PROJECTIVE SPACES AND ASSOCIATED MAPS

by

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ABSTRACT

The groups KO^i are computed for real, complex and quaternionic spaces. A study is made of which elements in $\pi_n S^{n-i}$ can be represented by a map f such that $f(\tau x) = f(x)$ for a given involution τ on S^n, for i=0,1,2,3. Certain elements in arbitrarily high stems are shown not to be represented by any such map. A computation is also made of the number of homotopy classes of multiplications on P^3 and P^7, this had been done for P^3 by Naylor but the method used here is much simpler.

INTRODUCTION

The groups of stable equivalence classes of vector bundles on spaces have proved to be of considerable importance in algebraic topology. They were introduced in the late 1950's from ideas of Grothendieck, Atiyah and Hirzebruch. A study of these groups for projective spaces led to the solution of the vector fields problem by Adams and via the theorem of Hirsch to information about immersions of projective spaces in Euclidean spaces by Atiyah and by Sanderson. Atiyah and Hirzebruch made generalized cohomology theories from these groups and this has led to their study from a homotopy point of view by several authors.

Here we compute the groups KO^i for projective spaces. This had already been done in some cases by Toda [29] (for RP^{8n}), and of course by Adams [2] for i=0. Adams used arguements involving spectral sequences, whereas Toda used direct obstruction theory techniques together with the Bott sequence which links up the real and complex K-theories. In [5] Anderson considered the Bott sequence and suggested that it could prove very useful to compute the KO groups of a space. We adopt this approach as far as possible by using the spectral sequence arguements as little as possible, but they cannot be dispensed with completely (without using the obstruction theory as in [29], but this really amounts to the same thing).

In 1944, J.H.C.Whitehead showed that if a map $f : S^n \to S^{n-1}$ is essential and is such that $f(x) = f(-x)$ for every $x \in S^n$ then $n \equiv 3 \bmod 4$. We apply the K-theory of projective spaces to extend this type of result to maps between spheres with a drop of two in the dimension. We can also say whether some elements discussed by Adams in [3] can be represented by such maps. The behaviour of elements in the 3-stem is discussed by studying the cohomotopy groups of projective spaces.

Recently Naylor computed the number of homotopy classes of multiplications on P^3. Using the cohomotopy of projective spaces his result is proved in a simple fashion in Chapter III and also the similar result for P^7.

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Chapter I. THE K-THEORY OF PROJECTIVE SPACES

§1 Preliminaries

In this section we give a review of a few concepts from homotopy theory that we will need in this chapter.

All our topological spaces are provided with a fixed base-point, usually denoted by o. All our maps and homotopies preserve base-points. $[X,Y]$ denotes the set of homotopy classes of maps from X to Y. If $A \subset X$ then X/A denotes the space obtained from X by identifying all the points of A with the base-point. CX will denote the reduced cone on X, i.e. the space $X \times I / X \times \{1\} \cup o \times I$ where I denotes the unit interval $[0,1]$. SX will denote the reduced suspension of X, i.e. the space $CX/X \times \{0\}$. $X \vee Y$ will denote the disjoint union of X and Y, with the two base-points identified. There is an obvious inclusion of $X \vee Y$ in the Cartesian Product $X \times Y$ (whose base-point is (o,o)) and the quotient space $X \times Y / X \vee Y$ is the smash product $X \wedge Y$. We note that there are homeomorphisms $S^1 \wedge X \to SX$ and $I \wedge X \to CX$. If $f : X \to Y$ is any map, the mapping cone of f, C_f is the space $Y \cup_f CX = CX \vee Y$ with identifications $x \times \{0\} \sim fx$. The mapping cylinder of f is the space $M_f = (X \wedge I) \vee Y$ with identifications $x \times \{0\} \sim fx$. We see that there is an inclusion of X as $X \times \{1\}$ and that $M_f / X = C_f$.

We say that a map $f : A \to X$ is a cofibration if, given any homotopy $g_t : A \to Z$ and a map $h_0 : X \to Z$ such that $h_0 f = g_0$, then there is a homotopy $h_t : X \to Z$ such that $h_t f = g_t$.

Examples of cofibrations are inclusions of CW complexes and the inclusion of the end X in the mapping cylinder M_f of any map $f: X \to Y$. This latter example shows in fact, that every map is a cofibration "up to homotopy" i.e. given a map $f: X \to Y$ there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & M_f \arrow[ur, "h"'] & \end{tikzcd}$$

with i a cofibration and h a homotopy equivalence.

If $f: A \to X$ is a cofibration, the space X/fA is called the cofibre of f. If $f: X \to Y$ is any map, the cofibre of its equivalent cofibration is (up to homotopy type) the mapping cone C_f.

Cofibrations (and hence maps) are studied by means of the Puppe sequence [23]. This is constructed as follows. Let $f: X \to Y$ be any map, we have an inclusion $i: Y \to C_f$ which is a cofibration, and a map $p: C_f \to SX$ onto the cofibre of i, by collapsing Y to the base-point.

By iterating this procedure we get, following [23], an infinite sequence which up to homotopy can be written

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{p} SX \xrightarrow{Sf} SY \xrightarrow{Si} SC_f \xrightarrow{Sp} S^2X \to \dots$$

It is immediate from the definition of a cofibration that for any space A, the following induced sequence of based sets is exact

$$\dots \to [S^{n+1}X,A] \xrightarrow{S^np^*} [S^nC_f,A] \xrightarrow{S^ni^*} [S^nY,A] \to \dots$$
$$\dots \to [C_f,A] \xrightarrow{i^*} [Y,A] \xrightarrow{f^*} [X,A]$$

This sequence is a direct generalisation of the cohomology exact sequence for a pair.

The dual of a cofibration is the more familiar fibration. There is an analogous Puppe sequence

$$\dots \to \Omega^{n+1}B \to \Omega^nF \to \Omega^nE \to \Omega^nB \to \dots$$
$$\dots \to \Omega B \to F \to E \to B, \text{ for a fibration}$$

$\pi : E \to B$ with fibre F. For any space X the following induced sequence is an exact sequence of based sets

$$\dots \to [X,\Omega^{n+1}B] \to [X,\Omega^nF] \to [X,\Omega^nE] \to [X,\Omega^nB] \to \dots$$
$$\dots \to [X,\Omega B] \to [X,F] \to [X,E] \to [X,B].$$

This sequence is a direct generalisation of the homotopy exact sequence of a fibration.

Another important case of this latter Puppe sequence is the Bott sequence [11], obtained by looking at the fibration $O \to O/U$, where O and U are the stable orthogonal and unitary groups respectively. This fibration has fibre U and the inclusion of the fibre is induced by the inclusion $U(n) \subset O(2n)$. By Bott periodicity O/U is homotopically equivalent to the space $\Omega^2 BO$, where BO is the classifying space for the stable orthogonal group. By interpreting the Puppe sequence of this fibration in K-theory, we get the sequence

$$\ldots \to K^n(X) \xrightarrow{r} KO^n(X) \xrightarrow{p} KO^{n-1}(X) \xrightarrow{\partial} K^{n+1}(X) \to \ldots$$

Here $K^n(X)$ is the Grothendieck group of complex vector bundles on $S^{-n}X$ $(n \leqslant 0)$ and $KO^n(X)$ the corresponding group of real vector bundles. r is the map induced by the inclusion $U \subset O$ and so is induced by forgetting the complex structure on vector bundles.

By studying the Puppe sequence corresponding to the fibration $U \to U/O$ and using the Bott homotopy equivalence $BO \times Z \to \Omega(U/O)$ we get another Bott sequence

$$\ldots \to KO^n(X) \xrightarrow{c} K^n(X) \to KO^{n+2}(X) \to KO^{n+1}(X) \to \ldots$$

and the map c is induced by complexification on vector bundles; by comparing these two sequences we see that the boundary map ∂ in the first sequence is (up to sign) the map $\beta c : KO^{n-1}(X) \to K^{n+1}(X)$ where $\beta : K^{n-1}(X) \to K^{n+1}(X)$ is the Bott isomorphism.

This identification of the boundary map in the first sequence will be important in our calculations. This fact is brought out clearly in Atiyah's new approach to Bott periodicity and the Bott sequence[8].

We also need to know that the composition $rc : KO^n(X) \to KO^n(X)$ is multiplication by two. This follows immediately from the corresponding fact about vector spaces.

The reader will have noticed that we have disregarded signs in this section. This is because we do not need to know them in the applications.

§2 The groups KO^iP^n

In this section we shall calculate the groups given in table 2.1 . All the groups shown are reduced and P^n denotes real projective n-dimensional space. A meaning is given to the symbol KO^iP^n for $i > 0$ by extending the eightfold periodicity.

We note that some of these groups were already known. KO^0P^n can be found in [2, Theorem 7.4] and the values of KO^iP^{8n} in [29]. I have learnt since doing this work that the results of this section and of §3 have been also done by Fujii [32].

TABLE 2.1

i	KO^iP^{8n+1} $(n>0)$	KO^iP^{8n+2}	KO^iP^{8n+3}	KO^iP^{8n+4}
0	$Z_{2^{4n+1}}$	$Z_{2^{4n+2}}$	$Z_{2^{4n+2}}$	$Z_{2^{4n+3}}$
1	Z	Z_2	Z_2+Z_2	Z_2
2	0	Z_2	Z_2+Z_2	Z_2
3	0	0	Z	0
4	$Z_{2^{4n}}$	$Z_{2^{4n}}$	$Z_{2^{4n}}$	$Z_{2^{4n+1}}$
5	Z	0	0	0
6	Z_2	Z_2	Z_2	Z_2
7	Z_2	Z_2	$Z+Z_2$	Z_2

i	KO^iP^{8n+5}	KO^iP^{8n+6}	KO^iP^{8n+7}	KO^iP^{8n+8}
0	$Z_{2^{4n+3}}$	$Z_{2^{4n+3}}$	$Z_{2^{4n+3}}$	$Z_{2^{4n+4}}$
1	Z	0	0	0
2	0	0	0	0
3	0	0	Z	0
4	$Z_{2^{4n+2}}$	$Z_{2^{4n+3}}$	$Z_{2^{4n+3}}$	$Z_{2^{4n+4}}$
5	Z	Z_2	Z_2+Z_2	Z_2
6	Z_2	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2
7	Z_2	Z_2	$Z+Z_2$	Z_2

We assume the results on K^0P^n and KO^0P^n proved by Adams in [2]. We proceed by induction on the dimension n. The induction step uses the Puppe sequence of the covering map $\pi: S^{n-1} \to P^{n-1}$ whose cofibre is P^n. This gives us some information about KO^iP^n. We supplement this information with the Bott sequence for the space P^n. As mentioned in §1 this links up the real and the complex K-theories. We also need to use the Atiyah-Hirzebruch spectral sequence [9], which links up the cohomology and the K-theory of a space.

We remark that $K^0P^n = Z_{2[n/2]}$ see [2, Theorem 7.3].

We first calculate K^1P^n.

2.2 <u>Lemma</u> $\quad K^1P^n = Z$ if n is odd

$\qquad\qquad\quad = 0$ if n is even.

<u>Proof</u> When $n = 1$ this is true because $P^1 = S^1$ and $K^1S^1 = K^0S^0 = Z$.

Suppose that n is even and $n > 0$ then we have the Puppe sequence

$$\begin{array}{ccccccccc} K^0S^{n-1} & \to & K^1P^n & \to & K^1P^{n-1} & \xrightarrow{\pi^*} & K^1S^{n-1} & \to & K^0P^n \\ 0 & & & & Z & & Z & & \text{finite} \end{array}$$

(Here as always we write the values of groups already known, underneath)

Hence π^* is a monomorphism and so $K^1P^n = 0$.

Now suppose that n is odd and $n \geqslant 3$, then we have the Puppe sequence

$$\begin{array}{ccccccc} K^0P^{n-1} & \to & K^0S^{n-1} & \to & K^1P^n & \to & K^1P^{n-1} \\ \text{finite} & & Z & & & & 0 \end{array}$$

Hence $K^1P^n = Z$.

2.3 Lemma The values of the groups KO^iP^2 are as shown in table 2.1 .

Proof We are assuming (from [2]) that $KO^0P^2 = Z_4$. All the other groups follow trivially either from the Puppe sequence of the double covering map $S^1 \to S^1$ or alternatively from the Atiyah-Hirzebruch spectral sequence.

Note The results for P^3 are an immediate consequence of 2.3 and the fact that $S^2P^3 \simeq S^2P^2 \vee S^5$.

There now follows a series of lemmas, one for each step in our eight-fold induction together with a few others into which we have put the more difficult steps.

2.4 Lemma The results for KO^iP^{8n+2} imply those for KO^iP^{8n+3} .

Proof From the Puppe sequence

$$S^{8n+2} \to P^{8n+2} \to P^{8n+3} \to S^{8n+3} \to \ldots$$

we see that we have the following exact sequences

$$\begin{array}{ccccccc} KO^2P^{8n+2} & \to & KO^2S^{8n+2} & \to & KO^3P^{8n+3} & \to & KO^3\ P^{8n+2} \\ Z_2 & & Z & & & & 0 \end{array}$$

so $KO^3P^{8n+3} = Z$

$$(0=)KO^3S^{8n+2} \to KO^4P^{8n+3} \to KO^4P^{8n+2} \to KO^4S^{8n+2}(=0)$$

$$\text{So } KO^4\,P^{8n+3} = KO^4P^{8n+2} = Z_{2^{4n}}\,.$$

$$(0=)\,KO^4S^{8n+2} \to KO^5P^{8n+3} \to KO^5P^{8n+2}\ (=0)$$

$$\text{so } KO^5P^{8n+3} = 0\,.$$

$$\underset{0}{KO^5S^{8n+2}} \to KO^6P^{8n+3} \to \underset{Z_2}{KO^6P^{8n+2}} \to \underset{Z}{KO^6S^{8n+2}}$$

$$\text{so } KO^6P^{8n+3} = Z_2\,.$$

$$\underset{Z_2}{KO^6P^{8n+2}} \to \underset{Z}{KO^6S^{8n+2}} \to KO^7P^{8n+3} \to \underset{Z_2}{KO^7P^{8n+2}} \to \underset{0}{KO^7S^{8n+2}}$$

$$\text{and so } KO^7P^{8n+3} = Z \text{ or } Z + Z_2\,.$$

However from the Bott sequence, we have

$$\underset{Z_{2^{4n+1}}}{K^0P^{8n+3}} \to \underset{Z_{2^{4n+2}}}{KO^0P^{8n+3}} \to KO^7P^{8n+3}$$

hence KO^7P^{8n+3} has 2-torion, so it is $Z + Z_2$.

$$\underset{0}{KO^5P^{8n+3}} \to \underset{Z_{2^{4n}}}{KO^4P^{8n+3}} \xrightarrow{c} \underset{Z_{2^{4n+1}}}{K^0P^{8n+3}} \to KO^6P^{8n+3} \to \underset{0}{KO^5P^{8n+3}}$$

and so we have that $KO^6P^{8n+3} = Z_2$.

The map c in the last sequence was a monomorphism and so by the discussion in §1 the map r in the following sequence is an epimorphism

$$K^0P^{8n+3} \xrightarrow{r} \underset{Z_{2^{4n}}}{KO^4P^{8n+3}} \to KO^3P^{8n+3} \to \underset{Z}{K^1P^{8n+3}} \to \underset{0}{KO^5P^{8n+3}}$$

$$\text{so } KO^3P^{8n+3} = Z$$

Now we see that $r : \underset{Z}{K^1P^{8n+3}} \to \underset{Z}{KO^3P^{8n+3}}$ is multiplication by two and then we deduce from the Bott sequence and the results that we have already that KO^2P^{8n+3} has four elements. Similarly, we see that in the following sequence the map c is an epimorphism and r a monomorphism

$$\underset{Z_{2^{4n+1}}}{KO^0P^{8n+3}} \xrightarrow{c} \underset{Z_{2^{4n}}}{K^0P^{8n+3}} \to KO^2P^{8n+3} \to KO^1P^{8n+3} \to \underset{Z}{K^1P^{8n+3}} \xrightarrow{r} \underset{Z}{KO^3P^{8n+3}}$$

$$\text{so } KO^2P^{8n+3} \cong KO^1P^{8n+3} .$$

Now $K^1P^{8n+3} = Z$ and we have that the following composition is multiplication by two

$$\underset{\text{finite}}{KO^1P^{8n+3}} \xrightarrow{c} \underset{Z}{K^1P^{8n+3}} \xrightarrow{r} \underset{\text{finite}}{KO^1P^{8n+3}}$$

and so it is also zero , therefore KO^1P^{8n+3} must have exponent 2 and so it is $Z_2 + Z_2$.

We gave the proof of 2.4 in complete detail, however in the following lemmas, we will omit routine procedure with Puppe sequences.

2.5 <u>Lemma</u> The results for KO^iP^{8n+3} imply those for KO^iP^{8n+4}.

<u>Proof</u> Immediately from the Puppe sequence we see that $KO^5P^{8n+4} = 0$, $KO^6P^{8n+4} = Z_2$ and $KO^7P^{8n+4} = Z_2$.

From the Bott sequence we have

$$\underset{0}{KO^5P^{8n+4}} \to KO^4P^{8n+4} \to \underset{Z_{2^{4n+2}}}{K^0P^{8n+4}} \to \underset{Z_2}{KO^6P^{8n+4}} \to \underset{0}{KO^5P^{8n+4}}$$

$$\text{and so } KO^4P^{8n+4} = Z_{2^{4n+1}} .$$

In the following sequence the map c is an epimorphism

$$\underset{0}{K^1P^{8n+4}} \to KO^1P^{8n+4} \to \underset{Z_{2^{4n+3}}}{KO^0P^{8n+4}} \xrightarrow{c} \underset{Z_{2^{4n+2}}}{K^0P^{8n+4}}$$

$$\to KO^2P^{8n+4} \to KO^1P^{8n+4} \to K^1P^{8n+4}\ (=0)$$

$$\text{Hence } KO^2P^{8n+4} \cong KO^1P^{8n+4} = Z_2 .$$

Now in the Puppe sequence we have

$$\underset{Z_2}{KO^2P^{8n+4}} \to \underset{Z_2+Z_2}{KO^2P^{8n+3}} \to \underset{Z_2}{KO^2S^{8n+3}} \to KO^3P^{8n+4}$$

$$\to \underset{Z}{KO^3P^{8n+3}} \to \underset{Z}{KO^3S^{8n+3}} \to \underset{Z_2}{KO^2P^{8n+4}}$$

$$\text{so } KO^3P^{8n+4} = 0 .$$

To do the next inductiom step it does not seem sufficient to just look at the Bott and Puppe sequences. We will work out one of the differentials in the KO-theory spectral sequence. The following is part of the induction

2.6 <u>Lemma</u> $KO^3P^{8n+5} = 0$.

<u>Proof</u> The E_2-term of the KO-theory spectral sequence for P^4 is as follows

$$\begin{array}{ccccc}
Z & 0 & Z_2 & 0 & Z_2 \\
Z_2 & Z_2 & Z_2 & Z_2 & Z_2 \\
Z_2 & Z_2 & Z_2 & Z_2 & Z_2 \\
0 & 0 & 0 & 0 & 0 \\
Z & 0 & Z_2 & 0 & Z_2
\end{array}$$

We know that $KO^3P^4 = 0$, from 2.5 . This group is calculated from the indicated diagonal. The only differential that can kill the term $E_2^{4,-1} = Z_2$ is d_2. Hence $d_2 : E_2^{2,0} \to E_2^{4,-1}$ is non-zero. Similarly because $KO^2P^4 = Z_2$ we must have that $d_2 : E_2^{2,-1} \to E_2^{4,-2}$ is non-zero. However the differentials are stable cohomology operations, hence the differential $d_2 : E_2^{p,0} \to E_2^{p+2,-1}$ is $Sq^2\rho_2$ (where ρ_2 is reduction mod 2) and the differential $d_2 : E_2^{p,-1} \to E_2^{p+2,-2}$ is Sq^2 .

We know by induction that $KO^3P^{8n+4} = 0$ and so $KO^3P^{8n+5} = E_\infty^{8n+5,-2} = E_3^{8n+5,-2}$. However the differential

$$d_2 : E_2^{8n+3,-1} \to E_2^{8n+5,-2} \text{ is } Sq^2 : H^{8n+3}(P^{8n+5};Z_2) \to H^{8n+5}(P^{8n+5};Z_2)$$

which is an isomorphism by [26,page5] .

Hence $E_3^{8n+5,-2} = 0$.

2.7 <u>Lemma</u> The results for KO^iP^{8n+4} imply those for KO^iP^{8n+5} .

<u>Proof</u> Immediately from the Puppe sequence we have that $KO^5P^{8n+5} = Z$, $KO^6P^{8n+5} = Z_2$, $KO^7P^{8n+5} = Z_2$ and $KO^2P^{8n+5} = 0$.

From the Bott sequence we see that

$$r : K^0P^{8n+5} \to KO^4P^{8n+5} \text{ and } c : KO^1P^{8n+5} \to K^1P^{8n+5}$$

are isomorphisms.

2.8 <u>Lemma</u> The results for KO^iP^{8n+5} imply those for KO^iP^{8n+6}, apart from KO^6P^6. KO^6P^6 has four elements.

<u>Proof</u> Immediately from the Puppe sequence we see that $KO^7P^{8n+6} = Z_2$ and $KO^3P^{8n+6} = 0$.

From the Bott sequence we get

$$\begin{array}{ccccccc} K^1P^{8n+6} & \to & KO^7P^{8n+6} & \to & KO^6P^{8n+6} & \to & K^0P^{8n+6} \\ 0 & & Z_2 & & & & Z_{2^{4n+3}} \end{array}$$

$$\begin{array}{ccccccc} & \to & KO^0P^{8n+6} & \to & KO^7P^{8n+6} & \to & K^1P^{8n+6} \\ & & Z_{2^{4n+3}} & & Z_2 & & 0 \end{array}$$

and so KO^6P^{8n+6} has order four.

In the following themap c is an isomorphism

$$\begin{array}{ccccccc} K^1P^{8n+6} & \to & KO^1P^{8n+6} & \to & KO^0P^{8n+6} & \xrightarrow{c} & K^0P^{8n+6} \\ 0 & & & & Z_{2^{4n+3}} & & Z_{2^{4n+3}} \end{array}$$

$$\to KO^2P^{8n+6} \to KO^1P^{8n+6}$$

so $KO^1P^{8n+6} = 0$ and $KO^2P^{8n+6} = 0$.

$$\begin{array}{ccccccc} KO^2P^{8n+6} & \to & K^0P^{8n+6} & \to & KO^4P^{8n+6} & \to & KO^3P^{8n+6} \\ 0 & & Z_{2^{4n+3}} & & & & 0 \end{array}$$

hence $KO^4P^{8n+6} = Z_{2^{4n+3}}$.

The map c in the following sequence is multiplication by two

$$\begin{array}{ccccccc} K^1P^{8n+6} & \to & KO^5P^{8n+6} & \to & KO^4P^{8n+6} & \xrightarrow{c} & K^0P^{8n+6} \\ 0 & & & & Z_{2^{4n+3}} & & Z_{2^{4n+3}} \end{array}$$

so $KO^5P^{8n+6} = Z_2$.

It now remains to show that $KO^6P^{8n+6} = Z_2 + Z_2$ when $n \geqslant 1$, we already know that it has order four. $K^0P^{8n+6} = Z_{2^{4n+3}}$ and so if $n \geqslant 1$ the composition $rc : KO^6P^{8n+6} \to KO^6P^{8n+6}$ is zero, which shows that $KO^6P^{8n+6} = Z_2 + Z_2$.

We do not seem to be able to show that $KO^6P^6 = Z_2 + Z_2$ at this stage. So we assume only that it has order four and return later to show that it is in fact $Z_2 + Z_2$.

2.9 Lemma The results for KO^iP^{8n+6} imply those for KO^iP^{8n+7} (again, apart from KO^6P^7).

Proof Immediately from the Puppe sequence,

$$KO^1P^{8n+7} \cong KO^2P^{8n+7} = 0 \text{ and } KO^3P^{8n+7} = Z.$$

From the Bott sequence we have

$$\begin{array}{ccccccc} KO^2P^{8n+7} & \to & K^0P^{8n+7} & \to & KO^4P^{8n+7} & \to & KO^3P^{8n+7} \\ 0 & & Z_{2^{4n+3}} & & & & Z \end{array}$$

and from the Puppe sequence KO^4P^{8n+7} is finite, hence it is $Z_{2^{4n+3}}$; $c : KO^0P^{8n+7} \to K^0P^{8n+7}$ is an isomorphism, so in the sequence

$$\begin{array}{ccccccccc} K^0P^{8n+7} & \xrightarrow{r} & KO^0P^{8n+7} & \to & KO^7P^{8n+7} & \to & K^1P^{8n+7} & \to & KO^1P^{8n+7} \\ Z_{2^{4n+3}} & & Z_{2^{4n+3}} & & & & Z & & 0 \end{array}$$

the map r is multiplication by two and so $KO^7P^{8n+7} = Z + Z_2$. Also the maps r in the following sequence are multiplication by two

$$\begin{array}{ccccccccc} K^1P^{8n+7} & \xrightarrow{r} & KO^7P^{8n+7} & \to & KO^6P^{8n+7} & \to & K^0P^{8n+7} & \xrightarrow{r} & KO^0P^{8n+7} \\ Z & & Z + Z_2 & & & & Z_{2^{4n+3}} & & Z_{2^{4n+3}} \end{array}$$

and so KO^6P^{8n+7} has order eight.

From the Puppe sequence we have

$$\begin{array}{ccccccccc} KO^3S^{8n+6} & \to & KO^4P^{8n+7} & \to & KO^4P^{8n+6} & \to & KO^4S^{8n+6} & \to & KO^5P^{8n+7} \\ 0 & & Z_{2^{4n+3}} & & Z_{2^{4n+3}} & & Z_2 & & \end{array}$$

$$\begin{array}{cccccccc} \to & KO^5P^{8n+6} & \to & KO^5S^{8n+6} & \to & KO^6P^{8n+7} & \to & KO^6P^{8n+6} \\ & Z_2 & & Z_2 & & \text{order } 8 & & \text{order } 4 \end{array}$$

hence KO^5P^{8n+7} has order four, but $r\dot{c} : KO^5P^{8n+7} \to KO^5P^{8n+7}$ factors through $K^1P^{8n+7} = Z$ and so $KO^5P^{8n+7} = Z_2 + Z_2$. Also as $K^0P^{8n+7} = Z_{2^{4n+3}}$, $KO^6P^{8n+7} = Z_2 + Z_2 + Z_2$ for $n \geq 1$.

For the purposes of the induction, we will assume only that KO^6P^{8n+7} has order eight and return later to show that $KO^6P^7 = Z_2 + Z_2 + Z_2$.

2.10 <u>Lemma</u> The results for KO^iP^{8n-1} imply those for KO^iP^{8n} $(n \geq 1)$.

<u>Proof</u> Immediately from the Puppe sequence we have

$$KO^1P^{8n} \cong KO^2P^{8n} = 0.$$

From the Bott sequence

$$(0=)K^1P^{8n} \to KO^3P^{8n} \to KO^2P^{8n}(=0) \quad \text{so } KO^3P^{8n} = 0.$$

$$\begin{array}{ccccccc} KO^2P^{8n} & \to & K^0P^{8n} & \to & KO^4P^{8n} & \to & KO^3P^{8n} \\ 0 & & Z_{2^{4n}} & & & & 0 \end{array} \quad \text{so } KO^4P^{8n} = Z_{2^{4n}}.$$

In the following sequence :

$$\begin{array}{ccccc} K^1P^{8n} & \to & KO^5P^{8n} & \to & KO^4P^{8n} \\ 0 & & & & Z_{2^{4n}} \end{array}$$

$$\begin{array}{cccccccc} \xrightarrow{c} & K^0P^{8n} & \to & KO^6P^{8n} & \to & KO^5P^{8n} & \to & K^1P^{8n} \\ & Z_{2^{4n}} & & & & & & 0 \end{array}$$

the map c is multiplication by two, so $KO^5P^{8n} = Z_2$ and KO^6P^{8n} has order four.

In the sequence

$$\begin{array}{ccccccccccccc} K^1P^{8n} & \to & KO^7P^{8n} & \to & KO^6P^{8n} & \to & K^0P^{8n} & \xrightarrow{r} & KO^0P^{8n} & \to & KO^7P^{8n} & \to & K^1P^{8n} \\ 0 & & & & \text{order } 4 & & Z_{2^{4n}} & & Z_{2^{4n}} & & & & 0 \end{array}$$

the map r has kernel at most Z_2 so KO^7P^{8n} has order two or four but it is the cokernel of r so $KO^7P^{8n} = Z_2$.

$rc : KO^6P^{8n} \to KO^6P^{8n}$ factors through $K^0P^{8n} = Z_{2^{4n}}$ and as $n \geqslant 1$ it must be zero, so $KO^6P^{8n} = Z_2 + Z_2$.

2.11 <u>Lemma</u> The results for KO^iP^{8n} imply those for KO^iP^{8n+1} ($n \geqslant 1$).

<u>Proof</u> Immediately from the Puppe sequence we have that $KO^2P^{8n+1} \cong KO^3P^{8n+1} = 0$, $KO^1P^{8n+1} = Z$ and $KO^4P^{8n+1} = Z_{2^{4n}}$.

In the Bott sequence we have

$$\begin{array}{ccccccccc} K^0P^{8n+1} & \xrightarrow{r} & KO^0P^{8n+1} & \to & KO^7P^{8n+1} & \to & K^1P^{8n+1} & \xrightarrow{r} & KO^1P^{8n+1} \\ Z_{2^{4n}} & & Z_{2^{4n+1}} & & & & Z & & Z \end{array}$$

and both the maps r must be monomorphisms, so $KO^7P^{8n+1} = Z_2$.

In the Puppe sequence we have

$$\begin{array}{ccccccccc} KO^5S^{8n} & \to & KO^6P^{8n+1} & \to & KO^6P^{8n} & \to & KO^6S^{8n} & \to & KO^7P^{8n+1} \\ 0 & & & & Z_2 + Z_2 & & Z_2 & & Z_2 \end{array}$$

$$\begin{array}{cccccccc} \to & KO^7P^{8n} & \to & KO^7S^{8n} & \to & KO^0P^{8n+1} & \to & KO^0P^{8n} \\ & Z_2 & & Z_2 & & Z_{2^{4n+1}} & & Z_{2^{4n}} \end{array}$$

so $KO^6P^{8n+1} = Z_2$.

$r : K^0P^{8n+1} \to KO^4P^{8n+1}$ is an isomorphism, hence in the sequence

$$\begin{array}{ccccccc} KO^4P^{8n+1} & \xrightarrow{c} & K^0P^{8n+1} & \to & KO^6P^{8n+1} & \to & KO^5P^{8n+1} \\ Z_{2^{4n}} & & Z_{2^{4n}} & & Z_2 & & \end{array}$$

$$\begin{array}{ccccc} \to & K^1P^{8n+1} & \to & KO^7P^{8n+1} & \\ & Z & & Z_2 & . \end{array}$$

c is multiplication by two and so $KO^5P^{8n+1} = Z$.

2.12 Lemma The results for KO^iP^{8n+1} imply those for KO^iP^{8n+2}. ($n \geq 1$).

Proof Immediately from the Puppe sequence we have $KO^3P^{8n+2} = 0$ and $KO^4P^{8n+2} = Z_{2^{4n}}$.

From the Bott sequence we have

$$\begin{array}{ccccccc} K^0P^{8n+2} & \xrightarrow{r} & KO^0P^{8n+2} & \to & KO^7P^{8n+2} & \to & K^1P^{8n+2} \\ Z_{2^{4n+1}} & & Z_{2^{4n+2}} & & & & 0 \end{array}$$

and the map r is a monomorphism, so $KO^7P^{8n+2} = Z_2$.

$$\begin{array}{ccccccccc} K^1P^{8n+2} & \to & KO^7P^{8n+2} & \to & KO^6P^{8n+2} & \to & K^0P^{8n+2} & \xrightarrow{r} & KO^0P^{8n+2} \\ 0 & & Z_2 & & & & Z_{2^{4n+1}} & & Z_{2^{4n+2}} \end{array}$$

so $KO^6P^{8n+2} = Z_2$.

Similarly $c : KO^4P^{8n+2} \to K^0P^{8n+2}$ is a monomorphism and $KO^5P^{8n+2} = 0$; $c : KO^0P^{8n+2} \to K^0P^{8n+2}$ is an epimorphism, $KO^1P^{8n+2} = Z_2$ and $KO^2P^{8n+2} = Z_2$.

We have now completed our induction and have proved all the results in table 2.1, except for KO^6P^6 and KO^6P^7 .

2.13 Lemma $\quad KO^6P^6 = Z_2 + Z_2$.

Proof $\quad$ We look at the Atiyah-Hirzebruch KO-theory spectral sequence for P^6 . There is an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & E_2^{6,-8} & \to & KO^6P^6 & \to & E_2^{2,-4} & \to & 0 \\ & & Z_2 & & & & Z_2 & & \end{array}$$

Now in the spectral sequence for P^5 , $KO^6P^5 \cong E_2^{2,-4}$ and in that for P^9, $KO^6P^9 \cong E_2^{2,-4}$. By the naturality of the spectral sequence the inclusion map $P^5 \to P^9$ induces an isomorphism of $E_2^{2,-4}$ and so of KO^6P^9 with KO^6P^5. This isomorphism factors through KO^6P^6, and as we already know that it has four elements, we have the result.

2.14 Lemma $\quad KO^6P^7 = Z_2 + Z_2 + Z_2$.

Proof $\quad$ By the previous proof, $KO^6P^7 = Z_2 + G_4$, where G_4 is either Z_4 or $Z_2 + Z_2$. The Z_2-summand corresponds to $E_2^{2,-4}$ in the spectral sequence.

Also, from the Puppe sequence, we have the short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & KO^6P^8 & \to & KO^6P^7 & \to & KO^6S^7 & \to & 0 \\ & & Z_2+Z_2 & & Z_2+G_4 & & Z_2 & & \end{array}$$

and from the spectral sequence

$$0 \to E_2^{7,-9} \to G_4 \to E_2^{6,-8} \to 0 \ .$$

The $E_2^{7,-9}$ corresponds to the KO^6S^7 and so we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & E_2^{7,-9} & \to & G_4 & \to & E_2^{6,-8} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_2^{7,-9} & \to & Z_2 & \to & 0 & \to & 0 \end{array}$$

where the top line comes from the spectral sequence of P^7 and the bottom from that for S^7, the vertical maps are induced by the covering $S^7 \to P^7$.

Hence $G_4 = Z_2 + Z_2$.

We now summarise a few of the results of this section that we will need for the applications.

2.15 <u>Proposition</u>

Let $\pi : S^n \to P^n$ be the covering map, then the induced map $\pi^! : KO^i P^n \to KO^i S^n$ is zero when

either $n-i \equiv 1 \bmod 8$ and $n \not\equiv 3 \bmod 4$

or $n-i \equiv 2 \bmod 8$ and $n \equiv 1$ or $2 \bmod 4$.

<u>Proof</u> Implicit in the proofs of this section.

§3 The groups KO^iCP^n

In this section, we use very similar methods to those used in the previous section, to compute the groups KO^iCP^n where CP^n is complex projective space of real dimension 2n. The results are given in the following table. The symbol rZ denotes the direct sum of r copies of the integers.

TABLE 3.1

$$KO^iCP^{2n} = nZ \quad \text{for } i \text{ even}$$
$$= 0 \quad \text{for } i \text{ odd.}$$

i	KO^iCP^{4n+1}	KO^iCP^{4n+3}
0	$2nZ + Z_2$	$2n+1\,Z$
1	Z_2	0
2	$2n+1\,Z$	$2n+2Z$
3	0	0
4	$2nZ$	$2n+1\,Z + Z_2$
5	0	Z_2
6	$2n+1\,Z$	$2n+2Z$
7	0	0

We note that we will not assume any of these results. However some of them are known already. KO^0CP^n may be found in [24, theorem 3.9]. We will use the Bott sequence and so will need to know the values of the groups K^iCP^n. For i = 0, these can be found in [10] or [24, theorem 3.10] and for both i = 0 and 1 in [7]. However we reprove them in the following

3.2 Lemma $\quad K^0CP^n = nZ$ and $K^1CP^n = 0$.

Proof We induct on n. The result is clearly true when n = 1 because $CP^1 = S^2$.

For the induction step we use the Puppe sequence for the covering map $S^{2n-1} \to CP^{n-1}$ whose mapping cone is CP^n.

$$(0 =)\ K^0S^{2n-1} \to K^1CP^n \to K^1CP^{n-1}\ (= 0)$$ shows that $K^1CP^n = 0$, and

$$\underset{0}{K^1CP^{n-1}} \to \underset{Z}{K^1S^{2n-1}} \to K^0CP^n \to \underset{n-1Z}{K^0CP^{n-1}} \to \underset{0}{K^0S^{2n-1}}$$

shows that $K^0CP^n = nZ$.

We now start our induction, which has four steps. It starts easily with CP^1.

3.3 Lemma $\quad KO^2 CP^{4n+2} = 2n+1\, Z$

Proof We know by induction that $KO^2 CP^{4n+1} = 2n+1\, Z$. We work in the KO-theory spectral sequence for CP^{4n+2}. $E_2^{8n+3,-(8n+1)} = 0$ and $E_2^{8n+4,-(8n+2)} = Z_2$.

$$KO^2 CP^{4n+2} = 2n+1\, Z + E_\infty^{8n+4,-(8n+2)}$$

However in the proof of 2.6 we showed that the differential

$$d_2 : E_2^{8n+2,-(8n+1)} \to E_2^{8n+4,-(8n+2)}$$

was Sq^2, hence in this case $E_\infty^{8n+4,-(8n+2)} = 0$.

3.4 Lemma $\quad$ The results for $KO^i CP^{4n+1}$ imply those for $KO^i CP^{4n+2}$.

Proof From the Puppe sequence we see that $KO^5 CP^{4n+2} = 0$, $KO^6 CP^{4n+2} = 2n+1\, Z$, $KO^7 CP^{4n+2} = 0$, $KO^1 CP^{4n+2} = 0$ and $KO^4 CP^{4n+2} = 2n+1\, Z$.

From the Bott sequence

$$\begin{array}{ccccccccc} KO^1 CP^{4n+2} & \to & KO^0 CP^{4n+2} & \to & K^0 CP^{4n+2} & \to & KO^2 CP^{4n+2} & \to & KO^1 CP^{4n+2} \\ 0 & & & & 4n+2Z & & 2n+1\, Z & & 0 \end{array}$$

so $KO^0 CP^{4n+2} = 2n+1\, Z$

and similarly

$$\begin{array}{ccccccccc} K^1 CP^{4n+2} & \to & KO^3 CP^{4n+2} & \to & KO^2 CP^{4n+2} & \to & K^0 CP^{4n+2} & \to & KO^4 CP^{4n+2} \\ 0 & & & & 2n+1\, Z & & 4n+2Z & & 2n+1\, Z \end{array}$$

so $KO^3 CP^{4n+2} = 0$.

3.5 Lemma The results for KO^iCP^{4n+2} imply those for KO^iCP^{4n+3}.

Proof From the Puppe sequence, we see that $KO^0CP^{4n+3} = 2n+1\,Z$, $KO^1CP^{4n+3} = 0$, $KO^2CP^{4n+3} = 2n+2Z$, $KO^3CP^{4n+3} = 0$, $KO^6CP^{4n+3} = 2n+2Z$ and $KO^7CP^{4n+3} = 0$.

Also, from

$$\begin{array}{ccccccc} KO^3CP^{4n+2} & \to & KO^3S^{8n+5} & \to & KO^4CP^{4n+3} & \to & KO^4CP^{4n+2} \\ 0 & & Z_2 & & & & 2n+1\,Z \end{array}$$

$$\begin{array}{ccccccc} \to & KO^4S^{8n+5} & \to & KO^5CP^{4n+3} & \to & KO^5CP^{4n+2} \\ & Z_2 & & & & 0 \end{array}$$

we see that $KO^4CP^{4n+3} = 2n+1\,Z + Z_2$ and $KO^5CP^{4n+3} = 0$ or Z_2.

However in the Bott sequence for CP^{4n+3}, we have

$$\begin{array}{ccccccc} K^0CP^{4n+3} & \to & KO^5CP^{4n+3} & \to & KO^4CP^{4n+3} & \to & K^0CP^{4n+3} \\ 0 & & & & 2n+1\,Z + Z_2 & & 4n+3Z \end{array}$$

and so $KO^5CP^{4n+3} = Z_2$.

To complete the induction we first show that $KO^6CP^{4n+4} = 2n+2Z$ by using the spectral sequence as in 3.3, then proceed as in 3.4 and 3.5. However as the proofs are identical, we omit them.

Again we summarize some of the results that we will need in the next chapter :

3.6 <u>Proposition</u> Let η denote the Hopf map from S^3 to S^2, or a suspension of it. Then if $r-i \equiv 1 \bmod 8$,

$$\eta^! : \underset{Z}{KO^i S^{r-1}} \rightarrow \underset{Z_2}{KO^i S^r} \quad \text{is an epimorphism}$$

$$\text{and } \eta^! : \underset{Z_2}{KO^i S^r} \rightarrow \underset{Z_2}{KO^i S^{r+1}} \quad \text{is an isomorphism .}$$

§4 <u>The groups $KO^i HP^n$</u>

HP^n denotes quaternionic projective space of real dimension 4n. In this section we prove the results given in the following

<u>TABLE</u> 4.1

i	0	1	2	3
$KO^i HP^{2n}$	$2nZ$	0	nZ_2	nZ_2
$KO^i HP^{2n+1}$	$2n{+}1\,Z$	0	$n{+}1\,Z_2$	$n{+}1\,Z_2$

i	4	5	6	7
$KO^i HP^{2n}$	$2nZ$	0	nZ_2	nZ_2
$KO^i HP^{2n+1}$	$2n{+}1\,Z$	0	nZ_2	nZ_2

For $i = 0$, these are given in [24, Theorem 3.11]

The following is obvious (compare 3.2 or [7,page80])

4.2 Lemma $\quad K^0 HP^n = nZ$ and $K^1 HP^n = 0$.

We now prove the results given in 4.1 by induction on n, they are true for n = 1 because $HP^1 = S^4$.

4.3 Lemma $\quad$ The results for $KO^i HP^{2n-1}$ imply those for $KO^i HP^{2n}$ $\quad (n \geq 1)$.

Proof $\quad$ The mapping cone of the projection map $S^{8n-1} \to HP^{2n-1}$ is HP^{2n} and from the Puppe sequence of this map we see that $KO^0 HP^{2n} = 2nZ$, $KO^1 HP^{2n} = 0$, $KO^2 HP^{2n} = nZ_2$, $KO^3 HP^{2n} = nZ_2$, $KO^4 HP^{2n} = 2nZ$ and $KO^5 HP^{2n} = 0$.

From the Bott sequence

$$\begin{array}{ccccccccc} KO^1 HP^{2n} & \to & KO^0 HP^{2n} & \xrightarrow{c} & K^0 HP^{2n} & \to & KO^2 HP^{2n} & \to & KO^1 HP^{2n} \\ 0 & & 2nZ & & 2nZ & & nZ_2 & & 0 \end{array}$$

we see that the cokernel of

$r: K^0 HP^{2n} \to KO^0 HP^{2n}$ is nZ_2, and so $KO^7 HP^{2n} = nZ_2$.

The map r in the following sequence is a monomorphism

$$\begin{array}{ccccccccc} K^1 HP^{2n} & \to & KO^7 HP^{2n} & \to & KO^6 HP^{2n} & \to & K^0 HP^{2n} & \xrightarrow{r} & KO^0 HP^{2n} \\ 0 & & nZ_2 & & & & 2nZ & & 2nZ \end{array}$$

and so $KO^6 HP^{2n} = nZ_2$.

The results are completed by another induction step which is virtually identical.

For completeness we state the following whose proof is easy.

4.4 Theorem

Let X denote the Cayley projective plane.

Then

$$KO^0X = KO^4X = Z + Z$$

$$KO^6X = KO^7X = Z_2 + Z_2$$

and $KO^iX = 0$ otherwise.

Chapter II. EQUIVARIANT MAPS BETWEEN SPHERES

In this chapter we consider questions of the following type.

Let G be a group acting on a sphere S^n in a fixed way. Which homotopy classes $\alpha \in \pi_n S^m$ can be represented by maps $f : S^n \to S^m$ that take G-orbits to points?

We treat the cases $G = Z_2$, S^1 or S^3.

Clearly such maps f factor through the quotient space S^n/G. We use the results of the previous chapter to find invariants of a given homotopy class and show that they must vanish (in certain cases) if they factor through S^n/G.

When $G = Z_2$, Bredon [11] has recently studied this question in more generality (when S^m also has a Z_2-action). His techniques can also be used when $G = Z_p$, S^1 or S^3. However they only apply in the "stable range" and to fairly low stems. By using results of Adams [3] we can also consider some elements in arbitrarily high stems.

This question arose from the particular case in [14, page 228], which had in fact previously been considered by Whitehead [31].

I would like to thank Professor Bredon for sending me fuller details and extensions of the work announced in [11], which have helped me to check some of my results.

§1 Immediate Applications

We fix our notation such that τ_r denotes an involution on $S^n \subset R^{n+1}$ that changes the sign of $r+1$ co-ordinates, the quotient space is then $S^{n-r}P^r$.

The symbols $f : (S^n, \tau_r) \to S^m$ denote a map $f : S^n \to S^m$ such that $f(\tau_r x) = fx$ for all $x \in S^n$.

For completeness we give the following elementary result

1.1 Theorem   Let $f : (S^n, \tau_r) \to S^m$, then $f : S^n \to S^n$ has even degree. If r is even then degree $f = 0$. If r is odd, such f exist with any even degree.

Proof

$f : S^n \to S^n$ factors as $S^n \xrightarrow{S^{n-r}\pi} S^{n-r}P^r \to S^n$.

In cohomology we have the following diagram

$$\begin{tikzcd}
Z = H^nS^n & & H^nS^n = Z \arrow[ll, "f^*"'] \arrow[dl] \\
& H^rP^r \arrow[ul, "\pi^*"] &
\end{tikzcd}$$

If r is even $H^rP^r = Z_2$ so $F^* = 0$ i.e. degree $f = 0$

If r is odd $H^rP^r = Z$ and π^* takes a generator of

H^rP^r to twice a generator of H^nS^n, hence the degree of f is even.

Now let $p : P^r \to S^r$ be the map that collapses $P^{r-1} \subset P^r$ to a point. If r is odd $p^* : H^rS^r \to H^rP^r$ is an isomorphism and so the composite map

$$S^n \xrightarrow{S^{n-r}\pi} S^{n-r}P^r \xrightarrow{S^{n-r}p} S^n$$ has degree two.

By composing this map with a map of degree $d : S^n \to S^n$ we have a map : $(S^n, \tau_r) \to S^n$ of degree 2d.

We now prove the following theorem due to J.H.C.Whitehead [31, Theorem 7] (see also Conner and Floyd [14, page 228]) by using the results of Chapter I.

1.2 <u>Theorem</u> Let $f : (S^n, \tau_r) \to S^{n-1}$.

If $r \not\equiv 3 \bmod 4$, then $f \simeq o$.

If $r \equiv 3 \bmod 4$, then every element in $\pi_n S^{n-1}$ can be represented by such a map.

<u>Proof</u> We remind ourselves that $\pi_n S^{n-1} = 0$ when $n < 3$, $\pi_3 S^2 = Z$ generated by the Hopf map η and $\pi_n S^{n-1} = Z_2$ generated by the suspension of the Hopf map for $n > 3$. We also denote this element by η .

By assumption the map f factors through $S^{n-r}P^r$ and so in KO-theory we have a diagram

$$\begin{tikzcd} KO^iS^r & & KO^iS^{r-1} \arrow[ll, "f^!"'] \arrow[dl] \\ & KO^iP^r \arrow[ul, "\pi^!"] & \end{tikzcd}$$

Now if $r > 3$ and $r-i \equiv 1 \bmod 8$, by I 2.15 if $r \not\equiv 3 \bmod 4$ $\pi^!$ is zero, but $\eta^!$ is non-zero by I 3.6. This proves the theorem when $r \not\equiv 3 \bmod 4$.

It is clear that the standard representation of the Hopf map $\eta : S^3 \to S^2$ factors through $\pi : S^3 \to P^3$. The following homotopy commutative diagram shows that every multiple of it also does

$$\begin{tikzcd} S^3 \arrow[r, "n"] \arrow[d, "\pi"'] & S^3 \arrow[r, "\eta"] \arrow[d, "\pi"'] & S^2 \\ P^3 \arrow[r, "f"] & P^3 \arrow[ur, "g"'] & \end{tikzcd}$$

Here g is the factorisation map for η, $n : S^3 \to S^3$ is any map of degree n and f is the map $x \longmapsto x^n$ ($P^3 = SO(3)$ is a group). It is well known that the square is homotopy commutative (e.g. [16])

It now remains to show that $\eta : S^{4n+3} \to S^{4n+2}$ factors through P^{4n+3} for every $n > 0$. We give two explicit representatives which do, one is constructed geometrically and the other by homotopy.

1) We make a few remarks about the Hopf construction. It assigns to every map $S^m \times S^n \to S^p$ a map $S^{m+n+1} \to S^{p+1}$ by a modified suspension. An involution τ_r on S^{m+n+1}

is induced by any involution $\tau_{r-k} \times \tau_{k-1}$ on $S^m \times S^n$.

Now let $f : S^1 \times S^{4n+1} \to S^{4n+1}$ be the map given in complex co-ordinates by

$$(z, z_1, z_2, \ldots, z_{2n+1}) \longmapsto (zz_1, zz_2, \ldots, zz_{2n+1})$$

It is well known that the Hopf construction applied to this map gives a representative of the homotopy class η. It is also clear that it is a map

$$(S^{4n+3}, \tau_{4n+3}) \to S^{4n+2} .$$

2) Let $\pi : S^{4n+3} \to P^{4n+3}$ be the covering map and $q : P^{4n+3} \to P^{4n+3}/P^{4n+1}$ the collapsing map. Now $P^{2m+1}/P^{2m-1} \simeq S^{2m+1} \vee S^{2m}$ because it is of the form $S^{2m} \cup_\alpha e^{2m+1}$ and the attaching map α is the composite $S^{2m} \overset{\pi}{\to} P^{2m} \overset{q}{\to} P^{2m}/P^{2m-1}$ which is trivial by 1.1 . Hence we have a map $f : S^{4n+3} \to S^{4n+3} \vee S^{4n+2}$. By collapsing the S^{4n+3} to a point we get a map $g : S^{4n+3} \to S^{4n+2}$ which factors through P^{4n+3} by construction, we show that it is essential. When $n > 1$, there is a decomposition

$$\pi_{4n+3}(S^{4n+3} \vee S^{4n+2}) \cong \pi_{4n+3}(S^{4n+3}) + \pi_{4n+2}(S^{4n+2})$$

so if $g \simeq o$, the mapping cone of f would be $S^{4n+2}P^2 \vee S^{4n+2}$, however we know that it is P^{4n+4}/P^{4n+1} and Sq^2 is different in these spaces.

1.3 <u>Theorem</u> Let $f : (S^n, \tau_r) \to S^{n-2}$

If $r \equiv 1,2 \bmod 4$ then $f \simeq o$.

If $r \equiv 3 \bmod 4$, every element in $\pi_n S^{n-2}$ can be represented by such a map.

<u>Proof</u> $\pi_n S^{n-2} = Z_2$ if $n > 3$ and is generated by the element $\eta \circ \eta = \eta^2$. If $n \leqslant 3$ then the group is zero.

The second statement is an immediate consequence of Theorem 1.2.

The first statement follows from I 2.15 and I 3.6 as in 1.2.

<u>Remark</u> We will show in §2 that if $r \equiv 0 \bmod 4$, then η^2 can be represented by an equivariant map if $n > 4$.

In [3] Adams introduces an element μ_{8s+1} in the stable 8s+1 stem and an element μ_{8s+2} in the stable 8s+2 stem that are generalisations of η and η^2 respectively. They generate Z_2 summands in the stable stems and induce some non-zero maps on KO. Hence we can deduce similar results for these elements.

1.4 <u>Theorem</u> i) $\mu_{8s+1} : S^{n+8s+1} \to S^n$ can not be represented by a map $f : (S^{n+8s+1}, \tau_r) \to S^n$ if $r \not\equiv 3 \bmod 4$.

ii) $\mu_{8s+2} : S^{n+8s+2} \to S^n$ can not be represented by a map $f : (S^{n+8s+2}, \tau_r) \to S^n$ if $r \equiv 1,2 \mod 4$ and can if $r \equiv 3 \mod 4$.

Proof i) and the first statement of ii) are obvious because the elements μ are non-zero on KO in exactly the same dimensions as η is non-zero.

The second statement of ii) is a consequence of 1.2 and of Proposition 12.14 of [3] which implies that μ_{8s+2} can be taken to be $\mu_{8s+1} \circ \eta$.

Let σ_r denote an action of S^1 on S^n ($n \geqslant 2r+1$) that multiplies in r+1 complex co-ordinates. The quotient space will be $S^{n-2r-1}CP^r$. Any map $f : (S^n, \sigma_r) \to S^m$ is clearly a map $f : (S^n, \tau_{2r+1}) \to S^m$. From this we can immediately deduce similar results for S^1-actions, but for all the cases already considered the results are identical so we will not state them fully.

Let ω_r denote an action of S^3 on S^n, whose quotient space is $S^{n-4r-3}HP^r$. Then we have the following result

1.5 <u>Theorem</u>. Let $s \geq 1$ then $\mu_{8s+1} : S^{n+8s+1} \to S^n$ can not be represented by a map

$$f : (S^{n+8s+1}, \omega_r) \to S^n \quad \text{for any } r.$$

The same result holds for the element μ_{8s+2}.

<u>Proof</u> This is an immediate consequence of the facts $KO^1 HP^r = 0$ and $KO^5 HP^r = 0$ that were proved in I §4.

§2 The stable cohomotopy spectral sequence

In this section we use the stable cohomotopy spectral sequence to study the problem of which homotopy classes of maps between spheres can be represented by equivariant maps.

Let $\{X, Y\} = \varinjlim [S^r X, S^r Y]$ with the direct limit maps being suspensions. It is well known see e.g. [30] that we get a (generalised) cohomology theory by setting $h^{-n}(X) = \{S^n X, S^0\}$ and so we get an Atiyah-Hirzebruch spectral sequence with $E_2^{p,q} = H^p(X; \pi^s_{-q})$ and $E_\infty = \{h^{-n}(X)\}$ where $\pi^s_{-q} = \{S^q, S^0\}$ the stable q-stem.

This spectral sequence is discussed in Massey [18] and Peterson [22]. The use of the spectral sequence for computations is equivalent to studying the Postnikov decomposition of a high dimensional sphere, for this see Maunder [19].

We are now in a position to prove the following, which completes the results about Z_2-actions on the 2-stem .

2.1 <u>Theorem</u> $\eta^2 : S^n \to S^{n-2}$ can be represented by a map $f : (S^n, \tau_{4r}) \to S^{n-2}$ if and only if $n > 4$.

<u>Proof</u> It suffices to look at maps $S^{4r} \to S^{4r-2}$.

Let $r > 1$, then we are in the stable range. From the Puppe sequence of the map $\pi : S^{4r} \to P^{4r}$ we get

$$[SP^{4r}, S^{4r-1}] \xrightarrow{S\pi^*} [S^{4r+1}, S^{4r-1}] \to [P^{4r+1}, S^{4r-1}]$$

$$\to [P^{4r}, S^{4r-1}] \xrightarrow{\pi^*} [S^{4r}, S^{4r-1}]$$

We want to show that the map $S\pi^*$ is an epimorphism. To do this we calculate the various groups that appear in the sequence by means of the spectral sequence.

By [22,page 459], the initial differential $d_2 : E_2^{n,0} \to E_2^{n+2,-1}$ is $Sq^2\rho_2 : H^n(X;Z) \to H^{n+2}(X;Z_2)$. We now give an ad hoc proof that $d_2 : E_2^{n,-1} \to E_2^{n+2,-2}$ is Sq^2 . We compute $\{P^4, S^2\} = [S^2P^4, S^4]$ from the spectral sequence. However we can calculate its value independently as follows. $[S^2P^4, S^4] = [S^2P^4, BSp]$ because the 7-skeleton of BSp is S^4. By Bott periodicity

$BSp \simeq \Omega^4 BO$, so $[S^2P^4, S^4] = KO^2P^4 = Z_2$ by I 2.1 .

The relevant part of the E_2 term of the spectral sequence for P^4 is

$$\begin{tikzcd}[row sep=small]
Z & 0 & Z_2 \arrow[drr, "d_2'"] & 0 & Z_2 \\
Z_2 & Z_2 & Z_2 \arrow[drr, "d_2"] & Z_2 & Z_2 \\
Z_2 & Z_2 & Z_2 & Z_2 & Z_2
\end{tikzcd}$$

Because $\{P^4, S^2\} = Z_2$, both d_2 and d_2' are isomorphisms, but they are stable cohomology operations, hence d_2 is Sq^2 (and this checks that d_2' is $Sq^2\rho_2$).

We now compute $[P^{4r+1}, S^{4r-1}] = \{P^{4r+1}, S^{4r-1}\}$ $(r > 1)$

The relevant part of the E_2-term of the spectral sequence for P^{4r+1} is

$$\begin{tikzcd}[row sep=small]
4r-2 & 4r-1 & 4r & 4r+1 \\
Z_2 \arrow[drr] & 0 & Z_2 & Z \\
Z_2 & Z_2 \arrow[drr] & Z_2 & Z_2 \\
Z_2 & Z_2 & Z_2 & Z_2
\end{tikzcd}$$

Both the marked differentials are isomorphisms and so $[P^{4r+1}, S^{4r-1}] = 0$, which proves the result for $r > 1$.

It remains to look at the involution τ_4 . We first consider the case $n \geq 5$.

We want to show that the map $S\pi^*$ in the following exact sequence is an epimorphism

$$[S^{n-4}P^4, S^{n-2}] \xrightarrow{S\pi^*} [S^n, S^{n-2}] \to [S^{n-5}P^5, S^{n-2}]$$

By Freudenthal's suspension theorem $[SP^5, S^4] \cong [S^{n-5}P^5, S^{n-2}]$ for $n \geq 6$. However from the exact sequence of the Hopf fibration $S^7 \to S^4$ we see that $[SP^5, S^4] \to [P^5, S^3]$ is an epimorphism. But $[SP^5, S^4] \cong KO^3P^5 = 0$ by I 2.1.

The following will complete the proof of the theorem

2.2 <u>Lemma</u> $[P^4, S^2] = 0$.

<u>Proof</u>

From the Hopf fibration $S^3 \to S^2$, we have the following exact sequence of based sets

$$[P^4, S^3] \to [P^4, S^2] \to [P^4, BS^1] \xrightarrow{f} [P^4, BS^3]$$

Now $[P^4, S^3] \cong [SP^4, S^4] \cong KO^3P^4 = 0$. So it will be enough to show that f is injective.

$BS^1 = BU(1)$ and $BS^3 = BSU(2)$. The map f is induced by the usual inclusion of S^1 in S^3 and clearly takes a bundle ξ to the bundle $\xi \oplus \bar{\xi}$ where $\bar{\xi}$ denotes the conjugate bundle. There is just one non-trivial line bundle on P^4 and its first Chern class is the generator of $H^2(P^4;Z)$. We show that $\xi \oplus \bar{\xi}$ is non-trivial by computing its second Chern class. Clearly $c_1(\bar{\xi}) = c_1(\xi)$ and by the product formula [20, Theorem 26] we have $c_2(\xi\oplus\bar{\xi}) = c_1(\xi).c_1(\bar{\xi})$, and this is non-trivial on P^4.

2.3 <u>Theorem</u> Let $n \geqslant 8$, the elements in $\pi_n(S^{n-3})$ (which is Z_{24}, generated by a suspension of the Hopf map $\nu : S^7 \to S^4$) that can be represented by a map $f : (S^n, \tau_r) \to S^{n-3}$ are precisely (for $r \geqslant 8$)

Z_{12}	when	$r \equiv 1 \bmod 4$
0		$r \equiv 2 \bmod 4$
Z_{12} or Z_{24}		$r \equiv 3 \bmod 4$
Z_2		$r \equiv 0 \bmod 4$

<u>Proof</u> As we are taking $r \geqslant 8$, we are in the stable range and so it is enough to look at the diagram

$$\begin{tikzcd} S^r \arrow[rr] \arrow[dr, "\pi"'] & & S^{r-3} \\ & P^r \arrow[ur] & \end{tikzcd}$$

Case 1. $r \equiv 1 \bmod 4$

We show that the map $S\pi^*$ in the following exact sequence has image Z_{12}.

$$[SP^{4m+1}, S^{4m-1}] \xrightarrow{S\pi^*} [S^{4m+2}, S^{4m-1}]$$
$$\to [P^{4m+2}, S^{4m-1}] \to [P^{4m+1}, S^{4m-1}]$$

As in the proof of 2.1 we see that $[P^{4m+2}, S^{4m-1}] = Z_2$

and $[P^{4m+1}, S^{4m-1}] = 0$, which proves the result.

Case 2. $r = 4m+2$

As before it is easily checked from the spectral sequence that $[P^{4m+2}, S^{4m}]$ has order eight. However to compute $[P^{4m+3}, S^{4m}]$ we must evaluate a differential in the E_3-term :

$$d_3 : E_3^{4m,0} \to E_3^{4m+3,-2}$$

and $E_3^{4m,0} = H^{4m}(P^{4m+3};Z)$, $E_3^{4m+3,-2} = H^{4m+3}(P^{4m+3};Z_2)$ the differential is Adem's stable secondary operation Φ, see [4]. Using 1.3 we check that $[P^{4m+3}, S^{4m+1}]$ has order four and so d_3 is zero. So the order of $[P^{4m+3}, S^{4m}]$ is 8.24 which implies the result.

Case 3. $r = 4m+3$

The result stated is an immediate consequence of 1.1 . By using this sort of method it does not seem possible to settle this case (we would have to evaluate a differential in the E_4-term of the spectral sequence). However Professor Bredon has pointed out to me that it is a consequence of theorem 5.4 of his paper [12] that the image of the map $S\pi^*$ is Z_{12} when $r \equiv 3 \bmod 8$ and Z_{24} when $r \equiv 7 \bmod 8$. His theorem also implies the result for case 1.

Case 4. $r = 4m$

It follows easily from the spectral sequence that $[P^{4m+1}, S^{4m-2}]$ is either Z_{24} or Z_{12}, in which case the image of $S\pi^*$ is either 0 or Z_2. however the non-zero element of $Z_2 \subset Z_{24}$ is η^3 and this is in the image by 2.1 . Hence the result and also $[P^{4m+1}, S^{4m-2}] = Z_{12}$.

We turn our attention now to some cases outside the stable range.

$\pi_5 S^2 = Z_2$ generated by the element η^3 .

2.4 <u>Theorem</u> $\eta^3 : S^5 \to S^2$ can be represented by a map $f : (S^5, \tau_r) \to S^2$ if $r = 3,4$ but not if $r = 2,5$.

<u>Proof</u> The cases $r = 3,4$ follow immediately from 1.2 and 2.1 .

By an identical proof to that of 2.2, we can show that $[P^5, S^2] = 0$.

It is well known that if $\pi : S^2 \to P^2$ is the covering map then $S^2\pi \simeq o$ (this will be proved in the next section) . This completes the proof of 2.4 .

§3 Further non-stable results

First, we prove the following lemma. I am very grateful to Dr. B.J.Sanderson for showing me this result.

3.1 Lemma Let M^m be a manifold and M_o be M with an open disc removed. Let $f : M \to S^{n+m}$ be a differentiable embedding of M in S^{n+m} with trivial normal bundle, then $S^nM \simeq S^nM_o \vee S^{n+m}$.

Proof Let N denote the tubular neighbourhood of the embedding and T the Thom complex of the normal bundle. Then $T \simeq S^nM$, and $T = N/\partial N$. By removing a small disc of dimension $n+m$ from the interior of N, we can get the space $T - D$ which is clearly homotopically equivalent to S^nM_o. The attaching map of the $n+m$ disc D can be homotoped to zero over the sphere and so over the Thom complex T, which proves the lemma.

This lemma replaces a rather complicated direct proof of the following

3.2 Corollary The maps $\pi : S^2 \to P^2$ and $\pi : S^6 \to P^6$ are stably trivial.

Proof We prove that $\pi : S^6 \to P^6$ is stably trivial (the other case is similar).

We embed P^7 in S^{15}. P^7 is parallelisable and so the normal bundle is trivial . P^7 with an open disc removed is homotopically equivalent to P^6 and the attaching map for this disc is easily seen to be π, hence the result.

In fact we have shown that the attaching map for the top cell of any π-manifold is stably trivial.

We also have the following well known result

3.3 Corollary P^3 does not embed in R^4.

Proof Suppose it does, the normal bundle is either the trivial or Hopf line bundle. But it is stably trivial, so it is trivial. However we show that $SP^3 \not\simeq SP^2 \vee S^4$.

Let $\theta \in H^4(K(Z_2,2);Z_4) = Z_4$ be a generator.

By looking at the spectral sequence of the relative fibration $(PX,\Omega X) \to (X,o)$ with Z_4 coefficients, where PX is the space of based paths on X and $X = K(Z_2,2)$ we can check that the suspension map

$$H^4(K(Z_2,2);Z_4) \to H^3(K(Z_2,1);Z_4) = Z_2$$

is an epimorphism. $H^3(K(Z_2,1);Z_4)$ is generated by the third power which is non-zero on P^3. Hence θ is non-zero on SP^3 but is zero on $SP^2 \vee S^4$,

Alternatively we could have identified θ with the Pontryagin square operation.

A similar proof would show that P^7 does not embed in R^8.

We now discuss $\pi_6 S^3 = Z_{12}$. A generator of this group can be described as follows, see [27].

Let $f : S^3 \times S^2 \to S^2$ be defined by :
Take $q \in S^3$ as a quaternion and $q' \in S^2$ as a purely imaginary quaternion , then let $f(q,q') = qq'q^{-1}$.

The Hopf construction applied to f gives a representative of the generator of $\pi_6 S^3$.

3.4 <u>Theorem</u> The elements of $\pi_6 S^3$ that can be represented by a map $f : (S^6, \tau_r) \to S^3$ are

0	when	$r = 2$
Z_{12}		$r = 3$
Z_2		$r = 4$
Z_6		$r = 5$
0		$r = 6$

<u>Proof</u> Because the group $\pi_6 S^3$ is not killed by arbitrary many suspensions, the cases r = 2 , 6 are immediate from 3.2 .

The case r = 3 follows immediately from the description of the generator.

From the Puppe sequence $S^3 \xrightarrow{\pi} P^3 \to P^4$ we then deduce that $[S^2P^4, S^3] = Z_2$, however, from 1.2 we know that the element η^3 which has order two is in the image of $S^2\pi^* : [S^2P^4, S^3] \to [S^6, S^3]$. Hence the image is exactly Z_2 .

It now follows from the Puppe sequence

$S^4 \xrightarrow{\pi} P^4 \rightarrow P^5$ that $[SP^5, S^3] = Z_6$. It follows from 1.1 that the image is at least Z_6.

$\pi_7 S^4 = Z + Z_{12}$, the Z summand is generated by the Hopf map $\nu : S^7 \rightarrow S^4$ and the Z_{12} summand is in the image of the suspension map $\pi_6 S^3 \rightarrow \pi_7 S^4$.

3.5 <u>Theorem</u> The elements of $\pi_7 S^4$ that can be represented by a map $f : (S^7, \tau_r) \rightarrow S^4$ are

0	when $r = 2$
$Z + Z_{12}$	$r = 3$
Z_2	$r = 4$
$2Z + Z_6$	$r = 5$
0	$r = 6$
at least $2Z + Z_6$	$r = 7$

<u>Proof</u> The cases $r = 2,6$ are immediate from 3.2. The case $r=3$ is shown by explicit construction.

By 1.2 the elements in the Z_2 can be represented by such maps when $r = 4$. From the Puppe sequence

$$[S^7, S^4] \xrightarrow{S^3p^*} [S^3P^4, S^4] \xrightarrow{S^3i^*} [S^3P^3, S^4] \xrightarrow{S^3\pi^*} [S^6, S^4]$$

and the facts $[S^3P^3, S^4] \cong KO^1P^3 = Z_2 + Z_2$ from I 2.1, S^3p^* is zero from the case $r = 3$ of this theorem, $S^3\pi^*$ is an epimorphism from 1.3, we deduce that

$[S^3P^4, S^4] = Z_2$ which implies the result for $r = 4$.

Similarly, from the Puppe sequence $S^4 \to P^4 \to P^5$ we can show that $[S^2P^5, S^4] = Z + Z_6$ and that the map $S^2p^* : [S^7, S^4] \to [S^2P^5, S^4]$ is an epimorphism, but the composite $S^7 \xrightarrow{S^2\pi} S^2P^5 \xrightarrow{S^2p} S^7$ has degree two, which implies the result for $r = 5$.

The result stated for $r = 7$ is an immediate consequence of 1.1 . However there are other elements that can be represented by equivariant maps, clearly the Hopf map ν is such a map and as P^7 is an H-space every multiple of it is also (cf. the proof of 1.2) . However it is not clear that the set of such elements forms a subgroup.

The following theorem completes the results for the 3-stem

3.6 <u>Theorem</u> The elements of $\pi_n S^{n-3}$ ($n \geq 8$) ($\pi_n S^{n-3} = Z_{24}$, generated by the suspension of the Hopf map $\nu : S^7 \to S^4$) that can be represented by a map $f : (S^n, \tau_r) \to S^{n-3}$ are

0 when $r = 2$

Z_{24} $r = 3$

.

$$\begin{array}{ll} Z_2 & \text{when } r = 4 \\ Z_{12} & \qquad r = 5 \\ 0 & \qquad r = 6 \\ Z_{24} & \qquad r = 7\,. \end{array}$$

Proof These results follow immediately from the previous proof and the Freudenthal suspension theorem.

Chapter III. MULTIPLICATIONS ON PROJECTIVE SPACES

Let $\varphi : X \vee X \to X$ denote the 'folding' map.

A multiplication on a space X is a map $\mu : X \times X \to X$ such that $\mu | X \vee X = \varphi$.

By Adams [1] the only projective spaces that can have a multiplication are P^1, P^3 and P^7.

In this chapter we compute the number of homotopy classes of multiplications on P^3 and P^7. Two multiplications on a space are said to be homotopic if they are homotopic as maps relative to the wedge. This problem is hinted at in [17]. The general problem of finding the number of multiplications on an H-space is Problem 43 in Massey's list of problems (Ann. Math. vol. 62 (1955) p.327-359).

If a space X admits a homotopy associative multiplication and is such that $X \vee X \subset X \times X$ is a cofibration then Arkowitz and Curjel [6] set up a 1-1 correspondence between the set of multiplications on X and the homotopy set $[X \wedge X, X]$.

Using this result Naylor [21] showed that P^3 has exactly 768 different multiplications. We reprove his result in §1, in a much more elementary way and to stress thès we will not assume any results from the previous chapters.

In §2 we show that the number of multiplications on P^7 is 30,720. P^7 does not admit a homotopy associative multiplication, however the number of multiplications on P^7 is in 1-1 correspondence with the set $[P^7 \wedge P^7, P^7] \cong [P^7 \wedge P^7, S^7]$, because the proof in [6] only asumes that the multiplication has an inverse.

§1 Multiplications on P^3

As we have pointed out already, we must compute the order of the group $[P^3 \wedge P^3, S^3]$.

1.1 Lemma Let $\pi : S^2 \to P^2$ be the covering map, then $S^2\pi \simeq o$.

Proof cf. [21]

From the homotopy sequence of the pair (S^2P^2, S^3) it is seen that any map $S^4 \to S^2P^2$ factors through S^3. So if we assume that $S^2\pi \neq o$, it must factor as $S^4 \xrightarrow{\eta} S^3 \subset S^2P^2$. Then the mapping cone of $S^2\pi$ would contain SCP^2 and so Sq^2 would be non-zero on S^2P^2 a contradiction.

1.2 Lemma Let K be a five dimensional complex, then $[K^5, S^3] \cong [SK^5, S^4]$.

Proof The isomorphism is the boundary map in the Puppe sequence of the fibration $S^7 \to S^4$.

1.3 <u>Proposition</u> i) $[S^3P^2, S^3]$ has order four.

ii) $[S^3P^3, S^3]$ has order 48.

iii) $[P^2 \wedge P^2, S^3]$ has order four.

iv) $[P^3 \wedge P^2, S^3]$ has order 16.

<u>Proof</u> i) is immediate from the Puppe sequence of the map $S^1 \to S^1$ which is multiplication by two.

ii). From the Puppe sequence of the map $\pi : S^2 \to P^2$ we get

$$[S^4P^2, S^3] \to [S^6, S^3] \to [S^3P^3, S^3] \to [S^3P^2, S^3] \to [S^5, S^3]$$

both the maps at the ends are zero by 1.1, this together with i) gives the result.

iii). From the Puppe sequence of the map $f \wedge 1 : S^1 \wedge P^2 \to S^1 \wedge P^2$, where f has degree two, we get

$$[S^2 \wedge P^2, S^3] \to [S^2 \wedge P^2, S^3] \to [P^2 \wedge P^2, S^3]$$
$$\to [S^1 \wedge P^2, S^3] \to [S^1 \wedge P^2, S^3]$$

Both $[S^2 \wedge P^2, S^3]$ and $[S^1 \wedge P^2, S^3]$ are Z_2 and the end maps are multiplication by two, so $[P^2 \wedge P^2, S^3]$ has order four.

iv). From the Puppe sequence of $\pi \wedge 1 : S^2 \wedge P^2 \to P^2 \wedge P^2$, we get $[SP^2 \wedge P^2, S^3] \to [S^3 \wedge P^2, S^3] \to [P^3 \wedge P^2, S^3]$

$$\to [P^2 \wedge P^2, S^3] \to [S^2 \wedge P^2, S^3]$$

By 1.2, both the end maps are in the stable range and as $o \wedge 1$ is o, we have the result by 1.1, i) and iii).

Now, we look at the Puppe sequence of the map $\pi \wedge 1 : S^2 \wedge P^3 \to P^2 \wedge P^3$ and get

$$[SP^2 \wedge P^3, S^3] \xrightarrow{f} [S^3 \wedge P^3, S^3] \to [P^3 \wedge P^3, S^3]$$
$$\to [P^2 \wedge P^3, S^3] \xrightarrow{g} [S^2 \wedge P^3, S^3].$$

By the results in 1.3 it only remains to show that both f and g are zero. g is zero by 1.1 and 1.2 . There is a commuting diagram

$$\begin{tikzcd}
[S^2P^2 \wedge P^3, S^4] \arrow[r, "Sf"] \arrow[d] & [S^4 \wedge P^3, S^4] \arrow[d] \\
[SP^2 \wedge P^3, S^3] \arrow[r, "f"] & [S^3 \wedge P^3, S^3]
\end{tikzcd}$$

The vertical maps are induced by the boundary map in the Puppe sequence of the Hopf fibration $S^7 \to S^4$ and so are epimorphisms. Sf is zero by 1.1 , so $f = 0$ also. Hence $[P^3 \wedge P^3, S^3]$ has order $16.48 = 768$.

§2 Multiplications on P^7

The calculation for P^7 follows the same general pattern as that for P^3, although the details are obviously much more complicated.

Much use is made of the fact that the map $\pi : S^6 \to P^6$ is stably trivial, this was proved in II 3.2 .

We remind ourselves that $\pi_{13}S^7$ is Z_2 and that $\pi_{14}S^7$ is Z_{120} , [28] . The following analogue of 1.2 will also be used

2.1 Lemma Let K be a 13-dimensional complex, then $[K,S^7] \cong [SK,S^8]$.

Proof The isomorphism is induced by the boundary map in the Puppe sequence of the fibration $S^{15} \to S^8$.

For calculations with the stable cohomotopy spectral sequence, we will need the following

2.2 Lemma $d_2 : E_2^{n,-2} \to E_2^{n+2,-3}$ is $i_*Sq^2 : H^n(X;Z_2) \to H^{n+2}(X;Z_{24})$ where i_* is induced by the inclusion $Z_2 \subset Z_{24}$.

Proof By looking at the spectral sequence for P^5 and knowing from II 3.6 that $[S^5P^5,S^7] = Z_{12}$, it is clear that d_2 is non-zero. The only non-zero such stable cohomology operation is i_*Sq^2 .

2.3 Proposition

i) $[S^5P^6, S^7] = 0$, $[S^6P^6, S^7] = 0$ and $[S^7P^6, S^7]$ has order eight.

ii) $[P^6 \wedge P^5, S^7]$ has order four.

Proof

i) follows from the spectral sequence for P^6 in a straightforward way.

ii) follows from the spectral sequence for $P^6 \wedge P^5$. The only differentials that have to be evaluated are in the E_2-term. The evaluation of the differentials that are Sq^2 is done by means of the Cartan formula [26]. However to evaluate $Sq^2\rho_2$ one first has to look at the cell structure of $P^6 \wedge P^5$ and the results that are needed are that

$$Sq^2\rho_2 : H^n(P^6 \wedge P^5; Z) \to H^{n+2}(P^6 \wedge P^5; Z_2)$$

is an isomorphism on two summands from $3Z_2$ to $3Z_2$ when $n = 7$,

and an isomorphism on one summand from $2Z_2$ to $4Z_2$ when $n = 6$.

2.4 Proposition i) $[S^7P^7, S^7]$ has order 8.120

ii) $[P^6 \wedge P^6, S^7]$ has order four

iii) $[P^6 \wedge P^7, S^7]$ has order 2^5.

Proof

i). We have a Puppe sequence

$$[S^7 \wedge SP^6, S^7] \xrightarrow{f} [S^7 \wedge S^7, S^7] \to [S^7 \wedge P^7, S^7]$$
$$\to [S^7 \wedge P^6, S^7] \xrightarrow{g} [S^7 \wedge S^6, S^7]$$

The map f is induced by $1 \wedge S\pi : S^7 \wedge S^7 \to S^7 \wedge SP^6$, so is zero by II 3.2 . g is also zero for the same reason.

ii). We have a Puppe sequence

$$[S^6 \wedge P^6, S^7] \to [P^6 \wedge P^6, S^7] \to [P^6 \wedge P^5, S^7] \to [P^6 \wedge S^5, S^7]$$

Both the end groups are zero by 2.3 i), hence $[P^6 \wedge P^6, S^7] \cong [P^6 \wedge P^5, S^7]$ which has order four by 2.3 ii).

iii) Similarly we have

$$[P^6 \wedge SP^6, S^7] \xrightarrow{f} [P^6 \wedge S^7, S^7] \to [P^6 \wedge P^7, S^7]$$
$$\to [P^6 \wedge P^6, S^7] \xrightarrow{g} [P^6 \wedge S^6, S^7]$$

From the results in 2.3 i) and 2.4 ii) we must show that both f and g are zero. g is in the stable range and so is zero by II 3.2. An application of 2.1 brings f also into the stable range.

2.5 Theorem The number of distinct homotopy classes of multiplications on P^7 is 30,720.

<u>Proof</u> We look at the sequence

$$[SP^6 \wedge P^7, S^7] \xrightarrow{f} [S^7 \wedge P^7, S^7] \to [P^7 \wedge P^7, S^7]$$

$$\to [P^6 \wedge P^7, S^7] \xrightarrow{g} [S^6 \wedge P^7, S^7]$$

The map g is zero by II 3.2 and 2.1 .

By the results in 2.4, it remains only to show that f is zero. Suppose f is non-zero, then there is an element $x \in [SP^6 \wedge P^7, S^7]$ such that $fx \neq 0$. Then because the sequence of the Hopf fibration $S^{15} \to S^8$ splits (the splitting map is the suspension) $Sfx \neq 0$

We know that $S^2 fx = 0$ by II 3.2 and

$Sfx \in [S^8 P^7, S^8] \cong [S^8 P^6 \vee S^{15}, S^8]$. The kernel of $S : [S^8 P^7, S^8] \to [S^9 P^7, S^9]$ is generated by the element $2\sigma - \sigma'$ where $\pi_{15} S^8 = Z + Z_{120}$ and σ generates the Z summand and σ' generates the Z_{120} summand , [28].

So the element Sx has infinite order. However it is an element of $[S^2 P^6 \wedge P^7, S^8]$ which sits in the exact sequence

$$[S^2 P^6 \wedge S^7, S^8] \to [S^2 P^6 \wedge P^7, S^8] \to [S^2 P^6 \wedge P^6, S^8]$$

both the end groups in this sequence are finite and so we have a contradiction because $Sfx \in \text{Ker } S$.

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