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INEQUALITIES FOR THE NUMBER OF MONOTONIC FUNCTIONS OF PARTIAL ORDERS

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Abstract

Let P be a finite poset and let $x,y \in P$. Let C be a chain. Define $N(i,j)$ to be the number of strict order-preserving maps $\omega : P \to C$ satisfying $\omega(x) = i$ and $\omega(y) = j$. Various inequalities are proved, commencing with Theorem 3. If r,s,t,u,v,w are non-negative integers then $N(r, u+v+w)N(r+s+t, u) \leqslant N(r+t, u+v)N(r+s, u+w)$. The case $v = w = 0$ is a theorem of Daykin, Daykin and Paterson, which is an analogue of a theorem of Stanley for linear extensions.

1. Introduction

Let P be a poset (= partially ordered set) with n elements and C a chain with elements $1 < 2 < \ldots < c$. Monotonic mappings from the elements of P into C are defined as follows.

For (P,C), a map $\rho : P \to C$ is <u>order-preserving</u> if, for all $x,y \in P$, $x < y$ implies $\rho(x) \leqslant \rho(y)$. Let $R = R(P,C)$ be the set of all such ρ. (Some authors require $|P| = |C|$, but we do not need this restriction).

For (P,C), a map $\omega : P \to C$ is <u>strict order-preserving</u> if, for all $x,y \in P$, $x < y$ implies $\omega(x) < \omega(y)$. Note that ω need not be 1-1. Let $\Omega = \Omega(P,C)$ be the set of all such ω.

A map $\lambda : P \to [n] \equiv \{1,2,\ldots,n\}$ is a <u>linear extension</u> of P if λ is 1-1 and, for all $x,y \in P$, $x < y$ implies $\lambda(x) < \lambda(y)$. Let Λ be the set of all such λ.

A sequence $a_o, a_1, \ldots$ of non-negative real numbers is said to be <u>log concave</u> if $a_{i-1}\, a_{i+1} \leqslant a_i^2$ for $1 \leqslant i$. In particular, a log concave sequence is <u>unimodal</u>, i.e. for some j we have $a_o \leqslant a_1 \leqslant \ldots \leqslant a_j$ and $a_j \geqslant a_{j+1} \geqslant \ldots$. Log concave sequences can be proved (see [A]) to satisfy the more general inequality,

$$a_r\, a_{r+s+t} \leqslant a_{r+s}\, a_{r+t} \quad \text{for non-negative integers } r,s,t.$$

We adopt the following notation. Let Z^+ denote the non-negative integers. If $x_1,\ldots,x_k$ is a fixed subset in P and $i_1,\ldots,i_k \in Z^+$ then define $N^{**}(i_1,\ldots,k_k)$ to be the number of order-preserving maps $\rho : P \to C$ such that $\rho(x_j) = i_j$ for $1 \leqslant j \leqslant k$; and define $N(i_1,\ldots,i_k)$ to be the number of strict order-preserving maps $\omega : P \to C$ such that $\omega(x_j) = i_j$ for $1 \leqslant j \leqslant k$; also define $N^*(i_1,\ldots,i_k)$ to be the number of linear extensions $\lambda : P \to [n]$ such that $\lambda(x_j) = i_j$ for $1 \leqslant j \leqslant k$. Further, if $i_j \notin C$ for any j then $N(i_1,\ldots,i_k) = 0$ and similarly for N^{**}, N^*. Also we will write $x=x_1, y=x_2$, and we put $i_1,\ldots,i_k,\ j_1,\ldots,j_k \in C$ throughout.

A fundamental result is

Theorem 1. (Stanley [S]). Let $x_1,\ldots,x_k$ be a fixed subset in P. If $r,s,t \in Z^+$ and $i_h \notin [r,r+s+t]$ for $2 \leq h \leq k$, then

(1) $N^*(r,i_2,\ldots,i_k)N^*(r+s+t,i_2,\ldots,i_k) \leq N^*(r+t,i_2,\ldots,i_k)N^*(r+s,i_2,\ldots,i_k).$

Recently Daykin, Daykin and Paterson [DDP] established the analogue of Stanley's result for both strict order-preserving and order-preserving maps. In other words they proved that (1) holds with each N^* replaced by N, and with each N^* replaced by N^{**}.

Their proofs entailed defining an injection. This injection consists of constructing, for each pair of strict order-preserving maps (or order-preserving maps) with $\omega_1(x) = r$ and $\omega_2(x) = r+s+t$, a unique pair of maps with $\omega_3(x) = r+t$ and $\omega_4(x) = r+s$. That is if two ordered pairs of the form (ω_1,ω_2) are distinct, then their two associated (ω_3,ω_4) pairs are distinct, thus ensuring the inequality.

The results in this paper are motivated by these log concave sequences for partial orders. The reader will find, for example by looking at Theorem 3, that we are here basically concerned not with a single element $x \in P$ but with a pair of elements $x,y \in P$. However we extended the above injection technique to this more general situation, to obtain many results of a new kind.

2. Strict Order-Preserving Maps

We first state a generalization of the theorem of Daykin, Daykin and Paterson, which is proved in [D].

Theorem 2. Let $x_1,\ldots,x_k$ be a fixed subset in P. If $r,s,t \in Z^+$ and $j_h \leq i_h+s$ for $2 \leq h \leq k$, then

$$N(r,i_2,\ldots,i_k)N(r+s+t,j_2,\ldots,j_k) \leq N(r+t,i_2,\ldots,i_k)N(r+s,j_2,\ldots,j_k).$$

One of our main results is

Theorem 3. Let $x_1,\ldots,x_k$ be a fixed subset in P. If $r,s,t,u,v,w \in Z^+$ and $i_h - v \leq j_h \leq i_h+s$ for $3 \leq h \leq k$, then

$$N(r,u+v+w,i_3,\ldots,i_k)N(r+s+t,u,j_3,\ldots,j_k) \leq N(r+t,u+v,i_3,\ldots,i_k)N(r+s,u+w,j_3,\ldots,j_k).$$

Each map ω counted by the function N has $\omega(x_h)$ fixed for $3 \leq h \leq k$ in the respective factors. From now on we will simplify such expressions to omit any i_h,j_h. Hence the statement of this theorem abbreviates to

Theorem 3. If $r,s,t,u,v,w \in Z^+$ and $i_h - v \leq j_h \leq i_h+s$ for $3 \leq h \leq k$, then

$$N(r,u+v+w)N(r+s+t,u) \leq N(r+t,u+v)N(r+s,u+w).$$

Proof Suppose that the L.H.S. of the inequality is not zero.

Case $r,t,u,w > 0$. For the time being ignore elements $x_3,\ldots,x_k$.

Given any pair of strict order-preserving maps $\omega_1,\omega_2 : P \to C$ with $\omega_1(x) = r$, $\omega_1(y) = u+v+w$ and $\omega_2(x) = r+s+t$, $\omega_2(y) = u$, we will construct a unique pair of strict order-preserving maps $\omega_3,\omega_4 : P \to C$ with $\omega_3(x) = r+t$, $\omega_3(y) = u+v$ and $\omega_4(x) = r+s$, $\omega_4(y) = u+w$.

Now ω_3,ω_4 will depend on subsets D,E of P. We define $\delta : (\omega_1,\omega_2) \to (\omega_3,\omega_4,D,E)$ by

$$\begin{aligned}\omega_3(p) &= -s + \omega_2(p) && \text{if } p \in D\\ &= v + \omega_2(p) && \text{if } p \in E\\ &= \omega_1(p) && \text{if } p \in P\setminus(D\cup E),\end{aligned}$$

$$\text{and}\quad\begin{aligned}\omega_4(p) &= s + \omega_1(p) && \text{if } p \in D\\ &= -v + \omega_1(p) && \text{if } p \in E\\ &= \omega_2(p) && \text{if } p \in P\setminus(D\cup E).\end{aligned}$$

Initially let $D = \{x\}$ and $E = \{y\}$ and then D,E are constructed iteratively. We stop adjoining elements to D and E as soon as $\omega_3, \omega_4 \in \Omega$, and then the construction is complete.

When $d = x$ and $e = y$ we have

$$(3.1) \qquad 1 + \omega_1(d) \leqslant - s + \omega_2(d) = \omega_3(d),$$

$$(3.2) \qquad 1 + \omega_2(e) \leqslant - v + \omega_1(e) = \omega_4(e).$$

Assume for the moment that (3.1), (3.2) are invariants for any $d \in D$, $e \in E$ respectively. From (3.1) we deduce that $d \in D$ implies that $\omega_3(d)$, $\omega_4(d) \in C$, i.e.

$$(3.3) \qquad 2 \leqslant 1 + \omega_1(d) \leqslant \omega_3(d) \quad \text{and} \quad \omega_4(d) = s+\omega_1(d) \leqslant - 1 + \omega_2(d) \leqslant - 1+c.$$

Similarly from (3.2) we deduce that for $e \in E$,

$$(3.4) \qquad 2 \leqslant \omega_4(e) \quad \text{and} \quad \omega_3(e) \leqslant - 1+c.$$

Now suppose we have constructed some D and E but $\omega_3, \omega_4 \notin \Omega$. Also suppose there exists $p \in P\backslash(D \cup E)$ for which ω_3 or ω_4 loses order between some $d \in D$ and p, or between some $e \in E$ and p. Assume the former and then one of four cases holds.

<u>Case</u> 1. $p < d$ and $\omega_3(p) \geqslant \omega_3(d)$.

Bu using $\omega_1 \in \Omega$, (3.1) and the definition of ω_3 we get

$$\omega_1(p) < \omega_1(d) < - s + \omega_2(d) = \omega_3(d) \leqslant \omega_3(p) = \omega_1(p).$$

Since this is impossible this case cannot arise.

<u>Case</u> 2. $p < d$ and $\omega_4(p) \geqslant \omega_4(d)$.

By using $\omega_1, \omega_2 \in \Omega$ and the definition of ω_4 we get

$$1 \leqslant \omega_1(p) < \omega_1(d) = - s + \omega_4(d) \leqslant - s + \omega_4(p) = - s + \omega_2(p) \leqslant - s+c.$$

Hence d forces p to join D, so let $D' = D \cup p$. Notice that (3.1), (3.3) hold for $p \in D'$.

<u>Case</u> 3. $p > d$ and $\omega_3(p) \leqslant \omega_3(d)$.

Similarly to Case 2, d forces p to join D with (3.1), (3.3) holding for p in D'.

<u>Case</u> 4. $p > d$ and $\omega_4(p) \leqslant \omega_4(d)$.

Similarly to Case 1 this is impossible.

The latter four cases follow by the symmetry $d \leftrightarrow e$, $s \leftrightarrow v$ and $\omega_1 \leftrightarrow \omega_2$, $\omega_3 \leftrightarrow \omega_4$.

Now we have shown that since (3.1) holds for some $d \in D$ then it holds for any p forced to join D by d. Similarly (3.2) is invariant for any $e \in E$.

<u>Lemma</u> 1. Let $d \in D$, $e \in E$ and define $f, f' : D \to C$ by $f(d) = \omega_3(d)$, $f'(d) = \omega_4(d)$, and $g, g' : E \to C$ by $g(e) = \omega_3(e)$, $g'(e) = \omega_4(e)$, and $h : (D,E) \to C^2$ by $h(f,g) = (\omega_3(d), \omega_3(e))$ and $h(f',g') = (\omega_4(d), \omega_4(e))$, and $h', h'' : P \backslash (D \cup E) \to C$ by $h'(p) = \omega_3(p)$ and $h''(p) = \omega_4(p)$ for $p \in P \backslash (D \cup E)$. Then f, f', g, g', h, h', h'' are strict order-preserving.

<u>Proof</u> Let $d_1, d_2 \in D$ with $d_1 < d_2$. Then $\omega_2(d_1) < \omega_2(d_2)$ implies $f(d_1) = -s + \omega_2(d_1) < -s + \omega_2(d_2) = f(d_2)$. Also $\omega_1(d_1) < \omega_1(d_2)$ implies $f'(d_1) = s + \omega_1(d_1) < s + \omega_1(d_2) = f'(d_2)$. And similarly for $g, g,', h', h''$.

To see that h is strict order-preserving let $d \in D$, $e \in E$.

<u>Case</u> $d < e$. By using $\omega_2 \in \Omega$ and (3.1), (3.2) we get

$$\omega_3(d) = -s + \omega_2(d) < v + \omega_2(e) = \omega_3(e),$$

and $$\omega_4(d) = s + \omega_1(d) < \omega_2(d) < \omega_2(e) < -v + \omega_1(e) = \omega_4(e).$$

<u>Case</u> $d > e$ follows by symmetry. □

<u>Lemma</u> 2. $D \cap E = \phi$.

<u>Proof</u> From (3.1) and the definitions of ω_3, ω_4 we deduce for $d \in D$ that

(2.1) $\omega_1(d) < \omega_3(d) = -s + \omega_2(d) \leqslant \omega_2(d)$,

(2.2) $\omega_1(d) \leqslant \omega_4(d) = s + \omega_1(d) < \omega_2(d)$.

Similarly from (3.2) we deduce for $e \in E$ that

(2.3) $\omega_2(e) \leqslant \omega_3(e) = v + \omega_2(e) < \omega_1(e)$,

(2.4) $\omega_2(e) < \omega_4(e) = -v + \omega_1(e) \leqslant \omega_1(e)$.

<u>Corollary</u> 1. If $p \in P$ with $\omega_2(p) < -v + \omega_1(p)$ then this implies $p \notin D$.

<u>Corollary</u> 2. If $p \in P$ with $\omega_1(p) < -s + \omega_2(p)$ then this implies $p \notin E$.

Now suppose $d \in D$, $e \in E$. Since h in Lemma 1 is strict order-preserving, this means that neither d causes e to join D, nor e causes d to join E. □

From Lemmas 1,2 it follows that if $\omega_3, \omega_4 \notin \Omega$ then order must be lost between either $P\backslash(D \cup E)$ and D or $P\backslash(D \cup E)$ and E, that is Cases 1-4 along with the symmetric ones. Since P is finite the iterative construction of D and E must halt (possibly with $D \cup E = P$). When it halts we deduce from Lemmas 1,2 that $\omega_3, \omega_4 \in \Omega$. It remains to show

<u>Lemma</u> 3. δ is injective.

<u>Proof</u> Suppose $(\omega_1, \omega_2) \neq (\omega_1', \omega_2')$.

<u>Case</u> $\delta(\omega_1, \omega_2) = (\omega_3, \omega_4, D, E) = \delta(\omega_1', \omega_2')$.

This is clearly contradictory from the definitions.

<u>Case</u> $\delta(\omega_1, \omega_2) = (\omega_3, \omega_4, D, E) \neq (\omega_3, \omega_4, D', E') = \delta(\omega_1', \omega_2')$.

Without loss of generality assume $D \neq D'$ and also that there exists $p \in D\backslash D'$. Let $d \in D \cap D'$.

<u>Case</u> $p < d$. Now we adjoined p only to D and hence,

$$\omega_3(p) = -s + \omega_2(p) = \omega_1'(p).$$

Using this along with $\omega_1' \in \Omega$ and Case 2 we have

$$\omega_1'(d) = \omega_1(d) \leqslant -s + \omega_2(p) = \omega_1'(p) \leqslant -1 + \omega_1'(d),$$

giving a contradiction.

<u>Case</u> $p > d$ follows similarly.

If we now assume $E \neq E'$ then this follows by symmetry. We conclude that δ is injective. □

Finally consider elements $x_3, \ldots, x_k$. For x_h with $3 \leqslant h \leqslant k$ we have

$$\omega_1(x_h) = i_h \geqslant -s + \omega_2(x_h) = -s + j_h,$$
$$\omega_2(x_h) = j_h \geqslant -v + \omega_1(x_h) = -v + i_h.$$

From (2.1), (2.4) we deduce that $x_h \notin D$ and $x_h \notin E$ giving $\omega_3(x_h) = i_h$ and $\omega_4(x_h) = j_h$ as required, which completes the proof of this case.

Case Not $r,t,u,w > 0$. If $r = 0$ or $u = 0$ the result is trivial because the L.H.S. is zero. If $t = 0$ or $w = 0$ the theorem reduces to Theorem 2. □

One might think that if $u < v$ then $N(r,u)N(r+s+t,v) \leqslant N(r+t,u)N(r+s,v)$. However that this is not true is shown by

Example 1. Let $P = \{x < p < y\}$. Then

$$2.2 = N(1,4)N(5.8) \nleqslant N(4,4)N(2.8) = 0.5.$$

From Theorem 2 we have $N(r,u+w)N(r+t,u) \leqslant N(r+t,u+w)N(r,u)$. It would seem possible for such an inequality to be bijective. Nevertheless we give

Example 2. Let $P = \{x < p,\ y < p\}$. Then

$$(c-2)^2 = N(1,2)N(2,1) < N(2,2)N(1,1) = (c-2)(c-1).$$

We now consider extending each of the elements $x,y \in P$ in Theorem 3 to subsets of P.

Theorem 4. Let $k', k'' \in Z^+$ with $k' \leqslant k'' \leqslant k$. If $s,v,t_1,\ldots,t_{k''} \in Z^+$ and $i_h - v \leqslant j_h \leqslant i_h + s$ for $k'' < h \leqslant k$, then

$$N(i_1,\ldots,i_{k'},i_{k'+1}+v+t_{k'+1},\ldots,i_{k''}+v+t_{k''})N(i_1+s+t_1,\ldots,i_{k'}+s+t_{k'},i_{k'+1},\ldots,i_{k''}) \leqslant$$
$$N(i_1+t_1,\ldots,i_{k'}+t_{k'},i_{k'+1}+v,\ldots,i_{k''}+v)N(i_1+s,\ldots,i_{k'}+s,i_{k'+1}+t_{k'+1},\ldots,i_{k''}+t_{k''}) .$$

Proof Suppose that the L.H.S. of the inequality is not zero. Assume that not all of $t_1,\ldots,t_{k''}$ are zero for otherwise the result is obvious. If only $t_1 > 0$ or $t_{k'+1} > 0$ then this follows by Theorem 2 or Theorem 3. Hence without loss of generality assume $k' \geqslant 2$ and $t_1,t_2 > 0$.

Consider first the elements $x_1,\ldots,x_{k'}$. We follow the proof of Theorem 3, except that we define $\delta:(\omega_1,\omega_2) \to (\omega_3,\omega_4,D)$ by

$$\omega_3(p) = -s + \omega_2(p) \text{ if } p \in D$$
$$= \omega_1(p) \quad \text{otherwise,}$$

and
$$\omega_4(p) = s + \omega_1(p) \quad \text{if } p \in D$$
$$= \omega_2(p) \quad \text{otherwise.}$$

Initially let $D = \{x\}$. Then $\omega_3(x) = i_1 + t_1$ and $\omega_4(x) = i_1 + s$ as required. Now consider y.

<u>Case</u> $y \in D$ implies $\omega_3(y) = i_2 + t_2$ and $\omega_4(y) = i_2 + s$.

<u>Case</u> $y \notin D$. Now define $\delta':(\omega_3,\omega_4) \to (\omega_5,\omega_6,D')$ by

$$\omega_5(p) = -s + \omega_4(p) \quad \text{if } p \in D'$$
$$= \omega_3(p) \quad \text{otherwise,}$$

and
$$\omega_6(p) = s + \omega_3(p) \quad \text{if } p \in D'$$
$$= \omega_4(p) \quad \text{otherwise.}$$

And initially let $D' = \{y\}$. Now we must show that $x \notin D'$. If there is no path of elements between x and y then clearly $x \notin D'$. So suppose there exists a path of elements $x = q_1,q_2,\ldots,q_h = y$. Then for some $l \in \{2,\ldots,h\}$, $q_l \notin D$ with $q_{l-1} \in D$.

Firstly suppose by (3.1) that $\omega_1(q_l) \geqslant -s + \omega_2(q_l)$, which implies $l \neq h$. And hence $\omega_3(q_l) \geqslant -s + \omega_4(q_l)$, which implies $q_l \notin D'$. Thus q_l prevents x from being adjoined to D' via this path, and similarly for any path.

Otherwise suppose that $\omega_1(q_l) < -s + \omega_2(q_l)$, and that $q_l \in D'$.

<u>Case</u> $q_{l-1} < q_l$. Then we have

$$\omega_3(q_{l-1}) = -s + \omega_2(q_{l-1}) < -s + \omega_2(q_l) = \omega_5(q_l),$$

and $\omega_4(q_{l-1}) = s + \omega_1(q_{l-1}) < s + \omega_1(q_l) = \omega_6(q_l)$.

Therefore q_l does not foce q_{l-1} to join D'.

<u>Case</u> $q_{l-1} > q_l$ follows similarly.

We may conclude that $x \notin D'$. Further, since this analysis holds for any $d \in D$, $d' \in D'$ we deduce that $D \cap D' = \phi$. Also for x_h with $k' < h \leqslant k''$ we have $\omega_1(x_h) = i_h + v + t_h$ and $\omega_2(x_h) = i_h$. By (2.1) we know that $x_h \notin D$, $x_h \notin D'$.

This process is iterated for elements $x_3,\ldots,x_{k'}$, except when $3 \leqslant h \leqslant k'$ and $t_h = 0$. By (2.1) for any integer $l \geqslant 1$, $\omega_{2l+1}(x_h) = \omega_1(x_h) = i_h$ and $\omega_{2l+2}(x_h) = \omega_2(x_h) = i_h + s$.

By repeatedly using δ, depending on which cases apply, we are using Theorem 2 consecutively some number of times, thus resulting in an injection.

Now consider the elements $x_{k'+1},\ldots,x_{k''}$. As with the previous subset, we repeatedly apply δ but with $s = -v$ and $D = E$.

Let $\mathcal{D} = D,D',\ldots,D''$ be the set of disjoint subsets of P generated by δ for $x_1,\ldots,x_{k'}$ and similarly $\mathcal{E} = \{E,E',\ldots,E''\}$ for $x_{k'+1},\ldots,x_{k''}$. By Lemma 2 we have that $\mathcal{D}$, $\mathcal{E}$ are pairwise disjoint sets.

Also by the proof of Lemma 2 we deduce that $x_h \notin D$, $x_h \notin E$ for $k'' < h < k$, and any $D \in \mathcal{D}$, $E \in \mathcal{E}$. So if we used the injection a total of l times say, then $\omega_{2l+1}(x_h) = \omega_1(x_h) = i_h$ and $\omega_{2l+2}(x_h) = \omega_2(x_h) = j_h$ as required. □

One may hope that if $s \neq v$ then
$N(r,u)N(r+s+t,u+v+w) \leqslant N(r+t,u+w)N(r+s,u+v)$. However we have

<u>Example</u> 3. Let $P = \{x < p,\ y < p\}$.

$$(c-1)(c-8) = N(1,1)N(8,8) \nleqslant N(6,5)N(3,4) = (c-6)(c-4).$$

Further by Theorem 2, $N(3,5)N(6,4) \leqslant N(6,5)N(3,4)$.

Under certain conditions one may have $s \neq v$ in this context, as shown by

<u>Theorem</u> 5. If $r,s,t,u,v,w \in Z^+$ satisfy $t < s \leqslant v \leqslant w$ and $i_h \leqslant j_h \leqslant i_h + s$ for $3 \leqslant h \leqslant k$, then

$$N(r,u)N(r+s+t,u+v+w) \leqslant N(r+t,u+v)N(r+s,u+w).$$

Proof Suppose that the L.H.S. of the inequality is not zero. Assume $s < v$ for otherwise this follows by Theorems 4, 2.

Case $r,s,t,u,v,w > 0$. Define $\delta : (\omega_1,\omega_2) \to (\omega_3,\omega_4,D)$ as in Theorem 4. Initially let $D = \{x\}$. Then $\omega_3(x) = r + t$ and $\omega_4(y) = r + s$ as required.

Case $y \in D$ implies $\omega_3(y) = - s + u + v + w$ and $\omega_4(y) = s + u$. By $s < v$ we have $s + u < v - s + \omega_4(y) \leq - (v - s) + \omega_3(y) < - s + u + v + w$. This means we can define $\delta' : (\omega_3,\omega_4) \to (\omega_5,\omega_6,E)$ by

$$\begin{aligned}
\omega_5(p) &= \alpha + \omega_4(p) \quad \text{if } p \in E \\
&= \omega_3(p) \quad \text{otherwise,} \\
\omega_6(p) &= - \alpha + \omega_3(p) \text{ if } p \in E \\
&= \omega_4(p) \quad \text{otherwise,}
\end{aligned}$$

where $\alpha = v - s$, and initially let $E = \{y\}$. Then $\omega_5(y) = u + v$ and $\omega_6(y) = u + w$. From $s < v$, $\omega_4(y) < - \alpha + \omega_3(y)$ and hence by (2.3) for $e \in E$ we have $\omega_4(e) < \omega_3(e)$ and therefore $x \notin E$.

Case $y \notin D$. Define ω_5,ω_6 as above with $\alpha = - v$, resulting in $\omega_5(y) = u + w$ and $\omega_6(y) = u + v$. Assume for the moment that $x \notin E$, and then we apply Theorem 2 to ω_5,ω_6. The argument for $x \notin E$ runs very similarly as in the Case $y \notin D$ in Theorem 4.

To see that the construction is injective notice that in either case we are making several applications of Theorem 2.

For x_h with $3 \leq h \leq k$ we have $\omega_1(x_h) = i_h$, $\omega_2(x_h) = j_h$. Now $i_h - v + s < i_h \leq j_h \leq i_h + s < i_h + v$ and hence by (2.1), (2.4) for any of the applications of Theorem 2 the mappings of x_h remain fixed as required.

Case Not $r,s,t,u,v,w > 0$. If $r = 0$ or $u = 0$ the result is trivial because the L.H.S. is zero. If $s = 0$ then $t = 0$ and the theorem reduces to Theorem 2. If $v = 0$ or $w = 0$ then $s = 0$. If $t = 0$ then $1 + r > - w + r + s$ and again this reduces to Theorem 2, via (3.1). □

We now give examples to show that the condition $t \leqslant s \leqslant v \leqslant w$ in Theorem 5 is necessary.

With $t \leqslant s \leqslant v > w$ we have

Example 4. Let $P = \{x < p, y < p\}$. Then

$$(c-1)(c-5) = N(1,1)N(5,4) \nleqslant N(3,4)N(3,1) = (c-4)(c-3).$$

With $t \leqslant s > v \leqslant w$ we have

Example 5. Let $P = \{x < p < y\}$. Then

$$2.2 = N(1,4)N(5,8) \nleqslant N(3,5)N(3,7) = 1.3.$$

Swopping x and y in the last example shows the necessity for $t \leqslant s$.

Special cases of Theorem 2 along with Theorems 3, 5 can be stated as

Theorem 6. If $r,s,t,u,v,w \in Z^+$ satisfy $s \leqslant t \leqslant v \leqslant w$ and $i_h \leqslant j_h \leqslant i_h + s$ for $3 \leqslant h \leqslant k$, then

$$\begin{array}{ccc} N(r,u)N(r+s+t,u+v+w) & \leqslant & N(r+s,u+v)N(r+t,u+w) \\ \veebar & & \veebar \\ N(r,u+v+w)N(r+s+t,u) & \leqslant & N(r+t,u+v)N(r+s,u+w). \end{array}$$

A different kind of result for a subset in P is

Theorem 7. Let $k' \in Z^+$ with $k' \leqslant k$. Suppose $s_1,\ldots,s_{k'} \in Z^+$ satisfy

(7.1) $\quad 0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_{k'}$, and

(7.2) $\quad i_h - \beta \leqslant j_h \leqslant i_h + \alpha$ for $k' < h \leqslant k$,

where $\alpha = \min\{s_h - s_{h-1} : 1 \leqslant h \leqslant k', h \text{ odd}\}$,

$\beta = \min\{s_h - s_{h-1} : 2 \leqslant h \leqslant k', h \text{ even}\}$.

Then

$$N(i_1,\ldots,i_{k'})N(i_1+2s_1,\ldots,i_{k'}+2s_{k'}) \leqslant N(i_1+s_1,\ldots,i_{k'}+s_{k'})N(i_1+s_1,\ldots,i_{k'}+s_{k'}).$$

Proof Suppose that the L.H.S. of the inequality is not zero and that some $s_h > 0$ with $1 \leqslant h \leqslant k'$ for otherwise the result clearly holds. We make k' applications of Theorem 4 to the fixed subset $x_1,\ldots,x_{k'}$ in P.

Putting $s = s_1$ we get

$$N(i_1,i_2,\ldots,i_{k'})N(i_1+2s_1,i_2+2s_2,\ldots,i_{k'}+2s_{k'}) \leqslant$$
$$N(i_1+s_1,i_2+2s_2-s_1,\ldots,i_{k'}+2s_{k'}-s_1)N(i_1+s_1,i_2+s_1,\ldots,i_{k'}+s_1),$$ when all $s_h > 0$. If $s_h = 0$ for any h then by the proof of Lemma 2

$\omega_3(x_h) = \omega_1(x_h) = \omega_2(x_h) = \omega_4(x_h)$.

Subsequently if $k' \geqslant 2$ put $s = s_2 - s_1, s_3 - s_2,\ldots,s_{k'} - s_{k'-1}$.

This produces the sequence of mappings $\omega_1,\omega_2 \to \omega_3,\omega_4,D_1 \to \ \ldots \to \omega_{2k'+1},\omega_{2k'+2},D_{k'}$.

By (7.1) for $1 \leqslant h, l \leqslant k'$ if h is odd then

(7.3) $\omega_{2h+1}(x_l) \geqslant i_l + s_l \geqslant \omega_{2h+2}(x_l)$,

and if h is even then

(7.4) $\omega_{2h+1}(x_l) \leqslant i_l + s_l \leqslant \omega_{2h+2}(x_l)$.

Equality in (7.3) or (7.4) implies by the proof of Lemma 2 that $x_l \notin D_{l'}$ with $1 \leqslant l < l' \leqslant k'$.

For elements x_h with $k' < h \leqslant k$ and $1 \leqslant l \leqslant k'$, from (7.2), if l is odd then $\omega_2(x_h) = j_h \leqslant \omega_1(x_h) + \alpha = i_h + \alpha \leqslant i_h + s_l - s_{l-1}$, and if l is even then $\omega_2(x_h) = j_h \geqslant \omega_1(x_h) - \beta = i_h - \beta \geqslant i_h - (s_l - s_{l-1})$. Hence by (2.1), (2.4) in either case $x_h \notin D_l$. □

We now give a higher order inequality.

<u>Theorem</u> 8. Let $h \in Z^+$ with $h \geqslant 1$. Let $r_1,\ldots,r_h,u_1,\ldots,u_h$ be integers and $i = i_1, j = j_1$. Suppose

(8.1) $\sum_{1\leqslant l\leqslant h} (i+r_l) = hi$ and $\sum_{1\leqslant l\leqslant h} (j + u_l) = hj$, then

(8.2) $N(i+r_1,j+u_1) \ldots N(i+r_h,j+u_h) \leqslant N(i,j)^h$,

with $\omega(x_3),\ldots,\omega(x_k) = i_3,\ldots,i_k$ in every factor.

Proof Suppose that the L.H.S. of the inequality is not zero. Suppose also that $h \geq 2$ and not all of $r_1,\dots,r_h,u_1,\dots,u_h$ equal zero for otherwise the result clearly holds.

Without loss of generality assume that some $r_l > 0$ with $1 \leq l \leq h$. Then by (8.1) there exists a distinct pair $N(i+r_l,j+u_t)N(i+r_{l'},j+u_{t'})$, $1 \leq l,l',t,t' \leq h$, on the L.H.S. of (8.2) where r_l is negative and $r_{l'}$ is positive. And in view of (3.1), $1 + r_l < r_{l'}$.

Now by applying Theorem 2 to this pair we obtain

(8.3) $N(i+r_l,j+u_t)N(i+r_{l'},j+u_{t'}) \leq N(i+r_{l'}-\alpha,\sigma)N(i+r_l+\alpha,\tau)$,

where $\alpha = \min\{|r_l|,|r_{l'}|\}$. Hence $r_{l'} - \alpha = 0$ or $r_l + \alpha = 0$, and

$r_{l'} - \alpha \geq r_l + \alpha$.

Now either $\sigma = j + u_t$ and $\tau = j + u_{t'}$,

or $\sigma = j + u_{t'} - \alpha$ and $\tau = j + u_t + \alpha$.

The latter case implies $u_t + \alpha < u_{t'}$ by (3.1), and therefore

$j + u_t < \sigma,\tau < j + u_{t'}$.

We make the substitution of (8.3) in the L.H.S. of (8.2) and note that (8.1) still holds.

Repeated substitutions of this kind result in all of the x components of (8.2) being equal to i. If at this stage some y components of (8.2) are not equal to j then we make analagous applications of Theorem 2. And by (2.1) the images of x remain equal to i under the injection.

Also by (2.1) the image of x_l remains equal to i_l for $3 \leq l \leq k$ under any injection. □

Using the ideas developed here other results are proved in [D], for example

Theorem 9. If $r,s,u,v,r',s',t' \in Z^+$ satisfy $s \leq v \leq s' \leq t'$ and $i_h \leq j_h \leq i_h + s$ for $4 \leq h \leq k$, then

$$N(r,u,r'+s'+t')N(r+2s, u+2v, r') \leq N(r+s, u+v, r'+s')N(r+s, u+v, r'+t').$$

In the following inequality we let each of the elements $x,y \in P$ map to intervals in C. Hence define $N([i_1,i_2], [j_1,j_2])$ to be the number of strict order-preserving maps $\omega : P \to C$ such that $\omega(x) \in [i_1,i_2]$ and $\omega(y) \in [j_1,j_2]$.

<u>Theorem</u> 10. If $r,r',s,t,t',u,v,w,w' \in Z^+$ and $w' \leqslant v$, then

(10.1) $N([r',r],[u,v])N([r+s+t,t'],[u+s+w,w']) \leqslant$

$$N([r+t,t'-s],[u+w,v])N([r'+s,r+s],[u+s,w']).$$

<u>Proof</u> Suppose that the L.H.S. of the inequality is not zero. Thus we assume that the intervals on the L.H.S. are non-empty, i.e. $r' \leqslant r$, $u \leqslant v$, $r+s+t \leqslant t'$ and $u+s+w \leqslant w'$. Clearly on the R.H.S. we then have $r+t \leqslant t'-s$, $r'+s \leqslant r+s$, $u+s \leqslant w'$ and also $u+w \leqslant u+s+w \leqslant w' \leqslant v$.

Suppose $r' \leqslant h \leqslant r$ and $r+s+t \leqslant l \leqslant t'$, then we must show that

(10.2) $(N(h,u)+\ldots+N(h,v))(N(l,u+s+w)+\ldots+N(l,w')) \leqslant$

$$(N(l-s,u+w)+\ldots+N(l-s,v))(N(h+s,u+s)+\ldots+N(h+s,w')).$$

First we will establish that

(10.3) $N(h,j')N(l,j'') \leqslant N(l-s,j''-s)N(h+s,j'+s)$

when $u \leqslant j' < u+w$ and $u+s+w \leqslant j'' \leqslant w'$.

In view of (3.1) we have $j' + s < j''$. Also $h+s < l$ except when $t = 0$ and $\omega_1(x) = r$ and $\omega_2(x) = r+s+t$. Hence (10.3) follows by Theorem 4, where $r+t \leqslant l-s \leqslant t'-s$ and $r'+s \leqslant h+s \leqslant r+s$. When $h+s \geqslant l$ then $\omega_3(x) = \omega_1(x) = r$ and $\omega_4(x) = \omega_2(x) = r+s$, since here we in effect use Theorem 2 on y.

We will prove that

(10.4) $N(h,[u+w,v])N(l,[u+s+w,w']) \leqslant N(l-s,[u+w,v])N(h+s,[u+s+w,w'])$.

Summing (10.3) over j',j'' and adding (10.4) gives (10.2). Then summing (10.2) over h,l gives (10.1) as required.

We prove (10.4) as follows. Given any ordered pair (ω_1,ω_2) of maps counted by the L.H.S. we construct a unique pair (ω_3,ω_4) counted by the R.H.S. So we have

$\omega_1(x) = h, \quad \omega_2(x) = l,$

$\omega_3(x) = l-s, \ \omega_4(x) = h+s,$

$u+w \leqslant \omega_1(y), \omega_3(y) \leqslant v,$

$u+s+w \leqslant \omega_2(y), \omega_4(y) \leqslant w'$.

If $t = 0$ and $\omega_1(x) = r, \omega_2(x) = r+s+t$ then let $\omega_3(x) = \omega_1(x)$, $\omega_4(x) = \omega_2(x)$, also let $\omega_3(y) = \omega_1(y), \omega_4(y) = \omega_2(y)$.

Otherwise with $h+s < l$, by (3.1) we may apply Theorem 2 to the L.H.S. of (10.4), giving $\omega_3(x) = l-s$ and $\omega_4(x) = h+s$.

Consider the element y.

<u>Case</u> 1. $\omega_2(y) \leqslant s + \omega_1(y)$. Now by (2.1) we deduce that $y \notin D$ and thus $\omega_3(y) = \omega_1(y)$ and $\omega_4(y) = \omega_2(y)$. In other words for

(10.5) $u+w \leqslant j' \leqslant v, \quad u+s+w \leqslant j'' \leqslant s+j',$

$N(h,j')N(l,j'') \leqslant N(l-s,j')N(h+s,j'').$

<u>Case</u> 2. $\omega_2(y) > s + \omega_1(y)$. Now if $y \notin D$ then for

(10.6) $u+w \leqslant j' \leqslant v, \quad s+j' < j'' \leqslant w',$

$N(h,j')N(l,j'') \leqslant N(l-s,j')N(h+s,j'').$

However if $y \in D$ then for (10.6) we have

$N(h,j')N(l,j'') \leqslant N(l-s,j''-s)N(h+s,j'+s).$

We must show that $\omega_3(y), \omega_4(y)$ belong to the specified intervals, namely

$u+w \leqslant -s+\omega_2(y) = \omega_3(y) \leqslant -s+w' \leqslant -s+v \leqslant v,$

$u+s+w \leqslant s+\omega_1(y) = \omega_4(y) < \omega_2(y) \leqslant w'.$

Also we can deduce that $\omega_4(y) = j'+s < s+j''-s = s+ \omega_3(y)$.

This means that when $y \in D$ we are mapping into the area given by (10.5). Hence we require

<u>Lemma</u> 4. If $i_1 + s < i_2$ and $j_1 + s < j_2$ then

$$N(i_1,j_1)N(i_2,j_2)N(i_1,j_2-s)N(i_2,j_1+s) \leqslant (N(i_2-s,j_2-s)N(i_1+s,j_1+s))^2.$$

Proof Apply Theorem 4 to the first pair and Theorem 2 to the second pair. □

Suppose in Case 1 that $\delta(\omega_1,\omega_2) = (\omega_3,\omega_4,D)$ with $y \notin D$ and in Case 2 that $\delta(\omega_1',\omega_2') = (\omega_3',\omega_4',D')$ with $y \in D'$. Then Lemma 4 ensures that $(\omega_1,\omega_2) \neq (\omega_1',\omega_2')$ implies $(\omega_3,\omega_4) \neq (\omega_3',\omega_4')$.

We remark that similarly to the previous theorems we may extend this result to a fixed subset $x_1,\ldots,x_k$ in P. For $3 \leqslant h \leqslant k$ let $\alpha_h,\beta_h,\gamma_h,\delta_h \in Z^+$. Then in (10.1) we put $\omega(x_h) \in [\alpha_h,\beta_h]$ in the first and third factors, and $\omega(x_h) \in [\gamma_h,\delta_h]$ in the second and fourth factors. With $\delta_h \leqslant \alpha_h+s$ this follows using Theorems 2,4. □

The following shows the necessity for the condition $w' \leqslant v$ in Theorem 10.

Example 6. Let $P = \{x < p,\ y < p\}$. Then

$(2c-3)(2c-7) = N(1,[1,\ 2])N(3,[3,4]) \nleqslant N(2,2)N(2,[2,4]) = (c-2)(3c-9)$.

3. Order-Preserving Maps

We will employ a corresponding injection to δ in order to show that the preceding inequalities also hold for order-preserving maps.

Theorem 11. Theorems 2-10 hold with N replaced by N^{**}.

Proof The proofs follow a parallel course to those for strict order-preserving maps. For example, Cases 1 and 2 of Theorem 3 are modified as follows.

Case 1. $p < d$ and $\rho_3(p) > \rho_3(d)$.

By using $\rho_1 \in R$, (3.1) and the definition of ρ_3 we get
$\rho_1(p) \leq \rho_1(d) < -s+\rho_2(d) = \rho_3(d) < \rho_3(p) = \rho_1(p)$. Since this is impossible this case cannot arise.

Case 2. $p < d$ and $\rho_4(p) > \rho_4(d)$.

By using $\rho_1, \rho_2 \in R$ and the definition of ρ_4 we get
$1 \leq \rho_1(p) \leq \rho_1(d) = -s+\rho_4(d) < -s+\rho_4(p) = -s+\rho_2(p) \leq -s+c$.
Hence d forces p to join D, so let $D' = D \cup p$. Notice that (3.1), (3.3) hold for $p \in D'$.

(See [DDP] for the analogue of Theorem 1 for order-preserving maps). □

Notice that Examples 1-6 serve the same purpose in this section as for strict order-preserving maps because the result in each example is the same although some numerical values are different.

4. Linear Extensions

We would not expect an immediate analogue of Theorem 3 for linear extensions and the following example supports this view.

Example 7. Let $P = \{p_1 < x, y < p_2\}$. Then

$$1.2 = N^*(2,3)N^*(4,1) \nleq N^*(3,2)^2 = 1^2.$$

The following theorems appear in [D].

Theorem 12. Let $x,y \in P$. If $N^*(i_1,i_2) \neq 0$ and $N^*(i_1+2,i_2+2) \neq 0$ then $N^*(i_1+1,i_2+1) \neq 0$.

Theorem 13. Let $x,y \in P$. If $i_1 \neq i_2$ and $N^*(i_1,i_2+2) \neq 0$ and $N^*(i_1+2,i_2) \neq 0$ then $N^*(i_1+1,i_2+1) \neq 0$.

Theorem 14. Let ν_c^* be the total number of order-preserving injections from P into C. Then $\nu_1,\nu_2,\ldots$ is log concave and strict increasing.

The next example shows that the special case $N(r,u+w)N(r+t,u) \leqslant N(r+t,u+w)N(r,u)$ of Theorem 2 does not hold for linear extensions.

Example 8. Let $P = \{x < p, y\}$. Then

$$1.1 = N^*(1,2)N^*(2.1) \nleq N^*(2,2)N^*(1,1) = 0.$$

We also mention

Example 9. Let $P = \{y < p_1 < x, p_2 < p_3\}$. Then

$$1.3 = N^*(4,2)N^*(5,1) < N^*(5,2)N^*(4,1) = 2.2.$$

Corresponding examples for linear extensions to Examples 1-6 are given in [D].

Question 1. Does Theorem 10 hold for linear extensions?

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