**A Thesis Submitted for the Degree of PhD at the University of Warwick**

# Some Problems in Topology

Thesis submitted by

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## ABSTRACT

This thesis consists of four papers, entitled

(i) Homotopy Links,

(ii) A Note on Piecewise-linear Immersions,

(iii) Open and Closed Disc Bundles,

(iv) The Space of Homeomorphisms of a 2-manifold.

In (i), we define homotopy links and calculate them in the metastable range.

In (ii), we prove the Haefliger-Poenaru immersion theorem, using block bundles.

In (iii), we prove that $O_2 \simeq PL_2(I) \simeq PL_2$.

In (iv), we prove that the space of PL homeomorphisms of a 2-manifold, fixed on the boundary and an interior point, has contractible identity component unless the manifold is $S^2$ or $P^2$.

CONTENTS

# CHAPTER I

## HOMOTOPY LINKS

A link is the oriented image of an embedding of two spheres $S^p$, $S^q$ in the sphere $S^m$. Two links are equivalent if they are concordant oriented submanifolds of $S^m$. In [6], Haefliger showed that, under suitable conditions — smooth or piecewise-linear (PL), and codimension three - these equivalence classes form a group $L^m_{p,q}$.

A homotopy link is a pair of maps of two spheres $S^p$, $S^q$ into $S^m$ with disjoint images. Two homotopy links are equivalent if they are homotopic through homotopy links. These equivalence classes also form a group $HL^m_{p,q}$, provided we are in the metastable range, $p+2q \leq 2m-4$ or $q+2p \leq 2m-4$. These groups are the objects of study in this paper. Our theorem determines $HL^m_{p,q}$ in this range. There is a natural homomorphism $\psi: L^m_{p,q} \to HL^m_{p,q}$ as concordance implies isotopy [17] implies homotopy. $\psi$ is not an isomorphism, in general, as any link with one linking class zero has zero image under $\psi$. It is interesting that $\psi$ is not even an epimorphism, in general.

I am grateful to B. J. Sanderson for suggesting this idea.

### §0. Notation

H denotes the Hilbert space of sequences $(x_1,\ldots,x_i,\ldots)$ of real numbers. $R^n$ denotes Euclidean n-space, and will be considered as the subset of H of vectors such that $x_i=0$, for $i>n$. $S^n$ denotes the unit sphere in $R^{n+1}$. $D^n$ denotes the unit ball in $R^n$, and $\mathring{D}^n$ denotes the interior of $D^n$. We define $D^n_+ = \{\underline{x}\in S^n \mid x_1\geqslant 0\}$, and $D^n_- = \{\underline{x}\in S^n \mid x_1\leqslant 0\}$.

Let T be the rotation of H whose restriction to $R^2$ is a rotation through $\pi$, and which leaves fixed the orthogonal complement of $R^2$. Let $\sigma_i$ be the symmetry of H with respect to the hyperplane $x_i=0$.

Define $u = (0,0,1,0,\dots)$ and $v = (0,0,-1,0,\dots)$. Both are points of $S^n$, for all $n\geqslant 2$.

## §1. The group $HL^m_{p,q}$

In the following, we have $p,q\leqslant m-3$.

Define $X^m_{p,q} = \{ (f,g) \mid f\colon S^p \to S^m,\ g\colon S^q \to S^m,\ f(S^p)\cap g(S^q)=\emptyset\}$
Define an equivalence relation $\rho$ on $X^m_{p,q}$ by $(f_0,g_0)\rho(f_1,g_1)$ if and only if there are homotopies $f_t\colon S^p \to S^m$, $g_t\colon S^q \to S^m$ such that $f_t(S^p)\cap g_t(S^q)=\emptyset$.

Define $HL^m_{p,q} = X^m_{p,q}|\rho$.

Wenote that, as $f(S^p)$ is compact and hence closed in $S^m$, we can homotop g to a PL map in the complement of $f(S^p)$, by doing a small enough homotopy in $S^m$. Thus any element $\alpha$ of $HL^m_{p,q}$ has a representative $(f,g)$ in which both f and g are PL, and all such representatives are homotopic by PL homotopies.

A representative $(f,g)$ of $\alpha$ is in <u>good position</u> if

(i) f and g are PL,

(ii) $f(D^p_-) = u$ and $g(D^q_-) = v$,

(iii) $f(\mathring{D}^p_+) \subset \mathring{D}^m_+$ and $g(\mathring{D}^q_+) \subset \mathring{D}^m_+$.

$\alpha$ always has such a representative. Also two such representatives of $\alpha$ are homotopic through good position representatives of $\alpha$. For, let $F: S^p \times I \to S^m \times I$, $G: S^q \times I \to S^m \times I$ be a PL homotopy between two good position representatives of $\alpha$. Denote the point $(-1,0,0,..,0)$ in $D^p_-$ by $w$. $F|\{w\}\times I$ determines an element of $\pi_1(S^m)$ which is zero, as $m \geq 3$ by hypothesis. Therefore there is an ambient isotopy $H_t$ of $S^m \times I$, fixed on $S^m \times \partial I$ and commuting with projection onto I, such that $H_0$ is the identity and $H_1 F(\{w\}\times I) = u \times I$. Now $G|\{w\}\times I$ determines an element of $\pi_1(S^m - u) = \pi_1(R^m) = 0$, and hence there is an ambient isotopy $H'_t$ of $S^m \times I$, fixed on $S^m \times \partial I \cup \{u\} \times I$ and commuting with projection onto I, such that $H'_0$ is the identity and $H'_1 G(\{w\}\times I) = v \times I$. It is now easy to arrange $F(D^p_- \times I) = u \times I$ and $G(D^q_- \times I) = v \times I$, and then to arrange $F(\mathring{D}^p_+ \times I) \subset \mathring{D}^m_+ \times I$, $G(\mathring{D}^q_+ \times I) \subset \mathring{D}^m_+ \times I$.

We can now define the sum operation in $HL^m_{p,q}$. Let $\alpha$, $\beta$ be elements of $HL^m_{p,q}$. Take good position representatives $(f,g)$, $(f',g')$ of them. We define $(F,G)$ by

$$F|D^p_+ = f|D^p_+, \quad F|D^p_- = Tf'T|D^p_-,$$

$$G|D^q_+ = g|D^q_+, \quad G|D^q_- = Tg'T|D^q_-.$$

From the preceding work, $(F,G)$ represents a well-defined element of $HL^m_{p,q}$, which we denote by $\alpha+\beta$. This addition is commutative, associative and has identity given by a pair of point maps.

We now prove the existence of inverses if $p+2q \leq 2m-4$ or $q+2p \leq 2m-4$. Let $\alpha$ be an element of $HL^m_{p,q}$, and let $(f,g)$ be a good position representative of $\alpha$. Consider $(\sigma_2 f \sigma_2, \sigma_2 g \sigma_2)$, which is also in good position and represents $\beta$ say. We define $\alpha+\beta$ using these representatives to obtain a representative $(f',g')$ of $\alpha+\beta$. $f'$ extends to a map $F: D^{p+1} \to D^{m+1}$, by mapping the line segment $[x, \sigma_1 x]$ of $D^{p+1}$ linearly onto the segment $[fx, \sigma_1 fx]$ of $D^{m+1}$, for $x$ in $D^p_+$. Similarly, $g'$ extends to a map $G: D^{q+1} \to D^{m+1}$, and $F(D^{p+1}) \cap G(D^{q+1}) = \phi$.

Now consider $F: D^{p+1} \to D^{m+1} - G(D^{q+1})$. We can homotop $F$ to an embedding, keeping $F|S^p$ fixed, by [21, Chapter 8], if the connectivity of $D^{m+1} - G(D^{q+1})$ exceeds $(2p+2)-(m+1)+1$, i.e. if $(m+1)-(q+1)-2 \geq 2p-m+2$, i.e. if $2p+q \leq 2m-4$. (First, homotop $F|S^p: S^p \to S^m - G(S^q)$ to an embedding, as the connectivity of $S^m - G(S^q)$ exceeds $2p-m+1$, if $2p+q \leq 2m-3$.) By Zeeman's Unknotting Theorem for ball pairs [21, Chapter 4], we can ambient isotop $F$ to standard position $F'$ say. Now $S^m - F'(S^p) \simeq D^{m+1} - F'(D^{p+1})$, so it follows that $G|: S^q \to S^m - F(S^p)$ is null homotopic, as we have $G: D^{q+1} \to D^{m+1} - F(D^{p+1})$. This immediately implies that $(f',g')$ is null homotopic and hence $\alpha+\beta = 0$. This completes the proof that $HL^m_{p,q}$ is a group if $p+2q \leq 2m-4$ or $q+2p \leq 2m-4$.

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§2.

We have a natural map $\psi: L^m_{p,q} \to HL^m_{p,q}$, as concordance implies homotopy. $\psi$ is a homomorphism. We also have a homomorphism $\lambda: L^m_{p,q} \to \pi_p(S^{m-q-1})$, which associates to a link the homotopy class of $S^p$ in $(S^m-S^q) \simeq S^{m-q-1}$. We define $\varphi: \pi_p(S^{m-q-1}) \to HL^m_{p,q}$ by taking the standard inclusion of $S^q$ in $S^m$ and mapping $S^p \to (S^m-S^q)$ by an element of $\pi_p(S^{m-q-1})$. Clearly, this is well-defined and a homomorphism. We have the commutative diagram

$$\begin{array}{ccc} L^m_{p,q} & \xrightarrow{\psi} & HL^m_{p,q} \\ & \searrow^{\lambda} & \uparrow \varphi \\ & & \pi_p(S^{m-q-1}) \end{array}$$

We define $S: HL^m_{p,q} \to \pi_{p+q}(S^{m-1})$ as follows. Clearly, homotopy links in $S^m$ are the same as in $R^m$. Now, given $(f,g): (S^p,S^q) \to R^m$, such that $f(S^p) \cap g(S^q) = \phi$, we define $S(f,g): S^p \times S^q \to S^{m-1}$ by

$$S(f,g)(x,y) = \frac{f(x) - g(y)}{\|f(x) - g(y)\|} .$$

This gives a well-defined map $HL^m_{p,q} \to [S^p \times S^q,\ S^{m-1}]$. But, from the Puppe sequence

$$\to S^p \vee S^q \to S^p \times S^q \to S^p \wedge S^q \to S^{p+1} \vee S^{q+1} \to$$

we get the exact sequence

$$\pi_{p+1}(S^{m-1}) \oplus \pi_{q+1}(S^{m-1}) \to \pi_{p+q}(S^{m-1}) \to [S^p \times S^q, S^{m-1}] \to \pi_p(S^{m-1}) \oplus \pi_q(S^{m-1}).$$

As $p,\ q \leqslant m-3$, the first and last terms are zero, thus $\pi_{p+q}(S^{m-1}) \to [S^p \times S^q,\ S^{m-1}]$ is an isomorphism. (Note that $[S^p \times S^q,\ S^{m-1}]$ is a group, for $[X,\ S^{m-1}]$ is a group if $\dim X \leqslant 2m-4$, when $X$ is a CW-complex.) This defines $S$.

<u>Lemma 1</u> $S$ is a homomorphism.

Proof: Let $\alpha,\ \beta \in HL^m_{p,q}$, and choose good position representatives for them. In our addition construction $f_{\alpha+\beta}(\partial D^p_+) = u$, thus $f_{\alpha+\beta}$ factors through a wedge of p-spheres. We have the commutative diagram

$$\begin{tikzcd}
S^p \wedge S^q \arrow[r, "S f_{\alpha+\beta}"] \arrow[d] & S^{m-1} \\
(S^p_\alpha \vee S^p_\beta) \wedge (S^q_\alpha \vee S^q_\beta) = \{(S^p_\alpha \wedge S^q_\beta) \vee (S^p_\beta \wedge S^q_\alpha) \vee (S^p_\alpha \wedge S^q_\alpha) \vee (S^p_\beta \wedge S^q_\beta)\}. \arrow[ur, "h"'] &
\end{tikzcd}$$

$h$ is defined by the $S$ construction, hence $h | \{(S^p_\alpha \wedge S^q_\beta) \vee (S^p_\beta \wedge S^q_\alpha)\}$ is null homotopic. Therefore, up to homotopy, $S(\alpha+\beta)$ is determined by $h | \{(S^p_\alpha \wedge S^q_\alpha) \vee (S^p_\beta \wedge S^q_\beta)\} = S(\alpha) \vee S(\beta)$. Therefore $S(\alpha+\beta) = S(\alpha) + S(\beta)$. This completes the proof of the lemma.

<u>Lemma 2</u> $S\varphi\colon \pi_p(S^{m-q-1}) \to \pi_{p+q}(S^{m-1})$ equals $(-1)^q \Sigma^q$, where $\Sigma^q$ denotes q-fold suspension.

Proof: This is trivial if $p+q < m-1$. If $p+q = m-1$, then $\pi_{p+q}(S^{m-1}) = Z$, and it suffices to show that $S\varphi$ takes

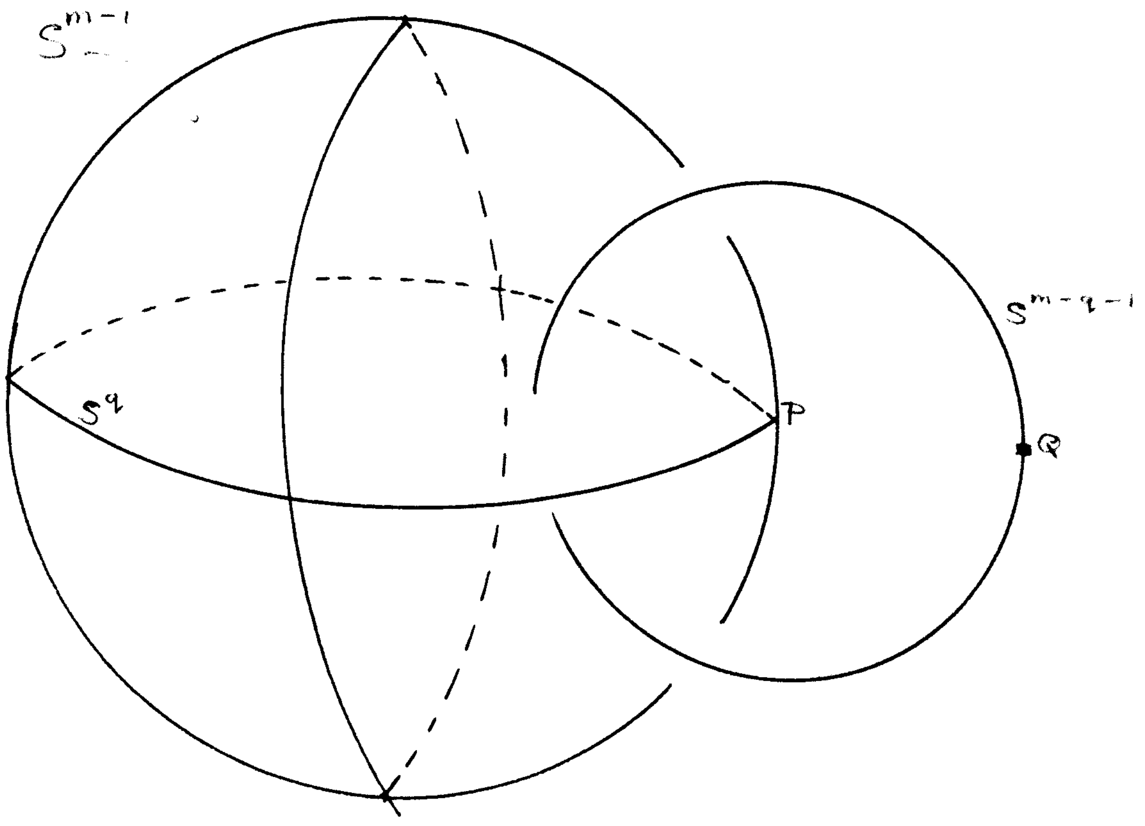


Fig. 1.

generator to generator, as $S\varphi$ is a homomorphism. Let $\mu$ denote the orientation preserving generator of $\pi_{p+q}(S^{m-1})$. Consider the standard $S^q \subset S^m$, and choose a "nice" embedding of a complementary $S^{m-q-1}$ to represent $\varphi(\mu)$, with the orientation induced from the standard choice of orientations of $S^q$ and $S^m$. (See Fig.1)

Fig.1 on opposite page.

P is the point of intersection of the disc spanning $S^{m-q-1}$ with $S^q$. P is a point of $S^{m-1}$, and is a regular point of $S\varphi(\mu)$, and has inverse image one point of $S^{m-q-1} \times S^q$, the point $Q \times P$. Therefore the degree of $S\varphi(\mu)$ is $\pm 1$. In fact the degree of $S\varphi(\mu)$ is $(-1)^q$. This is because of the minus sign in the formula for S. This proves the lemma if $p+q = m-1$.

Now we consider the case when $p+q > m-1$. Let $\alpha \in \pi_p(S^{m-q-1})$, and choose a representative of $\varphi(\alpha)$ which maps $S^p$ onto the "nice" $S^{m-q-1}$ of the first part of the proof. Then we have the commutative diagram,

$$\begin{tikzcd} S^p \wedge S^q \arrow[rr, "S\varphi(\alpha)"] \arrow[dr, "\alpha\wedge 1"'] & & S^{m-1} \\ & S^{m-q-1} \wedge S^q \arrow[ur, "S\varphi(\mu)"'] & \end{tikzcd}$$

Now $\alpha\wedge 1 = \Sigma^q\alpha$, thus $S\varphi(\alpha) = (-1)^q\Sigma^q\alpha$. (When q is odd, $S\varphi(\mu)\Sigma^q\alpha = -\Sigma^q\alpha$ as $\Sigma^q\alpha$ is a suspension.) This completes the proof that $S\varphi(\alpha) = (-1)^q\Sigma^q\alpha$.

The following commutative diagram sums up the work so far,

$$\begin{tikzcd} L^m_{p,q} \arrow[r, "\psi"] \arrow[dr, "\lambda"'] & HL^m_{p,q} \arrow[r, "S"] & \pi_{p+q}(S^{m-1}) \\ & \pi_p(S^{m-q-1}) \arrow[u, "\varphi"] \arrow[ur, "(-1)^q\Sigma^q"'] & \end{tikzcd}$$

<u>Theorem</u> $\varphi: \pi_p(S^{m-q-1}) \to HL^m_{p,q}$ is an isomorphism if $p+2q \leq 2m-4$, and an epimorphism if $p+2q = 2m-3$.

Proof: $\Sigma^q: \pi_p(S^{m-q-1}) \to \pi_{p+q}(S^{m-1})$ is an isomorphism if $p+2q \leq 2m-4$. Therefore $\varphi$ is a monomorphism if $p+2q \leq 2m-4$.

$\varphi$ is epi if, given any element $\alpha$ of $HL^m_{p,q}$ there is a representative $(f,g)$ of $\alpha$, where g is the standard inclusion. Take a PL representative of $\alpha$. By a general position argument, $S^m - f(S^p)$ is $(m-p-2)$-connected. Now, from [18, Chapter 8] any map $S^q \to M^m$, where M is $(m-p-2)$--connected, is homotopic to an embedding if $d+1 \leq m-p-2$, where $d = 2q-m$. Thus we can homotop g to an embedding in the complement of $f(S^p)$, if $p+2q \leq 2m-3$. By Zeeman's Unknotting Theorem, [28, Chapter 4], we can ambient isotop g to the standard inclusion. Therefore $\varphi$ is epi if $p+2q \leq 2m-3$, proving the required result.

# CHAPTER II

## CHAPTER II

## A NOTE ON PIECEWISE-LINEAR IMMERSIONS

Let V, X be piecewise-linear(PL) manifolds and TV, TX denote their tangent micro-bundles. Let Im(V,X) denote the space of PL immersions of V in X and R(TV,TX) the space of bundle monomorphisms of TV in TX. In [8], Haefliger and Poenaru defined "topologies" for these spaces, by making them into semi-simplicial complexes, and showed that they were weakly homotopy equivalent. The work of Rourke and Sanderson, [18] and [19], has shown that block bundles are more natural tools for use in the PL category than micro-bundles. The purpose of this note is to prove the corresponding result to [8], using block bundles instead of micro-bundles. To do this, we have to define a new "topology" on Im(V,X). A result of Haefliger's, [7, §9.2], shows that the new space has the same number of components as the old, but, in general, the higher homotopy groups will not be the same. The proofs follow those of [8] and use the main result of [8]. I am grateful to B. J. Sanderson for suggesting this work and for much helpful advice.

### §0. Definitions

All the work is in the PL category. $\Delta^k$, $I^k$ denote the standard k-simplex and k-cube respectively.

Let V, X be PL manifolds.

Def.1 A submanifold $V^n$ of $X^{n+q}$ is locally flat if, given a point x of V, there is a neighbourhood, N, of x in X and a homeomorphism $(N, N\cap V) \to (R^{n+q}, R^n)$.

Def.2 A map $f: V \to X$ is an embedding if f is a homeomorphism onto a locally flat submanifold of X.

Def.3 A map $f: V \to X$ is an immersion if f is locally an embedding.

Def.4 A map $f: \Delta^k \times V \to \Delta^k \times X$ is block preserving if $f^{-1}(\sigma\times X) = \sigma\times V$ for any face, $\sigma$, of $\Delta^k$.

We make the same definition if we replace $\Delta^k$ by $I^k$.

Def.5 A concordance of two embeddings ( immersions) $f, g: V \to X$ is a block preserving embedding ( immersion) $F: V\times I \to X\times I$ such that $F_0=f$ and $F_1=g$.

If such an F exists, f and g are said to be concordant.

The definition and some of the theory of block bundles will be assumed. If $\xi$ is a block bundle over a polyhedron K and L is a subpolyhedron of K, then $\xi|L$ denotes the restriction of $\xi$ to L.

Let $\xi^n$, $\eta^{n+q}$ be block bundles over simplicial complexes K, L respectively.

Def.6 A map $f: (E(\xi), K) \to (E(\eta), L)$, such that $f|K$ is simplicial, is a block bundle map if, for any simplex, $\sigma$, of K, there are charts $\varphi$, $\psi$ for $\xi|\sigma$, $\eta|f\sigma$ such that the following diagram commutes.

$$\begin{array}{ccc} E(\xi|\sigma) & \xrightarrow{f} & E(\eta|f\sigma) \\ \varphi\downarrow\cong & & \cong\downarrow\psi \\ \sigma\times I^n & \xrightarrow[f\times i]{} & f\sigma\times I^{n+q} \end{array}$$

i is the natural inclusion map $I^n \to I^{n+q}$.

Def.7 If $\xi$ is a block bundle over K and L is another polyhedron, then $\underline{\xi\times L}$ denotes $\pi^*\xi$ where $\pi$: $K\times L \to K$ is projection.

Def.8 Two block bundle maps f, g: $\xi|K \to \eta|L$ are homotopic if there is a block bundle map F: $\xi\times I \to \eta\times I$ such that $F|K\times I$ is block preserving, and $F_0=f$, $F_1=g$.

Def.9 Let f: $V \to X$ be an immersion. Then we define the homotopy class of df: $TV \to TX$ as follows. The map $f\times f$: $V\times V \to X\times X$ induces a map of the tangent micro-block bundles of V and X. As the natural map $\underline{PL}_q \to \underline{PL}_q(\mu)$ is a homotopy equivalence, see [14], this defines a homotopy class of block bundle maps $TV \to TX$. We choose a map from this homotopy class and call it df. As we only ever want the homotopy class of df, no ambiguity occurs.

It is obvious that if f is concordant to g then df is homotopic to dg.

## §1. The semi-simplicial complexes Im(V,X), Pl(V,X), R(TV,TX).

We use Im rather than Im, to distinguish our space from that used in [5].

Im(V, X) ( Pl(V, X)) has, as k-simplices, block preserving immersions ( embeddings) $\Delta^k\times V \to \Delta^k\times X$, such that

$\Delta^k \times o \to \Delta^k \times X$ is fibrewise over $\Delta^k$. ( From now on, all manifolds will have a base point in each component.) Both have the obvious boundary maps.

$\underline{R}(TV, TX)$ has, as k-simplices, block bundle maps $\Delta^k \times TV \to TX$. It also has the obvious boundarymaps.

As with $\underline{PL}_q$, see [18], these complexes have no degeneracies, but, from results of [19], we see that this does not matter for our purposes. All three complexes satisfy the Kan condition as $\Delta^k \cong \Lambda^k \times I$. Thus we can define their homotopygroups. $\pi_0(\underline{Im}(V, X))$ is the set of concordance classes of immersionsof V in X.

Given a block preserving immersion $f: \Delta^k \times V \to \Delta^k \times X$, define $f': \Delta^k \times V \times V \to X \times X$ by $f'(t,u,v) = (\pi f(t,u), \pi f(t,v))$, where $\pi: \Delta^k \times X \to X$ is projection. As in Def.9 this defines $df: \Delta^k \times TV \to TX$ up to homotopy. By an induction on k, we can choose representatives so as to define a semi-simplicial map $d: \underline{Im}(V, X) \to \underline{R}(TV, TX)$. d is unique up to homotopy.

The main theorem of this paper is

__Theorem__ $d: \underline{Im}(V, X) \to \underline{R}(TV, TX)$ is a weak homotopy equivalence, (w.h.e.), if $\dim X > \dim V$.

A w.h.e. is a map inducing bijections on $\pi_0$ and isomorphisms of all homotopy groups of corresponding components.

The scheme of the proof is the same as in [8] . Thus we must prove the following three lemmas.

<u>Lemma 1</u> d: $\underline{\text{Im}}(I^n, X) \to \underline{R}(TI^n, TX)$ is a w.h.e.

<u>Lemma 2</u> If V' is a submanifold of V, then the restriction map $\underline{R}(TV, TX) \to \underline{R}(TV', TX)$ is a fibration.

<u>Lemma 3</u> If V' is a locally flat submanifold of V, then the restriction map $\underline{\text{Im}}(V, X) \to \underline{\text{Im}}(V', X)$ is a fibration if dimX > dimV, or dimX = dimV and every component of V has non-empty boundary and every component of V-V' meets this boundary.

<u>Remark</u> The condition of Lemma 3 is equivalent to requiring that V equals V' union handles of index strictly less than dimX.

~~A fibration is a map with the covering homotopy property for cubes.~~

The proof for the case when V is compact now follows by an induction on the number of handles in some handle decomposition of V, and Lemma 1 provides the starting point. The induction uses the exact sequences of the fibrations of Lemmas 2 and 3. This is justified by the results of [19]. We can now extend to the general case by using the results of [16].

## <u>§2. Proof of the main theorem</u>

Lemma 1 is proved exactly as in [8] using the equivalence of block bundles and micro-block bundles.

<u>Lemma 2</u> The restriction map $\underline{R}(TV, TX) \to \underline{R}(TV', TX)$ is a fibration.

Proof: Factor the map, in the obvious way, as follows;

$$\underline{R}(TV, TX) \xrightarrow{\varphi} \underline{R}(TV|V', TX) \xrightarrow{\psi} \underline{R}(TV', TX).$$

It suffices to show that both $\varphi$ and $\psi$ are fibrations. Let $I \times I^h \times V' \cup 0 \times I^h \times V = P$. $\varphi$ is a fibration if, given a block bundle map $I \times I^h \times TV|P \to TX$, we can extend to a block bundle map $I \times I^h \times TV \to TX$. But $I \times I^h \times V$ deformation retracts onto $P$. Thus we can use the covering homotopy property for block bundles to obtain the required extension. Therefore $\varphi$ is a fibration.

To show $\psi$ is a fibration we first prove the following

Sub-lemma Let $m < n$, $I^m \subset I^n$ be the standard inclusion. Then the restriction map $\underline{Pl}(I^n, I^{n+q}) \to \underline{Pl}(I^m, I^{n+q})$ is a fibration.

Remark $\underline{Pl}(I^n, I^{n+q})$ is called $\tilde{V}_{n+q,n}$, the $\underline{PL}$ Stiefel manifold, by Rourke and Sanderson in [18, Part 3].

Proof of sub-lemma: Given a block preserving embedding $f: I \times I^h \times I^m \cup 0 \times I^h \times I^n \to I \times I^h \times I^{n+q}$, we want to extend $f$ to an embedding $I \times I^h \times I^n \to I \times I^h \times I^{n+q}$. Let $\eta^{n-m}$ be the standard trivial normal bundle of the standard inclusion $I^h \times I^m \subset I^h \times I^n$. Let $\nu^q$ denote the restriction to $0 \times I^h \times I^m$ of the normal block bundle of $f(0 \times I^h \times I^n) \subset 0 \times I^h \times I^{n+q}$. Then $\eta \oplus \nu$ is the normal block bundle of $f(0 \times I^h \times I^m) \subset 0 \times I^h \times I^{n+q}$. From results of [18], we can choose $\zeta^{n+q-m}$ to be the normal block bundle of $f(I\times I^h\times I^m) \subset I\times I^h\times I^{n+q}$ so that $\zeta|(0\times I^h\times I^m) = \eta\oplus\nu$. Again, by a result of [18], this implies $\zeta \cong (\eta\oplus\nu)\times I \cong (\eta\times I)\oplus(\nu\times I)$. $\eta\times I$ defines the required embedding of $I\times I^h\times I^n$ in $I\times I^h\times I^{n+q}$. This completes the proof of the sub-lemma.

To prove $\psi$ is a fibration, we must show that, given a block bundle map $0\times I^h\times(TV|V')\cup I\times I^h\times TV' \to TX$, we can extend to a block bundle map of $I\times I^h\times(TV|V')$. Triangulate $I\times I^h\times V'$ and $I\times I^h\times X$ so that the base map is simplicial. Now subdivide our triangulation of $I\times I^h\times V'$ so that it collapses simplicially to $0\times I^h\times V'$. We can now apply our sub-lemma to extend as required simplex by simplex, using the fact that block bundles over simplices are trivial. This completes the proof of Lemma 2.

<u>Lemma 3</u> The restriction map $\underline{\mathrm{Im}}(V, X) \to \underline{\mathrm{Im}}(V', X)$ is a fibration if $\dim X > \dim V$, or $\dim X = \dim V$ and every component of $V$ has non-empty boundary and every component of $V-V'$ meets this boundary.

Proof: We first note that it suffices to demonstrate the CHP for $I^1$. For suppose we have CHP for $I^1$, and suppose given a block preserving immersion $0\times I^h\times V\cup I\times I^h\times V' \to I\times I^h\times X$. We use our hypothesis applied to the restriction map $\underline{\mathrm{Im}}(I^h\times V, I^h\times X) \to \underline{\mathrm{Im}}(I^h\times V', I^h\times X)$ to obtain an immersion of $I\times I^h\times V$ in $I\times I^h\times X$ with all the required properties.

Now suppose given a block preserving immersion $F\colon 0\times V \cup I\times V' \to I\times X$. By [8, §7], an immersion $f\colon V^n \to X^{n+q}$ has a regular neighbourhood. i.e. there is an abstract regular neighbourhood $\Omega^q$ of $V$ and a commutative diagram,

$$\begin{tikzcd} V^n \arrow[r, "i"] \arrow[dr, "f"'] & \Omega^q \arrow[d, "\varphi"] \\ & X^{n+q} \end{tikzcd}$$

where $i$ is inclusion and $\varphi$ is an immersion. Using the same method as in the proof of the sub-lemma of Lemma 2, we see that $F$ can be extended to a block preserving immersion $F'\colon 0\times V \cup I\times N \to I\times X$, where $N$ is a regular neighbourhood of $V'$ in $V$.

Let $N'$ be a regular neighbourhood of $N$ in $V$, and let $\Omega^q$ be a regular neighbourhood of $F_0'|N'\colon N' \to X$ such that $\partial N' \subset \partial\Omega$. We have the commutative diagram

$$\begin{tikzcd} N' \arrow[r, "i"] \arrow[dr, "F_0'|"'] & \Omega \arrow[d, "\varphi"] \\ & X \end{tikzcd}$$

where $i$ is inclusion and $\varphi$ is an immersion. There exists $\varepsilon > 0$, and an embedding $j$ such that the following diagram commutes

$$\begin{tikzcd} {[0,\varepsilon]\times N} \arrow[r, "j"] \arrow[dr, "F'|"'] & I\times\Omega \arrow[d, "1\times\varphi"] \\ & I\times X \end{tikzcd}$$

where $j|(0\times N) = i|N$. By the uniqueness theorem for collars, [24, Chapter 5], there is an ambient isotopy $H$ of $I\times\Omega$ such that $H_0 = 1$, $H_1 j = 1\times i$ and $H_t|(0\times\Omega\cup I\times\partial\Omega) = 1$. The map $(1\times\varphi)H_1^{-1}(1\times i)$: $[0,\varepsilon]\times N' \to I\times X$ is an immersion which extends $F'$ and equals $1\times F_0'$ on $[0,\varepsilon]\times\partial N'$. Thus we can extend $F'$ to an immersion of $[0,\varepsilon]\times N'\cup I\times N$ in $I\times X$, and can extend this at once to an immersion of $[0,\varepsilon]\times V\cup I\times N$. Now there is a homeomorphism of $I\times V$ with $[0,\varepsilon]\times V\cup I\times N$, fixed on $0\times V\cup I\times V'$. Thus we can extend $F$ to an immersion $G$: $I\times V \to I\times X$, but $G$ is not necessarily block preserving.

Now, keeping $0\times V\cup I\times V'$ fixed, homotop $G$ to $G'$, where $G'(1\times V) \subset (1\times X)$. Cover this homotopy by a micro-bundle homotopy of $dG$ to obtain a new micro-bundle map $\Phi$: $T(I\times V) \to T(I\times X)$ covering $G'$. (We use micro-bundles for this part of the proof as we want to quote the result of [8].) Restricting $\Phi$ to $T(I\times V)|(1\times V)$ we have $\varphi=\Phi|$: $TV\oplus\varepsilon^1 \to TX\oplus\varepsilon^1$ where $\varphi = dG\oplus 1$ when we restrict to $T(I\times V)|(1\times V')$. As the natural map $\pi_i(V^{PL}_{n+q,n}) \to \pi_i(V^{PL}_{n+q+1,n+1})$ is an isomorphism if $i \leqslant n$, see [8], we can bundle homotop $\varphi$ to $\varphi_1 = \psi\oplus 1$, keeping our maps fixed on the base and over $(1\times V')$. Now apply the relative form of the main theorem of [8] to the bundle map $\psi$: $TV \to TX$. We can homotop $G'|(1\times V)$ to an immersion, keeping $(1\times V')$ fixed. Extend this homotopy and apply the theorem again to homotop the new map to an immersion of $I \times V$ in $I \times X$ which is block preserving and extends $F$ as required. This completes the proof of the main theorem. The relative form of the theorem is an immediate consequence by using a Five Lemma argument applied to the exact sequences of the restriction fibrations.

Remark The proof in [8] of the fibration lemma for immersion spaces cannot be adapted to this case as it uses the covering isotopy theorem of Hudson and Zeeman, and the fact that a subcube of a cube of embeddings is itself a cube of embeddings. The covering concordance theorem is false in codimension two by the example of a slice knot in $R^3$, and a subcube of a cube of embeddings which is not fibre-wise over the cube is meaningless.

Remark One consequence of this result is that the natural map $\pi_n(PL_{n+1}) \to \pi_n(\underline{PL}_{n+1})$ is an isomorphism. For both groups are isomorphic to the set of regular homotopy classes of immersions of $S^n \times I$ in $S^{n+1}$. This isomorphism can also be proved directly using the braid of the triple $(\underline{PL}_{n+1}, PL_{n+1}, O_{n+1})$. See Chapter 3, §3 of this thesis.

# CHAPTER III

# CHAPTER III

## OPEN AND CLOSED DISC BUNDLES

In this paper, we are concerned with the relationship between $R^n$ and $D^n$ fibre bundles in the piecewise-linear (PL) category, and n-plane vector bundles. In [2], W. Browder showed that the theories of $R^n$ and $D^n$ bundles (the open and closed bundles of the title) are not equivalent for all n, though no example of an $R^n$ bundle which is not a $D^n$ bundle is known. We prove that the theories of all three types of bundle are equivalent in the case $n = 2$, by proving that the groups of the bundles are homotopy equivalent. This has also been proved by Akiba [1], by a different proof. The case $n = 1$ is trivial.

The paper falls into four sections. §0 contains the definitions and basic results and the statement of the main theorem. §1 contains some useful lemmas. In §2, we prove the main theorem. §3 contains a proof that the natural map $\pi_n(PL_{n+1}) \to \pi_n(\underline{PL}_{n+1})$ is an isomorphism and some easy deductions from this fact. After it was written, I discovered that much of the material in §0 appears in [14].

## §0. Definitions and Basic Results

All manifolds, homeomorphisms and embeddings will be PL. $R^n$ is Euclidean n-space, $o$ denotes the origin of $R^n$ unless otherwise stated. $D^n$ is the standard n-cube $[-1,1]^n \subset R^n$. $S^{n-1}$ is the boundary of $D^n$, also denoted by $\partial D^n$. I is the unit interval $[0,1]$. $\Delta^n$ is the standard n-simplex embeddedin $R^n$. The vertices are numbered 0 up to n so that the 0 vertex lies at the origin and the r vertex lies on the $r^{th}$ axis, unit distance from the origin. For $m < n$, we have standard embeddings $R^m \subset R^n$, $D^m \subset D^n$, $\Delta^m \subset \Delta^n$, as the first m coordinates of $R^n$, all of which commute with the inclusions just defined, and respect the ordering of the vertices of $\Delta^m$ and $\Delta^n$.

$\phi$ denotes the empty set. $*$ denotes the topological space with one point. If Y is a topological subspace of X, then $\overline{Y}$ denotes the closure of Y in X. If X is a manifold then $\mathring{X}$ denotes $X-\partial X$.

Let Y be a submanifold of the manifold X.

<u>Def. 0.1</u> $H_Y(X)$ is defined to be the semi-simplicial (s.s.) complex whose k-simplices are homeomorphisms $\Delta^k \times X \to \Delta^k \times X$ which commute with projection onto $\Delta^k$, and such that the restriction to $\Delta^k \times Y$ is the identity. It has the obvious boundary and degeneracy maps.

If $X$ is orientable, $SH_Y(X)$ denotes the subcomplex of $H_Y(X)$ of orientation preserving homeomorphisms. If $Y = \phi$, we write $H(X)$ and $SH(X)$.

$H_Y(X)$ is a Kan complex, see [22], hence we can define its homotopy groups. We can now state the main result of the paper.

Theorem 2.1

$O_2$, $H_o(D^2)$, $H_o(R^2)$ are all homotopy equivalent.

$H_o(D^n)$, $H_o(R^n)$ are usually denoted by $PL_n(I)$, $PL_n(R)$ respectively. See [18]. They are the groups of $D^n$ and $R^n$ bundles, respectively, in the PL category.

$O_n$ denotes the singular complex of the group of isometries of $R^n$ keeping the origin fixed. We have maps $O_n \to H_o(D^n) \to H_o(R^n)$. The second map is defined in a natural way by choosing a homeomorphism of $R^n$ with $\mathring{D}^n$ once and for all. The first map is not naturally well-defined, but belongs to a well-defined homotopy class of maps. To obtain this homotopy class, we define a s.s. complex $PD_n$. There is a natural map $O_n \to PD_n$ and a natural map $H_o(D^n) \to PD_n$ which is a homotopy equivalence. The homotopy inverse of the second map defines the required homotopy class by composition with the first. For details, see [15].

Remark 1 $O_n$, $H_o(D^n)$, $H_o(R^n)$ all have two components and these two are homotopy equivalent. Thus we need only consider the identity components.

Remark 2 It is trivial to prove that $SO_1$, $SH_o(D^1)$, $SH_o(R^1)$ are all contractible.

For the following definitions suppose $Y^n$ is a submanifold of $X^m$, and $i: Y \to X$ the inclusion map.

Def. 0.2 $Y^n$ is a locally flat submanifold of $X^m$ if, given $y \in Y$, there is a neighbourhood $N$ of $y$ in $X$ and a homeomorphism $(N, N \cap Y) \to (D^m, D^n)$, the standard ball pair.

From now on, all submanifolds will be locally flat. By Zeeman's Theorem on unknotting ball pairs, [26, Chapter 4], this condition is automatically satisfied if $m \geqslant n+3$.

Let $Z$ be a submanifold of $Y$.

Def. 0.3 $E_Z(Y^n, X^m)$ is defined to be the s.s. complex whose k-simplices are embeddings $f: \Delta^k \times Y \to \Delta^k \times X$ which commute with projection onto $\Delta^k$ and such that

(i) $f|(\Delta^k \times Z) = (1 \times i)|(\Delta^k \times Z)$,

(ii) $f^{-1}(\Delta^k \times \partial X) = \Delta^k \times i^{-1}(\partial X)$,

(iii) given $(t,y) \in \Delta^k \times Y$, there is a closed neighbourhood $U$ of $t$ in $\Delta^k$, a closed neighbourhood $V$ of $y$ in $Y$, and an embedding $\alpha: U \times V \times D^{m-n} \to \Delta^k \times X$ such that the image of $\alpha$ is a closed neighbourhood of $f(t,y)$ in $\Delta^k \times X$ and the following diagram commutes, where $\pi$, $\pi'$ are projections onto the first factor.

$$\begin{array}{ccccc} U\times(V\times o) & \subsetneq & U\times(V\times D^{m-n}) & \xrightarrow{\pi} & U \\ \downarrow\subset & & \downarrow a & & \downarrow\subset \\ \Delta^k\times Y & \xrightarrow{f} & \Delta^k\times X & \xrightarrow{\pi'} & \Delta^k \end{array}.$$

<u>Remark</u> Condition (iii) is a local flatness condition, and, again by Zeeman's theorem on unknotting of ball pairs, is automatically satisfied if $m \geqslant n+3$.

$E_Z(Y,X)$ has the obvious boundary and degeneracy maps. If $Z = \phi$, we write $E(Y,X)$. $E_Z(Y,X)$ also satisfies the Kan condition and thus we can do homotopy theory with it. In particular we will make use of the following

<u>Theorem 0.4</u> Let $f: K \to L$ be a map of connected Kan complexes such that $f_*: \pi_i(K) \to \pi_i(L)$ is an isomorphism for $i \geqslant 1$. Then $f$ is a homotopy equivalence.

The following theorem of Hudson, see [13], and its Corollary will play an important part in the proofs.

Given $f: \Delta^k\times Y \to \Delta^k\times X$, a k-simplex of $E_Z(Y,X)$, we write $f_t$ for $f|: t\times Y \to t\times X$.

<u>Def. 0.5</u> $Y^n$ is an allowable submanifold of $X$ if $i^{-1}(\partial X)$ is a $(n-1)$-submanifold of $\partial Y$ or is empty.

This condition is trivially satisfied if $\partial X$ is empty.

<u>Theorem 0.6</u> (Hudson) If $Y$ is an allowable submanifold of $X$ and $f: \Delta^k\times Y \to \Delta^k\times X$ is a k-simplex of $E_{i^{-1}(\partial X)}(Y,X)$, then there is $h: \Delta^k\times X \to \Delta^k\times X$, a k-simplex of $H_{\partial X}(X)$, such that $h(1\times f_0) = f$ and $h_0 = 1_X$.

Corollary 0.7 If $Y$ is an allowable submanifold of $X$, then the restriction map $H_{\partial X}(X) \to E_{i^{-1}(\partial X)}(Y,X)$ is a fibration.

Proof: Suppose given $G: I^k \times X \to I^k \times X$ given by a map $I^k \to H_{\partial X}(X)$, and $F: I \times I^k \times Y \to I \times I^k \times X$ given by a map $I \times I^k \to E_{i^{-1}(\partial X)}(Y,X)$ such that $F(o,\tau,y) = G_\tau F(o,*,y)$ where $\tau \in I^k$ and $*$ is a vertex of $I^k$.

By Theorem 0.6, there is $H: I \times I^k \times X \to I \times I^k \times X$ given by a map $I \times I^k \to H_{\partial X}(X)$ such that $F(t,\tau,y) = H_{t,\tau}F(o,*,y)$ where $t \in I$, $\tau \in I^k$. Consider $H': I \times I^k \times X \to I \times I^k \times X$ defined by $H'_{t,\tau} = H_{t,\tau}H^{-1}_{o,\tau}G_\tau$. $H'$ is given by a map $I \times I^k \to H_{\partial X}(X)$, $H'_{o,\tau} = G_\tau$ and $F(t,\tau,y) = H'_{t,\tau}F(o,*,y)$. This completes the proof of the Corollary.

Remark The restriction map $H(X) \to H(\partial X)$ is also a fibration, as the analogue of Theorem 0.6 holds when $Y = \partial X$. For let $f: \Delta^k \times \partial X \to \Delta^k \times \partial X$ be given by a map $\Delta^k \to H(\partial X)$. Of course this map is null-homotopic and the null-homotopy $\Delta^k \times I \to H(\partial X)$ gives us $h: \Delta^k \times I \times \partial X \to \Delta^k \times I \times \partial X$ such that $h(1 \times f_0) = f$, $h_0 = 1$ and $h|(\Delta^k \times 1 \times \partial X) = 1$. By the Collaring Theorem for boundaries of manifolds, see [26, Chapter 5], which says that $\partial X$ has a neighbourhood in $X$ homeomorphic to $\partial X \times I$, we see that $h$ extends to $h': \Delta^k \times X \to \Delta^k \times X$ defined to be the identity outside $\partial X \times I$, and $h'$ satisfies the conditions of Theorem 0.6.

The existence of compatible collars, see [26] again, implies that if Y is an allowable submanifold of X, then the restriction map $H_Y(X) \to H_{i^{-1}(\partial X)}(\partial X)$ is also a fibration.

We can now generalise Corollary 0.7 to the following statement: If M is a union of components of $\partial X$ and $N = i^{-1}(M)$, then the restriction map $H_M(X) \to E_N(Y,X)$ is a fibration.

For examples of all the types of fibration mentioned here, see Diagram 2 in §1.

## §1 Preliminary Lemmas

We will need the following lemmas.

**Lemma 1.1** $SH_{D^m \cup \partial D^n}(D^n) \simeq *$, for $-1 \leq m < n$, where $D^{-1}$ denotes the empty set.

Proof: We use the Alexander trick as follows.

Let $h: \Delta^k \times D^n \to \Delta^k \times D^n$ represent an element of $\pi_k(H_{D^m \cup \partial D^n}(D^n))$. We want to define $H: I \times \Delta^k \times D^n \to I \times \Delta^k \times D^n$ to be a homotopy of h to the identity homeomorphism.

Embed $I \times \Delta^k \times D^n$ in $R^1 \times R^k \times R^n \cong R^{n+k+1}$ in the standard way to give us a linear structure.

Define $h': \partial(I \times \Delta^k \times D^n) \to \partial(I \times \Delta^k \times D^n)$ by $h'|(0 \times \Delta^k \times D^n) = h$, h' is the identity on the rest. These are compatible as $h|\partial(\Delta^k \times D^n) = 1$.

Define $u \in I \times \Delta^k \times D^n$ to be the point $(\frac{1}{2}, x, o)$, where x is an interior point of $\Delta^k$. Then $I \times \Delta^k \times D^n \cong u * \partial(I \times \Delta^k \times D^n)$, where * denotes join. Triangulate $\partial(I \times \Delta^k \times D^n)$ so that h' is simplicial. now define $H(u) = u$ and complete the definition of $H$ by first defining $h'$ to be extended linearly to $u^*$( the zero skeleton of $\partial(I\times\Delta^k\times D^n)$, and then extended linearly to all of $I\times\Delta^k\times D^n$, by working up the skeletons. Clearly $H$ is a homeomorphism. Also $H$ is the identity on $1\times\Delta^k\times D^n \cup I\times\partial\Delta^k\times D^n \cup I\times\Delta^k\times\partial D^n$ by definition. Finally, by the linearity of the construction, $H$ commutes with projection onto $I\times\Delta^k$ and is the identity on $I\times\Delta^k\times D^m$ as required. This completes the proof.

Corollary 1.2 $SH_{D^m}(D^n) \simeq SH_{S^{m-1}}(S^{n-1})$, for $-0\leq m<n$, where $S^{-1}$ denotes the empty set.

Proof: Consider the exact sequence of the fibration $H_{D^m\cup\partial D^n}(D^n) \to SH_{D^m}(D^n) \to SH_{S^{m-1}}(S^{n-1})$, and use the fact that the spaces are connected. See [21, Chapter 8].

Corollary 1.3 The standard component of $E_{\partial D^m}(D^m, D^n)$ is contractible.

The standard component means the component of $i$, the standard inclusion $Y \to X$.

Proof: Consider the exact sequence of the fibration $H_{D^m\cup\partial D^n}(D^n) \to H_{\partial D^n}(D^n) \to E_{\partial D^m}(D^m,D^n)$, and apply Lemma 1.1 to the total space and fibre.

Lemma 1.4 (Hirsch) If $K$ is a simplicial complex, then

$$H_{K\times 0}(K\times[0,\infty)) \simeq *.$$

Proof: See [11].

<u>Lemma 1.5</u> $SH(D^n) \simeq SH(R^n)$ if and only if

$$H_{S^{n-1}\times 0}(S^{n-1}\times I) \simeq *.$$

Proof: We define a restriction map $SH(D^n) \to E(D^n,R^n)$, by restricting our attention to $[-\frac{1}{2},\frac{1}{2}]^n \subset D^n$. Consider the commutative diagram of fibrations,

$$\begin{array}{ccccc}
H_{S^{n-1}\times 0}(S^{n-1}\times I) & \to & SH(D^n) & \to & E(D^n,R^n) \\
\downarrow & & \downarrow & & \downarrow = \\
H_{S^{n-1}\times 0}(S^{n-1}\times[0,\infty)) & \to & SH(R^n) & \to & E(D^n,R^n)
\end{array}$$

From Lemma 1.4, we see that $SH(R^n) \simeq SE(D^n,R^n)$. This proves the result by considering the top exact sequence.

<u>Remark</u> $SH(D^n)$, $SH(R^n)$ are homotopy equivalent to $SH_o(D^n)$, $SH_o(R^n)$ respectively. For consider the exact sequence of the fibration $SH_o(D^n) \to SH(D^n) \to E(o,R^n)$. It is trivial that, for any X, $E(*,X)$ has the same homotopy groups as X. Hence $E(o,R^n)$ is contractible.

<u>Lemma 1.6</u> $SO_n \simeq SH_o(R^n)$ if and only if $SO_{n+1} \simeq SH_o(D^{n+1})$.

Proof: Let o be a point of $S^n$, and consider the following commutative diagram of fibration exact sequences,

$$\begin{array}{ccccccc}
\to & \pi_r(SO_n) & \to & \pi_r(SO_{n+1}) & \to & \pi_r(S^n) & \to \\
 & \downarrow & & \downarrow & & \downarrow\cong & \\
\to & \pi_r(SH_o(S^n)) & \to & \pi_r(SH(S^n)) & \to & \pi_r(E(o,S^n)) & \to
\end{array}$$

By the Five Lemma, it follows that $SO_n \simeq SH_o(S^n)$ if and only if $SO_{n+1} \simeq SH(S^n)$. Now Corollary 1,2 says that $SH(S^n) \simeq SH(D^{n+1})$, and we use the result of [11], whose proof we give below, that $SH_o(S^n) \simeq SH(R^n)$ to complete the proof of Lemma 1.6.

There is a restriction map $SH_o(S^n) \to SH(R^n)$, as $S^n - o \cong R^n$. Let $D^n$ be a hemisphere of $S^n$ with $o \notin D^n$, and consider the commutative diagram of fibrations below,

$$\begin{tikzcd}
SH_{o \cup D^n}(S^n) \arrow[r] \arrow[d] & SH_o(S^n) \arrow[r] \arrow[d] & E(D^n, R^n) \arrow[d, "="] \\
H_{S^{n-1} \times 0}(S^{n-1} \times [0, \infty)) \arrow[r] & SH(R^n) \arrow[r] & E(D^n, R^n)
\end{tikzcd}$$

Now $SH_{o \cup D^n}(S^n)$ is isomorphic to $SH_{\partial D^n \cup o}(D^n)$, which is contractible by Lemma 1.1. Therefore by the Five Lemma and Lemma 1.4, we see that $SH_o(S^n) \simeq SH(R^n)$. This proof is in [U].

Now consider the following commutative diagram. Each row and column is a fibration.

$$\begin{tikzcd}
H_{S^n \times \partial I \cup I}(S^n \times I) \arrow[r] \arrow[d] & H_{S^n \times o \cup I}(S^n \times I) \arrow[r] \arrow[d] & SH_1(S^n) \simeq SH(R^n) \arrow[d] \\
H_{S^n \times \partial I}(S^n \times I) \arrow[r] \arrow[d] & H_{S^n \times o}(S^n \times I) \arrow[r] \arrow[d] & SH(S^n) \simeq SH(D^{n+1}) \arrow[d] \\
E_{\partial I}(I, S^n \times I) \arrow[r] & E_o(I, S^n \times I) \arrow[r, "\varrho"] & E(1, S^n)
\end{tikzcd}$$

Diagram 2

The left and middle vertical fibrations restrict attention to $o \times I \subset S^n \times I$, where $o$ is a point of $S^n$. The three horizontal fibrations restrict attention to $S^n \times 1 \subset S^n \times I$.

The map $\pi_r(E_o(I, S^n \times I)) \to \pi_r(S^n)$ is zero. For take a representative $\alpha$ of an element of $\pi_r(E_o(I, S^n \times I))$ and project onto $S^n$. This defines the required null-homotopy, as $\pi(\alpha|): \Delta^r \times \theta \to \Delta^r \times S^n \times I \to S^n$ is a point map.

All the spaces are connected except for the lower two in the left hand column.

We prove our results by considering the fibration exact sequences obtained from the diagram and proving that $H_{S^n \times o \cup I}(S^n \times I)$ and $E_o(I, S^n \times I)$ are contractible for $n = \cancel{2}\,1$ or $\cancel{3}\,2$. We then apply our previous Lemmas.

## §2. The case n = 2

This section is devoted to proving thefollowing

<u>Theorem 2.1</u> $O_2 \simeq H_o(D^2) \simeq H_o(R^2)$.

Proof: $SO_2 \simeq SH_o(D^2)$. This is a trivial consequence of Lemma 1.6 and the fact that $SO_1 \simeq SH_o(R^1) \simeq *$.

To complete the proof of the theorem it suffices to show that $H_{S^1 \times o}(S^1 \times I)$ is contractible, by Lemma 1.5.

Consider Diagram 2 in the case n = 1. $H_{S^1 \times \partial I \cup I}(S^1 \times I)$ is isomorphic to $H_{\partial D^2}(D^2)$, by "cutting along I", and is therefore contractible, by Lemma 1.1. As $SH(R^1)$ is contractible, we have $H_{S^1 \times o \cup I}(S^1 \times I) \simeq *$.

We now use

<u>Theorem 2.2</u> The standard component of $E_{\partial I}(I, S^1 \times I)$ is contractible.

This implies that $E_o(I, S^1 \times I)$ is contractible, as the map $\pi_r(E_0(I,S^1\times I)) \to \pi_r(S^1)$ is zero for all r. Hence $H_{S^1\times 0}(S^1\times I)$ is contractible. This completes the proof of Theorem 2.1 apart from the proof of Theorem 2.2.

Proof of Theorem 2.2

Define $I_* \subset S^1\times I$ by $I_* = (-o)\times I$.

A representative of an element of $\pi_k(E_{\partial I}(I,S^1\times I))$ is an embedding $\alpha$: $\Delta^k\times I \to \Delta^k\times S^1\times I$. We will show that $\alpha$ can be homotoped so that $\alpha(\Delta^k\times I)\cap(\Delta^k\times I_*) = \phi$. Such an embedding is a representative of $\pi_k(E_{\partial I}(I,D^2))$, as the closure of the complement of a regular neighbourhood of $I_*$ in $S^1\times I$ is homeomorphic to $D^2$. This group is zero by Corollary 1.3, as $k\geq 1$. This provides us with a null-homotopy of $\alpha$ in $E_{\partial I}(I,D^2)$ and hence clearly in $E_{\partial I}(I,S^1\times I)$, which completes the proof.

The proof that we can suppose that $\alpha(\Delta^k\times I)\cap(\Delta^k\times I_*) = \phi$ falls into two lemmas.

Firstly, we can suppose that, in every fibre, $\alpha I\cap I_*$ is a finite number of points. We use the linear structure of $\Delta^k\times S^1\times I\subset R^k\times R^2 \cong R^{k+2}$. Let $\pi$ denote projection of $R^{k+2}$ onto a $(k+1)$-hyperplane perpendicular to $I_*$. Then $\pi\alpha$: $\Delta^k\times I \to R^{k+1}$. By a small, level-preserving, isotopy of $\alpha$ we can make $\pi\alpha$ non-degenerate. i.e. $(\pi\alpha)^{-1}(y)$ is a finite number of points. To do this isotopy, work up the skeletons of a triangulation of $\Delta^k\times I$ in which $\alpha$ is linear on each simplex, moving the barycentre of each simplex and extending linearly.

We can now assign one of the numbers +1, -1, 0 to each point y of $\alpha(\Delta^k\times I)\cap(\Delta^k\times I_*)$. For y lies in a certain fibre and is an isolated point of $\alpha I \cap I_*$ in that fibre. We assign $\pm 1$ to y if I crosses $I_*$ at y, the sign depending on the orientation, and we assign 0 to y if I does not cross $I_*$ at y.

<u>Remark</u> As we are considering the standard component of $E_{\partial I}(I, S^1 \times I)$, the algebraic sum of these numbers in any fibre is zero.

For x in $\Delta^k$, define $n_x$ to be the number of points in $(I \cap I_*)_x$ to which we assign + 1. If $n_x$ is everywhere zero, we use

<u>Lemma 2.2.1</u>

If $n_x$ is everywhere zero, we can homotop $\alpha$ so that

$$\alpha(\Delta^k \times I) \cap (\Delta^k \times I_*) = \phi.$$

Proof: As $n_x$ is everywhere zero, all the intersection points are zero points. Consider a regular neighbourhood of $I_*$ in $S^1 \times I$, which does not meet $o \times I \subset S^1 \times I$. This neighbourhood is separated into two parts by $I_*$, and each point of intersection is associated to one of these parts- the side of $I_*$ on which I lies in a small neighbourhood of the point. Choose one side of $I_*$ and choose a small ambient isotopy of the regular neighbourhood on this side which pulls away from $I_*$. Do this isotopy in every fibre simultaneously and this will induce a homotopy of $\alpha$ as we do not disturb $\alpha$ on $\partial\Delta^k \times I$, as our regular neighbourhood does not meet $o \times I$. Do the same on the other side of $I_*$. Now $\alpha(\Delta^k \times I) \cap (\Delta^k \times I_*) = \phi$.

If $n_x$ is not everywhere zero, we use Lemma 2.2.2 and induction to reduce ourselves to that case.

Lemma 2.2.2

If $n_x$ is not everywhere zero, we can reduce $\max(n_x | x \text{ in } \Delta^k)$ by one.

Proof: Denote this number by n. We note that, for any point of $\Delta^k$, the property of having a signed point of intersection in the corresponding fibre is an open property. To put it more precisely, if x in $\Delta^k$ and y in $(I \cap I_*)_x$ is a +1 point, there is a neighbour hood U of x in $\Delta^k$ and a section s of the trivial line bundle $U \times I$ over U such that $s(x) = y$ and $s(z)$ is a +1 point of $(I \cap I_*)_z$, for any point z in U. We say that y persists over U. It may be convenient to denote $s(z)$ by y also.

Define $N = \{x \text{ in } \Delta^k | \; n_x = n\}$. Then N is an open subset of $\Delta^k$. Note that $\bar{N} \subset \mathring{\Delta}^k$. Also note that s of the last paragraph is unique in a neighbourhood of a point x in N.

We remark that if x in N and y in $(I \cap I_*)_x$ is a signed intersection point, then y persists over the compnnent C of N in which x lies. For let $U \subset \Delta^k$ be the maximal connected open neighbourhood of x over which y persists and suppose $C \not\subset U$. Then $(\bar{U}-U) \cap C \neq \phi$. Suppose z in $(\bar{U}-U) \cap C$. As z in C, $n_z = n$, and all the signed intersection points in the fibre over z persist over a neighbourhood V of z in C. But this implies that if w in V, $w \neq z$, then $n_w > n_z$, as y persists over w, which contradicts the maximality of $n_z$. Thus $C \subset U$.

Proposition 2.2.3 $\bar{N}$ is a subpolyhedron of $\Delta^k$.

Proof: Let $U = \alpha(\Delta^k \times I) \cap (\Delta^k \times I_*)$ and triangulate everything so that $\alpha$ is simplicial, $\Delta^k \times I_*$ is a subcomplex of $\Delta^k \times S^1 \times I$ which is a subcomplex of $R^{k+2}$, and so that the projection map $\pi: U \to \Delta^k$ is simplicial. Of course $\overline{N} \subset \pi U$. If $x \in N$, then $x$ lies in the interior of a unique simplex, $\tau^r$, of our triangulation of $\Delta^k$. Now the inverse image under $\pi$ of a point in $\Delta^k$ consists of a finite number of points. Therefore $\pi^{-1}(\tau^r)$ consists of a finite set of r-simplices, which contains n simplices $\sigma_i$ each with the property that $\mathring{\sigma}_i \cap x \times I_*$ is a signed point of intersection in the fibre over x. Using the linear structure of $R^{k+2}$, we see that any interior point t of $\sigma_i$ is a crossing point in the fibre over $\pi(t)$. Thus $\mathring{\tau}^r \subset N$, and hence $\tau^r \subset \overline{N}$. Thus $\overline{N}$ is a union of closed simplices of our triangulation of $\Delta^k$, as required.

<u>Proposition 2.2.4</u> We can suppose there are no zero points of intersection in the fibre over any point of N.

Proof: Choose a real valued function $\varphi$ on $\Delta^k$, zero outside N and positive but small on N. As in Lemma 2.2.1, we now push all the zero points away from $I_*$ in the obvious direction, pushing a distance $\varphi(x)$ in the fibre over x. This does not alter the value of $n_x$ for any point $x \in \Delta^k$. This is obvious outside N, and for points in N uses the fact that no signed intersection point "turns into" a zero point over N.

Without loss of generality, we may suppose that $\overline{N}$ has one component. For if not, we deal with each component in turn without disturbing the others.

Choose $x \in N$. As $n_x \neq 0$, and the number of $-1$ points in the fibre over $x$ equals the number of $+1$ points, we can find two points $y$, $y' \in I$ such that $\alpha y$ is a $+1$ point, $\alpha y'$ is a $-1$ point and $\alpha\{(y,y')\}$ does not meet $I_*$. ( Without loss of generality, we may suppose $y < y'$.) $y$ and $y'$ persist over $N$ and are consecutive intersection points ( in $I$) in every fibre. Now the two intervals spanned by $\alpha y$, $\alpha y'$, one in $\alpha I$ the other in $I_*$, form an embedded $S^1$ in $S^1 \times I$. This uses Prop. 2.2.4. This $S^1$ spans an embedded $D^2$. We now see that we can find $y$, $y'$ so that the interval $(\alpha y, \alpha y') \subset I_*$ also contains no intersection points. For the intersection points in $(\alpha y, \alpha y')$ occur in pairs of opposite sign, and a nested set of intervals of this sort gives us anested set of embedded 2-discs. We simply choose the pair of points corresponding to the innermost 2-disc. It now follows that $\mathring{D}^2$ does not meet $\alpha I$ or $I_*$.

Using this 2-disc, we could define an isotopy of $\alpha|[y,y']$ across the disc to the interval $[\alpha y, \alpha y'] \subset I_*$, and extend to an isotopy of $\alpha|(x \times I)$ by the identity outside $[y,y']$. The idea of the proof is to show that we can do this simultaneously in every fibre over a point of $N$.

$y$, $y'$ determine unique sections $s$, $s'$ of $N \times I \xrightarrow{p} N$, which by continuity extend to unique sections over $\bar{N}$.

Define $Z \subset \overline{N}-N$, by $Z = \{z \in \overline{N} \mid s(z) = s'(z)\}$.

Proposition 2.2.5  $Z$ is a subpolyhedron of $\overline{N}$.

Proof: Triangulate as in the proof of Prop. 2.2.3. Let $x \in Z$, then $x$ lies in the interior of a unique r-simplex $\sigma$ of our triangulation of $\Delta^k$, every point of which lies in $\overline{N}$, from the proof of Prop. 2.2.3. Hence, using the linear structure of $R^{k+2}$, we see that every point of $\sigma$ lies in $Z$. This proves the proposition.

Prop. 2.2.5 shows that thereis a regular neighbourhood $U$ of $Z$ in $\overline{N}$ and a function $\varphi: U \to [0,1]$ such that $\varphi^{-1}(0)=\partial U$, $\varphi^{-1}(1) = Z$. As $E(*,I)$ is contractible, we can suppose that, for all points $z \in Z$, $s(z)$ is the midpoint of $I_*$.

Choose an embedded 2-disc in $S^1\times\mathring{I}$, which meets $I_*$ in a 1-disc in its boundary containing the midpoint of $I_*$. Choose a pseudo-radial contraction of this 2-disc over itself to the midpoint of $I_*$. We can arrange that, with respect to this choice, $\alpha$ is standard over $U$. This means that if $u \in U$ and $\varphi(u) = t$, then $\alpha_u|[y,y']$ looks like part of the boundary of our 2-disc at level $t$ of our contraction. For every point $u \in U$, we can define an isotopy, as required at the bottom of P.33, by using our contraction. This gives the identity isotopy over $Z$. Our problem is to extend our isotopy of $\alpha|U\times I$ over the rest of $\overline{N}$.

Triangulate $\overline{N}$ with U as a subcomplex. Suppose we have extended over the r-skeleton of $\overline{N}-\mathring{U}$ and let $\sigma^{r+1} \in \overline{N}-\mathring{U}$. To each point of $\sigma^{r+1}$ we have associated a proper embedding $D^1 \to D^2 = S^1 \times I$ - a one sided neighbourhood of $I_*$, and to each point of $\partial\sigma^{r+1}$ we have associated an embedding of a 2-disc whose boundary is an arc in $I_*$ together with $D^1$. As $\sigma^{r+1}$ is contractible, we can put our embeddings $D^1 \to D^2$ standard over σ. ∂σ now gives us an element of $\pi_r(E_{\partial D^2}(D^2,D^2))$. Clearly $E_{\partial D^2}(D^2,D^2) \cong H_{\partial D^2}(D^2)$, which is contractible by Lemma 1.1. Thus we can put our 2-discs standard over ∂σ and extend in the obvious way to get an embedded 2-disc for every point of σ. Thus we can indeed extend our isotopy of $\alpha|U\times I$ to an isotopy of $\alpha|\overline{N}\times I$, which in every fibre does as required at the bottom of P. 33. It is easy to extend to an isotopy of α which is the identity outside a small neighbourhood of $\overline{N}$. This ensures that no new intersections are introduced and that we do not disturb any other components of $\overline{N}$ in the case when $\overline{N}$ is not connected.

We now have that in every fibre over a point of $\overline{N}$, $\alpha\{[y,y']\} \subset I_*$. We must push $\alpha|\,[y,y']$ slightly off $I_*$ in the obvious direction, keeping y,y' fixed. We have now achieved the result of Lemma 2.2.2, which completes the proof of Theorem 2.2.

## §3. $\pi_n(PL_{n+1}) \cong \pi_n(\underline{PL}_{n+1})$

This section is devoted to proving the following theorem and deducing some corollaries.

<u>Theorem 3.1</u> The natural map $\pi_n(PL_{n+1}) \to \pi_n(\underline{PL}_{n+1})$ is an isomorphism.

<u>Remark</u> It is well known that the natural map $\pi_n(PL_{n+q}) \to \pi_n(\underline{PL}_{n+q})$ is an isomorphism, if $q \geqslant 2$. See [9].

Proof: the proof will require the following three theorems.

<u>Theorem 3.2</u> (Hsiang, Levine, Sczarba)

A homotopy n-sphere smoothly embedded in $R^{n+k}$ has a trivial normal bundle, if $k \geqslant n-2$.

Proof: See [12].

<u>Theorem 3.3</u> $\pi_n(SO_{n+1}) \cong FCI_n^1$ = the set of regular homotopy classes of smooth immersions of $S^n \times R$ in $R^{n+1}$.

Proof: By standard obstruction theory, using the immersion theorem of Hirsch.

<u>Theorem 3.4</u> (Haefliger, Wall)

$\pi_n(PL_{n+1}) \cong FTI_n^1$ = the set of regular homotopy classes of smooth immersions of $S^n_\alpha \times R$ in $R^{n+1}$, where $\alpha$ ranges over all smoothings of $S^n$.

Proof: By standard obstruction theory, using the immersion theorem of Haefliger and Poenaru, [8], we see that $\pi_n(PL_{n+1}) \cong FBI_n^1$ = the set of regular homotopy classes of PL immersions of $S^n \times R$ in $R^{n+1}$. The Cairns-Hirsch Product

Theorem gives us an isomorphism between $FBI_n^1$ and $F\Gamma I_n^1$. For the details of this argument, see [8].

We now consider the homotopy exact sequences of the pairs $(PL_{n+1}, O_{n+1})$ and $(\underline{PL}_{n+1}, O_{n+1})$. We use the isomorphism $\pi_n(\underline{PL}_q, O_q) \cong \Gamma_n^q$ = the set of concordance classes of smooth embeddings of homotopy n-spheres in $R^{n+q}$, which are PD unknotted. See [18].

Theorem 3.2 can be interpreted as saying that the boundary map $\pi_n(\underline{PL}_q, O_q) \to \pi_{n-1}(O_q)$ is zero, if $q \geq n-2$. Thus the natural map $\pi_n(O_q) \to \pi_n(\underline{PL}_q)$ is mono, if $q \geq n-1$, and hence so is the map $\pi_n(O_q) \to \pi_n(PL_q)$.

We have the pair of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \pi_n(O_{n+1}) & \to & \pi_n(PL_{n+1}) & \xrightarrow{j_1} & \pi_n(PL_{n+1}, O_{n+1}) & \to & 0 \\ & & \downarrow = & & \downarrow i_3 & & \downarrow i_4 & & \\ 0 & \to & \pi_n(O_{n+1}) & \to & \pi_n(\underline{PL}_{n+1}) & \xrightarrow[j_2]{} & \pi_n(\underline{PL}_{n+1}, O_{n+1}) & \to & 0 \end{array}.$$

$j_2 i_3$ can be identified with the natural map $F\Gamma I_n^1 \to \Gamma_n$ by means of the commutative diagram

$$\begin{array}{ccc} \pi_n(PL_{n+1}) & \xrightarrow{j_2 i_3} & \pi_n(\underline{PL}_{n+1}, O_{n+1}) \cong \Gamma_n^{n+1} \\ \downarrow \cong & & \downarrow \cong \\ F\Gamma I_n^1 & \to & \Gamma_n \end{array}$$

and this map is onto, as every homotopy n-sphere is a $\pi$-manifold, by a theorem of Adams, and hence immerses in codimension one. Hence, by commutativity, $i_4$ is onto.

Now suppose that $i_4(x) = 0$, and hence $i_4 j_1(y) = 0$, for some $y \in \pi_n(PL_{n+1})$. From the above remark about $j_2 i_3$, we see that y lies in the image of the natural map $FCI_n^1 \to F\Gamma I_n^1$. But this map can be identified with the natural map $\pi_n(O_{n+1}) \to \pi_n(PL_{n+1})$, by Theorems 3.3 and 3.4. Hence $x = 0$, and $i_4$ is an isomorphism. The result now follows by the Five Lemma.

Corollary 1 Any normal open tube on $S^n \subset S^{2n}$ is trivial as a bundle.

We now quote the following theorem of Hirsch.

Theorem 3.5 The natural map $\pi_n(PL_k, O_k) \to \Gamma_n$ is onto if $k \geq n-1$.

An easy consequence of this theorem is

Corollary 2 The natural map $\pi_n(PL_n) \to \pi_n(\underline{PL}_n)$ is onto, if $n \geq 4$.

Proof: We use the homotopy exact sequences of the pairs $(PL_n, O_n)$ and $(\underline{PL}_n, O_n)$ and the fact that $\Gamma_n \cong \Gamma_n^n \cong \pi_n(\underline{PL}_n, O_n)$, if $n \geq 4$.

Now consider the two homotopy exact sequences

$$\begin{array}{ccccccc}
\pi_{n+1}(S^q) & \to & \pi_n(PL_q) & \to & \pi_n(PL_{q+1}I) & \to & \pi_n(S^q) \\
\downarrow\oplus & & \downarrow & & \downarrow & & \downarrow\oplus \\
\pi_{n+1}(S^q)\oplus A^q_{n+1} & \to & \pi_n(\underline{PL}_q) & \to & \pi_n(\underline{PL}_{q+1}) & \to & \pi_n(S^q)\oplus A^q_n .
\end{array}$$

For the definition of $A^q_n$, see [18]. We only use the fact that $A^q_n = 0$, if $n < 2q-3$. It is now easy to prove our remaining Corollaries.

Corollary 3 $\pi_n(PL_{n+2}I) \cong \pi_n(\underline{PL}_{n+2})$, for all n.

<u>Corollary 4</u> $\pi_n(PL_{n+1}I) \to \pi_n(\underline{PL}_{n+1})$ is onto, for all n.

Proof: This is known for n = 2 and 3. Use the above for $n \geq 4$.

<u>Corollary 5</u> $\pi_n(PL_{n+1}I) \to \pi_n(PL_{n+1})$ is onto, and hence every PL manifold has a tangent disc bundle.

<u>Corollary 6</u> Open normal bundles exist and are unique for embeddings of $M^n$ in $Q^{2n}$.

Closed normal bundles exist for embeddings of $M^n$ in $Q^{2n}$.

# CHAPTER IV

# CHAPTER IV

## THE SPACE OF HOMEOMORPHISMS OF A 2-MANIFOLD

Let $M^2$ be a compact, connected, piecewise-linear (PL) 2-manifold, and let $*$ be a point of $\mathrm{int}(M^2)$. If $N$ is a subset of $M^2$, then $H_N(M^2)$ denotes the space of PL homeomorphisms of $M^2$ which leave points of $N$ fixed. This paper proves that, if $M^2$ is not $S^2$ or $P^2$, then the identity component (and hence each component) of $H_{*\cup\partial}(M^2)$ is contractible. If $M^2$ is $S^2$ or $P^2$, then the identity component of $H_{*\cup\partial}(M^2)$ has the homotopy type of a circle.

The analogous results to the above in the smooth category have been proved for orientable 2-manifolds by Barden, using results of Eells and Earle [4] and Cerf [3]. In the topological category, the analogous result has been proved by Hamstrom [10].

In §0, we prove some use ful results and state the main theorem. In §1, we give a proof of the main theorem assuming Theorem 2.1 which is proved in §2. In §3, we consider the relationship between the spaces $H_{*\cup\partial M}(M)$, $H_{\partial M}(M)$, $H_*(M)$ and $H(M)$.

### §0. Some useful results

We use the definitions and notation of §0 of Chapter III. In addition, $P^2$ denotes the projective plane, $K$ is the Klein bottle $P \# P$, Moeb. is the Moebius band and $T$ is the torus $S^1 \times S^1$. ( $\#$ denotes connected sum.) The main theorem is

Theorem 1.1

(i) If $M^2$ is a compact, connected 2-manifold, not $S^2$ or $P^2$, and $*$ is a point of $\mathring{M}^2$, then $H_{*\cup\partial M}(M^2)$ has contractible identity component.

(ii) The identity components of $H_*(S^2)$ and $H_*(P^2)$ are homotopy equivalent to the circle $S^1$.

We will need the following results.

Lemma 0.1 If $M^2$ is the Moebius band, then $H_{\partial M}(M)$ is contractible.

Proof: It is easy to show that $H_{\partial M}(M)$ is connected. To prove that all the higher homotopy groups are zero, apply the methods of Chapter III.

Lemma 0.2 If $M$ is a connected manifold and $*$ is a point of $\mathring{M}$, then the image of the natural map $f: \pi_1(H(M)) \to \pi_1(E(*,\mathring{M})) \cong \pi_1(M)$ is central in $\pi_1(M)$.

Proof: Let $\alpha$ represent an element of $\pi_1(H(M))$ and $\beta: (I,\partial I) \to (M,*)$ represent $f([\alpha])$ in $\pi_1(M)$. Let $\gamma: (I,\partial I) \to (M,*)$ represent an element of $\pi_1(M)$ also.

Consider the composite map

$g: I\times I \xrightarrow{1\times\gamma} I\times M \xrightarrow{\alpha} I\times M \xrightarrow{\pi} M$, where $(1\times\gamma)(x,y) = (x, \gamma y)$, and $\pi$ is projection onto the second factor. Note that $g|(I\times\{i\}) = \beta$ and $g|(\{i\}\times I) = \gamma$, where $i = 0$ or $1$. Thus $g$ defines a homotopy between $\beta\gamma$ and $\gamma\beta$. Therefore $[\beta][\gamma] = [\gamma][\beta]$ for all $[\gamma]$ in $\pi_1(M)$ and hence $[\beta]$ is central in $\pi_1(M)$.

Lemma 0.3 If $M$ is a compact, connected 2-manifold, not $P^2$, $K$, $T$, $S^1\times I$ or Moeb, then $\pi_1(M)$ has trivial centre.

Proof: See [5].

## §1. Proof of The Main Theorem

This section will be devoted to a proof of

**Theorem 1.1**

(i) If $M^2$ is a compact, connected 2-manifold, not $S^2$ or $P^2$, and $*$ is a point of $\mathring{M}^2$, then $H_{*\cup\partial M}(M^2)$ has contractible identity component.

(ii) The identity components of $H_*(S^2)$ and $H_*(P^2)$ are homotopy equivalent to the circle $S^1$.

**Remark 1.2** If $\partial M \neq \phi$, it follows that the identity components of $H_{*\cup\partial M}(M)$ and $H_{\partial M}(M)$ are homotopy equivalent, by considering the fibration

$$H_{*\cup\partial M}(M) \to H_{\partial M}(M) \to E(*,\mathring{M}).$$

$\pi_k(E(*,\mathring{M})) = \pi_k(M) = 0$ if $k \geq 2$, and the map $\pi_1(H_{\partial M}(M)) \to \pi_1(E(*,\mathring{M}))$ can be seen to be zero by choosing a path in $M$ from $*$ to a point of $\partial M$.

The crucial step in the proof of this theorem is

**Theorem 2.1**

If $M^2$ is a compact, connected 2-manifold, not $S^2$ or $P^2$, $S^1 \subset \mathring{M}^2$ is essential, and $*$ is a point of $S^1$, then $E_*(S^1,M^2)$ has contractible identity component.

**Proof of Theorem 1.1(i) assuming Theorem 2.1**

If $M^2$ is a compact, connected 2-manifold, then $M^2 \cong (S^2 \# tT \# pP) - k$ disjoint open 2-discs, where $t$, $p$ are non-negative integers denoting repeated connected sum. From the relation $T \# P \cong P \# P \# P$, we see that if $M$ is non-orientable, then $M \cong pP$-$k$ holes, where $p \geq 1$. Of course, if $M$ is orientable, then $M \cong (S^2 \# tT)$-$k$ holes. (In the orientable case, $t$ is called the genus of $M$, and in the non-orientable case, $p$ is the genus.) We proceed by induction on $t$ in the orientable case or $p$ in the non-orientable case and reduce the case $p = 1$ to the orientable case.

If $M$ is orientable and $t = 0$, $k = 1$, we apply Lemma 1.1 of Chapter III. If $k \geq 2$, we apply Theorem 1.3 below. This starts the induction.

<u>Theorem 1.3</u>

$H_{*\cup\partial}(S^1 \times I - r \text{ holes})$, $r \geq 0$, has contractible identity component.

The induction step itself is as follows. Let $M$ be a compact 2-manifold, not $S^2$ or $P^2$. Then $M \cong tT - k$ holes, $t \geq 1$, or $M \cong pP - k$ holes, $p \geq 1$. In the orientable case, choose $S^1 \subset M^2$ to be a transverse circle of a torus. In the non-orientable case, when $p \geq 2$, choose $S^1 \subset M^2$ to be the attaching circle of a Moebius band, and in the case $p = 1$, when $M \cong$ Moeb $- (k-1)$ holes, choose $S^1$ to be the centre circle of $M$. In all cases, if we cut $M$ along $S^1$ we obtain a connected manifold $N$ or two connected manifolds $N_1$, $N_2$ of smaller genus than $M$.

Now, in all cases, $S^1$ is essential in $M$, hence we can apply Theorem 2.1. Consider the fibration

$$H_{\partial M \cup S^1}(M) \to H_{*\cup\partial M}(M) \to E_*(S^1, M^2).$$

From Theorem 2.1, we see that the identity components of $H_{\partial M \cup S^1}(M)$ and $H_{*\cup\partial M}(M)$ are homotopy equivalent. But the identity components of $H_{\partial M \cup S^1}(M)$ and $H_{\partial N}(N)$, are isomorphic where $N$ is the manifold ( possibly not connected) obtained by cutting $M$ along $S^1$. As each component of $N$ has non-empty boundary, Remark 1.2

implies that, if $*$ is an interior point of a component $C$ of $N$, then the identity components of $H_{\partial C}(C)$ and $H_{*\cup\partial C}(C)$ are homotpy equivalent. Now our induction assumption implies that the identity component of $H_{*\cup\partial C}(C)$ is contractible. Hence the identity component of $H_{*\cup\partial M}(M)$ is contractible. This completes the induction step and hence the proof of Theorem 1.1(i).

Proof of Theorem 1.3

Denote $(S^1\times I)$ - r holes by $F_r$. The proof is by induction on r. The case r = 0 is proved by using Remark 1.2 and the results of Chapter III.

The induction step is as follows. Consider the fibrations

(A) $H_{*\cup o\cup\partial}(F_r) \to H_{*\cup\partial}(F_r) \to E(o, \mathring{F}_r - *)$, where $*$, o are distinct points of $F_r$,

(B) $H_{*\cup D^2\cup\partial}(F_r) \to H_{*\cup o\cup\partial}(F_r) \to E_o(D^2, \mathring{F} - *)$, where o in $D^2 \subset \mathring{F}_r - *$.

As in Remark 1.2, it follows from our induction hypothrsis and (A) that the identity component of $H_{*\cup o\cup\partial}(F_r)$ is contractible.

Now there is a map $\alpha$: $E_o(D^2,R^2) \to E_o(D^2, \mathring{F}_r - *)$ obtained by choosing a neighbourhood of $D^2$ in $\mathring{F}_r - *$ homeomorphic to $R^2$. $\alpha$ is a homotopy equivalence. For let $f: \Delta^k\times D^2 \to \Delta^k\times F_r$ represent an element of $\pi_k(E_o(D^2, \mathring{F}_r - *))$. By contracting each $D^2$ over itself keeping o fixed, we can pull the image under f into a "small" neighbourhood of o and hence, a priori, into $R^2 \subset F_r - *$. Thus $\alpha_*$ is onto. Similarly $\alpha_*$ is a monomorphism, and therefore an isomorphism.

Now the identity components of $E_o(D^2,R^2)$ and $H_o(R^2)$ are homotopy equivalent. See P.26, line 6. As the identity component of $H_o(R^2)$ is homotopy equivalent to a circle, Theorem 2.1 of Chapter III, it follows that the identity component of $E_o(D^2, \mathring{F}_r - *)$ is homotopy equivalent to $S^1$. Hence (B) implies that the identity component of $H_{*\cup D^2\cup\partial}(F_r)$ is contractible. But $H_{*\cup D^2\cup\partial}(F_r) \cong H_{*\cup\partial}(F_{r+1})$. This completes the induction step and hence the proof of Theorem 1.3.

Proof of Theorem 1.1(ii)

If $M$ is $S^2$, the result follows as $H_*(S^2) \simeq H(R^2) \simeq S^1$. See the proof of Lemma 1.6 in Chapter III, and Theorem 2.1 of that chapter.

If $M$ is $P^2$, consider the fibration

$$H_{D^2}(P^2) \to H_*(P^2) \to E_*(D^2,P^2),$$

where $*$ in $D^2 \subset P^2$.

Now $H_{D^2}(P^2) \cong H_\partial(\mathrm{Moeb})$ which is contractible by Lemma 0.1. Also $E_*(D^2,P^2)$ has identity component homotopy equivalent to $S^1$, by the same argument as in the proof of Theorem 1.3. The result now follows.

§2.

This section is devoted to proving

Theorem 2.1

If $M^2$ is a compact 2-manifold, not $S^2$ or $P^2$, $S^1 \subset \mathring{M}^2$ is essential, and $*$ is a piont of $S^1$, then $E_*(S^1,M^2)$ has contractible identity component.

Proof: This is simiłar to the proof of Theorem 2.2 in Chapter III.

Let $\alpha: \Delta^k \times S^1 \to \Delta^k \times M^2$ represent an element of $\pi_k(E_*(S^1, M^2)$, $k \geq 1$. We prove that $\alpha$ is null homotopic.

Choose a regular neighbourhood V of $S^1$ in $M^2$, (V will be homeomorphic to $S^1 \times I$ or Moeb) and consider $\alpha_x(S^1) \cap \partial V$, where x in $\Delta^k$ and $\alpha_x$ denotes $\alpha | \{x\} \times S^1$. We now proceed as follows:

(i) Homotp $\alpha$ so that $\alpha_x(S^1) \cap \partial V$ consists of a finite number of points, for all x in $\Delta^k$.

(ii) Homotop $\alpha$ to remove all the crossing points in this intersection, for all x in $\Delta^k$.

(iii) We are left with an embedding $\beta: \Delta^k \times S^1 \to \Delta^k \times V$ and, by using a collar of $\partial V$ in V, we can push $\beta$ away from $\partial V$ so that $\beta_x(S^1) \cap \partial V = \phi$. Now $\beta$ represents an element of $\pi_k(E_*(S^1, V)$, where $S^1$ is the central circle of V, and we apply Theorem 2.2 or 2.3 to show that $\beta$ is null homotopic and hence so is $\alpha$.

<u>Theorem 2.2</u>

If $S^1 \times \{\frac{1}{2}\} = S^1 \subset S^1 \times I$ and $*$ in $S^1$, then the identity component of $E_*(S^1, S^1 \times I)$ is contractible.

<u>Theorem 2.3</u>

If $M^2$ is Moeb, $S^1 \subset M^2$ is the central circle, and $*$ in $S^1$, then the identity component of $E_*(S^1, M^2)$ is contractible.

<u>Proof of Theorem 2.2</u>

From the results of Chapter III and Remark 1.2 it follows that the identity component of $H_{S^1 \times \partial I \cup *}(S^1 \times I)$ is contractible, where $*$ in

$S^1\times I$. Now consider the fibration

$$H_{S^1\times\partial I\cup S^1}(S^1\times I) \to H_{S^1\times I\cup *}(S^1\times I) \to E_*(S^1, S^1\times I).$$

As the fibre is isomorphic to $H_{S^1\times\partial I}(S^1\times I)^2$, the result follows.

Proof of Theorem 2.3

From Lemma 0.1 and Remark 1.2 it follows that the identity component of $H_{\partial M\cup *}(M)$ is contractible. Now consider the fibration

$$H_{\partial M\cup S^1}(M) \to H_{\partial M\cup *}(M) \to E_*(S^1, M).$$

By "cutting along $S^1$", it is clear that the fibre is isomorphic to $H_{S^1\times\partial I}(S^1\times I)$ which has contractible identity component. The result now follows.

To complete the proof of Theorem 2.1, we must show that steps (i) and (ii) at the beginning of this section can be carried out.

We carry out step (i) by arguing as follows. $\partial V$ is either one circle or two circles. In either case, a component $C$ of $\partial V$ has a cylinder $S^1\times I$ as regular neighbourhood in $M$. Let $\pi$ denote projection $S^1\times I \to I$. By a small homotopy of $\alpha$ fixed outside a small neighbourhood of $\alpha^{-1}(\Delta^k\times C)$, we can arrange that $\pi\alpha$ is non-degenerate on a neighbourhood $W$ of $\alpha^{-1}(\Delta^k\times C)$. i.e. $(\pi\alpha)^{-1}(x)$ is a finite number of points, if $x$ in $\alpha(W)$. The required result follows immediately.

Now consider Step (ii). The points of $\alpha_x(S^1)\cap\partial V$ fall into two classes - crossing points and non-crossing points. Let $n_x$ be the number of crossing points in the fibre over x, and define $n = \max\{n_x \mid x \text{ in } \Delta^k\}$. If $n = 0$, we apply Step (iii) and what follows of the above work to complete the proof that $\alpha$ is null homotopic. If $n \neq 0$, we apply the following lemma and induction to reduce ourselves to the case $n = 0$.

Lemma 2.4

If $n(\alpha) > 0$, there is a homotopy of $\alpha$ to $\alpha'$, where $\alpha'$ has the property that $\alpha'_x(S^1) \cap V$ is a finite number of points, and such that $n(\alpha') < n(\alpha)$.

Proof: We define $N = \{x \text{ in } \Delta^k \mid n_x = n\}$. Choose x in N. As in Chapter III, all the crossing points over x persist over the component of N in which x lies. We may suppose N is connected.

As $N \neq \Delta^k$, $\bar{N} - N$ is non-empty. Thus there is a pair of crossing points y, y' over x which coalesce at some point z of $\bar{N} - N$.

If p, q are points of $S^1$, not $*$, denote by $[p,q]$ the arc of $S^1$ with endpoints p and q which does not contain $*$. Then we may suppose that $[y,y']$ contains no other crossing points.

The points $\alpha y$, $\alpha y'$ in $\partial V$ determine two arcs A and B with them as endpoints, and $A \cup \alpha[y,y']$, $B \cup \alpha[y,y']$ are both circles embedded in $M^2$. It can be seen, by considering the situation in a neighbourhood of z, that one of these two circles bounds a 2-disc embedded in $M^2$. (The possibility that both circles bound a 2-disc is eliminated later.)

As in Chapter III, our problem reduces to the following.

Divide the boundary of the 2-disc $D^2$ into two arcs P,Q. Let

$E(D^2, M - \mathring{V})$ denote the space of embeddings of $D^2$ in $M - \mathring{V}$ such that $D^2 \cap \partial V = P$, and let $E(Q, M - \mathring{V})$ denote the space of embeddings of $Q$ ($\cong I$) in $M - \mathring{V}$ such that $Q \cap \partial V = \partial Q$. Then we have to show that the natural map $\lambda$: $E(D^2, M-\mathring{V}) \to E(Q, M-\mathring{V})$ induces an epimorphism of all homotopy groups. (If $M-\mathring{V}$ is disconnected, we shall restrict attention to one component.)

Now $\lambda$ is a fibration and the fibre is $E_Q(D^2, M-\mathring{V})$. All the components of the fibre are isomorphic to $H_Q(D^2)$, which is contractible by applying the Alexander trick twice. Hence $\lambda_*$ is an isomorphism of all homotopy groups except possibly $\pi_1$.

Consider $\pi_0(E_Q(D^2, M-\mathring{V}))$. Suppose this has two distinct elements with representatives $f$ and $g$ say. Either $f(P) = g(P)$ or $f(P) \cup g(P)$ is a component $T$ of $\partial V$. If $f(P) = g(P)$, then $f(D^2) = g(D^2)$ and $f$ and $g$ are isotopic by the Alexander trick. Therefore $f(P) \cup g(P) = T$, and $f(D^2) \cap g(D^2) = Q$. Now $f(D^2) \cup g(D^2)$ is a 2-disc with boundary $T$. It follows that the component of $M - \mathring{V}$ of which $T$ is a boundary component must be a 2-disc. If $V$ is a cylinder, then $S^1$ is inessential as it bounds a 2-disc. If $V$ is a Moebius band, then $M$ is the projective plane $P^2$. As both these cases are excluded in our hypotheses, we see that $\lambda_*$ is an isomorphism on all homotopy groups. The result of Lemma 2.4 follows.

## §3. The spaces $H_{*\cup\partial}(M)$, $H_\partial(M)$, $H_*(M)$, $H(M)$.

In this section, we consider the relationships between the above four spaces of homeomorphisms of a 2-manifold M, where $*\in M$, and calculate their homotopy groups. Of course, we have a commutative square of inclusion maps

$$\begin{array}{ccc} H_{*\cup\partial}(M) & \xrightarrow{\delta} & H_\partial(M) \\ \gamma\downarrow & & \downarrow\beta \\ H_*(M) & \xrightarrow{\alpha} & H(M) \end{array} .$$

The following theorem contains the main results of the section.

Theorem 3.1

(a) If M is a compact, connected 2-manifold, not $S^2$, $P^2$, K, T, $S^1\times I$, Moeb or $D^2$, then $\alpha$, $\beta$, $\gamma$, $\delta$ are all homotopy equivalences on identity components. Hence all four spaces have contractible components.

(b) non-closed manifolds

(i) If $M = D^2$, $\alpha$ and $\delta$ are both homotopy equivalences. All homotopy groups of all four spaces are zero, except for $\pi_1(H_*(D^2)) \cong \pi_1(H(D^2)) \cong Z$.

(ii) If M = Moeb, $\gamma$ and $\delta$ are homotopy equivalences on identity components. $\pi_i(H(\text{Moeb})) = 0$, $i \geq 2$ and $\pi_1(H(\text{Moeb})) \cong Z$.

The natural map $Z \cong \pi_1(H(\text{Moeb})) \to \pi_1(E(*, \text{Moeb})) \cong Z$ is multiplication by 2.

(iii) If $M = S^1 \times I$, $\gamma$ and $\delta$ are homotopy equivalences on identity components. The natural map $\pi_i(H(S^1 \times I)) \to \pi_i(S^1 \times I)$ is an isomorphism, $i \geq 1$.

(c) <u>closed manifolds</u> : $\beta$ and $\gamma$ are automatically isomorphisms.

(i) If $M = T$, the natural map $\pi_i(H(T)) \to \pi_i(T)$ is an isomorphism, $i \geq 1$.

(ii) If $M = K$, $\pi_i(H(K)) = 0$, $i \geq 2$ and $\pi_1(H(K)) = Z$.

(iii) If $M = P^2$, the natural map $\pi_i(H(P^2)) \to \pi_i(P^2)$ is an isomorphism if $i \geq 3$, $\pi_2(H(P^2)) = 0$ and $\pi_1(H(P^2)) = Z_2$.

(iv) If $M = S^2$, $H(S^2) \simeq 0_3$.

<u>Remark</u> The results for $D^2$ and $S^2$ are contained in Chapter III, §1.

The rest of the section is devoted to proving the above results. We consider the maps $\alpha$, $\beta$, $\gamma$, $\delta$ in turn.

<u>The map $\alpha$</u>

Consider the fibration $H_*(M) \xrightarrow{\alpha} H(M) \to E(*, \mathring{M}) \simeq M$.

If $M \neq S^2$ or $P^2$, $\pi_i(M) = 0$ for $i \geq 2$. Thus $\alpha_* : \pi_i(H_*(M)) \to \pi_i(H(M))$ is an isomorphism if $i \geq 2$ and $M \neq S^2$ or $P^2$.

Lemmas 0.2 and 0.3 imply that, if $M \neq P$, $K$, $T$, $S^1 \times I$ or Moeb, then the map $\pi_1(H(M)) \to \pi_1(M)$ is zero.

Hence, if $M \neq S^2, P^2$, $K$, $T$, $S^1 \times I$ or Moeb, it follows that $\alpha$ is a homotopy equivalence on identity components.

<u>The map $\beta$</u>

If $\partial M = \phi$, $\beta$ is trivially an isomorphism, so we consider the case $\partial M = C_1 \cup \ldots \cup C_n$ where each $C_i$ is a circle. Let $o \in C_1$, and consider the fibration

$$H_0(M) \xrightarrow{\beta'} H(M) \to E(\bullet, C_1).$$

As $\pi_i(S^1) = 0$, $i \geq 2$, $\beta'_*$: $\pi_i(H_0(M)) \to \pi_i(H(M))$ is an isomorphism if $i \geq 2$.

Lemmas 0.2 and 0.3 imply that, if $M \neq S^1 \times I$ or Moeb, then the composite map

$$\pi_1(H(M)) \to \pi_1(C_1) \xrightarrow{i_*} \pi_1(M)$$

is zero. ( $i$: $C_1 \to M$ is the inclusion map.) Now $i_*$ is a monomorphism unless $M = D^2$. Hence, if $M \neq S^1 \times I$, Moeb or $D^2$, it follows that $\beta'$ is a homotopy equivalence on identity components.

Now consider the fibration

$$H_{C_1}(M) \to H_0(M) \to H_0(C_1).$$

The identity component of $H_0(C_1)$ is isomorphic to $H_{\partial I}(I)$ which is contractible by the Alexander trick. Thus $H_{C_1}(M)$ and $H_0(M)$ have homotopy equivalent identity components.

It now follows by induction, that $\beta$ is a homotopy equivalence on identity components, if $M \neq S^1 \times I$, Moeb or $D^2$.

The map $\gamma$

If $\partial M = \phi$, $\gamma$ is trivially an isomorphism, so we consider the case $\partial M = C_1 \cup \ldots \cup C_n$, where each $C_i$ is a circle. Let $o \in C_1$ and consider the fibration

$$H_{*\cup o}(M) \xrightarrow{\gamma'} H_*(M) \to E(o, C_1)$$

As above, $\gamma'_*$: $\pi_i(H_{*\cup o}(M)) \to \pi_i(H_*(M))$ is an isomorphism if $i \geq 2$.

Now consider the composite map

$$g\colon \pi_1(H_*(M)) \to \pi_1(C_1) \to \pi_1(M - *).$$

The method of proof of Lemma 0.2 shows that the image of $g$ is central in $\pi_1(M - *)$. But Lemma 0.3 shows that $\pi_1(M - *)$, which is isomorphic to $\pi_1(M - \mathring{D}^2)$, has trivial centre if $M - \mathring{D}^2 \ncong S^1 \times I$ or Moeb. i.e. if $M \ncong D^2$.

As before, it now follows by induction that $\gamma$ is a homotopy equivalence on identity components if $M \ncong D^2$.

<u>The map $\delta$</u>

Remark 1.2 shows that $\delta$ is a homotopy equivalence on identity components if $\partial M \neq \phi$.

<u>The exceptional cases: $S^1 \times I$, Moeb, T, K, $P^2$.</u>

(a) If $M = S^1 \times I$, the natural map $H(S^1 \times I) \to E(*, S^1 \times \mathring{I}) \simeq S^1$ is a homotopy equivalence of identity components.

As $\gamma$ is a homotopy equivalence, it is certainly true that $H_*(S^1 \times I)$ has contractible identity component. Now consider the fibration

$$H_*(S^1 \times I) \to H(S^1 \times I) \to E(*, S^1 \times \mathring{I}) \simeq S^1.$$

It is obvious that the map $\pi_1(H(S^1 \times I)) \to \pi_1(S^1)$ is onto. The result now follows.

(b) If $M = T$, paragraph (a) applies.

(c) If $M =$ Moeb, $\pi_i(H(\text{Moeb})) = 0$ if $i \geq 2$, $\pi_1(H(\text{Moeb})) = Z$, and the natural map $Z \cong \pi_1(H(\text{Moeb})) \to \pi_1(E(*, \mathring{\text{Moeb}})) \cong Z$ is multiplication by 2.

As $\gamma$ is a homotopy equivalence, it is certainly true that $H_*(\text{Moeb})$

has contractible identity component. Now consider the fibration

$$H_*(\mathrm{Moeb}) \to H(\mathrm{Moeb}) \to E(*, \mathring{\mathrm{Moeb}}) \simeq S^1.$$

It is immediate that $\pi_i(H(\mathrm{Moeb})) = 0$ if $i \geqslant 2$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_1(H(\mathrm{Moeb})) & \xrightarrow{f} & \pi_1(E(*, \mathring{\mathrm{Moeb}})) & \cong & \pi_1(\mathrm{Moeb}) & \cong & Z \\
\downarrow & & & & \uparrow & & \\
Z \cong \pi_1(H(\partial\mathrm{Moeb})) & & \to & & \pi_1(\partial\mathrm{Moeb}) & \cong & Z
\end{array}$$

The right hand map is multiplication by 2, thus the image of f lies in 2Z. But 2 is clearly in the image of f. The result follows as f is a monomorphism.

(d) If $M = K$, $\pi_i(H(K)) = 0$, $i \geqslant 2$ and $\pi_1(H(K)) = Z$.

The first part of the statement is clear from consideration of the fibration

$$H_*(K) \to H(K) \to E(*, K) \simeq K,$$

and the facts that $H_*(K)$ has contractible identity component and $\pi_i(K) = 0$ if $i \geqslant 2$.

From the well-known presentation of K as a square with edges identified, Fig.1, it follows that $\pi_1(K)$ is the free group on two generators a, b with the relation $abab^{-1} = 1$, i.e. $bab^{-1} = a^{-1}$.

Lemma 3.2

The centre of the group $G = \{a,b : bab^{-1} = a^{-1}\}$ is generated by $b^2$ and has infinite order.

Proof: Certainly $b^2$ is in the centre of $G$, as $b^2$ commutes with a and b. It commutes with a because $b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^{-1}b^{-1} = (bab^{-1})^{-1} = a$.

Note that a and b both have infinite order, as the infinite dihedral group $D(\infty)$ has the relations $bab^{-1} = a^{-1}$, $b^2 = 1$ and a has infinite order, and $Z+Z_2$ has the relations $bab^{-1} = a^{-1}$, $a^2 = 1$ with b of infinite order.

Now any element of G can be written in the form $a^rb^s$ by using the relation $ab = ba^{-1}$. Suppose $a^rb^s$ is central in G. Then

$$a^rb^sb = ba^rb^s.$$

Therefore $a^rb = ba^r \Rightarrow a^r = ba^rb^{-1} = (bab^{-1})^r = a^{-r} \Rightarrow r=0$, as a has infinite order.

Now b is not central in G, as $ab = ba \Rightarrow a = bab^{-1} = a^{-1} \Rightarrow a^2 = 1$.

This completes the proof of the lemma.

Now consider the map $f: \pi_1(H(K)) \to \pi_1(K)$, which is a monomorphism as $\pi_1(H_*(K)) = 0$. The image of f is central in $\pi_1(K)$, by Lemma 0.2. But $b^2$ is in the image of f. For consider the isotopy obtained by sliding K over itself parallel to the b generator of homotopy. "If we go twice round", we end up with the identity homeomorphism of K, so we have defined an element of $\pi_1(H(K))$ whose image in $\pi_1(K)$ is clearly $b^2$. Therefore $\pi_1(H(K)) = Z$.

(e) If $M = P^2$, the natural map $\pi_i(H(P^2)) \to \pi_i(P^2)$ is an isomorphism if $i \geq 3$, $\pi_2(H(P^2)) = 0$ and $\pi_1(H(P^2)) = Z_2$.

Consider the fibration

$$H_*(P^2) \to H(P^2) \to E(*, P^2) \simeq P^2.$$

As $\pi_i(H_*(P^2)) = 0$ if $i \geq 2$, the first result follows if $i \geq 3$.

Consider the homotopy exact sequence

C) $0 \to \pi_2(H(P)) \to \pi_2(P) \to \pi_1(H_*(P)) \to \pi_1(H(P)) \to \pi_1(P) \to \pi_0(H_*(P)) \to \pi_0(H(P))$.

We will prove that the map $\pi_1(P) \to \pi_0(H_*(P))$ is a monomorphism, and that the map $Z \cong \pi_2(P) \to \pi_1(H_*(P)) \cong Z$ is multiplication by 2. This will prove that $\pi_2(H(P)) = 0$ and $\pi_1(H(P)) = Z_2$.

<u>Lemma 3.3</u>

The map $\pi_1(P) \to \pi_0(H_*(P))$ is a monomorphism.

Proof: $\pi_0(H(P)) = 1$ and $\pi_0(H_*(P)) = Z_2$. As $\pi_1(P) = Z_2$, the exactness of C) gives the required result.

To see that $\pi_0(H(P)) = 1$, take a homeomorphism h of P. Ambient isotop h to leave a point * fixed and then so that a 2-disc containing * in its interior is mapped onto itself. If the map is orientation preserving, we can isotop h to be fixed on the disc, by the Alexander trick. We would then be left with a homeomorphism of Moeb fixed on the boundary, which must be isotopic to the identity. If h reverses the orientation of the disc, do an isotopy of h which takes * round an orientation reversing path.

This proof also shows that $\pi_0(H_*(P)) = Z_2$.

<u>Lemma 3.4</u>

The map $Z \cong \pi_2(P) \to \pi_1(H_*(P)) \cong Z$ is multiplication by 2.

Proof: Let $\pi$: $S^2 \to P^2$ be the standard projection and $o \cup o' = \pi^{-1}(*)$.

Consider the diagram

$$\begin{array}{ccccccc}
Z \cong \pi_2(P) & \to & \pi_1(H_*(P)) & & & \xrightarrow{k} & \pi_1(E(pt, P^2 - *)) \cong Z \\
\cong\uparrow\pi_* & & & & & & \times 2 \uparrow \pi|_* \\
Z \cong \pi_2(S^2) & \xrightarrow{f} & \pi_1(H_o(S^2)) \cong Z & \xleftarrow{g} & \pi_1(H_{o\cup o'}(S^2)) & \xrightarrow{h} & \pi_1(E(pt, S^2 - o\cup o')) \cong Z
\end{array}$$

Note that f is multiplication by 2. See Chapter III, §1.

g is an isomorphism because of the fibration

$$H_{o\cup o'}(S^2) \xrightarrow{g} H_o(S^2) \to E(o', S^2 - o) \simeq *.$$

h is an isomorphism as it is clearly onto.

Finally k is multiplication by 2. To see this, consider the diagram

$$\begin{array}{ccccccc}
\pi_1(H_*(P)) & \xrightarrow{l} & \pi_1(E_*(D^2, P^2)) & \xleftarrow{m} & \pi_1(E_*(D^2, R^2)) & \xleftarrow{n} & \pi_1(H_*(R^2)) \\
k\downarrow & & & & & & p\downarrow \\
\pi_1(E(pt, P^2 - *)) & & & \xleftarrow{q} & & & \pi_1(E(pt, R^2 - *))
\end{array}$$

The maps are all natural maps obtained by choosing an embedding $(R^2, *) \subset (P^2, *)$. l,m,n are all isomorphisms. See the proof of Theorem 1.1(ii) and Chapter III, §1. p is also an isomorphism. q is multiplication by2. The lemma now follows.

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