Research Report 222

The Topology of Partial Metric Spaces

Matthews SG

RR222

The T_0 world of Scott's topological models used in the denotational semantics of programming languages may at first sight appear to have nothing whatever in common with the Hausdorff world of metric space theory. Can this be true though when the notion of "distance" is so important in the application of inductive proof theory to recursive definitions? This paper shows that existing work on the application of quasi metrics to denotational semantics can be taken much further than just describing Scott topologies. Using our "partial metric" we introduce a new approach by constructing each semantic domain as an Alexandrov topology "sandwiched" between two metric topologies.

to be presented at the Eighth Summer Conference on General Topology and Applications, June 18-20 1992, Queens College, New York City.

Department of Computer Science
University of Warwick
Coventry CV4 7AL
United Kingdom

June 1992

The Topology of Partial Metric Spaces

Steve Matthews

Dept. of Computer Science
University of Warwick
Coventry, CV4 7AL
email *sgm@dcs.warwick.ac.uk*

1. Introduction

In the study of the denotational semantics of programming languages topological models are created for programming languages represented by systems of logic. More often than not this means a T_0 model for the lambda calculus in the spirit of Scott. However, the inherent assumption in this approach that all suitable models must be T_0 appears to remove any possibility that the theory of metric spaces (which are all T_2) can be applied in any way to domain theory in Computer Science. Rare exceptions to this rule are the use of quasi metrics in [Sm87] and metrics in [La87]. If metrics are to be used at all then the more conventional wisdom in Computer Science would dismiss Scott's T_0 approach in favour of a purely metric approach [dB&Z82]. Unfortuantely the latter T_2 approach pays the price of loosing any notion of partial order, a concept of fundamental importance in any Tarskian approach to fixed point semantics in denotational semantics. The distinct advantage of using quasi metrics is that such generalised metrics can be used to define T_0 topologies with partial orders, and so allowing Tarskian semantics. Quasi metrics are not without their problems though. Being non-symmetric a quasi metric is arguably an "unnatural" notion of distance. Perhaps a more important crticism is that they shed little light on how to develop proof theory for programs based upon metric reasoning. In [Sm87] the phrase *Reconciling Domains with Metric Spaces* is used to suggest that conventional wisdom from the theory of metric spaces can be applied to programming language semantics.

[Ko88] contains a proof that *All Topologies come from Generalised Metrics*. This may or may not be of interest to topologists in general as many of the more pleasant T_2 properties usually associated with metric spaces may be lost in a process of generalisation. However, the point that metrics can, if only in principle, be generalised to explore non - T_2 topologies is established. Combining this with Smyth's work on quasi metrics we are naturally lead to the following question.

To what extent can the T_2 topological methods of metric space theory be applied to programming language semantics ?

This report aims to demonstrate that a positive answer to this question can be found by showing that the existing work on quasi metrics can be taken much further than has previously been suggested. Not only can a generalised metric such as a quasi metric be used to describe Scott T_0 topologies but we can also use metrics as well. We introduce here the *weighted metric*, a version of a metric algebraically equivalent to a restricted form of quasi metric.

> *Defined using* ***partial metrics*** *we propose here a method of constructing domains as Alexandrov sub-topologies of weighted metric topologies.*

The purpose of this work is to bring the theory of metric spaces ever closer to Scott's theory of domains, and so make the reconciliation of domains with metric spaces a reality.

2. Backgound Definitions & Results

Definition 2.1
A ***Topology*** over a set U is a set $\mathcal{T} \subseteq 2^U$ such that ,

(T1) $\emptyset \in \mathcal{T}$
(T2) $U \in \mathcal{T}$
(T3) $\forall\ S \subseteq \mathcal{T}\ .\ \cup S \in \mathcal{T}$
(T4) $\forall\ O\ ,\ O' \in \mathcal{T}\ .\ O \cap O' \in \mathcal{T}$

Members of $\mathcal{T}$ are called ***open sets*** and their complements ***closed sets*** .

Definition 2.2
The T_0, T_1, and T_2 (i.e. ***Hausdorff***) ***Separation Axioms*** for a topology $\mathcal{T}$ are,

$$T_0 \quad = \quad \forall\ x \neq x' \in U\ .\quad \exists\, O \in \mathcal{T}\ .\ (\, x \in O \ \wedge\ x' \notin O\,) \\ \vee (\, x' \in O \ \wedge\ x \notin O\,)$$

$$T_1 \quad = \quad \forall\ x \neq x' \in U\ .\quad (\, \exists\, O, O' \in \mathcal{T}\ .\ x \in O - O' \\ \wedge\ x' \in O' - O\,)$$

$$T_2 \quad = \quad \forall\ x \neq x' \in U\ .\quad (\, \exists\, O, O' \in \mathcal{T}\ .\ x \in O \ \wedge\ x' \in O' \\ \wedge\ O \cap O' = \emptyset\,)$$

If $\mathcal{T}$ is T_2 then $\mathcal{T}$ is T_1, and if $\mathcal{T}$ is T_1 then $\mathcal{T}$ is T_0.

Definition 2.3

A ***Basis*** for a topology $\mathcal{T}$ is a set $\mathcal{B} \subseteq \mathcal{T}$ such that,

$$\mathcal{T} = \{ \; \bigcup S \; / \; S \subseteq \mathcal{B} \; \}$$

Lemma 2.1

$\mathcal{B} \subseteq 2^U$ is a basis for a topology $\mathcal{T}$ over U if,

$$U = \bigcup \mathcal{B} \quad \text{and} \quad \forall\, O, O' \in \mathcal{B} \;\; \exists\, S \subseteq \mathcal{B} \; . \; O \cap O' = \bigcup S$$

Lemma 2.2

$\mathcal{B}\,,\,\mathcal{B}' \subseteq 2^U$ are each a basis for the same topology on U if,

$$\forall\, O \in \mathcal{B} \;\; \exists\, S' \subseteq \mathcal{B}' \; . \; O = \bigcup S' \quad \text{and},$$
$$\forall\, O' \in \mathcal{B}' \;\; \exists\, S \subseteq \mathcal{B} \; . \; O' = \bigcup S$$

Definition 2.4

A ***Partial Order*** (***poset***) is a pair $\langle\, U \,,\, \ll \,\subseteq U \times U \,\rangle$ such that,

(PO1) $\quad \forall\, x \in U \quad . \quad x \ll x$

(PO2) $\quad \forall\, x, y \in U \quad . \quad x \ll y \;\wedge\; y \ll x \;\Rightarrow\; x = y$

(PO3) $\quad \forall\, x, y, z \in U \quad . \quad x \ll y \;\wedge\; y \ll z \;\Rightarrow\; x \ll z$

Definition 2.5

An ***Alexandrov Topology*** over a poset $\langle U , \ll \rangle$ is a topology $\mathcal{T}$ over U such that,

$$\forall\, O \in \mathcal{T} \;\; \forall\, x, y \in U \; . \; x \in O \;\wedge\; x \ll y \;\Rightarrow\; y \in O$$

The topology, $\mathcal{T} = \{ \; \{ \, n \,,\, n{+}1 \,,\, \ldots \,,\, \infty \, \} \; / \; n \in \omega \; \}$ over $\langle\, \omega \cup \{ \infty \} \,,\, \leq \,\rangle$ is an example of an Alexandrov topology. The ***Full Alexandrov Topology*** $\mathcal{T}[\,\ll\,]$ over a poset $\langle U , \ll \rangle$ is the topology

$$\{ \; O \subseteq U \; / \; \forall x \in O \,,\, y \in U \; . \; x \ll y \;\Rightarrow\; y \in O \; \}$$

of <u>all</u> ***upward closures*** [Vi89] . The topology

$$\mathcal{T}[\,\leq\,] \;\; ::= \;\; \{ \; \{ \, n \,,\, n{+}1 \,,\, \ldots \,,\, \infty \, \} \; / \; n \in \omega \cup \{ \infty \} \; \}$$

is the Full Alexandrov Topology over $\langle\, \omega \cup \{ \infty \} \,,\, \leq \,\rangle$. The Full Alexandrov Topology is always T_0 , however , not every Alexandrov Topology is T_0 as the indiscrete topology

$\mathcal{T} = \{ \varnothing , \{0,1\} \}$ shows.

Definition 2.6
A ***Scott Topology*** is an Alexandrov Topology $\mathcal{T}$ over a chain complete poset (cpo) $\langle U , \ll \rangle$ such that for each chain $X \in {}^{\omega}U$,

$$\forall O \in \mathcal{T} \; . \; lub\ X \in O \;\Rightarrow\; \exists k \geq 0 \; . \; \forall n > k \; . \; X_n \in O$$

The only T_2 Scott topologies are the "flat" trivial ones in which,

$$\forall x,y \in U \; . \; x \ll y \;\Leftrightarrow\; x = y$$

The least element $\perp$ (bottom) is not included here in our definition of a Scott topology as it is not essential to the development of the work presented in this report. Later work using our methods for fixed point semantics and reflexive domains will need to include $\perp$.

Definition 2.7
A ***Metric*** [Su75] is a function $d : U \times U \rightarrow \Re$ such that,

$$\begin{array}{lll} (M1) & \forall x,y \in U \; . & x = y \;\Leftrightarrow\; d(x,y) = 0 \\ (M2) & \forall x,y \in U \; . & d(x,y) = d(y,x) \\ (M3) & \forall x,y,z \in U \; . & d(x,z) \leq d(x,y) + d(y,z) \end{array}$$

Definition 2.8
An ***Open Ball*** for a metric $d : U \times U \rightarrow \Re$ is a set of the form,

$$B^d{}_{\varepsilon}(x) \;::=\; \{\; y \in U \;/\; d(x,y) < \varepsilon \;\}$$

for any $x \in U$ and $\varepsilon > 0$.

Lemma 2.3
The set of all open balls of a metric $d : U \times U \rightarrow \Re$ is a basis for a T_2 topology $\mathcal{T}[\,d\,]$ on U.

Definition 2.9
A ***Quasi Metric*** is a function $q : U \times U \rightarrow \Re$ such that,

$$\begin{array}{lll} (Q1) & \forall x,y \in U \; . & x = y \;\Leftrightarrow\; q(x,y) = q(y,x) = 0 \\ (Q2) & \forall x,y,z \in U \; . & q(x,z) \leq q(x,y) + q(y,z) \end{array}$$

Lemma 2.4
For each quasi metric $q : U \times U \rightarrow \Re$ the relation $\ll_q \subseteq U \times U$ defined by ,

$$\forall\ x , y \in U \ .\ x \ll_q y \iff q(x , y) = 0$$

is a partial ordering.

Lemma 2.5
For each quasi metric $q : U \times U \rightarrow \Re$ the set of all ***open balls*** of the form,

$$B^q{}_\varepsilon(x) \ ::= \ \{ \ y \in U \ / \ q(x , y) < \varepsilon \ \}$$

for any $x \in U$ and $\varepsilon > 0$ is the basis for an Alexandrov T_0 topology $\mho[\, q \,]$ over $\langle\, U \,,\, \ll_q \,\rangle$.

Example 2.1
For each complete metric $d : U \times U \rightarrow \Re$ and for each collection of closed sets $U^C \subseteq 2^U$ the function $q : U^C \times U^C \rightarrow \Re$ where ,

$$\forall\ X, Y \in 2^U \ .\ q(X , Y) ::= sup\, \{ \ inf\{ \ d(x,y) / (1 + d(x,y)) \ / \ y \in Y \ \} \ / \ x \in X \ \}$$

is a quasi metric such that, $\forall\ X , Y \in U^C \ .\ X \subseteq Y \iff X \ll_q Y$

Lemma 2.6
For each quasi metric $q : U \times U \rightarrow \Re$ the function $q^m : U \times U \rightarrow \Re$ where,

$$\forall\ x , y \in U \ .\ q^m (x , y) ::= q(x , y) + q(y , x)$$

is a metric such that $\mho[\, q \,] \subseteq \mho[\, q^m \,]$.

Lemma 2.7 (***Quasi Metric Contraction Mapping Theorem***)
For each quasi metric $q : U \times U \rightarrow \Re$ such that q^m is complete , and for each function $f : U \rightarrow U$ such that ,

$$\exists\ \ 0 \leq c < 1 \ \ \forall\ x , y \in U \ .\ q(f(x) , f(y)) \leq c \times q(x , y)$$

there exists a unique $a \in U$ such that $a = f(a)$.

Note that the unique fixed point of a quasi metric contraction mapping need not be maximal as the following simple example shows.

$$q : \{a,b\}^2 \rightarrow \{0,1\}$$
$$q(a,b) = 0 \ , \quad q(b,a) = 1$$
$$f : \{a,b\} \rightarrow \{a,b\}$$
$$f(a) = f(b) = a$$

Thus this theorem cannot be used to prove that recursive definitions define ***total objects***. In Section 5 we present a second theorem specifically designed for this purpose.

3. Partial Metrics

Definition 3.1
A ***Partial Metric*** (***Pmetric***) is a function $p : U \times U \rightarrow \Re$ such that ,

(P 1)	$\forall\ x,y \in U$	$.\ x = y \iff p(x,x) = p(x,y) = p(y,y)$
(P2)	$\forall\ x,y \in U$	$.\ p(x,x) \leq p(x,y)$
(P3)	$\forall\ x,y \in U$	$.\ p(x,y) = p(y,x)$
(P4)	$\forall\ x,y,z \in U$	$.\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

The pmetric axioms P1 thru P4 are intended to be a "minimal" generalisation of the metric axioms M1 thru M3 such that objects do not necessarily have to have zero distance from themselves. In this generalisation we manage to preserve the symmetry axiom M2 to get P3 , but have to "massage" the transitivity axiom M3 to produce the generalisation P4 . Consequently a metric is precisely a pmetric $p : U \times U \rightarrow \Re$ in which,

$$\forall\ x \in U \quad . \quad p(x,x) = 0$$

For each pmetric $p : U \times U \rightarrow \Re$ we preserve "half" of the metric reflexive axiom M1 giving ,

$$\forall\ x,y \in U \quad . \quad p(x,y) = 0 \quad \Rightarrow \quad x = y$$

Example 3.1
The poset $\langle \Re - \{0\} , \leq \rangle$ of positive real numbers can be defined by the pmetric
$p : (\Re - \{0\})^2 \rightarrow \Re$ where, $\forall\ x,y \in U \ . \ p(x,y) ::= max\{\ 1/x\ ,\ 1/y\ \}$
as $\forall\ x,y \in U \ . \ x \leq y \iff p(x,x) = p(x,y)$.

Example 3.2
The well known ***Baire Metric*** $d^B : {}^{\omega}S \times {}^{\omega}S \rightarrow \Re$ of ω - sequences over a set S is defined by ,

$$\forall\ x \neq y \in {}^{\omega}S \ .\ d^B(x, y) = 2^{-min\{\ n \in \omega\ /\ x(n) \neq y(n)\ \}}$$

This definition can be extended to the set $(\ \cup\{\ ^{\{0,\dots,n-1\}}S\ /\ n \in \omega\ \}\ ) \cup {}^{\omega}S$ of all finite and ω - sequences over S by the ***Baire Pmetric*** ,

$$p^B\ :\ (\ (\ \cup\{\ ^{\{0,\dots,n-1\}}S\ /\ n \in \omega\ \}\ ) \cup {}^{\omega}S\ )^2 \rightarrow \Re$$

where, $\forall\ n \in \omega\ ,\ x \in {}^{\{0,\dots,n-1\}}S\ .\ p^B(x, x) ::= 2^{-n}$. In this example we can use the condition $p(x, x) = 0$ to distinguish between those x which are in the Baire Space and those which are not.

Example 3.3

A ***Flat Pmetric*** is a pmetric $p_\perp\ :\ S_\perp \times S_\perp \rightarrow \{0, 1\}$ where, $S_\perp ::= S \cup \{\perp\}$ for a set S, and $\perp \notin S$, and ,

$$\forall\ x, y \in S_\perp\ .\quad p_\perp(x, y) = 0 \Leftrightarrow x = y \in S$$

Here an object $x \in S_\perp$ is ***totally defined*** if and only if $p_\perp(x, x) = 0$.

Definition 3.2

The ***Open Balls*** for a pmetric $p\ :\ U \times U \rightarrow \Re$ are the sets of the form,

$$B^p{}_\varepsilon(x) ::= \{\ y \in A\ /\ p(x, y) < \varepsilon\ \}$$

for each $\varepsilon > 0$ and $x \in A$.

Theorem 3.1

The set of all open balls of a pmetric $p\ :\ U \times U \rightarrow \Re$ is the basis of a topology $\mathcal{T}[p]$ on U

Proof :

Proof using Lemma 2.1

Suppose $p\ :\ U \times U \rightarrow \Re$ is a pmetric .

Then, $A = \cup_{x \in A}\ B^p{}_{p(x,x)+1}(x)$ and,

for any balls $B^p{}_\varepsilon(x)$ and $B^p{}_\delta(y)$,

$$B^p{}_\varepsilon(x) \cap B^p{}_\delta(y) = \cup\{\ B^p{}_\eta(z)\ /\ z \in B^p{}_\varepsilon(x) \cap B^p{}_\delta(y)\ \} \quad where,\ \eta ::= p(z, z) + min\{\ \varepsilon - p(x, z)\ ,\ \delta - p(y, z)\ \}$$

❑

Theorem 3.2
For each pmetric $p : U \times U \rightarrow \Re$, open ball $Bp_{\varepsilon}(a)$, and $x \in U$,

$$x \in Bp_{\varepsilon}(a) \quad \Rightarrow \quad \exists \ \delta > 0 \ . \ x \in Bp_{\delta}(x) \subseteq Bp_{\varepsilon}(a)$$

Proof :

Suppose $p : U \times U \rightarrow \Re$ is a pmetric .
Suppose $x \in Bp_{\varepsilon}(a)$
Then $p(x, a) < \varepsilon$
Let $\delta ::= \varepsilon - p(x, a) + p(x, x)$
Then $\delta > 0$ as $\varepsilon > p(x, a)$
Also, $p(x, x) < \delta$ as $\varepsilon > p(x, a)$
Thus $x \in Bp_{\delta}(x)$

Suppose now that $y \in Bp_{\delta}(x)$
$\therefore \ p(y, x) < \delta$
$\therefore \ p(y, x) < \varepsilon - p(x, a) + p(x, x)$
$\therefore \ p(y, x) + p(x, a) - p(x, x) < \varepsilon$
$\therefore \ p(y, a) < \varepsilon$ (by P4)
$\therefore \ y \in Bp_{\varepsilon}(a)$
Thus $Bp_{\delta}(x) \subseteq Bp_{\varepsilon}(a)$.

❑

Theorem 3.3
Pmetric topologies are T_0 .

Proof :

Suppose $p : U \times U \rightarrow \Re$ is a pmetric .
Suppose $x \neq y \in U$.

Then from P1 & P2 $p(x, x) < p(x, y)$ or $p(y, y) < p(x, y)$.

Wlog suppose $p(x, x) < p(x, y)$ then ,
$x \in Bp_{\varepsilon}(x) \wedge y \notin Bp_{\varepsilon}(x)$ where, $\varepsilon ::= (\ p(x, x) + p(x, y)\ ) / 2$

❑

Definition 3.3
For each pmetric $p : U \times U \rightarrow \Re$, $\ll_p \ \subseteq U \times U$ is the binary relation defined by,

$$\forall \ x, y \in A \ . \quad x \ll_p y \quad \Leftrightarrow \quad p(x, x) = p(x, y)$$

Theorem 3.4
For each pmetric $p : U \times U \to \Re$, $\ll_p$ is a partial order.

Proof :

Suppose $p : U \times U \to \Re$ is a pmetric .
We prove PO1 thru PO3

(PO1) $\forall x \in U \ . \quad x \ll_p x \quad$ as $\quad p(x,x) = p(x,x)$

(PO2) $\forall x, y \in U \ . \quad x \ll_p y \ \wedge \ y \ll_p x$
$\Rightarrow \ p(x,x) = p(x,y) = p(y,y)$ (by P3)
$\Rightarrow \ x = y$ (by P1)

(PO3) $\forall x, y, z \in A \ . \quad x \ll_p y \ \wedge \ y \ll_p z$
$\Rightarrow \ p(x,x) = p(x,y) \ \wedge \ p(y,y) = p(y,z)$
but, $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$ (by P4)
$\therefore \ p(x,z) \le p(x,x)$
$\therefore \ p(x,z) = p(x,x)$ (by P2)
$\therefore \ x \ll_p z$ (by definition of $\ll_p$)

❑

Theorem 3.5
Every pmetric topology is an Alexandrov Topology.

Proof:

Suppose $p : U \times U \to \Re$ is a pmetric ,
then we have to show that $\mathcal{T}[p] \subseteq \mathcal{T}[\ll_p]$.

It is sufficient to show that ,

$$\forall x \in U \ , \ \varepsilon > 0 \ . \quad Bp_\varepsilon(x) = \cup \{ \{ z \mid y \ll_p z \} \mid y \in Bp_\varepsilon(x) \}$$

Suppose $x, y, z \in U$ and $\varepsilon > 0$ are such that $y \in Bp_\varepsilon(x)$ and $y \ll_p z$.

Then , $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$ (by P4)
$= p(x,y)$ as $y \ll_p z$
$< \varepsilon$ as $y \in Bp_\varepsilon(x)$

Thus , $z \in Bp_\varepsilon(x)$.

❑

Theorem 3.6
For each pmetric $p : U \times U \rightarrow \Re$, $\mathcal{T}[p] = \mathcal{T}[\ll_p]$ if and only if ,

$$\forall\ x \in U \quad \exists\ \varepsilon > 0 \quad . \quad Bp_\varepsilon(x) = \{ y \mid x \ll_p y \}$$

Proof :

Suppose first that , $\forall\ x \in U \quad \exists\ \varepsilon > 0 \ . \ Bp_\varepsilon(x) = \{ y \mid x \ll_p y \}$
Then , $\forall\ O \in \mathcal{T}[\ll_p] \ . \ O = \bigcup_{x \in O} \{ y \mid x \ll_p y \}$

$$= \bigcup_{x \in O} Bp_\varepsilon(x)$$
$$\in \mathcal{T}[p]$$

$\therefore \quad \mathcal{T}[\ll_p] \subseteq \mathcal{T}[p]$
$\therefore \quad \mathcal{T}[p] = \mathcal{T}[\ll_p]$ (by Theorem 3.5)

Suppose now that , $\mathcal{T}[p] = \mathcal{T}[\ll_p]$
Then , $\forall\ x \in U \ . \ \{ y \mid x \ll_p y \} \in \mathcal{T}[p]$
Thus , by Theorem 3.2 ,

$$\forall\ x \in U \quad \exists\ \varepsilon > 0 \quad . \quad x \in Bp_\varepsilon(x) \subseteq \{ y \mid x \ll_p y \}$$

But , if $x \in Bp_\varepsilon(x)$ then $\{ y \mid x \ll_p y \} \subseteq Bp_\varepsilon(x)$
Thus , $\forall\ x \in U \quad \exists\ \varepsilon > 0 \ . \ Bp_\varepsilon(x) = \{ y \mid x \ll_p y \}$

❑

4. Partial & Quasi Metrics

Theorem 4.1
For each pmetric $p : U \times U \rightarrow \Re$ the function $q : U \times U \rightarrow \Re$ where ,

$$\forall\ x, y \in U \quad . \quad q(x, y) ::= p(x, y) - p(x, x)$$

is a quasi metric such that $\mathcal{T}[p] = \mathcal{T}[q]$ and $\ll_p = \ll_q$.

Proof :

Suppose $p : U \times U \rightarrow \Re$ is a pmetric .

Let $q : U \times U \rightarrow \Re$ be such that ,

$$\forall\ x, y \in U \quad . \quad q(x, y) = p(x, y) - p(x, x)$$

We show first that q is a quasi metric by proving Q1 and Q2 .

(Q1 $\Rightarrow$) $\quad \forall\, x, y \in U \,.\; x = y \;\Rightarrow\; q(x,y) = 0$ (by definition of q)

(Q1 $\Leftarrow$) $\quad \forall\, x, y \in U \,.\; q(x,y) = q(y,x) = 0$

$$\Rightarrow\; p(x,y) - p(x,x) = p(y,x) - p(y,y) = 0$$

$$\Rightarrow\; p(x,x) = p(x,y) = p(y,y) \qquad \text{(by P3)}$$

$$\Rightarrow\; x = y \qquad \text{(by P1)}$$

(Q2) $\quad \forall\, x, y, z \in U \,.\; p(x,z) \le p(x,y) + p(y,z) - p(y,y)$

$$\Rightarrow\; q(x,z) \le q(x,y) + q(y,z)$$

Thus q is a quasi metric.

Now, $\mathcal{T}[p] = \mathcal{T}[q]$ as,

$$\forall\, x \in U \,,\; \varepsilon > p(x,x) \;.\; B^p_\varepsilon(x) = B^q_{\varepsilon - p(x,x)}(x)$$

$$\forall\, x \in U \,,\; 0 < \varepsilon \le p(x,x) \;.\; B^p_\varepsilon(x) = \varnothing$$

$$\forall\, x \in U \,,\; \varepsilon > 0 \;.\; B^q_\varepsilon(x) = B^p_{\varepsilon + p(x,x)}(x)$$

and by Lemma 2.2 .

Finally, $\ll_p = \ll_q$ as,

$$\forall\, x, y \in U \,.\; p(x,x) = p(x,y) \;\Leftrightarrow\; q(x,y) = 0$$

❑

Definition 4.1

A ***Weighted Quasi Metric*** over a set U is a pair $\langle q , |\,| \rangle$ consisting of a quasi metric $q : U \times U \to \Re$ and a ***Weight Function*** $|\,| : U \to \Re$ where,

(WQ) $\quad \forall\, x, y \in U \,.\; q(x,y) + |x| = q(y,x) + |y|$

A quasi metric q is ***Weightable*** if there exists a weight function $|\,|$ such that $\langle q , |\,| \rangle$ is a weighted quasi metric.

Theorem 4.2

For each weighted quasi metric $< q , |\,| >$ over a set U the function $p : U \times U \to \Re$ where,

$$\forall\, x, y \in U \,.\; p(x,y) ::= q(x,y) + |x|$$

is a pmetric such that $\mathcal{T}[p] = \mathcal{T}[q]$ and $\ll_p = \ll_q$.

Proof :

Suppose $\langle q , | | \rangle$ is a weighted quasi metric over a set U.

Let $p : U \times U \to \Re$ be such that ,

$$\forall\ x, y \in U \ .\quad p(x, y) \ ::= \ q(x, y) + |x|$$

We show first that p is a pmetric by proving P1 thru P4 .

(P1 $\Rightarrow$) Trivial

(P1 $\Leftarrow$)

$$\begin{aligned}
&\forall\ x, y \in U \ .\quad p(x, x) = p(x, y) = p(y, y) \\
&\quad\Rightarrow\ |x| = q(x, y) + |x| = |y| && \text{(by Q1)} \\
&\quad\Rightarrow\ |x| = q(x, y) + |x| = q(y, x) + |y| = |y| && \text{(by WQ)} \\
&\quad\Rightarrow\ q(x, y) = q(y, x) = 0 \\
&\quad\Rightarrow\ x = y && \text{(by Q1)}
\end{aligned}$$

(P2)

$$\begin{aligned}
&\forall\ x, y \in U \ .\quad 0 \le q(x, y) \\
&\therefore\ \forall\ x, y \in U \ .\quad |x| \le q(x, y) + |x| \\
&\therefore\ \forall\ x, y \in U \ .\quad p(x, x) \le p(x, y) && \text{(by Q1)}
\end{aligned}$$

(P3)

$$\begin{aligned}
&\forall\ x, y \in U \ .\quad q(x, y) + |x| = q(y, x) + |y| && \text{(by WQ)} \\
&\therefore\ \forall\ x, y \in U \ .\quad p(x, y) = p(y, x)
\end{aligned}$$

(P4)

$$\begin{aligned}
&\forall\ x, y, z \in U \ .\quad q(x, z) \le q(x, y) + q(y, z) && \text{(by Q2)} \\
&\therefore\ \forall\ x, y, z \in U \ .\quad q(x, z) + |x| \\
&\qquad \le (\, q(x, y) + |x| \,) + (\, q(y, z) + |y| \,) - |y| \\
&\therefore\ \forall\ x, y, z \in U \ .\quad p(x, z) \\
&\qquad \le p(x, y) + p(y, z) - p(y, y) && \text{(by Q1)}
\end{aligned}$$

Thus p is a pmetric .

Now , $\mathcal{T}[\, p \,] = \mathcal{T}[\, q \,]$ as by Lemma 2.2 ,

$$\forall\ x \in U ,\ \varepsilon > |x| \ .\quad B^p{}_{\varepsilon}(x) = B^q{}_{\varepsilon - |x|}(x)$$

$$\forall\ x \in U ,\ 0 < \varepsilon \le |x| \ .\quad B^p{}_{\varepsilon}(x) = \varnothing$$

$$\forall\ x \in U ,\ \varepsilon > 0 \ .\quad B^q{}_{\varepsilon}(x) = B^p{}_{\varepsilon + |x|}(x)$$

Finally, $\ll_p = \ll_q$ as,

$$\forall\, x, y \in U \,.\quad p(x,x) = p(x,y) \iff q(x,y) = 0$$

❑

Note that Theorems 4.1 & 4.2 establish an algebraic equivalence between the *partial metric* and a restricted form of the *quasi metric*.

Theorem 4.3
Not every quasi metric is weightable

Proof :

Let $q : \{a,b,c\}^2 \to \{0,1,2,3\}$ be the unique quasi metric such that,

$$\begin{array}{ll} q(a,b) = 0 & q(b,a) = 2 \\ q(a,c) = 1 & q(c,a) = 1 \\ q(b,c) = 3 & q(c,b) = 0 \end{array}$$

Suppose that there exists a weight function $|\,| : \{a,b,c\} \to \{0,1,2,3\}$ for q, then,

$$\begin{array}{lll} |b| + q(b,c) & & \\ \quad = (|b| + q(b,a)) + 1 & & \\ \quad = (|a| + q(a,b)) + 1 & & \text{(by WQ)} \\ \quad = |a| + q(a,c) & & \\ \quad = |c| + q(c,a) & & \text{(by WQ)} \\ \quad = (|c| + q(c,b)) + 1 & & \\ \quad = (|b| + q(b,c)) + 1 & & \text{(by WQ)} \end{array}$$

But this is impossible.

❑

Unfortunately Theorem 4.3 does not answer the question of whether or not every quasi metric topology can be defined using a partial metric. Theroem 4.3 only shows that the method of defining a pmetric for a quasi metric topology using a weight function will not always work. The next result does answer the question for finite quasi metric topologies.

Theorem 4.4
For each quasi metric $q : U \times U \to \Re$ over a finite set U there exists a pmetric $p : U \times U \to \Re$ such that $\mathcal{T}[p] = \mathcal{T}[q]$ *and* $\ll_p = \ll_q$.

Proof Outline :

Let $P ::= \langle U , \ll_q \rangle$ be the partial order induced by a quasi metric $q : U \times U \rightarrow \Re$ over a finite set U.

Let $P^* \subseteq 2^U$ be the set of all chains in P (i.e. all non - empty totally ordered subsets of U).

Let $\$: U \rightarrow \omega$ be the function where ,

$$\forall x \in U . \quad \$(x) ::= 2^{\max\{ |c| \ / \ c \in P^* \wedge lub\, c \ll_q x \}}$$

Then it can be shown that the function $p : U \times U \rightarrow \omega$ where ,

$$\forall x , y \in U . \quad p(x,y) ::=$$

$$2^{|U|+1} - \max\{ \textstyle\sum_{x \in c} \$(x) \ / \ c \in P^* \wedge lub\, c \ll_q x \wedge lub\, c \ll_q y \}$$

is a pmetric such that $\mho[p] = \mho[q]$ and $\ll_p = \ll_q$.

❑

An important implication of Theorem 4.4 is that any finite partial order can be defined by a partial metric .

5. Metrics & Partial Metrics

Definition 5.1

A ***Weighted Metric*** over a set U is a pair $\langle d , || \rangle$ consisting of a metric $d : U \times U \rightarrow \Re$ and a ***Weight Function*** $|| : \rightarrow \Re$ where ,

$$(\mathrm{WM}) \qquad \forall x , y \in U . \quad d(x,y) \geq |x| - |y|$$

A metric d is ***Weightable*** if there exists a weight function $||$ such that $\langle d , || \rangle$ is a weighted metric.

The next two results show the algebraic equivalence between the *partial metric* and the *weighted metric* .

Theorem 5.1

For each pmetric $p : U \times U \rightarrow \Re$ the pair $\langle p^m : U \times U \rightarrow \Re , || : U \rightarrow \Re \rangle$ where ,

$$\forall\ x, y \in U \ .\ \ p^m(x,y) \ ::= \ 2 \times p(x,y) - p(x,x) - p(y,y)$$
$$\forall\ x \in U \ .\ \ |x| \ ::= \ p(x,x)$$

is a weighted metric such that ,

$$\mathcal{T}[p] \subseteq \mathcal{T}[p^m] \ , \text{ and },$$
$$\forall\ x, y \in U \ .\ \ p(x,y) \ ::= \ (\, p^m(x,y) + |x| + |y| \,) / 2$$

Proof :

Suppose $p : U \times U \to \Re$ is a pmetric .
Then , by Theorem 4.1 , the function $q : U \times U \to \Re$ where ,

$$\forall\ x, y \in U \ .\ \ q(x,y) \ ::= \ p(x,y) - p(x,x)$$

is a quasi metric such that $\mathcal{T}[p] = \mathcal{T}[q]$.

Thus , by Lemma 2.6 , the function $p^m : U \times U \to \Re$ where ,

$$\forall\ x, y \in U \ .\ \ p^m(x,y) \ ::= \ 2 \times p(x,y) - p(x,x) - p(y,y)$$

is a metric such that , $\forall\ x, y \in U \ .\ \ p^m(x,y) = q(x,y) + q(y,x)$

Thus , $\mathcal{T}[p] \subseteq \mathcal{T}[p^m]$.

Finally , WM holds as by P2 $\forall\ x, y \in U \ .\ \ p^m(x,y) \geq |x| - |y|$

❑

Theorem 5.2
For each weighted metric $\langle\, d \,,\, |\,| \,\rangle$ over a set U the function $p : U \times U \to \Re$ where ,

$$\forall\ x, y \in U \ .\ \ p(x,y) \ ::= \ (\, |x| + |y| + d(x,y) \,) / 2$$

such that $d = p^m$ and $\forall\ x \in U \ .\ |x| = p(x,x)$.

Proof :

as M1 $\Rightarrow$ P1 and ,
WM $\Rightarrow$ P2 and ,
M2 $\Rightarrow$ P3 and ,
M1 & M3 $\Rightarrow$ P4

❑

We now turn to constructing a contraction mapping theorem for partial metrics. The beauty of the result in Theorem 5.3 is that its formulation is virtually the same as Banach's original theorem for metric spaces.

Definition 5.2
For each pmetric $p : U \times U \rightarrow \Re$, and for each $X \in {}^{\omega}U$, X is a ***Cauchy Sequence*** if,

$$\forall\ \varepsilon > 0\ \ \exists\ k \in \omega\ \ \forall\ n, m > k\ \ .\ \ p(X_n, X_m) < \varepsilon$$

Definition 5.3
A pmetric $p : U \times U \rightarrow \Re$ is ***Complete*** if for each Cauchy sequence $X \in {}^{\omega}U$ there exists $a \in U$ such that, $\exists\ \lim_{n \rightarrow \infty} p(X_n, a) = 0$.

Note that these definitions for *Cauchy sequence* and *complete pmetric* are consistent with the usual metric definitions.

Theorem 5.3 (***Partial Metric Contraction Mapping Theorem***)
For each complete pmetric $p : U \times U \rightarrow \Re$, and for each function $f : U \rightarrow U$ such that,

$$\exists\ 0 \leq c < 1\ \ \forall\ x, y \in U\ \ .\ \ p(f(x), f(y)) \leq c \times p(x, y)$$

firstly, there exists a unique $a \in U$ such that $a = f(a)$, and secondly $p(a, a) = 0$.

Proof:

Suppose p, f, & c are as in the statement of the Theorem.
Suppose $u \in U$.
We begin by showing that $\lambda\ n \in \omega\ .\ f^n(u)$ is a Cauchy sequence.

$$\begin{aligned} \forall\ n, k \in \omega\ .\ & p(f^{n+k+1}(u), f^n(u)) \\ & \leq p(f^{n+k+1}(u), f^{n+k}(u)) \\ & \qquad + p(f^{n+k}(u), f^n(u)) - p(f^{n+k}(u), f^{n+k}(u)) \\ & \leq c^{n+k} \times p(f(u), u) + p(f^{n+k}(u), f^n(u)) \end{aligned}$$

Thus,

$$\begin{aligned} \forall\ n, k \in \omega\ .\ & p(f^{n+k+1}(u), f^n(u)) \\ & \leq (c^{n+k} + \ldots + c^n) \times p(f(u), u) + p(f^n(u), f^n(u)) \\ & \leq c^n \times ((1 - c^{k+1}) / (1 - c)) \times p(f(u), u) + c^n \times p(u, u) \\ & \leq c^n \times ((p(f(u), u) / (1 - c)) + p(u, u)) \end{aligned}$$

Thus as , $\forall\ n \in \omega\ .\ p(\ f^n(u)\ ,\ f^n(u)\ )\ \leq\ c^n \times p(u,u)$
we see that $\lambda\ n \in \omega\ .\ f^n(u)$ is a Cauchy sequence .

Thus as p is complete we can choose $a \in U$ such that ,

$$lim_{n \to \infty}\ p(\ f^n(u)\ ,\ a\ )\ =\ 0$$

But , $p(\ f(a)\ ,\ a\ )\ =\ 0$ as ,

$$\begin{aligned} \forall\ n \in \omega\ .\ & p(\ f(a)\ ,\ a\ ) \\ & \leq\ p(\ f(a)\ ,\ f^{n+1}(u)\ )\ +\ p(\ f^{n+1}(u)\ ,\ a\ ) \\ & \qquad -\ p(\ f^{n+1}(u)\ ,\ f^{n-1}(u)\ ) \\ & \leq\ c \times p(\ a\ ,\ f^n(u)\ )\ +\ p(\ f^{n+1}(u)\ ,\ a\ ) \end{aligned}$$

Thus $a\ =\ f(a)$.
Suppose $b \in U$ is such that $b\ =\ f(b)$, then ,

$$p(a,b)\ =\ p(\ f(a)\ ,f(b)\ )\ \leq\ c \times p(a,b)$$

Thus , $p(a,b) = 0$, and so $a = b$.
Thus the fixed point of f is unique .

Finally , $p(a,a) = 0$ as, $p(a,a)\ =\ p(\ f(a)\ ,f(a)\ )\ \leq\ c \times p(a,a)$

❑

6. Conclusions & Further Work

The principle conclusion from the research in this report is that generalised metric topology has a largely unexplored potential in the field of partial order denotational semantics. This conclusion is justified for the following reasons. Firstly , this research supports the quasi metric approach used in [Sm87] and the metric approach in [La87] to model partial order topologies. Secondly , we have shown that such work can be conducted using a symmetric distance function. Thirdly , we have shown that such work can be conducted within metric topology itself by adding the concept of a *weight* to points in a metric space. Finally , we have shown that Banach's Contraction Mapping Theorem can be generalised for proof theory applications in Scott topologies.

Whether or not an arbitrary quasi metric topology can be defined by a weighted metric is an open question which needs an answer in order to decide exactly which topologies can be defined using our approach. However , it is unlikely that there exists any simple algebraic method for converting an arbitrary quasi metric into a weighted quasi metric. Similarly , it is not yet known exactly which Scott topologies can be defined using this approach. If a favourable answer to this question can be established then perhaps weighted metric models for the lambda calculus could be feasible.

The Wadge *Cyclesum Test* [Wa81] for proving Kahn data flow networks free of deadlock has yet to receive a proper denotational proof within Kahn's fixed point semantics. By formulating the domain of all finite & infinite streams as a partial metric space we plan to use the *Partial Metric Contraction Mapping Theorem* to prove Wadge's test. Beyond this we hope to use partial metrics to provide a denotational semantics for intensional programming languages such as *LUCID* [Wa&A85] which have been inspired by Kahn's work [Ka74]. For more results on partial metrics such as ***partial metric products***, ***partial metric sums***, and ***partial metric function spaces*** see [Ma92] .

7. References

[dB&Z82] ***Processes and the Denotational Semantics of Concurrency***, J. W. de Bakker & J. I. Zucker, Dept. of Computer Science report IW 209 / 82, Stichting Mathematisch Centrum .

[Ka74] ***The Semantics of a Simple Language for Parallel Processing***, Gilles Kahn, Proc. IFIP Conf. 1974, pp. 471 – 475 .

[Ko88] ***All Topologies come from Generalised Metrics***, Ralph Kopperman, American Mathematical Monthly, Vol. 95, No. 2, February 1988 .

[La87] ***The Versatile Continuous Order***, J. D. Lawson, Mathematical Foundations of Programming Language Semantics, 3rd Workshop, Tulane 1987, LNCS 298, eds. M. Main et. al., Springer 1988 .

[Ma92] ***Partial Metric Spaces***, S. G. Matthews, 8th British Colloquium for Theoretical Computer Science, March 1992, in Research Report 212, Dept. of Computer Science, University of Warwick .

[Sm87] ***Quasi - Uniformities : Reconciling Domains with Metric Spaces***, M. B. Smyth, Mathematical Foundations of Programming Language Semantics, 3rd Workshop, Tulane 1987, in LNCS 298, eds. M. Main et. al., Springer 1988 .

[Su75] ***Introduction t o Metric and Topological Spaces***, W. A. Sutherland, Clarendon Press, Oxford 1975 .

[Wa81] ***An Extensional Treatment of Dataflow Deadlock***, W. W. Wadge, Theoretical Computer Science 13, pp. 3 – 15 .

[Wa85] ***Lucid , the Dataflow Programming Language*** ,
W . W. Wadge & E. A. Ashcroft ,
APIC Studies in Data Processing No. 22 , Academic Press 1985 .

[Vi89] ***Topology via Logic*** , S . Vickers , Cambridge University Press .