Phase transitions and the random-cluster representation for Delaunay Potts models with geometry-dependent interactions

by

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Declaration

The work presented here is my own, except where specifically stated otherwise, and was performed in the Department of Mathematics at the University of Warwick under the supervision of Dr Stefan Adams during the period October 2009 to September 2013. I confirm that this thesis has not been submitted for a degree at another university.

Abstract

We investigate the existence of phase transitions for a class of continuum multi-type particle systems. The interactions act on hyperedges between the particles, allowing us to define a class of models with geometry-dependent interactions. We establish the existence of stationary Gibbsian point processes for this class of models. A phase transition is defined with respect to the existence of multiple Gibbs measures, and we establish the existence of phase transitions in our models by proving that multiple Gibbs measures exist.

Our approach involves introducing a random-cluster representation for continuum particle systems with geometry-dependent interactions. We then argue that percolation in the random-cluster model corresponds to the existence of a phase transition. The originality in this research is defining a random-cluster representation for continuum models with hyperedge interactions, and applying this representation in order to show the existence of a phase transition.

We mainly focus on models where the interaction is defined in terms of the Delaunay hypergraph. We find that phase transitions exist for a class of models where the interaction between particles is via Delaunay edges or Delaunay triangles.

1 Introduction

Equilibrium statistical mechanics aims to describe the behaviour of thermodynamic systems in equilibrium. The approach is to analyse the macroscopic properties of systems, based on the microscopic interactions between individual elements. The interaction between these elements determines how the system behaves on a larger scale. The main aim of the present study is to investigate models that describe continuum systems where particles interact with one another geometrically.

The present study focuses on systems modelled by a continuum point process, a type of random process for which a given realisation consists of a set of points in d-dimensional continuous space, for $d \geq 2$. Specifically, since we are dealing with marked particle systems, we analyse marked point processes. This means that each member of the point distribution is assigned a random mark (or type) from a finite mark space, which gives the random process two levels of randomness: the positions and marks of the particles. The marks and positions are not independent. We investigate particle systems where the position and mark of any given particle can depend on the positions and marks of other particles in the system.

A phase transition occurs in a thermodynamic system when the system transforms from one equilibrium state to another. For example, consider the magnetisation of a ferromagnetic material, such as cobalt or iron. The Ising model can be used to describe this material as follows. A square lattice of fixed sites corresponds to the positions of the atoms of the material. We can then assign these sites marks, -1 or $+1$, corresponding to the magnetic moments. The particle interaction is defined such that particles have a higher probability of having a neighbour with the same mark. This results in configurations where there is a tendency for adjacent particles to align their marks in parallel. If the temperature of the system is below a certain critical value, the interaction is strong enough to result in the domination of one mark over the lattice. This corresponds to magnetisation, a phenomenon which occurs spontaneously as the temperature crosses the critical threshold. This is a rather simple example, where particles are located at the fixed sites of a lattice and only interact in pairs. The aim of the present study is to investigate phase transitional behaviour of continuum models where marked particles interact with one another via the geometry of the configuration. This is a key feature of the results, the particles are not restricted to interacting in pairs or on a lattice. We consider Potts models, an extension of the Ising model, allowing particles to be assigned any mark from a finite mark space. In particular, we analyse Potts models where the interaction between particles occurs on the Delaunay triangulation, which is one way to describe nearest-neighbour interactions in continuum models. We refer to such models as Delaunay Potts models. We find that phase transitions exist in a class of Delaunay Potts models.

There are different ways to mathematically define a phase transition. For example, one can analyse the singularity of the thermodynamic functions. These functions are determined by a finite number of thermodynamic parameters, which parametrise the equilibrium states of the system. When a ferromagnetic system undergoes a phase transition, the net magnetisation jumps between zero and some non-zero value. The net magnetisation is a thermodynamic function determined by parameters, such as the temperature. However, in the present study, we investigate phase transitions from a probabilistic point of view, analysing point processes and Gibbs measures. A Gibbs measure is a distribution of a countably infinite family of random variables which admit some prescribed conditional probabilities. For a given interaction model, we define a phase transition as the existence of multiple Gibbs measures in the model. Taking this approach gives us two main aims. The first is to investigate the existence of Gibbs measures for our class of models, i.e. for a given type interaction model, does at least one Gibbs measure exist? The second aim is to then determine, given the existence of at least one Gibbs measure for the model, do multiple Gibbs measures exist? These two questions are the key to achieving our results.

Percolation theory is useful for the investigation of phase transitional behaviour in type interaction models. For example, in the Ising model, one may compare a percolating network of particles with matching microscopic states to the system being in a macroscopic state of magnetisation. The random-cluster model is our tool for applying percolation theory to the analysis of multiple Gibbs measures. It is via the random-cluster model that the characterisation of phases is described in percolation terms. A novel feature of the present study is the definition of a multi-body continuum random-cluster model that can be used to analyse continuum models with geometry-dependent interaction.

One reason that continuum models are useful is that natural patterns and structures tend to have a level of randomness and are usually not perfect lattices. For example, take a triangular lattice that could be used to model the surface of a crystal. In reality, a perfect lattice is not observed. In fact it is more accurate to model the surface by allowing the vertices of the lattice to take small perturbations in space, which gives a slightly more random graph. In order to define such a model, we must allow particle positions to be distributed in the continuum. Using a geometry-dependent interaction, we can then assign some geometry to the configuration.

We are modelling infinite systems of marked particles in the continuum, interacting with one another geometrically. It is natural to ask why model an infinite system when all systems found in nature are finite? Infinite systems actually work as ideal approximations to very large finite systems. By taking the infinite limit of a finite region (with a fixed boundary layer) one can model a very large finite system. We also find that in finite systems there are certain mathematical phenomena that are not present in very large finite systems in nature. Therefore we use infinite systems, or more accurately, we take the infinite limit of a finite system with boundary conditions. This limit is known as the thermodynamic limit, which we will later discuss in detail.

The way in which systems interact is described through a key object known as the Hamiltonian. The interaction then determines the relative energies between different configurations. Crucially, microscopic changes in the system will alter the energy of the system. In the present study, we provide formulation of the Hamiltonian energy when the particles are marked and interact geometrically, as opposed to in pairs. Expressing the Hamiltonian in such a way enables us to investigate the existence (and non-uniqueness) of Gibbs measures.

Section 2 contains the introduction and necessary preliminaries regarding the definition of Gibbsian point processes for systems with geometry-dependent interactions. We first present some notation, originally introduced by Dereudre et. al. [DDG11], regarding unmarked particle systems. We give the reader an idea of the class of models for which Gibbs measures have been shown to exist for such models. We then extend the notation to describe marked particle systems and conclude the section with our first main result. This states under what conditions Gibbs measures exist for marked continuum models with geometry-dependent interactions. This is a prerequisite for the analysis of multiple Gibbs measures. In Section 3, we define a phase transition for this class of models and introduce the multi-body continuum random-cluster model. This is our main tool for showing the existence of phase transitions. We define the models and present the main phase transition results in Section 4. These results are for the existence of Gibbs measures and the non-uniqueness of the Gibbs measure (i.e. evidence of a phase transition) for a class of type interaction models with geometry-dependent interactions. Detailed proofs are provided for the main theorems in Section 5. In Section 6, we pick one of the models from Section 4 and use numerical analysis to provide some simulations of a particle configuration generated according to the model. By varying the parameters of the model, we support our main theorem showing that a phase transition occurs. We summarise our main results and conclusions in Section 7.

2 Gibbsian Point Processes

In this section, we define Gibbsian point processes for continuum systems where particles interact geometrically. We first introduce the unmarked case using the notation of Dereudre et. al. [DDG11]. This is done in Section 2.1. In Section 2.2, we discuss the class of models for which the existence of Gibbsian point processes has been previously established. Then, in Section 2.3, we adapt the notation for unmarked systems to the marked case, this is an extension of Dereudre et. al. [DDG11]. We conclude by stating the first main result: the existence of marked Gibbsian point processes with geometry-dependent interactions.

2.1 Preliminaries

Our object of study is physical systems consisting of many individual particles with prescribed positions in space. Let $R := \mathbb{R}^d$, let $\mathcal{B}_R$ be the Borel σ-field on R, and let $\mathrm{Leb}(\,\cdot\,)$ be the Lebesgue measure on $(R, \mathcal{B}_R)$. For a bounded region $\Lambda \in \mathcal{B}_R$, we write $|\Lambda|$ for $\mathrm{Leb}(\Lambda)$. We let ω denote a configuration of particles in $\mathbb{R}^d$, and

$$\begin{aligned} \Omega &:= \{\omega \subset \mathbb{R}^d : \omega \text{ is locally finite}\}, \\ \Omega_f &:= \{\omega \in \Omega : \omega \text{ is finite}\}. \end{aligned}$$

For $\omega \in \Omega$, denote the counting variable $N_\Lambda(\omega) := \#(\omega \cap \Lambda)$. The set Ω is equipped with the σ-field $\mathcal{F}$ generated by the counting variables N_Λ for all bounded $\Lambda \in \mathcal{B}_R$. We write $\omega_\Lambda := \omega \cap \Lambda$ and $\Omega_\Lambda := \{\omega \in \Omega : \omega \subset \Lambda\}$ with $\mathcal{F}_\Lambda$ the associated σ-field. For configurations $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$, we denote $\zeta\omega_{\Lambda^c} := \zeta \cup \omega_{\Lambda^c} \in \Omega$. Let $\Theta = (\vartheta_x)_{x \in \mathbb{R}^d}$ be the shift group, where $\vartheta_x : \Omega \to \Omega$ is the translation by the vector $-x \in \mathbb{R}^d$. For any $x \in \mathbb{R}^d$, the mapping ϑ_x is measurable, as shown by Matthes et. al. [MKM78].

In the present study, we analyse configurations of particles that are distributed in the continuum, with the position of each particle dependent on its local geometry. In order to describe such systems mathematically, the geometry of a configuration is characterised by a hypergraph structure. This formulation was introduced by Dereudre et. al. [DDG11]. For a configuration $\omega \in \Omega$, a measurable subset $\mathcal{E} \subset \Omega_f \times \Omega$ is a *hypergraph structure* if $\eta \subset \omega$ for any $(\eta, \omega) \in \mathcal{E}$. If $(\eta, \omega) \in \mathcal{E}$, we say that η is a *hyperedge* of ω, and we write $\eta \in \mathcal{E}(\omega)$. The interaction between particles in a configuration $\omega \in \Omega$ can be expressed in terms of their geometry because we can define a suitable hypergraph structure $\mathcal{E}$ such that the hyperedges $\eta \in \mathcal{E}(\omega)$ interact with the rest of the configuration ω. This is done by defining an *interaction potential*, a measurable function $\psi : \mathcal{E} \to \mathbb{R} \cup \{\infty\}$.

The majority of our models, for which we investigate the existence of phase transitions, are defined in terms of the Delaunay hypergraph structure. We focus on the planar case, i.e. $d = 2$. For convenience, we write $|\eta| := \#(\eta)$, where $\eta \in \Omega_f$. Let $\partial\Lambda$ be the boundary of any bounded region $\Lambda \subset \mathbb{R}^2$. For $|\eta| = 2, 3$, let $B(\eta)$ be an open ball, such that the members of η lie on the boundary $\partial B(\eta)$. For $|\eta| = 3$, the ball $B(\eta)$ is uniquely determined by η.

Definition 2.1. *Let $\omega \in \Omega$. For $k = 2, 3$, the Delaunay hypergraph structure $\mathcal{E}^{\mathrm{D}_k}$ is the set of all pairs (η, ω) with $|\eta| = k$ and $\eta \subset \omega$, for which there exists an open ball $B(\eta)$ with $\partial B(\eta) \cap \omega = \eta$ that contains no particles of ω.*

Interactions via the edges of the Delaunay graph is one way of defining nearest-neighbour interactions in the continuum. Analysis of interactions on Delaunay hypergraphs is also useful because the Delaunay triangulation is the geometric dual of the Voronoi tessellation. For a particle configuration $\omega \in \Omega$, the Voronoi cell of a point $x \in \omega$ is the set

$$V(x) := \{y \in \mathbb{R}^2 : d(x, y) = \inf_{x' \in \omega} d(x', y)\},$$

where $d(x, y)$ is the Euclidean metric. The Voronoi tessellation, of a distribution of points $\omega \in \Omega$, is the set of Voronoi cells $\mathcal{V}(\omega) = \{V(x) : x \in \omega\}$. This tessellation divides the continuum into regions so that if $y \in V(x)$, then x is the closest point of ω to y. For two vertices $x, y \in \omega$, the Voronoi cells $V(x), V(y)$ share a face if and only if $\{x, y\} \in \mathcal{E}^{\mathrm{D}_2}(\omega)$.

Since we shall mainly be dealing with Delaunay hyperedges, and these consist of either two or three points, we shall use the notation $\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)$ for a pair in the Delaunay hypergraph and $\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)$ for a triangle in the Delaunay hypergraph. For a particle configuration $\omega \in \Omega$, defined in terms of a Delaunay hypergraph structure, if $\tau \subset \omega$ and $|\tau| = 3$, let $B(\tau)$ be the open ball that circumscribes the triangle with vertices τ. Let $\delta(\tau)$ be the diameter of $B(\tau)$. Similarly, if $\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)$, let $B(\eta)$ be an open ball with the 2 points of η lying on the boundary.

A hyperedge potential $\psi : \Omega_f \times \Omega \to \mathbb{R} \cup \{\infty\}$ is called *shift-invariant* if

$$\begin{aligned} &(\vartheta_x\eta, \vartheta_x\omega) \in \mathcal{E} \text{ and } \psi(\vartheta_x\eta, \vartheta_x\omega) = \psi(\eta, \omega) \\ &\text{for all } x \in \mathbb{R}^d \text{ and } (\eta, \omega) \in \Omega_f \times \Omega \text{ such that } (\eta, \omega) \in \mathcal{E}. \end{aligned}$$

We can sum the interaction potential over the entire configuration,

$$H^{\psi}(\omega) \;:=\; \sum_{\eta \in \mathcal{E}(\omega)} \psi(\eta, \omega), \tag{2.1}$$

to obtain the *Hamiltonian energy.* Note that in the above sum, it possible for ψ to take values in $\mathbb{R} \cup \{\infty\}$. For a system described by $\mathcal{E}$ and ψ, (2.1) is known as the formal Hamiltonian because the sum is infinite. To make sense of this formal sum, it is necessary to define a Hamiltonian energy for a configuration within a finite region, with prescribed boundary condition outside of this region. Before introducing this Hamiltonian, we provide some further definitions. For some bounded $\Lambda \subset \mathbb{R}^2$, let

$$\mathcal{E}_\Lambda(\omega) := \{\eta \in \mathcal{E}(\omega) : \psi(\eta, \zeta\omega_{\Lambda^c}) \neq \psi(\eta, \omega) \text{ for some } \zeta \in \Omega_\Lambda\}. \tag{2.2}$$

This set of hyperedges $\mathcal{E}_\Lambda(\omega)$ represents all hyperedges $\eta \in \mathcal{E}(\omega)$ such that $\psi(\eta, \omega)$ may have different values if any member of ω_Λ is changed. If a hyperedge $\eta \in \mathcal{E}(\omega)$ is not in $\mathcal{E}_\Lambda(\omega)$ then $\psi(\eta, \omega)$ will be the same, no matter how much the configuration within Λ is altered. Specifically, for $\zeta \in \Omega_\Lambda$,

$$\mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c}) = \{\eta \in \mathcal{E}(\zeta\omega_{\Lambda^c}) : \psi(\eta, \zeta\omega_{\Lambda^c}) \neq \psi(\eta, \kappa\omega_{\Lambda^c}) \text{ for some } \kappa \in \Omega_\Lambda\}. \tag{2.3}$$

The set of hyperedges given by (2.2) is useful for analysing particle configurations where the configuration is random within some bounded region and fixed outside this region. Consider a configuration $\zeta\omega_{\Lambda^c} \in \Omega$, where $\zeta \in \Omega_\Lambda$ and $\omega \in \Omega$. For a given hyperedge $\eta \in \mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})$, the interaction potential $\psi(\eta, \zeta\omega_{\Lambda^c})$ may have different values if any member of ζ changes, whereas a given hyperedge $\eta \in \mathcal{E}(\zeta\omega_{\Lambda^c}) \backslash \mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})$ will always return the same value for $\psi(\eta, \zeta\omega_{\Lambda^c})$, despite variation of ζ. Therefore we use the set $\mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})$ to analyse random configurations $\zeta\omega_{\Lambda^c} \in \Omega$ with fixed boundary conditions outside Λ. The *Hamiltonian in* Λ *with configurational boundary condition* ω is given by the formula

$$H^{\psi}_{\Lambda,\omega}(\zeta) := \sum_{\eta \in \mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})} \psi(\eta, \zeta\omega_{\Lambda^c}) \quad \text{for } \zeta \in \Omega_\Lambda, \tag{2.4}$$

provided the sum is well-defined.

It is through these definitions of a hypergraph structure, hyperedge interaction potential and Hamiltonian that we are able to move away from pure particle interaction to systems depending on the geometry of particle configurations. One may describe a particle interaction system using a pairwise interaction potential that acts between nearest-neighbours of some graph, usually a function of the distance. By introducing the hypergraph structure, we are able to analyse a much more sophisticated class of models. Next, we explain how the Hamiltonian (2.4) is used in order to distribute particles within Λ according to ψ and $\mathcal{E}$.

In order to distribute particles in the continuum, we require a reference measure. This is some underlying probability measure to describe a particle distribution. For a model described by ψ and $\mathcal{E}$, we define a probability measure in terms of the Hamiltonian (2.4), with respect to some reference measure. The underlying distribution of the particles comes from the reference measure, and there is additional interaction between the particles due to ψ. If we set $\psi = 0$, the particle distribution will be identical to the distribution using the reference measure alone.

It is common to use the Poisson point process as the reference measure for a particle interaction system. A stationary Poisson point process, with intensity $z > 0$, is characterised by two fundamental properties:

(i) For any bounded $\Lambda \in \mathcal{B}_R$, N_Λ is a Poisson distribution random variable with mean $z|\Lambda|$.

(ii) For any $k = 1, 2, \ldots$, let $\Lambda_1, \ldots, \Lambda_k \in \mathcal{B}_R$ be disjoint. Then $N_{\Lambda_1}, \ldots, N_{\Lambda_n}$ are independent random variables.

This is a standard definition, for example see Stoyan et. al. [SKM95], and is a unique description of a Poisson point process. We denote by Π^z a Poisson point process on Ω with intensity $z > 0$. This is a common choice of reference measure because it is a completely random measure; (ii) is the property of complete randomness. We choose Π^z as our reference measure on Ω, and $\Pi^z_\Lambda := \Pi^z \circ \mathrm{pr}_\Lambda^{-1}$ as our reference measure on Ω_Λ, where $\mathrm{pr}_\Lambda : \omega \to \omega \cap \Lambda$ is the projection onto Λ.

Consider the *partition function* associated to the Hamiltonian (2.4),

$$Z^z_{\Lambda,\omega} := \int_{\Omega_\Lambda} e^{-H^\psi_{\Lambda,\omega}(\zeta)}\, \Pi^z_\Lambda\,(d\zeta).$$

Let ψ^- be the negative part of ψ. An unmarked configuration $\omega \in \Omega$ is called *admissible* for a bounded region $\Lambda \subset \mathbb{R}^d$ and an activity $z > 0$ if

$$H^{\psi^-}_{\Lambda,\omega}(\zeta) = \sum_{\eta \in \mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})} \psi^-(\eta, \zeta\omega_{\Lambda^c}) < \infty \quad \text{for } \Pi^z_\Lambda\text{-almost all } \zeta \in \Omega_\Lambda,$$

and $0 < Z^z_{\Lambda,\omega} < \infty$. We write $\Omega^{\Lambda,z}_*$ for the set of all these ω. For $\omega \in \Omega^{\Lambda,z}_*$, we can define the *Gibbs distribution* for $(\mathcal{E}, \psi, z)$ in a bounded region $\Lambda \subset \mathbb{R}^d$ with boundary condition ω by

$$G^z_{\Lambda,\omega}(F) = \frac{1}{Z^z_{\Lambda,\omega}} \int_{\Omega_\Lambda} \mathbb{I}_F(\zeta\omega_{\Lambda^c})\, e^{-H^\psi_{\Lambda,\omega}(\zeta)}\, \Pi^z_\Lambda(d\zeta), \tag{2.5}$$

where $F \in \mathcal{F}$ is arbitrary, and $\mathbb{I}$ is the standard indicator function.

For a hypergraph structure $\mathcal{E}$, a potential ψ, and an activity $z > 0$, a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called a *Gibbs measure* for $\mathcal{E}$, ψ, and z if $\mathbb{P}(\Omega_*^{\Lambda,z}) = 1$ and

$$\int_\Omega f \, d\mathbb{P} = \int_{\Omega_*^{\Lambda,z}} \frac{1}{Z^z_{\Lambda,\omega}} \int_{\Omega_\Lambda} f(\zeta\omega_{\Lambda^c}) \, e^{-H^\psi_{\Lambda,\omega}(\zeta)} \, \Pi^z_\Lambda(d\zeta) \, \mathbb{P}(d\omega), \tag{2.6}$$

for every bounded region $\Lambda \subset \mathbb{R}^d$ and every measurable $f : \Omega \to [0, \infty)$. The equations (2.6) are known as the Dobrushin-Lanford-Ruelle (DLR) equations. They are an extension of the work by Dobrushin [Dob68] and Lanford and Ruelle [LR69]. These equations express that $G^z_{\Lambda,\omega}(F)$ is a version of the conditional probability $\mathbb{P}(F|\mathcal{F}_{\Lambda^c})(\omega)$. For a probability measure P on Ω, we define

$$i(P) := |\Lambda|^{-1} \int_\Omega N_\Lambda \, dP$$

as the intensity of P. We write $\mathscr{P}_\Theta$ for the set of all Θ-invariant probability measures P on Ω with finite intensity $i(P)$, and $\mathscr{G}_\Theta(\psi, z)$ for the set of all Gibbs measures for ψ and z that belong to $\mathscr{P}_\Theta$. We investigate the case of multiple solutions to (2.6) in Section 3.

One can see from the definition of the Hamiltonian (2.4) that, given a bounded region $\Lambda \in \mathcal{B}_R$ and a boundary condition $\omega \in \Omega$, configurations $\zeta \in \Omega_\Lambda$ that return a high value of $H^\psi_{\Lambda,\omega}(\zeta)$ are less probable under the Gibbs distribution. If a given hyperedge $\eta \in \mathcal{E}(\omega)$ returns a high value of the potential $\psi(\eta, \omega)$, then a hyperedge of this form is rare throughout a configuration $\zeta\omega_{\Lambda^c}$ distributed according to the Gibbs distribution.

We will later include a parameter $\beta > 0$, factored with the Hamiltonian. So $H^\psi_{\Lambda,\omega}(\zeta)$ is replaced with $\beta H^\psi_{\Lambda,\omega}(\zeta)$ in the above definitions. This parameter β is known as the inverse temperature, and is a very common feature of models in statistical mechanics. It has the feature of controlling the strength of the interaction. High values of β correspond to low temperatures and configurations with high Hamiltonian energies (which are improbable under the Gibbs distribution) become even more improbable, as the system's favouritism towards low-energy configurations becomes even stronger. Likewise, a low value of β means that the interaction potential does not affect the system so much and high-energy configurations become more probable than when β is high. We now discuss some examples of models for which Gibbs measures have been shown to exist.

2.2 Some models

We now present a concise review of Gibbs models with geometry-dependent interaction for which results have been achieved. Recall that Gibbs measures are solutions to the Dobrushin-Lanford-Ruelle equations (2.6). Ruelle [Rue70] shows that if ψ is a pair interaction, where any particle can interact with any other particle, and ψ satisfies the the assumptions of superstability and lower regularity, then Gibbs measures exist. The assumptions of superstability and lower regularity are rather technical, and so we suppress the definitions in this section as we aim to provide the reader with a concise summary of the results, without the technicalities. However, it is worth noting that the assumptions of superstability and lower regularity mean that particles are generally repelled from one another for close distances and the interaction between the particles is weak over long distances.

Preston [Pre76] derives an existence theorem for models with interactions satisfying the assumptions of superstability and lower regularity. Bertin et. al. [BBD99a, BBD99b] adapt the work of Preston to form simpler sufficient conditions under which Preston's existence theorem is satisfied and provide a class of models which satisfy these conditions, therefore proving the existence of Gibbs measures. These models are all for nearest-neighbour interactions, as opposed to being restricted to pair interactions. For $d = 2$, Bertin et. al. [BBD99a] show that Gibbs measures exist for a nearest-neighbour model with a bounded interaction potential that is a function of the distance between a particle and its nearest neighbour of the configuration.

Dereudre et. al. [DDG11] show the existence of Gibbs measures for a slightly more advanced version, the k-nearest neighbour model. For $k \geq 1$ and $d \geq 2$, the hyperedges are singletons that interact with the k-nearest neighbours of the configuration. The interaction potential is a forced-clustering mechanism, such that it must be bounded if the k-nearest neighbours are within some finite range $\delta > 0$ of the hyperedge, but otherwise the potential is infinity. Gibbs measures are shown to exist for such a model. Multi-body interaction models such as the finite-range k-body potential have also been investigated by Kutoviy and Rebenko [KR04], who prove existence of at least one Gibbs state, and Belitsky and Pechersky [BP02], who show multiple Gibbs measures exist given a stabilising assumption on the interaction.

An interesting problem is to find out if there is a percolating path in the k-nearest-neighbour graph, as this helps determine whether or not a phase transition exists. For the case without forced-clustering, Häggström and Meester [HM96] show that there is no percolation for $d = 1$, but for any $d \geq 2$, there is a critical value $k_c = k_c(d)$ such that if $k > k_c$ then there is percolation. This helps determine whether multiple Gibbs measures exist for the model, we discuss this in Section 3.

Another class of models investigated by Meester and Roy [HM96] is the class of hard sphere Boolean models. A hard sphere Boolean model is a stationary (and ergodic) probability measure on the space of configurations of balls, where the balls are centred at Poisson points and the balls do not overlap. Consider the following model. Let points be distributed according to a Poisson point process. To each point, assign a ball of radius zero. Then let the radii of the balls grow linearly, with the same speed, such that each ball continues to grow until it touches another ball (i.e. the boundary of the ball touches the boundary of another ball). We then consider any two points to be connected if their respective balls have touching boundaries. Meester and Roy find that there is no percolation for this model. We exploit this result in Theorem 4.2. Now consider a sphere model where, given a Poisson point distribution, each point is the centre of a ball radius $R > 0$. Points with overlapping balls are said to be connected. The interaction potential is a function of (i) the total volume of the balls in the maximal component, and (ii) the total length of the maximal component. Bertin et. al. [BBD99a] show that, if this potential is bounded, Gibbs measures exist for this model.

A major focus of the present study is models with an interaction potential dependent on the structure of the Delaunay graph. Bertin et al. [BBD99a] investigate the case for an underlying hypergraph $\mathcal{E}^{\mathrm{D}_3}$, where the interaction potential is a bounded function of the triangle hyperedges, $\psi(\delta(\tau))$, and only acts on triangles where the smallest angle is sufficiently large. They derive a similar result for the set of Delaunay pairs $\mathcal{E}^{\mathrm{D}_2}$. These results show that Gibbs measures exist in Delaunay models, and are a positive step to showing that multiple Gibbs measures exist.

Another existence theorem for Delaunay interactions is presented by Dereudre and Georgii [DG09], who examine a planar point process with point interaction depending upon a bounded triangle potential. Again, if the triangle potential is bounded, then Gibbs measures exist. Besides this, there are some interesting remarks in [DG09]. We will later investigate type-dependent models, and Dereudre and Georgii remark that their results for a bounded triangle potential can also be applied directly to the case for marked particles, which we discuss in Section 2.3.

The class of models so far is for bounded interaction potentials. Dereudre et. al. [DDG11] improve upon the result of [DG09] by taking a triangle potential that is not bounded, but polynomially increasing. For $\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)$, and some constants $\kappa_0, \kappa_1 \geq 0$, and $\alpha > 0$, Dereudre et. al. show that if the interaction potential is polynomially increasing, $\psi(\tau, \omega) \leq \kappa_0 + \kappa_1 \delta(\tau)^\alpha$, then Gibbs measures exist for sufficiently large z. They find a similar result for pairwise Delaunay potentials. They also show that Gibbs measures exist when the interaction potential is a Delaunay triangle potential and depends on the smallest and largest angles of a triangle. This is a model that controls the shapes of the Delaunay triangles. We will later investigate such models.

Other unbounded interaction potentials include long-edge exclusion models. Dereudre et. al. [DDG11] show that such Gibbs measures do exist when the interaction potential is on Delaunay edges and is infinity if the edges are too large. This defines a model where the configuration will form a Delaunay hypergraph where all edges are sufficiently short. Such models are also discussed by Dereudre [Der08], who shows the existence of Gibbs measures for double hard-core interaction models. A geometric hard-core condition is introduced preventing Delaunay cells from being too small or too large. This builds upon previous results, such as those in [BBD99a], defining the energy from the local intrinsic geometry of the tessellation.

Dereudre and Lavancier [DL11] use the existence results from [Der08] and [BBD99a] to provide examples of Delaunay hard-core models for which Gibbs measures exist. They consider Delaunay triangle interaction potentials. Examples of such potentials include small-angle exclusion and large-cell exclusion. They also provide examples for Voronoi cell interactions, where the potential is dependent on the geometry of the cell. Interaction potentials defined in terms of the Voronoi tessellation are a useful way to describe a system according to its geometry. Bertin et. al. [BBD99a] show that Gibbs measures exist when the interaction potential is a bounded funtion of the area surrounding the nucleus of a Voronoi cell. Dereudre et. al. [DDG11] improve upon this result. They show that Gibbs measures exist if the interaction potential is a function of single Voronoi cells. The potential may be bounded, polynomially increasing or exclude cells with too many edges. This last condition is similar to the short-angle exclusion interaction. They show that it is also possible to define a bounded interaction in terms of neighbouring Voronoi cells, and Gibbs measures exist.

We now extend our current description of Gibbsian point processes to the mark space, and following this, we state what restrictions are required on the interaction in order for Gibbs measures to exist. We use the assumptions presented by Dereudre et. al. [DDG11], adapted for marked systems.

2.3 Type interactions

In this section, we extend our class of geometry-dependent Gibbs measures to the case where each particle can be assigned a mark. It is an extension of the notation presented by Dereudre et. al. [DDG11], with an adaptation for marked particle systems using counting measures. This description of marked particle systems is standard; for example, see Georgii and Zessin [GZ93].

Let S be the mark space for the configuration. Note that for the models presented in Section 4, for which we have phase transition results, we have S finite. However, our notation and definitions are perfectly valid for a general mark space S. Let $\mathcal{B}_S$ be the Borel σ-field associated to S. The space S is also equipped with a finite a priori measure μ on S with $\mu(S) > 0$. In our results, we require μ to be the uniform distribution. Note that other distributions are possible. The phase space for a particle is $X := \mathbb{R}^d \times S$, equipped with the Borel σ-field $\mathcal{B}_X := \mathcal{B}_R \otimes \mathcal{B}_S$.

We now extend the theory of Section 2.1 to let ω represent a configuration of marked particles. For the remainder of this study, ω denotes a marked configuration. A configuration of marked particles in $\mathbb{R}^d$ is described by a pair $\big(\xi_\omega, (u_x^\omega)_{x\in\xi_\omega}\big)$, where $\xi_\omega \subset \mathbb{R}^d$ is the set of occupied positions, and $u_x^\omega \in S$ is the mark of the particle at position $x \in \xi_\omega$. We can describe such a configuration by the counting measure

$$\omega = \sum_{x\in\xi_\omega} \delta_{(x,u_x^\omega)},$$

on $(X, \mathcal{B}_X)$. Note that there is a one-to-one correspondence between the marked particle configuration $\big(\xi_\omega, (u_x^\omega)_{x\in\xi_\omega}\big)$, and the counting measure ω. Therefore we can write ω for any marked configuration in $\mathbb{R}^d \times S$. The marked configuration space is the set Ω of all simple counting measures on $(X, \mathcal{B}_X)$,

$$\begin{aligned}\Omega := \{\omega \subset \mathbb{R}^d \times S :\ & \omega \text{ countable, having a locally} \\ & \text{finite projection onto } \mathbb{R}^d\}.\end{aligned}$$

Also, define $\Omega_f := \{\omega \in \Omega : \omega \text{ is finite}\}$. Let $\omega_\Lambda := (\xi_\omega \cap \Lambda, (u_x^\omega)_{x\in\xi_\omega\cap\Lambda})$, and let Ω_Λ be the set of all such configurations of marked particles located in Λ and $\mathcal{F}_\Lambda$ the associated σ-field. We let $\mathcal{K}_f$ and $\mathcal{K}$ denote the sets of all finite and locally finite sets of $\mathbb{R}^d$:

$$\begin{aligned}\mathcal{K} &:= \{\xi \subset \mathbb{R}^d : \xi \text{ is locally finite}\},\\ \mathcal{K}_f &:= \{\xi \in \mathcal{K} : \xi \text{ is finite}\}.\end{aligned}$$

For each $B \in \mathcal{B}_X$, the counting variable $N(B) : \omega \to \omega(B)$ on Ω describes the number of particles such that the pair (position, mark) belongs to B. The space Ω is equipped with the σ-field $\mathcal{F} := \sigma(N(B) : B \in \mathcal{B}_X)$. For $\Lambda \in \mathcal{B}_R$, we write $N_\Lambda = N(\Lambda \times S)$ for the number of particles located in Λ. Define

$$N_\Lambda^h = N_\Lambda(h) : \omega \to \int_{\Lambda \times S} h(u)\, \omega(dx, du),$$

for any measurable function $h : S \to [0, \infty)$ and bounded $\Lambda \in \mathbb{R}^d$. It is obvious that if we let $h = 1$ then $N_\Lambda^h = N_\Lambda$. Let P be a probability measure on $(\Omega, \mathcal{F})$. If $\int_\Omega N_\Lambda^h dP$ is finite for any bounded $\Lambda \subset \mathbb{R}^d$, then we define the h-intensity of P as

$$i^h(P) := |\Lambda|^{-1} \int_\Omega N_\Lambda^h \, dP.$$

For $h = 1$,

$$i(P) := |\Lambda|^{-1} \int_\Omega N_\Lambda \, dP$$

is the intensity of P.

Let $\xi \subset \Lambda$ and $\xi' \subset \Lambda^c$ be sets of occupied places in $\mathbb{R}^d$. For configurations $\big(\xi, (u_x)_{x\in\xi}\big)$ and $\big(\xi', (u_x)_{x\in\xi'}\big)$, denoted by $\zeta \in \Omega_\Lambda$ and $\omega \in \Omega_{\Lambda^c}$ respectively, we denote the combined configuration $\big(\xi \cup \xi', (u_x)_{x\in\xi\cup\xi'}\big)$ as

$$\zeta + \omega := \sum_{x \in \xi \cup \xi'} \delta_{(x, u_x)}.$$

The one-to-one correspondence between a counting measure and the marked configuration means that we can also express the marked configuration $\big(\xi, (u_x)_{x\in\xi}\big) \cup \big(\xi', (u_x)_{x\in\xi'}\big)$ as $\zeta \cup \omega \in \Omega$. For notational convenience, we will sometimes write $\zeta\omega \in \Omega$.

For a marked configuration, a measurable subset $\mathcal{E} \subset \Omega_f \times \Omega$ is a hypergraph structure if $\eta \subset \omega$ for any $(\eta, \omega) \in \mathcal{E}$. As in Section 2.1, if $(\eta, \omega) \in \mathcal{E}$, we say η is a hyperedge of ω and write $\eta \in \mathcal{E}(\omega)$. We can express a hyperedge $\eta \in \mathcal{E}(\omega)$ as $\eta = (\xi_\eta, (u_x^\eta)_{x\in\xi_\eta})$, where $\xi_\eta \subset \mathbb{R}^d$ is finite. Note that $\xi_\eta \subset \xi_\omega$ and $u_x^\eta = u_x^\omega$ for all $x \in \xi_\eta$. For geometry-dependent type interaction systems, we will define two kinds of interaction potential. There is the background interaction, as in Section 2, that acts on hyperedges but does not take into account the marks of the particles. There is also a type interaction that acts on hyperedges but also depends on the marks assigned to the particles. Sometimes we require the background interaction to act on a different hypergraph structure to the type interaction, and for this reason we denote the background hypergraph structure as $\mathcal{E}^B$ and the type hypergraph structure as $\mathcal{E}^T$. The formal Hamiltonian for marked particle systems reads as

$$H(\omega) \quad := \quad \sum_{\eta\in\mathcal{E}^B(\omega)} \psi(\eta,\omega) + \sum_{\eta\in\mathcal{E}^T(\omega)} \phi(\eta,\omega), \tag{2.7}$$

where $\psi : \mathcal{E}^B \to \mathbb{R}\cup\{\infty\}$ and $\phi : \mathcal{E}^T \to \mathbb{R}\cup\{\infty\}$ are measurable functions, known as the *background interaction* and *type interaction*, respectively. We must emphasise that the background interaction ψ does not depend on the marks, and $\psi(\eta,\omega)$ can always be expressed in terms of ξ_η and ξ_ω. Note that we can express (2.7) in the form

$$H(\omega) = \sum_{\eta\in\mathcal{E}(\omega)} g(\eta,\omega),$$

for some function $g : \mathcal{E} \to \mathbb{R}\cup\{\infty\}$, where $\mathcal{E}(\omega) := \mathcal{E}^B(\omega)\cup\mathcal{E}^T(\omega)$. In many cases the background interaction and type interaction act on the same hypergraph structure, i.e. $\mathcal{E}^B = \mathcal{E}^T$. In this context, $\mathcal{E}^B = \mathcal{E}^T = \mathcal{E}$ and $g(\eta,\omega) = \psi(\eta,\omega) + \phi(\eta,\omega)$.

Note how (2.7) compares to the Hamiltonian (2.1) defined for unmarked configurations. We have simply added an extra term in order to allow interaction on the marks. We can express (2.7) as $H^\psi(\omega)+H^\phi(\omega)$, where H^ψ is the background Hamiltonian and H^ϕ is the type Hamiltonian. The Hamiltonian in Λ with configurational boundary condition ω reads as

$$\begin{aligned} H_{\Lambda,\omega}(\zeta) \quad &:= \quad \sum_{\eta\in\mathcal{E}^B_\Lambda(\zeta\omega_{\Lambda^c})} \psi(\eta,\zeta\omega_{\Lambda^c}) + \sum_{\eta\in\mathcal{E}^T_\Lambda(\zeta\omega_{\Lambda^c})} \phi(\eta,\zeta\omega_{\Lambda^c}) \\ &= \quad H^\psi_{\Lambda,\omega}(\zeta) + H^\phi_{\Lambda,\omega}(\zeta), \end{aligned} \tag{2.8}$$

for $\zeta \in \Omega_\Lambda$, and

$$\begin{aligned} \mathcal{E}^B_\Lambda(\omega) \quad &:= \quad \{\eta \in \mathcal{E}^B(\omega) : \psi(\eta,\zeta\omega_{\Lambda^c}) \neq \psi(\eta,\omega) \text{ for some } \zeta \in \Omega_\Lambda\}, \\ \mathcal{E}^T_\Lambda(\omega) \quad &:= \quad \{\eta \in \mathcal{E}^T(\omega) : \phi(\eta,\zeta\omega_{\Lambda^c}) \neq \phi(\eta,\omega) \text{ for some } \zeta \in \Omega_\Lambda\}. \end{aligned}$$

We can now define marked Gibbsian point processes with geometry-dependent interactions. Let $\Pi^{z,\mu}$ be the Poisson point random field on X with intensity measure $z\,\mathrm{Leb}(\,\cdot\,) \otimes \mu$. For $\Lambda \in \mathcal{B}_R$, let $\Pi^{z,\mu}_\Lambda := \Pi^{z,\mu} \circ \mathrm{pr}_\Lambda^{-1}$ be the projection onto $(\Omega_\Lambda, \mathcal{F}_\Lambda)$. For our main results in Section 4, we only require the simplest case and assume that the measure μ is the uniform distribution on S, and S is finite. However, note that our definitions hold for $\Pi^{z,\nu}$, where the mark intensity ν is some measure different to the uniform distribution μ. Later in this section, we discuss examples of models for different forms of μ and S.

From now on, we include the inverse temperature $\beta > 0$, factored with the Hamiltonian in our definition of a Gibbs measure. This controls the strength of the interaction, and is a critical parameter for the existence of phase transitions, as we shall see later. Consider the partition function associated to the Hamiltonian (2.8),

$$Z_{\Lambda,\omega}^{z,\mu} = Z_{\Lambda,\omega}^{z,\mu}(\beta) := \int_{\Omega_\Lambda} e^{-\beta H_{\Lambda,\omega}(\zeta)} \, \Pi_\Lambda^{z,\mu}(d\zeta). \tag{2.9}$$

A marked configuration $\omega \in \Omega$ is called *admissible* for a bounded region $\Lambda \subset \mathbb{R}^d$ and an activity $z > 0$ if

$$H_{\Lambda,\omega}^{-}(\zeta) := \sum_{\eta \in \mathcal{E}_\Lambda^B(\zeta\omega_{\Lambda^c})} \psi^{-}(\eta, \zeta\omega_{\Lambda^c}) + \sum_{\eta \in \mathcal{E}_\Lambda^T(\zeta\omega_{\Lambda^c})} \phi^{-}(\eta, \zeta\omega_{\Lambda^c}) < \infty$$

for $\Pi_\Lambda^{z,\mu}$-almost all $\zeta \in \Omega_\Lambda$, and $0 < Z_{\Lambda,\omega}^{z,\mu} < \infty$. We write $\Omega_*^{\Lambda,z}$ for the set of all these ω.

For $\omega \in \Omega_*^{\Lambda,z}$, we can define the *marked Gibbs distribution*, for $\mathcal{E}^B$, $\mathcal{E}^T$, ψ, ϕ and z, in a bounded region $\Lambda \subset \mathbb{R}^d$ with boundary condition ω by

$$G_{\Lambda,\omega}^{z,\mu}(F) = \frac{1}{Z_{\Lambda,\omega}^{z,\mu}} \int_{\Omega_\Lambda} \mathbb{I}_F(\zeta\omega_{\Lambda^c}) \, e^{-\beta H_{\Lambda,\omega}(\zeta)} \, \Pi_\Lambda^{z,\mu}(d\zeta), \tag{2.10}$$

where $F \in \mathcal{F}$ is arbitrary. For hypergraph structures $\mathcal{E}^B$, $\mathcal{E}^T$, interaction potentials ψ, ϕ, and an activity $z > 0$, a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called a *marked Gibbs measure* for $\mathcal{E}^B$, $\mathcal{E}^T$, ψ, ϕ, and z if $\mathbb{P}(\Omega_*^{\Lambda,z}) = 1$ and

$$\int_\Omega f \, d\mathbb{P} = \int_{\Omega_*^{\Lambda,z}} \frac{1}{Z_{\Lambda,\omega}^{z,\mu}} \int_{\Omega_\Lambda} f(\zeta\omega_{\Lambda^c}) \, e^{-\beta H_{\Lambda,\omega}(\zeta)} \, \Pi_\Lambda^{z,\mu}(d\zeta) \, \mathbb{P}(d\omega), \tag{2.11}$$

for every bounded region $\Lambda \subset \mathbb{R}^d$ and every measurable $f : \Omega \to [0, \infty)$.

As in Section 2.1, let $\Theta = (\vartheta_x)_{x \in \mathbb{R}^d}$ be the shift group, where $\vartheta_x : \Omega \to \Omega$ is the translation by the vector $-x \in \mathbb{R}^d$. The translations ϑ_x act only on the positions of the particles and leave their marks untouched. We write $\mathscr{P}_\Theta$ for the set of all Θ-invariant probability measures P on $(\Omega, \mathcal{F})$ with finite intensity $i(P)$, and $\mathscr{G}_\Theta(\psi, \phi, z)$ for the set of all Gibbs measures for ψ, ϕ and z that belong to $\mathscr{P}_\Theta$.

We now consider some further definitions, regarding a hypergraph structure $\mathcal{E}$ and the associated interaction potential $g : \mathcal{E} \to \mathbb{R} \cup \{\infty\}$, that are required for the existence theorem. The definition of shift-invariance remains the same for marked systems because the translation vector leaves the marks untouched.

A hyperedge potential $g : \Omega_f \times \Omega \to \mathbb{R} \cup \{\infty\}$ is called shift-invariant if

$$(\vartheta_x \eta, \vartheta_x \omega) \in \mathcal{E} \text{ and } g(\vartheta_x \eta, \vartheta_x \omega) = g(\eta, \omega)$$
$$\text{for all } x \in \mathbb{R}^d \text{ and } (\eta, \omega) \in \Omega_f \times \Omega \text{ such that } (\eta, \omega) \in \mathcal{E}.$$

A hyperedge potential $g : \Omega_f \times \Omega \to \mathbb{R} \cup \{\infty\}$ satisfies the *finite horizon property* if for each $(\eta, \omega) \in \Omega_f \times \Omega$ such that $(\eta, \omega) \in \mathcal{E}$, there exists some bounded $\Delta \subset \mathbb{R}^d$ such that

$$(\eta, \tilde{\omega}) \in \mathcal{E} \text{ and } g(\eta, \tilde{\omega}) = g(\eta, \omega) \text{ when } \tilde{\omega} = \omega \text{ on } \Delta \times S. \tag{2.12}$$

For a bounded region $\Lambda \subset \mathbb{R}^d$ and a marked configuration $\omega \in \Omega$, we now introduce the set $\partial\Lambda(\omega) \subset \mathbb{R}^d$, called the ω-boundary of Λ. We assume that $\partial\Lambda(\omega)$ is some bounded region on the boundary of Λ, dependent on the configuration ω. Specifically, we assume $\partial\Lambda(\omega) = \Lambda^r \setminus \Lambda$, where Λ^r is the closed r-neighbourhood of Λ and $r = r(\Lambda, \omega)$ is chosen to be as small as possible. The ω-boundary is a region surrounding Λ such that, for any $\eta \in \mathcal{E}_\Lambda(\omega)$, the configuration outside $\Lambda \cup \partial\Lambda(\omega)$ does not affect the interaction potential $g(\eta, \omega)$. For some bounded region $\Lambda \subset \mathbb{R}^d$, a marked configuration denoted by $\omega \in \Omega$ is said to *confine the range of* g *from* Λ if there exists a bounded set $\partial\Lambda(\omega) \subset \mathbb{R}^d$ such that $g(\eta, \zeta\tilde{\omega}_{\Lambda^c}) = g(\eta, \zeta\omega_{\Lambda^c})$ whenever $\tilde{\omega} = \omega$ on $\partial\Lambda(\omega) \times S$, $\zeta \in \Omega_\Lambda$ and $\eta \in \mathcal{E}_\Lambda(\zeta\omega_{\Lambda^c})$. In this case we write $\omega \in \Omega_{\mathrm{cr}}^\Lambda$. Note that $\Omega_{\mathrm{cr}}^\Lambda \in \mathcal{F}$. For the elements of ω within $\partial\Lambda(\omega) \times S$, we use the abbreviation $\partial_\Lambda \omega = \omega \cap (\partial\Lambda(\omega) \times S)$.

Let $\mathsf{M} \in \mathbb{R}^{d \times d}$ be an invertible $d \times d$ matrix and consider for each $k \in \mathbb{Z}^d$ the cell

$$C(k) := \{\mathsf{M}x \in \mathbb{R}^d : \, x - k \in [-\tfrac{1}{2}, \tfrac{1}{2})^d\}.$$

These cells together constitute a periodic partition of $\mathbb{R}^d$ into parallelotopes. Let $C := C(0)$. Let Γ be a measurable subset of $\Omega_C \setminus \{\emptyset\}$ and define

$$\bar{\Gamma} := \big\{\omega \in \Omega : \, \vartheta_{\mathsf{M}k}(\omega_{C(k)}) \in \Gamma \text{ for all } k \in \mathbb{Z}^d\big\}.$$

This is the set of all marked configurations whose restriction to an arbitrary cell $C(k)$, when shifted back to $C(0)$, belong to Γ. Each $\omega \in \bar{\Gamma}$ denotes a pseudo-periodic marked configuration.

We now discuss different forms for the mark space S and the associated mark measure μ. If the state space has cardinality 1, i.e. $S = \{s\}$ for some $s \in \mathbb{R}$, then the phase space $\mathbb{R}^d \times S$ can be identified with $\mathbb{R}^d$, and we are describing models of the form of Section 2.1, where particles have no mark. For a Poisson random field $\Pi^{z,\mu}$ on $\mathbb{R}^d \times S$, the intensity measure is $z\,\mathrm{Leb}(\,\cdot\,) \otimes \mu$, so the the total mass $\mu(S)$ of μ is an intensity parameter. Therefore the choice of μ for $S = \{s\}$ only affects the density of the particles in $\mathbb{R}^d$. For the case of S finite and μ a uniform distribution, we have a model where particles are distributed in space and then randomly (uniformly) chosen a mark from the set S. If $S = \{1, \ldots, q\}$ for some $q \geq 2$, then the Poisson random field $\Pi^{z,\mu}$ is the reference measure for continuum q-type Potts models, for example [GH96, BBD04]. Models of this type are a major focus of the present study, and this form of S and μ is our choice for the mark space and reference measure.

It is possible to take S to be an infinite uncountable set, for example $S = \mathbb{R}$. Such a set creates a rather complex model, even if the positions of the particles are fixed, for example set $\xi_\omega = \mathbb{Z}^d$. This set-up assigns each site of $\mathbb{Z}^d$ a value from $S = \mathbb{R}$, creating a random interface. The mark measure μ on S can be interpreted as a random field of heights, see Dembo and Funaki [DF05] for further details. Another example of an uncountable and infinite mark space is $S = \mathbb{R}^d$. This could correspond to a model where particles are distributed in space and assigned velocities. The measure μ can then be defined as some appropriate distribution on particle velocities. For example, Maxwell famously describes the velocities of particles in a gas by a normal distribution. We can apply this to our mark space $S = \mathbb{R}^d$ by defining

$$\mu(du) := \frac{1}{(2\pi)^{d/2}} \exp\left\{ -\frac{|u|^2}{2} \right\} du.$$

A model with this form of S and μ describes an ideal gas, and is an example of how we can take S to be infinite and uncountable. However, we shall focus on the more simple case of $S = \{1, \ldots, q\}$ for our models in Section 4. An interesting topic of study is type interaction models where the mark measure μ also depends on the positions of the particles. However, models of this type are complex and in order to study the phase transitional behaviour of our models, we shall focus on the case where the mark reference measure μ is a uniform distribution on $S = \{1, \ldots, q\}$.

Before ending our introduction to type interaction systems, we provide the reader with some further details regarding the Ising and Potts model and their extension to the continuum. We do so because these models are fundamental to the multi-body continuum versions that we will focus on later. For this reason, we now give a brief description of these models using the notation of this section. The Ising model was introduced by Lenz [Len20], and later analysed by Ising [Isi25]. This model is defined on the integer lattice $\mathbb{Z}^d$, so the set of particle positions ω is not a random set in $\mathbb{R}^d$, but the fixed sites of $\mathbb{Z}^d$. The marked configuration $\omega \in \Omega$ can be expressed as $(\mathbb{Z}^d, (u_x)_{x \in \mathbb{Z}^d})$, where the state space $S = \{-1, 1\}$. The interaction is between neighbouring sites of $\mathbb{Z}^d$, and we let $\eta = \{x, y\}$ denote a pair $x, y \in \mathbb{Z}^d$

that share an edge. We define $\mathcal{E}(\mathbb{Z}^d)$ as the set of all edges $\eta \subset \mathbb{Z}^d$. The formal Hamiltonian (2.7) is expressed as

$$H(\omega) = \sum_{\{x,y\} \in \mathcal{E}(\mathbb{Z}^d)} -u_x u_y.$$

This describes the (ferromagnetic) Ising model, where the interaction penalises configurations with many neighbours that have opposite types. The strength of the interaction is controlled by the inverse temperature β. Potts [Pot52] introduced an extension to the Ising model, he presented a generalisation where each particle on the lattice could be assigned $q \geq 2$ different marks. In this case, the state space is $S = \{1, \ldots, q\}$, and the formal Hamiltonian is defined

$$H(\omega) = \sum_{\{x,y\} \in \mathcal{E}(\mathbb{Z}^d)} 1 - 2\mathbb{I}\{u_x = u_y\}.$$

The inverse temperature $\beta > 0$ is also included in this model, which controls the strength of the interaction.

We now discuss extensions of the Ising/Potts model to the continuum, which is the main focus of the present study. The following model, introduced by Widom and Rowlinson [WR70], can be thought of as an extension of the Ising model to the continuum. There are two types of particle, distributed in $\mathbb{R}^d$ for $d \geq 2$, there is no interaction between particles of the same type, and a hard-core repulsion between particles of opposite types. We can write $S = \{1, 2\}$ and the formal Hamiltonian, for a configuration $\omega \subset \mathbb{R}^d \times S$, is

$$H(\omega) = \sum_{\{x,y\} \subset \omega} \phi(|x - y|)\, \mathbb{I}\{u_x^\omega \neq u_y^\omega\}, \tag{2.13}$$

where

$$\phi(r) = \begin{cases} \infty & \text{if } r < r_0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

for some parameter $r_0 > 0$. Alternatively, there is the soft-core version, where

$$\phi(r) = \begin{cases} A & \text{if } r < r_0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.15}$$

for some parameter $A > 0$. Note how the Widom-Rowlinson model compares to the ferromagnetic Ising model: there are two types of particle and the interaction potential assigns a high penalty for two particles that are close together and of opposite type.

Georgii and Häggström [GH96] generalise this idea further and introduce continuum Potts models. These are models for marked particles distributed in $\mathbb{R}^d$ for $d \geq 2$, where the mark space is $S = \{1, \ldots, q\}$. The configuration is described by a formal Hamiltonian

$$H(\omega) = \sum_{\{x,y\} \subset \omega} \psi(|x-y|) + \sum_{\{x,y\} \subset \omega} \phi(|x-y|)\, \mathbb{I}\{u_x^\omega \neq u_y^\omega\}, \tag{2.16}$$

with a number of assumptions on the interaction potentials $\psi, \phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ which are much less restrictive than the functions (2.14) and (2.15) used by Widom and Rowlinson. A major focus of the present study is extend this work of Georgii and Häggström to the case of geometric interaction, i.e. replacing the pairwise interaction potentials with hyperedge potentials.

2.4 Existence of Gibbsian point processes

Recall that we can rewrite the formal Hamiltonian (2.7) as

$$H(\omega) = \sum_{\eta \in \mathcal{E}(\omega)} g(\eta, \omega),$$

for some function $g : \mathcal{E} \to \mathbb{R} \cup \{\infty\}$, where $\mathcal{E}(\omega) := \mathcal{E}^B(\omega) \cup \mathcal{E}^T(\omega)$. For a given hypergraph structure $\mathcal{E}$, interaction potential g and activity $z > 0$, we consider three conditions based on the analogous conditions, introduced by Dereudre et al. [DDG11], for unmarked systems. Our main theorem states that if these three conditions are satisfied for $(\mathcal{E}, g, z)$, then Gibbs measures exist. Note that $(\mathcal{E}, g, z)$ is equivalent to $(\mathcal{E}^B, \mathcal{E}^T, \psi, \phi, z)$.

The first condition states that hyperedges with a large finite horizon, defined by (2.12), require the existence of a large ball containing only a few points of the configuration ω.

(R) *The range condition.* There exist constants $l_R, n_R \in \mathbb{N}$ and $\delta_R < \infty$ such that for all $(\eta, \omega) \in \mathcal{E}$, one can find a horizon Δ satisfying the following. For every $x, y \in \Delta$, there exist l open balls $B_1, \ldots, B_l$ (with $l \leq l_R$) such that

- (i) the set $\cup_{i=1}^{l} \bar{B}_i$ is connected and contains x and y, and
- (ii) for each i, either $\mathrm{diam} B_i \leq \delta_R$ or $N_{B_i}(\omega) \leq n_R$.

The next condition is essentially equivalent to the classical concept of stability in statistical mechanics (we discuss this in Remark 2.1). For a finite marked configuration $\zeta = (\xi_\zeta, (u_x^\zeta)_{x \in \xi_\zeta}) \in \Omega_f$, we write $|\zeta| = |\xi_\zeta| := \#(\xi_\zeta)$ for the number of particles in ζ. Similarly for a configuration $\omega \in \Omega$, we write $|\omega_\Lambda| := \#(\xi_\omega \cap \Lambda)$, for bounded $\Lambda \in \mathcal{B}_R$ and $\omega_\Lambda := \omega \cap (\Lambda \times S)$.

(S) *Stability.* The hyperedge potential g is called *stable* if there exists a constant $c_s \geq 0$ such that

$$H_{\Lambda,\omega}(\zeta) \geq -c_s|\zeta \cup \partial_\Lambda \omega|$$

for all $\zeta \in \Omega_\Lambda$, $\omega \in \Omega_{\mathrm{cr}}^\Lambda$ and bounded $\Lambda \in \mathcal{B}_R$.

(U) *Upper Regularity.* M and Γ can be chosen so that the following holds.

(U1) *Uniform confinement:* $\bar{\Gamma} \subset \Omega_{\mathrm{cr}}^\Lambda$ for all bounded $\Lambda \in \mathcal{B}_R$, and

$$r_\Gamma := \sup_{\Lambda \subset \mathbb{R}^d} \sup_{\omega \in \bar{\Gamma}} r(\Lambda, \omega) < \infty.$$

(U2) *Uniform summability:*

$$c_\Gamma^+ := \sup_{\omega \in \bar{\Gamma}} \sum_{\eta \in \mathcal{E}(\omega) : \xi_\eta \cap C \neq \emptyset} \frac{g^+(\eta, \omega)}{|\hat{\xi}_\eta|} < \infty,$$

where $\hat{\xi}_\eta := \{k \in \mathbb{Z}^d : \xi_\eta \cap C(k) \neq \emptyset\}$ and g^+ is the positive part of g.

(U3) *Strong non-rigidity:* $e^{z|C|}\Pi_C^{z,\mu}(\Gamma) > e^{\beta c_\Gamma}$, where c_Γ is defined as c_Γ^+ with g in place of g^+.

We now state our existence theorem. This is the same theorem as the main result of Dereudre et. al. [DDG11], with the slight adaptation to allow the description of marked particle systems. The proof is provided in Section 5.1.

Theorem 2.1. *For every hypergraph structure $\mathcal{E}$, hyperedge potential g, and activity z, satisfying* (**R**), (**S**) *and* (**U**), *there exists at least one Gibbs measure $P \in \mathscr{G}_\Theta(g, z)$.*

We will see (Remark 2.5) that for Delaunay hypergraphs, it is useful to define $\Gamma = \Gamma^A$, where

$$\Gamma^A := \{\zeta \in \Omega_C : \xi_\zeta = \{x\} \text{ for some } x \in A\},$$

for some set $A \in \mathcal{B}_R$ such that $A \subset C$. The assumption **(U)** is then called **(U^A)**. We write c_Γ as c_A when $\Gamma = \Gamma^A$.

This simplifies the assumptions of **(U)** because we can replace (U2) and (U3) with

($U2^A$)

$$c_A^+ := \sup_{\omega \in \bar{\Gamma}^A} \sum_{\eta \in \mathcal{E}(\omega) : \xi_\eta \cap C \neq \emptyset} \frac{g^+(\eta, \omega)}{|\eta|} < \infty,$$

($U3^A$) $z|A| > e^{\beta c_A}$.

This gives the following corollary to Theorem 2.1:

Corollary 2.2. *For every hypergraph structure $\mathcal{E}$, hyperedge potential g, and activity z, satisfying* (**R**), (**S**) *and* ($\mathbf{U}^A$), *there exists at least one Gibbs measure $P \in \mathscr{G}_\Theta(g, z)$.*

Remark 2.1. *Stability.* Consider the following locally complete hypergraph structure of finite range,

$$\mathcal{E}^{\mathrm{C}_r} := \{(\eta, \omega) : \eta \subset \omega, \, \mathrm{diam}(\eta) \leq r, \, \omega \in \Omega\},$$

for $r > 0$. For a model with an interaction potential $g : \mathcal{E}^{\mathrm{C}_r} \to \mathbb{R}\{\infty\}$ such that $g(\eta, \omega)$ is only dependent on the first entry, it is useful to write $g(\eta, \omega) = g(\eta)$ and define the energy of a finite configuration $\zeta \in \Omega_f$ as

$$H(\zeta) = \sum_{\eta \in \mathcal{E}^{\mathrm{C}_r}(\zeta)} g(\eta). \tag{2.17}$$

The classical concept of stability in statistical mechanics (for example, see Ruelle [Rue69]) alleges that

$$H(\zeta) \geq -c_S|\zeta|, \qquad \forall \, \zeta \in \Omega_f. \tag{2.18}$$

Comparing (2.17) to our definition (2.4) of a Hamiltonian with configurational boundary condition, one can see that assumption **(S)** is equivalent to the classical case of stability (2.18).

Remark 2.2. *Bounded horizons.* For a hyperedge $\eta \in \mathcal{E}(\omega)$, we can analyse $g(\eta, \omega)$ by looking at the configuration ω_Δ, where Δ is some bounded neighbourhood of ξ_η, i.e. Δ is the horizon of η. In general, we can take Δ to be some closed ball with radius $r_{\eta,\omega}$, that contains all points of η, where $r_{\eta,\omega}$ is chosen as small as possible. Suppose an interaction potential g and its associated hypergraph structure $\mathcal{E}$ satisfy:

(*i*) $\sup_{u \in S} g\big(\{(0, u)\}, \{(0, u)\}\big) < \infty$, and

(*ii*) $(\mathcal{E}, g)$ has bounded horizons, i.e. $\exists \, r_g < \infty : \, r_{\eta,\omega} \leq r_g \quad \forall \, (\eta, \omega) \in \mathcal{E}$.

The second property (ii) means that for any hyperedge $\eta \in \mathcal{E}(\omega)$, the points of ξ_η fit in a ball radius r_g, hence **(R)** is satisfied with $\delta_R = 2r_g$. We now show how (i) and (ii) relate to assumption $(\mathbf{U}^A)$. Let $\mathsf{M} = a\mathsf{I}_d$, where $a > r_g$ and I_d is the $d \times d$ identity matrix. Let $A = B(0, b)$ be a centred ball of radius $b < (a/2) - r_g$. From (ii), we have know that any hyperedge is contained within a ball diameter $2r_g$. So we can choose $r(\Lambda, \omega) < 2r_g$ for any $\omega \in \bar{\Gamma}^A$ and bounded $\Lambda \subset \mathbb{R}^d$. Therefore $(\mathrm{U1}^A)$ is satisfied with $r_\Gamma = 2r_g$. Since there is one particle in each ball A and the distance between each ball is at least $2r_g$, property (ii) tells us that each $\eta \in \mathcal{E}(\omega)$ must be a single marked particle $\{(x, u_x^\omega)\}$ and hence $g(\eta, \omega) = g\big(\{(x, u_x^\omega)\}, \{(x, u_x^\omega)\}\big)$. Since we assume g to be shift invariant, assumption $(\mathrm{U2}^A)$ is satisfied with

$$c_A^+ = \sup_{u \in S} g^+\big(\{(0, u)\}, \{(0, u)\}\big) < \infty.$$

To satisfy $(\mathrm{U3}^A)$, we require

$$z\pi b^2 > \exp\left\{\beta \sup_{u \in S} g\big(\{(0, u)\}, \{(0, u)\}\big)\right\}.$$

Since we can choose a and b arbitrarily large, $(\mathrm{U3}^A)$ holds for any $z > 0$. Therefore, if we can show that a geometry-dependent type interaction model satisfies (i) and (ii), then **(R)** and $(\mathbf{U}^A)$ are satisfied automatically.

Remark 2.3. *Scale-invariance.* In the present study, we focus on scale-invariant potentials, which means that

$$(r\eta, r\omega) \in \mathcal{E} \text{ and } g(r\eta, r\omega) = g(\eta, \omega), \quad \forall\, (\eta, \omega) \in \mathcal{E},\ r > 0,$$

where $r\omega = (r\xi_\omega, (u_x^{r\omega})_{x \in r\xi_\omega})$, $r\xi_\omega := \{rx : x \in \xi_\omega\}$ and $u_{rx}^{r\omega} := u_x^\omega$ for any $x \in \xi_\omega$. Consider a model that describes the distribution of a marked particle configuration $\omega \in \Omega$. Assume the distribution of $r\omega \in \Omega$ can be described by a model with interaction potential g and intensity z. Scale invariance means that if we have the existence of a Gibbs measure for g and z, then existence is implied for a Gibbs measure for g and $r^{-d}z$. Therefore for scale-invariant potentials, it is sufficient to show existence of Gibbs measures for large z.

Remark 2.4. *Finite horizons for Delaunay models.* Our main results are for interaction potentials acting on the Delaunay hypergraphs, $\mathcal{E}^{\mathrm{D}_2}$ and $\mathcal{E}^{\mathrm{D}_3}$. For the case that $g(\eta, \omega) = g(\eta)$, the range condition **(R)** is satisfied as each hyperedge η has the finite horizon $\bar{B}(\eta)$. We now provide justification that this horizon is finite. For a positional configuration $\xi \in \mathcal{K}$, distributed according to a Poisson point process, every Voronoi cell $V \in \mathcal{V}(\xi)$ is bounded. For a detailed proof of this, the reader may refer to Møller [Møl94]. Since the Delaunay triangulation is the dual graph of the Voronoi tessellation, if every Voronoi cell $V \in \mathcal{V}(\xi_\omega)$ is bounded, then every Delaunay edge $\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)$ and triangle $\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)$ must also be bounded. This means the horizons $\bar{B}(\eta)$ and $\bar{B}(\tau)$ are finite.

Remark 2.5. *Upper regularity for Delaunay models.* For our Delaunay Potts models in Section 4, we show the existence of Gibbs measures via Corollary 2.2 and choose M and Γ as follows. Let M be such that $|\mathsf{M}_i| = a > 0$ for $i = 1, 2$ and $\sphericalangle(\mathsf{M}_i, \mathsf{M}_j) = \pi/3$ for $i \neq j$, and let $\Gamma = \Gamma^A := \big\{\zeta \in \Omega_C : \zeta = \{x\} \text{ for some } x \in A\big\}$ where $A = B(0, b)$ and $b \leq (\sqrt{3}/6)a$. This ensures that the neighbourhood of a particle at position $x \in \xi_\omega$ in a configuration $\omega \in \bar{\Gamma}$ contains 6 points. This is due to the fact that particles are attached to their nearest neighbours in the Delaunay graph. For example, a point $x \in \xi_\omega \cap C((0,0))$ has neighbours in all 4 adjacent boxes $C((0,1)), C((1,0)), C((0,-1)), C((-1,0))$, and 2 of the 4 corner boxes, $C((-1,1)), C((1,-1))$. Due to the fact that $\sphericalangle(\mathsf{M}_i, \mathsf{M}_j) = \pi/3$ and $b \leq (\sqrt{3}/6)a$, the points inside $C((1,1))$ and $C((-1,-1))$ cannot be attached to x. To see this, note that the shortest possible distance between x and a point of $C((1,1))$ is larger than the farthest possible distance between a point of $C((1,0))$ and a point of $C((0,1))$. Therefore, if the points are joined by Delaunay edges, there will be an edge from a point of $C((1,0))$ to a point of $C((0,1))$ and an edge between x and a point of $C((1,1))$ is not possible. Similarly for the edge between $C((-1,0))$ and $C((0,-1))$ eliminating the possibility of an edge between x and $C((-1,-1))$.

3 Phase transitions

In this section, we discuss phase transitions in terms of marked Gibbsian point processes. In Section 3.1, we give a mathematical definition for a phase transition and explain our approach for determining the existence of a phase transition for our class of models. Then, in Section 3.2, we provide details of this approach and introduce the random-cluster model. We state the conditions that our class of models must satisfy in order to apply the random-cluster model and compare this model to the original random-cluster model.

3.1 Multiple Gibbs measures

As we discussed in the introduction, there are different ways to approach the analysis of phase transitions. One approach is to analyse the partition function (2.9); for example, see Lee and Yang [LY52]. However, this approach does not concern us. We determine the existence of a phase transition for a given model by finding if there are multiple solutions to the Dobrushin-Lanford-Ruelle equations (2.11). This definition is widely used; for example, see Georgii [Geo88].

A phase transition is the transition of a system from one state to another. One can see how this is related to the existence of multiple Gibbs measures by considering percolation theory. In a Delaunay Potts model, if there is positive probability of percolation of particles with matching types (under the Gibbs measure), then there exist q distinct states for the system. There is an equilibrium state for each mark that can dominate. This therefore means that there exist q distinct Gibbs measures and hence the existence of a phase transition. If there is zero probability of percolation, then no mark dominates the system and the system is described by one Gibbs measure, the unique solution to (2.11). Realisations of a type interaction model will have different properties depending on whether or not a phase transition exists. The reader should note that there are further potential phase transitions than dominance of one type. It is possible to analyse, for example, liquid-gas phase transitions; see Lebowitz et. al. [LMP99]. In this case, the equilibrium states are states where the particle system either behaves as a liquid or a gas. However, in the present study we are dealing with marked particle systems and focus on phase transition due to the dominance of one type.

Different choices of boundary conditions in the limiting Gibbs distribution can affect the uniqueness of the limiting distribution, dependent on the parameters of the model. To compare this to percolation, if the boundary contains many type 1 particles, and the interaction potential strongly penalises neighbours sharing the same type, then neighbours near the boundary are more likely to be of type 1, and this neighbourhood dependence may be carried throughout the whole system so that we see large clusters of the same type. If the interaction is strong enough then there will be a cluster reaching to the boundary, and over the thermodynamic limit, we will see an infinite cluster. If there is a phase transition, then we tend to see the domination of one type, with clusterings of other types appearing throughout the configuration. A realisation of the particle distribution looks like a "sea" of the dominating type with "islands" of the other types appearing sporadically. How much one type appears to dominate depends upon how strong the interaction is, and how close the parameters are to their critical values. If there is no phase transition present, realisations of the model show all types to be equally distributed throughout the system, with no single type dominating. We shall discuss further details of the relationship between percolation and multiple Gibbs measures by introducing the random-cluster model in Section 3.2.

To define a Gibbs measure, we investigate the Gibbs distribution in a bounded region $\Lambda \subset \mathbb{R}^d$ over the thermodynamic limit $\Lambda \uparrow \mathbb{R}^d$. A Gibbs measure can be defined as any accumulation point of a sequence of Gibbs distributions. If this sequence converges to a unique limit, then this limit is the unique solution to the DLR equations (2.11). If there is more than one accumulation point, then the solution is not unique. This means that multiple Gibbs measures exist and therefore we have the existence of a phase transition. Our strategy is to construct different sequences, differing in boundary condition, and to show that their limits (accumulation points) are different. In Section 4, we provide the reader with some examples of models for which we have applied this method to obtain results. In Section 5, we provide the mathematical details.

For our class of models, we must first investigate the existence of Gibbs measures. Once this has been established, we can then determine under what conditions the Gibbs measure is non-unique. Our approach requires the random-cluster model, which we will discuss in Section 3.2. Defining the random-cluster model enables us to determine whether or not several solutions exist to (2.11), for given $\mathcal{E}^B$, $\mathcal{E}^T$, ψ, ϕ, q and z.

It is via the random-cluster model that the characterisation of phases is described in percolation terms. In Section 3.2 we adapt the original random-cluster model (used for lattice systems of interacting pairs) to analyse geometry-dependent continuum models. For a given model, we compare the probability of percolation of matching spins to the probability of percolation in the associated random-cluster model. The originality in the present study is that percolation is defined in terms of hyperedges in the continuum, rather than edges on a lattice. This enables us to obtain phase transition results for a class of continuum models with geometry-dependent interactions.

For our class of models in Section 4, we show that the Gibbs distribution has an accumulation point, which in turn is a Gibbs measure for the model. We show that this limit (i.e. accumulation point) exists, and is therefore a Gibbs measure. We do so via Theorem 2.1. This theorem states that Gibbs measures exist, but makes no comment on the uniqueness of the limit. To prove there are multiple Gibbs measures for a model, we show the existence of percolation in the associated random-cluster model, which we will prove in Section 3.2 implies the existence of multiple Gibbs measures. To summarise, percolation in the random-cluster model implies that the Gibbs measure found in Theorem 2.1 is non-unique.

Ruelle [Rue71] was the first to show the existence of a phase transition in a classical continuum system with finite range interaction potential. He did so for the hard-core Widom-Rowlinson model, described by (2.13) with interaction (2.14). Lebowitz and Lieb [LL72] extend the work of Ruelle to show that the soft-core interaction (2.15) is still strong enough to maintain a phase transition in the Widom-Rowlinson model. Georgii and Häggström [GH96] prove the existence of phase transitions for multi-type continuum Potts models in $\mathbb{R}^d$ with finite range repulsion between pairs of particles with different types. The Hamiltonian for their model is given by (2.16). We extend their result to the case of hyperedge interactions and we also consider the case of infinite-range interactions.

3.2 The random-cluster representation

In this section, we define a random-cluster model to describe continuum Potts models with geometry-dependent interactions. This model is an adaptation of the original random-cluster model, introduced by Fortuin and Kasteleyn [FK72] for the lattice case with pairwise interactions. We discuss how our random-cluster model compares to other random-cluster models at the end of this section. The originality of our random-cluster model is that it describes multi-body interactions in the continuum. Our definition of the random-cluster model is for general hypergraphs with $d \geq 2$, but in Section 5, we apply it specifically to Delaunay hypergraphs with $d = 2$.

We aim to formulate a random-cluster model for continuum systems of marked particles that interact via hyperedges. Let $d \geq 2$ and consider a marked configuration $\omega \in \Omega$ in $\mathbb{R}^d$. Recall that $\mathcal{E}^T$ is the hypergraph structure upon which the type interaction potential ϕ is defined. For any $\eta \in \mathcal{E}^T(\omega)$, particles in η interact with ω

according to position and mark. The basic idea of the random-cluster representation is to assign random states (open or closed) to hyperedges of $\mathcal{E}^T(\omega)$. We thus introduce the sets

$$H_{\mathbb{R}^d} := \{\xi \subset \mathbb{R}^d : \xi \text{ is finite}\}, \tag{3.1}$$
$$H_\Delta := \{\xi \in H_{\mathbb{R}^d} : \xi \subseteq \Delta\}, \tag{3.2}$$

for any $\Delta \in \mathcal{B}_R$. The set $H_{\mathbb{R}^d}$ is the set of all possible hyperedges in the space $\mathbb{R}^d$. The set H_Δ is the set of all possible hyperedges in $\mathbb{R}^d$ that are contained within Δ. Each hyperedge $\eta \in \mathcal{E}^T(\omega)$ can then be assigned a state open or closed. Note that this is a state assigned to each hyperedge for percolation terms and is unrelated to the topological definitions of open and closed. Let the set H represent the positions of the open hyperedges. The hyperedge $\eta \in \Omega_f$ is open if and only if the (unmarked) hyperedge $\xi_\eta \in \mathcal{K}_f$ is open, i.e. $\xi_\eta \in H$.

We define the random-cluster model for background and type interactions which only depend on the first entry, so that $\psi(\eta, \omega) = \psi(\eta, \eta)$, and similarly for ϕ. Therefore we suppress the dependence on the second entry and write $\psi(\eta, \omega)$ and $\phi(\eta, \omega)$ as $\psi(\eta)$ and $\phi(\eta)$, respectively. The background interaction acts on all hyperedges of $\mathcal{E}^B(\omega)$, regardless of marks. The type interaction depends on the marks, and we assume that it acts on the marks in such a way that

$$\phi(\eta) = \phi_0(\xi_\eta)\big(1 - \mathbb{I}\{\exists\, i \in S :\, u_x^\eta = i,\, \forall\, x \in \xi_\eta\}\big), \tag{3.3}$$

for some $\phi_0 : \mathcal{K}_f \to \mathbb{R} \cup \{\infty\}$. For our random-cluster model, we require the following definitions. These are adaptations of the definitions provided by Georgii and Häggström [GH96], extended to consider hyperedge interactions (as opposed to pair interactions).

- *Distribution of particle positions.* For boundary condition $\xi_\omega \in \mathcal{K}$, we distribute the positions of the configuration $\zeta = (\xi_\zeta, (u_x^\zeta)_{x\in\xi_\zeta})$ according to the following distribution on $\mathcal{K}_\Lambda$:

$$P^z_{\Lambda,\omega}(d\xi_\zeta) := \frac{1}{Z^z_{\Lambda,\omega}} \exp\Big(-\beta \sum_{\eta\in\mathcal{E}^B_\Lambda(\zeta\omega_{\Lambda^c})} \psi(\eta)\Big)\, \Pi^z_\Lambda(d\xi_\zeta), \tag{3.4}$$

where

$$Z^z_{\Lambda,\omega} := \int_{\Omega_\Lambda} \exp\Big(-\beta \sum_{\eta\in\mathcal{E}^B_\Lambda(\zeta\omega_{\Lambda^c})} \psi(\eta)\Big)\, \Pi^z_\Lambda(d\xi_\zeta)$$

is a normalisation constant.

- *Type-picking mechanism.* For fixed sets of positions $\xi_\omega \in \mathcal{K}$ and $\xi_\zeta \in \mathcal{K}_\Lambda$, we denote by $\lambda_{\zeta\omega_{\Lambda^c}}$ the distribution of

$$(\{x \in \xi_\zeta \xi_{\omega_{\Lambda^c}} : t_x = s\})_{1 \le s \le q}, \tag{3.5}$$

where $\{t_x : x \in \xi_\zeta\}$ are independent and uniformly distributed on the q spins, and $t_x = 1$ for any $x \in \xi_{\omega_{\Lambda^c}}$. Fixing the spin configuration outside Λ is known as the *wired* boundary condition. The boundary condition affects the way we count clusters in Λ. In the wired case, we fix all particles as type 1 outside the boundary. So if there is a cluster of type 1 particles within Λ, then if this cluster touches the boundary it is an infinite cluster. With boundary condition ξ_ω, we can form the configuration

$$\zeta\omega_{\Lambda^c} = \big(\xi_\zeta \xi_{\omega_{\Lambda^c}}, \{t_x : x \in \xi_\zeta \xi_{\omega_{\Lambda^c}}\}\big),$$

using the particle positional distribution $P^z_{\Lambda,\omega}$ and type-picking mechanism $\lambda_{\zeta\omega_{\Lambda^c}}$.

- *Hyperedge-drawing mechanism.* For a given configuration $\zeta\omega_{\Lambda^c} \in \Omega$, let $\mu_{\zeta\omega_{\Lambda^c}}$ denote the distribution of the random hyperedge configuration $\{\xi_\eta : \eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c}), \gamma_\eta = 1\}$, where $(\gamma_\eta)_{\eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c})}$ are independent $\{0,1\}$-valued random variables with

$$\text{Prob}(\eta \text{ is open}) = \text{Prob}(\gamma_\eta = 1) = p_\Lambda(\eta),$$

where

$$p_\Lambda(\eta) \quad := \quad \begin{cases} 1 - \exp\big(-\beta\phi_0(\xi_\eta)\big) & \text{if } \xi_\eta \in H_{\mathbb{R}^d} \setminus H_{\Lambda^c}, \\ 1 & \text{if } \xi_\eta \in H_{\Lambda^c}. \end{cases} \tag{3.6}$$

Let $\mathscr{H}$ be the set of all hyperedge configurations,

$$\mathscr{H} := \{H \subset H_{\mathbb{R}^d} : H \text{ is locally finite}\},$$

which comes equipped with the σ-field generated by the counting variables $H \to \#(H \cap G)$ with bounded measurable $G \subset H_{\mathbb{R}^d}$. For boundary condition $\xi_\omega \in \mathcal{K}$, the probability measure $\mathbb{P}^z_{\Lambda,\omega}$ on $\Omega \times \mathscr{H}$ is defined by

$$\mathbb{P}^z_{\Lambda,\omega}(d\omega', dH) := \frac{1}{\bar{Z}^z_{\Lambda,\omega}} P^z_{\Lambda,\omega}(d\xi_\zeta)\, \lambda_{\zeta\omega_{\Lambda^c}}(d\omega')\, \mu_{\zeta\omega_{\Lambda^c}}(dH), \tag{3.7}$$

where $\omega' := \zeta \cup \omega_{\Lambda^c} \in \Omega$,

$$\bar{Z}^z_{\Lambda,\omega} := \int_{\Omega_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta)\, \lambda_{\zeta\omega_{\Lambda^c}}(d\omega') \int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH)$$

is a normalisation constant. Note that $\zeta \in \Omega_\Lambda$ is the projection of $\omega' \in \Omega$ onto Λ. One can see that, given a set of occupied positions $\xi_\omega \in \mathcal{K}$, the measure $\mathbb{P}^z_{\Lambda,\omega}$ sets up a marked configuration of particles $\omega' = \zeta\omega_{\Lambda^c}$ and a configuration of open hyperedges H via the following steps:

1. Given the boundary condition ξ_ω, the distribution $P^z_{\Lambda,\omega}$ distributes particle positions of ζ in Λ. This forms ξ_ζ.
2. Given ξ_ζ, the type-picking mechanism $\lambda_{\zeta\omega_{\Lambda^c}}$ assigns marks to each particle in the configuration $\xi_\zeta\xi_{\omega_{\Lambda^c}} = \xi_{\zeta\omega_{\Lambda^c}}$. This forms $\zeta\omega_{\Lambda^c}$.
3. Given $\zeta\omega_{\Lambda^c}$, the hyperedge-drawing mechanism $\mu_{\zeta\omega_{\Lambda^c}}$ assigns types to the hyperedges of $\mathcal{E}^T(\zeta\omega_{\Lambda^c})$.

We now discuss the measurability of $\lambda_{\zeta\omega_{\Lambda^c}}$ and $\mu_{\zeta\omega_{\Lambda^c}}$. It is clear that $\lambda_{\zeta\omega_{\Lambda^c}}$ depends measurably on $\xi_\zeta\xi_{\omega_{\Lambda^c}}$, therefore the mapping $\xi_\zeta\xi_{\omega_{\Lambda^c}} \to \lambda_{\zeta\omega_{\Lambda^c}}$ is a probability kernel from $\mathcal{K}$ to Ω. Consider the Laplace transform $\mathcal{L}_{\zeta\omega_{\Lambda^c}}$ of $\mu_{\zeta\omega_{\Lambda^c}}$. For any measurable function $f : H_{\mathbb{R}^d} \to [0,\infty)$,

$$\begin{aligned} \mathcal{L}_{\zeta\omega_{\Lambda^c}}(f) &:= \int_{\mathscr{H}} \exp\Big\{ - \sum_{\xi_\eta \in H} f(\xi_\eta) \Big\} \mu_{\zeta\omega_{\Lambda^c}}(dH) \\ &= \prod_{\eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c})} \big(p_\Lambda(\eta)\, e^{-f(\xi_\eta)} + 1 - p_\Lambda(\eta) \big) \\ &= \exp\Big\{ - \sum_{\eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c})} \tilde{f}(\xi_\eta) \Big\}, \end{aligned}$$

where

$$\tilde{f}(\xi_\eta) := -\log\big\{ e^{-f(\xi_\eta)} + \mathbb{I}\{\xi_\eta \cap \Lambda \neq \emptyset\}\, e^{-\phi_0(\xi_\eta)}\, (1 - e^{-f(\xi_\eta)}) \big\}.$$

Since $\tilde{f}$ is measurable, the mapping $\xi_{\zeta\omega_{\Lambda^c}} \to \mathcal{L}_{\zeta\omega_{\Lambda^c}}$ is measurable, and hence the mapping $\xi_{\zeta\omega_{\Lambda^c}} \to \mu_{\zeta\omega_{\Lambda^c}}$ is a probability kernel from $\mathcal{K}$ to $\mathscr{H}$.

Consider the event $X \subset \Omega \times \mathscr{H}$, defined

$$X := \Big\{ (\omega, H) \in \Omega \times \mathscr{H} : \sum_{\xi_\eta \in H} \big(1 - \mathbb{I}\{\exists\, i \in S : u^\eta_x = i, \forall\, x \in \xi_\eta\} \big) = 0 \Big\}. \tag{3.8}$$

This is the event that the marks of the particles are the same on each connected component of the graph (ξ_ω, H). Equivalently, X can be described as the event that for any hyperedge $\eta \in \mathcal{E}^T(\omega)$, if the points of η are not of the same type, then η cannot be open. Define the random-cluster representation measure, on $\Omega \times \mathscr{H}$, as

$$\mathbb{P} := \mathbb{P}^z_{\Lambda,\omega}\big(\,\cdot\, \big| X\big). \tag{3.9}$$

Let pr be the projection from $\Omega \times \mathscr{H}$ onto Ω and sp the projection from $\Omega \times \mathscr{H}$ onto $\mathcal{K} \times \mathscr{H}$. For each $(\omega, H) \in \Omega \times \mathscr{H}$, let $K(\xi_\omega, H)$ denote the number of connected components in the graph (ξ_ω, H). If there exists a sequence $x_1, \xi_{\eta_1}, x_2, \xi_{\eta_2}, \dots, x_n$ of distinct $x_1, \dots, x_n \in \xi_\omega$ and distinct $\xi_{\eta_1}, \dots, \xi_{\eta_{n-1}} \in H$ such that $\{x_i, x_{i+1}\} \subseteq \xi_{\eta_i}$ for all $i = 1, \dots, n-1$, then we say $x_1, \dots, x_n$ are members of the same connected component. So $K(\xi_\omega, H)$ is the number of components of the hypergraph (ξ_ω, H), where H is the set of hyperedges (with no reference to the particle marks) that are open. Let $K_\Lambda(\xi_\omega, H)$ be the number of connected components completely contained within Λ. Given $\xi_{\zeta\omega_{\Lambda^c}} \in \mathcal{K}$, if the set of hyperedges H is distributed according to $\mu_{\zeta\omega_{\Lambda^c}}$, then $K(\xi_{\zeta\omega_{\Lambda^c}}, H)$ is the number of connected components of $(\xi_{\zeta\omega_{\Lambda^c}}, H)$ that are completely contained within Λ plus the infinite cluster outside Λ. We will sometimes write $K(\xi_{\zeta\omega_{\Lambda^c}}, H)$ as $K(\zeta\omega_{\Lambda^c}, H)$, but note that this function has no dependence on the marks of the particles. We discuss the measurability of $K(\,\cdot\,,\,\cdot\,)$ in Proposition 3.3.

For a bounded region $\Lambda \in \mathcal{B}_R$ and boundary condition $\xi_\omega \in \mathcal{K}$, define the multi-body continuum random-cluster distribution $C^{z,q}_{\Lambda,\omega}$ on $\mathcal{K}_\Lambda \times \mathscr{H}$ by

$$C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dH) := \frac{1}{\hat{Z}^{z,q}_{\Lambda,\omega}}\, q^{K(\zeta\omega_{\Lambda^c}, H)}\, P^{z/q}_{\Lambda,\omega}(d\xi_\zeta)\, \mu_{\zeta\omega_{\Lambda^c}}(dH), \tag{3.10}$$

where $P^{z/q}_{\Lambda,\omega}$ is defined by (3.4) with activity z/q, and

$$\hat{Z}^{z,q}_{\Lambda,\omega} := \int_{\Omega_\Lambda} P^{z/q}_{\Lambda,\omega}(d\xi_\zeta) \int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH)\, q^{K(\zeta\omega_{\Lambda^c}, H)}$$

is a normalisation constant. The following propositions state that

(i) if we disregard the hyperedges of the random-cluster representation, then we obtain a Gibbs distribution of the form (2.10) for the geometry-dependent continuum Potts model, and

(ii) if we disregard the particle types of the random-cluster representation measure, then we obtain the multi-body continuum random-cluster distribution (3.10).

Proposition 3.1. $\mathbb{P} \circ \mathrm{pr}^{-1} = G^{z,\mu}_{\Lambda,\omega}$.

Proof. Let $X_\omega \subset \mathscr{H}$ be the ω-section of $X \subset \Omega \times \mathscr{H}$, i.e. the event that for given $\omega \in \Omega$, any hyperedge of $\mathcal{E}^T(\omega)$ must be closed if the spins of the particles within the hyperedge do not match.

$$\begin{aligned}
\mu_{\zeta\omega_{\Lambda^c}}(X_{\zeta\omega_{\Lambda^c}}) &= \prod_{\substack{\eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c}) \\ \nexists i \in S:\, u^\eta_x = i,\, \forall x \in \xi_\eta}} (1 - p_\Lambda(\eta)) \\
&= \exp\Bigg\{ - \sum_{\substack{\eta \in \mathcal{E}^T(\zeta\omega_{\Lambda^c}) \\ \xi_\eta \cap \Lambda \neq \emptyset \\ \nexists i \in S:\, u^\eta_x = i,\, \forall x \in \xi_\eta}} \phi_0(\xi_\eta) \Bigg\} &\quad (3.11) \\
&= \exp\Big\{ - H^\phi_{\Lambda,\omega}(\zeta) \Big\}. &\quad (3.12)
\end{aligned}$$

Note that $\int \Pi^z_\Lambda(d\xi_\zeta)\lambda_{\zeta\omega_{\Lambda^c}}$ is the same as the measure $\Pi^{z,\mu}_\Lambda$ on Ω_Λ, where μ is a uniform distribution on $S = \{1, \ldots, q\}$. This is because both are distributing the positions in Λ according to a Poisson point process with activity z, and then assigning marks according to the a uniform distribution on $S = \{1, \ldots, q\}$. Therefore, for any bounded measurable function f on Ω,

$$\begin{aligned}
\int_{\Omega \times \mathscr{H}} f \circ \mathrm{pr}\, d\mathbb{P} &= \frac{1}{\mathbb{P}^z_{\Lambda,\omega}(X)} \int_X f \circ \mathrm{pr}\, d\mathbb{P}^z_{\Lambda,\omega} \\
&= \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)} \int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta) \int_\Omega \lambda_{\zeta\omega_{\Lambda^c}}(d\omega') f(\omega') \mu_{\zeta\omega_{\Lambda^c}}(X_\omega) \\
&= \frac{1}{Z^z_{\Lambda,\omega}\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)} \\
&\quad \times \int_\Omega \Pi^{z,\mu}_\Lambda(d\zeta)\, f(\zeta\omega_{\Lambda^c})\, \exp\{ - H^\psi_{\Lambda,\omega}(\zeta) - H^\phi_{\Lambda,\omega}(\zeta)\} \\
&\qquad \times \mathbb{I}\Big\{ \omega_{\Lambda^c} = \Big(\xi_{\omega_{\Lambda^c}}, \{u^{\omega_{\Lambda^c}}_x : x \in \xi_{\omega_{\Lambda^c}}\}\Big) : u^{\omega_{\Lambda^c}}_x = 1, \forall\, x \in \xi_{\omega_{\Lambda^c}} \Big\} \\
&= c_1 \int_\Omega f\, dG^{z,\mu}_{\Lambda,\omega},
\end{aligned}$$

where $c_1 = 1$ since both $\mathbb{P}$ and $G^{z,\mu}_{\Lambda,\omega}$ are probability measures. Note that the Gibbs distribution $G^{z,\mu}_{\Lambda,\omega}$ is that described by (2.10), with boundary condition ω consisting of the points of ξ_ω as positions and all marks of ω_{Λ^c} set as type 1. $\square$

Proposition 3.2. $\mathbb{P} \circ \mathrm{sp}^{-1} = C^{z,q}_{\Lambda,\omega}$.

Proof. For $(\xi_\omega, H) \in \mathcal{K} \times \mathscr{H}$, let

$$X_{(\xi_\omega,H)} := \{\omega \in \Omega : (\omega, H) \in X\}, \tag{3.13}$$

where $X \subset \Omega \times \mathscr{H}$ is defined by (3.8). Therefore

$$\lambda_{\zeta\omega_{\Lambda^c}}(X_{(\zeta\omega_{\Lambda^c},H)}) = \frac{q^{K_\Lambda(\zeta\omega_{\Lambda^c},H)}}{q^{\#(\xi_\zeta)}}. \tag{3.14}$$

For any measurable function f on $\mathcal{K}_\Lambda \times \mathscr{H}$,

$$\begin{aligned}\int_{\Omega\times\mathscr{H}} f \circ \mathrm{sp}\, d\mathbb{P} = \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)} &\int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}\,(d\xi_\zeta) \\ &\times \int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}\,(dH)\, f(\xi_\zeta, H)\, \lambda_{\zeta\omega_{\Lambda^c}}(X_{(\xi_{\zeta\omega_{\Lambda^c}},H)})\end{aligned}$$

$$= \frac{Z^{z/q}_{\Lambda,\omega}}{Z^z_{\Lambda,\omega}\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)} \int_{\mathcal{K}_\Lambda} P^{z/q}_{\Lambda,\omega}\,(d\xi_\zeta) \int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}\,(dH)\, f(\xi_\zeta, H)\, q^{K_\Lambda(\zeta\omega_{\Lambda^c},H)} \tag{3.15}$$

$$= c_2 \int_{\Omega_\Lambda\times\mathscr{H}} f(\xi_\zeta, H)\, C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dH). \tag{3.16}$$

Using definition (3.4), one can see that $P^z_{\Lambda,\omega}$ is absolutely continuous relative to $P^{z/q}_{\Lambda,\omega}$ with Radon-Nikodym density proportional to $q^{\#(\xi_\omega\cap\Lambda)}$. This gives line (3.15). Recall that $K_\Lambda(\zeta\omega_{\Lambda^c}, H)$ is the total number of clusters, i.e. connected components of $(\xi_{\zeta\omega_{\Lambda^c}}, H)$, that are contained within Λ. $K(\zeta\omega_{\Lambda^c}, H)$ is the total number of clusters within Λ plus the infinite cluster outside Λ. Line (3.16) comes from the definition of $C^{z,q}_{\Lambda,\omega}$ and the fact that $K(\zeta\omega_{\Lambda^c}, H) - K_\Lambda(\zeta\omega_{\Lambda^c}, H)$ is constant, equal to 0 or 1. To see this, note that there is just one infinite cluster outside Λ. If this outer cluster is attached to a cluster inside Λ, then the total number of clusters is equal to the number of clusters within Λ, i.e. $K(\zeta\omega_{\Lambda^c}, H) = K_\Lambda(\zeta\omega_{\Lambda^c}, H)$. If the outer cluster is separate to the clusters within Λ, then the total number of clusters consists of the clusters inside Λ plus the single outside cluster, hence $K(\zeta\omega_{\Lambda^c}, H) = K_\Lambda(\zeta\omega_{\Lambda^c}, H) + 1$. Since both $\mathbb{P}$ and $C^{z,q}_{\Lambda,\omega}$ are probability measures, we have $c_2 = 1$. $\square$

For any boxes $\Delta, \Lambda \subset \mathbb{R}^2$, configuration $\omega \in \Omega$, $H \in \mathscr{H}$, and $s \in S$, let

$$\begin{aligned} N_\Delta(\xi_\omega) &:= |\{x \in \xi_\omega \cap \Delta\}|, \\ N_{\Delta,s}(\xi_\omega) &:= |\{x \in \xi_\omega \cap \Delta : u_x^\omega = s\}|, \\ N_{\Delta \leftrightarrow \Lambda^c}(\xi_\omega, H) &:= |\{x \in \xi_\omega \cap \Delta : x \text{ belongs to a component} \\ &\qquad \text{connected to } \Lambda^c \text{ in } (\xi_\omega, H)\}|. \end{aligned}$$

For convenience, we often write $N_\Delta(\xi_\omega) = N_\Delta(\omega)$, and similarly for $N_{\Delta,s}$ and $N_{\Delta \leftrightarrow \Lambda^c}$.

Proposition 3.3. *For any Borel measurable $\Delta \subset \Lambda \subset \mathbb{R}^d$, the functions $N_{\Delta \leftrightarrow \Lambda^c}$ and $K(\,\cdot\,,\,\cdot\,)$ on $\mathcal{K} \times \mathscr{H}$ are measurable.*

Proof. Let B be the set of all $(x, y, \xi_{\zeta\omega_{\Lambda^c}}, H) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{K} \times \mathscr{H}$ which are such that if $x, y \in \xi_{\zeta\omega_{\Lambda^c}}$, $x \neq y$, then x is connected to y in the graph $(\xi_{\zeta\omega_{\Lambda^c}}, H)$. We can write $B = \cup_{k\geq 1} B_k$, where

$$B_1 := \{(x, y, \xi_{\zeta\omega_{\Lambda^c}}, H) :\; x, y \in \xi_{\zeta\omega_{\Lambda^c}}, \{x, y\} \subset \xi_\eta \text{ for some } \xi_\eta \in H\}$$

and, for $k \geq 1$,

$$\begin{aligned} B_{k+1} := \Big\{(x, y, \xi_{\zeta\omega_{\Lambda^c}}, H) : \sum_{z \in \xi_{\zeta\omega_{\Lambda^c}}} \Big(&\mathbb{I}\{(x, z, \xi_{\zeta\omega_{\Lambda^c}}, H) \in B_1\} \\ &\times \mathbb{I}\{(z, y, \xi_{\zeta\omega_{\Lambda^c}}, H) \in B_k\}\Big) > 0\Big\}. \end{aligned}$$

Since the functions $(x, \xi_{\zeta\omega_{\Lambda^c}}) \to \mathbb{I}\{x \in \xi_{\zeta\omega_{\Lambda^c}}\}$, $(x, y, H) \to \mathbb{I}\{\{x, y\} \subset \xi_\eta \in H\}$ and $\xi_{\zeta\omega_{\Lambda^c}} \to \sum_{z \in \xi_{\zeta\omega_{\Lambda^c}}} f(z, \xi_{\zeta\omega_{\Lambda^c}})$ are measurable (for measurable f), it follows by induction that B_k is measurable for any $k \geq 1$. Therefore B is measurable. Let $f(x, \xi_{\zeta\omega_{\Lambda^c}}, H) = 1$ when $\sum_{y \in \xi_{\omega_{\Lambda^c}}} \mathbb{I}\{(x, y, \xi_{\zeta\omega_{\Lambda^c}}, H) \in B\} > 0$ and $f(x, \xi_{\zeta\omega_{\Lambda^c}}, H) = 0$ otherwise.

$$N_{\Delta \leftrightarrow \Lambda^c}(\xi_{\zeta\omega_{\Lambda^c}}, H) = \sum_{x \in \xi_\zeta \cap \Delta} f(x, \xi_{\zeta\omega_{\Lambda^c}}, H),$$

therefore $N_{\Delta \leftrightarrow \Lambda^c}$ is measurable on $\mathcal{K} \times \mathscr{H}$.

For any $l \geq 1$, we have $K(\zeta\omega_{\Lambda^c}, H) \geq l$ if and only if

$$\sum_{x_1, \ldots, x_l \in \xi_{\zeta\omega_{\Lambda^c}}} \prod_{1 \leq i < j \leq l} \mathbb{I}\{x_i \neq x_j\} \mathbb{I}\{(x_i, x_j, \xi_{\zeta\omega_{\Lambda^c}}, H) \in B^c\} > 0.$$

The above expression depends measurably on $(\xi_{\zeta\omega_{\Lambda^c}}, H)$. Therefore $K(\,\cdot\,,\,\cdot\,)$ is measurable. $\square$

Proposition 3.4 relates the Gibbs measure for the model, given a boundary condition, to the connectivity probabilities of the random-cluster distribution.

Proposition 3.4. *For any measurable* $\Delta \subseteq \Lambda$,

$$\int_\Omega (q\, N_{\Delta,1} - N_\Delta)\, dG^{z,\mu}_{\Lambda,\omega} = (q-1)\int_{\mathcal{K}_\Lambda\times\mathscr{H}} N_{\Delta\leftrightarrow\Lambda^c}\, dC^{z,q}_{\Lambda,\omega}.$$

Proof. Define f on Ω as $f = q\, N_{\Delta,1} - N_\Delta$ and apply Proposition 3.1:

$$\begin{aligned}
\int_\Omega f\, dG^{z,\mu}_{\Lambda,\omega} &= \mathbb{P}^z_{\Lambda,\omega}(X)^{-1}\int_X f\circ \mathrm{pr}\, d\mathbb{P}^z_{\Lambda,\omega} \\
&= \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)}\int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta)\int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH)\int_{X_{(\xi_{\omega'},H)}} \lambda_{\zeta\omega_{\Lambda^c}}(d\omega')f(\omega'). \\
&= \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)}\int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta)\int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH) \\
&\qquad\times \sum_{x\in\xi_\zeta\cap\Delta}\int_{X_{(\xi_{\omega'},H)}} \lambda_{\zeta\omega_{\Lambda^c}}(d\omega')\Big(q\mathbb{I}\{u_x^{\zeta_\Delta}=1\}-1\Big).
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)}\int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta)\int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH) \\
&\qquad\times \sum_{x\in\xi_\zeta\cap\Delta} \lambda_{\zeta\omega_{\Lambda^c}}(X_{(\xi_{\omega'},H)})\,(q-1)\,\mathbb{I}\{x\leftrightarrow\Lambda^c \text{ in } (\xi_{\omega'},H)\} \qquad (3.17) \\
&= \frac{1}{\bar{Z}^z_{\Lambda,\omega}\mathbb{P}^z_{\Lambda,\omega}(X)}\int_{\mathcal{K}_\Lambda} P^z_{\Lambda,\omega}(d\xi_\zeta)\int_{\mathscr{H}} \mu_{\zeta\omega_{\Lambda^c}}(dH)\lambda_{\zeta\omega_{\Lambda^c}}(X_{(\xi_{\omega'},H)})\,(q-1)N_{\Delta\leftrightarrow\Lambda^c} \\
&= \int_{\mathcal{K}_\Lambda\times\mathscr{H}} (q-1)N_{\Delta\leftrightarrow\Lambda^c}\, dC^{z,q}_{\Lambda,\omega}. \qquad (3.18)
\end{aligned}$$

Line (3.17) holds because if x is connected to Λ^c in $(\xi_{\omega'},H) = (\xi_{\zeta\omega_{\Lambda^c}},H)$, then

$$\int_{X_{(\xi_{\omega'},H)}} \lambda_{\zeta\omega_{\Lambda^c}}(d\omega')\big(q\mathbb{I}\{u_x^{\zeta_\Delta}=1\}-1\big) = (q-1)\lambda_{\zeta\omega_{\Lambda^c}}(X_{(\xi_{\omega'},H)})$$

because all particles in the same cluster have the same type, and particles in Λ^c are of type 1. If x is not connected to Λ^c then

$$\int_{X_{(\xi_{\omega'},H)}} \lambda_{\zeta\omega_{\Lambda^c}}(d\omega')\big(q\mathbb{I}\{u_x^{\zeta_\Delta}=1\}-1\big) = 0$$

because the type of x is independent of $X_{(\xi_{\omega'},H)}$ under $\lambda_{\zeta\omega_{\Lambda^c}}$ and so the probability of x taking any given type from S is $1/q$. Line (3.18) follows as in the proof of Proposition 3.2, see (3.16). $\square$

If we can show that there exists some $\alpha > 0$ such that

$$\int_{\mathcal{K}_\Lambda \times \mathscr{H}} N_{\Delta \leftrightarrow \Lambda^c} \, dC^{z,q}_{\Lambda,\omega} \geq \alpha, \tag{3.19}$$

then Proposition 3.4 implies that

$$\int_\Omega (q \, N_{\Delta,1} - N_\Delta) \, dG^{z,\mu}_{\Lambda,\omega} \geq (q-1)\alpha. \tag{3.20}$$

This means that

$$\int_\Omega (qN_{\Delta,k} - N_\Delta) \, dG^{z,\mu}_{\Lambda,\omega^{(k)}} \geq (q-1)\,\alpha, \tag{3.21}$$

for $k = 1, \ldots, q$, where $\omega^{(k)} := \zeta \cup \omega^{(k)}_{\Lambda^c}$ is a configuration with tempered boundary condition that fixes all particles outside of Λ to be type k.

Once the existence of Gibbs measures for the model with such a boundary condition has been established, the existence of multiple Gibbs measures follows by the classical argument. There is a unique Gibbs measure if and only if

$$\int_\Omega qN_{\Delta,k} \, dG^{z,\mu}_{\Lambda,\omega^{(k)}} = \int_\Omega N_\Delta \, dG^{z,\mu}_{\Lambda,\omega^{(k)}} \quad \forall k = 1, \ldots, q. \tag{3.22}$$

Therefore (3.21) means there are at least q distinct measures with distinct density of marks. For a model given by interaction potentials ψ and ϕ, there will be a range for the modelling parameters such that (3.19) holds. Whenever (3.19) holds for specific parameters, the existence of multiple Gibbs measures for these parameters is given by Proposition 3.4.

When analysing configurations with boundary conditions, we fix conditions outside some bounded box and analyse the finite configuration within the box. We can then divide this box into three layers. The purpose of this is to partition the continuum in order to make comparisons to a lattice case, and also condition on the configuration within certain regions. We will see more details on this later. The first layer is the macro-box, $\Lambda \subset \mathbb{R}^d$. This is the bounded region of $\mathbb{R}^d$, within which we analyse a random configuration. Outside Λ, we assume some prescribed configuration (the boundary condition). This macro-box Λ is divided into a partition $\Lambda = \cup_{k,l} \Delta_{k,l}$, where the range of k and l depends on the number of boxes that make up the partition of Λ. The boxes $\Delta_{k,l}$ are meso-boxes of the configuration, and form a partition of micro-boxes.

Meso-boxes are introduced in order to divide the macro-box and compare it to a finite region of a lattice, each meso-box representing a site of the lattice. Each meso-box is then divided into a partition of micro-boxes so that we can analyse the configuration within a meso-box. By analysing the probability that each micro-box contains at least one particle, we can find the probability that the respective meso-box has a sufficiently consistent density of particles. The size of the micro-boxes depends upon the specific details of the model under investigation. For the models of Section 4, the meso-boxes comprise either 9 or 81 micro-boxes. The size of the meso-boxes also depends upon the details of the model under investigation, but they must be sufficiently large such that each micro-box may contain at least one particle.

The aim is to compare continuum site percolation to site percolation on a lattice. This is done by discretisation of the continuous space, we compare each meso-box to a site of the lattice. We show that if the configuration within each meso-box satisfies some percolation property, then there is a connected path of matching typed particles from one meso-box to another. One can think of a meso-box exhibiting the percolation property being analogous to a site of the lattice being open.

In order to show (3.19) holds, we shall introduce a measure $\widetilde{C}^{z,q}_{\Lambda,\omega}$ which is stochastically smaller than $C^{z,q}_{\Lambda,\omega}$ and, conditional on the particle configuration, has hyperedges drawn independently of one another. We then establish percolation for the new measure. However, the definition of $\widetilde{C}^{z,q}_{\Lambda,\omega}$ depends on the specific details of the model under investigation. In Section 5, we discuss different examples. If the probability, under $\widetilde{C}^{z,q}_{\Lambda,\omega}$, of the percolation event occurring for a given meso-box is greater than the critical probability of site percolation on the lattice, then there is a positive probability, under $\widetilde{C}^{z,q}_{\Lambda,\omega}$, of there being a path of connected meso-boxes to the boundary of the macro-box Λ. Since $\widetilde{C}^{z,q}_{\Lambda,\omega}$ is stochastically smaller than $C^{z,q}_{\Lambda,\omega}$, we can show there is a positive probability of percolation via hyperedges for the random-cluster distribution.

To obtain this percolation property required of each meso-box, we define micro-boxes $\Delta^{i,j}_{k,l}$ such that each $\Delta_{k,l}$ is divided into a partition $\Delta_{k,l} = \cup_{i,j}\Delta^{i,j}_{k,l}$. The range for i and j depend on how many micro-boxes make up the partition of $\Delta_{k,l}$. This choice will depend on the details of the model under investigation. We denote a general macro, meso and micro-box as Λ, Δ and ∇, respectively. Note that $\nabla \subset \Delta \subset \Lambda \subset \mathbb{R}^d$. In Section 5, we apply the multi-body continuum random-cluster distribution to some specific models in order to show that a phase transition exists in these models.

Remark 3.1. *Other random-cluster models.* Our multi-body continuum random-cluster model compares directly to the original random-cluster model, introduced by Fortuin and Kasteleyn [FK72]. Their model, also known as the Fortuin-Kasteleyn model, is used for analysing Ising and Potts models on a lattice. The Fortuin-

Kasteleyn model can be used to analyse the phase transitional behaviour of the Potts model on a lattice with a pairwise interaction between particles. Grimmett [Grim94] extends the work of Fortuin and Kasteleyn to a random-cluster model with many-body interactions on a lattice. This form of random cluster model is used to analyse lattice Potts models with hyperedge interactions. A continuum random-cluster model was introduced by Georgii and Häggström [GH96]. They extend the Fortuin-Kasteleyn model to the continuum, in order to show the existence of a phase transition in a continuum Potts model with pairwise interaction. Our multi-body continuum random-cluster model is designed for the analysis of continuum Potts models with many-body interactions. We see some applications in the next sections.

Remark 3.2. *The analysis of the Widom-Rowlinson model by stochastic geometric methods.* We have previously discussed the hard-core Widom-Rowlinson model. Recall that it is a two type interaction continuum model where there is no interaction between particles of the same type and hard-core exclusion between particles of different type. Ruelle [Rue71] showed that there is a phase transition for this model. There is another equivalent formulation for the Widom-Rowlinson model where particles have only one type (by integrating out the coordinates of one type, the effective diameter of the remaining particle positions is doubled). Chayes, Chayes and Kotecký [CCK95] study the Widom-Rowlinson model and introduce a new geometric representation for the model in order to prove the existence of a phase transition via a percolation based proof. The new representation for the model is equivalent to the random-cluster representation for the Potts model. We can compare the representation by Chayes et. al. to the random-cluster model. In [CCK95], percolation configurations of spherical particles are generated, the size of the spheres to be radius $r_0 > 0$, where r_0 is the hard-core exclusion parameter for opposite-type particles. Groups of particles with overlapping spheres are classified as being in the same cluster. This compares directly to the distribution of particle positions in our random-cluster model. Each particle is then coloured either type 1 or 2, conditioning on the event that particles within the same cluster must be of the same type. This is analogous to our method. All permissible configurations, according to the steps taken by Chayes et. al., have weights which depend exponentially on the number of clusters within them. This compares directly with the factor of $q^{K(\xi_\omega,H)}$ in our random-cluster measure.

4 Results

We now apply the theory of the previous sections in order to prove the existence and non-uniqueness of Gibbs measures for a class of Delaunay Potts models with geometry-dependent interactions. From now on, we assume $d = 2$ and our focus is on interaction potentials acting on a planar Delaunay hypergraph.

We split our results into three sections. First, we analyse type interaction models where both the background and type interaction act over a finite range. We look at an extension of the nearest-neighbour continuum Potts model introduced by Bertin at. al. [BBD04]. We keep the hard-core pairwise background assumption but now allow the type interaction to act on triangles of the Delaunay triangulation, as opposed to the edges. There is a penalty for Delaunay triangles that do not consist of particles with the same type. This is the first case of proving a phase transition in a type interaction continuum model where marked particles act in groups of three, as opposed to pairs. Our other example of a finite-range model is for an interaction potential acting on the lily-pond graph, formed by dynamically constructing a set of touching balls in space. We prove that the set of Gibbs measures is non-empty and that the Gibbs measure is unique.

The next section focuses on a class of models where the background interaction is strictly positive for all long-range interactions. The benefit of this is that particles distributed according to such a model will have positional configurations with a geometrical structure. The particle positions bear less resemblance to a Poisson point distribution. This is a major advantage since natural continuum systems often have some sort of geometrical structure, rather than being distributed according to a pure Poisson point process. There are two key models we analyse for such geometry-dependence in the background interaction. These models are similar to those introduced by Dereudre et. al., see Example 2.4 of [DG09]. We look at strict repulsion between particles over long range, with no interaction between particles over medium range (i.e. zero contribution from the interaction potential), and hard-core repulsion between particles over short range. Configurations distributed according to this kind of model have a more even density of particles, since large hyperedges are penalised. We also discuss a geometry-dependent model that favours equilateral Delaunay triangles. In the third section of our results, we remove the finite-range assumption on the type interaction and investigate how this affects the existence of a phase transition.

Recall that in order to define the random-cluster representation for a type interaction model, it is required that the background and type interactions can be expressed as $\psi(\eta,\omega) = \psi_0(\xi_\eta)$ and

$$\phi(\eta,\omega) \quad = \quad \phi_0(\xi_\eta)\big(1 - \mathbb{I}\{\exists\, i \in S : \, u_x^\eta = i, \, \forall\, x \in \xi_\eta\}\big),$$

respectively. We shall be using this notation for the remainder of the study. Let

$$p_0 := \frac{1}{3}\big(1 - p_c^{\text{site}}(\mathbb{Z}^2)\big), \tag{4.1}$$

where $p_c^{\text{site}}(\mathbb{Z}^2)$ is the critical probability for Bernoulli site percolation on the integer lattice. This constant plays an important role in the results below. Note that for $\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)$, $\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)$, we use the notation $\xi_\eta = \{x_\eta, y_\eta\}$, $\xi_\tau = \{x_\tau, y_\tau, z_\tau\}$. Also, for any $L > 0$, let $[L]$ be the largest integer not greater than L. For models with a pairwise hard-core assumption, define J_L as

$$J_L := \left[\frac{L^2}{\pi r_0^2}\right] + 1, \tag{4.2}$$

where $r_0 > 0$ is the hard-core distance parameter. This is the maximum number of particles that can fit in an $L \times L$ box, given that the hard-core assumption is satisfied.

4.1 Finite-range interactions

We are interested in a model where all particles are required to have at least some distance $r_0 > 0$. Marked particles interact in triads, through the hyperedges of the Delaunay hypergraph. The formal Hamiltonian (2.7) is expressed as follows:

$$H(\omega) \quad = \quad \sum_{\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)} \psi(\eta,\omega) + \sum_{\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)} \phi(\tau,\omega) \tag{4.3}$$

where

$$\psi(\eta,\omega) \quad := \quad \psi_0(|x_\eta - y_\eta|), \tag{4.4}$$

$$\phi(\tau,\omega) \quad := \quad \phi_0(\delta(\tau))\,\big(1 - \mathbb{I}\{\exists\, i \in S : \, u_x^\tau = i, \forall\, x \in \xi_\tau\}\big) \tag{4.5}$$

and

$$\psi_0(r) := \begin{cases} \infty & \text{if } r < r_0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

$$\phi_0(\delta) := \begin{cases} 1 & \text{if } \delta < 2r_1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

for some $r_1, r_0 > 0$ such that $r_0 < r_1/\sqrt{\pi}$. Theorem 4.1 below is our main result for the model described by the Hamiltonian (4.3).

Theorem 4.1 (Finite-range triangle interaction)**.** *For*

$$\begin{aligned} z &> \frac{81 q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(\sqrt{2\pi}-2)^2 r_0^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{162}}}{1-(1-p_0)^{\frac{1}{162}}}\right\}, \end{aligned}$$

there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.3).

Remark 4.1. We can compare the model of Theorem 4.1 to the nearest-neighbour continuum Potts (NNCP) model introduced by Bertin et al. [BBD04]. The one crucial difference between the NNCP model of [BBD04] and the above Delauany Potts model is the type interaction. In the NNCP model, there is repulsion between any opposite-type pairs that share a sufficiently short edge in the Delauany triangulation. In our model, there is repulsion between triads of particles that form the vertices of a triangle, of sufficiently small diameter, in the Delaunay triangulation. The background interaction is the same in both models. So our Delaunay Potts model can be thought of as an extension of the NNCP model, from interacting pairs to interacting triads. Our bounds for z and β are very similar to those found by Bertin et. al. [BBD04]. The fact that Bertin investigates a model for interacting pairs means that q^2 is replaced with q in the bound for β in Theorem 4.1. This is because our proof in Section 5.2 requires defining a measure that is stochastically dominated by the random-cluster distribution, and this is either defined in terms of q or q^2 for edges or triangle-hyperedges, respectively.

Remark 4.2. *The k-nearest-neighbour model.* Häggström and Meester [HM96] discuss percolation for nearest-neighbour and hard-sphere models. For $d \geq 1$, the k-nearest-neighbour graph is defined such that for an unmarked particle configuration $\xi_\omega \in \mathcal{K}$, distributed according to a Poisson point process with activity $z > 0$, each particle at some position $x \in \xi_\omega$ is connected to its k nearest neighbours by some undirected edge. Häggström and Meester show that for $d \geq 2$, there exists some $k = k(d) \in [2, \infty)$ such that there is some infinite cluster in the model. Using this result, one can use a similar argument to the proof of Theorem 4.1 in Section 5.2 to show that there exists a phase transition for a model with hard-core background interaction between all pairs and a finite-range, bounded type interaction acting between pairs on the k-nearest-neighbour graph.

We now consider a hard-sphere model known as the lily-pond model. The original version was introduced by Häggström and Meester [HM96] for unmarked particles. We now provide the details for a variation of their model suited for marked particle systems. Let $d = 2$. The lily-pond hypergraph structure $\mathcal{E}^{\mathrm{LP}}_{r_{\max}}$ is defined as follows. For a marked particle configuration $\omega \in \Omega$,

$$\mathcal{E}^{\mathrm{LP}}_{r_{\max}}(\omega) := \{\eta \subset \omega : \, |\eta| = 2, \, \bar{B}^{LP}_{r_{\max}}(x_\eta, \omega) \cap \bar{B}^{LP}_{r_{\max}}(y_\eta, \omega) \neq \emptyset\}.$$

For all $x \in \xi_\omega$, the closed balls $\bar{B}^{LP}_{r_{\max}}(x, \omega)$ are defined as follows. Consider balls of radius zero around every $x \in \xi_\omega$ and let the radii grow linearly in time until they either hit another ball or reach radius $r_{\max} > 0$, at which point they stop growing. So if $\eta \in \mathcal{E}^{\mathrm{LP}}_{r_{\max}}(\omega)$, then x_η and y_η have touching balls.

Marked particles interact with other marked particles that are part of the same edge $\eta \in \mathcal{E}^{\mathrm{LP}}_{r_{\max}}(\omega)$. Giving the balls a maximum radius $r_{\max}$ prevents marked particles from interacting with one another when there is a huge distance between them. This ensures finite range of the interaction. This condition also allows assumption (U1) to hold as we shall see later. This model is a q-typed particle system in $\mathbb{R}^2$ with soft-core exclusion between particles of different colour and hard-core pair interaction between all particles. The formal Hamiltonian is given by

$$H(\omega) = \sum_{\eta \in \mathcal{E}^{\mathrm{LP}}_{r_{\max}}(\omega)} \psi(\eta, \omega) + \sum_{\eta \in \mathcal{E}^{\mathrm{LP}}_{r_{\max}}(\omega)} \phi(\eta, \omega), \tag{4.8}$$

where $\psi(\eta, \omega)$ is the pairwise hard-core interaction defined by (4.4), and

$$\phi(\eta, \omega) \;\; := \;\; \mathbb{I}\{\sigma(x_\eta) \neq \sigma(y_\eta)\}. \tag{4.9}$$

The following theorem states that a phase transition does not occur for this model.

Theorem 4.2 (Lily-pond model)**.** *For the lily-pond model given by the Hamiltonian in (4.8), there exists exactly one Gibbs measure for every $z, \beta > 0$.*

4.2 Infinite-range of the background interaction

Consider an extension of Theorem 4.1. The type interaction remains the same, and we keep the hard-core assumption on the background interaction. The difference is an additional background interaction, acting on Delaunay triangles, that favours configurations with equilateral triangles. A higher penalty is paid to configurations with many flat triangles. This adds a geometric dependence to the previously pairwise background interaction. The formal Hamiltonian energy is given by

$$H(\omega) \;=\; \sum_{\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)} \psi(\eta, \omega) + \sum_{\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)} \big(\phi(\tau, \omega) + \psi_{\mathrm{tri}}(\tau, \omega)\big), \tag{4.10}$$

where ψ and ϕ are the interaction potentials defined by (4.4) and (4.5), respectively. Let

$$\psi_{\mathrm{tri}}(\tau, \omega) \;:=\; -\frac{A(\tau)}{\delta(\tau)^2}, \tag{4.11}$$

where $A(\tau)$ is defined as the area of the interior of the triangle with vertices ξ_τ. Once again, the distance parameters in ψ and ϕ satisfy $r_0 < r_1/\sqrt{\pi}$. Note that for any triangle τ, the background interaction ψ_{tri} is negative and minimised for equilateral triangles:

$$0 > -\frac{A(\tau)}{\delta(\tau)^2} \geq -\frac{3\sqrt{3}}{16}, \tag{4.12}$$

for any triangle τ. The following is our main result for such a model.

Theorem 4.3 (Equilateral Delaunay triangle interaction)**.** *If*

$$\begin{aligned} z &> \frac{81\, e^{\frac{33\sqrt{3}}{8}}\, q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(\sqrt{2\pi}-2)^2 r_0^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{162}}}{1-(1-p_0)^{\frac{1}{162}}}\right\}, \end{aligned}$$

then there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.10).

We now consider a Delaunay Potts model where the background interaction is between pairs, and attains only finite values for large distances. We keep the hard-core assumption on the background interaction, and we also keep the finite-range assumption on the type interaction. The formal Hamiltonian is given by

$$H(\omega) = \sum_{\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)} \psi(\eta, \omega) + \sum_{\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)} \phi(\tau, \omega), \tag{4.13}$$

where ϕ is defined by (4.5), and

$$\psi(\eta, \omega) := \psi_1(|x_\eta - y_\eta|), \tag{4.14}$$

where

$$\psi_1(r) := \begin{cases} \infty & \text{if } r < r_0, \\ 0 & \text{if } r_0 \le r < R_0, \\ K & \text{if } r \ge R_0, \end{cases} \tag{4.15}$$

for some $K, r_0, R_0 > 0$. We assume the parameters r_1, r_0 and R_0 satisfy:

$$\left(1 + \sqrt{1 + \frac{\pi}{8\beta K}}\right) r_0 < R_0 < \left(\sqrt{19\pi}\right) r_0, \tag{4.16}$$

and $R_0 < \sqrt{2} r_1$. Note that this also means β and K must satisfy

$$1 + \sqrt{1 + \frac{\pi}{8\beta K}} < \sqrt{19\pi}.$$

The following is our phase transition result for this model.

Theorem 4.4 (Infinite-range pairwise background interaction)**.** *Let $p_1 \in (1 - p_0, 1)$ be given. Then for*

$$z > \frac{p_1 \, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1 + p_0 - 1)(R_0 - 2r_0)^2},$$
$$\beta > \log\left\{ \frac{1 + (q^2 - 1)(1 - p_0)^{\frac{1}{180}}}{1 - (1 - p_0)^{\frac{1}{180}}} \right\},$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.13).

Remark 4.3. We assume $K > 0$. However, note that $K = 0$ implies $\psi_1 = \psi_0$, and we have the previous model described by the Hamiltonian in (4.3). We assume $K > 0$, because if $K = 0$ then the method we use to prove Theorem 4.4 is no longer valid. This is due to the fact that we require the probability that two particles are farther than R_0 to be sufficiently small.

Remark 4.4. If we were to allow R_0 to be larger than $(\sqrt{19\pi})r_0$, then this would increase the phase transition bound for β. The choice of factoring $\sqrt{\pi}r_0$ with $\sqrt{19}$ ensures $J_{R_0} < 20$. This choice of $\sqrt{19}$ is arbitrary and is chosen so that we can take R_0 sufficiently large in our model. If we choose, for example, $R_0 < (\sqrt{2\pi})r_0$, then $J_{R_0} < 2$ and this improves the bounds on z and β but means R_0 is very close to $2r_0$.

We now consider the case where the interaction is solely between pairs sharing an edge in the Delaunay graph. The formal Hamiltonian is expressed as follows:

$$H(\omega) \quad = \sum_{\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)} \Big(\psi(\eta,\omega) + \phi(\eta,\omega)\Big), \tag{4.17}$$

where ψ is defined by (4.14), and

$$\phi(\eta,\omega) \quad := \quad \phi_1(|x_\eta - y_\eta|)\,\mathbb{I}\{u_x^\eta \neq u_y^\eta\}, \tag{4.18}$$

where

$$\phi_1(r) \quad := \quad \begin{cases} 1 & \text{if } r < r_1, \\ 0 & \text{if } r \geq r_1, \end{cases} \tag{4.19}$$

for some $r_1 > 0$. We assume the parameters satisfy (4.16) and $2R_0 < \sqrt{2}r_1$. Note that R_0 is included in this model in the definition of ψ, which is given by (4.14). For this model we have the following result.

Theorem 4.5 (Infinite-range background interaction with pairwise type interaction)**.** *Let $p_1 \in (1 - p_0, 1)$ be given. Then for*

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1 + p_0 - 1)(R_0 - 2r_0)^2}, \\ \beta &> \log\left\{\frac{1 + (q-1)(1-p_0)^{\frac{1}{180}}}{1 - (1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.17).

Consider the following Hamiltonian energy, with a hard-core background interaction between pairs and an infinite-range background interaction on Delaunay triangles,

$$H(\omega) \;=\; \sum_{\eta\in\mathcal{E}^{\mathrm{D}_2}(\omega)} \psi(\eta,\omega) + \sum_{\tau\in\mathcal{E}^{\mathrm{D}_3}(\omega)} \Big(\psi_{\mathrm{tri}}(\tau,\omega) + \phi(\tau,\omega)\Big), \tag{4.20}$$

where ψ and ϕ are defined by (4.4) and (4.5), respectively. Let

$$\psi_{\mathrm{tri}}(\tau,\omega) \;:=\; \psi_2(\delta(\tau)), \tag{4.21}$$

where

$$\psi_2(\delta) \;:=\; \begin{cases} 0 & \text{if } \delta < D_0, \\ K & \text{if } \delta \geq D_0, \end{cases} \tag{4.22}$$

for some $D_0 > 0$. We assume the parameters satisfy:

$$\left(1+\sqrt{1+\frac{\pi}{8\beta K}}\right) r_0 < D_0 < \sqrt{19\pi}\, r_0, \tag{4.23}$$

and $D_0 < \sqrt{2} r_1$. Assumption (4.23) is assumption (4.16), replacing R_0 with D_0. The following is our main result for this model.

Theorem 4.6 (Infinite-range triangle background interaction)**.** *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(D_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.20).

4.3 Infinite-range of the type interaction

We now consider an extension of the models from the previous section. We keep the hard-core and infinite-range assumptions on the background interaction, but we now remove the finite-range assumption on the type interaction. Such a model is characterised by the following Hamiltonian energy:

$$H(\omega) \;=\; \sum_{\eta\in\mathcal{E}^{\mathrm{D}_2}(\omega)} \psi(\eta,\omega) + \sum_{\tau\in\mathcal{E}^{\mathrm{D}_3}(\omega)} \phi(\tau,\omega), \tag{4.24}$$

where ψ is defined by (4.14), and

$$\phi(\tau,\omega) \;:=\; 1-\mathbb{I}\{\exists\, i\in S:\, u_x^\tau = i, \forall\, x\in\xi_\tau\}. \tag{4.25}$$

Again, we assume the parameters satisfy (4.16).

Theorem 4.7 (Infinite-range type interaction)**.** *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$\begin{aligned} z &> \frac{p_1\, q^{161} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exist at least q distinct Gibbs measures for the model given by the Hamiltonian in (4.24).

Remark 4.5. For all our main results, the bounds for z and β are extremely high. We emphasise that these bounds are not critical thresholds. Our results state whether or not a phase transition occurs in each model, the bounds provided are to give the reader an idea of how the parameters may affect the model. For example, if the bound for z is increasing in q, this would suggest that allowing the particles to be assigned a wider selection of marks implies the particle density must be higher to maintain a phase transition (keeping other parameters the same). We will discuss this in more detail in the conclusion, Section 7.

5 Proofs of the theorems

We now present proofs of Theorems 2.1 and 4.1 - 4.7. Section 5.1 is the proof of our existence theorem, and Sections 5.2 - 5.8 prove our phase transition results. Sometimes the similarity of the models means the phase transition proofs follow very similar arguments, in which case we emphasise the key differences.

5.1 Existence of marked Gibbs measures (Theorem 2.1)

Please note that the following proof is a very slight adaptation to that provided by Dereudre et. al. [DDG11] for unmarked Gibbsian point processes. In order to prove our main phase transition results, we require an extension of their result to the marked case. We explain in detail the steps taken by Dereudre et. al. and discuss how their proof can be extended to the marked case. We are particularly interested in the simplest case of a finite mark space S and uniform distribution μ. This is because the models of Section 4 are defined with respect to some mark space $\{1, \ldots, q\}$, for $q \geq 2$, and the reference measure $\Pi^{z,\mu}$ is suitable when μ is the uniform distribution. However, we will discuss the effects of a different mark space S and a more biased distribution.

As we briefly discussed in Section 3, we determine the existence of Gibbs measures by finding the accumulation point of a sequence of Gibbs distributions. We first define a Gibbs distribution in a finite box and make this shift-invariant by spatial averaging. We show that the sequence of spatially averaged Gibbs distributions has an accumulation point in a suitable topology. By analysing the specific entropy of the Gibbs distribution, relative to the Poisson point process, we are able to show that the sequence of Gibbs distributions admits a subsequence that converges to some measure in the required topology. Finally, we show that by conditioning on this limiting measure, and applying the finite range condition **(R)**, we have the desired Gibbs measure.

An essential component to many of our proofs is the Gibbs consistency relation. We now prove this for the family of finite-volume Gibbs distributions $(G^{z,\mu}_{\Delta,\omega})_{\Delta \in \mathcal{B}_R}$, with boundary condition $\omega \in \Omega^{\Delta,z}_{*}$.

Lemma 5.1. *Let Λ and Δ be bounded subsets of $\mathbb{R}^d$ such that $\Lambda \subset \Delta$ and let $\omega \in \Omega_*^{\Delta,z}$. Then*

$$G_{\Delta,\omega}^{z,\mu}(\Omega_*^{\Delta,z}) = 1 \quad \textit{and} \quad \int_\Omega f\, dG_{\Delta,\omega}^{z,\mu} = \int_\Omega \Big(\int_\Omega f\, dG_{\Lambda,\tilde{\omega}}^{z,\mu} \Big) dG_{\Delta,\omega}^{z,\mu}(d\tilde{\omega})$$

for all measurable functions $f : \Omega \to [0, \infty)$.

Proof. Let Λ, Δ be fixed and let $\omega \in \Omega$ denote a fixed configuration. Consider any two configurations, $\zeta \in \Omega_\Lambda$ and $\kappa \in \Omega_\Delta$. By definition,

$$\begin{aligned} \mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) &= \{\eta \in \mathcal{E}(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) : g(\eta, \zeta'\kappa_{\Lambda^c}\omega_{\Delta^c}) \neq g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \\ &\qquad \text{for some } \zeta' \in \Omega_\Lambda\}, \\ \mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) &= \{\eta \in \mathcal{E}(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) : g(\eta, \kappa'\omega_{\Delta^c}) \neq g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \\ &\qquad \text{for some } \kappa' \in \Omega_\Delta\}. \end{aligned}$$

So $\mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \subset \mathcal{E}(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$ is the set of hyperedges that affect the interaction potential g when a mark or position of ζ is changed, and $\mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \subset \mathcal{E}(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$ is the set of hyperedges that affect the interaction potential g when a mark or position of ζ or κ_{Λ^c} is changed. This means that $\mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \subset \mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$, hence

$$\begin{aligned} H_{\Delta,\omega}(\zeta\kappa_{\Lambda^c}) &:= \sum_{\eta \in \mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})} g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \\ &= \sum_{\eta \in \mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})} g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) + \sum_{\eta \in \mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})} g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c}), \end{aligned} \tag{5.1}$$

where the first term in the above sum is $H_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta)$. Since $\mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$ is the set of hyperedges that affect $g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$ when $\zeta \in \Omega_\Lambda$ is changed, the set

$$\mathcal{E}_\Delta(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta\kappa_{\Lambda^c}\omega_{\Delta^c}) \tag{5.2}$$

only contains hyperedges that do not affect $g(\eta, \zeta\kappa_{\Lambda^c}\omega_{\Delta^c})$ when ζ is changed. Therefore the second term in the sum (5.1) is some term in $\mathbb{R} \cup \{\infty\}$, independent of ζ. Therefore

$$\begin{aligned} H^-_{\Delta,\omega}(\zeta\kappa_{\Lambda^c}) < \infty \quad &\Longrightarrow \quad H^-_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta) < \infty, \\ H_{\Delta,\omega}(\zeta\kappa_{\Lambda^c}) < \infty \quad &\Longrightarrow \quad H_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta) < \infty. \end{aligned}$$

Note that for all $\zeta, \zeta' \in \Omega_\Lambda$, $\kappa \in \Omega_\Delta$ and $\omega \in \Omega$,

$$\mathcal{E}_\Delta(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) = \mathcal{E}_\Delta(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c}),$$

which holds because (5.2) is independent of ζ, as explained above. So we have

$$g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) = g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}), \quad \forall\, \eta \in \mathcal{E}_\Delta(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c}).$$

Therefore

$$\begin{aligned}
&H_{\Delta,\omega}(\zeta \kappa_{\Lambda^c}) + H_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta') \\
&\quad = \sum_{\eta \in \mathcal{E}_\Delta(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\quad = \sum_{\eta \in \mathcal{E}_\Delta(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\qquad + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\quad = \sum_{\eta \in \mathcal{E}_\Delta(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \setminus \mathcal{E}_\Lambda(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\qquad + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\quad = \sum_{\eta \in \mathcal{E}_\Delta(\zeta' \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta' \kappa_{\Lambda^c} \omega_{\Delta^c}) + \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \kappa_{\Lambda^c} \omega_{\Delta^c})} g(\eta, \zeta \kappa_{\Lambda^c} \omega_{\Delta^c}) \\
&\quad = H_{\Delta,\omega}(\zeta' \kappa_{\Lambda^c}) + H_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta),
\end{aligned} \tag{5.3}$$

for all $\zeta, \zeta' \in \Omega_\Lambda$. This means that for fixed configurations $\kappa \in \Omega_\Delta$ and $\omega \in \Omega$, if we pick two different configurations $\zeta, \zeta' \in \Omega_\Lambda$, then the Hamiltonian energy of the configuration $\zeta \kappa_{\Lambda^c}$ in Δ (boundary condition ω) plus the Hamiltonian energy of the configuration ζ' in $\Lambda \subset \Delta$ (boundary condition $\kappa\omega_{\Lambda^c}$) is the same if we instead consider the energy of $\zeta' \kappa_{\Lambda^c}$ in Δ plus the energy of ζ in Λ. By multiplying both sides of (5.3) by β, then taking the exponential and integrating over ζ', we have

$$e^{-\beta H_{\Delta,\omega}(\zeta \kappa_{\Lambda^c})} Z^{z,\mu}_{\Lambda,\kappa\omega_{\Delta^c}}(\beta) = e^{-\beta H_{\Lambda,\kappa\omega_{\Delta^c}}(\zeta)} \int_{\Omega_\Lambda} e^{-\beta H_{\Delta,\omega}(\zeta' \kappa_{\Lambda^c})} \Pi^{z,\mu}_\Lambda(d\zeta'). \tag{5.4}$$

Note that $G^{z,\mu}_{\Delta,\omega}$ is defined for $H^-_{\Delta,\omega} < \infty$ and $H_{\Delta,\omega} < \infty$, and

$$\{H^-_{\Delta,\omega} < \infty, H_{\Delta,\omega} < \infty\} \subset \{H^-_{\Lambda,\kappa\omega_{\Delta^c}} < \infty, H_{\Lambda,\kappa\omega_{\Delta^c}} < \infty\}, \tag{5.5}$$

for any $\kappa \in \Omega_\Delta$. Using the above, we can show that for a fixed boundary condition

$\omega \in \Omega$, the set of configurations $\kappa \in \Omega_\Delta$ such that $Z^{z,\mu}_{\Lambda,\kappa\omega_{\Delta^c}} = 0$, under $G^{z,\mu}_{\Delta,\omega}$, is a zero set. To see this, observe that

$$\begin{aligned}
&G^{z,\mu}_{\Delta,\omega}(Z^{z,\mu}_{\Lambda,\kappa\omega_{\Delta^c}} = 0) \\
&\quad = G^{z,\mu}_{\Delta,\omega} \circ \mathrm{pr}^{-1}_{\Delta\setminus\Lambda}\Big(\int_{\Omega_\Lambda} e^{-\beta H_{\Delta,\omega}(\zeta'\kappa_{\Lambda^c})}\Pi^{z,\mu}_\Lambda(d\zeta') = 0\Big) \qquad (5.6)\\
&\quad = (Z^{z,\mu}_{\Delta,\omega})^{-1}\int_{\Omega_{\Delta\setminus\Lambda}}\int_{\Omega_\Lambda} \mathbb{I}\Big\{\int_{\Omega_\Lambda} e^{-\beta H_{\Delta,\omega}(\zeta'\kappa_{\Lambda^c})}\Pi^{z,\mu}_\Lambda(d\zeta') = 0\Big\} \\
&\qquad\qquad \times e^{-\beta H_{\Delta,\omega}(\zeta'\kappa_{\Lambda^c})}\, d\Pi^{z,\mu}_\Lambda\, d\Pi^{z,\mu}_{\Delta\setminus\Lambda} \qquad (5.7)\\
&\quad = 0, \qquad (5.8)
\end{aligned}$$

where $\mathrm{pr}_{\Delta\setminus\Lambda} : \omega \to \omega_{\Delta\setminus\Lambda}$ is the projection from Ω to $\Omega_{\Delta\setminus\Lambda}$. Line (5.6) comes directly from (5.5), and (5.7) comes from the definition of the Gibbs distribution $G^{z,\mu}_{\Delta,\omega}$. The final step (5.8) is trivial: the equation in line (5.7) is zero if the indicator is not satisfied, and if it is satisfied then the integral over Ω_Λ in (5.7) is zero. We also have

$$G^{z,\mu}_{\Delta,\omega}(Z^{z,\mu}_{\Lambda,\kappa\omega_{\Delta^c}} = \infty) = 0 \qquad (5.9)$$

because

$$\int_{\Omega_{\Delta\setminus\Lambda}}\int_{\Omega_\Lambda} e^{-\beta H_{\Delta,\omega}(\zeta'\kappa_{\Lambda^c})}\,\Pi^{z,\mu}_\Lambda(d\zeta')\, d\Pi^{z,\mu}_{\Delta\setminus\Lambda} = Z^{z,\mu}_{\Delta,\omega} < \infty.$$

Using (5.8), (5.9) and (5.4), we obtain the desired result. $\square$

We can now proceed with the proof of Theorem 2.1. Assume that (**R**), (**S**) and (**U**) are satisfied. Choose M and Γ as in **(U)**. For $n \geq 1$, let

$$\Lambda_n = \bigcup_{k\in\{-n,\ldots,n\}^d} C(k).$$

Let $\bar\omega \in \bar\Gamma$ denote a fixed pseudo-periodic marked configuration with

$$\sup_{k\in\mathbb{Z}^d} N_{C(k)} < \infty.$$

By (U1) we can find a number $m \geq 1$ such that $\partial\Lambda_n \subset \Lambda_{n+m}$ for all $n \geq 1$. Let $\zeta \in \Omega_{\Lambda_n}$ be such that $\zeta\bar\omega_{\Lambda_n^c} \in \bar\Gamma$. Recall that $\hat\xi_\eta$ is defined in assumption **(U)**.

We can write, for $L_n = \{-n, \dots, n\}^d$,

$$\begin{aligned}
H_{\Lambda_n,\bar{\omega}}(\zeta) &= \sum_{\eta \in \mathcal{E}_{\Lambda_n}(\zeta\bar{\omega}_{\Lambda_n^c})} g(\eta, \zeta\bar{\omega}_{\Lambda_n^c}) \\
&= \sum_{k \in L_{n+m}} \sum_{\substack{\eta \in \mathcal{E}_{\Lambda_n}(\zeta\bar{\omega}_{\Lambda_n^c}): \\ \hat{\xi}_\eta \ni k}} \frac{g(\eta, \zeta\bar{\omega}_{\Lambda_n^c})}{|\hat{\xi}_\eta|} \\
&= \sum_{k \in L_n} \sum_{\substack{\eta \in \mathcal{E}_{\Lambda_n}(\zeta\bar{\omega}_{\Lambda_n^c}): \\ \hat{\xi}_\eta \ni k}} \frac{g(\eta, \zeta\bar{\omega}_{\Lambda_n^c})}{|\hat{\xi}_\eta|} \\
&\qquad + \sum_{k \in L_{n+m} \setminus L_n} \sum_{\substack{\eta \in \mathcal{E}_{\Lambda_n}(\zeta\bar{\omega}_{\Lambda_n^c}): \\ \hat{\xi}_\eta \ni k}} \frac{g(\eta, \zeta\bar{\omega}_{\Lambda_n^c})}{|\hat{\xi}_\eta|} \\
&\le c_\Gamma \, |L_n| + c_\Gamma \, |L_{n+m} \setminus L_n| \qquad (5.10) \\
&< \infty,
\end{aligned}$$

so $\bar{\omega}$ is admissible for Λ_n and z.

Define the Gibbs distribution

$$G_n := G^{z,\mu}_{\Lambda_n,\bar{\omega}} \circ \mathrm{pr}^{-1}_{\Lambda_n},$$

in Λ_n with boundary condition $\bar{\omega}$ and activity z, projected onto Λ_n. Let P_n be the probability measure on $(\Omega, \mathcal{F})$ relative to which the configurations in the disjoint blocks $\Lambda_n + (2n+1)\mathsf{M}k$, $k \in \mathbb{Z}^d$, are independent with distribution G_n. This independence is ensured by the periodisation of the boundary condition $\bar{\omega}$ and $\vartheta_{\mathsf{M}k}(\omega_{C(k)}) \in \Gamma$, for any $k \in \mathbb{Z}^d$. Define

$$\hat{P}_n := \frac{1}{|\Lambda_n|} \int_{\Lambda_n} P_n \circ \vartheta_x^{-1} \, dx. \qquad (5.11)$$

The measure $\hat{P}_n$ is a simple spatial averaging of the measure P_n. We consider the case where μ is uniform on S; but for other distributions, the measure $\hat{P}_n$ is a spatial averaging with no ergodic averaging for the mark space. This, combined with the periodisation of the configuration, means $\hat{P}_n$ is shift-invariant with finite intensity. The intensity

$$i(\hat{P}_n) = \frac{1}{|\Lambda_n|} \int N_{\Lambda_n} \, dG_n$$

is finite because

$$\begin{aligned}\int N_{\Lambda_n}\,dG_n &= \frac{1}{Z^{z,\mu}_{\Lambda,\bar\omega}}\int N_{\Lambda_n} e^{-\beta H_{\Lambda,\bar\omega}(\zeta)}\,\Pi^{z,\mu}_{\Lambda_n}(d\zeta)\\ &\le e^{\beta c_S\#(\partial_{\Lambda_n}\bar\omega)}\int N_{\Lambda_n} e^{\beta c_S N_{\Lambda_n}}\,d\Pi^{z,\mu}_{\Lambda_n}<\infty, \end{aligned}\tag{5.12}$$

by **(S)**. From (5.12), we have $\hat P_n \in \mathscr{P}_\Theta$.

A measurable function $f:\Omega\to\mathbb{R}$ is called local if $f(\omega)=f(\omega_{\Lambda_n})$ for some $n\ge 1$. We say that f is tame if $|f(\omega)|\le aN_{\Lambda_n}+b$ for some $n\ge 1$ and suitable constants $a,b\ge 0$. Let $\mathcal{L}$ denote the linear space of all tame local functions. The topology of local convergence, or $\tau_{\mathcal{L}}$, on $\mathscr{P}_\Theta$ is defined as the smallest topology for which the mappings $P\to\int f\,dP$, for $f\in\mathcal{L}$, are continuous.

The relative entropy of two measures Q_1,Q_2 on the same measurable space is defined

$$I(Q_1|Q_2):=\begin{cases}\int f\ln f\,dQ_2 & \text{if } Q_1<<Q_2 \text{ with density } f,\\ \infty & \text{otherwise.}\end{cases}$$

For any stationary point random field $P\in\mathscr{P}_\Theta$, let $P_{\Lambda_n}:=P\circ \mathrm{pr}^{-1}_{\Lambda_n}$ be the projection of P onto Ω_{Λ_n}. For a Poisson point random field $\Pi^{z,\mu}$ on $X=\mathbb{R}^d\times S$ with intensity measure $z\,\mathrm{Leb}(\,\cdot\,)\otimes\mu$,

$$I\big(P_{\Lambda_n}|\Pi^{z,\mu}_{\Lambda_n}\big):=\begin{cases}\int f\ln f\,d\Pi^{z,\mu}_{\Lambda_n} & \text{if } P_{\Lambda_n}<<\Pi^{z,\mu}_{\Lambda_n} \text{ with density } f,\\ \infty & \text{otherwise,}\end{cases}$$

is the relative entropy of P_{Λ_n} with respect to $\Pi^{z,\mu}_{\Lambda_n}$. Note that we are dealing with a special case of the relative entropy where the mark intensity μ is a uniform distribution. If we instead consider the relative entropy of P_{Λ_n} with respect to $\Pi^{z,\nu}_{\Lambda_n}$, where ν is some finite measure (on S) different from μ, then we find that

$$I\big(P_{\Lambda_n}|\Pi^{z,\mu}_{\Lambda_n}\big)=I\big(P_{\Lambda_n}|\Pi^{z,\nu}_{\Lambda_n}\big)+|\Lambda_n|\,I(\mu|\nu).\tag{5.13}$$

In general, the relative entropy with respect to $\Pi^{z,\mu}_{\Lambda_n}$ is maximised when μ is a uniform distribution. The specific entropy is defined

$$I^{z,\mu}(P):=\lim_{n\to\infty}\frac{1}{|\Lambda_n|}I\big(P_{\Lambda_n}|\Pi^{z,\mu}_{\Lambda_n}\big).$$

Proposition 5.2. *For all $c_1, c_2 \geq 0$ and $z > 0$, the set*

$$\{P \in \mathscr{P}_\Theta : I^{z,\mu}(P) - c_1 i(P) \leq c_2\}$$

is relatively sequentially compact in $\tau_{\mathcal{L}}$.

Proof. The proof is a detailed explanation of the analogous lemma proven by Georgii [Geo94] (see Lemma 3.4), with some additional comments provided. This is not original work, but it is useful for the reader to understand these details in context of our existence result, Theorem 2.1.

The set $\{I^{z,\mu}(P) - c_1\, i(P) \leq c_2\}$ is closed because $i(P)$ is continuous and $I^{z,\mu}$ is lower semicontinuous. Georgii and Zessin [GZ93] (see Proposition 2.6) prove that the level sets $\{I^{z,\mu} \leq c\}$ are compact and sequentially compact in $\tau_{\mathcal{L}}$ and that I^z is lower semicontinuous relative to $\tau_{\mathcal{L}}$. $I^{z,\mu}$ has compact level sets, and so the same is true for $I^{\rho z,\mu}$, where

$$I^{\rho z,\mu}(P) := \lim_{n\to\infty} \frac{1}{|\Lambda_n|} I\big(P_{\Lambda_n} | \Pi^{\rho z,\mu}_{\Lambda_n}\big) \tag{5.14}$$

for $\rho > 0$, and the mark distribution μ is independent of the particle positions. If we were to take a more biased distribution ν, then due to (5.13), we have

$$I^{\rho z,\nu}(P) = I^{\rho z,\mu}(P) + I(\mu|\nu). \tag{5.15}$$

This gives the reader an idea of the role of the mark distribution. The more biased the mark distribution becomes, the less it resembles a uniform distribution, which increases the second term in the above sum. Considering a uniform distribution μ, we have

$$I^{\rho z,\mu}(P) = I^{z,\mu}(P) - i(P) \ln \rho + \rho - 1. \tag{5.16}$$

Choosing $c_1 = \ln \rho$,

$$I^{\rho z,\mu}(P) \leq c_2 + \rho - 1 \quad \Longrightarrow \quad I^{z,\mu}(P) - i(P)\, c_1 \leq c_2,$$

and so $\{I^{z,\mu}(P) - c_1\, i(P) \leq c_2\}$ is contained in the compact set $\{I^{z\rho,\mu} \leq c_2 + \rho - 1\}$. ☐

Proposition 5.3. *In the limit $n \to \infty$ we have*

$$I^{z,\mu}(\hat{P}_n) - \beta\, c_S\, i(\hat{P}_n) \leq |C|^{-1}\Big(\beta\, c_\Gamma - \ln \Pi^{z,\mu}_C(\Gamma)\Big) + o(1).$$

Proof. Due to the definition of $\hat{P}_n$, we have

$$I^{z,\mu}(\hat{P}_n) = \frac{1}{|\Lambda_n|} I\big(G_n | \Pi^{z,\mu}_{\Lambda_n}\big).$$

Also,

$$\begin{aligned} I\big(G_n|\Pi^{z,\mu}_{\Lambda_n}\big) &= \int_{\Omega_{\Lambda_n}} \frac{1}{Z^{z,\mu}_{\Lambda_n,\bar\omega}} e^{-\beta H_{\Lambda_n,\bar\omega}(\zeta)} \ln\left\{ \frac{1}{Z^{z,\mu}_{\Lambda_n,\bar\omega}} e^{-\beta H_{\Lambda_n,\bar\omega}(\zeta)}\right\} d\Pi^{z,\mu}_{\Lambda_n} \\ &= \int_{\Omega_{\Lambda_n}} \frac{1}{Z^{z,\mu}_{\Lambda_n,\bar\omega}} e^{-\beta H_{\Lambda_n,\bar\omega}(\zeta)} \Big(-\beta H_{\Lambda_n,\bar\omega}(\zeta) - \ln Z^{z,\mu}_{\Lambda_n,\bar\omega}\Big) \Pi^{z,\mu}_{\Lambda_n}(d\zeta) \\ &= -\beta \int_{\Omega_{\Lambda_n}} H_{\Lambda_n,\bar\omega}(\zeta)\, dG_n - \ln Z^{z,\mu}_{\Lambda_n,\bar\omega}. \qquad (5.17) \end{aligned}$$

We now estimate the terms on the right hand side of (5.17). From (**S**) and (U1), we have

$$\int_{\Omega_{\Lambda_n}} H_{\Lambda_n,\bar\omega}(\zeta)\, dG_n \geq -c_S \int_{\Omega_{\Lambda_n}} N_{\Lambda_n}\, dG_n - c_S \#(\partial^{\Gamma}_{\Lambda_n}\bar\omega) > -\infty, \qquad (5.18)$$

where $\partial^{\Gamma}_{\Lambda_n}\bar\omega := \bar\omega \cap \big((\Lambda_n^{r_\Gamma} \setminus \Lambda_n) \times S\big)$. For $\zeta \in \Omega_f$, recall that $|\zeta| := |\xi_\zeta|$. Note that $|\partial^{\Gamma}_{\Lambda_n}\bar\omega| = o(|\Lambda_n|)$. It remains to find an estimate for the second term, namely the partition function. Fix any $n \geq 1$ and let $\zeta \in \Omega_{\Lambda_n}$ be such that $\zeta\bar\omega_{\Lambda_n^c} \in \bar\Gamma$. Using (5.10), we have

$$H_{\Lambda_n,\bar\omega}(\zeta) \leq c_\Gamma\, |L_n| + o(|\Lambda_n|). \qquad (5.19)$$

Therefore

$$\begin{aligned} Z^{z,\mu}_{\Lambda_n,\bar\omega} &\geq \int \mathbb{I}_{\bar\Gamma}(\zeta\bar\omega_{\Lambda_n^c}) e^{-\beta H_{\Lambda_n,\bar\omega}(\zeta)}\, \Pi^{z,\mu}_{\Lambda_n}(d\zeta) \\ &\geq e^{-\beta c_\Gamma\, |L_n| - o(|\Lambda_n|)} \Pi^{z,\mu}_C\big(\Gamma\big)^{|L_n|}. \end{aligned}$$

Combining with (5.17) and (5.18), one can see that

$$\begin{aligned} I^{z,\mu}(\hat{P}_n) &= -\frac{\beta}{|\Lambda_n|} \int H_{\Lambda_n,\bar\omega}\, dG_n - \frac{1}{|\Lambda_n|} \ln Z^{z,\mu}_{\Lambda_n,\bar\omega} \\ \Rightarrow I^{z,\mu}(\hat{P}_n) - \beta c_S i(\hat{P}_n) &\leq -\frac{1}{|\Lambda_n|} \ln\Big(e^{-\beta c_\Gamma\, \# L_n - o(|\Lambda_n|)} \Pi^{z,\mu}_C(\Gamma)^{\# L_n}\Big) \\ &= \frac{1}{|\Lambda_n|}\big(\beta c_\Gamma\, \# L_n + o(|\Lambda_n|) - \# L_n \ln \Pi^{z,\mu}_C(\Gamma)\big) \\ &= |C|^{-1}\big(\beta c_\Gamma - \ln \Pi^{z,\mu}_C(\Gamma)\big) + o(1), \end{aligned}$$

as required. □

Propositions 5.2 and 5.3 imply that the sequence $(\hat{P}_n)$ admits a subsequence that converges to some $\hat{P} \in \mathscr{P}_\Theta$ in $\tau_{\mathcal{L}}$. Due to the continuity of the intensity i and lower semicontinuity of $I^{z,\mu}$, Proposition 5.3 implies

$$\begin{aligned} I^{z,\mu}(\hat{P}) - \beta\, c_S\, i(\hat{P}) &\leq |C|^{-1}\Big(\beta\, c_\Gamma - \ln \Pi_C^{z,\mu}(\Gamma)\Big) \\ &< |C|^{-1}\big(\beta\, c_\Gamma - \ln\{e^{\beta c_\Gamma - z|C|}\}\big) \qquad (5.20) \\ &= z, \end{aligned}$$

where (5.20) comes from assumption (U3). The following proposition completes the proof of Theorem 2.1.

Proposition 5.4. *Let S be a finite set and let μ be a uniform distribution. The conditional probability $P := \hat{P}\big(\,\cdot\,\big|\{\emptyset\}^c\big) \in \mathscr{P}_\Theta$ is a Gibbs measure for $\mathcal{E}$, g, z and q.*

Proof. $\hat{P} \in \mathscr{P}_\Theta$ and $\hat{P}(\{\emptyset\}^c) < 1$, so P is well-defined and in $\mathscr{P}_\Theta$. We now show that P is a Gibbs measure.

Let δ_- and δ_+ be the diameters of the largest open ball in C and the smallest closed ball containing C, respectively. Recall that δ_R, n_R, l_R are constants introduced in **(R)**. Fix some bounded region $\Lambda \subset \mathbb{R}^d$. Let $n_\Lambda \geq 1$ be the smallest number with $\Lambda \subset \Lambda_{n_\Lambda}$ and $n_\Lambda \geq \delta_R/6\delta_+$. Fix an integer $m \geq 6 l_R \delta_+/\delta_-$ and for $n \geq 1$, divide $\Lambda_{n+(2n+1)m} =: \hat{\Lambda}_n$ into $(2m+1)^d$ translates $\Lambda_n^k := \Lambda_n + (2n+1)\mathsf{M}k$ of Λ_n, where $k \in L_m$. Let

$$\hat{\Omega}_{\mathrm{cr}}^{\Lambda,n} := \big\{\zeta \in \Omega_{\hat{\Lambda}_n \setminus \Lambda} : \min_{0 \neq k \in L_m} N_{\Lambda_n^k} > n_R\big\}, \quad \forall n \geq n_\Lambda, \tag{5.21}$$

$$\hat{\Omega}_{\mathrm{cr}}^{\Lambda} := \bigcup_{n \geq n_\Lambda} \hat{\Omega}_{\mathrm{cr}}^{\Lambda,n}, \tag{5.22}$$

$$\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} := \bigcup_{n=n_\Lambda}^{p} \hat{\Omega}_{\mathrm{cr}}^{\Lambda,n} \quad \forall p \geq n_\Lambda. \tag{5.23}$$

We claim that the proof of Proposition 5.4 is complete if we can show that

$$\int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p}} f\, d\hat{P} = \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z}} f_\Lambda\, d\hat{P}, \tag{5.24}$$

for $f : \Omega \to [0,1]$ a local function, i.e. $f(\omega) = f\big(\omega \cap (\Lambda_n \times S)\big)$ for some $n \geq 1$, and $p \geq n_\Lambda$ so large that f is $\mathcal{F}_{\hat{\Lambda}_p}$-measurable, and

$$f_\Lambda(\omega) := \int_{\Omega_*^{\Lambda,z}} f\, dG_{\Lambda,\omega}^{z,\mu}. \tag{5.25}$$

To see this, note that

$$1 - P(\hat{\Omega}^{\Lambda}_{\mathrm{cr}}) = P\Big(\bigcap_{n \geq n_\Lambda} (\hat{\Omega}^{\Lambda,n}_{\mathrm{cr}})^c\Big) \tag{5.26}$$

$$= P\Big(\bigcap_{n \geq n_\Lambda} \big\{\zeta \in \Omega_{\hat{\Lambda}_n \setminus \Lambda} : \min_{0 \neq k \in L_m} N_{\Lambda^k_n} \leq n_R\big\}\Big) \tag{5.27}$$

$$\leq \inf_{n \geq n_\Lambda} \sum_{0 \neq k \in L_m} P(N_{\Lambda^k_n} \leq n_R)$$

$$= (\#L_m - 1) \inf_{n \geq n_\Lambda} P(N_{\Lambda_n} \leq n_R)$$

$$\to 0 \qquad \text{as } n \to \infty. \tag{5.28}$$

We have (5.26) and (5.27) by definitions (5.22) and (5.21), respectively. For any translation invariant point process P, we know that $P\big(N_{\mathbb{R}^d} \in (0,\infty)\big) = 0$, where $N_{\mathbb{R}^d} := \#(\omega \times S)$. For example, see Proposition 6.1.3 of [MKM78]. Therefore $P\big(N_{\mathbb{R}^d} \in (0,\infty)\big) = 0$, because P is translation invariant. This gives (5.28), because

$$P(N_{\Lambda_n} \leq n_R) \to P(N_{\mathbb{R}^d} \leq n_R) = P(\{\emptyset\}) = 0 \quad \text{ as } n \to \infty. \tag{5.29}$$

We have the above limit because $\Lambda_n \uparrow \mathbb{R}^d$ as $n \to \infty$.

Now let $p \to \infty$ and set $f = 1$. One can see that $\hat{P}(\hat{\Omega}^{\Lambda}_{\mathrm{cr}} \cap \Omega^{\Lambda,z}_*) = \hat{P}(\hat{\Omega}^{\Lambda}_{\mathrm{cr}})$, and $P(\Omega^{\Lambda,z}_*) = 1$ by (5.28). For arbitrary f,

$$P = \int_{\Omega^{\Lambda,z}_*} G^{z,\mu}_{\Lambda,\omega}\, P(d\omega). \tag{5.30}$$

Since Λ is chosen arbitrarily, (5.30) means that P is a Gibbs measure. So if we can show that (5.24) holds then we have (5.30) and the proof of Proposition 5.4 is complete.

We now proceed with the proof of (5.24). Let f and $p \geq n_\Lambda$ be fixed. Let n be large enough such that $\hat{\Lambda}_p \subset \Lambda_n$. Define

$$\bar{G}_n := \frac{1}{|\Lambda_n|} \int_{\Lambda^\circ_n} G^{z,\mu}_{\Lambda_n,\bar{\omega}} \circ \vartheta_x^{-1}\, dx = \frac{1}{|\Lambda_n|} \int_{\Lambda^\circ_n} G^{z,\mu}_{\Lambda_n - x, \vartheta_x \bar{\omega}}\, dx, \tag{5.31}$$

where

$$\Lambda^\circ_n := \{x \in \mathbb{R}^d : \hat{\Lambda}_p + x \subset \Lambda_n\}. \tag{5.32}$$

Lemma 5.7 of [GZ93] tells us that

$$\lim_{n \to \infty} \Big(\int_\Omega f\, d\hat{P}_n - \int_\Omega f\, d\bar{G}_n \Big) = 0 \tag{5.33}$$

for all $f \in \mathcal{L}$. Therefore $\hat{P}$ is an accumulation point of the sequence $(\bar{G}_n)$. Let $x \in \Lambda_n^\circ$, so $\hat{\Lambda}_p \subset \Lambda_n - x$. Since

$$\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \in \mathcal{F}_{\hat{\Lambda}_p \setminus \Lambda} \subset \mathcal{F}_{(\Lambda_n - x)\setminus\Lambda}, \tag{5.34}$$

we can apply Lemma 5.1 to find

$$\int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p}} f\, dG^{z,\mu}_{\Lambda_n - x, \vartheta_x \bar{\omega}} = \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z}} \Big(\int_{\Omega_\Lambda} f\, dG^{z,\mu}_{\Lambda,\omega} \Big)\, G^{z,\mu}_{\Lambda_n - x, \vartheta_x \bar{\omega}}(d\omega). \tag{5.35}$$

Averaging the left hand side over x,

$$\frac{1}{|\Lambda_n|} \int_{\Lambda_n^\circ} \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p}} f\, dG^{z,\mu}_{\Lambda_n - x, \vartheta_x \bar{\omega}}\, dx = \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p}} f\, d\bar{G}_n,$$

by definition (5.31). Similarly for the right hand side of (5.35),

$$\frac{1}{|\Lambda_n|} \int_{\Lambda_n^\circ} \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z}} \Big(\int_{\Omega_\Lambda} f\, dG^{z,\mu}_{\Lambda,\omega} \Big)\, G^{z,\mu}_{\Lambda_n - x, \vartheta_x \bar{\omega}}(d\omega) dx = \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z}} f_\Lambda\, d\bar{G}_n,$$

using the definition (5.25) of f_Λ. Therefore

$$\int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p}} f\, d\bar{G}_n = \int_{\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z}} f_\Lambda\, d\bar{G}_n, \tag{5.36}$$

for all $f \in \mathcal{L}$, the linear space of local tame functions. We know that $\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \in \mathcal{F}_{\hat{\Lambda}_p \setminus \Lambda}$, so the integrand on the left is measurable with respect to $\mathcal{F}_{\hat{\Lambda}_p \setminus \Lambda}$ and belongs to $\mathcal{L}$. Also, $\hat{\Omega}_{\mathrm{cr}}^{\Lambda,\leq p} \cap \Omega_*^{\Lambda,z} \in \mathcal{F}^*_{\hat{\Lambda}_p \setminus \Lambda}$, where $\mathcal{F}^*_{\hat{\Lambda}_p \setminus \Lambda}$ denotes the completion of $\mathcal{F}_{\hat{\Lambda}_p \setminus \Lambda}$. Thus the integrand on the right is measurable with respect to $\mathcal{F}^*_{\hat{\Lambda}_p \setminus \Lambda}$. Therefore if n forms a subsequence for which $\bar{G}_n$ tends to $\hat{P}$ in $\tau_{\mathcal{L}}$, then taking n over this limit, (5.36) gives (5.24). $\square$

5.2 Finite-range triangle interaction (Theorem 4.1)

In this section, we prove our first phase transition result, Theorem 4.1. The structure of the proof is as follows. First, we prove that Gibbs measures exist for our model, see Proposition 5.5, by applying Corollary 2.2. Recall that in Section 3.2, we explain that if (3.19) is satisfied for an appropriate partition of a box Λ, then multiple Gibbs measures exist. Therefore we prove Theorem 4.1 by utilising the random-cluster representation of Section 3.2 to show that (3.19) holds. This is given by Proposition 5.6 below. Theorem 4.1 follows directly from Proposition 5.6, we precisely explain how at the end of this subsection.

Proposition 5.5. *For any $z, \beta > 0$, there exists at least one Gibbs measure for the Delaunay Potts model given by the Hamiltonian in (4.3).*

Proof. We apply Corollary 2.2 and Remark 2.5. The range condition **(R)** is satisfied because each edge $\eta \in \mathcal{E}^{\mathrm{D}_2}(\omega)$ and hyperedge $\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega)$ have the finite horizons $\bar{B}(\eta)$ and $\bar{B}(\tau)$, respectively. The stability condition **(S)** clearly holds as ψ and ϕ are both non-negative. Finally, we show that the alternative upper regularity condition $(\mathbf{U}^A)$ holds. Let M be such that $|\mathsf{M}_1| = |\mathsf{M}_2| = a > 0$ and $\sphericalangle(\mathsf{M}_1, \mathsf{M}_2) = \pi/3$, and let $A = B(0, \sqrt{3}a/6)$, where a is specified later. Recall that a ball of radius $\sqrt{3}a/6$ is chosen as this means that any point x in a configuration $\omega \in \bar{\Gamma}^A$ has 6 neighbours in the Delaunay graph. Uniform confinement $(\mathrm{U1}^A)$ is satisfied with $r_{\Lambda,\omega} = 2r_1$. We also find that $(\mathrm{U2}^A)$ is satisfied with $c_A^+ = 1$. To satisfy assumption $(\mathrm{U3}^A)$, we require $z\pi a^2/12 > e^\beta$. Therefore $(\mathbf{U}^A)$ is satisfied for any $z, \beta > 0$ if we choose $a > (12e^\beta/(z\pi))^{1/2}$. □

We are applying the multi-body continuum random-cluster representation to Delaunay triangle hyperedge interactions. Our hypergraph structure here is given by Delaunay tessellations. We also use the notation T for a set of hyperedge triangles of unmarked particles, as opposed to H. Definitions (3.1) and (3.2) are replaced with

$$T_{\mathbb{R}^2} := \{\xi \subset \mathbb{R}^2 : \xi \text{ is a set of 3 distinct points}\}, \tag{5.37}$$

$$T_\Delta := \{\xi \in T_{\mathbb{R}^2} : \xi \subseteq \Delta\}, \tag{5.38}$$

for any measurable $\Delta \subseteq \mathbb{R}^2$. We also define

$$\mathscr{T} := \{T \subset T_{\mathbb{R}^2} : T \text{ is locally finite}\}$$

as the set of all possible hyperedge configurations. Let $\Lambda \in \mathcal{B}_R$ be a bounded set in

$\mathbb{R}^2$ with specific partition of meso-boxes

$$\Lambda = \bigcup_{\substack{k \in I_1 \\ l \in I_2}} \Delta_{k,l},$$

where $I_1, I_2 \subset \mathbb{Z}^2$ are appropriate index sets. Each meso-box is divided into a partition of micro-boxes, $\Delta_{k,l} = \cup_{i,j=0}^{8} \Delta_{k,l}^{i,j}$, where

$$\Delta_{k,l}^{i,j} := [9Lk + Li, 9Lk + L(i+1)] \times [9Ll + Lj, 9Ll + L(j+1)],$$

for some $L > 0$ satisfying

$$2r_0 < L < \sqrt{2\pi}\, r_0. \tag{5.39}$$

For brevity, we will often refer to a $9L \times 9L$ meso-box as Δ and a $L \times L$ micro-box as ∇. We require $L > 2r_0$ so that $|\nabla \ominus r_0| > 0$ in Lemma 5.10, below. The assumption $L < (\sqrt{2\pi})r_0$ is needed so that $J_L = 2$, see (5.87).

Proposition 5.6. *There exists $\alpha > 0$ such that*

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC_{\Lambda,\omega}^{z,q} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta_{k,l} \subset \Lambda$, and for all

$$z > \frac{81 q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(\sqrt{2\pi} - 2)^2 r_0^2}$$

and

$$\beta > \log\left\{\frac{1 + (q^2-1)(1-p_0)^{\frac{1}{162}}}{1 - (1-p_0)^{\frac{1}{162}}}\right\}.$$

The remainder of this section is dedicated to proving Proposition 5.6, which implies our phase transition result, Theorem 4.1. The proof of Proposition 5.6 is long and technical, so before embarking upon the proof, we begin by providing an outline of the structure.

First, we introduce some notation and definitions that are required for the proof. When proving the existence of a phase transition via random-cluster representation, it is standard procedure to rewrite the random-cluster distribution $C_{\Lambda,\omega}^{z,q}$ so that it is a factor of two measures. We define a measure $M_{\Lambda,\omega}$ for the distribution of particle positions and a measure $\mu_{\zeta\omega_{\Lambda^c}}^{(q)}$ for the distribution of the hyperedges, given the particle positions. We then introduce a new measure $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ for assigning hyperedges to a given particle configuration, and we prove that this is stochastically dominated by $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$. We define a new measure $\widetilde{C}^{z,q}_{\Lambda,\omega}$ for distributing positions and marks. Under $\widetilde{C}^{z,q}_{\Lambda,\omega}$, the particle positions are distributed in the same way as the random-cluster distribution (i.e. according to $M_{\Lambda,\omega}$). The particle mark distribution is defined in such a way that percolation under $\widetilde{C}^{z,q}_{\Lambda,\omega}$ implies that Proposition 5.6 is satisfied. This comes from the stochastic domination of $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$ by $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$.

We show that percolation occurs under $\widetilde{C}^{z,q}_{\Lambda,\omega}$ via the following four steps; (i)-(iv), below. The measure $M_{\Lambda,\omega}$ distributes particle positions in Λ given the positions of the configuration ω. We define $M_{\Lambda,\omega;\Delta,\zeta}$ by conditioning on ξ_ζ inside $\Lambda\setminus\Delta$, where $\xi_\zeta \in \mathcal{K}_\Lambda$. We define $M_{\Lambda,\omega;\Delta,\zeta}$ as the conditional distribution of the particle positions in a box $\Delta \subset \Lambda$, given the positional configuration ξ_ζ relative to $M_{\Lambda,\omega}$.

(i) *Gibbs consistency relation.* For $\Lambda \in \mathcal{B}_R$, the family $(M_{\Lambda,\omega;\Delta,\zeta})_{\Delta\subset\Lambda}$ satisfies the Gibbs consistency relation, see Lemma 5.8. This property is implied by the additivity of the Hamiltonian energy and is required in order to analyse the partition of Δ under $M_{\Lambda,\omega;\Delta,\zeta}$.

(ii) *Density quotient estimate.* We denote by $h_{\Lambda,\omega}$ the Radon-Nikodym density of $M_{\Lambda,\omega}$ with respect to $P^{z/q}_{\Lambda,\omega}$. The density quotient is the ratio between $h_{\Lambda,\omega}$ for a given configuration and $h_{\Lambda,\omega}$ for the same configuration with a particle removed. We derive a deterministic lower bound (uniformly) for the density quotient for all $\Lambda \in \mathcal{B}_R$, see Lemma 5.9.

(iii) *Ensuring micro-boxes are not empty.* The positivity of the density quotient allows us to find a lower bound on the probability under $M_{\Lambda,\omega;\nabla,\zeta}$ that a micro-box $\nabla = \Delta^{k,l}_{i,j}$ contains at least one particle, see Lemma 5.10.

(iv) *Percolation.* The Gibbs consistency relation means we can use Lemma 5.10 to deduce that the probability under $M_{\Lambda,\omega;\Delta,\zeta}$ that every micro-box of Δ contains at least one point is bounded. If this probability is strictly positive, then we argue that the probability of percolation under $\widetilde{C}^{z,q}_{\Lambda,\omega}$ is positive, see Lemma 5.11.

The above method, of introducing a new measure and showing percolation via steps (i)-(iv), is based on the techniques of Georgii and Häggström [GH96] and Bertin et. al. [BBD04]. Georgii and Häggström divide Λ into meso-boxes and do not further divide into micro-boxes, as for their pair interaction model it suffices to prove that each meso-box contains a sufficiently high number of particles. Further partitioning into micro-boxes, in the style of Bertin et. al., ensures that there are sufficiently many Delaunay hyperedges contained within any given meso-box. In our proofs, we apply the steps above for models where particles interact within hyperedges and over an infinite range, properties not possible in [GH96] and [BBD04]. We now provide the required preliminaries and prove Proposition 5.6 via the method discussed above.

Let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations of particles such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and let $T \in \mathscr{T}$ be a subset of $\{\xi_\tau : \tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})\}$. Recall that $\xi_\omega \in \mathcal{K}$ and $\xi_\zeta \in \mathcal{K}_\Lambda$ are the positional configurations of ω and ζ, respectively. The definition of the hyperedge-drawing mechanism implies that the event $\{\Delta \leftrightarrow \Lambda^c\}$ is also the event that there exists a particle position of $\xi_\zeta \cap \Delta$ connected to infinity in the random graph $(\xi_{\zeta\omega_{\Lambda^c}}, T)$. One can rewrite the random-cluster distribution (3.10) as follows:

$$C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dT) = M_{\Lambda,\omega}(d\xi_\zeta)\mu^{(q)}_{\zeta\omega_{\Lambda^c}}(dT), \tag{5.40}$$

where

$$\mu^{(q)}_{\zeta\omega_{\Lambda^c}}(dT) := \frac{q^{K(\zeta\omega_{\Lambda^c},T)}\,\mu_{\zeta\omega_{\Lambda^c}}(dT)}{\int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}}(dT)\,q^{K(\zeta\omega_{\Lambda^c},T)}}, \tag{5.41}$$

and $M_{\Lambda,\omega}$ is defined on $\mathcal{K}_\Lambda$ as the distribution of particle positions given by the marginal distribution $C^{z,q}_{\Lambda,\omega}(\,\cdot\,,\mathscr{T})$.

For boundary condition $\xi_\omega \in \mathcal{K}$, define $h_{\Lambda,\omega}$ on $\mathcal{K}_\Lambda$ by

$$h_{\Lambda,\omega}(\xi_\zeta) := \frac{1}{\hat{Z}^{z,q}_{\Lambda,\omega}} \int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}}(dT)\,q^{K(\zeta\omega_{\Lambda^c},T)}. \tag{5.42}$$

This is the Radon-Nikodym density of $M_{\Lambda,\omega}$ with respect to $P^{z/q}_{\Lambda,\omega}$. For a bounded set $\nabla \subset \Lambda$, consider the conditional distribution $M_{\Lambda,\omega;\nabla,\zeta}$ of the positional configuration in ∇ given the positional configuration $\xi_{\zeta_{\nabla^c}}\xi_{\omega_{\Lambda^c}}$ in ∇^c relative to $M_{\Lambda,\omega}$. The measure $M_{\Lambda,\omega;\nabla,\zeta}$ distributes particle positions inside the micro-box ∇, given the positions of some configuration $\zeta_{\nabla^c}\omega_{\Lambda^c}$ outside ∇. Let $\xi_\omega \in \mathcal{K}$ and $\xi_\zeta \in \mathcal{K}_\Lambda$ be fixed boundary conditions. Let $\xi_{\zeta'} \in \mathcal{K}_\Lambda$ and $\xi_\kappa := \xi_{\zeta'} \cap \nabla \in \mathcal{K}_\nabla$. Then

$$\begin{aligned}
M_{\Lambda,\omega;\nabla,\zeta}(d\xi_\kappa) &= M_{\Lambda,\omega}(d\xi_{\zeta'}|\xi_{\zeta'} = \xi_\zeta \text{ on } \Lambda\setminus\nabla) \\
&= \frac{1}{Z'_{\Lambda,\omega}} \int_{\mathscr{T}} C^{z,q}_{\Lambda,\omega}(d\xi'_\zeta, dT)\,\mathbb{I}\{\xi_{\zeta'} = \xi_\zeta \text{ on } \Lambda\setminus\nabla\} \\
&= \frac{1}{\hat{Z}^{z,q}_{\Lambda,\omega} Z'_{\Lambda,\omega}} \int_{\mathscr{T}} \mu_{\zeta'\omega_{\Lambda^c}}(dT)\,q^{K(\zeta'\omega_{\Lambda^c},T)}\,P^{z/q}_{\Lambda,\omega}(d\xi_{\zeta'}|\xi_{\zeta'} = \xi_\zeta \text{ on } \Lambda\setminus\nabla) \\
&= \frac{1}{\tilde{Z}^{z,q}_{\nabla,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\nabla^c}\omega_{\Lambda^c}}(dT)\,q^{K(\kappa\zeta_{\nabla^c}\omega_{\Lambda^c},T)}\,P^{z/q}_{\nabla,\zeta\omega_{\Lambda^c}}(d\xi_\kappa),
\end{aligned} \tag{5.43}$$

where $Z'_{\Lambda,\omega}$ is a normalisation constant,

$$\tilde{Z}^{z,q}_{\nabla,\zeta\omega_{\Lambda^c}} := \frac{Z_H Z^{z/q}_{\nabla,\zeta\omega_{\Lambda^c}}}{\hat{Z}^{z,q}_{\Lambda,\omega} Z'_{\Lambda,\omega} Z^{z/q}_{\Lambda,\omega}},$$

and

$$Z_H := \exp\Big\{ - \sum_{\eta \in \mathcal{E}_\Lambda(\kappa\zeta_{\nabla^c}\omega_{\Lambda^c}) \setminus \mathcal{E}_\nabla(\kappa\zeta_{\nabla^c}\omega_{\Lambda^c})} \psi(\eta, \kappa\zeta_{\nabla^c}\omega_{\Lambda^c}) \Big\}$$

is independent of κ, by the same argument that follows (5.1). To see this, observe that

$$\begin{aligned} &P^{z/q}_{\Lambda,\omega}(d\xi_{\zeta'} | \xi_{\zeta'} = \xi_\zeta \text{ on } \Lambda \setminus \nabla) \\ &\quad = \frac{1}{Z^{z/q}_{\Lambda,\omega}} e^{-H^\psi_{\Lambda,\omega}(\zeta')} \Pi^{z/q}_\Lambda (d\xi_{\zeta'} | \xi_{\zeta'} = \xi_\zeta \text{ on } \Lambda \setminus \nabla) \\ &\quad = \frac{Z_H}{Z^{z/q}_{\Lambda,\omega}} e^{-H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\kappa)} \Pi^{z/q}_\nabla (d\xi_\kappa) \\ &\quad = \frac{Z_H}{Z^{z/q}_{\Lambda,\omega}} Z^{z/q}_{\nabla,\zeta\omega_{\Lambda^c}} P^{z/q}_{\nabla,\zeta\omega_{\Lambda^c}}(d\xi_\kappa). \end{aligned}$$

The positions of the configuration $\zeta\omega_{\Lambda^c}$ are the boundary condition, since the positions of $\zeta \in \Omega_\Lambda$ are distributed according to $M_{\Lambda,\omega}$ and we are now looking at a random configuration of particle positions within $\nabla \subset \Lambda$, given the positions of ζ.

Let $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ denote the distribution of the random hyperedge configuration $\{\xi_\tau : \tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}),\ \tilde{\gamma}_\tau = 1\} \in \mathscr{T}$, where $(\tilde{\gamma}_\tau)_{\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})}$ are independent $\{0,1\}$-valued random variables with

$$\mathrm{Prob}(\tilde{\gamma}_\tau = 1) = \tilde{p} := \frac{1 - e^{-\beta}}{1 + (q^2 - 1)e^{-\beta}} \tag{5.44}$$

when $\delta(\tau) < 2r_1$, and $\mathrm{Prob}(\tilde{\gamma}_\tau = 1) = 0$ otherwise. We prove the stochastic domination of $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$ over $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ in Lemma 5.7, below.

For a boundary condition $\xi_\omega \in \mathcal{K}$ and given the positions $\xi_\zeta \in \mathcal{K}_\Lambda$ of a configuration ζ, let $\tilde{\lambda}_{\zeta\omega_{\Lambda^c}}$ distribute the marks of $\zeta\omega_{\Lambda^c} \in \Omega$ as follows. The type-picking mechanism $\tilde{\lambda}_{\zeta\omega_{\Lambda^c}}$ assigns each $x \in \xi_\zeta\xi_{\omega_{\Lambda^c}}$ the type $\tilde{t}_x$, where $\{\tilde{t}_x : x \in \xi_\zeta\xi_{\omega_{\Lambda^c}}\}$ are independent Bernoulli distributed:

$$\mathrm{Prob}(\tilde{t}_x = s) = \begin{cases} \tilde{p} & \text{if } s = 1, \\ 1 - \tilde{p} & \text{if } s \neq 1. \end{cases} \tag{5.45}$$

For a boundary condition $\xi_\omega \in \mathcal{K}$, fixing the positions of $\omega \in \Omega$, define $\widetilde{C}^{z,q}_{\Lambda,\omega}$ on Ω by

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(d\omega') := M_{\Lambda,\omega}(d\xi_\zeta)\, \tilde{\lambda}_{\zeta\omega_{\Lambda^c}}(d\omega'), \tag{5.46}$$

where $\omega' := \zeta\omega_{\Lambda^c} \in \Omega$.

For a hypergraph structure $\mathcal{E}$ and finite marked configuration $\omega \in \Omega_f$, let $|\mathcal{E}(\omega)| := |\{\xi_\tau : \tau \in \mathcal{E}(\omega)\}|$ be the number of hyperedges in $\mathcal{E}(\omega)$. For a marked hyperedge τ, recall that $|\tau| := |\xi_\tau| = 3$ is the number of particles in τ.

Lemma 5.7. *For all $q \geq 2$, $\zeta \in \Omega_\Lambda$ and $\omega \in \Omega$, we have $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$.*

Proof. Let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$, where Λ is a bounded region of $\mathbb{R}^2$. For brevity in the calculations below, we write $\mathcal{E} := \mathcal{E}^{\mathrm{D}_3}_\Lambda(\zeta\omega_{\Lambda^c})$, where

$$\mathcal{E}^{\mathrm{D}_3}_\Lambda(\zeta\omega_{\Lambda^c}) := \{\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}) : g(\tau, \zeta\omega_{\Lambda^c}) \neq g(\tau, \kappa\omega_{\Lambda^c}) \text{ for some } \kappa \in \Omega_\Lambda\}.$$

We wish to refer to the finite configuration of particles that make up the hyperedges of $\mathcal{E}$. We call this configuration $\zeta' \in \Omega_f$. Any hyperedge $\eta \in \mathcal{E}$ satisfies $\eta \subset \zeta'$. Define the configuration $\zeta' \subset \zeta\omega_{\Lambda^c}$ as

$$\zeta' := \{(x, u_x) \in \zeta\omega_{\Lambda^c} : \forall \tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}),\ \text{if } \xi_\tau \ni x \text{ then } \tau \in \mathcal{E}\}.$$

The reason for defining this finite configuration is so that we can refer to the set of hyperedges $\mathcal{E}$ and the configuration of particles ζ' that make up these hyperedges. Therefore we can discuss the finite hypergraph $(\zeta', \mathcal{E})$. We cannot describe this hypergraph with ζ, because ζ is the configuration inside Λ, and some hyperedges of $\mathcal{E}$ comprise particles that lie outside Λ. So ζ is completely contained within Λ and ζ' consists of the configuration ζ plus some extra particles just outside the boundary of Λ.

The hypergraph $(\zeta', \mathcal{E})$ is finite because $|\zeta'|$ and $|\mathcal{E}|$ are finite. $|\mathcal{E}| = |\mathcal{E}^{\mathrm{D}_2}_\Lambda(\zeta\omega_{\Lambda^c})|$ is finite because $\mathcal{E}^{\mathrm{D}_2}_\Lambda(\zeta\omega_{\Lambda^c})$ consists only of hyperedges that intersect Λ, of which there are a finite number (because Λ is bounded and we assume hard-core repulsion). Let each hyperedge $\tau \in \mathcal{E}$ be open or closed; i.e. each hyperedge is assigned state 1 or 0. We denote by $\Gamma = \{0,1\}^{\mathcal{E}}$ the set of hyperedge configurations, and for each $\gamma \in \Gamma$ and $\tau \in \mathcal{E}$, we have $\gamma(\tau) \in \{0,1\}$. For $p \in [0,1]$, $q \geq 2$, define the probability measure $\mu_{p,q} \in \mathscr{M}_1(\Gamma)$,

$$\mu_{p,q}(\gamma) := \frac{1}{Z_{p,q}} q^{k(\gamma)} \prod_{\tau \in \mathcal{E}} p^{\gamma(\tau)}(1-p)^{1-\gamma(\tau)},$$

where $Z_{p,q}$ is a normalisation constant,

$$Z_{p,q} := \sum_{\gamma\in\Gamma}\Big\{q^{k(\gamma)}\prod_{\tau\in\mathcal{E}}p^{\gamma(\tau)}(1-p)^{1-\gamma(\tau)}\Big\},$$

and $k(\gamma)$ is the number of connected components in the hypergraph $(\zeta', \{\tau \in \mathcal{E} : \gamma(\tau) = 1\})$. The probability measure $\mu_{p,q}$ independently assigns each hyperedge state 1 with probability p and 0 with probability $1-p$. We claim that

$$q_1 \geq q_2,\ q_1 \geq 1,\ \frac{p_1}{q_1^2(1-p_1)} \geq \frac{p_2}{q_2^2(1-p_2)} \quad \Rightarrow \quad \mu_{p_1,q_1} \succeq \mu_{p_2,q_2}. \tag{5.47}$$

Let $\gamma, \gamma' \in \Gamma$. We say that a function $f : \Gamma \to \mathbb{R}$ is increasing on Γ if $f(\gamma) \leq f(\gamma')$ whenever γ and γ' are such that $\gamma(\tau) \leq \gamma'(\tau)$ for all $\tau \in \mathcal{E}$. We now prove (5.47). Let $f : \Gamma \to \mathbb{R}$ be increasing. Then

$$\mu_{p_2,q_2}(f) = \frac{1}{Z_{p_2,q_2}}\sum_{\gamma\in\Gamma} f(\gamma) q_2^{k(\gamma)} p_2^{\sum_{\tau\in\mathcal{E}}\gamma(\tau)}(1-p_2)^{\sum_{\tau\in\mathcal{E}}(1-\gamma(\tau))} \tag{5.48}$$

$$= \frac{1}{Z_{p_2,q_2}}\Big(\frac{1-p_2}{1-p_1}\Big)^{|\mathcal{E}|}\sum_{\gamma\in\Gamma}\Big\{f(\gamma)\Big(\frac{q_2}{q_1}\Big)^{k(\gamma)}\Big(\frac{p_2}{1-p_2}\Big)^{\sum_{\tau\in\mathcal{E}}\gamma(\tau)} \Big(\frac{1-p_1}{p_1}\Big)^{\sum_{\tau\in\mathcal{E}}\gamma(\tau)}(1-p_1)^{\sum_{\tau\in\mathcal{E}}(1-\gamma(\tau))}p_1^{\sum_{\tau\in\mathcal{E}}\gamma(\tau)}q_1^{k(\gamma)}\Big\} \tag{5.49}$$

$$= \frac{1}{Z_{p_2,q_2}}\Big(\frac{1-p_2}{1-p_1}\Big)^{|\mathcal{E}|} \times\sum_{\gamma\in\Gamma} f(\gamma)g(\gamma)(1-p_1)^{\sum_{\tau\in\mathcal{E}}(1-\gamma(\tau))}p_1^{\sum_{\tau\in\mathcal{E}}\gamma(\tau)}q_1^{k(\gamma)}$$

$$= \frac{Z_{p_1,q_1}}{Z_{p_2,q_2}}\Big(\frac{1-p_2}{1-p_1}\Big)^{|\mathcal{E}|}\mu_{p_1,q_1}(fg), \tag{5.50}$$

where

$$g(\gamma) := \Big(\frac{q_2}{q_1}\Big)^{k(\gamma)}\prod_{\tau\in\mathcal{E}}\Big(\frac{p_2}{1-p_2}\frac{1-p_1}{p_1}\Big)^{\gamma(\tau)},$$

which can be rewritten as

$$g(\gamma) = \Big(\frac{q_2}{q_1}\Big)^{k(\gamma)+\sum_{\tau\in\mathcal{E}}(|\tau|-1)\gamma(\tau)}\prod_{\tau\in\mathcal{E}}\left(\frac{p_2/\{q_2^{|\tau|-1}(1-p_2)\}}{p_1/\{q_1^{|\tau|-1}(1-p_1)\}}\right)^{\gamma(\tau)}. \tag{5.51}$$

We have (5.48) and (5.50) by the definition of $\mu_{p,q}$, and (5.49) uses the fact that

$$|\mathcal{E}| = \sum_{\tau\in\mathcal{E}} \gamma(\tau) + \sum_{\tau\in\mathcal{E}} (1-\gamma(\tau)).$$

Setting $f = 1$, we obtain

$$\mu_{p_2,q_2}(1) = 1 = \frac{Z_{p_1,q_1}}{Z_{p_2,q_2}} \Big(\frac{1-p_2}{1-p_1}\Big)^{|\mathcal{E}|} \mu_{p_1,q_1}(g).$$

Substituting this into (5.50), we find

$$\mu_{p_2,q_2}(f) = \frac{\mu_{p_1,q_1}(fg)}{\mu_{p_1,q_1}(g)}. \tag{5.52}$$

Observe that $\sum_{\tau\in\mathcal{E}} \gamma(\tau)$ and $k(\gamma) + \sum_{\tau\in\mathcal{E}}(|\tau|-1)\gamma(\tau)$ are increasing functions of γ. The former is obvious. To see the latter, define the configurations γ^τ and γ_τ, for $\tau, \tau' \in \mathcal{E}$,

$$\gamma^\tau(\tau') := \begin{cases} \gamma(\tau') & \text{if } \xi_{\tau'} \neq \xi_\tau, \\ 1 & \text{if } \xi_{\tau'} = \xi_\tau, \end{cases} \tag{5.53}$$

$$\gamma_\tau(\tau') := \begin{cases} \gamma(\tau') & \text{if } \xi_{\tau'} \neq \xi_\tau, \\ 0 & \text{if } \xi_{\tau'} = \xi_\tau. \end{cases} \tag{5.54}$$

Let $k_\tau(\gamma)$ be the number of connected components in the hypergraph $(\zeta', \{\tau \in \mathcal{E} : \gamma(\tau) = 1\})$ that contain vertices belonging to the hyperedge τ. Consider the hypergraph when the hyperedge τ closed. The number of clusters removed from the configuration γ when τ is opened is given by the expression

$$k(\gamma_\tau) - k(\gamma^\tau) = k_\tau(\gamma_\tau) - 1. \tag{5.55}$$

No more than $|\tau| = 3$ open clusters can be attached to τ (one cluster for each vertex). Therefore

$$k(\gamma_\tau) - k(\gamma^\tau) = k_\tau(\gamma_\tau) - 1 \leq |\tau| - 1 = 2. \tag{5.56}$$

So for any $\tau \in \mathcal{E}$, changing $\gamma(\tau)$ from 0 to 1 will remove no more than 2 clusters. From this we can see that $k(\gamma) + \sum_{\tau\in\mathcal{E}}(|\tau|-1)\gamma(\tau)$ is increasing.

Assume that the conditions of (5.47) hold. Then, using the expression (5.51) and the fact that that $\sum_{\tau\in\mathcal{E}} \gamma(\tau)$ and $k(\gamma) + \sum_{\tau\in\mathcal{E}}(|\tau|-1)\gamma(\tau)$ are increasing, we can see that g is decreasing. For $q_1 \geq 1$, f increasing and g decreasing, we have

$$\mu_{p_1,q_1}(fg) \leq \mu_{p_1,q_1}(f)\mu_{p_2,q_2}(g). \tag{5.57}$$

Using (5.52), we have $\mu_{p_1,q_1}(f) \geq \mu_{p_2,q_2}(f)$ and the result (5.47) follows.

Under $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$, all hyperedges outside Λ are open. So any cluster attached to the boundary of Λ is an infinite cluster. Therefore we can apply (5.47), taking $p_1 = p_\Lambda(\tau)$, $q_1 = q$, $p_2 = \tilde{p}$ and $q_2 = 1$. We have $q \geq 1$ and

$$\frac{p_\Lambda(\tau)}{q^2(1 - p_\Lambda(\tau))} \geq \frac{\tilde{p}}{1 - \tilde{p}}, \tag{5.58}$$

for all $\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$. The inequality (5.58) holds because $\phi_0(\delta) \geq u$ for all $\delta < 2r_1$. Therefore, using (5.47) and (5.58), we have $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$. $\square$

Lemma 5.8. *For all bounded regions $\nabla, \Delta, \Lambda \in \mathcal{B}_R$ such that $\nabla \subset \Delta \subset \Lambda$, any local and measurable function f on $\mathcal{K}_\Delta$, and any $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$,*

$$\int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_\nabla} f(\xi_{\hat{\kappa}}\xi_{\kappa_{\nabla^c}}) M_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\xi_{\hat{\kappa}}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi_\kappa) = \int_{\mathcal{K}_\Delta} f(\xi_\kappa) M_{\Lambda,\omega;\Delta,\zeta}(d\xi_\kappa).$$

Proof. Let ∇, Δ, Λ be bounded regions of $\mathbb{R}^2$ satisfying $\nabla \subset \Delta \subset \Lambda$. Let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ be such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$. For brevity in the calculations below, we write $\xi := \xi_\kappa \in \mathcal{K}_\Delta$ and $\hat{\xi} := \xi_{\hat{\kappa}} \in \Omega_\nabla$ as the positions of configurations $\kappa \in \Omega_\Delta$ and $\hat{\kappa} \in \Omega_\nabla$. Note that since the background interaction $\psi(\eta, \omega)$ only depends on ξ_η, we can write $H^\psi_{\Lambda,\omega}(\zeta)$ as $H^\psi_{\Lambda,\omega}(\xi_\zeta)$. Using the definition (5.43) for the distribution of particle positions, the measure $M_{\Lambda,\omega;\Delta,\zeta}$ can be expressed as follows:

$$\begin{aligned} M_{\Lambda,\omega;\Delta,\zeta}(d\xi) &= \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT)\, q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)} P^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}}(d\xi) \\ &= \frac{1}{Z^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}}} \exp\Big\{ -\beta H^\psi_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\xi_\nabla) \\ &\qquad -\beta\Big(H^\psi_{\Delta,\zeta\omega_{\Lambda^c}}(\xi) - H^\psi_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\xi_\nabla)\Big)\Big\} M^0_{\Lambda,\omega;\Delta,\zeta}(d\xi), \end{aligned} \tag{5.59}$$

where $M^0_{\Lambda,\omega;\Delta,\zeta}$ is defined on $\mathcal{K}_\Delta$,

$$M^0_{\Lambda,\omega;\Delta,\zeta}(d\xi) := \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT)\, q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)} \Pi^{z/q}_\Delta(d\xi). \tag{5.60}$$

If f is some local and measurable function on $\mathcal{K}_\Delta$, then

$$\begin{aligned}
M_{\Lambda,\omega;\Delta,\zeta}(f) &= \int_{\mathcal{K}_\Delta} f(\xi) M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&= \int_{\mathcal{K}_\Delta} \left[\int_{\mathcal{K}_\Delta} f(\xi') M_{\Lambda,\omega;\Delta,\zeta}(d\xi' | \xi'_{\nabla^c} = \xi_{\nabla^c}) \right] M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&= \int_{\mathcal{K}_\Delta} \Bigg[\int_{\mathcal{K}_\Delta} f(\xi') \frac{1}{Z^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}}} \exp\Big\{ -\beta H^{\psi}_{\nabla,\kappa'\zeta_{\Delta^c}\omega_{\Lambda^c}}(\xi'_\nabla) \\
&\qquad -\beta\Big(H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\xi') - H^{\psi}_{\nabla,\kappa'\zeta_{\Delta^c}\omega_{\Lambda^c}}(\xi'_\nabla) \Big) \Big\} \\
&\qquad \times M^0_{\Lambda,\omega;\Delta,\zeta}(d\xi' | \xi'_{\nabla^c} = \xi_{\nabla^c}) \Bigg] M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&= \int_{\mathcal{K}_\Delta} \Bigg[\int_{\mathcal{K}_\nabla} f(\hat{\xi}\xi_{\nabla^c}) \frac{1}{Z^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}}} \exp\Big\{ -\beta H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \\
&\qquad -\beta\Big(H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\hat{\xi}\xi_{\nabla^c}) - H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \Big) \Big\} \\
&\qquad \times M^0_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) \Bigg] M_{\Lambda,\omega;\Delta,\zeta}(d\xi).
\end{aligned} \tag{5.61}$$

Note that

$$\begin{aligned}
&H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\hat{\xi}\xi_{\nabla^c}) - H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \\
&\quad = \sum_{\eta\in\mathcal{E}^{\mathrm{D}_2}_{\Delta}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|) - \sum_{\eta\in\mathcal{E}^{\mathrm{D}_2}_{\nabla}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|)
\end{aligned}$$

depends only on the configuration outside ∇. Similarly, since we can write

$$\begin{aligned}
Z^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}} &:= \int_{\mathcal{K}_\Delta} \exp\Big(-\beta \sum_{\eta\in\mathcal{E}^{\mathrm{D}_2}_{\Delta}(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|) \Big) \Pi^{z/q}_{\Delta}(d\xi) \\
&= \int_{\mathcal{K}_\Delta} \exp\Big(-\beta H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \\
&\qquad -\beta\Big(H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\hat{\xi}\xi_{\nabla^c}) - H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \Big) \Big) \Pi^{z/q}_{\Delta}(d\xi)
\end{aligned}$$

and

$$Z^{z/q}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}} := \int_{\mathcal{K}_\nabla} \exp\Big(-\beta H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \Big) \Pi^{z/q}_{\nabla}(d\hat{\xi}),$$

from (5.61), we have

$$M_{\Lambda,\omega;\Delta,\zeta}(f) = \int_{\mathcal{K}_\Delta} \Bigg[\int_{\mathcal{K}_\nabla} f(\hat{\xi}\xi_{\nabla^c}) \frac{1}{Z^{z/q}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}} \exp\Big\{ -\beta H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \Big\} \times M^0_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) \Bigg] M_{\Lambda,\omega;\Delta,\zeta}(d\xi)$$

$$= \int_{\mathcal{K}_\Delta} \int_{\mathcal{K}_\nabla} f(\hat{\xi}\xi_{\nabla^c}) M_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi),$$

and the proof is complete. □

Lemma 5.9. *For any $\Lambda \in \mathcal{B}_R$, let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$ be such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\zeta\omega_{\Lambda^c} \cup (\{x\},\{u_x\}) \in \Omega^{\Lambda,z}_*$ for any $u_x \in S$. Then there exists some $\alpha_1 > 0$ such that*

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq \alpha_1.$$

Proof. For a hypergraph structure $\mathcal{E}$, we will use the notation $\mathcal{E}\big(\omega \cup \{x\}\big) := \mathcal{E}\big(\omega \cup (\{x\},\{u_x\})\big)$, for any $u_x \in S$. Define

$$\begin{aligned} T^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}} &:= \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}) \cap \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c} \cup \{x\}), \\ T^{+}_{x,\zeta\omega_{\Lambda^c}} &:= \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c} \cup \{x\}) \setminus \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}), \\ T^{-}_{x,\zeta\omega_{\Lambda^c}} &:= \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}) \setminus \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c} \cup \{x\}), \end{aligned}$$

and $\mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}$, $\mu^{+}_{x,\zeta\omega_{\Lambda^c}}$ and $\mu^{-}_{x,\zeta\omega_{\Lambda^c}}$ as the hyperedge-drawing mechanisms on $T^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}$, $T^{+}_{x,\zeta\omega_{\Lambda^c}}$ and $T^{-}_{x\zeta\omega_{\Lambda^c}}$, respectively.

The hyperedge-drawing mechanism $\mu_{\zeta\omega_{\Lambda^c}}$ distributes hyperedges based on the *positions* of the particles $\zeta\omega_{\Lambda^c}$. Therefore, given an additional particle position $x \in \mathbb{R}^2 \setminus \xi_{\zeta\omega_{\Lambda^c}}$, we can define a hyperedge drawing mechanism based on the positional configuration $\xi_\zeta\xi_{\omega_{\Lambda^c}} \cup \{x\}$. We write this hyperedge-drawing mechanism as $\mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}$. We also write $K(\xi_\zeta\xi_{\omega_{\Lambda^c}} \cup \{x\}, T)$ as $K(\zeta\omega_{\Lambda^c} \cup \{x\}, T)$. Observe that

$$\begin{aligned} &\int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}(dT)\, q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},T)} \\ &= \int_{\mathscr{T}} \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(dT_1)\, q^{K(\zeta\omega_{\Lambda^c},T_1)} \int_{\mathscr{T}} \mu^{+}_{x,\zeta\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},T_1\cup T_2)}}{q^{K(\zeta\omega_{\Lambda^c},T_1)}}. \end{aligned} \tag{5.62}$$

This is because the definition (3.6) of the hyperedge-drawing probability p_Λ means that the hyperedge distribution on $T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}$ is independent of the distribution on $T^{+}_{x,\zeta\omega_{\Lambda^c}}$. To see this, consider hyperedge configurations $\gamma_{\mathrm{ext}} \in \{0,1\}^{T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}}$ and $\gamma_+ \in \{0,1\}^{T^{+}_{x,\zeta\omega_{\Lambda^c}}}$ for $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$. Let E_{ext} and E_+ be the events that γ_{ext} and γ_+ occur, respectively. Let $\gamma = \gamma_{\mathrm{ext}} \cup \gamma_+$.

$$
\begin{aligned}
\mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}(E_{\mathrm{ext}} \cap E_+) &= \prod_{\substack{\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}\cup\{x\}):\\ \gamma(\tau)=1}} p_\Lambda(\tau) \prod_{\substack{\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c}\cup\{x\}):\\ \gamma(\tau)=0}} \big(1 - p_\Lambda(\tau)\big) \\
&= \prod_{\substack{\tau \in T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}} \cup T^{+}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma(\tau)=1 \\ \tau\cap\Lambda\neq\emptyset}} \big(1 - e^{-\phi_0(\xi_\tau)}\big) \prod_{\substack{\tau \in T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}} \cup T^{+}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma(\tau)=0 \\ \tau\cap\Lambda\neq\emptyset}} e^{-\phi_0(\xi_\tau)} \\
&= \prod_{\substack{\tau \in T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma_{\mathrm{ext}}(\tau)=1 \\ \tau\cap\Lambda\neq\emptyset}} \big(1 - e^{-\phi_0(\xi_\tau)}\big) \prod_{\substack{\tau \in T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma_{\mathrm{ext}}(\tau)=0 \\ \tau\cap\Lambda\neq\emptyset}} e^{-\phi_0(\xi_\tau)} \\
&\quad \times \prod_{\substack{\tau \in T^{+}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma_{+}(\tau)=1}} \big(1 - e^{-\phi_0(\xi_\tau)}\big) \prod_{\substack{\tau \in T^{+}_{x,\zeta\omega_{\Lambda^c}}:\\ \gamma_{+}(\tau)=0}} e^{-\phi_0(\xi_\tau)} \\
&= \mu^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}(E_{\mathrm{ext}})\, \mu^{+}_{x,\zeta\omega_{\Lambda^c}}(E_+).
\end{aligned}
$$

Note that there are hyperedge triangles of $T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}$ that share edges with the triangles of $T^{+}_{x,\zeta\omega_{\Lambda^c}}$, so $T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}} \cap T^{+}_{x,\zeta\omega_{\Lambda^c}} \neq \emptyset$. Each triangle of $T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}$ that shares an edge with a triangle of $T^{+}_{x,\zeta\omega_{\Lambda^c}}$ will share exactly one edge. Although these triangles share an edge, this does not affect the dependence on the states (0 or 1) of the triangles. If the two vertices of the common edge have different colour, then it does not matter what colour the third vertex of the neighbour triangle has. If the colours are equal, then p_Λ only depends on the third vertex.

By a similar argument to (5.62), we also have

$$
\begin{aligned}
&\int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}}(dT)\, q^{K(\zeta\omega_{\Lambda^c},T)} = \\
&\qquad \int_{\mathscr{T}} \mu^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}(dT_1)\, q^{K(\zeta\omega_{\Lambda^c},T_1)} \int_{\mathscr{T}} \mu^{-}_{x,\zeta\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c},T_1\cup T_2)}}{q^{K(\zeta\omega_{\Lambda^c},T_1)}}.
\end{aligned}
\tag{5.63}
$$

Due to the definition of $h_{\Lambda,\omega}$ and the expressions (5.62) and (5.63),

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} = \frac{\int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}(dT)\, q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},T)}}{\int_{\mathscr{T}} \mu_{\zeta\omega_{\Lambda^c}}(dT)\, q^{K(\zeta\omega_{\Lambda^c},T)}}$$
$$= \frac{\int_{\mathscr{T}} \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(dT_1)\, q^{K(\zeta\omega_{\Lambda^c},T_1)} \int_{\mathscr{T}} \mu^{+}_{x,\zeta\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},T_1\cup T_2)}}{q^{K(\zeta\omega_{\Lambda^c},T_1)}}}{\int_{\mathscr{T}} \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(dT_1)\, q^{K(\zeta\omega_{\Lambda^c},T_1)} \int_{\mathscr{T}} \mu^{-}_{x,\zeta\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c},T_1\cup T_2)}}{q^{K(\zeta\omega_{\Lambda^c},T_1)}}}.$$

Recall that, due to the boundary condition, $K(\zeta\omega_{\Lambda^c}, T) - K_\Lambda(\zeta\omega_{\Lambda^c}, T)$ is constant, either 0 or 1. Therefore $K(\zeta\omega_{\Lambda^c}, T)$ is finite. Because $\phi_0(\delta) = 0$ for $\delta > 2r_1$ and $\psi_0(r) = \infty$ for $r < r_0$, we have

$$K(\zeta\omega_{\Lambda^c}, T_1) - K(\zeta\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) \leq \frac{4\pi r_1^2}{\pi r_0^2} \tag{5.64}$$
$$\Rightarrow \quad K(\zeta\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) - K(\zeta\omega_{\Lambda^c}, T_1) \geq -\frac{4\pi r_1^2}{\pi r_0^2}. \tag{5.65}$$

This is because the left hand side of (5.64) must be no greater than the maximum number of clusters that can be removed by adding a particle at position x into the configuration $\zeta\omega_{\Lambda^c}$. The maximum number is $(4\pi r_1^2)/(\pi r_0^2)$. This is because for the addition of one particle to remove as many clusters as possible, this particle must form a cluster which joins as many other particles from the initial configuration $\zeta\omega_{\Lambda^c}$ as possible.

Under the hyperedge-drawing mechanism, a particle at position x cannot form an open hyperedge with particles further than $2r_1$. Due to the hard-core condition, there must be a ball of radius r_0 around each particle of the configuration, within which are no other particles of the configuration. Therefore the maximum number of particles, that x may form a new hyperedge with, is $(4\pi r_1^2)/(\pi r_0^2)$.

Also,

$$K(\zeta\omega_{\Lambda^c}, T_1 \cup T_2) - K(\zeta\omega_{\Lambda^c}, T_1) \leq 0. \tag{5.66}$$

Therefore

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq q^{-\frac{4r_1^2}{r_0^2}} =: \alpha_1 > 0. \tag{5.67}$$

□

For any bounded Borel set ∇ and real $r > 0$, let

$$\nabla \ominus r := \bigcap_{y \in B(0,r)} \{x + y,\, x \in \nabla\}$$

denote the r-minus sampling of ∇.

Lemma 5.10. *For all cells $\nabla = \Delta_{k,l}^{i,j} \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\nabla$ be such that $\kappa\zeta_{\nabla^c}\omega_{\Lambda^c} \in \Omega_*^{\nabla,z}$. We have*

$$M_{\Lambda,\omega;\nabla,\zeta}(|\xi_\kappa| \geq 1) > 1 - \frac{p_0}{81},$$

for all

$$z > \frac{81q}{p_0\alpha_1(L - 2r_0)^2}.$$

Proof. We write $\xi := \xi_\kappa \in \mathcal{K}_\nabla$. Since $H^\psi(\omega)$ does not depend on the marks, we sometimes emphasise this and write $H^\psi(\xi_\omega)$. We have

$$\begin{aligned}
\frac{M_{\Lambda,\omega;\nabla,\zeta}(|\xi| = 1)}{M_{\Lambda,\omega;\nabla,\zeta}(|\xi| = 0)} &= \frac{\int_{\mathcal{K}_\nabla} h_{\nabla,\zeta\omega_{\Lambda^c}}(\xi) e^{-H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{|\xi| = 1\}\, \Pi_\nabla^{z/q}(d\xi)}{\int_{\mathcal{K}_\nabla} h_{\nabla,\zeta\omega_{\Lambda^c}}(\xi) e^{-H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{|\xi| = 0\}\, \Pi_\nabla^{z/q}(d\xi)} \\
&= \frac{e^{-z|\nabla|/q} z \int_\nabla h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) e^{-H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}\, dx}{q\, h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) \int_{\mathcal{K}_\nabla} \mathbb{I}\{|\xi| = 0\}\, \Pi_\nabla^{z/q}(d\xi)} \\
&= \frac{e^{-z|\nabla|/q} z \int_\nabla h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) e^{-H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}\, dx}{q\, h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) e^{-z|\nabla|/q}} \\
&= \frac{z}{q} \int_\nabla \exp\Big(- H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big) \frac{h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}{h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset)}\, dx.
\end{aligned} \tag{5.68}$$

The second line comes from the definition of a Poisson point process Π_Λ^z on $\mathcal{K}_\Lambda$ with activity $z > 0$,

$$\int_{\mathcal{K}_\Lambda} f\, d\Pi_\Lambda^z = e^{-z|\Lambda|} \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\})\, dx_1 \ldots dx_n,$$

for any bounded measurable function f on $\mathcal{K}_\Lambda$. From Lemma 5.9 we have

$$\begin{aligned}
\frac{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=1)}{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=0)} &\geq \frac{z}{q}\int_{\nabla}\alpha_1 \exp\Big(-H^{\psi}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big)\,dx \\
&\geq \frac{z}{q}\alpha_1\int_{\nabla\ominus r_0}\exp\Big(-H^{\psi}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big)\,dx \\
&\geq \frac{\alpha_1 z|\nabla\ominus r_0|}{q} \qquad (5.69)\\
\Longrightarrow \qquad M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=0) &\leq \frac{q}{\alpha_1 z|\nabla\ominus r_0|} \qquad (5.70)\\
\Longrightarrow \qquad M_{\Lambda,\omega;\nabla,\zeta}(|\xi|\geq 1) &> 1-\frac{q}{\alpha_1 z|\nabla\ominus r_0|} \qquad (5.71)\\
&> 1-\frac{p_0}{81}. \qquad (5.72)
\end{aligned}$$

The final inequality holds for all

$$z > \frac{81q}{p_0\alpha_1(L-2r_0)^2}.$$

Since $L\in(2r_0,\sqrt{2\pi}\,r_0)$, we have $|\nabla\ominus r_0| = (L-2r_0)^2 > 0$. □

Lemma 5.11. *For any $\Delta=\Delta_{k,l}\subset\Lambda$, there exists $\alpha_2>0$ such that*

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta\longleftrightarrow\Lambda^c\})\geq\alpha_2,$$

for all

$$z > \frac{81q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(\sqrt{2\pi}-2)^2r_0^2}$$

and

$$\beta\geq\log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{162}}}{1-(1-p_0)^{\frac{1}{162}}}\right\}.$$

Proof. Let $\nabla=\Delta^{i,j}_{k,l}$ for some $i,j\in\{0,\dots,8\}$. By Lemma 5.10, if $z>81/(p_0\alpha_1(L-2r_0)^2)$ then

$$\forall\,\omega\in\Omega,\zeta\in\Omega_\Lambda,\qquad M_{\Lambda,\omega;\nabla,\zeta}(|\xi_\kappa|=0)\leq\frac{p_0}{81}, \qquad (5.73)$$

where $\kappa\in\Omega_\nabla$. For a meso-box Δ, and micro-box $\nabla\subset\Delta$, let $A_\nabla\in\mathcal{K}_\nabla$ be the empty configuration on ∇, and define

$$A_{\Delta,\nabla} \;:=\; \big\{\xi\in\mathcal{K}_\Delta : |\xi\cap\nabla|=0\big\}\in\mathcal{K}_\Delta.$$

For convenience, we now write $\xi = \xi_\kappa \in \mathcal{K}_\Delta$ and $\hat{\xi} = \xi_{\hat{\kappa}} \in \mathcal{K}_\nabla$ for configurations $\kappa \in \Omega_\Delta$, $\hat{\kappa} \in \Omega_\nabla$. From (5.73), we have

$$\begin{aligned}
&\int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_\nabla} \mathbb{I}\{\hat{\xi}\xi_{\nabla^c} \in A_{\Delta,\nabla}\} M_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&\qquad = \int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_\nabla} \mathbb{I}\big\{\hat{\xi}\xi_{\nabla^c} \in A_\nabla \cap \mathcal{K}_\Delta\big\} M_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&\qquad \le \frac{p_0}{81}
\end{aligned} \tag{5.74}$$

for all $z > 81/(p_0\alpha_1(L-2r_0)^2)$. From Lemma 5.8, we have

$$\begin{aligned}
&\int_{\mathcal{K}_\Delta} \mathbb{I}\{\xi \in A_{\Delta,\nabla}\} M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&\qquad = \int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_\nabla} \mathbb{I}\{\hat{\xi}\xi_{\nabla^c} \in A_{\Delta,\nabla}\} M_{\Lambda,\omega;\nabla,\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi).
\end{aligned} \tag{5.75}$$

Combining (5.74) and (5.75), we have

$$\forall\, \omega \in \Omega, \zeta \in \Omega_\Lambda, \qquad M_{\Lambda,\omega;\Delta,\zeta}(|\xi_\kappa \cap \nabla| = 0) \le \frac{p_0}{81}, \tag{5.76}$$

where $\kappa \in \Omega_\Delta$. Note that in (5.73) and (5.76), $M_{\Lambda,\omega;\nabla,\zeta}$ is defined on $\mathcal{K}_\nabla$ and $M_{\Lambda,\omega;\Delta,\zeta}$ on $\mathcal{K}_\Delta$. For $\xi \in \mathcal{K}_{\Delta_{k,l}}$,

$$A_{k,l} := \bigcap_{i,j=0,\dots,8} (|\xi \cap \Delta_{k,l}^{i,j}| \ge 1),$$

we have

$$\begin{aligned}
M_{\Lambda,\omega;\Delta,\zeta}(A_{k,l}) &\ge 1 - \sum_{i,j=0,\dots,8} M_{\Lambda,\omega;\Delta,\zeta}(|\xi \cap \Delta_{k,l}^{i,j}| = 0) \\
&> 1 - 81 \cdot \frac{p_0}{81} = 1 - p_0 \\
&> p_c^{\text{site}}(\mathbb{Z}^2).
\end{aligned}$$

For the proof of Proposition 5.6 (and hence Theorem 4.1), this stage of the proof is the main instance where we use comparison with percolation. Let

$$B_{k,l} = \{\kappa\zeta_{\Delta_{k,l}^c}\omega_{\Lambda^c} : \forall\, \xi_\kappa \in A_{k,l}, \forall\, x \in \xi_\kappa, u_x^\kappa = 1\},$$

where $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $\kappa \in \Omega_{\Delta_{k,l}}$. Recall the definition (5.44) of $\tilde{p} = \tilde{p}(\beta)$,

$$\begin{aligned}
\tilde{p} &:= \frac{1-e^{-\beta}}{1+(q^2-1)e^{-\beta}} \in (0,1), \quad \forall\, q \geq 2,\, \beta > 0 \\
\implies \quad e^{\beta} &= \frac{1+\tilde{p}\,(q^2-1)}{1-\tilde{p}} \\
\implies \quad \beta &= \log\left\{\frac{1+\tilde{p}\,(q^2-1)}{1-\tilde{p}}\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{81J_L}}}{1-(1-p_0)^{\frac{1}{81J_L}}}\right\} \\
\iff \quad \tilde{p} &> (1-p_0)^{\frac{1}{81J_L}}, \qquad (5.77)
\end{aligned}$$

where J_L, see (4.2), is the maximum number of particles that can fit in a $L \times L$ box under the hard-core assumption on the background interaction. For $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $\kappa \in \Omega_\Delta$, we write $\omega'' := \kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega$. We have

$$\begin{aligned}
\widetilde{C}^{z,q}_{\Delta_{k,l},\zeta\omega_{\Lambda^c}}(B_{k,l}) = &\int_{\mathcal{K}_{\Delta_{k,l}}} M_{\Lambda,\omega;\Delta_{k,l},\zeta}(d\xi_\kappa) \\
&\times \int_{\Omega} \tilde{\lambda}_{\kappa\zeta_{\Delta^c_{k,l}}\omega_{\Lambda^c}}(d\omega'')\,\mathbb{I}\{\kappa\zeta_{\Delta^c_{k,l}}\omega_{\Lambda^c} \in B_{k,l}\} \\
= &\int_{\mathcal{K}_{\Delta_{k,l}}} M_{\Lambda,\omega;\Delta_{k,l},\zeta}(d\xi_\kappa)\,\tilde{p}^{|\xi_\kappa|}\,\mathbb{I}\{\xi_\kappa \in A_{k,l}\} \qquad (5.78) \\
\geq &\ (1-p_0)\tilde{p}^{81J_L} \qquad (5.79) \\
\geq &\ (1-p_0)^2 \qquad (5.80) \\
> &\ 1-2p_0 \\
> &\ p_c^{\text{site}}(\mathbb{Z}^2). \qquad (5.81)
\end{aligned}$$

We have (5.78) using the definition of $\tilde{\lambda}_{\zeta\omega_{\Lambda^c}}$, see (5.45). We have (5.79) because for an admissible configuration, J_L is the maximum number of points permissible in a $L \times L$ micro-box $\Delta^{i,j}_{k,l}$, and there are 81 micro-boxes in a meso-box $\Delta_{k,l}$. Inequality (5.80) comes from the inequality (5.77). Line (5.81) is due to the definition (4.1) of p_0 and the fact that $p_c^{\text{site}}(\mathbb{Z}^2) \in (0,1)$.

For Bernoulli site percolation on an infinite locally finite graph, $G = (V, E)$, there exists a critical value $p_c \in [0,1]$ such that, for any $x \in V$,

$$\mathbb{P}_p(x \leftrightarrow \infty) > 0 \iff p > p_c, \qquad (5.82)$$

see [Grim99], [GHM99]. If we compare the meso-boxes $\Delta_{k,l}$ to the sites of $\mathbb{Z}^2$, and the probability $\widetilde{C}^{z,q}_{\Delta_{k,l},\zeta\omega_{\Lambda^c}}(B_{k,l})$ to the probability of a site of $\mathbb{Z}^2$ being open (for Bernoulli site percolation), then we can use (5.82) to deduce that there exists a path of boxes $\Delta_{i,j}$ such that $B_{i,j}$ occurs, from any $\Delta_{k,l} \subset \Lambda$ to Λ^c.

Assume that $A_{k,l}$ and $A_{k+1,l}$ occur simultaneously. The central band of $\Delta_{k,l} \cup \Delta_{k+1,l}$ is defined

$$\mathrm{CB}_{k\,:\,k+1,\,l} = \Big(\bigcup_{i=0}^{4} \Delta_{k,l}^{4+i,4}\Big) \cup \Big(\bigcup_{i=0}^{4} \Delta_{k+1,l}^{i,4}\Big).$$

We keep in all the squares $\Delta_{k,l}$ and $\Delta_{k+1,l}$ the particles

$$H = \big\{(x, u_x^\omega) \in \omega \cap \big((\Delta_{k,l} \cup \Delta_{k+1,l}) \times S\big) \,:\, \xi_\tau \cap \mathrm{CB}_{k\,:\,k+1,\,l} \neq \emptyset,\, \forall \tau \ni (x, u_x^\omega)\big\}.$$

Let

$$\mathcal{E}^{\mathrm{D}_3}_{\mathrm{CB}}(\omega) := \{\tau \in \mathcal{E}^{\mathrm{D}_3}(\omega) : \tau \cap \mathrm{CB}_{k\,:\,k+1,\,l} \neq \emptyset\}. \tag{5.83}$$

The hyperedges of the restriction of the hypergraph $(\omega, \mathcal{E}^{\mathrm{D}_3}(\omega))$ to $(H, \mathcal{E}^{\mathrm{D}_3}_{\mathrm{CB}}(\omega))$ all have diameter less than $\sqrt{2}L < 2r_1$ because the little squares $\Delta_{k,l}^{i,j}$, $\Delta_{k+1,l}^{i,j}$, $i, j = 0, \ldots, 8$ contain at least one point and the circles circumscribed by the Delaunay triangles are empty. We will see later in (5.87) that it is necessary for $L < (\sqrt{2\pi})r_0$, therefore to ensure $L < \sqrt{2}r_1$, we assume $r_0 < r_1/\sqrt{\pi}$.

The Delaunay triangles of $(H, \mathcal{E}^{\mathrm{D}_3}_{\mathrm{CB}}(\omega))$ are all connected and completely cover the set $\mathrm{CB}_{k\,:\,k+1,\,l}$, i.e. they form a connected covering of $\mathrm{CB}_{k\,:\,k+1,\,l}$. Since any micro-box of $\mathrm{CB}_{k\,:\,k+1,\,l}$ has at least four other micro-boxes between itself and the boundary of $\Delta_{k,l} \cup \Delta_{k+1,l}$, we know that any hyperedge connected to $\mathcal{E}^{\mathrm{D}_3}_{\mathrm{CB}}(\omega)$ must be contained within $\Delta_{k,l} \cup \Delta_{k+1,l}$. This means every triangle of $(H, \mathcal{E}^{\mathrm{D}_3}_{\mathrm{CB}}(\omega))$, and its neighbouring triangles, are contained within $\Delta_{k,l} \cup \Delta_{k+1,l}$. Therefore we are able to connect any point of $\xi_\omega \cap \Delta_{k,l}^{4,4}$ to any point of $\xi_\omega \cap \Delta_{k+1,l}^{4,4}$ in the hypergraph $(\omega, \mathcal{E}^{\mathrm{D}_3}(\omega))$ inside $\Delta_{k,l} \cup \Delta_{k+1,l}$.

Line (5.81) tells us that the probability, under $\widetilde{C}^{z,q}_{\Lambda,\omega}$, of a meso-box $\Delta_{k,l}$ containing all type 1 points with at least one point in each micro-box $\Delta_{k,l}^{i,j}$, $i, j = 0, \ldots, 8$, is greater than the critical probability for site percolation on $\mathbb{Z}^2$. We can compare each meso-box to a site of $\mathbb{Z}^2$. When two neighbouring boxes, $\Delta_{k,l}$ and $\Delta_{k+1,l}$, satisfy $B_{k,l}$ and $B_{k+1,l}$, then the central micro-boxes of $\Delta_{k,l}$ and $\Delta_{k+1,l}$ can be connected within $\Delta_{k,l}$ and $\Delta_{k+1,l}$. This compares to site percolation on $\mathbb{Z}^2$: two neighbouring sites being open allows them to be connected. If the probability of a site being open is greater than $p_c^{\mathrm{site}}(\mathbb{Z}^2)$ then there is a positive probability of percolation.

Thus, using (5.81), there exists some $\alpha_2 > 0$ such that

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \leftrightarrow \Lambda^c\}) \geq \alpha_2, \tag{5.84}$$

for all

$$z > \frac{81q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(L-2r_0)^2}, \tag{5.85}$$

$$\beta \geq \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{81J_L}}}{1-(1-p_0)^{\frac{1}{81J_L}}}\right\}. \tag{5.86}$$

Note that

$$L \in (2r_0, \sqrt{2\pi}\,r_0) \quad \Rightarrow \quad J_L := \left[\frac{L^2}{\pi r_0^2}\right] + 1 = 2, \tag{5.87}$$

which gives the required bound for β. Since the right hand side of (5.85) is decreasing in L, we take L as large as possible in the interval $(2r_0, \sqrt{2\pi}\,r_0)$. This gives the required bound for z, and the proof is complete. □

We now provide the final step that shows how the previous lemmas complete the proof of Proposition 5.6, and hence Theorem 4.1. For $\Delta \subset \Lambda \subset \mathbb{R}^2$, let

$$\begin{aligned} D_{\Delta,\Lambda} \quad = \quad & \{\omega : \Delta \overset{(\omega,\mathcal{E}^{\mathrm{D}_3}(\omega))}{\longleftrightarrow} \Lambda^c, \text{ and all vertices of the component} \\ & \text{have the same type}\}. \end{aligned}$$

$$\begin{aligned}
\int_{\mathcal{K}_\Lambda \times \mathscr{T}} & C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dT)\, N_{\Delta\leftrightarrow\Lambda^c}(\zeta\omega_{\Lambda^c}, T) \\
&= \int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\mathscr{T}} \mu^{(q)}_{\zeta\omega_{\Lambda^c}}(dT) N_{\Delta\leftrightarrow\Lambda^c}(\zeta\omega_{\Lambda^c}, T) \\
&\geq \int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\mathscr{T}} \tilde{\mu}_{\zeta\omega_{\Lambda^c}}(dT) N_{\Delta\leftrightarrow\Lambda^c}(\zeta\omega_{\Lambda^c}, T) \\
&\geq \int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\mathscr{T}} \tilde{\mu}_{\zeta\omega_{\Lambda^c}}(dT)\, \mathbb{I}\{\Delta \overset{(\zeta\omega_{\Lambda^c},T)}{\longleftrightarrow} \Lambda^c\} \\
&\geq \int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\Omega} \tilde{\lambda}_{\zeta\omega_{\Lambda^c}}(d\omega')\, \mathbb{I}\{\omega' \in D_{\Delta,\Lambda}\} \qquad (5.88) \\
&= \widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \leftrightarrow \Lambda^c\}) \geq \alpha_2. \qquad (5.89)
\end{aligned}$$

The first inequality above is due to Lemma 5.7. As site percolation implies bond percolation (see Section 1.6 of [Grim99]), we have (5.88). Lemma 5.11 obviously gives (5.89), and the proof of Proposition 5.6 is complete.

5.3 Lily-pond model (Theorem 4.2)

We now prove Theorem 4.2. There is no phase transition and the proof is brief. We first use Corollary 2.2 to show that at least one Gibbs measure exists. We then apply a percolation result of Häggström and Meester [HM96] to determine the uniqueness of the Gibbs measure.

For given $(\eta, \omega) \in \mathcal{E}^{\mathrm{LP}}_{r_{\max}}$, the closed ball with centre $|x_\eta + y_\eta|/2$ and radius $3\,|x_\eta - y_\eta|/2$ can serve as a horizon of (η, ω). This ensures that if the particles of ω are altered outside of the horizon then the potentials ψ, ϕ are not affected (more specifically it ensures that η *exists*, so there is definitely an edge between x_η and y_η). Using this horizon assumption, **(R)** is obviously saitisfied. Assumption **(S)** also clearly holds with $c_{\mathrm{S}} = 0$. Concerning $\mathbf{(U}^A\mathbf{)}$, let $\mathsf{M} = a\mathsf{I}$ and $\Gamma = \Gamma^A$ with $A = B(0, b)$ where $b \le \rho_0 a$ for some sufficiently small constant $\rho_0 > 0$. The neighbourhood of a point x in $\omega \in \bar{\Gamma}$ contains a minimal number of points. Then $(\mathrm{U1}^A)$ holds with $r_\Gamma = 2r_{\max}$ and $(\mathrm{U2}^A)$ holds with $c_A^+ = 2$. $(\mathrm{U3}^A)$ holds when $z|A| > e^{2\beta}$ which implies $z > e^{2\beta}/(\pi\rho_0^2 a^2)$. So for any $z > 0$, $\mathbf{(U}^A\mathbf{)}$ will hold as long as we choose M and Γ^A such that $a > \left(e^{2\beta}/(\pi\rho_0^2 z)\right)^{1/2}$. So we can apply Corollary 2.2 to show that there exists at least one Gibbs measure for $z, \beta > 0$.

Theorem 5.2 of [HM96] tells us that the unmarked dynamic lily-pond model does not percolate. If there is no percolation in the unmarked case it is clearly impossible for the model to percolate in the marked case. Hence multiple Gibbs measures do not exist no matter how strong the interaction is due to $\beta > 0$.

5.4 Equilateral Delaunay triangle interaction (Theorem 4.3)

We proceed with the proof of Theorem 4.3 via the same argument as for Theorem 4.1. The set-up and notation remain the same, with the observation that the new definition (4.10) of $H(\omega)$ replaces (4.3). This accordingly alters the Hamiltonian with configurational boundary condition, and so the proof of Theorem 4.3 follows that for Theorem 4.1 with

$$\sum_{\eta \in \mathcal{E}_\Lambda^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c})} \psi(\eta, \omega),$$

now replaced with

$$\sum_{\eta \in \mathcal{E}_\Lambda^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c})} \psi(\eta, \omega) + \sum_{\tau \in \mathcal{E}_\Lambda^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})} \psi_{\mathrm{tri}}(\tau, \omega).$$

The proof of Theorem 4.3 is complete if we can prove Propositions 5.12 and 5.13.

Proposition 5.12. *There exists at least one Gibbs measure for the Delaunay Potts model given by the Hamiltonian in (4.10), for any $z, \beta > 0$.*

Proof. We apply Corollary 2.2 and Remark 2.5, exactly as in Proposition 5.5. The only difference is that we now have with $c_S = 3\sqrt{3}/16$. $\square$

Proposition 5.13. *There exists $\alpha > 0$ such that*

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC^{z,q}_{\Lambda,\omega} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta_{k,l} \subset \Lambda$, and all

$$z > \frac{81 e^{\frac{33\sqrt{3}}{8}} q^{\frac{4r_1^2}{r_0^2}} + 1}{p_0(\sqrt{2\pi} - 2) r_0^2}$$

and

$$\beta > \log \left\{ \frac{1 + (q^2 - 1)(1 - p_0)^{\frac{1}{162}}}{1 - (1 - p_0)^{\frac{1}{162}}} \right\}.$$

The proof of Proposition 5.13 follows the exact same argument as that for Proposition 5.6. We have stochastic domination, $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$, because the type interaction has not changed and the definitions of $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$ and $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ are not dependent on the background interaction. The Gibbs consistency relation is satisfied for $(M_{\Lambda,\omega;\Delta,\zeta})_{\Delta \subset \Lambda}$ for any $\Lambda \in \mathcal{B}_R$. The new background Hamiltonian does not violate the additivity property of the Hamiltonian between two regions, see (5.61). Therefore the new Hamiltonian does not affect the proof of the Gibbs consistency relation.

We also have the required density quotient bound, Lemma 5.14 below. The only difference between the proof of Theorem 4.3 and the proof of Theorem 4.1 is Lemma 5.15, obtaining the lower bound for the probability under $M_{\Lambda,\omega;\Delta,\zeta}$ that a micro-box is non-empty. Although the lower bound is the same, we require slightly more justification due to the fact that the background Hamiltonian can now be negative, which means the exponent of the negative Hamiltonian can be greater than 1. We can then complete our percolation argument exactly as in Section 5.2.

Lemma 5.14. *For any $\Lambda \in \mathcal{B}_R$, let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$ be such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\zeta\omega_{\Lambda^c} \cup (\{x\}, \{u_x\}) \in \Omega^{\Lambda,z}_*$ for any $u_x \in S$. Then there exists some $\alpha_1 > 0$ such that*

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq \alpha_1.$$

The proof of Lemma 5.14 is exactly the same as the proof of the analogous Lemma 5.9. Since the background interaction still contains the hard-core assumption, and the type interaction is still of finite-range, we again find $\alpha_1 := q^{-4r_1^2/r_0^2}$. The only difference is the addition of the geometry-dependence, and this does not affect the proof.

Lemma 5.15. *For all cells $\nabla = \Delta_{k,l}^{i,j} \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\nabla$ be such that $\kappa\zeta_{\nabla^c}\omega_{\Lambda^c} \in \Omega_*^{\nabla,z}$. We have*

$$M_{\Lambda,\omega;\nabla,\zeta}(|\xi_\kappa| \geq 1) > 1 - \frac{p_0}{81},$$

for all

$$z > \frac{81q\, e^{\frac{33\sqrt{3}}{8}}}{p_0\alpha_1(L-2r_0)^2}.$$

Proof. For a marked configuration $\kappa \in \Omega_\nabla$, we denote the positional configuration as $\xi := \xi_\kappa \in \mathcal{K}_\nabla$. Recall that since $H^\psi(\omega)$ does not depend on the marks, we sometimes write $H^\psi(\xi_\omega)$.

$$\begin{aligned}
\frac{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=1)}{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=0)} &= \frac{\int_{\mathcal{K}_\nabla} h_{\nabla,\zeta\omega_{\Lambda^c}}(\xi) e^{-\beta H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{|\xi|=1\}\, \Pi_\nabla^{z/q}(d\xi)}{\int_{\mathcal{K}_\nabla} h_{\nabla,\zeta\omega_{\Lambda^c}}(\xi) e^{-\beta H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{|\xi|=0\}\, \Pi_\nabla^{z/q}(d\xi)} \\
&\geq \frac{e^{-z|\nabla|/q} z \int_\nabla h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) e^{-\beta H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}\, dx}{q\, e^{\frac{33\sqrt{3}}{8}}\, h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) \int_{\mathcal{K}_\nabla} \mathbb{I}\{|\xi|=0\}\, \Pi_\nabla^{z/q}(d\xi)} \qquad (5.90) \\
&= \frac{e^{-z|\nabla|/q} z \int_\nabla h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) e^{-\beta H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}\, dx}{q\, e^{\frac{33\sqrt{3}}{8}}\, h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) e^{-z|\nabla|/q}} \\
&= \frac{z}{q}\, e^{-\frac{33\sqrt{3}}{8}} \int_\nabla \exp\Big(-\beta H^\psi_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big) \frac{h_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})}{h_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset)}\, dx, \qquad (5.91)
\end{aligned}$$

where, for $\omega \in \Omega, \zeta \in \Omega_\nabla$,

$$\begin{aligned}
H^\psi_{\nabla,\omega}(\zeta) &:= \sum_{\tau \in \mathcal{E}_\nabla^{\mathrm{D}_3}(\zeta\omega_{\nabla^c})} \psi_{\mathrm{tri}}(\eta,\omega) + \sum_{\eta \in \mathcal{E}_\nabla^{\mathrm{D}_2}(\zeta\omega_{\nabla^c})} \psi(|x_\eta - y_\eta|), \\
&= H^{\psi_{\mathrm{tri}}}_{\nabla,\omega}(\zeta) + H^{\psi_{\mathrm{hc}}}_{\nabla,\omega}(\zeta),
\end{aligned}$$

and

$$H^{\psi_{\mathrm{hc}}}_{\nabla,\omega}(\zeta) := \sum_{\eta \in \mathcal{E}_\nabla^{\mathrm{D}_2}(\zeta\omega_{\nabla^c})} \psi(|x_\eta - y_\eta|).$$

We now provide justification for the inequality (5.90). In Lemma 5.10, we exploited the positivity of the background Hamiltonian at this stage. However, now we use the fact that

$$H^{\psi_{\rm tri}}_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) \geq -\frac{33\sqrt{3}}{8}. \tag{5.92}$$

We now prove (5.92). We have

$$H^{\psi_{\rm tri}}_{\nabla,\zeta\omega_{\Lambda^c}}(\emptyset) \quad = \quad - \sum_{\tau \in \mathcal{E}^{\rm D_3}_{\nabla}(\zeta_{\nabla^c}\omega_{\Lambda^c})} \frac{A(\tau)}{\delta(\tau)^2}.$$

The set $\mathcal{E}^{\rm D_3}_{\nabla}(\zeta_{\nabla^c}\omega_{\Lambda^c})$ consists of all hyperedge triangles of $\mathcal{E}^{\rm D_3}(\zeta_{\nabla^c}\omega_{\Lambda^c})$ that can be affected by the configuration within ∇. When the number of triangles in this set is maximised, $\mathcal{E}^{\rm D_3}_{\nabla}(\zeta_{\nabla^c}\omega_{\Lambda^c})$ consists of both

(i) the triangles which cross the empty region ∇, i.e. triangles with edges touching ∇, but no points inside ∇; and

(ii) the triangles that share an edge with the triangles described by (i).

We now justify that the set of triangles described by (i) is finite. A triangle could either cut one corner of ∇ or one whole side of ∇. It is clear that for the maximum number of triangles intersecting the empty region ∇, each of the four corners are cut by a triangle. The Delaunay triangulation is defined such that any circle circumscribing a Delaunay triangle must be empty. Since ∇ is empty, there can be no more than 6 triangles (additional to the 4 corner triangles) intersecting ∇ and sharing vertices with the corner triangles. These 6 triangles, plus the 4 corner triangles, form a connected covering of ∇. So the set of triangles described by (i) is no greater than 10.

Therefore, there are a further 12 triangles of (ii), 2 for each of the 4 corner triangles (each of which has two edges not intersecting ∇), and 4 more for the other 4 triangles in (i) that each have one edge not intersecting ∇. This means that $|\mathcal{E}^{\rm D_3}_{\nabla}(\zeta_{\nabla^c}\omega_{\Lambda^c})| \leq 10 + 12 = 22$, and

$$\sum_{\tau \in \mathcal{E}^{\rm D_3}_{\nabla}(\zeta_{\nabla^c}\omega_{\Lambda^c})} \frac{A(\tau)}{\delta(\tau)^2} \leq 22 \cdot \frac{3\sqrt{3}}{16} = \frac{33\sqrt{3}}{8}, \tag{5.93}$$

which gives (5.92).

From (4.12), we have

$$
\begin{aligned}
& H^{\psi}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) \quad < \quad H^{\psi_{\rm hc}}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\}) \\
\Longrightarrow \quad & \exp\Big\{-\beta H^{\psi}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big\} \quad > \quad \exp\Big\{-\beta H^{\psi_{\rm hc}}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=1)}{M_{\Lambda,\omega;\nabla,\zeta}(|\xi|=0)} \quad & \geq \quad \frac{z}{q} e^{-\frac{33\sqrt{3}}{8}} \int_{\nabla} \alpha_1 \exp\Big(-\beta H^{\psi_{\rm hc}}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big)\, dx \\
& \geq \quad \frac{z}{q} e^{-\frac{33\sqrt{3}}{8}} \alpha_1 \int_{\nabla\ominus r_0} \exp\Big(-\beta H^{\psi_{\rm hc}}_{\nabla,\zeta\omega_{\Lambda^c}}(\{x\})\Big)\, dx \\
& \geq \quad \alpha_1 \frac{z}{q} e^{-\frac{33\sqrt{3}}{8}} |\nabla\ominus r_0|, \qquad (5.94)
\end{aligned}
$$

and we complete the proof as in Lemma 5.10 to find

$$
M_{\Lambda,\omega;\nabla,\zeta}(|\xi|\geq 1) \quad > \quad 1-\frac{p_0}{81}, \qquad (5.95)
$$

for all

$$
z \quad > \quad \frac{81 q\, e^{\frac{33\sqrt{3}}{8}}}{\alpha_1 p_0 (L-2r_0)^2},
$$

as required. $\square$

The next result follows from Lemmas 5.14 and 5.15 via precisely the same argument as in the proof of Lemma 5.11. This completes the proof of Theorem 4.3.

Lemma 5.16. *For any $\Delta=\Delta_{k,l}\subset\Lambda$, there exists $\alpha_2>0$ such that*

$$
\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta\longleftrightarrow\Lambda^c\})\geq\alpha_2,
$$

for all

$$
z > \frac{81 e^{\frac{33\sqrt{3}}{8}} q^{\frac{4r_1^2}{r_0^2}+1}}{p_0(\sqrt{2\pi}-2)^2 r_0^2}
$$

and

$$
\beta > \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{162}}}{1-(1-p_0)^{\frac{1}{162}}}\right\}.
$$

5.5 Infinite-range pairwise background interaction (Theorem 4.4)

We now consider type interaction models where the background interaction between all particles is of infinite range. This creates more balanced geometry of the particle positions, but the proof for phase transition is slightly more complicated. This is due to the fact that to analyse a configuration in any bounded region of space, we must consider the configuration in the boundary. If the particles only interact over a finite range, then there is only a certain range past the boundary that must be considered. For the infinite-range case we must apply a new technique.

The idea is to compute the lower bound of the conditional probability under $M_{\Lambda,\omega;\nabla,\zeta}$ that a micro-box ∇ is non-empty, given that the positions of $\zeta\omega_{\Lambda^c}$ satisfy certain criteria within a finite range of the boundary $\partial\nabla$. We provide an estimate on the overall weight these criteria have, see Lemma 5.19, and then apply the Gibbs consistency relation in order to achieve our percolation result as in Section 5.2. We therefore start by proving that there is at least one Gibbs measure, and then prove the existence of multiple Gibbs measures via the random-cluster representation and Proposition 5.18.

Proposition 5.17. *There exists at least one Gibbs measure for the Delaunay Potts model given by the Hamiltonian in (4.13), for every $z, \beta > 0$.*

Proof. Once again, we apply Corollary 2.2 and Remark 2.5, as in Proposition 5.5. The only difference this time is that $c_A^+ = K + 1$. $\square$

Define the square cell

$$\Delta(i) := [-3L/2, 3L/2)^2 + 3Li, \quad i \in \mathbb{Z}^2, \tag{5.96}$$

for some $L > 0$ such that

$$R_0 < L < \sqrt{2}r_1. \tag{5.97}$$

It is possible to choose such an L due to assumption (4.16). This also ensures $L > 2r_0$. Note that $\Delta(i)\cap\Delta(k) = \emptyset$ for $i, k \in \mathbb{Z}^2$, $i \neq k$. Let $I \subset \mathbb{Z}^2$ be an appropriate index set and let I be finite. Let $\Lambda \in \mathcal{B}_R$ with specific partition $\Lambda = \cup_{i\in I}\Delta(i)$. Let each $\Delta(i)$ be divided into 9 boxes,

$$\nabla_i(j) := [-L/2, L/2)^2 + 3Li + Lj, \quad i \in I,\, j \in \{-1, 0, 1\}^2. \tag{5.98}$$

Note that $\nabla_i(j) \cap \nabla_i(k) = \emptyset$ for $j, k \in \{-1, 0, 1\}^2$, $j \neq k$ and $i \in I$.

Proposition 5.18. *If for a given $p_1 \in (1-p_0, 1)$, and for*

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for $K > K_0$, there exists $\alpha > 0$ such that

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC^{z,q}_{\Lambda,\omega} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. Proposition 5.18 follows from Lemmas 5.19 - 5.22 below. □

For any admissible configuration $\omega \in \Omega$ and some box $\Delta = \Delta(i) \subset \Lambda$, define $E_{\Delta,\omega} \subset \mathcal{K}_\Delta$,

$$E_{\Delta,\omega} := \left\{\xi_\zeta \in \mathcal{K}_\Delta : \zeta\omega_{\Delta^c} \in \Omega^{\Delta,z}_* \,\middle|\, |\xi \cap \nabla_i(j)| > 0,\ \forall j \in \{-1,0,1\}^2 \setminus \{(0,0)\}\right\}.$$

Lemma 5.19. *For any $\Delta \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega^{\Delta,z}_*$. There exists some $p_1 \in (1-p_0, 1)$ and $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for any $K > K_0$,*

$$M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) > p_1.$$

Proof. For a configuration $\kappa \in \Omega_\Delta$, let $\xi := \xi_\kappa \in \mathcal{K}_\Delta$. Since $H^\psi_{\Lambda,\omega}(\zeta)$ depends only on ξ_ζ, we emphasise this dependence by writing $H^\psi_{\Lambda,\omega}(\xi_\zeta)$.

$$\begin{aligned} M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) &= 1 - M_{\Lambda,\omega;\Delta,\zeta}(E^c_{\Delta,\zeta\omega_{\Lambda^c}}) \\ &= 1 - \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT) \int_{\mathcal{K}_\Delta} \Pi^{z/q}_\Delta(d\xi) \\ &\qquad\qquad \times q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)}\, e^{-\beta H^\psi_{\Delta,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}\} \\ &> 1 - \frac{e^{-\beta K}}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT) \int_{\mathcal{K}_\Delta} \Pi^{z/q}_\Delta(d\xi) \\ &\qquad\qquad \times q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)}\, \mathbb{I}\{\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}\} &(5.99) \\ &> 1 - e^{-\beta K} q^{\frac{|\Delta|}{\pi r_0^2}+1} &(5.100) \\ &> 1 - e^{-\beta K} q^{\frac{18 r_1^2}{\pi r_0^2}+1}. &(5.101) \end{aligned}$$

So $M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) > p_1$ holds for all

$$K > \frac{1}{\beta}\log\left\{\frac{q^{\frac{18r_1^2}{\pi r_0^2}+1}}{1-p_1}\right\} =: K_0(\beta, q, r_0, r_1, p_1).$$

Note that (5.99) holds because if $\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}$, then there is at least one micro-box $\nabla \subset \Delta$ that contains no particles. Therefore there is at least one edge of length greater or equal to L, hence greater than R_0. Therefore, due to the long-range repulsion in the background interaction,

$$H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\xi_\zeta) > K,$$

which gives (5.99). For $\xi = \xi_\kappa \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}$,

$$K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}, T) \leq \frac{|\Delta|}{\pi r_0^2} + 1,$$

where $K(\,\cdot\,,\,\cdot\,)$ is the function determining the number of connected components. This gives (5.100). We have (5.101) because $|\Delta| = 9L^2 < 18r_1^2$, due to (5.97). $\square$

Lemma 5.20. *For any $\Lambda \in \mathcal{B}_R$, let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$ be such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\zeta\omega_{\Lambda^c} \cup (\{x\}, \{u_x\}) \in \Omega_*^{\Lambda,z}$ for any $u_x \in S$. Then there exists some $\alpha_1 > 0$ such that*

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq \alpha_1.$$

Proof. As in the proof of Lemma 5.9, we still have $\phi_0(\delta) = 0$ for $\delta > 2r_1$ and $\psi_0(r) = \infty$ for $r < r_0$. The fact that $\psi(r) = K$ for $r > R_0$ does not matter as we are considering the number of clusters drawn by the hyperedge distribution, which is independent of the background interaction. We rely on the hard-core condition to ensure there is a maximum number of particles within a bounded region, but the finite range assumption on the type interaction means we need not consider the infinite range assumption of the background. Therefore, we again find

$$\begin{aligned} K(\zeta\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) - K(\zeta\omega_{\Lambda^c}, T_1) &\geq -\frac{4\pi r_1^2}{\pi r_0^2}, \\ K(\zeta\omega_{\Lambda^c}, T_1 \cup T_2) - K(\zeta\omega_{\Lambda^c}, T_1) &\leq 0, \end{aligned}$$

and

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq q^{-\frac{4r_1^2}{r_0^2}} =: \alpha_1 > 0, \tag{5.102}$$

as required. $\square$

For a meso-box $\Delta = \Delta(i)$, comprising 9 micro-boxes $\nabla_i(j)$, $j \in \{-1, 0, 1\}^2$, we denote $\hat{\nabla}$ as the central micro-box $\nabla_i(0, 0)$. Note how the following lemma compares with the analogous Lemma 5.10 in Section 5.2. We now condition on the configuration outside the micro-box.

Lemma 5.21. *For all cells $\hat{\nabla} = \nabla_i(0, 0) \subset \Delta(i) \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\Delta,z}$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$. Let $p_1 \in (1 - p_0, 1)$ be given. If*

$$z \;\; > \;\; \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{8\beta J_L K}}{(p_1 + p_0 - 1)(L - 2r_0)^2},$$

then

$$M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| \geq 1) > \frac{1 - p_0}{p_1},$$

where $\hat{\xi} := \xi_{\hat{\kappa}} \in \mathcal{K}_{\hat{\nabla}}$ is the set of positions of some configuration $\hat{\kappa} \in \Omega_{\hat{\nabla}}$ such that $\hat{\kappa}\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\hat{\nabla},z}$.

Proof.

$$\begin{aligned}
&\frac{M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| = 1)}{M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| = 0)} \\
&\qquad = \frac{\int_{\mathcal{K}_{\hat{\nabla}}} h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) e^{-\beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi})}\, \mathbb{I}\{|\hat{\xi}| = 1\}\, \Pi^{z/q}_{\hat{\nabla}}(d\hat{\xi})}{\int_{\mathcal{K}_{\hat{\nabla}}} h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) e^{-\beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi})}\, \mathbb{I}\{|\hat{\xi}| = 0\}\, \Pi^{z/q}_{\hat{\nabla}}(d\hat{\xi})} \\
&\qquad = \frac{e^{-z|\hat{\nabla}|/q} z \int_{\hat{\nabla}} h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\}) e^{-\beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}\, dx}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset) \int_{\mathcal{K}_{\hat{\nabla}}} \mathbb{I}\{|\hat{\xi}| = 0\}\, \Pi^{z/q}_{\hat{\nabla}}(d\hat{\xi})} \\
&\qquad = \frac{e^{-z|\hat{\nabla}|/q} z \int_{\hat{\nabla}} h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\}) e^{-\beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}\, dx}{q\, h_{\hat{\nabla},\zeta\omega_{\Lambda^c}}(\emptyset) e^{-z|\hat{\nabla}|/q}} \\
&\qquad = \frac{z}{q} \int_{\hat{\nabla}} \exp\Big(- \beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})\Big) \frac{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset)}\, dx \\
&\qquad \geq \frac{z}{q} \int_{\hat{\nabla}} \alpha_1 \exp\Big(- \beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})\Big)\, dx \\
&\qquad \geq \frac{z}{q} \alpha_1 \int_{\hat{\nabla} \ominus r_0} \exp\Big(- \beta H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})\Big)\, dx \\
&\qquad \geq q^{-1} \alpha_1 z |\hat{\nabla} \ominus r_0| \exp\{-8\beta J_L K\},
\end{aligned} \tag{5.103}$$

where J_L is the maximum number of points that can fit in an $L \times L$ box. (5.103) holds because if $x \in \hat{\nabla} \ominus r_0$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}}$, then the distance between x and any points of $\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}$ is greater than r_0, and x has at most $8J_L$ neighbours, hence

$$
\begin{aligned}
H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\}) &= \sum_{\substack{(y,u_y)\in\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}: \\ \{(x,u_x),(y,u_y)\}\in\mathcal{E}^{\mathrm{D}_2}_{\hat{\nabla}}((x,u_x)\cup\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c})}} \psi(|x-y|) \\
&\le 8J_L K,
\end{aligned}
$$

for any $u_x \in S$. Using (5.103),

$$
\begin{aligned}
& M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}|=0) \le \frac{q\,e^{8\beta J_L K}}{\alpha_1 z|\hat{\nabla}\ominus r_0|} && (5.104)\\
\Longrightarrow \quad & M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}|\ge 1) > 1-\frac{q\,e^{8\beta J_L K}}{\alpha_1 z|\hat{\nabla}\ominus r_0|} && (5.105)\\
& \qquad > \frac{1-p_0}{p_1}, && (5.106)
\end{aligned}
$$

which holds if $p_1 > 1-p_0$ and

$$
\frac{q\,e^{8\beta J_L K}}{\alpha_1 z|\hat{\nabla}\ominus r_0|} < 1-\frac{1-p_0}{p_1},
$$

which is given if

$$
z > \frac{p_1 q\,e^{8\beta J_L K}}{\alpha_1(p_1+p_0-1)(L-2r_0)^2}.
$$

Note that since $L > 2r_0$, we have $|\hat{\nabla}\ominus r_0| = (L-2r_0)^2 > 0$. □

Lemma 5.22. *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$
\begin{aligned}
z &> \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2}, \\
\beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\},
\end{aligned}
$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exists $\alpha_2 > 0$ such that

$$
\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \ge \alpha_2,
$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. Let $\hat{\nabla} = \nabla_i(0,0)$ for any $i \in I$. Fix $p_1 \in (1-p_0, 1)$. By Lemma 5.21, if

$$z \quad > \quad \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1}\, e^{8\beta J_L K}}{(p_1+p_0-1)(L-2r_0)^2},$$

then for all admissible configurations $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$, $\kappa \in \Omega_\Delta$ such that $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}}$, we have

$$M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| \geq 1) > \frac{1-p_0}{p_1}, \tag{5.107}$$

where $\hat{\xi} = \xi_{\hat{\kappa}} \in \Omega_{\hat{\nabla}}$ is the positional configuration of the marked configuration $\hat{\kappa} \in \Omega_{\hat{\nabla}}$. Let

$$A_\Delta := \big\{\xi \in \mathcal{K}_\Delta : |\xi \cap \nabla_i(j)| \geq 1, \forall j \in \{-1,0,1\}^2\big\}.$$

The aim is to find a lower bound for $M_{\Lambda,\omega;\Delta,\zeta}(A_\Delta)$. Let

$$A_{\hat{\nabla}} := \{\xi \in \mathcal{K}_{\hat{\nabla}} : |\xi| \geq 1\}.$$

So we have the set $A_\Delta \subset \mathcal{K}_\Delta$ of (positional) configurations in Δ such that every micro-box of Δ contains at least one particle; the set $E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$ of configurations in Δ that are admissible with the boundary $\zeta\omega_{\Lambda^c}$, and contain at least one particle in each micro-box apart from the central micro-box $\hat{\nabla}$; the set $A_{\hat{\nabla}} \subset \mathcal{K}_{\hat{\nabla}}$ of configurations in the central micro-box of Δ that are non-empty. If a configuration $\kappa \in \Omega_\Delta$ is admissible with some boundary condition $\zeta\omega_{\Lambda^c} \in \Omega$, then:

$$\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}},\, \xi_\kappa \cap \hat{\nabla} \in A_{\hat{\nabla}} \quad \Rightarrow \quad \xi_\kappa \in A_\Delta. \tag{5.108}$$

Let $\xi = \xi_\kappa \in \mathcal{K}_\Delta$ and $\hat{\xi} = \xi_{\hat{\kappa}} \in \mathcal{K}_{\hat{\nabla}}$. From (5.107) and Lemma 5.19, we have

$$\begin{aligned}
&\int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_{\hat{\nabla}}} \mathbb{I}\{\hat{\xi}\xi_{\hat{\nabla}^c} \in A_\Delta\} M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&\quad = \int_{\mathcal{K}_\Delta}\int_{\mathcal{K}_{\hat{\nabla}}} \mathbb{I}\big\{\hat{\xi}\xi_{\hat{\nabla}^c} \in A_{\hat{\nabla}} \cap E_{\Delta,\zeta\omega_{\Lambda^c}}\big\} M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi) \\
&\quad > \Big(\frac{1-p_0}{p_1}\Big) \cdot p_1 \\
&\quad = 1 - p_0,
\end{aligned} \tag{5.109}$$

for all $\quad z \; > \; \dfrac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1}\, e^{8\beta J_L K}}{(p_1+p_0-1)(L-2r_0)^2} \quad$ and $\quad K > K_0$.

We can apply Lemma 5.8 to see that

$$\int_{\mathcal{K}_\Delta} \mathbb{I}\{\xi \in A_\Delta\} M_{\Lambda,\omega;\Delta,\zeta}(d\xi)$$
$$= \int_{\mathcal{K}_\Delta} \int_{\mathcal{K}_{\hat{\nabla}}} \mathbb{I}\{\hat{\xi}\xi_{\hat{\nabla}^c} \in A_\Delta\} M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(d\hat{\xi}) M_{\Lambda,\omega;\Delta,\zeta}(d\xi). \tag{5.110}$$

Combining (5.109) and (5.110), we therefore have the desired lower bound:

$$M_{\Lambda,\omega;\Delta,\zeta}(A_\Delta) > 1 - p_0. \tag{5.111}$$

Let $A_i := A_{\Delta(i)}$ and

$$B_i = \{\kappa \in \Omega_{\Delta(i)} : \xi_\kappa \in A_i \text{ and } \forall\, x \in \xi_\kappa, u^\kappa_x = 1\}.$$

As in Lemma 5.11, we have

$$\begin{aligned} \beta \;&\geq\; \log\left\{\frac{1 + (q^2-1)(1-p_0)^{\frac{1}{9J_L}}}{1 - (1-p_0)^{\frac{1}{9J_L}}}\right\} \\ \iff \quad \tilde{p} \;&\geq\; (1-p_0)^{\frac{1}{9J_L}}. \end{aligned} \tag{5.112}$$

We have established that we require $L > R_0$ so that Lemma 5.19 holds, and $L > 2r_0$ so that $|\nabla \ominus r_0| > 0$. We now explain why we require the assumption that $L < \sqrt{2}r_1$ in (5.97). If $2L < \sqrt{2}r_1$ then $\delta(\tau) < 2r_1$ for any hyperedge $\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$ for a configuration with at least one point in each micro-box. This relies on the property that the circle circumscribing any Delaunay triangle contains no other points of the configuration in its interior. Therefore the probability of the hyperedge τ being open under the hyperedge drawing mechanism $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ is $\tilde{p} \neq 0$. Let $\omega'' := \kappa\zeta_{\Delta(i)^c}\omega_{\Lambda^c} \in \Omega$. We have

$$\begin{aligned} \widetilde{C}^{z,q}_{\Delta(i),\zeta\omega_{\Lambda^c}}(B_i) \;&=\; \int_{\mathcal{K}_{\Delta(i)}} M_{\Lambda,\omega;\Delta(i),\zeta}(d\xi_\kappa) \int_\Omega \tilde{\lambda}_{\xi\zeta_{\Delta(i)^c}\omega_{\Lambda^c}}(d\omega'')\, \mathbb{I}\{\kappa \in B_i\} \\ &=\; \int_{\mathcal{K}_{\Delta(i)}} M_{\Lambda,\omega;\Delta(i),\zeta}(d\xi_\kappa)\, \tilde{p}^{|\xi_\kappa|}\, \mathbb{I}\{\xi \in A_i\} \\ &\geq\; (1-p_0)\tilde{p}^{9J_L} \\ &\geq\; (1-p_0)^2 \\ &>\; 1 - 2p_0 \\ &>\; p_c^{\mathrm{site}}(\mathbb{Z}^2). \end{aligned} \tag{5.113}$$

If B_i and B_j occur for two neighbouring boxes $\Delta(i)$ and $\Delta(j)$, then we are able to connect any point of $\nabla_i(0,0)$ to any point of $\nabla_j(0,0)$ via hyperedges that are completely contained within $\Delta(i) \cup \Delta(j)$. We can compare each box $\Delta(i)$ to the site $i \in I \subset \mathbb{Z}^2$ (recall I is the index set defined to partition Λ into meso-boxes). From (5.113), we know that the probability of each box $\Delta(i)$ satisfying B_i is greater than the critical probability for site percolation in $\mathbb{Z}^2$. Therefore, we have a positive probability for a path of neighbouring boxes $\Delta(i)$ satisfying B_i, hence

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq \alpha_2, \tag{5.114}$$

for some $\alpha_2 > 0$ and any $\Delta = \Delta(i)$. This holds for $p_1 \in (1 - p_0, 1)$ and any

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{8\beta J_L K}}{(p_1 + p_0 - 1)(L - 2r_0)^2}, \\ \beta &> \log\left\{\frac{1 + (q^2 - 1)(1 - p_0)^{\frac{1}{9J_L}}}{1 - (1 - p_0)^{\frac{1}{9J_L}}}\right\}, \\ K &> K_0, \end{aligned}$$

where $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ is established in Lemma 5.19.

To minimise the above bound on β, we pick L so that J_L is minimised. Recall that $L \in (R_0, \sqrt{2}r_1)$. Since we assume $R_0 < (\sqrt{19\pi})r_0$, we have

$$J_L|_{L=R_0} = \left[\frac{R_0^2}{\pi r_0^2}\right] + 1 < 20. \tag{5.115}$$

Because

$$R_0 > \left(1 + \sqrt{1 + \frac{\pi}{8\beta K}}\right) r_0,$$

the bound for z is minimised when L is minimised in $(R_0, \sqrt{2}r_1)$. This gives the desired bounds. $\square$

Now we complete the proof of Proposition 5.18 by the same argument as at the end of Section 5.2, with the observation that

$$\begin{aligned} &\int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\mathscr{T}} \tilde{\mu}_{\zeta\omega_{\Lambda^c}}(dT)\, \mathbb{I}\{\Delta \overset{(\zeta\omega_{\Lambda^c},T)}{\longleftrightarrow} \Lambda^c\} \\ &\qquad \geq \int_{\mathcal{K}_\Lambda} M_{\Lambda,\omega}(d\xi_\zeta) \int_{\Omega} \tilde{\lambda}_{\zeta\omega_{\Lambda^c}}(d\omega')\, \mathbb{I}\{\zeta \cup \omega_{\Lambda^c} \in D_{\Delta,\Lambda}\}, \end{aligned}$$

as site percolation implies tile percolation (for example, see [Grim94]).

5.6 Infinite-range background interaction with pairwise type interaction (Theorem 4.5)

This model is a version of the model discussed in Section 5.5, but purely for edge interactions. The proof for Theorem 4.5 therefore follows the exact same structure as that for Theorem 4.4, with some minor alterations, which we will now discuss.

Let $\Lambda = \cup_{i \in I} \Delta(i)$ where, $\Delta(i)$ is defined by (5.96) and let $\nabla_i(j)$ be defined by (5.98) where

$$R_0 < L < \frac{\sqrt{2}}{2} r_1. \tag{5.116}$$

We use the notation E for a set of edges, replacing the general hyperedge set H in Section 3.2. Definitions (3.1) and (3.2) are replaced with

$$\begin{aligned} E_{\mathbb{R}^2} &:= \{\xi \subset \mathbb{R}^2 : \xi \text{ is a set of 2 distinct points}\}, & (5.117)\\ E_{\Delta} &:= \{\xi \in E_{\mathbb{R}^2} : \xi \subseteq \Delta\}, & (5.118) \end{aligned}$$

for any measurable $\Delta \subseteq \mathbb{R}^2$. We also define

$$\mathscr{E} := \{E \subset E_{\mathbb{R}^2} : E \text{ is locally finite}\}.$$

So the definitions of $M_{\Lambda,\omega}$, $M_{\Lambda,\omega;\nabla,\zeta}$ and $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$ are as before, replacing T and $\mathscr{T}$ with E and $\mathscr{E}$, respectively. A crucial difference between this proof and that presented in Section 5.5 is the definition of $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$. We now let $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$ denote the distribution of the random edge configuration $\{\eta \in \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c}) : \tilde{\gamma}_\eta = 1\} \in \mathscr{E}$, where $(\tilde{\gamma}_\eta)_{\eta \in \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c})}$ are independent $\{0,1\}$-valued random variables with

$$\mathrm{Prob}(\tilde{\gamma}_\eta = 1) = \tilde{p} := \frac{1 - e^{-\beta}}{1 + (q-1)e^{-\beta}} \tag{5.119}$$

when $|x_\eta - y_\eta| < r_1$, and $\mathrm{Prob}(\tilde{\gamma}_\eta = 1) = 0$ otherwise. Note how this compares to the definition (5.44). The reason for this slight adaptation is so that $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$, which we prove in Lemma 5.25, below. This new definition of $\tilde{p}$ also applies to the definition (5.45) of $\tilde{\lambda}_{\zeta\omega_{\Lambda^c}}$. We now prove Theorem 4.5 by proving Propositions 5.23 and 5.24.

Proposition 5.23. *There exists at least one Gibbs measure for the Delaunay Potts model given by the Hamiltonian in (4.17), for every $z, \beta > 0$.*

Proof. The proof follows precisely the same argument as the proof for Proposition 5.17. $\square$

Proposition 5.24. *If for a given $p_1 \in (1-p_0, 1)$, and for*

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for $K > K_0$, there exists $\alpha > 0$ such that

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC^{z,q}_{\Lambda,\omega} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. Proposition 5.24 follows from Lemmas 5.25 - 5.29. □

Lemma 5.25. *For all $q \geq 2$, $\zeta \in \Omega_\Lambda$ and $\omega \in \Omega$, we have $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$.*

Proof. The proof of Lemma 5.25 follows exactly the same argument as the analogous Lemma 5.7 of Section 5.2. We now mention how the slight adaptation to the definition of $\tilde{p}$ affects the proof. The definition of $\mathcal{E}$ now becomes $\mathcal{E} := \mathcal{E}^{\mathrm{D}_2}_\Lambda(\zeta\omega_{\Lambda^c})$, since we are dealing with Delaunay edges η, rather than triangles τ. Claim (5.47) is replaced with

$$q_1 \geq q_2,\ q_1 \geq 1,\ \frac{p_1}{q_1(1-p_1)} \geq \frac{p_2}{q_2(1-p_2)} \quad \Rightarrow \quad \mu_{p_1,q_1} \succeq \mu_{p_2,q_2}, \tag{5.120}$$

which follows by the same argument, noting that $|\xi_\eta| = 2$. We can then apply (5.120), taking $p_1 = p_\Lambda(\tau)$, $q_1 = q$, $p_2 = \tilde{p}$ and $q_2 = 1$. We have $q \geq 1$ and

$$\frac{p_\Lambda(\eta)}{q(1-p_\Lambda(\eta))} \geq \frac{\tilde{p}}{1-\tilde{p}}, \tag{5.121}$$

for all $\eta \in \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c})$. Using (5.120) and (5.121), we have $\mu^{(q)}_{\zeta\omega_{\Lambda^c}} \succeq \tilde{\mu}_{\zeta\omega_{\Lambda^c}}$. □

We now prove that there is a positive probability of the event $E_{\Delta,\zeta\omega_{\Lambda^c}}$ occurring under $M_{\Lambda,\omega;\Delta,\zeta}$.

Lemma 5.26. *For any $\Delta \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega^{\Delta,z}_*$. There exists some $p_1 \in (1-p_0, 1)$ and $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for any $K > K_0$,*

$$M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) > p_1.$$

Proof. Crucially, the background interaction is still a function on the edges of the Delaunay triangulation, so the proof of this theorem is the same as that for Lemma 5.19. The type interaction now acts on edges, but this does not affect the proof. By the same method as in Lemma 5.19,

$$\begin{aligned} M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) &= 1 - \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dE) \int_{\mathcal{K}_\Delta} \Pi^{z/q}_{\Delta}(d\xi) \\ &\qquad \times q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},E)}\, e^{-\beta H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\xi)}\, \mathbb{I}\{\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}\} \\ &> p_1, \\ \text{for all} \quad K &> \frac{1}{\beta}\log\left\{\frac{q^{\frac{18r_1^2}{4\pi r_0^2}+1}}{1-p_1}\right\} =: K_0(\beta,q,r_0,r_1,p_1). \end{aligned}$$

□

Lemma 5.27. *For any $\Lambda \in \mathcal{B}_R$, let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$ be such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\zeta\omega_{\Lambda^c} \cup (\{x\},\{u_x\}) \in \Omega^{\Lambda,z}_*$ for any $u_x \in S$. Then there exists some $\alpha_1 > 0$ such that*

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq \alpha_1.$$

Proof. The proof of Lemma 5.20 follows the same structure as the analogous lemmas of previous sections. Note that we write $E^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}, E^{+}_{x,\zeta\omega_{\Lambda^c}}, E^{-}_{x,\zeta\omega_{\Lambda^c}}$ and $\mathcal{E}^{\mathrm{D}_2}$ in place of $T^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}, T^{+}_{x,\zeta\omega_{\Lambda^c}}, T^{-}_{x,\zeta\omega_{\Lambda^c}}$ and $\mathcal{E}^{\mathrm{D}_3}$, respectively.

The definition of the edge-drawing probability p_Λ means that the edge distribution on $E^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}$ is independent of the distribution on $E^{+}_{x,\zeta\omega_{\Lambda^c}}$. Unlike the case for hyperedges, we now have $E^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}} \cap E^{+}_{x,\zeta\omega_{\Lambda^c}} = \emptyset$. Together with the fact that $p_\Lambda(\eta_1)$ is independent of $p_\Lambda(\eta_2)$ for any edges $\eta_1, \eta_2 \in \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c})$, we have

$$\mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}(A_{\text{ext}} \cap A_+) = \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(A_{\text{ext}})\, \mu^{+}_{x,\zeta\omega_{\Lambda^c}}(A_+),$$

where A_{ext} and A_+ are events on $\{0,1\}^{E^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}}$ and $\{0,1\}^{E^{+}_{x,\zeta\omega_{\Lambda^c}}}$, respectively. We find

$$\begin{aligned} \frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} &= \frac{\int_{\mathscr{E}} \mu_{\zeta\omega_{\Lambda^c}\cup\{x\}}(dE)\, q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},E)}}{\int_{\mathscr{E}} \mu_{\zeta\omega_{\Lambda^c}}(dE)\, q^{K(\zeta\omega_{\Lambda^c},E)}} \\ &= \frac{\int_{\mathscr{E}} \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(dE_1)\, q^{K(\zeta\omega_{\Lambda^c},E_1)} \int_{\mathscr{E}} \mu^{+}_{x,\zeta\omega_{\Lambda^c}}(dE_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c}\cup\{x\},E_1\cup E_2)}}{q^{K(\zeta\omega_{\Lambda^c},E_1)}}}{\int_{\mathscr{E}} \mu^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}(dE_1)\, q^{K(\zeta\omega_{\Lambda^c},E_1)} \int_{\mathscr{E}} \mu^{-}_{x,\zeta\omega_{\Lambda^c}}(dE_2)\, \frac{q^{K(\zeta\omega_{\Lambda^c},E_1\cup E_2)}}{q^{K(\zeta\omega_{\Lambda^c},E_1)}}}. \end{aligned}$$

Again, $K(\zeta\omega_{\Lambda^c}, E)$ is finite because the boundary condition implies $K(\zeta\omega_{\Lambda^c}, E) - K_\Lambda(\zeta\omega_{\Lambda^c}, E)$ is constant. Because $\phi_0(r) = 0$ for $r > r_1$ and $\psi_0(r) = \infty$ for $r < r_0$, we have

$$\begin{aligned} K(\zeta\omega_{\Lambda^c} \cup \{x\}, E_1 \cup E_2) - K(\zeta\omega_{\Lambda^c}, E_1) &\geq -\frac{\pi r_1^2}{\pi r_0^2}, \\ K(\zeta\omega_{\Lambda^c}, E_1 \cup E_2) - K(\zeta\omega_{\Lambda^c}, E_1) &\leq 0. \end{aligned}$$

Therefore

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq q^{-\frac{r_1^2}{r_0^2}} =: \alpha_1 > 0, \tag{5.122}$$

as required. □

Lemma 5.28. *For all cells $\hat{\nabla} = \nabla_i(0,0) \subset \Delta(i) \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\Delta,z}$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$. Let $p_1 \in (1-p_0, 1)$ be given. If*

$$z \quad > \quad \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1} e^{8\beta J_L K}}{(p_1 + p_0 - 1)(L - 2r_0)^2},$$

then

$$M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| \geq 1) > \frac{1-p_0}{p_1},$$

where $\hat{\xi} := \xi_{\hat{\kappa}} \in \mathcal{K}_{\hat{\nabla}}$ is the set of positions of some configuration $\hat{\kappa} \in \Omega_{\hat{\nabla}}$ such that $\hat{\kappa}\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\hat{\nabla},z}$.

The proof of Lemma 5.28 follows exactly the same structure as that for Lemma 5.21. This works because the proof of Lemma 5.21 does not feature the type interaction or (hyper)edge-drawing mechanism.

Note that the Gibbs consistency relation, Lemma 5.8, is satisfied for the model described by Theorem 4.5. The proof is the same as we have

$$M_{\Lambda,\omega;\Delta,\zeta}(d\xi) = \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT)\, q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)} P^{z/q}_{\Delta,\zeta\omega_{\Lambda^c}}(d\xi).$$

The fact that we are integrating over E instead of T does not affect the additivity of the Hamiltonian energy, and therefore does not affect the proof.

Lemma 5.29. *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$
\begin{aligned}
z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2},\\
\beta &> \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\},
\end{aligned}
$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exists $\alpha_2 > 0$ such that

$$
\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq \alpha_2,
$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. By the same argument as in the start of the proof for Lemma 5.22, we can apply Lemmas 5.26 and 5.28 to deduce the following. If

$$
\begin{aligned}
z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1} e^{8\beta K J_L}}{(p_1+p_0-1)(L-2r_0)^2},\\
\beta &> \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{9J_L}}}{1-(1-p_0)^{\frac{1}{9J_L}}}\right\},
\end{aligned}
$$

then there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$,

$$
M_{\Lambda,\omega;\Delta,\zeta}(A_\Delta) > 1 - p_0. \tag{5.123}
$$

And once again, we have

$$
\widetilde{C}^{z,q}_{\Delta(i),\zeta\omega_{\Lambda^c}}(B_i) \;>\; p_c^{\text{site}}(\mathbb{Z}^2), \tag{5.124}
$$

for

$$
\beta \;\geq\; \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{9J_L}}}{1-(1-p_0)^{\frac{1}{9J_L}}}\right\}.
$$

Therefore

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq \alpha_2, \tag{5.125}$$

for some $\alpha_2 > 0$ and any $\Delta = \Delta(i)$. This holds for $p_1 \in (1-p_0, 1)$ and any

$$\begin{aligned} z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1}\, e^{8\beta J_L K}}{(p_1+p_0-1)(L-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{9J_L}}}{1-(1-p_0)^{\frac{1}{9J_L}}}\right\}, \\ K &> K_0, \end{aligned}$$

where $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ is established in Lemma 5.26. By the same argument as in the proof Lemma 5.22, we find the desired bounds for z and β. $\square$

Similarly to previous cases, we have

$$\int_{\mathcal{K}_\Lambda \times \mathscr{E}} C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dE)\, N_{\Delta\leftrightarrow\Lambda^c}(\zeta\omega_{\Lambda^c}, E) \geq \widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \leftrightarrow \Lambda^c\}) \geq \alpha_2,$$

which completes the proof of Proposition 5.24.

5.7 Infinite-range triangle background interaction (Theorem 4.6)

The proof of Theorem 4.6 follows the same structure and arguments as Sections 5.5 and 5.6. We still keep the pairwise hard-core assumption on the background interaction, but now there is also a background interaction on the Delaunay triangles. Again, the proof is complete if we can prove Propositions 5.30 and 5.31, below.

Proposition 5.30. *There exists at least one Gibbs measure for the Delaunay Potts model given by the Hamiltonian in (4.20), for every $z, \beta > 0$.*

Proof. This proof follows the same argument as our other infinite-range background models. $\square$

Let $\Lambda = \cup_{i\in I}\Delta(i)$ where, $\Delta(i)$ is defined by (5.96) and let $\nabla_i(j)$ be defined by (5.98) where, this time,

$$D_0 < L < \sqrt{2}r_1. \tag{5.126}$$

Proposition 5.31. *If for a given $p_1 \in (1-p_0, 1)$, and for*

$$\begin{aligned} z &> \frac{p_1 \, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(D_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for $K > K_0$, there exists $\alpha > 0$ such that

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC^{z,q}_{\Lambda,\omega} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. Proposition 5.31 follows from Lemmas 5.32 - 5.35 below. □

Lemma 5.32. *For any $\Delta \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega^{\Lambda,z}_*$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega^{\Delta,z}_*$. There exists some $p_1 \in (1-p_0, 1)$ and $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for any $K > K_0$,*

$$M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) > p_1.$$

Proof. The background Hamiltonian in Λ with boundary condition ω is given by

$$H^\psi_{\Lambda,\omega}(\zeta) = \sum_{\eta \in \mathcal{E}^{\mathrm{D}_2}_\Lambda(\zeta\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|) + \sum_{\tau \in \mathcal{E}^{\mathrm{D}_3}_\Lambda(\zeta\omega_{\Lambda^c})} \psi_2(\delta(\tau)),$$

where ψ_0 and ψ_2 are given by (4.6) and (4.22), respectively. For a configuration $\kappa \in \Omega_\Delta$, let $\xi := \xi_\kappa \in \mathcal{K}_\Delta$ and recall that since $H^\psi_{\Lambda,\omega}(\zeta)$ depends only on ξ_ζ, we can write $H^\psi_{\Lambda,\omega}(\xi_\zeta)$. As in Lemma 5.18,

$$\begin{aligned} M_{\Lambda,\omega;\Delta,\zeta}(E_{\Delta,\zeta\omega_{\Lambda^c}}) &= 1 - \frac{1}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT) \int_{\mathcal{K}_\Delta} \Pi^{z/q}_\Delta(d\xi) \\ &\qquad \times q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)} \, e^{-\beta H^\psi_{\Delta,\zeta\omega_{\Lambda^c}}(\xi)} \, \mathbb{I}\{\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}\} \\ &> 1 - \frac{e^{-\beta K}}{\tilde{Z}^{z,q}_{\Delta,\zeta\omega_{\Lambda^c}}} \int_{\mathscr{T}} \mu_{\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT) \int_{\mathcal{K}_\Delta} \Pi^{z/q}_\Delta(d\xi) \\ &\qquad \times q^{K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c},T)} \, \mathbb{I}\{\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}\}, \end{aligned}$$

since $L > D_0$ implies $H^\psi_{\Delta,\zeta\omega_{\Lambda^c}}(\xi) > K$ for $\xi \in E^c_{\Delta,\zeta\omega_{\Lambda^c}}$. The rest of the argument hold precisely as in the proof of Lemma 5.19. □

Lemma 5.33. *For any $\Lambda \in \mathcal{B}_R$, let $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and $x \in \Lambda \setminus \xi_\zeta$ be such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\zeta\omega_{\Lambda^c} \cup (\{x\}, \{u_x\}) \in \Omega_*^{\Lambda,z}$ for any $u_x \in S$. Then there exists some $\alpha_1 > 0$ such that*

$$\frac{h_{\Lambda,\omega}(\xi_\zeta \cup \{x\})}{h_{\Lambda,\omega}(\xi_\zeta)} \geq \alpha_1.$$

Proof. The hard-core assumption on the background interaction and finite-range type interaction mean that the proof is exactly the same as that for Lemma 5.20, with

$$\alpha_1 := q^{-\frac{4r_1^2}{r_0^2}}.$$

□

Lemma 5.34. *For all cells $\hat{\nabla} = \nabla_i(0,0) \subset \Delta(i) \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\Delta,z}$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$. Let $p_1 \in (1-p_0, 1)$ be given. If*

$$z \quad > \quad \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{8\beta J_L K}}{(p_1 + p_0 - 1)(L - 2r_0)^2},$$

then

$$M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| \geq 1) > \frac{1-p_0}{p_1},$$

where $\hat{\xi} := \xi_{\hat{\kappa}} \in \mathcal{K}_{\hat{\nabla}}$ is the set of positions of some configuration $\hat{\kappa} \in \Omega_{\hat{\nabla}}$ such that $\hat{\kappa}\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\hat{\nabla},z}$.

Proof. By the usual argument of applying the density quotient bound, Lemma 5.33, we find

$$\frac{M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| = 1)}{M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| = 0)} \quad \geq \quad zq^{-1} \int_{\hat{\nabla}} \alpha_1 \exp\Big(- \beta H^{\psi}_{\hat{\nabla},\xi\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})\Big)\, dx.$$

If $x \in \hat{\nabla} \ominus r_0$ and $\xi := \xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}}$, then the distance between x and any particles of $\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}$ is greater than r_0, and x has at most $8J_L$ neighbours, hence

$$\sum_{\substack{(y,u_y)\in\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}: \\ \{(x,u_x),(y,u_y)\}\in\mathcal{E}^{\mathrm{D}_2}_{\hat{\nabla}}((x,u_x)\cup\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c})}} \psi_0(|x-y|) \quad = \quad 0,$$

$$\sum_{\substack{\eta\in\mathcal{E}^{\mathrm{D}_2}_{\hat{\nabla}}((x,u_x)\cup\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}): \\ \xi_\eta\ni x}} \psi_2\big(\delta(\eta)\big) \quad \leq \quad 8J_L K,$$

and so

$$H^{\psi}_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\}) \leq 8J_L K.$$

This gives

$$\frac{M_{\Lambda,\omega;\hat{\nabla},\xi\zeta_{\Delta^c}}(|\hat{\xi}|=1)}{M_{\Lambda,\omega;\hat{\nabla},\xi\zeta_{\Delta^c}}(|\hat{\xi}|=0)} \quad \geq \quad \alpha_1 q^{-1} z |\hat{\nabla} \ominus r_0| \exp\{-8\beta J_L K\},$$

and the result follows as in Lemma 5.21. $\square$

Note that the Gibbs consistency relation, Lemma 5.8, still holds. The proof follows the same argument, with the observation that

$$\begin{aligned}
& H^{\psi}_{\Delta,\zeta\omega_{\Lambda^c}}(\hat{\xi}\xi_{\nabla^c}) - H^{\psi}_{\nabla,\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\hat{\xi}) \\
&\quad = \sum_{\eta \in \mathcal{E}^{\mathrm{D}2}_{\Delta}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|) - \sum_{\eta \in \mathcal{E}^{\mathrm{D}2}_{\nabla}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_0(|x_\eta - y_\eta|) \\
&\qquad + \sum_{\tau \in \mathcal{E}^{\mathrm{D}3}_{\Delta}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_2\big(\delta(\tau)\big) - \sum_{\tau \in \mathcal{E}^{\mathrm{D}3}_{\nabla}(\hat{\kappa}\kappa_{\nabla^c}\zeta_{\Delta^c}\omega_{\Lambda^c})} \psi_2\big(\delta(\tau)\big)
\end{aligned}$$

depends only on the configuration outside ∇.

Combining Lemmas 5.34 and 5.8, we can use the same arguments as the previous sections to obtain Lemma 5.35. This completes the proof of Proposition 5.31 and hence Theorem 4.6.

Lemma 5.35. *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$\begin{aligned}
z &> \frac{p_1\, q^{\frac{4r_1^2}{r_0^2}+1} e^{160\beta K}}{(p_1+p_0-1)(D_0-2r_0)^2}, \\
\beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\},
\end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exists $\alpha_2 > 0$ such that

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq \alpha_2,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

5.8 Infinite-range type interaction (Theorem 4.7)

In order to prove Theorem 4.7, we implement the usual percolation argument. This requires Proposition 5.37 and proof that at least one Gibbs measure exists, see Proposition 5.36. We are dealing with a type interaction acting on $\mathcal{E}^{\mathrm{D}_3}$, and so the preliminaries and definition of the random-cluster model are very similar to those in Section 5.5. The only difference is the definition of $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$. We now have

$$\mathrm{Prob}(\tilde{\gamma}_\tau = 1) = \tilde{p} := \frac{1 - e^{-\beta}}{1 + (q^2 - 1)e^{-\beta}}, \tag{5.127}$$

for any $\tau \in \mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$.

Proposition 5.36. *There exists at least one Gibbs measure for the Delaunay Potts model described by the Hamiltonian in (4.24), for every $z, \beta > 0$.*

Proof. We can again apply Corollary 2.2 and Remark 2.5. The Delaunay hypergraph structure means we have finite horizons, as in the previous sections. The type interaction is infinite-range, but it is still bounded, so we still have $c_A^+ = K + 1$ and the result holds. $\square$

Proposition 5.37. *If for a given $p_1 \in (1 - p_0, 1)$, and for*

$$\begin{aligned} z &> \frac{p_1\, q^{161} e^{160\beta K}}{(p_1 + p_0 - 1)(R_0 - 2r_0)^2}, \\ \beta &> \log\left\{\frac{1 + (q^2 - 1)(1 - p_0)^{\frac{1}{180}}}{1 - (1 - p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$ such that for $K > K_0$, there exists $\alpha > 0$ such that

$$\int_{\mathcal{K}_\Lambda \times \mathscr{T}} dC^{z,q}_{\Lambda,\omega} N_{\Delta \leftrightarrow \Lambda^c} \geq \alpha,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. Proposition 5.37 follows from Lemmas 5.38, 5.39 and 5.40 below. $\square$

The type interaction is no longer finite-range, so we cannot use the same argument for the density quotient estimate. Instead, we condition on the event $E_{\Delta,\omega}$.

Lemma 5.38. *For all cells $\hat{\nabla} = \nabla_i(0,0) \subset \Delta(i) \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\Delta,z}$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$. Let $x \in \hat{\nabla}$ such that $\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \cup (\{x\},\{u_x\}) \in \Omega_*^{\hat{\nabla},z}$ for any $u_x \in S$. There exists $\alpha_1 > 0$ such that*

$$\frac{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset)} \geq \alpha_1.$$

Proof. Similarly to the previous density quotient estimate lemmas, we have

$$\begin{aligned}
&\int_{\mathscr{T}} \mu_{\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\}}(dT)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\},T)} \\
&\quad = \int_{\mathscr{T}} \mu^{\text{ext}}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_1)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)} \\
&\qquad\quad \times \int_{\mathscr{T}} \mu^{+}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\},T_1\cup T_2)}}{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)}}.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathscr{T}} \mu_{\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T)} \\
&\quad = \int_{\mathscr{T}} \mu^{\text{ext}}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_1)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)} \\
&\qquad\quad \times \int_{\mathscr{T}} \mu^{-}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1\cup T_2)}}{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset)} = \frac{\int_{\mathscr{T}} \mu_{\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\}}(dT)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\},T)}}{\int_{\mathscr{T}} \mu_{\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T)}} \\
&= \Bigg(\int_{\mathscr{T}} \mu^{\text{ext}}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_1)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)} \\
&\qquad\quad \times \int_{\mathscr{T}} \mu^{+}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}\cup\{x\},T_1\cup T_2)}}{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)}} \Bigg) \\
&\quad \times \Bigg(\int_{\mathscr{T}} \mu^{\text{ext}}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_1)\, q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)} \\
&\qquad\quad \times \int_{\mathscr{T}} \mu^{-}_{x,\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}}(dT_2)\, \frac{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1\cup T_2)}}{q^{K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c},T_1)}} \Bigg)^{-1}.
\end{aligned}$$

We have arrived at the same problem of finding a lower bound and an upper bound for

$$K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}, T_1) - K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2)$$

and

$$K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) - K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}, T_1),$$

respectively. As before, we know that $K(\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}, T)$ is finite. Since we still have the hard-core assumption and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}}$,

$$K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}, T_1) - K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) \geq -8J_L. \tag{5.128}$$

The inequality (5.128) holds because if there is just one particle at position x in the central box $\hat{\nabla}$, and the positions ξ_κ inside Δ satisfy $E_{\Delta,\zeta\omega_{\Lambda^c}}$, then each of the 8 surrounding boxes of $\hat{\nabla}$ must contain at least one particle and no more than J_L particles. Since

$$K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \cup \{x\}, T_1 \cup T_2) - K(\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}, T_1)$$

is no greater than the maximum number of clusters that can be removed by adding a particle at position $x \in \hat{\nabla}$ into the configuration $\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c}$, left hand side of (5.128) is minimised when κ is such that each of the micro-boxes surrounding $\hat{\nabla}$ contain J_L particles. This gives

$$\frac{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset)} \geq q^{-8J_L} =: \alpha_1 > 0. \tag{5.129}$$

$\square$

Lemma 5.39. *For all cells $\hat{\nabla} = \nabla_i(0,0) \subset \Delta(i) \subset \Lambda$, let $\omega \in \Omega$ and $\zeta \in \Omega_\Lambda$ be configurations such that $\zeta\omega_{\Lambda^c} \in \Omega_*^{\Lambda,z}$ and $\kappa \in \Omega_\Delta$ be such that $\kappa\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\Delta,z}$ and $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}} \subset \mathcal{K}_\Delta$. Let $p_1 \in (1-p_0, 1)$ be given. If*

$$z \quad > \quad \frac{p_1\, q^{8J_L+1} e^{8\beta J_L K}}{(p_1 + p_0 - 1)(L - 2r_0)^2},$$

then

$$M_{\Lambda,\omega;\hat{\nabla},\kappa\zeta_{\Delta^c}}(|\hat{\xi}| \geq 1) > \frac{1-p_0}{p_1},$$

where $\hat{\xi} := \xi_{\hat{\kappa}} \in \mathcal{K}_{\hat{\nabla}}$ is the set of positions of some configuration $\hat{\kappa} \in \Omega_{\hat{\nabla}}$ such that $\hat{\kappa}\kappa_{\hat{\nabla}^c}\zeta_{\Delta^c}\omega_{\Lambda^c} \in \Omega_*^{\hat{\nabla},z}$.

Proof. The proof holds exactly as in Lemma 5.21, with the observation that

$$\frac{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\{x\})}{h_{\hat{\nabla},\kappa\zeta_{\Delta^c}\omega_{\Lambda^c}}(\emptyset)} \geq \alpha_1,$$

when the positions of $\kappa \in \Omega_\Delta$ satisfy $\xi_\kappa \in E_{\Delta,\zeta\omega_{\Lambda^c}}$. $\square$

Lemma 5.40. *Let $p_1 \in (1-p_0, 1)$ be given. Then for*

$$\begin{aligned} z &> \frac{p_1\, q^{161} e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2}, \\ \beta &> \log\left\{\frac{1+(q^2-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\}, \end{aligned}$$

there exists a $K_0 = K_0(\beta, q, r_0, r_1, p_1) > 0$, such that for $K > K_0$ and the given values of z and β, there exists $\alpha_2 > 0$ such that

$$\widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \longleftrightarrow \Lambda^c\}) \geq \alpha_2,$$

for any $\Delta = \Delta(i) \subset \Lambda$.

Proof. The result follows by the exact same argument as the analogous Lemma 5.22 in Section 5.5. We apply Lemmas 5.38 and 5.39, as well as the Gibbs consistency relation. We optimise the bounds over L to find the required bounds. $\square$

We complete the proof of Proposition 5.37 via the usual argument to find

$$\int_{\mathcal{K}_\Lambda\times\mathscr{T}} C^{z,q}_{\Lambda,\omega}(d\xi_\zeta, dT)\, N_{\Delta\leftrightarrow\Lambda^c}(\zeta\omega_{\Lambda^c}, T) \geq \widetilde{C}^{z,q}_{\Lambda,\omega}(\{\Delta \leftrightarrow \Lambda^c\}) \geq \alpha_2,$$

which holds since we have stochastic domination of $\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$ over $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$.

6 Numerical Analysis

We now present some simulations of the Delaunay Potts model, with infinite-range background interaction and pairwise type interaction, of Section 4.2. The aim of this section is to accompany the result in Theorem 4.5. It is important to stress that the simulations do not provide any definitive proof of a phase transition, this was done in Section 5.6. By varying the values of the parameters z and β, we observe what effects these parameters have on the model. In particular, we see that the values of these parameters are critical in determining whether or not a phase transition exists. This section provides the reader with some visualisations to give an idea of how the model behaves, but it only serves as an accompaniment to the main result. So instead of analysing simulations of every model of Section 4, we just provide one example; the focus of this entire section is the model described by the Hamiltonian in (4.17). For simplicity, we analyse this model for $q = 2$.

The approach we adopt is that presented by Geyer and Møller [GM94], also utilised by Bertin et. al [BBD04] for their finite-range continuum Potts model. Simulations are obtained by defining an appropriate Metropolis-Hastings algorithm. We do not go into details regarding the origin of the algorithm introduced by Geyer and Møller, we just state the algorithm defined in terms of our Delaunay Potts model. The simulations we obtain are not samples of the equilibrated dynamics for the model but visualisations of the dynamical evolution, which gives an indication of the phase transitional behaviour.

The following algorithm describes a Markov chain that converges to the desired particle distribution as the number of iterations $N \geq 1$ increases. It is defined for some bounded region $\Lambda \in \mathcal{B}_R$ and external boundary region $\Lambda^r \in \mathcal{B}_R$, where $\Lambda^r := (\Lambda \oplus r) \setminus \Lambda$ for some $r > 0$. The algorithm is given by steps 1 and 2, below. Before presenting the algorithm, we provide the reader with a brief description of how the algorithm generates the particle configuration, see (a)-(e).

The algorithm generates a marked configuration $\zeta^{(N)}$ in the box Λ, given some boundary condition ω_{Λ^r} and initial (empty) configuration $\zeta^{(0)}$ inside Λ. Step 1 of the algorithm fixes the boundary condition ω_{Λ^r} and initial configuration $\zeta^{(0)}$. Step 2 is iterated N times and produces the configuration $\zeta^{(N)}$. After the nth iteration, a configuration $\zeta^{(n)}$ inside Λ is produced. This configuration is based on the boundary condition ω_{Λ^r} and the configuration in Λ at the $(n-1)$-th step, $\zeta^{(n-1)}$. If $\zeta^{(n-1)}$ contains no particles, then $\zeta^{(n)}$ either

(a) remains an empty particle configuration, or

(b) becomes a single uniformly distributed particle inside Λ.

If $\zeta^{(n-1)}$ contains at least one particle, then $\zeta^{(n)}$ is formed by either

(c) leaving $\zeta^{(n-1)}$ unchanged, or

(d) deleting a randomly chosen particle of $\zeta^{(n-1)}$, or

(e) creating a new uniformly distributed particle at position $x \in \Lambda \setminus \xi_{\zeta^{(n-1)}}$, and adding this particle to the configuration. Recall that $\xi_{\zeta^{(n-1)}}$ is the point cloud of the marked configuration $\zeta^{(n-1)}$.

For configurations $\omega \in \Omega$, $\zeta \in \Omega_\Lambda$ and a marked particle $(x, u_x) \in (\Lambda \setminus \xi_\zeta) \times S$, let

$$\begin{aligned} V\big(x, \zeta\omega_{\Lambda^c}\big) &:= H\big(\zeta\omega_{\Lambda^c} \cup \big(\{x\}, \{u_x\}\big)\big) - H\big(\zeta\omega_{\Lambda^c}\big) \\ &= \sum_{\eta \in E^+_{x,\zeta\omega_{\Lambda^c}}} \Big(\psi(\eta,\omega) + \phi(\eta,\omega)\Big) \\ &\qquad - \sum_{\eta \in E^-_{x,\zeta\omega_{\Lambda^c}}} \Big(\psi(\eta,\omega) + \phi(\eta,\omega)\Big) \end{aligned}$$

be the energy required to insert the marked particle marked (x, u_x) into the configuration $\zeta\omega_{\Lambda^c}$. In the above expression, recall that $E^+_{x,\zeta\omega_{\Lambda^c}}$ and $E^-_{x,\zeta\omega_{\Lambda^c}}$ are defined in Section 5.6:

$$\begin{aligned} E^+_{x,\zeta\omega_{\Lambda^c}} &:= \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c} \cup (\{x\}, \{u_x\})) \setminus \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c}), \\ E^-_{x,\zeta\omega_{\Lambda^c}} &:= \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c}) \setminus \mathcal{E}^{\mathrm{D}_2}(\zeta\omega_{\Lambda^c} \cup (\{x\}, \{u_x\})). \end{aligned}$$

The algorithm is defined as follows:

1. Fix some admissible configuration in the external boundary region, where all particles are assigned type 1. This is the wired boundary condition. We denote this boundary condition as ω_{Λ^r}. The initial configuration $\zeta^{(0)}$ inside Λ starts off empty, $\xi_{\zeta^{(0)}} := \emptyset$.

2. Construct the configuration $\zeta^{(N)}$ by repeating steps 2.1 - 2.3, N times. For $n = 1, \ldots, N$,

   2.1. Randomly choose a number $p \in [0, 1]$ according to the uniform distribution $U[0, 1]$.

2.2. *Particle deletion.* If $p \le 1/2$,

(i) choose x uniformly in $\xi_{\zeta^{(n-1)}}$;

(ii) choose $\hat{p} \sim U[0,1]$, and if

$$\hat{p} \le \min\left\{1, \frac{|\xi_{\zeta^{(n-1)}}|}{z|\Lambda|} \exp\left\{V\big(x, \zeta^{(n-1)}\omega_{\Lambda^r} \setminus (\{x\},\{u_x\})\big)\right\}\right\},$$

then let $\zeta^{(n)} := \zeta^{(n-1)} \setminus \big(\{x\},\{u_x\}\big)$, otherwise $\zeta^{(n)} := \zeta^{(n-1)}$.

2.3. *Particle insertion.* If $p > 1/2$,

(i) choose x uniformly in $\Lambda \setminus \xi_{\zeta^{(n-1)}}$;

(ii) set $u_x = 1$ with probability $1/2$, otherwise set $u_x = 2$;

(iii) choose $\hat{p} \sim U[0,1]$, and if

$$\hat{p} \le \min\left\{1, \frac{z|\Lambda|}{|\xi_{\zeta^{(n-1)}}|+1} \exp\left\{-V\big(x, \zeta^{(n-1)}\omega_{\Lambda^r}\big)\right\}\right\},$$

then $\zeta^{(n)} := \zeta^{(n-1)} \cup \big(\{x\},\{u_x\}\big)$, otherwise $\zeta^{(n)} := \zeta^{(n-1)}$.

The above algorithm is applied to produce the simulations shown in Figures 6.1 - 6.9. Three separate simulations are observed for different parameters, and the time evolution of each system is illustrated. In each case we take $r_1 = 30$, $r_0 = 5$, $R_0 = 20$, $K = 0.25$, and observe the affect from varying the activity z and inverse temperature β. We take $\Lambda = [100, 600]^2$ and $r = 100$. So the boundary is fixed in $[0, 700]^2 \setminus [100, 600]^2$. Type 1 particles are plotted as empty discs and type 2 particles are solid discs. We take $N = 50,000$, the plots show the configurations $\zeta^{(N)}$ inside the box Λ and the state of the system $\zeta^{(n)}$ at $n = 15,000$ and $n = 30,000$. It is important to note that none of the figures are direct samples of the Gibbs measure. As the number of iterations increases, the samples converge to a stationary sample from the specified Gibbs measure for the given parameters. We show up to 50,000 iterations.

Figures 6.1 - 6.3 show the time evolution in the system for $z = 0.04$ and $\beta = 1$. As the number of iterations increases, we see an illustration of the dynamical evolution leading to phase separation, clusters of type 2 particles amongst a sea of type 1 particles. Due to the wired boundary condition of type 1 particles, we see that type 1 particles dominate and over time the type 2 clusters slowly shrink. This is what we expect to see when a phase transition occurs, as discussed in Section 3. However, these figures do not show a typical sample from this Gibbs measure. The more iterations we take, the more the configuration will resemble a typical sample.

One can see in Figures 6.4 - 6.6 that if we lower the inverse temperature β, the model behaves differently; neither one of the particle types dominates and there appears to be an even distribution of types 1 and 2 throughout Λ. Both types are present and mixed at all observation times. This implies that there is only one Gibbs measure for the system, and a phase transition does not occur when the inverse temperature is lowered.

Now consider reducing the activity z, see Figures 6.7 - 6.9. The low density of particles means there is no clear percolating path, and both particle types are present and mixed for all observation times with neither type dominating. This indicates that a phase transition is not present.

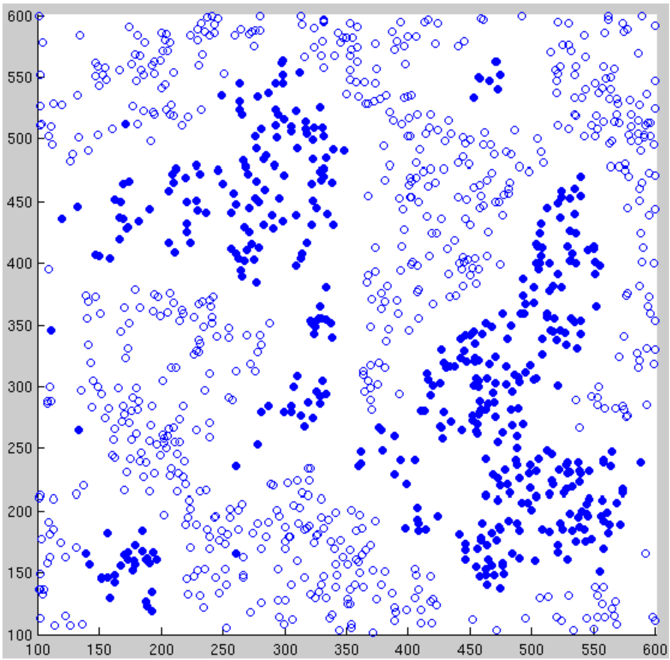

Figure 6.1: $z = 0.04$, $\beta = 1$; 15,000 iterations.

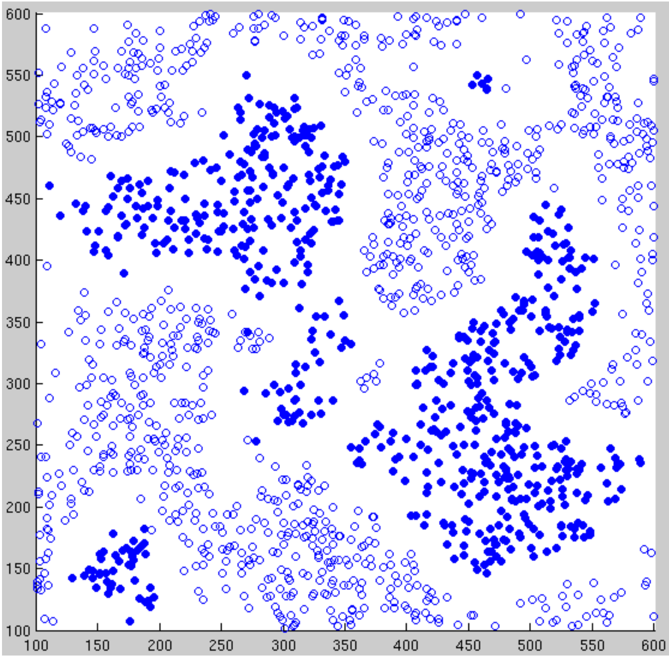

Figure 6.2: $z = 0.04$, $\beta = 1$; 30,000 iterations.

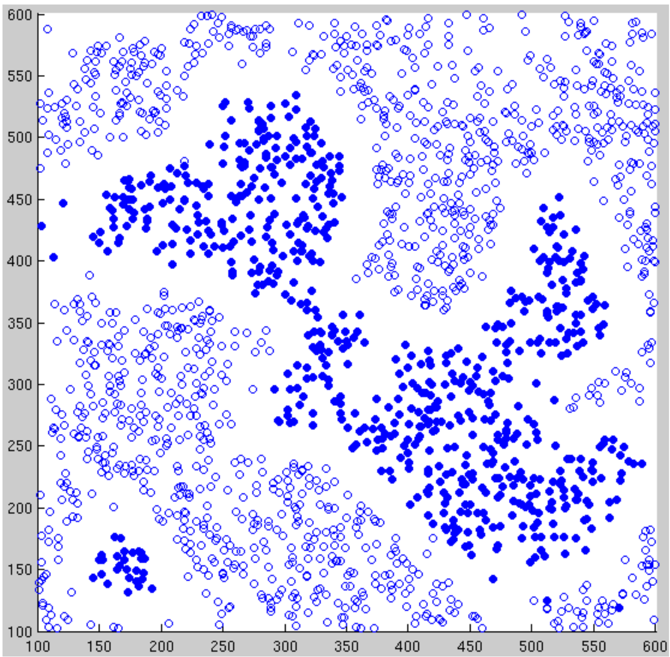

Figure 6.3: $z = 0.04$, $\beta = 1$; 50,000 iterations.

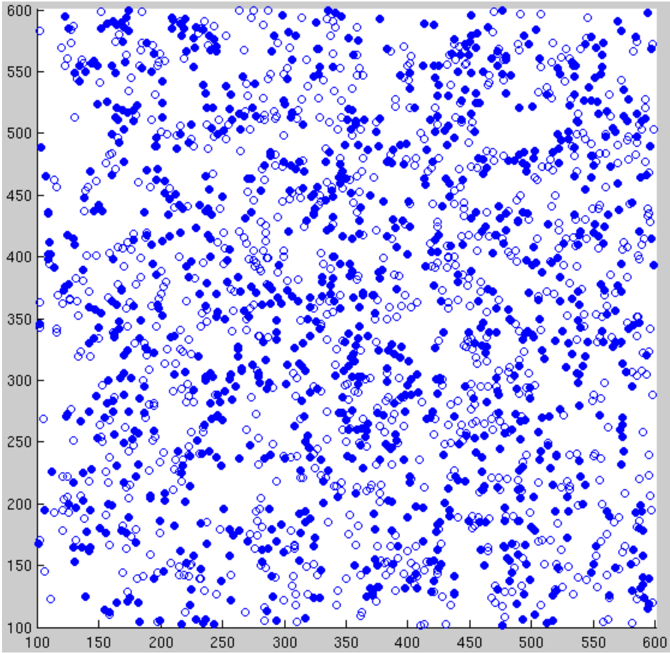

Figure 6.4: $z = 0.04$, $\beta = 0.2$; 15,000 iterations.

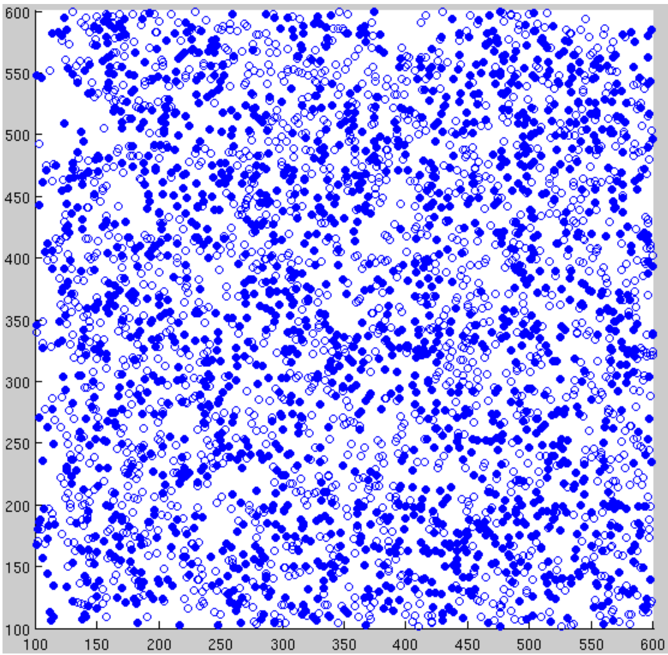

Figure 6.5: $z = 0.04$, $\beta = 0.2$; 30,000 iterations.

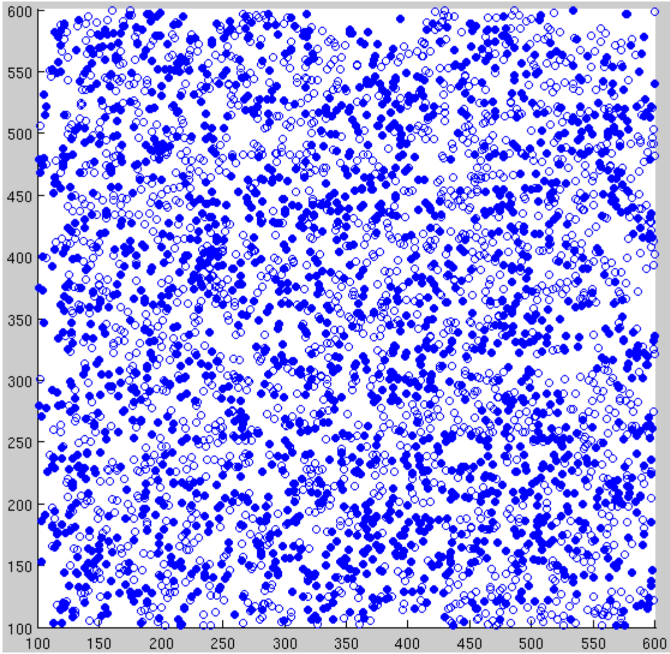

Figure 6.6: $z = 0.04$, $\beta = 0.2$; 50,000 iterations.

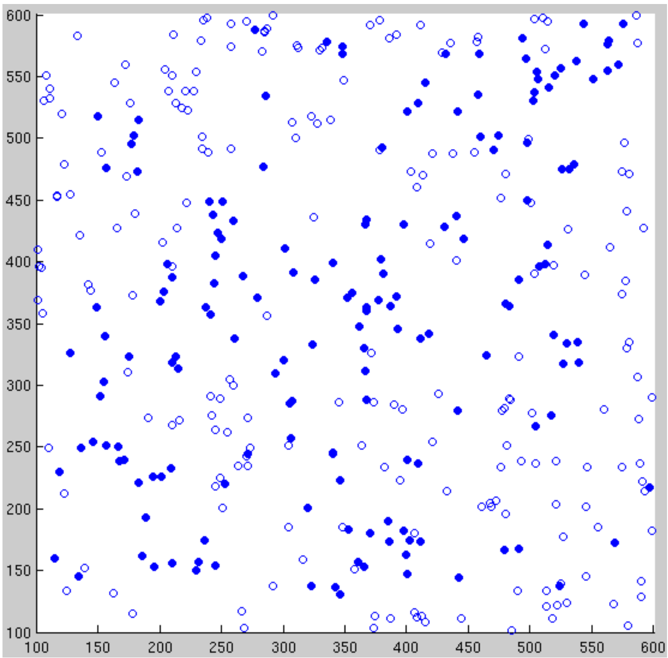

Figure 6.7: $z = 0.007$, $\beta = 1$; 15,000 iterations.

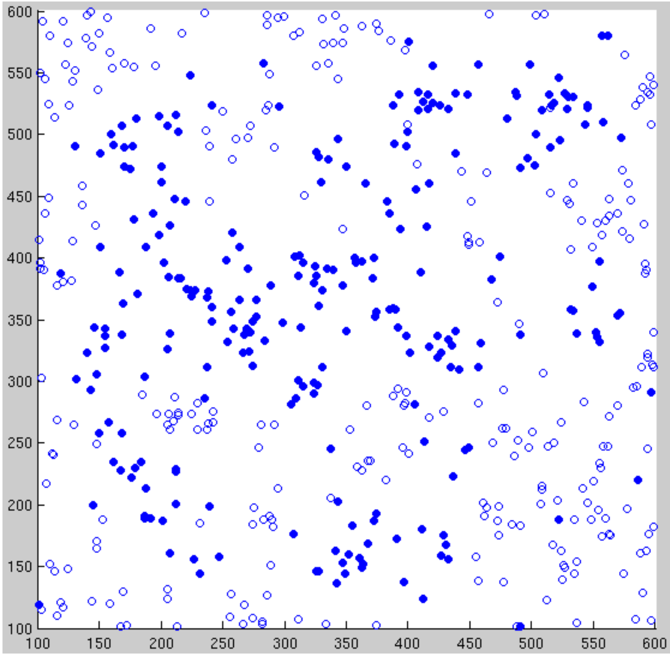

Figure 6.8: $z = 0.007$, $\beta = 1$; 30,000 iterations.

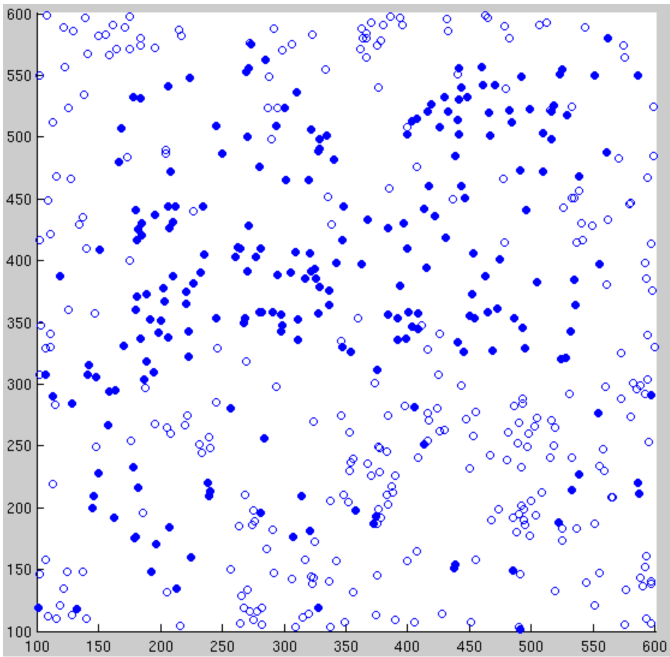

Figure 6.9: $z = 0.007$, $\beta = 1$; 50,000 iterations.

Remark 6.1. One may notice that the system appears to have a lower density of particles for early observation times (i.e. lower number of iterations). This is due to the insertion/deletion nature of the algorithm, and the fact that the initial configuration $\zeta^{(0)}$ inside Λ starts off empty, so it takes a lot of iterations to reach the specified activity z.

Remark 6.2. Note that although we are demonstrating the existence of a phase transition in our model, we are not doing so for z and β satisfying the bounds in Theorem 4.5:

$$\begin{aligned}\beta &> \log\left\{\frac{1+(q-1)(1-p_0)^{\frac{1}{180}}}{1-(1-p_0)^{\frac{1}{180}}}\right\} \simeq 7,\\ z &> \frac{p_1\, q^{\frac{r_1^2}{r_0^2}+1}\, e^{160\beta K}}{(p_1+p_0-1)(R_0-2r_0)^2} \simeq 2^{33}e^{280},\end{aligned}$$

where $q=2$, $r_1=30$, $r_0=5$, $R_0=20$ and $K=0.25$. The constant p_0 is given by (4.1) and we have $p_1 \in (1-p_0, 1)$. This bound for z is extremely high and therefore graphical simulations are unfeasible using the above bounds. However, running simulations for lower values of z and β can still be used to support our main result, which is that a phase transition exists.

Remark 6.3. Recall the assumptions on the model parameters,

$$\left(1+\sqrt{1+\frac{\pi}{8\beta K}}\right) r_0 < R_0 < \left(\sqrt{19\pi}\right) r_0 \quad \text{and} \quad R_0 < \frac{\sqrt{2}}{2} r_1,$$

see (4.16). Note that we choose $r_1=30$, $r_0=5$, $R_0=20$, $K=0.25$ and $\beta \geq 0.2$ for the simulations, values which ensure these assumptions are satisfied.

Remark 6.4. Using numerical simulations it is possible to find an expected phase diagram of the model. This could be done by running many separate simulations for different values of z and β. However, when z and β are close to the critical values, it can take millions of iterations before phase transitional behaviour is observable, for example see [BBD04]. However, it is not feasible to run this many iterations to find a phase diagram.

Remark 6.5. The simulations have been produced using MATLAB. In the pictures, it looks like the particles sometimes overlap. Note that the program used to perform the algorithm plots the particle positions as discs with a diameter larger than 5. Therefore the slight overlapping of the discs does not correspond to violation of the hard-core condition, the centres of each disc are at least 5 units apart.

7 Conclusions

In conclusion, we have shown that phase transitions occur for a class of continuum Potts models with geometry-dependent interactions. These are models where the interaction between particles occurs via hyperedges of the Delaunay hypergraph. Each particle can take a mark from a finite mark space. The choice of mark is dependent on the geometry and marks of the neighbouring particles. The originality in our work is showing that phase transitions exist for continuum systems of marked particles where the particles interact geometrically, as opposed to in pairs. Our tool for achieving this result is the adaptation of the random-cluster representation for multi-body continuum interactions. Using this, we show that multiple Gibbs measures exist for our models.

We find that for each model, a phase transition occurs for sufficiently large activity and sufficiently small temperature. In each case, there must be a pairwise hard-core interaction between any two particles sharing a Delaunay edge. The interaction between marked particles can either be described by an interaction on the Delaunay edges or an interaction on the Delaunay triangles. In the former case, interaction occurs whenever the particles are of opposite type. For Delaunay triangle interactions, there is type interaction whenever the three particles have not got the same type. The former case has been investigated by Bertin et. al. [BBD04], but only for a finite-range background interaction.

We consider a pairwise background interaction between any particles sharing a Delaunay edge or triangle. We show that a phase transition exists when there is hard-core interaction between small edges; for larger edges or hyperedges, the background may either remain zero (see Theorem 4.1) or take some finite value (see Theorems 4.4 - 4.7). The latter case results in configurations with a more even density of particles, since large edges are penalised. It also means that for a phase transition to occur, the activity must be higher. The argument behind this is that a more even density means more consistency in the size of the Delaunay triangles, and therefore there are fewer small triangles with high repulsion between opposite types. We also see that the strength and range of repulsion for large distances affects the activity required for a phase transition, as can be seen from the presence of K and R_0 in the activity bounds in Theorems 4.4 - 4.7. We can see that if the repulsion is high then a higher activity is required to maintain a phase transition, and if the repulsion is only for very long edges, then a lower activity is required.

Our simulations help to demonstrate to the reader why a higher activity leads to a phase transition. If there is a higher density of particles, then there is higher chance of a percolating path. In the main results of Section 4, we find bounds on the activity and inverse temperature that state how high these parameters need to be in order for a phase transition to occur. These bounds are in terms of the other parameters of the models, and give some indication of how the parameters of the model affect the critical activity and critical temperature. Although the reader should note that these bounds do not show the exact critical activity and critical temperature, so the parameters may not actually affect the critical values directly in the way that our bounds suggest. Our findings come from the different methods used in Section 5 to produce the bounds. In every case, we see that allowing the particles to take a wider selection of marks means that the activity must be higher in order to maintain a phase transition. This can be seen through the presence of q in the bounds on z in the main theorems.

We also discuss a geometry-dependent model that favours equilateral Delaunay triangles. This contribution to the energy is negative, decreasing for triangles that resemble an equilateral shape. In Theorem 4.3, we find that this negative interaction means a higher activity is required to maintain a phase transition, compared to when the negative interaction is not present. However, favouring equilateral triangles does not necessarily increase the critical activity. The bounds we find for the activity are not critical thresholds and appear due to the techniques and bounds within the proof, so may not correspond to the true behaviour of the system.

Besides the Delaunay graph, we also consider pairwise interaction according to the lily-pond model. In Theorem 4.2, we find that a phase transition does not occur in this case. For every model, we see that lower temperatures lead to a phase transition. This is a common feature of particle systems in statistical mechanics. The reasoning behind this is that higher temperatures mean the interaction between particles is weakened, so there is less bias towards matching type particles being together. For lower temperatures, the corresponding Gibbs measure has an intensity which is closer to the given intensity measure of the reference process, which is the Poisson point process with uniform mark distribution.

In every model, the type interaction must be strictly positive for small triangles, i.e. triangles circumscribed by a circle with sufficiently small diameter. For larger triangles, the type interaction can disappear or remain positive. We find that if there is a wider selection of marks for the particles to take, then the temperature must be even lower in order to maintain a phase transition. This can be seen by the presence of q in the bounds on β in the main results. We also find that the relationship between the number of marks and the temperature required for a phase transition is different depending on whether the marked particles interact in pairs or triangles. This can be seen, in the β bounds of the main results, from the presence of q^2 for triangle type interactions (see Theorems 4.1, 4.3, 4.4, 4.6 and 4.7) and q for pairwise type interactions (Theorem 4.5). The differences in the bounds suggests that if marked particles interact in triangles, then the number of marks affects the temperature required for phase transition more drastically than if they act in pairs.

Note that an infinite-range background interaction with a hard-core repulsion can be viewed as an approximation to a polynomial background interaction. The key difference between the polynomial background interaction and our approximation is that the polynomial interaction is unbounded for long edges. This means that additional techniques are required to those used in Section 5. The polynomial background interaction is desirable for describing particle systems as it gives an accurate representation of how particles behave in natural systems; for example, the famous Lennard-Jones potential [Jon24] is used to describe the intermolecular forces within a gas. Multi-type geometric interactions could be useful for modelling interacting molecules, where the particles represent molecules that interact via the geometry of their positions. Another possible application of modelling marked particle systems with geometric interaction is non-linear voter-type models, see Liggett [Lig99]. We can allow each particle to represent a voter, where the mark of the particle represents the voter's stance (e.g. yes/no). Allowing particles to interact geometrically means that opinions can be influenced within groups as opposed to pairs. This has been investigated Castellano et. al. [CMP09], who analyse the case where particles interact with random multiple neighbours, the phase transitional behaviour depends on how many neighbours are involved in the interaction. Using the results of the present study, we could analyse the phase transitional behaviour of multi-type voter models where voters interact within triads.

Further research in this area could be to investigate the possibility of removing the hard-core assumption from the background interaction. Our results suggest that a lower hard-core distance means a higher activity would be required for a phase transition to occur, this can be seen from the presence of r_0 in the activity bounds. So we would expect that removing the hard-core assumption entirely would also require a higher activity. Further refinement of the mark space is also an interesting topic for analysis. We are using a finite mark space, and the uniform distribution for the mark intensity. We have been looking at the simple case where the reference measure is a product with the mark space. It is possible to analyse a mark distribution that depends on the positions. The existence result (Theorem 2.1) could be extended for different mark spaces and underlying mark distributions. One could also analyse the effects of these changes to the existence of multiple Gibbs measures.

In the present study, we focus on continuum models with underlying interaction acting on the Delaunay hypergraph in two dimensions. However, our multi-body continuum random-cluster model is defined in terms of any general hypergraph structure. Therefore it is possible to investigate the existence of phase transitions for continuum type interaction models with interaction on hypergraphs other than Delaunay and in higher dimensions. The techniques used in the proofs of the main theorems are specific to planar Delaunay models, and rely on partitioning continuous space in order to compare to percolation on the square lattice. For $d \geq 2$, one may formulate a random-cluster model for any given d-dimensional hyperedge model using the theory of Section 3.2. This requires the interaction potential to depend only on the hyperedges and for the type interaction to only act on hyperedges where the particle marks differ, see (3.3). After formulating the random-cluster model, one could show that a phase transition exists by applying similar techniques to the proofs in Section 5. However, since these proofs are for planar Delaunay models, new techniques would be required for other hypergraph structures and higher dimensions. For example, proof of a phase transition for models in higher dimensions would require results involving higher dimension site percolation on a lattice. The critical site percolation probability for the square lattice with $d = 3$ has been evaluated numerically as $p_c^{\text{site}}(\mathbb{Z}^3) \simeq 0.3116$; for example, see Grassberger [Gra92].

To summarise, we have shown the existence of phase transitions for a class of Delaunay Potts models where the interaction can act on edges or triangles. Allowing the marked particles to interact within triangles is an important and original feature when showing the existence of phase transitions for our class of models. A key feature of the present study is permitting the background interaction to take negative values and to have infinite-range. The latter condition is a step towards analysing the case where particles interact according to a polynomial distribution.

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List of mathematical notation

Below is a comprehensive list of the mathematical symbols used. Each symbol is accompanied by a brief explanation and the page number where the symbol is introduced. Commonly used mathematical notation is not explained. The vast majority of definitions below are for marked particles. Note that in Sections 2.1 and 2.2, some definitions can also apply to unmarked particle configurations. Throughout the thesis, most symbols and abbreviations are uniquely defined. However, sometimes symbols may have different meanings in different sections. In these cases, the multiple definitions are listed in the table below. The choice of definition should be obvious from the context within which it is used.

Symbol	**Explanation**	**p.**
$A(\tau)$	area of the interior of triangle with vertices ξ_τ	42
β	inverse temperature	8
$\mathcal{B}_R$	Borel σ-field on the set R	4
$\mathcal{B}_S$	Borel σ-field on the set S	12
$\mathcal{B}_X$	$\mathcal{B}_R \otimes \mathcal{B}_S$	12
$B(\tau)$	open ball with positions of τ lying on boundary	5
$B(x,r)$	open ball centred at x with radius r	22
$\bar{B}$	closed ball	22
c_s	stability constant	20
c_Γ, c_Γ^+	uniform summability constants	20
c_A, c_A^+	uniform summability constants with $\Gamma = \Gamma^A$	20
$C(k)$	cell at position $k \in \mathbb{Z}^d$ after applying M to form periodic partition of $\mathbb{R}^d$	16
C	$C(0)$	16
$C^{z,q}_{\Lambda,\omega}$	random-cluster distribution	30
$\mathrm{CB}_{k:k+1,l}$	central band of $\Delta_{k,l} \cup \Delta_{k+1,l}$	75
d	dimension of space	1
$d(x,y)$	Euclidean metric	5
D_0	interaction potential parameter	45
$D_{\Delta,\Lambda}$	set of all configurations such that $\Delta \leftrightarrow \Lambda^c$ and particles in the component are of the same type	76
$\partial\Lambda$	boundary of bounded region $\Lambda \subset \mathbb{R}^d$	5
$\partial\Lambda(\omega)$	ω-boundary of Λ	16
$\partial_\Lambda\omega$	intersection of ω with $\partial\Lambda(\omega) \times S$	16
$\partial^\Gamma_{\Lambda_n}\bar{\omega}$	intersection of $\bar{\omega}$ with $(\Lambda_n^{r_\Gamma} \setminus \Lambda_n) \times S$	54

Symbol	Explanation	p.
$\delta(\tau)$	diameter of $B(\tau)$	5
$\delta_{(x,u_x^\omega)}$	Dirac measure	12
Δ	region of $\mathbb{R}^d$	16
$\Delta_{k,l}$	meso-box of Λ	35
$\Delta_{k,l}^{i,j}$	micro-box of Λ	36
$\Delta(i)$	$[-3L/2, 3L/2)^2 + 3Li$	82
$E_{\Delta,\omega}$	event that every micro-box (other than the centre) of Δ contains at least one particle	83
$\mathcal{E}$	hypergraph structure	4
$\mathcal{E}^{\mathrm{D}_2}$	Delaunay hypergraph structure (pairs)	5
$\mathcal{E}^{\mathrm{D}_3}$	Delaunay hypergraph structure (triangles)	5
$\mathcal{E}^B$	background hypergraph structure	14
$\mathcal{E}^T$	type hypergraph structure	14
$\mathcal{E}^{\mathrm{C}_r}$	locally complete hypergraph structure of finite range	21
$\mathcal{E}_{r_{\max}}^{\mathrm{LP}}$	lily-pond hypergraph structure	41
$\mathcal{E}(\omega)$	set of all hyperedges	4
$\mathcal{E}_\Lambda^B(\omega)$	set of all hyperedges such that ω_Λ affects $\psi(\eta, \omega)$	14
$\mathcal{E}_\Lambda^T(\omega)$	set of all hyperedges such that ω_Λ affects $\phi(\eta, \omega)$	14
$\mathcal{E}_{\mathrm{CB}}^{\mathrm{D}_3}(\omega)$	hyperedges of $\mathcal{E}^{\mathrm{D}_3}(\omega)$ intersecting $\mathrm{CB}_{k,k+1,l}$	75
$\mathscr{E}$	set of all Delaunay pair hyperedge configurations	90
$\mathcal{F}$, $\mathcal{F}_\Lambda$	σ-fields	4
g	interaction potential	14
$G_{\Lambda,\omega}^z$	unmarked Gibbs distribution	7
$G_{\Lambda,\omega}^{z,\mu}$	marked Gibbs distribution	15
G_n	projection of Gibbs distribution onto Λ_n	51
$\mathscr{G}_\Theta(\psi, z)$	set of unmarked Gibbs measures that belong to $\mathscr{P}_\Theta$	8
$\mathscr{G}_\Theta(\psi, \phi, z)$	set of marked Gibbs measures that belong to $\mathscr{P}_\Theta$	15
γ_η	state of η under $\mu_{\zeta\omega_{\Lambda^c}}$	28
$\tilde{\gamma}_\tau$	state of τ under $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$	62
Γ	measurable subset of $\Omega_C \setminus \{\emptyset\}$	16
$\bar{\Gamma}$	set of all marked configurations whose restriction to an arbitrary cell $C(k)$ belong to Γ when shifted back to $C(0)$	16
Γ^A	set of configurations $\zeta \in \Omega_C$ such that $\xi_\zeta = \{x\}$ for some $x \in A \subset C$	20
$h_{\Lambda,\omega}$	Radon-Nikodym derivative of $M_{\Lambda,\omega}$ w.r.t. $P_{\Lambda,\omega}^{z/q}$	61

Symbol	Explanation	p.
H	- formal Hamiltonian	14
	- positions of open hyperedges	27
H^{ψ}	formal background Hamiltonian	5
$H^{\psi}_{\Lambda,\omega}$	background Hamiltonian in Λ with configurational boundary condition ω	6
$H^{\phi}_{\Lambda,\omega}$	type Hamiltonian in Λ with configurational boundary condition ω	14
$H_{\Lambda,\omega}$	$H^{\psi}_{\Lambda,\omega} + H^{\phi}_{\Lambda,\omega}$	14
$H_{\mathbb{R}^d}$	set of all possible hyperedges in $\mathbb{R}^d$	27
H_{Δ}	set of all possible hyperedges within Δ	27
$\mathscr{H}$	set of all hyperedge configurations	28
$i(P)$	intensity of P	8
$i^h(P)$	h-intensity of P	13
$\mathbb{I}$	indicator function	7
$I(Q_1\|Q_2)$	relative entropy of Q_1 with respect to Q_2	52
$I^{z,\mu}(P)$	specific entropy of P	52
J_L	maximum number of particles that can fit in an $L \times L$ box under hard-core condition	39
K	interaction potential parameter	43
K_0	function of β, q, r_0, r_1, p_1	84
$K(\xi_\omega, H)$	number of connected components in the hypergraph (ξ_ω, H)	30
$K(\omega, H)$	same as $K(\xi_\omega, H)$	30
$K_\Lambda(\xi_\omega, H)$	number of connected components in the hypergraph (ξ_ω, H) completely contained within Λ	30
$K_\Lambda(\omega, H)$	same as $K_\Lambda(\xi_\omega, H)$	30
$\mathcal{K}$	set of all locally finite sets of $\mathbb{R}^d$	12
$\mathcal{K}_f$	set of all finite sets of $\mathbb{R}^d$	12
κ	marked configuration in Δ	48
L	size of micro-box	39
L_n	$\{-n, \ldots, n\}^d$	51
$\mathrm{Leb}(\,\cdot\,)$	Lebesgue measure	4
$\lambda_{\zeta\omega_{\Lambda^c}}$	type-picking mechanism	28
$\tilde{\lambda}_{\zeta\omega_{\Lambda^c}}$	alternative type-picking mechanism	62
Λ	region of $\mathbb{R}^d$	4
Λ^r	closed r-neighbourhood of Λ	16

Symbol	**Explanation**	**p.**
Λ^n	$\cup_{k\in L_n} C(k)$	50
$\Lambda \oplus r$	$\cup_{y\in \bar{B}(0,r)}\{x+y,\, x\in\Lambda\}$	103
M	invertible $d\times d$ matrix	16
$M_{\Lambda,\omega}$	distribution of particle positions	61
$M_{\Lambda,\omega;\nabla,\zeta}$	conditional distribution of particle positions in ∇ given configuration in ∇^c relative to $M_{\Lambda,\omega}$	61
μ	mark distribution	12
$\mu_{\zeta\omega_{\Lambda^c}}$	hyperedge-drawing mechanism	28
$\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$	alternative hyperedge-drawing mechanism	62
$\mu^{(q)}_{\zeta\omega_{\Lambda^c}}$	distribution of Delaunay triangle hyperedges	61
$\mu^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}$	hyperedge-drawing mechanism on $T^{\mathrm{ext}}_{x,\zeta\omega_{\Lambda^c}}$	68
$\mu^{+}_{x,\zeta\omega_{\Lambda^c}}$	hyperedge-drawing mechanism on $T^{+}_{x,\zeta\omega_{\Lambda^c}}$	68
$\mu^{-}_{x,\zeta\omega_{\Lambda^c}}$	hyperedge-drawing mechanism on $T^{-}_{x,\zeta\omega_{\Lambda^c}}$	68
η	hyperedge (often used to denote Delaunay pair)	4
N_Λ	number of particles within a finite box $\Lambda\subset\mathbb{R}^d$	4
$N_\Delta(\omega)$	number of particles of ω within Δ	33
$N_{\Delta,s}(\omega)$	number of type-s particles of ω within Δ	33
$N_{\Delta\leftrightarrow\Lambda}(\omega,H)$	number of particles of ω within Δ that belong to a component connected to Λ^c in (ξ_ω,H)	33
∇	micro-box of Λ	36
$\hat{\nabla}$	central micro-box in Δ	85
$\nabla\ominus r$	r-minus sampling of ∇	71
$\nabla_i(j)$	$[-L/2,L/2)^2+3Li+Lj$	82
ω	configuration of (marked) particles	4/12
ω_Λ	(marked) configuration within Λ	4/12
ω'	$\zeta\cup\omega_{\Lambda^c}$	28
Ω	(marked) configuration space	4/12
Ω_f	finite (marked) configuration space	4/12
Ω_Λ	(marked) configuration space for configurations within Λ	4/12 4/12
$\Omega^{\Lambda,z}_*$	set of all admissible configurations for bounded region Λ and activity z	15
$\Omega^{\Lambda}_{\mathrm{cr}}$	set of configurations that confine the range of Λ	16
p_0	$(1-p_c^{\mathrm{site}}(\mathbb{Z}^2))/3$	39
p_1	constant in interval $(1-p_0,1)$	83
$p_\Lambda(\eta)$	probability that η is open	28

Symbol	**Explanation**	**p.**
$p_c^{\text{site}}(\mathbb{Z}^2)$	critical probability for Bernoulli site percolation on integer lattice	39
$\tilde{p}$	parameter in definition of $\tilde{\mu}_{\zeta\omega_{\Lambda^c}}$	62
P	probability measure on $(\Omega, \mathcal{F})$	8
P_n	probability measure on $(\Omega, \mathcal{F})$	51
$\hat{P}_n$	spatial averaging of P_n	51
$P^z_{\Lambda,\omega}$	distribution of particle positions	27
$\mathbb{P}$	- Gibbs measure on $(\Omega, \mathcal{F})$	8
	- random-cluster representation measure	30
$\mathbb{P}^z_{\Lambda,\omega}$	probability measure on $\Omega \times \mathscr{H}$	28
$\mathscr{P}_\Theta$	set of all Θ-invariant probability measures on $(\Omega, \mathcal{F})$ with finite intensity	8
Π^z	Poisson point process with intensity z	7
Π^z_Λ	Poisson point process projected onto Λ	7
$\Pi^{z,\mu}$	Poisson point random field on X with intensity measure $z\,\mathrm{Leb}(\,\cdot\,) \otimes \mu$	14
$\Pi^{z,\mu}_\Lambda$	$\Pi^{z,\mu} \circ \mathrm{pr}_\Lambda^{-1}$	14
ϕ	type interaction potential	14
ϕ_0	- part of ϕ that only depends on position	27
	- finite range triangle type interaction	40
ϕ_1	finite range pairwise type interaction	44
ψ	background interaction potential	4
ψ^-	negative part of ψ	7
ψ_0	hard-core pairwise background interaction	40
ψ_1	infinite range hard-core pairwise background interaction	43
ψ_2	infinite range hard-core triangle background interaction	45
ψ_{tri}	equilateral triangle background interaction	42
pr	projection from $\Omega \times \mathscr{H}$ to Ω	30
pr_Λ	projection onto Λ	7
q	number of marks	17
$r(\Lambda, \omega)$	smallest possible r such that we can assume $\partial\Lambda(\omega) = \Lambda^r \setminus \Lambda$	16
r_Γ	$\sup_{\Lambda \subset \mathbb{R}^d} \sup_{\omega \in \bar{\Gamma}} r(\Lambda, \omega)$	20
r_0, r_1	interaction potential parameters	40

Symbol	Explanation	p.
R_0	interaction potential parameter	43
$\mathbb{R}^d$, R	d-dimensional Euclidean space	4
$(R, \mathcal{B}_R)$	measurable space	4
S	mark space	12
sp	projection from $\Omega \times \mathscr{H}$ to $\mathcal{K} \times \mathscr{H}$	30
$T^{\text{ext}}_{x,\zeta\omega_{\Lambda^c}}$	set of hyperedges in $\mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$ not affected by insertion of particle at position x	68
$T^{+}_{x,\zeta\omega_{\Lambda^c}}$	set of hyperedges not in $\mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$ that are created by insertion of particle at position x	68
$T^{-}_{x,\zeta\omega_{\Lambda^c}}$	set of hyperedges in $\mathcal{E}^{\mathrm{D}_3}(\zeta\omega_{\Lambda^c})$ that are removed by insertion of particle at position x	68
$\mathscr{T}$	set of all Delaunay triangle hyperedge configurations	58
τ	Delaunay triangle hyperedge	5
θ_x	translation by vector $-x \in \mathbb{R}^d$	4
Θ	shift group $(\vartheta_x)_{x\in\mathbb{R}^d}$	4
u^ω_x	mark of particle at position $x \in \xi_\omega$	12
X	- phase space for marked particles	12
	- event that the marks of particles are the same on each connected component	29
X_ω	ω-section of X	31
$X_{(\xi_\omega,H)}$	set of configurations such that $(\omega, H) \in X$	32
ξ_ω	set of positions occupied by a marked configuration ω	12
$\hat{\xi}_\eta$	set of $k \in \mathbb{Z}^d$ such that ξ_η intersects $C(k)$	20
z	intensity of Poisson point process	7
$Z^z_{\Lambda,\omega}$	- partition function associated to $H^\psi_{\Lambda,\omega}$ (Section 2)	7
	- normalisation constant for $P^z_{\Lambda,\omega}$ (Sections 3-5)	27
$Z^{z,\mu}_{\Lambda,\omega}$	partition function associated to $H_{\Lambda,\omega}$	15
$\bar{Z}^z_{\Lambda,\omega}$	normalisation constant for $\mathbb{P}^z_{\Lambda,\omega}$	29
$\hat{Z}^{z,q}_{\Lambda,\omega}$	normalisation constant for $C^{z,q}_{\Lambda,\omega}$	30
$\tilde{Z}^{z,q}_{\nabla,\zeta\omega_{\Lambda^c}}$	normalisation constant for $M_{\Lambda,\omega;\nabla,\zeta}$	62
ζ	configuration within Λ	4
$\zeta\omega_{\Lambda^c}$	$\zeta \cup \omega_{\Lambda^c}$	4
$\lvert\Lambda\rvert$	Lebesgue measure of a finite box Λ	4
$\lvert\eta\rvert$	number of particles in the hyperedge η	5
$[L]$	largest integer not greater than L	39
$\succeq$	stochastic domination	63