CATEGORIES OF OPERATORS AND H-SPACES

by

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Abstract

By introducing categories of operators the concept of an associative H-space is generalized. Each such category gives rise to a structure on a space X if it can be made to act on it. To each category $\underline{C}$ of operators a category $W\underline{C}^{\sim}$ of operators is associated which gives rise to a $\underline{C}$-structure up to higher homotopies and all possible coherence conditions. After introducing the notion of a structure map and of homotopies of structure maps the category of $W\underline{C}^{\sim}$-spaces and homotopy classes of structure maps is set up and studied. The theory is applied to prove a classification theorem.

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I am very much indebted to Dr. J. M. Boardman for his continuous help and encouragement during my research. He suggested the problem dealt with in this thesis and the particular approach chosen to solve it. His constant guidance kept me out of the traps an inexperienced research student is more than likely to fall into. For this, and in particular for the priviledge of working jointly on a paper dealing with the problems discussed in this thesis, I would like to express my best thanks.

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INTRODUCTION

The concept of an H-space, i.e. a space X with base point e and a multiplication map m: $X \times X \to X$ such that e is a homotopy identity, arose as a generalization of that of a topological group. It turned out to be of great importance in homotopy theory, especially in the study of extraordinary cohomology theories.

Generally speaking most of the techniques which apply to topological groups cannot be applied to H-spaces because of their lack of structure. From the homotopy theoretic point of view the distinguishing feature is the associativity (and commutativity) of the multiplication rather than the existence of a continuous inverse.[for example see [1], Satz 8.2 and 8.3]. Since many spaces of interest have no natural monoid or commutative monoid structure, such as the loop space ΩX or the stable orthogonal group, this led to an extensive study of H-spaces which almost have the structure of a topological monoid or a commutative topological monoid such as homotopy associative, homotopy commutative, strongly homotopy commutative [7] H-spaces, and A_∞-spaces [5]. In the last two cases, part of the structure consists of higher homotopies and coherence conditions, and important constructions like the classifying space construction turn out to hold for them.

A problem in the theory of H-spaces with additional structure has been to find the right concept of maps between them. The notion of a homomorphism, i.e. a map that preserves the multiplication and the coherence conditions, turned out to be too restrictive, while a notion of a map that commutes with the multiplication up to homotopy was too weak for many applications. The complexity of structures with higher homotopies and coherence conditions made it so far impossible to find a satisfactory definition of maps between such spaces, while Sugawara [7] succeeded in doing this for monoids. A study of the category of topological monoids and homotopy classes of such maps can be found in [8].

The purpose of this thesis is to develop a satisfactory theory - from the view of homotopy theory - of spaces with homotopy-associative (and homotopy-commutative) multiplication and all possible higher homotopies and coherence conditions and of structure maps between such spaces. A suitable definition of homotopy between such maps makes these spaces and the homotopy classes of structure maps between them into a category. We adopt following as test propositions:

(A) If X is a space in the category and Y is homotopy equivalent to X then Y is in the category.

(B) A structure map over a homotopy equivalence is an isomorphism in the category.

To avoid the difficulties arising from the complexity of the topological models used to define the higher homotopies (for example of an A_∞-space, such as the well known Stasheff pentagon), we approach the problem in a completely new way, which in addition provides us withresults for a much wider range of "structure" spaces than just A_∞-spaces or homotopy-commutative A_∞-spaces with suitable higher coherence conditions. Rather than speaking of a particular space with a given structure we introduce categories of operators which "act" on spaces and thus induce a structure on them. Such a category $\underline{B}$ basically consists of objects $0,1,2,3,\ldots$, a topological structure on each morphism set such that composition is continuous, a continuous bifunctor $\oplus\colon \underline{B} \times \underline{B} \to \underline{B}$ such that $m \oplus n = m + n$. An action of $\underline{B}$ on a space X associates with each morphism $f\colon m \to n$ a map $\alpha(f)\colon X^m \to X^n$ continuously in f and such that $\alpha(f \oplus g) = \alpha(f) \times \alpha(g)$ and α is functorial. For example the category consisting of exactly one morphism $n \to 1$ for each n gives rise to a topological monoid structure. With each category $\underline{B}$ of operators we associate another category $W\tilde{\underline{B}}$ of operators which gives rise to a structure that is a $\underline{B}$-structure up to higher homotopies and all possible coherence conditions of which the morphism spaces keep track. $W\tilde{\underline{B}}$ has a universal property such that a space with a $W(W\tilde{\underline{B}})^{\sim}$-structure can be given a $W\tilde{\underline{B}}$-structure. This universal property is the key for the development of our theory.

A slight generalization of the concept of categories of operators gives rise to the definition of structure maps.

In order to avoid spurious difficulties in our topological constructions, we work in the category CG of compactly generated Hausdorff spaces. For details see [6]. Two of the properties of CG which we frequently use without mentioning are full adjointness and the fact that the product of two quotient spaces is the quotient of the product.

This thesis is part of a joint work with my supervisor, Dr. J. M. Boardman, who applied the theory represented here to obtain results about the stable groups O, SO, F, SF, U, SU, PL, SPL, Top, STop etc. and their classifying spaces. A summary of this joint work is included at the end of this thesis.

In the first chapter we give the definition of categories of operators and list a few examples. In the second chapter we construct the category $W\tilde{\underline{B}}$ for each category $\underline{B}$ of operators and discuss its basic properties. Chapter III deals with the concept of structure maps and we set up the category of $W\tilde{\underline{B}}$-spaces and homotopy classes of structure maps. It includes the proofs for the test theorems. In Chapter IV we study spaces with $W\tilde{\underline{B}}$-structures and state a classification theorem.

Example 4 in §1 and the results of the second section of Chapter IV are entirely due to Dr. J.M. Boardman and we restrict ourselves to sketching the proofs.

CHAPTER I: DEFINITIONS AND EXAMPLES

§ 1 CATEGORIES OF OPERATORS

All our topological constructions will be in the category of compactly generated Hausdorff spaces. This means that we only need to check that the identification spaces constructed are Hausdorff. The rest is automatic.

Let N be the set of all sequences in n generators $0,\ldots,n-1$ including the empty sequence. N is a free monoid under juxtaposition.

Define a left action of $S(k)$, the symmetric group in k letters, on the sequences of length k by

$$\xi(i_1,\ldots,i_k) = (i_{\xi^{-1}(1)},\ldots,i_{\xi^{-1}(k)})$$

$\xi \in S(k)$.

We have two variants of categories of operators: with or without permutations.

Definition 1.1: In a category $\underline{B}$ of operators on n object generators

(a) the objects are elements of N

(b) the morphisms from $\underline{a}$ to $\underline{b}$ form a (compactly generated) topological space $\underline{B}(\underline{a},\underline{b})$ and composition is continuous

(c) we are given a strictly associative, continuous bifunctor $\oplus$: $\underline{B} \times \underline{B} \to \underline{B}$ such that $\underline{a} \oplus \underline{b} = \underline{ab}$

(d) if $\underline{B}$ has permutations we are also given a morphism $\overline{\xi}$: $\underline{a} \to \xi\underline{a}$ for each $\xi \in S(k)$ and each sequence $\underline{a}$ of length k such that

(i) $\overline{\xi \cdot \eta} = \overline{\xi} \circ \overline{\eta}$

(ii) if $\xi \in S(k)$ is the identity, then $\overline{\xi} = 1_{\underline{a}}$

(iii) if $\xi \in S(m)$ and $\eta \in S(k)$ then $\overline{\xi} \oplus \overline{\eta} = \overline{\xi \oplus \eta}$, where $\xi \oplus \eta \in S(m+k)$ is the usual sum permutation

(iv) given r morphisms α_i in $\underline{B}$ such that source α_i is a sequence of length m_i and target α_i one of length n_i, and $\xi \in S(r)$, then we have

$$\overline{\xi(n_1,\ldots,n_r)} \circ (\alpha_1 \oplus \ldots \oplus \alpha_r)$$
$$= (\alpha_{\xi^{-1}(1)} \oplus \ldots \oplus \alpha_{\xi^{-1}(r)}) \circ \overline{\xi(m_1,\ldots,m_r)}$$

where $\xi(n_1,\ldots,n_r) \in S(n_1+\ldots+n_r)$ is defined as follows:

Let $n_1+\ldots+n_{k-1}+1 = i \leqslant n_1+\ldots+n_k$, $i > 0$.

Then $\xi(n_1,\ldots,n_r)(i) = 1 + \Sigma_{j=1}^{\xi(k)-1} n_{\xi^{-1}(j)}$.

<u>Notation</u>: For $\xi \in S(k)$ we denote the induced morphisms $\overline{\xi}$ simply by ξ.

We call a category of operators on n object generators an M^nT-category if it has no permutations and an M^nTP-category if it has.

Unless otherwise stated we denote sequences of length

1 by small Latin letters and general sequences by underlined small Latin letters. For morphisms we use the letters α, β, γ and for permutations the letters ξ, η, ζ. Categories are denoted by underlined capital Latin letters.

Definition 1.2: Let $\underline{A}$ be an M^nT-(M^nTP) category and $\underline{B}$ an M^mT(resp.M^mTP)-category. An MT(resp.MTP)-functor $\delta: \underline{A} \to \underline{B}$ from $\underline{A}$ to $\underline{B}$ is a functor such that

(i) δ maps object generators into object generators, i.e. it maps sequences of length 1 into sequences of length 1

(ii) δ preserves sums, i.e. $\delta(\underline{a} \oplus \underline{b}) = \delta\underline{a} \oplus \delta\underline{b}$ and $\delta(\alpha \oplus \beta) = \delta\alpha \oplus \delta\beta$

(iii) $\delta: \underline{A}(\underline{a},\underline{b}) \to \underline{B}(\underline{a},\underline{b})$ is continuous

(iv) if $\underline{A}$ and $\underline{B}$ have permutations, then δ preserves permutations, i.e. $\delta(\xi) = \xi$

If $\underline{A}$ and $\underline{B}$ are categories on the same object generators and δ in addition preserves generators, i.e. $\delta(a) = a$, then δ is called an M^nT(resp. M^nTP)-functor.

Note that (iv) is equivalent to saying that δ is equivariant, i.e. $\delta(\alpha \circ \xi) = \delta(\alpha) \circ \xi$.

Example 1: $\mathrm{End}(X_1,\dots,X_n)$ for based spaces $X_1,\dots,X_n$ is an M^nTP-category. $\mathrm{End}\,(X_1,\dots,X_n)(\underline{a},\underline{b})$ is the space of all based maps $X_{\underline{a}} \to X_{\underline{b}}$ (see [6], chapter 5), where $X_{\underline{a}} = X_{i_1}\times\dots\times X_{i_k}$ if $\underline{a} = (i_1,\dots,i_k)$. The functor $\oplus$ is just $\times$. The permutations are the obvious shuffles.

Definition 1.3: An M^nT (resp. M^nTP)-category $\underline{B}$ is said to act on $(X_1,\dots,X_n)$ if we are given an M^nT (resp. M^nTP)-functor $\underline{B} \to \mathrm{End}(X_1,\dots,X_n)$.
If an M^1T (resp. M^1TP)-category $\underline{B}$ acts on X we call X a $\underline{B}$-space.

Example 2: $\underline{A}$, an M^1T-category: Denote the unique sequence of length m by $\underline{m}$. $\underline{A}(\underline{m},\underline{n})$ is the space of all order preserving functions $(1,\dots,m) \to (1,\dots,n)$ with the discrete topology. There is exactly one function $\lambda_n\colon \underline{n} \to \underline{1}$ for each n. An $\underline{A}$-space is a topological monoid (in CG, the category of compactly generated Hausdorff spaces).

Example 3: $\underline{S}$, an M^1TP-category: Again denote the unique sequence of length m by $\underline{m}$. $\underline{S}(\underline{m},\underline{n})$ is the set of all functions $(1,\dots,m) \to (1,\dots,n)$ with the discrete topology. The permutations are the ordinary permutations $(1,\dots,m) \to (1,\dots,m)$. An $\underline{S}$-space is an abelian topological monoid. Such a space X is known to have the homotopy type of a product of Eilenberg-Mac Lane spaces, if X is a connected CW-complex. (The proof of this is roughly as follows: Let G_n denote the nth homotopy group of X, and M(G,n) the Moore space with G as n-th homology group. For each n construct a map f_n: $M(G_n,n) \to X$ which induces an isomorphism of the n-th homotopy groups. The abelian monoid structure on X enables one to construct maps of the infinite symmetric products $SP(M(G_n,n))$ into X from the f_n's. These give rise to a map of the restricted product $\Pi_n SP(M(G_n,n))$ into X. $SP(M(G_n,n))$ is of the same homotopy type as the Eilenberg-Mac Lane complex $K(G_n,n)$, and the constructed map gives the required homotopy equivalence. For details see [3]).

<u>Definition 1.4</u>: An M^nTP-category <u>B</u> is in <u>normal form</u> if

(a) each morphism is expressible as

$\alpha = (\alpha_1 \oplus \ldots \oplus \alpha_k) \circ \xi$

where α_i is a morphism into an object generator for each i, $1 \leq i \leq k$, and ξ a permutation

(b) this expression is unique up to following equivalence

$(\alpha_1 \circ \eta_1 \oplus \ldots \oplus \alpha_k \circ \eta_k) \circ \xi$
$= (\alpha_1 \oplus \ldots \oplus \alpha_k) \circ [(\eta_1 \oplus \ldots \oplus \eta_k) \circ \xi]$

where η_i and ξ are permutations

(c) The morphism spaces of $\underline{B}$ have the quotient topology of the appropriate disjoint unions of product spaces of morphisms into a generator under the relation (b).

Analogously for M^nT-categories to be in normal form we demand that each morphism is uniquely expressible as a sum of morphisms into a generator and that the morphism spaces have the appropriate disjoint union topology of product spaces of morphisms into a generator.

The importance of categories in normal form is clear from

<u>Theorem 1.5</u>: Given an arbitrary M^nTP (resp. M^nT)-category $\underline{B}$ there exists another M^nTP (resp. M^nT)-category $\underline{C}$ in normal form and an M^nTP (resp. M^nT)-functor $\gamma: \underline{C} \to \underline{B}$ satisfying $\underline{C}(\underline{a},b) = \underline{B}(\underline{a},b)$, and $\gamma|\underline{C}(\underline{a},b)$ is the identity. $\underline{C}$ and γ are unique up to isomorphism. (Recall: b denotes an object generator)

<u>Proof</u>: Put $\underline{C}(\underline{a},b) = \underline{B}(\underline{a},b)$ as required. Construct the spaces of morphisms into larger sequences according to condition 1.4 (c): For $\underline{a} = (i_1,\dots,i_k)$ and $\underline{b} = (j_1,\dots,j_l)$

let $V(\underline{a},\underline{b}) = \bigcup \underline{C}(\underline{c}_1, j_1)\times\underline{C}(\underline{c}_2, j_2)\times\ldots\times\underline{C}(\underline{c}_l, j_l)\times\{\xi\}$ taken over all partitions of the sequence $\underline{c}_1\oplus\ldots\oplus\underline{c}_l =$ $(i_{\xi^{-1}(1)},\ldots,i_{\xi^{-1}(k)})$ into connected subsequences and all permutations $\xi\in S(k)$. $V(\underline{a},\underline{b})$ has the disjoint union topology of the products and hence is in CG.

Introduce the relation 1.4 (b) into $V(\underline{a},\underline{b})$:

$(\bar{\alpha}_1\circ\eta_1,\ldots,\bar{\alpha}_l\circ\eta_l;\xi) \sim (\bar{\alpha}_1,\ldots,\bar{\alpha}_l;(\eta_1\oplus\ldots\oplus\eta_l)\circ\xi)$

where $\bar{\alpha}_i$ is the element α_i of $\underline{B}$ considered as element of $\underline{C}$, and η_i a permutation in $S(\text{source } \alpha_i)$.

Let $\underline{C}(\underline{a},\underline{b}) = V(\underline{a},\underline{b})/\sim$. $\underline{C}(\underline{a},\underline{b})$ is Hausdörff and hence in CG. Composition with permutations is forced on us by 1.4 (b) and 1.1 (d): Let $\eta\in S(l)$ and $\zeta\in S(k)$ then

$\eta\circ(\bar{\alpha}_1,\ldots,\bar{\alpha}_l;\xi) = (\bar{\alpha}_{\eta^{-1}(1)},\ldots,\bar{\alpha}_{\eta^{-1}(l)};\eta(r_1,\ldots,r_l)\circ\xi)$

$(\bar{\alpha}_1,\ldots,\bar{\alpha}_l;\xi)\circ\zeta = (\bar{\alpha}_1,\ldots,\bar{\alpha}_l;\xi\circ\zeta)$

where r_p is the length of source $\bar{\alpha}_p$.

Denoting $(\bar{\alpha}_1,\ldots,\bar{\alpha}_l;\xi)$ by $(\bar{\alpha}_1\oplus\ldots\oplus\bar{\alpha}_l)\circ\xi$ we have defined a continuous associative sum in $\underline{C}$.

To define composition, note that it suffices to define it for $\bar{\alpha}\circ[\bar{\beta}_1\oplus\ldots\oplus\bar{\beta}_n]$, where $\bar{\alpha}$ is a morphism into a generator, and to check associativity and the existence of a unit just for these elements since we have taken care of the permutations already. Denote composition in $\underline{B}$ by $*$ and in $\underline{C}$ by $\circ$. Define

$\overline{\alpha}\circ[\overline{\beta}_1\oplus\ldots\oplus\overline{\beta}_n] = \overline{\alpha*(\beta_1\oplus\ldots\oplus\beta_n)}$

Since composition and sum in $\underline{B}$ are continuous and since this definition respects the identifications, this composition is well defined and continuous. It is associative and has $\overline{1}_{j_1}\oplus\ldots\oplus\overline{1}_{j_n}$ as identity for $(j_1,\ldots,j_n)$.

By construction $\oplus$ is a Bifunctor. The per mutations are represented by an element of the form (sum of identities)$\circ\xi$. By construction $\underline{C}$ is in normal form.

Define the functor γ: $\underline{C}\to\underline{B}$ by

$\gamma[(\overline{\alpha}_1\oplus\ldots\oplus\overline{\alpha}_k)\circ\xi] = (\alpha_1\oplus\ldots\oplus\alpha_k)*\xi$

Since $\oplus$ is a bifunctor the relation 1.4 (b) holds in any M^nTP-category. Hence γ is well defined. It is continuous, and preserves sums, permutations and identities. From the definition of composition it is clear that γ is a functor. Hence it is an M^nTP-functor.

The construction for M^nT-categories is completely analogous, but simpler.]]

We refer to the construction of morphism spaces into longer sequences once the ones into generators and their compositions with permutations are given as the normal form construction.

<u>Corollary 1.6</u>: Let $\underline{B}$ be an M^nTP (resp. M^nT)-category and $\underline{C}$ the associated category in normal form. If $\underline{B}$ acts

on $(X_1,\ldots,X_n)$ then we ~~can~~ can canonically make $\underline{C}$ act on $(X_1,\ldots,X_n)$.

This effects a welcome simplification in the theory. Of our examples, 2 and 3 are in normal form, but 1 is not.

Example 4: (This example is due to J. M. Boardman) $\underline{Q}_n$, an M^1TP-category operating on the n-th loop space $\Omega^n Y = X$. The space $\Omega^n Y$ is the space of all maps $(I^n, \partial I^n) \to (Y, o)$, where o is the basepoint of Y, I^n is the standard n-cube, and ∂I^n its boundary. A point $\alpha \in \underline{Q}_n(\underline{k}, \underline{1})$ where $\underline{k}$ is the unique sequence of length k, is an ordered collection α of k n-cubes I_i^n linearly embedded in I^n with their a xes parallel to those of I^n, having disjoint interiors. It acts on $\Omega^n Y$ as follows: Given $(f_1,\ldots,f_k) \in X^k$, i.e. maps $f_i\colon I^n \to Y$, we construct the map

$$\alpha(f_1,\ldots,f_k)\colon I^n \to Y$$

by using f_i on the little cube I_i^n and the zero map outside the little cubes. We topologize $\underline{Q}_n(\underline{k},\underline{1})$ as a subspace of R^{2kn}. The permutations permute the coordinates of X^k. $\underline{Q}(\underline{k},\underline{r})$ is now obtained by the normal form construction.

We observe that $\underline{Q}(\underline{k},\underline{1})$ is (n-2)-connected. We will make use of this fact in Chapter IV.

Example 5: Let $\underline{B}$ be an M^1TP (resp. M^1T)-category in normal form and $\underline{C}$ a topological category with n objects $0,\ldots,n-1$ (in a topological category the morphism sets are topological spaces and composition is continuous). Then $\underline{B}$ and $\underline{C}$ give rise to an M^nTP (resp. M^nT)-category $\underline{B}*\underline{C}$. Denote the unique sequence of length m in $\underline{B}$ by $\underline{m}$. A morphism from $\underline{a} = (i_1,\ldots,i_k)$ to j is a (k+1)-tuple $(\beta;f_1,\ldots,f_k)$ with $\beta \in \underline{B}(\underline{k},\underline{1})$ and $f_p \in \underline{C}(i_p,j)$. $\underline{B}*\underline{C}(\emptyset,j) = \underline{B}(\emptyset,\underline{1})$ where $\emptyset$ denotes the empty sequence. Denote the morphisms of $\underline{B}*\underline{C}(\emptyset,j)$ by $(\beta;\emptyset j)$. Give $\underline{B}*\underline{C}(\underline{a},j)$ the product topology of $\underline{B}(\underline{k},\underline{1}) \times \underline{C}(i_1,j) \times\ldots\times \underline{C}(i_k,j)$. Define composition with permutations on the right by

$$(\beta;f_1,\ldots,f_k)\circ\xi = (\beta\circ\xi;f_{\xi(1)},\ldots,f_{\xi(k)}), \qquad \xi \in S(k).$$

Define the morphisms into longer sequences by the normal form construction. Composition is given by

$$(\beta;f_1,\ldots,f_k)\circ(x_1\oplus\ldots\oplus x_k)$$
$$= (\beta\circ(\beta_1\oplus\ldots\oplus \beta_k);f_1\circ g_{11},\ldots,f_1\circ g_{1p_1},\ldots,f_k\circ g_{k1},\ldots,f_k\circ g_{kp_k})$$

where $x_i = (\beta_i;g_{i1},\ldots,g_{ip_i})$, with the convention that $f_i\circ\emptyset j$ drops out.

The composition is continuous and since it is induced by the compositions in $\underline{B}$ and $\underline{C}$ it is associative. $(1_{\underline{k}};1_{i_1},\ldots,1_{i_k})$ serves as identity and $\oplus$ is a bifunctor by construction. Hence $\underline{B}*\underline{C}$ is an $M^n TP$-category.

Note intuitively that if we denote $(\beta;1_j,\ldots,1_j)$ by $(\beta;j)$ and $(1_{\underline{1}};f)$ by f, we have $(\beta;f_1,\ldots,f_k) = (\beta;j)\circ(f_1\oplus\ldots\oplus f_k)$.

Observe that if $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ and $(\underline{C}(k,k), 1_k)$ are NDR-pairs, $1 \leq k \leq n$, then the $(\underline{B}*\underline{C}(\underline{a},\underline{a}), 1_{\underline{a}})$ are NDR-pairs too for all sequences $\underline{a}$. This follows from the fact that $(\underline{B}*\underline{C}(j,j), 1_j)$ is a NDR-pair for all j, $1 \leq j \leq n$ and $\underline{B}*\underline{C}$ is in normal form. See also [6; Lemma 7.3].

We have n canonical MTP-inclusion functors

$$\iota_p\colon \underline{B} \to \underline{B}*\underline{C}$$

given by $\iota_p[(\beta_1\oplus\ldots\oplus \beta_k)\circ\xi] = [(\beta_1;p)\oplus\ldots\oplus(\beta_k;p)]\circ\xi$ and a topological (i.e. continuous) inclusion functor

$$\Lambda\colon \underline{C} \to \underline{B}*\underline{C}$$

given by $\Lambda(f) = f$.

All functors embed the respective categories as closed (in the topological sense) subcategories in $\underline{B}*\underline{C}$. Hence their images have the relative topology in $\underline{B}*\underline{C}$.

The construction in the M^1T-case is completely analogous.

For illustration: If $\underline{A}$ is the category of Example 2 then an action of $\underline{A}*\underline{C}$ induces a functor from $\underline{C}$ into the category of topological monoids .]]

Let $\underline{\underline{Is}}_n$ be the category with n objects 0,...,n-1 and exactly one morphism between any two objects.

Lemma 1.7: Any M^nTP (M^nT)-category $\underline{B}$ in normal form is augmented over $\underline{S}*\underline{\underline{Is}}_n$ (resp. $\underline{A}*\underline{\underline{Is}}_n$) by a (necessarily) unique M^nTP (M^nT)-functor δ: $\underline{B} \to \underline{S}*\underline{\underline{Is}}_n$ ($\underline{A}*\underline{\underline{Is}}_n$)

Proof: There exists exactly one morphism from $\underline{a} = (i_1,\ldots,i_k)$ to j in $\underline{S}*\underline{\underline{Is}}_n$ uniquely represented by $(\lambda_k;(i_1,j),\ldots,(i_k,j))$ where λ_k is the unique function $(1,\ldots,k) \to (1)$ in $\underline{S}$ and (i,j) the unique morphism from i to j in $\underline{\underline{Is}}_n$. This determines δ uniquely on $\underline{B}(\underline{a},j)$. Using the normal form of $\underline{B}$ we get a necessarily unique extension of δ to $\underline{B}$. That δ is a functor follows again from the fact that there is exactly one morphism from $\underline{a}$ to j in $\underline{S}*\underline{\underline{Is}}_n$.]]

CHAPTER II: THE UNIVERSAL CONSTRUCTION

Unless otherw ise stated we only consider categories in normal form from now on.

The concept of monoid is not a good one from the point of view of homotopy theory, because the existence of a monoid structure on a space is not a homotopy invariant. For example, the loop space ΩX has no natural monoid structure, although it is a deformation retract of a natural monoid. Similarly for other categories of operators. For this reason we look for a "universal" structure.

Suppose given an M^nTP (M^nT)-category $\underline{B}$. We want to construct a "universal" M^nTP (M^nT)-category $\underline{U}$ with the following properties:

(U1) There exists an M^nTP (M^nT)-functor $\varepsilon: \underline{U} \to \underline{B}$, the <u>standard augmentation</u> of $\underline{B}$, and a collection ι of equivariant maps (not functors) $\iota: \underline{B}(\underline{a},b) \to \underline{U}(\underline{a},b)$ for all sequences $\underline{a}$ and all generators b, the <u>standard section</u> of $\underline{B}$, such that

$\varepsilon\circ\iota \;|\underline{B}(\underline{a},b) = 1 \;|\underline{B}(\underline{a},b)$

$\iota\circ\varepsilon \;|\underline{U}(\underline{a},b) \simeq 1 \;|\underline{U}(\underline{a},b)$ equivariantly (if $\underline{B}$ has permutations) and fibrewise

(U2) ($\underline{U}$, ε, ι) is universal with respect to (U1), i.e.

given an M^nTP (M^nT)-category $\underline{C}$, an M^nTP (M^nT)-functor $\delta : \underline{C} \to \underline{B}$ and a collection σ of equivariant maps $\sigma : \underline{B}(\underline{a},b) \to \underline{C}(\underline{a},b)$ for all sequences $\underline{a}$ and all generators b such that

$\delta \circ \sigma \;|\underline{B}(\underline{a},b) = 1 \;|\underline{B}(\underline{a},b)$

$\sigma \circ \delta \;|\underline{C}(\underline{a},b) \simeq 1 \;|\underline{C}(\underline{a},b)$ equivariantly and fibrewise, then there exists an M^nTP (M^nT)-functor $\nu : \underline{U} \to \underline{C}$ such that $\delta \circ \nu = \varepsilon$.

Notation: A collection of maps σ as given in (U2) such that $\delta \circ \sigma \;|\underline{B}(\underline{a},b) = 1 \;|\underline{B}(\underline{a},b)$ is called a (equivariant) section of δ.

A functor δ which has a section σ satisfying the requirements of (U2) is called fibre homotopically trivial.

We are going to give a construction W which associates with each M^nTP (M^nT)-category $\underline{B}$ such that $(\underline{B}(b,b), 1_b)$ is a NDR-pair for all generators b, an M^nTP (M^nT)-category $W\underline{B}$ together with an augmentation $\varepsilon_B : W\underline{B} \to \underline{B}$ and a section $\iota_B : \underline{B} \to W\underline{B}$ such that the triple $(W\underline{B}, \varepsilon_B, \iota_B)$ satisfies (U1). Furthermore for any M^nTP (M^nT)-category $\underline{B}$ we can find a triple $(\underline{B}^\sim, \varepsilon'_B, \iota'_B)$ such that $W\underline{B}^\sim$ exists and $(W\underline{B}^\sim, \varepsilon'_B \circ \varepsilon_{B^\sim}, \iota'_B \circ \iota_{B^\sim})$ satisfies (U1) and (U2).

§ 2 THE CONSTRUCTION W

Since the construction for the M^nT-case is completely analogous to the one for the M^nTP-case, rather more simple in fact, we restrict ourselves to the MTP-case.

Let $\underline{B}$ be an M^nTP-category such that $(\underline{B}(b,b), 1_b)$ is a NDR-pair for all generators b. To obtain the universal property we start off the free M^nTP-category in the discrete topology over $\underline{B}$. We then topologize the morphism sets and attach "cells" to them to obtain the property (U1).

We form a bar construction by considering words $[\alpha_0|\dots|\alpha_k]$ where $k \geq 0$, each α_i is a morphism in $\underline{B}$, and the composite $\alpha_0 \circ \dots \circ \alpha_k$ exists in $\underline{B}$.

Definition 2.1: The category $W^0\underline{B}$ has as morphisms from $\underline{a}$ to $\underline{b}$ those words $[\alpha_0|\dots|\alpha_k]$ for which $\alpha_0 \circ \dots \circ \alpha_k$ is a morphism in $\underline{B}$ from $\underline{a}$ to $\underline{b}$ subject to the following relations and their consequences:

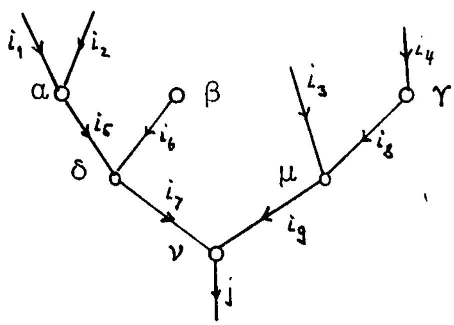
(a) $[\alpha \oplus \beta] = [\alpha \oplus 1|1 \oplus \beta] = [1 \oplus \beta|\alpha \oplus 1]$

(b) $[1]$ is an identity

(c) $[\alpha|\xi] = [\alpha \circ \xi]$, $[\xi|\beta] = [\xi \circ \beta]$ (if $\underline{B}$ has permutations).

Composition in $W^0\underline{B}$ is by juxtaposition.

Let us give an alternative pictorial description of $W^0\underline{B}$. A morphism in $W^0\underline{B}(\underline{a},b)$ is represented by a pair (θ,ξ) where $\xi \in S(r)$, r being the length of $\underline{a}$, and θ a finite tree in the plane with directed edges, labelled by $0,\ldots,n-1$, repetition is allowed, except that some edges do not join two vertices (see picture below). There is just one, called the <u>root</u>, labelled by b, that leaves a vertex and goes nowhere, and exactly r <u>twigs</u> labelled by $i_{\xi^{-1}(1)},\ldots,i_{\xi^{-1}(r)}$ if $\underline{a} = (i_1,\ldots i_r)$, that come from nowhere. The other edges are called links and join two vertices. Each vertex has exactly one outgoing edge and the vertex is labelled by a morphism in $\underline{B}(\underline{p},q)$ where q is the label of the outgoing edge and $\underline{p} = (j_1,\ldots,j_k)$ where k is the number of incoming edges and $j_1,\ldots,j_k$ their labels from left to right.



$(i_1,i_2,i_3,i_4)\to(j)$

Call the tree with no vertex consisting of a labelled edge only, a <u>trivial tree</u>.

The relation can now be described as follows:

(2.2) any vertex labelled by $1_b \in \underline{B}(b,b)$ may be suppressed

(2.3) if we obtain the tree θ by substituting a vertex

$\alpha \in \underline{B}(\underline{p},q)$ of the tree φ by the vertex $\alpha\circ\eta$, where η is a permutation, permute the incoming edges of α and the subtrees of φ sitting on them in such a fashion that the $\eta(i)$-th incoming edge of α is the i-th incoming edge of $\alpha\circ\eta$, then

$$(\varphi,\xi) \sim (\theta,\eta^{-1}(r_1,\ldots,r_k)\circ\xi)$$

where r_i is the number of twigs of the subtree of φ over the i-th incoming edge of α.

Define composition with permutations on the right by

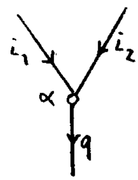
$$(\theta,\xi)\circ\zeta = (\theta,\xi\circ\zeta)$$

Now the sets $W^0\underline{B}(\underline{a},\underline{b})$ can be obtained by the normal form construction. A morphism of $W^0\underline{B}(\underline{a},\underline{b})$ with $\underline{a} = (i_1,\ldots,i_r)$ and $\underline{b} = (j_1,\ldots,j_s)$ is represented by a pair (θ,ξ) where $\xi \in S(r)$ and θ is an ordered collection of s such trees, called a copse, the twigs of this collection being labelled by $i_{\xi^{-1}(1)},\ldots,i_{\xi^{-1}(r)}$ in order (always from left to right) and the roots by $j_1,\ldots,j_s$, again subject to the relation (2.2) and a generalized version of (2.3): Let φ be the copse obtained from the copse θ by changing the tree the twigs of which are labelled by $i_{\xi^{-1}(t+1)},\ldots,i_{\xi^{-1}(t+q)}$ according to (2.3), and let $e \in S(t)$ and $e' \in S(r-t-q)$ be the identities, then

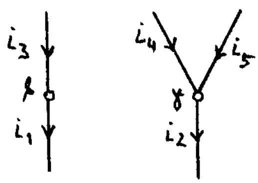
$$(\theta,\xi) \sim (\varphi,(e \oplus \eta^{-1}(r_1,\ldots,r_q) \oplus e')\circ\xi)$$

Composition $\theta\circ\varphi$ of copses θ,φ is obtained by attaching the roots of φ to the twigs of θ (see picture below). Since the roots of φ are labelled in the same way as the twigs of θ, $\theta\circ\varphi$ is a well defined copse. The sum $\theta \oplus \psi$ of the two copses θ and ψ is obtained by putting them side by side, thr trees of θ followed by the trees of ψ.

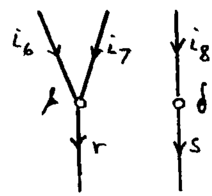
If



θ

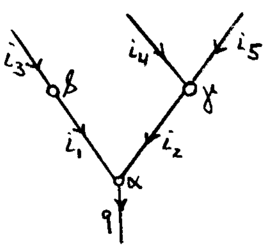


φ

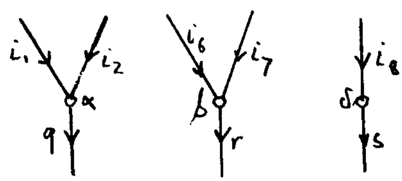


ψ

then



$\theta\circ\varphi$



$\theta \oplus \psi$

Let $\xi \in S(r)$ and let φ be a copse with r trees. Let $\xi.\varphi$ be the copse with the j-th tree being the $\xi^{-1}(j)$-th tree of φ. Sum and composition in $W^0\underline{B}$ are now given by

$$(\theta,\xi) \oplus (\varphi,\eta) = (\theta \oplus \varphi,\xi \oplus \eta)$$

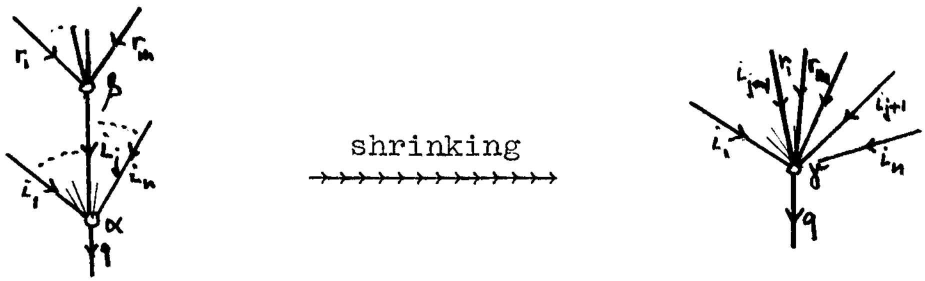
$$(\theta,\xi)\circ(\varphi,\eta) \quad = (\theta\circ(\xi.\varphi),\xi(r_1,\ldots,r_k)\circ\eta)$$

where r_q is the number of twigs of the q-th tree in φ.

If the edges of two copses are labelled in the same way they are said to have the same <u>type</u>. Give the set of all copses of one type the product topology of their vertex spaces, and give the set of all copses with a given source and target the union topology of the union of all types with the given source and target. The trivial copses, i.e. the copses consisting of trivial trees only, are their own open and closed components. Composition of copses is continuous, associative and the the trivial copses act as identities. $\oplus$ is continuous, associative, and a bifunctor. Hence, disregarding all relations , the copses over $\underline{B}$ form an M^nT-category, if we just consider the copses and leave the permutations out. Disregarding all relations, the spaces of all pairs (the topology is induced by the copse component) form an M^nTP-category. Including the relations they give rise to the M^nTP-category $W^0\underline{B}$.

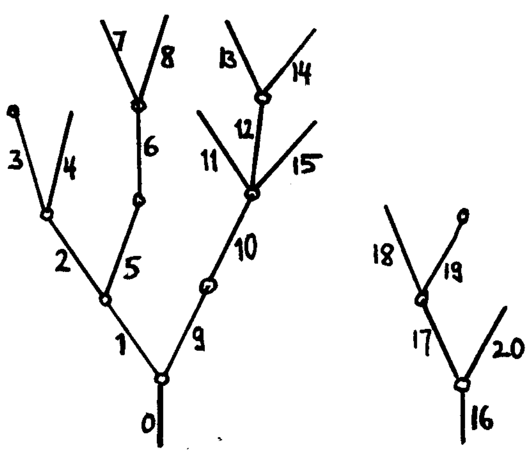
In the M^nT-case we would continue to work with the category of copses over $\underline{B}$, while the M^nTP-case requires the slightly more complicated category of pairs.

Let α and β be vertices of a copse θ joint by the j-th incoming edge of α. Let α and β have n resp. m incoming edges. Shrinking the link between α and β means substituting the subtreeof θ consisting of α and β and their and their edges by a vertex $\gamma = \alpha\circ(1 \oplus \beta \oplus 1)$ with m+n-1 incoming edges (see picture).



Let $TB(\underline{a},\underline{b}) = \{(\theta,\xi) \mid \xi \in S(k)$, where k = length $\underline{a}$, θ a copse with target $\underline{b}$ and source $\xi\underline{a}$ (the source of $\theta$is given by the labels of its twigs)$\}$ topologized by the topology inherited from the copses. Each type of copses with target $\underline{b}$ and source some $\eta\underline{a}$, $\eta \in S(k)$, determines an open and closed subset of $TB(\underline{a},\underline{b})$, called a component.

Index the edges of a copse by 0,1,..,k,... starting from the root of the first tree and going up the most left sequence of edges. Continue going upwards the next sequence of edges to the right, and continue (see picture). Call this the standard indexing.



Let $T_p\underline{B}(\underline{a},\underline{b}) \subset T\underline{B}(\underline{a},\underline{b})$ be the subspace of those elements the copses of which have exactly p links. Since $T_p\underline{B}(\underline{a},\underline{b})$ is a collection of components of $T\underline{B}(\underline{a},\underline{b})$ it is closed in $T\underline{B}(\underline{a},\underline{b})$ and hence has the relative topology.

If i indexes the i-th link of the copse θ in the standard indexing and $d^i\theta$ is the copse obtained from θ by shrinking the i-th link, the correspondence

$$(\theta,\xi) \rightsquigarrow (d^i\theta,\xi), \qquad (\theta,\xi) \in T_p\underline{B}(\underline{a},\underline{b}),$$

defines a continuous map, called a <u>face map</u>,

$$d^i\colon K \to T_{p-1}\underline{B}(\underline{a},\underline{b})$$

where K is a component such that the i-th edge is a link.

Let $s^i\theta$ be the copse obtained from the copse θ by inserting the vertex 1_c in the i-th edge where c is the label of this edge. The correspondence

$$(\theta,\xi) \rightsquigarrow (s^i\theta,\xi)$$

defines a continuous map, called a <u>degeneracy map</u>,

$$s^i\colon T_p\underline{B}(\underline{a},\underline{b}) \to T_{p+1}\underline{B}(\underline{a},\underline{b})$$

Call $x \in T_p\underline{B}(\underline{a},\underline{b})$ <u>degenerate</u> if it is in the image of some degeneracy map.

Following identities hold whenever the maps involved are defined:

(2.4) $d^j \circ d^i = d^i \circ d^{j+1} \qquad j \geq i$

(2.5) $s^{j+1} \circ s^i = s^i \circ s^j \qquad j \geq i$

$$(2.6)\quad d^j \circ s^i = 1 \qquad j = i,\ i+1$$

$$(2.7)\quad s^i \circ d^j = d^{j+1} \circ s^i \qquad i < j$$
$$= d^j \circ s^{i+1} \qquad i \geq j$$

For the time being we restrict ourselves to the case where $\underline{b} = b$, i.e. $\underline{b}$ is a sequence of length 1.

Lemma 2.8: Each $x \in Tp\underline{B}(\underline{a},b)$ can be written uniquely as

$$x = s^{k_m} \circ \ldots \circ s^{k_1} y$$

where $k_1 < \ldots < k_m$ and y is not degenerate.

Proof: y is uniquely determined by deleting all vertices labelled by an identity from the tree of x. Hence x is obtained from y by inserting identities, i.e. by applying degeneracy maps. By (2.5) we can choose them uniquely in the required fashion.]]

Let $\Gamma = (R,M,Q)$ be a gadget consisting of a topological monoid M with multiplication $*$ and unit e, a closed right "ideal" R and a closed left "ideal" Q, i.e. closed subspaces R and Q of M such that $R*M \subset R$ and $M*Q \subset Q$, satisfying

Axiom M1: (i) There exist no inverses in M, i.e. if

$$x*y = e \text{ then } x = y = e.$$

(ii) There exist $r_o \in R$ and $t_o \in Q$ such that $r_o * t_o$ is a right identity in Q and a left identity in R. And $r_o * t_o \neq e$.

(iii) (M, e) is a NDR-pair.

Put $u = r_o * t_o$. Then $u*u = u$. Hence without loss of generality we can assume that $r_o = t_o = u$. Since $R*M \subset R$ and $M*Q \subset Q$, $R*Q \subset R \cap Q$. Also if $x \in R \cap Q$, then $x = x*u \in R*Q$, and hence $R \cap Q \subset R*Q$.

<u>Example 2.9</u>: Let M be the unit interval with the multiplication $t_1 * t_2 = \max(t_1, t_2)$. Then e is $0 \in I$. Take $Q = R = 1 \in I$, a nd $u = 1$. Then (M1) is satisfied.

More examples will come in some later section.

Define maps $s^i: M^p \to M^{p-1}$ and $d^i: M^p \to M^{p+1}$, called <u>face</u> and <u>degeneracy maps</u>, by

$$s^i(t_o, \dots, t_{p-1}) = (t_o, \dots, t_{i-1}, t_i * t_{i+1}, t_{i+2}, \dots, t_{p-1})$$
$$d^i(t_o, \dots, t_{p-1}) = (t_o, \dots, t_{i-1}, e, t_i \quad, \dots, t_{p-1}).$$

The maps satisfy following identities:

(2.4')	$d^i \circ d^j = d^{j+1} \circ d^i$	$j \geq i$
(2.5')	$s^i \circ s^{j+1} = s^j \circ s^i$	$j \geq i$
(2.6')	$s^i \circ d^j = 1$	$j = i, i+1$
(2.7')	$d^j \circ s^i = s^i \circ d^{j+1}$	$i < j$
	$= s^{i+1} \circ d^j$	$i \geq j$

Call $\delta \in M^p$ degenerate if it is in the image of some degeneracy map.

<u>Lemma 2.10</u>: Each $\delta \in M^p$ can be expressed uniquely as

$$\delta = d^{l_n} \circ \ldots \circ d^{l_1} \partial$$

where $l_1 < \ldots < l_n$ and ∂ is not degenerate.

<u>Proof</u>: ∂ is uniquely determined by deleting the coordinates e of $\delta = (t_1, \ldots, t_p)$. δ is then obtained from ∂ by applying degeneracy maps. (2.4') allows us to choose them in the required fashion.]]

Let $(\theta,\xi) \in Tp(\underline{a},b)$. To each link of θ we assign an element of M, to each twig an element of Q, and to the root an element of R. In the case of a trivial tree root and twig coincide and we assign to it an element of $R \cap Q$. The elements of $Tp(\underline{a},b)$ together with all possible assignments of this form give rise to a topological space $C_p\underline{B}\Gamma(\underline{a},b) = Tp\underline{B}(\underline{a},b) \times (R \times M^p \times Q^k)$, where k = length $\underline{a}$, $p \geq 0$. Let $T\text{-}1\underline{B}(b,b)$ be the space consisting of the trivial tree with the edge labelled by b. Define $C_{-1}\underline{B}\Gamma(\underline{a},b) = \emptyset$ if $\underline{a} \neq b$ and $C_{-1}\underline{B}\Gamma(b,b) = T\text{-}1\underline{B}(b,b) \times (R \cap Q) = R \cap Q$. For convenience we denote $C_p\underline{B}\Gamma(\underline{a},b)$ simply by $C_p(\underline{a},b)$.

Let $V_p(\underline{a},b) = V_p\underline{B}\Gamma(\underline{a},b)$ be the disjoint union
$V_p(\underline{a},b) = C_{-1}(\underline{a},b) \cup C_0(\underline{a},b) \cup \ldots \cup C_p(\underline{a},b)$
and $V(\underline{a},b) = V\underline{B}\Gamma(\underline{a},b)$ the disjoint union of all $C_p(\underline{a},b)$, $p = -1,0,1,\ldots$. The $C_p(\underline{a},b)$'s are in CG, and hence the $V_p(\underline{a},b)$'s and $V(\underline{a},b)$ are in CG because they are Hausdorff [6; Lemma 9.2].

Introduce the following relations in $V(\underline{a},b)$:

(2.11) Each $x \in C_p(\underline{a},b)$ is given by a pair $(\theta,\xi) \in Tp(\underline{a},b)$ with an element of M assigned to each edge. Let $y \in C_p(\underline{a},b)$ be obtained from x by changing (θ,ξ) to (φ,η) according to relation (2.3). The elements of M assigned to each edge of φ are given by carrying the elements of M assigned to the edges of θ along during the permutation of edges which defines φ. Then $x \sim y$.

(2.12) $(d^i x,\delta) \sim (x,d^i\delta)$, i indexes a link in $x = (\theta,\xi)$

(2.13) $(s^i x,\partial) \sim (x,s^i\partial)$

where $(x,d^i\delta)$, $(x,s^i\partial) \in C_p(\underline{a},b)$.

Note that if i indexes a link then $(d^i x,\delta) \in C_{p-1}(\underline{a},b)$ iff $(x,d^i\delta) \in C_p(\underline{a},b)$, and $(s^i x,\partial) \in C_{p+1}(\underline{a},b)$ iff $(x,s^i\partial) \in C_p(\underline{a},b)$ since R is a right ideal and Q a left ideal of M.

Call a point $(x,\delta) \in C_p(\underline{a},b)$ <u>degenerate</u> if x or δ are degenerate.

<u>Lemma 2.14</u>: Each $(x,\delta) \in V(\underline{a},b)$ is related under (2.12) and (2.13) to a unique non-degenerate point.

<u>Proof</u>: Let $x = s^{k_m} \circ \ldots \circ s^{k_1} y$ be the unique expression for x given in Lemma 2.8. Define a function λ by

$$\lambda(x,\delta) = (y, s^{k_1} \circ \ldots \circ s^{k_m} \delta)$$

Define a function ρ by setting $\delta = d^{l_n} \circ \ldots \circ d^{l_1} \partial$ as uniquely given in Lemma 2.10. Define

$$\rho(x,\delta) = (d^{l_1} \circ \ldots \circ d^{l_n} x, \partial).$$

By Axiom M1 (ii), $e \in M$ cannot be assigned to a root or a twig since M does not have any inverses. Hence $l_1, \ldots, l_n$ index links in x and $d^{l_1} \circ \ldots \circ d^{l_n} x$ is defined.

$\lambda\rho(x,\delta)$ is not degenerate since ∂ is not degenerate and hence since M does not have inverses $s^i \partial$ is not degenerate for any i.

It is easily seen that $\lambda\rho(x_1,\delta_1) = \lambda\rho(x_2,\delta_2)$ if (x_1,δ_1) and (x_2,δ_2) are related under (2.12) and (2.13). Since $\lambda\rho$ is the identity on non-degenerate points, $\lambda\rho(x,\delta)$ is independent of the choice of (x,δ) in its equivalence class.]]

Let $W(\underline{a},b) = W\underline{B}\Gamma(\underline{a},b)$ be obtained from $V(\underline{a},b)$ by factoring out the relations (2.11), (2.12), and (2.13), and $\underline{W}(\underline{a},b)$ by factoring out the relations (2.12) and (2.13) only. Let $\underline{W}_p(\underline{a},b) = \underline{\pi}\,(V_p(\underline{a},b))$, where $\underline{\pi}\colon V(\underline{a},b) \to \underline{W}(\underline{a},b)$ is the projection, and let $\underline{\pi}_p = \underline{\pi}|V_p$.

Lemma 2.15: (a) $\underline{W}_p\underline{B}\Gamma(\underline{a},b)$ and $\underline{W}\underline{B}\Gamma(\underline{a},b)$ are in CG.

(b) $\underline{W}\underline{B}\Gamma(\underline{a},b)$ has the limit tpology from $\underline{W}_{-1}\underline{B}\Gamma(\underline{a},b) \subset \ldots \subset \underline{W}_p\underline{B}\Gamma(\underline{a},b) \subset \ldots$

(c) $(\underline{W}_p\underline{B}\Gamma(\underline{a},b),\ \underline{W}_{p-1}\underline{B}\Gamma(\underline{a},b))$ and $(\underline{W}\underline{B}\Gamma(\underline{a},b),\ \underline{W}_p\underline{B}\Gamma(\underline{a},b))$ are NDR-pairs for all p.

(d) $(\underline{W}\underline{B}\Gamma(b,b),\ 1_b)$ are NDR-pairs if $(R\cap Q, r_o * t_o)$ is a NDR-pair.

Since we are required to prove similar statements to those of Lemma 2.15 in the further development of our theory we analyse the general problem before we prove 2.15.

We are given a space X which is a disjoint union of spaces $X_0 \cup X_1 \cup X_2 \cup \ldots$ and an equivalence relation $\sim$ on X. Let $Y = X/\sim$ and $Y_n = (X_0 \cup \ldots \cup X_n)/\sim$, and let $\pi_n\colon (X_0 \cup \ldots \cup X_n) \to Y_n$ be the projection. Put $DX_n = \{x \in X_n |$ There exists $y \in X_i$, $i < n$, such that $y \sim x\}$.
We suppose:

(1) $\sim$ satisfies: if $x, y \in X_n - DX_n$ and $x \sim y$ then $x = y$.

(2) DX_n is a finite union of closed subspaces F_r and we are given continuous maps $f_r \colon F_r \to X_{i_r}$ such that $x \sim f_r(x)$ for all $x \in F_r$ and $i_r < n$ for all r.

(3) $f \colon DX_n \to Y_{n-1}$ given by $f|F_r = \pi_{n-1} \circ f_r$ is well defined.

Then Y_n is obtained from Y_{n-1} by attaching X_n to Y_{n-1} by the attaching map $f \colon DX_n \to Y_{n-1}$, and Y is the direct limit of $Y_o \subset Y_1 \subset \dots$.

Now we assume further:

(4) (X_n, DX_n) are NDR-pairs for all n and Y_o, X_n are in CG for all n.

Then by induction (Y_n, Y_{n-1}) are NDR-pairs for all n [6; Lemma 8.5] and hence Y is in CG and (Y, Y_n) are NDR-pairs for all n [6; Theorem 9.4 and Lemma 9.2].

<u>Proof of Lemma 2.15</u>: Let $X_p = C_p(\underline{a},b)$ and $\sim$ the equivalence relation generated by (2.12) and (2.13). Hence $Y_p = \underline{W}_p(\underline{A},b)$. DX_p is the space of all degenerate points of X_p, i.e. of those points $(x,\delta) = (\theta,\xi,\delta)$ where a vertex of θ is labelled by an identity or a coordinate of δ has the value e. By Lemma 2.14 two non-degenerate points cannot be related, and hence (1) holds.

Let F_i and G_i be the (closed) subspaces of DX_p consisting of those points (x,δ), $x = (\theta,\xi)$, where the vertex on top of the i-th link of θ is labelled by an identity resp. the i-th coordinate of δ has the value e. The maps $f_i\colon F_i \to X_{p-1}$ and $g_i\colon G_i \to X_{p-1}$, given by

$$f_i(x,\delta) = (y,s^i\delta) \quad\text{and}\quad g_i(x,\delta) = (d^i x,\partial)$$

where y and ∂ are the unique elements such that $x = s^i y$ and $\delta = d^i\partial$, are continuous (since s^i and d^i are continuous and since $x \to y$ and $\delta \to \partial$ are given by projections). $f\colon DX_p \to Y_{p-1}$, given by $f|F_i = \underline{\pi}_{p-1}\circ f_i$ and $f|G_i = \underline{\pi}_{p-1}\circ g_i$ is well defined by Lemma 2.14. Hence (2) and (3) hold.

Since X_p is the disjoint union of products arising from the different types of trees, and since (M, e) and $(\underline{B}(b,b), 1_b)$ are NDR-pairs for all generators b, (X_p, DX_p) is an NDR-pair for all p [6; Lemma 7.3]. Hence (4) holds .

Observe that $X_o = Y_o$ and we will show later that $r_o * t_o \in R * Q = \underline{W}_{-1}(b,b) \subset \underline{W}(b,b)$ will serve as identity. Hence Lemma 2.15 follows from our general consideration. For part (d) use [6; Lemma 7.2].]]

<u>Lemma 2.16</u>: If $(x,\delta) \sim (y,\partial)$ under (2.11) , then

$\lambda\rho(x,\delta) \sim \lambda\rho(y,\partial)$ under (2.11), where $\lambda\rho$ is the function constructed in the proof of Lemma 2.14 .

Proof: Picturing each element as a pair (θ,ξ) with elements of M assigned to each edge, the proof is immediate since (2.11) commutes with the shrinking of links and the deleting of vertices labelled by an identity.]]

Corollary 2.17: $W\underline{B}\Gamma(\underline{a},b) = \underline{W\underline{B}\Gamma}(\underline{a},b)/\sim$, where $\sim$ is the equivalence relation generated by (2.11) applied to non-degenerate points only.]]

Corollary 2.18: $W\underline{B}\Gamma(\underline{a},b)$ has the limit topology from $W_p\underline{B}\Gamma(\underline{a},b) = \underline{W}_p\underline{B}\Gamma(\underline{a},b)/(2.11)$ and is in CG.

Proof: $W(\underline{a},b)$ is Hausdorff since $\underline{W}(\underline{a},b)$ is. Hence it is in CG. If $q\colon \underline{W}(\underline{a},b) \to W(\underline{a},b)$ is the projection then $q^{-1}\circ q(\underline{W}_p(\underline{a}\_,b)) = \underline{W}_p(\underline{a},b)$. The corollary now follows from [6;Theorem 9.5].]]

Let $(x,\delta) = (\theta,\xi,\delta) \in C_p(\underline{a},b)$ be a representative of an element in $W(\underline{a},b)$. Define composition with permutations on the right by

$$(\theta,\xi,\delta)\circ\zeta = (\theta,\xi\circ\zeta,\delta)$$

This defines a continuous composition with permutations in $W(\underline{a},b)$.

Define the spaces of morphisms into longer sequences by the normal form construction. We can give an alternative description along the lines of copses. We have reduced our construction to trees because the trivial copses would have made the argument somewhat unclean. Let $\underline{a} = (i_1,\ldots,i_k)$ and $\underline{b} = (j_1,\ldots,j_l)$. Let $T_{-1}(\underline{a},\underline{a})$ denote the one point space of the trivial copse from $\underline{a}$ to $\underline{a}$. To each link of θ in $(\theta,\xi) \in Tp(\underline{a},\underline{b})$ assign an element of M, to each twig an element of Q, and to each root an element of R, thus constructing spaces $C_p(\underline{a},\underline{b})$. In case θ contains a trivial tree, assign to its only edge an element of $R \cap Q$. Introduce in

$$V(\underline{a},\underline{b}) = C_{-1}(\underline{a},\underline{b}) \cup C_0(\underline{a},\underline{b}) \cup C_1(\underline{a},\underline{b}) \cup \ldots$$

the product relations from (2.12) and (2.13) and denote the quotient of $V(\underline{a},\underline{b})$ under these relations by $\underline{W}(\underline{a},\underline{b})$. Applying our previous considerations to each tree individually we again find that each element of $\underline{W}(\underline{a},\underline{b})$ is uniquely represented by a non-degenerate triple (θ,ξ,δ). Here (θ,ξ) is called degenerate if θ contains a degenerate tree, while the definition for δ being degenerate is the old one. Let

$$W'(\underline{a},\underline{b}) = \underline{W}(\underline{a},\underline{b})/\sim$$

where the equivalence relation is generated as follows:

Let $\theta = \theta_1 \oplus \ldots \oplus \theta_l$ such that θ_i is a non-degenerate tree. An element of M is assigned to each edge of θ_i, thus giving rise to a representative $(\theta_i, \text{unit}, \delta_i)$ of $\underline{W}(\underline{a}, j_i)$. Let $(\varphi_i, \xi_i, \partial_i) \sim (\theta_i, \text{unit}, \delta_i)$ under (2.11). Then

$$(2.19) \quad (\theta, \zeta, \delta_1 \times \ldots \times \delta_l) \sim (\varphi_1 \oplus \ldots \oplus \varphi_l, (\xi_1 \oplus \ldots \oplus \xi_l) \circ \zeta, \partial_1 \times \ldots \times \partial_l)$$

Relation (2.19) can be formulated for any triple $(\theta, \xi, \delta) \in C_p(\underline{a}, \underline{b})$ and as in the previous case $W'(\underline{a}, \underline{b})$ is obtained from $V(\underline{a}, \underline{b})$ by factoring out the relations (2.12), (2.13), and (2.19).

<u>Lemma 2.20</u>: $W\underline{B}\Gamma(\underline{a}, \underline{b}) \cong W'\underline{B}\Gamma(\underline{a}, \underline{b})$

<u>Proof</u>: Let $x_p = (\theta_p, \xi_p, \delta_p) \in V(\underline{a}_p, j_p)$. Define

$$h: W(\underline{a}, \underline{b}) \to W'(\underline{a}, \underline{b}) \quad \text{and} \quad k: W'(\underline{a}, \underline{b}) \to W(\underline{a}, \underline{b})$$

by $h\{x_1, \ldots, x_n; \xi\} = \{\theta_1 \oplus \ldots \oplus \theta_n, (\xi_1 \oplus \ldots \oplus \xi_n) \circ \xi, \delta_1 \times \ldots \times \delta_n\}$

$$k\{\theta_1 \oplus \ldots \oplus \theta_n, \xi, \delta_1 \times \ldots \times \delta_n\}$$
$$= \{(\theta_1, \text{unit}, \delta_1), \ldots, (\theta_n, \text{unit}, \delta_n); \xi\}$$

Where $\{\ \}$ denotes the equivalence class. h and k are well defined and since they are continuous on representatives they are continuous. They are inverse to each others.]]

As a consequence of the lemma we find that

$$\oplus : W(\underline{a}, \underline{b}) \times W(\underline{c}, \underline{d}) \to W(\underline{a} \oplus \underline{c}, \underline{b} \oplus \underline{d})$$

is given by

$$\{\theta,\xi,\delta\} \oplus \{\varphi,\eta,\partial\} = \{ \theta \oplus \varphi, \xi \oplus \eta, \delta \times \partial\} .$$

$\oplus$ is continuous since $W(\underline{a},\underline{b})$ is obtained by the norma l form construction.

Because of the normal form construction it suffices to define composition for $\{\theta,\text{unit},\delta\}\circ\{\varphi,\eta,\partial\}$ and prove the associativity for this case. $\{\theta,\text{unit},\delta\}\circ\{\varphi,\eta,\partial\}$ is represented by the pair $(\theta\circ\varphi,\eta)$ to each link, root, or twig of which coming from θ or φ we assign the value of M which it had in θ or φ, a nd to each new link of which we assign the product in M of the elements assigned to the original twig in θ with the element assigned to the original root in φ. Since the multiplication in M is associative, composition factors through (2.12), (2.13), and from the intuitive idea of a tree it is clear that it factors through (2.19). Since the multiplication in M and the composition of copses are continuous and associative, the composition in W is continuous and associa tive. The triple consisting of the copse of trivial trees with labelled edges $i_1,\ldots,i_k$, the unit permutation, and the element $u = r_o * t_o$ assigned to each tree acts as identity. And from the intuitive idea of copses it follows that $\oplus$ is a bifunctor (this also follows from the fact that W is obtained by the normal form construction and that the definition of composition is extended to the whole of W using the normal form). Hence

<u>Theorem 2.21</u>: $W\underline{B}\Gamma$ is an M^nTP-category in normal form.]]

Suppose the links of $(\theta,\text{unit}) \in Tp(\underline{a},b)$ are indexed by $i_1 < \ldots < i_p$ in the standard indexing. Let

$$\varepsilon(\theta,\text{unit}) = d^{i_1}\circ\ldots\circ d^{i_p}\theta$$

which determines a unique element in $\underline{B}$, namely the label of the unique vertex of $d^{i_1}\circ\ldots\circ d^{i_p}\theta$. Putting $\varepsilon(\theta,\text{unit}) = 1_b \in \underline{B}$ if θ is the trivial tree from b to b, the correspondence

$$\varepsilon\{\theta_1\oplus\ldots\oplus\theta_q,\xi,\delta\} = [\varepsilon(\theta_1,\text{unit}) \oplus\ldots\oplus \varepsilon(\theta_q,\text{unit})]\circ\xi$$

defines a continuous map from $W(\underline{a},\underline{b})$ to $\underline{B}(\underline{a},\underline{b})$. Here as always in future $\{\ \}$ denotes the equivalence class of the element in question. Since the shrinking process is basicly composition in $\underline{B}$, it is associative. Hence, since trivial trees are mapped to the corresponding identities, ε is an object preserving continuous functor. By definition it preserves sums and permutations. Hence it is an M^nTP-functor. Since the definition of ε is independent of Γ we denote it by ε_B. We call it the <u>standard augmentation</u> of $\underline{B}$.

Remark 2.22: Let $\underline{B}$, $\underline{C}$, $\underline{D}$ be an M^nTP- , M^mTP- , M^kTP-category respectively, and let $\gamma: \underline{B} \to \underline{C}$ and $\delta: \underline{C} \to \underline{D}$ be MTP-functors. Then γ and δ determine canonical MTP-functors $W\gamma$ and $W\delta$ such that $\varepsilon_C \circ W\gamma = \gamma \circ \varepsilon_B$, similarly for δ, and $W(\delta\circ\gamma) = W\delta\circ W\gamma$. (You construct $W\gamma$ by applying γ to the vertex labels of each copse).

Definition 2.23: A CW-M^nTP-category $\underline{B}$ is an M^nTP-category such that the morphism spaces are CW-complexes, composition and sum are skeletal, and the identities are vertices.

Theorem 2.24: If $\underline{B}$ is a CW-M^nTP-category and $\Gamma = (R,M,Q)$ satisfies in addition to Axiom M1 following conditions: M is a CW-complex, R and Q are subcomplexes of M, r_0, t_0, and e are vertices, and the multiplication is skeletal. Then $W\underline{B}\Gamma$ is a CW-M^nTP-category.

Proof: Since the morphism spaces of $\underline{B}$ are CW-complexes, $W\underline{B}\Gamma$ exists. Since products in CG of CW-complexes are CW-complexes, $C_p(\underline{a},b)$ is a CW-complex for each p. Hence $\underline{W}_0(\underline{a},b)$ resp. $\underline{W}_{-1}(\underline{a},b)$ are CW-complexes. $Tp(\underline{a},b)$ has the product cell structure from $\bigcup_q [(\Pi_r \underline{B}(\underline{a}_r, b_r)) \times (\xi_q)]$, the vertex labels of its trees. Since the identities in $\underline{B}$ are vertices the degenerate points of $Tp(\underline{a},b)$ form a subcomplex. Analogously the degenerate points of M^p form a subcomplex. Hence $DC_p(\underline{a},b)$ is a subcomplex of $C_p(\underline{a},b)$.

Let $(x,\delta) \in DC_p(\underline{a},b)$, $\delta = d^{i_s} \circ \dots \circ d^{i_1}\partial$, $y = d^{i_1} \circ \dots \circ d^{i_s} x = s^{j_t} \circ \dots \circ s^{j_1} z$, as given by the Lemmas 2.8 and 2.10, and suppose that x is in the q-skeleton of $Tp(\underline{a},b)$ and δ in the r-skeleton of $R \times M^p \times Q^k$. Then ∂ is in the r-skeleton of $R \times M^{p-s} \times Q^k$ since e is a vertex and y is in the q-skeleton of $Tp\text{-}s(\underline{a},b)$ since composition in $\underline{B}$ is skeletal. Since the identities of $\underline{B}$ are vertices, z is in the q-skeleton of $Tp\text{-}s\text{-}t(\underline{a},b)$, and since multiplication in M is skeletal, $s^{j_1} \circ \dots \circ s^{j_t}\partial$ is in the r-skeleton of $R \times M^{p-s-t} \times Q^k$. Hence the attaching maps of Lemma 2.15 are skeletal and hence $\underline{W}(\underline{a},b)$ is a CW-complex.

Composition with permutations is cellular in $\underline{B}$. Hence the relation (2.11) induces a cellular identification in $\underline{W}(\underline{a},b)$, and therefore $W(\underline{a},b)$ is a CW-complex. Composition with permutations on the right is cellular since it is so in $V(\underline{a},b)$. Therefore $W(\underline{a},\underline{b})$ is a CW-complex. $\oplus$ is skeletal because $W(\underline{a},\underline{b})$ is obtained by the normal form construction.

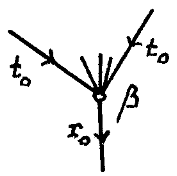
Composition is skeletal since it is induced by inclusions of factors into a product. Since $r_o * t_o$ is a vertex, the identities are vertices.]]

§ 3 THE CONTRACTABILITY OF $W\underline{B}\Gamma$ OVER $\underline{B}$

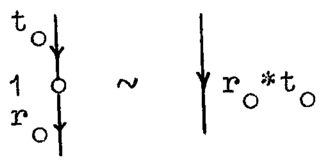
Define the standard section $\iota: \underline{B} \to W\underline{B}\Gamma$ of the standard augmentation (see p.39) by

$\iota(\beta) = \{\theta, \text{unit}, \delta\}$, β a morphism into a generator

where (θ, δ) is the tree with exactly one vertex which is labelled by β, and t_o assigned to each twig and r_o to the root (see picture).



ι is equivariant and since



it preserves identities.

Now suppose that Γ satisfies following additional axiom:

<u>Axiom M2</u>: There exists a homotopy $m_t: M \to M$ such that

$$m_t(v_1) * m_t(v_2) = m_t(v_1 * v_2)$$

$$m_t(e) = e \qquad \text{for all } t \in I$$

$$m_o = id_M \quad \text{and} \quad m_1(v) = e \quad \text{for all } v \in M \ .$$

Putting $u = r_o = t_o = r_o * t_o$ (see p.28), m_t induces homotopies $l_t: R \to R$ and $k_t: Q \to Q$ given by $l_t(x) = u*m_t(x)$, and $k_t(x) = m_t(x) * u$. Then $l_o = id_R$, $k_o = id_Q$, $l_1(x) = u = r_o$, and $k_1(y) = u = t_o$ for all $x \in R$ and $y \in Q$. Furthermore $l_t(x) * m_t(v) = l_t(x * v)$ and $m_t(v)*k_t(y)=k_t(v*y)$ for all $t \in I$, $v \in M$, $x \in R$, and $y \in Q$. ~~In addition we have~~ $\cancel{l_t(x) * k_t(y) = l_t(x * y) = k_t(x* y)}$.

Note that the monoid of example 2.9 satisfies Axiom M2 with the homotopy $m_t(v) = t.v$, $v \in M = I$, with the ordinary multiplication on the right of the equation.

Theorem 3.1: If Γ satisfies the Axioms M1 and M2, then $\varepsilon_B: W\underline{B}\Gamma \to \underline{B}$ is fibre homotopically trivial (see p.19 for the definition).

Proof: We have to construct equivariant fibrewise homotopies $H_t: \iota\circ\varepsilon| W(\underline{a},b) \simeq id| W(\underline{a},b)$.

Define $h_t: R \times M^p \times Q^k \to R \times M^p \times Q^k$ by

$$h_t(x,v_1,\dots,v_p,y_1,\dots,y_k)$$
$$= (l_t(x),m_t(v_1),\dots,m_t(v_p),k_t(y_1),\dots,k_t(y_k))$$

for each p and k. The h_t's induce homotopies

$\underline{H}_t = (1 \times h_t): C_p(\underline{a},b) = Tp(\underline{a},b) \times (R \times M^p \times Q^k) \to C_p(\underline{a},b)$, for each $p \geq 0$. For $p = -1$ define $\underline{H}_t: C_{-1}(a,a) \to C_{-1}(a,a)$

by $\underline{H}_t(x) = \overset{u * m_t(x) * u}{\cancel{1_t(x)}}$. (Recall that $C_{-1}(a,a) = R \cap Q$). The collection of the $\underline{H}_t$'s induces a homotopy

$$\underline{H}_t\colon V(\underline{a},b) \to V(\underline{a},b)\ .$$

$\underline{H}_t$ factors automatically through relation (2.11). Since $m_t(e) = e$ for all t it factors through (2.12) and because of M2 and the properties of 1_t and k_t it factors through (2.13). Hence it induces an equivariant and fibrewise homotopy

$$H_t\colon W(\underline{a},b) \to W(\underline{a},b)$$

such that $H_o = id|\ W(\underline{a},b)$ and $H_1 = \iota\circ\varepsilon|\ W(\underline{a},b)$ (by the properties of 1_t and k_t and the conditions on m_o and m_1).]]

<u>Lemma 3.2</u>: Under the assumptions of Theorem 2.24,

$\varepsilon|\ W\underline{B}\Gamma(\underline{a},\underline{b})$ and $\iota|\ \underline{B}(\underline{a},b)$ are skeletal.

<u>Proof</u>: ε is induced by the projection

$$C_p(\underline{a},b) = Tp(\underline{a},b) \times (R \times M^p \times Q^k) \to Tp(\underline{a},b)$$

followed by the shrinking of all links. Hence since composition in $\underline{B}$ is skeletal, ε is skeletal. Since ι is induced by the identity $\underline{B}(\underline{a},b) \to T_1'(\underline{a},b)$, where $T_1'(\underline{a},b)$ is the subspace of $T1(\underline{a},b)$ of all pairs of the form (θ,unit), it is skeletal.]]

We now give some further examples of systems $\Gamma = (R,M,Q)$.

Example 3.3: Let M be the unit interval with multiplication $t_1 * t_2 = t_1 . t_2$. Then $e = 1$. Take $Q = R = (0)$ and $u = r_o = t_o = 0$. Then M1 is satisfied.

M2 cannot be satisfied since $m_t(0.v) \neq m_t(0).m_t(v)$ necessarily sine $m_t = 1$ is required for all $t \in I$.

Example 3.4: Let M be an arbitrary topological monoid with an idempotent $u \neq e$. Put $R = u * M$ and $Q = M * u$, $r_o = t_o = u$. Then this data satisfies M1.

Example 3.5: Let K be the free tpological monoid over I, the unit interval, (K is the reduced product space I_∞ in the sense of James), i.e.

$$K = I^0 \cup I^1 \cup I^2 \cup \ldots / \sim$$

where the equivalence relation is given by

$$(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \sim (t_1, \ldots, t_{i-1}, t_i, \ldots, t_n) .$$

Hence $(0) = I^0$ is the identity in K.

Let J be the monoid obtained from K by introducing the relation

$$(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) \sim (1, t_{i+1}, \ldots, t_n) ,$$

i.e. $1 \in I$ acts as a right zero. In particular it is an idempotent. Clearly J is a monoid.

Let $L_n = I^1 \cup I^2 \cup \ldots \cup I^n$ and $J_n = \pi(L_n)$ where $\pi\colon L = \varinjlim_n L_n \to J$ is the projection. $J_1 = L_1$. Let $\underline{I}^n = \{(t_1,\ldots,t_n) \in I^n \mid t_i = 0$ some i, or $t_j = 1$ some $j > 1\}$ J_{n+1} is obtained from J_n by attaching I^{n+1} by an attaching map $f_{n+1}\colon \underline{I}^{n+1} \to J_n$, and J_n and J are in CG (It is easy to verify that the conditions (1),...,(4) of p. 33 hold with $X_p = I^p$ and $DX_p = \underline{I}^p$).

Since the attaching maps are skeletal, J is a CW-complex.

<u>Claim</u>: J_{n-1} is a strong deformation retract of J_n.

Proof: $J_n = J_{n-1} \cup_{f_n} I^n$. All faces of I^n become attached to J_{n-1} with exception of the face $t_1 = 1$. Hence the deformation retraction of I^n to the other faces induces a deformation retraction of J_n to J_{n-1}.

The multiplication of K induces the monoid structure in J. Since J is in CG and since it has an idempotent different from the identity it gives rise to a system satisfying M1 (see Example 3.4). Although J is contractible we cannot find a deformation satisfying M2, since any such deformation must be relative to $u = 1 \in I^1$ and to $0 \in I^1$, the identity in J.

Nevertheless the monoid J will be of some importance later on. For note that if $A \subset X$ is a strong deformation retract and $p_t\colon X \to X$ is a deforming homotopy such that $p_o = id_X$ and $p_1 = i \circ p$ where $i\colon A \to X$ is the inclusion and $p\colon X \to A$ the retraction, then the correspondence

$$(t_1, \dots, t_n) \to p_{t_1} \circ \dots \circ p_{t_n}$$

defines a continuous map of J into the space of maps from X to X.

§ 4 THE UNIVERSAL PROPERTY

From now on we restrict ourselves to the system Γ given in Example 2.9, and denote for this case $W\underline{B}\Gamma$ simply by $W\underline{B}$. Since R and Q are just points in this case we neglect them and consider δ in $(\theta,\xi,\delta) \in C_p(\underline{a},b)$ simply as a p-tuple of points in I.

Definition 4.1: $x \in W\underline{B}(\underline{a},\underline{b})$ is called indecomposable if it cannot be written as a composition $x = y \circ z$ such that y and z are not permutations. (Note that any identity is a permutation).

Lemma 4.2: (a) $\{\theta,\xi,\delta\}$ with $(\theta,\xi,\delta) \in C_p(\underline{a},b)$ non-degenerate and $\delta = (t_1,\ldots,t_p)$ is decomposable iff $p \geq 1$ and $t_i = 1$ for some i.

(b) $\{\theta,\xi,\delta\}$ with $(\theta,\xi,\delta) \in C_p(\underline{a},\underline{b})$ non-degenerate and $\delta = (t_1,\ldots,t_p)$ is decomposable iff $p \geq 1$ and $t_i = 1$ for some i.

Proof: (a) Suppose $\{\theta,\xi,\delta\}$ is decomposable,

$$\{\theta,\xi,\delta\} = \{\psi_1,\xi_1,\delta_1\} \circ \{\psi_2,\xi_2,\delta_2\}$$

with (ψ_i,ξ_i,δ_i), $i = 1,2$, non-degenerate and not trivial.

Then $\xi_1 \circ (\psi_2, \xi_2, \delta_2)$ is not degenerate and not trivial, and since $\max(1,1) = 1$, $\lambda\rho[(\psi_1, \mathrm{unit}, \delta_1)\circ(\xi_1\circ(\psi_2,\xi_2,\delta_2))]$ has at least one link to which $1 \in I$ is assigned. ($\lambda\rho$ is the function defined in Lemma 2.14).

Conversely suppose that there is a link in θ to which $1 \in I$ has been assigned, the i-th link in the standard indexing, say. Let ψ' be the subtree of θ sitting on the i-th link, and suppose that ψ' has q twigs. Let φ be the tree obtained from θ by deleting ψ', and suppose the twigs of φ indexed by $j < i$ are labelled by $i_{\xi^{-1}(1)}, \ldots, i_{\xi^{-1}(s)}$, then the twigs indexed by $j > i$ are labelled by $i_{\xi^{-1}(s+q+1)}, \ldots, i_{\xi^{-1}(k)}$, if $\underline{a} = (i_1, \ldots, i_k)$. Assign to the links of φ and ψ' the values in I inherited from θ. Let

$$\psi = 1_{\xi^{-1}(1)} \oplus \cdots \oplus 1_{\xi^{-1}(s)} \oplus \psi' \oplus 1_{\xi^{-1}(s+q+1)} \oplus \cdots \oplus 1_{\xi^{-1}(k)}$$

where 1_b is the trivial tree with labelled edge b. Then (φ, unit) and (ψ, ξ) with the values of I assigned to their links determine not trivial and not degenerate elements $(\varphi, \mathrm{unit}, \delta_1)$ and (ψ, ξ, δ_2) such that

$$\{\theta, \xi, \delta\} = \{\varphi, \mathrm{unit}, \delta_1\} \circ \{\psi, \xi, \delta_2\} .$$

(b) follows from (a) by applying (a) to each tree.]]

We refer to the process of "cutting up" a tree into two composable ones by cutting off the i-th link as <u>chopping the i-th link</u> .

<u>Lemma 4.3:</u> Each element $x \in \underline{WB}(\underline{a},\underline{b})$ which is not a permutation can be decomposed into indecomposable elements $x = x_1 \circ \ldots \circ x_p$. This decomposition is unique up to the equivalence generated by

(a) $x_1 \circ \ldots \circ (x_i^! \oplus 1) \circ (1 \oplus x_{i+1}^!) \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ (x_i^! \oplus x_{i+1}^!) \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ (1 \oplus x_{i+1}^!) \circ (x_i \oplus 1) \circ \ldots \circ x_p$

(b) $x_1 \circ \ldots \circ (x_i \circ \xi) \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ x_i \circ \xi \circ x_{i+1} \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ x_i \circ (\xi \circ x_{i+1}) \circ \ldots \circ x_p$
where ξ is a permutation.

<u>Proof</u>: Represent x by a non-degenerate triple (θ,ξ,δ) $= (\theta,\text{unit},\delta)\circ\xi$. This representative is unique up to the relation (2.19). Chop each link of θ to which $1 \in I$ is assigned. This decomposes this representative into non-degenerate elements each of which represents an indecomposable element in $\underline{WB}$. There are exactly three choices involved:

(1) the order in which we chop the links,

(2) the choice of the particular non-degenerate representative,

(3) in the chopping process the permutation ξ can be broken

up into a block permutation (as defined on p. 7) and another one such that the block permutation can be associated with the copse on the left.

Relation (a) takes care of (1), while relation (b) takes care of (2) and (3).]]

Let $W^p\underline{B}$ be the $M^n TP$-subcategory of $W\underline{B}$ generated by all $W_p\underline{B}(\underline{a},b)$, p fixed. (Note that $W_p\underline{B}$ is not even a category). Let $V^p(\underline{a},\underline{b})$ be the subspace of $V(\underline{a},\underline{b})$ of all those elements x such that $\{x\} = \{x_1\}\circ\ldots\circ\{x_m\}$, where $\{x\}$ denotes the equivalence class of x, i.e. its image in $W\underline{B}$, where each tree in x_k has at most p links, $1 \leq k \leq m$. Observe that we do not require that x is non-degenerate. $V^p(\underline{a},\underline{b})$ is closed in $V(\underline{a},\underline{b})$.

Let $\pi^p\colon V^p \to W^p\underline{B}$ be given by $\pi^p = \pi | V^p$, where $\pi\colon V \to W\underline{B}$ is the projection. Let $\underline{W}^p\underline{B}$ be the inverse image of $W^p\underline{B}$ under the projection $\underline{\pi}\colon \underline{W}\underline{B} \to W\underline{B}$ induced by the relation (2.19), and $\underline{\pi}^p$ its restriction to $\underline{W}^p\underline{B}$. $\underline{W}_p\underline{B}(\underline{a},b)$ is obtained from $\underline{W}_o\underline{B}(\underline{a},b)$ by attaching $C_1(\underline{a},b),\ldots,C_p(\underline{a},b)$ in order. Consequently $\underline{W}^p\underline{B}(\underline{a},b)$ is obtained from $\underline{W}^o\underline{B}(\underline{a},b)$ by attaching $V^p(\underline{a},b) \cap C_1(\underline{a},b),\ldots,V^p(\underline{a},b) \cap C_q(\underline{a},b),\ldots$ in order with $q = 1,2,3,\ldots$

For each type α of trees in $Tp(\underline{a},b)$ and each $\xi \in S(\text{length}\,\underline{a})$

we have a component $M_{\alpha,p}(\underline{a},b) = [\Pi_k \underline{B}(\underline{a}_k, j_k)] \times (\xi_\alpha)$. Denote the subspace of degenerate points of $Tp(\underline{a},b)$ by $T\dot{p}(\underline{a},b)$ and let $C_p'(\underline{a},b) = T\dot{p}(\underline{a},b)\times I^p \cup Tp(\underline{a},b)\times\partial I^p$,where ∂I^p denotes the boundary of the cube I^p. Set $Q_{\alpha,p}(\underline{a},b) = M_{\alpha,p}(\underline{a},b)\times I^p$, and $Q'_{\alpha,p}(\underline{a},b) = Q_{\alpha,p}(\underline{a},b)\cap C_p'(\underline{a},b)$. $C_p'(\underline{a},b)$ is the closed subspace of $C_p(\underline{a},b)$ consisting of the degenerate or decomposable points.

We have characteristic maps

$$\chi_{\alpha,p}: (Q_{\alpha,p}(\underline{a},b),\ Q'_{\alpha,p}(\underline{a},b)) \to (\underline{W}^p\underline{B}(\underline{a},b),\ \underline{W}^{p-1}\underline{B}(\underline{a},b))$$

which by Lemma 2.16 induce characteristic maps

$$\chi_{\alpha,p}: (Q_{\alpha,p}(\underline{a},b),\ Q'_{\alpha,p}(\underline{a},b)) \to (W^p\underline{B}(\underline{a},b),\ W^{p-1}\underline{B}(\underline{a},b)).$$

Let $\underline{D}$ be a subcategory of $W\underline{B}$, and let $D_{\alpha,p} \subset Q_{\alpha,p}$ be the subset of all those elements x such that $\pi(x) \in \underline{D}$. We assume that $D_{\alpha,p}$ is closed in $Q_{\alpha,p}$ (and hence has the relative topology) and that if $x \in \underline{D}$, $x = y\circ z$, then y and z are in $\underline{D}$.

<u>Definition 4.4</u>: Let $\underline{B}$ and $\underline{C}$ be topological categories and $\varphi_o, \varphi_1: \underline{B} \to \underline{C}$ continuous functors such that $\varphi_o(A) = \varphi_1(A)$ for all objects A in $\underline{B}$. Call φ_o and φ_1 <u>homotopic</u> if there exist continuous functors $\Theta_t: \underline{B} \to \underline{C}$ for all $t \in I$ such that $\Theta_t(A) = \varphi_o(A)$ for all $t \in I$ and for all objects A in $\underline{B}$, $\Theta_o = \varphi_o$,

$\Theta_1 = \varphi_1$, and Θ: $\underline{B}(A_1, A_2) \times I \to \underline{C}(\varphi_o(A_1), \varphi_o(A_2))$ given by $\Theta(\alpha, t) = \Theta_t(\alpha)$ is continuous. Θ_t is called a <u>homotopy of functors</u>. If φ_o, φ_1 are MTP-functors then Θ_t is called a homotopy of MTP-functors if Θ_t is an MTP-functor for each $t \in I$.

<u>Lemma 4.5</u>: Let $\underline{C}$ be an M^nTP-category and $\underline{D}$ a subcategory of $W\underline{B}$ as given above. Let δ_t: $\underline{D} \to \underline{C}$ be a homotopy of functors preserving objects, sums and permutations ($\underline{D}$ need not be an M^nTP-category).

(1) Given a homotopy of M^nTP-functors $\gamma_t^{p-1}: W^{p-1}\underline{B} \to \underline{C}$ and equivariant maps $f_{\alpha,p}$: $Q_{\alpha,p}(\underline{a}, b) \times I \to \underline{C}(\underline{a}, b)$ for all $\underline{a}$, b, and α such that

(i) $\gamma_t^{p-1} | \, W^{p-1}\underline{B} \cap \underline{D} = \delta_t | \, W^{p-1}\underline{B} \cap \underline{D}$

(ii) $f_{\alpha,p} | \, D_{\alpha,p}(\underline{a}, b) \times (t) = \delta_t \circ (\chi_{\alpha,p} | \, D_{\alpha,p}(\underline{a}, b))$

$f_{\alpha,p} | \, Q'_{\alpha,p}(\underline{a}, b) \times (t) = \gamma_t^{p-1} \circ (\chi_{\alpha,p} | \, Q'_{\alpha,p}(\underline{a}, b))$

(iii) $f_{\alpha,p}(x, t)$ factors through the relation (2.11) for each $t \in I$.

If x is a trivial tree representing the identity of b, then $f_{\alpha,-1}(x, t) = 1_b$.

Then there exists a unique homotopy of M^nTP-functors γ_t^p: $W^p\underline{B} \to \underline{C}$ extending γ_t^{p-1} such that

$\gamma_t^p | W^p\underline{B} \cap \underline{D} = \delta_t | W^p\underline{B} \cap \underline{D}$ and $\gamma_t^p \circ (\chi_{\alpha,p} | Q_{\alpha,p}(\underline{a},b))$
$= f_{\alpha,p} | Q_{\alpha,p}(\underline{a},b) \times (t)$.

(2) Given homotopies of M^nTP-functors $\gamma_t^p \colon W^p\underline{B} \to \underline{C}$ for all p such that $\gamma_t^p | W^{p-1}\underline{B} = \gamma_t^{p-1}$ and $\gamma_t^p | W^p\underline{B} \cap \underline{D} = \delta_t | W^p\underline{B} \cap \underline{D}$ then there exists a unique homotopy of M^nTP-functors $\gamma_t \colon W\underline{B} \to \underline{C}$ such that $\gamma_t | W^p\underline{B} = \gamma_t^p$ and $\gamma_t | \underline{D} = \delta_t$.

<u>Proof</u>: Let $\{\theta_1 \oplus \cdots \oplus \theta_n, \xi, \delta_1 \times \cdots \times \delta_n\} \in W^p\underline{B}(\underline{a},b)$ be indecomposable. Define

$$\gamma_t^p\{\theta_1 \oplus \cdots \oplus \theta_n, \xi, \delta_1 \times \cdots \times \delta_n\}$$
$$= [\gamma_t^p(\theta_1, \mathrm{unit}, \delta_1) \oplus \cdots \oplus \gamma_t^p(\theta_n, \mathrm{unit}, \delta_n)] \circ \xi$$

with
$$\gamma_t^p\{\theta_k, \mathrm{unit}, \delta_k\} = \begin{cases} \gamma_t^{p-1}\{\theta_k, \mathrm{unit}, \delta_k\} & \text{if } (\theta_k, \mathrm{unit}, \delta_k) \in V^{p-1}(\underline{a}_k, b_k) \\ f_{\alpha,p}(\theta_k, \mathrm{unit}, \delta_k; t) & \text{if } (\theta_k, \mathrm{unit}, \delta_k) \in Q_{\alpha,p}(\underline{a},b). \end{cases}$$

This definition of γ_t^p on indecomposables is forced upon us by the condition that γ_t^p is an M^nTP-functor satisfying the extension conditions of the lemma. Because of (i), (ii), and (iii) γ_t^p is well defined and compatible with δ. It is continuous since sum and composition in $\underline{C}$ are continuous. Extend γ_t^p to the whole of $W^p\underline{B}$ by

$$\gamma_t^p(x_1\circ\ldots\circ x_n) = \gamma_t^p(x_1)\circ\ldots\circ\gamma_t^p(x_n)$$

where the x_i's are indecomposables. By definition γ_t^p preserves sums and permutations. Since indecomposables in $\underline{D}$ are indecomposables in $W^p\underline{B}$ γ_t^p extends δ. Since the $f_{\alpha,p}$'s are equivariant, factor through (2.11), and preserve identities for $p = -1$, γ_t^p is a well defined functor by Lemma 4.3. Again this extension is forced upon us to make γ_t^p into a functor.

γ_t^p is continuous since the maps from $[V^p(\underline{a},\underline{b})\cap C_q(\underline{a},\underline{b})]\times I$ to $\underline{C}$ which induce γ_t^p are defined by projecting closed subspaces of $V^p(\underline{a},\underline{b}) \cap C_q(\underline{a},\underline{b})$ to some product of such spaces of lower filtration q (factoring out vertices labelled by identities and links to which $1 \in I$ has been assigned) and following by product maps involving $f_{\alpha,p}$ and $\gamma_t^{p-1}\circ\pi^p$. Different positions of identities in the copses and different assignments of elements $1 \in I$ require different projections, but since $f_{\alpha,p}$ extends $\gamma_t^{p-1}\circ\chi_{\alpha,p}|Q_{\alpha,p}(\underline{a},b)$ they coincide on their intersections.

Since $W\underline{B}$ has the limit topology from the $W^p\underline{B}$'s the second part is immediate.]]

<u>Remark</u>: By taking the functor homotopies to be the trivial ones we obtain the same results for M^nTP-functors (delete t and I wherever they occur).

<u>Theorem 4.6</u> (The universal property):

Given a commutative diagram

$$\begin{tikzcd}
 & \underline{D} \arrow[dl, "\rho"'] \arrow[dr, "\delta_t"] & \\
W\underline{B} \arrow[d, "\varepsilon"'] & & \underline{G} \arrow[d, "\mu"] \\
\underline{B} \arrow[rr, "\gamma"] & & \underline{C}
\end{tikzcd}$$

of M^nTP-categories $\underline{B}$, $\underline{C}$, $\underline{G}$ and a subcategory $\underline{D}$ of $W\underline{B}$, M^nTP-functors γ, μ, the standard augmentation $\varepsilon = \varepsilon_B$, the inclusion functor ρ, and a homotopy of functors δ_t preserving objects, sums, and permutations for each $t \in I$.

Assume:

(1) If $x \in \underline{D}$ is a composition in $W\underline{B}$, $x = y \circ z$, then y and z are in $\underline{D}$.

$D_{\alpha,p}(\underline{a},b)$ is closed in $Q_{\alpha,p}(\underline{a},b)$ (see p 53), and each connected component of $D_{\alpha,p}(\underline{a},b)$ containing a point $x \notin Q'_{\alpha,p}(\underline{a},b)$ is open and closed in $Q_{\alpha,p}(\underline{a},b)$

(2) For each generator b there exists a closed neighbourhood Z_b of 1_b in $\underline{B}(b,b)$ such that $(Z_b, 1_b \cup \operatorname{fr} Z_b)$ is a NDR-pair (fr = frontier), and $\gamma(Z_b) = 1_b \in \underline{C}(b,b)$

(3) μ is fibre homotopically trivial.

Then

I: There exists an M^nTP-functor $\nu_o \colon \underline{WB} \to \underline{G}$ such that $\mu \circ \nu_o = \gamma \circ \varepsilon$ and $\nu_o \circ \rho = \delta_o$.

II: Given any two M^nTP-functors $\nu_o, \nu_1 \colon \underline{WB} \to \underline{G}$ such that $\mu \circ \nu_o = \mu \circ \nu_1 = \gamma \circ \varepsilon$ and $\nu_o \circ \rho = \delta_o$, $\nu_1 \circ \rho = \delta_1$, then there exists a homotopy ν_t of M^nTP-functors between ν_o and ν_1 extending δ_t, and such that $\mu \circ \nu_t = \gamma \circ \varepsilon$.

For the proof of Theorem 4.6 another filtration (really double filtration, and we induct over the sum of both) of $\underline{WB}$ seems to be more suitable than the one used in Lemma 4.5.

$C_o(b,b) \cong B(b,b)$, and we can assume wlog that $Z_b \subset D_{\alpha,o}(b,b)$ if the latter is not empty, since $D_{\alpha,o}(b,b)$ is open and closed.

Let $Y_b = Z_b - (1_b \cup \mathrm{fr}\, Z_b)$. Let $F_p V\underline{B}(\underline{a},\underline{b})$ be the subspace of $V\underline{B}(\underline{a},\underline{b})$ of those elements x such that $\{x\} = \{x_1\} \circ \ldots \circ \{x_q\}$ and $\{x_i\}$ is a sum $\{y_1\} \oplus \ldots \oplus \{y_k\}$ of morphisms into a generator for each i such that $\{y_l\}_{1 \le l \le k}$ is in $\underline{D}$ or y_l has s links and t vertices labelled by elements in the Y_b's with $s+t \le p$. $F_p V\underline{B}(\underline{a},\underline{b})$ is closed in $V\underline{B}(\underline{a},\underline{b})$. $F_p W\underline{B} = \pi(F_p V\underline{B})$ is an M^nTP-subcategory of $W\underline{B}$ containing $\underline{D}$ since it is closed under composition and sum and since it contains all permutations. We denote the M^nTP-subcategory of $\underline{WB}$ generated by $\underline{D}$ and the identities by $F_{-1}\underline{WB}$.

Let $D^{\sim}_{\alpha,p}$ be the union of all those connected components of $D_{\alpha,p}(\underline{a},b)$ which contain an element $x \in Q'_{\alpha,p}(\underline{a},b)$. Then by assumption (1) $D^{\sim}_{\alpha,p}(\underline{a},b)$ is a product $D'_{\alpha,p}(\underline{a\_},b)\times I^p$. Let $P_{\alpha,p,k}(\underline{a},b)$ be the subspace of $M_{\alpha,p}(\underline{a},b) - D'_{\alpha,p}(\underline{a},b)$ of all those pairs (θ,ξ) such that at least $k - p$ vertices of θ are labelled by elements in the Z_b's and at most $k - p$ ones by elements in the $(Y_b\cup 1_b)$'s. Denote the closed subspace of those points of $P_{\alpha,p,k}(\underline{a},b)$ with less than k-p vertices labelled by elements in the Y_b's by $P'_{\alpha,p,k}(\underline{a},b)$. Note that for k=0, $P'_{\alpha,p,o}(\underline{a},b) = \emptyset$ unless p=-1 and $\underline{a}$ = b, when it contains the representative of 1_b.

Let $R_{\alpha,p,k}(\underline{a},b) = P_{\alpha,p,k}(\underline{a},b)\times I^p$ and

$R'_{\alpha,p,k}(\underline{a},b) = P_{\alpha,p,k}(\underline{a},b) \times \partial I^p \cup P'_{\alpha,p,k}(\underline{a},b) \times I^p$.

$R'_{\alpha,p,k}(\underline{a},b)$ cosists exactly of those points of $R_{\alpha,p,k}(\underline{a},b)$ that are equivalemt to a point in $F_{k-1}V\underline{B}(\underline{a},b)$. We have characteristic maps $\varkappa_{\alpha,p,k} = \chi_{\alpha,p}| R_{\alpha,p,k}$

$$\varkappa_{\alpha,p,k}: (R_{\alpha,p,k}(\underline{a},b),\ R'_{\alpha,p,k}(\underline{a},b)) \to (F_k W\underline{B}(\underline{a},b),\ F_{k-1}W\underline{B}(\underline{a},b))$$

In a completely analogous way to Lemma 4.5 we can prove

Lemma 4.7: Let $\underline{C}$ be an $M^n TP$-category, $\underline{D}$ a subcategory of $\underline{WB}$ satisfying the requirements of Theorem 4.6, and

$\delta_t: \underline{D} \to \underline{C}$ a homotopy of functors preserving objects sums and permutations. Then

(1) δ_t determines a unique M^nTP-functor $\gamma_t^{-1}: F_{-1}W\underline{B} \to \underline{C}$ extending δ_t.

(2) Given a homotopy of M^nTP-functors $\gamma_t^{k-1}: F_{k-1}W\underline{B} \to \underline{C}$ $k \geq 0$, and equivariant maps $f_{\alpha,p,k}: R_{\alpha,p,k}(\underline{a},b)\times I \to \underline{C}(\underline{a},b)$ for all $\alpha,p,k,\underline{a},b$ and such that $f_{\alpha,p,k}|R'_{\alpha,p,k}(\underline{a},b)\times(t) = \gamma_t^{k-1}\circ\varkappa_{\alpha,p,k}|R'_{\alpha,p,k}(\underline{a},b)$ $f_{\alpha,p,k}(x,t)$ factors through relation (2.11) for each t.

Then there exists a unique homotopy of M^nTP-functors $\gamma_t^k: F_kW\underline{B} \to \underline{C}$ extending γ_t^{k-1} and the maps $f_{\alpha,p,k}$.

(3) Given a sequence of homotopies of M^nTP-functors $\gamma_t^k: F_kW\underline{B} \to \underline{C}$ such that $\gamma_t^k| F_{k-1}W\underline{B} = \gamma_t^{k-1}$ for all k and γ_t^{-1} extends δ_t, then there exists a unique M^nTP-functor $\gamma_t: W\underline{B} \to \underline{C}$ extending δ_t and such that $\gamma_t| F_kW\underline{B} = \gamma_t^k$.]]

<u>Proof of Theorem 4.6</u>: We are going to prove the statements I and II simultaneously.

δ_o resp. δ_t determine ν_o^{-1} (resp. ν_t^{-1}): $F_{-1}W\underline{B} \to \underline{G}$.

Inductively suppose that we have defined ν_o^{k-1} (resp. ν_t^{k-1}): $F_{k-1}W\underline{B} \to \underline{G}$ such that $\mu\circ\nu_o^{k-1}, (\mu\circ\nu_t^{k-1}) = \gamma\circ\varepsilon|\ F_{k-1}W\underline{B}$ and $\nu_o^{k-1}, (\nu_t^{k-1})|\ F_iW\underline{B} = \nu_o^i, (\nu_t^i)$ for all $i < k$. We have to define maps $f_{\alpha,p,k}$: $R_{\alpha,p,k}(\underline{a},b) \to \underline{G}(\underline{a},b)$ (resp. $R_{\alpha,p,k}(\underline{a},b)\times I \to \underline{G}(\underline{a},b)$) satisfying the requirements of Lemma 4.7 for $t=0$ (resp. for all $t \in I$), and such that $\mu\circ f_{\alpha,p,k} = \gamma\circ\varepsilon\circ\varkappa_{\alpha,p,k}$. The Theorem then follows from Lemma 4.7 (2) and (3).

Since we work with fixed α, p, k, $\underline{a}$, and b during the construction of any particular map $f_{\alpha,p,k}$ we denote $R_{\alpha,p,k}(\underline{a},b)$, $P_{\alpha,p,k}(\underline{a},b)$, $\underline{G}(\underline{a},b)$, $f_{\alpha,p,k}$, and $\varkappa_{\alpha,p,k}$ simply by R, P, G, f, and $\varkappa$.

Let σ be the equivariant section of μ and H: $\sigma\circ\mu|G \simeq id_G$ the equivariant fibrewise homotopy given by assumption (3). Consider I^p as a cone on ∂I^p, i.e. $I^p = \{(d,u)|\ d \in \partial I^p,\ u \in I,\ (d_1,0) \sim (d_2,0)$ for $d_1, d_2 \in \partial I^p\}$. Identify $\partial I^p \subset I^p$ with $\partial I^p \times 1$.

<u>Case I</u>: The homotopy F': $P \times \partial I^p \times I \to G$ given by $F'(x,d,u) = H(\nu_o^{k-1}\circ\varkappa(x,d),\ u)$ for $p > 0$ factors through the cone point since $F'(x,d,0) = \sigma\circ\mu\circ\nu_o^{k-1}\circ\varkappa(x,d) = \sigma\circ\gamma\circ\varepsilon\circ\varkappa(x,d)$ which is independent of $d \in \partial I^p$. Hence F' induces a map

$$F\colon R = P \times I^p \to G$$

such that $F| P \times \partial I^p = \nu_0^{k-1} \circ \varkappa | P \times \partial I^p$. Since H is an equivariant fibre wise homotopy, F is equivariant and $\mu \circ F(x,d,u)$ is independent of u. Hence

$\mu \circ F(x,d,u) = \mu \circ F(x,d,0) = \mu \circ \sigma \circ \gamma \circ \varepsilon \circ \varkappa(x,d) = \gamma \circ \varepsilon \circ \varkappa(x,d)$.

Suppose $(x',d') \sim (x,d)$ under (2.11), then $F'(x,d,u) = F'(x',d',u)$. Hence $F(x,d,u) = F(x',d',u)$ and since each permutation of coordinates of (d,u), now considered as a p-tuple, is induced by the same permutation of the p-tuple $d \in \partial I^p$, F factors through the relation (2.11).

For p=0 define $F: R \to G$ by $F(x) = \sigma \circ \gamma \circ \varepsilon \circ \varkappa(x)$.

<u>Case II</u>: Define $F': P \times (\partial I^p \times I \cup I^p \times \partial I) \times I \to G$ by

$$F'(x,u,t) = H(g(x,u),\ t)$$

where
$$\begin{aligned} g(x,u) &= \nu_t \circ \varkappa(x,u') \quad \text{if} \quad u = (u',t) \in \partial I^p \times I \\ &= \nu_\varepsilon \circ \varkappa(x,u') \quad \text{if} \quad u = (u',\varepsilon) \in I^p \times \partial I,\ \varepsilon=0,1 \end{aligned}$$

Using the same argument as above we obtain a map

$$F: R \times I \to G$$

which factors through the relation (2.11) and which satisfies:
$$\begin{aligned} F| P \times \partial I^p \times I &= \nu_t^{k-1} \circ \varkappa | P \times \partial I^p \times I \\ F| P \times I^p \times \varepsilon &= \nu_\varepsilon \circ \varkappa | P \times I^p \quad , \ \varepsilon=0,1 \\ \mu \circ F(r,t) &= \gamma \circ \varepsilon \circ \varkappa(r) \quad \text{for all } (r,t) \in R \times I \end{aligned}$$

If $P' = \emptyset$, F serves as f (resp. f_t) because then $R' = P \times \partial I^p$.

P is the union of the closed product spaces of $M_{\alpha,p}(\underline{a},b) - D'_{\alpha,p}(\underline{a},b)$ with exactly k-p factors being some of the neighbourhoods Z_b and the other factors being

$\underline{B}(\underline{a}_k, b_k)$ if $\underline{a}_k \neq b_k$ or the closure of $\underline{B}(b,b) - Z_b$. The intersection of two summands of P is by definition in P'. After reshuffling the factors each summand is of the form $Z_{b_1} \times Z_{b_2} \times \ldots \times Z_{b_{k-p}} \times X$. We denote it by $Z \times X$. Let $Y \subset Z$ be the closed subspace of those points with at least one coordinate in some $(1_b \cup \mathrm{fr}\, Z_b)$. (Z,Y) is a NDR-pair by assumption (2) and [6; Lemma 7.3]. Note that $(Z \times X) \cap P' = Y \times X$. ν_o^{k-1} (resp. ν_t^{k-1}) determine f on a subspace of $Z \times X \times I^p$ (resp. $Z \times X \times I^p \times I$) and to prove the theorem it now suffices to extend f over each individual summand $Z \times X \times I^p$ of R such that the required identities hold.

Case I: By induction f is determined on R' and hence on

$$Z \times X \times \partial I^p \cup Y \times X \times I^p \text{ in any reshuffled summand } \quad Z \times X \times I^p.$$

Case II: f is determined on $R' \times I \cup R \times \partial I$ and hence on

$$Z \times X \times \partial I^p \cup Y \times X \times I^p \times I \cup Z \times X \times I^p \times \partial I = Z \times X \times \partial I^{p+1} \cup Y \times X \times I^{p+1}$$

The maps F (we delete the shuffling maps) satisfy for $(z,x) \in Z \times X$

$$F \,|Z \times X \times \partial I^p = f|\, Z \times X \times \partial I^p$$

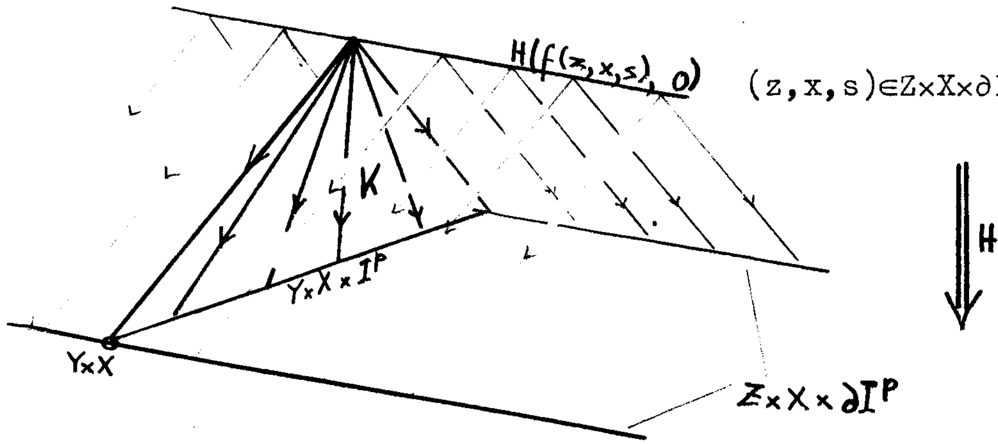
resp. $$F|\, Z \times X \times \partial I^{p+1} = f|\, Z \times X \times \partial I^{p+1}$$

since ∂I^p (resp. ∂I^{p+1}) are identified with the level 1 in the cones I^p (resp. I^{p+1}).

We restrict ourselves to case I for the rest of the proof since case II differs from it only in the number of cube coordinates. Otherwise the proofs are from now on the same.

$K(y,x,s,t) = H(f(y,x,s),\ t)$ with $(y,x,s) \in Y\times X\times I^p$ defines a homotopy $K\colon Y\times X\times I^p\times I \ \to\ G$ such that

K:

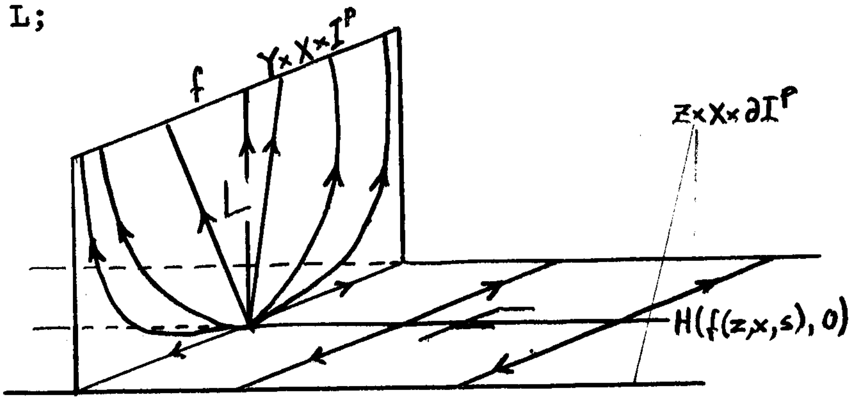


such that $K(y,x,s,1) = f(y,x,s)$, $K(y,x,s,0) = \sigma\circ\mu\circ f(y,x,s)$ $= \sigma\circ\gamma\circ\varepsilon\circ\varkappa(y,x,s)$ which is independent of s. Since f factors trough (2.11) by induction hypothesis and since f and H are equivariant, K is equivariant, factors through (2.11), and $\mu\circ K(y,x,s,t)$ is independent of t. K induces a homotopy

$$L\colon Y \times X \times I^p \times I \ \to\ G$$

such that $L\colon F|\ Y\times X\times I^p \simeq f|\ Y\times X\times I^p$ rel $Y\times X\times\partial I^p$ by definition of F (see picture next page). L is equivariant and factors through (2.11) since K does. Furthermore $\mu\circ L(y,x,s,t) = \mu\circ L(y,x,s,1) = \mu\circ f(y,x,s)$.

L;



Define a map

$N: Y\times X\times I^p\times I \cup Z\times X\times\partial I^p\times I \cup Z\times X\times I^p\times 0 \rightarrow G$

by $N| Y\times X\times I^p\times I = L$

$N| Z\times X\times\partial I^p\times I$ = constant on $f| Z\times X\times\partial I^p$

$N| Z\times X\times I^p\times 0 = F| Z\times X\times I^p$

Then $\mu\circ N(z,x,s,t) = \gamma\circ\varepsilon\circ\varkappa(z,x,s)$ which is independent of $s \in I^p$. N is equivariant and factors through (2.11) since F, f, and L do.

$(Z\times I^p,\ Y\times I^p \cup Z\times\partial I^p)$ is a NDR-pair [6; Lemma 7.3]. Hence [6; Theorem 7.1] there exists a retraction

$r': Z\times I^p\times I \rightarrow Y\times I^p\times I \cup Z\times\partial I^p\times I \cup Z\times I^p\times 0$

which extends to a retraction

$r: Z\times X\times I^p\times I \rightarrow Y\times X\times I^p\times I \cup Z\times X\times\ I^p\times I \cup Z\times X\times I^p\times 0$

given by $r(z,x,s,t) = (z',x,s',t')$ where $(z',s',t')=r'(z,s,t)$

(Here we actually require that r' is symmetric in the p coordinates of I^p. Lemma 7.18, p. 121 of this thesis shows the existence of such an r')

Define $f|\ Z\times X\times I^p = N\circ r|\ Z\times X\times I^p\times 1$. Then f extends $f|\ Y\times X\times I^p \cup Z\times X\times \partial I^p$ and $\mu\circ f(z,x,s) = \mu\circ N\circ r(z,x,s,1) = \mu\circ N(z',x,s',1')$ with $(z',s',1') = r'(z,s,1)$. Hence

$$\mu\circ f(z,x,s) = \gamma\circ\varepsilon\circ\varkappa(z',x,s')$$

which is independent of $s' \in I^p$. $\varepsilon\circ\varkappa(z,x,s)$ is an expression in the coordinates of z and x involving composition and sum in $\underline{B}$. $\varepsilon\circ\varkappa(z',x,s')$ can be obtained from this expression by substituting the coordinates of z by the corresponding ones of z', since only one type of tree is involved. Each coordinate of z and its corresponding one of z' are in the same neighbourhood Z_b. Since γ is an M^nTP-functor it preserves the expressions for $\varepsilon\circ\varkappa(z,x,s)$ and $\varepsilon\circ\varkappa(z',x,s')$, and since $\gamma(Z_b) = 1_b$ we obtain $\gamma\circ\varepsilon\circ\varkappa(z,x,s) = \gamma\circ\varepsilon\circ\varkappa(z',x,s')$. Hence $\mu\circ f(z,x,s) = \gamma\circ\varepsilon\circ\varkappa(z,x,s)$.

Since the retraction r effects vertices of the trees involved which lie in some $\underline{B}(b,b)$ on which the trivial permutation group operates and no others, $N\circ r$ is equivariant and factors through (2.11). Hence so does f.]]

<u>Lemma 4.8</u>: Suppose that in addition to the assumptions of Theorem 4.6 we are given homotopies

$$\tau_{\underline{a},b}:\underline{B}(\underline{a},b)\times I \to \underline{G}(\underline{a},b)$$

for some $\underline{a}$, b such that

$\tau_{\underline{a},b}(x,t) = \delta_t(\iota_B x)$ if $\iota_B x \in \underline{D}$ (see p.43 for the definition of ι_B)

$\tau_{b,b}(1_b,t) = 1_b$ if $\tau_{b,b}$ is defined, for all $t \in I$

$\mu \circ \tau_{\underline{a},b}(x,t) = \gamma(x)$ whenever it is defined.

Then there exists a homotopy of $M^n TP$-functors $\nu_t\colon W\underline{B} \to \underline{G}$ such that $\nu_t \circ \rho = \delta_t$, $\mu \circ \nu_t = \gamma \circ \varepsilon$, and $\nu_t \circ \iota_B(x) = \tau_{\underline{a},b}(x,t)$ for $x \in \underline{B}(\underline{a},b)$ if $\tau_{\underline{a},b}$ is defined

Proof: The $\tau_{\underline{a},b}$ determine some of the $f_{\alpha,p,k}$'s for $p=0$ and $k=0,1$ compatibly with the boundary conditions. The Lemma now follows from Theorem 4.6.]]

Theorem 4.9: Let $\underline{D}$ be a subcategory of $W\underline{B}$ satisfying 4.6 (1), $\underline{G}$ an $M^n TP$-category, $\mu\colon \underline{G} \to \underline{B}$ a fibre homotopically trivial $M^n TP$-functor, and $\delta_t\colon \underline{D} \to \underline{G}$ a homotopy of functors preserving objects sums and permutations and such that $\mu \circ \delta_t = \varepsilon_B \circ \rho$ where $\rho\colon \underline{D} \to W\underline{B}$ is the inclusion functor. Suppose the identities of $\underline{B}$ are isolated. Then there exists a homotopy of $M^n TP$-functors $\nu_t\colon W\underline{B} \to \underline{G}$ such that $\nu_t \circ \rho = \delta_t$ and $\mu \circ \nu_t = \varepsilon_B$. ν_0 and ν_1 compatible with δ_0 resp. δ_1 and satisfying $\mu \circ \nu_i = \varepsilon_B$, $i=0,1$ may be given in advance.

Proof: Since $1_b \in \underline{B}(b,b)$ is isolated, $(\underline{B}(b,b), 1_b)$ is a NDR-pair. Hence $W\underline{B}$ exists. Apply Theorem 4.6 with $Z_b = (1_b)$, $\gamma = \mathrm{id}_{\underline{B}}$, and $\underline{B} = \underline{C}$.]]

Lemma 4.10: Given any M^nTP-category $\underline{B}$ (in normal form), then there exists an M^nTP-category $\underline{B}^{\sim}$ (in normal form) such that

(1) $(\underline{B}^{\sim}(b,b),\ 1_b)$ is a NDR-pair for all generators b.

(2) There exists a fibre homotopically trivial M^nTP-functor $\varepsilon_B^{!}\colon \underline{B}^{\sim} \to \underline{B}$.

(3) Each $1_b \in \underline{B}^{\sim}(b,b)$ has a closed neighbourhood Z_b such that $(Z_b,\ 1_b \cup \mathrm{fr}\ Z_b)$ is a NDR-pair.

(4) $\varepsilon_B^{!}(Z_b) = 1_b \in \underline{B}(b,b)$.

Proof: Let $\underline{B}^{\sim}(b,b) = \underline{B}(b,b) \cup I/\sim$ where $\underline{B}(b,b) \ni 1_b \sim 1 \in I$, and $\underline{B}^{\sim}(\underline{a},b) = \underline{B}(\underline{a},b)$ for $\underline{a} \neq b$. Composition with permutations on the right is the one in $\underline{B}$. $\underline{B}^{\sim}(\underline{a},\underline{b})$ is now obtained by the normal form construction. Define composition as follows: Let β be a morphism into a generator, β not contained in one of the attached whiskers, let $\circ_B$ and $\oplus_B$ be the composition and sum in $\underline{B}$. Then

$$\beta\circ(\alpha_1\oplus\cdots\oplus\alpha_k) = \beta\circ_B(\alpha_1' \oplus_B\cdots\oplus_B\alpha_k')$$

where α_i is a morphism into a generator, $\alpha_i' = \alpha_i$ if α_i is not contained in one of the attached whiskers, and $\alpha_i' = 1_b$, if $\alpha_i \in I \subset \underline{B}^{\sim}(b,b)$.

For $\beta = t \in I \subset \underline{B}^{\sim}(b,b)$ define

$\beta\circ\alpha = \alpha$ if α is not contained in one of the attached whiskers, and $\beta\circ\alpha = \max(t,u)$ if $\alpha = u \in I \subset \underline{B}^{\sim}(b,b)$.

The composition is well defined, continuous and associative. $0 \in I$ serves as identity. By construction $\oplus$ is a bifunctor. Hence $\underline{B}^{\sim}$ is an M^nTP-category.

Clearly (1) is satisfied and with $Z_b = I \subset \underline{B}^{\sim}(b,b)$ (3) holds. Define

$$\varepsilon_B^!: \underline{B}^{\sim} \to \underline{B}$$

by $\varepsilon_B^!(\alpha) = \alpha$ if is not contained in an attached whisker, and $\varepsilon_B^!(\alpha) = 1_b$ if $\alpha \in I \subset \underline{B}^{\sim}(b,b)$. Extend $\varepsilon_B^!$ to the whole of $\underline{B}^{\sim}$ using its normal form (this is possible since $\varepsilon_B^!$ is equivariant where it is defined already).

The section $\iota_B^!: \underline{B} \to \underline{B}^{\sim}$ is given by $\iota_B^!(\alpha) = \alpha$. Then $\iota_B^!\circ\varepsilon_B^!|\underline{B}^{\sim}(\underline{a},b) \simeq \mathrm{id}|\underline{B}^{\sim}(\underline{a},b)$ equivariantly and fibrewise by shrinking the whiskers to $1 \in I$, and $\varepsilon_B^!\circ\iota_B^!|\underline{B}(\underline{a},b) = \mathrm{id}|\underline{B}(\underline{a},b)$ Condition (4) follows from the definition of $\varepsilon_B^!$.]]

<u>Remark 4.11</u>: Of course, it would have sufficed to attach a whisker only to those morphism spaces $\underline{B}(b,b)$ for which 1_b is not isolated to obtain a category with the properties required in Lemma 4.10.

<u>Notation</u>: Denote $\varepsilon_B^!\circ\varepsilon_{B^{\sim}}$ and $\iota_{B^{\sim}}\circ\iota_B^!$ by $\varepsilon_B^{\sim}$ resp. $\iota_B^{\sim}$, where $\varepsilon_{B^{\sim}}$ and $\iota_{B^{\sim}}$ are the standard augmentation and standard section of $\underline{B}^{\sim}$.

<u>Theorem 4.12</u>: Given any M^nTP-category (resp. M^nT-category) $\underline{B}$, then the triple $(W\underline{B}^\sim, \varepsilon_B^\sim, \iota_B^\sim)$ satisfy the conditions (U1) and (U2) of p. 18 .

<u>Proof</u>: $\varepsilon_B^\sim$ is fibre homotopically trivial with a section $\iota_B^\sim$. Hence (U1) holds. (U2) follows from Theorem 4.6 with $\underline{B} = \underline{B}^\sim$, $\underline{C} = \underline{B}$, and $\gamma = \varepsilon_B'$.]]

<u>Theorem 4.13</u>: Let $\underline{B}$ be an M^nTP-category such tha t $W\underline{B}$ exists. Then there exists an M^nTP-functor $\sigma: W\underline{B} \to W\underline{B}^\sim$ such that $\varepsilon_B = \varepsilon_B^\sim \circ \sigma$ iff $\underline{B}$ has isolated identities.

<u>Proof</u>: If $\underline{B}$ has isolated identities then σ exists by Theorem 4.9 .

Suppose σ exists. $\varepsilon_B' \circ \varepsilon_{B^\sim} \circ \sigma \circ \iota_B = \varepsilon_B \circ \iota_B = id_B$

$$\begin{tikzcd}
W\underline{B} \arrow[r, "\sigma"'] \arrow[d, shift right, "\varepsilon_B"'] & W\underline{B}^\sim \arrow[d, shift right, "\varepsilon_{B^\sim}"'] \\
\underline{B} \arrow[u, shift right, "\iota_B"'] \arrow[r, shift right, "\iota_B'"'] & \underline{B}^\sim \arrow[u, shift right, "\iota_{B^\sim}"'] \arrow[l, shift right, "\varepsilon_B'"']
\end{tikzcd}$$

Recall that ι_B preserves identities. $\varepsilon_{B^\sim} \circ \sigma \circ \iota_B$ defines a section of ε_B', which preserves identities. Since $\varepsilon_B' | \underline{B}^\sim(b,b)$ is given by the identity outside the attached whisker this section can only be continuous if the identity 1_b in $\underline{B}(b,b)$ is isolated.]]

Remark 4.14: (1) In the case that $\underline{B}$ has isolated identities it is easy to define $\sigma\colon W\underline{B} \to W\underline{B}^{\sim}$ without referring to Theorem 4.9 by constructing a functor $\underline{B} \to \underline{B}^{\sim}$ and using Remark 2.22

(2) A similar theorem can be stated for the category obtained from $\underline{B}$ by attaching a whisker to those morphism spaces $\underline{B}(b,b)$ only for which 1_b is not isolated.

(3) Theorem 4.13 shows that Theorem 4.6 is false without some condition like 4.6 (2).

Lemma 4.15: Let G be a discrete topological group, X and Y G-spaces with a free G-action, Y a CW-complex and assume that G acts freely on the cells of Y (i.e. if $g \neq 1$, $g \in G$, then x and gx always lie in different cells). Let $p\colon X \to Y$ be an equivariant map and $s\colon Y \to X$ a section (not necessarily equivariant) of p such that there exists a fibrewise homotopy $H\colon id_X \simeq s\circ p$. Then there exists an equivariant section $\tau\colon Y \to X$ and an equivariant homotopy $T\colon id_X \simeq \tau\circ p$ which is fibrewise.

Proof: We construct a "regular" neighbourhood V_n of the n-skeleton Y^n of Y which is invariant under the action of

G and such that the part Q of V_n over an open n-cell $\overset{\circ}{e}$ does not intersect gQ over $g\overset{\circ}{e}$ if $g \neq 1$, $g \in G$. We then construct a map $u: V_n \to I$ which is 1 outside V_{n-1}, 0 on Y^n, and satisfies $u(gx) = u(x)$. The section is then constructed by induction over the skeletons of Y. Assume we have constructed $\tau_{n-1}: V_{n-1} \to X$. We extend it to V_n using s on those points x with $u(x) = 1$ and $H(\tau_{n-1}(x), u(x))$ on the others. The equivariant deformation is constructed analogously. Now the details:

Let $Z = Y/G$ and $\pi: Y \to Z$ the projection. Since the action of G on Y is free on cells Z is a CW-complex, such that π is cellular. Consider each cell e^n as cone over its boundary, $e^n = \{(x,i) \in \dot{e}^n \times [0,2] \mid (x_1,2) \sim (x_2,2),\ x_1, x_2 \in \dot{e}^n\}$ Let V be any subset of Z. We are going to construct a "regular" neighbourhood N(V) of V. Let Z^q be the q-skeleton of Z. Define

$$U_{p,q}(V) = Z^q \cap V \qquad \text{for} \quad q \leq p$$

Define $U_{p,q}(V) \subset Z^q$ for $q > p$ inductively by

$$U_{p,q} \cap \chi e^q_\alpha = \chi\{(\chi^{-1} U_{p,q-1} \cap \dot{e}^q_\alpha) \times [0,1]\}$$

where χ is the characteristic map, e^q_α a q-cell and $\dot{e}^q_\alpha$ its boundary. Let

$$U_p(V) = \bigcup_q U_{p,q}(V) \quad \text{and} \quad N(V) = \bigcup_p U_p(V).$$

Let e be an n-cell of Z. Any two lifts of $N(\overset{\circ}{e})$, where

$\mathring{e}$ denotes the interior of e, cannot intersect each others for the two lifts of $\mathring{e}$ itself are disjoint since the action of G is free on cells.

<u>Claim</u>: $N(\bigcup_\rho V_\rho) = \bigcup_\rho N(V_\rho)$

This follows immediately from the definition.

<u>Claim</u>: $N(V \cap W) = N(V) \cap N(W)$

We first prove that $U_{p,k}(V) \cap U_{q,k}(W) = U_{q,k}(V \cap W)$ for $p \leq q$.

For $k \leq p$: $U_{p,k}(V) \cap U_{q,k}(W) = V \cap Z^k \cap W \cap Z^k$

$$= U_{q,k}(V \cap W)$$

$$= U_{q,k}(V) \cap W$$

For $p < k \leq q$ we get inductively:

$$U_{p,k}(V) \cap U_{q,k}(W) \cap \chi e^k_\alpha = \chi\{(\chi^{-1} U_{p,k-1}(V) \cap \dot{e}^k_\alpha) \times [0,1]\} \cap W \cap \chi e^k_\alpha$$

$$= \chi\{(\chi^{-1} U_{p,k-1}(V) \cap \dot{e}^k_\alpha) \times [0,1] \cap \chi^{-1} W \cap e^k_\alpha\}$$

$$= \chi\{(\chi^{-1}(U_{p,k-1}(V) \cap W) \cap \dot{e}^k_\alpha) \times [0,1]\}$$

$$= \chi\{(\chi^{-1}(U_{q,k-1}(V \cap W)) \cap \dot{e}^k_\alpha) \times [0,1]\}$$

$$= U_{q,k}(V \cap W) \cap \chi e^k_\alpha$$

Again by induction we obtain for $k > q$:

$$U_{p,k}(V) \cap U_{q,k}(W) \cap \chi e^k_\alpha$$

$$= \chi\{(\chi^{-1} U_{p,k-1}(V) \cap \dot{e}^k_\alpha) \times [0,1]\} \cap \chi\{(\chi^{-1} U_{q,k-1}(W) \cap \dot{e}^k_\alpha) \times [0,1]\}$$

$$= \chi\{(\chi^{-1}(U_{p,k-1}(V) \cap U_{q,k-1}(W)) \cap \dot{e}^k_\alpha) \times [0,1]\}$$

$$= \chi\{(\chi^{-1} U_{q,k-1}(V \cap W) \cap \dot{e}^k_\alpha) \times [0,1]\}$$

$$= U_{q,k}(V \cap W) \cap \chi e^k_\alpha .$$

$$\begin{aligned}\text{Now } N(V)\cap N(W) &= (\bigcup_{p,k} U_{p,k}(V))\cap(\bigcup_{q,l} U_{q,l}(W))\\ &= \bigcup_{p,q,k,l}(U_{p,k}(V)\cap U_{q,l}(W))\\ &= \bigcup_{p,q,k}(U_{p,k}(V)\cap U_{q,k}(W)) \text{ since } U_{p,k}\subset Z^k, \text{ and } U_{q,l}\subset Z^l\\ &= \bigcup_{p,k} U_{p,k}(V\cap W)\\ &= N(V\cap W) .\end{aligned}$$

Hence in particular $N(\overset{\circ}{e}{}^n_\alpha)\cap N(\overset{\circ}{e}{}^n_\beta) = \emptyset$ for $\alpha \neq \beta$.

We furthermore define a set $M(Z^n)$ for each n.
$M(Z^n) = \bigcup_{q\geq n+1} M_q(Z^n)$, where $M_q(Z^n) = M(Z^n)\cap Z^q$ is defined inductively by $M_{n+1}(Z^n) = Z^n$, and given a k-cell e^k_α, $k > n+1$, then

$$M_k(Z^n)\cap\chi e^k_\alpha = \chi\{(\chi^{-1}M_{k-1}(Z^n)\cap\dot{e}^k_\alpha)\times[0,1]\} .$$

The difference between $M(Z^n)$ and $N(Z^n)$ is that the "collar" part over the points of $N(Z^n)$ which lie in the (n+1)-skeleton has been omitted in $M(Z^n)$. It follows from the construction that $N(Z^n) = M(Z^n) \cup \bigcup_\alpha N(\overset{\circ}{e}{}^n_\alpha)$.

Let $PX^n = p^{-1}(\pi^{-1}(N(Z^n)))$ and $X^n = p^{-1}(Y^n)$, let $PY^n = \pi^{-1}(N(Z^n))$. X^n and PX^n are in CG since they are closed. We are going to define equivariant sections $\tau_n\colon PY^n \to PX^n$ of $p|PX^n$ and equivariant fibrewise homotopies $T_n\colon id_{PX^n} \simeq \tau_n\circ(p|PX^n)$.

For each $e^o_\alpha \in Z^o$ choose a lift $l(N(e^o_\alpha))$ in Y. Define

$$\tau_o | l(N(e_\alpha^o)) = s | l(N(e_\alpha^o))$$

Since $N(e_\alpha^o) \cap N(e_\beta^o) = \emptyset$ for $\alpha \neq \beta$ this is well defined. Now extend it to the whole of PY^o by

$$\tau_o(x) = \tau_o(x').\xi \quad \xi \in G$$

(we write the action of G on the right), where $x = x'.\xi$, $x' \in l(N(e_\alpha^o))$ some α. Since the action is free this is well defined. Define

$$T_o(x,t) = H(x',t).\xi$$

if $x = x'.\xi$ and $p(x') \in l(N(e_\alpha^o))$, some α. T_o is well defined equivariant and fibrewise. Since p is an equivariant map τ_o is a section, and T_o: $id_{PX^o} \simeq \tau_o \circ (p|PX^o)$.

Suppose inductively that $\tau_{n-1}: PY^{n-1} \to PX^{n-1}$ and $T_{n-1}: id_{PX^{n-1}} \simeq \tau_{n-1} \circ (p|PX^{n-1})$ have been defined. Define a map $u: N(Z^{n-1}) \to I$ as follows: Let $(x,t) \in \chi(\dot{e}^n \times [0,1]) \subset U_{n-1,n}(Z^{n-1})$. Set $u(x,t) = t$. Extend u inductively by $u(y,t) = u(y)$ for $(y,t) \in \chi\{(\chi^{-1}U_{n-1,k-1}(Z^{n-1}) \cap \dot{e}_\alpha^k) \times [0,1]\} \subset U_{n-1,k}(Z^{n-1})$, $k > n$. u is well defined and continuous. Extend u to $u: N(Z^n) \to I$ by $u(x) = 1$ for $x \in N(Z^n) - N(Z^{n-1})$. Since $u(x) = 1$ for $x \in fr_{N(Z^n)}N(Z^{n-1})$ this is well defined. Notice that $u(x) = 0$ iff $x \in M(Z^{n-1})$. For each n-cell $e_\alpha^n \in Z^n$ choose a lift $l(N(e_\alpha^n))$. Define

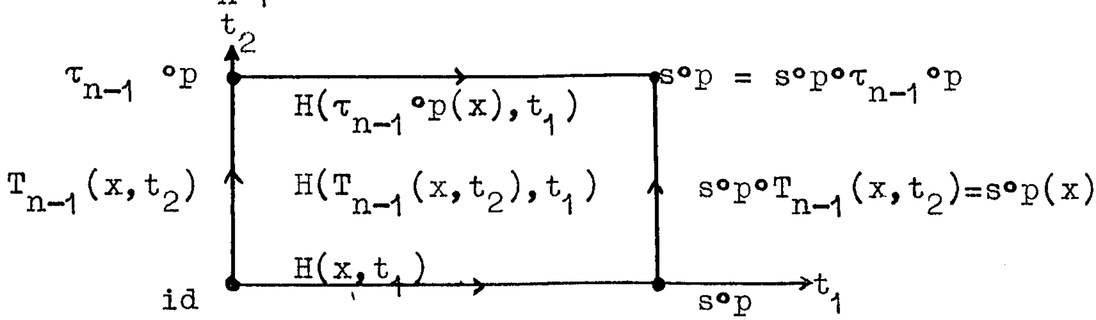
$$\tau_n(x) = \begin{cases} s(x) & u \circ \pi(x) = 1 \\ H(\tau_{n-1}(x), u \circ \pi(x)) & 0 \leq u \circ \pi(x) < 1 \end{cases}$$

$x \in l(N(e^n_\alpha))$. τ_n is well defined and continuous since it is independent of α on possible intersections $l(N(e^n_\alpha)) \cap l(N(e^n_\beta))$. Since furthermore $\pi^{-1}M(Z^{n-1}) \cap \bigcup_\alpha \pi^{-1}N(\overset{\circ}{e}{}^n_\alpha) = \emptyset$, any two lifts of $N(\overset{\circ}{e}{}^n)$ are disjoint, the action of G is free on cells, and $\tau_n | \pi^{-1}M(Z^{n-1}) \cap \bigcup_\alpha l(N(e^n_\alpha)) = \tau_{n-1} | \pi^{-1}M(Z^{n-1}) \cap \bigcup_\alpha l(N(e^n_\alpha))$ w e can extend τ_n over the whole of PY^n by

$$\tau_n(x) = \tau_n(x').\xi$$

if $x = x'.\xi$, $x' \in l(N(\overset{\circ}{e}{}^n_\alpha))$, some α, $\xi \in G$.

H and T_{n-1} define a product homotopy which is fibrewise



Hence there exists a homotopy $K_{n-1} : PX^{n-1} \times I \times I \to PX^{n-1}$ which is fibrewise, such that

$K_{n-1}(x, 0, t_2) = T_{n-1}(x, t_2)$

$K_{n-1}(x, 1, t_2) = H(x, t_2)$

$K_{n-1}(x, t_1, 0) = x$

$K_{n-1}(x, t_1, 1) = H(\tau_{n-1} \circ p(x), t_1)$.

Define

$$T_n(x, t) = \begin{cases} H(x,t) & u\circ\pi\circ p(x) = 1 \\ K_{n-1}(x, u\circ\pi\circ p(x), t) & 0 \leq u\circ\pi\circ p(x) < 1 \end{cases}$$

$x \in p^{-1}l(N(e^n_\alpha))$. Then T_n is well defined, continuous, and fibrewise, and since $T_n(x,t) = T_{n-1}(x,t)$ for $x \in p^{-1}\circ\pi^{-1}M(Z^{n-1})$ we can extend T_n over the whole of $PX^n \times I$ by

$$T_n(x'.\xi, t) = T_n(x', t).\xi$$

for $x' \in p^{-1}\circ\pi^{-1}l(N(\mathring{e}^n_\alpha))$, some α , $\xi \in G$. Then

T_n: $id_{PX^n} \simeq \tau_n\circ p|\ PX^n$ equivariantly and fibrewise.

Finally define τ: $Y \to X$ and T: $X\times I \to X$ by $\tau|\ Y^n = \tau_n|\ Y^n$ and $T|\ X^n\times I = T_n|\ X^n\times I$. Claerly τ is a continuous equivariant section of p. Since we work in CG, T is continuous if it is continuous on each compact subset of $X\times I$. Each compact subset of $X\times I$ is contained in a product $C\times I$ where C is a compact subset of X. p(C) is compact and hence it is contained in some Y^n. Hence $C\times I$ is contained in $X^n\times I$ on which T is continuous.]]

<u>Proposition 4.16</u>: Given a CW-M^nTP-category $\underline{B}$ such that composition with permutations is free on the cells of the morphism spaces of $\underline{B}$. Let $\underline{C}$ be an M^nTP-category and γ:$\underline{C} \to \underline{B}$ an M^nTP-functor such that there

exist maps $s\colon \underline{B}(\underline{a},b) \to \underline{C}(\underline{a},b)$ satisfying $\gamma\circ s|\ \underline{B}(\underline{a},b) = id_{\underline{B}(\underline{a},b)}$ and fibrewise homotopies $H\colon s\circ\gamma|\ \underline{C}(\underline{a},b) \simeq id_{\underline{C}(\underline{a},b)}$. Then γ is fibre homotopically trivial.

Proof: Put $Y = \bigcup_{\xi\in S(k)}\underline{B}(\underline{a},b)$ and $X = \bigcup_{\xi\in S(k)}\underline{C}(\underline{a},b)$ for each sequence $\underline{a} = (i_1,\ldots,i_k)$. Now apply Lemma 4.15 to X and Y. We only have to make sure that the constructed new maps and homotopies map the morphism spaces of $\underline{B}$ resp. $\underline{C}$ into the corresponding morphism spaces of $\underline{C}$. Since s and H respect the morphism spaces a quick investigation of the proof of Lemma 4.15 shows that τ and T do too.]]

CHAPTER III: STRUCTURE MAPS

§5 GENERALIZED HOMOTOPY B-MAPS

Suppose the category $W\underline{B}$ of operators acts on the spaces X and Y, we want to give an appropriate definition of morphism between them. In fact there are various possibilities.

Definition 5.1: Let $\underline{B}$ be an M^1TP (resp M^1T)-category and (X, α), (Y, β) be $\underline{B}$-spaces, i.e. spaces in CG and we are given actions $\alpha: \underline{B} \to \operatorname{End} X$ and $\beta: \underline{B} \to \operatorname{End} Y$. A map $f: X \to Y$ is called a B-homomorphism if for each $x \in \underline{B}(\underline{n}, \underline{m})$, where $\underline{k}$ is the unique sequence of length k, $f^m \circ \alpha(x) = \beta(x) \circ f^n$.

We are more interested in a definition in which f merely commutes with the action up to coherent homotopies. This is more complicated and appears to be new.

Let L_n be the "linear" category with objects $0, \ldots, n$ and one morphism $i \to j$ whenever $i \leqslant j$.

Definition 5.2: Suppose (X, γ) and (Y, δ) are $W\underline{B}$ -spaces A map $f: X \to Y$ is a generalized homotopy B-map if we are given an action $\rho: W(\underline{B} * L_1) \to \operatorname{End}(X, Y)$ that induces the given $W\underline{B}$-actions on X and Y and the given map $f: X \to Y$ (for the definition of $B * L_1$ see p.15)

Later on we give a more precise definition of a generalized homotopy $\underline{B}$-map.

If we attempt to construct the category of $W\underline{B}$-spaces and generalized homotopy $\underline{B}$-maps we find that it is not possible. The composite of two generalized homotopy $\underline{B}$ -maps is not defined, except up to a homotopy, which is itself defined only up to a homotopy, which is ..... . Instead we form a semisimplicial complex GMap $\underline{B}$, whose n-simplexes are actions of $W(\underline{B} * L_n)$ on $(n + 1)$-tuples of spaces.

<u>Lemma 5.3</u>: Let $\underline{B}$, $\underline{C}$ be M^1TP-categories in normal form and $\gamma: \underline{B} \to \underline{C}$ an M^1TP-functor. Let $\underline{D}$ and $\underline{F}$ be topological categories with objects $0, \ldots, n$ and $0, \ldots, m$ respectively, and $\delta: \underline{D} \to \underline{F}$ a continuous functor. Then there exists a unique MTP-functor

$$\nu = \gamma * \delta: \underline{B} * \underline{D} \to \underline{C} * \underline{F}$$

such that the following diagram commutes for all p, $0 \leqslant p \leqslant n$

$$\begin{tikzcd}
\underline{B} \arrow[r, "\iota_p"] \arrow[d, "\gamma"'] & \underline{B} * \underline{D} \arrow[d, "\nu"] & \underline{D} \arrow[l, "\Lambda"'] \arrow[d, "\delta"] \\
\underline{C} \arrow[r, "\iota_{\delta(p)}"] & \underline{C} * \underline{F} & \underline{F} \arrow[l, "\Lambda"']
\end{tikzcd}$$

(For ι_p and Λ see p.16)

Proof: On object generators ν is given by $\nu(i) = \delta(i)$. Adopting the intuitive description of p.16 the morphisms in $\underline{B} * \underline{D}$ from $\underline{a} = (i_1, \ldots, i_k)$ to b are given by a pair $(\beta; b) \circ f$ where $\beta \in \underline{B}(\underline{k}, \underline{1})$, ($\underline{k}$ is the unique sequence of length k in $\underline{B}$) and f is a k-fold sum $f_1 \oplus \ldots \oplus f_k$ of morphisms $f_q \in \underline{D}(i_q, b)$. Define

$$\nu[(\beta; b) \circ f] = (\gamma(\beta); \delta(b)) \circ \delta(f)$$

where $\delta(f_1 \oplus \ldots \oplus f_k) = \delta f_1 \oplus \ldots \oplus \delta f_k$. ν is continuous and equivariant. Hence we can extend it to the whole of $\underline{B} * \underline{D}$ using the normal form. This automatically makes ν commute with sums and permutations. Since γ and δ are functors, ν preserves identities, and it follows immediately from the definition that ν preserves compositions. Hence it is an MTP-functor.

$$\nu(\beta, b) = \nu \circ \iota_b(\beta)$$

$$(\gamma(\beta); \delta(b)) = \iota_{\delta(b)} \circ \gamma(\beta)$$

Hence $\nu \circ \iota_b(\beta) = \iota_{\delta(b)} \circ \gamma(\beta)$.

$$\nu(f) = \nu \circ \Lambda(f) \qquad\qquad f \in \underline{D}(i, j)$$

$$= \Lambda \circ \delta(f)$$

Hence $\nu[(\beta; b) \circ f] = \nu(\beta; b) \circ \nu(f)$

$$= (\gamma(\beta); \delta(b)) \circ \delta(f)$$

from which the commutativity of the diagram respectively the uniqueness of ν follow.]]

Each monotonically increasing map $f: (0, \ldots, n) \to (0, \ldots, m)$ gives rise to a unique functor $\underline{f}: L_n \to L_m$ such that $\underline{f}(i) = fi$ for all objects $i \in L_n$. Since f is monotonically increasing, $\underline{f}(i, j) = (fi, fj)$ is defined, where $i \leq j$ and $(i, j): i \to j$ is the unique map from i to j.

Let $f^i: (0, \ldots, n - 1) \to (0, \ldots, n)$ and $g^i: (0, \ldots, n+1) \to (0, \ldots, n)$ $i = 0, \ldots, n$, be given by

$$f^i(j) = \begin{cases} j & 0 \leq j < i \\ j + 1 & i \leq j \leq n \end{cases}$$

$$g^i(j) = \begin{cases} j - 1 & i < j \leq n + 1 \\ j & 0 \leq j \leq i \end{cases}$$

i.e. $i \in (0, \ldots, n)$ is not in the image of f^i and its counter image under g^i consists of two points

By Lemma 5.3 we have induced functors

$$\underline{\partial}^i = 1 * \underline{f}^i \colon \underline{B} * L_{n-1} \to \underline{B} * L_n$$
$$\underline{s}^i = 1 * \underline{g}^i \colon \underline{B} * L_{n+1} \to \underline{B} * L_n$$

satisfying following identities:

$$\begin{array}{lll} \underline{\partial}^i \circ \underline{\partial}^{j-1} & = \underline{\partial}^j \circ \underline{\partial}^i & i < j \\ \underline{s}^{j-1} \circ \underline{s}^i & = \underline{s}^i \circ \underline{s}^j & i < j \\ \underline{s}^j \circ \underline{\partial}^i & = \underline{\partial}^i \circ \underline{s}^{j-1} & i < j \\ & = 1 & i = j,\ j+1 \\ & = \underline{\partial}^{i-1} \circ \underline{s}^j & i > j+1 \end{array}$$

By Remark 2.22 the same identities hold for $W(\underline{\partial}^i) = \partial^i$ and $W(\underline{s}^i) = s^i$

Let
$$\partial^i \colon \mathrm{End}(X_o, \ldots, \hat{X}_i, \ldots, X_n) \to \mathrm{End}(X_o, \ldots, X_n),$$
where "^" means "delete", be the inclusion functor and $s^i \colon \mathrm{End}(X_o, \ldots, X_{i-1}, X_i, X_i, X_{i+1}, \ldots, Xn) \to \mathrm{End}(X_o, \ldots, Xn)$ be the projection functor induced by the identity on the mapping spaces. They are MTP-functors. Let
$$\rho \colon W(\underline{B} * L_n) \to \mathrm{End}(X_o, \ldots, X_n)$$
be an M^{n+1}TP-functor. Then ρ induces unique functors ρ_1 and ρ_2 such that the following diagrams commute:

$$\begin{array}{ccc} W(\underline{B} * L_{n+1}) & \xrightarrow{s^i} & W(\underline{B} * L_n) \\ \downarrow \rho_1 & & \downarrow \rho \\ \mathrm{End}(X_o, \ldots, X_i, X_i, \ldots, X_n) & \xrightarrow{s^i} & \mathrm{End}(X_o, \ldots, X_n) \end{array}$$

$$\begin{array}{ccc} W(\underline{B} * L_{n-1}) & \xrightarrow{\partial^i} & W(\underline{B} * L_n) \\ \downarrow \rho_2 & & \downarrow \rho \\ \mathrm{End}(X_o, \ldots, \hat{X}_i, \ldots, X_n) & \xrightarrow{\partial^i} & \mathrm{End}(X_o, \ldots, X_n) \end{array}$$

ρ_1 and ρ_2 are understood to be the actions $\rho \circ s^i$ and $\rho \circ \partial^i$.

Hence GMap$\underline{B}$ indeed is a semi simplicial complex, the n-simplexes of which are the actions of $W(\underline{B} * L_n)$ on $(n + 1)$-tuples of spaces and the face and degeneracy operators are induced by composition with ∂^i respectively s^i.

Definition 5.2*: Let (X, γ) and (Y, δ) be W$\underline{B}$-spaces. A pair (f, ρ), where $f\colon X \to Y$ is a map and $\rho\colon W(\underline{B} * L_1) \to \mathrm{End}\,(X, Y)$ an action, is called a generalized homotopy B-map if

$$\begin{tikzcd}
W\underline{B} = W(\underline{B} * L_o) \arrow[r, "\partial^1"] \arrow[d, "\gamma"] & W(\underline{B} * L_1) \arrow[d, "\rho"] & W(\underline{B} * L_o) = W\underline{B} \arrow[l, "\partial^0"'] \arrow[d, "\delta"'] \\
\mathrm{End}X \arrow[r, "\partial^1"] & \mathrm{End}(X, Y) & \mathrm{End}\ Y \arrow[l, "\partial^0"']
\end{tikzcd}$$

commutes and

$$\rho \circ \iota \circ \Lambda(0, 1) = f$$

where $\iota: \underline{B} * L_1 \to W(\underline{B} * L_1)$ is the standard section and $\Lambda: L_1 \to \underline{B} * L_1$ the inclusion functor.

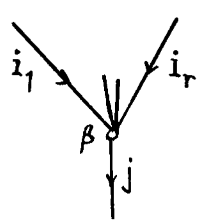
<u>Definition 5.4</u>: A semi simplicial complex K satisfies the <u>restricted Kan extension condition</u> if given n $(n-1)$-simplexes σ_i, $i \in (0, \ldots, n)$, $i \neq k$, where $0 \leqslant k \leqslant n$, $k \neq 0, 1$ fixed, such that

$$\partial^{j-1}\sigma_i = \partial^i \sigma_j \qquad\qquad 0 \leqslant i < j \leqslant n,\ i,j \neq k$$

then there exists an n-simplex σ such that $\partial^i\sigma = \sigma_i$, $i \neq k$. (i.e. it satisfies the Kan extension condition with the restriction that the omitted face is not the first or the last).

<u>Theorem 5.5</u>: The semi simplicial complex GMap $\underline{B}^{\sim}$ satisfies the restricted Kan extension condition.

Before we start proving this theorem let us give a description of the trees representing the elements of $W(\underline{B} * L_n)$ which is simpler than the description in the general case. We make use of the fact that there is exactly one morphism from i to j in L_n if $i \leqslant j$. In the general case we labelled the vertices by morphisms $(\beta, j) \circ (f_1 \oplus \dots \oplus f_k)$ of $\underline{B} * L_n$ into generator, the incoming edges by source $(f_1), \dots,$ source (f_k), and the outgoing edge by target $(f_1) = \dots =$ target $(f_k) = j$. Since in L_n the morphisms f_i are uniquely determined by their source and target it suffices to label the vertices by a morphism of $\underline{B}$ into a generator. A typical vertex now looks like



$\beta: \underline{r} \to \underline{1}$ in $\underline{B}$

$i_1, \dots, i_r \leqslant j$

of course, we again have elements of I assigned to each link. Note that in this representation a vertex labelled by 1 may only be suppressed if the incoming and outgoing edge are labelled by the same object generator.

<u>Proof of Theorem 5.5</u>: Given $k \neq 0, n$, $0 \leqslant k \leqslant n$, and for all $i \in (0,\ldots, n)$, an action $\rho_i : W(\underline{B}^{\sim} * L_{n-1}) \to \mathrm{End}(X_o,\ldots, \hat{X}_i,\ldots,X_n)$ such that

$$\rho_i \circ \partial^{j-1} = \rho_j \circ \partial^i \qquad\qquad 0 \leqslant i < j \leqslant n, \quad i, j \neq k$$

We have to construct an action ρ: $W(\underline{B}^{\sim} * L_n) \to \mathrm{End}(X_o,\ldots, X_n)$ such that $\rho \circ \partial^i = \rho_i$.

For this we construct an action of a M^{n+1}TP-subcategory of $W(\underline{B}^{\sim} * L_n)$ on $(X_o,\ldots, X_n)$ which extends the actions of the ρ_i's and which is fibre homotopically trivial over $\underline{B} * L_n$. We then apply the Universal Theorem.

The elements of $\partial^i(W(\underline{B}^{\sim} * L_{n-1}))$ are represented by trees none of the edges of which has the label i. On those elements ρ has to be given by ρ_i for $i \neq k$ because of the condition that $\rho \circ \partial^i = \rho_i$. Since $\rho_i \circ \partial^{j-1} = \rho_j \circ \partial^i$, $0 \leqslant i < j \leqslant n$, $i, j \neq k$, ρ_i and ρ_j agree on the elements in $\partial^i(W(\underline{B}^{\sim} * L_{n-1})) \cap \partial^j(W(\underline{B}^{\sim} * L_{n-1}))$. Hence ρ is well defined on all elements of $\partial^i(W(\underline{B}^{\sim} * L_{n-1}))$ for each $i \in (0,\ldots, n)$, $i \neq k$. This, of course, determines ρ on all those elements of $W(\underline{B}^{\sim} * L_n)$ that are compositions of sums of elements in the $\partial^i W(\underline{B}^{\sim} * L_{n-1})$, $i \in (0,\ldots, n)$, $i \neq k$.

Let $\underline{C}$ be the M^{n+1}TP-subcategory of $W(\underline{B}^{\sim} * L_n)$ generated by $\partial^i W(\underline{B}^{\sim} * L_{n-1})$ $i \in (0, \ldots, n)$, $i \neq k$. By our consideration above the ρ_i define an action

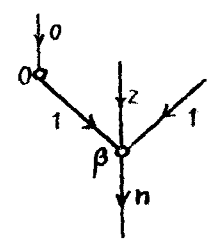
$$\eta\colon \underline{C} \to \mathrm{End}(X_o, \ldots, X_n)$$

by $\eta(\partial^i W(\underline{B}^{\sim} * L_{n-1}) = \rho_i$.

If a representing tree θ of a morphism of $\underline{C}$ into a generator has all object generators $0, \ldots, n$ as labels for its edges, θ contains a collection of edges to which $1 \in I$ is assigned (twigs may be included) and which separate θ into a tree φ and a copse of trees ψ_q such that there exist i, $j_q \neq k$ such that none of the edges of φ and ψ_q are labelled by i respectively j_q.

Note that the subspace of the representing trees of the elements in $\underline{C}$ is closed in the space of the representing trees of the elements of $W(\underline{B}^{\sim} * L_n)$. Furthermore if $x \in \underline{C}$ is indecomposable in $\underline{C}$ then it is indecomposable in $W(\underline{B}^{\sim} * L_n)$, for if $y \circ z \in \underline{C}$ is such that none of the edges of its representing tree is labelled by i then none of the edges of the representing trees of y and z is labelled by i and hence y and z are in $\underline{C}$. Since with $\{\theta, \xi, \delta\}$ all elements $\{\varphi, \xi, \delta\}$ are in $\underline{C}$ where φ is a tree of the same type as θ, $\underline{C}$ satisfies the requirements for the category $\underline{D}$ in Theorem 4.6.

The standard augmentation $\varepsilon = \varepsilon_{\underline{B}^{\sim} * L_n}$ reduced to $\underline{C}$ augments $\underline{C}$ over $\underline{B} * L_n$. Define a section $\sigma\colon \underline{B}^{\sim} * L_n \to \underline{C}$ of $\varepsilon \mid \underline{C}$ by $\sigma|\underline{C}(\underline{a}, b)$ = standard section if $0 \notin \underline{a} = (i_1, \ldots, i_k)$ or $b \neq n$. $\sigma(\beta; (i_1, n), \ldots, (i_k, n)) = \{\theta, \text{unit}, \delta\}$ where θ is the tree with the vertex at the root labelled by β, the q-th incoming edge labelled by i_q if $i_q \neq 0$, and by $i_q + 1$ if $i_q = 0$, vertices labelled by $0 \in I \subset \underline{B}^{\sim}(\underline{1}, \underline{1})$ on top of the q-th edge if it is labelled by $i_q + 1$ and their incoming edges labelled by 0. Assign $1 \in I$ to each link.

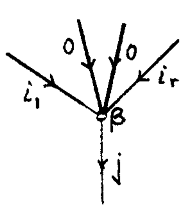
$\sigma(\beta; (0, n), (2, n), (1, n)) =$ 

The standard deformation (see p.44) gives the required deformation of $\underline{C}(\underline{a}, b)$ with $b \neq n$, into the section.

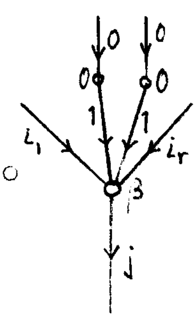
The equivariant fibrewise deformations of $\underline{C}(\underline{a}, n)$ into the section are given in steps. We first shrink all links labelled by 0, we then introduce new vertices 0 (recall that $0 \in I \subset \underline{B}^{\sim}(\underline{1}, \underline{1})$ is the unit) on top of each twig labelled by 0. Change the labels of the newly created links to 1 and label the new twigs 0. We get $1 \in I$ assigned to the new links by a deformation and then we shrink all links that are not a new link. For each deformation we have to make sure that we stay in $\underset{\sim}{C}$.

Now the details:

$H_t^1\{\theta,\xi,\delta\} = \{\theta,\xi,H_t^1(\delta)\}$ with $H_t^1(u_1,\dots, u_p)=(t_1.u_1,\dots,t_p.u_p)$ where $t_i = t$ if u_i is assigned to a link labelled by 0 and $t_i = 1$ otherwise. H^1 is well-defined, continuous, equivariant and fibrewise. If all i, $0 \leqslant i \leqslant n$ occur as labels of links in θ then in the collection of edges to which $1 \in I$ has been assigned and which decompose $\{\theta,\xi,\delta\}$ as mentioned at the beginning of the proof, none of the edges may be labelled by 0. Hence this deformation stays in $\underset{\sim}{C}$. Denote $H_0^1\underset{\sim}{C}$ by C_1. Each element of C_1 can be represented by a tree none of the edges of which with exception, may be, of some twigs is labelled by 0. The space of those trees is closed. If we stick a vertex labelled by $0 \in I \subset \underset{\sim}{B}^{\sim}(1, 1)$ on top of each twig of those trees labelled by 0, change the label of the newly created link from 0 to 1, and assign to it the value $0 \in I$, and label the twigs over the new vertices by 0, we obtain a related representative (see picture).



, $i_1, i_r, j \neq 0$ is changed to



0 assigned to the new links.

The next homotopy only affects the newly created links. Define
$H_t^2\{\theta,\xi,\delta\} = \{\theta,\xi,H_t^2(\delta)\}$ with $H_t^2(u,\dots, u_p)=(\max(t_1,u_1),\dots, \max(t_p, u_p))$ where $t_i = t$ if u_i is assigned to an outgoing edge of a vertex the incoming edge of which is labelled by 0 (such a vertex, of course, is labelled by 0). Since the multiplication $t_1 * t_2 = \max(t_1, t_2)$ is associative, H_t^2 is well-defined. It clearly is continuous, equivariant and fibrewise. By the same consideration as above, H_t^2 stays in $\underline{C}$. Denote $H_t^2\, C_1$ by C_2.

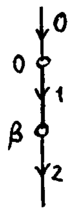
Finally define $H_t^3\{\theta,\xi,\delta\} = \{\theta,\xi,H_t^3(\delta)\}$ with $H_t^3(u_1,\dots, u_p) = (t_1.u.,\dots, t_p.u_p)$, where $t_i = 1$ if u_i is assigned to an outgoing edge of a vertex the incoming edge of which is labelled by 0 in the representation chosen above. $t_i = t$ otherwise. H_t^3 is well-defined, continuous, equivariant and fibrewise. Since $H_t^3\{\theta,\xi,\delta\}$ is a composition of an element represented by a tree the edges of which are not labelled by 0, and an element which is a sum of elements $\iota_{\underline{B}^\sim}(0; (0, 1))$, ($\iota_{\underline{B}^\sim}$ is the standard section), H_t^3 stays in $\underline{C}$. $H_0^3(C_2(\underline{a}, b)) = \sigma(\underline{B}^\sim * L_n(\underline{a}, b))$.

Hence $\underline{C}$ is fibre homotopically trivially augmented over

$\underline{B}^{\sim} * L_n$ and hence over $\underline{B} * L_n$. Now apply Theorem 4.6 with $\underline{B} = \underline{B}^{\sim} * L_n$, $\underline{C} = \underline{B} * L_n$, $\underline{G} = \underline{D} = \underline{C}$, $\delta = id_{\underline{C}}$ and $\gamma = \varepsilon'_{\underline{B} * L_n}$, which is possible since $B^{\sim} * L_n = (B * L_n)^{\sim}$.]]

Remark 5.6: If $\underline{B}$ has isolated identities we get the same result for GMap $\underline{B}$ using the Theorem 4.9 instead of 4.6

Remark 5.7: If $n = 2$ let $(f, \rho_2): (X, \alpha_0) \to (Y, \alpha_1)$ and $(g, \rho_0): (Y, \alpha_1) \to (z, \alpha_2)$ be generalized homotopy $\underline{B}^{\sim}$- maps. Then there exists an extension $\rho: W(\underline{B}^{\sim} * L_2) \to \mathrm{End}(X, Y, Z)$ such that $\rho \circ \partial^0 = \rho_0$, $\rho \circ \partial^2 = \rho_2$ and $\rho \circ \partial^1 \circ \iota_{\underline{B}^{\sim} * L_1} \circ \Lambda(0, 1) = g \circ f$. (This follows from Lemma 4.8 choosing $\tau : \underline{B}^{\sim} * L_2\ (0, 2) \to \underline{C}(0, 2)$ to be

$\tau(\beta;\ (0, 2)) =$ 

and the value 1 assigned to the link)

The same holds for generalized homotopy $\underline{B}$-maps if $\underline{B}$ has isolated identities.

For most purposes the concept of a generalized homotopy $\underline{B}$-map has undesirable complications arising from the existence of mixed maps, such as $X \times Y \to Y$. For this reason we discontinue to study them, although the Theorem 5.5 provides us with a good starting point for the development of the theory.

To be able to give some other definition for structure maps we have to introduce a new type of category of operators.

§6 REDUCED CATEGORIES OF OPERATORS

Definition 6.1: A reduced M^nTP-category $\underline{B}$ has as objects finite sequences $\underline{a} = (i_1, \ldots, i_k)$ of integers $0, \ldots, n-1$ such that $i_1 = \ldots = i_k$, the empty sequence is included. The morphisms between two objects form a topological space in CG and composition is continuous. We are given a multiplicative structure $\oplus$ on $\underline{B}$ such that

$$(i_1, \ldots, i_m) \oplus (j_1, \ldots, j_k) = (i_1, \ldots, i_m, j_1, \ldots, j_k)$$

whenever $i_1 = \ldots = i_m = j_i = \ldots = j_k$. It induces a strictly associative map of the corresponding morphism spaces and behaves like a functor whenever it is defined, i.e.

$$(\beta \oplus \gamma) \circ (\beta' \oplus \gamma') = (\beta \circ \beta') \oplus (\gamma \circ \gamma')$$

$$1_{\underline{a}} \oplus 1_{\underline{b}} = 1_{\underline{a} \oplus \underline{b}}$$

Furthermore we are given permutations satisfying the conditions (d) of Definition 1.1.

Analogously we can define reduced M^nT-categories

Each M^nTP-category $\underline{B}$ gives rise to a reduced M^nTP-category $R\underline{B}$, the subcategory of $\underline{B}$ consisting of all objects $(i_1, \ldots, i_k)$ of $\underline{B}$ such that $i_1 = \ldots = i_k$ and all morphisms between such objects. Note that for $n = 1$ the definition of an M^1TP-category and a reduced M^1TP-category coincide.

<u>Definition 6.2</u>: A <u>reduced MTP-functor</u> between a reduced M^nTP-category $\underline{B}$ and a reduced M^mTP-category $\underline{C}$ is a continuous functor mapping object gnerators into object generators and preserving sums and permutations. If it in addition preserves object generators if $m = n$, it is called a <u>reduced M^nTP-functor</u>.

If $\gamma\colon \underline{B} \to \underline{C}$ is an MTP-functor then its restriction $\gamma\colon R\underline{B} \to R\underline{C}$ is a reduced MTP-functor.

We say that a reduced M^nTP-category $\underline{B}$ acts on $(X_0,\ldots,X_{n-1})$ if we are given a reduced MTP-functor

$$\gamma\colon \underline{B} \to \mathrm{REnd}(X_0,\ldots,X_{n-1})$$

In order to develop a theory for actions of reduced M^nTP-categories we are going to prove a universal theorem equivalent to 4.6 for $RW\underline{B}$. Clearly the notion of a fibre homotopically trivial augmentation holds for reduced M^nTP-categories too, as well as the notion of a section.

<u>Lemma 6.3</u>: Each element $x \in RW\underline{B}(\underline{a}, \underline{b})$ can be decomposed into indecomposable elements (in the sense of Definition 4.1), $x = x_1\circ\ldots\circ x_p$. This decomposition is unique up to the equivalence generated by

(a) $x_1 \circ \ldots \circ (x_i' \oplus 1) \circ (1 \oplus x_{i+1}') \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ (x_i' \oplus x_{i+1}') \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ (1 \oplus x_{i+1}') \circ (x_i \oplus 1) \circ \ldots \circ x_p$

(b) $x_1 \circ \ldots \circ (x_i \circ \xi) \circ x_{i+1} \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ x_i \circ \xi \circ x_{i+1} \circ \ldots \circ x_p$
$= x_1 \circ \ldots \circ x_i \circ (\xi \circ x_{i+1}) \circ \ldots \circ x_p$
where ξ is a permutation.

Proof: In view of Lemma 4.2 an element $x \in RWB(\underline{a}, \underline{b})$ is decomposable in $RW\underline{B}(\underline{a}, \underline{b})$ iff there exists a collection of edges in a non-degenerate representing copse labelled by the same object generator and the values $1 \in I$ assigned to them which separate the copse into two copses (here we again suppose that 1 is assigned to the twigs and the roots. "Separate" means that each complete edge path runs through exactly one edge of this collection). Chop all edges of any such collection (chopping a twig or a root gives rise to a trivial tree) to obtain indecomposable elements. As in Lemma 4.3 there are three choices involved which are taken care of by the relations (a) and (b):

(1) the order in which we chop these collections

(2) the choice of the particular non-degenerate representative

(3) the choice of the position of permutations]]

Since a morphism in $R\mathcal{W}\underline{B}$ can be decomposable in $\mathcal{W}\underline{B}$ even if it is indecomposable in $R\mathcal{W}\underline{B}$, we have to refine the filtration $R\mathcal{W}^p\underline{B}$ of $R\mathcal{W}\underline{B}$: For any $M^n TP$-category $\underline{B}$, for which the construction $\mathcal{W}$ is defined, let $R\mathcal{W}^{p,q}\underline{B}$ be the subcategory of $R\mathcal{W}^p\underline{B}$ generated by $R\mathcal{W}^{p-1}\underline{B}$ and all those elements $x = \{\theta,\xi,\delta\}$ such that $(\theta,\xi) \in Tp\underline{B}(\underline{a}, b)$ and $\delta \in I^p$ has a collection β of p-q coordinates with value 1. Denote the (closed) subspace of $Q_{\alpha,p}(\underline{a},b)$ consisting of these representatives (θ,ξ,δ) by $Q_{\alpha,p,\beta,q}(\underline{a}, b)$. More precisely speaking, $Q_{\alpha,p,\beta,q}$ is the subspace of $Q_{\alpha,p}$ of those elements (θ,ξ) such that to a chosen collection β of p-q links of θ the value 1 has been assigned. Note that if the collection β separates the tree θ into a tree and a copse representing elements in $R\mathcal{W}\underline{B}$ (we might have to add some twigs to the collection)then each element in $Q_{\alpha,p,\beta,q}$ represents a composition. Let $Q'_{\alpha,p,\beta,q} = Q_{\alpha,p,\beta,q}$ if β is a collection that separates θ into a tree and a copse representing elements in $R\mathcal{W}\underline{B}$. Otherwise let $Q'_{\alpha,p,\beta,q} \subset Q_{\alpha,p,\beta,q}$ be the (closed) subspace of those representatives (θ,ξ,δ) that are either degenerate or $1 \in I$ has been assigned to more links of θ than just to the ones in the collection β. $Q'_{\alpha p,\beta, q}$ consists of all those elements of $Q_{\alpha,p,\beta,q}$ that are related to some element of lower filtration p or q, or that represent composites of elements, that can be represented by elements of some lower filtration p. ~~If~~ $\cancel{(\theta,\xi,\delta) \in Q_{\alpha,p,\beta,0}(\underline{a},b)}$ ~~then~~ $\cancel{\delta = (1,\dots,1)}$. ~~Hence~~ $\cancel{Q_{\alpha,p,\beta,0} = Q'_{\alpha,p,\beta,0}}$ ~~and hence~~ $\cancel{RW^{p,0}\underline{B} = RW^{p-1}\underline{B}}$.

Let $\underline{D}$ be a subcategory of $RW\underline{B}$ such that $D_{\alpha,p}(\underline{a},b)$ is closed in $Q_{\alpha,p}(\underline{a}, b)$ for all $\alpha,p,\underline{a},b$ (see p.53) and such that if $x \in \underline{D}$ is a composition $x = y \circ z$ with $y,z \in RW\underline{B}$ then y and z are in $\underline{D}$. Let $D_{\alpha,p,\beta,q}(\underline{a}, b) = D_{\alpha,p}(\underline{a},b) \cap Q_{\alpha,p,\beta,q}(\underline{a},b)$.

Lemma 4.5 can now be stated for reduced M^nTP-categories and in view of Lemma 6.3 the proof goes over:

<u>Lemma 6.4</u>: Let $\underline{C}$ be a reduced M^nTP-category and $\underline{D}$ a subcategory of $RW\underline{B}$ as given above. Let δ_t: $\underline{D} \to \underline{C}$ be a homotopy of functors preserving objects, sums and permutations.

(1) Given a homotopy of reduced M^nTP-functors

$$\gamma_t^{p,q-1}: RW^{p,q-1}\underline{B} \to \underline{C}$$

and equivariant maps

$$f_{\alpha,p,\beta,q}: Q_{\alpha,p,\beta,q}(\underline{a}, b)\times I \to \underline{C}(\underline{a}, b)$$

for all α,β, $\underline{a},b$ such that

(a) $\gamma_t^{p,q-1}|RW^{p,q-1}\underline{B} \cap \underline{D} = \delta_t|RW^{p,q-1}\underline{B} \cap \underline{D}$

(b) $f_{\alpha,p,\beta,q}|D_{\alpha,p,\beta,q}(\underline{a}, b)\times(t)$

$= \delta_t\circ(\chi_{a,p}|D_{\alpha,p,\beta,q}(\underline{a}, b))$

$f_{\alpha,p,\beta,q}|Q'_{\alpha,p,\beta,q}(\underline{a}, b)\times(t)$
$= \gamma_t^{p,q-1}\circ(\chi_{a,b}|Q'_{\alpha,p,\beta,q}(\underline{a}, b))$

(c) $f_{\alpha,p,\beta,q}(x, t)$ factors through the relation (2.11) for each $t \in I$.

If x is a trivial tree representing the identity of b, then $f_{\alpha,-1,\beta,0}(x) = 1_b$.

Then there exists a unique homotopy of reduced M^nTP-functors

$$\gamma_t^{p,q}\colon RW^{p,q}\underline{B} \to \underline{C}$$

extending $\gamma_t^{p,q-1}$ and $\delta_t|\underline{D} \cap RW^{p,q}\underline{B}$ such that

$\gamma_t^{p,q}\circ\chi_{\alpha,p}|Q_{\alpha,p,\beta,q}(\underline{a}, b)$
$= f_{\alpha,p,\beta,q}|Q_{\alpha,p,\beta,q}(\underline{a}, b)\times(t)$.

If $q-1=p$ we can substitute $(p,q-1)$ by $(p+1,0)$

(2) Given homotopies of reduced M^nTP-functors $\gamma_t^{p,q}$: $RW^{p,q}\underline{B} \to \underline{C}$ for all p and q such that $\gamma_t^{p,q}|RW^{p,q}\underline{B} \cap RW^{s,t}\underline{B} = \gamma_t^{s,t}|RW^{p,q}\underline{B} \cap RW^{s,t}\underline{B}$ and $\gamma_t^{p,q}|RW^{p,q}\underline{B} \cap \underline{D} = \delta_t|RW^{p,q}\underline{B} \cap \underline{D}$, then there exists a unique homotopy of reduced M^nTP-functors

$$\gamma_t\colon RW\underline{B} \to \underline{C} \text{ extending } \delta_t \text{ such that}$$
$$\gamma_t | RW^{p,q}\underline{B} = \gamma_t^{p,q}. \qquad]]$$

In the same manner we can state and prove the analogue of Lemma 4.7 with a refinement of the filtration F_p in the spaces $R_{\alpha,p,k}$. Let $R_{\alpha,p,k,\beta,q}$ be the (closed) subspace of $R_{\alpha,p,k}$ of those elements (θ,ξ,δ) such that $1 \in I$ has been assigned to a collection β of p-q links of θ. If the collection β separates the tree θ then put $R'_{\alpha,p,k,\beta,} = R_{\alpha,p,k,\beta,q}$. Otherwise let $R'_{\alpha,p,k,\beta,q}$ consist of those elements that are related to some element of lower filtration p,k, (see p.59) or that have $1 \in I$ assigned to more than just the links of the collection β. We refrain from stating the analogue of Lemma 4.7.

<u>Theorem 6.5</u> (The universal property):

Given a commutative diagram

$$\begin{tikzcd}
 & \underline{D} \arrow[dl, "\rho"'] \arrow[dr, "\delta_t"] & \\
RW\underline{B} \arrow[d, "\varepsilon"'] & & \underline{G} \arrow[d, "\mu"] \\
R\underline{B} \arrow[rr, "\gamma"] & & \underline{C}
\end{tikzcd}$$

of reduced M^nTP-categories $RW\underline{B}$, $R\underline{B}$, $\underline{C}$, $\underline{G}$, where $\underline{B}$ is an M^nTP-category, and a subcategory $\underline{D}$ of $W\underline{B}$, reduced M^nTP-functors γ,μ, the standard augmentation $\varepsilon = \varepsilon_B$, the inclusion functor ρ and a homotopy of functors δ_t preserving objects, sums and permutations for each $t \in I$.

Assume

(1) If $x \in \underline{D}$ is a composition in $RW\underline{B}$, $x = y \circ z$, then y and z are in $\underline{D}$, $D_{\alpha,p}(\underline{a}, b)$ is closed in $Q_{\alpha,p}(\underline{a},b)$, and each connected component of $D_{\alpha,p,\beta,q}(\underline{a},b)$ containing a point $x \notin Q'_{\alpha,p,\beta,q}(\underline{a},b)$ is open and closed in $Q_{\alpha,p,\beta,q}(\underline{a},b)$

(2) $\underline{B}$ and γ and μ satisfy the conditions (2) and (3) of Theorem 4.6

Then

I: There exists a reduced M^nTP-functor $\nu_0 : RW\underline{B} \to \underline{G}$ such that $\mu \circ \nu_0 = \gamma \circ \varepsilon$ and $\nu_0 \circ \rho = \delta_0$

II: Given any two reduced M^nTP-functors $\nu_0, \nu_1 : RW\underline{B} \to \underline{G}$ such that $\mu \circ \nu_0 = \mu \circ \nu_1 = \gamma \circ \varepsilon$ and $\nu_0 \circ \rho = \delta_0$, $\nu_1 \circ \rho = \delta_1$, then there exists a homotopy of reduced M^nTP-functors $\nu_t : RW\underline{B} \to \underline{G}$ between ν_0 and ν_1 such that $\nu_t \circ \rho = \delta_t$ and $\mu \circ \nu_t = \gamma \circ \varepsilon$.

<u>Proof</u>: The proof proceeds on the same lines as the proof of Theorem 4.6. We again construct compatible functors $\nu_0^{k,q}: F_{k,q}RW\underline{B} \to \underline{G}$ extending δ_0, respectively $\nu_t^{k,q}: F_{k,q}RW\underline{B} \to \underline{G}$ extending δ_t. We restrict ourselves to proving I. The proof of II is similar . The differences have been described in the proof of Theorem 4.6.

$\nu_0^{-1,q}$ and $\nu_t^{-1,q}$ are uniquely determined by δ_0 respectively δ_t.

Suppose inductively that we have defined $\nu_0^{p,q-1}: F_{p,q-1}RW\underline{B} \to \underline{G}$ such that

$$\nu_0^{p,q-1}|F_{r,s}RW\underline{B} = \nu_0^{r,s} \text{ for } r = p \text{ and } s<q-1$$

or r<p and

$$\mu\circ\nu_0^{p,q-1} = \gamma\circ\varepsilon|F_{p,q-1}RW\underline{B}.$$

Recall that $\nu_0^{p,p}$ induces $\nu_0^{p+1,0}$. We have to define equivariant maps

$$f = f_{\alpha,p,k,\beta q}: R = R_{\alpha,p,k,\beta,q}(\underline{a}, b) \to \underline{G}(\underline{a},b)$$

which factor through (2.11)

(We omit the indices whenever there is no danger of confusion) satisfying

$$f|R'(\underline{a},b) = \nu_0^{p,q-1}\circ\varkappa_{\alpha,p,k}|R'(\underline{a},b)$$

and

$$\mu\circ f = \gamma\circ\varepsilon\circ\varkappa_{\alpha,p,k}|R$$

If $\nu^{p,q-1}$ does not determine f on the whole of $R = P\times I_\beta^q$, i.e. if $R\neq R'$, (recall $R_{\alpha,p,k} = P_{\alpha,p,k}\times I^p$. Hence $P = P_{\alpha,p,k}$. $R_{\alpha,p,k,\beta} = P_{\alpha,p,k} \times I^q_\beta$, where $I^q_\beta \subset I^p$ is the face with $t_i = 1$ for each i in the collection β.)

$\cancel{P = P_{\alpha,p,k})}$ then it determines it exactly on $P\times\partial I^q_\beta \cup P'\times I^q_\beta$
Now we can proceed in exactly the same way as in the proof of Theorem 4.6 using I^q_β instead of I^p.]]

<u>Remark 6.6</u>: The analogues of the Lemma 4.8, the Theorems 4.9, 4.12, and the Proposition 4.16 hold for reduced M^nTP-categories and reduced M^nTP-funcotors.

§7 HOMOTOPY B-MAPS

To simplify the notation we denote the sequences in $RW(\underline{B}*L_n)$ of length m in the generators 0, 1, or 2 by $\underline{m}$, $\underline{m}'$, $\underline{m}''$ respectively. We hardly ever deal with $\underline{B}*L_n$ where $n>2$.

Definition 7.1: Let $\underline{B}$ be a category of operators and (X,γ), (Y, δ) $W\underline{B}$-spaces. A pair (f,ρ), where $f:X\to Y$ is a map and $\rho:RW(\underline{B}*L_1) \to REnd(X,Y)$ a reduced M^2TP-functor, is called a homotopy B-map between (X,γ) and (Y,δ) if

$$\begin{tikzcd} W\underline{B} = RW(\underline{B}*L_0) \arrow[r, "\partial^1"] \arrow[d, "\gamma"] & RW(\underline{B}*L_1) \arrow[d, "\rho"] & RW(\underline{B}*L_0) = W\underline{B} \arrow[l, "\partial^0"'] \arrow[d] \\ EndX \arrow[r, "\partial^1"] & REnd(X,Y) & EndY \arrow[l, "\partial^0"'] \end{tikzcd}$$

commutes (where ∂^i is the restriction of the face operator ∂^i to the restricted subcategories) and

$$\rho\circ\iota_{\underline{B}*L_1} \circ \Lambda(0,1) = f$$

where $\iota_{\underline{B}*L_1}$ is the reduction of the standard section to $R(\underline{B}*L_1)$ and $\Lambda:L_1 \to R(\underline{B}*L_1)$ is the canonical inclusion functor.

<u>Remark</u>: Although we will distinguish between an M^nTP-category $\underline{B}$ and its reduced subcategory $R\underline{B}$ we use the same symbol for an M^nTP-functor and its restriction to the reduced subcategory.

<u>Definition 7.2</u>: Let $(f, \rho), (g, \varkappa): (X, \gamma) \to (Y, \delta)$ be homotopy $\underline{B}$-maps. We call (f, ρ) and $(g, \varkappa)$ <u>homotopic</u> and write $(f, \rho) \cong (g, \varkappa)$ if there exists a homotopy of reduced M^2TP-functors $\lambda_t: RW(\underline{B}*L_1) \to REnd(X,Y)$ such that $\lambda_0 = \rho$ and $\lambda_1 = \varkappa$, and $\lambda_t \circ \partial^0 = \delta$, $\lambda_t \circ \partial^1 = \gamma$ for all $t \in I$. Analogously define "homotopic" for generalized homotopy $\underline{B}$-maps.

A generalized homotopy $\underline{B}$-map $(f,\rho): (X,\gamma) \to (Y,\delta)$ canonically induces a homotopy $\underline{B}$-map $(f,\rho'): (X,\gamma) \to (Y,\delta)$ by restricting the functor $\rho: W(\underline{B}*L_1) \to End(X,Y)$ to the reduced M^2TP-subcategory $RW(B*L_1)$.

<u>Theorem 7.3</u>: Let $(f,\rho): (X,\gamma) \to (Y,\delta)$ be a homotopy $B^\sim$-map. Then ρ induces an action $\nu: W(\underline{B}^\sim * L_1) \to End(X,Y)$ such that $(f,\nu): (X,\gamma) \to (Y,\delta)$ is a generalized homotopy $\underline{B}^\sim$-map. Furthermore if (f,ρ) is the canonical homotopy $\underline{B}^\sim$-map obtained from a generalized homotopy

$\underline{B}^{\sim}$-map $(f,\rho'): (X,\gamma) \to (Y,\delta)$, then $(f,\nu) \cong (f,\rho')$.

<u>Proof</u>: $RW(\underline{B}^{\sim}*L_1)$ generates an M^2TP-subcategory $\underline{C}$ of $W(\underline{B}^{\sim}*L_1)$, i.e. each morphism of $\underline{C}$ is a composition of sums of elements in $RW(\underline{B}^{\sim}*L_1)$ and of permutations. Let $y = x_1 \oplus \ldots \oplus x_p$ be an element in $\underline{C}$, $x_i \in RW(\underline{B}^{\sim}* L_1)$. Define $\eta(x_1\oplus\ldots\oplus x_p) = \rho(x_1) \times\ldots\times \rho(x_p)$ and $\eta(\xi) = \xi$, where ξ is a permutation. Extend η to an action of $\underline{C}$ by

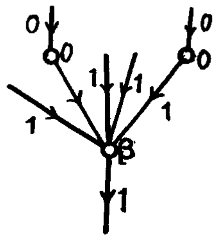
$$\eta(y_1 \circ\ldots\circ y_n) = \eta(y_1) \circ\ldots\circ \eta(y_n)$$

where y_i is a sum of morphisms in $RW(\underline{B}^{\sim}* L_1)$.

$\varepsilon = \varepsilon_{\underline{B}^{\sim}*L_1}|\underline{C}$ augments $\underline{C}$ over $\underline{B}^{\sim}*L_1$. We will show that ε is fibre homotopically trivial and then we will apply the Universal Theorem.

Note that $\underline{C}(\underline{n},1) = W(\underline{B}^{\sim}*L_1)(\underline{n},\underline{1})$, $\underline{C}(\underline{n},\underline{1}') = W(\underline{B}^{\sim}*L_1)(\underline{n}, \underline{1}')$ and $\underline{C}(\underline{n}',\underline{1}') = W(\underline{B}^{\sim}*L_1)(\underline{n}',\underline{1}')$. Hence the standard section and the standard deformation guarantee that ε is fibre homotopically trivial on these morphism spaces. So we can restrict our attntion to $\underline{C}(\underline{a},\underline{1}')$ where $\underline{a} = (i_1,\ldots, i_k)$ with $0, 1 \in \underline{a}$. As in §5 we use the simplified description for the trees.

Define a section $\sigma: B^{\sim}*L_1(\underline{a},\underline{1}') \to \underline{C}(\underline{a},\underline{1}')$ by

$\sigma(\beta; (i_1,1),\ldots, (i_k,1))=$ 

The value 1 is assigned to each link. More precisely $\sigma(\beta;(i_1,1),\dots,(i_k,1)) = \{\theta,\mathrm{unit},\delta\}$ where θ is the tree with the vertex at the root labelled by β, all incoming and outgoing edges labelled by 1. If $i_q = 0$, then on top of the q-th incoming edge sits a vertex labelled by 0 (the identity of $\underline{B}^{\sim}$), and its incoming edge is labelle d by 0.

Each representing tree of $\underline{C}(\underline{a},\underline{1}')$ has a collection of edges to which $1 \in I$ is assigned, and which decompose the tree into a tree all twigs of which are labelled by 1, and a copse all the twigs of each individual tree of which are labelled by 0 or 1 only. Conversely each tree with such a collection of edges represents an element in $\underline{C}$.

Define the equivariant fibrewise deformation into the section in steps:

$H_t^1\{\theta,\xi,\delta\} = \{\theta,\xi,H_t^1(\delta)\}$ with $H_t^1(u_1,\dots,u_p)=(t_1\cdot u_1,\dots,t_p\cdot u_p)$ where $t_i = t$ if u_i is assigned to a link labelled by 0, and $t_i = 1$ otherwise. Since each link in the separating collection of θ is labelled by 1, this homotopy stays in $\underline{C}$. It certainly is well defined, continuous, equivariant, and fibrewise. Each element in $H_0^1\underline{C}(\underline{a},\underline{1}')$ can be represented by a tree such that only its twigs are labelled by 0, and its vertices at the bottom of twigs labelled by 0 are labelled by 0 (the identity in $\underline{B}^{\sim}$). Now define

$H_t^2\{\theta,\xi,\delta\} = \{\theta,\xi,H_t^2(\delta)\}$ with $H_t^2(u_1,\ldots,u_p)=(\max(t_1,u_1),\ldots,$
$\max(t_p,u_p))$,

where $t_i = t$ if u_i is assigned to a link that is preceded by a twig labelled by 0 in our chosen representation, and $t_i = 0$ otherwise. Since the multiplication map "max" is associative, H_t^2 is well defined.It is continuous, equivariant, and fibrewise. Since links to which the value 1 is assigned are not affected, H_t^2 stays in $\underline{C}$. Each element of $H_1^2 \circ H_0^1(\underline{C}(\underline{a},\underline{1}'))$ is a composition $y \circ z$, where $y \in W(\underline{B}^{\sim}*L_1)(\underline{k}',\underline{1}')$ and $z = x_1 \oplus \ldots \oplus x_k$ with $x_q = 1_{\underline{1}'}$ if $i_q = 1$, or

$$x_q = \{\theta,\mathrm{unit},I^0\} \qquad \text{with } \theta = \begin{array}{c} 0\downarrow \\ \bullet\, 0 \\ 1\uparrow \end{array} \qquad \text{if } i_q = 0.$$

Hence z is uniquely determined by $\underline{a}$. If K_t is the standard deformation of $W(\underline{B}^{\sim}*L_1)(\underline{k}',\underline{1}')$ into the standard section, then the deformation H_t^3, given by $H_t^3(y \circ z) = K_t(y) \circ z$, deforms $H_1^2 \circ H_0^1(\underline{C}(\underline{a},\underline{1}'))$ into the given section.

Let $\underline{D}$ be the subcategory $\partial^0 W\underline{B}^{\sim} \cup \partial^1 W\underline{B}^{\sim}$ of $W(\underline{B}^{\sim}*L_1)$. $\underline{D}$ satisfies the reqirements of the Universal Theorem. Define $\varkappa: \underline{D} \to \underline{C}$ to be the inclusion. Define

$$\tau = \sigma: \underline{B}^{\sim}*L_1(\underline{1},\underline{1}') \to \underline{C}(\underline{1},\underline{1}').$$

By the Universal Theorem and Lemma 4.8 we obtain an M^2TP-functor $\mu: W(\underline{B}^{\sim}*L_1) \to \underline{C}$ extending $\varkappa$, and such that

$\varepsilon'_{\underline{B}*L_1} \circ \varepsilon \circ \varkappa = \varepsilon^{\sim}_{\underline{B}*L_1}$, and $\eta \circ \varkappa \circ \iota_{\underline{B}^{\sim}*L_1} \circ \Lambda(0,1) = f$. (For ε' and $\varepsilon^{\sim}$ see p. 69). Hence

$$(f, \eta \circ \varkappa): (X, \gamma) \to (Y, \delta)$$

gives the required generalized homotopy $\underline{B}^{\sim}$-map.

Now suppose that ρ has been obtained by restricting $\rho': W(\underline{B}^{\sim}*L_1) \to End(X,Y)$. Let $\lambda: \underline{C} \to W(\underline{B}^{\sim}*L_1)$ be the inclusion functor. Then $\eta = \rho' \circ \lambda$ since ρ' is an M^2TP-functor. By definition $\varepsilon = \varepsilon_{\underline{B}^{\sim}*L_1} \circ \lambda$, and hence $\varepsilon'_{\underline{B}*L_1} \circ \varepsilon \circ \varkappa =$ $\varepsilon'_{\underline{B}*L_1} \circ \varepsilon_{\underline{B}^{\sim}*L_1} \circ \lambda \circ \varkappa = \varepsilon^{\sim}_{\underline{B}*L_1}$. Let $\delta_t: \underline{D} \to W(\underline{B}^{\sim}*L_1)$ be the inclusion functor for all t. Then by Theorem 4.6 II there exists a homotopy of M^2TP-functors $F_t: W(\underline{B}^{\sim}*L_1) \to W(\underline{B}^{\sim}*L_1)$ extending δ_t, and such that $F_0 = \lambda \circ \varkappa$ and $F_1 = id$. $\rho' \circ F_t$ gives the required homotopy $(f, \eta \circ \varkappa) \simeq (f, \rho')$.]]

Remark 7.4:

(1) Since morphisms that are indecomposable in $\underline{C}$ can be decomposable in $W(\underline{B}^{\sim}*L_1)$ we cannot expect that the reduced action ρ induces a canonical action $\nu: W(\underline{B}^{\sim}*L_1) \to End(X,Y)$.

(2) Theorem 7.3 can be proved for actions $\rho: RW(\underline{B}^{\sim}*L_n) \to REnd(X_0, \ldots, X_n)$ with n arbitrary. But since the obtained $M^{n+1}TP$-functor

$$\nu\colon W(\underline{B}^{\sim} * L_n) \to \mathrm{End}(X_0, \ldots, X_n)$$

is not canonically induced by ρ it is not very interesting.

(3) An analogue theorem holds for homotopy $\underline{B}$-maps if $\underline{B}$ has isolated identities. Just replace $\underline{B}^{\sim}$ by $\underline{B}$.

<u>Definition and Lemma 7.5</u>: Let $f\colon (X,\nu) \to (Y,\mu)$ be a $W\underline{B}$-homomorphism (see Definition 5.1). The <u>induced homotopy B-map</u> $f_* = (f, f_*)\colon (X,\nu) \to (Y,\mu)$ is defined by

(1) $f_* | \partial^0 W\underline{B} = \mu \quad , f_* | \partial^1 W\underline{B} = \nu$

(2) $f_* | RW(\underline{B} * L_1)(\underline{n}, \underline{1}')$ is given by the composite

$$RW(\underline{B} * L_1)(\underline{n}, \underline{1}') \xrightarrow[s^0]{} W\underline{B}(\underline{n}, \underline{1}) \xrightarrow[\nu]{} \mathrm{End}\, X \xrightarrow[f\circ -]{} R\mathrm{End}(X,Y)$$

where s^0 is the degeneracy functor, and $(f\circ -)g = f^n \circ g$ for $g\colon X^m \to X^n$ (we frequently shorten $f^n = f\times\cdots\times f$, n times, to f).

Conversely each homotopy $\underline{B}$-map $(f,\varkappa)\colon (X,\nu) \to (Y,\mu)$ such that $\varkappa$ satisfies (2) is induced by a $W\underline{B}$-homomorphism.

<u>Proof</u>: f_* is continuous, by the normality of $RW(\underline{B} * L_1)$ well defined, and preserves sums, permutations, and identities. Since s^0 and ν are functors we only have to show that composition of $x \in RW(\underline{B} * L_1)(\underline{n}', \underline{1}')$ with $y \in RW(\underline{B} * L_1)(\underline{m}, \underline{n}')$ is preserved:

$$\begin{aligned} f_*(x \circ y) &= f \circ (\nu \circ s^0(x)) \circ (\nu \circ s^0(y)) \\ &= f \circ \nu(x) \circ (\nu \circ s^0(y)) \\ &= \mu(x) \circ f \circ (\nu \circ s^0(y)) \\ &= f_*(x) \circ f_*(y) \quad . \end{aligned}$$

Conversely given a homotopy $\underline{B}$-map $(f,\varkappa)$ such that $\varkappa$ satisfies (2). Then $\varkappa \circ \iota_{\underline{B}*L_1} \circ \Lambda(0,1) = f$, and $s^0 \circ \iota_{\underline{B}*L_1} \circ \Lambda(0,1) = 1 \in W\underline{B}$. Hence for $x \in W\underline{B}$ considered embedded in $RW(\underline{B}*L_1)$ by ∂^0 or ∂^1:

$$\begin{aligned} \varkappa(x \circ (\iota_{\underline{B}*L_1} \circ \Lambda(0,1) \oplus \dots \oplus \iota_{\underline{B}*L_1} \circ \Lambda(0,1))) &= f \circ \nu(x) \circ 1_X \\ &= \varkappa(x) \circ f \circ 1_X \end{aligned}$$

Hence $f \circ \nu(x) = \mu(x) \circ f$.

This also follows from the tree representation.]]

Clearly composites of $\underline{B}$-homomorphisms are $\underline{B}$-homomorphisms. Neither do we have any problems in defining composites of $W\underline{B}$-homomorphisms with homotopy $\underline{B}$-maps:

<u>Definition and Lemma 7.6</u>: Let $(f,\rho)\colon (X,\mu) \to (Y,\nu)$ be a homotopy $\underline{B}$-map and $g\colon (Y,\nu) \to (Z,\lambda)$ a $W\underline{B}$-homomorphism. Then there exists a canonical composite homotopy $\underline{B}$-map $g \circ (f,\rho) = (g \circ f, \varkappa)\colon (X,\mu) \to (Z,\lambda)$ defined by

$\varkappa |\ \partial^0 W\underline{B} = \lambda \quad , \quad \varkappa |\ \partial^1 W\underline{B} = \mu$

$\varkappa |\ RW(\underline{B}*L_1)(\underline{n},\underline{1}')$ is defined by $\varkappa(x) = g \circ \rho(x)$.

<u>Proof</u>: Again we have to show that $\varkappa$ is a functor. Since ρ extends μ it suffices to show that $\varkappa$ preserves compositions of $x \in \partial^0 W\underline{B}$ with $y \in RW(\underline{B}*L_1)(\underline{n},\underline{m}')$:
$\varkappa(x \circ y) = g \circ \rho(x) \circ \rho(y) = \lambda(x) \circ g \circ \rho(y) = \varkappa(x) \circ \varkappa(y)$.]]

<u>Remark</u>: Analogously we can define compositions $(f,\rho)\circ h$ where $h: (W,\sigma) \to (X,\mu)$ is a $W\underline{B}$-homomorphism.

Again we run into trouble if we attempt to construct the category of $W\underline{B}$-spaces and homotopy $\underline{B}$-maps, for as in the case of the generalized homotopy $\underline{B}$-maps the composite is only defined up to a homotopy, which is itself defined only up to a homotopy, which is ... . To get around this difficulty we again form a semi simplicial complex Map$\underline{B}$, the n-simplexes of which are actions of $RW(\underline{B}*L_n)$ on (n+1)-tuples of spaces. The face and degeneracy operators are induced by the compositions

$$\rho\circ\partial^i: RW(\underline{B}*L_{n-1}) \longrightarrow RW(\underline{B}*L_n) \longrightarrow REnd(X_0,\ldots,X_n)$$

$$\rho\circ s^i: RW(\underline{B}*L_{n+1}) \longrightarrow RW(\underline{B}*L_n) \longrightarrow REnd(X_0,\ldots,X_n)$$

(compare p. 84).

<u>Theorem 7.7</u>: The semi simplicial complex Map$\underline{B}^{\sim}$ satisfies the restricted Kan extension condition.

If $\underline{B}$ has isolated identities, then Map$\underline{B}$ satisfies the restricted Kan extension condition.

The proof is exactly the same as the one of Theorem 5.5 with the exception that we use $RW(\underline{B}^{\sim}*L_n)$ instead of $W(\underline{B}^{\sim}*L_n)$ and Theorem 6.5 instead of Theorem 4.12.]]

The Remark 5.7 applies to the reduced case too.

Definition 7.8: Let $(f_i,\rho_i)\colon (X,\mu) \to (Y,\nu)$, $i = 0,1$, be homotopy $\underline{B}$-maps. Then we call (f_0,ρ_0) and (f_1,ρ_1) s-homotopic and write $(f_0,\rho_0) \simeq (f_1,\rho_1)$ if there exists a reduced M^3TP-functor $\sigma\colon RW(\underline{B}*L_2) \to REnd(X,Y,Y)$ such that $\sigma\circ\partial^2 = \rho_0$, $\sigma\circ\partial^1 = \rho_1$, and $\sigma\circ\partial^0 = 1_{Y*}$.

The condition $\sigma\circ\partial^0 = (1_Y)_*$ is equivalent to saying that $\sigma\circ\partial^0$ is degenerate. It is easy to show that a homotopy $\underline{B}$-map is degenerate iff it is the homotopy $\underline{B}$-map induced by the identity.

Lemma 7.9: Let $(f,\rho)\colon (X,\mu) \to (Y,\nu)$ be a homotopy $\underline{B}$-map, $g\colon (Y,\nu) \to (Z,\lambda)$ a $W\underline{B}$-homomorphism, $(g\circ f,\varkappa)\colon(X,\mu)\to(Z,\lambda)$ their canonical composite. Then there exists an action $\sigma\colon RW(\underline{B}*L_2) \to REnd(X,Y,Z)$ such that $\sigma\circ\partial^1 = \varkappa$, $\sigma\circ\partial^0 = g_*$, and $\sigma\circ\partial^2 = \rho$.

Proof: Define σ as follows:

$\sigma|\ \partial^2 RW(\underline{B}*L_1) = \rho$

$\sigma|\ \partial^0 RW(\underline{B}*L_1) = g_*$

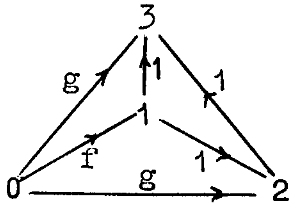
$\sigma|\ RW(\underline{B}*L_2)(\underline{n},\underline{1}") = \varkappa \circ s^1$.

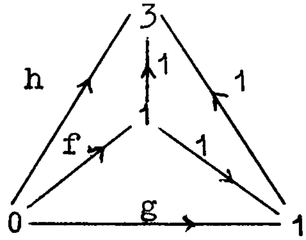
σ is continuous, well defined, preserves sums, permutations, and identities. It satisfies the statement of the Lemma. It remains to show that σ is a functor, and for this it suffices to show that σ preserves compositions of $x \in \partial^0 RW(\underline{B}*L_1)$ with $y \in \partial^2 RW(\underline{B}*L_1)$:

$$\begin{aligned} \sigma(x\circ y) &= \varkappa(s^1(x)) \circ \varkappa(s^1(y)) \\ &= \lambda(s^0(x)) \circ g \circ \rho(y) \\ &= g \circ \nu(s^0(x)) \circ \sigma(y) \\ &= \sigma(x) \circ \sigma(y) \quad . \end{aligned}$$]]

Remark: We can prove an analogous lemma for compositions $(f\circ h,\zeta)$ of (f,ρ) with a $W\underline{B}$-homomorphism $h: (W,\omega) \to (X,\mu)$.

Clearly "$\cong$" is an equivalence relation. From Theorem 7.7 and Lemma 7.9 we can immediately deduce that "$\simeq$" is an equivalence relation for homotopy $\underline{B}^{\sim}$-maps. For reflexivity follows from Lemma 7.9, while symmetry and transitivity follow from Theorem 7.7 and a trivial version of Lemma 7.9 by considering the following 3-simplexes: All maps are supposed to be homotopy $\underline{B}^{\sim}$-maps or homotopy $\underline{B}$-maps and $\underline{B}$ has isolated identities.

I 

II 

I symmetry: The bottom is given by the homotopy $(f,\rho) \simeq (g,\varkappa)$. The front and the right hand side are given by reflexivity (i.e. by Lemma 7.9). Since the second face is missing we can fill in the 3-simplex by Theorem 7.7. The resulting left hand sides provides us with a homotopy $(g,\varkappa) \simeq (f,\rho)$.

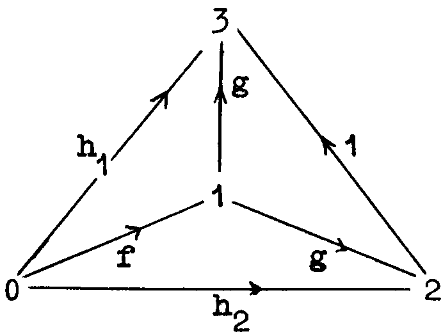
II transitivity: The homotopies $(f,\rho) \simeq (g,\varkappa)$ and $(g,\varkappa) \simeq (h,\lambda)$ give the bottom and the front. The right hand side is given by reflexivity. Since the second face is missing the 3-simplex can be filled by Theorem 7.7. The resulting left hand side provides us with a homotopy $(f,\rho) \simeq (h,\lambda)$.

<u>Definition 7.10</u>: Let $(f,\rho)\colon (X,\mu) \to (Y,\nu)$ and $(g,\sigma)\colon (Y,\nu) \to (Z,\lambda)$ be homotopy $\underline{B}$-maps. Then the homotopy $\underline{B}$-map $(h,\zeta)\colon (X,\mu) \to (Z,\lambda)$ is called a <u>composite</u> of (f,ρ) with (g,σ) if there exists an action

η: $RW(\underline{B}*L_2) \to REnd(X,Y,Z)$ such that $\eta\circ\partial^0 = \sigma$, $\eta\circ\partial^1 = \zeta$, $\eta\circ\partial^2 = \rho$.

<u>Lemma 7.11</u>: Let $(f,\rho)\colon (X,\mu) \to (Y,\nu)$ and $(g,\sigma)\colon (Y,\nu) \to (Z,\lambda)$ be homotopy $\underline{B}^\sim$-maps. Then there exists a composite of (f,ρ) with (g,σ) and it is unique up to s-homotopy. If $\underline{B}$ is an M^1TP-category with isolated identities then the same holds if we substitute $\underline{B}^\sim$ by $\underline{B}$.

<u>Proof</u>: The first part follows from Theorem 7.7. Now suppose that $(h_i,\zeta_i)\colon (X,\mu) \to (Z,\lambda)$ are two composites of (f,ρ) with (g,σ), $i = 0,1$. Let $\eta_i\colon RW(\underline{B}^\sim * L_2) \to REnd(X,Y,Z)$ be the actions defining them. Consider the following 3-simplex:



The bottom and the left hand side are given by the actions η_1 and η_2. By Lemma 7.9 there exists an action determining the right hand side. Since the first face is missing we can apply Theorem 7.7 and fill in the 3-simplex. The resulting front face gives the required s-homotopy.]]

<u>Lemma 7.12</u>: Let (f,ρ), $(h,\varkappa)$: $(X,\mu) \to (Y,\nu)$ be homotopy $\underline{B}^{\sim}$-maps. Then $(f,\rho) \simeq (h,\varkappa)$ iff there exists an action σ: $RW(\underline{B}^{\sim} * L_2) \to REnd(X,X,Y)$ such that $\sigma \circ \partial^0 = \rho$, $\sigma \circ \partial^1 = \varkappa$, $\sigma \circ \partial^2 = (1_X)_*$. (Recall that s-homotopy is defined by an action $RW(\underline{B}^{\sim} * L_2) \to REnd(X,Y,Y)$). If $\underline{B}$ is an M^1TP-category with isolated identities then the same holds if we substitute $\underline{B}^{\sim}$ by $\underline{B}$.

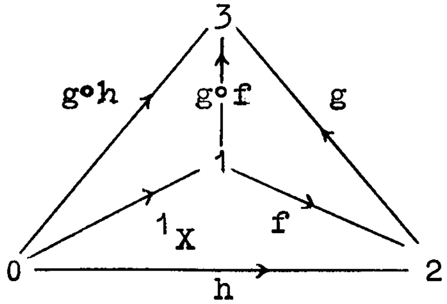
<u>Proof</u>: The canonical composites $(f,\rho)\circ(1_X)_*$ and $(1_Y)_*\circ(f,\rho)$ are equal. From Lemma 7.9 and the uniqueness of composition of homotopy $\underline{B}^{\sim}$-maps it follows that $(f,\rho) \simeq (h,\varkappa)$ iff $(h,\varkappa) \simeq (1_Y)_*\circ(f,\rho)$, i.e. $(h,\varkappa)$ is a (not canonical) composite of $(1_Y)_*$ with (f,ρ), and hence a composite of (f,ρ) with $(1_X)_*$, which proves the Lemma one way. The converse follows in the same manner.]]

<u>Lemma 7.13</u>: Let (f,ρ), $(h,\varkappa)$: $(X,\mu) \to (Y,\nu)$ be s-homotopic homotopy $\underline{B}^{\sim}$-maps and (g,ζ): $(Y,\nu) \to (Z,\lambda)$, (k,γ): $(W,\omega) \to (X,\mu)$ homotopy $\underline{B}^{\sim}$-maps. Then $(g,\zeta)\circ(f,\rho) \simeq (g,\zeta)\circ(h,\varkappa)$ and $(f,\rho)\circ(k,\gamma) \simeq (h,\varkappa)\circ(k,\gamma$
If $\underline{B}$ is an M^1TP-category with isolated identities, then the same holds if we replace $\underline{B}^{\sim}$ by $\underline{B}$.

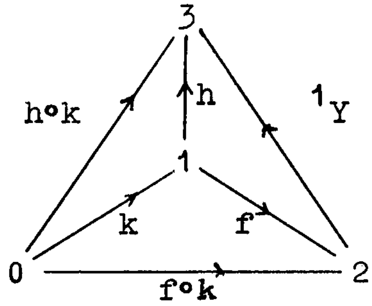
<u>Proof</u>: By Lemma 7.12 we have an action

$\sigma\colon RW(\underline{B}^{\sim} * L_2) \to REnd(X,X,Y)$ such that $\sigma\circ\partial^0 = \rho$, $\sigma\circ\partial^1 = \varkappa$, and $\sigma\circ\partial^2 = (1_X)_*$. In the following 3-simplexes

I



II



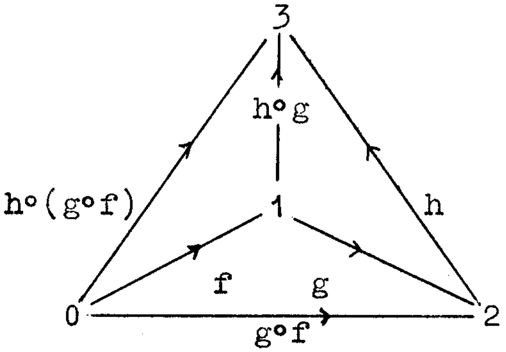
in I the bottom is given by σ, the front and the right face by composition (Lemma 7.11), in II the bottom and the left face are given by composition, the right face by the given s-homotopy. Now apply Theorem 7.7. The resulting left face of I and front face of II give the required s-homotopies.]]

<u>Theorem 7.14</u>: The $W\underline{B}^{\sim}$-spaces and s-homotopy classes of homotopy $\underline{B}^{\sim}$-maps form a category.

If $\underline{B}$ is an M^1TP-category with isolated identities then the $W\underline{B}$-spaces and s-homotopy classes of homotopy $\underline{B}$-maps form a category.

<u>Proof</u>: By Lemma 7.11 and Lemma 7.13 we have a well defined composition. By Lemma 7.9 the $W\underline{B}^{\sim}$-homomorphisms $(1_X)_*\colon (X,\mu) \to (X,\mu)$ provide the identities. Associativity

is obtained from Theorem 7.7 by considering the following 3-simplex:



The bottom face defines g∘f, the front defines h∘(g∘f), the right face defines h∘g. Since the second face is missing we can fill in this 3-simplex. We find that the representative (composition is unique up to s-homotopy) for h∘(g∘f) represents (h∘g)∘f , too.]]

We next discuss the connection between the two definitions of homotopy between structure maps (see Definition 7.2 and 7.8). For this we first have to side track and study "equivariant" NDR-pairs of spaces of representing trees.

<u>Definition 7.15</u>: Call a subspace A of $M = M_{\alpha,p}(\underline{a},b)$, (see p. 53), an <u>equivariant NDR</u>, if the maps $u:M \to I$ and $h: M\times I \to M$ representing A as a NDR in M (see [6;Definition 6.2]) satisfy :

$u(x) = u(y)$ if $x \sim y$ under (2.3)

$u(x \circ \xi) = u(x)$, where ξ is a permutation.

$h(x,t) \sim h(y,t)$ under (2.3) if $x \sim y$ under (2.3)

$h(x \circ \xi, t) = h(x,t) \circ \xi$, where ξ is a permutation.

By taking a radial map and a radial deformation $v\colon I^n \to I$ and $j\colon I^n \times I \to I^n$, (radial from the point $(\frac{1}{2}, \ldots, \frac{1}{2})$), we can represent ∂I^n as a NDR in I^n in such a manner that v and j are symmetric in the n coordinates of I^n.

Now suppose we are given an equivariant NDR A in $M = M_{\alpha,p}(\underline{a}, b)$ represented by u and h.

<u>Lemma 7.16</u>: $A \times I^n \cup M \times \partial I^n$ can be represented as NDR in $M \times I^n$ by maps $w\colon M \times I^n \to I$ and $k\colon M \times I^n \times I \to M \times I^n$ such that

(A) $w(\theta, \xi, \delta) = w(\varphi, \eta, \partial)$ if $(\theta, \xi, \delta) \sim (\varphi, \eta, \partial)$ under (2.11)

$w(\theta, \xi, \delta) = w(\theta, \mathrm{unit}, \delta)$

(B) $k(\theta, \xi, \delta, t) \sim k(\varphi, \eta, \partial, t)$ under (2.11) if $(\theta, \xi, \delta) \sim (\varphi, \eta, \partial)$ under (2.11)

$k(\theta, \xi, \delta, t) = k(\theta, \mathrm{unit}, \delta, t) \circ \xi$.

<u>Proof</u>: Define $w(\theta, \xi, \delta) = u(\theta, \xi) . v(\delta)$. From the definition of u and v (A) follows immediately.

Define

$$k(\theta,\xi,\delta,t) = (\theta,\xi,\delta) \qquad \text{if } (\theta,\xi) \in A, \text{ and } \delta \in \partial I^n$$
$$= [h(\theta,\xi,t),\ j[\delta,(u(\theta,\xi)/v(\delta)).t]]$$
$$\text{if } v(\delta) \geq u(\theta,\xi) \text{ and } v(\delta) > 0$$
$$= [h[\theta,\xi,(v(\delta)/u(\theta,\xi)).t],\ j(\delta,t)]$$
$$\text{if } u(\theta,\xi) \geq v(\delta) \text{ and } u(\theta,\xi) > 0 \quad .$$

By [6; Theorem 6.3], k is continuous. It follows directly from the definition that it satisfies the condition (B).]]

<u>Remark 7.17</u>: Let $K \subset I^n$ be a NDR such that the representing maps $v'\colon I^n \to I$ and $j'\colon I^n \times I \to I^n$ are symmetric in certain subsets U_i of the n coordinates of I^n. Then by the same construction $M\times K \cup A\times I^n$ can be represented as a NDR in $M\times I^n$ by maps satisfying (A) and (B) of Lemma 7.16, if the coordinates of I^n in $M\times I^n$ are only permuted inside the U_i under the relation (2.11).

<u>Lemma 7.18</u>: Let K and M and A be as in Remark 7.17, and suppose (2.11) permutes the coordinates of I^n in $M\times I^n$ inside the subsets U_i of the coordinates of I^n only. Then there exists a retraction

$$r\colon M\times I^n\times I \to M\times I^n\times 0 \cup (M\times K \cup A\times I^n)\times I$$

such that

(A) Let $x = (\theta,\xi,\delta)$, $y = (\varphi,\eta,\partial)$, $r(x,t) = (x',t')$, $r(y,t) = (y',t'')$. If $x \sim y$ under (2.11), then $t' = t''$ and $x' \sim y'$ under (2.11).

(B) $r(\theta,\xi,\delta,t) = r(\theta,\mathrm{unit},\delta,t)\circ\xi$.

<u>Proof</u>: r is defined by

$$
\begin{aligned}
r(\theta,\xi,\delta,t) &= (\theta,\xi,\delta,t) \quad \text{if } t = 0 \text{ and } (\theta,\xi,\delta)\in M\times K \cup A\times I^n \\
&= [k(\theta,\xi,\delta,1),\ l[t,w(\theta,\xi,\delta)/s(t)]] \\
&\qquad \text{if } s(t) \geqslant w(\theta,\xi,\delta) \text{ and } s(t) > 0 \\
&= [k[\theta,\xi,\delta,s(t)/w(\theta,\xi,\delta)],\ l(t,1)] \\
&\qquad \text{if } w(\theta,\xi,\delta) \geqslant s(t) \text{ and } w(\theta,\xi,\delta) > 0 \quad ,
\end{aligned}
$$

where $s\colon I \to I$ and $l\colon I\times I \to I$ are defined by $s(t) = t/2$ and $l(t_1,t_2) = (1-t_2).t_1$. w and k are the maps of Lemma 7.16. By [6; Theorem 6.3], r is continuous.]]

Let $\underline{B}$ be an $M^n TP$-category such that $(\underline{B}(b,b), 1_b)$ is a NDR-pair for all object generators b. Let $\underline{D}$ be a subcategory of RW$\underline{B}$ satisfying:

(1) If $x \in \underline{D}$ is a composite in RW$\underline{B}$, $x = y\circ z$, then y and z are in $\underline{D}$.

(2) Suppose $D_{\alpha,p}(\underline{a},b)$ contains trees that do not represent decomposable elements of $\underline{D}$, then $D_{\alpha,p}(\underline{a},b)$ is a product, $D_{\alpha,p}(\underline{a},b) = D'\times I^p$, and $D' \cup M'_{\alpha,p}(\underline{a},b) \subset M_{\alpha,p}(\underline{a},b)$

is an equivariant NDR, where $M'_{\alpha,p}(\underline{a},b) \subset M_{\alpha,p}(\underline{a},b)$ is the subspace of those trees that contain a vertex labelled by an identity.

<u>Lemma 7.19</u>: Given an action ρ_0: $RW\underline{B} \to REnd(X_0,\ldots,X_{n-1})$ and a homotopy of functors δ_t: $\underline{D} \to REnd(X_0,\ldots,X_{n-1})$ preserving objects, sums, and permutations, such that $\rho_0 | \underline{D} = \delta_0$. Then there exists a homotopy of reduced M^nTP-functors ρ_t: $RW\underline{B} \to REnd(X_0,\ldots,X_{n-1})$ extending ρ_0 and δ_t .

<u>Proof</u>: By Lemma 6.4 we have to construct homotopies of reduced M^nTP-functors $\gamma_t^{p,q}: RW^{p,q}\underline{B} \to \underline{E}$, where $\underline{E} = REnd(X_0,\ldots,X_{n-1})$, such that $\gamma_t^{p+r,q+s}$ extends $\gamma_t^{p,q}$, $r,s > 0$, and such that $\gamma_t^{p,q}$ is compatible with δ_t. For this we have to construct maps

$$f_{\alpha,p,\beta,q}: Q_{\alpha,p,\beta,q}(\underline{a},b) \to \underline{E}(\underline{a},b)$$

satisfying the requirements of Lemma 6.4 .

Since $(\underline{B}(b,b), 1_b)$ is a NDR-pair for all object generators, and since the trivial group of permutations acts on it, $M'_{\alpha,p}(\underline{a},b) \subset M_{\alpha,p}(\underline{a},b)$ is an equivariant NDR. (We know that it is a NDR. The representing maps are induced by those of the NDR-pairs $(\underline{B}(b,b), 1_b)$. Hence it trivially

is an equivariant NDR-pair).
Induction start: For $p = -1$, $\gamma_t^{-1,q}$ is uniquely determined since $RW^{-1,q}\underline{B}$ consists of identities only.
Induction step from $(p,q-1)$ to (p,q): We drop indices whenever there is no danger of confusion.

Suppose $Q = Q_{\alpha,p,\beta,q} \neq Q'$. Then $f = f_{\alpha,p,\beta,q}$ is determined on $(M\times\partial I^q_\beta \cup M'\times I^q_\beta)\times I$ by $\gamma_t^{p,q-1}$, where $I^q_\beta \subset I^p$ is the cube determined by the collection β of links in the trees of M. f furthermore is given on $Q\times 0 = M\times I^q_\beta\times 0$ (compare p. 97) by ρ_0. If $D = D_{\alpha,p} \cap Q$ contains an element which is not in Q', then $D = D'\times I^q_\beta$, and $D' \cup M'$ is an equivariant NDR in M. f is determined on $D'\times I^q_\beta\times I$ by δ_t. Denote $f|\ \{[M\times\partial I^q_\beta \cup (M'\cup D')\times I^q_\beta]\times I \cup Q\times 0\}$ by g. Define
$f\colon Q\times I \to \underline{E}$ by $f = g\circ r$, where

$$r\colon Q\times I = M\times I^q\times I \;\to\; [M\times\partial I^q_\beta \cup (M'\cup D')\times I^q_\beta]\times I \cup M\times I^q_\beta\times 0$$

is the retraction of Lemma 7.18. Since r is equivariant and factors through (2.11) for each fixed t, $f = g\circ r$ satisfies the requirements of Lemma 6.4 .

<u>Lemma 7.20</u>: Let $(g_i,\rho_i)\colon (X,\mu) \to (Y,\nu)$, $i = 0,1$, be homotopy $\underline{B}$-maps such that $(g_0,\rho_0) \cong (g_1,\rho_1)$. If $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ is a NDR-pair, then $(g_0,\rho_0) \simeq (g_1,\rho_1)$.

<u>Proof</u>: Let $\varkappa_t:(g_0,\rho_0) \cong (g_1,\rho_1)$ be the given homotopy of reduced M^2TP-functors. Let $\underline{D}$ be the reduced M^3TP-subcategory of $RW(\underline{B}*L_2)$ generated by $\partial^i RW(\underline{B}*L_1)$, $i = 0,1,2$. By Lemma 7.9 there exists an action σ_0: $RW(\underline{B}*L_2) \to REnd(X,Y,Y)$ such that $\sigma_0 \circ \partial^0 = (1_Y)_*$, $\sigma_0 \circ \partial^1 = \rho_0$, $\sigma_0 \circ \partial^2 = \rho_0$. Define δ_t: $\underline{D} \to REnd(X,Y,Y)$ by

$$\delta_t | \partial^0 RW(\underline{B}*L_1) = (1_Y)_* \quad \text{for all } t \in I$$
$$\delta_t | \partial^1 RW(\underline{B}*L_1) = \varkappa_t$$
$$\delta_t | \partial^2 RW(\underline{B}*L_1) = \rho_0 \quad \text{for all } t \in I .$$

$$\begin{tikzcd} & 1 \arrow[dr, "(1_Y)_*"] & \\ 0 \arrow[ur, "\rho_0"] \arrow[rr, "\varkappa_t"'] & & 2 \end{tikzcd}$$

δ_t is a well defined homotopy of functors since ρ_0, $\varkappa_t$, and $(1_Y)_*$ extend the actions μ and ν. $\underline{D}$ satisfies the requirements of Lemma 7.19. Hence there exists a homotopy σ_t: $RW(\underline{B}*L_2) \to REnd(X,Y,Y)$ of reduced M^3TP-functors extending δ_t and σ_0. σ_1 defines the required s-homotopy.]]

<u>Theorem 7.21</u>: Let (g_0,ρ_0): $(X,\mu) \to (Y,\nu)$ be a homotopy $\underline{B}$-map and g_1: $X \to Y$ a map homotopic to g_0. If $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ is a NDR-pair, then g_1 can be made into

a homotopy $\underline{B}$-map $(g_1,\rho_1)\colon (X,\mu) \to (Y,\nu)$ such that $(g_0,\rho_0) \cong (g_1, \rho_1)$ and hence $(g_0,\rho_0) \simeq (g_1,\rho_1)$.

Proof: Let g_t be the homotopy between g_0 and g_1. Let $\underline{D}$ be the subcategory of $RW(\underline{B}*L_1)$ consisting of the identities $1_{\underline{1}}$ and $1_{\underline{1}'}$ and of the morphism $j = \iota_{\underline{B}*L_1}(1;(0,1))$ only. Define a homotopy of functors $\delta_t\colon \underline{D} \to REnd(X,Y)$ by $\delta_t(j) = g_t$. Since $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ is a NDR-pair, $\underline{D}$ satisfies the requirements of Lemma 7.19. Hence there exists a homotopy of reduced M^2TP-functors $\rho_t\colon RW(\underline{B}*L_1) \to REnd(X,Y)$ extending ρ_0 and δ_t. Since $\rho_1 \circ \iota_{\underline{B}*L_1} \circ \Lambda(0,1) = g_1$, ρ_t is a homotopy $(g_0,\rho_0) \cong (g_1,\rho_1)$.]]

§ 8 HOMOTOPY EQUIVALENCES AND HOMOTOPY TYPE

The aim of this chapter is to prove the following two theorems:

Theorem 8.1: Let $\underline{B}$ be an M^1TP-category with isolated identities. Let (f,ρ): $(X,\alpha) \to (Y,\beta)$ be a homotopy $\underline{B}$-map and $f: X \to Y$ a homotopy equivalence. Then (f,ρ) is a s-homotopy equivalence, i.e. it is an isomorphism in the category of $W\underline{B}$-spaces and s-homotopy classes of homotopy $\underline{B}$-maps.

Theorem 8.2: Let (X,α) be a $W\underline{B}^{\sim}$-space and $f: X \to Y$ a homotopy equivalence. Then Y can be made into a $W\underline{B}^{\sim}$-space (Y,β) and f into a s-homotopy equivalence (f,ρ): $(X,\alpha) \to (Y,\beta)$.
If $\underline{B}$ is an M^1TP-category with isolated identities, the same holds if we replace $\underline{B}^{\sim}$ by $\underline{B}$.

By using the mapping cylinder these theorems reduce to proving the statements for strong deformation retracts, and this can be reduced in the case of Theorem 8.1 to proving that it holds if f is the identity.

In the proof that homotopy $\underline{B}$-maps are s-homotopy equivalences it is often easier to work with the category $RW(\underline{B}*Is_1)$ rather than the category $RW(\underline{B}*L_2)$. Recall that Is_1 is the category with two objects and exactly one morphism between any two objects. We again can use the simplified description for the trees representing the elements of $RW(\underline{B}*Is_1)$, (see p. 86).

The inclusion functors $d^i\colon L_0 \to Is_1$, $i = 0,1$, given by $d^0(0) = 1$, and $d^1(0) = 0$, induce inclusion functors

$$\partial^i = W(1 * d^i)\colon W\underline{B} = W(\underline{B}*L_0) \to RW(\underline{B}*Is_1).$$

As in §5 each action $\rho\colon RW(\underline{B}*Is_1) \to REnd(X_0,X_1)$ induces actions ρ_i such that

$$\begin{tikzcd} W\underline{B} \arrow[r, "\partial^i"] \arrow[d, "\rho_i"'] & RW(\underline{B}*Is_1) \arrow[d, "\rho"] \\ End\ X_j \arrow[r, "\partial^i"] & REnd(X_0,X_1) \end{tikzcd}$$

commutes for $i \neq j$, $i,j = 0,1$.

The inclusion functors $u,v\colon L_1 \to Is_1$ given by $u(0) = 1$, $u(1) = 0$, and $v(i) = i$, $i = 0,1$, induce inclusion functors

$$W(1 * u),\ W(1 * v)\colon RW(\underline{B}*L_1) \to RW(\underline{B}*Is_1)$$

<u>Lemma 8.3</u>: Any action ρ: $RW(\underline{B}*Is_1) \to REnd(X,Y)$ induces actions

$$\nu: RW(\underline{B}*L_2) \to REnd(X,Y,X) \qquad \text{and}$$
$$\mu: RW(\underline{B}*L_2) \to REnd(Y,X,Y)$$

such that

$$\nu\circ\partial^0 = \rho\circ W(1 * u) \qquad \mu\circ\partial^0 = \rho\circ W(1 * v)$$
$$\nu\circ\partial^1 = \rho\circ\partial^1\circ s^0 \qquad \mu\circ\partial^1 = \rho\circ\partial^0\circ s^0$$
$$\nu\circ\partial^2 = \rho\circ W(1 * v) \qquad \mu\circ\partial^2 = \rho\circ W(1 * u).$$

In particular, if $\Lambda_1: L_1 \to \underline{B}*L_1$ and $\Lambda_2: Is_1 \to \underline{B}*Is_1$ are the canonical inclusions,

$$\nu\circ\partial^0\circ\iota_{B*L_1}\circ\Lambda_1(0,1) = \rho\circ\iota_{B*Is_1}\circ\Lambda_2(1,0)$$
$$\nu\circ\partial^2\circ\iota_{B*L_1}\circ\Lambda_1(0,1) = \rho\circ\iota_{B*Is_1}\circ\Lambda_2(0,1).$$

Hence the actions $\nu\circ\partial^0$ and $\nu\circ\partial^2$ determine homotopy $\underline{B}$-maps that are s-homotopy inverse to each others.

<u>Proof</u>: Define functors k,l: $L_2 \to Is_1$ by

$$k(i) = 0 \quad i = 0,2 \qquad\qquad l(i) = 1 \quad i = 0,2$$
$$ = 1 \quad i = 1 \qquad\qquad = 0 \quad i = 1 .$$

k and l induce reduced MTP-functors $\varkappa = W(1*k)$ and $\lambda = W(1 * l)$ from $RW(\underline{B}*L_2)$ to $RW(\underline{B}*Is_1)$ which satisfy

$$\varkappa\circ\partial^0 = W(1 * (k\circ\underline{f}^0)) = W(1 * u)$$
$$\varkappa\circ\partial^1 = W(1 * (k\circ\underline{f}^1)) = W(1 * (\underline{f}^1\circ\underline{g}^0)) = \partial^1\circ s^0$$
$$\varkappa\circ\partial^2 = W(1 * (k\circ\underline{f}^2)) = W(1 * v) .$$

For $\underline{f}$ and $\underline{g}$ see p. 82. Similarly for λ we obtain

$\lambda\circ\partial^0 = W(1 * v)$, $\lambda\circ\partial^1 = \partial^0\circ s^0$, $\lambda\circ\partial^2 = W(1 * u)$. Now define $\nu = \rho\circ\varkappa$, $\mu = \rho\circ\lambda$.]]

<u>Lemma 8.4</u>: Suppose $\underline{B}$ has isolated identities. Let $(1_X,\nu)\colon (X,\mu) \to (X,\lambda)$ be a homotopy $\underline{B}$-map. Then $(1_X,\nu)$ is a s-homotopy equivalence.

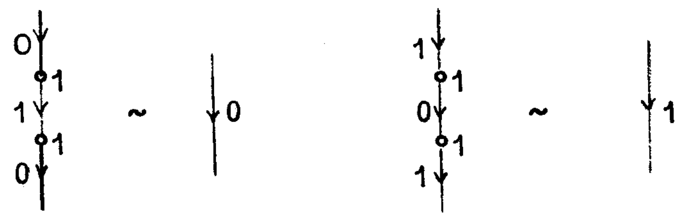
<u>Proof</u>: Let $\underline{C}'$ be the reduced M^2TP-subcategory of $RW(\underline{B}*Is_1)$ generated under $\oplus$ and composition by all those elements the representing trees θ of which are either of the following forms:

(A) In each complete directed edge path of θ the label of the edges changes <u>at most</u> once, and then from 1 to 0.

(B) θ is of the form $\quad \begin{matrix} 0_{\circ 1} \\ 1 \end{matrix}$

(As in §5 and §6 the pictures give the labelling of the edges and <u>not</u> the value of I assigned to them).

The space of representing trees of $\underline{C}'$ is closed in the space of the representing trees of $RW(\underline{B}*Is_1)$. Introduce a relation among the trees of $\underline{C}'$ by

(R) 

and its consequences (i.e. if any such sequence of edges occurs in a tree representing an element of $\underline{C}'$, and if $1 \in I$ is assigned to its incoming and outgoing edge, then this tree may be reduced under (R)). Let $\underline{C}$ be the reduced M^2TP-quotient category of $\underline{C}'$ obtained by factoring out these relations. Using the general construction of p. 33 it is easy to show that the morphism spaces of $\underline{C}$ are in CG.

Define an action $\eta\colon \underline{C} \to \mathrm{REnd}(X,X)$ as follows: Each morphism of $\underline{C}$ can be represented as a composition of sums of elements which are represented by trees of the form (A) or (B), or which are permutations. Define

$$\begin{aligned} \eta\{\theta,\xi,\delta\} &= 1_X \quad \text{if } \theta \text{ is of the form (B)} \\ &= \nu\{\theta,\xi,\delta\} \quad \text{if } \theta \text{ is of the form (A).} \end{aligned}$$

This determines η uniquely on $\underline{C}$. Since $\nu\{\theta,\mathrm{unit},I^0\} = 1_X$ if $\theta = \begin{array}{c} 1\downarrow \\ \circ\, 1 \\ 0\downarrow \end{array}$, η is compatible with the relation (R).

$\varepsilon_{B*Is_1}|\underline{C}'$ induces an augmentation functor

$$\varepsilon\colon \underline{C} \to R(\underline{B}*Is_1)$$

<u>Claim</u>: ε is fibre homotopically trivial.

<u>Proof</u>: Call the vertices with the labels

$$\begin{array}{c} 0\downarrow \\ \circ\, 1 \\ 1\downarrow \end{array} \qquad \text{and} \qquad \begin{array}{c} 1\downarrow \\ \circ\, 1 \\ 0\downarrow \end{array}$$

a g-vertex, resp. an f-vertex, and denote them by $\begin{array}{c} \downarrow \\ g\circ \\ \downarrow \end{array}$ resp.

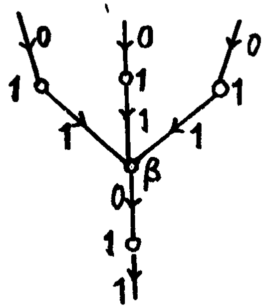
f . Then (R) means $g \circ f = 1$, $f \circ g = 1$, where f and g are the elements of $\underline{C}$ represented by a tree consisting of an f-vertex, resp. a g-vertex only.

The standard section ι_{B*Is_1} induces a section

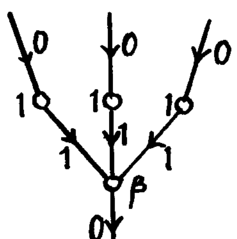
$$\sigma: R(\underline{B}*Is_1)(\underline{n}',\underline{1}) \to \underline{C}(\underline{n}',\underline{1}) .$$

(As usually $\underline{n}$ and $\underline{n}'$ denote the sequences of length n in the generator 0 resp. 1).

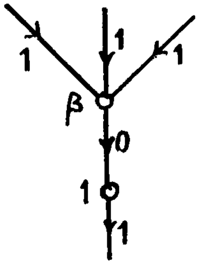
For the other morphism spaces we construct a different section. For $(\beta;(0,1),\ldots,(0,1)) \in R(\underline{B}*Is_1)(\underline{n},\underline{1}')$ define $\sigma(\beta;(0,1),\ldots,(0,1)) = \{\theta,\text{unit},\delta\}$, where θ is the tree with exactly three vertices on each (directed) edge path, labelled by 1, β, 1 in order, the edges change their label after each vertex, and the value 1 is assigned to each link:



Similarly define the section on $R(\underline{B}*Is_1)(\underline{n},\underline{1})$ by such a tree, deleting the g-vertex at the root:



and on $R(\underline{B}*Is_1)(\underline{n}',\underline{1}')$ by deleting the g-vertices at the twigs:



We have four kinds of trees, namely those representing morphisms $\underline{n} \to \underline{1}'$, $\underline{n}' \to \underline{1}$, $\underline{n} \to \underline{1}$, $\underline{n}' \to \underline{1}'$. Using (R) we can choose the representatives such that each represents a composition of elements of the first two kinds. Replace

$\alpha: \underline{n} \to \underline{1}$ by $\alpha \circ f^n \circ g^n: \underline{n} \to \underline{n}' \to \underline{n} \to \underline{1}$

$\beta: \underline{n}' \to \underline{1}'$ by $\beta \circ f \circ g: \underline{n}' \to \underline{1}' \to \underline{1} \to \underline{1}'$.

We furthermore replace $\alpha: \underline{n} \to \underline{1}'$ by

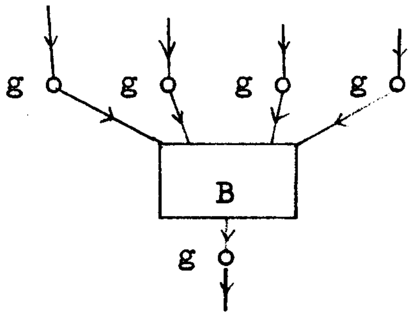
$g \circ f \circ \alpha \circ f^n \circ g^n: \underline{n} \to \underline{n}' \to \underline{n} \to \underline{1}' \to \underline{1} \to \underline{1}'$

Let Y be the space of those representing trees. Since the identities in $\underline{B}$ are isolated, we can assume that none of the representatives in Y can be reduced under the relation (2.13). In addition we can assume that sequences

do not occur in any tree in Y, unless this tree consists of this sequence only.

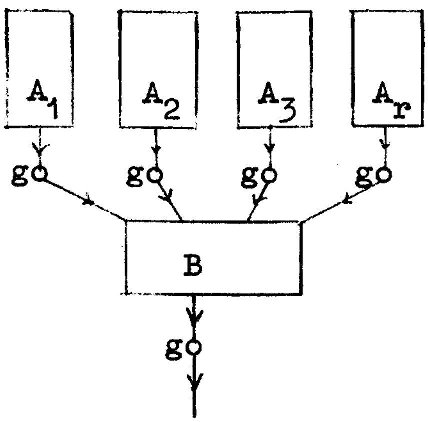
We are now going to construct the equivariant, fibrewise deformation of $C = \underline{C}(\underline{n},\underline{1}')$ into the section. The deformation of the other morphism spaces is constructed analogously and therefore is omitted.

Filter C as follows: F_mC consists of all those elements that can be represented by a point in Y which has at most m g-vertices on any edge path. Then the lowest filtration is two, and each element of F_2C can be represented by a tree



such that B is a subtree that does not have any g-vertex. Deform F_2C into the section by mapping the values t_i of the links in B to $u.t_i$ at the time u, $1 \geq u \geq 0$. At time 0 the tree represents an element in the section.

We now want to deform F_nC strongly into $F_{n-1}C$. Consider a typical representative of F_nC (in Y):



$= \theta$

Here B is a tree which does not contain a g-vertex, and each A_i does not have more than n-1 g-vertices in any edge path (we consider the g-vertices below the A_i as a part of them). Let N be the space of the trees of the form B, and M_i the space of the trees of the form A_i, $i = 1,\ldots,r$. Index the A_i by $1,\ldots,r$ in such a manner that $A_1,\ldots,A_k$ contain an edge path with n-1 g-vertices while $A_{k+1},\ldots,A_r$ do not. Index the twig of B on which A_i sits by i. Let $M = M_1 \times\ldots\times M_r$.

θ can only represent an element of lower filtration if an f-vertex is at a twig of B indexed (not labelled) by $i \in (1,\ldots,k)$, or if we have link combinations

or

which do not include a twig, in $A_1,\ldots,A_k$. t is the value to the particular incoming or outgoing edge of the f-vertex. Call such a link combination a critical sequence with value t, if it is part of a (directed) edge path through n-1 g-vertices in some A_i, and if it does not contain a twig.

Let M_i', $i = 1,\ldots,k$, be the (closed) subspace of M_i of those trees that contain a critical sequence in each edge path that runs through $n-1$ g-vertices. Let $\underline{s}$ be a subsequence of $(1,\dots,k)$. Let $N\underline{s}$ be the (closed) subspace of those trees of N that have an f-vertex on the bottom of the i-th twig for all $i \in \underline{s}$, but not for $i \in (1,\dots,k) - \underline{s}$.

If a tree θ of $Z = N \times M$ represents an element of lower filtration, then θ is in some $N\underline{s}\times R_1\times\dots\times R_k\times M_{k+1}\times\dots\times M_r$, where $R_i = M_i$ if $i \in \underline{s}$ and $R_i = M_i'$ if $i \notin \underline{s}$.

We are now going to deform Z into the subspace of those trees representing an element of lower filtration. We do that by a triple induction: (A) on the number k of trees A_i that contain $n-1$ g-vertices in some edge pa th, (B) on the length of $\underline{s}$, and (C) on the total number of critical sequences in the A_i. Notice that if $\underline{s}$ and $\underline{r}$ are subsequences of $(1,\dots,k)$ such that none is contained in the other, and $\theta \in N\underline{s}$ and $\varphi \in N\underline{r}$ are related under (2.12), then both are related to a tree $\psi \in N\underline{s}\cap\underline{r}$. Hence they are dealt with in an earlier induction step. Similar arguments apply to the other induction sta ges.

Start (A): For $k = 0$, θ (see two pages ago) represents an element of lower filtration.

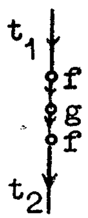
Induction step (A): Suppose we have constructed the deformation for $k-1$.

We first construct a strong deformation retraction into the subspace of those elements that are either of lower filtration, or that can be represented by a tree in $N\underline{i} \times M$, where $\underline{i} = (1,\dots,k)$, or by a tree such that less than k of the A_i's contain a sequence through n-1 g-vertices.

Start (B): $\underline{s} = \emptyset$. Let $\underline{u} = (u_1,\dots,u_p)$ be the collection of values assigned to the links of the tree B labelled by 1, which lie on an edge path starting in a tree A_i, $i \in (1,\dots,k)$.

Start (C): Let q be the total number of critical sequences in $A_1,\dots,A_k$. Let q = 0. Define H to be the deformation which changes the value $u_i \in \underline{u}$ to $t.u_i$ at the time t, $1 \geq t \geq 0$. Then H is well defined and compatible with the previous induction steps. H_1 is the identity.

Induction step (C): Suppose q > 0. Let $\underline{t} = (t_1,\dots,t_q)$ be the collection of values assigned to the incoming, resp. outgoing edges of the f-vertices of the q critical sequences in question, (i.e. the values of the critical sequences). Note that



counts as two critical sequences and that we neglect the values assigned to the links that start or end in a g-vertex (they are always 1).

H is by induction defined on all trees θ for which $\underline{t} \in \partial I^q$. For ~~on~~ the lower faces θ is related to a tree with q-1 critical sequences, and on the upper faces one of the critical sequences can be reduced by the relation (R). Since our aim is to ~~get out of $N_\emptyset \times M$ keeping the elements of $N_{\underline{s}} \times M$ fixed for $\underline{s} \neq \emptyset$~~ deform $N_\emptyset \times M$ into $N_i \times M$, $i = (1,\dots,k)$, compatibly with the deformations of the other spaces $N_{\underline{s}} \times M$, we want to construct a strong deformation retraction

$$I^{p+q} \to I^p \times \partial I^q \cup \underline{0} \times I^q$$

where $\underline{0} = (0,\dots,0) \in I^p$. Since this deformation retraction has to be compatible with relation (2.11) we want it to be symmetric in the coordinates of I^p and I^q (recall p is the number of coordinates of $\underline{u}$). We construct such a deformation later on in the proof.

Induction step (B): Length $\underline{s}$ = m.

Again we induct on q. Let $\underline{v} = (v_1,\dots,v_m)$ be the collection of values that are assigned to the outgoing edges of the m f-vertices at the bottom of the twigs of the tree B indexed by the elements of $\underline{s}$. Let $\underline{t}$ and $\underline{u}$ be as above.

Start (C): q = 0. H is defined exactly on all those trees θ for which $\underline{v} \in \partial I^m$. On the lower faces θ is related to a tree on which H has been defined by induction step $(B, m-1)$, and on the upper ones to a tree on which H has been defined by induction step $(A, k-1)$. Again we want to define a strong deformation retraction, this time for

$$I^{p+m} \rightarrow I^p \times \partial I^m \cup \underline{0} \times I^m$$

which is symmetric in the coordinates of I^p and I^m.

Induction step (C): Suppose $q > 0$. Then H has been defined exactly for $\underline{t} \in \partial I^q$ or $\underline{v} \in \partial I^m$, and hence for $(\underline{t}, \underline{v}) \in \partial I^{q+m}$. Hence again we want a strong deformation retraction

$$I^{p+q+m} \rightarrow I^p \times \partial I^{q+m} \cup \underline{0} \times I^{q+m}$$

which is symmetric in the coordinates of I^p, I^q, and I^m.

This defines H on the whole of $N \times M$. $H_0(N \times M)$ consists of trees that are related to a tree in $N\underline{i} \times M$, where $\underline{i} = (1, \ldots, k)$. We are now going to construct a strong deformation retraction of $N\underline{i} \times M$ into the closed subspace of all those elements which represent an element in C of lower filtration and such that this deformation extends the deformation given by induction step $(A, k-1)$.

Let $\underline{v}$ and $\underline{t}$ be as before.

Start (C): Denote the new deformation by K. By induction (A), K has been defined on those trees for which $\underline{v}$ is in an upper face of I^m. Hence we want a symmetric strong deformation retraction

$$I^m \to UI^m$$

where UI^m is the collection of upper faces of I^m.

Induction step (C): Suppose $q > 0$. Then K is determined on those trees θ for which $\underline{t} \in \partial I^q$ or $\underline{v} \in UI^m$. Hence we want a strong deformation retraction

$$I^{m+q} \to I^m \times \partial I^q \cup UI^m \times I^q$$

which is symmetric in the coordinates of I^m and I^q.

Since all deformations constructed are well defined, continuous, equivariant and fibrewise, the claim is proved if we can find the required deformations

$$F_s : I^{p+r} \to \underline{0} \times I^r \cup I^p \times \partial I^r$$

$$G_s : I^{p+r} \to I^p \times \partial I^r \cup UI^p \times I^r$$

Let $\underline{u} = (u_1, \ldots, u_p)$, $\underline{v} = (v_1, \ldots, v_r)$, $0 \leq u_i, v_j \leq 1$, $i = 1, \ldots, p$, $j = 1, \ldots, r$. Let

$$t(s) = t(s, \underline{u}, \underline{v}) = \min[s, \max(u_1/(2-u_1), \ldots, u_p/(2-u_p)), n(v_1), \ldots, n(v_r)]$$

where $s \in I$ and

$$\begin{aligned} n(v_i) &= v_i/(\tfrac{1}{2}-v_i) && \text{if } v_i < \tfrac{1}{2} \\ &= 1 && \text{if } v_i = 1 \\ &= (1-v_i)/(v_i-\tfrac{1}{2}) && \text{if } v_i > \tfrac{1}{2} \end{aligned} .$$

Note that $t(s, \underline{u}, \underline{v})$ is continuous in s, $\underline{u}$, and $\underline{v}$.

Now define:

$$F_s(\underline{u},\underline{v}) = \{\max[0,u_1+t(s).(u_1-2)],\dots,\max[0,u_p+t(s).(u_p-2)], m_s(v_1),\dots,m_s(v_r)\}$$

where

$$\begin{aligned} m_s(v_i) &= \max[0,v_i+t(s).(v_i-\tfrac{1}{2})] \qquad && \text{if} \quad v_i \leqslant \tfrac{1}{2} \\ &= \min[1,v_i+t(s).(v_i-\tfrac{1}{2})] && \text{if} \quad v_i > \tfrac{1}{2}. \end{aligned}$$

Define

$$q(s)=q(s,\underline{u},\underline{v})= \min[s,(1-u_1)/(2-u_1),\dots,(1-u_p)/(2-u_p),n(v_1),\dots,n(v_r)],$$

where $s \in I$ and $n(v)$ as above. Note that q is continuous in s, $\underline{u}$, $\underline{v}$. Define

$$G_s(\underline{u},\underline{v}) = \{\min[1,q(s).(2-u_1)+u_1],\dots,\min[1,q(s).(2-u_p)+u_p], m_s(v_1),\dots,m_s(v_r)\},$$

where $m_s(v)$ is defined as above, substitute only $t(s)$ by $q(s)$.

F_s, and G_s satisfy our requirements on the deformations.

Let $\underline{D}$ be the subcategory of $RW(\underline{B}*Is_1)$ generated by all those elements that can be represented by a tree of form (A). Since the projection $\underline{C}' \to \underline{C}$ is one-one on the trees of this form, there exists an inclusion functor $\delta\colon \underline{D} \to \underline{C}$. Now apply the reduced version of Theorem 4.9.

We obtain a reduced M^2TP-functor

$$\rho\colon RW(\underline{B}*Is_1) \to \underline{C}$$

extending δ. By the choice of the section we have $\eta\circ\rho\circ\iota_{B*Is_1}\circ\Lambda(i,j) = 1_X$ for all $i,j = 0,1$. Hence $\eta\circ\rho$ provides us with an action which in view of Lemma 8.3 gives the required result.]]

<u>Lemma 8.5</u>: Let $\underline{B}$, $\underline{C}$ be M^1TP-categories and $\underline{D}$ a topological category with n objects. Let $\gamma\colon \underline{B} \to \underline{C}$ be a fibre homotopically trivial M^1TP-functor. Then $\gamma*1\colon \underline{B}*\underline{D} \to \underline{C}*\underline{D}$ is fibre homotopically trivial.

The proof is immediate.]]

<u>Lemma 8.6</u>: Let A be a strong deformation retract of X. $A \xrightarrow{i} X \xrightarrow{p} A$, $p\circ i = 1_A$, and $H_t\colon 1_X \simeq i\circ p$ rel A. If (A,α) is a $W\underline{B}^{\sim}$-space, then X can be made into a $W\underline{B}^{\sim}$-space (X,β) and p and i into s-homotopy equivalences $(p,\pi)\colon (X,\beta) \to (A,\alpha)$, $(i,\gamma)\colon (A,\alpha) \to (X,\beta)$ which are inverse to each other.
If $\underline{B}$ is an M^1TP-category with isolated identities the same holds if we replace $\underline{B}^{\sim}$ by $\underline{B}$.

<u>Proof</u>: For the time being put $\underline{M} = W\underline{B}^{\sim}$. For each $b \in \underline{M}(\underline{m},\underline{n})$ the action α induces a map $\bar{b} = i\circ\alpha(b)\circ p$:

$$X^m \xrightarrow{p} A^m \xrightarrow{\quad \alpha(b) \quad} A^n \xrightarrow{i} X^n \quad .$$

(We write p instead of p^m as long as it is clear what we mean). $\overline{b\circ a} = \bar{b} \circ \bar{a}$, because $p\circ i = 1_A$, but unfortunately $\bar{1} = i\circ p \neq 1_X$. This can be corrected by bringing in the homotopy H_t. Using this data we define a category $\underline{G}$ which acts on REnd(X,A).

Let J be the monoid of Example 3.5, denote the multiplication in J by * and $1 \in J_1$ by u. Using J we are going to construct an M^1TP-category $\underline{C}$, which in addition with $\underline{M}$ gives rise to the category $\underline{G}$.

Let $\underline{C}(\underline{n},\underline{1}) = \underline{M}(\underline{n},\underline{1}) \times J^n$, where J^n is the n-fold product of J, $n \neq 1$. Define $\underline{C}(\underline{1},\underline{1}) = (\underline{M}(\underline{1},\underline{1})\times J \cup J)/\sim$ where the equivalence relation is generated by

$$(1,u*w) \sim (u*w)$$

with $1 \in \underline{M}(\underline{1},\underline{1})$ being the identity of $\underline{M}$. Since the attaching map $f\colon u*J \to \underline{M}(\underline{1},\underline{1})\times J$ given by $f(u*w) = (1,u*w)$ is continuous and since $(J, u*J)$ is a NDR-pair, $\underline{C}(\underline{1},\underline{1})$ is in CG. (Recall: $u*J$ is the image of the upper faces of the cubes I^n under the attaching maps $I^n \to J_{n-1}$ and hence is a subcomplex of the CW-complex J).

Define an action of $S(n)$ on $\underline{C}(\underline{n},\underline{1})$ as follows: Let $a \in \underline{M}(\underline{n},\underline{1})$ and $(v_1,\ldots,v_n) \in J^n$, $\xi \in S(n)$, $n > 1$. Then

$$(a;v_1,\ldots,v_n)\circ\xi = (a\circ\xi;v_{\xi(1)},\ldots,v_{\xi(n)}) .$$

Define $\underline{C}$ now by the normal form construction.

Composition in $\underline{C}$ is given as follows, motivated by the action on $\mathrm{REnd}(X,A)$, see below :

Let $(b;v_1,\ldots,v_n) \in \underline{M}(\underline{n},\underline{1})\times J^n$, $(x_1 \oplus\ldots\oplus x_n)\circ\xi \in \underline{C}(\underline{m},\underline{n})$ with $x_i = (b_i;w_i) \in \underline{M}(\underline{m}_i,\underline{1})\times J^{m_i}$, or $x_i = u_i \in J \subset \underline{C}(\underline{1},\underline{1})$, $i = 1,\ldots,n$. Then

(C1) $(b;v_1,\ldots,v_n) \circ [(x_1 \oplus\ldots\oplus x_n)\circ\xi]$

$= [b\circ(b_1'\oplus\ldots\oplus b_n');w_1'\times\ldots w_n']\circ\xi$

where $(b_i',w_i') = (b_i,w_i)$ if $x_i = (b_i,w_i)$ and

$(b_i',w_i') = (1;v_i * u_i)$ if $x_i = u_i$.

If $v \in J \subset \underline{C}(\underline{1},\underline{1})$ and $w \in J$, then define

(C2) $v\circ(b;v_1,\ldots,v_n) = (b;v_1,\ldots,v_n)$

$v\circ w = v*w$

This definition factors through the relation imposed on $\underline{M}(\underline{1},\underline{1})\times J \cup J$ and hence is well defined. Since it is induced by the compositions in $\underline{M}$ and in J it is continuous and associative. $0 \in J_1 \subset J$ serves as identity. Hence by the normal form construction, $\underline{C}$ is an M^1TP-category.

Let $\underline{G}$ be the reduced M^2TP-category given by

$\underline{G}(\underline{m},\underline{n}) = \underline{C}(\underline{m},\underline{n})$, $\underline{G}(\underline{m}',\underline{n}') = \underline{M}(\underline{m},\underline{n})$, $\underline{G}(\underline{m}',\underline{n}) = \underline{M}(\underline{m},\underline{n})$, and $\underline{G}(\underline{m},\underline{1}') = \underline{C}(\underline{m},\underline{1})$ for $m \neq 1$, and $\underline{G}(\underline{1},\underline{1}') = \underline{M}(\underline{1},\underline{1})\times J \subset \underline{C}(\underline{1},\underline{1})$. Define the remaining morphism spaces by a reduced version of the normal form construction.

To define composition in $\underline{G}$ we embed $\underline{M}$ into $\underline{C}$ by

$$b \to (b;0,\ldots,0) \quad ,$$

$b \in \underline{M}(\underline{n},\underline{1})$. Since

$(b;0,..,0)\circ[(c_1;0,..,0)\oplus\ldots\oplus(c_n;0,..,0)]\circ\xi$

$= (b\circ(c_1\oplus\ldots\oplus c_n)\circ\xi;0,\ldots,0)$

the composition in $\underline{M}$ is "induced" by the one in $\underline{C}$. Hence composition in $\underline{G}$ can now be defined to be the one in $\underline{C}$, and hence is associative and continuous and has identities. Note that $(1;0,\ldots,0)$ serves as identity in $\underline{G}(\underline{1}',\underline{1}')$. It remains to check that for $a \in \underline{G}(\underline{n},\underline{1}')$, $b \in \underline{G}(\underline{m}',\underline{n})$, and $c \in \underline{G}(\underline{n},\underline{1})$, $a\circ b$ and $c\circ b$ are in the subcategory $\underline{M}$ of $\underline{C}$. But this follows immediately from (C1) and (C2).

Define an action η: $\underline{G} \to \mathrm{REnd}(X,A)$ as follows:

$$\eta(b;v_1,\ldots,v_n) = i\circ\alpha(b)\circ p\circ(H_{v_1}\times\ldots\times H_{v_n}) \quad \text{for } (b;v_1,\ldots,v_n) \in \underline{G}(\underline{n},\underline{1})$$

$$\eta(v) = H_v \quad \text{for } v \in J \subset \underline{G}(\underline{1},\underline{1})$$

$$\eta(b;v_1,\ldots,v_n) = \alpha(b)\circ p\circ(H_{v_1}\times\ldots\times H_{v_n}) \quad \text{for } (b;v_1,\ldots,v_n) \in \underline{G}(\underline{n},\underline{1}')$$

$$\eta(b;0,\ldots,0) = \alpha(b) \qquad \text{for } (b;0,\ldots,0) \in \underline{G}(\underline{n}',\underline{1}')$$

$$= i\circ\alpha(b) \qquad \text{for } (b;0,\ldots,0) \in \underline{G}(\underline{n}',\underline{1})$$

where $H_v = H_{t_1} \circ \ldots \circ H_{t_n}$ if $v = (t_1, \ldots, t_n) \in J$.

Since $\underline{G}$ is in normal form as reduced M^2TP-category, η is uniquely determined on the whole of $\underline{G}$. It is continuous and by definition preserves sums and permutations. Since $H_0 = 1_X$, and $H_v \circ i = i$, and $p \circ i = 1_A$ it preserves identities and compositions.

Define an augmentation $\varkappa\colon \underline{G} \to R(\underline{M}{*}Is_1)$ by

$\varkappa(b;v_1,\ldots,v_n) = (b;(i,j),\ldots,(i,j))$ if $(b;v_1,\ldots,v_n) \in \underline{G}(\underline{a},b)$ with $\underline{a} = (i,\ldots,i)$ and $b = j$, and

$\varkappa(v) = (1;(0,0))$ if $v \in J \subset G(\underline{1},\underline{1})$.

$\varkappa$ is well defined because of the normal form of $\underline{G}$. It is continuous, and from the definition of composition in $\underline{C}$ it follows immediately that $\varkappa$ is a reduced M^2TP-functor. Define a section σ of $\varkappa$ by

$$\begin{aligned}(b;(i,j),\ldots,(i,j)) &= (b;u,\ldots,u) \in \underline{G}(\underline{a},j),\ \underline{a} = (i,\ldots,i), \\ &\qquad (i,j) \neq (1,0),(1,1). \\ &= (b;0,\ldots,0) \in \underline{G}(\underline{a},j),\ \underline{a} = (i,\ldots,i), \\ &\qquad (i,j) = (1,0),(1,1).\end{aligned}$$

We have shown (p.47) that $I = J_1$ is a strong deformation retract of J. Hence $u \in J_1$ is a strong deformation retract of J. Applying the product of the deformation of J to $\underline{C}(\underline{n},\underline{1})$ and the identity deformation to $\underline{M}(\underline{n},\underline{1})$, we obtain an equivariant fibrewise deformation of $\underline{G}$ into

$\sigma(R(\underline{M}*Is_1))$. Hence $\varkappa$ is fibre homotopically trivial.

Now resubstitute $\underline{M}$ by $W\underline{B}^{\sim}$. Since $\varepsilon = \varepsilon_{\underline{B}}^{\sim} * 1\colon R(W\underline{B}^{\sim}*Is_1) \to R(\underline{B}*Is_1)$ is fibre homotopically trivial by Lemma 8.5, $\varepsilon\circ\varkappa\colon \underline{G} \to R(\underline{B}*Is_1)$ is fibre homotopically trivial.

Let $r\colon W\underline{B}^{\sim} \to \underline{G}$ be the embedding given by $W\underline{B}^{\sim}(\underline{n},\underline{m}) \to \underline{G}(\underline{n}',\underline{m}')$. Let $\underline{D}$ be the subcategory of $RW(\underline{B}^{\sim}*Is_1)$ given by $\partial^0 W\underline{B}^{\sim}$. Define $\delta\colon \underline{D} \to \underline{G}$ to be the embedding r. δ and $\underline{D}$ satisfy the requirements of Theorem 6.5. Define $\tau_1\colon R(\underline{B}^{\sim}*Is_1)(\underline{1},\underline{1}') \to \underline{G}(\underline{1},\underline{1}')$, $\tau_2\colon R(\underline{B}^{\sim}*Is_1)(\underline{1}',\underline{1}) \to \underline{G}(\underline{1}',\underline{1})$ by

$$\tau_1(b;(0,1)) = \sigma(\iota_{B^{\sim}}b;(0,1))$$

$$\tau_2(b;(1,0)) = \sigma(\iota_{B^{\sim}}b;(1,0))$$

where $\iota_{B^{\sim}}$ is the standard section. By the Theorem 6.5 there exists a reduced M^2TP-functor $\rho\colon RW(\underline{B}^{\sim}*Is_1) \to \underline{G}$ extending δ and such that

$$\eta\circ\rho\circ\iota_{B^{\sim}*Is_1}\circ\Lambda(0,1) = p\circ i\circ p = p \quad \text{and}$$

$$\eta\circ\rho\circ\iota_{B^{\sim}*Is_1}\circ\Lambda(1,0) = i$$

where $\iota_{B^{\sim}*Is_1}$ is the standard section and $\Lambda\colon Is_1 \to R(\underline{B}^{\sim}*Is_1)$ the canonical inclusion.

By lemma 8.3 the lemma is proved putting $\beta = \eta\circ\rho\circ\partial^1$ $\gamma = \eta\circ\rho\circ W(1 * u)$, $\pi = \eta\circ\rho\circ W(1 * v)$.]]

Lemma 8.7: Let A be a strong deformation retract of X.

$A \xrightarrow{i} X \xrightarrow{p} A$, $p \circ i = 1_A$, and suppose $H_t\colon 1_X \simeq i \circ p$ rel A satisfies $H_{t_1} \circ H_{t_2} = H_{\max(t_1,t_2)}$. If (X,ζ) is a $W\underline{B}^{\sim}$-space then A can be made into a $W\underline{B}^{\sim}$-space (A,α) and p and i into s-homotopy equivalences $(p,\pi)\colon (X,\zeta) \to (A,\alpha)$ and $(i,\gamma)\colon (A,\alpha) \to (X,\zeta)$ inverse to each other.

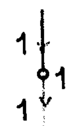
Proof: Put $\underline{M} = W\underline{B}^{\sim}$. The action ζ induces a map

$$\bar{x}\colon A^n \xrightarrow{i} X^n \xrightarrow{\zeta(x)} X^m \xrightarrow{p} A^m$$

for each $x \in \underline{M}(\underline{n},\underline{m})$. Although this time $\bar{1} = 1_A$, we have $\bar{x} \circ \bar{y} \neq \overline{x \circ y}$. We again correct this by bringing in the homotopy H_t. The condition on the homotopy H_t provides us with the condition we need for the degenerate trees.

Let $L(\underline{a},b)$ be the subspace of all those representing trees (θ,ξ,δ) of $RW(\underline{M}*Is_1)(\underline{a},b)$ such that all links of θ are labelled by 1, and θ is not a trivial tree. If any edge of θ is labelled by 0, then it is either the root or a twig. $L(\underline{a},b)$ is closed in the space of the representatives of $RW(\underline{M}*Is_1)(\underline{a},b)$. Hence introducing the relations (2.11), (2.12), (2.13) in $L(\underline{a},b)$ we obtain a space $\underline{G}(\underline{a},b)$ in CG. The composition with permutations on the right is the one induced from the composition of the representing trees with permutations. By applying the reduced version of the normal form construction we obtain morphism spaces into longer sequences. Define composition in $\underline{G}$ as follows: Use the ordinary tree composition but assign the value 1 to the newly created links iff these are labelled by 1. If they are labelled by 0, shrink the new links (see p. 25) to obtain a representative in L. This composition is well defined, continuous, and associative since the composition in $\underline{M}$ is. Again the trees

and



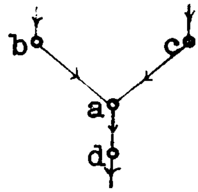
serve as identities. It follows from the tree representation that $\oplus$ is bifunctorial whenever it is defined (see also Lemma 2.20). Hence $\underline{G}$ is a reduced M^2TP-category.

Define an action ν: $\underline{G} \to \mathrm{REnd}(X,A)$ as follows: Given a representative (θ,ξ,δ) of an element in $\underline{G}$. Replace each vertex v labelled by $b \in \underline{M}$ by $H_t \circ \zeta(b)$, where t is the value of the link below v. If v is at the root simply replace it by $\zeta(b)$. Shrinking all links as defined on p. 25, using $\times$ instead of $\oplus$, we obtain maps

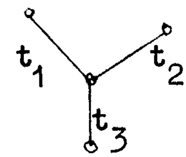
$$m(\theta,\xi,\delta)\colon X^n \xrightarrow{\ \xi\ } X^n \longrightarrow X^m$$

where n is the number of twigs and m the number of roots in θ .

For example



with the values



gives rise to the map $\zeta(d) \circ H_{t_3} \circ \zeta(a) \circ [H_{t_1} \circ \zeta(b) \times H_{t_2} \circ \zeta(c)]$

Define

$$\begin{aligned}
\nu\{\theta,\xi,\delta\} &= p^{\mathbf{m}} \circ \mathbf{m}(\theta,\xi,\delta) && \text{if } \{\theta,\xi,\delta\} \in \underline{G}(\underline{n},\underline{\mathbf{m}}') \\
&= p^{\mathbf{m}} \circ \mathbf{m}(\theta,\xi,\delta) \circ i^{n} && \text{if } \{\theta\ \xi\ \delta\} \in \underline{G}(\underline{n}',\underline{\mathbf{m}}') \\
&= \mathbf{m}(\theta,\xi,\delta) \circ i^{n} && \text{if } \{\theta\ \xi\ \delta\} \in \underline{G}(\underline{n}',\underline{\mathbf{m}}) \\
&= \mathbf{m}(\theta,\xi,\delta) && \text{if } \{\theta\ \xi\ \delta\} \in \underline{G}(\underline{n},\underline{\mathbf{m}})
\end{aligned}$$

Since ζ and H are continuous, $\mathbf{m}$ is continuous. $\mathbf{m}$ factors through the relations since ζ is an M^1TP-functor, $H_0 = 1_X$, and $H_{t_1} \circ H_{t_2} = H_{\max(t_1,t_2)}$. Hence ν is well defined and continuous. Since ζ is an M^1TP-functor ν preserves sums and permutations, and since $p \circ i = 1_A$ it preserves identities. From the definition of composition in $\underline{G}$ it follows immediately that ν is a functor because $H_1 = i \circ p$.

The standard augmentation ε_{M*Is_1} induces an augmentation functor $\varkappa \colon \underline{G} \to R(\underline{M}*Is_1)$. The standard section ι_{M*Is_1} induces a section of $\varkappa$ and the standard deformation of $RW(\underline{M}*Is_1)$ induces a deformation of $\underline{G}$ into the section. Hence $\varkappa$ is fibre homotopically trivial.

Resubstitute $W\underline{B}^{\sim}$ for $\underline{M}$. Since

$$\varepsilon = \varepsilon_{\underline{B}}^{\sim} * 1 \;:\; R(W\underline{B}^{\sim} * Is_1) \to R(\underline{B} * Is_1)$$

is fibre homotopically trivial (Lemma 8.5), so is $\varepsilon \circ \chi$.

Let $\underline{D} = \partial^1 W\underline{B}^{\sim} \subset RW(\underline{B}^{\sim} * Is_1)$. Define $\delta\colon \underline{D} \to \underline{G}$ as follows: For $x \in \partial^1 W\underline{B}^{\sim}(\underline{n},\underline{1})$ let $\delta(x) = \iota_{W\underline{B}^{\sim}}(x) \in \underline{G}(\underline{n},\underline{1})$. Extend δ over $\partial^1 W\underline{B}^{\sim}$ using the normal form. It follows immediately that δ is an MTP-functor. $\underline{D}$ and δ satisfy the requirements of the Universal Theorem. Now define

$$\tau_1\colon R(\underline{B}^{\sim} * Is_1)(\underline{1},\underline{1}') \to \underline{G}(\underline{1},\underline{1}') \quad \text{and}$$
$$\tau_2\colon R(\underline{B}^{\sim} * Is_1)(\underline{1}',\underline{1}) \to \underline{G}(\underline{1}',\underline{1}) \quad \text{by}$$

$$\tau_1(b;(0,1)) = \{(\iota_{\underline{B}^{\sim}} b;(0,1)), \text{unit}, I^0\}$$
$$\tau_2(b;(1,0)) = \{(\iota_{\underline{B}^{\sim}} b;(1,0)), \text{unit}, I^0\} \; .$$

τ_1 and τ_2 satisfy the requirements of Lemma 4.8. Hence there exists a functor $\rho\colon RW(\underline{B}^{\sim} * Is_1) \to \underline{G}$ extending δ and such that

$$\nu \circ \rho \circ \iota_{\underline{B}^{\sim} * Is_1} \circ \Lambda(0,1) = p \quad \text{and} \quad \nu \circ \rho \circ \iota_{\underline{B}^{\sim} * Is_1} \circ \Lambda(1,0) = i \; .$$

From the tree representation and the choice of $\delta |\, \partial^1 W\underline{B}^{\sim}$ it follows that $\nu \circ \rho \circ \partial^1 = \zeta$. Let $\alpha = \nu \circ \rho \circ \partial^0$. Then by Lemma 8.3, putting $\gamma = \nu \circ \rho \circ W(1 * u)$ and $\pi = \nu \circ \rho \circ W(1 * v)$, we obtain homotopy $\underline{B}^{\sim}$-maps $(p,\pi)\colon (X,\zeta) \to (A,\alpha)$ and $(i,\gamma)\colon (A,\alpha) \to (X,\zeta)$, which are s-homotopy equivalences inverse to each other.]]

<u>Remark</u>: If $\underline{B}$ is an M^1TP-category with isolated identities then Lemma 8.7 holds also if we replace $\underline{B}^\sim$ by $\underline{B}$.

<u>Proof of the Theorems 8.1 and 8.2</u>:

Let (X,α) be a $W\underline{B}^\sim$-space (or a $W\underline{B}$-space and $\underline{B}$ has isolated identities), and $f\colon X \to Y$ be a homotopy equivalence. Let M be the mapping cylinder of f, $M = (X\times I \cup Y)/[(x,1)\sim fx]$. Let $i\colon X \to M$ and $j\colon Y \to M$ be the natural inclusions, and $p\colon M \to Y$ the natural projection. Define $H\colon 1_M \simeq j\circ p$ by

$$H_u(x,t) = (x,\max(u,t))$$
$$H_u(y) \quad = y$$

Then $H_{u_1}\circ H_{u_2} = H_{\max(u_1,u_2)}$. Since f is a homotopy equivalence, $i(X)$ is a strong deformation retract of M (see Appendix). In the following diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(X,\alpha) \arrow[rr, "{(f,\rho)}"] \arrow[d, shift left, "{(i,\gamma)}"] & & (Y,\beta) \arrow[d, shift right, "{(1_Y,\omega)}"'] \\
(M,\alpha^*) \arrow[u, shift left, "{(k,\delta)}"] \arrow[urr, "{(p,\lambda)}"] \arrow[rr, shift left, "{(p,\varkappa)}"] & & (Y,\alpha^{**}) \arrow[u, shift right, "{(1_Y,\mu)}"'] \arrow[ll, shift left, "{(j,\nu)}"]
\end{tikzcd}$$

let $k\colon M \to X$ be the retraction, α^* the $W\underline{B}^\sim$-structure induced on M by α and (k,δ), (i,γ) the s-homotopy equivalences given by Lemma 8.6. Let α^{**} be the $W\underline{B}^\sim$-structure induced on Y by α^* and $(p,\varkappa)$, (j,ν) the s-homotopy equivalences given by Lemma 8.7. The composite $(p,\varkappa)\circ(i,\gamma) \simeq (p\circ i,\chi) = (f,\chi)$ is a s-homotopy equivalence

$$(f,\chi)\colon (X,\alpha) \to (Y,\alpha^{**}) \ ,$$

which proves Theorem 8.2 .

Now suppose f is given as a homotopy $\underline{B}$-map (f,ρ) and the identities of $\underline{B}$ are isolated. Since $p \simeq f\circ k$, and since $(f,\rho)\circ(k,\delta)$ is a homotopy $\underline{B}$-map, there exists an action λ such that $(p,\lambda) \simeq (f,\rho)\circ(k,\delta)$ by Theorem 7.21. Define $(1_Y,\mu)$ to be the composite $(p,\lambda)\circ(j,\nu)$. By Lemma 8.4 there exists a s-homotopy inverse $(1_Y,\omega)$ of $(1_Y,\mu)$. Now

$$\begin{aligned}(f,\rho)\circ(k,\delta)\circ(j,\nu)\circ(1_Y,\omega) &\simeq (p,\lambda)\circ(j,\nu)\circ(1_Y,\omega)\\ &\simeq (1_Y,\mu)\circ(1_Y,\omega)\\ &\ (1_Y)_*\end{aligned}$$

$$\begin{aligned}&(k,\delta)\circ(j,\nu)\circ(1_Y,\omega)\circ(f,\rho)\\ &\quad\simeq (k,\delta)\circ(j,\nu)\circ(1_Y,\omega)\circ(f,\rho)\circ(k,\delta)\circ(i,\gamma)\\ &\quad\simeq (k,\delta)\circ(j,\nu)\circ(1_Y,\omega)\circ(p,\lambda)\circ(i,\gamma)\\ &\quad\simeq (k,\delta)\circ(j,\nu)\circ(1_Y,\omega)\circ(p,\lambda)\circ(j,\nu)\circ(p,\varkappa)\circ(i,\gamma)\\ &\quad\simeq (k,\delta)\circ(j,\nu)\circ(p,\varkappa)\circ(i,\gamma)\\ &\quad\simeq (k,\delta)\circ(i,\gamma)\\ &\quad\simeq (1_A)_* \quad .\end{aligned}$$

Hence $(k,\delta)\circ(j,\nu)\circ(1_Y,\omega)$ is a s-homotopy inverse of (f,ρ).]]

CHAPTER IV: STRUCTURE THEORY

§ 9 STRUCTURE THEORY I

Throughout this chapter we assume that $\underline{B}$ is an M^1T-category (so without permutations), such that $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ is a NDR-pair. We choose an M^1T-category instead of an M^1TP-category, because the proofs are then slightly simpler. A refinement of the methods used in this chapter and the use of equivariant NDR's as studied in §7 should give the same results for categories with permutations.

Again we denote the sequences of length n in the object generators 0 or 1 of $RW(\underline{B}*L_1)$ by $\underline{n}$, resp. $\underline{n}'$. Denote $RW(\underline{B}*L_1)(\underline{m},\underline{1}')$ by $C_{\underline{m}}$, and regard $W\underline{B}$ embedded in $RW(\underline{B}*L_1)$ by ∂^0 and ∂^1, so that composition of elements of $RW(\underline{B}*L_1)$ with elements of $W\underline{B}$ makes sense.

For each $W\underline{B}$-space (X,γ) we are going to construct a $W\underline{B}$-space UX, a $\underline{B}$-space MX, which is a quotient of UX, and a $\underline{B}$-space NX, which is a subspace of both UX and MX. All three spaces have the same homotopy type as X. In addition UX and MX satisfy certain universal properties with regard to homotopy $\underline{B}$-maps.

Let $\beta \in W\underline{B}(\underline{n},\underline{m})$, and (X,γ) be a $W\underline{B}$-space. We denote $\gamma(\beta)(x_1,\ldots,x_n)$ by $\beta.(x_1,\ldots,x_n)$ or simply by $\beta.\underline{x}$.

Let $K_{p,q} = Tp(\underline{q},\underline{1}')\times I^p \times X^q$, where $Tp = Tp(\underline{B}*L_1)$, (see p. 26). Let K_p be the disjoint union $K_p = \cup_{q=0}^{\infty} K_{p,q}$, and let K be the disjoint union $K = \cup_{p=-1}^{\infty} K_p$.

Introduce an equivalence relation on K by

(9.1) $(\theta,\delta,\underline{x}) \sim (\varphi,\partial,\underline{x})$ if $\{\theta,\delta\} = \{\varphi,\partial\}$, where $\{x\}$ denotes the equivalence class of x in $RW(\underline{B}*L_1)$, i.e. if $(\theta,\delta) \sim (\varphi,\partial)$ under (2.12) and (2.13). ((2.11) does not apply).

(9.2) Suppose θ in (θ,δ) has the value 1 assigned to a link labelled by 0. Let (φ_1,δ_1) and (φ_2,δ_2) be obtained by chopping this link (see p. 50). Then $(\theta,\delta,\underline{x}) \sim (\varphi_1,\delta_1,\{\varphi_2,\delta_2\}\cdot\underline{x})$.

In (9.2) it suffices to restrict our attention to non-degenerate elements (θ,δ). Hence after having factored out (9.1), the relation (9.2) reads:

(9.2)* $\qquad (c\circ\beta,\underline{x}) \sim (c,\beta\cdot\underline{x})$,

where $c \in C_{\underline{m}}$, and $\beta \in W\underline{B}(\underline{n},\underline{m})$.

Let $UX = K/\sim$, $\pi\colon K \to UX$ the projection, $U_pX = \pi(\cup_{-1}^{p}K_i)$ and $\pi_p = \pi| \cup_{-1}^{p}K_i$. Note that $U_0X = K_0$.

Call $(\theta,\delta,\underline{x})$ degenerate if

(A1) (θ,δ) is degenerate (see p. 31)

(A2) the value $1 \in I$ is assigned to a link of θ labelled by 0 .

Denote the closed subspace of the degenerate points of $K_{p,q}$ by $DK_{p,q}$ and of K_p by DK_p. DK_p consists of exactly those points of K_p that are related to a point in some K_r with $r < p$. Note that if $x,y \in K_p - DK_p$, $x \sim y$, then $x = y$.

<u>Claim</u>: Each $(\theta,\delta,\underline{x}) \in K_p$ is related to a unique non-degenerate point.

<u>Proof</u>: Let λ be the function associating with (θ,δ) a unique non-degenerate related point (Lemma 2.14). From the tree representation it follows that (θ,δ) can be decomposed uniquely into $(\varphi_1,\delta_1)\circ(\varphi_2,\delta_2)$, such that the value 1 is not assigned to any link in φ_1 labelled by 0. Define $\rho(\theta,\delta,\underline{x}) = (\varphi_1,\delta_1,\{\varphi_2,\delta_2\}\cdot\underline{x})$. The correspondence $(\theta,\delta,\underline{x}) \to \rho(\lambda(\theta,\delta),\underline{x})$ associates with each element of $K_{p,q}$ a related non-degenerate one. It preserves non-degenerate elements and factors through the relations (9.1) and (9.2), which proves the claim.

Let $Y_{p,q} \subset K_{p,q}$ be the (closed) subspace of all those points $(\theta,\delta,\underline{x})$, for which (θ,δ) is degenerate. Let $Z_{\alpha,p,q} \subset Tp(\underline{q},\underline{1}')\times I^p$ be the subspa ce of all those trees of one type for which the value 1 is assigned to a particular link labelled by 0. Chopping this link induces a projection $\varkappa\colon Z_{\alpha,p,q} \to Z'_{\alpha,p,q} \times Z''_{\alpha,p,q}$. $DK_{p,q}$ is a finite union of spaces $Y_{p,q}$ and $Z_{\alpha,p,q}$. By Lemma 2.15 we have continuous maps $f\colon Y_{p,q} \to U_{p-1}X$ for all p and q. Since

$$Z_{\alpha,p,q} \times X^q \xrightarrow{\varkappa\times 1} Z'_{\alpha,p,q} \times Z''_{\alpha,p,q} \times X^q \xrightarrow{1\times\text{action}}$$

$$Z'_{\alpha,p,q} \times (\{Z''_{\alpha,p,q}\} \cdot X^q) \xrightarrow{\pi_{p-1}} U_{p-1}X$$

is continuous, the conditions (1), (2), (3) of p. 33 are satisfied. Since $(I, \partial I)$, $(I, 0)$, and $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ are NDR-pairs, $(K_{p,q}, DK_{p,q})$ and hence (K_p, DK_p) are NDR-pairs. Hence by the construction of p. 33 we obtain

__Lemma 9.3__: (a) UX, U_pX are in CG, p=0,1,2,...

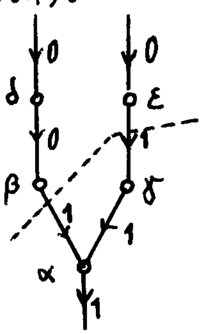
(b) UX is the direct limit of $U_0X \subset U_1X \subset \ldots$

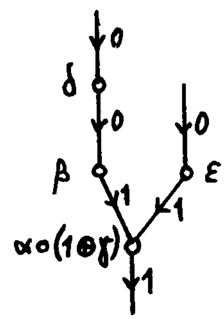
(c) (UX, U_pX), $(U_{p+1}X, U_pX)$ are NDR-pairs for all $p \geqslant 0$.]]

To construct MX, we introduce a further relation in K, which is independent of (9.1) and (9.2). Hence MX is a quotient of UX.

(9.4) Suppose θ in (θ,δ) has the value 1 assigned to a collection of links labelled by 1, which separates the tree θ into a tree φ the edges of which are labelled by 1 only, and into a copse ψ . Let (θ',δ') be the pair obtained from (θ,δ) by shrinking (see p. 25) all links in φ . Then $(\theta,\delta,\underline{x}) \sim (\theta',\delta',\underline{x})$.

For example, suppose that in the following picture the links on the separating line have the value 1 assigned to them, while the values of the links above the line are the same. Then the elements $(\theta,\delta,\underline{x})$ and $(\varphi,\partial,\underline{x})$ are related under (9.4).

$\theta =$ 

$\varphi =$ 

Let $MX = K/\sim$, ω: $K \to MX$ be the projection, $M_pX = \omega(\cup_{-1}^{p} K_i)$ and $\omega_p = \omega| \cup_{-1}^{p} K_i$. Note that $M_0X = K_0$.

Call $(\theta,\delta,\underline{x})$ degenerate, if it satisfies (A1) or (A2) of p. 155, or if

(A3) the value 1 is assigned to a separating collection of links labelled by 1, and chopping these links decomposes (θ,δ) into a tree at least one link and a copse.

Let $RK_{p,q}$ be the subspace of the degenera te points of $K_{p,q}$, and RK_p of those of K_p. RK_p consists of exactly those points of K_p , that are related to a point in some

K_r , $r < p$. Notice that if $x,y \in K_p - RK_p$ and $x \sim y$, then $x = y$.

Suppose (θ,δ) is degenerate under (A3). From the tree representation it follows, that we can decompose (θ,δ) into $(\varphi_1,\delta_1)\circ(\varphi_2,\delta_2)$ such that the edges of φ_1 are labelled by 1 only, and (φ_2,δ_2) is not degenerate under (A3). Let $(\theta,\delta)^*$ be the pair obtained by substituting the values of the links in θ which come from φ_1 by 0. Then $(\theta,\delta)^*$ is not degenerate under (A3). The correspondence $(\theta,\delta,\underline{x}) \to \rho(\lambda[(\theta,\delta)^*],\underline{x})$ associates with each element of K_p a unique non-degenerate related one. Let $C_{\alpha,p,q}$ be the subspace of $Tp(\underline{q},\underline{1}')\times I^p$ consisting of one type of trees, such that the value 1 is assigned to a separating collection of links labelled by 1 such that the tree below this separating collection has at least one link. Let $x \in C_{\alpha,p,q}$. The correspondence $x \to x^*$ as defined above, induces a continuous map g of C into some $Tr(\underline{q},\underline{1}')\times I^r$ with $r < p$. Hence the composite $\omega_{p-1}\circ(g\times 1)$: $C_{\alpha,p,q}\times X^q \to M_{p-1}X$ is continuous. By the consideration of p. 157 we furthermore have continuous maps

$$Y_{p,q} \to M_{p-1}X$$
$$Z_{\alpha,p,q} \to M_{p-1}X \ .$$

Since $RK_{p,q}$ is a finite union of spaces $Y_{p,q}$, $Z_{\alpha,p,q}$, and $C_{\alpha,p,q}$ the conditions (1), (2), (3) of p.33 are satisfied. Using [6; Lemma 7.3] again, we find that $RK_{p,q}$ is a NDR in $K_{p,q}$. Hence by the construction of p.33 we get:

<u>Lemma 9.5</u>: (a) MX, M_pX are in CG, $p = 0,1,2,\ldots$

(b) MX is the direct limit of $M_0X \subset M_1X \subset \ldots$

(c) (MX, M_pX), $(M_{p+1}X, M_pX)$ are NDR-pairs for all $p \geq 0$.]]

NX is the subspace of MX and UX represented by all points $(\theta,\delta,\underline{x})$ of K, such that all edges of θ with exception of the root are labelled by 0. On this set of representatives the relations defining MX and UX coincide. Hence, if ζ: UX $\rightarrow$ MX is the projection induced by the relation (9.4), $\zeta|$ NX is the identity.

If $(\theta,\delta,\underline{x}) \in K_p$ represents an element of NX, then so does $\rho(\lambda(\theta,\delta),\underline{x})$. Furthermore if NK_p is the subspace of those elements in K_p that represent an element of NX, then $DK_p \cap NK_p$ is a NDR in NK_p. Hence NX is in CG.

<u>Definition 9.6</u>: Let (Z,δ) be a $\underline{B}$-space. Then the $W\underline{B}$-structure on Z given by $\delta\circ\varepsilon_{\underline{B}}$: $W\underline{B} \rightarrow \mathrm{End}Z$ is called the $W\underline{B}$-structure on Z <u>induced by δ</u> .

<u>Lemma 9.7</u>: (a) UX is a $W\underline{B}$-space (UX,χ) and there exists a homotopy $\underline{B}$-map $(u,\mu)\colon (X,\gamma) \to (UX,\chi)$.

(b) MX is a $\underline{B}$-space $(MX,\varkappa)$ and there exists a homotopy $\underline{B}$-map $(m,\nu)\colon (X,\gamma) \to (MX,\varkappa^*)$, where $\varkappa^*$ is the $W\underline{B}$-structure on MX induced by $\varkappa$.

(c) NX is a $\underline{B}$-space .

(d) $\zeta\colon (UX,\chi) \to (MX,\varkappa^*)$ is a $W\underline{B}$-homomorphism .

<u>Proof</u>: We use the relation (9.2)*. Let $\alpha \in W\underline{B}(\underline{n},\underline{1})$, and let $y_i \in U_{p_i}X$ be represented by $(c_i,\underline{x}_i) \in C_{\underline{m}_i} \times X^{\underline{m}_i}$, $i=1,\ldots,n$. (Recall that $C_{\underline{m}} = RW(\underline{B}*L_1)(\underline{m},\underline{1}'))$. Define
$$\alpha\cdot(y_1,\ldots,y_n) = \{\alpha\circ(c_1\oplus\ldots\oplus c_n),\underline{x}_1\times\ldots\times\underline{x}_n\} \in U_{p_1+\ldots+p_n}X$$
Extend this definition to actions of $\alpha' \in W\underline{B}(\underline{n},\underline{m})$ using the normal form of $W\underline{B}$, and taking the m-fold product of the above definition. Since $(UX)^n$ is filtered by $(UX)^n_p = \bigcup_{p_1+,\ldots+p_n=p}(U^{p_1}X \times\ldots\times U^{p_n}X)$ and since the topology of UX is the quotient topology from the disjoint union of the $C_q \times X^q$ under (9.2)*, this defines a continuous action χ of $W\underline{B}$ on UX. To define the action $\mu\colon RW(\underline{B}*L_1) \to REnd(X,UX)$ it suffices to define the action of elements of $C_{\underline{m}}$ compatibly with the action γ on X and χ on UX. Let $\beta \in C_q$. Define

$$\beta.(x_1,\dots,x_q) = \{\beta; x_1,\dots,x_q\} \quad .$$

Use the normal form of $RW(\underline{B}*L_1)$ to extend this definition over the whole of $RW(\underline{B}*L_1)$. It clearly is compatible with the action χ. For $\alpha \in W\underline{B}(\underline{n},\underline{m})$ and $\beta \in C_m$ we have

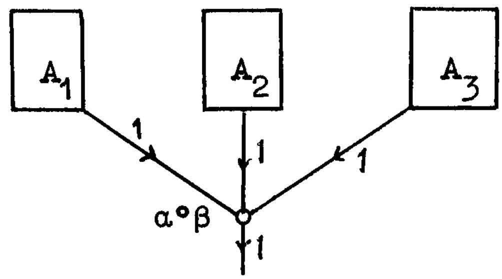
$$\begin{aligned}(\beta\circ\alpha).(x_1,\dots,x_n) &= \{\beta\circ\alpha; x_1,\dots,x_n\} \\ &= \{\beta; \alpha.(x_1,\dots,x_n)\} \qquad \text{by } (9.2)^* \\ &= \beta.(\alpha.(x_1,\dots,x_n)) \ .\end{aligned}$$

Hence μ is a continuous functor. By definition it preserves sums. Hence it is a reduced M^2T-functor, extending γ and $\dot{\chi}$. $u = \mu\circ\iota_{\underline{B}*L_1}\circ\Lambda(0,1)$ is given by $u(x) = \{\iota_{\underline{B}*L_1}(1;(0,1));x\}$. Note that $u\colon X \to UX$ is an inclusion ($u(X)$ is closed in UX).

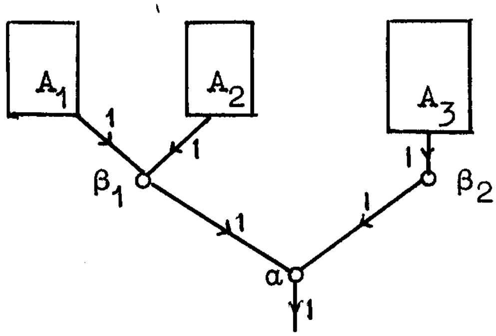
(b) Since MX is a quotient of UX, it is also a quotient of the disjoint union of the $C_q \times X^q$. Let $\beta \in \underline{B}(\underline{n},\underline{1})$, and let $y_i \in M_{p_i}X$ be represented by $(c_i,\underline{x}_i) \in C_{q_i}\times X^{q_i}$. Define the action $\varkappa$ by

$$\beta.(y_1,\dots,y_n) = \{(\iota_{\underline{B}}\beta)\circ(c_1\oplus\dots\oplus c_n); \underline{x}_1\times\dots\times\underline{x}_n\}$$

where $\{x\}$ as usually denotes the equivalence class of x. Using the normal form of $\underline{B}$, we can extend this definition uniquely over the whole of $\underline{B}$. The relation (9.4) assures that $\varkappa$ is a functor. For if $\alpha \in \underline{B}(\underline{m},\underline{1})$ and $\beta \in \underline{B}(\underline{n},\underline{m})$, $\beta = \beta_1\oplus\dots\oplus\beta_m$, then $(\alpha\circ\beta).(y_1,\dots,y_n)$ is represented by $(\theta,\delta,\underline{x}_1\times\dots\times\underline{x}_n)$, where θ is the tree with the vertex at the root labelled by $(\alpha\circ\beta)$, the representing trees $A_1,\ldots,A_n$ of $c_1,\ldots,c_n$ on its incoming edges, and the value 1 assigned to each of these edges,

$\theta =$ 

while $\alpha.(\beta.(y_1,\ldots,y_n))$ is represented by $(\varphi,\partial,\underline{x}_1\times\ldots\times\underline{x}_n)$, where φ is the tree with the vertex α at the root, the vertices on top of its incoming edges labelled by $\beta_1,\ldots,\beta_m$, and the trees $A_1,\ldots,A_n$ sitting on the incoming edges of the β_i's.

$\varphi =$ 

The value 1 is assigned to the links ending in some β_i or in α. By relation (9.4) we can shrink the links below the vertices β_i. But then we obtain the representative for $(\alpha\circ\beta).(y_1,\ldots,y_n)$.

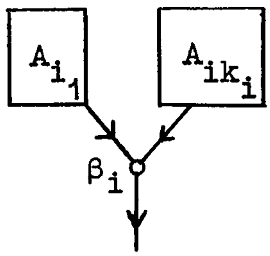
Since the filtration of MX induces the one of $(MX)^n$, $\varkappa$ is an M^1T-functor.

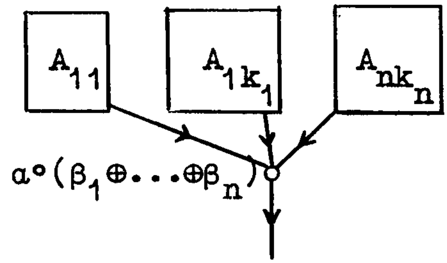
To construct the homotopy $\underline{B}$-map $(m,\nu)\colon (X,\gamma) \to (MX,\varkappa^*)$, we have to extend the actions γ and $\varkappa^*$ of $\partial^i W\underline{B} \subset RW(\underline{B}*L_1)$, $i = 0,1$, over the whole of $RW(\underline{B}*L_1)$. Let $\beta \in C_n$. Define ν by

$$\beta.(x_1,\dots,x_n) = \{\beta;x_1,\dots,x_n\}$$

and extend this to an action of $RW(\underline{B}*L_1)$ using the normal form. As in part (a) it follows that ν extends γ and $\varkappa^*$, and is functorial. Hence ν is a reduced M^2T-functor. $m = \nu\circ\iota_{\underline{B}*L_1}\circ\Lambda(0,1)$ is given by $m(x) = \{\iota_{\underline{B}*L_1}(1;(0,1));x\}$. Note that $m\colon X \to MX$ is an inclusion.

(c) The $\underline{B}$-structure λ of NX is defined on representatives as follows: Let $\alpha \in \underline{B}(\underline{n},\underline{1})$, and let $(\theta_i,\delta_i,\underline{x}_i) \in K_{p_i}$, $i = 1,\dots,n$, be representatives of elements in NX. Each edge of θ_i with exception of the root is labelled by 0. Let β_i be the label of the vertex at the root of θ_i, and let $A_{i1},\dots,A_{ik_i}$ be the subtrees of θ_i sitting on the incoming edges of β_i. Let φ be the tree with the vertex at the root labelled by $\alpha\circ(\beta_1\oplus\dots\oplus\beta_n)$, and the trees $A_{11},\dots,A_{1k_1},\dots,A_{n1},\dots,A_{nk_n}$ sitting on its incoming edges. Assign the values of the links of the θ_i's to the links of φ (the incoming edges of $\alpha\circ(\beta_1\oplus\ldots\oplus\beta_n)$ have the values of the incoming edges of the β_i in θ_i).

θ_i = 

φ = 

Let (φ,∂) be the pair thus obtained. Define λ by

$$\alpha.(\{\theta_1,\delta_1,\underline{x}_1\},\ldots,\{\theta_n,\delta_n,\underline{x}_n\}) = \{\varphi,\partial,\underline{x}_1\times\ldots\times\underline{x}_n\} \ .$$

From the tree representation it is clear that λ is an M^1T-functor.

(d) Let $\alpha \in W\underline{B}(\underline{n},\underline{1})$, and let $y_i \in UX$ be represented by $(c_i,\underline{x}_i) \in C_{\mathbf{m}_i}\times X^{\mathbf{m}_i}$. Then

$$\alpha.(y_1,\ldots,y_n) = \{\alpha\circ(c_1\oplus\ldots\oplus c_n),\underline{x}_1\times\ldots\times\underline{x}_n\}$$

Under relation (9.4) the representative $[\alpha\circ(c_1\oplus\ldots\oplus c_n),\underline{x}_1\times\ldots\times\underline{x}_n]$ of MX is related to $[(\iota_{\underline{B}}\circ\varepsilon_{\underline{B}}(\alpha))\circ(c_1\oplus\ldots\oplus c_n);\underline{x}_1\times\ldots\times\underline{x}_n]$, which represents $\varepsilon_{\underline{B}}(\alpha).(\{y_1\},\ldots,\{y_n\})$, where $\{y_i\}$ is the equivalence class of y_i in MX. Hence, since $W\underline{B}$ is in normal form, ζ is a $W\underline{B}$-homomorphism.]]

<u>Remark 9.8</u>: (1) From the tree representation it follows immediately that the homotopy $\underline{B}$-map

$(m,\nu):(X,\gamma) \to (MX,\varkappa^*)$ is the canonical composite of the $W\underline{B}$-homomorphism ζ with the homotopy $\underline{B}$-map (u,μ): $(X,\gamma) \to (UX,\chi)$.

(2) $u(X)$ and $m(X)$ are subspaces of NX. Hence u and m factor into $X \subset NX \subset UX$ and $X \subset NX \subset MX$. The images of u and m in NX agree.

(3) We can construct $W\underline{B}$-homomorphisms $(UX,\chi) \to (NX,\lambda^*)$, where λ^* is the $W\underline{B}$-structure on NX induced by λ, and a $\underline{B}$-homomorphism $(MX,\varkappa) \to (NX,\lambda)$. Since we do not use them we refrain from giving the definitions.

<u>Theorem 9.9</u>:

(a) Each homotopy $\underline{B}$-map (f,ρ): $(X,\gamma) \to (Y,\delta)$ factors uniquely as $(f,\rho) = Uf\circ(u,\mu)$, where Uf: $(UX,\chi) \to (Y,\delta)$ is a $W\underline{B}$-homomorphism and $Uf\circ(u,\mu)$ the canonical composite. Further, Uf is a continuous function of (f,ρ).

(b) Let (Z,η) be a $\underline{B}$-space and η^* the $W\underline{B}$-structure on Z induced by η. Then each homotopy $\underline{B}$-map (f,ρ): $(X,\gamma) \to (Z,\eta^*)$ factors uniquely as $(f,\rho) = Mf\circ(m,\nu)$, where Mf: $(MX,\varkappa) \to (Z,\eta)$ is a $\underline{B}$-homomorphism (and hence a $W\underline{B}$-homomorphism

$(MX,\varkappa^*) \to (Z,\eta^*))$ and $Mf\circ(m,\nu)$ the canonical composite. Furthermore, Mf is a continuous function of (f,ρ).

Proof: ρ induces maps $f_n\colon C_n\times X^n \to Y$, which in turn determine maps $h_{n,p}\colon Tp(\underline{n},\underline{1}')\times I^p\times X^n \xrightarrow[\chi\times 1]{} C_n\times X^n \xrightarrow[f_n]{} Y$, where

χ is the characteristic map for $Tp(\underline{n},\underline{1}')\times I^p$. μ is induced by the identities $Tp(\underline{n},\underline{1}')\times I^p\times X^n = K_{p,n}$. Hence if Uf with the required properties exists, then Uf must be induced by the collection of the $h_{n,p}$'s, $h_{n,p}\colon K_{p,n} \to Y$. Since ρ is a functor, $h_{n,p}$ respects the relations (9.1) and (9.2). Hence it indeed induces a map $Uf\colon UX \to Y$, and the part (a) is proved if we can show, that Uf is a $W\underline{B}$-homomorphism: Let $\beta \in W\underline{B}(\underline{n},\underline{1})$ be represented by (θ,δ) in $Tp(\underline{n},\underline{1})\times I^p$, and $y_i \in UX$ by $(\varphi_i,\delta_i,\underline{x}_i) \in K_{p_i,q_i}$, $i=1,\dots,n$.

Then

$Uf[\beta.(y_1,\dots,y_n)]$

$= h_{m,r}[(\theta,\delta)\circ[(\varphi_1,\delta_1)\oplus\dots\oplus(\varphi_n,\delta_n)];\underline{x}_1\times\dots\times\underline{x}_n]$ some r,m

$= f_m(\{\theta,\delta\}\circ[\{\varphi_1,\delta_1\}\oplus\dots\oplus\{\varphi_n,\delta_n\}];\underline{x}_1\times\dots\times\underline{x}_n)$

$= \{\theta,\delta\}.[f_{q_1}(\{\varphi_1,\delta_1\};\underline{x}_1)\times\dots\times f_{q_n}(\{\varphi_n,\delta_n\},\underline{x}_n)]$ since f_m is induced by an action

$= \beta.(Uf(y_1)\times\dots\times Uf(y_n))$.

From the definition it is obvious that Uf is a continuous function of (f,ρ).

(b) ρ induces maps $f_n\colon C_n \times X^n \to Z$, which in turn induce maps $h_{p,n}\colon K_{p,n} = Tp(\underline{n},\underline{1}') \times I^p \times X^n \to C_n \times X^n \xrightarrow[f_n]{} Z$. If Mf exists it must be induced by the collection of maps $h_{p,n}$. Since Z is a $\underline{B}$-space and ρ an action, $h_{p,n}$ respects (9.2) and (9.4). By definition it respects (9.1). Hence the collection of the $h_{p,n}$ indeed induces a map $Mf\colon MX \to Z$. It remains to show that Mf is a $\underline{B}$-homomorphism. Let $\beta \in \underline{B}(\underline{n},\underline{1})$, and let $y_i \in MX$ be represented by $(\varphi_i, \delta_i, \underline{x}_i)$ in K_{p_i,q_i}, $i = 1,\ldots,n$. Then

$Mf[\beta.(y_1,\ldots,y_n)]$

$= h_{r,m}[\iota_{\underline{B}}(\beta;(1,1))\circ[(\varphi_1,\delta_1)\oplus\ldots\oplus(\varphi_n,\delta_n)], \underline{x}_1 \times \circ\circ\circ \times \underline{x}_n]$

$= f_m(\iota_{\underline{B}} \circ[\{\varphi_1,\delta_1\}\oplus\ldots\oplus\{\varphi_n,\delta_n\}], \underline{x}_1 \times \ldots \times \underline{x}_n)$

$= (\varepsilon_{\underline{B}} \circ \iota_{\underline{B}}(\beta)).(f_{q_1}\{\varphi_1,\delta_1,\underline{x}_1\} \times \ldots \times f_{q_n}\{\varphi_n,\delta_n,\underline{x}_n\})$ since f_m is induced by an action and Z has the induced W$\underline{B}$-structure

$= \beta.(Mf(y_1) \times \ldots \times Mf(y_n))$.

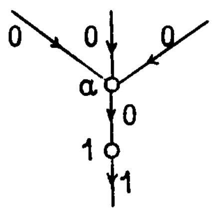
From the definition it is clear that Mf is a continuous function in (f,ρ) .]]

<u>Theorem 9.10</u>: X is a deformation retract of NX (strongly).
NX is a strong deformation retract of UX.
NX is a strong deformation retract of MX.

<u>Proof</u>: $X \subset NX$ is the subspace of $K_0 = U_0X = M_0X \subset NX$ of all triples (θ, I^0, x), where

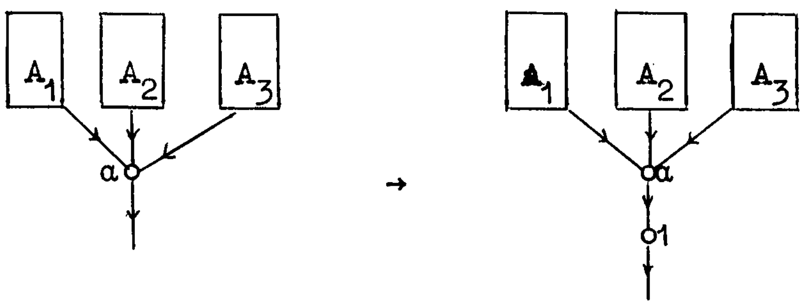
$\theta =$ 

Recall that each element of NX is represented by a triple $(\theta, \delta, \underline{x})$ such that each edge of θ is labelled by 0 with exception of the root. Using the relation (9.1) we can choose the representatives such that the vertex at the root of θ is labelled by $1 \in \underline{B}(\underline{1},\underline{1})$. (Substitute the vertex at the root by the subtree



α is the label of the vertex at the root of θ

and assign the value 0 to the new link between the vertices labelled by α and 1. Hence



The strong deformation retraction is induced by a deformation of the space of these representatives: Define $H_t\{\theta, \delta, \underline{x}\} = \{\theta, H_t(\delta), \underline{x}\}$ with

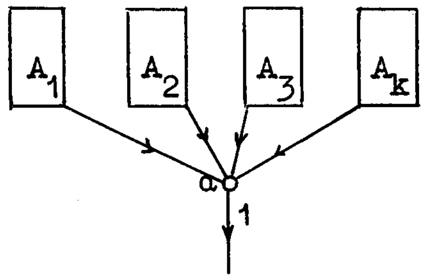
$H_t(u_1,\dots,u_p) = (\max(t_1,u_1),\dots,\max(t_p,u_p))$, where $t_i = t$ if u_i is assigned to a link ending in the vertex at the root, and $t_i = 0$ otherwise. Since links with the value 1 are not affected H_t preserves the relation (9.2). Since the multiplication "max" on I is associative, H_t also preserves the relation (9.1). $H_1\{\theta,\delta,\underline{x}\}$ can be represented by a triple $(\varphi,\partial,\underline{x})$ such that (φ,∂) represents a composition $(\iota_{\underline{B}*L_1}(1;(0,1)))\circ z$. Hence by the relation (9.2), $H_1\{\theta,\delta,\underline{x}\} \in X$. Note that throughout the deformation the elements of X stay fixed.

Define the strong deformation retraction of UX into NX by
$H_t\{\theta,\delta,\underline{x}\} = \{\theta,H_t(\delta),\underline{x}\}$ with $H_t(u_1,\dots,u_p)=(t_1\cdot u_1,\dots,t_p\cdot u_p)$ where $t_i = t$ if u_i is assigned to a link labelled by 1, and $t_i = 1$ otherwise. Since links labelled by 0 are not affected, (9.2) is preserved, and it follows immediately that (9.1) is preserved. Notice that H_t keeps the elements of NX fixed since only the roots of their representing trees are labelled by 1. $H_0\{\theta,\delta,\underline{x}\} \in NX$.

The deformation retraction of MX into NX is more complicated. Filter MX as follows: F_nMX is the subspace of MX of those elements that can be represented by a triple $(\theta,\delta,\underline{x})$ such that at most n links of θ are labelled by 1.

Notice that the subspace of the representing elements of F_nMX is closed in the space of the representing elements of MX. $F_0MX = NX$. We are now going to define a strong deformation retraction of F_nMX into $F_{n-1}MX$.

Consider a typical representative $(\theta,\delta,\underline{x})$ of F_nMX with

$\theta =$  $\alpha \in \underline{B}(\underline{k},\underline{1})$

The roots of $A_1,\ldots,A_k$ can be labelled by 0 or 1. Index $A_1,\ldots,A_k$ such that $A_1,\ldots,A_r$ have their roots labelled by 1 and $A_{r+1},\ldots,A_k$ by 0. Index the incoming edges of α by the indices of the trees sitting on them. We consider one type of trees only.

Let P_i be the space of the trees of the type of A_i. Let P_i' be the subspace of those trees A_i of P_i such that the value 1 is assigned to a collection of links of A_i labelled by 1, which separates A_i into a tree each edge of which is labelled by 1 and a copse. Since $(I, 1)$ is a NDR-pair, P_i' is a NDR in P_i. Let $Q \subset P_1\times\ldots\times P_r = P$ be the subspace of all those copses $A_1\oplus\ldots\oplus A_r$ such that the value 0 is assigned to a link labelled by 1, or a vertex with label $1 \in \underline{B}(\underline{1},\underline{1})$ has the incoming and outgoing edge labelled by 1. Since $(I, 0)$ and $(\underline{B}(\underline{1},\underline{1}), 1_{\underline{1}})$ are NDR-pairs, so is (P, Q).

Let $\underline{t} = (t_1,\ldots,t_r)$ be the collection of values assigned to the incoming edges of α indexed by $1,\ldots,r$.

<u>Case I</u>: $r \neq k$. Then $(\theta,\delta,\underline{x})$ represents an element of $F_{n-1}MX$ iff $(A_1\oplus\ldots\oplus A_r) \in Q$ or $\underline{t} \in LI^r$ where $LI^r \subset I^r$ is the collection of lower faces of I^r. Hence we want a strong deformation retraction

$$P_1\times\ldots\times P_k\times I^r \rightarrow Q\times P_{r+1}\times\ldots\times P_k\times I^r \cup P_1\times\ldots\times P_k\times LI^r$$

Since $Q\times P_{r+1}\times\ldots\times P_k \subset P_1\times\ldots\times P_k$ is a NDR and since LI^r is a strong deformation retract of I^r, such a deformation retraction exists by [6; Theorem 6.3].

<u>CaseII</u>: $r = k$. Then $(\theta,\delta,\underline{x})$ represents an element of $F_{n-1}MX$ iff one of the following conditions holds:

(1) $A = A_1\oplus\ldots\oplus A_k \subset Q$

(2) $\underline{t} \in LI^k$

(3) for each i either $t_i = 1$, $t_i \in \underline{t}$, or $A_i \in P_i'$. But at least one A_i is in P_i' for some i.

(4) $\alpha = 1$

Construct the deformation H of F_nMX into $F_{n-1}MX$ by induction on the number of trees in $A = A_1\oplus\ldots\oplus A_k$ that are in some P_i'. Let P^q be the subspace of P of those copses, such that at least q of their trees are in some P_i'. ~~Then each element of~~ Hence, if $A \in P^k$, then θ represents an element of lower filtration.

Suppose inductively that H has been defined on all elements $\{\theta, \delta, \underline{x}\}$, for which the subcopse A is contained in P^{q+1}.

Let $A \in P^q$; wlog $A \in P_1' \times \ldots \times P_q' \times P_{q+1} \times \ldots \times P_k$, which we denote by R. H has been defined on $\{\theta, \delta, \underline{x}\}$ iff

(5) $A_1 \oplus \ldots \oplus A_k \in Q$

(6) $A_i \in P_i'$ for some $i \neq 1, \ldots, q$

(7) $\underline{t} \in LI^k \cup \{(t_1, \ldots, t_k) \in I^k \mid t_{q+1} = \ldots = t_k = 1\}$ for $q \neq 0$, which we denote by GI^k, for $q \neq 0$ or $\underline{t} \in LI^k$ which we again denote by GI^k for $q = 0$

(8) $\alpha = 1$

Let $B = \underline{B}(\underline{k}, \underline{1})$ and $B' = \emptyset$ if $k \neq 1$, $B' = (1_{\underline{1}})$ if $k = 1$. Let $R' \subset R$ be the subspace of all those copses A satisfying (5) or (6). Since $(I, 0)$, $(I, \partial I)$, and $(\underline{B}(\underline{1}, \underline{1}), 1_{\underline{1}})$ are NDR-pairs, so are (R, R') and (B, B') and hence $(R \times B, R \times B' \cup R' \times B)$. We want a deformation retraction

$$R \times B \times I^k \to (R' \times B \cup R \times B') \times I^k \cup R \times B \times GI^k .$$

By [6; Theorem 6.3] it suffices to show that there exists a deformation retraction

$$I^k \to GI^k .$$

If $q \neq 0$, then $GI^k = 0 \times I^{k-1} \cup I \times G' I^{k-1}$, where

$G'I^{k-1} = LI^{k-1} \cup \{(t_2,\dots,t_k) \in I^{k-1} \mid t_{q+1}=\dots=t_k=1\}$. $(I^{k-1}, G'I^{k-1})$ is a NDR-pair. Since 0 is a deformation retraction of I, there exists a deformation retraction $I^k \to GI^k$ for $q \neq 0$.

In view of condition (3), GI^k reduces to LI^k if $q = 0$, and LI^k is a deformation retraxt of I^k.]]

<u>Corollary 9.11</u>: UX and MX have the same homotopy type as X.

<u>Corollary 9.12</u>: If <u>B</u> is an M^1T-category with isolated identities, and (X,γ) a W<u>B</u>-space, then

$$(u,\mu)\colon (X,\gamma) \to (UX,\chi) \qquad \text{and}$$
$$(m,\nu)\colon (X,\gamma) \to (MX,\chi^*)$$

are s-homotopy equivalences.

This follows from Theorem 9.10 and Theorem 8.1]]

<u>Corollary 9.13</u>: Let <u>A</u> be the M^1T-category of Example 2, p.9. Then any W<u>A</u>-space is of the same homotopy type as a topological monoid.

The last result has been known to J.F. Adams and J.D. Stasheff (unpublished), but their topological monoid seems to be different from our monoid MX.

§10 STRUCTURE THEORY II

The results of this chapter are entirely due to Dr J.M. Boardman. We enclose them to give some indication of applications of the theory we have developed.

Definition 10.1: A space X is called an E-space if it is given an E-structure, which consists of an M^1TP-category $\underline{B}$, acting on X, for which $\underline{B}(\underline{n}, \underline{1})$ is contractible for all n.

Main Theorem 10.2: A CW-complex X admits an E-structure with $\pi_0(X)$ a group, if and only if it is an infinite loop space.

Sketch proof.

X is an infinite loop space if and only if there is a sequence of spaces X_n and homotopy equivlences $X_n \simeq \Omega X_{n+1}$ for $n \geqslant 0$, with $X = X_0$. Careful use of mapping cylinders and telescopes enables us to find a space Y homotopy-equivalent to X, and spaces Y_1, Y_2, ... such that

$$Y = \Omega Y_1,\ Y_1 = \Omega Y_2,\ Y_2 = \Omega Y_3,\ \ldots$$

Example 4(p.14) shows that the space $Y = \Omega^n Y_n$ admits a category of operators $\underline{Q}_n$, which becomes highly connected for n large. Moreover, we can include $\underline{Q}_n$ in $\underline{Q}_{n+1}$ as a subcategory of operators, so that the union $\bigcup_n \underline{Q}_n$ acts on Y. This is an E-structure on Y. By Theorem 8.2, given a category $\underline{B}$ acting on Y, we can make $W\underline{B}^{\sim}$ act on X, and this is another E-structure on X.

Conversely, suppose we are given an E-structure on X. In this direction the theorem reduces to the following theorem, as induction step:

<u>Theorem 10.3</u>: Given an E-space X, where X is a CW-complex, for which $\pi_0(X)$ is a group (by means of the E-structure) then there exists a "classifying space" BX such that $X \simeq \Omega BX$, and BX is an E-space, and BX is a CW-complex.

The first step is the construction of a good category to act on E-spaces. What we need is a category $W\underline{B}^{\sim}$, in which each space $\underline{B}(\underline{n}, \underline{1})$ is a contractible CW-complex on which the symmetric group S_n acts freely and cellularly.

We now return to the given E-structure on X, and deduce from it by Theorem 4.9 an action of $W\underline{A}$ on X, where $\underline{A}$ is the category of Example 2, p.9. Then $W\underline{A}$ also acts on X^n. We now use the relative universal property many times.

For each point $\alpha \in \underline{B}(\underline{m}, \underline{n})$ we construct a homotopy $\underline{A}$-map $f_\alpha : X^m \to X$, n continuous in α. These must behave properly with respect to products $\oplus$ and permutations. However, we cannot compose homotopy $\underline{A}$-maps. Whenever α and β are composable, we construct a 2-simplex of the semi simplicial complex Map$\underline{A}$ with faces f_α, f_β and $f_{\beta\alpha}$. This corresponds to an "edge" of $W\underline{B}^\sim$. Similarly for higher-dimensional simplexes, although the details become vastly more complicated. What we now have is a kind of E-space in the "category" of $W\underline{A}$-spaces.

The next step is to reduce all the $W\underline{A}$-actions to $\underline{A}$-actions, the homotopy $\underline{A}$-maps to $\underline{A}$-homomorphisms, etc. The main tool for achieving this is Theorem 9.9 and Corollary 9.11 which first replaces X by the universal monoid MX, and continues with the help of the restricted Kan extension condition. Much complication is caused by the fact that the natural map $M(X\times Y) \to MX\times MY$ is only a homotopy equivalence, so that homotopy inverses have to be chosen. Now monoid homomorphisms can be composed, which enables us to replace the semi-simplicial gadget by an E-space in the category of monoids, in which all the actions are monoid homomorphisms.

Finally we apply a suitable classifying space functor B, and define $BX = BMX$. The most convenient is Milgram's functor [4], because it has the property $B(MX)^n = (BMX)^n$. Thus BX becomes an E-space, as required. Further, it is a CW-complex. Milgram [4] proves that $\Omega BMX \simeq MX$ which with $MX \simeq X$ (Corollary 9.13) shows that $\Omega BX \simeq X$, provided that $\pi_0(X)$ is a group.

Of course the theorem can be strengthened in all the obvious ways. The homotopy equivalence between the given E-space and the constructed infinite loop space can be made into an equivalence of E-spaces. Also we can consider higher-dimensional "simplexes" of E-actions, in the spirit of Map$\underline{B}$, and prove results about these.

APPENDIX

The following lemma has been stated by A. Dold [1; Satz 3.6]. A proof of the dual situation can be found in [2; Theorem 6.1]. The lemma holds under slightly weaker conditions.

Lemma (Dold): Given cofibrations i, i' and a homotopy equivalence f

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

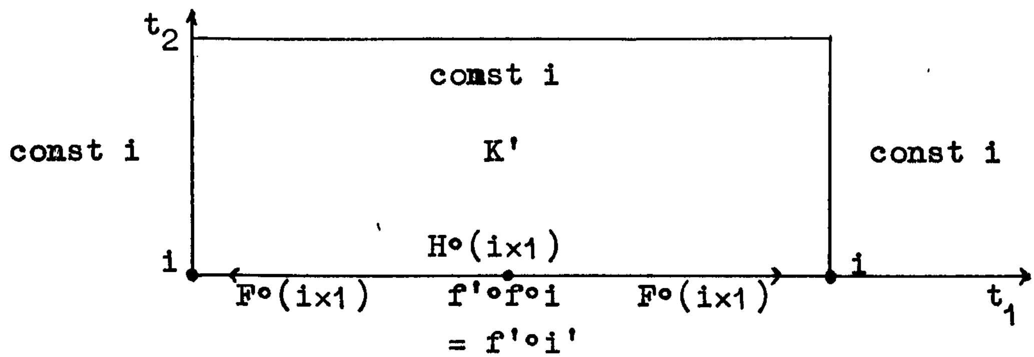
such that $f \circ i = i'$. Then we can choose a homotopy inverse f'' and homotopies $D_t\colon X \to X$, $D_t'\colon Y \to Y$, such that $D_t\colon f'' \circ f \simeq id_X$ rel iA, and $D_t'\colon f \circ f'' \simeq id_Y$ rel $i'A$.

Proof: Let f' be any homotopy inverse of f, and $F\colon X \times I \to X$ a homotopy between $f' \circ f$ and id_X. Since $F \circ (i \times 1) | A \times 0 = f' \circ f \circ i = f' \circ i'$, and since i' has the HEP (homotopy extension property), there exists an extension of $F \circ (i \times 1)$ over $Y \times I$, i.e. a map $G\colon Y \times I \to X$ such that $G_0 = G | Y \times 0 = f'$, and $G \circ (i' \times 1) = F \circ (i \times 1)$. Let

$f'' = G_1 = G|\,Y\times 1$. Since $f'' \simeq f'$, it is a homotopy equivalence. $f''\circ i' = F_1\circ i = i$. Hence f'' is a map "under" A.
Let $H: X\times[0,2] \to X$ be given by

$$H(x,t) = \begin{cases} G(fx,\ 1-t) & 0 \leqslant t \leqslant 1 \\ F(x,\ t-1) & 1 \leqslant t \leqslant 2 \end{cases} .$$

Since $G(fix,\ 1-t) = G(i'x,\ 1-t) = F(ix,\ 1-t)$, we have $H\circ(i\times 1) = F\circ(i\times 1) - F\circ(i\times 1)$, (on the right side we have the addition of homotopies). Hence there exists a homotopy $K': A\times[0,2]\times[0,1] \to X$ such that $K': H\circ i \simeq$ (constant on i) rel ((0) ∪ (2)), i.e.



$K'(a,0,t_2) = ia$

$K'(a,2,t_2) = ia$

$K'(a,t_1,0) = H(ia,t_1)$

$K'(a,t_1,1) = ia$.

$A\times[0,2] \xrightarrow[i\times 1]{} X\times[0,2]$ has the HEP. Hence there exists a map $K: X\times[0,2]\times[0,1] \to X$, such that $K\circ(i\times 1\times 1) = K'$ and $K|\,X\times[0,2]\times 0 = H$. Now define $D: X\times[0,4] \to X$ by

$$D(x,t) = \begin{cases} K(x,0,t) & 0 \leq t \leq 1 \\ K(x,t-1,1) & 1 \leq t \leq 3 \\ K(x,2,4-t) & 3 \leq t \leq 4 \end{cases}$$

Then $D(ia,t) = ia$, since we move along the "boundary" parts of K' which are constant on i.

$D(x,0) = K(x,0,0) = H(x,0) = f''\circ f$

$D(x,4) = K(x,2,0) = H(x,2) = id_X$.

Hence D: $f''\circ f \simeq id_X$ rel iA.

Apply the procedure to f'' to obtain a homotopy inverse g and a homotopy L: $g\circ f'' \simeq id_Y$ rel $i'A$. Let D' be following combined homotopy:

$$f\circ f'' \underset{-L\circ f\circ f''}{\simeq} (g\circ f'')\circ(f\circ f'') = g\circ(f''\circ f)\circ f'' \underset{g\circ D\circ f''}{\simeq} g\circ f'' \underset{L}{\simeq} id_Y \quad .$$

Since $f\circ f''\circ i' = f\circ i = i'$, and $g\circ f''\circ i' = g\circ i = i'$, this combined homotopy is a homotopy rel $i'A$.]]

Corollary: Let $A \overset{i}{\subset} X$ be a cofibration which is a homotopy equivalence. Then there exists a retraction $p: X \to A$ and a homotopy H_t: $i\circ p \simeq id_X$ rel iA, i.e. iA is a strong deformation retract of X.

Proof: Use the previous Lemma with $f = i$, $i = id_A$, and $i' = i$.]]

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The following is a summary of joint work with my supervisor, Dr J.M. Boardman. We enclose it here to illustrate the position of the theory in a more general context.

HOMOTOPY-EVERYTHING H-SPACES

by

J.M. Boardman and R.M. Vogt March 1968

An H-space is a space X with basepoint e and multiplication map $m\colon X^2 = X \times X \longrightarrow X$, such that e is a homotopy identity - the maps $x \rightsquigarrow m(x,e)$ and $x \rightsquigarrow m(e,x)$ are homotopic to the identity map 1 of X. (We take all maps and homotopies in the based sense. We use the k-topologies (i.e. compactly generated) throughout in order to avoid spurious topological difficulties. Then function spaces have a canonical topology, obtained from the compact-open topology.) We call X a monoid if e is a strict identity and m is associative.

In the literature there are various kinds of H-space: homotopy-associative, homotopy-commutative, strongly homotopy-commutative [4], and A_∞-spaces [3]. In the last two cases, part of the structure consists of higher coherence conditions and homotopies. In this note we introduce in §2 the concept of homotopy-everything H-space, in which all possible coherence conditions hold; we abbreviate this to E-space. We also define E-maps, in §4. Our two main theorems are Theorem A, which is the structure theorem for E-spaces, and Theorem C, which shows that many familiar spaces such as BPL are in fact E-spaces. We sketch few of the proofs. Full details will appear elsewhere, in due course. Many of the results are folk theory.

A space X is called an <u>infinite loop space</u> if there is a sequence of spaces X_i and homotopy equivalences $X_i \simeq \Omega X_{i+1}$ for $i \geq 0$, such that $X = X_0$.

<u>Theorem A</u>

A CW-complex X admits an E-space structure with $\pi_0(X)$ a group, if and only if it is an infinite loop space. (Any multiplication on X induces a multiplication on $\pi_0(X)$). Every E-space X has a "classifying space" BX which is also an E-space.

1. The machine

Here we develop a machine for constructing numerous E-spaces.

We consider the category $\underline{I}$ of real inner-product spaces of countable (algebraic) dimension, and linear isometric maps between them. As examples we have $\mathbb{R}^\infty$, with orthonormal base $\{e_1, e_2, e_3, \ldots\}$, and its subspaces $\mathbb{R}^n$ with base $\{e_1, e_2, \ldots, e_n\}$, for n finite. Every such space is isomorphic to one of these; in particular $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$ We topologize $\underline{I}(A,B)$, the set of all isometric linear maps from A to B, by first giving A and B the <u>finite</u> topology, which makes A the topological direct limit of its finite-dimensional subspaces. (The obvious metric topology on $\underline{I}(A,B)$ is not acceptable.)

<u>Lemma</u> The space $\underline{I}(A,\mathbb{R}^\infty)$ is contractible.

<u>Proof</u> This result is a consequence of two easy homotopies:

(a) $i_1 \simeq i_2 : A \longrightarrow A \oplus A$

(b) $i_1 \simeq a : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \oplus \mathbb{R}^\infty$, where a is an isomorphism.

To obtain (b), we first construct a homotopy $1 \simeq f: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$, where f is defined by $fe_n = e_{2n}$, by applying the Gram-Schmidt orthogonalization process to the obvious linear homotopy f_t given by $f_t e_n = (1 - t)e_n + te_{2n}$. Then we compose with an isomorphism $a\colon \underset{\sim}{R}^\infty \cong \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty$ chosen to make $a \circ f = i_1$.

Now fix $g\colon A \longrightarrow \underset{\sim}{R}^\infty$, and let $h\colon A \longrightarrow \underset{\sim}{R}^\infty$ be a typical linear isometric embedding. We construct a contraction homotopy Q_t of $\underline{I}(A,\underset{\sim}{R}^\infty)$. For the first half, we take $Q_0 h = h = a^{-1} \circ a \circ h$, $Q_{\frac{1}{2}} h = a^{-1} \circ i_1 \circ h$, and use homotopy (b). For the second half we rewrite $Q_{\frac{1}{2}} h = a^{-1} \circ i_1 \circ h = a^{-1} \circ (h \oplus g) \circ i_1$, take $Q_1 h = a^{-1} \circ (h \oplus g) \circ i_2$, and use homotopy (a). But $Q_1 h = a^{-1} \circ i_2 \circ g$, which is independent of h. Thus $\underline{I}(A,\underset{\sim}{R}^\infty)$ is contractible.

Assume we have a functor T defined on the category $\underline{I}$, taking topological spaces as values, and a continuous natural transformation called <u>Whitney sum</u> $w\colon TA \times TB \longrightarrow T(A \oplus B)$, such that

(a) Tf is continuous in $f \in \underline{I}(A,B)$,

(b) $T\underset{\sim}{R}^0$ consists of one point (which will serve as basepoint of TA for all A),

(c) w preserves associativity, commutativity, and unit for $\times$ and $\oplus$,

(d) $T\underset{\sim}{R}^\infty$ is the direct limit of the spaces $T\underset{\sim}{R}^n$ for n finite.

<u>Theorem B</u>

$T\underset{\sim}{R}^\infty$ is an E-space. If T is also monoid-valued (e.g. group-valued), the resulting classifying space $BT\underset{\sim}{R}^\infty$ agrees with that given by Theorem A.

As a (non-canonical) multiplication on $T\underset{\sim}{R}^\infty$, we take

$$T\underset{\sim}{R}^\infty \times T\underset{\sim}{R}^\infty \xrightarrow[w]{} T(\underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty) \xrightarrow[Tf]{} T\underset{\sim}{R}^\infty,$$

where $f: \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty \longrightarrow \underset{\sim}{R}^\infty$ is some linear isometric embedding. It is homotopy-commutative, because if $s: \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty \cong \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty$ is the map interchanging factors, $f \simeq f\circ s$ by the Lemma, and then $Tf \simeq Tf\circ Ts$ by the axioms. Similarly, homotopy-associativity reduces to the existence of a homotopy

$$f\circ(f \oplus 1) \simeq f\circ(1 \oplus f): \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty \oplus \underset{\sim}{R}^\infty \longrightarrow \underset{\sim}{R}^\infty.$$

It is fairly clear that the Lemma will provide all the coherence homotopies we could possibly desire.

In the examples we give below, we define TA and w explicitly only for <u>finite-dimensional</u> A, and note that axiom (d) allows us to extend the definition to $\underset{\sim}{R}^\infty$ and hence to the whole of $\underline{I}$. In each case the maps Tf are obvious, in view of the inner-product structure.

<u>Examples</u>

1. $TA = \underset{\sim}{O}(A)$, the orthogonal group of A. Then $T\underset{\sim}{R}^n = \underset{\sim}{O}(n)$ and $T\underset{\sim}{R}^\infty = \underset{\sim}{O}$.
2. $TA = \underset{\sim}{U}(A \otimes \underset{\sim}{C})$, the unitary group of the complex vector space $A \otimes \underset{\sim}{C}$. Then $T\underset{\sim}{R}^n = \underset{\sim}{U}(n)$ and $T\underset{\sim}{R}^\infty = \underset{\sim}{U}$.
3. $TA = B\underset{\sim}{O}(A)$, a suitable classifying space of the group $\underset{\sim}{O}(A)$. Then $T\underset{\sim}{R}^n = B\underset{\sim}{O}(n)$ and $T\underset{\sim}{R}^\infty = B\underset{\sim}{O}$. Some care is needed in the choice of $B\underset{\sim}{O}(A)$, if we are to obtain a Whitney sum map. We could take the Grassmannian of all k-planes in $A \otimes \underset{\sim}{R}^\infty$, where $k = \dim A$.

4. $TA = F(A)$, the space of based homotopy equivalences of the sphere S^A. Here, S^A is the one-point compactification $A \cup \infty$ of A, with ∞ as basepoint. The Whitney sum is the smash product, since $S^A \wedge S^B \cong S^{A \oplus B}$. Then $F(\underset{\sim}{R}^\infty) = F$.

There is also a semisimplicial analogue, in which T takes semisimplicial complexes as values, and $\underline{I}(A,B)$ is replaced by its singular complex.

5. $TA = \mathrm{Top}(A)$. A k-simplex of $\mathrm{Top}(A)$ is a fibre-preserving homeomorphism of $A \times \Delta^k$ over Δ^k, where Δ^k is the standard k-simplex. Then $T\underset{\sim}{R}^n = \mathrm{Top}(n)$, and $T\underset{\sim}{R}^\infty = \mathrm{Top}$.

6. The semisimplicial analogues of examples 1 - 4.

7. The orientation-preserving versions of the other examples, namely $S\underset{\sim}{O}$, $S\underset{\sim}{U}$, $BS\underset{\sim}{O}$, SF, $S\mathrm{Top}$.

8. $TA = PL(A)$, defined as $\mathrm{Top}(A)$ but allowing only piecewise linear homeomorphisms of $A \times \Delta^k$. This <u>fails</u>, because the only singular simplexes of $\underline{I}(A,B)$ that map $PL(A)$ into $PL(B)$ are the constant ones! Thus the homotopies requird for Theorem B are not allowed. Instead we must revise the machine, which turns out to be rather complicated. Suffice it to say that for a k-simplex of $\underline{P}(A,B)$ we take a pair (ξ, f), where ξ is a p.l. sub-bundle of the product bundle $B \times \Delta^k$ over Δ^k, and $f\colon \xi \oplus (A \times \Delta^k) \cong B \times \Delta^k$ is a p.l. fibre-homeomorphism that extends the inclusion of ξ.

Further, there are obvious natural transformations $\underset{\sim}{O}(A) \longrightarrow F(A)$, etc. The only one that causes difficulty is the construction of a suitable map $\underset{\sim}{O}(A) \longrightarrow PL(A)$, which is extremely awkward (compare §4).

<u>Theorem C</u> We have E-spaces

$\underset{\sim}{O}$, $S\underset{\sim}{O}$, F, SF, $\underset{\sim}{U}$, $S\underset{\sim}{U}$, PL, SPL, Top, $STop$, Γ = "$PL/\underset{\sim}{O}$", F/PL, etc.,

and all their iterated classifying spaces. The natural maps between these are all E-maps, including $\underset{\sim}{O} \longrightarrow PL$ and $PL \longrightarrow \Gamma$.

2. Categories of operators

There are two variants: <u>with</u> or <u>without</u> permutations.

<u>Definition</u> In a <u>category</u> $\underline{B}$ of <u>operators</u>

(a) the objects are $0,1,2,\ldots$;

(b) the morphisms from m to n form a topological <u>space</u> $\underline{B}(m,n)$, and composition is continuous;

(c) we are given a strictly associative continuous functor $\oplus: \underline{B} \times \underline{B} \longrightarrow \underline{B}$ such that $m \oplus n = m + n$,

(d) if $\underline{B}$ has permutations, we are also given for each n a homomorphism $S(n) \longrightarrow \underline{B}(n,n)$, where $S(n)$ is the symmetric group on n letters. We neglect any symbol for it.

In the case with permutations we demand two further conditions:

(i) if $\pi \in S(m)$ and $\rho \in S(n)$, then $\pi \oplus \rho$ lies in $S(m + n)$ and is the usual sum permutation;

(ii) given any r morphisms $\alpha_i : m_i \longrightarrow n_i$ and $\pi \in S(r)$, we have

$$\pi(\underline{n})\circ(\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_r) = \pi(\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_r)\circ\pi(\underline{m}),$$

where $m = \Sigma m_i$, $n = \Sigma n_i$, π acts on $\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_r$ by permuting the factors, and the permutation $\pi(\underline{n}) \in S(n)$ is obtained from π by "thickening" - we replace $i \in \{1,2,\ldots,r\}$ by a block of n_i elements, and let π permute these blocks.

All functors between such categories are required to preserve the objects, the functor $\oplus$, the topology, and the permutations (if any).

<u>Examples</u>

1. $\underline{\mathrm{End}}_X$, for a space X with basepoint. $\underline{\mathrm{End}}_X(m,n)$ is the space of all (based) maps $X^m \longrightarrow X^n$, where X^n is the nth power of X. The functor $\oplus$ is just $\times$. This example has permutations.

<u>Definition</u> A category $\underline{B}$ of operators <u>acts on X</u> if we are given a functor $\underline{B} \longrightarrow \underline{\mathrm{End}}_X$. We then call X a <u>B-space</u>.

2. $\underline{A}$. $\underline{A}(m,n)$ is the set of all order-preserving maps $\{1,2,\ldots,m\} \longrightarrow \{1,2,\ldots,n\}$. There is one map $\lambda_n : n \longrightarrow 1$ for each n. Then an $\underline{A}$-space is a monoid.

3. $\underline{S}$. For $\underline{S}(m,n)$ we take the set of <u>all</u> maps $\{1,2,\ldots,m\} \longrightarrow \{1,2,\ldots n\}$ This includes permutations. Then an $\underline{S}$-space is an abelian monoid. Such a space X is known to have the homotopy type of a product of Eilenberg-MacLane spaces, if X is a connected CW-complex.

<u>Definition</u> A space X is called an <u>E-space</u> if it is given an <u>E-structure</u>, which consists of a category $\underline{B}$ of operators with permutations, acting on X, for which $\underline{B}(n,1)$ is contractible for all n. (We do not single out any canonical category $\underline{B}$.)

4. $\underline{I}$. Define $\underline{I}(m,n) = \underline{I}((\mathbb{R}^\infty)^m, (\mathbb{R}^\infty)^n)$ as in §1. By the Lemma in §1, $\underline{I}(m,1)$ is contractible, so that any $\underline{I}$-space, such as $T\mathbb{R}^\infty$, is an E-space. Hence part of Theorem B.

5. $\underline{Q}_n$, a category of operators on the nth loop space $\Omega^n Y = X$. The space $\Omega^n Y$ is the space of all maps $(I^n, \partial I^n) \longrightarrow (Y, o)$, where o is the basepoint of Y, I^n is the standard n-cube, and ∂I^n its boundary. A point $\alpha \in \underline{Q}_n(k,1)$ is a collection α of k n-cubes I_i^n linearly embedded in I^n with their axes parallel to those of I^n, having disjoint interiors. It acts on $\Omega^n Y$ as follows: given $(f_1, f_2, \ldots, f_k) \in X^k$, i.e. maps $f_i: I^n \longrightarrow Y$, we construct the map $\alpha(f_1, f_2, \ldots, f_k): I^n \longrightarrow Y$ by using f_i on the little cube I_i^n and the zero map outside the little cubes. We topologize $\underline{Q}_n(k,1)$ as a subspace of $\mathbb{R}^{2kn}$. To define $\underline{Q}_n(k,r)$ for general r, we use r range cubes instead of one. We observe that $\underline{Q}_n(k,1)$ is $(n-2)$-connected, so that as n tends to ∞, Theorem A becomes plausible.

We say that a category $\underline{B}$ of operators is <u>in standard form</u> if there exists a (necessarily unique) augmentation functor $\underline{B} \longrightarrow \underline{A}$ if $\underline{B}$ is without permutations ($\underline{B} \longrightarrow \underline{S}$ if $\underline{B}$ has permutations), such that every morphism α in $\underline{B}$ over $\lambda_{m_1} \oplus \lambda_{m_2} \oplus \ldots \oplus \lambda_r : m \longrightarrow r$

is uniquely expressible in the form $\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_r$, where $\alpha_i : m_i \longrightarrow 1$, and we have the appropriate product topology. The importance of categories in standard form is that given an arbitrary category of operators $\underline{B}$ there is another category $\underline{B}'$ in standard form and a functor $\underline{B}' \longrightarrow \underline{B}$ satisfying $\underline{B}'(n,1) = \underline{B}(n,1)$. Hence if $\underline{B}$ acts on X, we can canonically make $\underline{B}'$ act on X. This effects a welcome simplification in the theory. Of our examples, 2,3, and 5 are in standard form, but 1 and 4 are not.

3. The bar construction

The concept of monoid is not a good one from the point of view of homotopy theory, because the existence of a monoid structure on a space is not a homotopy invariant. For example, the loop space ΩX has no natural monoid structure, although it is a deformation retract of a natural monoid. Similarly for other categories of operators.

Suppose given a category $\underline{B}$ of operators, in standard form. We form a bar construction, by considering words $[\alpha_0|\alpha_1|\ldots|\alpha_k]$, where $k \geqslant 0$, each α_i is a morphism in $\underline{B}$, and the composite $\alpha_0 \circ \alpha_1 \circ \ldots \circ \alpha_k$ exists in $\underline{B}$.

<u>Definition</u> The category $W^0\underline{B}$ has as morphisms from m to n those words $[\alpha_0|\alpha_1|\ldots|\alpha_k]$ for which the composite $\alpha_0 \circ \alpha_1 \circ \ldots \circ \alpha_k$ is a morphism in $\underline{B}$ from m to n, subject to the following relations and their consequences:

$$[\alpha \oplus \beta] = [\alpha \oplus 1 | 1 \oplus \beta] = [1 \oplus \beta | \alpha \oplus 1],$$

$[1]$ is an identity,

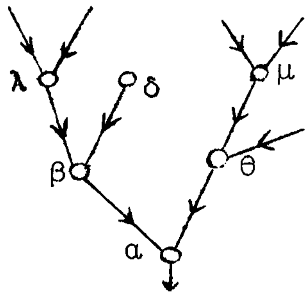
$[\alpha|\pi] = [\alpha \circ \pi]$, $[\pi|\beta] = [\pi \circ \beta]$ if $\underline{B}$ has permutations π

Composition in $W^0\underline{B}$ is by juxtaposition.

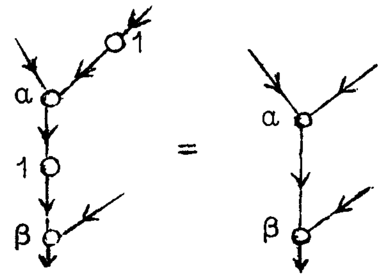
To form the category $W\underline{B}$, we take for each morphism x in $W^0\underline{B}$ a cube $C(x)$ of suitable dimension, having x as one vertex, and identify the faces not containing x with certain cubes $C(x_i)$ of lower dimension, where x_i runs through the words formed from x by one "amalgamation". (We give an alternative description below.) The categories $W^0\underline{B}$ and $W\underline{B}$ inherit obvious identification topologies. For composition we have $C(x)\circ C(y) \subset C(x\circ y)$ as a face, and $\oplus$: $C(x) \times C(y) \cong C(x \oplus y)$. The <u>augmentation</u> ε: $W\underline{B} \longrightarrow \underline{B}$ is defined by $\varepsilon C(x) = \varepsilon x$, and $\varepsilon[\alpha_0|\alpha_1|\ldots|\alpha_k] = \alpha_0\circ\alpha_1\circ\ldots\circ\alpha_k$.

In particular, the familiar pentagon in $W\underline{A}(4,1)$ is now subdivided into 5 squares.

Let us give an alternative pictorial description of $W^0\underline{B}$ and $W\underline{B}$, in the case <u>without</u> permutations (for simplicity). A morphism in $W^0\underline{B}(n,1)$ is represented by a finite tree with directed edges, except that some edges do not join two vertices (see pictures). There is just one, called the <u>root</u>, that leaves a vertex and goes nowhere; there are exactly n <u>twigs</u> that come from nowhere; the other edges are called <u>links</u> and join two vertices. Each vertex has a label $\alpha \in \underline{B}(r,1)$, where r is the number of incoming edges, and has exactly one outgoing edge. The only relation is that a vertex labelled with $1 \in \underline{B}(1,1)$ may be suppressed.



$5 \longrightarrow 1$



$3 \longrightarrow 1$

A morphism in $W^0\underline{B}(m,n)$ is an ordered collection of n such trees, called a <u>copse</u>. Composition $x \circ y$ is obtained by attaching the roots of y in order to the twigs in x.

To describe a morphism in $C(x) \subset W\underline{B}$ we simply assign a real number t_i to each link of the copse x, $(0 \leqslant t_i \leqslant 1)$, and add the relations:

(i) When we suppress a vertex labelled 1, if it separates links with values t and u, we give the new link that appears the value $\max(t,u)$.

(ii) A link with value 0 joining α to β may be shrunk to form a new copse having one fewer vertex; the vertices α and β are <u>amalgamated</u> to form γ, which is obtained from α and β by using the composition in $\underline{B}$.

When we compose copses, we assign the value 1 to each new link that appears. Consistency is assured by the tree differential calculus. Putting copses side by side describes the functor $\oplus$.

To make the following theorems true, we need to replace $\underline{B}$ by a slightly different category $\tilde{\underline{B}}$ augmented over $\underline{B}$, which is obtained from $\underline{B}$ by growing a whisker on $\underline{B}(1,1)$ rooted at 1, and taking the outer end as a new identity morphism. However, we can replace $\tilde{\underline{B}}$ by $\underline{B}$ in all the results if we know that the identity $1 \in \underline{B}(1, 1)$ is isolated.

We call an augmentation functor $\theta: \underline{C} \longrightarrow \underline{B}$ <u>fibre-homotopically trivial</u> if for each n there exists a section $X: \underline{B}(n,1) \longrightarrow \underline{C}(n,1)$ and a fibrewise homotopy $X \circ \theta \simeq 1$, $S(n)$-equivariantly if $\underline{B}$ and $\underline{C}$ have permutations.

<u>Theorem D</u>

(a) $\varepsilon: W\tilde{\underline{B}} \longrightarrow \underline{B}$ is fibre-homotopically trivial.

(b) Given any category of operators $\underline{C}$ augmented over $\underline{B}$ by a fibre-homotopically trivial augmentation, there exists a functor $F: W\tilde{\underline{B}} \longrightarrow \underline{C}$ that lifts ε (not uniquely).

The superiority of our definition is clear from:

<u>Theorem E</u>

Suppose X and Y have the same homotopy type, and $W\tilde{\underline{B}}$ acts on X. Then we can make $W\tilde{\underline{B}}$ act on Y.

4. Maps between H-spaces

Suppose the category of operators $W\underline{B}$ acts on the spaces X and Y. We need an appropriate definition of morphism between them. In fact there are two. If the map $f: X \longrightarrow Y$ commutes strictly with the actions, we call f a WB-homomorphism. We are more interested in the appropriate definition in which f merely commutes with the actions up to coherent homotopies; this is more complicated and appears to be new.

Let $\underline{L}_n$ be the "linear" category with objects $a_0, a_1, \ldots, a_n$ and one morphism $a_i \longrightarrow a_j$ whenever $i \leqslant j$. We can generalise the bar construction in §3 to form $W(\underline{B} \times \underline{L}_n)$, a category which we make act on $(n + 1)$-tuples of spaces, $(X_0, X_1, \ldots, X_n)$. (In $\underline{B} \times \underline{L}_n$ $\oplus$ is no longer a functor, so that the first relation makes sense only inside each copy $\underline{B} \times a_i$ of $\underline{B}$.)

Definition We say the map $f: X \longrightarrow Y$ is a homotopy B-map if we are given an action of $W(\underline{B} \times \underline{L}_1)$ on the pair (X,Y) that induces the given $W\underline{B}$-structures on X and Y and the given map $f: X \longrightarrow Y$.

Similarly we say that a map $f: X \longrightarrow Y$ between E-spaces is an E-map if there exists some suitable category of operators $\underline{C}$ on the pair (X,Y) that induces the given E-structures on X and Y, such that f lies in $\underline{C}(X,Y)$, and each space $\underline{C}(X^n,Y)$ is contractible. We call two E-structures on X <u>equivalent</u> if the identity map between the two structures admits an E-structure.

<u>Theorem F</u>

Let X and Y be $W\tilde{\underline{B}}$-spaces, and $f\colon X \longrightarrow Y$ a homotopy $\tilde{\underline{B}}$-map which is also a homotopy equivalence. Then any homotopy inverse $g\colon Y \longrightarrow X$ can be made into a homotopy $\tilde{\underline{B}}$-map.

<u>Example</u> Under suitable semisimplicial interpretations we have inclusions i: $\underset{\sim}{O}(n) \subset PD(n)$ and $PL(n) \subset PD(n)$. As is well known, PL(n) is a deformation retract of PD(n), with a retraction $p\colon PD(n) \longrightarrow PL(n)$, say. The only other fact we need is that PD(n) admits an action of $\underset{\sim}{O}(n)$ on the left and of PL(n) on the right. Then it is obvious that $p \circ i\colon \underset{\sim}{O}(n) \longrightarrow PL(n)$ is a homotopy homomorphism (in the usual sense): take $x, y \in \underset{\sim}{O}(n)$, then

$$p(x.y) \simeq p(x.py) \simeq p(px.py) = px.py.$$

In fact it can be shown from the above information that $p \circ i$ admits the structure of homotopy $\underline{A}$-map.

When we attempt to construct the category of $W\underline{B}$-spaces and homotopy $\underline{B}$-maps, we find that it is not possible. The composite of two homotopy $\underline{B}$-maps is not defined unless one of them is induced from a $W\underline{B}$-homomorphism, except up to a homotopy, which is itself defined only up to a homotopy, which is itself defined only up to a homotopy, which is ... Instead we form a semisimplicial complex

K, whose n-simplexes are actions of $W(\underline{B} \times \underline{L}_n)$ on $(n + 1)$-tuples of spaces.

<u>Theorem G</u>

This complex K satisfies the restricted Kan extension condition (in which the omitted face is not allowed to be the first or last).

This result provides all we need for composition up to homotopy, etc.

5. Structure theory

We consider $W\underline{A}$-spaces, with $\underline{A}$ as in §2. We first note that if X and Y are $W\underline{A}$-spaces, so are $X \times Y$ and the powers X^n. The following theorem is essentially due to Adams.

<u>Theorem H</u>

Given a $W\underline{A}$-space X, there is a universal monoid MX with a homotopy $\underline{A}$-map $i: X \longrightarrow MX$, such that any $W\underline{A}$-map $f: X \longrightarrow Y$ to a monoid Y factors uniquely as $g \circ i$, where $g: MX \longrightarrow Y$ is a monoid homomorphism, Moreover, if X is a CW-complex the map i is a homotopy equivalence.

We know [2] that MX has a classifying space BMX, which is functorial, connected, and satisfies $MX \simeq \Omega BMX$ provided $\pi_0(MX)$ is a group. Further, we have $B(G \times H) \cong BG \times BH$. In one direction, the main theorem A follows from the more detailed theorem, by putting $BX = BMX$.

Theorem J

Let X be a E-space, so that in particular it supports a W$\underline{A}$-structure by Theorem D. Then the classifying space BMX is an E-space. If Y is another E-space and f: $X \longrightarrow Y$ an E-map, then f admits a homotopy $\underline{A}$-map sturcture, and we find an E-map BMf: $BMX \longrightarrow BMY$ (not well defined).

Consider the E-spaces X^n. We can make each operator $\alpha: X^n \longrightarrow X$ into a homotopy $\underline{A}$-map. This induces by Theorem H a monoid homomorphism $M\alpha: (MX)^n \simeq MX^n \longrightarrow MX$, and hence $BM\alpha: (BMX)^n \longrightarrow BMX$. Along these lines we construct an E-structure on BMX, which makes it an E-space. The details are considerable.

6. Cohomology theories

Assume that the CW-complex Y is an E-space such that $\pi_0(Y)$ is a group; then by Theorem A, Y is an infinite loop space. Explicitly, put $Y_n = B^nY = B(B^{n-1}Y)$ by Theorem A and $Y_{-n} = \Omega^nY$, for $n \geqslant 0$; then we have homotopy equivalences $Y_n \simeq \Omega Y_{n+1}$ for all integers n, and we can define a graded cohomology theory [1] by setting

$$t^n(X,A) = [X/A, Y_n],$$

the set of homotopy classes of based maps from X/A to Y_n, for any CW-pair (X,A). The coefficient groups are the groups t^nP, where P is a point. Here they are zero for $n > 0$. Let us call such a theory connective.

Theorem K

Every connective graded additive cohomology theory t on CW-pairs arises from some E-space Y, which is uniquely defined up to homotopy equivalence of E-spaces.

In particular the E-space $\mathbb{Z} \times BU$ gives rise to the connective K-theory cK. This is more usually obtained by appealing to Bott periodicity and killing off the unwanted coefficient groups. In other cases we cannot appeal to Bott periodicity, for example

Definition We define connective p.l. K-theory by using the E-space $\mathbb{Z} \times BPL$: for $n > 0$ we put

$$cK^n_{PL}(X,A) = [X/A,\ B^n(\mathbb{Z} \times BPL)].$$

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