. Lehmann
Department of Computer Science
University of Warwick
COVENTRY CV4 7AL U.K.

For Yael

A dissertation submitted for the degree of Doctor of Philosophy.
1976.

P R E F A C E

I am extremely grateful to my thesis advisor Professor Eliahu Shamir for his constant encouragement and amicable help during my stay at the Hebrew University.

I have been, during the elaboration of this work, in constant and fruitful discussions with Michael B. Smyth and it is my pleasure to thank him for the patient care he took to re-introduce me into the intricacies of continuous partial orders.

I have benefitted from conversations with David Park, Michael Paterson, W. Wadge, M. Beynon and A. Shamir.

I am also grateful to Mrs. Josie Lloyd for her typing and the Science Research Council for its financial support.

A B S T R A C T

A precise meaning is given to general recursive definitions of functionals of arbitrarily high type, including non-deterministic definitions. Domain equations involving products, sums, powers and functor domains are solved.

The use of categories with ω-colimits as semantic domains is investigated and it is shown that such categories provide a general construction for power-domains and that no such construction can be obtained with partial orders.
Initial fixpoints of continuous functors on such categories are defined and studied. They provide a meaning for recursive definitions of the type $x{:=}f(x)$.
The category of domains is defined and shown to possess ω-colimits. Initial fixpoints of continuous functors on the category of domains provide the solution to domain equations.
The product, sum, power and functor domain of domains are defined and studied. Product, sum, power and functor domain are proved to be continuous functors in the category of domains.

INTRODUCTION

This work defines the mathematical semantics of recursive non-deterministic programs and provides the techniques necessary for handling the semantics of programming languages exhibiting non-deterministic features such as parallelism.
It should also be a first step towards a general theory of computability including non-determinism and functionals of arbitrarily high type, generalizing Kleene's attempt [6].
The second step in that direction could be the definition of a suitable category of effectively given domains and the third the elaboration of a theory of computable objects.

The central idea in this work is that when considering non-deterministic programs the notions of complete partial order, least fixpoints of continuous functions and domain equations have to be generalized.
It is not sufficient any more, when considering the process of successive approximations converging to the final value, to look at the sequence of objects but it is also necessary to consider the way in which each approximation is related to the preceding one, thus replacing a partial order by a category and a least upper bound by a colimit.

If f is a recursively defined non-deterministic function, f(x) will be the colimit of a sequence of approximations.
Typically these approximations will give partial information of the type: "There is a possible branch of the computation which gives a result approximated by y_0 and there is another branch giving a result approximated by y_1,... and those are the only possible branches."

Given two successive such approximations it is vital indeed, if one wants a clear picture of $f(x)$, to know how to relate the different branches talked about in the two successive approximations.

If one is not interested in non-deterministic computable functions an adequate theory of computability can be described using complete partial orders and so, one could question the interest of using categories as domains. But multi-valued functions are a very natural object for a theory of computation, quite independently of non-determinism, as was pointed out by Martin Hyland [4]. The use of categories as domains, by the generality it introduces, should also have a beneficial heuristic effect in the choice of definitions.

This work is by no means self-contained but a consistent effort has been made to follow the notation and terminology of Mac Lane [7].

Previous related work

Two previous attempts to define a mathematical semantics for non-determinism have been made: the first, by R. Milner [10], uses the notion of an oracle which still has an operational flavour, and the second by G. Plotkin [11] which defines a restricted category of complete partial orders, those which are colimits of finite ones (called SFP objects) and defines the power of such objects to be certain c.p.o's (themselves SFP objects). This last attempt gives the best results that may be obtained in the framework of partial orders and, though bold and elegant, is quite difficult to follow, only partially motivated and does not give a semantics as precise as should be desired because many different sets of possible values are identified (see next chapter).

M. Smyth [15] generalized Plotkin's construction to algebraic domains and noticed that a quasi-order coarser than Plotkin's could be defined which would make the whole treatment much simpler but also the semantics less precise and so, less interesting.

The present power-domain construction is a categorical version of Smyth's proposal which keeps the conceptual and technical simplicity of Smyth but remedy the imprecision in the semantics and gives a fully precise semantics in which no two different sets are identified.

The idea of using categories instead of partial orders was probably first advocated by H. Egli.

On why it is imperative to solve domain equations, see Plotkin [11] or Smyth [15].

The categorical approach to the solution of domain equations (for c.p.o.'s) appears in one sentence of Scott [13] and has been developed by Reynolds [12].

Plan

Chapter 1 reviews fixpoint semantics and the problems involved in domain equations and non-deterministic definitions. It sets the case for using categories as domains.

Chapter 2 recalls some definitions about categories and proves the existence of an initial fixpoint for every continuous functor.

Chapter 3 defines the category of domains: *Dom* in which domain equations are solved and proves the existence of colimits in *Dom*. The colimits in *Dom* may be seen as both direct and inverse limits.

Chapter 4 defines the usual sum and product of domains and proves their continuity.

Chapter 5 defines the power of a domain and proves continuity for the power functor.

Chapter 6 defines the functor space of two domains and proves continuity for the arrow functor.

Chapter 7 is a conclusion.

Chapter I

Fixpoint semantics, domain equations and non-determinism.

I.1 Fixpoint semantics

The problem to which fixpoint semantics is an answer is the following: how can we make sense, in a consistent and meaningful way, of general recursive equations of the type $x=f(x)$? Typically the preceding equation may be thought of as defining a function when f is a functional, but more involved cases should be considered.

1) When f is a non-deterministic functional the equation should define a non-deterministic function.
2) In most programming languages, procedures which take procedures as parameters may be defined (even recursively) and then the type of the function defined is not clear any more and the distinction between function and functional fades out. To give mathematical sense to such a phenomenon it is necessary to find a semantic domain (meanings for programs) D which satisfies the equation $D\simeq[D\to D]$ ($\simeq$ means isomorphic) where $[D\to D]$ should be a substantial subset of D^D, containing at least the "computable" functions. If $[D\to D]$ is written $\to(D,D)$ it becomes obvious that the above equation is also of the form $x=f(x)$.

The message of fixpoint semantics is that those equations should be solved and not considered as operational definitions of a process. The advantage of such a solution is two-fold (the second reason given here has not yet received the consideration it deserves).

1) Such a solution would provide a criterion against which to judge the correctness of implementations.

2) Fixpoint semantics allows the recursive definitions to be considered as equations and this is the only way towards proofs of correctness in complex situations, particularly with non-deterministic programs which tend to be more complex than deterministic ones.

The main tool, and until [13] the only one, to solve equations of the type mentioned above was Tarski's least fixpoint theorem and some variations on the same theme. The successes of this least-fixpoint semantics will be rapidly reviewed now.

I.2 Least-fixpoint semantics

The message here is: all interesting equations of the type $x=f(x)$ are such that x varies over an ω-complete partial order D which has a least element, f is an ω-continuous endo-function $D \to D$, and the interesting solution is the least-fixpoint of f.

<u>Definition 1</u> : A partial order D is ω-complete iff every denumerable directed $S \subseteq D$ has a least upper bound (l.u.b.)

<u>Definition 2</u> : Let A and B be partial orders, $f: A \to B$ is ω-continuous iff f preserves all existing l.u.b.'s of denumerable directed subsets.

The theorem that asserts the existence of a least fixpoint under the conditions above is a variation on Tarski's fixpoint theorem. There the assumption on D is stronger but the assumption on f weaker (in fact this variation is easier to prove than Tarski's original result and is quite trivial).

Some people have preferred to use Tarski's result about monotone functions but the use of monotone non-ω-continuous functions does not seem convincing to the author.

Least-fixpoint semantics have proved to be extremely successful in defining the meaning of a large class of recursive programs, essentially due to the fact that ω-complete partial orders are preserved by many constructions and that many useful functions are ω-continuous. We shall recall (with the notation of [13]).

Theorem 1 : If A and B are ω-complete partial orders with least elements then so are $A\times B$, $A+B$ and $[A\to B]$.

$[A\to B]$ is the set of all ω-continuous functions: $A\to B$ with the pointwise ordering.

Theorem 2 : $f:A\times B\to C$ is ω-continuous iff it is separately ω-continuous in each variable, and $[A\times B\to C]\simeq[A\to[B\to C]]$.

Theorem 3 : The evaluation map : eval : $A\times[A\to B]\to B$ is ω-continuous.

The abstraction map : lambda : $[A\times B\to C]\to[A\to[B\to C]]$ is ω-continuous.

The least-fixpoint map : ℓfix : $[A\to A]\to A$ is ω-continuous.

The composition of two ω-continuous functions is ω-continuous.

The composition map : $\circ$: $[A\to B]\times[B\to C]\to[A\to C]$ is ω-continuous.

The projection maps : p_1 : $A\times B\to A$ and p_2 : $A\times B\to B$ are ω-continuous.

Constant functions are ω-continuous.

There are two dark spots left in this rosy picture: non-deterministic programs and domain equations.

I.3 Domain equations

The necessity of solving domain equations was explained above in relation with the equation $D=[D\to D]$ which is a preliminary to any semantics for untyped procedures but other similar examples are found.

In [8] Plotkin shows that, when dealing with parallel processes, the programs should be given as meanings resumptions, elements of a domain R which satisfies $R \simeq [S \to P[S+(S \times R)]]$ where S is the domain of final values and P is the power-domain constructor.

The first failure of fixpoint semantics is that such interesting equations cannot be solved by least-fixpoint methods for the obvious reason that no reasonable partial order can be defined on domains. What would it mean for a domain to be less than another one?

For the first time, in [13], Scott solved the equation $D=[D \to D]$ and his method was generalized to other equations involved: $\to$, $\times$ and $+$ (with the exception of P) by Reynolds [12].

The method used there is categorical: the class of domains is category and if the arrows are carefully selected (they have to be pairs of continuous projections) the category may be proved to have directed colimits and $\to$, $\times$ and $+$ may be seen to be continuous functors.

Domain equations may then be solved by initial-fixpoint methods in categories which generalize the least-fixpoint theorem.

I.4 Non-deterministic programs and power-domains

The neat way to fixpoint semantics for non-deterministic programs is the definition of a power-constructor P which acts on domains to give a domain reasonably close to what could be expected for a power-set. Non-deterministic continuous functions from A to B are then elements in $[A \to P(B)]$.

Unfortunately the problems are numerous when one tries to define the power-domain of an ω-complete partial order to be an ω-complete partial order.

Let us first list two conditions that should be fulfilled by P to be semantically acceptable.

1) The union map $\cup$: $P(A)\times P(A)\to P(A)$ is ω-continuous.

2) The singleton map : {} : $A\to P(A)$ is ω-continuous.

The reason why these conditions are imperative is that the rule of the game is that all semantically meaningful functions should be ω-continuous and union will be used to translate the non-deterministic <u>or</u> and the singleton map to translate deterministic functions.
The problem of finding an acceptable constructor P has been solved (independently) by Milner and Egli for a very special case: for flat domains (those domains where $x\neq y$ and $x\sqsubseteq y \Rightarrow x=\bot$, $\bot$ is the least element). This solution is too restricted to be of real interest because even if A and B are flat [A→B] and $P(A)$ are not flat any more and the constructions cannot be iterated. Plotkin [11] has a more useful construction which, though not really general, is general enough for iteration of constructions to be possible. The author thinks that his construction should be preferred to Plotkin's on three counts.

1) It is mathematically simpler (much simpler).

2) It is fully general: it gives a power domain to any ω-complete-poset whereas Plotkin defines power-domains only for algebraic countably based posets.

3) It preserves the identity of every subset of possible values whereas Plotkin defines the elements of his power-domain to be only equivalence classes of subsets and so identifies many different subsets.

An example of the problems arising when one tries to solve equations on non-flat domains will be given now. It shows quite conclusively that no satisfying partial order may be defined on the power-set of a domain.

Let E be the domain consisting of an infinite countable ascending chain with a top element. E is a non-flat continuous lattice (see Fig.1)

∞
⋮
$.n$

E $\qquad\qquad$ $i \sqsubseteq j$ iff $i \leq j$

$.2$
$.1$ $\qquad\qquad$ $i \sqsubseteq \infty$
$.0=\perp$ $\qquad\qquad$ $\infty \sqsubseteq \infty$

Fig. 1

Let s: $E \to E$ be defined by : $s(n)=n+1$ and $s(\infty)=\infty$.

s is a continuous function totally acceptable semantically.

Let now S_1 and S_2 be the two following recursive definitions:

S_1 : $\quad x::=s(x)$

S_2 : $\quad x::=s(x)$ <u>or</u> 0

S_1 and S_2 may be considered as recursive definitions of constant functions. Any reasonable semantics should associate with S_1 an element of E and with S_2 a subset of E.

As far as S_1 is concerned it is a deterministic definition and there is no question about its meaning if we stick to a fixpoint-semantics, there is a unique fixpoint : ∞ . Operationally we could say that S_1 computes the l.u.b. of the sequence : $0 \sqsubseteq s(o) \sqsubseteq s(s(o)) \sqsubseteq \dots\ s^n(o) \sqsubseteq \dots$ which is ∞ .

For S_2 things are not so simple. The semantic interpretation of <u>or</u> should be union and so a fixpoint-semantics should provide as a meaning for S_2 an $A \subseteq E$ such that $A=\{o\} \cup s(A)$.

Clearly there are two such sets : E and E_o=E-{∞}.

Which one of them should be chosen?

If the semantics has to have any operational relevance at all the set defined by S_2 should contain the singleton defined by S_1 because S_2 is richer than S_1 in possible computations. The only acceptable meaning for S_2 is then E.

If E is to be in some sense the least-fixpoint of S_2 then we must have $E \sqsubseteq E_o$.

But clearly in any reasonable order (in particular in Milner-Egli's order defined by $A \sqsubseteq_M B$ iff $\forall a \in A\ \exists b \in B\ a \sqsubseteq b$ and $\forall b \in B\ \exists a \in A\ a \sqsubseteq b$) $E_o \sqsubseteq E$.

In fact in Milner-Egli's order $E \not\sqsubseteq E_o$.

At this point only three possible ways seem open :

1) Abandon the idea of least-fixpoint semantics and adopt a "best" fixpoint semantics as one of those studied by A. Shamir [14] (for a preview of these results see [8]). For the moment not enough is known on continuous best fixpoints on non-flat domains to see whether this is a promising avenue for further research.

2) Decide that no difference should be made between E and E_o. Plotkin [11] and Smyth [15] develop such ideas. It works but the treatment is mathematically difficult and, most important, many identifications are made for which no convincing non-technical reason can be given.

3) Abandon the idea that domains are partial orders and admit that they are categories on which every denumerable chain has a colimit.

This third proposal is the one which is developed here.

A fringe benefit of this idea is that now domain equations fall into the same basket as meanings of programs.

In other terms now the equation D=[D→D] is a recursive program.

Before the technical results one word is in order on why colimits are better than l.u.b.'s for semantics of non-deterministic programs. In a category there are directed arrows between objects with an associative composition of arrows and suitable identity arrows. A partial order is a category in which there is at most one arrow between any two objects. The notion of a colimit which generalizes that of l.u.b. has the following distinctive feature.

Let C be a chain of arrows and objects :

$$C = \quad a_o \xrightarrow[f_o]{} a_1 \xrightarrow[f_1]{} a_2 \xrightarrow[f_2]{} \cdots\cdots a_i \xrightarrow[f_i]{} a_{i+1} \quad \cdots\cdots$$

Its colimit depends not only on the objects $a_o,\ldots a_i,\ldots$ but also on the arrows $f_o,\ldots,f_i,\ldots$

By contrast in a partial order the colimit (l.u.b.) cannot depend on the arrows because there is no possible choice for the arrows (at most one between a_i and a_{i+1}). In our semantic interpretations the objects will represent partial information and the arrows possible ways in which two successive pieces of information may be related. In the case of non-deterministic functions an object will consist of partial information concerning each possible computation and when two such objects follow each other it is indeed of vital importance to know how they relate, how do the possible computations described in the first object relate to those described in the second.

In the above example the computation defined by S_2 will be represented by :

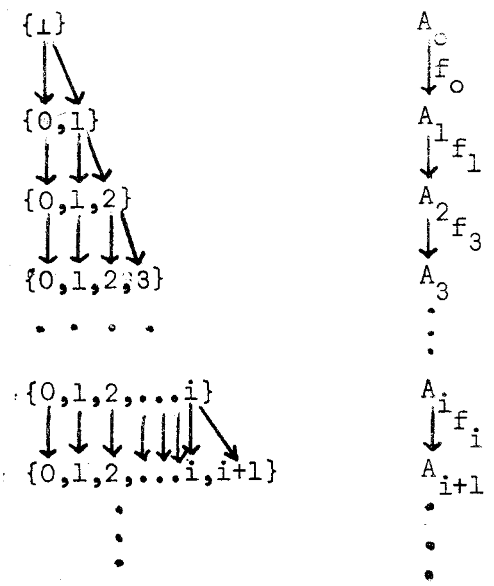


Fig. 2

which will be seen to be a chain in $P(E)$ and the meaning of S_2 will be the colimit of this chain, proved to be E.

Due to our will to treat domain equations, the detailed description of the category $P(D)$ will come only in Chapter 5. The first part of Chapter 5 may be read immediately by the reader too curious to wait.

Chapter II

Categories and initial fixpoints.

Two words of caution are needed here, the first on foundations and the second on isomorphisms. One point, on which our terminology differs from Mac Lane's [7] is that the word "set" will always be used in its strict sense (say in Zermello-Fraenkel axiomatic set theory). The word "class" will denote a collection of sets satisfying a certain property (definable in the language of set theory); when an object is said to be a subset of a class the word "set" has to be understood strictly : not all sub-classes are subsets, and it should come as no surprise that solutions to the equation $D=P(D)$ are found, but obviously such solutions are proper classes.

All the categories used in this paper as domains are large categories : the objects and morphisms form a class (proper or otherwise).

<u>Definition 1</u> : A category is small iff the class of its objects and the class of its arrows are sets.

Cat will denote the category of all small categories (it is a proper large category).

Cat' will denote the category of all large categories (it is not a large category).

Now comes the second of our words of caution.

An important difference between partial orders and categories is the existence, in categories, of non-trivial isomorphisms.

<u>Definition 2</u> : In a category C an arrow $f:a\to b$ is an isomorphism iff there is an arrow $g:b\to a$ such that $g\circ f=1_a$ and $f\circ g=1_b$.

In this case g is also an isomorphism and a and b are said to be isomorphic (noted a~b).

The image of an isomorphism by a functor is an isomorphism.

Clearly the identities (1_a for each object a) are isomorphisms, but there could be other ones (called above non-trivial).

Isomorphisms in the functor category B^A are called natural equivalences or better natural isomorphisms and noted $\tau : S \cong T$.

A universal arrow, when it exists, is always unique only up to isomorphism (in the comma category); in particular initial objects, products, co-products, limits and colimits, left and right adjoints are defined only up to isomorphism.

This is a fact that we shall have to bear in mind and we shall try to use the definite article only for objects which are uniquely determined.

On the other hand isomorphic objects are indistinguishable and when we shall look for solutions of equations in categories we shall be satisfied with a solution up to isomorphism.

In the categories which will be used as domains it is critical that the colimits (the object) are defined uniquely and not only up to isomorphism (we want the left-adjoint right-inverse of a functor to be uniquely determined) and we recall the following definition :

Definition 3 : A category is skeletal iff any two isomorphic objects are identical.

The term skeletal should not frighten anybody, the author thinks that those categories are quite pleasant to work with.

If C is a category any skeletal full sub-category of C is called a skeleton of C. C is equivalent to any of its skeletons and any two skeletons of C are isomorphic. We shall admit that any large category has a large skeleton.

In a skeletal category the limits and colimits are uniquely determined as far as the objects are concerned, the arrows of the limiting cones being determined only up to isomorphism (even in a skeletal category there are non-trivial isomorphisms).

The theorems about initial fixpoints that will be proved in the sequel of this chapter are formulated for arbitrary categories (not necessarily skeletal), a slightly sharper version may be obtained for skeletal categories if one remembers that isomorphic objects are identical.

<u>Definition 4</u> : The category ω is the category whose objects are the natural numbers and such that there is an arrow : $m \to n$ iff $m \leq n$ and in this case there is exactly one arrow : $m \to n$.

ω is the set of natural numbers ordered by the usual ordering.

Pictorially :

$$\omega = \quad 0 \to 1 \to 2 \to \ldots \; i \to i+1 \to \ldots$$

where identities and arrows obtained by composition are not drawn.

Partial orders are exactly the categories in which there is at most one arrow between two objects.

Colimits in partial orders are exactly l.u.b.

ω is a partial order. Partial orders are skeletal.

Posets are the small partial orders and a poset has an initial element iff it has a least element.

<u>Definition 5</u> : A category C is an ω-category iff every functor $F:\omega \to C$ has a colimit.

In the sequel only ω-categories will be considered, but any directed category containing ω as a sub-category could have been chosen. ω-posets with initial element, in the present terminology, are exactly the ω-chain-complete posets of Markowsky & Rosen [9].

<u>Definition 6</u> : A functor $H:A\to B$ is an ω-functor iff it preserves ω-colimits.

<u>Caution</u> : For every $F:\omega\to A$ which has a colimit, H has to preserve the colimiting cones, not only the colimit objects.

<u>Lemma 1</u> : If $H:A\to B$ and $G:B\to C$ are ω-functors, so is $G\circ H$.

<u>Proof</u> : Obvious.

<u>Lemma 2</u> : If $H:A\times B\to C$ is a bi-functor, H is an ω-functor iff for all objects b in B the restriction of H, $H_b:A\to C$ ($H_b(f)=H(f,1_b)$) is an ω-functor and for all objects a in A $H_a:B\to C$ is an ω-functor.

In short a bi-functor is jointly continuous iff it is separately continuous. The proof is obvious.

The fundamental fixpoint theorem of category theory shall be proved now. Its present form is due to M. Smyth.

<u>Theorem 1</u> : Let C be an ω-category, F and ω-endo-functor $F:C\to C$ and $h:a\to Fa$ be an arrow of C, then there are arrows $\eta:Fb\to b$ and $g:a\to b$ such that :

1) η is an isomorphism

   $g=\eta\circ Fg\circ h$

3) For any arrows $k:a\to c$ and $m:Fc\to c$ such that $k=m\circ Fk\circ h$ there is a unique $\alpha:b\to c$ such that $\alpha\circ\eta=m\circ F\alpha$.

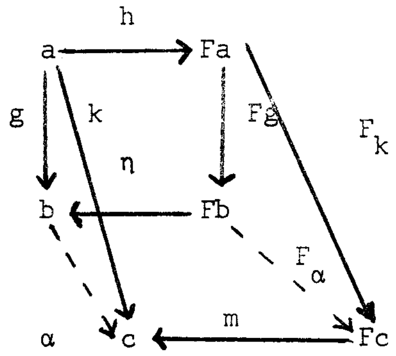


Fig. 1

Proof: Let $H:\omega \to C$ be the following :

$$a \xrightarrow{h} Fa \xrightarrow{Fh} F^2a \longrightarrow \ldots\ldots F^ia \xrightarrow{F^ih} F^{i+1}a \longrightarrow \ldots\ldots$$

C is an ω-category and H has a colimit.

Let $j:H \dot{\to} b$ be the colimiting cone and g be its first arrow $j_o:a\to b$.

F being an ω-functor Fj is a colimiting cone for FH.

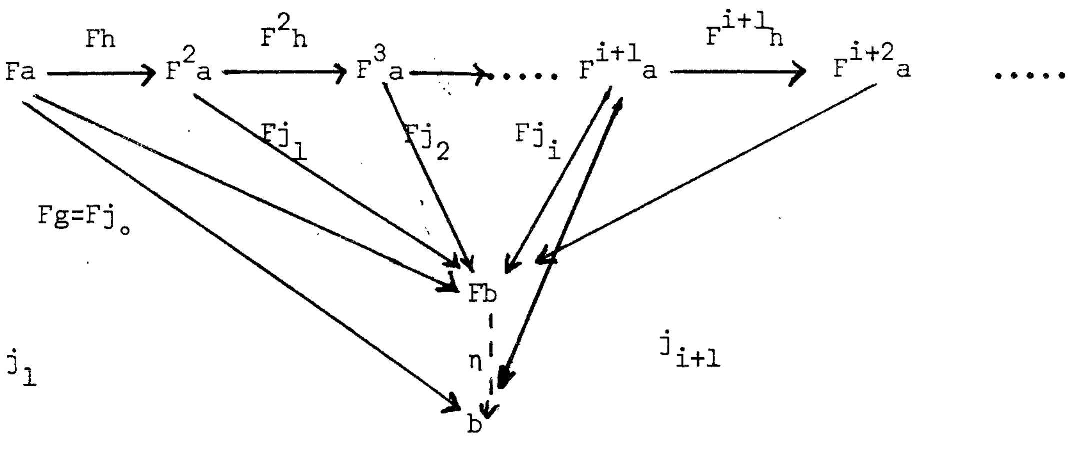


Fig. 2

Let $\eta:Fb\to b$ be the unique arrow such that $j_i=\eta\circ Fj_{i-1}$ for $i\geq 1$.

Clearly $g=j_o=j_1\circ h=\eta\circ Fg\circ h$.

To see that η is an isomorphism just observe that $\mu:H\dot{\to}Fb$ defined by $\mu_o=Fj_o\circ h$ and $\mu_i=Fj_{i-1}$ for $i\geq 1$ is a cone and implies the existence of a unique $\lambda:b\to Fb$ such that $\mu_i=\lambda\circ j_i$.

Clearly then $\eta\circ\lambda:b\to b$ such that $\eta\circ\lambda\circ j_i=\eta\circ\mu_i=j_i$ which implies $\eta\circ\lambda=1_b$ and similarly $\lambda\circ\eta:Fb\to Fb$ is such that $\lambda\circ\eta\circ Fj_i=Fj_i$ which implies $\lambda\circ\eta=1_{Fb}$.

Let us now prove the universal property 3)

The diagram
$$\begin{tikzcd} a \arrow[r, "h"] \arrow[dr, "k"'] & Fa \arrow[d, "m\circ Fk"] \\ & c \end{tikzcd}$$
commutes.

A commuting cone $e:H\dot{\to}c$ may be defined by $e_o=k$ and $e_{i+1}=m\circ Fe_i$

$(e_{i+1}\circ F^ih=m\circ Fe_i\circ F^ih=m\circ F(e_i\circ F^{i-1}h)=m\circ Fe_{i-1}=e_i)$

Then there exists a unique $\beta:b\to c$ such that $e_i=\beta\circ j_i$.

But $m\circ F\beta\circ\eta^{-1}\circ j_i=m\circ F\beta\circ\eta^{-1}\circ\eta\circ Fj_{i-1}=m\circ F\beta\circ Fj_{i-1}=m\circ F(\beta\circ j_{i-1})=m\circ Fe_{i-1}=e_i$

$m\circ F\beta\circ\eta^{-1}$ is then also a solution to $e_i=x\circ j_i$ and $\beta=m\circ F\beta\circ\eta^{-1}$, $\beta\circ\eta=m\circ F\beta$.

For uniqueness suppose that $\alpha\circ\eta=m\circ F\alpha$, then $e_o=k=m\circ Fk\circ h=m\circ F\alpha\circ Fg\circ h=\alpha\circ\eta\circ Fg\circ h=\alpha\circ g=\alpha\circ j_o$ and by induction for $i\geq 1$:

$e_i=m\circ Fe_{i-1}=m\circ F\alpha\circ Fj_{i-1}=\alpha\circ\eta\circ Fj_{i-1}=\alpha\circ j_i$, which implies $\alpha=\beta$.

Q.E.D.

If C is a partial order Theorem 1 states that if C is an ω-complete partial order and $f:C\to C$ is an ω-continuous function, for each $a\in C$ such that $a\sqsubseteq f(a)$ (a is a pre-fixpoint) there is a fixpoint b of f which is the least post-fixpoint greater or equal to a.

For categories a theorem with a similar diagram appears in Wand [16] but in a different setting.

When C has an initial element initial fixpoints may be defined.

<u>Definition 7</u> : $\perp$ is an initial object in C iff there exists a unique arrow $\perp_a : \perp \to a$ from $\perp$ to every object a in C.

<u>Definition 8</u> : A category is an initial category iff it has an initial object.

<u>Theorem 2</u> : Let C be an initial ω-category, F an ω-endo-functor $F:C\to C$ then there is an arrow $\eta : Fb \to b$ such that:

1) η is an isomorphism

2) for any arrow $m:Fc\to c$ there is a unique $\alpha : b \to c$ such that $\alpha \circ \eta = m \circ F\alpha$.

b is called an initial fixpoint of F; it is unique up to isomorphism.

<u>Proof</u> : Apply Theorem 1 to $f=\perp_{F\perp}$.

If C is a skeletal category then the b of Theorems 1 and 2 verifies b=Fb, and is uniquely determined which allows the definition of an initial fixpoint functor.

Theorem 2 says that (b,η) is an initial element in a suitable category. It is the author's guess that such categories are related to those described in [3].

<u>Definition 9</u> : infix : $[C\to C]\to C$ is defined by

- infix $F=b_F$, the unique b implied by Theorem 2
- infix $\begin{array}{c}F\\ \downarrow\tau\\ F\end{array} = \alpha_\tau$, the unique $\alpha : b_F \to b_{F'}$ such that

$$\alpha \circ \eta_F = \eta_{F'} \circ \tau_{b_{F'}} \circ F\alpha \ .$$

The main claim of this paper is that all recursive programs, even non-deterministic ones, can straightforwardly be considered as ω-endo-functors on initial categories and that the meaning of such a program is its initial fixpoint as defined in Theorem 2.

Nevertheless there are cases when other fixpoints of the type considered in Theorem 1 will be of interest. It will be shown that a domain equation may be considered as an ω-endo-functor in the category of domains, which is an ω-category, but in certain cases the initial fixpoint is too trivial to be of interest and other fixpoints will have to be considered; a good example of this fact is the equation $D=[D\to D]$ whose initial fixpoint is the one point domain.

Chapter III

The category of domains

The purpose of this chapter is to define the category of domains, in which domain equations will be solved and to prove that it is an ω-category with an initial object.

The objects of the category of domains, which shall be called domains, should be the structures in which to give meaning to programs. There are many properties that one could think of and which are probably necessary if one wants to have a reasonable theory of computation, but as it is not yet clear what properties exactly are needed or what properties should be helpful, this work is aiming at the broadest possible notion of a domain which can support the product, sum, functor domain and power domain constructions and with which domain equations can be solved. It is clear that the notion of a domain presented here is too broad for a theory of computation because domains are not necessarily effectively given, but the right category for a theory of computation is certainly a sub-category of the one which is defined here. One of the aims of this work is to show that domain equations may be solved without bothering whether the domains involved are effectively given or even "continuous" in the sense of Scott's continuous lattices, but obviously any reasonable notion of a "continuous" domain would involve a sub-category of ours, closed under colimits of countable chains.

The following definition is the broadest the author could think of.

<u>Definition 1</u> : A domain is a large skeletal initial ω-category.

A domain has to be a category because the power domain of a partial order is not a partial order and it has to be an initial ω-category because initial fixpoints of ω-functors will be the meanings of programs. It is reasonable to suppose that domains are skeletal because isomorphic objects cannot be distinguished and should have the same semantic interpretation. There is also a compelling technical reason for considering only skeletal categories for domains : it is only on skeletal categories that the left-adjoint right-inverse of a functor is uniquely determined as will be seen in the sequel.

One could also wish domains to be small but that would lead to some slight technical problems in the definition of power domains. The objects of a domain may be seen as pieces of incomplete information and the morphisms as possible ways in which two items of information may be related. The initial object is the absence of information and ω-colimits represent the information gathered through an infinite sequence of experiments including the way successive items relate to each other. The morphisms of the category of domains should allow the solution of domain equations, that is to say they should make the category of domains an ω-category and they should make the product, sum, functor domain and power domain operations ω-functors.

When one looks at the methods used to solve domain equations, in Scott [13] and Reynolds [12], one sees that they amount to the construction of larger and larger domains, each domain in the sequence being a sub-domain of the next one. As a consequence, an arrow $F:A\to B$ should ensure that A is a sub-category of B. In fact A should even be a full sub-category of B, because B should be richer in objects but not in arrows between the old objects. In other words $f:A\to B$ should yield a functor $F:A\to B$ both full and faithful. To preserve the colimit structure F should also be an ω-functor. This unfortunately does not ensure that the functor domain constructor is a functor in the category of domains. (We want a functor covariant in both variables). More precisely to any couple of arrows in the category of domains: $f:A\to A'$ and $g:B\to B'$ an arrow h should be associated: $h:[A\to B]\to[A'\to B']$. The way to ensure that is to ask that an arrow $f:A\to A'$, yields not only a functor $F:A\to A'$ but also a functor $H:A'\to A$.

If $f:A\to A'$ yields $F:A\to A'$ and $H:A'\to A$ and $g:B\to B'$ yields $G:B\to B'$ and $L:B'\to B$ then $M:\lambda f\ G\circ f\circ H$ is a functor: $[A\to B]\to[A'\to B']$ and $N:\lambda g\ L\circ g\circ F$ is a functor $[A'\to B']\to[A\to B]$. To recapitulate $f:A\to A'$ should be a pair of functors (F,H) $F:A\to A'$ and $H:A'\to A$ such that F and H are ω-functors (both should preserve the structure) and F is full and faithful. This does not make the category of domains an ω-category and to ensure the existence of ω-colimits some conditions on the relation between the two functors F and G are needed. Returning to the basic intuition that $f:A\to A'$ should make A a sub-category of A' one may see that in the construction of solutions to domain equations it will be used in the way that the objects of A are approximating those of A', A is a sub-category of A' on which A' may be projected. Now only a small leap is needed to conceive that A should be a co-reflective sub-category of A' (see Freyd [2] p.79). This means that given an object b in A' there is an object $\bar{b}$ in A which best approximates b by an arrow $m_b:\bar{b}\to b$ in the sense that for any object a in A and for any morphism $f:a\to b$ in A' there is a unique $g:a\to\bar{b}$ in A such that $f=m_b\circ g$.

Equivalently (see MacLane [7] p.88-90), in terms of the functors F and H above, (F,H) should be a pair of adjoint functors such that $H \circ F = I_A$ the identity functor on A, and the unit η of the adjunction : $I_A \dot{\rightarrow} H \circ F$ the identity natural transformation. In the terminology of MacLane [7] (p.92) F is a left-adjoint right-inverse for H or there is an adjunction $\langle F,H;1,\varepsilon \rangle$ with unit the identity.

The notion of a pair of adjoint functors has been defined by Kan [5] in 1958 and has been since then recognized as the most important concept of category theory. The best up to date summary on the subject is probably MacLane [7] chapters IV and V. If A and B are partial orders, $F:A \rightarrow B$ and $H:B \rightarrow A$ (F,H) is a pair of adjoint functors iff F and H are monotone functions such that: $H \circ F \sqsubseteq id_A$ and $F \circ H \sqsubseteq id_B$ (Galois connection).

Three facts about adjunctions will be recalled.

<u>Fact 1</u>: The composition of two adjunctions is an adjunction (MacLane [7] Theorem 1 p. 101).

If $\langle F,G,\eta,\varepsilon \rangle : X \rightarrow A$ and $\langle \overline{F},\overline{G},\overline{\eta},\overline{\varepsilon} \rangle : A \rightarrow D$ are two adjunctions then the composite functors yield an adjunction : $\langle \overline{F}F, G\overline{G}, G\overline{\eta}F.\eta, \overline{\varepsilon}.\overline{F}\varepsilon\overline{G} \rangle : X \rightarrow D$. Note that if $\eta = I_X$ and $\overline{\eta} = I_A$ $\quad G\eta F.\overline{\eta} = I_X$.

<u>Fact 2</u>: If (F,G) is a pair of adjoint functors $F:A \rightarrow B$, then F preserves all colimits existing in A and G preserves all limits existing in B. (MacLane [7] Theorem 1 p. 114).

This makes the condition that F be an ω-functor redundant, but the condition that G is an ω-functor is still necessary and not implied by the other conditions. In the category of domains an arrow is a "continuous" co-projector and if $f:A \rightarrow A'$ then A is a "continuous" co-reflective sub-category of A'. The fact that G preserves all existing limits will not be used in the sequel and the author has no intuitive explanation as to why it should be so.

<u>Fact 3</u>: If (F,G) is a pair of adjoint functors then each one of them determines the other up to natural equivalence (MacLane [7] Corollary 1 p. 83).

This is not sufficient for our purpose and we need :

<u>Theorem 1</u> : Let B be a skeletal category and $G:B \to A$ a functor which has a left-adjoint right-inverse then this left-adjoint right-inverse F is uniquely determined by G.

<u>Proof</u> : By fact 3 $F:A \to B$ is determined up to natural equivalence; B being skeletal the effect of F on objects is uniquely determined and MacLane [7] Theorem 2(ii) (p.81) implies that an adjunction is completely determined by its right functor G, the effect of its left functor on objects and its unit. In our case the unit being the identity, the effect of F on arrows is defined by $FGh=h$.

There is then no need to consider the morphism $f:A \to A'$ as being a couple (F,G) and f may be defined to be a G such that a suitable F exists. Contrary to Smyth [15], the right adjoint will be emphasized here, both for the lack of an acceptable term for the left-adjoint (embedding is used with another meaning by MacLane), and because the left-adjoint does not seem to determine uniquely its right-adjoint left-inverse.

<u>Definition 2</u> : A functor $G:B \to A$ is a co-reflector iff it has a left-adjoint right-inverse, that is to say that there is an adjunction $\langle F,G;1_A,\varepsilon\rangle$.

The term ω-co-reflector will be used for such functors which are ω-functors. If A and B are partial orders, $G:B\to A$ is an ω-co-reflector iff it is an ω-continuous projection in the sense of Scott [13] (Definition 3.6). A characterization of ω-co-reflectors shall be proved now :

Theorem 2 : If A and B are categories, G a functor $B\to A$, G is a co-reflector iff to each object $a\in A$ may be associated an object $F_o a\in B$ such that $GF_o a=a$ and for any arrow f in A : $a\to Gb$ there exists a unique arrow $\overline{f}$ in B : $F_o a\to b$ such that $f=G\overline{f}$.

Proof : only if : let F_o be the left-adjoint right-inverse of G; by MacLane [7] Theorem 1 p. 80 $\phi:\overline{f}\mapsto Gf$ is an isomorphism from $B(F_o a,b)$ to $A(a,Gb)$.

if : the sentence "for any arrow .... $f=G\overline{f}$" is equivalent to "the couple $(F_o a,1_a)$ is universal from a to G", in the presence of $GF_o a=a$. By MacLane [7] Theorem 2(ii) p.81 it defines a left-adjoint right-inverse for G.

The category of domains may now be defined :

Definition 3 : The category of domains, *Dom*, is the category which has as objects the domains and as arrows the ω-co-reflectors. In *Dom* a morphism $f:A\to B$ is an ω-co-reflector $G:B\to A$.

There are two natural forgetful functors that one can define from *Dom* to Cat' the category of large categories. The left-forgetful functor (For_L) is a covariant functor that sends an ω-co-reflector G to its left-adjoint right-inverse F and the right-forgetful functor (For_R) is a contravariant functor that sends an ω-co-reflector G to the functor G. The main Theorem will now give the important properties of *Dom*.

Theorem 3 : The category *Dom* is an initial ω-category, and the right-forgetful functor For_R transforms colimits on ω into limits on ω^{op}.

Before we proceed to the proof of Theorem 3 some technical lemmas.

Lemma 1 : If A and B are skeletal initial categories and $G:B\to A$ a co-reflector then $G\perp=\perp$, $G\perp_b=\perp_{Gb}$, and if F is the left-adjoint right-inverse of G, $F\perp=\perp$ and $F\perp_a=\perp_{Fa}$.

Proof : $A(\perp,Gb)\cong B(F\perp,b)$ implies that $F\perp$ is an initial object, B being skeletal $F\perp=\perp$. $F\perp_a$ has to be an arrow : $\perp=F\perp\to Fa$ but there is only one such arrow $\perp_{Fa}$. $G\perp=GF\perp=\perp$ because $G\circ F=I_A$. $G\perp_b$ has to be an arrow from $\perp=G\perp\to Gb$ and there is only one such arrow $\perp_{Gb}$.

Q.E.D.

Lemma 2 : If A and B are categories, $G:B\to A$ a co-reflector with left-adjoint right-inverse F and $\varepsilon : F\circ G\overset{\bullet}{\to}I_B$ the counit of the adjunction, then $G\circ\varepsilon=G$, and $\varepsilon\circ F=F$.

Remark : Following MacLane [7], G denotes both the functor and the natural transformation : $G\overset{\bullet}{\to}G$ consisting of identity arrows; thus $G\circ\varepsilon=G$ is equivalent to : for all objects $b\in B$, $G\varepsilon_b=1_{Gb}$, or $\varepsilon_b=\overline{1}_{Gb}$ in the notations of Theorem 2.

Proof : MacLane [7] Theorem 1 p.80(ii) implies $G=(G\circ\varepsilon).(\eta\circ G)$ and $F=(F\circ\eta).(\varepsilon\circ F)$. Here $\eta=I_A$, $\eta\circ G=G$, and $F\circ\eta=F$.

Q.E.D.

Lemma 3 : If A,B,F,G and ε are as in Lemma 2, and $g:b\to b'$ an arrow in B, then $\overline{Gg}=g\circ\varepsilon_b$

Proof : $G(g\circ\varepsilon_b)=Gg\circ G\varepsilon_b=Gg$ by Lemma 2.

Proof of Theorem 3 :

Let us show first that *Dom* has an initial object : the category 1 with one object (.) and one (identity) arrow. Clearly, 1 is a domain and given any domain A there is a unique functor $G_A:A\to 1$. G_A is an ω-functor. It is left to prove that G_A is a co-reflector. Let $F_A:1\to A$ be the functor that sends the unique object of 1 to the initial object of A : $\perp$ and its only arrow to $1_{\perp_A}$. Given any arrow f in $1(.,Ga)$ there exists a unique arrow $\overline{f}$ in $A(\perp_A,a)$ and $Gf=f=1$. . Clearly $G_A\circ F_A=I_1$. By Theorem 2 G is a co-reflector.

Let us now show that *Dom* is an ω-category. Let $\Phi:\omega\to Dom$.

$$\Phi : \quad A_o \xleftarrow{G_o} A_1 \xleftarrow{G_1} A_2 \longleftarrow \cdots\cdots A_i \xleftarrow{G_i} A_{i+1} \longleftarrow \cdots\cdots$$

Let F_i be the left-adjoint right-inverse of G_i : $\begin{tikzcd} A_i \arrow[r, shift right, "F_i"'] & A_{i+1} \arrow[l, shift right, "G_i"'] \end{tikzcd}$

If, as asserted in the Theorem, For_R transforms colimits into limits then the colimiting cone $\nu:\Phi\overset{\bullet}{\to}A_\infty$ should be the limiting cone in Cat'. Let A_∞ then be the category with objects the infinite sequences : $\langle a_o,\ldots,a_i,\ldots\rangle$ such that for $i\in\mathbb{N}$ $a_i\in A_i$ and $G_i a_{i+1}=a_i$, and with arrows the infinite sequences : $\langle f_o,f_1,\ldots,f_i,\ldots\rangle$ such that for $i\in\mathbb{N}$, $f_i\in A_i$ and $G_i f_{i+1}=f_i$. Let $G_{\infty i}:A_\infty\to A_i$ be the functor that projects on the i^{th} coordinate.

$$\Phi : \quad \begin{tikzcd} A_o & A_1 \arrow[l, "G_o"'] & A_2 \arrow[l, "G_1"'] & \cdots\cdots A_i \arrow[l] & A_{i+1} \arrow[l, "G_i"'] & \cdots\cdots \arrow[l] \\ & & A_\infty \arrow[ull, "G_{\infty o}"] \arrow[ul, "G_{\infty 1}"'] \arrow[u, "G_{\infty 2}"'] \arrow[ur, "G_{\infty i}"'] \arrow[urr, "G_{\infty i+1}"'] & & & \end{tikzcd}$$

The above cone is a limiting cone in Cat' and the theorem asserts that : A_∞ is a domain, $G_{\infty i}$ is an ω-co-projector and given a cone $\mu: B \dot{\to} \Phi$, if B is a domain and μ composed of ω-co-projectors then the unique functor $H:B \to A_\infty$ such that $\mu = \nu \circ H$ is also an ω-co-projector.

Let us prove that A_∞ is a domain. A_∞ is large. A_∞ is skeletal because isomorphisms in A_∞ are sequences of isomorphisms. A_∞ is initial because $\langle \bot_{A_0}, \bot_{A_1}, \ldots, \bot_{A_i}, \ldots \rangle$ is an initial object by Lemma 1. It is easy to see that A_∞ is an ω-category where the colimits are taken coordinatewise (remember that the G_i's preserve colimits).

Let us prove now that the $G_{\infty i}$'s are ω-co-projectors. $G_{\infty i}$ is an ω-functor because the colimits in A_∞ may be computed coordinatewise. If f_i is an arrow in A_i, let $F_{i\infty} f_i = \langle f_0, f_1, \ldots, f_i, f_{i+1}, \ldots \rangle$ where for $j<i$ $f_j = G_j f_{j+1}$ and for $j>i$ $f_j = F_{j-1} f_{j-1}$. Clearly $F_{i\infty}$ is a functor : $A_i \to A_\infty$ ($G_j \circ F_j = I_{A_j}$), and $G_{\infty i} \circ F_{i\infty} = I_{A_i}$. Suppose $f_i : A_i \to G_{\infty i} b$ is an arrow in A_i. Then $b = \langle b_0, \ldots, b_i, \ldots \rangle$ with $b_i = G_{\infty i} b$.
Let $F_{i\infty} a_i = \langle a_0, \ldots, a_i, \ldots \rangle$. Let $g: F_{i\infty} a_i \to b$ be an arrow in A_∞ : $g = \langle g_0, \ldots, g_i, \ldots \rangle$. $G_{\infty i} g = f_i$ iff $g_i = f_i$. Suppose $g_i = f_i$, then for $j<i$ $g = G_j \ldots G_{i-1} f_i$ and for $j>i$ $g_j : a_j \to b_j$ such that $G_{j-1} g_j = g_{j-1}$.
But $a_j = F_{j-1} a_{j-1}$ and by Theorem 2 there is exactly one arrow $\overline{g}$ such that $G_{j-1} \overline{g}_{j-1} = g_{j-1}$. The arrow g defined by :
$g_i = f_i$, for $j<i$ $g_j = G_j \ldots G_{i-1} f_i$ and for $j>i$ $g_j = \overline{g}_{j-1}$ is the only arrow in $A_\infty : F_{i\infty} a_i \to b$, whose i^{th} coordinate is f_i. Theorem 2 now asserts that $G_{\infty i}$ is an ω-co-reflector.

We have shown that ν is a cone in *Dom*, let us show now that it enjoys the universal property. Suppose B is a domain and $\mu:B\dot{\to}\Phi$ a cone with arrows ω-co-projectors; ν being a limiting cone in *Cat'* there exists a unique $H_\infty:B\to A_\infty$ such that $\mu=\nu\circ H_\infty$. This certainly implies that there is at most one ω-co-reflector with this property.

We shall prove that H_∞ is an ω-co-reflector. Let μ be composed of arrows $H_i:B\to A_i$. We know that if f is an arrow in B: $H_\infty f=\langle H_o f, H_1 f,\ldots,H_i f,\ldots\rangle$. H_∞ is an ω-functor because all H_i's are ω-functors and in A_∞ the colimits are coordinatewise. Let L_i be the left-adjoint right-inverse of H_i. $L_i:A_i\to B$. To clarify the situation a lemma will be proved now.

<u>Lemma 4</u>: $H_\infty\circ L_i=F_{i\infty}$ for all $i\in N$.

<u>Proof</u>: Let f_i be an arrow in A_i. $H_\infty(L_i f_i)=\langle H_o L_i f_i,\ldots,H_j L_i f_i,\ldots\rangle$

But $H_k\circ L_k=I_{A_k}$ and $L_k=L_{k+1}\circ F_k$

$(H_\infty\circ L_i)f_i=\langle\ldots\ldots,G_{i-1}f_i,f_i,F_i f_i, F_{i+1}F_i f_i,\ldots\ldots\rangle = F_{i\infty}f_i$.

Q.E.D.

From now on the composition sign ($\circ$) will be omitted whenever possible. Define $K_i:A_\infty\to B$ by $K_i=L_i G_{\infty i}$. Then $K_i=L_{i+1}F_i G_i G_{\infty i+1}$.
The co-unit of the adjunction $(F_i,G_i),\varepsilon_i$ is a natural transformation: $F_i G_i\dot{\to}I_{A_{i+1}}$ and $\tau_i=L_{i+1}\varepsilon_i G_{\infty i+1}$ is a natural transformation: $K_i\dot{\to}K_{i+1}$.

<u>Lemma 5</u>: For $j\geq i$ $H_i K_j=G_{\infty i}$ and $H_i\tau_j=G_{\infty i}$.

<u>Proof</u>: $K_j=L_j G_{\infty j}$, $H_i=G_i\ldots\ldots G_{j-1}H_j$, $H_j L_j=I_{A_j}$ and $G_{\infty i}=G_i\ldots\ldots G_{j-1}G_{\infty j}$.

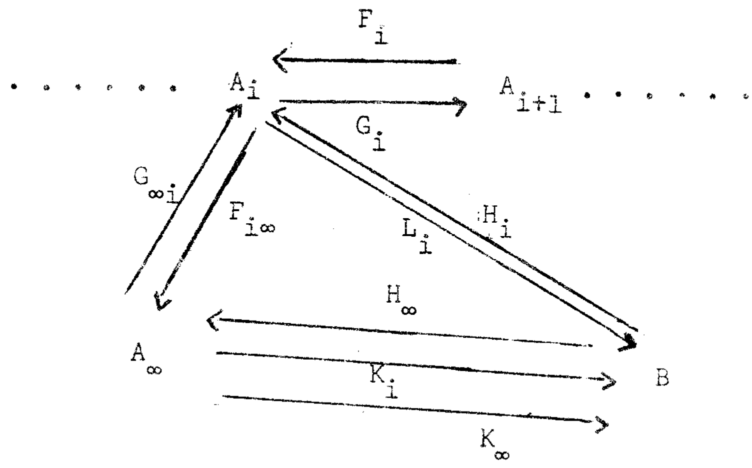
$\tau_i=L_{j+1}\varepsilon_j G_{\infty j+1}$, $H_i=G_i\ldots\ldots G_j H_{j+1}$, $H_{j+1}L_{j+1}=I_{A_{j+1}}$,

$G_j\varepsilon_j=G_j$ by Lemma 2 and $G_{\infty i}=G_i\ldots\ldots G_j G_{\infty j+1}$.

Back to the proof of Theorem 3. By Theorem 2 it is enough for us to define for each object $a\epsilon A_\infty$ an object $K_\infty a\epsilon B$ with the appropriate property.

$$K_0 a \xrightarrow[\tau_0 a]{} K_1 a \xrightarrow[\tau_1 a]{} \cdots\cdots \longrightarrow K_i a \xrightarrow[\tau_i a]{} K_{i+1} a \longrightarrow \cdots\cdots$$

is a functor $\psi_a:\omega\to B$, it has a colimit because B is an ω-category and even if the colimiting cone is not uniquely determined its vertex is uniquely determined because B is skeletal. Let us define $K_\infty a$=colimit ψ_a.



Recapitulating diagram. The diagram does not commute, but the following sub-diagrams commute.

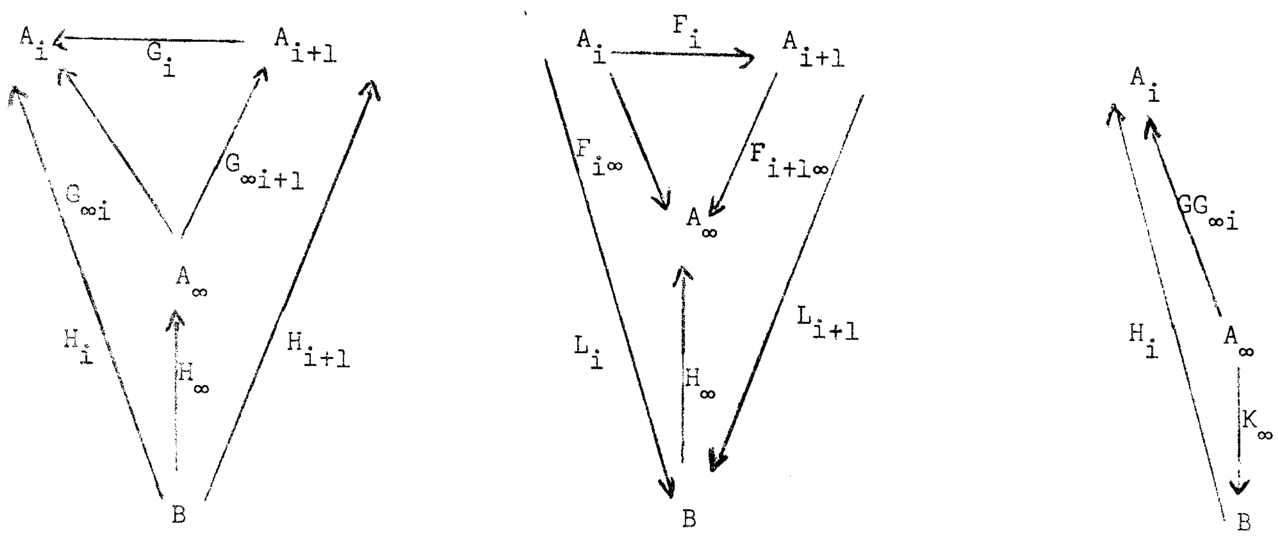


Fig. 1

Lemma 6: $\forall i\in N$, $\forall a\in A_\infty$ $\quad H_i(K_\infty a)=G_{\infty i}a$

Proof: $H_i(K_\infty a)=H_i(\text{colimit }\psi_a)=\text{colimit}(H_i\psi_a)$

because H_i preserves ω-colimits and the categories are skeletal. But by Lemma 5, $H_i\psi_a$ consists of a fixed object $G_{\infty i}a$ and identity arrows after a certain point and its colimit is $G_{\infty i}a$.

Lemma 7: $\forall a\in A_\infty$ $\quad H_\infty K_\infty a=a$

Proof: $H_\infty K_\infty a=\langle H_o K_\infty a,\dots H_i K_\infty a,\dots\rangle=\langle G_{\infty o}a,\dots G_{\infty i}a,\dots\rangle=a$

Before we proceed further let us study the colimiting cone

$$\begin{tikzcd}
K_o a \arrow[r, "\tau_o a"'] \arrow[drr, "\sigma_o a"'] & K_1 a \arrow[r] \arrow[dr, "\sigma_1 a"'] & \dots \arrow[r] & K_i a \arrow[r, "\tau_i a"'] \arrow[dl, "\sigma_i a"] & K_{i+1} a \arrow[r] & \dots \\
 & & K_\infty a & & &
\end{tikzcd}$$

The arrows $\sigma_i a$ are not uniquely determined but we shall use any colimiting cone. The a being fixed we shall drop it from the notation. Clearly $\forall i\in N$ $\sigma_i=\sigma_{i+1}\tau_i$. H_i preserves colimits and:

$$\begin{tikzcd}
H_iK_ia=G_{\infty i}a \arrow[r, "1_{G_{\infty i}a}"'] \arrow[dr, "H_i\sigma_i a"'] & H_iK_{i+1}a=G_{\infty i}a \arrow[r, "1_{G_{\infty i}a}"'] \arrow[d, "H_i\sigma_{i+1}a"] & \dots\dots \\
 & H_iK_\infty a=G_{\infty i}a &
\end{tikzcd}$$

Fig. 2

is a colimiting diagram. This implies $\forall j\geq i$ $H_i\sigma_j=H_i\sigma_i$. Moreover the identity cone being obviously a colimiting cone $H_i\sigma_i$ is an isomorphism: $G_{\infty i}a\to G_{\infty i}a$. As $G_iH_{i+1}\sigma_{i+1}=H_i\sigma_{i+1}=H_i\sigma_i$, $\psi_a=\langle H_o\sigma_o a,\dots,H_i\sigma_i a,\dots\rangle$ is an isomorphism in $A_\infty: a\to a$.

Now to end the proof of Theorem 3, let us show that (K_∞, H_∞) satisfies the universal property used in Theorem 2.

Suppose $f:a\to H_\infty b$ is an arrow in A_∞. Then $G_{\infty i}f:G_{\infty i}a\to H_i b$ is an arrow in A_i.
$H_i\sigma_i a:H_iK_ia\to H_iK_\infty a$ but $H_iK_ia=H_iK_\infty a=G_{\infty i}a$ by Lemmas 5 and 6.
$G_{\infty i}f\circ H_i\sigma_i a:G_{\infty i}a\to H_i b$. L_i is the left-adjoint right-inverse of H_i and by Theorem 2 there exists a unique $h_i:L_iG_{\infty i}a\to b$ such that $H_ih_i=G_{\infty i}f\circ H_i\sigma_i a$.

$$\begin{tikzcd}
L_iG_{\infty i}a=K_ia \arrow[rr, "\tau_i a"'] \arrow[dr, "h_i"'] & & K_{i+1}a \arrow[dl, "h_{i+1}"] \\
& b &
\end{tikzcd}$$

Fig. 3

The diagram of Fig. 3 commutes because $H_i(h_{i+1}\circ\tau_i a)=H_ih_{i+1}\circ H_i\tau_i a=G_iH_{i+1}h_{i+1}\circ 1_{G_{\infty i}a}$ by Lemma 5 $= G_i(G_{\infty i+1}f\circ H_{i+1}\sigma_{i+1}a)=G_{\infty i}f\circ H_i\sigma_{i+1}a=G_{\infty i}f\circ H_i\sigma_i a$ as noticed above. Then $H_i(h_{i+1}\circ\tau_i a)=H_ih_i\Rightarrow h_{i+1}\circ\tau_i a=h_i$ because H_i is faithful. The universal property of the colimiting cone $\sigma_a:\psi_a\to K_\infty a$ implies: $\exists!\ \alpha:K_\infty a\to b$ such that $\forall i\in N\ h_i=\alpha\circ\sigma_i a$. The proof of Theorem 3 is closed by the next two lemmas.

Lemma 8: $H_\infty\alpha=f$

Proof: $\forall i\in N\ h_i=\alpha\circ\sigma_i a\Rightarrow\forall i\in N\ H_ih_i=H_i\alpha\circ H_i\sigma_i a\Rightarrow\forall i\in N\ G_{\infty i}f\circ H_i\sigma_i a=H_i\alpha\circ H_i\sigma_i a$ by construction of h_i. But we noticed above that $H_i\sigma_i a$ is an isomorphism and we have: $\forall i\in N\ G_{\infty i}f=H_i\alpha\Leftrightarrow f=H_\infty\alpha$

Lemma 9: Let $\beta:K_\infty a\to b$ be such that $H_\infty\beta=f$ then $\beta=\alpha$.

Proof: $H_\infty\beta=f \Leftrightarrow \forall i\in N\ H_i\beta=G_{\infty i}f$. Then $H_i(\beta\circ\sigma_i a)=G_{\infty i}f\circ H_i\sigma_i a$ and by definition of $h_i : h_i=\beta\circ\sigma_i a$. But $\forall i\in N\ h_i=\beta\circ\sigma_i a \Rightarrow \beta=\alpha$ by the universal property for α.

Q.E.D.

End of proof of Theorem 3.

Remark: The existence of an initial object in a domain has not been used in the proof of the Theorem and clearly an extension of $\mathcal{D}om$ where objects are not necessarily initial is also an ω-category. Before we conclude this chapter, two lemmas which will explain the proof of Theorem 3.

Lemma 10: If a is an object in A_∞ then the following diagram is a colimiting cone:

$$\begin{tikzcd}
F_{0\infty}G_{\infty 0}a \arrow[r, "(F_{1\infty}\varepsilon_0 G_{\infty 1})a"] \arrow[drr] & F_{1\infty}G_{\infty 1}a \arrow[r] \arrow[dr] & F_{i\infty}G_{\infty i}a \arrow[r, "(F_{i+1\infty}\varepsilon_i G_{\infty i+1})a"] \arrow[d, "\varepsilon_i^\infty a"'] & F_{i+1\infty}G_{\infty i+1}a \arrow[dl] \\
 & & a &
\end{tikzcd}$$

Fig. 4.

where $\varepsilon_i : F_iG_i \dot{\to} I_{A_{i+1}}$ is the co-unit of $\langle F_i, G_i ; I_{A_i}, \varepsilon_i\rangle$

and $\varepsilon_i^\infty : F_{i\infty}G_{\infty i} \dot{\to} I_A$ is the co-unit of $\langle F_{i\infty}, G_{\infty i} ; I_{A_i}, \varepsilon_i^\infty\rangle$

Intuitively Lemma 10 says that a is the colimit of its successive projections.

Proof: The diagram commutes for: $G_{\infty i}\varepsilon_i^\infty a=1_{G_{\infty i}a}$ by Lemma 2 and

$$G_{\infty i}\varepsilon_{i+1}^\infty a\circ G_{\infty i}F_{i+1\infty}\varepsilon_i G_{\infty i+1}a=G_iG_{\infty i+1}\varepsilon_{i+1}^\infty a\circ G_i\varepsilon_i G_{\infty i+1}a$$

$$= G_i 1_{G_{\infty i+1}a}\circ 1_{G_{\infty i}a}=1_{G_{\infty i}a}$$ by Lemma 2.

For the universal property, suppose the following diagram commutes:

$$\begin{tikzcd}
F_{0\infty}G_{\infty 0}a \arrow[r, "F_{1\infty}\varepsilon_0 G_{\infty 1}a"'] \arrow[drr, "h_0"'] & F_{1\infty}G_{\infty 1}a \arrow[r, dotted] \arrow[dr, "h_1"] & F_{i\infty}G_{\infty i}a \arrow[r, "F_{i+1\infty}\varepsilon_i G_{\infty i+1}a"'] \arrow[d, "h_i"] & F_{i+1\infty}G_{\infty i+1}a \\
 & & b &
\end{tikzcd}$$

Fig.5

For unicity suppose $\alpha:a\to b$ such that $\forall i\in N: h_i=\alpha\circ\varepsilon_i^{\infty}a$. Then clearly

$G_{\infty i}h_i=G_{\infty i}\alpha\circ G_{\infty i}\varepsilon_i^{\infty}a=G_{\infty i}\alpha$ and $\alpha=\langle G_{\infty 0}h_0,\dots,G_{\infty i}h_i,\dots\rangle$ (1)

To prove the existence of such an α let α be defined by (1).

$\alpha:a\to b$. To show $\forall i\in N$ $h_i=\alpha\circ\varepsilon_i^{\infty}a$ it is enough to show that

$\forall j\in N$ $\forall i\in N$ $G_{\infty j}h_i=G_{\infty j}\alpha\circ G_{\infty j}\varepsilon_i^{\infty}a$.

For $j<i$ $G_{\infty j}\alpha\circ G_{\infty j}\varepsilon_i^{\infty}a=G_jG_{j+1}\dots G_{\infty i}\alpha\circ G_jG_{j+1}\dots G_{\infty i}\varepsilon_i^{\infty}a$

$= G_jG_{j+1}\dots G_{i-1}G_{\infty i}h_i\circ G_jG_{j+1}\dots G_{i-1}1_{G_{\infty i}a}=G_{j\infty}h_i$

For $j=i$ $G_{\infty i}\alpha\circ G_{\infty i}\varepsilon_i^{\infty}a=G_{\infty i}\alpha\circ 1_{G_{\infty i}a}=G_{\infty i}h_i$

For $j>1$ let us prove our claim by induction $|j-1|$.

$G_{\infty j}h_i=G_{\infty j}h_{i+1}\circ G_{\infty j}F_{i+1\infty}\varepsilon_i G_{\infty i+1}a$ by commutativity of the above diagram (Fig.5)

$=G_{\infty j}\alpha\circ G_{\infty j}\varepsilon_{i+1}^{\infty}a\circ G_{\infty j}F_{i+1\infty}\varepsilon_i G_{\infty i+1}a$ by the induction hypothesis

$=G_{\infty j}\alpha\circ G_{\infty j}(\varepsilon_{i+1}^{\infty}a\circ F_{i+1\infty}\varepsilon_i G_{\infty i+1}a)$

But $G_{\infty i}\varepsilon_{i+1}^{\infty}a\circ G_{\infty i}F_{i+1\infty}\varepsilon_i G_{\infty i+1}a=G_iG_{\infty i+1}\varepsilon_{i+1}^{\infty}a\circ G_i\varepsilon_i G_{\infty i+1}a=G_i1_{G_{\infty i+1}a}\circ 1_{G_{\infty i}a}=1_{G_{\infty i}a}$

which proves $\varepsilon_i^{\infty}a=\varepsilon_{i+1}^{\infty}a\circ F_{i+1\infty}\varepsilon_i G_{\infty i+1}a$.

Q.E.D.

Lemma 11: If $f:a \to a'$ is an arrow in A_∞ then it is the unique arrow implied by the following diagram:

$$\begin{tikzcd}
F_{i\infty}G_{\infty i}a \arrow[r, "F_{i+1}\varepsilon G_{\infty i+1}a"'] \arrow[d, "F_{i\infty}G_{\infty i}f"'] & F_{i+1\infty}G_{\infty i+1}a \arrow[r] \arrow[d, "F_{i+1\infty}G_{\infty i+1}f"] & \cdots & a \arrow[d, dashed, "f"] \\
F_{i\infty}G_{\infty i}a' \arrow[r, "F_{i+1}\varepsilon G_{\infty i+1}a'"'] & F_{i+1\infty}G_{\infty i+1}a' \arrow[r] & \cdots & a'
\end{tikzcd}$$

Fig.6

Proof: The above squares commute because $F_{i+1\infty}\varepsilon G_{\infty i+1}$ is a natural transformation: $F_{i\infty}G_{\infty i} \overset{\bullet}{\to} F_{i+1\infty}G_{\infty i+1}$ and there is a unique $g:a \to a'$ to make the whole diagram commute. But clearly f does for:

$$\begin{tikzcd}
F_{i\infty}G_{\infty i}a \arrow[r, "\varepsilon_i^\infty a"] \arrow[d, "F_{i\infty}G_{\infty i}f"'] & a \arrow[d, "f"] \\
F_{i\infty}G_{\infty i}a' \arrow[r, "\varepsilon_i^\infty a'"'] & a'
\end{tikzcd}$$

commutes because $\varepsilon_i^\infty a$ is a natural transformation: $F_{i\infty}G_{\infty i} \overset{\bullet}{\to} I_A$

Q.E.D.

Some lemmas will be proved now to help showing that certain functors are ω-functors.

Lemma 12: Let A and B be categories and $G:B \to A$ be a co-projector. G is an isomorphism iff its left-adjoint right-inverse F is surjective on objects.

Proof: only if part: If G is an isomorphism then F is its inverse and is surjective on objects.

if part: Suppose F surjective on objects. Let b be an object in B; there is an object a in A such that $b=Fa \Rightarrow Gb=GFa=a$ $b=FGb$

Let $f:b\to b'$ be an arrow in B. $FGf:FGb\to FGb'$ is the unique arrow $\alpha:FGb\to FGb'$ such that $G\alpha=GFGf=Gf$ and $FGf=f$.

Q.E.D.

Lemma 13: Let G be a graph, C a category admitting colimits on G and D a category. $F:D\to C$ preserves G-colimits iff for any $H:G\to D$ with colimiting cone $\nu:H\dot{\to}a$, if $\mu:F\circ H\dot{\to}b$ is the colimiting cone from $F\circ H$ then the unique arrow $\phi:b\to Fa$ such that $F\nu=\phi\circ\mu$ is an isomorphism.

Proof: only if part: Suppose F preserves G-colimits. $F\nu$ is then a colimiting cone. ϕ is the unique arrow between two colimiting cones: it is an isomorphism.

if part: if ϕ is an isomorphism and μ a colimiting cone then $\phi\circ\mu$ is also a colimiting cone.

The next lemma, combining Lemmas 12 and 13 and Theorem 3 will be useful for proving that functors in *Dom* are ω-functors.

Lemma 14: Let D be a category, M a functor: $D\to$*Dom*, $L:\omega\to D$ be a functor with colimit $\nu:L\dot{\to}a$ (in D), $M\nu_i$ be the adjunction $(K_i,H_i;1_{M(L(i))},\delta_i):M(L(i))\to Ma$, and $M(L(i\to i+1))$ be the adjunction $(F_i,G_i;1_{M(L(i))},\varepsilon_i)$: $M(L(i))\to M(L(i+1))$.

With the above notations, M is an ω-functor iff for any object e in Ma (Ma is a domain), e is the colimit vertex of:

$$K_oH_oe \xrightarrow[K_1\circ\varepsilon_o\circ H_{1\,e}]{} K_1H_1e \longrightarrow K_iH_ie \xrightarrow[K_{i+1}\circ\varepsilon_i\circ H_{i+1\,e}]{} K_{i+1}H_{i+1}e \longrightarrow \cdots$$

Fig. 7

<u>Proof</u>: By Lemma 13, Lemma 12 and the definition of K_α on objects during the proof of Theorem 3.

The colimits in *Dom* have the curious property of being inverse limits following the G's and nearly direct limits following the F's. This is the property referred to in the literature as "coincidence of direct and inverse limits". To be totally precise A_∞ is an inverse limit in Cat' by the forgetful functor For_R, but is not quite a direct limit because For_L does not preserve colimits, nevertheless if For_L is restricted to the sub-category of Cat' where the only functors are full, faithful and have a continuous right adjoint then For_L preserves colimits.

A number of interesting sub-categories of *Dom* whose objects are partial orders are closed under ω-colimits and themselves ω-categories.
CPO: the category of complete partial orders is a full sub-category of *Dom*, closed under ω-colimits and so an ω-category, so are CLAT the category of complete lattices, CCP the category of ω-chain continuous posets, SFP the category of partial orders which are ω-colimits of finite partial orders (a small amount of work is needed here), CPOBJ the category of complete partial orders admitting bounded joins. These last results are proved, by a method which specializes ours when the domains are partial orders, by Wand [16] and Plotkin [11].

The above theorem 3 is much more general and its proof shows that adjunctions are essential but that order-enrichment can be dispensed with. Other ω-sub-categories of *Dom* are:

DCC the full sub-category whose objects are co-complete domains,

DBC the full sub-category whose objects are domains admitting finite bounded co-products. CONLAT the full sub-category of continuous lattices can be seen to be an ω-category quite easily using Lemma 10.

Chapter IV

Products and Sums

The usual constructions of products and sums of domains will be presented and it will be shown that they are ω-bi-functors in the category $\mathcal{Dom}$. Some interesting ω-functors will be exhibited.

A word of caution could be helpful here: these are <u>not</u> products and co-products in $\mathcal{Dom}$. Clearly $\mathcal{Dom}$ admits neither of them.

<u>Product of domains</u>

The category Cat' has products and a bi-functor $\pi:\mathrm{Cat}'\times\mathrm{Cat}'\to\mathrm{Cat}'$ may be defined by: $F:A\to A'$, $G:B\to B'$ $\pi(F,G):A\times B\to A'\times B'$ such that $\pi(F,G)\ (a,b)=(Fa,Gb)$.

$$
\begin{tikzcd}
 & A \\
A\times B \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & \\
 & B
\end{tikzcd}
$$

Fig.1

<u>Lemma 1</u>: If A and B are domains, $A\times B$ (their product in Cat') is a domain.

<u>Proof</u>: A product of skeletal categories is skeletal.

A product of initial categories is initial: $(\perp_A,\perp_B)$ is an initial object in $A\times B$.

A product of ω-categories is an ω-category and the colimits are computed componentwise.

One may also notice that the product of small domains is a small domain.

Lemma 2: The projections $p_1:A\times B\to A$ and $p_2:A\times B\to B$ are ω-functors.

Proof: The colimits in $A\times B$ are computed componentwise.

Lemma 3: The projections p_1 and p_2 are ω-co-projectors.

Proof: The left-adjoint right-inverse of p_1 is: $p_1^+:A\to A\times B$ defined by: $p_1^+(f)=\left(f,1_{\perp_B}\right)$. [As a corollary p_i^+ is an ω-functor] .

Lemma 4: If $G_1:B_1\to A_1$ and $G_2:B_2\to A_2$ are ω-co-projectors then $\pi(G_1,G_2):B_1\times B_2\to A_1\times A_2$ is an ω-co-projector.

Proof: $\pi(G_1,G_2)$ preserves ω-colimits in each variable separately and by Lemma 2 of Chapter II it is an ω-functor. If F_1 and F_2 are the left-adjoints right-inverses of G_1 and G_2 $\pi(F_1,F_2):A_1\times A_2\to B_1\times B_2$ is easily seen to be a left-adjoint right-inverse of $\pi(G_1,G_2)$.

Definition 1: $\times:\mathcal{D}om\times\mathcal{D}om\to\mathcal{D}om$ is defined by:

for A and B domains $\times(A,B)=A\times B$

for G_1 and G_2 ω-co-projectors $\times(G_1,G_2)=\pi(G_1,G_2)$.

The infix notation will often be preferred. Clearly $\times$ is not a product in $\mathcal{D}om$: $A\times B$ is not a sub-category of A, but:

Lemma 5: $For_R\circ\times=\pi\circ(For_R,For_R)$

$For_L\circ\times=\pi\circ(For_L,For_L)$

Proof: By Definition 1 and the proof of Lemma 4.

The next theorem enables us to solve domain equations involving $\times$.

Theorem 1: $\times : \mathcal{D}om\times\mathcal{D}om \to \mathcal{D}om$ is an ω-functor.

Proof: By Lemma 14 of Chapter III with $D=\mathcal{D}om\times\mathcal{D}om$ and $M=\times$.

Let the following be a colimiting cone in $\mathcal{D}om\times\mathcal{D}om$:

$$
\begin{tikzcd}
\cdots & (A_i,B_i) \arrow[r, shift left, "{(F_i,N_i)}"] \arrow[dr, shift left, "{(F_{i\infty},N_{i\infty})}"] & (A_{i+1},B_{i+1}) \arrow[l, shift left, "{(G_i,P_i)}"] & \cdots \\
& & (A_\infty,B_\infty) \arrow[ul, shift left, "{(G_{i\infty},P_{\infty i})}"] &
\end{tikzcd}
$$

Fig.2

Then all we have to prove is that, if e is an object in $A_\infty \times B_\infty$ it is the colimit vertex of:

$$
(F_{i\infty}\times N_{i\infty})\,(G_{\infty i}\times P_{\infty i})e \xrightarrow[(F_{i+1\infty}\times N_{i+1\infty})\circ(\varepsilon_i\times\delta_i)\circ(G_{\infty i+1}\times P_{\infty i+1})e]{} (F_{i+1\infty}\times N_{i+1\infty})(G_{\infty i+1}\times P_{\infty i+1})e
$$

where $\varepsilon_i : F_iG_i \dot{\to} I_{A_{i+1}}$ and $\delta_i : N_iP_i \dot{\to} I_{B_{i+1}}$ are the co-units of the adjunctions. The result is obvious because colimits in products are computed componentwise, and by Lemma 10 of Chapter III.

Sums of Domains

Scott [13], defined the sum of two continuous lattices to be their foreign union where the bottom and top elements are identified; it does not seem possible to generalize this notion of a sum and a separated sum will be opted for.

Definition 2: A bi-functor $+ : Cat' \times Cat' \to Cat'$ may be defined the following way:

- if A and B are categories, A+B is their foreign union with a new initial element ($\perp$) and the corresponding arrows ($1_\perp$, $\perp_a$, $\perp_b$ for $a \in A, b \in B$) added
- if $F : A \to A'$ and $G : B \to B'$, then $F+G : A+B \to A'+B'$ preserves $\perp$ and acts as F on A and as G on B.

Remark: In A+B there is no arrow the co-domain of which is $\perp$.

Lemma 6: If A and B are domains, A+B is a domain.

Proof: The sum of two skeletal categories is skeletal.

A+B is always initial.

The sum of two ω-categories is an ω-category because the ω-sequences in A+B are in A or in B after a certain point, except the trivial sequence of initial elements.

One may notice that the sum of small domains is a small domain.

Lemma 7: The injections $i_1 : A \to A+B$ and $i_2 : B \to A+B$ are ω-functors.

Proof: Obvious

Lemma 8: Let us define $j_1 : A+B \to A$ by:

$j_1(a)=a$ for $a \in A$		$j_1(f)=f$ for $f \in A$
$j_1(b)=\perp_A$ for $b \in B$	and	$j_1(g)=1_{\perp_A}$ for $g \in B$
$j_1(\perp_{A+B})=\perp_A$		$j_1(\perp_a)=\perp_a$ for $a \in A$
		$j_1(\perp_b)=1_{\perp_A}$ for $b \in B$
		$j_1\left(1_{\perp_{A+B}}\right)=1_{\perp_A}$,

then j_1 is an ω-functor which is a left-inverse to i_1.

Proof: Check that the definition of j_1 respects the composition of arrows (there are no arrows to $\perp_{A+B}$).

j_1 is an ω-functor because in $A+B$ all sequences are in A or B after a certain point except the trivial sequence on $\perp_{A+B}$.

$j_1 \circ i_1 = I_A$ by construction.

Remark: (i_1, j_1) is not a pair of adjoint functors.

Lemma 9: If $G_1:B_1 \to A_1$ and $G_2:B_2 \to A_2$ are ω-co-projectors then $G_1+G_2:B_1+B_2 \to A_1+A_2$ is an ω-co-projector.

Proof: G_1+G_2 is an ω-functor because G_1 and G_2 are such and a sequence in B_1+B_2 is in B_1 or in B_2 after a certain point except the trivial sequence on $\perp_{B_1+B_2}$.

If $F_1:A_1 \to B_1$ and $F_2:A_2 \to B_2$ are the left-adjoint right-inverses of G_1 and G_2 respectively then $(G_1+G_2)\circ(F_1+F_2)=I_{A_1+A_2}$.

Suppose $f:a \to (G_1+G_2)b$ is an arrow in A_1+A_2. If $a \in A_1$ then $b \in B_1$ and $(G_1+G_2)b = G_1 b$ and there is a unique $\overline{f}:F_1 a = (F_1+F_2)a \to b$ such that $G_1\overline{f} = (G_1+G_2)\overline{f} = f$. Similarly if $a \in B_1$.

If $a=\perp$, $(F_1+F_2)a=\perp$ and the result is obvious.

Theorem 2: $+:\mathit{Dom}\times\mathit{Dom}\to\mathit{Dom}$ is an ω-functor.

Proof: We shall use Lemma 14 of Chapter III, for $D=\mathit{Dom}\times\mathit{Dom}$ and $M=+$.

Let the following be a colimiting cone in $\mathit{Dom}\times\mathit{Dom}$:

$$\begin{tikzcd}
\cdots & (A_i,B_i) \arrow[r, shift left, "{(F_i,N_i)}"] \arrow[dr, shift left, "{(F_{i\infty},N_{i\infty})}"] & (A_{i+1},B_{i+1}) \arrow[l, shift left, "{(G_i,P_i)}"] & \cdots \\
& & (A_\infty,B_\infty) \arrow[ul, shift left, "{(G_{\infty i},P_{\infty i})}"] &
\end{tikzcd}$$

Fig. 3

Then all we have to prove is that, if e is an object in $A_\infty+B_\infty$ it is the colimit vertex of:

$$(F_{i\infty}+N_{i\infty})(G_{\infty i}+P_{\infty i})e \xrightarrow{(F_{i+1\infty}+N_{i+1\infty})\circ(\varepsilon_i+\delta_i)(G_{\infty i+1}+P_{\infty i+1})e} (F_{i+1\infty}+N_{i+1\infty})(G_{\infty i+1}+P_{\infty i+1})e$$

where $\varepsilon_i:F_iG_i\dot{\to}I_{A_{i+1}}$ and $\delta_i:N_iP_i\dot{\to}I_{B_{i+1}}$ are the co-units of the corresponding adjunctions. The result is obvious because:

1) if $e\epsilon A_\infty$ then the whole sequence is in A_∞ and by Lemma 10 of Chapter III e is its colimit in A_∞ and therefore in $A_\infty+B_\infty$.

2) if $e\epsilon B_\infty$ symmetrically

3) if $e=\perp_{A_\infty+B_\infty}$ then $(G_{\infty i+1}+P_{\infty i+1})e=\perp_{A_{i+1}+B_{i+1}}$

$(\varepsilon_i+\delta_i)(G_{\infty i+1}+P_{\infty i+1})e=1_{\perp_{A_{i+1}+B_{i+1}}}$

and $(F_{i+1\infty}+N_{i+1\infty})\circ(\varepsilon_i+\delta_i)\circ(G_{\infty i+1}+P_{\infty i+1})e=1_{\perp_{A_\infty+B_\infty}}$

and the colimit vertex is $\perp_{A_\infty+B_\infty}$.

Q.E.D.

Chapter V

Power domains

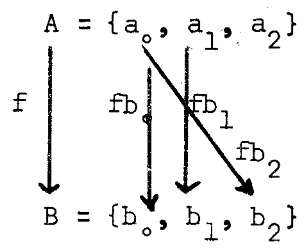
The preceding chapters should have convinced the reader that domains which are ω-categories are no more difficult to handle than complete partial orders or complete lattices, the present chapter will hopefully convince him that ω-categories are the most natural power domains even for partial orders. A power domain $P(D)$ will be defined for every domain D and the construction presented here is thus more general than Plotkin's [11] which is defined only on SFP objects.

Given a domain D, what should an object of $P(D)$ be? Naturally one thinks of sets of objects of D, representing a set of possible values. Unfortunately this is not quite satisfactory. Looking deeper into the problem one may see that the objects of $P(D)$ will represent sets of possible values and morphisms ways by which sets of possible values may arise from each other. Certainly the same value may arise in different ways, possibly in an infinite number of different ways and it is reasonable to suppose that the objects of $P(D)$ should reflect this fact in including possibly a number of copies of the same value, one for each way of obtaining the value. That is why the objects of $P(D)$ are the multi-sets (or sets with repetitions) on D.

One may notice that the power of a small category will not be small and that is the reason why we considered non-small domains. However a simple technical trick could do if one wants only small domains, and this would be a first, quite insignificant, step towards the definition of effectively given domains.

At this stage the purist would perhaps welcome a formal definition of a multi-set, but, to avoid lengthening this already long paper and choosing between equivalent ways of defining multi-sets, such a formal definition will be left to the reader. The intuitive notion of a set with repetitions being clear enough for the sequel.

In $P(D)$ an arrow $f:A \to B$ should express the way the elements of B arise from those of A and, the objects of D being repeated in B as many times as necessary, it is reasonable to ask that f associates with each $b \in B$ a unique arrow of $D: fb:a \to b$ such that $a \in A$.



Example of an arrow in the power domain.

Fig. 1

An extremely important remark is that $f:A \to B$ does not imply that every element of A is the domain of an arrow in f. For example in the preceding example $a_2 \in A$ is not the source of any arrow. The operational interpretation of such a remark is not totally clear: should computations that may erase some of their intermediate results or throw them off be considered or should we accept that not all ω-sequences represent interesting computations? In [1] Robert Floyd argued in favour of programming languages for non-determinism with a failure option on grounds of usefulness and the semantic counter-part of this failure option crops up here unexpectedly as a must. As will be seen further, the category where arrows are restricted to those for which $\forall a \in A\ \exists b\ Dom(fb)=a$ is not an ω-category. A bright point is that the arrows verifying the above condition have a universal characterization, they are the monics, and so $P(D)$ is an ω-category where only the monics have a clear operational meaning, but this is another story.

Definition 1: Let D be a category, $P(D)$ is the following category:

- A is an object of $P(D)$ iff it is a multi-set on D
- $f:A \to B$ is an arrow of $P(D)$ iff f associates with each element b of B a unique arrow (of D) $fb:a \to b$ of domain a, an element of A.
- the composition of arrows is defined by $(g \circ f)b = (gb) \circ (f \text{ dom } gb)$.
- $1_A : A \to A$ is such that $1_A a = 1_a$.

Lemma 1: If D is a skeletal category, $P(D)$ is skeletal.

Proof: Suppose $f:A \to B$ and $g:B \to A$ are such that $g \circ f = 1_A$, $f \circ g = 1_B$.

$\forall b \in B\ 1_b = (f \circ g)(b) = (fb) \circ (g \text{ dom } fb) \Rightarrow (\text{dom } g \text{ dom } fb) = b$

$\forall a \in A\ 1_a = (g \circ f)(a) = (ga) \circ (f \text{ dom } ga) \Rightarrow (\text{dom } f \text{ dom } ga = a)$

In particular
$$1_{\text{dom } fb} = (g \text{ dom } fb) \circ (f \text{ dom } g \text{ dom } fb) = (g \text{ dom } fb) \circ (fb)$$
$$1_{\text{dom } ga} = (f \text{ dom } ga) \circ (g \text{ dom } f \text{ dom } ga) = (f \text{ dom } ga) \circ (ga)$$

Then g dom fb, fb, f dom ga and ga are all isomorphisms and D being skeletal: dom fb=codomain fb=b and dom ga=a. It follows that A and B are multi-sets containing the same elements repeated the same number of times: A=B.

Lemma 2: If D is an initial category, $P(D)$ is an initial category.

Proof: Let $\perp$ be the initial object of D. Clearly $\{\perp\}$ is initial in $P(D)$.

Lemma 3: If D is an ω-category, $P(D)$ is an ω-category.

Proof: Suppose $H:\omega\to P(D)$ is a functor.

$$H: \quad A_o \xrightarrow{f_o} A_1 \xrightarrow{f_1} \cdots \longrightarrow A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

More pictorially:

H:

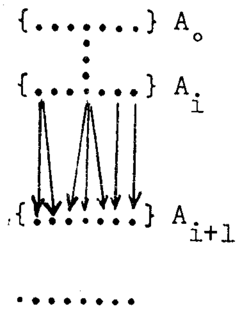


Fig.2

By definition of the arrows in $P(D)$, H may be considered as a set of possibly infinite trees (there are as many trees as elements in A_o) the nodes of which are labelled by objects of D and the edges by arrows of D. In such a tree some branches are finite, others are infinite. The colimit of H will be the multi-set containing the colimits of all the infinite branches. Let A_∞ be the multi-set on D whose cardinality is that of the infinite branches of the forest H and which for each infinite branch, contains a copy of its colimit in D. Let $f_{i\infty}:A_i\to A_\infty$ be the arrow in $P(D)$ which joins every element a of A_∞ to the element b of A_i through which the infinite branch whose colimit is a passes, by the arrow present in the colimiting cone.

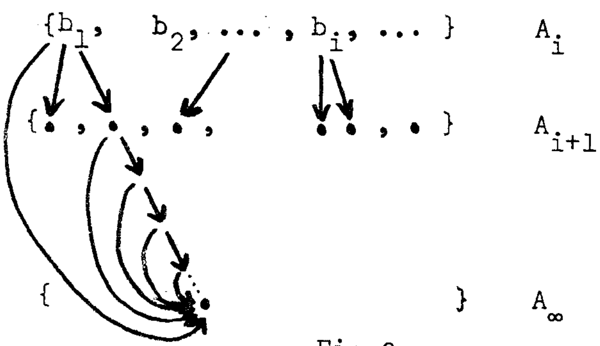


Fig.3

Clearly each element in A_∞ is the co-domain of exactly one arrow of $f_{i\infty}$ and $f_{i\infty}=f_{i+1\infty}\circ f_i$.

Let us prove that $\nu:H\overset{\bullet}{\to}A_\infty$ formed of the $f_{i\infty}$ enjoys the universal property. Suppose $\mu:H\overset{\bullet}{\to}C$ is a cone.

$$\begin{tikzcd}
A_o \arrow[r, "f_o"] \arrow[drr] & A_1 \arrow[r] \arrow[dr] & \cdot\ \cdot & A_i \arrow[r, "f_i"] \arrow[dl, "g_i"'] & A_{i+1} \arrow[r] \arrow[dll] & \cdot\ \cdot\ \cdot \\
 & & C & & &
\end{tikzcd}$$

Fig.4

Let us prove that there is a unique $g_\infty:A_\infty\to C$ such that $g_i=g_\infty\circ f_{i\infty}$. Let c be an element of C. Let $h_i=g_ic$ and $b_i=\mathrm{dom}\ h_i$. $h_i:b_i\to c$, where b_i is an element of A_i. $g_i=g_{i+1}\circ f_i$ implies that $h_i=g_ic=(g_{i+1}c)\circ(f_i\ \mathrm{dom}\ g_{i+1}c)=h_{i+1}\circ f_ib_{i+1}$.

The diagram of Fig. 5 commutes.

$$\begin{tikzcd}
B: & b_o \arrow[r, "f_1b_1"] \arrow[drrr, "h_o"'] & b_1 \arrow[r] \arrow[drr] & \cdots \arrow[r] & b_i \arrow[r, "f_ib_{i+1}"] \arrow[d, "h_i"'] & b_{i+1} \arrow[r] \arrow[dl] & \cdot\ \cdot\ \cdot \\
 & & & & c & &
\end{tikzcd}$$

Fig.5

Let $\alpha:B\overset{\bullet}{\to}a$ be the colimiting cone from B. $f_{i\infty}a=\alpha_i:b_i\to a$. There is a unique arrow $\phi_c:a\to c$ such that $\forall i\ \phi_c\circ\alpha_i=h_i$. But B is an infinite branch in H and by definition of A_∞ , a is an element of A_∞, it is then possible to define $g_\infty:A_\infty\to C$ by $g_\infty c=\phi_c$.

$(g_\infty\circ f_{i\infty})c=(g_\infty c)\circ(f_{i\infty}\ \mathrm{dom}\ g_\infty c)=\phi_c\circ(f_{i\infty}a)=\phi_i\circ\alpha_i=h_i=g_ic$, and $g_\infty\circ f_{i\infty}=g_i$.

For unicity, just notice that $\ell \circ f_{i\infty} = g_i$ implies

$h_i = g_i c = (\ell \circ f_{i\infty})c = (\ell c) \circ (f_{i\infty}\ \mathrm{dom}\ \ell c)$.

Let $e = \mathrm{dom}\ \ell c$.

$$\begin{tikzcd} b_i \arrow[rr, "f_{i\infty}e"] \arrow[dr, "h_i"'] & & e \arrow[dl, "\ell c"] \\ & c & \end{tikzcd}$$

Fig.6

$e \in A_\infty$ and as such is the colimit of an infinite branch in H.

Let E: $e_o \xrightarrow{m_o} e_1 \xrightarrow{m_1} \dots\ e_i \xrightarrow{m_i} e_{i+1} \rightarrow \dots$ be this branch.

$\ell \circ f_{i\infty} = g_i$ implies that the diagram of Fig.7 commutes.

$$\begin{tikzcd} b_i \arrow[rr, "f_{i\infty}e"] \arrow[dr, "h_i = g_i c"'] & & e \arrow[dl, "\ell c"] \\ & c & \end{tikzcd}$$

Fig.7

But $f_{i\infty}e : e_i \to e$ and $b_i = e_i$ $\forall i \in N$ and E=B. Then e=a and $\forall i \in N$ the diagram of Fig. 8 commutes.

$$\begin{tikzcd} b_i \arrow[rr, "f_{i\infty}a"] \arrow[dr, "h_i"'] & & a \arrow[dl, "\ell c"] \\ & c & \end{tikzcd}$$

Fig.8

This implies $\ell c = \phi_c = g_\infty c$ and $\ell = g_\infty$.

Q.E.D.

<u>Theorem 1</u>: If D is a domain, $P(D)$ is a domain.

<u>Proof:</u> By Lemmas 1,2,3.

Note that the colimit object A_∞ is what could be guessed: the sets of colimits following the infinite branches. Note also that if all the arrows f_i in the sequence are Milner-like (for any $a \in A_i$ there is at least one arrow in f_i of domain a) then the colimiting cone ν contains only such arrows (the $f_{i\infty}$) but given a cone $\mu:H\dot{\to}c$ composed of such arrows the unique g_∞ such that $g_\infty \circ \nu = \mu$ is not necessarily Milner-like.

An example of this type should clarify the ideas. Let D be the poset consisting of the natural numbers and infinity ordered by the usual relation (this is the domain considered in Chapter I).

Let $A_i = \{j \mid j \leq i\}$ and $f_i : A_i \to A_{i+1}$ be defined

by: for any $j \leq i+1$ $\quad f_i j = \begin{cases} i & \text{if } j=i+1 \\ j & \text{else} \end{cases}$

Pictorially:

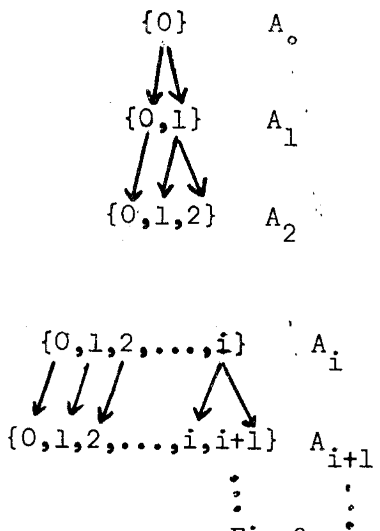


Fig.9

The colimit object A_∞ is the set of all colimits following the infinite branches and $A_\infty = D$. The colimiting cone is such that $f_{i\infty} : A_i \to A_\infty$ defined by dom $f_{i\infty}\ j = \begin{cases} i & \text{if } j \geq i \\ j & \text{else} \end{cases}$

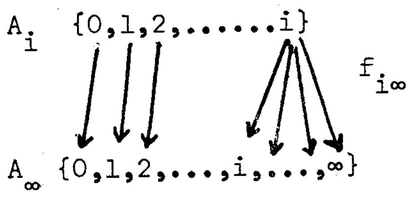


Fig. 10

Let C be $D'=D-\{\infty\}=\{0,1,2,\ldots,i,\ldots\}$

and $g_i: A_i \to C$ be dom $g_i\ j = \begin{cases} i \text{ if } j>i \\ j \text{ else} \end{cases}$

Clearly $g_i = g_{i+1} \circ f_i$

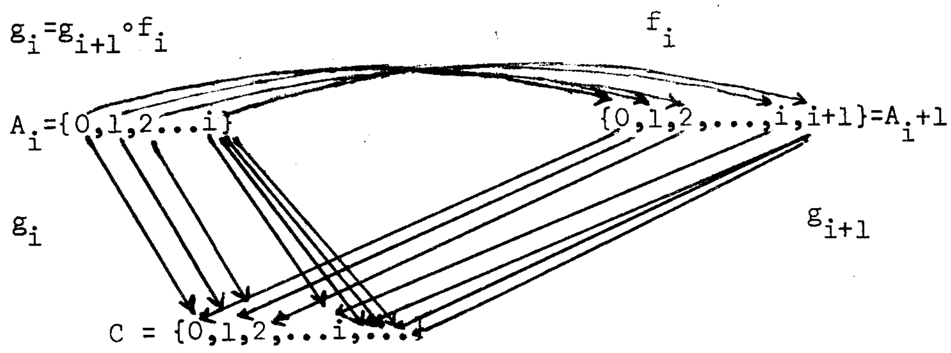


Fig.11

The unique $\phi: A_\infty \to C$ such that $\forall i\ \phi \circ f_{i\infty} = g_i$ is defined by: dom $\phi j = j$, and $\infty \in A_\infty$ is not the domain of any arrow of ϕ.

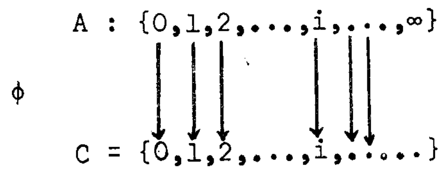


Fig. 12

This exemplifies why arrows which are not Milner-like have to be introduced.

As in Chapter IV for + and ×, P will be made a functor $Dom \to Dom$ and proved an ω-functor, but before getting to that some functors related with P will be proved to be ω-functors and this should help clarify our construction.

<u>Definition 2</u>: The singleton functor $\{\}:D \to P(D)$

is defined by: $\{\}d=\{d\}$ and $\{\}\begin{array}{c} d \\ \downarrow f \\ d' \end{array} = \begin{array}{c} \{d\} \\ \downarrow f \\ \{d'\} \end{array}$

<u>Lemma 4</u>: If D is a category then the singleton functor $\{\}:D \to P(D)$ is an ω-functor.

The proof follows immediately from that of Lemma 3.

<u>Definition 3</u>: The union functor $\cup : P(D) \times P(D) \to P(D)$ is defined by $\cup(A,B)=A \oplus B$ foreign union and

$$f:A \to A' \quad g:B \to B' \quad \cup(f,g)(c)= \begin{cases} fc \text{ if } c \in A' \\ gc \text{ if } c \in B' \end{cases}$$

<u>Remark 1</u>: $\cup$ is commutative and associative but is not idempotent: $A \cup A$ is not equal to A, it is a multi-set containing two copies of A. This certainly is slightly annoying but does not seem to be a serious drawback.

<u>Remark 2</u>: $\cup$ has a universal characterization: it is the product in P(D).

$$\begin{tikzcd} & & A \\ C \arrow[r, dashed, "f \cup g"] \arrow[urr, "f"] \arrow[drr, "g"'] & A \cup B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & & B \end{tikzcd}$$

where i_1 and i_2 are composed only of identity arrows.

Fig. 13

P(D) then has arbitrary small products even when D does not.

Lemma 5: $\cup$ is an ω-functor.

Proof: Obvious from the proof of Lemma 3.

Definition 4: The "big union" functor $\uplus: P(P(D)) \to P(D)$ is defined by:

$$\uplus A = \bigoplus_{a \in A} a \qquad \text{the foreign union of the elements of } A.$$

$$\uplus \begin{array}{c} A \\ \downarrow f \\ B \end{array} = \begin{array}{c} \uplus A \\ \downarrow \uplus f \\ \uplus B \end{array} \quad \text{where} \quad (\uplus f)c = fb \qquad \text{for } c \in b \in B$$

Lemma 6: The "big union" functor is an ω-functor.

Proof: Obvious from the proof of Lemma 3.

Let us now show how domain equations involving P may be solved.

P defines a functor: $\mathrm{Cat}' \to \mathrm{Cat}'$

Definition 5: $P: \mathrm{Cat}' \to \mathrm{Cat}'$ is defined by:

- $P(C)$ is the category defined in Definition 1
- $P \begin{array}{c} C \\ \downarrow F \\ C' \end{array} = \begin{array}{c} P(C) \\ \downarrow \hat{F} \\ P(C) \end{array}$ where $\hat{F}(A) = \{F(a) \mid a \in A\}$

and $\hat{F}\begin{pmatrix} A \\ \downarrow F \\ A' \end{pmatrix} = \begin{array}{c} \hat{F}(A) \\ \downarrow \hat{F}f \\ \hat{F}(A') \end{array}$ defined by $(\hat{F}f)(Fa') = F(fa')$ for $a' \in A'$.

Lemma 7: If $G: B \to A$ is an ω-functor then $\hat{G}: P(B) \to P(A)$ is an ω-functor.

Proof: By the proof of Lemma 3 the colimits in power-domains are "elementwise".

Lemma 8: If $G: C \to B$ and $G': B \to A$ then $\widehat{G' \circ G} = \hat{G}' \circ \hat{G}$.

Proof: Obvious.

<u>Lemma 9</u>: Let $F:A\to B$, $G:B\to A$. If (F,G) is a pair of adjoint functors, then $(\hat{F},\hat{G})$ is a pair of adjoint functors.

<u>Proof</u>: $P(A)(M,\hat{G}N)\simeq P(B)(\hat{F}M,N)$

<u>Lemma 10</u>: If $G:B\to A$ is an ω-co-reflector then $\hat{G}:P(B)\to P(A)$ is an ω-co-reflector.

<u>Proof:</u> $\hat{G}$ is an ω-functor by Lemma 7. Let F be the left-adjoint of G. $\hat{G}\circ\hat{F}=\widehat{G\circ F}=\hat{1}_A=1_{P(A)}$ by Lemma 8. Let $f:C\to\hat{G}(D)$ be an arrow in $P(A)$. Let ϕ be the isomorphism $\phi:A(a,Gb)\to B(Fa,b)$ then $\phi(f)$ is clearly the unique arrow $\bar{f}:\hat{F}(c)\to D$ in $P(B)$ such that $\hat{G}\bar{f}=f$. By Theorem 2 of Chapter III $\hat{G}$ is an ω-co-projector.

<u>Theorem 2</u>: $P_{|Dom}:Dom\to Dom$ is an ω-functor.

<u>Proof</u>: By Theorem 1 and Lemma 10 $P_{|Dom}$ is a functor of the type: $Dom\to Dom$. To show that it is an ω-functor we shall use Lemma 14 of Chapter III, with $D=Dom$ and $M=P_{|Dom}$. We shall abbreviate $P_{|Dom}$ to P.

Suppose that the following is a colimiting cone in Dom:

$$\begin{tikzcd}
A_i \arrow[r, shift left, "F_i"] \arrow[dr, shift left, "F_{i\infty}"] & A_{i+1} \arrow[l, shift left, "G_i"] \\
 & A_\infty \arrow[ul, shift left, "G_{\infty i}"]
\end{tikzcd}$$

Fig.14

and let $\varepsilon_i : F_i G_i \overset{\bullet}{\to} I_{A_{i+1}}$ be the co-unit of the adjunction. Then all we have to prove is that, if e is an object in $P(A_\infty)$ it is the colimit vertex of:

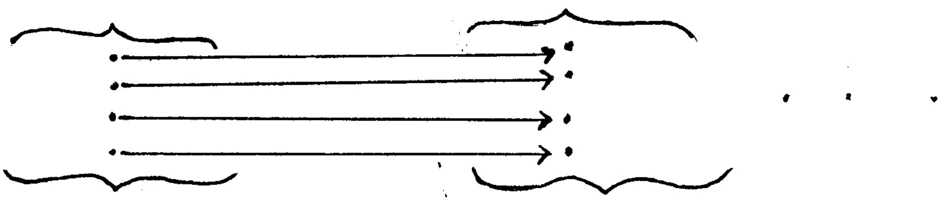
$$\cdots\cdots\; \hat{F}_{i\infty}\hat{G}_{\infty i}e \xrightarrow[(\hat{F}_{i+1\infty}\circ\hat{\varepsilon}_i\circ\hat{G}_{\infty i+1})e]{} \hat{F}_{i+1\infty}\hat{G}_{\infty i+1}e \;\cdots\cdots$$

But e is a multi-set on A_∞: $e=\{a|a\epsilon e\}$ $\hat{F}_{i\infty}\hat{G}_{\infty i}e=\{F_{i\infty}G_{\infty i}a|a\epsilon e\}$; $\hat{F}_{i+1\infty}\hat{G}_{\infty i+1}e=\{F_{i+1\infty}G_{\infty i+1}a|a\epsilon e\}$ and $(\hat{F}_{i+1\infty}\circ\hat{\varepsilon}_i\circ\hat{G}_{\infty i+1})e$ is the arrow h in $P(A_\infty)$ for which $h(F_{i+1\infty}G_{\infty i+1}a)=(F_{i+1\infty}\circ\varepsilon_i\circ G_{\infty i+1})a$. Pictorially:

$$\begin{tikzcd}[column sep=small]
h & \{F_{i\infty}G_{\infty i}a \arrow[d, "(F_{i+1\infty}\circ\varepsilon_i\circ G_{\infty i+1})a"'] , & F_{i\infty}G_{\infty i}a_1 \arrow[d], \ldots\ldots \} \\
 & \{F_{i+1\infty}G_{\infty i+1}a , & F_{i+1\infty}G_{\infty i+1}a_1, \ldots\ldots \}
\end{tikzcd}$$

Fig. 15

The above sequence in A_∞ is a sequence of the form:



where all the multi-sets have the same cardinality and the arrows are parallel. In Theorem 1 we proved that the colimit of such a sequence is the multi-set composed of the colimits following the infinite branches: e.

Q.E.D.

Chapter VI

Functor domains

Domains of functors will be defined, many related functors will be proved to be ω-functors and the construction of functor domain will be proved to be an ω-functor in the category *Dom*.

Definition 1: Given two categories A and B, their functor category $[A \to B]$ is the category whose objects are the ω-functors: $A \to B$, and whose arrows are the natural transformations.

Notice that only ω-functors are taken as objects and that the composition of natural transformations is the "vertical" composition, denoted "." . The "horizontal" composition is denoted $\circ$, or by juxtaposition when the meaning is clear. $[A \to B]$ is a full sub-category of B^A.

Lemma 1: If B is initial then $[A \to B]$ is initial.

Proof: The constant functor which sends every arrow in A to the identity $1_\perp$ on the initial object of B is obviously an ω-functor initial in $[A \to B]$ (it is also initial in B^A).

Lemma 2: If B is an ω-category then $[A \to B]$ is an ω-category.

Proof: B^A is an ω-category where colimits are computed pointwise (see MacLane [7] p.111-112) and the colimit of a sequence of ω-functors is an ω-functor. To see that suppose that the F_i's are ω-functors and that

$$\begin{tikzcd}
F_o \arrow[r, "\tau_o"] \arrow[drr, "\tau_{o\infty}"'] & F_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] & F_i \arrow[r, "\tau_i"] \arrow[dl, "\tau_{i\infty}"] & F_{i+1} \arrow[r] & \cdots \\
 & & F_\infty & & &
\end{tikzcd}$$

Fig. 1

and

$$
\begin{tikzcd}
a_o \arrow[r, "f_o"] \arrow[drr, "f_{o\infty}"'] & a_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] & a_i \arrow[r, "f_i"] \arrow[dl, "f_{i\infty}"] & a_{i+1} \arrow[r] \arrow[dll] & \cdots \\
 & & a_\infty & & &
\end{tikzcd}
$$

Fig.2

are colimiting cones, in B^A and A respectively. Then if we let $b_j^i = F_i a_j$, $\sigma_j^i = \tau_i a_j$, $g_j^i = F_i f_j$ then

$$
\begin{tikzcd}
b_o^o \arrow[r, "\sigma_o^o"'] \arrow[d, "g_o^o"'] & b_o^1 \arrow[r, "\sigma_o^1"'] \arrow[d, "g_o^1"] & b_o^2 \arrow[d] \cdots & b_o^j \arrow[r, "\sigma_o^j"'] \arrow[d, "g_o^j"'] & b_o^{j+1} \arrow[d] \cdots \\
b_1^o \arrow[r, "\sigma_1^o"'] \arrow[d] & b_1^1 \arrow[r, "\sigma_1^1"'] \arrow[d] & b_1^2 \cdots & b_1^j \arrow[r, "\sigma_1^j"'] & b_1^{j+1} \cdots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots & \vdots & \vdots \\
b_i^o \arrow[r, "\sigma_i^o"] \arrow[d, "g_i^o"'] & b_i^1 \arrow[d, "g_i^1"] \cdots & & b_i^j \arrow[r, "\sigma_i^j"] \arrow[d, "g_i^j"'] & b_i^{j+1} \arrow[d, "g_i^{j+1}"] \cdots \\
b_{i+1}^o \arrow[r, "\sigma_{i+1}^o"] & b_{i+1}^1 \cdots & & b_{i+1}^j \arrow[r, "\sigma_{i+1}^j"'] & b_{i+1}^{j+1} \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

7.50
1.65
5.85

Fig. 3

is a two-dimensional infinite diagram. Its colimit may be taken either by rows or by columns. By rows its colimit is that of:

$$
H: \quad F_\infty a_o \xrightarrow[F_\infty f_o]{} F_\infty a_1 \longrightarrow \cdots \quad F_\infty a_i \xrightarrow[F_\infty f_i]{} F_\infty a_{i+1} \longrightarrow \cdots
$$

By columns it is:

$$
\begin{tikzcd}
F_o a_\infty \arrow[r] \arrow[drr, "\tau_o a_\infty"] & F_1 a_\infty \arrow[r] \arrow[dr] & \cdots & F_i a_\infty \arrow[r, "\tau_j a_\infty"'] \arrow[dl, "\tau_{i\infty} a_\infty"] & F_{i+1} a_\infty \arrow[r] \arrow[dll] & \cdots \\
 & & F_\infty a_\infty & & &
\end{tikzcd}
$$

This implies that $F_\infty a_\infty$ is the colimit of H. As for the colimiting cone

$$\begin{tikzcd}
F_\infty a_o \arrow[r, "F_\infty f_o"'] & F_\infty a_1 \arrow[r] & \cdots & F_\infty a_i \arrow[r, "F_\infty f_i"'] \arrow[dl, "\mu_i"] & F_\infty a_{i+1} \arrow[r] & \cdots \\
& & F_\infty a_\infty & & &
\end{tikzcd}$$

μ_i is the unique arrow making the following diagram commute:

$$\begin{tikzcd}
F_o a_i \arrow[r, "\tau_o a_i"'] \arrow[dd, "F_o f_{o\infty}"'] & F_1 a_i \arrow[r] & \cdots & F_j a_i \arrow[r, "\tau_j a_i"'] \arrow[dl, "\tau_{j\infty} a_i"] \arrow[ddl, bend left, "\tau_{j\infty} a_\infty \circ F_j f_{i\infty}"] & F_{j+1} a_i \arrow[r] & \cdots \\
& & F_\infty a_i \arrow[d, dashed, "\mu_i"] & & & \\
F_o a_\infty \arrow[rr, "\tau_{o\infty} a_\infty"'] & & F_\infty a_\infty & & &
\end{tikzcd}$$

Fig. 4

$\mu_i = F_\infty f_{i\infty}$ because $\tau_{j\infty}$ is a natural transformation. This proves that F_∞ is an ω-functor.

Lemma 3: If A and B are large categories then $[A \to B]$ is a large category.

Proof: Obvious.

Theorem 1: If A and B are domains then a skeleton of $[A \to B]$ is a domain.

The proof is trivial. (We admitted that a skeleton of a large category is large).

Even when A and B are skeletal, $[A \to B]$ is not because there could be naturally equivalent functors which are not identical. A skeleton of $[A \to B]$ will be denoted $SK[A \to B]$. Taking $SK[A \to B]$ as a functor space just means that we consider ω-functors only up to natural equivalence and this is reasonable for all semantic purposes. To be precise we shall suppose that $SK[A \to B]$ comes equipped with a specific equivalence of categories $[A \to B] \to SK[A \to B]$: $\langle T, K; \eta, 1 \rangle$ where K is the inclusion. This way any functor to $[A \to B]$ may be interpreted as a functor to $SK[A \to B]$.

The following results will be expressed in terms of whole functor spaces rather than skeletons; their implications in terms of domains are obvious.

Lemma 4: The composition map $\circ : [A \to B] \times [B \to C] \to [A \to C]$ is an ω-functor.

In other words $\circ$ is an object in $[[A \to B] \times [B \to C] \to [A \to C]]$.

Proof: By Lemma 2 of Chapter II it is enough to prove that it is separately an ω-functor. Suppose $\mu : H \dot{\to} F$ is a colimiting cone in $[A \to B]$ and G is an object in $[B \to C]$ then $G\mu : GH \dot{\to} GF$ is a colimiting cone because G is an ω-functor and in $[A \to B]$ the colimits are pointwise. Suppose $\nu : K \dot{\to} G$ is a colimiting cone in $[B \to C]$ and F is an object in $[A \to C]$ then $\nu F : KF \to GF$ is a colimiting cone.

Lemma 5: The evaluation map, $\mathrm{eval} : A \times [A \to B] \to B$ is an ω-functor.

Proof: By Lemma 2 of Chapter II it is enough to prove that it is an ω-functor separately on both arguments.

If f is an object of $[A \to B]$ then $\mathrm{eval}_f = f$ which is an ω-functor.
If a is an object in A then $\mathrm{eval}_a(f) = f(a)$ which preserves ω-colimits because ω-colimits in $[A \to B]$ are pointwise.

__Lemma 6:__ The abstraction map, lambda:[A×B→C]→[A→[B→C]] is an ω-functor.

__Proof:__ The colimits in the functor spaces are pointwise.

__Lemma 7:__ The initial fixpoint map, infix:[A→A]→A is an ω-functor (see Def.9 in Chapter II).

__Proof:__ Let the following be a colimiting cone in [A→A].

$$\begin{tikzcd}
f_0 \arrow[r, "\cdot", "\tau_0"'] & f_1 \arrow[r, "\cdot"] & \cdots & f_i \arrow[r, "\cdot", "\tau_i"'] \arrow[dl, "\tau_{i\infty}"] & f_{i+1} \arrow[r, "\cdot"] & \cdots \\
 & & f_\infty & & &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, bend left, "f_i"] \arrow[r, bend right, "f_{i+1}"'] \arrow[r, phantom, "\Downarrow\tau_i"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left, "f_i"] \arrow[r, bend right, "f_{i+1}"'] \arrow[r, phantom, "\Downarrow\tau_i"] & A
\end{tikzcd}
\quad \text{and} \quad \tau_i^2=\tau_i\circ\tau_i:f_i^2\overset{\cdot}{\to}f_{i+1}^2$$

More generally let for $j\geq 1$ $\tau_i^j=\underbrace{\tau_i\circ\ldots\circ\tau_i}_{j\ \text{times}}:f_i^j\to f_{i+1}^j$ and let $f_i^0=1_A$ and $\tau_i^0=1_A$.

Let us prove that the following diagram commutes for $i\geq 0$ $j\geq 0$

$$\begin{tikzcd}
f_i^j\bot \arrow[r, "\tau_i^j\bot"'] \arrow[d, "f_i^j\bot_{f_i\bot}"'] & f_{i+1}^j\bot \arrow[d, "f_{i+1}^j\bot_{f_{i+1}\bot}"] \\
f_i^{j+1}\bot \arrow[r, "\tau_i^{j+1}\bot"'] & f_{i+1}^{j+1}\bot
\end{tikzcd}$$

Fig. 5

$\tau_i^{j+1}=\tau_i^j\circ\tau_i$ and $\tau_i^{j+1}\bot=\tau_i^j f_{i+1}\bot\circ f_i^j\tau_i\bot$

and $\tau_i^{j+1}\bot\circ f_i^j\bot_{f_i\bot}=\tau_i^j f_{i+1}\bot\circ f_i^j\left(\tau_i\bot\circ\bot_{f_i\bot}\right)$

$=\tau_i^j f_{i+1}\bot\circ f_i^j\bot_{f_{i+1}\bot}$

$=f_{i+1}^j\bot_{f_{i+1}\bot}\ \tau_i^j\bot$ because τ_i^j is a natural transformation.

By using methods already used in the proof of Lemma 1 one can show that:

$$
\begin{tikzcd}
f^j_0\perp \arrow[r, "\tau^j_0\perp"'] & f^j_1\perp \arrow[r] & \cdots & f^j_i\perp \arrow[r, "\tau^j_i\perp"'] \arrow[dll, "\tau^j_{i\infty}\perp"] & f^j_{i+1}\perp \arrow[r] & \cdots \\
 & f^j_\infty\perp & & & &
\end{tikzcd}
$$

is a colimiting cone and that the colimit of the two-dimensional infinite diagram obtained from Diag. 1 by taking $i\geq 0$, $j\geq 0$ is (by rows) infix f_∞ and (by columns) the colimit of:

$$
\text{infix } f_0 \xrightarrow{\text{infix } \tau_0} \text{infix } f_1 \longrightarrow \cdots \text{infix } f_i \xrightarrow{\text{infix } \tau_i} \text{infix } f_{i+1} \longrightarrow \cdots
$$

Q.E.D.

We may now get to the final point of this work and the reason why the definition of morphisms in *Dom* has to involve pairs of functors. $\rightarrow$ will be defined as a bi-functor in *Dom* and it will be proved to be an ω-functor.

Before we proceed to the definition of $\rightarrow$ as a bi-functor in *Dom* some notations. A,A',B,B' are domains. $G:B\rightarrow A$ and $G':B'\rightarrow A'$ are ω-co-projectors whose respective adjunctions are $<F,G;1_A,\varepsilon>$ and $<F',G';1_{A'},\varepsilon'>$. τ varies over the arrows of $SK[B\rightarrow B']$, σ over arrows of $SK[A\rightarrow A']$ $K:SK[B\rightarrow B']\rightarrow[B\rightarrow B']$ is the injection and the corresponding adjoint equivalence: $[B\rightarrow B']\rightarrow SK[B\rightarrow B']$ is $<T,K;\eta,1>$. Similarly $I:SK[A\rightarrow A']\rightarrow[A\rightarrow A']$ is the injection and the adjoint equivalence: $[A\rightarrow A']\rightarrow SK[A\rightarrow A']$ is: $<S,L,;\theta,1>$.

<u>Definition 2</u>: $\rightarrow$: *Dom*$\times$*Dom*$\rightarrow$*Dom* is defined by:

$\rightarrow(A,A')$ is $SK[A\rightarrow A']$

$\rightarrow(G,G')$ is $\lambda\tau.S(G'\circ K\tau\circ F)$.

Let us check the correctness of the definition. By Theorem 1 SK[A→A'] is a domain. If τ is an arrow in SK[B→B'] $K\tau$ is an arrow in [B→B'], a natural transformation in [B→B'].

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\downarrow F"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\downarrow K\tau"] & B' \arrow[r, bend left, "G'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow G'"] & A' \end{tikzcd}$$

G' and F being ω-functors. $G'\circ K\tau\circ F$ is an arrow in [A→A'] and $S(G'\circ K\tau\circ F)$ is an arrow in SK[A→A']. The following lemma will ensure correctness for the definition.

<u>Lemma 8</u>: $\hat{G}:\lambda\tau.S(G'\circ K\tau\circ F)$ is a ω-co-projector whose left-adjoint right-inverse is $\hat{F}:\lambda\sigma.T(F'\circ L\sigma\circ G)$.

<u>Proof</u>: We shall use Theorem 2 of Chapter III. $\hat{G}$ is an ω-functor by Lemma 3 and because K and S are ω-functors.

Suppose h is an object of SK[A→A']

$\hat{G}\hat{F}h=S(G'\circ K(T(F'\circ Lh\circ G))\circ F)$

$\hat{G}\hat{F}h\cong G'\circ F\circ h\circ G\circ F=h$

SK[A→A'] is skeletal and $\hat{G}\hat{F}h=h$

Suppose now that $\sigma:h\dot{\to}S(G'\circ K\ell\circ F)$ is an arrow in SK[A→A'];

ℓ is an object in SK[B→B'].

We have to show that there is a unique $\bar{\sigma}:T(F'\circ Lh\circ G)\dot{\to}\ell$ in SK[B→B']

such that $\sigma=S(G'\circ K\bar{\sigma}\circ F)$.

<u>Unicity</u>: Suppose $\bar{\sigma}_1$ and $\bar{\sigma}_2$ both satisfy the above conditions. Then $\sigma=S(G'\circ K\bar{\sigma}_1\circ F)=S(G'\circ K\bar{\sigma}_2\circ F)$ but $G'\circ K\bar{\sigma}_1\circ F$ and $G'\circ K\bar{\sigma}_2\circ F$ are natural transformations between the same functors and S is faithful (see MacLane [7] Theorem 1 p.91) and $G'\circ K\bar{\sigma}_1\circ F=G'\circ K\bar{\sigma}_2\circ F$. (1)

Let b be an object in B.

$K\bar{\sigma}_1 b$ is an arrow in B': $[T(F'\circ Lh\circ G)]b\to\ell b$ but $T(F'\circ Lh\circ G)\cong F'\circ Lh\circ G$ and $[T(F'\circ Lh\circ G)]b\sim(F'\circ Lh\circ G)b$ and B' being skeletal: $K\bar{\sigma}_1 b:(F'\circ Lh\circ G)b\to\ell b$.

Similarly for $\bar{\sigma}_2$.

(1) implies, for any object a in A: $G'(K\bar{\sigma}_1(Fa))=G'(K\bar{\sigma}_2(Fa))=\alpha$.

(F',G') being a pair of adjoint functors there is a unique $\bar{\alpha}:(F'\circ Lh\circ G)(Fa)\to \ell Fa$ such that $G'\bar{\alpha}=\alpha$, then $K\bar{\sigma}_1(Fa)=K\bar{\sigma}_2(Fa)$ and $K\bar{\sigma}_1\circ F=K\bar{\sigma}_2\circ F$. $K\bar{\sigma}_1$ is a natural transformation in $[B\to B']$ and the following diagram commutes:

$$\begin{tikzcd}
(F'\circ Lh\circ G)b \arrow[d, "K\bar{\sigma}_1 b"'] & & (F'\circ Lh\circ G)\ (FGb) \arrow[ll, "(F'\circ Lh\circ G)\varepsilon_b=1_{(F'\circ Lh\circ G)b}"'] \arrow[d, "(K\bar{\sigma}_1)(FGb)"] \\
\ell b & & \ell FGb \arrow[ll, "\ell\varepsilon_b"]
\end{tikzcd}$$

(Lemma 2, Chapter III)

Fig. 6

and similarly for $\bar{\sigma}_2$. But $(K\bar{\sigma}_1)(FGb)=(K\bar{\sigma}_1\circ F)(Gb)=(K\bar{\sigma}_2\circ F)(Gb)$ and $K\bar{\sigma}_1 b=K\bar{\sigma}_2 b$. This is true for all b and $K\bar{\sigma}_1=K\bar{\sigma}_2$ which implies $\bar{\sigma}_1=\bar{\sigma}_2$ because K is the injection.

Existence: As a matter of convenience let:

$p=G'\circ KT(F'\circ Lh\circ G)\circ F$; p is an object in $[A\to A']$

Let $\alpha:p\dot{\to}G'\circ K\ell\circ F$ be an arrow in $[A\to A']$which will be precised later.

$$\begin{tikzcd}
B \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow G"] & A \arrow[r, bend left, "p"] \arrow[r, bend right, "G'\circ K\ell\circ F"'] \arrow[r, phantom, "\downarrow\alpha"] & A' \arrow[r, bend left, "F'"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\downarrow F'"] & B'
\end{tikzcd}$$

$F'\circ\alpha\circ G : F'\circ p\circ G \dot{\to} F'\circ G'\circ K\ell\circ F\circ G$

$$\begin{tikzcd}
B \arrow[r, bend left, "F\circ G"] \arrow[r, bend right, "I_B"'] \arrow[r, phantom, "\downarrow\varepsilon"] & B \arrow[r, bend left, "K\ell"] \arrow[r, bend right, "K\ell"'] \arrow[r, phantom, "\downarrow K\ell"] & B' \arrow[r, bend left, "F'\circ G'"] \arrow[r, bend right, "I_{B'}"'] \arrow[r, phantom, "\downarrow\varepsilon'"] & B'
\end{tikzcd}$$

$\varepsilon'\circ K\ell\circ\varepsilon : F'\circ G'\circ K\ell\circ F\circ G \dot{\to} K\ell$

Let $\bar{\sigma}:T(\varepsilon'\circ K\ell\circ\varepsilon) \, . \, T(F'\circ\alpha\circ G) \, . \, \bar{\sigma}:T(F'\circ p\circ G)\dot{\to}TK\ell=\ell$ is an arrow in $SK[B\to B']$

But $KT(F'\circ p\circ G)\simeq F'\circ G'\circ F'\circ Lh\circ G\circ F\circ G=F'\circ Lh\circ G$ and $SK[B\to B']$ being skeletal

$T(F'\circ p\circ G)=T(F'\circ Lh\circ G)$. $\quad\bar{\sigma}:T(F'\circ Lh\circ G)\dot{\to}\ell$ as wished.

$K\bar{\sigma}=KT(\varepsilon'\circ K\ell\circ\varepsilon) \, . \, KT(F'\circ\alpha\circ G)=\eta_{K\ell}.(\varepsilon'\circ K\ell\circ\varepsilon) \, . \, (F'\circ\alpha\circ G) \, . \, \eta^{-1}_{F'\circ p\circ G}$

$$G'\circ K\bar{\sigma}\circ F=(G'\circ\eta_{K\ell}\circ F) \, . \, (G'\circ\varepsilon'\circ K\ell\circ\varepsilon\circ F) \, . \, (G'\circ F'\circ\alpha\circ G\circ F) \, . \, (G'\circ\eta^{-1}_{F'\circ p\circ G}\circ F)$$

$$=(G'\circ\eta_{K\ell}\circ F) \, . \, \alpha \, . \, (G'\circ\eta^{-1}_{F'\circ p\circ G}\circ F)$$

Because $G'\circ\varepsilon'=G'$ and $\varepsilon\circ F=F$ by lemma 2 Chapter III.

$\eta:I_{[B\to B']}\dot{\to}KT$ is a natural equivalence and $\eta_{K\ell}$ and $\eta^{-1}_{F'\circ p\circ G}$ are natural equivalences in $[B\to B']$, $(G'\circ\eta_{K\ell}\circ F)$ and $(G'\circ\eta^{-1}_{F'\circ p\circ G}\circ F)$ are then natural equivalences in $[A\to A']$.

$(G'\circ\eta_{K\ell}\circ F) \, : \, G'\circ K\ell\circ F\dot{\to}G'\circ K\ell\circ F$

$(G'\circ\eta^{-1}_{F'\circ p\circ G}\circ F) \, : \, G'\circ KT(F'\circ p\circ G)\circ F=G'\circ KT(F'\circ Lh\circ G)=p\dot{\to}p$

Until now no assumption was made on α. Let us define α:

$\alpha=(G'\circ\eta_{K\ell}\circ F)^{-1} \, . \, \beta \, . \, (G'\circ\eta^{-1}_{F'\circ p\circ G}\circ F)^{-1}$ for some $\beta:p\dot{\to}G'\circ K\ell\circ F$.

We have $G'\circ K\bar{\sigma}\circ F=\beta$.

$\theta:I_{[A\to A']}\dot{\to}LS$ is a natural equivalence and $LS(G'\circ K\bar{\sigma}\circ F)=LS\beta=\theta_{G'\circ K\ell\circ F}.\beta.\theta^{-1}_{p}$

$\theta_p:LSp\dot{\to}p$, $\theta_{G'\circ K\ell\circ F} \, : \, G'\circ K\ell\circ F\dot{\to}LS(G'\circ K\ell\circ F)$

But $LSp\simeq G'\circ F'\circ Lh\circ G\circ F=Lh$ and $Sp=h$. $\quad\theta_p \, : \, p\dot{\to}Lh$

If we define β by: $\beta=\theta^{-1}_{G'\circ K\ell\circ F}.L\sigma.\theta_p$ we get $LS(G'\circ K\bar{\sigma}\circ F)=L\sigma$ which implies $S(G'\circ K\bar{\sigma}\circ F)=\sigma$.

Q.E.D.

<u>Theorem 2</u>: $\to:\mathcal{D}om\times\mathcal{D}om\to\mathcal{D}om$ is an ω-functor.

<u>Proof</u>: We shall use Lemma 14 of Chapter III, with $D=\mathcal{D}om\times\mathcal{D}om$ and $M=\to$.

Let the following be a colimiting cone in $\mathcal{D}om\times\mathcal{D}om$:

$$\begin{tikzcd}
\cdots & (A_i,B_i) \arrow[r, shift left, "{(F_i,N_i)}"] \arrow[d, shift left, "{(F_{i\infty},N_{i\infty})}"] & (A_{i+1},B_{i+1}) \arrow[l, shift left, "{(G_i,P_i)}"] & \cdots \\
& (A_\infty,B_\infty) \arrow[u, shift left, "{(G_{\infty i},P_{\infty i})}"] & &
\end{tikzcd}$$

and let $\varepsilon_i : F_iG_i \overset{\bullet}{\to} I_{A_{i+1}}$ and $\delta_i : N_iP_i \overset{\bullet}{\to} I_{B_{i+1}}$ be the corresponding co-units of adjunctions. It is then enough to prove that if e is an object in $[A_\infty\to B_\infty]$ it is the colimit vertex of:

$$\cdots \quad N_{i\infty}\circ P_{\infty i}\circ e\circ F_{i\infty}\circ G_{\infty i} \xrightarrow[\alpha_i]{\bullet} N_{i+1\infty}\circ P_{\infty i+1}\circ e\circ F_{i+1\infty}\circ G_{\infty i+1} \quad \cdots$$

with $\alpha_i = N_{i+1\infty}\circ\varepsilon'_i\circ P_{\infty i+1}\circ e\circ F_{i+1\infty}\circ\varepsilon_i\circ G_{\infty i+1}$. The reader should check that α_i is the arrow appearing in the text of Lemma 14 and that skeletons and adjoint equivalences may be freely ignored. (The colimit in the skeleton will be the object isomorphic to the colimit in $[A_\infty\to B_\infty]$. But by Lemma 11 of Chapter III:

$$\cdots \quad F_{i\infty}\circ G_{\infty i} \xrightarrow[F_{i+1\infty}\circ\varepsilon_i\circ G_{\infty i+1}]{\bullet} F_{i+1\infty}\circ G_{\infty i+1} \quad \cdots$$

has, as colimit vertex in $[A_\infty\to A_\infty]$ the identity $I_{[A_\infty\to A_\infty]}$ and:

$$\cdots \quad N_{i\infty}\circ P_{\infty i} \xrightarrow[N_{i+1\infty}\circ\varepsilon'_i\circ P_{\infty i+1}]{\bullet} N_{i+1\infty}P_{\infty i+1} \quad \cdots$$

has $I_{[B_\infty\to B_\infty]}$ as colimit. By Lemma 3 the composition $\circ$ is an ω-functor and the above sequence has colimit: $I_{[B_\infty\to B_\infty]}\circ e\circ I_{[A_\infty\to A_\infty]}=e$.

Q.E.D.

Chapter VII

Remarks and Conclusion

Small or large categories

To avoid unnecessary technicalities domains have been defined to be large categories; it is not difficult to restrict the constructions to small categories. The only non-trivial point is the functor P. There the simplest way of cutting down the number of objects is to limit $P(D)$ to those multi-sets of cardinality less or equal to the continuum: C. Clearly the colimit of a sequence of such sets has still cardinality<C, the corresponding tree having at most $C^{\aleph_0}=C$ branches.

Finite domains

Unfortunately there does not seem to be a way of making the power of a finite domain finite and this keeps us from generalizing Plotkin's SFP objects, those domains which are colimits of sequences of finite domains.

Further research

It would be interesting to know how colimits in $\mathcal{D}$om may be used to construct new models for the λ-calculus or other structures verifying interesting equations. The other direction of research which is obviously open is to develop a theory of computation on these generalized domains defining effectively given domains and computable objects. Towards that goal it would be useful to have a good notion of a basis for a domain and also to study the structure of $\mathcal{D}$om itself. The relation between the functor P in $\mathcal{D}$om and powers in topoi is certainly worth investigating. Further restrictions on domains (categorical properties preserved by colimits, $+,\times,\to$ and P) or on arrows should probably be introduced and the definition of P perhaps modified. A different area of study could be to look for rules to prove correctness of non-deterministic programs.

Conclusion

Many questions are left unanswered but the author hopes he has shown that a natural and precise semantics for non-deterministic programs is possible and that the notion of continuity which is essential in mathematical semantics and theory of computation should be defined and studied categorically and not topologically.

Bibliography

[1] R.W. Floyd.
Non-deterministic algorithms. JACM Vol.14, No.4 October 1969.

[2] P. Freyd.
Abelian Categories. Harper & Row 1964.

[3] J.A. Goguen, J.W. Thatcher, E.G. Wagner, J.B. Wright.
Initial algebra semantics and continuous algebras.
Computer Science Report RC 5701. March 1975.
I.B.M. Thomas J. Watson Research Center.

[4] M. Hyland.
Private communication.

[5] D.M. Kan.
Adjoint functors. Trans.Amer.Math.Soc. 87 p.294-329 (1958).

[6] S.C. Kleene.
Recursive functionals and quantifiers of finite types.
I Trans.Amer.Math.Soc. 91 (1959) 1-52.
II " " " " 108 (1963) 106-142.

[7] S. Mac Lane.
Categories for the working mathematician. Springer Verlag 1971.

[8] Z. Manna and A. Shamir.
The optimal fixedpoint of recursive programs in Proc. of 7th Annual ACM Symposium on the Theory of Computation.
Alburquerque 1975.

[9] G. Markowsky and B. Rosen.
Bases for chain-complete posets. I.B.M. Thomas J. Watson Research Center RC 5363. April 1975.

[10] R. Milner.
An approach to the semantics of parallel programs in Proc. Convegno di Informatica Teorica, I.E.I., Pisa. March 1973.

[11] G.D. Plotkin.
A power-domain construction. D.A.I. Research Report No. 3, University of Edinburgh, Dept. of A.I. April 1975.

[12] J. Reynolds.
Notes on a lattice-theoretic approach to the theory of computation. Syracuse University 1971.

[13] D.S. Scott.
Continuous lattices, in Toposes, Algebraic Geometry and Logic. Springer Verlag Lecture Notes 274. 1971.

[14] A. Shamir.
Ph.D. thesis Weizman Institute, Rehovoth, Israel, 1976.

[15] M.B. Smyth.
Power-domains. Theory of Computation Report No. 12,
Department of Computer Science, University of Warwick, May 1976.
(see also Mathematical Foundations of Computer Science 1976,
Lecture Notes in Computer Science No. 45 p.537).

[16] M. Wand.
Fixed-point constructions in order-enriched categories.
Technical Report No. 23, Computer Science Dept.,
Indiana University. April 1976.