Research report 160

NOTES ON THE SEPARABILITY IN METRIC SETS

STEPHEN G MATTHEWS

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These notes consider the various topologies over metric sets generated by the selection of the metric axioms given below. In particular we study the sparability properties of quasi and partial metric spaces.

Department of Computer Science
University of Warwick
Coventry CV4 7AL
United Kingdom

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Notes on Separability in Metric Sets

Stephen G. Matthews

Dept. of Computer Science
University of Warwick
Coventry, CV4 7AL
UK

email sgm@uk.ac.warwick.cs

ABSTRACT

These notes consider the various topologies over metric sets generated by a selection of the metric axioms given below. In particular we study the separability properties of quasi & partial metric spaces.

Introduction

A *metric set* is defined to be a set S together with a function $d : S \times S \rightarrow [0,\infty)$ which satisfies a selection of the axioms given in the next section. Such d are called *generalised metrics*, and have been studied by mathematicians [Ko88] as well as computer scientists. de Bakker & Zucker [dB&Z82] use ordinary metric spaces to provide an alternative to partial order fixed point semantics. This approach uses Banach's unique fixed point theorem instead. Smyth [Sm87] generalises symmetry by removing the axiom (M6) in order to combine both partial order & metric semantics (see below). In [Ma85] a generalised metric was used to reason about the *completeness* of data objects by generalising the reflexive axiom (M1) to get (M2). These notes are a study of the seperability properties of such generalisations.

Axioms for Metric Sets For any $x , y , z \in S$,

$(M1)$ $\quad d(x,y) = 0 \Leftrightarrow x = y$ $\quad$ (*reflexivity*)

$(M2)$ $\quad d(x,y) = 0 \Rightarrow x = y$ $\quad$ (*partial reflexivity*)

$(M3)$ $\quad d(x,x) = d(x,y) = d(y,y) \Leftrightarrow x = y$ $\quad$ (*iso reflexivity*)

$(M4)$ $\quad x = y \Rightarrow d(x,y) = 0$ $\quad$ (*pseudo reflexivity*)

$(M5)$ $\quad d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$ $\quad$ (*Smyth reflexivity*)

$(M6)$ $\quad d(x,y) = d(y,x)$ $\quad$ (*symmetry*)

$(M7)$ $\quad d(x,z) \leq d(x,y) + d(y,z)$ $\quad$ (*transitivity*)

$(M8)$ $\quad d(x,z) \leq \max\{ d(x,y) , d(y,z) \}$ $\quad$ (*ultra transitivity*)

A *metric* satisfies axioms M1, M6, & M7. An *ultrametric* satisfies axioms M1, M6, & M8. A *quasimetric* satisfies axioms M1 & M7. A *Smyth-metric* satisifes axioms M5 & M7. A *semimetric* satisfies axioms M1 & M6. A *pseudometric* satisfies axioms M4, M6, & M7. A *partial-metric* satisifies axioms M2, M6, & M7. A *partialultrametric* satisfies axioms M2, M6, & M8. An *iso-metric* satisfies axioms M3, M6, & M8.

Separation Properties

The separation properties to be used here for a toplogical space $< S , \tau >$ are,

$$(T_0) \quad \forall\ x \neq y \in S \ .\ \exists\ G \in \tau \ .\ x \in G \text{ and } y \notin G$$
$$\text{or } \exists\ G' \in \tau \ .\ y \in G' \text{ and } x \notin G'$$

$$(T_1) \quad \forall\ x \neq y \in S \ .\ \exists\ G \in \tau \ .\ x \in G \text{ and } y \notin G$$
$$\text{and } \exists\ G' \in \tau \ .\ y \in G' \text{ and } x \notin G'$$

$$(T_2) \quad \forall\ x \neq y \in S \ .\ \exists\ G , G' \in \tau \ .$$
$$x \in G \text{ and } y \in G' \text{ and } G \cap G' = \phi$$

These properties hold for τ iff they hold for any base for τ.

Metric Spaces

First we consider the standard *open ball* topology $\tau_d \subseteq 2^S$ induced by a metric d on a set S †. τ_d has the basis,

$$\{\ B_\varepsilon(x) \mid x \in S \text{ and } \varepsilon > 0\ \}$$

where the ε-ball $B_\varepsilon(x)$ is defined to be,

$$\{\ y \in S \mid d(x,y) < \varepsilon\ \}$$

To see that the ε-balls do indeed form a basis for a topology use the following standard result [Si76].

Theorem (Base Theorem)
Let $S \neq \phi$, and $\Omega \subseteq 2^S$. Then Ω is a base for a toplogy on S iff, $S = \bigcup \Omega$, and,

$$\forall\ B_1 , B_2 \in \Omega \ .\ x \in B_1 \cap B_2 \quad => \quad \exists\ B_3 \in \Omega \ .\ x \in B_3 \subseteq B_1 \cap B_2$$

Theorem
The open balls induced by a metric form a base for a topology.
Proof: first note that (in a metric space) any ε-ball $B_\varepsilon(x)$ contains x, and so, $S = \bigcup \Omega$. Secondly, note that, if

$$z \in B_\varepsilon(x) \cap B_\delta(y)$$

then,

$$z \in B_\gamma(z) \subseteq B_\varepsilon(x) \cap B_\delta(y)$$

where,

$$\gamma ::= \min\{\ \varepsilon - d(x,z)\ ,\ \delta - d(y,z)\ \}$$

† for a standard introduction to metric spaces see [Su75].

□

The open ball topology for metric spaces is T_2 (i.e. Hausdorff), as for any $x \neq y \in S$,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \in B_\varepsilon(y) \quad \text{and} \quad B_\varepsilon(x) \cap B_\varepsilon(y) = \phi$$

where,

$$\varepsilon ::= \frac{d(x,y)}{2}$$

In fact, the separability of metric spaces is stronger than T_2, however, for our purposes we do not need to consider any notion of separability stronger than T_2. The purpose of these notes is to consider the questions of whether d generates a topology when the metric axioms are relaxed, and, if so, what are the resultant separation properties.

Quasimetric Spaces

Using the same proof as above for metrics, we can show that a quasimetric induces an open ball topology†. An example of a quasimetric space is the unit interval $<[0,1], d>$, where,

$$d(x,y) ::= \begin{cases} y - x & \text{if } x \leq y \\ 1 & \text{if } y < x \end{cases}$$

In this example the ε-balls have the forms,

$$B_\varepsilon(x) = \{ y \mid x \leq y < x + \varepsilon \} \qquad (\varepsilon \leq 1)$$

and,

$$B_\varepsilon(x) = [0,1] \qquad (\varepsilon > 1)$$

Under this topology the unit interval is actually T_2, as, if $x < y$, and,

$$\varepsilon ::= \frac{y - x}{2}$$

then $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint intervals containing x and y respectively. However, the separability properties of quasimetric spaces are a little different from those of metric spaces.

Theorem

Quasimetric Spaces are T_1.

Proof: Suppose $x \neq y \in S$. Then (by M1), $d(x,y) \neq 0$, and $d(y,x) \neq 0$.

And so,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \notin B_\varepsilon(x)$$

where,

$$\varepsilon ::= d(y,x)$$

Similarly,

$$y \in B_\delta(y) \quad \text{and} \quad x \notin B_\delta(y)$$

where,

$$\delta ::= d(x,y)$$

† Quasimetrics are defined in [Si76]

Thus quasmiteric spaces are T_1. □

Theorem
Any finite quasimetric space is T_2.
Proof: Suppose,

$$S = \{ x_1 , x_2 , \cdots x_n \}$$

For each $1 \leq i \leq n$, let

$$\varepsilon_i ::= \frac{\min \{ d(x_i, y) \mid y \in S - \{x_i\} \}}{2}$$

Then,

$$\forall \; 1 \leq i \leq n \quad . \quad x_i \in B_{\varepsilon_i}(x_i)$$

and,

$$\forall \; i \neq j \quad . \quad B_{\varepsilon_i}(x_i) \cap B_{\varepsilon_j}(x_j) = \phi$$

Thus, finite quasimetric spaces are T_2. □

Theorem
Not all Quasimetric spaces are T_2.
Proof: Consider the quasimetric space $< S , d >$ where,

$$S := \{ 0 , 1 , 2 , \ldots \}$$

and,

$$d : S \times S \rightarrow \{ 1 , 1/2 , 1/3 , 1/4 , \ldots , 0 \}$$

where,

$$\forall \; x , y \in S \quad . \quad d(x, y) := \begin{cases} 0 & x = y \\ 1/(y+2) & x < y \\ 1 & x > y \end{cases}$$

Suppose that $x \neq y \in S$, and that $\varepsilon , \delta > 0$.
Let,

$$\gamma := \min \{ \varepsilon , \delta \}$$

Then we can always find $z \in S$ such that,

$$x < z \quad \text{and} \quad y < z \quad \text{and} \quad \frac{1}{z+2} < \gamma$$

that is, such that,

$$z \in B_\varepsilon(x) \cap B_\delta(y)$$

Thus this space cannot be T_2. □

Smyth-metric Spaces

Mike Smyth has studied a weaker form of quasimetric space which we shall call a *Smyth-metric Space*. These spaces have the same open ball topology as quasi spaces. A Symth-metric is Smyth reflexive (M5) and transitive (M7). The earlier example of the quasi metric over the unit interval is Smyth as it is quasi. Two examples of Smyth-metrics which are not quasi are the following. A partial order $\leq$ on a set S is equivalent to the Smyth-metric,

$$d(x, y) \quad ::= \quad \begin{cases} 0 & \text{if } x \leq y \\ 1 & \textit{otherwise} \end{cases}$$

The usual open ball topology is the *Alexandroff topology,*

$$\{ \uparrow A \mid A \subseteq S \}$$

where the *upper closure* $\uparrow A$ of a set $A \subseteq S$ is given by,

$$\uparrow A \quad ::= \quad \{ y \in S \mid \exists x \in A \,.\, x \leq y \}$$

To see this note that for each $x \in S$ and $\varepsilon > 0$,

$$B_\varepsilon(x) \quad = \quad \begin{cases} \uparrow\{x\} & \text{if } \varepsilon \leq 1 \\ \uparrow S & \text{if } \varepsilon > 1 \end{cases}$$

and that for each $A \subseteq S$,

$$\uparrow A \quad ::= \quad \cup\{ B_1(x) \mid x \in A \}$$

This example is Smyth but not quasi as, if $x < y$, then $d(x, y) = 0$ does not imply $x = y$.

Another example is a Smyth-metric on Σ^∞, the set of all finite and infinite sequences over an alphabet Σ under the initial segment ordering.

$$d(x, y) \quad ::= \quad \begin{cases} 0 & \text{if } x \leq y \\ 2^{-|y|} & \text{if } y < x \\ 2^{-n} & \textit{otherwise}, \ \textit{where} \ n ::= \min\{ n \mid x_n \neq y_n \} \end{cases}$$

Here the open ball topology is a refinement of the Alexandroff topology, and has the following as a base.

$$\{ \uparrow\{x\} \mid x \in S \text{ and } |x| < \infty \}$$

This is an example of a Scott topology. A *Scott topology* τ on a set S induced by a complete partial order $\leq$ on S is a sub-topology of the Alexandroff topology such that,

$$\forall \ G \in \tau \,.\, \forall \ X \subseteq S \,.\, \ \exists \cap X \ \Rightarrow \ X \cap G \neq \phi$$

Theorem

Every Symth space is T_0.

Proof: Suppose $x \neq y$ in a Smyth space $< S, d >$. Then,

$$d(x, y) \neq 0 \quad \textit{or} \quad d(y, x) \neq 0$$

Suppose (wlog) that $d(x, y) \neq 0$. Then,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \notin B_\varepsilon(x)$$

where,

$$\varepsilon \quad ::= \quad d(x, y)$$

Thus every Smyth space has been shown to be T_0. □

Theorem

A Smyth space is quasi iff it is T_1.
Let $< S , d >$ be a Smyth space, and suppose $x \neq y \in S$.
First, suppose $< S , d >$ is quasi, then by M1,

$$d(x,y) \neq 0 \quad \text{and} \quad d(y,x) \neq 0$$

thus,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \notin B_\varepsilon(x) \quad \text{and}$$
$$y \in B_\delta(y) \quad \text{and} \quad x \notin B_\delta(y)$$

where,

$$\varepsilon \ ::= \ d(x,y) \quad \text{and} \quad \delta \ ::= \ d(y,x)$$

Thus $< S , d >$ has been shown to be T_1.
Suppose now that the Smyth space $< S , d >$ is T_1, then,

$$\exists \ B_\varepsilon(v) \ . \ x \in B_\varepsilon(v) \ \text{and} \ y \notin B_\varepsilon(v)$$

but,

$$d(v,y) \leq d(v,x) + d(x,y)$$

thus $d(x,y) \neq 0$.
Thus it has been shown that, $x \neq y \Rightarrow d(x,y) \neq 0$, i.e. M2. Thus M1 follows from M5.
Thus $< S , d >$ has been shown to be quasi. □

This last theorem shows that quasimetric & Smyth-metric are equivalent notions if we are working in a universe of T_1 spaces, however, when we come to discuss Scott topologies we are in general dealing with T_0 spaces.

Ultrametric Spaces

A non-archimedean space (introduced by A.F. Monna [M50]) is one for which there exists a basis Ω such that,

$$\forall \ B_1 , B_2 \in \Omega \ . \ B_1 \subseteq B_2 \ \ or \ \ B_2 \subseteq B_1 \ \ or \ \ B_1 \cap B_2 = \phi$$

It can easily be shown that the ultrametric spaces are non-archimedean.

Partial-metrics

A *partial-metric* [Ma85] is partially reflexive, symmetric, and transitive (M2+M6+M7). It was introduced as a method of distinguishing between "partial" and "complete" objects in the context of programming language semantics. An object x is said to be *complete* if $d(x,x) = 0$, otherwise it is said to be partial. And so, this distinction was introduced as the "partial reflexive" axiom. We shall see later that, for separability reasons, this decision needs to be refined. First, consider the following *partial-metric* on Σ^∞, the set of all finite and infinite sequences over the alphabet Σ.

$$d(x,y) \ ::= \ \begin{cases} 2^{-|x|} & \text{if } x \leq y \\ 2^{-M} & otherwise \end{cases}$$

where, $\leq$ is the initial segment ordering on Σ^∞, and,

$$\min\{ n \mid x_n \neq y_n \} \ ::= \ \min\{ \ n \ \mid \ x_n \neq y_n \ \}$$

Here the complete objects are precisely the infinite sequences. $< \Sigma^{\infty}, d >$ can be used to model dataflow semantics [Ka74] [Wa85]. This partial metric is in fact a *partial-ultrametric*, as d is ultra transitive (i.e. M8). An intuitively pleasing property of partial-ultrametrics is that no point can be closer to another than the point itself, i.e. for all $x, y \in S$,

$$d(x, x) \leq d(x, y)$$

as d is ultra transitive. Not all partial-metrics are ultra transitive, e.g. consider the following partial metric defined on the set $\{ a, b \}$.

$$d(a, a) ::= 1 \quad , \quad d(a, b) ::= ½ \quad , \quad d(b, b) ::= 0$$

Theorem
The set of open balls defined by a partial ultrametric forms a base for the usual open ball topology, also, this topology is non-archimedean.
Proof: We use the "Base Theorem" employed earlier to show that the collection of all open balls forms a base. Suppose that d is a partial-ultrametric on S. First note that S is the union of all the balls as, for each $x \in S$,

$$x \in B_{\varepsilon}(x)$$

where,

$$\varepsilon ::= d(x, x) + 1$$

Instead of showing that,

$$z \in B_1 \cap B_2 \quad => \quad \exists B_3 \; . \; z \; B_3 \subseteq B_1 \cap B_2$$

For any open balls B_1, B_2, and B_3, we prove the stronger non-archimedean property,

$$B_1 \subseteq B_2 \quad or \quad B_2 \subseteq B_1 \quad or \quad B_1 \cap B_2 = \phi$$

Suppose that,

$$B_{\varepsilon}(x) \cap B_{\delta}(y) \neq \phi$$

and that (wlog) $\varepsilon \leq \delta$. Then we can choose,

$$z \in B_{\varepsilon}(x) \cap B_{\delta}(y)$$

Suppose $p \in B_{\varepsilon}(x)$, then (as d is ultra transitive),

$$d(y, p) \leq \max \{ d(y, z) , d(z, x) , d(x, p) \}$$

thus, $d(y, p) < \delta$,
thus, $p \in B_{\delta}(y)$,
thus, $B_{\varepsilon}(x) \subseteq B_{\delta}(y)$,
Thus we have shown that a partial-ultrametric defines a non-archimedean open ball topology. □

Corollary
Ultrametric spaces are non-archimedean.

These notes are an investigation into the separability properties of particular metric sets. They are motivated by a belief that certain metric sets may be useful in programming language semantics. To this end we are assuming that any such set must be at least T_0. That is, any two distinct points must be separable by some open set, where we understand an open set to be a *finite property*. Unfortunately, partial-ultrametric spaces are not always T_0. For example, consider the following partial-ultrametric on the set $\{ a , b \}$.

$$d(x,x) ::= d(x,y) ::= d(y,y) ::= 1$$

However, we can define the following equivalence relation on partial-ultrametric spaces.

$$x \equiv y \quad ::= \quad d(x,x) = d(x,y) = d(y,y)$$

Theorem
Two points in a partial-ultrametric space are equivalent iff they are inseparable (i.e. every open set containing one point must contain the other).
Proof: We have to show that for any $x , y \in S$,

$$x \equiv y \quad iff \quad \forall\, z \in S\ ,\ \varepsilon > 0 \ .\ x \in B_\varepsilon(z) \Rightarrow y \in B_\varepsilon(z)$$
$$\text{and} \quad y \in B_\varepsilon(z) \Rightarrow x \in B_\varepsilon(z)$$

Suppose first that $x \equiv y$, and that $x \in B_\varepsilon(z)$.
then, $d(x,x) < \varepsilon$ as $d(x,x) \le d(x,z)$.
Thus, $d(x,y) < \varepsilon$ as $d(x,x) = d(x,y)$.
Now, $d(z,y) \le \max\{ d(z,x) , d(x,y) \}$.
Thus, $d(z,y) < \varepsilon$.
Thus, $y \in B_\varepsilon(z)$.
Similarly, we can show that,

$$\forall\, z \in S\ ,\ \varepsilon > 0 \ .\ y \in B_\varepsilon(z) \Rightarrow x \in B_\varepsilon(z)$$

Thus, x and y have been shown to be inseparable.
Now for the second half of the theorem. Suppose that x and y are inseparable. Then,

$$\forall\, z \in S\ ,\ \varepsilon > 0 \ .\ x \in B_\varepsilon(z) => y \in B_\varepsilon(z)$$

but, if $d(x,x) \ne d(x,y)$ then $d(x,x) < d(x,y)$, giving,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \notin B_\varepsilon(x)$$

where, $\varepsilon ::= d(x,y)$, a contradiction. And so, $d(x,x) = d(x,y)$.
Similarly we can show that $d(y,y) = d(y,x)$, and so, we have shown that $x \equiv y$. □

If we regard $\equiv$ as *computational equivalence*, then the topology can be regarded as one describing the computational structure of some class of objects. However, if the topology is to be one describing the structure of a class of computable objects then it is reasonable to assume that objects are completely specified by their computability properties. And so, we consider a more restrictive version of partial-ultrametric spaces called *iso-metric* spaces, where the reflexive axiom M2 is tightened to get M3,

$$x = y \quad \Leftrightarrow \quad d(x,x) = d(x,y) = d(y,y)$$

We can thus define the following partial order for iso-metric spaces.

$$x \le y \quad ::= \quad d(x,x) = d(x,y)$$

Theorem

The open ball topology τ_d induced by a partial-ultrametric d is a sub-topology of the Alexandroff topology τ_A .
Proof: Suppose $< S , d >$ is a partial-ultrametric space.
We show that,

$$\forall \ G \in \tau_d \quad . \quad G = \cup \{ \uparrow \{ x \} \mid x \in G \}$$

Suppose $x , y \in S$ & $G \in \tau_d$ are such that $x \le y$ and $x \in G$.
Then there exist $z \in S$ and $\varepsilon > 0$ such that,

$$x \in B_\varepsilon(z) \subseteq G$$

Now, $d(x , x) < \varepsilon$ as $d(x , x) \le d(x , z) < \varepsilon$.
thus, $d(y , x) < \varepsilon$ as $x \le y$.
Thus, $d(y , z) \le \max \{ d(y , x) , d(x , z) \} < \varepsilon$.
Thus, $y \in B_\varepsilon(z)$, and so $y \in G$.
Thus, $G \in \tau_A$. □

Theorem
Not every iso-metric space has the full Alexandroff topology.
Proof: In the iso-metric space $< \Sigma^\infty , d >$ given above the singleton sets consisting of maximal objects (i.e. the infinite sequences) are open in the Alexandroff topology, but not in the open ball topology. □

Theorem
In an *iso-poset* (i.e. a poset with an ordering derivable from an iso-metric) objects are consistent iff they are comparable. That is,

$$\forall \ x , y \quad . \quad (\exists \ z \ . \ x \le z \text{ and } y \le z) \Leftrightarrow x \le y \ \text{ or } \ y \le x$$

Proof: Comparability trivially implies consistency.
Suppose $x , y \in S$ are consistent objects in an iso-poset derived from an iso-metric d . Then we can choose $z \in S$ such that $x \le z$ and $y \le z$.
Thus,

$$d(x , x) = d(x , z) \quad \text{and} \quad d(y , y) = d(y , z)$$

Thus,

$$d(x , y) \le \max \{ d(x , z) , d(y , z) \} = \max \{ d(x , x) , d(y , y) \}$$

Thus,

$$d(x , y) = d(x , x) \quad \text{or} \quad d(x , y) = d(y , y)$$

Thus, $x \le y$ or $y \le x$. □
Note that the consistency relation is not transitive as the following example shows.

$$d(y , y) = d(z , z) = 0 \quad , \quad d(y , z) = 1/2$$

$$d(y , x) = d(x , x) = d(x , z) = 1$$

y is not consistent with z although x is consistent with y , and y is consistent with z . In general though consistency is relfexive and symmetric.

For each point x , $\downarrow \{ x \}$ is a total order.

Theorem

Iso-metric spaces are T_0.

Proof: Suppose that $< S , d >$ is an iso-metric space, and that $x \neq y \in S$. Then (by M3),

$$d(x,x) < d(x,y) \quad or \quad d(y,y) < d(x,y)$$

Suppose (wlog) $d(x,x) < d(x,y)$.

Then,

$$x \in B_\varepsilon(x) \quad \text{and} \quad y \notin B_\varepsilon(x)$$

where, $\varepsilon ::= d(x,y)$. □

Theorem

An iso-metric space $< S , d >$ is T_1 iff

$$\forall\, x , y \in S \quad . \quad x \leq y \quad \Leftrightarrow \quad x = y$$

Proof: as for all $x < y \in S$, and balls $B_\varepsilon(z)$,

$$x \in B_\varepsilon(z) \quad \Rightarrow \quad y \in B_\varepsilon(z)$$

□

Corollary

Any T_1 iso-metric space is T_2.

Iso-metric spaces are *first countable*, that is, there is a countable local base for each point, e.g. for each $x \in S$, the set of all ε-balls for rational ε forms a local base at x .

A topological space is said to be *separable* if it has a countable dense subset, while it is said to be *second countable* if it has a countable base. A space $< S , \tau >$ is said to be *first countable* if for each $x \in S$ there exists a *local base* $\tau_x \subseteq \tau$ such that,

$$\forall\, G \in \tau \quad . \quad \exists\, G' \in \tau_x \quad . \quad x \in G' \subseteq G$$

A standard result is that every second countable topological space is separable. Suppose that $\{ B_n \mid n \geq 0 \}$ is a base for a second countable space. Let, $\{ x_n \mid n \geq 0 \}$ be such that for each $n \geq 0$, $x_n \in B_n$. Then it can be shown [Si76] that this is a countable dense subset.

Theorem

Iso-metric spaces are first countable.

Proof: let $x \in S$ be a point in an iso-metric space $< S , \tau >$. Then it can be shown that,

$$\{ B_q(x) \mid q \in Q^{\dagger} \text{ and } q > 0 \}$$

is a local base for x . □

† "Q" denotes the set of all non-negative rational numbers. We assume that "q" always denotes such a rational.

Theorem
A set $A \subseteq S$ in an isometric space $< S , \tau >$ is dense in S (i.e. $\bar{A} = S$) iff

$$\forall \; x \in S \;, \; \varepsilon > 0 \;\; . \;\; \exists \; a \in A \;\; . \;\; d(x , a) \; < \; d(x , x) + \varepsilon$$

Proof: Note the following standard property of any topological space with a base Ω.

$$x \in \bar{A} \quad \textit{iff} \quad \forall \; B \in \Omega \;\; . \;\; x \in B \;\Rightarrow\; A \cap B \neq \phi$$

For open ball metric set topologies this is equivalent to saying that for any $x \in S$ and $B_\delta(y) \in \tau$,

$$x \in B_\delta(y) \quad \Rightarrow \quad \exists \; a \in A \;\; . \;\; a \in B_\delta(y)$$

Suppose first that A is dense in S.
Let $x \in S$, and $\varepsilon > 0$.
Let, $\delta \;::=\; d(x , x) + \varepsilon$.
Then, $x \in B_\delta(x)$.
Thus we can choose $a \in A$ such that $a \in B_\delta(x)$.
Thus $d(a , x) < \delta$ i.e. $d(a , x) < d(x , x) + \varepsilon$.
Thus the first part of the theorem is proved.
Suppose now that the following condition in the theorem holds.

$$\forall \; x \in S \;, \; \varepsilon > 0 \;\; . \;\; \exists \; a \in A \;\; . \;\; d(a , x) \; < \; d(x , x) + \varepsilon$$

We will show that A is dense in S.
Suppose that for some $x \in S$ and $B_\delta(y)$, $x \in B_\delta(y)$.
Let $\varepsilon \;::=\; \delta - d(x , x)$.
Then we can choose $a \in A$ such that,

$$d(a , x) \; < \; d(x , x) + \varepsilon$$

i.e. $d(a , x) < \delta$.
but,

$$d(a , y) \; \leq \; \max\{ \, d(a , x) \;,\; d(x , y) \, \}$$

Thus, $d(a , y) < \delta$, and so, $a \in B_\delta(y)$. □

Convergence
In [Ma85] we used a partial-metric (i.e. a metric set with M2, M6, & M7) to generalise Banach's contraction mapping theorem. For this the following notion of convergence used in metric spaces was sufficient.

$$\lim_{n \to \infty} d(X_n , y) \quad = \quad 0$$

This served the purpose for sequences whose limits are intended to be complete i.e. $d(y , y) = 0$. In these notes we generalise the metric space notion of convergence to isometric spaces so that it is the same as topological convergence, and so allows convergence to partial limits as well as complete ones.

A sequence $X \in {}^\omega S$ is said to *converge* to a point $y \in S$ in a topological space $< S , \tau >$ if,

$$\forall \; G \in \tau \;\; . \;\; y \in G \;\Rightarrow\; \exists \; k \;\; . \;\; \forall \; n > k \;\; . \;\; X_n \in G$$

For an iso-metric space $< S, d >$ this is equivalent to,

$$\forall\ \varepsilon > 0 \quad . \quad \exists\, k \quad . \ \forall\ n > k \quad . \quad d(X_n, y) \ < \ d(y, y) + \varepsilon$$

i.e.

$$\lim_{n \to \infty} d(X_n, y) \quad = \quad d(y, y)$$

All T_2 spaces have unique limit points, however, the following example shows that not all T_1 spaces have unique limits.
Let,

$$< \{ 0, 1, 2, \ \ldots \ \}, \ \tau >$$

be the T_1 space with base,

$$\{ \ \{0\} \cup \{ \ n \ \mid n \ge k \ \} \ \mid \ n \ge 2 \ \}$$

$$\cup \{ \ \{1\} \cup \{ \ n \ \mid n \ge k \ \} \ \mid \ n \ge 2 \ \}$$

$$\cup \{ \ \{ \ n \ \mid n \ge k \ \} \ \mid \ n \ge 2 \ \}$$

In this T_1 topology the sequence $\lambda\, n \ . \ n + 2$ converges to both 0 and 1.

The set of limit points of a space X is denoted $lim(X)$. Although the above definition of convergence for iso-metrics is the natural anologue to the usual one for metric spaces it's properties are weaker. For example, as metric spaces are T_2 limits of sequences are unique, however, iso-metric spaces are in general only T_0. The following example shows that limits of sequences in iso-metric spaces are not necessarily unique. Let $< \{ a, b \}, d >$ be such that,

$$d(a, a) \ = \ d(a, b) \ = \ 1 \quad , \quad d(b, b) \ = \ 0$$

(i.e. $a < b$). Let $X ::= \lambda n \, . \, b$. Then X converges to both a and b. We do, however, have the following results to relate limit points.

Theorem
For each sequence X in a topological space, $lim(X)$ is closed.
Proof: Suppose X is a sequence in a topological space $< S, \tau >$, and that $y' \in S \ \& \ G \in \tau$ are such that,

$$y \ \in \ \overline{lim(X)} \cap G$$

Then $lim(X) \cap G \ \ne \ \phi$.
Thus we can choose $y \ \in \ lim(X) \cap G$.
Thus $\exists\ k \ . \ \forall\ n > k \quad . \ X_n \in G \quad$, as $y \in lim(X)$.
Thus $y' \in lim(X)$.
Thus $\overline{lim(X)} \ \subseteq \ lim(X)$.
Thus $\overline{lim(X)} \ = \ lim(X)$. □

Theorem
Two points belonging to the same limit set are consistent.
Proof: Suppose $y, y' \in lim(X)$.
Then (wlog) we can assume that $d(y, y) \ge d(y', y')$.
Let $\varepsilon > 0$. Then we can choose k such that for each $n > k$,

$$d(X_n, y) < d(y, y) + \varepsilon \quad \text{and} \quad d(X_n, y') < d(y', y') + \varepsilon$$

thus for each $n > k$,

$$\begin{aligned} d(y, y') &\leq \max\{ d(y, X_n), d(X_n, y') \} \\ &< \max\{ d(y, y) + \varepsilon, d(y', y') + \varepsilon \} \\ &= d(y, y) + \varepsilon \end{aligned}$$

Thus, $d(y, y) = d(y, y')$ as $d(y, y) \leq d(y, y')$.
Thus $y \leq y'$, and so y and y' are consistent. □

Theorem
x and y are consistent iff they have a common limit set i.e. if there exists an X such that $x, y \in lim(X)$.
Proof: The previous theorem shows that any pair of points in a common limit set must be consistent. Suppose that x and y are consistent, then (wlog) $x \leq y$. It just remains to show that x and y have a common limit set.
Let $X ::= \lambda\, n\, .\, y$.
Then it can be shown that $x, y \in lim(X)$. □

Theorem
Any approximation to a limit is also a limit i.e. if $y' \in lim(X)$ and $y \leq y'$, then $y \in lim(X)$, i.e. $lim(X) = \downarrow lim(X)$.
Proof: Suppose $y' \in lim(X)$, and $\varepsilon > 0$.
Then we can choose k such that,

$$\forall\, n > k \quad . \quad d(X_n, y') < d(y', y') + \varepsilon$$

Suppose also that y is such that $y \leq y'$.
Then, $d(y, y) = d(y, y') \geq d(y', y')$.
Thus, for each $n > k$,

$$\begin{aligned} d(X_n, y) &\leq \max\{ d(X_n, y'), d(y', y) \} \\ &\leq \max\{ d(y', y') + \varepsilon, d(y, y) \} \\ &\leq \max\{ d(y, y) + \varepsilon, d(y, y) \} \\ &= d(y, y) + \varepsilon \end{aligned}$$

Thus, $y \in lim(X)$. □

Theorem
If the lub of a set (in an iso-metric space) exists then the set is a total order, and so is directed (i.e. every finite subset has a lub).

The range of limits which may exist in a limit set can be deceptively large, and so of little use. For example, if an iso-metric space has a least element then every sequence converges to that least element. If the ordering is to be understood as an information ordering then limits should preserve information content, i.e. if $y \in lim(X)$ then we should also insist on,

$$\exists \lim_{n \to \infty} d(X_n, X_n) = d(y, y)$$

in other words w also need,

$$\forall\ \varepsilon > 0\ .\ \exists\ N\ .\ \forall\ n > N\ .\ d(y,y) - \varepsilon < d(X_n, X_n) < d(y,y) + \varepsilon$$

We say that y is a *proper limit* of X. A sequence having a proper limit is said to be *properly convergent.*

Theorem
Proper limits are unique.
Proof: Suppose that y , $y' \in lim(X)$ are both proper limits.
Then (by a previous result) y and y' are comparable.
if $y = y'$ there is nothing to prove, thus suppose (wlog) that $y < y'$.
Then $d(y,y) > d(y',y')$.
Thus y and y' cannot both be proper limits for X , a contradiction. □

Theorem
If a sequence X properly converges to y then $y = lub(lim(X))$.
Proof:
Suppose that X properly converges to y . Suppose also that $y' \in lim(X)$.
We show that $y' \leq y$.
Suppose this is not the case, then $y \leq y'$.
Now, let $\varepsilon > 0$, then we can choose N such that for each $n > N$,

$$d(X_n, y) < d(y,y) + \varepsilon$$

and,

$$d(y,y) - \varepsilon < d(X_n, X_n) < d(y,y) + \varepsilon$$

And so,

$$\begin{aligned} d(y,y) - \varepsilon &< d(X_n, X_n) \\ &\leq d(X_n, y') \\ &\leq \max\{\ d(X_n, y)\ ,\ d(y, y')\ \} \\ &< \max\{\ d(y,y) + \varepsilon\ ,\ d(y,y)\ \} \\ &= d(y,y) + \varepsilon \end{aligned}$$

Thus,

$$\lim_{n \to \infty} d(X_n, y') = d(y,y)$$

Thus, $d(y,y) = d(y',y')$.
Thus $y = y'$. □

Theorem
Not every convergent sequence in an iso-metric space is properly convergent.
Proof:
Consider the space of finite sequences over ω with the usual iso-metric. Every sequence converges to the null sequence, however, the sequence given by,

$$X_n ::= \begin{cases} <> & \text{if } n = 0 \\ <1, 2, \cdots, n> & \textit{otherwise} \end{cases}$$

X has no proper limit as there are no infinite sequences in this space. □

Proper limits give us a natural way to define the iso-metric analogue to the Cauchy sequence found in metric spaces. A *Cauchy Sequence* is a sequence X such that,

$$\exists \lim_{n,m \to \infty} d(X_n, X_m)$$

Theorem
Every chain in an iso-metric space is Cauchy
Proof:
Suppose X is a chain, and $n \leq m$.
Then $X_n \leq X_m$.
Thus, $d(X_n, X_n) \geq d(X_m, X_m)$.
Thus $\lambda i \ . \ d(X_i, X_i)$ is a decreasing sequence, and so converges as it is bounded below by 0.
But $d(X_n, X_m) = d(X_n, X_n)$,
Thus,

$$\exists \lim_{n,m \to \infty} d(X_n, X_m)$$

□

Theorem
Every properly convergent sequence is Cauchy.
Proof:
Suppose X is properly convergent, then,

$$\lim_{n \to \infty} d(X_n, y) = \lim_{n \to \infty} d(X_n, X_n) = d(y, y)$$

Let $\varepsilon > 0$, then we can choose N sucht that for each $n > N$,

$$d(X_n, y) < d(y, y) + \varepsilon$$

$$d(y, y) - \varepsilon < d(X_n, X_n) < d(y, y) + \varepsilon$$

Thus for all $n, m > N$,

$$\begin{aligned} d(y, y) - \varepsilon &< d(X_n, X_n) \\ &\leq d(X_n, X_m) \\ &\leq \max\{ d(X_n, y) , d(y, X_m) \} \\ &\leq d(y, y) + \varepsilon \end{aligned}$$

Thus,

$$\exists \lim_{n,m \to \infty} d(X_n, X_m)$$

□

Completeness
Unfortunately, the connection between the $\leq$-convergence properties of a chain and it's τ_d-convergence properties is rather weak. Consider the iso-metric space $< \omega^{\infty}, d >$ of all finite and infinite sequeces over ω .

$$d(x, y) \quad ::= \quad \begin{cases} 0 & \text{if } x = y \text{ , and } |x| = \infty \\ 1 + 2^{-n} & \textit{otherwise} \end{cases}$$

where $n ::= \min\{ m \mid x_m \neq y_m \}$.
Let $X : \omega \to \omega^{\infty}$ be defined by,

$$X_n \quad ::= \quad \begin{cases} <> & \text{if } n = 0 \\ < 1 , 2 , \cdots , n > & \textit{otherwise} \end{cases}$$

Let, $y ::= < 1 , 2 , \cdots >$. Then X is a chain $\leq$-converging to y . However, X does not τ-converge to y as

$$y \in B_1(y) \quad , \quad \text{and} \quad \forall\ n \geq 0 \quad . \quad X_n \notin B_1(y)$$

Thus chain convergence does not always imply topological convergence. The problem with this example is that the convergence of X is not proper, i.e.

$$\lim_{n \to \infty} d(X_n , X_n) \quad \neq \quad d(y , y)$$

The following two theorems show that if we restrict ourselves to proper convergence then we can describe chain completeness in terms of the proper convergence of chains.

Theorem
If a chain in an iso-metric space properly converges then the proper limit is an upper bound of the chain.
Proof:
Suppose X is a chain properly converging to y .
Let $n \geq 0$.
As X is a chain with proper convergence we can show that,

$$\lambda\ i \ .\ d(X_i , X_i)$$

decreases to y , and so,

$$d(y , y) \quad \leq \quad d(X_n , X_n)$$

Suppose first that,

$$d(y , y) \quad < \quad d(X_n , X_n)$$

Then as X converges to y we can choose $m > n$ such that,

$$d(y , y) \quad \leq \quad d(X_m , y) \quad \leq \quad d(X_n , X_n)$$

thus,

$$\begin{aligned} d(X_n , y) \quad &\leq \quad \max\{ d(X_n , X_m) , d(X_m , y) \} \\ &= \quad \max\{ d(X_n , X_n) , d(X_m , y) \} \\ &= \quad d(X_n , X_n) \end{aligned}$$

thus, $d(X_n , y) = d(X_n , X_n)$.
thus, $X_n \leq y$.
Suppose now that, $d(y , y) = d(X_n , X_n)$.
Let $m \geq n$, then,

$$d(X_m , y) \quad \leq \quad \max\{ d(X_n , X_m) , d(X_n , y) \}$$

$$
\begin{aligned}
&= \max\{\ d(\ X_n\ ,X_n\ ,)\ ,\ d(\ X_n\ ,y\ )\ \} \\
&= d(\ X_n\ ,y\ ) \\
&\le \max\{\ d(\ X_n\ ,X_m\ )\ ,\ d(\ X_m\ ,y\ )\ \} \\
&= \max\{\ d(\ X_n\ ,X_n\ )\ ,\ d(\ X_m\ ,y\ )\ \} \\
&= \max\{\ d(\ y\ ,y\ )\ ,\ d(\ X_m\ ,y\ )\ \} \\
&= d(\ X_m\ ,y\ )
\end{aligned}
$$

Thus,

$$\forall\ m \ge n \quad . \quad d(\ X_n\ ,y\ ) = d(\ X_m\ ,y\ )$$

thus,

$$d(\ X_n\ ,y\ ) = d(\ y\ ,y\ )$$

as X converges to y . Thus,

$$d(\ y\ ,y\ ) = d(\ X_n\ ,X_n\ ) = d(\ X_n\ ,y\ )$$

Thus, $X_n = y$, and so $X_n \le y$. □

We can now define the notion of completion for iso-metric spaces. A *complete iso-metric space* is one in which every Cauchy sequence properly converges. We thus have the following interesting result.

Theorem
Complete iso-metric spaces are chain complete.
Proof:
By previous results, every chain is Cauchy, and every properly converging chain is ≤-convergent.
□

Lucid & Iso-metrics

Much of this work on metric sets is motivated by the need to obtain a better understanding of proof theory for lazy dataflow languages such as the functional programming language *Lucid* [W&A85]. This language has sequences (i.e. mebers of ${}^{\omega}S$ as it's primitive data objects, and so suggests a metric treatment. However, topologically it is a T_0 space, and so requires a generalised metric. The Lucid model of computation is a hybrid. It uses both Kahn's model as well as lazy evaluation. Although operationally both models appear to be very different we can show that it does make sense to embed Kahn dataflow into lazy evaluation. Let D be our domain of atomic objects (think of D as ω). The *Kahn Domain* (i.e. the Kahn dataflow domain) is the poset $< D^{\infty}, \le >$ poset of all finite and infinite sequences over D under the initial segment ordering. let $D_{\perp}$ be the poset $< D \cup \{\perp\}\ ,\ \le >$, where,

$$x \le y \quad ::= \quad x = \perp \quad or \quad x = y$$

Then the *Lucid Domain* is the poset $<\ {}^{\omega}(D_{\perp})\ ,\ \le\ >$, where,

$$x \le y \quad ::= \quad \forall\ n \ge 0\ .\ x_n \le y_n$$

It is easy to embed the Kahn Domain into the Lucid Domain (wrt partial orderings) by,

$$i(x) \quad ::= \quad \begin{cases} x & |x| = \infty \\ x\ @ < \perp, \perp,\ \cdots > & otherwise \end{cases}$$

Operationally we can understand Kahn dataflow as a lazy evaluation algorithm with the additional constraint that each daton must be fully evaluated before we can begin evaluating the next one. Conversely, we can regard lazy evaluation as an extended form of dataflow in which we now allow datons to be evaluated in any order.

The *Cycle Sum Test* of Ashcroft & Wadge [W&A85] is used to prove that certain Kahn dataflow networks [Ka74] will not deadlock. A proof of this test was given in [Ma85] using the iso-metric approach, in essence a generalisation of the Banach Contraction Mapping Theorem to iso-metric spaces. We cannot however extend this test to the Lucid Domain, as it is not an iso-set, e.g. consider,

$$x \quad ::= \quad < \perp , 2 , 3 \ \cdots \ >$$

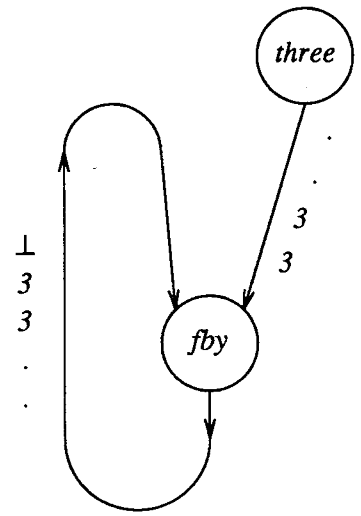
$$y \quad ::= \quad < 1 , \perp , 3 \ \cdots \ >$$

x and y are consistent but not comparable, and so this ordering is not derivable from an iso-metric. So can we redefine the ordering on $^{\omega}(D_{\perp})$ in order to make the Lucid Domain an iso-set ?

Lucid is a lazy language, and so has total results derived from partially defined objects, e.g.

$$\mathit{next}(\ x\ ) \qquad \textbf{where} \qquad x = x \ \mathit{fby}\ 3\ ; \qquad \textbf{end}$$

has a well defined result. In other words, As we are not interested in the first value of x it is reasonable to use a model of computation in which we can compute with undefined values i.e. compute with $\perp$. What we would like to be able to do is have an extended Kahn model in which "somehow" we have undefined elements in a network as oppossed to Kahn's model where deadlock occurs. For example, we need a lazy operational semantics for,



The *lazon* (lazy daton) $\perp$ is not produced by any node, but is somehow "produced" as the meet of all possible solutions to the network. Instead of regarding the meaning of the network as the limit of finite approximations generated by processes, we take it to be the meet of all possible solutions.

Further Work 1

The following question needs to be answered. Is there a non-trivial iso-metric for the Lucid Domain which induces a sub-ordering of the usual pointwise one ? If so, then we have a *lazyflow* semantics for Kahn networks, and so formalised an intuitively useful programming notion. If not then we have clarified a fundamental distinction between the concepts of *data flow* and *lazy evaluation.*

Further Work 2

This work on iso-metrics has shown that the usual definition of a sequence X converging to a point y,

$$\forall\ G \in \tau \ .\ y \in G \Rightarrow \exists\ k\ .\ \forall\ n > k\ .\ X_n \in G$$

needs to be looked at again. Presumably it was originally introduced for work in T_2 spaces such as the real numbers where limits would be unique. However, we have shown above that T_1 spaces do not (in general) have unique limits, let alone T_2 spaces. It would be desirable though to have a topological definition of convergence for T_0 spaces which does give unique limits. The following is a possible candidate which needs to be examined more closely, especially with respect to partial metrics.

$$\forall\ G \in \tau \ .\ y \in G \Leftrightarrow \exists\ k\ .\ \forall\ n > k\ .\ X_n \in G$$

This adds the following constraint to the earlier condition. If X is eventually in an open set then it's limit must also be in that set. This is too strong for the reals, for example, consider the sequence,

$$\forall\ n \geq 0\ .\ X_n ::= \frac{1}{n+1}$$

This extra constraint does not hold for $G = (0,1)$, as $0 \notin (0,1)$. However, Scott topologies are certainly not T_2, and not even T_1. Under this new definition of convergence limits are unique for any T_0 space. This seems appropriate for iso-metric spaces, however, this has yet to be proved.

Further Work 3

This work on iso-metric spaces needs to be extended to partial metric spaces, in particular to the Lucid domain. This domain has the usual (Tychonoff) product. For example, the ω product ${}^{\omega}(D_{\perp})$ over the flat domain $D_{\perp}$ has the base with sets of the form,

$$\{\ x \mid \forall\ 1 \leq i \leq k\ .\ x_{n_i} = d_i\ \}$$

for each finite set of $d_i \neq \perp$.

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