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# THE EXISTENCE AND CHARACTERISATION OF DUALITY OF MARKOV PROCESSES IN THE EUCLIDEAN SPACE

RUI XIN LEE

A thesis submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Statistics

September, 2013

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*Captain, the most elementary*
*and valuable statement in science,*
*the beginning of wisdom, is,*
*"I do not know".*
*I do not know what that is, sir.*

— Lt. Cmdr. Data, Entrprise, Star Trek

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Lastly but certainly not least, I have been very privileged to have the unbounded support and faith from my parents, even though they have not been able to be beside me since I was 17. This thesis is dedicated to my parents.

# DECLARATION

The work in Chapter 4, 5, 6, 7 is the result of collaboration with Professor Vassili Kolokoltsov and has been published in the following paper:

- Vassili Kolokoltsov and Rui Xin Lee. *Stochastic duality of Markov processes: a study via generators*. Stochastic Analysis and Application. 31: 992-1023. 2013.

I declare that, to the best of my knowledge, all material contained in this thesis is my own original work, unless otherwise stated, cited or commonly known. No part of this thesis has been submitted for a degree at any other university.

*United Kingdom, September, 2013*

Rui Xin Lee

# ABSTRACT

This thesis examines the existence of dual Markov processes and presents the full characterization of Markov processes in Euclidean space equipped with the natural order (the Pareto order).

Considering the theory of Siegmund's duality for real-valued Markov Processes, we have presented an alternative proof to Siegmund [72] using Lebesgue–Stieltjes integration by parts to show the existence of a Markov dual process in several one dimensional cases, including the real space and closed intervals. Assuming that a dual process exists, we also provided a straightforward method, using duality relation, to compute explicitly the dual generator to a Feller process of the usual Lévy-Khintchine type.

We extended Siegmund's duality to finite dimensional space equipped with the Pareto order. The existence of a dual Markov process on an arbitrary Euclidean space is shown using Fubini's Theorem applied to Siegmund's approach. Given a pre-generator of the general Lévy-Khintchine type, we were able to construct a Feller process with an invariant core under some conditions assumed on the pre-generator. Furthermore, we also showed the criterion for the Feller process to have a dual Markov process.

We then studied the relationship between intertwining and duality for two processes in the sense of $\mathbb{E}f(X_t^x, y) = \mathbb{E}f(x, Y_t^y)$ for a certain function $f$. Of most interest are shift-invariant functions (functions which depend on the difference of their arguments). To explore this, we developed a systematic approach to duality using the analysis of the generators of dual Markov processes, then illustrated this approach using various examples. In particular, we gave a full characterization of duality arising from Pareto order in $\mathbb{R}^d$ in terms of generators for basic classes of Feller processes.

Lastly, we initiate the application of intertwining to the study of duality of Markov processes in domains with a boundary. To circumvent specific difficulties arising from the boundary, we introduce an additional tool of a regularized dual.

# 1 INTRODUCTION

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## 1.1 INTRODUCTION

There are different notions for duality in stochastic analysis. For instance, the Markov processes $(X_t^x)_{t\geqslant 0}$ and $(Y_t^y)_{t\geqslant 0}$ (small $x$, $y$ here, and in what follows, denote the initial points of respective processes) which take values in the same Borel space $\mathsf{E}$ are called dual with respect to the reference measure $\nu$ on $\mathsf{E}$, if the duality equation

$$\int_{\mathsf{E}} \mathbb{E}h(X_t^x)g(x)\nu(dx) = \int_{\mathsf{E}} h(x)\mathbb{E}g(Y_x^t)\nu(dx) \tag{1.1}$$

holds for the appropriate class of functions $h$ and $g$.

Alternatively, suppose that $f$ is a Borel function on the product $\mathsf{E} \times \mathsf{F}$ of two Borel spaces. One says that an $\mathsf{F}$-valued Markov process $Y_t^y$ is an $f$-dual (or dual with respect to function $f$) to an $\mathsf{E}$-valued Markov process $X_t^x$, if

$$\mathbb{E}f(x, Y_t^y) = \mathbb{E}f(X_t^x, y) \tag{1.2}$$

for all $x \in \mathsf{E}, y \in \mathsf{F}$. In this definition, $\mathbb{E}$ on the right hand side and the left hand side of (1.2) correspond to the distributions of processes $X_t^x$ and $Y_t^y$ respectively.

A particular case of (1.2) is the duality of one-dimensional processes (spaces $\mathbf{E}$ and $\mathbf{F}$ are real-valued) arising from stochastic monotonicity, where $f(x, y) = \mathbf{1}_{\{x \geqslant y\}}$ and hence (1.2) becomes

$$\mathbf{P}(Y_t^y \leqslant x) = \mathbf{P}(X_t^x \geqslant y). \tag{1.3}$$

This is Siegmund's duality. In the classical Lévy's example of this duality, $X_t^x$ and $Y_t^y$ are the reflected and absorbed Brownian motions on $\mathbb{R}_+$.

Duality of the first kind in (1.1) is not the focus of this thesis and we refer readers to e.g. work by Angiuli et al. [4] and references therein for a detailed survey of the theory. Instead, we are concerned with the $f$-duality, in particular, the existence and characterisation of dual Markov processes in the sense of (1.2).

This thesis presents three approaches to the problem of finding the existence of duality, namely the Lebesgue-Stieltjes integration by parts, Fubini's Theorem and intertwining. Part I of this thesis (chapters 2 and 3) focuses on the existence of dual Markov processes in finite dimensional real space. This part begins by revisiting the existence of Siegmund's duality in different scenarios in the real space $\mathbb{R}$. In contrast to the method given by Siegmund [72], our proof to the existence of duality employs Lebesgue-Stieltjes integration by parts. This notion of duality is extended to finite-dimensional spaces, where the existence of a dual Markov process in $\mathbb{R}^d$ is established via Fubini's Theorem.

In Part II (chapters 4, 5, 6 and 7), we study the tool of intertwining and its role in generalising the theory of duality to $f$-duality on $\mathbb{R}^d$ arising from Pareto and similar partial orders. Our objective here is to characterise classes of dual Markov processes with respect to shift-invariant functions (functions which depend on the difference of its arguments). The full characterization of duality is given in terms of generators for basic classes of Feller processes. When considering the duality of Markov processes in domains with a boundary, we introduce the concept of a regularised dual to overcome difficulties which arose in this scenario.

## 1.2 A BRIEF SURVEY OF THE THEORY

In the early fifties, the notion of a duality relation was implicitly described by Lindley [57, 58] in the application of random walks in queuing theory. In both papers he used the notion of duality to transform solutions to problems for the "absorbing walk" into solutions to problems for the "reflecting walk". The word "dual" was believed to be first used in the late fifties in the work by Karlin and Mcgregor [47], where the properties of ergodicity, recurrence and transience of birth and death process were characterised. The duality relation of absorbing and non-absorbing processes was discussed in section 6 of the paper by Karlin and Mcgregor [47].

The general concept of duality has been formalised during the following decades. In general, there are three main approaches to duality, namely the classical Markovian approach developed by Siegmund, stochastic recursion and intertwining. In 1976 Siegmund [72] studied duality on the positive half line. Siegmund showed, using Fubini's Theorem, that the dual of a Markov process, reflected at the origin, is uniquely determined by (1.3) and is also a Markov process absorbed at the origin. The existence of a dual Markov process is conditional on the original Markov process being stochastically monotone, which was first defined by Daley [22].

In 1996, Asmussen and Sigman [11] developed another approach to duality using stochastic recursion. In the paper, the authors considered stochastic sequences $(V_t)_{t\geqslant 0}$ defined via general recursion $V_{t+1} = f(V_t, U_t)$. Here, $(U_t)_{t\geqslant 0}$ is a stationary driving sequence and the function $f$ is non-negative, continuous and monotone in its first variable. A dual function $g$ of $f$ is constructed such that $g(\cdot.u)$ is the generalised inverse of $f(\cdot, u)$. One way to achieve this is to obtain $U_t$ in $g$ by time-reversing $U_t$ in the original function $f$. This kind of duality coincides with Siegmund's duality for Markov chains with discrete time when $(U_t)_{t\geqslant 0}$ are uniformly, independently and identically distributed on the interval $(0, 1)$.

Both approaches developed by Siegmund [72] and Asmussen and Sigman [11] were restricted to one dimensional cases with discrete or continuous time. Błaszczyszyn and Sigman [14] extended both methods to general state-space with discrete time. For the case of stochastic recursion, Błaszczyszyn and Sigman [14] introduced a set-value dual function to allow for unique inversion in general state-space. On the other hand, Choquet's Theorem was employed in their paper to construct Markovian duality on general state-space. Sigman and Ryan [73] studied the theory of duality for continuous-time, real-valued stochastic processes that were defined via general recursive functions driven by processes with stationary increments.

In the paper by Holley and Stroock [40], Siegmund's theory was generalised to duality with respect to a function, in the sense of $E(f(X_t^x, y)) = E(f(x, Y_t^y))$ for a certain function $f$. The textbook by Liggett [56] gave a detail survey on this kind of duality. This notion of duality is mainly studied at the level of Markovian semi-group, using tools in functional analysis such as intertwining (see e.g. [65] by Pal and Shkolnikov) or "dressing operators" (see e.g. [80] by Takasaki). Generally speaking, two Markov semi-groups $(T_t)_{t\geqslant 0}$ and $(S_t)_{t\geqslant 0}$ on $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are said to be intertwined via a Markov kernel $Q : (E, \mathcal{E}) \to (F, \mathcal{F})$ if $S_t Q = QT_t$ (see [13] by Baine and references therein for a detailed survey on the tool). The notion of intertwining can be seen as a transfer of spectral information between semi-groups, or a link between Markov processes.

Section 5 of the paper by Carmona et al. [16] provides assumptions allowing the properties of intertwining and duality to be equivalent. The duality relationship between Markov chain kernels was established via intertwining in the work by Huillet and Martinez [43]. In the paper, Huillet presented duality between stochastic matrices, in discrete time and space, and revisited Siegmund's duality of monotone chains, birth and death processes and the non-neutral Moran model.

The properties of duality have also attracted significant interest by researchers since the early 1980s. Cox and Rösler [20] studied duality in the sense of (1.2) and its relation to time reversal when reversing the role of entrance and exit laws. Clifford and Sudbury [19] explained Siegmund's duality for absorbing and reflecting Markov processes and identified the sample paths of their dual by using a graphical representation similar to that used in the study of infinite particle systems. Strong stationary duality was discussed by Diaconis and Fill [26]. In particular, they showed that strong stationary times could be studied by constructing an absorbing dual process in such a way that the strong stationary time for the original process is equivalent to the absorption time of the dual process. A simulation procedure via Siegmund's duality was suggested and discussed in detail by Asmussen and Rubinstein [10]. In the textbook by Anderson [3], stochastic monotonicity and duality for Markov chains were surveyed in detail. The properties of stochastic monotonicity and duality of Markov chains form an important tool in the work by Kolokoltsov [54]. In the paper, Kolokoltsov developed the theory of monotonicity and duality for one-dimensional Feller processes via the approximations of Markov chains. He then studied local monotonicity conditions to prove the well-posedness of the corresponding Markov semi-group. Möhle [62] worked on "cone duality" and its relation to the duality in the sense of (1.2).

Apart from developing the tools to construct dual Markov processes and studying their properties, many research projects have also been conducted to discover applications for duality. For instance, Siegmund's duality applied in the context of queuing and storage systems or birth and death chains yields the relationship between the probability of the considered process and the ruin probability of the dual process. Suppose that $(X_t^x)_{t\geqslant 0}$ and $(Y_t^y)_{t\geqslant 0}$ are Markov processes on the positive half line satisfying Siegmund's duality relation in (1.3). Then $P(X_T^0 \geqslant y) = P(\tau \leqslant T)$, where $\tau$ is the first hitting time when the dual process $(Y_t^y)$ reaches $0$. For duality in queueing theory, readers are referred to papers by Ghosal [34], Loynes [59], Bekker and Zwart [12] and section 9.4 of work by Asmussen [7]. For birth and death chains, readers are referred to work by Callaert [15], Anderson [3], van Doorn [81], van Doorn [82], Dette et al. [25], Daley [21].

In the field of interacting particle systems, the theory of duality was studied to obtain solutions to particular problems (see for example work by Kipnis et al. [50] and Spohn [75]). In this field, stochastic monotonicity is known as being "attractive". Giardinà et al. [36] studied the deep connection between duality and symmetry of generators and gave a general scheme using intertwining to construct a dual stochastic process for continuous time Markov processes whose generator has a symmetry.

Another example of the applications for duality can be also seen in superprocesses (see Ethier and Kurtz [32], Mytnik [64]). In Mytnik [64] the theory of duality was extended to investigate the limiting behaviour of branching particle systems which undergo random migration and critical branching. In insurance mathematics, there is also duality between queuing theory and the risk processes (see papers by Asmussen and Pihlsgård [9], Djehiche [29] for duality in financial ruin probabilities). In the field of population dynamics and branching processes, Alkemper and Hutzenthaler [1] presented a stochastic picture of moment duality and Laplacian duality to the processes in the field of population dynamics, by studying the graphical representation of the approximation forward and backward particle processes.

## 1.3 MAIN CONTRIBUTION

This thesis focuses on time-homogenous Markov processes. We begin by presenting the criterion for the existence of dual Markov processes in finite dimensional Euclidean space. We then consider

$f$-duality for the case of shift-invariant functions $f$ and give a systematic study of the theory via the analysis of the generators of dual Markov processes. Our study is inspired by the analysis of one-dimensional processes by Kolokoltsov [52, 54]. The main result of this thesis (illustrating this approach) will be a complete characterization, in terms of their generators, of Markov processes in $\mathbb{R}^d$, which are dual with respect to Pareto order. This characterization is seemingly new even for the one-dimensional case, i.e. for Siegmund's duality. Additionally we illustrate this approach using other examples. We also address some difficulties arising from the conditions of Siegmund's duality (1.3) at boundary points. For instance, these boundary conditions prevent the second dual to reflected Brownian motion to coincide with itself. In order to overcome such difficulties, we introduce the notion of a regularized dual that can correct this and similar issues.

Let us emphasise that, in this thesis, we are not aiming to produce any new applications of duality, instead we look at its characterisation. With this characterisation at hand, we greatly extend the applicability of many known results on dual processes. For example, the transience-recurrence duality (Theorem 3.5 in textbook by Liggett [56]) was essentially given without any examples. Also, the example of exit-entrance duality in the paper Cox and Rösler [20] were reduced to Brownian motion and Ornstein-Uhlenbeck process. Another example includes the ruin probability calculations of the work by Asmussen and Pihlsgård [9].

## 1.4 THESIS OUTLINE

### PART I

CHAPTER 2 This chapter introduces the notion of duality on one-dimensional real space. We begin by extending Siegmund's theory of stochastically monotone Markov processes on the positive half line to the real space by presenting an alternative proof to Theorem 1 in Siegmund [72]. When considering duality on closed intervals we study the boundary conditions for both the original and the dual Markov processes. This chapter is concluded by computing an explicit form for the dual generator of a Feller process generated by the usual Lévy-Khintchine type generator, given that the dual process exists.

CHAPTER 3 This chapter is devoted to generalising the concept of Siegmund duality to Pareto-ordered finite dimensional space $\mathbb{R}^d$. The criterion for the existence of a Markov dual on

$\mathbb{R}^d$ are given. We also consider the example of an integro-differential (Lévy-Khintchine type) pre-generator. Following the approach of Kolokoltsov [52, 55], we give the criterion to construct a stochastically monotone Markov process from the pre-generator which satisfies the criterion for its Markov dual process to exist in $\mathbb{R}^d$.

## PART II

CHAPTER 4 As the beginning of Part II, this chapter lays the analytical foundation with the objective of characterising the duality of Markov processes. Considering the $f$-duality in the sense of (1.2), we give the basic tools of intertwining, and explain its role in deriving the $f$-duality. Brief ideas on the applications of the tools to the theories of differential equations and stochastic processes are discussed before we give detailed descriptions and examples in the following chapters.

CHAPTER 5 In this chapter we deal with duality on $\mathbb{R}^d$ arising from Pareto and similar partial-order. After examining the characterisation for each case of diffusion and jump processes, full characterisation of duality is given in terms of generators for basic Feller processes by applying the tools discussed in chapter 4. This chapter is concluded by giving a pathwise example of the study of duality via stochastic differential equations.

CHAPTER 6 This chapter focusses on the duality analysis of translation-invariant functions $f$, in other words, functions depending only on the difference of their arguments, $f(x, y) = \tilde{f}(y - x)$. We shall give several examples of duality for such instances.

CHAPTER 7 In this chapter we address some of the difficulties arising from the condition (1.3) at boundary points, which, for instance, prevents the second dual of a reflected Brownian motion to coincide with itself. The tools proposed in Chapter 4 are utilised to study the theory of duality for processes in domains with a boundary. To circumvent the difficulties arising from the boundary conditions, we introduce an additional tool - the regularised dual.

# 2 SIEGMUND'S DUALITY OF MARKOV PROCESSES AND GENERATORS

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## 2.1 INTRODUCTION

In this chapter the concept of stochastic monotonicity, in one dimensional real space, and its role in defining Siegmund's duality is introduced. The relationship between stochastic monotonicity and duality was first explored by Siegmund [72], where the necessary and sufficient conditions for a Markov dual process to exist on the positive half line ($\mathbb{R}_+$) were presented. This notion of duality is generally known as Siegmund's duality and has been used by e.g. Piau [67], Huillet [42], Dette et al. [24].

More recently, Asmussen and Sigman [11] approached duality in discrete time using the stochastic recursive method while Sigman and Ryan [73] presented the duality via stochastic recursion for continuous-time processes. In both papers, the stochastic recursive approach to duality involves an elementary sample-path analysis and is, so far, constrained to one dimension. Both Siegmund's and the stationary recursive duality method on discrete time are extended to general state-space by

Błaszczyszyn and Sigman [14]. In Anderson's textbook [3], the theory is systematically presented for Markov chains on discrete state-space.

We first introduce Siegmund's duality in section 2.3. We then extend the theory from $\mathbb{R}_+$ to real space by presenting an alternative proof to Theorem 1 in paper [72] by Siegmund. Additionally, we discuss the boundary conditions of Siegmund duality's on a closed interval on $\mathbb{R}$. Lastly we consider the example of a Feller process of the usual Lévy-Khintchine type which is assumed to have a Markov dual process. We show an alternative method to that given by Kolokoltsov [54], via direct computation using the duality relation to write the explicit form of the corresponding dual generator.

## 2.2 BASIC NOTATIONS

Let $(X_t^x)_{t\geqslant 0}$ be a time-homogeneous Markov process on the real line $\mathbb{R}$ which is endowed with Borel sigma-algebra $\mathcal{B}(\mathbb{R})$. This Markov process is characterised by a family of transition probability distribution functions $P_t^X(x, A)$, which describe the probability of $(X_t^x)_{t\geqslant 0}$ arriving in a real subset $A$ at time $t \geqslant 0$ given that it starts from some $x \in \mathbb{R}$.

Suppose that Borel sets $A_1$ and $A_2$ are intervals $(-\infty, y]$ and $[y, \infty)$ respectively for some $y \in \mathbb{R}$. Then the transition probabilities for process $\mathbf{X}$ starting from $x \in \mathbb{R}$ to arrive in the set $A_1$and $A_2$ at time $t$ can be written respectively as

$$\begin{aligned} P_t^X(x, A_1) &= P(X_t^x \leqslant y) = F_{x,t}^X(y), \\ P_t^X(x, A_2) &= P(X_t^x \geqslant y). \end{aligned}$$

The function $P_t^X(x, A_1) = P(X_t^x \leqslant y)$ is a cumulative distribution function of the random variable $(X_t^x)_{t\geqslant 0}$ at $y \in \mathbb{R}$. If this function is absolutely continuous in $y$, then its probability transition density function, denoted as $p_t^X(x, y)$, exists. In this case, the probability transition measure $P_t^X(x, dy)$ can also be expressed as $p_t^X(x, y)dy$. For the remainder of the section, let us assume that process $\mathbf{X}$ is honest, that is $P_t^X(x, \mathbb{R}) = 1$ for all $x \in \mathbb{R}$, $t \geqslant 0$.

A Markov process $(X_t^x)_{t\geqslant 0}$ is said to be stochastically monotone if the function $P(X_t^x \geqslant y)$ is a non-decreasing in $x$ for all fixed $y$. This also means that $(X_t^x)_{t\geqslant 0}$ is stochastically monotone if, and only if, the expectation $\mathbb{E}\big(f(X_t^x)\big)$, or correspondingly the semi-group $T^X f(x)$ is a non-decreasing function in $x$ for every non-decreasing function $f$.

*Remark* 1. The definition of stochastic monotonicity for Markov processes first appears in work by Daley [22] where the notion of stochastic comparisons between Markov chains were discussed. In general, we say that a Markov process $(X_t^x)_{t\geqslant 0}$ stochastically dominates $(Y_t^y)_{t\geqslant 0}$ if the semi-group $T_t^X f(x) \geqslant T_t^Y f(y)$ for any bounded increasing functions $f$ when $x \geqslant y$. Therefore, a Markov process $(X_t^x)_{t\geqslant 0}$ is stochastically monotone if, and only if, it stochastically dominates itself (see e.g. papers by Keilson and Kester [48], Wang [84], Chen and Wang [18] and references therein for more on stochastic comparisons). Stochastic monotonicity and related duality are well developed for Markov chains, see e.g. work by Anderson [3] and van Doorn [81], for birth and death processes and for one-dimensional diffusions see work by Cox and Rösler [20].

We will now give some examples of stochastically monotone Markov processes.

**Example 1.** Let $(X_t^x)_{t\geqslant 0}$ be a Poisson process with intensity $c$. First, we consider the transition probabilities $P(X_t^x \geqslant y)$ when the process starts at $x \leqslant y$, both $x, y \in \mathbb{N}$. The function

$$P(X_t^x \geqslant y) = \sum_{i=y}^{\infty} \frac{e^{-ct}(ct)^{(i-x)}}{(i-x)!}$$

is non-decreasing in $x$ since the summation $\sum_{i=y-x}^{\infty} \frac{e^{-ct}(ct)^i}{i!}$ becomes larger when $i$ begins at a smaller number. On the other hand, if the process starts at an $x$ which is larger than $y$, the probability distribution $P(X_t^x \geqslant y) = P(X_t^x \geqslant x) = 1$ since a Poisson process has non-increasing sample paths by definition. Thus any Poisson process on $\mathbb{N}_+ \cup \{0\}$ is stochastically monotone.

**Example 2.** A Brownian Motion on $\mathbb{R}$ is also stochastically monotone. This is because the function

$$P(X_t^x \geqslant y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(z-x)^2} dz \tag{2.1}$$

is non-decreasing in $x \in \mathbb{R}$, since on the right hand side $(z-x)^2$ decreases in $x \in \mathbb{R}$.

**Example 3.** All time-homogeneous Lévy Processes are stochastically monotone. Let us consider some $x_1$ and $x_2 \in \mathbb{R}$ such that $x_2 \geqslant x_1$. By the property of translation invariance,

$$\begin{aligned} P(X_t^{x_2} \geqslant y) &= P(X_t^{x_2+x_1-x_1} \geqslant y) \\ &= P(X_t^{x_1} \geqslant y - x_2 + x_1) \\ &\geqslant P(X_t^{x_1} \geqslant y) \end{aligned}$$

since $x_2 - x_1$ is positive.

## 2.3 SIEGMUND'S DUALITY RELATION ON $\mathbb{R}$.

Let us consider a stochastically monotone Markov process $(X_t^x)_{t\geqslant 0}$ on $\mathbb{R}$ such that for all $y \in \mathbb{R}$, the function $P(X_t^x \geqslant y)$ is right continuous in $x \in \mathbb{R}$ and tends to $0$ and $1$ as $x$ tends to $-\infty$ and $\infty$ respectively. Let us denote $F_{y,t}^Y(x)$ as $P(X_t^x \geqslant y)$. It is clear that $F_{y,t}^Y(x)$ is also a cumulative distribution function. Therefore we are able to define a corresponding collection of random variables $(Y_t^y)_{t\geqslant 0}$ on $\mathbb{R}$ such that their cumulative distribution functions satisfy:

$$\mathbf{P}(Y_t^y \leqslant x) = \mathbf{P}(X_t^x \geqslant y) = F_{y,t}^Y(x). \tag{2.2}$$

We call (2.2) Siegmund's duality relation to emphasise an important contribution from Siegmund [72]. The family of random variables $(Y_t)_{t\geqslant 0}$ that satisfies the duality relation is said to be a dual to the Markov Process $(X_t^x)_{t\geqslant 0}$. Conversely, if a Markov process $(X_t^x)_{t\geqslant 0}$ has a dual with cumulative distribution function $F_{y,t}^Y(x)$, the relation in (2.2) ensures $(X_t^x)_{t\geqslant 0}$ is stochastically monotone.

In this section, we are interested in the existence and uniqueness of dual Markov processes. Since the dual is defined by equalities between probability distribution functions, a dual Markov process must be unique. To show its existence in Theorem 2, we will use the following well-known integration by parts methods (see for example, work by Hewitt [38] and de Barra [23]).

**Theorem 1.** *(Lebesgue-Stielges integration by parts) Let* $f$ *and* $g$ *be right-continuous functions on the finite interval* $[a, b]$ *such that both of them have bounded variation on* $[a, b]$*. Then*

$$\int_{[a,b]} f(x-)d\mu_g + \int_{[a,b]} g(x)d\mu_f = f(b)g(b) - f(a-)g(a-) \tag{2.3}$$

*where* $f(x-) = \lim_{y\uparrow x} f(y)$ *and* $\mu_f$, $\mu_g$*are the signed measures induced by* $f$ *and* $g$ *respectively.*

In the next proposition we extend the analogue of the above theorem from an arbitrary interval to the real line.

**Proposition 1.** *Let* $f$ *and* $g$ *be right-continuous, non-negative and monotonically increasing functions on* $\mathbb{R}$ *such that both of them are bounded on* $\mathbb{R}$*. Then*

$$\int_{\mathbb{R}} f(x-)d\mu_g + \int_{\mathbb{R}} g(x)d\mu_f = \lim_{b\to\infty} f(b)g(b) - \lim_{a\to-\infty} f(a)g(a) \tag{2.4}$$

*where* $f(x-) = \lim_{y\uparrow x} f(x)$ *and* $\mu_f$, $\mu_g$*are the signed measures induced by* $f$ *and* $g$ *respectively.*

The proofs to Theorem 1 and Proposition 1 are adaptations from those given by Hewitt [38] and de Barra [23]. Both proofs are included in section 2.6 for completeness. With both tools we show in the following theorem the necessary conditions for a dual Markov process to exist.

**Theorem 2.** *Suppose that* $(X_t^x)_{t\geqslant 0} \in \mathbb{R}$ *is a stochastically monotone Markov process such that its transition probability distribution functions* $P(X_t^x \geqslant y)$ *are right continuous in* $x \in \mathbb{R}$ *and tend to* $1$ *and* $0$ *as* $x$ *tends to* $\infty$ *and* $-\infty$ *respectively. Then its dual* $(Y_t^y)_{t\geqslant 0}$ *is a Markov process.*

*Proof.* It is enough to show that the family of random variables $(Y_t^y)_{t\geqslant 0}$ defined by the duality relation satisfies the Chapman-Kolmogorov equation.

Let us denote $F_{x,t}^X(z)$ as $P(X_t^x \leqslant z)$. Then $dF_{x,t}^X(z)$ is the probability measure that induces the probability transition distribution of $(X_t^x)_{t\geqslant 0}$ for a given starting point $x \in \mathbb{R}$. The Chapman-Kolmogorov equation of process $(X_t^x)_{t\geqslant 0}$ for all $x, y \in \mathbb{R}$, and time $u,\ s \geqslant 0$ is

$$P(X_{s+u}^x \geqslant y) = \int_{\mathbb{R}} P(X_u^z \geqslant y)dF_{x,s}^X(z) \tag{2.5}$$

where the right hand side is also equal to $\int_{\mathbb{R}} F^Y_{y,u}(z) dF^X_{x,s}(z)$ according to duality relation in (2.2). Observe that both $F^Y_{y,u}(z)$ and $F^X_{x,s}(z)$ are right continuous, monotonically increasing and bounded by $0$ and $1$ for all $z \in \mathbb{R}$. Using Proposition 1, the equation above becomes

$$\begin{aligned}P(X^x_{s+u} \geqslant y) &= \int_{\mathbb{R}} F^Y_{y,u}(z) dF^X_{x,s}(z) \\ &= \lim_{z\to\infty} F^Y_{y,u}(z) F^X_{x,s}(z) - \lim_{z\to-\infty} F^Y_{y,u}(z) F^X_{x,s}(z) - \int_{\mathbb{R}} F^X_{x,s}(z-) dF^Y_{y,u}(z)\end{aligned}$$

where $F^X_{x,s}(z-) = \lim_{w\uparrow z} F^X_{x,s}(w)$. The first and second term on the right hand side of the equation above tend to $1$ and $0$ respectively. Therefore we get

$$\begin{aligned}P(X^x_{s+u} \geqslant y) =& 1 - \int_{\mathbb{R}} F^X_{x,s}(z-) dF^Y_{y,u}(z) \\ =& 1 - \int_{\mathbb{R}} P(X^x_s < z) dF^Y_{y,u}(z).\end{aligned}$$

Clearly, $\int_{\mathbb{R}} dF^Y_{y,u}(z) = 1$, as $dF^Y_{y,u}(z)$ is a probability measure on $\mathbb{R}$. Substituting this into the above,

$$\begin{aligned}P(X^x_{s+u} \geqslant y) =& \int_{\mathbb{R}} dF^Y_{y,u}(z) - \int_{\mathbb{R}} F^X_{x,s}(z-) dF^Y_{y,u}(z) \\ =& \int_{\mathbb{R}} \Big(1 - P(X^x_s < z)\Big) dF^Y_{y,u}(z) \\ =& \int_{\mathbb{R}} P(X^x_s \geqslant z) dF^Y_{y,u}(z).\end{aligned} \tag{2.6}$$

Applying the duality relation on both left and right hand sides of (2.6), we can write

$$P(Y^y_{s+u} \leqslant x) = \int_{\mathbb{R}} P(Y^z_s \leqslant x) dF^Y_{y,u}(z). \tag{2.7}$$

We can conclude that $(X^x_t)_{t\geqslant 0}$ has a dual Markov process $(Y^y_t)_{t\geqslant 0}$ that corresponds to the family of cumulative distribution functions $F^Y_{y,t}$. □

In Siegmund's paper [72] Fubini's theorem was used to show that the same necessary and sufficient conditions are required for a dual to form a Markov process on the positive half line. This proof can be easily adapted to Markov processes on the real line. This adaptation is presented in the appendix section 2.6.

Next, we show an example to compute the dual of a Lévy process.

**Example 4.** Let $(X_t)_{t\geqslant 0}$ be a Lévy process. Then according to Theorem 2, its dual Markov process exists. Furthermore, $(Y_t)_{t\geqslant 0}$ also has an independent incremental property. This is because

$$\begin{aligned} P(X_t^x \geqslant y) &= P(X_t^0 \geqslant y - x) \\ &= P(-X_t^0 \leqslant x - y) \\ &= P(-X_t^y \leqslant x). \end{aligned}$$

Hence, the dual of Lévy process $X_t$ is $-X_t$. In other words, the dual is also a Lévy process with sample paths of the opposite direction.

Now, we look at some analytical properties of Siegmund's duality. Let $B(\mathbb{R})$ be the Banach space of bounded Borel-measurable functions equipped with the supremum norm. Also, let $C_\infty^k(\mathbb{R})$ be the space of $k$ times differentiable functions on $\mathbb{R}$ with all these derivatives vanishing at infinity. By $(T_t^X)_{t\geqslant 0}$ we denote a Markov semi-group on $B(\mathbb{R})$ corresponding to Markov process $(X_t)_{t\geqslant 0}$. Then for any $f \in B(\mathbb{R})$ we write

$$\begin{aligned} T_t^X f(x) &= \mathbb{E}\Big(f(X_t)|X_0 = x\Big) \\ &= \mathbb{E}\Big(f(X_t^x)\Big). \end{aligned}$$

The transition probability distribution function $P(X_t^x \geqslant y)$ can also be represented as $T_t^X \mathbf{1}_{\geqslant y}(x)$. Let us also denote $(T_t^Y)_{t\geqslant 0}$ to be the Markov semi-group for the dual process $(Y_t)_{t\geqslant 0}$. Then the duality relation can be written as

$$T_t^Y \mathbf{1}_{\leqslant x}(y) = T_t^X \mathbf{1}_{\geqslant y}(x). \tag{2.8}$$

We say that $(T_t)_{t\geqslant 0}$ is a C-Feller (or Feller continuous) semi-group on the Banach space of bounded continuous functions $C_b(\mathbb{R})$ if the function $T_t f(x) = \mathbb{E}(f(X_t^x))$ is a continuous function of $x$ for all bounded continuous $f$. A C-Feller semi-group is a sub-Markov semi-group in $C_b(\mathbb{R})$, in other words, for any $0 \leqslant u \leqslant 1$, we have $0 \leqslant T_t u \leqslant 1$.

*Remark* 2. It is worth noting that a C-Feller semi-group is not necessarily Feller and vice versa. Recall that a Feller semi-group is a strongly continuous semi-group of positive linear contraction on $C_\infty(\mathbb{R})$ ($T_t f \in C_\infty(\mathbb{R})$ if $f \in C_\infty(\mathbb{R})$). Contrary to the definitions of a Feller semi-group, strong continuity is not a requirement for the definition of C-Feller. In particular, a Feller semi-group is C-Feller if, and only if, the corresponding semi-group applied to a constant is a continuous function for all $t \geqslant 0$. On the other hand, a C-Feller semi-group is Feller if $C_\infty(\mathbb{R})$ is invariant under the semi-group and the corresponding restriction is strongly continuous (see chapter 3 and 4, in textbook [55] by Kolokoltsov and work by Schiling and Wang [71] and references therein for more explanation on the matter).

**Proposition 2.** *Suppose* $(X_t^x)_{t\geqslant 0}$ *is a* C*-Feller process such that* $P(X_t^x \geqslant y)$ *is continuous in* $y \in \mathbb{R}$. *Then* $P(X_t^x \geqslant y)$ *must also be continuous in* $x \in \mathbb{R}$.

*Proof.* Let $(x_n)_{n\geqslant 0} \in \mathbb{R}$ be a sequence that converges to $x$ as $n$ tends to $\infty$. Since $(X_t^x)_{t\geqslant 0}$ is a C-Feller process, its transition probability measure initiating at $x_n$, $P(X_t^{x_n} \in dy) = \mu_n$ converges weakly to $P(X_t^x \in dy) = \mu$ as $n \to \infty$. Denote the set $[y, \infty)$ as $A$ for some $y \in \mathbb{R}$. Then its boundary point $(\partial A)$ is $y$ . Since $P(X_t^x \geqslant y)$ is continuous in $y$, the measure admits no atoms, in other words, $\mu(\partial A) = P(X_t^x = y) = 0$. Then by Portmanteau Theorem, $\mu_n(A) \to \mu(A)$, meaning that $P(X_t^x \geqslant y)$ is continuous in $x$. $\square$

**Proposition 3.** *Let* $(T_t^X)_{t\geqslant 0}$ *be a* C*-Feller semi-group such that* $T_t^X \mathbf{1}_{\geqslant y}(x)$ *is monotonically increasing in* $x$ *and continuous in* $y \in \mathbb{R}$. *Then, its dual semi-group also forms a* C*-Feller process.*

*Proof.* By Proposition 2, $T_t^X \mathbf{1}_{\geqslant y}(x)$ is also continuous (and therefore right continuous) in all $x \in \mathbb{R}$. Since the corresponding Markov process is stochastically monotone, by Theorem 2 it has a dual Markov process $(Y_t^y)_{t\geqslant 0}$ and a corresponding dual semi-group $(T_t^Y)_{t\geqslant 0}$. Furthermore, $T_t^Y \mathbf{1}_{\leqslant x}(y) = P(Y_t^y \leqslant x)$ is a continuous function in $x$ as well as $y$. By Portmanteau Theorem, its distribution measure $P(Y_t^{y_n} \in dx)$ converges weakly to $P(Y_t^y \in dx)$. This is precisely the definition of the dual $(Y_t^y)_{t\geqslant 0}$ being a C-Feller process. $\square$

## 2.4 SIEGMUND'S DUALITY FOR PROCESSES ON AN INTERVAL.

In this section we investigate the theory of Siegmund's duality on an arbitrary interval $[a, b] \subset \mathbb{R}$. In particular, we study the behaviours of Markov process $(X_t)_{t \geqslant 0}$ and its dual $(Y_t)_{t \geqslant 0}$ at both barriers $\{a\}$ and $\{b\}$. This section serves as an overview of the more detailed survey on the theory of duality on an interval, which is presented in chapter 7.

Similar to before, let us suppose that $(X_t)_{t \geqslant 0}$ is a stochastically monotone Markov Process evolving on an interval $[a, b] \subset \mathbb{R}$ such that its probability distribution function $P(X_t^x \geqslant y)$ is right continuous in $x \in [a, b)$ and monotonically increasing in $x \in [a, b]$. Also, assume that for all $y \in [a, b]$, $\lim_{x \uparrow b} P(X_t^x \geqslant y) = 1$ and $\lim_{x \downarrow a} P(X_t^x \geqslant y) = 0$. Define a set of functions $F_{y,t}^Y(x)$ , $x, y \in [a, b]$ such that

$$F_{y,t}^Y(x) = P(X_t^x \geqslant y). \tag{2.9}$$

Let us recall that (2.9) is Siegmund's duality relation.

Clearly, for all $t \geqslant 0$ and $y \in [a, b]$, the function $F_{y,t}^Y(x)$ is monotonically increasing in $x \in [a, b]$ and right continuous in $x \in [a, b)$. With this set-up, the set of functions $F_{y,t}^Y(x)$ forms a family of probability distribution functions that correspond to a family of random variables, $(Y_t)_{t \geqslant 0}$. Furthermore if we let $y$ in (2.9) equal to $\{a\}$,

$$F_{a,t}^Y(x) = P(X_t^x \geqslant a) = 1$$

for all $x \in [a, b]$ and any $t \geqslant 0$. In other words, the probability of $Y_t^a$ being smaller than or equal to any $x \in [a, b]$ equals to 1.

**Proposition 4.** *If* $(X_t)_{t \geqslant 0}$ *is a stochastically monotone Markov process on* $[a, b]$ *such that* $P(X_t^x \geqslant y)$ *is right-continuous in* $x \in [a, b)$, *then it has a dual Markov process* $(Y_t)_{t \geqslant 0}$ *on* $[a, b]$ *which is absorbed at* $\{a\}$.

*Proof.* Similar to Theorem 2, we need to show that the family of cumulative distribution functions $F_t^Y(y)$ satisfies the Chapman-Kolmogorov equation. By the Chapman-Kolmogorov equation for $(X_t)_{t\geqslant 0}$ and duality relation,

$$
\begin{aligned}
P(Y_{t+s}^y \leqslant x) =& P(X_{t+s}^x \geqslant y) \\
&= \int_{[a,b]} P(X_s^z \geqslant y) dF_{x,t}^X(z) \\
&= \int_{[a,b]} F_{y,s}^Y(z) dF_{x,t}^X(z)
\end{aligned}
\tag{2.10}
$$

for all $x, y \in [a, b]$ and $t \geqslant 0$. Since both $F_{y,s}^Y$ and $F_{x,t}^X$ are right-continuous functions on $[a, b)$, we apply Lebesgue-Stieltjes integration by parts in Theorem 1 to (2.10) to obtain

$$
\begin{aligned}
P(Y_{t+s}^y \leqslant x) =& P(X_{t+s}^x \geqslant y) \\
&= F_{y,s}^Y(b) F_{x,t}^X(b) - F_{y,s}^Y(a-) F_{x,t}^X(a-) - \int_{[a,b]} F_{x,t}^X(z-) dF_{y,s}^Y(z).
\end{aligned}
$$

In the equation above the second term equals to $0$ by definition. Also, $F_{x,t}^X(z-) = \lim_{x\uparrow z} F_{x,t}^X(x)$, which also equals $P(X_t^x < z)$. Therefore, the equation above can be rewritten as

$$
\begin{aligned}
P(Y_{t+s}^y \leqslant x) =& P(X_{t+s}^x \geqslant y) \\
&= 1 - \int_{[a,b]} P(X_t^x < z) dF_{y,s}^Y(z) \\
&= \int_{[a,b]} \big(1 - P(X_t^x < z)\big) dF_{y,s}^Y(z) \\
&= \int_{[a,b]} P(X_t^x \geqslant z) dF_{y,s}^Y(z).
\end{aligned}
$$

By duality relation,

$$
\begin{aligned}
P(Y_{t+s}^y \leqslant x) &= \int_{[a,b]} P(X_t^x \geqslant z) dF_{y,s}^Y(z) \\
&= \int_{[a,b]} P(Y_t^z \leqslant x) dF_{y,s}^Y(z).
\end{aligned}
$$

Hence, $(X_t)_{t\geqslant 0}$ has a dual Markov process in $[a,b]$. Lastly since $F^Y_{a,t}(x) = 1$ for all $x \in [a,b]$, the dual Markov process is absorbed at the barrier $\{a\}$. □

Evidently, one can construct a dual Markov process $(Y_t)_{t\geqslant 0}$ on $[a,b]$ if $(X_t)_{t\geqslant 0}$ is a stochastic monotone Markov process on $[a,b]$ such that $P(X^x_t \geqslant y)$ is right continuous in $x \in [a,b)$. Clearly if $\{a\}$ and $\{b\}$ are unattainable, both $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ evolve on $(a,b)$ in a similar way to an honest process on $\mathbb{R}$. Otherwise, $\{a\}$ is an absorbing barrier for the dual process $(Y_t)_{t\geqslant 0}$ and $\{b\}$ is an absorbing barrier for the original process $(X_t)_{t\geqslant 0}$. The latter can be shown by letting $x$ in (2.9) equal to $b$.

One example that illustrates the behaviour of Markov process $(X_t)_{t\geqslant 0}$ and its dual process $(Y_t)_{t\geqslant 0}$ is the Brownian Motion on $[0,\infty)$.

**Example 5.** (Reflected Brownian Motion at 0). Suppose that $(X_t)_{t\geqslant 0}$ is a Brownian Motion on $(-\infty,\infty)$. Then $(|X_t|)_{t\geqslant 0}$ is a Brownian Motion on the half line $[0,\infty)$ reflected at 0. Its transition probability distribution function at some initial point $x \in [0,\infty)$ can be written as

$$\begin{aligned} P(|X_t|^x \geqslant y) &= P(X^x_t \geqslant y) + P(X^x_t \leqslant -y) \\ &= \frac{1}{\sqrt{2\pi t}}\Big(\int_y^\infty e^{-\frac{(z-x)^2}{2t}}\,dz + \int_{-\infty}^{-y} e^{-\frac{(z-x)^2}{2t}}\,dz\Big) \\ &= \frac{1}{\sqrt{2\pi t}}\Big(\int_y^\infty e^{-\frac{(z-x)^2}{2t}}\,dz + \int_y^{\infty} e^{-\frac{(z+x)^2}{2t}}\,dz\Big) \end{aligned} \tag{2.11}$$

By symmetry and translation of the normal density function, the probability transition cumulative distribution function in (2.11) becomes

$$P(|X_t|^x \geqslant y) = \frac{1}{\sqrt{2\pi t}}\Big(\int_{-\infty}^{x-y} e^{-\frac{z^2}{2t}}\,dz + \int_{-\infty}^{-y-x} e^{-\frac{z^2}{2t}}\,dz\Big).$$

Next, split both the integrals above to obtain

$$\begin{aligned}
P(|X_t|^x \geqslant y) =& \frac{1}{\sqrt{2\pi t}}\Big(\int_{-y}^{x-y} e^{-\frac{z^2}{2t}}\,dz + \int_{-\infty}^{-y} e^{-\frac{z^2}{2t}}\,dz \\
&+ \int_{-\infty}^{-y} e^{-\frac{z^2}{2t}}\,dz - \int_{y}^{y+x} e^{-\frac{z^2}{2t}}\,dz\Big) \\
=& \frac{1}{\sqrt{2\pi t}}\Big(\int_0^x e^{-\frac{(z-y)^2}{2t}} - e^{-\frac{(z+y)^2}{2t}}\,dz + 2\int_{-\infty}^{-y} e^{-\frac{z^2}{2t}}\,dz\Big). \qquad (2.12)
\end{aligned}$$

By Proposition 4, since the reflected Brownian Motion is stochastically monotone and $P(|X_t|^x \geqslant y)$ is right-continuous in $x \in [0, \infty)$, (2.12) is also the transition cumulative distribution function, $F^Y_{y,t}(x)$ for the dual Markov process $(Y_t)_{t\geqslant 0}$.

Suppose that the dual process starts at $y = 0$. Then for all $x \in [0, \infty)$, the probability distribution function can be written as

$$P(Y^0_t \leqslant x) = \frac{2}{\sqrt{2\pi t}}\int_{-\infty}^{0} e^{-\frac{z^2}{2t}}\,dz = 1,$$

confirming that state $\{0\}$ is an absorbing barrier for the dual process.

Next let $x = 0$. Then the probability of the dual process $\mathbf{Y}$ starting from any $y \in [0, \infty)$ arriving at the absorbing barrier is

$$\begin{aligned}
P(Y^y_t = 0) &= P(Y^y_t \leqslant 0) \\
&= 2\int_{-\infty}^{-y} e^{-\frac{z^2}{2t}}\,dz.
\end{aligned}$$

Therefore its probability transition density on $(0, \infty)$ starting at $y \in (0, \infty)$ is

$$P^Y_t(y, dz) = \frac{1}{\sqrt{2\pi t}}\Big(e^{-\frac{(z-y)^2}{2t}} - e^{-\frac{(z+y)^2}{2t}}\Big)\,dz.$$

This density coincides with that of an absorbing Brownian Motion on the half line absorbed at the origin (see textbook by Knight [51] on Brownian Motion that is absorbed at the origin).

**Example 6.** (Subordinator) Let $(X_t)_{t\geqslant 0}$ be a subordinator. Recall that $(X_t)_{t\geqslant 0}$ has non-decreasing sample paths on $\mathbb{R}_+$. By Example 4, the dual of $X_t$ is $-X_t$ and has a non-increasing sample path which is absorbed at 0.

## 2.5 DUAL GENERATORS

In this section, we consider the example of an arbitrary one dimensional Feller process $(X_t)_{t\geqslant 0}$ in $\mathbb{R}$ with a generator of the usual Lévy-Khintchine form:

$$\begin{aligned} L^X f(x) &= \frac{1}{2} G(x) f''(x) + b(x) f'(x) \\ &\quad + \int_{\mathbb{R}_+} \Big[ f(x+\omega) - f(x) - f'(x)\omega \mathbf{1}_{B_1}(\omega) \Big] \nu(x, d\omega). \end{aligned} \tag{2.13}$$

This generator was discussed in detail in work by Kolokoltsov [54]. A criterion for stochastic monotonicity in terms of this generator is given by the following theorem.

**Theorem 3.** *(presented as Theorem 2.1, Kolokoltsov [54]) Let $X_t$ be a Feller process in $\mathbb{R}$ with a generator of the usual Lévy-Khintchine form in (2.13) with continuous $G$, $b$, $\nu$, and let the space $C_c^2(\mathbb{R})$ of twice continuously differentiable functions with compact support be a core. For simplicity assume also that the coefficients are bounded, that is*

$$\sup_x \Big( G(x) + |b(x)| + \int_{\mathbb{R}_+} (1 \wedge \omega^2) \nu(x, d\omega) \Big) < \infty.$$

*If the Lévy measure $\nu$ is such that for any $a > 0$, the functions*

$$\int_a^\infty \nu(x, d\omega), \int_{-\infty}^{-a} \nu(x, d\omega)$$

*are non-decreasing and non-increasing functions of $x$ respectively, then the process $(X_t)_{t\geqslant 0}$is stochastically monotone. Moreover, the dual Markov process exists.*

The proof to Theorem 3 in Kolokoltsov's paper [54] utilises discrete approximation of transition probabilities using the theory of stochastic monotonicity for Markov Chains (for example, see textbook by Anderson [3] ). Using the same approach of discretization, the explicit form for the dual generator was also computed in Proposition 3.1 in the same paper by Kolokoltsov.

In this section, we present an alternative approach to Proposition 3.1 in [54] to compute an explicit form of generator corresponding to the dual Markov process using the duality relation in (2.2) directly. Let us first recall the definition of an infinitesimal generator of a Feller process. Let us denote $C_\infty(\mathbb{R})$ as the space of continuous functions on $\mathbb{R}$ vanishing at $\pm\infty$. Also, let $C^1_\infty(\mathbb{R})$ be the space of continuously differentiable functions with both the original functions and their derivatives vanishing at $\pm\infty$. Let $(T^X_t)_{t\geqslant 0}$ be a Feller semi-group of $(X_t)_{t\geqslant 0}$ on $C_\infty(\mathbb{R})$. A function $f \in C_\infty(\mathbb{R})$ belongs to the domain $\mathcal{D}_{L^X}$ of the infinitesimal generator of $(X_t)_{t\geqslant 0}$ if the limit

$$L^X f(x) = \lim_{t\downarrow 0} \frac{1}{t}(T^X_t f - f) = \frac{d}{dt}\bigg|_{t=0} T^X_t f(x)$$

exists in $C_\infty(\mathbb{R})$. The operator $L^X : \mathcal{D}_{L^X} \to C_\infty(\mathbb{R})$ is called the infinitesimal generator of $X_t$ or of the semi-group $T^X_t$. Suppose that $\mathbf{1}_{\leqslant y} \in \mathcal{D}_{L^X}$. Clearly, to write the duality relation in terms of infinitesimal generators, we just differentiate both sides of the duality relation in (2.8) with respect to $t$:

$$\begin{aligned} L^Y \mathbf{1}_{\leqslant y}(x) &= \frac{d}{dt}\bigg|_{t=0} T^Y_t \mathbf{1}_{\leqslant y}(x) \\ &= \frac{d}{dt}\bigg|_{t=0} T^X_t \mathbf{1}_{\geqslant y}(x) \\ &= L^X \mathbf{1}_{\geqslant y}(x) \end{aligned} \tag{2.14}$$

where $L^Y$ is the infinitesimal generator of the dual process $(Y_t)_{t\geqslant 0}$.

For any $f \in C^1_\infty(\mathbb{R})$, we can write the expression of dual generator $L^Y f$ in terms of the original generator $L^X$, as shown in the following:

$$\begin{aligned} L^Y f(y) &= -L^Y\Big(\int_{\mathbb{R}} \mathbf{1}_{\geqslant(\cdot)}(z) f'(z)dz\Big)(y) \\ &= -L^Y\Big(\int_{\mathbb{R}} \mathbf{1}_{\leqslant z}(\cdot) f'(z)dz\Big)(y) \\ &= -\int_{\mathbb{R}} \Big(L^Y \mathbf{1}_{\leqslant z}(\cdot)\Big)(y) f'(z)dz \\ &= -\int_{\mathbb{R}} \Big(L^X \mathbf{1}_{\geqslant y}(\cdot)\Big)(z) f'(z)dz. \end{aligned} \tag{2.15}$$

In the following lemmas, we consider the jump and diffusion parts of the generator (2.13) separately.

**Lemma 1.** *Consider a one-dimensional Feller process generated by*

$$L_1^X f(x) = \int_0^\infty \Big(f(x+\omega) - f(x)\Big)\nu(x, d\omega) \tag{2.16}$$

*where* $f \in C_\infty^1(\mathbb{R})$ *and* $\nu$ *is a Lévy measure supported on* $\mathbb{R}_+$ *such that* $\int_a^\infty \nu(x, d\omega)$ *is a non-decreasing function. Also, assume that either (i)* $\nu(x, d\omega) = \nu(x, \omega)\mu(d\omega)$ *with* $\nu(x, \omega)$ *being continuously differentiable in* $x$ *or (ii)* $\nu(x, d\omega) = \alpha(x)\nu(d\omega)$ *with* $\alpha(x)$ *being continuously differentiable in* $x$*. Then the dual generator takes the explicit form:*

$$L_1^Y f(x) = \int_0^\infty \Big[f(x-\omega) - f(x)\Big]\tilde{\nu}(x, d\omega)$$

*where in*

*(i)* $\tilde{\nu}(x, d\omega) = \nu(x-\omega, \omega)\mu(d\omega) + \int_\omega^\infty \frac{\partial}{\partial x}\nu(x-\omega, y)\mu(dy)d\omega$*;*

*(ii)* $\tilde{\nu}(x, d\omega) = \alpha(x-\omega)\nu(d\omega) + \frac{\partial}{\partial x}\alpha(x-\omega)\int_\omega^\infty \nu(dy)d\omega$.

*Proof.* Let us first consider case (i). Applying results (2.15) and given any $f \in C_\infty^1(\mathbb{R})$ we can write

$$\begin{aligned} L_1^Y f(x) &= -\int_{\mathbb{R}} L^X \mathbf{1}_{\geqslant x}(y) f'(y)dy \\ &= -\int_{\mathbb{R}} \Big[\int_0^\infty \Big(\mathbf{1}_{\geqslant x}(y+\omega) - \mathbf{1}_{\geqslant x}(y)\Big)\nu(y, \omega)\mu(d\omega)\Big] f'(y)dy. \end{aligned}$$

Since variable $\omega$ considered in the second integral is always non-negative, $\mathbf{1}_{\geqslant x}(y+\omega) - \mathbf{1}_{\geqslant x}(y) = \mathbf{1}_{[x-\omega, x)}(y)$. Therefore by interchanging the integral term

$$\begin{aligned} L_1^Y f(x) &= -\int_{\mathbb{R}} \Big[\int_0^\infty \mathbf{1}_{[x-\omega, x)}(y)\nu(y, \omega)\mu(d\omega)\Big] f'(y)dy \\ &= -\int_0^\infty \Big[\int_{\mathbb{R}} \mathbf{1}_{[x-\omega, x)}(y) f'(y)\nu(y, \omega)dy\Big]\mu(d\omega) \\ &= -\int_0^\infty \Big[\int_{x-\omega}^x f'(y)\nu(y, \omega)dy\Big]\mu(d\omega). \end{aligned} \tag{2.17}$$

Applying integration by parts to the second integral,

$$
\begin{aligned}
L_1^Y f(x) &= \int_0^\infty \Big[ \int_{x-\omega}^x f(y) \frac{\partial}{\partial y} \nu(y, \omega) dy + f(x-\omega)\nu(x-\omega, \omega) - f(x)\nu(x, \omega) \Big] \mu(d\omega) \\
&= \int_0^\infty \Big[ \int_{x-\omega}^x f(y) \frac{\partial}{\partial y} \nu(y, \omega) dy + \Big[ f(x-\omega) - f(x) \Big] \nu(x-\omega, \omega) \\
&\quad + f(x)[\nu(x-\omega, \omega) - \nu(x, \omega)] \Big] \mu(d\omega).
\end{aligned} \tag{2.18}
$$

Notice that in the third term above, $\nu(x-\omega, \omega) - \nu(x, \omega)$ equals to $\int_\omega^{x-\omega} \frac{\partial}{\partial y} \nu(y, \omega) dy$. Rewriting $L^Y f(x)$ gives

$$
\begin{aligned}
L_1^Y f(x) &= \int_0^\infty \Big[ \int_{x-\omega}^x f(y) \frac{\partial \nu(y, \omega)}{\partial y} dy + \Big[ f(x-\omega) - f(x) \Big] \nu(x-\omega, \omega) \\
&\quad + \int_{x-\omega}^x f(x) \frac{\partial \nu(y, \omega)}{\partial y} dy \Big] \mu(d\omega) \\
&= \int_0^\infty \Big[ f(x-\omega) - f(x) \Big] \nu(x-\omega, \omega) \mu(d\omega) \\
&\quad + \int_0^\infty \int_{x-\omega}^x \Big[ f(y) - f(x) \Big] \frac{\partial \nu(y, \omega)}{\partial y} dy \mu(d\omega).
\end{aligned} \tag{2.19}
$$

But in the second term, changing the order of integration and by letting $y = x - z$ give

$$
\begin{aligned}
&\int_0^\infty \int_{x-\omega}^x \Big[ f(y) - f(x) \Big] \frac{\partial}{\partial y} \nu(y, \omega) dy \mu(d\omega) \\
&= \int_{-\infty}^x \int_{x-y}^\infty \Big[ f(y) - f(x) \Big] \frac{\partial}{\partial y} \nu(y, \omega) \mu(d\omega) dy \\
&= \int_0^\infty \int_z^\infty \Big[ f(x-z) - f(x) \Big] \frac{\partial}{\partial x} \nu(x-z, \omega) \mu(d\omega) dz \\
&= \int_0^\infty \Big[ f(x-\omega) - f(x) \Big] \Big( \int_\omega^\infty \frac{\partial}{\partial x} \nu(x-\omega, y) \mu(dy) \Big) d\omega.
\end{aligned}
$$

Substituting the above back into (2.19), we conclude that the dual generator for $L_1^Y$ has the form

$$
L_1^Y f(x) = \int_0^\infty \Big[ f(x-\omega) - f(x) \Big] \left( \nu(x-\omega, \omega) \mu(d\omega) + \int_\omega^\infty \frac{\partial}{\partial x} \nu(x-\omega, y) \mu(dy) \right) d\omega.
$$

Similar computation can be applied to case (ii). Equation (2.17) becomes

$$\begin{aligned}
L_1^Y f(x) =& -\int_0^\infty \Big[\int_{x-\omega}^x f'(y)\alpha(y)dy\Big]\nu(d\omega) \\
=& \int_0^\infty \Big[\int_{x-\omega}^x f(y)\alpha'(y)dy + \Big[f(x-\omega) - f(x)\Big]\alpha(x-\omega) \\
& + f(x)[\alpha(x-\omega) - \alpha(x)]\Big]\nu(d\omega) \\
=& \int_0^\infty \Big[f(x-\omega) - f(x)\Big]\alpha(x-\omega)\nu(d\omega) + \int_0^\infty \int_{x-\omega}^x \Big[f(y) - f(x)\Big]\alpha'(y)dy\nu(d\omega).
\end{aligned} \tag{2.20}$$

Exchanging the integral in the second term yields

$$\int_0^\infty \int_{x-\omega}^x \Big[f(y) - f(x)\Big]\alpha'(y)dy\nu(d\omega) = \int_0^\infty \Big[f(x-\omega) - f(x)\Big]\int_\omega^\infty \frac{\partial}{\partial x}\alpha(x-\omega)\nu(dy)d\omega.$$

Therefore

$$\begin{aligned}
L_1^Y f(x) =& \Big(\int_0^\infty \Big[f(x-\omega) - f(x)\Big]\alpha(x-\omega)\nu(d\omega)\Big) \\
& + \Big(\int_0^\infty \Big[f(x-\omega) - f(x)\Big]\frac{\partial}{\partial x}\alpha(x-\omega)\int_\omega^\infty \nu(dy)d\omega\Big).
\end{aligned}$$

□

The argument above can also be made when the measure $\nu$ is supported on $\mathbb{R}_-$. Lemma 1 leads to the following:

**Lemma 2.** *Consider a one-dimensional deterministic process generated by*

$$L_2^X f(x) = a(x)f'(x), \tag{2.21}$$

*where* $f \in C^1_\infty(\mathbb{R})$. *Then the explicit form of the dual generator is*

$$L_2^Y f(x) = -a(x)f'(x).$$

*Proof.* Let

$$\begin{aligned} L^X_{\delta_h} f(x) &= \int_0^\infty \Big(f(x+\omega) - f(x)\Big)\delta_h(d\omega) \\ &= \Big[f(x+h) - f(x)\Big]. \end{aligned}$$

Observe that the equation above is precisely $L^X_1$ if we let $\nu(x, d\omega)$ equal to a Dirac measure $\delta_h(d\omega)$ for some given $h$ which is greater than 0. Therefore by letting $\alpha(x)\nu(d\omega)$ in (2.20) equal to $\alpha(x)\delta_h(d\omega)$ (where $\alpha(x)$ is a constant function taking 1) its dual operator, $L^Y_{\delta_h}$ acting on indicator function $\mathbf{1}_{\leqslant y}$, has the form

$$\begin{aligned} L^Y_{\delta_h}\mathbf{1}_{\leqslant y}(x) = &\int_0^\infty \Big[\int_{x-\omega}^x \mathbf{1}_{\leqslant y}(z)\alpha'(z)dz\Big]\delta_h(d\omega) \\ &+ \int_0^\infty \Big[\mathbf{1}_{\leqslant y}(x-\omega) - \mathbf{1}_{\leqslant y}(x)\Big]\alpha(x-\omega)\delta_h(d\omega). \end{aligned}$$

Since $\alpha(z) = 1$ does not depend on $z$ and the $\alpha'(z)$ in the first term vanishes, $L^Y_{\delta_h}\mathbf{1}_{\leqslant y}(x)$ can be simplified as

$$L^Y_{\delta_h}\mathbf{1}_{\leqslant y}(x) = \mathbf{1}_{\leqslant y}(x-h) - \mathbf{1}_{\leqslant y}(x), \tag{2.22}$$

Now, we write the generator $L^X_2$ as

$$\begin{aligned} L^X_2 f(x) &= a(x)\lim_{h\downarrow 0}\frac{\Big[f(x+h) - f(x)\Big]}{h} \\ &= a(x)\lim_{h\downarrow 0}\frac{L^X_{\delta_h} f(x)}{h}. \end{aligned}$$

We substitute this into the expression of the dual generator shown in (2.15),

$$\begin{aligned} L^Y_2 f(x) &= -\int_{\mathbb{R}} L^X_2 \mathbf{1}_{\geqslant x}(y) f'(y)dy \\ &= -\lim_{h\downarrow 0}\frac{1}{h}\int_{\mathbb{R}} a(y) L^X_{\delta_h}\mathbf{1}_{\geqslant x}(y) f'(y)dy \\ &= -\lim_{h\downarrow 0}\frac{1}{h}\int_{\mathbb{R}} a(y) L^Y_{\delta_h}\mathbf{1}_{\leqslant y}(x) f'(y)dy. \end{aligned} \tag{2.23}$$

Let us first consider the integral $\int_{\mathbb{R}} a(y) L^{Y}_{\delta_h} \mathbf{1}_{\leqslant y}(x) f'(y) dy$. Since by previous calculation,

$$L^{Y}_{\delta_h} \mathbf{1}_{\leqslant y}(x) = \mathbf{1}_{\geqslant x-h}(y) - \mathbf{1}_{\geqslant x}(y) = \mathbf{1}_{[x-h,x)}(y).$$

Substituting this into the integral in (2.23), we obtain,

$$\begin{aligned}
\int_{\mathbb{R}} a(y) L^{Y}_{\delta_h} \mathbf{1}_{\leqslant y}(x) f'(y) dy &= \int_{\mathbb{R}} a(y) \Big[\mathbf{1}_{\geqslant x-h}(y) - \mathbf{1}_{\geqslant x}(y)\Big] f'(y) dy \\
&= \int_{\mathbb{R}} a(y) \mathbf{1}_{[x-h,x)}(y) f'(y) dy \\
&= \int_{x-h}^{x} a(y) f'(y) dy.
\end{aligned}$$

Now let us denote $\frac{\partial g}{\partial y}(y) = a(y) f'(y)$. Then the dual generator to $L^X_2$ is written as

$$\begin{aligned}
L^{Y}_{2} f(x) &= -\lim_{h \downarrow 0} \frac{1}{h} \int_{x-h}^{x} \frac{\partial g(y)}{\partial y} dy \\
&= -\lim_{h \downarrow 0} \frac{g(x) - g(x-h)}{h} \\
&= -\frac{\partial g}{\partial x}(x) \\
&= -a(x) f'(x).
\end{aligned}$$

□

**Lemma 3.** *Let* $L^X_3 f(x) = \int_0^\infty f'(x) y \mathbf{1}_{B_1}(y) \nu(x, dy)$ *be a generator of a Feller process. Then its dual operator is*

$$L^{Y}_{3} f(x) = -\int_0^\infty f'(x) y \mathbf{1}_{B_1}(y) \nu(x, dy).$$

*Proof.* The proof is straight forward by letting $\int_0^\infty y \mathbf{1}_{B_1}(y) \nu(x, dy) = a(x)$ in $L^X_2 f$ in Lemma 2.

**Lemma 4.** *Let* $L^X_4 f(x) = \frac{1}{2} G(x) f''(x)$ *for some* $f \in C^2_\infty(\mathbb{R})$ *and* $G \in C^1_\infty(\mathbb{R})$*. Then its dual operator is*

$$L^{Y}_{4} f(x) = \frac{1}{2}\Big[G(x) f''(x) + G'(x) f'(x)\Big].$$

*Proof.* Define the following operators:

$$
\begin{aligned}
L^X_{\delta_h} f(x) &= \int_0^\infty \Big(f(x+\omega) - f(x)\Big)\delta_h(d\omega) \\
&= f(x+h) - f(x); \\
L^X_{\delta_{-h}} f(x) &= \int_{-\infty}^0 \Big(f(x+\omega) - f(x)\Big)\delta_{-h}(d\omega) \\
&= f(x-h) - f(x). \qquad (2.24)
\end{aligned}
$$

Clearly, $L^X_{\delta_h}$ and $L^X_{\delta_{-h}}$ equal to $L^X_1$ in Lemma 1 by letting $\nu(x, d\omega) = a(x)\nu(d\omega)$ in $L^X_1$ be $\delta_h(d\omega)$ and $\delta_{-h}(d\omega)$ respectively. In this proof we denote their dual operators by $L^Y_{\delta_h}$ and $L^Y_{\delta_{-h}}$.

Now, rewriting generator $L^X_4 f(x)$, we obtain

$$
\begin{aligned}
L^X_4 f(x) &= \frac{1}{2} G(x) f''(x) \\
&= \frac{G(x)}{2} \lim_{h\downarrow 0} \frac{1}{h^2}\Big[f(x+h) + f(x-h) - 2f(x)\Big] \\
&= \frac{G(x)}{2} \lim_{h\downarrow 0} \frac{1}{h^2}\Big[L^X_{\delta_h} f(x) + L^X_{\delta_{-h}} f(x)\Big].
\end{aligned}
$$

The dual generator $L^X f(x)$, denoted as $L^Y f(x)$ is

$$
\begin{aligned}
L^Y_4 f(x) &= -\int_{\mathbb{R}} L^X_4 \mathbf{1}_{\geqslant x}(y) f'(y) dy \\
&= -\int_{\mathbb{R}} \frac{G(y)}{2} \lim_{h\downarrow 0} \frac{1}{h^2}\Big[L^X_{\delta_h} \mathbf{1}_{\geqslant x}(y) + L^X_{\delta_{-h}} \mathbf{1}_{\geqslant x}(y)\Big] f'(y) dy
\end{aligned}
$$

Substituting the indicator function into $L^X_{\delta_h} f(x)$ and $L^X_{\delta_{-h}} f(x)$ in (2.24), the dual operator

$$
\begin{aligned}
L^Y_4 f(x) &= -\lim_{h\downarrow 0} \frac{1}{h^2} \int_{\mathbb{R}} \frac{G(y)}{2}\Big[\mathbf{1}_{\geqslant x}(y+h) - \mathbf{1}_{\geqslant x}(y) + \mathbf{1}_{\geqslant x}(y-h) - \mathbf{1}_{\geqslant x}(y)\Big] f'(y) dy \\
&= -\lim_{h\downarrow 0} \frac{1}{h^2} \int_{\mathbb{R}} \frac{G(y)}{2}\Big[\mathbf{1}_{\geqslant x-h}(y) - \mathbf{1}_{\geqslant x}(y) + \mathbf{1}_{\geqslant x+h}(y) - \mathbf{1}_{\geqslant x}(y)\Big] f'(y) dy \\
&= -\lim_{h\downarrow 0} \frac{1}{2h^2} \Big[\int_{x-h}^{x} G(y) f'(y) dy - \int_x^{x+h} G(y) f'(y) dy\Big].
\end{aligned}
$$

Letting function $H'(y) = G(y)f'(y)$,

$$\begin{aligned}
L_4^Y f(x) &= \lim_{h \downarrow 0} \frac{H(x+h) + H(x-h) - 2H(x)}{2h^2} \\
&= \frac{1}{2} \frac{\partial^2 H(x)}{\partial x^2} \\
&= \frac{1}{2} \frac{\partial [G(x) \frac{\partial f}{\partial x}]}{\partial x} \\
&= \frac{G(x)}{2} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \frac{\partial G(x)}{\partial x} \frac{\partial f}{\partial x}.
\end{aligned}$$

$\square$

By linearity of generators we have proven the following theorem using the duality relation directly in Lemma 1 - 4.

**Proposition 5.** *(presented as Proposition 3.1, Kolokoltsov [54] ) Under the assumptions in Theorem 3 suppose additionally that the Lévy measures are supported on* $\mathbb{R}_+$ *and either (i)* $\nu(x, d\omega) = \nu(x, \omega)d\omega$ *with* $\nu(x, \omega)$ *being continuously differentiable in* $x$ *or (ii)* $\nu(x, \omega) = a(x)\nu(d\omega)$ *with a certain Lévy measure* $\nu$ *and a continuously differentiable function* $a$*. Then the generator of the dual Markov process acts by*

$$\begin{aligned}
L^Y f(x) =& \frac{1}{2} G(x) f''(x) + \Big( \frac{1}{2} G'(x) - b(x) + \int_0^1 y(\nu - \tilde{\nu})(x, d\omega) \Big) f'(x) \\
&+ \int_0^\infty \Big[ f(x - \omega) - f(x) + f'(x) \mathbf{1}_{B_1}(\omega) \Big] \tilde{\nu}(x, \omega) \mu(d\omega)
\end{aligned} \tag{2.25}$$

*where in*

*(i)* $\tilde{\nu}(x, d\omega) = \Big( \nu(x - \omega, \omega) + \int_\omega^\infty \frac{\partial}{\partial x} \nu(x - \omega, y) dy \Big) \mu(d\omega)$;

*(ii)* $\tilde{\nu}(x, d\omega) = a(x - \omega)\nu(d\omega) + \frac{\partial}{\partial x} a(x - \omega) \int_\omega^\infty \nu(dy) d\omega$.

## 2.6 APPENDIX

### THEOREMS OF INTEGRATIONS BY PARTS

**Theorem.** *(Lebesgue-Stielges integration by parts) Let* $f$ *and* $g$ *be right-continuous functions on the finite interval* $[a,b]$ *such that both of them are of bounded variation on* $[a,b]$*. Then*

$$\int_{[a,b]} f(x-)d\mu_g + \int_{[a,b]} g(x)d\mu_f = f(b)g(b) - f(a-)g(a-) \tag{2.26}$$

*where* $f(x-) = \lim_{y\uparrow x} f(y)$ *and* $\mu_f$, $\mu_g$ *are the signed measures induced by* $f$ *and* $g$ *respectively.*

*Proof.* Without loss of generality, we assume that $f$ and $g$ are non-negative and monotonically increasing functions, otherwise we may decompose $f$ and $g$ with $f = f_1 - f_2$ and $g = g_1 - g_2$ as above then combine the resulting equations in the form (2.3) to obtain the result of the theorem.

Since $f$ and $g$ are right continuous and monotonically increasing, we choose some increasing sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_m\}_{m=1}^{\infty}$ of measurable simple functions of the form $\sum_{i=1}^{k} \lambda_i \chi_{[a_i, a_{i+1})}$ where $\lambda_i < \lambda_{i+1}$, such that each of them tends to $f$ and $g$ respectively. We show that (2.3) holds for $f$ and $g$ replaced by $f_n$ and $g_m$. Since $f_n$ and $g_m$ are right continuous and monotonically increasing Borel measurable (step) functions, the integrals are defined.

As $f$ and $g$ are right continuous, let us assume them defined on an interval $[a-\epsilon, b+\epsilon)$, for some $\epsilon > 0$ and constant on the intervals $[a-\epsilon, a]$ and $[b, b+\epsilon)$. Then if $g_m = \sum_{i=1}^{s-1} \zeta_i \chi_{[a_i, a_{i+1})}$, we have $a_0 < a < a_1 < \ldots < a_s \leqslant b < a_{s+1}$. Clearly $\mu_{g_m}(a_i, a_{i+1}] = g_m(a_{i+1}) - g_m(a_i)$ for each $i$ and $\mu_{g_m}(x_i, x_{i+1}] = 0$ if $x_i$ and $x_{i+1}$ belong to the same partition. Then if $h$ is any finite-valued Borel-measurable function,

$$\int_{[a,b]} h(x-)d\mu_{g_m} = \sum_{i=1}^{s-1} h(a_i-)\big[g_m(a_i) - g_m(a_{i-1})\big]$$

and similarly,

$$\int_{[a,b]} h(x)d\mu_{f_n} = \sum_{i=1}^{r-1} h(a_i')\big[f_n(a_i') - f_n(a_{i-1}')\big]$$

Let $\{c_0, c_1, \ldots, c_p\}$ be the union of the points of the two partitions $\{a_i\}_{i=1}^{s-1}$ of $g_m$, $\{a_i'\}_{i=1}^{r-1}$ of $f_n$ and the endpoints $\{a\}$ and $\{b\}$, such that $c_0 < a = c_1 < c_2 < \ldots < c_{p-1} = b < c_p$. Then,

$$\begin{aligned}\int_{[a,b]} f_n(x-)d\mu_{g_m} &= \sum_{i=1}^{p-1} f_n(c_i-)\left[g_m(c_i) - g_m(c_{i-1})\right] \\ &= \sum_{i=1}^{p-1} f_n(c_{i-1})\left[g_m(c_i) - g_m(c_{i-1})\right] \\ &= - f_n(c_0)g_m(c_0) + g_m(c_{p-1})f_n(c_{p-2}) \\ &\quad - \sum_{i=1}^{p-1} g_m(c_i)\left[f_n(c_i) - f_n(c_{i-1})\right] \end{aligned} \tag{2.27}$$

On the other hand,

$$\int_{[a,b]} g_m(x)d\mu_{f_n} = \sum_{i=1}^{p-1} g_m(c_i)\left[f_n(c_i) - f_n(c_{i-1})\right].$$

By observing that $f_n(c_0)g_m(c_0) = f_n(a-)g_m(a-)$ and that $c_{p-1} = b$, we rewrite (2.27) as

$$\int_{[a,b]} f_n(x-)d\mu_{g_m} + \int_{[a,b]} g_m(x)d\mu_{f_n} = g_m(b)f_n(b) - f_n(a-)g_m(a-), \tag{2.28}$$

i.e. (2.3) holds for $f_n$ and $g_m$.

Next, suppose that $h$ is any non-negative monotonically increasing Borel measurable function. Then given some $\epsilon > 0$, by Theorem 5, page 58 in the text book by de Barra [23], we can find a step function $\phi$, $0 \leqslant \phi \leqslant h$ such that $|h - \phi| < \epsilon$ uniformly on $(a - \epsilon, b + \epsilon)$. Then for all $n$ greater than some fixed $n_0$,

$$\begin{aligned}\left|\int h d\mu_{f_n} - \int h d\mu_f\right| &\leqslant \int (h-\phi)d\mu_{f_n} + \int (h-\phi)d\mu_f \\ &\quad + \left|\int \phi d\mu_{f_n} - \int \phi d\mu_f\right|\end{aligned}$$

where the first and second terms on the right hand side are less than $\epsilon\left(\mu_{f_n}[a,b]+\mu_f[a,b]\right) < K\epsilon$ while the third term clearly tends to zero as $n \to \infty$. Therefore we must have $\int h d\mu_{f_n} \to \int h d\mu_f$ as $n$ tends to $\infty$. Same result holds for $\mu_{g_m}$ and $\mu_g$.

Now let $n \to \infty$ in (2.28). By (Lebesgue's) monotone convergence theorem and the fact that $f_n \uparrow f$ uniformly (and hence point-wise), we get

$$\int_{[a,b]} f(x-)d\mu_{g_m} + \int_{[a,b]} g_m(x)d\mu_{f_n} = g_m(b)f(b) - f(a-)g_m(a-).$$

Similarly, by letting $m \to \infty$ in equation above we show that (2.3) holds. $\square$

**Proposition.** *Let* $f$ *and* $g$ *be right-continuous, non-negative and monotonically increasing functions on* $\mathbb{R}$ *such that both of them are bounded on* $\mathbb{R}$*. Then*

$$\int_{\mathbb{R}} f(x-)d\mu_g + \int_{\mathbb{R}} g(x)d\mu_f = \lim_{b\to\infty} f(b)g(b) - \lim_{a\to-\infty} f(a)g(a) \tag{2.29}$$

*where* $f(x-) = \lim_{y\uparrow x} f(x)$ *and* $\mu_f$, $\mu_g$ *are the signed measures induced by* $f$ *and* $g$ *respectively.*

*Proof.* Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_l\}_{l=1}^{\infty}$ be the sequences of functions supported on $(-k-2\epsilon, k]$ and $(-l-2\epsilon, l]$ respectively such that for some small $\epsilon > 0$,

$$f_k(x) = \begin{cases} f(x) & ,-k \leqslant x \leqslant k \\ f(-k) & ,-(k+2\epsilon) < x < -k, \end{cases}$$

and

$$g_l(x) = \begin{cases} g(x) & ,-l \leqslant x \leqslant l \\ g(-l) & ,-(l+2\epsilon) < x < -l. \end{cases}$$

Then both $\{f_k\}_{k=1}^{\infty}$ and $\{g_l\}_{l=1}^{\infty}$ converge uniformly to $f$ and $g$ respectively and all $f_k$ are right continuous on $[-k, k)$, monotonically increasing and non-negative functions $[-k-\epsilon, k]$, and similar observation holds for $g_k$. Assume without loss of generality that $k < l$. By Theorem 1,

$$\begin{aligned} \int_{\mathbb{R}} f_k(x-)d\mu_{g_l} + \int_{\mathbb{R}} g_l(x)d\mu_k =& f_k(k)g_l(k) - f(-k-\epsilon)g_l(-k-\epsilon) \\ =& f(k)g(k) - f(-k)g(-k). \end{aligned}$$

Now, similar to the argument in Theorem 1, if $h$ is any non-negative monotonically increasing Borel measurable functions, $\int h d\mu_{f_k} \to \int h d\mu_f$. Letting $l \to \infty$, by (Lebesgue's) monotone convergence theorem and the fact that $\{f_{k_a}\}_{k=1}^{\infty}$ and $\{g_{k_a}\}_{k=1}^{\infty}$ are increasing sequences that converge uniformly to $f$ and $g$ respectively,

$$\int_{\mathbb{R}} f_k(x-)d\mu_g + \int_{\mathbb{R}} g(x)d\mu_k = f(a+k)g(a+k) - f(-k)g(-k).$$

Then letting $k \to \infty$, we show that (2.4) holds. □

Next, we give an alternative proof to Theorem 2 using Fubini's theorem.

**Theorem.** *(copy of Theorem 2) Suppose that* $(X_t^x)_{t\geqslant 0} \in \mathbb{R}$ *is a stochastically monotone Markov process such that* $P(X_t^x \geqslant y)$ *is right continuous in all* $x \in \mathbb{R}$ *and tends to* $1$ *and* $0$ *as* $x$ *tends to* $\infty$ *and* $-\infty$ *respectively. Then its dual* $(Y_t^y)_{t\geqslant 0}$ *is a Markov process.*

*Alternative Proof to Theorem 2.* We have to show that the dual process $(Y_t)_{t\geqslant 0}$ satisfies the Chapman-Kolmogorov equation. By the Chapman-Kolmogorov equation of $(X_t)_{t\geqslant 0}$, the duality relation in (2.2) can be written as

$$\begin{aligned} P(Y_{t+s}^y \leqslant x) &= P(X_{t+s}^x \geqslant y) \\ &= \int_{z\in\mathbb{R}} P_t^X(x, dz)P(X_s^z \geqslant y) \\ &= \int_{z\in\mathbb{R}} P(Y_s^y \leqslant z)P_t^X(x, dz). \end{aligned}$$

The cumulative distribution function $P(Y_s^y \leqslant z)$ can be written as $\int_{w\leqslant z} P_s^Y(y, dw)$, where $P_s^Y(y, dw)$ is the measure that induces the corresponding distribution function. Therefore

$$\begin{aligned} P(X_{t+s}^x \geqslant y) &= \int_{z\in\mathbb{R}} \int_{w\leqslant z} P_s^Y(y, dw)P_t^X(x, dz) \\ &= \int_{z\in\mathbb{R}} \left\{ \int_{w\in\mathbb{R}} \mathbb{I}_{\leqslant z}(w)P_s^Y(y, dw) \right\} P_t^X(x, dz). \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}
&\int_{z\in\mathbb{R}}\left\{\int_{w\in\mathbb{R}}\mathbb{I}_{\leqslant z}(w)P_s^Y(y,dw)\right\}P_t^X(x,dz)\\
&=\int_{w\in\mathbb{R}}\left\{\int_{z\in\mathbb{R}}\mathbb{I}_{\leqslant z}(w)P_t^X(x,dz)\right\}P_s^Y(y,dw).
\end{aligned}$$

Since $\mathbb{I}_{\leqslant z}(w)=\mathbb{I}_{\geqslant w}(z)$, we get

$$\begin{aligned}
P(X_{t+s}^x\geqslant y) &= \int_{w\in\mathbb{R}}\left\{\int_{z\in\mathbb{R}}\mathbb{I}_{\geqslant w}(z)P_t^X(x,dz)\right\}P_s^Y(y,dw)\\
&= \int_{w\in\mathbb{R}}\int_{z\geqslant w}P_t^X(x,dz)P_s^Y(y,dw)\\
&= \int_{w\in\mathbb{R}}P_t(X_t^x\geqslant w)P_s^Y(y,dw)\\
&= \int_{w\in\mathbb{R}}P_t(Y_t^w\leqslant x)P_s^Y(y,dw).
\end{aligned}$$

Therefore, there is a dual Markov process to $(X_t)_{t\geqslant 0}$. $\square$

# 3 PARETO-ORDERED DUALITY IN EUCLIDEAN SPACE I.

CONTENTS


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## 3.1 INTRODUCTION

In chapter 2, the theory of monotonicity and duality was introduced via the transition probability distributions of two Markov processes. However, in many applications the distributions of Markov processes are given in terms of its infinitesimal generator. Therefore in this chapter we consider Lévy-Khintchine type pre-generators and take an analytic approach to the theory of duality for Markov processes in finite dimensional $\mathbb{R}^d$. In section 3.3, we use Fubini's theorem to show the necessary and sufficient conditions for a Markov dual process to exist in $\mathbb{R}^d$. Aiming at an instance of integro-differential operators of the form

$$Lf(x) = \sum_{i=1}^{d} b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} G_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d_+} [f(x+y) - f(x)]\nu(x, dy) \tag{3.1}$$

we construct a Feller process generated by (3.1). Such constructions of Markov processes can be done using standard stochastic calculus (see chapter 4 of textbook by Kolokoltsov [55]) or by analysing well-posedness of the corresponding Markov semi-group (as described by Kolokoltsov [53] and in chapter 5 of Kolokoltsov's textbook [55]). The latter, which is the approach we are taking in this chapter, requires some regularity conditions on (3.1) and involves some analysis of the corresponding evolution equations for the derivatives of a Markov evolution. Some conditions are needed so that the Feller processes generated by (3.1) are stochastically monotone. Moreover, with additional regularity assumptions, we show that a dual Markov process to the considered Feller process exists on $\mathbb{R}^d$.

## 3.2 BASIC NOTATIONS

Let $(X_t)_{t\geqslant 0} = (X_{1,t}, \dots, X_{d,t})_{t\geqslant 0}$ be a time-homogenous Markov process in $d$–dimensional real space ($\mathbb{R}^d$). Consider some $\mathbf{a} = (a_1, \dots, a_d)$ and $\mathbf{b} = (b_1, \dots b_d)$ in $\mathbb{R}^d$. A $d$–dimensional Euclidean space is said to be equipped with Pareto order when $\mathbf{a} \geqslant \mathbf{b}$ if and only if $a_i \geqslant b_i$ for all $i \in \{i, \dots, d\}$. Similarly, given a closed and non-empty convex cone $c \subset \mathbb{R}^d$ with a vertex at the origin, we say that $\mathbf{a}$ $c$-dominates $\mathbf{b}$ (written as $\mathbf{a} \geqslant_c \mathbf{b}$) if $\mathbf{a} - \mathbf{b} \in c$.

In this chapter, we assume Pareto order in $\mathbb{R}^d$ for simplicity, noting that the following analysis holds true for all similar orders generated by linear transformations of Pareto order.

Suppose that the Markov process $(X_t)_{t\geqslant 0}$ is characterised by a family of transition probability distributions $P_t^X(\mathbf{x}, A)$, at time $t$. Here $\mathbf{x} = (x_1, \dots x_n) \in \mathbb{R}^d$ denotes a starting point while $A$ is a Borel subset in $\mathcal{B}(\mathbb{R}^d)$. If sets $A$ and $B$ are intervals $[\mathbf{y}, \infty)$ and $(-\infty, \mathbf{y}] \subset \mathbb{R}^d$ respectively, we write

$$P_t^X(\mathbf{x}, A) = P(X_t^x \geqslant \mathbf{y}) = P(X_{1,t}^x \geqslant y_1, \dots, X_{d,t}^x \geqslant y_d)$$
$$P_t^X(\mathbf{x}, B) = P(X_t^x \leqslant \mathbf{y}) = P(X_{1,t}^x \leqslant y_1, \dots, X_{d,t}^x \leqslant y_d) = F_{\mathbf{x},t}^{\mathbf{X}}(\mathbf{y})$$

One can interpret the transition probability distribution $P(X_t^x \leqslant \mathbf{y})$ as a joint cumulative distribution of random variables $X_{1,t}^x, \dots, X_{d,t}^x$. A Markov process $\mathbf{X}$ is said to be stochastically monotone if $P(X_t^x \leqslant \mathbf{y})$ is a non-decreasing function (in the Pareto order) in $\mathbf{x} \in \mathbb{R}^d$.

Assume that :

(P1) $(X_t)_{t\geqslant 0}$ is stochastically monotone;
(P2) $P(X_t^x \leqslant \mathbf{y})$ is right-continuous in $\mathbf{x}$, i.e. $\lim_{z_i \downarrow x_i,\ \text{for all}\ 0\leqslant i\leqslant d} P(X_t^z \geqslant \mathbf{y}) = P(X_t^x \geqslant \mathbf{y})$;

(P3) For $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbb{R}^d$ such that $\mathbf{x}^{(1)} \leqslant \mathbf{x}^{(2)}$,

$$\triangle L(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \sum_{\mathbf{a} \in A} (-1)^{s(\mathbf{a})} P(X_t^{\mathbf{a}} \geqslant \mathbf{y}) \geqslant 0 \tag{3.2}$$

such that

1. set $A = \left\{ \mathbf{a} = (a_1, \dots, a_d) : a_i \in \{x_i^{(1)}, x_i^{(2)}\} \right\}$,
2. for any $\mathbf{a} \in A$, $s(\mathbf{a}) = \#\left| i : a_i = x_i^{(1)}, i = 1, 2, \dots, d \right|$ is the number of indices $i$ for which $a_i = x_i^{(1)}$

(P4) $\lim_{\mathbf{x}_i \to \infty, \text{ for all } i=1,\dots,d} P(X_t^{\mathbf{x}} \leqslant \mathbf{y}) = 1$;

Now, let us define a function

$$F_{\mathbf{y},t}^{Y}(x) = P(X_t^{\mathbf{x}} \geqslant \mathbf{y}) \tag{3.3}$$

By assumptions P1 and P2, the left hand side of (3.3) is a non-decreasing and right continuous function in $\mathbf{x}$. Also, condition P4 ensures that the function $F_{\mathbf{y},t}^{Y}(x)$ tends to 1 as all $x_i$ tend to $\infty$. Furthermore, by condition P3 $F_{\mathbf{y},t}^{Y}(x)$ is a function with non-negative increments in $\mathbf{x}$. With these four properties, $F_{\mathbf{y},t}^{Y}(x)$ forms a family of joint cumulative distribution functions $F_{\mathbf{y},t}^{Y}(x) = P(\mathbf{Y}_t^{\mathbf{y}} \leqslant \mathbf{x})$. Similar to before, equation (3.3) is called the duality relation. The family of random variables $(Y_t)_{t \geqslant 0}$ with cumulative distribution functions satisfying (3.3) is said to be the dual of $(X_t)_{t \geqslant 0}$.

In the next theorem we present a natural extension of Theorem 2 in chapter 2. We use the same approach developed by Siegmund [72] to show the necessary and sufficient condition for the existence of a dual Markov process to $(X_t)_{t \geqslant 0}$ on $\mathbb{R}^d$.

## 3.3 PARETO DUALITY IN EUCLIDEAN SPACE

**Theorem 4.** *Suppose that $(X_t)_{t \geqslant 0}$ is a Markov process on $\mathbb{R}^d$ equipped with Pareto order such that all conditions P1 to P4 are satisfied. Then its dual Markov process$(Y_t)_{t \geqslant 0}$ exists on $\mathbb{R}^d$.*

*Proof.* It remains to show that $P(Y_s^{\boldsymbol{y}} \leqslant \boldsymbol{x})$ satisfies the Chapman-Kolmogorov equation. By the Chapman-Kolmogorov equation of $(X_t)_{t\geqslant 0}$ and the duality relation in (3.3):

$$\begin{aligned} P(Y_{t+s}^{\boldsymbol{y}} \leqslant \boldsymbol{x}) &= P(X_{t+s}^{\boldsymbol{x}} \geqslant \boldsymbol{y}) \\ &= \int_{\boldsymbol{z}\in\mathbb{R}^d} P_t^X(\boldsymbol{x}, d\boldsymbol{z}) P(X_s^{\boldsymbol{z}} \geqslant \boldsymbol{y}) \\ &= \int_{\boldsymbol{z}\in\mathbb{R}^d} P(Y_s^{\boldsymbol{y}} \leqslant \boldsymbol{z}) P_t^X(\boldsymbol{x}, d\boldsymbol{z}). \end{aligned}$$

Since $P(Y_s^{\boldsymbol{y}} \leqslant \boldsymbol{z})$ can be written as $\int_{\boldsymbol{w}\leqslant\boldsymbol{z}} P_s^Y(\boldsymbol{y}, d\boldsymbol{w})$, we obtain

$$\begin{aligned} P(X_{t+s}^{\boldsymbol{x}} \geqslant \boldsymbol{y}) &= \int_{\boldsymbol{z}\in\mathbb{R}^d} \int_{\boldsymbol{w}\leqslant\boldsymbol{z}} P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) P_t^X(\boldsymbol{x}, d\boldsymbol{z}) \\ &= \int_{\boldsymbol{z}\in\mathbb{R}^d} \left\{ \int_{\boldsymbol{w}\in\mathbb{R}^d} \mathbb{I}_{\leqslant\boldsymbol{z}}(\boldsymbol{w}) P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) \right\} P_t^X(\boldsymbol{x}, d\boldsymbol{z}). \end{aligned}$$

Applying Fubini's theorem to the equation above, the first and second integration are interchangeable. Therefore,

$$\begin{aligned} P(X_{t+s}^{\boldsymbol{x}} \geqslant \boldsymbol{y}) &= \int_{\boldsymbol{z}\in\mathbb{R}^d} \left\{ \int_{\boldsymbol{w}\in\mathbb{R}^d} \mathbb{I}_{\leqslant\boldsymbol{z}}(\boldsymbol{w}) P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) \right\} P_t^X(\boldsymbol{x}, d\boldsymbol{z}) \\ &= \int_{\boldsymbol{w}\in\mathbb{R}^d} \left\{ \int_{\boldsymbol{z}\in\mathbb{R}^d} \mathbb{I}_{\leqslant\boldsymbol{z}}(\boldsymbol{w}) P_t^X(\boldsymbol{x}, d\boldsymbol{z}) \right\} P_s^Y(\boldsymbol{y}, d\boldsymbol{w}). \end{aligned}$$

Since $\mathbb{I}_{\leqslant\boldsymbol{z}}(\boldsymbol{w}) = \mathbb{I}_{\geqslant\boldsymbol{w}}(\boldsymbol{z})$, the Chapman-Kolmogorov equation becomes

$$\begin{aligned} P(X_{t+s}^{\boldsymbol{x}} \geqslant \boldsymbol{y}) &= \int_{\boldsymbol{w}\in\mathbb{R}^d} \left\{ \int_{\boldsymbol{z}\in\mathbb{R}^d} \mathbb{I}_{\geqslant\boldsymbol{w}}(\boldsymbol{z}) P_t^X(\boldsymbol{x}, d\boldsymbol{z}) \right\} P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) \\ &= \int_{\boldsymbol{w}\in\mathbb{R}^d} \int_{\boldsymbol{z}\geqslant\boldsymbol{w}} P_t^X(\boldsymbol{x}, d\boldsymbol{z}) P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) \\ &= \int_{\boldsymbol{w}\in\mathbb{R}^d} P_t(X_t^{\boldsymbol{x}} \geqslant \boldsymbol{w}) P_s^Y(\boldsymbol{y}, d\boldsymbol{w}) \\ &= \int_{\boldsymbol{w}\in\mathbb{R}^d} P_t(Y_t^{\boldsymbol{w}} \leqslant \boldsymbol{x}) P_s^Y(\boldsymbol{y}, d\boldsymbol{w}). \end{aligned}$$

Therefore, $(X_t)$ has a dual Markov process on $\mathbb{R}^d$. □

## 3.4 WELL-POSEDNESS

In this section we analyse the well-posedness of Markov semi-groups by analysing the corresponding evolution equations for the derivatives of a Markov evolution. In particular, we study the conditions of integro-differential operators of the form

$$Lf(x) = \sum_{i=1}^{d} b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} G_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d_+} [f(x+y) - f(x)]\nu(x, dy), \tag{3.4}$$

where the function $f \in C^2_\infty(\mathbb{R}^d)$, so that we can reconstruct a Feller process from L. The construction of stochastically monotone Markov processes from this pre-generator was briefly discussed in chapter 5, Kolokoltsov [55], but the details of such a construction will be presented in this chapter.

Before we proceed, let us first introduce the notion of conditionally positive operator. Let $C(\mathbb{R}^d)$ be a space of bounded continuous functions that map $\mathbb{R}^d$ to $\mathbb{R}$. We say that an operator L in $C(\mathbb{R}^d)$ defined on a domain $D_L$ is conditionally positive if $Lf(x) \geqslant x$ for any $f \in D_L$ s.t. $f(x) = \max_y f(y) \geqslant 0$.

Let us also recall the perturbation theory. In summary, this theory can be applied when the generator under consideration can be represented as a sum of two operators, one is bounded and another generates a semi-group. The next theorem is a simple form of the perturbation theory presented in chapter 1 of the textbook by Kolokoltsov [55] (for a detailed study see work done by Maslov [60] and Simon and Reed [74]).

**Theorem 5.** *(Perturbation theory) Let an operator* $L$ *with domain* $D_L$ *generate a strongly continuous semi-group* $(T_t)_{t\geqslant 0}$ *on a Banach Space* $B$*, and let* $A$ *be a bounded operator on* $B$*. Then*

*(i)* $L + A$ *also generates a strongly continuous semi-group* $(\tilde{T}_t)_{t\geqslant 0}$ *on* $B$ *given by the following series converging in the operator norm:*

$$\tilde{T}_t = T_t + \sum_{m=1}^{\infty} \tilde{T}_t^{(m)}$$

*where* $\tilde{T}_t^{(m)} = \int_0^t T_{t-s_m} A \tilde{T}_{s_m}^{(m-1)} ds_m$.

*(ii)* $\tilde{T}_t f$ *is the unique (bounded) solution of the integral equation*

$$\tilde{T}_t f = T_t f + \int_0^t T_{t-s} A \tilde{T}_s f ds$$

*with a given* $f_0 = f$.

*(iii) If additionally,* $D$ *is an invariant core for* $L$ *that is itself a Banach space under the norm* $\|\cdot\|_D$*, the* $T_t$ *are uniformly (for* $t$ *from a compact interval) bounded operators* $D \to D$ *and* $A$ *is a bounded operator* $D \to D$*, then* $D$ *is an invariant core for* $L + A$ *and* $\tilde{T}_t$ *are uniformly bounded operators in* $D$.

Suppose that $\int_{\mathbb{R}^d_+} \nu(x, dy)$ in (3.4) is bounded. Then by Theorem 5, it is straight forward to conclude that the operator in (3.4) generates a Feller process. For the case where $\int_{\mathbb{R}^d_+} \nu(x, dy)$ is not assumed to be bounded, the following theorem (which was also briefly discussed in Theorem 5.9.4 by Kolokoltsov [55]) gives the criterion for (3.4) to generate a Feller process.

**Theorem 6.** *Consider an operator* $L$ *of the form in (3.4) for all* $x \in \mathbb{R}^d$ *such that:*

*(1)* $\nu$ *is a Lévy measure with support on* $\mathbb{R}^d_+$*, with a finite first moment*

$$\sup_x \int_{\mathbb{R}^d_+} |y| \nu(x, dy) < \infty,$$

*and that* $\nu$ *is twice continuous differentiable in* $x$ *with*

$$\sup_{x, 1 \leqslant j \leqslant d} \int_{\mathbb{R}^d_+} |y| \frac{\partial \nu(x, dy)}{\partial x_j} < \infty, \quad \sup_{x, 1 \leqslant j, k \leqslant d} \int_{\mathbb{R}^d_+} |y| \frac{\partial^2 \nu(x, dy)}{\partial x_j \partial x_k} < \infty;$$

*(2)* $G, b \in C^2(\mathbb{R}^d)$ *are twice differentiable such that their first and second derivatives are bounded;*

*(3) The matrix* $G = (G_{i,j})_{i,j=1}^d$ *is positive definite and its element* $G_{i,j}$ *depends only on* $x_i, x_j$*;*

*Then* $L$ *generates a Feller process. Moreover the space* $C^2_\infty(\mathbb{R}^d)$ *is an invariant core for the semi-group of this process.*

*Proof.* First, observe that for any $h \geqslant 0$, assumption (1) implies that

$$\begin{aligned}
\sup_x \int_{\mathbb{R}^d_+ \setminus B_h} \nu(x, dy) &\leqslant \sup_x \frac{1}{h} \int_{\mathbb{R}^d_+} |y| \nu(x, dy) < \infty; \\
\sup_{x, 1 \leqslant j \leqslant d} \int_{\mathbb{R}^d_+ \setminus B_h} \frac{\partial \nu(x, dy)}{\partial x_j} &\leqslant \sup_{x, 1 \leqslant j \leqslant d} \frac{1}{h} \int_{\mathbb{R}^d_+} |y| \frac{\partial \nu(x, dy)}{\partial x_j} < \infty; \\
\sup_{x,, 1 \leqslant j, k \leqslant d} \int_{\mathbb{R}^d_+ \setminus B_h} \frac{\partial^2 \nu(x, dy)}{\partial x_j \partial x_k} &\leqslant \sup_{x, 1 \leqslant j \leqslant d} \frac{1}{h} \int_{\mathbb{R}^d_+} |y| \frac{\partial^2 \nu(x, dy)}{\partial x_j \partial x_k} < \infty. \qquad (3.5)
\end{aligned}$$

In (3.4), $L$ consists of a second-order differential operator $\sum_{i=1}^d b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d G_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}$ and an operator $\int_{\mathbb{R}^d_+} \Big[ f(x+y) - f(x) \Big] \nu(x, dy)$, where the latter may, or may not, be bounded. Let us introduce an approximation operator $L^h$ defined as:

$$L^h f(x) = \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d G_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d_+ \setminus B_h} [f(x+y) - f(x)] \nu(x, dy). \qquad (3.6)$$

In the equation above, the first two terms on the right-hand side represent a second-order differential operator. The third term is a bounded operator in the Banach space $C_\infty(\mathbb{R}^d)$, $C^1_\infty(\mathbb{R}^d)$ and $C^2_\infty(\mathbb{R}^d)$ by observations in (3.5). Since, in the first two terms each $b_i$ and $G_{i,j}$ are bounded continuous functions with $G$ being a positive definite matrix, the second-order differential operator is a diffusion operator and generates a conservative Feller semi-group in $C_\infty(\mathbb{R}^d)$ with an invariant core $C^2_\infty(\mathbb{R}^d)$ (see chapter 6 of the book by Applebaum [5] and Kolokoltsov's book [55]). By perturbation theory in Theorem 5, $L^h f$ as the sum of diffusion and bounded operators also generates a conservative Feller semi-group $(T^h_t)_{t \geqslant 0} \in C_\infty(\mathbb{R}^d)$, since the conservativeness is preserved through the perturbation series representation. Now, we consider the following equation corresponding to (3.4):

$$\begin{aligned}
\dot{f}^h_t &= L^h f_t \\
f^h_0 &= f \text{ where } f \in C_\infty(\mathbb{R}^d). \qquad (3.7)
\end{aligned}$$

Taking the first derivative of $\dot{f}_t^h$ in (3.7) with respect to $x_k$ for all $k \in \{1, \ldots, d\}$ for $f \in C^2_\infty(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x_k}\dot{f}_t^h(x) =& \frac{d}{d_t}\Big(\frac{\partial f_t^h(x)}{\partial x_k}\Big) \\
=& L^h \frac{\partial f_t}{\partial x_k} + \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_k}\frac{\partial f_t}{\partial x_i} + \sum_{i \neq k} \frac{\partial G_{ik}}{\partial x_k}\frac{\partial^2 f_t}{\partial x_i \partial x_k} + \frac{1}{2}\frac{\partial G_{kk}}{\partial x_k}\frac{\partial^2 f_t}{\partial x_k^2} \\
&+ \int_{\mathbb{R}^d_+ \setminus B_h} \Big[f_t(x+y) - f_t(x)\Big] \frac{\partial \nu}{\partial x_k}(x, dy)
\end{aligned} \tag{3.8}$$

Let us consider this as an evolution equation for $g = \frac{\partial f_t^h}{\partial x_k}$, $k = 1 \ldots d$. On the right-hand side of (3.8), the first term $L^h$ generates a conservative Feller semi-group in $C_\infty(\mathbb{R}^d)$ as the result of previous analysis in (3.6). The second term is a bounded and non-homogeneous operator. The third and fourth terms in (3.8) are conditionally positive order-one differential operators. The final term represents a bounded operator since

$$\int_{\mathbb{R}^d_+ \setminus \epsilon} |y| \frac{\partial \nu(x, dy)}{\partial x_k} < \int_{\mathbb{R}^d} |y| \frac{\partial \nu(x, dy)}{\partial x_k} < \infty \tag{3.9}$$

by assumption 2. Hence by perturbation theory again, (3.8) also generates a bounded and positivity preserving conservative Feller semi-group in $C_\infty(\mathbb{R}^d)$. This implies that the first derivative with respect to $x$ of the function $T_t^h f(x)$ is bounded uniformly in $h$ for $f \in C_\infty(\mathbb{R}^d)$. We conclude that evolution in (3.7) is well-posed.

Now let us choose a sequence of $(L^{h_j})_{j \in \mathbb{N}}$ such that $h_j \downarrow 0$ as $j \to \infty$. By lemma 6.2 in Ethier and Kurtz [32] we can write

$$\big(T_t^{h_i} - T_t^{h_j}\big) f = \int_0^t T_{t-s}^{h_j}\big(L^{h_i} - L^{h_j}\big) T_s^{h_i}\, ds \tag{3.10}$$

for any $i < j$. Substituting $L^{h_i}$ and $L^{h_j}$ in (3.6) and by mean value theorem and triangular inequality, we can estimate

$$\begin{aligned}
\Big|\big(L^{h_i} - L^{h_j}\big)T_s^{h_i} f(x)\Big| &\leqslant \int_{B_{h_j} \setminus B_{h_i}} \Big|T_s^{h_i} f(x+y) - T_s^{h_i} f(x)\Big| \nu(x, dy) \\
&\leqslant \int_{B_{h_j}} \Big\| \sum_{i=1}^{d} \nabla_{x_i} T_s^{h_i} f \Big\| \Big|y\Big| \nu(x, dy) \\
&\leqslant \int_{B_{h_j}} d \cdot \Big\| \nabla_{x_i} T_s^{h_i} f \Big\| \Big|y\Big| \nu(x, dy) \\
&= o(1) d \Big\| f \Big\|_{C^1_\infty(\mathbb{R}^d)} \text{ as } h_i \to 0
\end{aligned} \tag{3.11}$$

Moreover, we can also conclude that

$$\begin{aligned}
\big\|\big(T_t^{h_i} - T_t^{h_j}\big) f\big\| &= \Big\| \int_0^t T_{t-s}^{h_j} \big(L^{h_i} - L^{h_j}\big) T_s^{h_i} ds \Big\| \\
&\leqslant \int_0^t \Big\| T_{t-s}^{h_j} \big(L^{h_i} - L^{h_j}\big) T_s^{h_i} ds \Big\| \\
&= o(1) t \Big\| f \Big\|_{C^1_\infty(\mathbb{R}^d)} \text{ as } h_i, h_j \to 0.
\end{aligned}$$

Hence, as $h$ tends to $0$, the family $T_t^h f$ converges uniformly to a family $T_t f$ which specifies a strongly continuous semi-group in $C_\infty(\mathbb{R}^d)$.

Finally we show that the operator L has an invariant core $C^2_\infty(\mathbb{R}^d)$. Taking the second derivatives of $\dot{f_t} = Lf_t$, we get

$$
\begin{aligned}
\frac{\partial^2}{\partial x_k \partial x_l}\dot{f}_t(x) =& \frac{d}{d_t}\frac{\partial^2 f_t}{\partial x_k \partial x_l} \\
=& L\frac{\partial^2 f_t}{\partial x_k \partial x_l} + \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_k}\frac{\partial^2 f_t}{\partial x_i \partial x_l} + \sum_{i=1}^{d} \frac{\partial b_i}{\partial x_l}\frac{\partial^2 f_t}{\partial x_i \partial x_k} && (3.12)\\
&+ \frac{1}{2}\left(\sum_{i\neq l}\left(\frac{\partial G_{il}}{\partial x_l} + \frac{\partial G_{li}}{\partial x_l}\right) + \sum_{i\neq k}\left(\frac{\partial G_{ik}}{\partial x_k} + \frac{\partial G_{ki}}{\partial x_k}\right)\right)\frac{\partial^3 f_t}{\partial x_i \partial x_k \partial x_l} && (3.13)\\
&+ \frac{1}{2}\left(\frac{\partial G_{kk}}{\partial x_k}\frac{\partial^3 f_t}{\partial x_k^2 \partial x_l} + \frac{\partial G_{ll}}{\partial x_i}\frac{\partial^3 f_t}{\partial x_l^2 \partial x_k}\right) && (3.14)\\
&+ \int_{\mathbb{R}^d_+}\left(\frac{\partial f_t(x+y)}{\partial x_l} - \frac{\partial f_t(x)}{\partial x_l}\right)\frac{\partial \nu}{\partial x_k}(x, dy) && (3.15)\\
&+ \int_{\mathbb{R}^d_+}\left(\frac{\partial f_t(x+y)}{\partial x_k} - \frac{\partial f_t(x)}{\partial x_k}\right)\frac{\partial \nu}{\partial x_l}(x, dy) && (3.16)\\
&+ \sum_{i=1}^{d}\frac{\partial^2 b_i}{\partial x_k \partial x_l}\frac{\partial f_t}{\partial x_i} && (3.17)\\
&+ \frac{1}{2}\left(\frac{\partial^2 G_{lk}}{\partial x_l \partial x_k}\frac{\partial^2 f_t}{\partial x_l \partial x_k} + \frac{\partial^2 G_{kl}}{\partial x_k \partial x_l}\frac{\partial^2 f_t}{\partial x_k \partial x_l}\right) && (3.18)\\
&+ \int_{\mathbb{R}^d_+}\Big(f_t(x+y) - f_t(x)\Big)\frac{\partial^2 \nu}{\partial x_k \partial x_l}(x, dy). && (3.19)
\end{aligned}
$$

In (3.12), L is a conditionally positive operator that generates a conservative Feller semi-group. Following L are two bounded operators. The terms (3.13) and (3.14) consist of first order differential operators while (3.15) , (3.16) and (3.19) are bounded and since assumption (1) ensures that for all $k \in \{1, \dots d\}$

$$
\begin{aligned}
0 < \int_{\mathbb{R}^d_+}\left(\frac{\partial f_t(x+y)}{\partial x_l} - \frac{\partial f_t(x)}{\partial x_l}\right)\frac{\partial \nu}{\partial x_k}(x, dy) &\leqslant \int_{\mathbb{R}^d_+}\|f''\||y|\frac{\partial \nu}{\partial x_k}(x, dy) < \infty \\
0 < \int_{\mathbb{R}^d_+}\Big(f_t(x+y) - f_t(x)\Big)\frac{\partial^2 \nu}{\partial x_k \partial x_l}(x, dy) &\leqslant \int_{\mathbb{R}^d_+}\|f'\||y|\frac{\partial^2 \nu}{\partial x_k \partial x_l}(x, dy) < \infty
\end{aligned}
$$

The term (3.17) is a bounded and non-homogeneous term, while (3.18) forms a multiplicative operator which is bounded and conditionally positive. Hence by perturbation theory, $T_t$ preserves $C^2_\infty(\mathbb{R}^d)$. Since the generator L is well defined there too, the space is an invariant core to L. □

In addition, assuming the same conditions (1) to (3) in Theorem 6 and another two assumptions (listed in the following theorem), the Feller processes corresponding to the pre-generator L in (3.4) are stochastically monotone as we will show in the following theorem.

**Theorem 7.** *Consider a Feller process* $(X_t)_{t\geqslant 0} \in \mathbb{R}^d$ *generated by the operator in (3.4). Suppose that operator* $L$ *satisfies all assumptions (1) to (3), in Theorem 6. In addition, further assume that*

*(4) the their first and second derivatives of* $G, b \in C^2(\mathbb{R}^d)$ *are positive;*

*(5)* $\frac{\partial \nu}{\partial x_i}(x, dy)$ *are non-negative measures on* $\mathbb{R}^d_+$.

*Then the process* $(X_t)_{t\geqslant 0}$ *is stochastically monotone.*

*Proof.* Differentiating the equation $\dot{f_t} = Lf$ for any $j \in \{1, \ldots d\}$. We get

$$\begin{aligned}\frac{\partial}{\partial x_j}\dot{f_t}(x) =&\frac{d}{dt}\frac{\partial f_t}{\partial x_j}\\ =&L\frac{\partial f_t}{\partial x_j} + \sum_{i=1}^{d}\frac{\partial b_i}{\partial x_j}\frac{\partial f_t}{\partial x_i} + \sum_{i\neq j}\frac{\partial G_{ij}}{\partial x_j}\frac{\partial^2 f_t}{\partial x_i \partial x_j} + \frac{1}{2}\frac{\partial G_{jj}}{\partial x_j}\frac{\partial^2 f_t}{\partial x_j^2}\\ &+ \int_{\mathbb{R}^d}\Big[f_t(x+y) - f_t(x)\Big]\frac{\partial \nu}{\partial x_j}(x, dy).\end{aligned} \tag{3.20}$$

Let us consider this as an evolution equation for $g = \frac{\partial f_t}{\partial x_j}, j = 1 \ldots d$. The left hand side of (3.20) is a differentiation operator. On the right hand side, the operator L in the first term generates a conservative Feller semi-group and is conditionally positive. The second term is a bounded, positive, non-homogeneous term. The third and fourth term represent a first order operator, which is also conditionally positive. The fourth term is a positive term which is bounded by assumption (5) and assumption (1) in the following sense:

$$\int_{\mathbb{R}^d}\Big[f_t(x+y) - f_t(x)\Big]\frac{\partial \nu}{\partial x_k}(x, dy) < \sup_{x,j\in\{1,\ldots d\}}\int |y|\frac{\partial \nu(x, dy)}{\partial x_j} < \infty$$

where $k \in \{1, \ldots, d\}$. Since (3.20) is a sum of an operator that generates a conservative Feller Process and some bounded positive generators, it generates a conservative Feller semi-group. Since on the right hand side of (3.20) L is also a conditionally positive operator, while the other terms are positive and bounded, the operator on the left hand side of (3.20) acting on functions

$\nabla_{x_j} f$ for all $j \in \{1, \dots, d\}$ and $f \in C_\infty(\mathbb{R}^d)$ is also conditionally positive. Then the solution to the corresponding evolution equation preserves positivity, i.e for any $t \geqslant 0$

$$\frac{\partial T_0 f(x)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} \geqslant 0 \Rightarrow \frac{\partial T_t f(x)}{\partial x_j} \geqslant 0. \tag{3.21}$$

Now choose an increasing and bounded sequence of functions $f_n \in C^1_\infty(\mathbb{R}^d), n \in \mathbb{N}$ such that $\nabla_{x_j} f_n(x) \geqslant 0$ for all $j \in \{i, \dots d\}$ and $f_n \uparrow \mathbf{1}_{[y,\infty]}$ as $n \to \infty$ point-wise. Since all $f_n$ have a non-negative first derivative, by conditional positivity in (3.21), we have for all natural numbers $n$

$$\frac{\partial T_t f_n(x)}{\partial x_j} \geqslant 0$$

i.e. $T_t f_n(x)$ is non-decreasing in $x$ for all $n$.

Because $(T_t, t \geqslant 0)$ is a Markov semi-group, by dominated convergence theorem the point-wise convergence of $f_n \uparrow \mathbf{1}_{[y,\infty)}$ ensures that $T_t f_n \uparrow T_t \mathbf{1}_{[y,\infty)}$ point-wise too, as $n \to \infty$. Clearly, the limit of $T_t f_n$ in $n$ is the probability distribution function of the Feller process $X_t$, i.e.

$$\lim_{n \to \infty} T_t f_n(x) = P(X_t \geqslant y | X_0 = x). \tag{3.22}$$

Since the convergence of $T_t f_n(x)$ in $n$ preserves monotonicity, the limit of $T_t f_n$, $\mathbb{P}(X_t \geqslant y | X_0 = x)$ is non-decreasing in $x$, which is essentially the definition of stochastic monotonicity of process $(X_t, t \geqslant 0)$. $\square$

In the following theorem, we impose stronger regularity assumptions on the coefficients of operator L to see that its corresponding Feller process has probability distribution functions that satisfy property P3 .

**Theorem 8.** *Let $(X_t)_{t \geqslant 0} \in \mathbb{R}^d$ be a Feller Process generated by (3.4). Assume that*

*1.* $b(x) \in C_\infty(\mathbb{R}^d)$ *such that for all* $1 \leqslant k \leqslant d$,

$$\frac{\partial^k b(x)}{\partial x_1 \dots \partial x_k} \geqslant 0 \textit{ and bounded above}$$

2. $G_{i,j}(x) \in C_\infty(\mathbb{R}^d)$ *only depends on* $x_i$ *and* $x_j$ *and has mixed derivatives satisfying*

$$\frac{\partial^2 G_{i,j}(x)}{\partial x_i \partial x_j} \geqslant 0 \text{ and bounded above}$$

3. $\nu(x, dy)$ *is a Lévy measure with support on* $\mathbb{R}^d_+$ *such that its first moment is finite:*

$$0 < \sup_y \int_{\mathbb{R}^d} |y|\nu(x, dy) < \infty \tag{3.23}$$

*and for all* $1 \leqslant k \leqslant d$ *the following is satisfied:*

$$0 < \sup_y \int_{\mathbb{R}^d} |y| \frac{\partial^k \nu(x, dy)}{\partial x_1 \ldots \partial x_k} < \infty \tag{3.24}$$

*Then the function* $P(X_t^x \geqslant y)$ *satisfies condition* P3 *in section 3.2.*

*Proof.* For notational simplicity, we prove the case of a three-dimension Feller process, noting that the proof follows analogously for any finite number of dimensions. In what follows, for $f \in C_\infty(\mathbb{R}^d)$ we consider the operator L as the following:

$$Lf(x) = \sum_{i=1}^{3} b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{3} G_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d_+} [f(x+y) - f(x)]\nu(x, dy) \tag{3.25}$$

Recall the equations in (3.7)

$$\begin{aligned} \dot{f_t} &= Lf \\ f_0 &= f \text{ where } f \in C_\infty(\mathbb{R}^d) \end{aligned}$$

Taking the third derivatives for $\dot{f_t}$ gives

$$\begin{aligned}\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\dot{f_t}(x) =& \frac{d}{dt}\frac{\partial^3 f_t(x)}{\partial x_1 \partial x_2 \partial x_3} \\ =& L\frac{\partial^3 f_t(x)}{\partial x_1 \partial x_2 \partial x_3} \\ &+ \sum_{\sigma\{a_1,a_2,a_3\}} L^1_{a_1}\frac{\partial^3 f_t(x)}{\partial x_{a_1} \partial x_{a_2} \partial x_{a_3}} \\ &+ \sum_{\sigma\{a_1,a_2,a_3\}} L^2_{a_1,a_2}\frac{\partial^3 f_t(x)}{\partial x_{a_1} \partial x_{a_2} \partial x_{a_3}} \\ &+ \sum_{\sigma\{a_1,a_2,a_3\}} L^3_{a_1,a_2,a_3}\frac{\partial^3 f_t(x)}{\partial x_{a_1} \partial x_{a_2} \partial x_{a_3}}\end{aligned} \tag{3.26}$$

where $\sigma\{a_1, a_2, a_3\}$ denotes all permutations of $\{a_1, a_2, a_3\}$.

We can write $L^1$, $L^2$ and $L^3$ as

$$L^1_{a_1}\frac{\partial^3 f_t(x)}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} = \sum_{i=1}^{3}\frac{\partial b_i}{\partial x_{a_1}}\frac{\partial^3 f_t(x)}{\partial x_i \partial x_{a_2}\partial x_{a_3}} + \frac{1}{2}\sum_{i\neq a_i}\Big(\frac{\partial G_{i,a_1}}{\partial x_{a_1}} + \frac{\partial G_{a_1,i}}{\partial x_{a_1}}\Big)\frac{\partial^4 f_t(x)}{\partial x_i \partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} + \frac{1}{2}\frac{\partial G_{a_1,a_1}}{\partial x_{a_1}}\frac{\partial^4 f_t(x)}{\partial x^2_{a_1}\partial x_{a_2}\partial x_{a_3}\partial x_{a_4}} + \int_{\mathbb{R}^d}\frac{\partial^2\big(f_t(x+y)-f_t(x)\big)}{\partial x_{a_2}\partial x_{a_3}}\frac{\partial \nu(x,dy)}{\partial x_{a_1}} \tag{3.27}$$

$$L^2_{a_1,a_2}\frac{\partial^3 f_t(x)}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} = \sum_{i=1}^{3}\frac{\partial^2 b_i}{\partial x_{a_1}\partial x_{a_2}}\frac{\partial^2 f_t(x)}{\partial x_i \partial x_{a_3}} + \frac{1}{2}\Big(\frac{\partial^2 G_{a_1,a_2}}{\partial x_{a_1}\partial x_{a_2}} + \frac{\partial^2 G_{a_2,a_1}}{\partial x_{a_2}\partial x_{a_1}}\Big)\frac{\partial^3 f_t(x)}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} + \int_{\mathbb{R}^d}\frac{\partial\big(f_t(x+y)-f_t(x)\big)}{\partial x_{a_3}}\frac{\partial^2 \nu(x,dy)}{\partial x_{a_1}\partial x_{a_2}} \tag{3.28}$$

$$L^3_{a_1,a_2,a_3}\frac{\partial^3 f_t(x)}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} = \sum_{i=1}^{3}\frac{\partial^3 b_i}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}}\frac{\partial f_t(x)}{\partial x_i} + \int_{\mathbb{R}^d}\big(f_t(x+y)-f_t(x)\big)\frac{\partial^3 \nu(x,dy)}{\partial x_{a_1}\partial x_{a_2}\partial x_{a_3}} \tag{3.29}$$

The equation in (3.27) consists of a positive bounded, non-homogeneous operator, a first-order operator (hence conditionally positive) and a bounded, positive operator ensured by (3.23) in assumption 3. Similarly, both equations in (3.28) and (3.29) consist of positive bounded non-homogeneous operators and another bounded positive operator ensured by (3.24) in assumption 3. Therefore, (3.26) is the sum of an operator L (that generates a conservative Feller Process) and bounded positive operator acting on $\nabla_{x_1,x_2,x_3} f_t$, for $f_t \in C^3_\infty(\mathbb{R}^d)$. Hence (3.26) generates a conservative semi-group. Since it is also a conditionally positive operator, the corresponding evolution problem has a solution that preserves positivity, i.e

$$\frac{\partial^3 T_0 f(x)}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial^3 f(x)}{\partial x_1 \partial x_2 \partial x_3} \geqslant 0 \Rightarrow \frac{\partial^3 T_t f(x)}{\partial x_1 \partial x_2 \partial x_3} \geqslant 0 \tag{3.30}$$

Now choose an increasing sequence of bounded functions $f_n \in C^3(\mathbb{R}^d)$ such that the derivatives $\nabla_{x_1,x_2,x_3} f_n(x) \geqslant 0$ and $f_n(x) \uparrow \mathbf{1}_{[y,\infty)}(x)$ point-wise as $n$ tends to infinity. By conditional positivity, we have $\nabla_{x_1,x_2,x_3} T_t f_n(x) \geqslant 0$ , which also means that for $x_1 \leqslant x_2$

$$\triangle K_n(x_1, x_2) = \sum_{a \in A} (-1)^{s(a)} T_t f_n(a) \geqslant 0 \tag{3.31}$$

where set $A = \Big\{a = (a_1, a_2, a_3) : a_i \in \{x_{1i}, x_{2i}\}\Big\}$, and for $a \in A$, $s(a)$ is the number of indices $i$ for which $a_i = x_{1i}$.

Then by dominated convergence theorem, we have $T_t f_n(x) \uparrow T_t \mathbf{1}_{[y,\infty)}(x)$ point-wise as $n \to \infty$, and the monotonicity in (3.31) is preserved. Hence

$$\begin{aligned}\lim_{n\to\infty} K_n(x_1, x_2) &= \lim_{n\to\infty} \sum_{a \in A} (-1)^{s(a)} T_t f_n(a) \geqslant 0 \\ &= \sum_{a \in A} (-1)^{s(a)} P(X_t^a \geqslant y) \geqslant 0\end{aligned}$$

□

Theorem 6, 7 and 8 hence lead to proof of the existence of a dual generator to a given multi-dimensional Feller Process.

**Theorem 9.** *Let $(X_t)_{t\geqslant 0} \in \mathbb{R}^d$ be a Feller Process generated by (3.4) where all assumptions (1) – (3) in Theorem 8 are satisfied. Suppose additionally that for $i \in \{1, \dots d\}$, all $\frac{\partial \nu}{\partial x_i}(x, dy)$ are non-negative measures on $\mathbb{R}^d_+$. Then $(X_t)_{t\geqslant 0}$ has a dual Markov process in $\mathbb{R}^d$*

*Proof.* We have shown that the operator L generates a conservative Feller semi-group and has an invariant core $C^2_\infty(\mathbb{R}^d)$. Furthermore, its corresponding Feller process $X_t$ is stochastically monotone and its transition probabilities $P(X_t^x \geqslant \mathbf{y})$ are superadditive in x. Hence by Theorem 4, its Markov dual process exists in $\mathbb{R}^d$. □

# 4 PARETO-ORDERED DUALITY IN EUCLIDEAN SPACE II

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## 4.1 INTRODUCTION

Part I of this thesis presented the criterion for a dual Markov process to exist in a finite dimensional Euclidean space. At this point a natural question is; are we able to explicitly characterise the dual Markov process? We have seen a couple of methods which compute the explicit form of dual generators corresponding to the general real-valued Feller processes, namely the direct computational method (see section 2.5) and the method of discrete approximation using Markov chains (see paper by Kolokoltsov [54]). The extension of both methods to characterise the duality for multidimensional cases can be somewhat messy. Therefore in this chapter we propose a more systematic analysis to duality via semi-groups and generators.

This chapter focuses on the theory of duality for Markov processes in the sense that

$$\mathsf{E}f(X_t^x, y) = \mathsf{E}f(x, Y_t^y) \tag{4.1}$$

for a certain $f$. In the following sections we lay the foundations of our analytical approach to the theory of duality. The basic tools (intertwining of operators) are given in section 4.3. This is followed by a discussion of its application to the theory of differential equations and stochastic processes.

## 4.2 ON THE GENERAL NOTION OF SEMI-GROUP DUALITY

For a topological (e.g. metric) space $X$ we denote $B(X)$ and $C_b(X)$ to be the spaces of bounded Borel measurable, and bounded continuous functions, respectively. Equipped with the sup-norm $\|f\| = \sup_x |f(x)|$ both these spaces become Banach spaces. Bounded signed measures on $X$ are defined as bounded $\sigma$-additive functions on the Borel subsets of $X$. The set of such measures $\mathcal{M}(X)$ equipped with the total variation norm is also a Banach space. The standard duality between $B(X)$ and $\mathcal{M}(X)$ is given by the integration:

$$(f, \mu) = \int_X f(x)\mu(dx).$$

Let $X$ and $Y$ be two topological spaces. A signed (stochastic) kernel from $X$ to $Y$ is a function of two variables $p(x, A)$, where $x \in X$ and $A$ are Borel subsets of $Y$ such that $p(x, .)$ is a bounded signed measure on $Y$ for any $x$ and $p(., A)$ is a Borel function for any Borel set $A$. We say that this kernel is bounded if $\sup_x \|p(x, .)\| < \infty$. We say that this kernel is weakly continuous if the mapping $x \mapsto p(x, .)$ is continuous with measures $\mathcal{M}(Y)$ considered in their weak topology. If all measures $p(x, .)$ are positive, the corresponding kernel is called a stochastic kernel.

Any bounded kernel specifies a bounded linear operator $B(Y) \to B(X)$ via the formula

$$Tg(x) = \int_Y g(z)p(x, dz).$$

We call $T$ the integral operator with the kernel $p$. The standard dual operator $T'$ is defined as the operator $\mathcal{M}(X) \to \mathcal{M}(Y)$ specified by the duality relation

$$(f, T'\mu) = (Tf, \mu),$$

or explicitly as

$$T'\mu(dy) = \int_X p(x, dy)\mu(dx).$$

Clearly the kernel $p(x, dz)$ is weakly continuous if and only if $T$ acts on continuous functions, that is, $T : C(Y) \to C(X)$.

Let $p(x, dz)$ be a bounded signed kernel from $X$ to itself, $T$ be the corresponding integral operator, and $f(x, y)$ be a bounded measurable function on $X \times Y$.

Let us say that the operator $T^{D(f)} : B(Y) \to B(X)$ is $f$-*dual* to $T$, if

$$(T^{D(f)} f(x, .))(y) = (Tf(., y))(x) \tag{4.2}$$

for any $x$ and $y$. In other words, the application of $T^D$ to the second argument of $f$ is equivalent to the application of $T$ to its first argument. Of course, if $T^{D(f)}$ is $f$-dual to $T$, then $T$ is $\tilde{f}$-dual to $T^{D(f)}$ with $\tilde{f}(y, x) = f(x, y)$.

We say that $f$ *separates points of* $X$ if, for any $x_1$ and $x_2 \in X$, there exists $y \in Y$ such that $f(x_1, y) \neq f(x_2, y)$. The following is a more non-trivial notion. We say that $f$ *separates measures on* $X$ if, for any $Q_1, Q_2 \in \mathcal{M}(X)$, there exists $y \in Y$ such that $\int f(x, y) Q_1(dx) \neq \int f(x, y) Q_2(dx)$. If this is the case, the integral operator $F = F_f : \mathcal{M}(X) \to B(Y)$ given by

$$(FQ)(y) = \int f(x, y) Q(dx) \tag{4.3}$$

is an injective bounded operator, so that the linear inverse $F^{-1}$ is defined on the image $F(\mathcal{M}(X))$. Let us say that the function $FQ$ is $f$-generated by $Q$.

*Remark* 3. Cox and Rösler [20] say that a function $g$ is representable by $f$, if there exists a unique $Q$ such that $g = FQ$. This paper [20] deals with the application of duality to exit and entrance laws of Markov processes.

## 4.3 BASIC TOOLS

**Proposition 6.** *Let* $f$ *be a bounded measurable function separating measures on* $X$ *and* $T$ *be an integral operator in* $B(X)$ *with a bounded signed kernel* $p$. *Then* $T^{D(f)}$ *is well defined on* $F(\mathcal{M}(X))$ *and its action on the* $f$*-generated functions coincides with* $T'$, *that is*

$$T^{D(f)} = F \circ T' \circ F^{-1}, \tag{4.4}$$

*or equivalently*

$$F^{-1} \circ T^{D(f)} = T' \circ F^{-1}. \tag{4.5}$$

*In other words, the* $f$*-dual operator* $T^{D(f)}$ *is obtained by the "dressing" of the standard dual* $T'$ *by the operator* $F$.

*Proof.* Let $g \in F(\mathcal{M}(X))$ be given by $g(y) = \int f(x,y) Q_g(dx)$. Then

$$T^{D(f)} g(y) = \int_X (T^{D(f)} f(x,.))(y) Q_g(dx)$$

$$= \int_X (Tf(.,y))(x) Q_g(dx) = \int_X \int_Y f(z,y) p(x,dz) Q_g(dx) = \int_Y f(z,y) \tilde{Q}(dz),$$

with

$$\tilde{Q}(dz) = \int p(x,dz) Q_g(dx).$$

Thus $T^{D(f)} g$ is $f$-generated by $\tilde{Q} = T' Q_g$, as required. □

*Remark* 4. Equation (4.5) is a particular case of intertwining. Readers are referred to work by Biane [13], Dubédat [30], Patie and Simon [66], Hirsch and Yor [39] and Carmona et al. [16]for exciting recent developments. Applications of (4.5) in the case of discrete Markov chains are analysed in detail in the paper Huillet and Martinez [43].

Representation in the form of (4.4) has a direct implication for the theory of semi-groups.

**Proposition 7.** *Let* $f$ *be a bounded measurable function separating measures on* $X$. *Also, let* $T_t$ *be a semigroup of integral operators in* $B(X)$ *specified by the family of bounded signed kernels* $p_t(x, dz)$ *from* $X$ *to* $X$, *so that* $T_0$ *is the identity operator and* $T_t T_s = T_{t+s}$, *which, in terms of kernels, rewrites as the Chapman-Kolmogorov equation*

$$\int_X p_t(x,dz) p_s(z,dw) = p_{t+s}(x,dw).$$

*Then the dual operators* $T_t^{D(f)}$ *in* $F(\mathcal{M}(X))$ *also form a semigroup, so that*

$$T_t^{D(f)} = F \circ T_t' \circ F^{-1}. \tag{4.6}$$

*Proof.* This is straightforward from (4.4) and the fact that $T'_t$ forms a semi-group in $\mathcal{M}(X)$. $\square$

*Remark* 5. The duality (1.1) is, of course, also included in the general scheme above, that is, the dual can still be expressed as (4.4). For instance, if $\nu(dx)$ has a density $\nu(x)$ with respect to the Lebesgue measure and $T'$ can be reduced to the action on functions, then $F^{-1}$ is the multiplication on $nu(x)$ and $f(x,y) = \delta(x-y)\nu^{-1}(x)$.

It is also worth noting that the assumption of boundedness of $f$ is not essential. If it is not bounded (we shall discuss interesting examples of such situations later), the integral operator $F$ will not be defined on all bounded measures, but only on its subspace. This will be reflected in the domain of $T^{D(f)}$, but the overall scheme of Proposition 6 still remains valid.

## 4.4 LINKS WITH DIFFERENTIAL EQUATIONS AND STOCHASTIC PROCESSES

Let us explain briefly the main ideas behind the application of the above results to the theory of differential equations and stochastic processes. Precise details for particular situations will be discussed in the following chapters.

Let a semi-group $T_t$ in $B(X)$ be generated by a (possibly unbounded) operator $L$ in $B(X)$ defined on an invariant (under all $T_t$) domain $D \subset B(X)$, so that

$$\left.\frac{d}{dt}\right|_{t=0} T_t h = \lim_{t\to 0} \frac{1}{t}(T_t h - h) = Lh, \quad h \in D,$$

with convergence in some appropriate topology (say, strongly or point-wise) and thus $T_t$ represent resolving operators for the Cauchy problem of the equation $\dot{h} = Lh$. Then (4.4) implies that

$$\left.\frac{d}{dt}\right|_{t=0} T_t^{D(f)} g = F \circ \left.\frac{d}{dt}\right|_{t=0} T' \circ F^{-1} g = F \circ L' \circ F^{-1} g,$$

that is, the generator of the semi-group $T_t^{D(f)}$ is

$$L^{D(f)} = F \circ L' \circ F^{-1}, \tag{4.7}$$

so that $T_t^{D(f)}$ represent resolving operators for the Cauchy problem of the equation $\dot{g} = L^{D(f)}g$. Here $L'$ is naturally the standard dual operator to L. Thus duality can yield explicit solutions for equations of this kind. Of course, our arguments were heuristic as we did not pay attention to the domain of definition of $L'$, which should be done in practical situations. The main difficulty here is to characterise the operator $F_f$.

Next, in order to be able to fill the duality equation (4.2) with probabilistic content, i.e. to rewrite it as (1.2), the semi-groups $T_t$ and $T_T^{D(f)}$ should be positivity preserving and generate some Markov processes.

This line of investigation reduces to the question of whether, for a given conditionally positive operator L, the corresponding dual $L^{D(f)}$ is also conditionally positive.

It can now be seen that the basic issues which must be addressed to make the theory work for general functions f are (i) the characterisation of the operators F and $F^{-1}$ (for the analytic part of the story) and (ii) the criteria for conditional positivity of $L^{D(f)}$ (for its probabilistic content).

As we shall see it is often convenient to reduce the operator F to some subclass of Borel measures Q, where its inverse can be explicitly found. For instance, it is often easier to work with Q having density with respect to some reference measure.

# 5 PARETO-ORDERED DUALITY IN EUCLIDEAN SPACE III

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## 5.1 INTRODUCTION

In this chapter, we apply the analytical tools (formulas (4.6) and (4.7)) discussed in chapter 4 to characterise classes of dual Markov processes with respect to various functions $f$ depending on the difference of their arguments. Here, we deal with duality in $\mathbb{R}^d$ arising from Pareto and similar partial orders. After examining the characterisation for each case of diffusion and jump processes individually, the full characterization of duality is given in terms of generators for basic classes of Feller processes. This chapter is concluded by giving a path-wise example to the study of duality using stochastic differential equations.

## 5.2 BASIC NOTIONS

As our basic example we consider $f$-duality for functions $f$ arising from translation-invariant partial orders, or more generally, from translation-invariant binary relations. Namely, let $X$ be a topological linear space and $M$ a Borel subset of $X$. Then $M$ defines a translation-invariant binary relation $R_M$ on $X$ such that $xR_My$ means, by definition, that $x-y \in M$, or $x \in y+M$.

Let $\tilde{M} = \{(x,y) \in X \times X : xR_My\}$. Let us say that the duality (4.2) arises from the binary relation $M$, if

$$f(x,y) = f_M(x,y) = \mathbf{1}_{\tilde{M}}(x,y) = \mathbf{1}_{x-M}(y) = \mathbf{1}_{y+M}(x). \tag{5.1}$$

*Remark* 6. Suppose that $f$-duality arises from a translation-invariant binary relation $R_M$ and both $T_t$ and $T_t^{D(f)}$ are known to be integral operators with kernels $p_t(x,dz)$ and $p_t^{D(f)}(y,dw)$ respectively. One can give another instructive proof of Proposition 7 bypassing representation (4.4) and using instead Fubini's theorem, as was done by Siegmund [72] for one-dimensional duality. Namely, it is sufficient to show the semigroup identity $T_{t+s}^{D(f)} = T_s^{D(f)}T_t^{D(f)}$ applied to the functions $f(x,.) = \mathbf{1}_{x-M}$, as it then extends to the whole $F(\mathcal{M}(X))$ by linearity. And for these functions we have

$$(T_{t+s}^{D(f)}\mathbf{1}_{x-M})(y) = (T_{t+s}\mathbf{1}_{y+M})(x) = (T_t(T_s\mathbf{1}_{y+M}))(x) = \int p_t(x,dz)(T_s\mathbf{1}_{y+M})(z)$$

$$= \int p_t(x,dz)(T_s^{D(f)}\mathbf{1}_{z-M})(y) = \int p_t(x,dz)\left(\int \mathbf{1}_{z-M}(w)p_s^{D(f)}(y,dw)\right).$$

Applying Fubini's theorem this rewrites as

$$\begin{aligned}\int p_s^{D(f)}(y,dw)\int \mathbf{1}_{w+M}(z)p_t(x,dz) &= \int (T_t\mathbf{1}_{w+M})(x)p_s^{D(f)}(y,dw)\\ &= T_s^{D(f)}(T_t^{D(f)}\mathbf{1}_{x-M})(y),\end{aligned}$$

as required.

If $M$ contains the origin and is closed under the addition of vectors, then the relation $R_M$ is a pre-order (i.e. it is reflexive and transient) and can be naturally denoted by $\geqslant_M$. If this is the case and $T_t$ and $T_t^{D(f)}$ are integral operators with positive stochastic kernels thus specifying Markov processes, then duality relation (4.2) or equivalently (1.2) corresponds to the equation

$$\mathbf{P}(X_t^x \geqslant_M y) = \mathbf{P}(Y_t^y \leqslant_M x), \tag{5.2}$$

extending one-dimensional duality (1.3).

The basic example we are going to analyse now is the Pareto partial order in $X = \mathbb{R}^d$, i.e. $\geqslant_M$ with $M = \mathbb{R}_+^d$, and its natural extension with $M = C(e_1, \cdots, e_d)$ the cone generated by $d$ linear independent vectors $\{e_1, \cdots, e_d\}$ in $\mathbb{R}^d$:

$$C(e_1, \cdots, e_d) = \{x = \sum_{j=1}^{d} \alpha_j e_j : \quad \alpha_j \geqslant 0,\ j = 1, \cdots, d\}. \tag{5.3}$$

Of course the relation $\geqslant_M$ with such $M$ is again a Pareto order, but in a transformed system of coordinates.

Let us start with $M = \mathbb{R}_+^d$ corresponding to the Pareto order, which we shall denote just by $\geqslant$ omitting the subscript $M$. The corresponding dual semi-groups or processes (if exist) will be referred to as Pareto dual. In this case

$$(FQ)(y) = \int f_M(x, y) Q(dx) = \int_{x \geqslant y} Q(dx) \tag{5.4}$$

is just the usual multidimensional distribution function for the measure $Q$ on $\mathbb{R}^d$. It is known (and easy to see) that $FQ$ characterizes $Q$ uniquely implying that $F$ is injective and thus $f_M$ separates measures on $\mathbb{R}^d$ yielding the main condition of Proposition 6. Moreover, if $Q$ has a density $q$ with respect to the Lebesgue measure, then $q$ can be found from $FQ = g$ by differentiation:

$$q(y_1, \cdots, y_d) = F^{-1} g(y) = (-1)^d \frac{\partial^d g(y)}{\partial y_1 \cdots \partial y_d}. \tag{5.5}$$

Thus, in this case, for the Pareto order, the operator $F^{-1}$ has a simple explicit expression.

In the case of orders arising from the cones $M = C(e_1, \cdots, e_d)$ given by (5.3) this formula generalises to

$$q(y_1, \cdots, y_d) = (F^{-1}g)(y) = (-1)^d \frac{\frac{\partial^d g}{\partial y^d}(y)[e_1, e_2, \cdots, e_d]}{|\det(e_1, e_2, \cdots, e_d)|}, \tag{5.6}$$

where $\det(e_1, e_2, \cdots, e_d) = \det(e_i^j)$ is the determinant of the matrix whose $i$th columns consist of the coordinates of the vector $e_i$ and

$$\frac{\partial^d g}{\partial y^d}(y)[e_1, e_2, \cdots, e_d] = \sum_{i_1, i_2, \cdots, i_d} \frac{\partial^d g}{\partial y_{i_1} \cdots \partial y_{i_d}}(y) e_1^{i_1} e_2^{i_2} \cdots e_d^{i_d}.$$

*Remark 7.* For completeness, let us sketch a proof of this formula. If a measure $Q$ on $\mathbb{R}^d$ has a continuous density $q$, so that

$$g(x) = FQ(x) = \int_{y+C(e_1,\cdots,e_d)} q(z) dz,$$

the function $q$ can be clearly found as the limit

$$q(y) = \lim_{h \to 0} \int_{y+\Pi(he_1,\cdots,he_d)} q(z)\, dz |\Pi(he_1, \cdots, he_d)|^{-1}, \tag{5.7}$$

where

$$\Pi(he_1, \cdots, he_d) = \{x = \sum_j \alpha_j he_j, \quad \alpha_j \in [0,1]\}$$

is the parallelepiped built on the vectors $\{he_1, \cdots, he_d\}$ and

$$|\Pi(he_1, \cdots, he_d)| = h^d |\det(e_i^j)|$$

is its Euclidean volume.

From simple combinatorics it follows (see e.g. textbook by Kallenberg [46]) that

$$\int_{y+\Pi(he_1,\cdots,he_d)} q(z)\, dz$$

$$= g(y) - \sum_j g(y + he_j) + \sum_{i<j} g(y + he_i + he_j) + \cdots + (-1)^d g(y + he_1 + \cdots + he_d).$$

Let us expand all terms in Taylor series up to the derivatives of order $d$. As the final expression should be of order $h^d$ (to get a limit in (5.7)) we conclude that all terms with the derivatives of orders less than $d$ necessarily cancel, so that

$$\begin{aligned}
&\int_{y+\Pi(he_1,\cdots,he_d)} q(z)\, dz \\
=&\frac{1}{d!}h^d \left( -\sum_j \frac{\partial^d g}{\partial y^d}[e_j] + \sum_{i<j} \frac{\partial^d g}{\partial y^d}[e_i + e_j] + \cdots + (-1)^d \frac{\partial^d g}{\partial y^d}[e_1 + \cdots + e_d] \right) \\
&+ O(h^{d+1}),
\end{aligned} \tag{5.8}$$

where $O(h^{d+1})$ denotes the expression of order $h^{d+1}$ that does not contribute to the limit in (5.7), and where we use the well established (though a bit ambiguous) notation for the action of the higher order derivative on equal vectors:

$$\frac{\partial^d g}{\partial y^d}(y)[v] = \frac{\partial^d g}{\partial y^d}(y)[v, \cdots, v].$$

It remains to note that all terms in expansion (5.8) containing products of coordinates of coinciding vectors should vanish (otherwise, using different scaling on $e_i$ we would arrive to a contradiction with the existence of the limit in (5.7)). The only non-vanishing terms should contain the products of $d$ coordinates of all $d$ vectors. All these products comes from the last term in the sum (5.8) leading to (5.6).

For instance, let us consider a 'two-dimensional light cone'

$$C(e_1, e_2) = \{(x, y) : y \geqslant |x|\} \in \mathbb{R}^2, \tag{5.9}$$

corresponding to vectors $e_1 = (1, 1), e_2 = (-1, 1)$. Then formula (5.6) for the inverse operator turns to the simple wave operator

$$q(x, y) = F^{-1} g(x, y) = \frac{1}{2} \left( \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial x^2} \right)(x, y). \tag{5.10}$$

## 5.3 DUALITY FROM PARETO ORDER: GLOBAL ANALYSIS

Let us now make the detailed analysis of the duality arising from the standard Pareto order in $\mathbb{R}^d$, i.e. with $M = \mathbb{R}^d_+$. We aim at (i) finding explicitly the dual operator $L^{D(f)}$ for the main classes of the generators of Feller processes in $\mathbb{R}^d$ including diffusions and jump processes and (ii) establishing criteria (in terms of the initial operator L) ensuring that this dual operator is conditionally positive and specifies a Markov process, so that the duality relation (5.2) holds that we shall write simply as

$$\mathbf{P}(X_t^x \geqslant y) = \mathbf{P}(Y_t^y \leqslant x) \tag{5.11}$$

for the case of the Pareto partial order.

Let us analyse formula (4.4) from Proposition 6. In the case of duality arising from Pareto order and the operator T being integral with a probability kernel $p(x, dz)$ (i.e. all measures $p(x, .)$ are probability measures, as is the case for transition operators of Markov processes) it states that for a distribution function g of a measure Q on $\mathbb{R}^d$. i.e. $g(x) = \int_{z \geqslant x} Q(dz)$ we have

$$T^{D(f)} g(x) = F \circ T' \circ F^{-1} g(x) = \int_{y \geqslant x} \int_{\mathbb{R}^d} p(z, dy) Q(dz). \tag{5.12}$$

We are interested in the question of when this operator can be extended to all bounded measurable g as a positive operator preserving constants, i.e. as an integral operator with a probability kernel.

Assume first that the measure Q has a continuous density q so that (5.5) holds, i.e.

$$q(x) = (-1)^d \frac{\partial g^d}{\partial x_1 \cdots \partial x_d}.$$

In this case

$$T^{D(f)} g(x) = (-1)^d \int_{y \geqslant x} \int_{\mathbb{R}^d} p(z, dy) \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \tag{5.13}$$

We like to get rid of the derivatives of g. To be able to do it, let us assume that the kernel $p(x, dz)$ is weakly continuous and has weakly continuous mixed derivatives, that is, for any $I \subset \{1, \cdots, d\}$ (including $\{1, \cdots, d\}$ itself) the mixed derivative

$$\frac{\partial p^{|I|}}{\partial z_I}(z, dy) \tag{5.14}$$

is a well defined weakly continuous kernel (possibly signed). Then, integrating the integral over $z$ in (5.13) by parts $d$ times and assuming that all boundary terms vanish, we get

$$T^{D(f)}g(x) = \int_{\mathbb{R}^d} \left( g(z) \int_{y \geqslant x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \right) dz. \tag{5.15}$$

This is an integral operator with the integral kernel (more precisely its density)

$$p^D(x, z) = \int_{y \geqslant x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy).$$

For this operator to be positive and constant preserving, necessary conditions are that, for all $x \in \mathbb{R}^d$,

$$\int_{y \geqslant x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \geqslant 0, \tag{5.16}$$

$$\int \left( \int_{y \geqslant x} \frac{\partial p^d}{\partial z_1 \cdots \partial z_d}(z, dy) \right) dz = 1. \tag{5.17}$$

From the integration by parts it is seen that for the last condition to hold it is sufficient to assume that for any subset $I \subset \{1, \cdots, d\}$ excluding the whole set $\{1, \cdots, d\}$,

$$\lim_{z_{\bar{I}} \to -\infty} \int_{\mathbb{R}^{|I|}} dz_I \int_{y \geqslant x} \frac{\partial p^{|I|}}{\partial z_I}(z_I, z_{\bar{I}}, dy) = 0, \tag{5.18}$$

and there exists a finite limit

$$\lim_{z_{\bar{I}} \to \infty} \int_{\mathbb{R}^{|I|}} dz_I \int_{y \geqslant x} \frac{\partial p^{|I|}}{\partial z_I}(z_I, z_{\bar{I}}, dy), \tag{5.19}$$

which equals 1 for the empty set $I$. Moreover, one sees by inspection that this condition also ensures that integrating by parts (5.13) for a $g$ having finite density (5.5), all boundary terms will in fact vanish, justifying equation (5.15).

Thus we have proved the following statement.

**Proposition 8.** *Suppose an integral operator* $T$ *in* $B(\mathbb{R}^d)$ *is given by a probability kernel* $p(x, dy)$ *having all mixed derivatives* (5.14) *well defined and weakly continuous and such that* (5.16) *holds,* (5.18) *holds for any subset* $I \subset \{1, \cdots, d\}$ *excluding the whole set* $\{1, \cdots, d\}$*, and there exists a* *finite limit* (5.19)*, which equals* $1$ *for the empty set* I*. Then the Pareto dual operator* $T^{D(f)}$ *is also an integral operator with a probability kernel.*

Condition (5.16) is of course not directly verifiable. Therefore we shall see how it can be read from the generator of the process.

## 5.4 DUALITY FROM PARETO ORDER: DIFFUSION PROCESSES

We plan now to find the generators of the dual processes, when they exist. Let us start with the simplest case of deterministic processes generated by the first order differential operators of the form

$$L\phi(x) = (b(x), \nabla\phi(x)) = \sum_{j=1}^{d} b_j(x)\frac{\partial\phi}{\partial x_j}. \tag{5.20}$$

In this case the dual operator is well defined on functions and

$$L'g(x) = -\mathrm{div}(gb)(x) = -\sum_j \frac{\partial}{\partial x_j}[b_j(x)g(x)].$$

For a vector $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ let us denote $\check{x}_i$ the vector in $\mathbb{R}^{d-1}$ obtained from $x$ by deleting the coordinate $x_i$. For a function $g(x)$ let us write $g(\check{z}_i, x_i)$ for the value of $g$ on the vector, whose $i$th coordinate is $x_i$, and other coordinates are those of the vector $z$. Let us write $d\check{z}_j$ for the product of differentials $dz_k$ with all $k = 1, \cdots, d$ excluding $j$.

Integrating by parts and assuming that $g$ decays quickly enough so that the boundary terms at infinity vanish, we have

$$L^{D(f)}g(x) = FL'F^{-1}g(x) = (-1)^{d+1}\int_{z\geqslant x}\sum_j \frac{\partial}{\partial z_j}\left[b_j(z)\frac{\partial^d g(z)}{\partial z_1\cdots\partial z_d}\right] dz_1\cdots dz_d$$

$$= (-1)^d \sum_j \int_{\check{z}_j\geqslant\check{x}_j} b_j(\check{z}_j, x_j)\frac{\partial^d g(z)}{\partial z_1\cdots\partial z_d}(\check{z}_j, x_j)\, d\check{z}_j. \tag{5.21}$$

In general one cannot simplify this expression much further, and this is not a conditionally positive operator (it does not have a Lévy-Khintchine form with variable coefficients) without further assumptions.

**Proposition 9.** *Let* $L$ *have form* (5.20) *with all* $b_j \in C^1(\mathbb{R}^d)$ *(the space of bounded continuous functions with bounded continuous derivatives). Then* $L^{D(f)}$ *is given by* (5.21)*, so that the solution to the Cauchy problem of the equation* $\dot{g} = L^{D(f)} g$ *is given by the corresponding formula* (4.4) *with* $F$ *and* $F^{-1}$ *given by* (5.4) *and* (5.5)*. Moreover, if each* $b_j$ *depends only on the coordinate* $x_j$*, then*

$$L^{D(f)} g(x) = -b_j(x_j) \frac{\partial g}{\partial x_j}, \tag{5.22}$$

*that is,* $L^{D(f)}$ *coincides with* $L$ *up to a sign and the dual process exists and is just the deterministic motion in the opposite direction to the original one.*

*Proof.* Formula (5.22) is straightforward from (5.21) and the assumptions made on $b_j$. This makes the last statement plausible. However, strictly speaking, having the generator calculated on some subclass of functions does not directly imply that the semigroup $T_t^{D(f)}$ coincides with the semigroups on $C(\mathbb{R}^d)$ generated by operator (5.22). The simplest way to see that this is in fact the case is via direct calculations with the semigroup $T_t^{D(f)}$ itself. Namely, if the deterministic Markov process $X_t^x$ with generator (5.20) can be expressed as $X_t^x = X^t(x)$ via the solutions $X^t(x)$ of the Cauchy problem for the ODE $\dot{x} = b(x)$, its transition kernel takes the form $p_t(z, dy) = \delta(y - X^t(z))$. Then (5.13) becomes

$$T_t^{D(f)} g(x) = (-1)^d \int_{X^t(z) \geqslant x} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \tag{5.23}$$

Under the assumption that $b_i$ depend only on $x_i$, the coordinates of $X^t(z)$ are themselves solutions $X_i^t(z_i)$ of the one-dimensional ODE $\dot{x}_i = b_i(x_i)$, so that one has

$$T_t^{D(f)} g(x) = (-1)^d \int_{X_i^t(z_i) \geqslant x_i} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz. \tag{5.24}$$

From the obvious monotonicity of one-dimensional ODE this rewrites as

$$T^{D(f)} g(x) = (-1)^d \int_{z_i \geqslant (X_i^t)^{-1}(x_i)} \frac{\partial g^d}{\partial z_1 \cdots \partial z_d} dz = g((X^t)^{-1}(x)), \tag{5.25}$$

which is of course the semigroup generated by the operator (5.22). □

Let us turn to a diffusion operator having the form

$$L\phi(x) = (a(x)\nabla, \nabla)\phi(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \tag{5.26}$$

with a positive definite diffusion matrix $a(x) = (a_{ij}(x))$.

In this case

$$L'g(x) = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}[a_{ij}(x)g(x)],$$

and consequently

$$L^{D(f)}g(x) = FL'F^{-1}g(x) = (-1)^d \int_{z \geqslant x} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial z_i \partial z_j}\left[a_{ij}(z)\frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d})\right] dz_1 \cdots dz_d.$$

Let us integrate twice by parts the terms containing mixed derivatives and integrate once by parts the remaining terms. This yields

$$L^{D(f)}g(x) = (-1)^{d-1} \sum_{j=1}^{d} \int_{\check{z}_j \geqslant \check{x}_j} \frac{\partial}{\partial x_j}\left[a_{jj}(\check{z}_j, x_j)\frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d}(\check{z}_j, x_j)\right] d\check{z}_j$$

$$+2(-1)^d \sum_{i<j} \int_{\check{z}_{ij} \geqslant \check{x}_{ij}} \left[a_{ij}\frac{\partial^d g}{\partial z_1 \cdots \partial z_d}\right](\check{z}_{ij}, x_i, x_j)\, d\check{z}_{ij},$$

where $\check{z}_{ij}$ denotes the vector in $\mathbb{R}^{d-2}$ obtained from $z$ by deleting $i$th and $j$th coordinates, and $(\check{z}_{ij}, x_i, x_j)$ is the vector with $i$th and $j$th coordinates taken from the vector $x$, and other coordinates taken from the vector $z$. In case $d = 1$, the second sum in this expression is of course empty.

Again in general case one cannot simplify this expression essentially. However, assuming additionally that the coefficients $a_{ij}$ depends only on the coordinates $x_i, x_j$ (in particular, $a_{ii}$ depends only on $x_i$), we have

$$L^{D(f)}g(x) = (-1)^{d-1}\sum_{j=1}^{d}\int_{\check{z}_j \geqslant \check{x}_j} \frac{\partial}{\partial x_j}\left[a_{jj}(x_j)\frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d}(\check{z}_j, x_j)\right] d\check{z}_j$$

$$+2(-1)^d \sum_{i<j}\int_{\check{z}_{ij} \geqslant \check{x}_{ij}} a_{ij}(x_i, x_j)\frac{\partial^d g}{\partial z_1 \cdots \partial z_d}(\check{z}_{ij}, x_i, x_j)\, d\check{z}_{ij}.$$

Integrating by parts with respect to the variables $\check{z}_j$ in the first sum and the variables $\check{z}_{ij}$ in the second, yields (assuming the boundary terms at infinity vanish)

$$L^{D(f)}g(x) = \sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left[a_{jj}(x_j)\frac{\partial g(x)}{\partial x_j}\right] + 2\sum_{i<j} a_{ij}(x_i, x_j)\frac{\partial^2 g}{\partial x_i \partial x_j}, \tag{5.27}$$

or

$$L^{D(f)}g(x) = Lg(x) + \sum_{j=1}^{d}\frac{\partial a_{jj}(x_j)}{\partial x_j}\frac{\partial g(x)}{\partial x_j}. \tag{5.28}$$

**Proposition 10.** *Let* $L$ *have form* (5.26) *with a positive definite diffusion matrix* $a(x) = (a_{ij}(x))$ *and with all* $a_{ij} \in C^1(\mathbb{R}^d)$*, so that* $L$ *generates a Feller diffusion in* $\mathbb{R}^d$ *that we denote* $X_t^x$*. If the coefficients* $a_{ij}$ *depends only on the coordinates* $x_i, x_j$*, then* $L^{D(f)}$ *is given by* (5.28) *and it also generates a diffusion process in* $\mathbb{R}^d$ *that we denote* $Y_t^y$*, and the duality relation* (5.11) *holds.*

*Proof.* Again formula (5.28) makes the statement very plausible, but to deduce (4.4) from (4.7) additional argument is of course needed. This goes as follows.

But notice first that it is sufficient to prove the statement under additional assumption that coefficients $a_{ij}$ are infinitely smooth with all derivatives bounded (actually we need twice differentiability for the above calculation of $L^{D(f)}$ and $d$ times differentiability for the formulas of Proposition 8 to make sense) and the operator $L$ is strictly elliptic, because any $L$ of type (5.26) can be approximated by the sequence of $L$ of the same form but strictly elliptic and with smooth coefficients. Passing to the limit in the duality equation allows one to prove its validity for the general case.

Next, under this smoothness and non-degeneracy assumption, it is well known from the standard theory of diffusions (or Ito's processes) that operator (5.28) generates a unique Feller process such that its semigroup $T_t^{D(f)}$ preserves the space $C^2_\infty(\mathbb{R}^d)$ of twice continuously differentiable functions vanishing at infinity with all its derivatives up to order two. Hence, the Cauchy problem for the equation

$$\dot{g} = L^{D(f)} g$$

is well posed in classical sense for initial functions $g_0$ from $C^2_\infty(\mathbb{R}^d)$. It is then straightforward to see (4.7) that both functions $T_t^{D(f)} g_0$ and $F \circ T'_t \circ F^{-1} g_0$ satisfies this equation. Consequently these two functions coincide implying (4.4) for the semigroups $T_t$ and $T_t^{D(f)}$, as required. □

Thus we have shown that under appropriate assumptions the $f$-dual operators to the first order and diffusion operators respectively are again first order and diffusion operators respectively defining the $f$-dual or Pareto dual processes.

It is instructive to see which diffusions are self-dual. This is given by the following result that is a direct consequence of Propositions 10 and 9.

**Proposition 11.** *Let*

$$L\phi(x) = \sum_{i,j=1}^{d} a_{ij}(x_i, x_j) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \frac{1}{2} \sum_{j=1}^{d} \frac{\partial a_{jj}}{\partial x_j}(x_j) \frac{\partial \phi}{\partial x_j}(x) \tag{5.29}$$

*with a positive definite (possibly not strictly) diffusion matrix* $a(x) = (a_{ij}(x))$ *such that* $a_{ij}$ *depend only on* $x_i, x_j$ *and are continuously differentiable (with bounded derivatives). Then the diffusion generated by* $L$ *is self-dual in the Pareto sense.*

## 5.5 APPLICATION TO OTHER CONES

Generalization of our results to orders arising from cones $C(e_1, \cdots, e_d)$ can be obtained by the change of variables, though the calculations quickly become rather cumbersome. Let us consider only the simple example of the two-dimensional cone (5.9). The question we are going to answer is as follows: under what conditions the diffusion operator

$$Lg(x,y) = a(x,y)\frac{\partial^2 g}{\partial x^2} + 2b(x,y)\frac{\partial^2 g}{\partial x \partial y} + c(x,y)\frac{\partial^2 g}{\partial y^2} \tag{5.30}$$

generates a diffusion that has a dual in the sense of the order generated by C, and how the dual generator looks like. Having in mind the relation with the standard Pareto order we can expect that the coefficients should depend in certain way on two arbitrary functions of one variable and one arbitrary function of two variables. This is in fact the case as the following result shows.

**Proposition 12.** *Let* $L$ *be of form* (5.30) *with smooth coefficients generate a Feller diffusion* $X_t^x$*. If the coefficients have the form*

$$\begin{aligned} a(x,y) &= \alpha(x+y) + \beta(x-y) + \omega(x,y), \\ c(x,y) &= \alpha(x+y) + \beta(x-y) - \omega(x,y), \\ b(x,y) &= \alpha(x+y) - \beta(x-y) \end{aligned} \tag{5.31}$$

*with some smooth functions* $\alpha, \beta, \omega$*, then* $X_t^x$ *has the dual diffusion* $Y_t^y$ *so that* (5.2) *holds with* $M = C(e_1, e_2)$ *of form* (5.9)*, where* $Y_t^y$ *is generated by the operator*

$$L^{D(f)}g = Lg + 4(\alpha'(x+y)+\beta'(x-y))\frac{\partial g}{\partial x}(x,y) + 4(\alpha'(x+y)-\beta'(x-y))\frac{\partial g}{\partial y}(x,y). \tag{5.32}$$

*Proof.* Formulas (5.31) are obtained from Proposition 10 by rotation of coordinates, that is by change $x' = x+y, y' = x-y$. □

## 5.6 DUALITY FROM PARETO ORDER: JUMP PROCESSES

Let us now turn to the generators L of pure jump processes, that is

$$L\phi(x) = \int_{\mathbb{R}^d} (\phi(w) - \phi(x))\nu(x, dw) \tag{5.33}$$

with some bounded stochastic kernel $\nu$. For a measure $Q$ having a density with respect to Lebesgue measure, let us write shortly $L'q$ for the measure $L'Q$. We have

$$L'q(dz) = \int_{\mathbb{R}^d} q(x)\nu(x, dz)dx - q(z)dz \int_{\mathbb{R}^d} \nu(z, dw).$$

Consequently, relabeling the variables of integration, we have

$$F \circ L'(q) = (-1)^d \int_{z \geqslant y} (L'q)(dz)$$

$$= (-1)^d \int_{w \geqslant y} \int_{\mathbb{R}^d} q(z)\nu(z, dw)\, dz - (-1)^d \int_{z \geqslant y} \int_{\mathbb{R}^d} q(z)\nu(z, dw)\, dz.$$

The integrals in the two terms partially cancel. Namely, we can write

$$F \circ L'(q) = (-1)^d \int q(z) \left( \mathbf{1}_{z \geqslant y} \left[ \int_{w \geqslant y} \nu(z, dw) - \int \nu(z, dw) \right] + \mathbf{1}_{z \not\geqslant y} \int_{w \geqslant y} \nu(z, dw) \right) dz,$$

implying

$$F \circ L'(q) = (-1)^d \int q(z) \left[ \mathbf{1}_{z \not\geqslant y} \int_{w \geqslant y} \nu(z, dw) - \mathbf{1}_{z \geqslant y} \int_{w \not\geqslant y} \nu(z, dw) \right] dz.$$

Hence, for a smooth ($d$ times differentiable) function $g$ we can write either

$$\begin{aligned} L^{D(f)} g &= F \circ L' \circ F^{-1} g(y) \\ &= (-1)^d \int \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \left[ \mathbf{1}_{z \not\geqslant y} \int_{w \geqslant y} \nu(z, dw) - \mathbf{1}_{z \geqslant y} \int_{w \not\geqslant y} \nu(z, dw) \right] dz, \end{aligned} \tag{5.34}$$

or

$$\begin{aligned} L^{D(f)} g =& F \circ L' \circ F^{-1} g(y) \\ =& (-1)^d \int_{w \geqslant y} \int_{\mathbb{R}^d} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \nu(z, dw)\, dz \\ &- (-1)^d \int_{z \geqslant y} \int_{\mathbb{R}^d} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \nu(z, dw)\, dz \end{aligned} \tag{5.35}$$

If $\nu(z, dw)$ depends smoothly on $z$, this expression can be rewritten by moving the derivatives from $g$ to $\nu$. For this transformation expression (5.35) is more handy than (5.34). To perform the integration by parts in its second term we shall use the following simple formula (with a straightforward proof by mathematical induction)

$$\int_{z\geqslant y} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \phi(z) dz = (-1)^d \sum_{I\subset\{1,\cdots,d\}} \int_{z_I \geqslant y_I} g(y_{\bar{I}}, z_I) \frac{\partial^{|I|}\phi}{\partial z_I}(y_{\bar{I}}, z_I) dz_I, \qquad (5.36)$$

which is valid when the boundary terms at infinity vanish, for instance if either $\phi$ or $g$ vanish at infinity with all its derivatives. Here $|I|$ is the number of indices in $I$, the integral over the set $\{z_I \geqslant y_I\}$ is $|I|$-dimensional and $(y_{\bar{I}}, z_I)$ denotes the vector whose coordinates with indices from $I$ are those of the vector $z$ and other coordinates are from the vector $y$.

Using this formula we transform (5.35) into the expression

$$L^{D(f)} g(y) = F \circ L' \circ F^{-1} g(y)$$

$$= \int_{w\geqslant y} \int_{\mathbb{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw)\, dz - \sum_{I\subset\{1,\cdots,d\}} \int_{z_I \geqslant y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbb{R}^d} \frac{\partial^{|I|}\nu}{\partial z_I}(y_{\bar{I}}, z_I, dw).$$

Singling out from the sum the terms corresponding to $I$ being empty and $I$ being the whole set $\{1, \cdots, d\}$, this rewrites as

$$\int_{w\geqslant y} \int_{\mathbb{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw)\, dz - \int_{z\geqslant y} \int_{\mathbb{R}^d} g(z) \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw)\, dz$$

$$-{\sum}'_{I\subset\{1,\cdots,d\}} \int_{z_I \geqslant y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbb{R}^d} \frac{\partial^{|I|}\nu}{\partial z_I}(y_{\bar{I}}, z_I, dw) - g(y) \int_{\mathbb{R}^d} \nu(y, dw),$$

where $\sum'$ denotes the sum over all proper subsets $I$, i.e. all subsets $I$ excluding empty set and the whole set $\{1, \cdots, d\}$. Performing the cancellation in the first two terms yields finally (see the trick leading to (5.34))

$$L^{D(f)} g(y) = F \circ L' \circ F^{-1} g(y) = -g(y) \int_{\mathbb{R}^d} \nu(y, dw)$$

$$- \sum_{I\subset\{1,\cdots,d\}}{}' \int_{z_I \geqslant y_I} dz_I g(y_{\bar{I}}, z_I) \int_{\mathbb{R}^d} \frac{\partial^{|I|}\nu}{\partial z_I}(y_{\bar{I}}, z_I, dw)$$

$$+\int_{\mathbb{R}^d} g(z)dz \left[ \mathbf{1}_{z \ngeq y} \int_{w \geqslant y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) - \mathbf{1}_{z \geqslant y} \int_{w \ngeq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) \right]. \quad (5.37)$$

For instance, for $d = 1$

$$L^{D(f)} g(y) = \int_{-\infty}^{y} g(z)\, dz \int_{w \geqslant y} \frac{\partial \nu}{\partial z}(z, dw) - \int_{y}^{\infty} g(z)\, dz \int_{w < y} \frac{\partial \nu}{\partial z}(z, dw) - g(y) \int \nu(y, dw), \quad (5.38)$$

which is the formula essentially obtained by Kolokoltsov [52, 54], and for $d = 2$

$$L^{D(f)} g(y) = -g(y_1, y_2) \int \nu(y, dw)$$

$$-\int_{z_1 \geqslant y_1} g(z_1, y_2) dz_1 \int \frac{\partial \nu}{\partial z_1}(z_1, y_2, dw) - \int_{z_2 \geqslant y_2} g(y_1, z_2) dz_2 \int \frac{\partial \nu}{\partial z_2}(y_1, z_2, dw)$$

$$+\int g(z_1, z_2) dz_1 dz_2 \left[ \mathbf{1}_{z \ngeq y} \int_{w \geqslant y} \frac{\partial^2 \nu}{\partial z_1 \partial z_2}(z, dw) - \mathbf{1}_{z \geqslant y} \int_{w \ngeq y} \frac{\partial^2 \nu}{\partial z_1 \partial z_2}(z, dw) \right]. \quad (5.39)$$

*Remark* 8. It is worth stressing that one should be cautious in using these formulas as they may not be true for $f$ not vanishing at infinity, say even for a constant function $f$ (so that these formulas cannot be used even for checking conservativity condition $L^{D(f)}\mathbf{1} = 0$). Generally one has to use the following extension of (5.36) (also proved by direct induction) that is valid whenever $g$, $\phi$ are smooth and such that for all $I \subset \{1, \cdots, d\}$ and $y_{\bar{I}}$ there exist finite limits of the functions $g(y_{\bar{I}}, z_I)$, $\phi(y_{\bar{I}}, z_I)$ and their derivatives in $z_I$, as $z_I \to \infty$ (here $\infty$ means precisely $+\infty$):

$$\int_{z \geqslant y} \frac{\partial^d g(z)}{\partial z_1 \cdots \partial z_d} \phi(z) dz$$

$$= (-1)^d \sum_{I \subset \{1, \cdots, d\}} \int_{z_I \geqslant y_I} \left[ \sum_{J \subset \bar{I}} (-1)^{|J|} g(y_{\bar{I} \setminus J}, \infty_J, z_I) \frac{\partial^{|I|} \phi}{\partial z_I}(y_{\bar{I} \setminus J}, \infty_J, z_I) \right] dz_I, \quad (5.40)$$

where $(y_{\bar{I} \setminus J}, \infty_J, z_I)$ denotes the vector with $\bar{I} \setminus J$ -coordinates from $y$, $I$-coordinates from $z$ and other coordinates being $+\infty$. For instance, in case $d = 2$ we have

$$\int_{y_1}^{\infty} \int_{y_2}^{\infty} \frac{\partial^2 g(z)}{\partial z_1 \partial z_2} \phi(z)\, dz = \int_{y_1}^{\infty} \int_{y_2}^{\infty} \frac{\partial^2 \phi(z)}{\partial z_1 \partial z_2} g(z)\, dz$$

$$+\int_{y_1}^{\infty}\left[g(z_1,y_2)\frac{\partial^2\phi}{\partial z_1}(z_1,y_2)-g(z_1,\infty)\frac{\partial^2\phi}{\partial z_1}(z_1,\infty)\right]dz_1$$
$$+\int_{y_2}^{\infty}\left[g(y_1,z_2)\frac{\partial^2\phi}{\partial z_2}(y_1,z_2)-g(\infty,z_2)\frac{\partial^2\phi}{\partial z_2}(\infty,z_2)\right]dz_2$$
$$+g(y_1,y_2)\phi(y_1,y_2)-g(\infty,y_2)\phi(\infty,y_2)-g(y_1,\infty)\phi(y_1,\infty)+g(\infty,\infty)\phi(\infty,\infty). \tag{5.41}$$

Assuming that for all $y$

$$\lim_{z\to-\infty}\int_{w\geqslant y}\nu(z,dw)=0,\quad \lim_{z\to\infty}\int_{w<y}\nu(z,dw)=0, \tag{5.42}$$

equation (5.38) rewrites in the equivalent conservative form

$$L^{D(f)}g(y)=\int_{-\infty}^{y}(g(z)-g(y))\,dz\int_{w\geqslant y}\frac{\partial\nu}{\partial z}(z,dw)-\int_{y}^{\infty}(g(z)-g(y))\,dz\int_{w<y}\frac{\partial\nu}{\partial z}(z,dw). \tag{5.43}$$

**Proposition 13.** *Let* $L$ *have form* (5.33) *with a bounded weakly continuous stochastic kernel* $\nu$*, so that* $L$ *generates a* $C$*-Feller (i,e. its semigroup preserves continuous functions) jump process in* $\mathbb{R}^d$ *that we denote* $X_t^x$*. Then* $L^{D(f)}$ *is given by* (5.34)*. If the kernel* $\nu$ *has continuous bounded mixed derivatives, so that*

$$\frac{\partial^{|I|}\nu}{\partial z_I}(z,dw)$$

*is again a bounded kernel (possibly signed) for any nonempty subset* $I\in\{1,\cdots d\}$ *(including the whole set* $\{1,\cdots d\}$*), then* $L^{D(f)}$ *can be rewritten as* (5.37)*. Finally* $L^{D(f)}$ *generates itself a* $C$*-Feller Markov process that we denote* $Y_t^y$ *if and only if the following conditions hold:*

*All mixed derivatives of orders from* $1$ *to* $d-1$ *of the jump rates are non-positive, i.e.*

$$\int_{\mathbb{R}^d}\frac{\partial^{|I|}\nu}{\partial z_I}(z,dw)\leqslant 0 \tag{5.44}$$

*for any proper subset* I *of* $\{1, \cdots d\}$*; and*

$$\begin{aligned}
&\int_{w \geqslant y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) \geqslant 0, \quad z \ngeq y, \\
&\int_{w \ngeq y} \frac{\partial^d \nu}{\partial z_1 \cdots \partial z_d}(z, dw) \leqslant 0, \quad z \geqslant y.
\end{aligned} \tag{5.45}$$

*If this is the case, the duality relation* (5.11) *holds.*

*Proof.* Everything is proved apart from the criterion for the generation of a Markov process. To get it one only has to note that the operator $\int g(z)\mu(y, dz) - \alpha(y)g(y)$ with given kernel $\mu$ and function $\alpha$ is conditionally positive (and generates a process) if and only if the kernel $\mu$ is stochastic (i.e. positive), and that the kernels from various terms in (5.37) are mutually singular, so that this positivity condition should be applied separately to each term.

One completes the proof by the same argument as used at the end of the proof of Proposition 10. □

A couple of remarks are in order here. Condition (5.45) is not very transparent. A simple particular case to have in mind is when the kernel $\nu$ decomposes into a sum of kernels depending on all variables but for one, i.e.

$$\nu(z, dw) = \sum_j \nu_j(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_d, dw),$$

in which case the condition (5.45) becomes void (thus trivially satisfied). On the other hand, conditions (5.44) are easy to check. To visualize this condition it is instructive to observe that if $q$ is a density of a positive measure on $\mathbb{R}^d$, then the distribution function

$$g(x) = \int_{z \ngeq x} q(z) dz$$

is positive, but has all mixed derivatives negative. Even more specifically, if $\nu$ decomposes into a sum of kernels depending on one variable only, that is

$$\nu(z, dw) = \sum_j \nu_j(z_j, dw),$$

all conditions of Proposition 13 are reduced to an easy to check requirement that all rates $\int \nu_j(z_j, dw)$ are decreasing functions of $z_j$.

Let us note that the method of the calculation of dual used above can still be used for processes with a boundary. For instance, let us consider a process on $\mathbb{R}_+$ with the generator

$$L\phi(x) = \int_{\mathbb{R}_+} (\phi(w) - \phi(x))\nu(x, dw). \tag{5.46}$$

The operator $L'$ takes the form

$$L'q(dz) = \int_{\mathbb{R}_+} q(x)\nu(x, dz)dx - q(z)dz \int_{\mathbb{R}_+} \nu(z, dw)$$

and the same calculations as above yield

$$L^{D(f)}g(y) = \int_0^y g(z)\, dz \int_{w \geqslant y} \frac{\partial \nu}{\partial z}(z, dw) - \int_y^\infty g(z)\, dz \int_{0 \leqslant w < y} \frac{\partial \nu}{\partial z}(z, dw)$$
$$- g(y) \int \nu(y, dw) + g(0) \int_{w \geqslant y} \nu(0, dw), \tag{5.47}$$

that is, an additional term appears arising from additional boundary taken into account while integrating by parts. Under assumption (5.42), this rewrites in the equivalent conservative form

$$L^{D(f)}g(y) = \int_0^y (g(z) - g(y))\, dz \int_{w \geqslant y} \frac{\partial \nu}{\partial z}(z, dw)$$
$$- \int_y^\infty (g(z) - g(y))\, dz \int_{0 \leqslant w < y} \frac{\partial \nu}{\partial z}(z, dw) + \int_{w \geqslant y} (g(0) - g(y))\nu(0, dw). \tag{5.48}$$

We assume strong smoothness condition for $\nu$, which forces the dual Lévy kernel to have a density. This is not necessary. Just assuming monotonicity of $\int_{w \geqslant y} \nu(z, dw)$ and $\int_{w<y} \nu(z, dw)$ (and

thus the existence almost sure of non-negative derivatives of these functions of $z$), we obtain, instead of (5.49), the formula

$$L^{D(f)}g(y) = \int_{-\infty}^{y} (g(z) - g(y))\, d_z \int_{w \geqslant y} \nu(z, dw) - \int_{y}^{\infty} (g(z) - g(y))\, d_z \int_{w < y} \nu(z, dw), \tag{5.49}$$

with similar modifications for (5.48) and analogously for $d$-dimensional case.

Let us mention the link with the theory of stochastic monotonicity. Recall from chapter 2 that a Markov process $X_t^x$ is called stochastically monotone with respect to Pareto ordering if the function $\mathbf{P}(X_t^x \geqslant y)$ is a monotone function of $x$ for any $y$. Stochastic monotonicity is studied for various classes of processes, see work by Chen and Wang [18], Chen [17], Kolokoltsov [52, 55], Zhang [87], Wang [84], Rabehasaina [69] and references therein.

If duality (5.11) holds, then $X_t^x$ is obviously stochastically monotone, but, generally speaking, this condition is too weak to ensure duality, because stochastic monotonicity of a positive function on $\mathbb{R}^d$ does not imply (apart from one-dimensional case) that it is the multi-dimensional distribution function for some positive measure (see chapter 3). Therefore it is remarkable enough that for diffusion processes with generators (5.26) the conditions of stochastic monotonicity and of the existence of Pareto dual coincide. Even for deterministic processes this is already not so, as for stochastic monotonicity of processes generated by operators (5.20), $b_j$ are allowed to depend on other coordinates $x_k$ (in a monotone way, see e.g. paper by Chen and Wang [18] and references therein to previous works).

We assumed boundedness of all coefficients involved. This simplification leads to the most straightforward formulations that catch up the essence of duality. Of course, extensions to unbounded kernel rates, diffusion coefficients, etc, are possible under the conditions that ensure that all processes involved are well defined.

## 5.7 ARBITRARY FELLER PROCESSES

We have analysed three classes of the generators $L$ separately. But it is clear that if we consider a process with the generator being the sum of the generators of different classes, then applying conditions of the results above to each term separately will ensure that the dual to the sum is also conditionally positive and generates a process leading to the duality relation (5.11). We refer to textbooks by Applebaum [5] or Jacob [44] for a general introduction to Feller processes with arbitrary pseudo-differential generators.

For simplicity, we shall give the corresponding result for one-dimensional Feller processes, but extension to higher dimensions is straightforward. For this case, the generators of the dual were obtained by Kolokoltsov [54] by approximating continuous state space generators by discrete Markov chains and in chapter 2, via the direct method on duality relation. The method of this chapter will give the same result without any technical restrictions used by Kolokoltsov [54].

**Proposition 14.** *Let a Feller process $X_t^x$ in $C_\infty(\mathbb{R})$ have a generator*

$$\begin{aligned} Lg(x) &= a(x)\frac{d^2}{dx^2}g(x) + b(x)\frac{d}{dx}g(x) \\ &\quad + \int_{-\infty}^{\infty}(g(z) - g(x) - (z-x)g'(x)\mathbf{1}_{|z-x|\leqslant 1})\nu(x, dz) \end{aligned} \tag{5.50}$$

*with $a, b \in C^2(\mathbb{R})$, $a$ being non-negative, and with the weakly continuous Lévy kernel $\nu$ such that, for any $y$, conditions* (5.42) *hold and the functions*

$$\int_{w\geqslant y}\nu(z, dw), \quad -\int_{w<y}\nu(z, dw) \tag{5.51}$$

*are non-decreasing in z, for $z < y$ and $z > y$ respectively, so that their derivatives exist almost surely and are non-negative. Moreover*

$$\mathbf{1}_{z<y}\, d_z\int_{w\geqslant y}\nu(z, dw) + \mathbf{1}_{z>y}\, d_z\int_{w<y}\nu(z, dw) \tag{5.52}$$

*is a Lévy kernel (it integrates* $\min(1, (w-z)^2)$ *and the integral*

$$\int_{y-1}^{y+1}(z-y)\left[\mathbf{1}_{z<y}(\nu(y, dz) + d_z\int_{w\geqslant y}\nu(z, dw)) + \mathbf{1}_{z>y}(\nu(y, dz) - d_z\int_{w<y}\nu(z, dw))\right]$$

*exists, at least in the sense of the main (or the Cauchy) value. Then the dual process* $Y_t^y$ *exists (in the sense of* (5.11)*) and has the generator*

$$\begin{aligned} L^{D(f)}g(y) =& a(y)\frac{d^2}{dy^2}g(y) + (a'(y) - b(y))\frac{d}{dy}g(y) \\ &+ \int_{-\infty}^{y} (g(z) - g(y) - (z-y)g'(y)\mathbf{1}_{|z-y|\leqslant 1})d_z \left(\int_{w\geqslant y} \nu(z, dw)\right) \\ &- \int_{y}^{\infty} (g(z) - g(y) - (z-y)g'(y)\mathbf{1}_{|z-y|\leqslant 1})d_z \left(\int_{w<y} \nu(z, dw)\right) \\ &+ g'(y)\int_{y-1}^{y+1} (z-y)\Big[\mathbf{1}_{z<y}(\nu(y, dz) \\ &+ d_z\int_{w\geqslant y} \nu(z, dw)) + \mathbf{1}_{z>y}(\nu(y, dz) - d_z\int_{w<y} \nu(z, dw))\Big] \end{aligned} \tag{5.53}$$

*Proof.* Formula (5.50) is obtained by combining (5.49), (5.28) and (5.22). Conditions given ensure that the dual operator is well defined as a Lévy-Khintchine type operator with variable coefficients. □

*Remark* 9. As shown by Kolokoltsov's paper [54] and Theorem 5.9.2 in the Kolokoltsov's textbook [55], conditions of stochastic monotonicity (monotonicity of functions (5.51)) are sufficient for the operator (5.50) to generate a Feller process, so that this condition can be dispensed with.

As a corollary of Proposition 14, we can get now the full characterization of self–duality.

**Proposition 15.** *Let a Feller process* $X_t^x$ *in* $C_\infty(\mathbb{R})$ *have a generator* (5.50)*. Then it is self dual (in the sense of* (5.11)*) if and only if the following conditions holds:*

$$b(x) = a'(x)/2, \quad d_y\nu(y, dz) + d_z\nu(z, dy) = 0. \tag{5.54}$$

*In particular, if* $\nu$ *has a density* $\nu(z, w)$*, which is differentiable with respect to the first argument, then the second equation of* (5.54) *rewrites as*

$$\frac{\partial \nu}{\partial y}(y, z) + \frac{\partial \nu}{\partial z}(z, y) = 0. \tag{5.55}$$

*Clearly, this condition is satisfied for* $\nu(y, z) = g(|y - z|)$ *with a smooth* $g$*, which corresponds to symmetric Lévy generators.*

*Proof.* The condition on $b$ follows from Proposition 11. The condition on $\nu$ arises by the comparison of the integral terms of (5.50) with (5.50) separately for $y > z$ and $y < z$. □

## 5.8 PARETO DUALITY VIA SDES

Duality of processes is defined via their distributions, and not pathwise. It is not clear in general whether any canonical pathwise connection or a natural coupling between dual processes exists. We shall consider some examples when it does.

First of all, if $X_t$ is a Lévy process in $\mathbb{R}^d$, then $-X_t$ is its Pareto-dual, because

$$\mathbf{P}(X_t^x \geqslant y) = \mathbf{P}(X_t^0 \geqslant y - x) = \mathbf{P}(-X_t^0 \leqslant x - y) = \mathbf{P}(-X_t^y \leqslant x).$$

Next, if $X_t$ is a diffusion generated by the SDE

$$dX_t = dW_t + b(X_t)\,dt,$$

where $W_t$ is a d-dimensional standard Wiener process and $b = (b_1(x_1), \cdots, b_d(x_d))$ with even functions $b_j$, then $Y_t = -X_t$ is again Pareto-dual, as it satisfies the SDE

$$dY_t = -dW_t - b(Y_t)\,dt,$$

and hence has the generator of the dual process (by Propositions 10 and 9).

Finally we shall prove the following characterization of Pareto-duality of diffusions in terms of Stratonovich SDEs.

**Proposition 16.** *Let a Feller diffusion $X_t^x$ be generated by the Stratonovich SDE of the form*

$$dX_t = \sigma(X_t) \circ dW_t + b(X_t)\,dt, \tag{5.56}$$

*where* $b = (b_1(x_1), \cdots, b_d(x_d))$ *and* $\sigma = (\sigma_{jk})$ *with elements* $\sigma_{jk}$ *depending only on* $x_j$. *Then the diffusion generated by the SDE with inverted drift, that is*

$$dY_t = \sigma(Y_t) \circ dW_t - b(Y_t)\,dt, \tag{5.57}$$

*is Pareto-dual to* $X_t$.

*Proof.* Notice that in Ito's form the SDE for $X_t$ reads as

$$dX_t^i = \sum_j \sigma_{ij}(X_t)dW_j + \left[\sum_j \frac{1}{2}\sigma'_{ij}(X_t^i)\sigma_{ij}(X_t^i) + b_i(X_t^i)\right] dt. \tag{5.58}$$

The generator of $X_t$ is

$$Lf(x) = \frac{1}{2}\sum_{i,j,k} \sigma_{ik}(x_i)\sigma_{jk}(x_j)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i \left[\sum_j \frac{1}{2}\sigma'_{ij}(x_i)\sigma_{ij}(x_i) + b_i(x_i)\right] \frac{\partial f}{\partial x_j}.$$

Consequently, by the assumptions above and by Propositions 10 and 9, $X_t$ has a dual process $Y_t$ generated by the operator

$$L^D f(x) = \frac{1}{2}\sum_{i,j,k} \sigma_{ik}(x_i)\sigma_{jk}(x_j)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i \left[\sum_j \frac{1}{2}\sigma'_{ij}(x_i)\sigma_{ij}(x_i) - b_i(x_i)\right] \frac{\partial f}{\partial x_j},$$

which is the generator of a process solving SDE (5.57). □

# 6 STOCHASTIC f-DUALITY FROM TRANSLATION INVARIANT f

CONTENTS

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## 6.1 INTRODUCTION

In chapter 4 and 5, we have analysed in some detail of the duality arising from Pareto ordering. In the general case, explicit calculations are not always available. However, we propose in this chapter some general schemes for the analysis of translation-invariant functions f, that is functions depending only on the difference of their arguments:

$$f(x, y) = f(y - x),$$

with some other function f that we still denote by f (with some ambiguity). Specifically, we discuss several examples of duality with operator $F^{-1}$ being the Laplacian or a fractional Lapacian.

## 6.2 f-DUALITY FROM TRANSLATION INVARIANT f

Recall the operator F from (4.3)

$$(FQ)(y) = \int f(x, y)Q(dx)$$

When applied to a measure Q with density q, the operator F takes the form

$$g(y) = (FQ)(y) = \int_{\mathbb{R}^d} f(y - x)q(dx), \tag{6.1}$$

i.e. it becomes a convolution operator. It is then well known that under appropriate regularity assumptions, f is the fundamental solution of the pseudo-differential operator $L_f$ with the symbol

$$L_f(p) = \frac{1}{\hat{f}(p)}, \tag{6.2}$$

where

$$\hat{f}(p) = \int e^{-ixp} f(x) dx$$

is the Fourier transform of f.

*Remark* 10. In fact, by the definition of the fundamental solution,

$$L_f \left( \frac{1}{i} \frac{\partial}{\partial x} \right) f(x) = \delta(x),$$

which by taking the Fourier transform from both sides rewrites as

$$L_f(p)\hat{f}(p) = 1,$$

as claimed.

Hence $g(y)$ from (6.1) solves the equation $L_f g = q$, so that $F^{-1} = L_f$. Of course, for an arbitrary f, the operator $L_f$ can be quite complicated and the identification of the appropriate classes of functions q and g can be non-trivial. Let us consider the simplest example where $L_f$ is Laplacian, or more generally, the fractional power of a Laplacian $L_f$.

It is well known that the fundamental solution for the Laplace operator $\Delta$ in dimension $d \geqslant 3$ is the function

$$f(x) = -\frac{1}{(d-2)\sigma_{d-1}} \frac{1}{|x|^{d-2}},$$

where $\sigma_{d-1}$ is the area of a unit sphere in $\mathbb{R}^d$. Hence the dual operator (4.4) takes the form

$$T^{D(f)} = \Delta^{-1} \circ T' \circ \Delta, \tag{6.3}$$

and the generator for the corresponding dual semigroup becomes

$$L^{D(f)} = \Delta^{-1} \circ L' \circ \Delta. \tag{6.4}$$

Let L be a diffusion operator of the special kind:

$$Lg(x) = a(x)\Delta g(x)$$

with a non-negative bounded smooth function $a(x)$. Then $L' = \Delta \circ a(x)$ and thus

$$L^{D(f)} = \Delta^{-1} \circ L' \circ \Delta = L, \tag{6.5}$$

so that L is self f-dual.

Noting that in two dimensions ($d = 2$) the fundamental solution for the Laplacian is known to be $\log|x|/2\pi$. We then get the following.

**Proposition 17.** *Let $X_t^x$ be the Feller diffusion generated by the operator $Lg(x) = a(x)\Delta g(x)$ in $\mathbb{R}^d$ with a non-negative bounded smooth function $a(x)$. Then, for all $x, y \in \mathbb{R}^d$, we have*

$$\mathbb{E}\frac{1}{|X_t^x - y|^{d-2}} = \mathbb{E}\frac{1}{|X_t^y - x|^{d-2}}, \tag{6.6}$$

$$\mathbb{E}\log|X_t^x - y| = \mathbb{E}\log|X_t^y - x|, \quad d = 2 \tag{6.7}$$

*for $d \geqslant 3$ and $d = 2$ respectively.*

Turning to the fractional Laplacian $|\Delta|^{\alpha/2}$ in $\mathbb{R}^d$ with $\alpha \in (0, 2)$, $d \geqslant 2$, let us recall that the inverse operator is given by the Riesz potential

$$|\Delta|^{-\alpha/2} g(x) = I^\alpha g(x) = \frac{1}{H_d(\alpha)} \int_{\mathbb{R}^d} \frac{g(y)\, dy}{|x-y|^{d-\alpha}},$$

where

$$H_d(\alpha) = 2^\alpha \pi^{d/2} \frac{\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)},$$

see e.g. work by Helgason [37]. Hence, the operator $|\Delta|^{\alpha/2}$ is $L_f$ for

$$f(x) = \frac{1}{H_d(\alpha)} \frac{1}{|x|^{d-\alpha}}.$$

Let us consider a stable-like process generated by the operator

$$Lg(x) = -a(x)|\Delta|^{\alpha/2} g(x)$$

with a positive smooth function $a(x)$. Then $L' = |\Delta|^{\alpha/2} \circ a(x)$ and thus

$$L^{D(f)} = |\Delta|^{-\alpha/2} \circ L' \circ |\Delta|^{\alpha/2} = L, \tag{6.8}$$

so that $L$ is self $f$-dual. Thus we proved the following extension of Proposition 17:

**Proposition 18.** *Let $X_t^x$ be the stable-like process generated by the operator $Lg(x) = a(x)|\Delta|^{\alpha/2} g(x)$ in $\mathbb{R}^d$ with $d \geqslant 2$, $\alpha \in (0, 2]$ excluding the case $d = \alpha = 2$ (for which (6.7) holds), and with a non-negative bounded smooth function $a(x)$. Then, for all $x, y \in \mathbb{R}^d$,*

$$\mathbb{E} \frac{1}{|X_t^x - y|^{d-\alpha}} = \mathbb{E} \frac{1}{|X_t^y - x|^{d-\alpha}}. \tag{6.9}$$

# 7 STOCHASTIC DUALITY FOR MARKOV PROCESSES IN ONE DIMENSIONAL REAL INTERVALS.

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## 7.1 INTRODUCTION

Recall that in section 2.4, we introduced Siegmund's duality on real-valued intervals, giving particular attention to the barrier conditions for both the original Markov process and its dual. In this section we study this kind of duality in detail. We begin by deducing the consequences of applying the general approach using formalas (4.6) and (4.7) to the study of duality for Markov processes in domains with a boundary. Furthermore, we discuss some difficulties arising from condition (1.3) at boundary points, which, for instance, prevents the second dual to reflected Brownian motion to coincide with itself. This chapter is concluded by introducing the notion of a regularised dual which addresses this and similar issues.

## 7.2 REFLECTED AND ABSORBED PROCESSES IN $\bar{\mathbb{R}}_+$

Let $C^k_\infty(\mathbb{R}^d)$ be the space of $k$ times differentiable functions on $\mathbb{R}^d$ with all these derivatives vanishing at infinity. Also, $C^k_\infty(\bar{\mathbb{R}}_+)$ will denotes the restriction of functions from $C^k_\infty(\mathbb{R})$ on $\bar{\mathbb{R}}_+ = \{x \geqslant 0\}$.

Consider a Feller process $\mathbf{X} = (X_t^x)_{t\geqslant 0}$ on $\mathbb{R}$ generated by operator

$$
\begin{aligned}
Lg(x) =& a(x)\frac{d^2}{dx^2}g(x) + b(x)\frac{d}{dx}g(x) \\
&+ \int_{-\infty}^{\infty} (g(z) - g(x) - (z-x)g'(x)\mathbf{1}_{|z-x|\leqslant 1})\nu(x, dz)
\end{aligned}
$$

under the conditions of Proposition 14 assuming additionally that

(A) $a \in C^2(\mathbb{R})$ and is an even function such that $a(x) \geqslant 0$, $b \in C^2(\mathbb{R})$ and is an odd function (implying $b(0) = 0$), the support of $\nu$ is in $\mathbb{R}_+$ for $x \geqslant 0$ and $\nu(-x, dy) = R\nu(x, dy)$, where $R$ denotes the reflection of the measure with respect to the origin (so that, by definition, $\int \phi(y)R\nu(x, dy) = \int \phi(-y)\nu(x, dy)$).

Then, as is well known (see e.g. Theorem 6.8.1 in Kolokoltsov's textbook [55]) the magnitude $|X_t^x|$ is itself a Markov process on $\mathbb{R}_+$, also referred to as $X_t^x$ reflected at the origin. Moreover, if the transition probabilities of $X_t^x$ are $p_t(x, dy)$, then $|X_t^x|$ has the transition density

$$
p_t^{\mathrm{ref}}(x, dy) = p_t(x, dy) + Rp_t(x, dy),
$$

and the semi-group $T_t^{\mathrm{ref}}$ of $|X_t^x|$ can be obtained from the semi-group $T_t$ of $X_t^x$ by the restriction to even functions.

*Remark* 11. Assuming that the kernel $\nu$ is twice smooth would imply that the space $C_\infty^2(\mathbb{R})$ is an invariant core for $X_t^x$ and consequently that the subspace of functions $f$ from $C_\infty^2(\bar{\mathbb{R}}_+)$ such that $f'(0) = 0$ is an invariant core for $|X_t^x|$.

*Remark* 12. If $X_t^x$ were a diffusion, the process $|X_t^x|$ on $\bar{\mathbb{R}}_+$ would be stochastically monotone by the coupling argument, see e.g. Sect II,2 of textbook by Liggett [56]) and hence by Siegmund's theorem in [72] or Theorem 2 in this thesis, it had a Markov dual $Y_t^y$ on $\bar{\mathbb{R}}_+$ (in the sense (1.3)) with absorbtion at the origin. In our case monotonicity follows from the construction of the dual below, which turns out to be given by a semi-group with a conditionally positive generator.

**Proposition 19.** *Under the conditions of Proposition 14 and assumption (A) above, the dual process $Y_t^y$ is Feller on $\bar{\mathbb{R}}_+$ absorbed at the origin and generated by the operator*

$$
\begin{aligned}
L^D g(y) =& a(y)\frac{d^2}{dy^2}g(y) + (a'(y) - b(y))\frac{d}{dy}g(y) + \int_{w\geqslant y}(g(0) - g(y)\nu(0, dw)\\
&+ \int_0^y (g(z) - g(y) - (z-y)g'(y)\mathbf{1}_{|z-y|\leqslant 1})d_z\left(\int_{w\geqslant y}\nu(z, dw)\right)\\
&- \int_y^\infty (g(z) - g(y) - (z-y)g'(y)\mathbf{1}_{|z-y|\leqslant 1})d_z\left(\int_{w<y}\nu(z, dw)\right)\\
&+ g'(y)\int_{y-1}^{y+1}(z-y)\Bigg[\mathbf{1}_{z<y}(\nu(y, dz) + d_z\int_{w\geqslant y}\nu(z, dw))\\
&+ \mathbf{1}_{z>y}(\nu(y, dz) - d_z\int_{w<y}\nu(z, dw))\Bigg]
\end{aligned}
\tag{7.1}
$$

*The semi-group $T_t^D$ of $Y_t^y$ is given explicitly by the formula*

$$
(T_t^D g)(y) = g(0)\int_y^\infty p_t^{ref}(0, dz) + \int_0^\infty g(x)\left(\int_y^\infty \frac{\partial}{\partial x}p_t^{ref}(x, dz)\right)dx. \tag{7.2}
$$

*Proof.* Using (4.6) with $F^{-1}g(x) = -g'(x)$ we get for $g \in C^1_\infty(\bar{\mathbb{R}}_+)$

$$
(T_t^D g)(y) = -\int_y^\infty dz\int_0^\infty g'(x)p_t^{ref}(x, dz)\,dx, \tag{7.3}
$$

and hence

$$
(T_t^D g)(y) = g(0)\int_y^\infty p_t^{ref}(0, dz) + \int_0^\infty dx\int_y^\infty g(x)\frac{\partial}{\partial x}p_t^{ref}(x, dz), \tag{7.4}
$$

yielding (7.2) as required.

It is worth stressing that this formula implies the conservativity condition $T_t^D\mathbf{1} = \mathbf{1}$ (preservation of constants by $T_t^D$), because

$$
\lim_{x\to\infty}\int_y^\infty p_t^{ref}(x, dz) = 1
$$

by the Feller property and hence

$$
\int_0^\infty \frac{\partial}{\partial x}\left(\int_y^\infty p_t^{ref}(x, dz)\right)dx = 1 - \int_y^\infty p_t^{ref}(0, dz). \tag{7.5}
$$

Operators $T_t^D$ form a semi-group by Proposition 7. The form of the generator follows from (5.48). As it is conditionally positive, the semigroup $T_t^D$ preserves positivity and preserves constants thus being a semi-group of a Markov process. Moreover, as also seen directly from (7.2), $T_t^D f(0) = f(0)$, so that the value at the origin is preserved meaning that this process is absorbing at the origin. □

*Remark* 13. (i) Formula (7.3) is valid only for $g$ vanishing at infinity, and (7.2) extends it (yields a minimal extension) to bounded functions on $\bar{\mathbb{R}}_+$. Plugging $g = 1$ into (7.3) yields zero, not 1.

*Remark* 14. (ii) The attempt to use integration in (7.5) in the opposite direction, at least when $p_t(x, dz)$ has a density $p_t(x, z)$, and using $\lim_{x\to\infty} p_t^{\mathrm{ref}}(x, z) = 0$ would give

$$\int_y^\infty dz \left( \int_0^\infty \frac{\partial}{\partial x} p_t^{\mathrm{ref}}(x, z)\, dx \right) = -\int_y^\infty p_t^{\mathrm{ref}}(0, z)\, dz,$$

which is different from the r.h.s. of (7.5).

It is worth noting additionally that if $a(0) \neq 0$ and $\nu = 0$, then the subspace of functions $g$ from $C^2_\infty(\bar{\mathbb{R}}_+)$ such that $g''(0) = 0$ is an invariant core for $Y_t^y$. In fact, the condition $L^D g(0) = 0$ (following from $T_t^D g(0) = g(0)$) implies $g''(0) = 0$. In particular, the integral $\int_{w\geqslant y}(g(0) - g(y)\nu(0, dw)$ from (7.1) is well-defined for such $g$. On the other hand, if $a(0) = 0$ and $\nu = 0$, then $a(x) = ax^2(1 + o(1))$, $b(x) = bx(1 + o(1))$ as $x \to 0$ with $a \geqslant 0$, $b \in \mathbb{R}$ implying that 0 is an inaccessible boundary point, so that $X_t^x = |X_t^x|$ for $x > 0$. In this case nothing comes out of the origin, so that $p_t^{ref}(0, z) = 0$ for all $z > 0$ implying that the first term on the r.h.s. of (7.2) vanishes and hence that 0 is also inaccessible for $Y_t^y$ (which follows also from its generator). In particular, if additionally $b(x) = a'(x)/2$, the process $|X_t^x|$ is self-dual on $\mathbb{R}_+$.

There is an extensive literature on the absorption - reflection link presented in Proposition 19, mostly because of its natural interpretation in terms of ruin probabilities having important applications in insurance mathematics. For piecewise deterministic Markov processes it was obtained in paper by Asmussen and Peterson [8] (see also work by Asmussen [6]) and used effectively by Djehiche [29] to assess ruin probabilities via large deviations. Then it was extended to diffusions with jumps by Sigman and Ryan [73], and to Lévy processes by Asmussen and Pihlsgård [9]. Our result is an extension of the corresponding result from [73] by Sigman and Ryan, as we do it for arbitrary stochastically monotone processes. Our proof is quite different, as it is more elementary, using effectively only formula (4.6).

## 7.3 SECOND DUAL AND REGULARISED DUAL

Extension of the previous result to processes with a boundary from the right or with two boundaries is if course natural, see work by Asmussen and Pihlsgård [9], but not quite straightforward. We shall clarify the aspects of duality (even the definition has to be modified), needed for these cases reducing our attention, for simplicity, to Feller processes with transition probabilities having no atoms, that is to processes such that the function $\mathbf{P}(X_t^x \geqslant y)$ is continuous in $y$ and hence (by Feller property) also in $x$. For such process,

$$\mathbf{P}(X_t^x \geqslant y) = \mathbf{P}(Y_t^y \leqslant x) \Longleftrightarrow \mathbf{P}(X_t^x \leqslant y) = \mathbf{P}(Y_t^y \geqslant x). \tag{7.6}$$

It is natural to ask whether the second dual coincides with the original process. For diffusions on $\mathbb{R}^d$ this is in fact the case, as is seen from Proposition 10 or, in one-dimensional case ($d = 1$), directly from (7.6). However, for processes on $\mathbb{R}_+$ this does not hold, as seen already from Lévy's example of reflected Brownian motion in example 5 of chapter 2 in this thesis. In fact, reflected Brownian Motion cannot be dual to absorbing Brownian Motion, as any dual process on $\mathbb{R}_+$ should be absorbing at the left end, that is at the origin, as seen directly from (1.3). However, the reflected Brownian Motion is "almost dual" to the absorbing Brownian Motion in the sense that $\mathbf{P}(Y_t^y \leqslant x) = \mathbf{P}(X_t^x \geqslant y)$ (with $Y$ reflected and $X$ absorbing Brownian Motion) holds for all $y \neq 0$ and all $x$. This suggests that the usual definition of duality imposes unnatural restrictions on the boundary.

Consequently, we shall give the following definition. Let $X_t^x$ be a stochastically monotone process on $[a, \infty)$ such that $\mathbf{P}(X_t^x \geqslant y)$ is right continuous in $x$. A process $Y_t^y$ on $[a, \infty)$ will be called a *regularised dual* to a process $X_t^x$ on $[a, \infty)$ if (1.3) holds for all $x \geqslant a, y > a$, and the distribution for $y = a$ is defined by continuity as

$$\mathbf{P}(Y_t^a \leqslant x) = \mathbf{P}(Y_t^{a+} \leqslant x) = \lim_{z \to a} \mathbf{P}(Y_t^z \leqslant x). \tag{7.7}$$

*Remark* 15. One could also relax the condition for $x = a$ by defining $\mathbf{P}(Y_t^y \leqslant a) = \lim_{z\to a} \mathbf{P}(Y_t^z \leqslant x)$. This would lead to the same result, as for usual definition, due to the right continuity of $\mathbf{P}(X_t^x \geqslant y)$ in $x$.

*Remark* 16. If one only assumes monotonicity of the function $\mathbf{P}(X_t^x \geqslant y)$, it would become natural to define the dual distribution $\mathbf{P}(Y_t^y \leqslant x)$ as the right continuous modification of the function $\mathbf{P}(X_t^x \geqslant y)$.

The following statement is now clear.

**Proposition 20.** *Under the assumptions of Proposition 19 and assuming the function* $\mathbf{P}(X_t^x \geqslant y)$ *is continuous in* $y$ *for* $y > 0$*, the reflected process* $|X_t^x|$ *is a regularised dual to* $Y_t^y$*. Thus the second regularised dual to* $|X_t^x|$ *coincides with* $|X_t^x|$*.*

*Remark* 17. The usual (not regularised) dual of $Y_t^x$ from Proposition 19 is a rather pathological process $Z_t^z$, whose distributions coincides with that of $|X_t^z|$ for $z \neq 0$, but the origin is an unattainable point without escape from it. In other words, its distribution is the same as the process reflected from the boundary, but only on the outside of the boundary. This means discontinuity on its pathology. Thus $Z_t^z$ should be "reflected from the origin" without touching it.

## 7.4 PROCESSES ON INTERVALS

The theory still does not allow to treat reflected diffusions on $\bar{\mathbb{R}}_-$, even reflected Brownian motion, since equation (1.3) implies that, in order to have a dual, a process on $\bar{\mathbb{R}}_-$ should be absorbing at the origin.

A natural extension of the definition of regularised dual given above for processes on a half-line turns out to be the following. Let $X_t^x$ be a stochastically monotone process on $[a, b]$ (meaning $[a, \infty)$ or $(-\infty, b]$ in case $b = \infty$ or $a = -\infty$ respectively) such that $\mathbf{P}(X_t^x \geqslant y)$ is right continuous in $x$. A process $Y_t^y$ on $[a, b]$ will be called a *regularised dual* to a process $X_t^x$ on $[a, b]$ if (1.3) holds for all $x, y \in [a, b]$ excluding $y = a$ and $x = b$, where additional conditions are imposed: equation (7.7) for $y = a$ and equation

$$\mathbf{P}(Y_t^y = b) = P(Y_t^y \geqslant b) = \mathbf{P}(Y_t^y \geqslant b_-) = \lim_{z \to b} \mathbf{P}(Y_t^y \geqslant z) \tag{7.8}$$

for $x = b$. Notice that this latter condition is equivalent to

$$P(Y_t^y < b) = \mathbf{P}(Y_t^y < b_-) = \lim_{z \to b} \mathbf{P}(Y_t^y < z). \tag{7.9}$$

If $a = -\infty$ or $b = \infty$ the corresponding conditions involving $a$ or $b$ are considered to be void, so that for a process on $\mathbb{R}$ the definition reduces to a usual one.

As an example, let us consider an arbitrary one-dimensional diffusion on an interval $[\alpha, \beta]$ (assuming, for definiteness, that $\alpha$, $\beta$ are finite points) reflected at both boundaries. More precisely, let $X_t^x$ be a diffusion on $\mathbb{R}$ generated by operator (5.50) with vanishing $\nu$ assuming that

(B) $a, b \in C^2(\mathbb{R})$ are $2(\beta - \alpha)$ periodic functions which are symmetric and antisymmetric respectively with respect to reflections $R_\alpha$ ($R_\alpha(x) = 2\alpha - x$) and $R_\beta$ ($R_\beta(x) = 2\beta - x$) around points $\alpha$ and $\beta$, implying in particular $a'(\alpha) = a'(\beta) = b(\alpha) = b(\beta) = 0$; for simplicity (though this is not very essential) assume also that $a > 0$ everywhere, so that smooth transition densities $p_t(x, y)$ of $X_t^x$ are well defined.

Then the corresponding diffusion $(X^{\text{ref}})_t^x$ on $[\alpha, \beta]$ obtained by reflecting $X_t^x$ at both boundary points is well-defined (see e.g. Theorem 6.8.1 in Kolokoltsov [55]) as a Markov process. Moreover, the transition densities of $(X^{\text{ref}})_t^x$ are clearly given by

$$p_t^{ref}(x, y) = \sum_{k=-\infty}^{\infty} [p_t(x, y + 2k(\beta - \alpha)) + p_t(x, 2\alpha - y + 2k(\beta - \alpha))],$$

and the semi-group $T_t^{\text{ref}}$ of $(X^{\text{ref}})_t^x$ can be obtained from the semi-group $T_t$ of $X_t^x$ by the restriction to functions symmetric with respect to reflections $R_\alpha$ and $R_\beta$.

Finally, assumed smoothness of $a, b$ implies that the space $C_\infty^2(\mathbb{R})$ is an invariant core for $X_t^x$ and consequently the subspace of functions $f$ from $C_\infty^2([\alpha, \beta])$ such that $f'(\alpha) = f'(\beta) = 0$ is an invariant core for $(X^{\text{ref}})_t^x$.

**Proposition 21.** *Under assumption (B) above the regularised dual process* $Y_t^y$ *to* $(X^{ref})_t^x$ *is a diffusion on* $[\alpha, \beta]$ *absorbed at both boundaries and generated by the operator* (5.50) *with vanishing* $\nu$ *on the invariant core of functions* $g$ *from* $C_\infty^2([\alpha, \beta])$ *such that* $g''(\alpha) = g''(\beta) = 0$. *Finally, the semi-group* $T_t^D$ *of* $Y_t^y$ *is given explicitly by the formula*

$$(T_t^D g)(y) = g(\alpha) \int_y^\beta p_t^{ref}(\alpha, z)\, dz + g(\beta) \int_\alpha^y p_t^{ref}(\alpha, z)\, dz + \int_\alpha^\beta g(x) \left( \int_y^\beta \frac{\partial}{\partial x} p_t^{ref}(x, z)\, dz \right) dx. \tag{7.10}$$

*Proof.* It is similar to the proof of Proposition 19 above. The only difference is that, taking into account (7.8) we can define the action of $T_t^D$ on constants by the conservativity condition $T_t^D \mathbf{1} = \mathbf{1}$ (rather then deduce it). For smooth $g$ vanishing at $\beta$ we get similar to (7.3) and (7.4) that

$$(T_t^D g)(y) = -\int_y^\beta dz \int_\alpha^\beta g'(x) p_t^{\mathrm{ref}}(x, z)\, dx, \tag{7.11}$$

and hence

$$(T_t^D g)(y) = g(\alpha) \int_y^\infty p_t^{\mathrm{ref}}(\alpha, z)\, dz + \int_y^\beta \left( \int_\alpha^\beta g(x) \frac{\partial}{\partial x} p_t^{\mathrm{ref}}(x, z) dx \right) dz. \tag{7.12}$$

Combining this equation with $T_t^D \mathbf{1} = \mathbf{1}$ we get for any smooth function $g$ on $[\alpha, \beta]$ that

$$(T_t^D g)(y) = g(\beta) + (g(\alpha) - g(\beta)) \int_y^\infty p_t^{\mathrm{ref}}(\alpha, z)\, dz + \int_y^\beta \left( \int_\alpha^\beta (g(x) - g(\beta)) \frac{\partial}{\partial x} p_t^{\mathrm{ref}}(x, z) dx \right) dz,$$

yielding (7.10) as required. The rest of the proof is literally the same as for Proposition 19. □

*Remark* 18. Of course one can deal with reflected processes on $\bar{\mathbb{R}}_-$ by introducing a symmetric notion of duality. Namely, for a process $X_t^x$ on an interval of $\mathbb{R}$ let us say that $Y_t^y$ is its *right dual*, if $\mathbf{P}(Y_t^y \leqslant x) = \mathbf{P}(X_t^x \geqslant y)$ holds for all $x, y$ (that is, it is the usual duality used above) and *left dual* if $\mathbf{P}(Y_t^y < x) = \mathbf{P}(X_t^x > y)$ holds for all $x, y$, which is equivalent to $\mathbf{P}(Y_t^y \geqslant x) = \mathbf{P}(X_t^x \leqslant y)$. Thus, by definition, $Y_t^y$ is right dual to $X_t^x$ if and only if $X_t^x$ is left dual to $Y_t^y$. The theory of left dual processes on $\mathbb{R}_-$ (and their regularised version) is completely analogous to the theory of right dual process on $\mathbb{R}_+$.

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