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ORDER-PRESERVING AND
ORDER-NON-REVERSING
MAPS OF PARTIAL ORDERS

BY

D.E. DAYKIN & J.W. DAYKIN & M.S. PATERSON

Department of Computer Science
University of Warwick
Coventry CV4 7AL
England.

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D.E. Daykin[*] and J.W. Daykin[+] and M.S. Paterson[+]

Abstract. Stanley used the Aleksandrov-Fenchel inequalities from the theory of mixed volumes to prove the following result. Let P be a partially ordered set with n elements, and let $x \in P$. If N_i^* is the number of linear extensions $\lambda : P \to \{1, 2, \ldots, n\}$ satisfying $\lambda(x) = i$, then the sequence $N_1^*, \ldots, N_n^*$ is log concave (and therefore unimodal). Here the analogous results for both order-preserving and order-non-reversing maps are proved using an explicit injection. Further, if ν_c is the number of order-preserving maps of P into a chain of length c, then ν_c is shown to be log concave, and the corresponding result is established for order-non-reversing maps.

1. Introduction

Let P be a poset (= partially ordered set) with n elements and C a chain with c elements. We are interested in certain log concavity properties of order-related mappings from the elements of P into C. Definitions of 3 classes of such maps are as follows.

* Dept of Mathematics
University of Reading
Reading
U.K.

+ Dept of Computer Science
University of Warwick
Coventry
U.K.

A map $\lambda : P \to [n] \equiv \{1,2,\ldots,n\}$ is a linear extension of P if λ is 1-1 and, for all $x,y \in P$, $x < y$ implies $\lambda(x) < \lambda(y)$.

For (P,C), a map $\omega : P \to C$ is order-preserving if, for all $x,y \in P$, $x < y$ implies $\omega(x) < \omega(y)$. Note that ω need not be 1-1. (Some authors require $|P| = |C|$, but we do not need this restriction).

For (P,C), a map $\rho : P \to C$ is order-non-reversing if, for all $x,y \in P$, $x < y$ imples $\rho(x) \leq \rho(y)$.

A sequence $a_o, a_1, \ldots$ of non-negative real numbers is said to be log concave if $a_{i-1}\, a_{i+1} \leq a_i^2$ for $1 \leq i$. In particular, a log concave sequence is unimodal, i.e. for some j we have $a_o \leq a_1 \leq \ldots \leq a_j$ and $a_j \geq a_{j+1} \geq \ldots$

In 1980 Stanley [10] used the Aleksandrov-Fenchel inequalities (which guarantee the logarithmic concavity of coefficients arising from the volume of weighted sums of n-dimensional polytopes) to prove that certain sequences of combinatorial interest are log concave (surveys of mixed volumes appear in [3], [5]). One such result is:

Theorem 1 (Stanley). Let $x_1 < \ldots < x_k$ be a fixed chain in P. If $1 \leq i_1 < \ldots < i_k \leq n$, then define $N^*(i_1,\ldots,i_k)$ to be the number of linear extensions $\lambda : P \to [n]$ such that $\lambda(x_j) = i_j$ for $1 \leq j \leq k$. Suppose $1 \leq j \leq k$ and $i_{j-1} + 1 < i_j < i_{j+1} - 1$, where we set $i_o = 0$ and $i_{k+1} = n+1$. Then

$$N^*(i_1,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k)N^*(i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_k) \leq N^*(i_1,\ldots,i_k)^2.$$

In particular, the case k=1 yields $N^*_{i-1} N^*_{i+1} \leq N^{*2}_i$, confirming a conjecture of Chung, Fishburn and Graham [4], which strengthened an unpublished conjecture of R. Rivest that the sequence $N^*_1,\ldots,N^*_n$ is unimodal. Chung et al. established log concavity for the case that P is a union of two linear orders, i.e. chains.

Graham [8] asked whether the analogue of Stanley's Theorem is true for order-preserving maps, and noted that the FKG inequality [6] can be used very naturally to prove the log concavity of various sequences of a combinatorial nature (e.g.see [9]). He suggests [7] that Stanley's result, and the analogue conjectured result for order-preserving maps, should have proofs based on the FKG inequality or the more general AD inequality [1], but these have as yet eluded discovery.

We present an injective proof for the corresponding result for order-preserving maps. The injection consists of constructing, for each pair of maps with $\omega_1(x) = r$ and $\omega_2(x) = r+s+t$, a unique pair of maps with $\omega_3(x) = r+s$ and $\omega_4(x) = r+t$. That is if two ordered pairs of the form (ω_1,ω_2) are distinct, then their two associated (ω_3,ω_4) pairs are distinct, thus ensuring the inquality. With minor changes the injection yields log concavity for order-non-reversing maps.

Log concave sequences can be proved (see [2]) to satisfy the more general inequality,

$$a_r \, a_{r+s+t} \leqslant a_{r+s} \, a_{r+t} \quad \text{for natural numbers } r,s,t \geqslant 0.$$

Finally, if ν_c is the number of order-preserving maps of P into a chain of length c, then ν_c is shown to be log concave. The analogue is established for order-non-reversing maps.

2. Order-Preserving Maps

<u>Theorem 2</u> Let C be a finite chain, and let $x_1,\ldots,x_k$ be a fixed subset in P. Define $N(i_1,\ldots,i_k)$ to be the number of order-preserving maps $\omega : P \to C$ such that $\omega(x_j) = i_j$ for $1 \leqslant j \leqslant k$. Let r,s,t be natural numbers, then

$$N(r,i_2,\ldots,i_k)N(r+s+t,i_2,\ldots,i_k) \leqslant N(r+s,i_2,\ldots,i_k)N(r+t,i_2,\ldots,i_k).$$

In particular, the case k = 1 and s = t = 1 yields $N_r N_{r+2} \leqslant N_{r+1}^2$.

Proof Suppose that the L.H.S. of the inquality is not equal to 0, and that $s,t > 0$, for otherwise the result clearly holds. Since only the height in C, of x_1 changes, for brevity denote x_1 by x. We will first prove the result for $k = 1$ namely, $N(r)N(r+s+t) \leqslant N(r+s)N(r+t)$, and then show how it easily extends to $k > 1$.

Given any pair of order-preserving maps $\omega_1, \omega_2 : P \to C$ with $\omega_1(x) = r$ and $\omega_2(x) = r+s+t$, we will construct a unique pair of order-preserving maps $\omega_3, \omega_4 : P \to C$ with $\omega_3(x) = r+s$ and $\omega_4(x) = r+t$.

We may assume C is $1<2<\ldots<c$ and write $C+t$ for $1+t<2+t<\ldots<c+t$. Now the pair ω_1, ω_2 may equally be regarded as an order-preserving map B into the direct product $(C+t) \times C = \{(\gamma,\gamma') : \gamma \in C+t, \gamma' \in C\}$, with the partial ordering: $(\gamma_1,\gamma_2) < (\delta_1,\delta_2)$ if both $\gamma_1 < \delta_1$ in $C+t$ and $\gamma_2 < \delta_2$ in C. Thus $B = B_1 \times B_2 : P \to (C+t) \times C$ where, for $p \in P$, $B_1(p) = t + \omega_1(p)$ and $B_2(p) = \omega_2(p)$. In particular, we have $B(x) = (r+t, r+s+t)$.

Now define the operation $\text{flip}(j,k) = (k,j)$. We will say p forces q for $p,q \in P$ if,
either $p < q$ and $\text{flip}(B(p)) \nless B(q)$, i.e. $B_2(p) \geqslant B_1(q)$ or $B_1(p) \geqslant B_2(q)$,
or $p > q$ and $\text{flip}(B(p)) \ngtr B(q)$.

Also define $D_B = \{p : p \in P, x \text{ (forces)}^* p\}$, where "(forces)*" is the reflexive and transitive closure of "forces". That is, $x \in D_B$ and the forcing procedure propagates from x to form the subset D_B of P. Since P is finite the propagation must halt (possibly with $D_B = P$), and then we let $\delta(B) : P \to \mathbb{Z} \times \mathbb{Z}$ be defined by

$$\begin{aligned} \delta(B)(p) &= \text{flip}(B(p)) \quad \text{if } p \in D_B \\ &= B(p) \quad \text{if } p \notin D_B . \end{aligned}$$

Lemma 1 $\delta(B)$ is order-preserving.

Proof Immediate from the definitions. □

Lemma 2 $B_1(d) < B_2(d)$ for $d \in D_B$.

Proof We have $B_1(x) < B_2(x)$, since $s > 0$. So it is sufficient to show that if $B_1(d) < B_2(d)$ for some $d \in D_B$, then this relation holds for any p in D_B forced by d. Suppose first that $d < p$, and so $B_1(p) \leqslant B_2(d) < B_2(p)$ or $B_2(p) \leqslant B_1(d) < B_2(d) < B_2(p)$. The latter is impossible and the former establishes the claim. The proof if $d > p$, is similar. □

Lemma 3 $\delta(B)(p) \in (C+t) \times C$ for $p \in P$.

Proof If $p \notin D_B$ then it is clearly true. Now for $p \in D_B$ we have $1 < 1+t \leqslant B_1(p) < B_2(p) \leqslant c < c+t$. Hence $\mathrm{flip}(B(d)) \in (C+t) \times C$. □

Lemma 4 $\delta(\delta(B)) = B$.

Proof It is sufficient to show that $D_{\delta(B)} = D_B$. Suppose $d \in D_B$, then $\mathrm{flip}(\delta(B)(d)) = B(d)$ by the definition of $D_{\delta(B)}$. Therefore d forces p with respect to $\delta(B)$ if

either $d < p$ and $B(d) \not< \delta(B)(p)$,

or $d > p$ and $B(d) \not> \delta(B)(p)$.

If $p \notin D_B$ then $\delta(B)(p) = B(p)$ and so, since B is order-preserving, d does not force p.

If $p \in D_B$ then $\delta(B)(p) = \mathrm{flip}(B(p))$ and in this case d forces p with respect to $\delta(B)$ iff d forces p with respect to B.

Hence $D_{\delta(B)} = D_B$. □

Corollary δ is injective. □

Now ω_3 and ω_4 are given by $\delta(B) = (t + \omega_3, \omega_4)$ concluding the case $k = 1$.

Finally, we show how the result extends to a subset $\{x_1, \ldots, x_k\} \subset P$ where $k > 1$. For x_i with $2 \leqslant i \leqslant k$, we have $\omega_1(x_i) = \omega_2(x_i)$ and $B_1(x_i) = t + \omega_1(x_i) > B_2(x_i) = \omega_2(x_i)$. From Lemma 2 we deduce that $x_i \notin D_B$ giving $\delta(B)(x_i) = B(x_i)$ as required. □

Theorem 3 Let ν_c be the total number of order-preserving maps $\omega : P \to C$. Then $\nu_1, \nu_2, \ldots$ is log concave.

Proof Given P, let Q be the poset $P \cup q$ where $p < q$, $\forall p \in P$. Then note that ν_i equals N_{i+1} for $x = q$ in Q and $1 \leqslant i$. The result follows by log concavity of the sequence $N_{i+1}, N_{i+2}, \ldots$ □

3. Order-non-reversing maps

We will employ a corresponding injection to show log concavity for order-non-reversing maps. Define $N^{**}(i_1, \ldots, i_k)$ to be the number of order-non-reversing maps $\rho : P \to C$ such that $\rho(x_j) = i_j$ for $1 \leqslant j \leqslant k$.

Theorem 4 The analogue of Theorem 2 holds for $N^{**}(i_1, \ldots, i_k)$.

Proof The proof follows a parallel course to that of Theorem 2, but $(C + t) \times C$ now takes the usual product ordering, and p forces q if $p < q$ and $B_2(p) > B_1(q)$ or $B_1(p) > B_2(q)$ and similarly when $p > q$. □

Define ν_c^{**} to be the total number of order-non-reversing maps $\rho : P \to C$.

Theorem 5 The analogue of Theorem 3 holds for ν_c^{**}.

Proof The proof follows that of Theorem 3, with ν_i^{**} equal to N_i^{**} for $x = q$ in Q and $1 \leqslant i$.

4. Remarks

It appears unlikely that Stanley's Theorem for linear extensions quoted earlier can be proved using the kind of injection presented here. We may however easily strengthen his result to bring it into line with our Theorems 2 and 4, by removing the condition that the x's form a chain in P. Suppose $\{x_1,\ldots,x_k\}$ is an arbitrary subset of P, then without loss of generality assume $i_1 < \ldots < i_k$. If we augment P with the new relations $x_1 < \ldots < x_k$ then N^* is unchanged and Stanley's Theorem applies to the new partial order.

As $|C|$ increases, the proportion of order-preserving maps with $\omega(y) = \omega(z)$ for some $y \neq z$ diminishes. We note that

$$\nu_c \sim \nu_c^{**} \text{ and } N_i \sim N_i^{**} \text{ as } c \to \infty.$$

We may define a real-valued function $f_{P,x}$ on the unit interval by

$$f_{P,x}(u) = \lim_{c\to\infty} c \;\; N_{\lfloor uc\rfloor}/\nu_c = \lim_{c\to\infty} c \;\; N^{**}_{\lfloor uc\rfloor}/\nu_c^{**} \quad .$$

$f_{P,x}$ represents the probability distribution of the value of $\omega(x)$ with respect to the uniform distribution of order-preserving maps over the convex region of a unit n-dimensional cube defined by P. For further details on these convex regions see [10].

An easy consequence of our theorems is

<u>Corollary</u> $f_{P,x}$ is a log-concave (real-valued) function, i.e. $\log(f_{P,x})$ is concave.

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REFERENCES

1. R. Ahlswede, D.E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, Z. Wahrscheinlichkeitstheorie und Verw. Gebeite 43 (1978),183-185.

2. M. Aigner, Combinatorial Theory, Grundlehren Math. 234 (Springer-Verlag, Berlin, 1979).

3. H. Busemann, Convex Surfaces, Interscience, New York, 1958.

4. F.R.K. Chung, P.C. Fishburn and R.L. Graham, On unimodality for linear extensions of partial orders, SIAM J. Algebraic and Discrete Methods 1 (1980), 405-410.

5. H.G. Eggleston, Convexity, Cambridge Univ. Press, Cambridge, 1958.

6. C.M. Fortuin, J. Ginibre and P.N. Kasteleyn, Correlation inequalities on some partially ordered sets, Commun. Math. Phys. 22 (1971), 89-103.

7. R.L. Graham, Applications of the FKG inequality and its relatives, to appear.

8. ____________________, Linear extensions of partial orders and the FKG inequality, in Ordered Sets, I. Rival, ed., 213-236, D. Reidal, Dordrecht-Holland, 1982.

9. P.D. Seymour, D.J.A. Welsh, Combinatorial applications of an inequality from statistical mechanics, Math. Proc. Camb. Phil. Soc. 77 (1975), 485-495.

10. R.P. Stanley, Two combinatorial applications of the Aleksandrov-Fenchel inequalities, J. Combinatorial Theory (Ser. A) 31 (1981), 56-65.