Deformations of $\mathbb{Q}$-Fano 3-folds and weak Fano manifolds

by

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Contents

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Declarations

I declare that, to the best of my knowledge and unless otherwise stated, all the work in this thesis except Chapters 1 and 2 is original. I confirm that this thesis has not been submitted for a degree at another university.

Abstract

Fano varieties are one of important classes in the classification of algebraic varieties. In this thesis, we mainly study problems on deformations of Fano varieties motivated by the classification problems. In particular, we study Fano 3-folds with terminal singularities and weak Fano manifolds.

In Chapter 2, we prepare necessary notions on deformation theory and singularities. We also explain about the orbifold Riemann-Roch formula and computation of numerical data of a K3 surface with Du Val singularities and a $\mathbb{Q}$-Fano 3-fold.

In Chapter 3, we study the deformation theory of a $\mathbb{Q}$-Fano 3-fold with only terminal singularities. First, we show that the Kuranishi space of a $\mathbb{Q}$-Fano 3-fold is smooth. Second, we show that every $\mathbb{Q}$-Fano 3-fold with only "ordinary" terminal singularities is $\mathbb{Q}$-smoothable, that is, it can be deformed to a $\mathbb{Q}$-Fano 3-fold with only quotient singularities. Finally, we prove $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold assuming the existence of a Du Val anticanonical element. As an application, we get the genus bound for primary $\mathbb{Q}$-Fano 3-folds with Du Val anticanonical elements.

In Chapter 4, we prove that a weak Fano manifold has unobstructed deformations. For a general variety, we investigate conditions under which a variety is necessarily obstructed.

In Chapter 5, we investigate a certain coboundary map associated to a 3-fold terminal singularity which is important in the study of deformations of singular 3-folds. We determine when this map vanishes. As an application, we prove that almost all $\mathbb{Q}$-Fano 3-folds have $\mathbb{Q}$-smoothing. We also treat the $\mathbb{Q}$-smoothability problem on $\mathbb{Q}$-Calabi-Yau 3-folds.

In Chapter 6, we study deformations of a pair of a $\mathbb{Q}$-Fano 3-fold X with its elephant $D \in |-K_X|$. We prove that, if X has only quotient singularities and there exists D with only isolated singularities, there is a deformation $\mathcal{X} \to \Delta^1$ of X over a unit disc such that $|-K_{\mathcal{X}_t}|$ has a Du Val element for $t \in \Delta^1 \setminus 0$. We also give several examples of $\mathbb{Q}$-Fano 3-folds without Du Val elephants.

Chapter 1

Introduction

In this thesis, we mainly study deformations of Fano varieties. Fano varieties are important in the classification of algebraic varieties as we explain in the following.

One of the central topics in complex algebraic geometry is the classification of algebraic varieties over $\mathbb{C}$. For the classification, the Minimal Model Program (= **MMP**) suggests to take a "good model" among varieties with a fixed function field. In dimension 1, there is only one choice. In dimension 2, we can take a smooth minimal model by contracting unnecessary -1-curves. In dimension 3, we can still take a minimal model of a variety, but it admits mild singular points, called **terminal singularities**. In the 1980s, the MMP is established in dimension 3 by many people including Kawamata, Kollár, Mori, Reid, Shokurov (cf. [35]). There is a recent big development in the MMP in higher dimension and many consequences are obtained (cf. [5]).

The end products of the MMP are roughly divided into three cases according to the positivity of the canonical divisor K_X of a variety X. They are called **of general type, Calabi–Yau** or **Fano** if K_X is positive, trivial or negative respectively. In dimension 1 and 2, there is satisfactory classification of these varieties. However, in dimension 3, although the general framework of the MMP is established, the precise classification of these three classes is far from completion. Actually, it seems impossible to have classification of Calabi–Yau 3-folds or 3-folds of general type.

In this thesis, we focus on Fano varieties. They are considered to be simplest among the three classes. Indeed, there are only finitely many families of smooth Fano n-folds for fixed n ([32]). Up to dimension 3, smooth Fano varieties are already classified. However, in dimension 3, we should consider Fano varieties with terminal singularities (= **$\mathbb{Q}$-Fano 3-folds**). There are also finitely many families of them ([33]), but the classification gets complicated due to singularities and is not completed. It seems that the classification problem of $\mathbb{Q}$-Fano 3-folds is near the boundary of possible tasks and impossible tasks. In this thesis, we aim to overcome this subtlety by studying **deformations** and **elephants** of $\mathbb{Q}$-Fano 3-folds.

Deformation theory is an old subject in algebraic geometry. The general notions and tools of deformation theory were developed by people including Kodaira–Spencer, Grothendieck and others. (We refer to [56] and references therein). However, for concrete geometric problems, there is no universal method and we should find ways to treat each problem. In the following, we explain two fundamental geometric problems in deformations of algebraic varieties.

One is to determine the structure of the Kuranishi space of a projective variety. The Kuranishi space of a variety parametrizes all small deformations of the variety and is also called as the *semi-universal deformation space*. It is important in the moduli problem for varieties. For a general variety, the Kuranishi space has bad singularities even for smooth surfaces. However, it is also known that the Kuranishi space of a Fano manifold or a Calabi–Yau manifold is smooth. In order to show the smoothness of the Kuranishi space, we should show that all obstruction classes of deformations vanish. Thus the smoothness of the Kuranishi space is also called the *unobstructedness*. Obstruction classes for a smooth projective variety X are defined as elements of $H^2(X, \Theta_X)$, where Θ_X is the tangent sheaf. We can show $H^2(X, \Theta_X) = 0$ for a Fano manifold by the Kodaira–Akizuki–Nakano vanishing theorem. However, for a Calabi–Yau manifold X such that $\dim X \geq 3$, we have $H^2(X, \Theta_X) \neq 0$ and should use some technique called *T^1-lifting* to treat this case. We apply this technique to show the unobstructedness for a weak Fano manifold in Chapter 4. We also show the unobstructedness for a $\mathbb{Q}$-Fano 3-fold in Chapter 3. We introduce a different method that consists of explicitly interpreting obstruction classes as complexes.

Another geometric problem is to find a smoothing of a projective variety with singularities, that is, to determine whether a given variety has a deformation to a smooth variety. In this thesis, we only treat isolated hypersurface singularities or their quotients by cyclic group action. Smoothing problems are delicate and related with topological conditions. For example, Friedman ([13]) showed that; a Calabi–Yau threefold with ordinary double points has a smoothing if and only if, on the small resolution of the Calabi–Yau, the exceptional curves have a non-trivial relation in the homology group. Reid ([53]) made a speculation to connect Calabi–Yau threefolds by birational contractions and smoothings. Inspired by this, smoothing problems on Calabi–Yau threefolds were investigated by several people including Namikawa and Gross. Namikawa–Steenbrink ([45]) proved that a Calabi–Yau threefold with isolated rational hypersurface singularities can be deformed to one with only ordinary double points. They gave two methods in the paper. One is to use the coboundary map of some local cohomology group. Another is to use some Hodge theoretic invariant of a singularity. We apply the former method to the $\mathbb{Q}$-smoothing problem of a $\mathbb{Q}$-Fano 3-fold.

Now, we explain about $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold. By the classification of 3-fold terminal singularities, they are quotients of isolated hypersurface singularities by cyclic group action. Locally, we can deform a neighborhood of a terminal singularity to one with only quotient singularities. Such a deformation is called a **$\mathbb{Q}$-smoothing**. Altınok–Brown–Reid ([3]) conjectured that **a $\mathbb{Q}$-Fano 3-fold admits a $\mathbb{Q}$-smoothing globally** as a part of their program on the classification of $\mathbb{Q}$-Fano 3-folds. The $\mathbb{Q}$-smoothability is meaningful in the classification since, in some papers as [60] and [6], they gave classification by assuming that the $\mathbb{Q}$-Fano 3-folds have only quotient singularities. Note that the numerical type of a $\mathbb{Q}$-Fano 3-fold does not change by a $\mathbb{Q}$-smoothing. We treat this $\mathbb{Q}$-smoothability problem in Chapters 3 and 5. We actually solve this in most of the cases.

Another fundamental problem in the classification of $\mathbb{Q}$-Fano 3-folds is to find anticanonical elements with only mild singularities. An anticanonical element is called an *elephant*. A Fano 3-fold with only canonical Gorenstein singularities has an elephant with only Du Val singularities ([57], [50]). By using this fact, Mukai classified "indecomposable" Fano 3-folds with canonical Gorenstein singularities in [41]. Hence the existence of a Du Val elephant is useful in the classification. However a $\mathbb{Q}$-Fano 3-fold may not have such a good element in general. There exist examples of $\mathbb{Q}$-Fano 3-folds with empty anticanonical linear systems or with only non Du Val elephants as in [3, 4.8.3]. We can still hope to have a Du Val elephant for some $\mathbb{Q}$-Fano 3-folds with large $\dim |-K_X|$. Takagi proved the existence of a Du Val elephant for a $\mathbb{Q}$-Fano 3-fold with only index 2 singularities and with $\dim |-K_X| \geq 3$. There is a birational result (cf. [2]) which states; if a $\mathbb{Q}$-Fano 3-fold X does not have a Du Val elephant and $\dim |-K_X| \geq 1$, there exists another $\mathbb{Q}$-Fano 3-fold X' which is birational to X and has a Du Val elephant. The birational map between X and X' arises from the Sarkisov program. Altınok–Brown–Reid ([3]) conjectured the following: *Let X be a $\mathbb{Q}$-Fano 3-fold with an elephant D which is possibly very singular. Then the pair (X, D) can be deformed to a pair of a $\mathbb{Q}$-Fano 3-fold with quotient singularities and its Du Val elephant.* Such a deformation is called a **simultaneous $\mathbb{Q}$-smoothing**. We solve this problem when there exists an elephant with only isolated singularities in Chapter 6. Our result does not use conditions on $\dim |-K_X|$.

Chapter 2

Preliminaries

2.1 Deformation functors

First, we introduce a deformation functor of an algebraic scheme.

Definition 2.1.1. (cf. [56, 1.2.1]) Let X be an algebraic scheme over an algebraically closed field k and S an algebraic scheme over k with a closed point $s \in S$. A *deformation* of X over S is a pair $(\mathcal{X}, i)$, where $\mathcal{X}$ is a scheme flat over S and $i\colon X \hookrightarrow \mathcal{X}$ is a closed immersion such that the induced morphism $X \to \mathcal{X} \times_S \{s\}$ is an isomorphism.

Two deformations $(\mathcal{X}_1, i_1)$ and $(\mathcal{X}_2, i_2)$ over S are said to be *equivalent* if there exists an isomorphism $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$ over S which makes the following diagram commute;

$$\begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[dr, hook, "i_2"'] & \mathcal{X}_1 \arrow[d, "\varphi"] \\ & \mathcal{X}_2. \end{tikzcd}$$

Let Art_k be the category of Artin local k-algebras with residue field k. We define the functor $\mathrm{Def}_X\colon \mathrm{Art}_k \to (Sets)$ by setting

$$\mathrm{Def}_X(A) := \{(\mathcal{X}, i) : \text{deformation of } X \text{ over } \operatorname{Spec} A\}/(\text{equiv}), \tag{2.1}$$

where (equiv) means the equivalence introduced in the above.

We also introduce the deformation functor of a closed immersion.

Definition 2.1.2. (cf. [56, 3.4.1]) Let $f\colon D \hookrightarrow X$ be a closed immersion of algebraic schemes over an algebraically closed field k and S an algebraic scheme over k with a closed point $s \in S$. A *deformation* of a pair (X, D) over S is a data (F, i_X, i_D) in the cartesian diagram

$$\begin{tikzcd} D \arrow[r, hook, "i_D"] \arrow[d, "f"] & \mathcal{D} \arrow[d, "F"] \\ X \arrow[r, hook, "i_X"] \arrow[d] & \mathcal{X} \arrow[d, "\Psi"] \\ \{s\} \arrow[r, hook] & S, \end{tikzcd} \tag{2.2}$$

where Ψ and $\Psi \circ F$ are flat and i_D, i_X are closed immersions. Two deformations (F, i_D, i_X) and (F', i'_D, i'_X) of (X, D) over S are said to be *equivalent* if there exist isomorphisms $\alpha\colon \mathcal{X} \to \mathcal{X}'$ and $\beta\colon \mathcal{D} \to \mathcal{D}'$ over S which makes the following diagram commute;

$$\begin{tikzcd} D \arrow[r, hook, "i_D"] \arrow[dr, "i'_D"'] & \mathcal{D} \arrow[r] \arrow[d, "\beta"] & \mathcal{X} \arrow[d, "\alpha"] & X \arrow[l, hook', "i_X"'] \arrow[dl, "i'_X"] \\ & \mathcal{D}' \arrow[r] & \mathcal{X}'. & \end{tikzcd}$$

We define the functor $\mathrm{Def}_{(X,D)}\colon \mathrm{Art}_k \to (Sets)$ by setting

$$\mathrm{Def}_{(X,D)}(A) := \{(F, i_D, i_X) : \text{deformation of } (X, D) \text{ over } \operatorname{Spec} A\}/(\text{equiv}), \tag{2.3}$$

where (equiv) means the equivalence introduced in the above.

Next, we introduce the notion of a deformation of a pair of a variety and its effective Cartier divisors.

Definition 2.1.3. Let X be an algebraic variety and D_j for $j \in J$ a finite number of effective Cartier divisors. Set $D := \sum_{j\in J} D_j$. We can define a functor $\mathrm{Def}^J_{(X,D)}\colon \mathrm{Art}_k \to (Sets)$ by setting $\mathrm{Def}^J_{(X,D)}(A)$ to be the equivalence classes of deformations of a closed immersion $i\colon D \hookrightarrow X$ induced by deformations of each irreducible components $D_j \hookrightarrow X$ for $A \in \mathrm{Art}_k$.

We skip the script J when $D = \sum_{j\in J} D_j$ is the decomposition into irreducible components and there is no confusion.

Remark 2.1.4. If X is smooth and $D = \sum_{j\in J} D_j$ is a SNC divisor, $\mathrm{Def}^J_{(X,D)}(A)$ does not include an element which induces a smoothing of D.

Let X be a reduced algebraic scheme over an algebraically closed field k and D its effective Cartier divisor. We define the tangent space of the deformation functor Def_X and $\mathrm{Def}_{(X,D)}$ by setting

$$T^1_X := \mathrm{Def}_X(A_1),$$

$$T^1_{(X,D)} := \mathrm{Def}_{(X,D)}(A_1).$$

Moreover, we have an isomorphism

$$T^1_X \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

(cf. [56, Corollary 1.1.11])

In the study of deformations of $\mathbb{Q}$-Fano 3-folds, we often need the following lemma about the behavior of flatness under taking complements of codimension 3 closed subsets.

Proposition 2.1.5. *Let X be an algebraic scheme over k and $Z \subset X$ a closed subset such that $\operatorname{codim}_X Z \geq 3$. Assume that X is Cohen Macaulay. Let $U := X \setminus Z$ and $\mathcal{U} \to \operatorname{Spec} A$ a deformation of U over $A \in \mathrm{Art}_k$. Let $\mathcal{X} := (X, i_*\mathcal{O}_{\mathcal{U}})$ be the scheme induced from $\mathcal{U}$. Let $\mathcal{M}$ be a coherent sheaf on $\mathcal{U}$, flat over A.*

Then a coherent sheaf $i_*\mathcal{M}$ on $\mathcal{X}$ is flat over A.

Proof. This is a special case of [31, Theorem 12]. □

From this proposition, we have the following equivalence of deformation functors.

Corollary 2.1.6. *Let X be a normal variety over k which is Cohen Macaulay. Let $Z \subset X$ be a closed subset such that $\operatorname{codim}_X Z \geq 3$. Let $i\colon U := X \setminus Z \hookrightarrow X$ be an open immersion.*

(i) The restriction morphism $i^*\colon \mathrm{Def}_X \to \mathrm{Def}_U$ is an isomorphism.

(ii) Let D be an effective Cartier divisors. Then the restriction morphism $\mathrm{Def}_{(X,D)} \to \mathrm{Def}^J_{(U,D_U)}$ is an isomorphism, where $D_U := D \cap U$.

Proof. (i) Let $(\mathcal{U} \to \operatorname{Spec} A) \in \mathrm{Def}_U(A)$ be a deformation of U. By Proposition 2.1.5, we see that the sheaf $i_*\mathcal{O}_{\mathcal{U}}$ is a sheaf of flat A-algebras. Thus we can define a functor $i_*\colon \mathrm{Def}_U \to \mathrm{Def}_X$. We can check that i_* and i^* are converse to each other.

(ii) Let $(\mathcal{U}, \mathcal{D}_U) \to \operatorname{Spec} A$ be a deformation of (U, D_U) over $A \in \mathrm{Art}_k$. We see that $i_*\mathcal{O}_{\mathcal{U}}$ and $i_*\mathcal{I}_{\mathcal{D}_U}$ are sheaves of flat A-algebras by Proposition 2.1.5. Thus we can define an inverse functor $i_*\colon \mathrm{Def}_{(U,D_U)} \to \mathrm{Def}_{(X,D)}$ of i^*. □

In several parts of this thesis, we study unobstructedness of some deformation functors. Unobstructedness is defined as follows.

Definition 2.1.7. We say that deformations of X are *unobstructed* if, for all $A, A' \in \mathrm{Art}_k$ with an exact sequence

$$0 \to J \to A' \to A \to 0$$

such that $\mathfrak{m}_{A'} \cdot J = 0$, the natural restriction map of deformations

$$\mathrm{Def}_X(A') \to \mathrm{Def}_X(A)$$

is surjective, that is, Def_X is a smooth functor.

Proposition 2.1.8. *Let X be an algebraic scheme with a versal formal couple $(R, \hat{u})$ in the sense of [56, Definition 2.2.6]. Set $A_m := k[t]/(t^{m+1})$ for all integers $m \geq 0$. Assume that*

$$\mathrm{Def}_X(A_{n+1}) \to \mathrm{Def}_X(A_n)$$

are surjective for all non-negative integers $n \geq 0$.

Then deformations of X are unobstructed.

Proof. For $A \in \mathrm{Art}_k$, let $h_R(A)$ be the set of local k-algebra homomorphisms from R to A. This rule defines a functor

$$h_R \colon \mathrm{Art}_k \to (\mathrm{Sets}).$$

Since $(R, \hat{u})$ is versal, we have a smooth morphism of functors

$$\phi_{\hat{u}} \colon h_R \to \mathrm{Def}_X$$

defined by $\hat{u}$.

Then we can see that

$$h_R(A_{n+1}) \to h_R(A_n)$$

are surjective for all n by the assumption and the versality.

By [12, Lemma 5.6] and the assumption, we can see that h_R is a smooth functor. This implies that Def_X is smooth. □

2.2 Complex analytic situations

Deformation theory was first developed by Kodaira–Spencer for complex manifolds. We can define a deformation functor of a complex analytic space and a functor for a closed immersion of complex analytic spaces similarly as in Definition 2.1.1 and 2.1.2. Moreover, we can define a deformation of a germ of a complex analytic space as follows.

Definition 2.2.1. Let $(V, p), (S, 0)$ be germs of complex analytic spaces.

A *deformation* of (V, p) over $(S, 0)$ is a pair (f, i) of the following data;

- $f\colon (\mathcal{V}, p) \to (S, 0)$ is a flat morphism of germs of complex analytic spaces.
- $i\colon (V, p) \hookrightarrow (\mathcal{V}, p)$ is a closed immersion such that the induced morphism $(V, p) \to (\mathcal{V} \times_S 0, p)$ is an isomorphism of germs.

We can define equivalence between two deformations of (V, p) similarly as in Definition 2.1.1.

The following notion is an origin of infinitesimal semi-universal family.

Theorem 2.2.2. *Let X be a compact complex analytic space. Then there exists a deformation $f\colon \mathcal{X} \to S \ni 0$ of X with the following properties.*

(i) Let $f'\colon \mathcal{X}' \to S' \ni 0$ be a deformation of X. Then there exists an open neighborhood $U \subset S'$ of 0 and a holomorphic map $\varphi\colon U \to S$ such that $\varphi(0) = 0$ and two deformations $f_\varphi\colon \mathcal{X} \times_S U \to U$ and $(f')^{-1}(U) \to U$ are equivalent as deformations of X.

(ii) Let $(d\varphi)_0\colon T_{S',0} \to T_{S,0}$ be the homomorphism between the tangent spaces of S, S' induced by $\varphi\colon U \to S$. Then $(d\varphi)_0$ is uniquely determined by f'.

Definition 2.2.3. We call the base space S of the family $f\colon \mathcal{X} \to S$ in Theorem 2.2.2 as the *Kuranishi space* of X and denote it as $\mathrm{Def}(X)$. We call the family $f\colon \mathcal{X} \to S$ as the *Kuranishi family*.

Moreover, if φ is unique on the level of germs, we call $f\colon \mathcal{X} \to S$ as the *universal family*.

We also have the Kuranishi space for a germ of an isolated singularity as follows.

Theorem 2.2.4. *Let (V, p) be a germ of a complex analytic space V with an isolated singularity $p \in V$.*

Then there exists a germ $(S, 0)$ and a deformation $(\mathcal{V}, p) \to (S, 0)$ of (V, p) with the similar properties as in Theorem 2.2.2.

We call $(S, 0)$ the Kuranishi space of (V, p) and denote it by $\mathrm{Def}(V, p)$.

Example 2.2.5. Let (V,p) be a germ of an isolated singularity in $(\mathbb{C}^n, 0)$ defined by $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Let $g_1, \dots, g_\tau \in \mathcal{O}_{\mathbb{C}^n,0}$ be elements which induce a $\mathbb{C}$-basis of

$$T^1_{(V,p)} := \frac{\mathcal{O}_{\mathbb{C}^n,0}}{\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle}$$

Set $F(x_1, \dots, x_n, t_1, \dots, t_\tau) := f(x) + \sum_{i=1}^{\tau} t_j g_j(x)$. Let $(\mathcal{V}, p) := ((F = 0), 0) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, 0)$ be a family over $(\mathbb{C}^\tau, 0)$ induced by the projection $\mathbb{C}^n \times \mathbb{C}^\tau \to \mathbb{C}^\tau$. Then we have the following:

(i) $T^1_{(V,p)} \simeq \mathrm{Def}_{(V,p)}(A_1)$.

(ii) $(\mathcal{V}, p) \to (\mathbb{C}^\tau, 0)$ is the Kuranishi family of (V,p).

2.3 Terminal singularities

In the rest of this chapter, we assume that $k = \mathbb{C}$ unless otherwise stated.

We encounter several singularities in this thesis. Quotients of $\mathbb{C}^n$ or hypersurfaces $(f = 0) \subset \mathbb{C}^n$ give examples of singularities.

Example 2.3.1. Let $\mathbb{Z}_r$ be a cyclic group of order r. Consider the action of $\mathbb{Z}_r$ on $\mathbb{C}^n$ given by, for $a_1, \dots, a_n \in \mathbb{Z}$,

$$\sigma \colon (x_1, \dots, x_n) \mapsto (\zeta_r^{a_1} x_1, \dots, \zeta_r^{a_n} x_n),$$

where $\sigma \in \mathbb{Z}_r$ is a generator, $x_1, \dots, x_n$ are the coordinates on $\mathbb{C}^n$ and ζ_r is the primitive r-th root of unity. We write the quotient variety as $\mathbb{C}^n/\mathbb{Z}_r(a_1, \dots, a_n)$. If the quotient is singular, we call it a *quotient singularity*.

Consider the above $\mathbb{Z}_r$-action on $\mathbb{C}^n$. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a $\mathbb{Z}_r$-semi-invariant polynomial, that is, $\sigma \cdot f = \zeta f$ for some $\zeta \in \mathbb{C}$. Then the hypersurface $(f = 0) \subset \mathbb{C}^n$ is preserved by the $\mathbb{Z}_r$-action and we can take the quotient variety $(f = 0)/\mathbb{Z}_r$. If it has a singularity, we call it a *hyperquotient singularity* and write this $(f = 0)/\mathbb{Z}_r(a_1, \dots, a_n)$.

Definition 2.3.2. Let X be a normal variety of dimension n and $X^0 \overset{i}{\hookrightarrow} X$ the smooth locus of X. The canonical sheaf $\omega_{X^0} := \bigwedge^n \Omega^1_{X^0}$ on X^0 can be extended to a reflexive sheaf $\omega_X := i_* \omega_{X^0}$ on X. This corresponds to a divisor class K_X and we call it the *canonical divisor* of X.

Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier, that is, for some positive integer m, the divisor mK_X is Cartier. We call such X $\mathbb{Q}$-*Gorenstein*.

Definition 2.3.3. Let X be a normal $\mathbb{Q}$-Gorenstein variety and $\mu\colon Y \to X$ a good resolution of singularities, that is, a proper birational morphism such that Y is smooth and the exceptional locus $\operatorname{Exc}\mu$ is a SNC divisor. Let $\operatorname{Exc}\mu = \bigcup_{i=1}^{l} E_i$ be the decomposition into irreducible components. Then we can write

$$mK_Y = \mu^* mK_X + \sum_{i=1}^{l} ma(E_i, X)E_i$$

for some rational numbers $a(E_i, X)$ for $i = 1, \ldots, l$. We set

$$\operatorname{discrep}(X) := \inf\{a(E_i, X) \mid \mu\colon Y \to X\colon \text{resolution}\}.$$

It is known that $\operatorname{discrep}(X) \geq -1$ or $\operatorname{discrep}(X) = -\infty$ ([35, Corollary 2.31]).

We say that X has only *terminal* (resp. *canonical*) singularities if $\operatorname{discrep}(X) > 0$ (resp. ≥ 0).

In the most part of this thesis, we are interested in singularities in dimension 2 or 3. It is known that, if X is a normal variety with only terminal singularities, then X is regular in codimension 2, that is, $\operatorname{codim}_X \operatorname{Sing} X \geq 3$ (cf. [35, Corollary 5.18]). In particular, 2-dimensional terminal singularities are smooth.

We have the classification of 2-dimensional canonical singularities as follows.

Proposition 2.3.4. *Let (V, p) be a germ of a 2-dimensional canonical singularity over the complex number field $\mathbb{C}$. Then (V, p) is isomorphic to a hypersurface singularity $((f = 0), 0) \subset (\mathbb{C}^3, 0)$, where f is one of the polynomials given as follows;*

A_n: $x^2 + y^2 + z^{n+1}$ *(for $n \geq 1$)*

D_n: $x^2 + y^2 z + z^{n-1}$ *(for $n \geq 4$)*

E_6: $x^2 + y^3 + z^4$

E_7: $x^2 + y^3 + yz^3$

E_8: $x^2 + y^3 + z^5$.

These singularities are called **Du Val** *singularities.*

Remark 2.3.5. We can check that the minimal resolution $\tilde{V} \to V$ of a Du Val singularity satisfies that $\mu^* K_V = K_{\tilde{V}}$

In 3-fold case, there are plenty of non-Gorenstein terminal singularities. Quotient singularities of special type give such examples.

Example 2.3.6. Let r, a be positive integers such that $r \geq 2$ and $(r, a) = 1$. Then the quotient $U := \mathbb{C}^3/\mathbb{Z}_r(1, a, r-a)$ has a terminal singularity at the origin 0. Conversely, it is known that a 3-fold terminal quotient singularity is of this form.

Now, we study a 3-fold terminal singularity in general. Let U be a Stein neighborhood of a 3-fold terminal singularity p. We assume that $\mathcal{O}_U(mK_U) \simeq \mathcal{O}_U$ for some positive integer m. Let $r \in \mathbb{Z}$ be the minimal integer among such integers. We call r the *Gorenstein index* of the point $p \in U$.

We can construct a finite morphism

$$\pi_U \colon V := \operatorname{Spec} \bigoplus_{i=0}^{r-1} \mathcal{O}_U(iK_U) \to U,$$

where we put a k-algebra structure on $\bigoplus_{i=0}^{r-1} \mathcal{O}_U(iK_U)$ by the isomorphism $\mathcal{O}_U(-rK_U) \simeq \mathcal{O}_U$. π_U is called the index one cover of U. Using the following proposition, we see that V has only terminal Gorenstein singularity at $\pi^{-1}(p) =: \{q\}$.

Proposition 2.3.7. *The index one cover π_U satisfies the following:*

(i) π_U *is étale on* $U \setminus \{p\}$*;*

(ii) $K_V = \pi_U^* K_U$ *is a Cartier divisor;*

(iii) $\operatorname{discrep}(V) \geq \operatorname{discrep}(U)$. *In particular,* V *is terminal.*

Proof. (i),(ii) follow from [52, (3.6) Proposition].

(iii) is a special case of [35, Proposition 5.20]. □

It is known that a 3-fold terminal Gorenstein singularity is an isolated cDV singularity, that is, a hypersurface singularity $(F = 0) \subset \mathbb{A}^4$ defined by a polynomial F of the form

$$F(x, y, z, u) = f(x, y, z) + uh(x, y, z, u),$$

where f is one of the polynomials in Proposition 2.3.4 and $h \in k[x, y, z, u]$.

Thus we finally obtain the following.

Theorem 2.3.8. *([52, (3.2) Theorem]) Let (U, p) be a germ of a 3-fold terminal singularity. Then $(U, p) \simeq (V, q)/\mathbb{Z}_r$, where (V, q) is an isolated cDV singularity with a $\mathbb{Z}_r$-action which is free outside q.*

When $r > 1$, we can give the explicit description of (U, p) as follows. ([40], [36, Theorem 6.4]) The cyclic group acts on coordinates x, y, z, u of $\mathbb{C}^4$ in this order.

(i) $(U,p) \simeq ((xy + h(z, u^r) = 0)/\mathbb{Z}_r(a, -a, 0, 1), 0)$ *such that* r*: arbitrary,* $(r, a) = 1$.

(ii) $(U,p) \simeq ((x^2 + y^2 + h(z, u) = 0)/\mathbb{Z}_2(1, 0, 1, 1), 0)$ *such that* $h \in \mathfrak{m}^3$.

(iii) $(U,p) \simeq ((h(x, y, z) + u^2 = 0)/\mathbb{Z}_2(1, 1, 0, 1), 0)$ *such that* $h \in \mathfrak{m}^3$*,* xyz *or* y^2z *appears in* h *with non-zero coefficient.*

(iv) $(U,p) \simeq ((u^2+x^3+h_1(y, z)x+h_2(y, z) = 0)/\mathbb{Z}_2(0, 1, 1, 1), 0)$ *such that* $h_2 \notin \mathfrak{m}^5$.

(v) $(U,p) \simeq ((u^2 + h(x, y, z) = 0)/\mathbb{Z}_3(1, 2, 2, 0), 0)$ *such that* $h \in \mathfrak{m}^3$*, cubic terms of* F *is* $x^3 + y^3 + z^3, x^3 + yz^2$ *or* $x^3 + y^3$.

(vi) $(U,p) \simeq ((x^2 + y^2 + h(z, u^2) = 0)/\mathbb{Z}_4(1, 3, 2, 1), 0)$.

Remark 2.3.9. Consider the case $r > 1$.

In the cases (i) to (v), the $\mathbb{Z}_r$-action on $\mathbb{A}^4$ preserves the defining equation F, that is, $g \cdot F = F$ for $g \in \mathbb{Z}_r$. If this condition is satisfied, we say that (U, p) is an *ordinary* singularity.

In (vi), we have $g \cdot F = -F$ for a generator $g \in \mathbb{Z}_4$. We say that (U, p) in (vi) is an *non-ordinary* singularity.

By the classification in Theorem 2.3.8, we obtain the following properties of 3-fold terminal singularities.

Corollary 2.3.10. *Let* (U, p) *be a germ of a 3-fold terminal singularity.*

(i) There exists a deformation $\pi\colon (\mathcal{U}, p) \to (\Delta^1, 0)$ *of* (U, p) *over an open unit disk such that the general fiber* $\mathcal{U}_t := \pi^{-1}(t)$ *for* $t \neq 0$ *has only quotient singularities.*

(ii) For small U*, there exists a member* $D \in |-K_U|$ *with only a Du Val singularity at* p.

We recall the following result on rigidity of an isolated quotient singularity of dimension 3 or higher by Schlessinger.

Theorem 2.3.11. *([54]) Let* U *be a Stein neighborhood of an isolated quotient singularity* p *such that* $\dim U \geq 3$.

Then U *is infinitesimally rigid, that is, for any* $A \in \mathrm{Art}_k$ *and* $\mathcal{U} \in \mathrm{Def}_U(A)$*, we have* $\mathcal{U} \simeq U \times \operatorname{Spec} A$.

Remark 2.3.12. Let $U := (F = 0)/\mathbb{Z}_r \subset \mathbb{A}^4/\mathbb{Z}_r$ be an affine variety with an ordinary terminal singularity as one of (i)–(v) in Theorem 2.3.8. Set $\mathcal{U} := (F + t = 0)/\mathbb{Z}_r \subset \mathbb{A}^4 \times \mathbb{A}^1/\mathbb{Z}_r$, where t is the coordinate on $\mathbb{A}^1$ and $\mathbb{Z}_r$ acts trivially on $\mathbb{A}^1$. Then we have $\pi\colon \mathcal{U} \to \mathbb{A}^1$ induced by the 2nd projection $\mathbb{A}^4 \times \mathbb{A}^1 \to \mathbb{A}^1$. We see that the general fiber $\mathcal{U}_t$ has only quotient singularities. For example, if $F = xy + z^2 + u^2$ with the $\mathbb{Z}_2$-action of weights $(1,1,0,1)$, then $\mathcal{U}_t$ has two $1/2(1,1,1)$-singularities.

Let $U := (x^2 + y^2 + h(z,u^2) = 0)/\mathbb{Z}_4$ be an exceptional terminal singularity with weights $(1,3,2,1)$ as in Theorem 2.3.8 (vi). Set $\mathcal{U} := (x^2+y^2+h(z,u^2)+tz = 0)/\mathbb{Z}_4 \subset \mathbb{A}^4 \times \mathbb{A}^1/\mathbb{Z}_4$, where t is the same variable as above. Then we have $\pi\colon \mathcal{U} \to \mathbb{A}^1$ as above and $\mathcal{U}_t$ has only quotient singularities. For example, for $h(z,u^2) = z^3 + u^2$, the general fiber $\mathcal{U}_t$ has a $1/4(1,3,1)$-singularity and two $1/2(1,1,1)$-singularity.

More precisely, we have the following description of the first order deformations of a 3-fold terminal singularity.

Proposition 2.3.13. *Let $p \in U$ be a Stein neighborhood of a 3-fold terminal singularity with the Gorenstein index r. Let $\pi_U\colon V \to U$ be the index one cover and $q := \pi_U^{-1}(p)$. Then we have*

$$T^1_{(U,p)} \simeq (T^1_{(V,q)})^{\mathbb{Z}_r},$$

where the R.H.S is the invariant part with respect to the $\mathbb{Z}_r$-action on $T^1_{(V,q)}$ is induced by π_U.

Proof. We have an isomorphism $T^1_{(U,p)} \simeq H^1(U', \Theta_{U'})$, where $U' := U \setminus \{p\}$ by Corollary 2.1.6 since U' is smooth. Let $\pi_{U'}\colon V' := \pi_U^{-1}(U') \to U'$ be the restriction of π_U. We see that the $\mathbb{Z}_r$-invariant part of $(\pi_{U'})_*\Theta_{V'}$ is isomorphic to Θ'_U. Thus we see that the $\mathbb{Z}_r$-invariant part of $H^1(V', \Theta_{V'})$ is isomorphic to $H^1(U', \Theta_{U'})$. By Corollary 2.1.6, we see that $T^1_{(V,q)} \simeq H^1(V', \Theta_{V'})$. Hence we obtain isomorphisms

$$T^1_{(U,p)} \simeq H^1(U', \Theta_{U'}) \simeq (H^1(V', \Theta_{V'}))^{\mathbb{Z}_r} \simeq (T^1_{(V,q)})^{\mathbb{Z}_r}.$$

Thus we finish the proof of Proposition 2.3. $\square$

Example 2.3.14. Let $U := (x^2 + y^2 + z^2 + u^2 = 0)/\mathbb{Z}_2 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)$ be the ordinary singularity appeared in Remark 2.3.12. Then we have $T^1_U \simeq \mathbb{C}\eta_1$, where η_1 corresponds to a deformation $f_{\eta_1}\colon (x^2 + y^2 + z^2 + u^2 + t = 0)/\mathbb{Z}_2 \to \mathbb{C}$.

Let $U := (x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$ be the non-ordinary singularity appeared in Remark 2.3.12. Then we have $T^1_U \simeq \mathbb{C}\eta_z$, where η_z corresponds to a deformation $f_{\eta_z}\colon (x^2 + y^2 + z^3 + u^2 + tz = 0)/\mathbb{Z}_4 \to \mathbb{C}$.

2.4 Obstruction theory of deformation functors

Let X be an algebraic scheme. The functor Def_X is often obstructed, that is, there is some element $\xi_n \in \mathrm{Def}_X(A_n)$ which is not in the image of $\mathrm{Def}_X(A_{n+1}) \to \mathrm{Def}_X(A_n)$

as in the following example.

Example 2.4.1. For a Fano 3-fold X with canonical singularities, its deformation functor Def_X is not smooth in general. For example, let X be a cone over the del Pezzo surface S of degree 6. Then X has 2 different smoothings, one with a general fiber $\mathbb{P}(\Theta_{\mathbb{P}^2})$ and one with a general fiber $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This implies that Def_X is obstructed.

We have the following criterion for unobstructedness.

Proposition 2.4.2. *([20, Theorem 10.2]) Let X be a reduced algebraic scheme over an algebraically closed field k. Let $\mathcal{T}^1_X := \underline{\mathrm{Ext}}^1_X(\Omega^1_X, \mathcal{O}_X)$, $\mathcal{T}^2_X := \underline{Ext}^2_X(\Omega^1_X, \mathcal{O}_X)$. Assume that $H^2(X, \Theta_X) = 0, H^1(X, \mathcal{T}^1_X) = 0$ and $H^0(X, \mathcal{T}^2_X) = 0$.*

Then the functor Def_X is unobstructed.

By this criterion, we obtain the following unobstructedness.

Corollary 2.4.3. *Assume either of the following;*

(i) Let X be a projective curve with only hypersurface singularities.

(ii) Let X be a normal projective surface with only Du Val singularities such that $-K_X$ is ample.

(iii) Let X be a normal projective surface with only Du Val singularities such that $-K_X$ is trivial, that is, X is a K3 surface with Du Val singularities.

Then we have $H^2(X, \Theta_X) = H^1(X, \mathcal{T}^1_X) = H^0(X, \mathcal{T}^2_X) = 0$. In particular, the functor Def_X is unobstructed.

Proof. The conditions $H^1(X, \mathcal{T}^1_X) = H^0(X, \mathcal{T}^2_X) = 0$ follow since X has only hypersurface singularities.

In (i), we obtain $H^2(X, \Theta_X) = 0$ since $\dim X = 1$.

In (ii), we obtain $H^2(X, \Theta_X) = 0$ by the following argument as in [17]. Since $H^2(X, \Theta_X) = \mathrm{Hom}(\Theta_X, \omega_X)^*$ by the Serre duality, it is enough to show that $\mathrm{Hom}(\mathcal{O}_X(-K_X), (\Omega^1_X)^{**}) = 0$. By the Bogomolov-Sommese vanishing theorem, the Kodaira-Iitaka dimension $\kappa(X, L)$ of a rank one reflexive sheaf $L \subset (\Omega^1_X)^{**}$ satisfies $\kappa(X, L) \leq 1$. Since $\kappa(X, -K_X) = 2$, we obtain $\mathrm{Hom}(\mathcal{O}_X(-K_X), (\Omega^1_X)^{**}) = 0$ as required.

In (iii), we have

$$H^2(X, \Theta_X) \simeq \mathrm{Hom}(\Theta_X, \omega_X)^* \simeq H^0(X, (\Omega^1_X)^{**})^*$$

by the Serre duality and we obtain $H^0(X, (\Omega^1_X)^{**}) = 0$ as follows. Since X has only quotient singularities, we have the Hodge decomposition

$$H^1(X, \mathbb{C}) \simeq H^0(X, (\Omega^1_X)^{**}) \oplus H^1(X, \mathcal{O}_X).$$

Since $H^1(X, \mathcal{O}_X) = 0$, we obtain $H^0(X, (\Omega^1_X)^{**}) = 0$ by the Hodge symmetry. $\square$

2.5 Local to global obstructions of deformations

Let X be a normal proper variety. We have a spectral sequence

$$H^i(X, \mathcal{E}xt^j(\Omega^1_X, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{i+j}(\Omega^1_X, \mathcal{O}_X)$$

of Ext groups. This spectral sequence induces an exact sequence

$$\begin{aligned} 0 \to H^1(X, \Theta_X) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\pi} H^0(X, \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) \\ \to H^2(X, \Theta_X) \to \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X). \end{aligned} \quad (2.4)$$

This sequence controls the local to global behavior of deformations in the following way.

Assume that X has only isolated singularities. Then we have

$$\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \simeq \mathrm{Def}_X(A_1),$$

$$H^0(X, \mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) \simeq \bigoplus_{i=1}^{l} T^1_{(X,p_i)},$$

where we set $\mathrm{Sing}\, X = \{p_1, \ldots, p_l\}$ and (X, p_i) is a germ around p_i. Hence π can be regarded as a projection map

$$T^1_X \to \bigoplus_{i=1}^{l} T^1_{(X,p_i)}.$$

Hence, if $H^2(X, \Theta_X) = 0$, the map π is surjective and we can lift local deformations of singularities to a deformation of X. If the singularities of X have smoothings in that case and Def_X is a smooth functor, then we obtain a global smoothing of X. Here, a *smoothing* of an algebraic scheme X means a deformation of X such that the general fiber of the deformation is smooth. For example, if X is a variety as in (i), (ii) or (iii) in Corollary 2.4.3, then X has a smoothing.

However, for a Fano 3-fold with terminal Gorenstein singularities, it may happen that $H^2(X, \Theta_X) \neq 0$ (cf. [43, Example 5]). Nevertheless, Namikawa proved that they admit smoothings ([43]). Our main object in this thesis is to generalize this result to the non-Gorenstein case.

2.6 The orbifold Riemann-Roch formula and Hilbert series

The orbifold Riemann-Roch formula is useful in the computation of the graded ring of a variety with only hyperquotient singularities. We state the theorem and compute the numerical data on K3 surfaces with Du Val singularities and $\mathbb{Q}$-Fano 3-folds.

Let X be a normal projective variety and A an ample Weil divisor on X. For $n \geq 0$, let

$$H^0(X, nA) := \{f \in \mathbb{C}(X) \mid \operatorname{div} f + nA \geq 0\}$$

be the Riemann-Roch space and $R(X, A) := \oplus_{n \geq 0} H^0(X, nA)$ the graded ring associated to the pair (X, A). Since we have $X \simeq \operatorname{Proj} R(X, A)$, an explicit description of $R(X, A)$ in terms of generators and their relations gives an explicit description of an embedding of X to a certain weighted projective space.

Reid [52] gave the orbifold Riemann-Roch formula for a projective variety X with only isolated cyclic quotient singularities and a divisor D on X. The formula describes the Euler characteristic $\chi(X, \mathcal{O}_X(D))$ of (X, D) as the sum

$$\chi(X, \mathcal{O}_X(D)) = (\text{RR-type expression in } D) + \sum_Q c_Q(D),$$

where the RR expression is the usual $\operatorname{ch}(D) \cdot \operatorname{Td}_X$ and the sum $\sum_Q c_Q(D)$ is the contribution from quotient singularities on X. See [52, p.407, Corollary] for the statement.

We first give the formula for surfaces. For that purpose, we introduce the notion of baskets.

Definition 2.6.1. Let U be a Stein neighborhood of a Du Val singularity p and D a divisor on U. By [52, (9.4)], there exists a deformation $(\mathcal{U}, \mathcal{D}) \to \Delta^1$ of the pair (U, D) over a unit disc such that $\mathcal{U}_t$ has only cyclic quotient singularities $p_1, \ldots, p_l$ for $t \neq 0$. Let $U_{p_j} \subset \mathcal{U}_t$ be a Stein neighborhood of p_j for $j = 1, \ldots, l$. Let r_j, i_j for $j = 1, \ldots, l$ be the integers such that $(\mathcal{U}_t, p_j) \simeq (\mathbb{C}^2/\mathbb{Z}_{r_j}(1, -1), 0)$ and $\mathcal{D}_t = i_j A_j \in \operatorname{Cl} U_{p_j}$ for a generator $A_j \in \operatorname{Cl} U_{p_j}$ corresponding to $(x = 0)/\mathbb{Z}_{r_j} \subset \mathbb{C}^2_{x,y}/\mathbb{Z}_{r_j}(1, -1)$.

In this situation, we say that the pair (U, D) at p is of type $\{_{i_j}(\frac{1}{r_j}(1,-1))\}_{j=1}^{l}$. We call the data $\{_{i_j}(\frac{1}{r_j}(1,-1))\}_{j=1}^{l}$ the *basket* of the pair (U, D) at p.

Let a_j be a integer such that $0 < a_j < r_j$ and $(r_j, a_j) = 1$. We also write the basket as $\{_{a_j i_j}(\frac{1}{r_j}(a_j, -a_j))\}_{j=1}^{l}$.

By using this notion, we state the orbifold Riemann-Roch for surfaces with Du Val singularities.

Theorem 2.6.2. *Let X be a projective surface with only Du Val singularities and D a divisor on X. Then we have*

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \frac{D(D - K_X)}{2} + \sum_Q c_Q(D),$$

where the sum is taken over the points Q at which D is not Cartier and the term $c_Q(D)$ is written as follows;

Let $\{_{i_j}(\frac{1}{r_j}(1,-1))\}_{j=1}^{l}$ be the basket of the pair (X, D) at Q. Then we define

$$c_Q(D) := -\sum_{j=1}^{l} \frac{i_j(r_j - i_j)}{2r_j}.$$

In order to state the formula on a 3-fold with only terminal singularities, we need the following notion of baskets which can be defined similarly as in the surface case.

Definition 2.6.3. Let U be a Stein neighborhood of a 3-fold terminal singularity p and D a $\mathbb{Q}$-Cartier divisor on U. By [52, (6.4)], there exists a deformation $(\mathcal{U}, \mathcal{D}) \to \Delta^1$ of the pair (U, D) over a unit disc such that $\mathcal{U}_t$ has only cyclic quotient singularities $p_1, \ldots, p_l$. Let $U_{p_j} \subset \mathcal{U}_t$ be a Stein neighborhood of p_j for $j = 1, \ldots, l$. Let r_j, i_j for $j = 1, \ldots, l$ be the integers such that $(\mathcal{U}_t, p_j) \simeq (\mathbb{C}^3/\mathbb{Z}_{r_j}(1, a_j, -a_j), 0)$ for some integer a_j and $\mathcal{D}_t|_{U_{p_j}} = i_j K_{U_{p_j}} \in \operatorname{Cl} U_{p_j}$.

In this situation, we say that the pair (U, D) at p is of type $\{_{i_j}(\frac{1}{r_j}(1, a_j, -a_j))\}_{j=1}^{l}$. We call the data $\{_{i_j}(\frac{1}{r_j}(1, a_j, -a_j))\}_{j=1}^{l}$ (resp. $\{(\frac{1}{r_j}(1, a_j, -a_j))\}_{j=1}^{l}$) the *basket* of the pair (U, D) (resp. *basket* of U) at p.

By using this notion, we state the orbifold Riemann-Roch formula for a 3-fold.

Theorem 2.6.4. *([52, (10.2)]) Let X be a projective 3-fold with only terminal* *singularities and D a $\mathbb{Q}$-Cartier divisor on X. Then we have*

$$\chi(X,\mathcal{O}_X(D)) = \chi(X,\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}D\cdot c_2(X) + \sum_Q c_Q(D), \tag{2.5}$$

where the sum is taken over the points at which D is not Cartier. The term $c_Q(D)$ can be defined as follows;

Suppose that (X,D) is of type $\{{}_{i_j}(\frac{1}{r_j}(1,a_j,-a_j))\}_{j=1}^l$ at Q. For $1 \le j \le l$, let

$$c(i_j,r_j,a_j) := -i_j\frac{r_j^2-1}{12r_j} + \sum_{k=1}^{i_j-1}\frac{\overline{b_jk}(r_j-\overline{b_jk})}{2r_j},$$

where $0<b_j<a_j$ satisfies that $a_jb_j \equiv 1 \mod r_j$, $\overline{x}$ for $x\in\mathbb{Z}$ denotes the smallest residue mod r_j and the sum is zero when $i_j = 1$.

To describe a graded ring explicitly, the notion of the Hilbert series is useful.

Definition 2.6.5. Let $R = \oplus_{n\ge 0}R_n$ be a graded ring. Its *Hilbert series* $P(t)$ is defined by setting

$$P_n := \dim_{\mathbb{C}} R_n, \quad P(t) := \sum_{n\ge 0} P_n t^n.$$

If R is generated by finitely many homogeneous elements $x_1,\dots,x_h$ of positive degrees over $R_0 = \mathbb{C}$, the Hilbert series is a rational function of the form

$$P(t) = \frac{Q(t)}{\prod_{i=1}^h (1-t^{d_i})},$$

where $Q(t)\in\mathbb{C}[t]$ is a polynomial.

By the orbifold Riemann-Roch formula, we obtain the following description of the Hilbert series on a K3 surface with Du Val singularities.

Corollary 2.6.6. *Let S be a K3 surface with only Du Val singularities and D an ample divisor on S. Let $P_{(S,D)}(t) := \sum_{n\ge 0} h^0(S,nD)t^n$ be the Hilbert series of the pair (S,D). Then we can write*

$$P_{(S,D)}(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{(1-t)^3}\frac{D^2}{2} - \sum_{\mathcal{B}_Q}\frac{1}{1-t^r}\sum_{k=1}^{r-1}\frac{\overline{ik}(r-\overline{ik})}{2r}t^k,$$

where the sum is taken over the points Q at which D is not Cartier and $\mathcal{B}_Q = \{{}_i(1/r(1,-1))\}$ is the basket of (S,D) at Q.

Remark 2.6.7. In the setting of Corollary 2.6.6, we define the genus $g = g(S, D)$ by

$$g(S, D) := h^0(S, D) - 1.$$

The genus and the baskets $\mathcal{B}_Q$ of the singularities of (S, D) determines the Hilbert series $P_{(S,D)}(t)$. Indeed, we have

$$D^2 = 2g - 2 + \sum_{\mathcal{B}_Q} \frac{i(r-i)}{r}.$$

Remark 2.6.8. The numerical date $(g, \mathcal{B})$ for a polarized K3 surface (S, D) should satisfy the inequalities

$$g \geq -1, \quad 2g - 2 + \sum_{\mathcal{B}} \frac{i(r-i)}{r} > 0.$$

Since we have $H^2(S, \Theta_S) = 0$ by Theorem 2.4.3, we can deform S to a surface S_t with only cyclic quotient singularities with the same numerical data $(g, \mathcal{B})$. The minimal resolution $\tilde{S}_t \to S_t$ has $\sum_{\mathcal{B}}(r-1)$ exceptional -2-curves. Since the Picard number $\rho(\tilde{S}_t)$ and $\rho(S_t)$ should satisfy $\rho(\tilde{S}_t) - \rho(S_t) \leq 19$, we obtain

$$\sum_{\mathcal{B}}(r-1) \leq 19. \tag{2.6}$$

In fact, we just need a local $\mathbb{Q}$-smoothing $(\mathcal{U}, \mathcal{D}_U) \to \Delta^1$ of the pair (U, D_U) of a Stein neighborhood of a Du Val singularity and its divisor. Let $\mu\colon \tilde{U} \to U$ and $\mu_t\colon \tilde{\mathcal{U}}_t \to \mathcal{U}_t$ be the minimal resolutions and let m_0 and m_t be the number of exceptional curves of μ and μ_t respectively. We can check that $m_0 \geq m_t$ by calculating case by case with the list in [52, (4.10)]. Thus we obtain the same inequality (2.6).

Let X be a $\mathbb{Q}$-Fano 3-fold, that is, a projective 3-fold X with only terminal singularities such that $-K_X$ is ample. On a $\mathbb{Q}$-Fano 3-fold, we have the following description.

Corollary 2.6.9. *Let X be a $\mathbb{Q}$-Fano 3-fold and $R_X := \oplus_{n \geq 0} H^0(X, -nK_X)$ the anticanonical ring. The Hilbert series $P_X(t)$ of the anticanonical ring R_X can be written as follows;*

$$P_X(t) = \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{(1-t)^4}\frac{(-K_X)^3}{2} - \sum_{\mathcal{B}_Q} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r - \overline{bi})}{2r} t^i,$$

where the sum is taken over non-Gorenstein points Q with the baskets $\mathcal{B}_Q = \{1/r(1, a, -a)\}$ and $0 < b < r$ satisfies $ab \equiv 1 \mod r$.

Remark 2.6.10. In the setting of Corollary 2.6.9, we define the genus $g = g(X)$ by

$$g(X) := h^0(X, -K_X) - 2.$$

The genus and the baskets of the singularities determines the Hilbert series $P_X(t)$. Indeed, we have

$$(-K_X)^3 = 2g - 2 + \sum_{\mathcal{B}_Q} \frac{b(r-b)}{r}.$$

Let X be a $\mathbb{Q}$-Fano 3-fold with the numerical data $(g, \mathcal{B})$. We have the inequalities

$$g \geq -2, \quad 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r} > 0$$

since $h^0(X, -K_X) \geq 0$ and $(-K_X)^3 > 0$.

By [27, (2.2)], we have

$$1 = \chi(X, \mathcal{O}_X) = -\frac{1}{24} K_X \cdot c_2(X) + \sum_{\mathcal{B}} \frac{r^2 - 1}{24r}.$$

Since $-K_X \cdot c_2(X) > 0$ by [39, Theorem 6.1], we obtain

$$\sum_{\mathcal{B}} \frac{r^2 - 1}{r} < 24.$$

Now we give another constraint for the numerical data $(g, \mathcal{B})$ to be the data of an existing polarized K3 surface or $\mathbb{Q}$-Fano 3-fold.

Let (S, D) be a polarized K3 surface with Du Val singularities of type A and the numerical data $(g, \mathcal{B} = \{_1(\frac{1}{r_j}(a_j, -a_j))\}_{j=1}^{l})$. Let $f\colon T \to S$ be the minimal resolution and N the l.c.m. for all r_j. Set $Nd := f^*(ND)$ and $e_i^{(j)}$ for $1 \leq j \leq l$ and $1 \leq i \leq r_j - 1$. Then these elements satisfy the following;

$$(Nd)^2 = N^2 \left(2g - 2 + \sum_{j=1}^{l} \frac{b_j(r_j - b_j)}{r_j} \right), \quad Nd \cdot e_i^{(j)} = 0,$$

$$e_i^{(j)} \cdot e_{i'}^{(j')} = \begin{cases} -2 & (i = i', j = j') \\ \delta_{j,j'} & (|i - i'| = 1, j = j') \\ 0 & (\text{otherwise}). \end{cases}$$

More generally, for a numerical data $(g, \mathcal{B} = \{_1(1/r_j(a_j, -a_j))\})$, we can define the lattice $L(g, \mathcal{B})$ with a basis Nd and $\{e_i^{(j)} \mid 1 \leq i \leq r_j - 1, 1 \leq j \leq l\}$ with the relations as above. If the data $(g, \mathcal{B})$ comes from a polarized K3 surface, the lattice $L(g, \mathcal{B})$ should be a sublattice of the K3 lattice Λ_{K3}.

For a numerical data $(g, \mathcal{B} = \{_1(\frac{1}{r_j}(1, a_j, -a_j))\}_{j=1}^{l})$, we can define the lattice $L(g, \mathcal{B})$ by the same rule from the data $(g, \{_1(\frac{1}{r_j}(a_j, -a_j))\}_{j=1}^{l})$. If $(g, \mathcal{B})$ is a numerical data of a $\mathbb{Q}$-Fano 3-fold with a Du Val elephant, the lattice $L(g, \mathcal{B})$ should be a sublattice of Λ_{K3}. If $g \geq -1$ and $(g, \mathcal{B})$ is a numerical data of a $\mathbb{Q}$-Fano 3-fold, Conjecture 3.1.7 suggests that the lattice $L(g, \mathcal{B})$ should be a sublattice of Λ_{K3} without assuming the existence of a Du Val elephant.

Chapter 3

Deformations of Fano threefolds with terminal singularities

3.1 Introduction

All algebraic varieties in this chapter are defined over $\mathbb{C}$.

3.1.1 Background and our results

Let us begin with the definition of a $\mathbb{Q}$-Fano 3-fold.

Definition 3.1.1. Let X be a normal projective variety. We say that X is *a* $\mathbb{Q}$*-Fano* 3*-fold* if $\dim X = 3$, X has only terminal singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

$\mathbb{Q}$-Fano 3-folds are important objects in the classification of algebraic varieties. Toward the classification of $\mathbb{Q}$-Fano 3-folds, it is fundamental to study their deformations.

Definition 3.1.2. Let X be an algebraic variety and Δ^1 an open unit disc of dimension 1. A $\mathbb{Q}$*-smoothing* of X is a flat morphism of complex analytic spaces $f\colon \mathcal{X} \to \Delta^1$ such that $f^{-1}(0) \simeq X$ and $f^{-1}(t)$ has only quotient singularities of codimension at least 3.

If X is proper, we assume that f is a proper morphism.

Remark 3.1.3. Schlessinger [54] proved that an isolated quotient singularity of dimension ≥ 3 is infinitesimally rigid under small deformations.

Reid ([51], [52]) and Mori [40] showed that a 3-dimensional terminal singularity can be written as a quotient of an isolated cDV hypersurface singularity by a finite cyclic group action and it admits a $\mathbb{Q}$-smoothing.

In general, a local deformation may not lift to a global deformation. However, Altınok–Brown–Reid conjectured the following in [3, 4.8.3].

Conjecture 3.1.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then X has a $\mathbb{Q}$-smoothing.*

The following theorem is an answer to their question in the ordinary case.

Theorem 3.1.5. *Let X be a $\mathbb{Q}$-Fano 3-fold with only ordinary terminal singularities (See Definition 3.3.2). Then X has a $\mathbb{Q}$-smoothing.*

We prove a more general statement in Theorem 3.3.5 that implies Theorem 3.1.5.

Previously, Namikawa [43] proved that a Fano 3-fold with only terminal Gorenstein singularities admits a smoothing, that is, it can be deformed to a smooth Fano 3-fold. Minagawa [37] proved $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold of Fano index one, that is, it has a global index one cover. Takagi also treated some cases in [61, Theorem 2.1]. Note that the singularities on a $\mathbb{Q}$-Fano 3-fold in their cases are ordinary.

In order to prove the $\mathbb{Q}$-smoothablity, we need the following theorem on the unobstructedness of deformations of a $\mathbb{Q}$-Fano 3-fold.

Theorem 3.1.6. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then the deformations of X are unobstructed.*

Namikawa [43] proved the unobstructedness in the Gorenstein case and Minagawa [37] proved it for a $\mathbb{Q}$-Fano 3-fold of Fano index one. We show it for any $\mathbb{Q}$-Fano 3-fold. This theorem reduces the problem of finding good deformations to that of 1st order infinitesimal deformations.

As explained in Chapter 1, an elephant is important in the classification of Fano 3-folds. Altınok–Brown–Reid [3] gave the following conjecture about deformation of an elephant of a $\mathbb{Q}$-Fano 3-fold.

Conjecture 3.1.7. *Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that $|-K_X|$ contains an element D.*

1. *Then there exists a deformation $f\colon \mathcal{X} \to \Delta^1$ of X such that $|-K_{\mathcal{X}_t}|$ contains an element D_t with only Du Val singularities for general $t \in \Delta^1$.*
2. *Moreover, a divisor $D_t \subset \mathcal{X}_t$ is locally isomorphic to $\frac{1}{r}(a, r-a) \subset \frac{1}{r}(1, a, r-a)$, where both sides are corresponding cyclic quotient singularities for some coprime integers r and a around each Du Val singularities of D_t.*

We call a deformation as above a *simultaneous* $\mathbb{Q}$*-smoothing* of a pair (X, D). If we first assume the existence of a Du Val elephant, we get the following result.

Theorem 3.1.8. *Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that $|-K_X|$ contains an element D with only Du Val singularities.*

Then X has a simultaneous $\mathbb{Q}$-smoothing. In particular, X has a $\mathbb{Q}$-smoothing.

Note that we do not need the assumption of ordinary singularities as in Theorem 3.1.5. The motivation of Conjecture 3.1.7 is to treat a $\mathbb{Q}$-Fano 3-fold with only non Du Val elephants. We investigate this case in Chapter 6.

A $\mathbb{Q}$-Fano 3-fold is called *primary* if its canonical divisor generates the class group mod torsion elements. Takagi [60] studied primary $\mathbb{Q}$-Fano 3-folds with only terminal quotient singularities and established the genus bound for those with Du Val elephants. Hence Theorems 3.1.5 and 3.1.8 are useful for the classification. Actually, as an application of Theorem 3.1.8, we can reprove his bound as follows.

Corollary 3.1.9. *Let X be a primary $\mathbb{Q}$-Fano 3-fold. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

Takagi expected the existence of a Du Val elephant for X such that $h^0(X, -K_X)$ is appropriately big ([60, p.37]). If we assume the expectation, Corollary 3.1.9 implies the genus bound as above for every primary $\mathbb{Q}$-Fano 3-fold.

3.1.2 Outline of the proofs

We sketch the proof of the above theorems on a $\mathbb{Q}$-Fano 3-fold X.

First, we explain how to prove the unobstructedness briefly. If X is Gorenstein, we have

$$\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X) \simeq \operatorname{Ext}^2(\Omega^1_X \otimes \omega_X, \omega_X) \simeq H^1(X, \Omega^1_X \otimes \omega_X)^*$$

since ω_X is invertible and the unobstructedness is reduced to the Kodaira-Nakano type vanishing of the cohomology. However, if X is non-Gorenstein, that is, ω_X is not invertible, we can not reduce the vanishing of the Ext group to the vanishing of cohomology groups a priori and we do not have a direct method to prove the vanishing of the Ext group. Moreover, since we do not have a branched cover of a $\mathbb{Q}$-Fano 3-fold which is Fano or Calabi–Yau in the general case, we can not reduce the unobstructedness to that of such cover. We solve this difficulty by considering the obstruction classes rather than the ambient obstruction space Ext^2 and considering the smooth part. The important point is that deformations of X are bijective to deformations of the smooth part as in [34, 12.1.8] or [31, Theorem 12]. The description of the obstruction by a 2-term extension as in Proposition 3.2.2 is a crucial tool.

In order to find a good deformation of first order, we follow the line of the proof in the case of Fano index 1 by Minagawa [37] which used [45, Theorem 1] of Namikawa-Steenbrink on the non-vanishing of the homomorphism between cohomology groups. We need a generalization of this theorem to the non-Gorenstein setting which is Proposition 3.3.4. We can generalize this lemma provided that the singularity is ordinary. The generalization of this lemma for general terminal singularities implies Conjecture 3.1.4.

Now, in order to find a good deformation of first order under the assumption of a Du Val elephant, we use the deformation theory of the pair of X and D where $D \in |-K_X|$. The smoothness of the Kuranishi space of X implies that the smoothness of the Kuranishi space of the pair (X, D) for $D \in |-K_X|$ (Theorem 3.2.9). The important point in the proof is that an elephant contains the non-Gorenstein points of X. By this, in order to see that a deformation of X is a $\mathbb{Q}$-smoothing, it is enough to see that the singularities of D deforms non trivially. Here we adapt the diagram of [45, Theorem 1.3] to the case (X, D). Instead of the Namikawa-Steenbrink's proposition [45, Theorem 1.1] on non-vanishing of a certain cohomology map, we use the coboundary map of the local cohomology sequence for the pair. To use such a map, we arrange a resolution of singularities of the pair which has non-positive discrepancies as in Proposition 3.4.1. Moreover we refine the Lefschetz theorem for class groups by Ravindra-Srinivas [48] for our cases (Proposition 3.4.5) and this Lefschetz statement plays an important role for lifting.

3.2 Unobstructedness of deformations of a $\mathbb{Q}$-Fano 3-fold

3.2.1 Preliminaries on infinitesimal deformations

We use the following lemma about an isomorphism of some Ext groups.

Lemma 3.2.1. *Let X be an algebraic scheme over an algebraically closed field k. Let $\mathcal{X} \in \mathrm{Def}_X(A)$ be a deformation of X over $A \in \mathrm{Art}_k$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathcal{X}}$-module which is flat over A. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module which is also an $\mathcal{O}_{\mathcal{X}}$-module by the canonical surjection $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_X$. Then we have the following;*

(i) $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \simeq \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{F} \otimes_A k, \mathcal{G})$ *for all* i*, where* $\underline{\mathrm{Ext}}^i$ *is a sheaf of* Ext *groups.*

(ii) $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F} \otimes_A k, \mathcal{G})$ *for all* i*.*

Proof. (i) Let $\mathcal{E}_\bullet \to \mathcal{F} \to 0$ be a resolution of $\mathcal{F}$ by a complex $\mathcal{E}_\bullet$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules. By [19, Proposition 6.5], we see that

$$\mathcal{H}^i(\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}_\bullet, \mathcal{G})) \simeq \underline{\mathrm{Ext}}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}), \tag{3.1}$$

where $\mathcal{H}^i$ is a cohomology sheaf and $\underline{\mathrm{Hom}}$ is a sheaf of Hom groups. Since $\mathcal{F}$ is flat over A, we see that $\mathcal{E}_\bullet \otimes_A k \to \mathcal{F} \otimes_A k \to 0$ is still a resolution of the sheaf $\mathcal{F} \otimes_A k$. Hence we have

$$\mathcal{H}^i(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}_\bullet \otimes_A k, \mathcal{G})) \simeq \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{F} \otimes_A k, \mathcal{G}). \tag{3.2}$$

Note that $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}_\bullet, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}_\bullet \otimes_A k, \mathcal{G})$ since $\mathcal{G}$ is an $\mathcal{O}_X$-module. By this and isomorphisms (3.1) and (3.2), we obtain the required isomorphism in (i).

(ii) This follows from (i) and the local-to-global spectral sequence of Ext groups;

$$H^i(\mathcal{X}, \underline{\mathrm{Ext}}^j_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G})$$

□

We need the following description of the obstruction space for deformations of l.c.i. schemes.

Proposition 3.2.2. *Let* k *be an algebraically closed field of characteristic* 0*. Let* X *be a reduced scheme of finite type over* k*. Let* $U \subset X$ *be an open subset with only l.c.i. singularities and* $\iota \colon U \to X$ *an inclusion map. Assume that* $\mathrm{codim}_X X \setminus U \geq 3$ *and* $\mathrm{depth}_p X \geq 3$ *for all closed point* $p \in X$*. Let* Ω^1_U *be the Kähler differential sheaf on* U*. Set* $A_n := k[t]/(t^{n+1})$ *and let*

$$\xi_n := (f_n \colon \mathcal{X}_n \to \mathrm{Spec}\, A_n)$$

be a deformation of X*.*

Then the obstruction to lift $\mathcal{X}_n$ *over* A_{n+1} *lies in* $\mathrm{Ext}^2(\Omega^1_U, \mathcal{O}_U)$*.*

Proof. We need to define an element

$$o_{\xi_n} \in \mathrm{Ext}^2(\Omega^1_U, \mathcal{O}_U)$$

which has a property that $o_{\xi_n} = 0$ if and only if there is a deformation $\xi_{n+1} = (f_{n+1} \colon \mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$ which sits in the following cartesian diagram;

$$\begin{tikzcd} \mathcal{X}_{n+1} \arrow[d] & \mathcal{X}_n \arrow[l] \arrow[d] \\ \operatorname{Spec} A_{n+1} & \operatorname{Spec} A_n. \arrow[l] \end{tikzcd} \tag{3.3}$$

Since the characteristic of k is zero, we have

$$\Omega^1_{A_n/k} \simeq A_{n-1}$$

as A_n-modules and an exact sequence

$$0 \to (t^{n+1}) \xrightarrow{d} \Omega^1_{A_{n+1}/k} \otimes_{A_{n+1}} A_n \to \Omega^1_{A_n/k} \to 0.$$

Let $f_{\mathcal{U}_n} \colon \mathcal{U}_n \to \operatorname{Spec} A_n$ be the flat deformation of U induced by f_n. By pulling back the above sequence by the flat morphism $f_{\mathcal{U}_n}$, we get the following exact sequence;

$$0 \to \mathcal{O}_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \to f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \to 0. \tag{3.4}$$

Then, there is the relative cotangent sequence of a relative l.c.i. morphism $f_{\mathcal{U}_n}$ (cf. [56, Theorem D.2.8]);

$$0 \to f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \to \Omega^1_{\mathcal{U}_n/k} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0. \tag{3.5}$$

By combining the sequences (3.4), (3.5), we get the following exact sequence;

$$0 \to \mathcal{O}_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \to \Omega^1_{\mathcal{U}_n/k} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0. \tag{3.6}$$

Let

$$o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U) \simeq \operatorname{Ext}^2(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n}, \mathcal{O}_U)$$

be the element corresponding to the exact sequence (3.6). Note that we have the isomorphism of Ext^2 by Lemma 3.2.1.

We check that this o_{ξ_n} is the obstruction to the existence of lifting of ξ_n over A_{n+1}.

Suppose that we have a lifting $\xi_{n+1} = (f_{n+1} \colon \mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$ with the diagram (3.3). Then we can see that $o_{\xi_n} = 0$ as in [56, Proposition 2.4.8].

Conversely, suppose that $o_{\xi_n} = 0$. Consider the following exact sequence

$$\mathrm{Ext}^1(\Omega^1_{\mathcal{U}_n/k}, \mathcal{O}_U) \xrightarrow{\epsilon} \mathrm{Ext}^1(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k}, \mathcal{O}_U) \xrightarrow{\delta} \mathrm{Ext}^2(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n}, \mathcal{O}_U)$$

which is induced by the exact sequence (3.5). Consider

$$\gamma \in \mathrm{Ext}^1(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k}, \mathcal{O}_U)$$

which corresponds to the exact sequence (3.4). It is easy to see that $\delta(\gamma) = o_{\xi_n}$. Hence there exists $\gamma' \in \mathrm{Ext}^1(\Omega^1_{\mathcal{U}_n/k}, \mathcal{O}_U)$ such that $\epsilon(\gamma') = \gamma$. The class γ' corresponds to the following short exact sequence

$$0 \to \mathcal{O}_U \to \mathcal{E} \to \Omega^1_{\mathcal{U}_n/k} \to 0$$

for some coherent sheaf $\mathcal{E}$ on $\mathcal{U}_n$. We can construct a sheaf of A_n-algebras $\mathcal{O}_{\mathcal{U}_{n+1}}$ by $\mathcal{O}_{\mathcal{U}_{n+1}} := \mathcal{E} \times_{\Omega^1_{\mathcal{U}_n/k}} \mathcal{O}_{\mathcal{U}_n}$ as in [56, Theorem 1.1.10] with an exact sequence of sheaves of A_n-algebras

$$0 \to \mathcal{O}_U \to \mathcal{O}_{\mathcal{U}_{n+1}} \to \mathcal{O}_{\mathcal{U}_n} \to 0. \tag{3.7}$$

Set $\mathcal{O}_{\mathcal{X}_{n+1}} := \iota_*\mathcal{O}_{\mathcal{U}_{n+1}}$.

We need the following claim.

Claim 3.2.3. (i) $R^1\iota_*\mathcal{O}_U = 0$.

(ii) Let M be a finite A_n-module. Then

$$R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{M}) = 0,$$

where $\widetilde{M}$ is a coherent sheaf on $\operatorname{Spec} A_n$ associated to M.

Proof of Claim. (i) Let $p \in X \setminus U$ be a point and U_p a small affine neighborhood of p. Put $Z_p := U_p \cap (X \setminus U)$. It is enough to show that $H^1(U_p \setminus Z_p, \mathcal{O}_{U_p \setminus Z_p}) = 0$. We have $H^2_{Z_p}(U_p, \mathcal{O}_{U_p}) = 0$ since $\operatorname{depth}_q \mathcal{O}_{X,q} \geq 3$ for all scheme-theoretic point $q \in Z_p$ by the hypothesis. Since $H^i(U_p, \mathcal{O}_{U_p}) = 0$ for $i = 1, 2$, we have $H^1(U_p \setminus Z_p, \mathcal{O}_{U_p}) \simeq H^2_{Z_p}(U_p, \mathcal{O}_{U_p}) = 0$.

(ii) We proceed by induction on $\dim_k M$.

If $M \simeq k$, then this is the first claim.

Now assume that there is an exact sequence

$$0 \to k \to M \to M' \to 0$$

of A_n-modules and the claim holds for M'. Then we have an exact sequence

$$R^1\iota_*(f_{\mathcal{U}_n}^*\widetilde{k}) \to R^1\iota_*(f_{\mathcal{U}_n}^*\widetilde{M}) \to R^1\iota_*(f_{\mathcal{U}_n}^*\widetilde{M'})$$

and the left and right hand sides are zero by the induction hypothesis. Hence $R^1\iota_*(f_{\mathcal{U}_n}^*\widetilde{M}) = 0$. □

Note that $\iota_*\mathcal{O}_U \simeq \mathcal{O}_X, \iota_*\mathcal{O}_{\mathcal{U}_n} \simeq \mathcal{O}_{\mathcal{X}_n}$ by Claim 3.2.3. By taking ι_* of (3.7), we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathcal{X}_{n+1}} \to \mathcal{O}_{\mathcal{X}_n} \to 0 \tag{3.8}$$

since $R^1\iota_*\mathcal{O}_U = 0$.

We can regard $\mathcal{O}_{\mathcal{X}_{n+1}}$ as a sheaf of A_{n+1}-algebras by the homomorphism $A_{n+1} \to A_n$. We can see that $\mathcal{O}_{\mathcal{X}_{n+1}}$ is a sheaf of flat A_{n+1}-algebras by the local criterion of flatness (cf. [20, Proposition 2.2]) and the exact sequence (3.8). Let $\mathcal{X}_{n+1} := (X, \mathcal{O}_{\mathcal{X}_{n+1}})$ be the scheme defined by the sheaf $\mathcal{O}_{\mathcal{X}_{n+1}}$. Then the morphism $\mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1}$ is flat and

$$\xi_{n+1} := (\mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$$

is a lifting of ξ_n. □

Remark 3.2.4. The author does not know whether the above construction of obstruction classes works for general A, A' as in Definition 2.1.7. However Proposition 2.1.8 reduces the study of unobstructedness to the case $A = A_n, A' = A_{n+1}$.

3.2.2 Proof of the theorem

We need the following Lefschetz type theorem.

Theorem 3.2.5. *([15, Chapter 3.1. Theorem]) Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n and $L \subset \mathbb{P}^N$ a linear subspace of codimension $d \leq n$. Assume that $X \setminus (X \cap L)$ has only l.c.i. singularities. Then the relative homotopy group satisfies*

$$\pi_i(X, X \cap L) = 0 \qquad (i \leq n - d).$$

In particular, the restriction map $H^i(X, \mathbb{C}) \to H^i(X \cap L, \mathbb{C})$ is injective for $i \leq n - d$.

By using the obstruction class in Proposition 3.2.2, we can show the following theorem.

Theorem 3.2.6. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then deformations of X are unobstructed.*

Proof. Let U be the smooth part of X. Note that $\operatorname{codim}_X X \setminus U \geq 3$ and X is Cohen-Macaulay since X has only terminal singularities. Hence X and U satisfy the assumption of Proposition 3.2.2. Set $k := \mathbb{C}$.

Let $\xi_n \in \operatorname{Def}_X(A_n)$ be a deformation of X

$$f_n \colon \mathcal{X}_n \to \operatorname{Spec} A_n$$

and $o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$ the obstruction class defined in the proof of Proposition 3.2.2. We show that $o_{\xi_n} = 0$ in the following.

Let ω_X be the dualizing sheaf on X. By taking the tensor product of the sequence (3.6) with the relative dualizing sheaf $\omega_{\mathcal{U}_n/\operatorname{Spec} A_n}$ of $f_{\mathcal{U}_n}$, we have an exact sequence

$$\begin{aligned} 0 \to \omega_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \\ \to \Omega^1_{\mathcal{U}_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0. \quad (3.9) \end{aligned}$$

By taking ι_* of the above sequence, we get a sequence

$$\begin{aligned} 0 \to \omega_X \to \iota_*(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \\ \to \iota_*(\Omega^1_{\mathcal{U}_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \to \iota_*(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \to 0. \quad (3.10) \end{aligned}$$

This sequence is exact by the following claim.

Claim 3.2.7. (i) $R^1\iota_*\omega_U = 0$.

(ii) $R^1\iota_*(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) = 0$.

Proof of Claim. (i) Let $p \in X \setminus U$ be a singular point and U_p a small affine neighborhood at p. It is enough to show that $H^2_p(U_p, \omega_{U_p}) = 0$. Let $\pi_p \colon V_p \to U_p$ be the index 1 cover of U_p. Then we have $(\pi_p)_*\mathcal{O}_{V_p} \simeq \oplus_{i=0}^{r-1}\mathcal{O}_{U_p}(iK_{U_p})$ where r is the index of the singularity $p \in X$. Hence

$$H^2_q(V_p, \mathcal{O}_{V_p}) \simeq \bigoplus_{i=0}^{r-1} H^2_p(U_p, \mathcal{O}_{U_p}(iK_{U_p})),$$

where $q := \pi^{-1}(p)$. L.H.S. is zero by the same argument as in Claim 3.2.3 since $\operatorname{depth}_q \mathcal{O}_{V_p,q} = 3$. Hence we proved the first claim.

(ii) Let $f_{(n,p)}\colon \mathcal{U}_{(n,p)} \to \operatorname{Spec} A_n$ be the deformation of U_p induced from f_n. It is enough to show that

$$H^2_p(\mathcal{U}_{(n,p)}, f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_{(n,p)}/A_n}) = 0.$$

Set $\omega^{[i]}_{\mathcal{U}_{(n,p)}/A_n} := \iota_*\omega^{\otimes i}_{\mathcal{U}'_{(n,p)}/A_n}$, where $\mathcal{U}'_{(n,p)} := \mathcal{U}_{(n,p)} \setminus \{p\}$. We can take an index 1 cover $\phi_{(n,p)}\colon \mathcal{V}_{(n,p)} \to \mathcal{U}_{(n,p)}$ which is determined by an isomorphism $\omega^{[r_p]}_{\mathcal{U}_{(n,p)}/A_n} \simeq \mathcal{O}_{\mathcal{U}_{(n,p)}}$, where r_p is the Gorenstein index of U_p. Set $g_{(n,p)} := f_{(n,p)} \circ \phi_{(n,p)}$. Note that

$$(\phi_{(n,p)})_*(g^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k}) \simeq \bigoplus_{i=0}^{r-1} f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega^{[i]}_{\mathcal{U}_{(n,p)}/A_n}.$$

We can see that $H^2_p(\mathcal{U}_{(n,p)}, f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_{(n,p)}/A_n})$ is a direct summand of

$$H^2_q(\mathcal{V}_{(n,p)}, g^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k}) \simeq H^2_q(\mathcal{V}_{(n-1,p)}, \mathcal{O}_{\mathcal{V}_{(n-1,p)}})$$

and this is zero by Claim 3.2.3(ii).

□

Note that we have an isomorphism

$$\operatorname{Ext}^2(\iota_*(\Omega^1_{\mathcal{U}_n/A_n} \otimes \omega_{\mathcal{U}_n/A_n}), \omega_X) \simeq \operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X)$$

by Lemma 3.2.1. By using this isomorphism, we define $o'_{\xi_n} \in \operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X)$ to be the element corresponding to the sequence (3.10). Let $r_2\colon \operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X) \to \operatorname{Ext}^2(\Omega^1_U \otimes \omega_U, \omega_U)$ be the natural restriction map and $T\colon \operatorname{Ext}^2(\Omega^1_U \otimes \omega_U, \omega_U) \to \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$ be the map induced by tensoring ω_U^{-1}. Then we have

$$T(r_2(o'_{\xi_n})) = o_{\xi_n}.$$

Hence it is enough to show that $\operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X) = 0$. By the Serre duality, we have $\operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X)^* \simeq H^1(X, \iota_*(\Omega^1_U \otimes \omega_U))$, where * is the dual.

In the following, we show that

$$H^1(X, \iota_*(\Omega^1_U \otimes \omega_U)) = 0.$$

Let m be a positive integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth member D_m which is disjoint with the singular points of X. Let $\pi_m\colon Y_m := \operatorname{Spec} \oplus_{i=0}^{m-1}\mathcal{O}_X(iK_X) \to X$ be a cyclic cover determined by D_m. Note that Y_m has only terminal Gorenstein singularities.

There is the residue exact sequence

$$0 \to \Omega^1_U \to \Omega^1_U(\log D_m) \to \mathcal{O}_{D_m} \to 0$$

By tensoring this sequence with ω_U and taking the push-forward of the sheaves by ι, we obtain an exact sequence

$$0 \to \iota_*(\Omega^1_U \otimes \omega_U) \to \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U) \to \iota_*(\omega_U|_{D_m}).$$

The last homomorphism is surjective and $\iota_*(\omega_U|_{D_m}) \simeq \omega_X|_{D_m}$ since $\iota_*(\omega_U|_{D_m})$ is supported on $D_m \subset U$. Hence we obtain an exact sequence

$$0 \to \iota_*(\Omega^1_U \otimes \omega_U) \to \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U) \to \omega_X|_{D_m} \to 0 \tag{3.11}$$

It induces an exact sequence

$$H^0(X, \omega_X|_{D_m}) \to H^1(X, \iota_*(\Omega^1_U \otimes \omega_U)) \to H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U)).$$

We have $H^0(X, \omega_X|_{D_m}) = 0$ since $-K_X$ is ample. Therefore, it is enough to show that

$$H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U)) = 0.$$

Put $D' := \pi_m^{-1}(D_m)$ which satisfies that $D' \simeq D_m$ and $\pi_m^* D_m = mD'$. By using the isomorphism

$$(\pi_m)_* \left(\Omega^1_{Y_m}(\log D')(-D')\right) \simeq \bigoplus_{i=0}^{m-1} \iota_* \left(\Omega^1_U(\log D_m) \otimes \mathcal{O}_U\left((i+1)\, K_U\right)\right),$$

we can see that $H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U))$ is a direct summand of

$$H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')).$$

We can show that

$$H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')) = 0$$

as follows. There is an exact sequence

$$0 \to \Omega^1_{Y_m}(\log D')(-D') \to \Omega^1_{Y_m} \to \Omega^1_{D'} \to 0$$

and it induces an exact sequence

$$H^0(D', \Omega^1_{D'}) \to H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')) \to H^1(Y_m, \Omega^1_{Y_m}) \xrightarrow{\beta} H^1(D', \Omega^1_{D'}).$$

We can see that $H^1(D', \mathcal{O}_{D'}) = 0$ since $D_m \simeq D'$ and we have an exact sequence

$$0 \to \mathcal{O}_X(-D_m) \to \mathcal{O}_X \to \mathcal{O}_{D_m} \to 0.$$

This and the Hodge symmetry imply $H^0(D', \Omega^1_{D'}) = 0$. Hence it is enough to show that β is injective. We use the following commutative diagram

$$\begin{tikzcd}
H^1(Y_m, \Omega^1_{Y_m}) \arrow[r, "\beta"] & H^1(D', \Omega^1_{D'}) \\
H^1(Y_m, \mathcal{O}^*_{Y_m}) \otimes \mathbb{C} \arrow[r, "\gamma"] \arrow[u, "\phi"] \arrow[d, "\beta_1"] & H^1(D', \mathcal{O}^*_{D'}) \otimes \mathbb{C} \arrow[u, "\psi"'] \arrow[d, "\beta_2"] \\
H^2(Y_m, \mathbb{C}) \arrow[r, "\delta"] & H^2(D', \mathbb{C}).
\end{tikzcd}$$

We can see that δ is injective by Theorem 3.2.5 since Y_m has only l.c.i. singularities. Note that β_1 is an isomorphism since $H^i(Y_m, \mathcal{O}_{Y_m}) = 0$ for $i = 1, 2$. Hence $\delta \circ \beta_1 = \beta_2 \circ \gamma$ is injective. This implies that γ is injective. We can show that ϕ is surjective by an argument which is similar to that in [42, (2.2)]. Note that ψ is injective since D' is a smooth surface and $H^1(D', \mathcal{O}_{D'}) = 0$. Hence $\psi \circ \gamma = \beta \circ \phi$ is injective. Therefore β is injective.

Hence we proved $o_{\xi_n} = 0$. It is enough for unobstructedness by Proposition 2.1.8 since X is a projective variety and has a semi-universal deformation space. □

Remark 3.2.8. For a Fano 3-fold X with canonical singularities, its Kuranishi space $\mathrm{Def}(X)$ is not smooth in general. For example, let X be a cone over the del Pezzo surface of degree 6. Then X has 2 different smoothings.

Next, we study deformations of a $\mathbb{Q}$-Fano 3-fold with its pluri-anticanonical element.

Theorem 3.2.9. *Let X be a $\mathbb{Q}$-Fano 3-fold and m a positive integer. Assume that $|-mK_X|$ contains an element D. Let $\mathrm{Def}_{(X,D)}$ and Def_X be the deformation functors of the pair (X, D) and X respectively.*

Then the forgetful map $\mathrm{Def}_{(X,D)} \to \mathrm{Def}_X$ is a smooth morphism of functors. In particular, the deformations of the pair (X, D) are unobstructed.

Proof. Set $k := \mathbb{C}$. Let A be an Artin local k-algebra, $e = (0 \to k \to \tilde{A} \to A \to 0)$ a small extension and $\zeta := (f\colon (\mathcal{X}, \mathcal{D}) \to \operatorname{Spec} A)$ a flat deformation of the pair (X, D). Assume that we have a lifting $\tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{A}$ of $f\colon \mathcal{X} \to \operatorname{Spec} A$. It is enough to show that there exists a lifting $\tilde{\mathcal{D}} \subset \tilde{\mathcal{X}}$ of $\mathcal{D} \subset \mathcal{X}$. Let $\mathcal{N}_{D/X}$ be the normal sheaf of $D \subset X$. Since an obstruction to the existence of such a lifting lies in $H^1(D, \mathcal{N}_{D/X})$, it is enough to show that

$$H^1(D, \mathcal{N}_{D/X}) = 0.$$

There is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_{D/X} \to 0$$

and this induces an exact sequence

$$H^1(X, \mathcal{O}_X(D)) \to H^1(D, \mathcal{N}_{D/X}) \to H^2(X, \mathcal{O}_X).$$

The L.H.S and R.H.S. are zero by the Kodaira vanishing theorem. Hence we have $H^1(D, \mathcal{N}_{D/X}) = 0$. $\square$

3.3 A $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold: the ordinary case

3.3.1 Stratification on the Kuranishi space of a singularity

First, we recall a stratification on the Kuranishi space of an isolated singularity introduced in the proof of [45, Theorem 2.4].

Let V be a Stein space with an isolated hypersurface singularity $p \in V$. Then we have its semi-universal deformation space $\operatorname{Def}(V)$ and the semi-universal family $\mathcal{V} \to \operatorname{Def}(V)$. It has a stratification into Zariski locally closed and smooth subsets $S_k \subset \operatorname{Def}(V)$ for $k \geq 0$ with the following properties;

- $\operatorname{Def}(V) = \amalg_{k \geq 0} S_k$.
- S_0 is a non-empty Zariski open subset of $\operatorname{Def}(V)$ and $\mathcal{V}$ is smooth over S_0.
- S_k are of pure codimension in $\operatorname{Def}(V)$ for all $k > 0$ and $\operatorname{codim}_{\operatorname{Def}(V)} S_k < \operatorname{codim}_{\operatorname{Def}(V)} S_{k+1}$.
- If $k > l$, then $\overline{S_k} \cap S_l = \emptyset$.
- $\mathcal{V}$ has a simultaneous resolution on each S_k, that is, there is a resolution of $\mathcal{V} \times_{\operatorname{Def}(V)} S_k$ which is smooth over S_k.

3.3.2 A useful homomorphism between cohomology groups

Let us explain the homomorphism which we need for finding $\mathbb{Q}$-smoothings. For that purpose, we explain the index one cover and the ordinariness of a terminal singularity again (See Proposition 2.3.7, Remark 2.3.9).

Let $p \in U$ be a 3-dimensional Stein neighborhood of a terminal singularity p of index r, that is, r is the minimal positive integer such that rK_U is Cartier. Fix a positive integer m such that $r|m$. Let

$$\pi_U \colon V := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}(iK_U) \to U$$

be the finite morphism defined by the isomorphism $\mathcal{O}_U(rK_U) \simeq \mathcal{O}_U$. Note that V is a disjoint union of several copies of the index 1 cover of U. Let $G := \mathbb{Z}/m\mathbb{Z}$ be the Galois group of π_U. Set $Q := \pi_U^{-1}(p)$.

We consider the case $m = r$ to explain the ordinariness of a terminal singularity. In this case, V is called the *index one cover* of U. The germ (V, Q) is a germ of a terminal Gorenstein singularity and it is known that (V, Q) is a cDV singularity and that (V, Q) is a hypersurface in the germ $(\mathbb{C}^4, 0)$. Let f_V be the defining equation of (V, Q) in $(\mathbb{C}^4, 0)$. f_V is a $\mathbb{Z}_r$-semi-invariant function with respect to the action of $G = \mathbb{Z}_r$. Let $\zeta_U \in \mathbb{C}$ be the eigenvalue of the action on f_V, that is, ζ_U satisfies that $g \cdot f_V = \zeta_U f_V$, where $g \in G$ is the generator. We have the following fact by the classification of 3-dimensional terminal singularities by Reid and Mori.

Fact 3.3.1. *Let (U, p) be a germ of a 3-dimensional terminal singularity. Then ζ_U is 1 or -1.*

By this fact, we introduce the following notions on terminal singularities.

Definition 3.3.2. Let (U, p) be a germ of 3-dimensional terminal singularity. We say that (U, p) is *ordinary* (resp. *non-ordinary*) if $\zeta_U = 1$ (resp. $\zeta_U = -1$).

Now we go back to general m which is some multiple of r. Let $\nu_V \colon \tilde{V} \to V$ be a G-equivariant good resolution, $F_V := \nu_V^{-1}(Q) = \operatorname{Exc}(\nu_V)$ its exceptional locus which has normal crossing support and $\tilde{U} := \tilde{V}/G$ the quotient. So we have a diagram

$$\begin{tikzcd} \tilde{V} \arrow[r, "\tilde{\pi}_U"] \arrow[d, "\nu_V"] & \tilde{U} \arrow[d, "\mu_U"] \\ V \arrow[r, "\pi_U"] & U. \end{tikzcd} \tag{3.12}$$

Let $\mathcal{F}_U^{(0)}$ be the $\mathbb{Z}_m$-invariant part of $(\tilde{\pi}_U)_*(\Omega^2_{\tilde{V}}(\log F_V)(-F_V - \nu_V^* K_V))$. Set $V' := V \setminus Q$. We have the coboundary map of the local cohomology group

$$\tau_V \colon H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1}) \to H^2_{F_V}(\tilde{V}, \Omega^2_{\tilde{V}}(\log F_V)(-F_V - \nu_V^* K_V)).$$

This is same as the homomorphism used by Namikawa–Steenbrink [45] and Minagawa [37].

Lemma 3.3.3. *([45, Theorem 1.1], [37, Lemma 4.1]) Let V be a Stein space as above. Assume that V is not rigid. Then $\tau_V \neq 0$.*

We see that the cohomology groups appearing in τ_V are $\mathcal{O}_{V,Q}$-modules. Moreover, τ_V is an $\mathcal{O}_{V,Q}$-module homomorphism. Note that $T^1_{(V,Q)} \simeq H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1})$ is generated by one element η_V as an $\mathcal{O}_{V,Q}$-module. Actually $\eta_V \in T^1_{(V,Q)}$ corresponds to a deformation $(f_V + t = 0) \subset (\mathbb{C}^4, 0) \times \Delta^1$, where t is the coordinate on Δ^1. Hence we see that $\tau_V(\eta_V) \neq 0$.

The G-invariant part of τ_V is

$$\phi_U \colon H^1(U', \Omega^2_{U'} \otimes \omega_{U'}^{-1}) \to H^2_{E_U}(\tilde{U}, \mathcal{F}_U^{(0)}),$$

where $U' := U \setminus \{p\}$ is the punctured neighborhood and $E_U \subset \tilde{U}$ is the exceptional locus of μ_U.

If (U,p) is ordinary, we see that η_V is contained in $H^1(U', \Omega^2_{U'} \otimes \omega_{U'}^{-1}) \subset H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1})$ since η_V induces a deformation $(f_V + t = 0)/\mathbb{Z}_r \subset \mathbb{C}^4/\mathbb{Z}_r \times \Delta^1$ of the germ (U,p). Hence we obtain the following.

Lemma 3.3.4. *Let (U,p) be a germ of an ordinary terminal singularity. Then $\phi_U \neq 0$.*

3.3.3 Proof of the theorem

We can find good first order deformations as follows.

Theorem 3.3.5. *Let X be a $\mathbb{Q}$-Fano 3-fold.*

Then X has a deformation $f \colon \mathcal{X} \to \Delta^1$ over an unit disc such that the singularities on $\mathcal{X}_t$ for $t \neq 0$ satisfy the following condition;

Let $p_t \in \mathcal{X}_t$ be a singular point and U_{p_t} its Stein neighborhood. Then $\phi_{U_{p_t}} = 0$, where $\phi_{U_{p_t}}$ is the homomorphism defined in Section 3.3.2.

Remark 3.3.6. We first explain the strategy of our proof. Let $p_i \in U_i$ be a Stein neighborhood of a singularity on X. In order to find a good deformation direction, we study the restriction homomorphism $p_{U_i}\colon T^1_X \to T^1_{U_i}$. The problem is that this is not always surjective. Actually there is an example of a $\mathbb{Q}$-Fano 3-fold X such that $H^2(X, \Theta_X) \neq 0$ ([43, Example 5]). So we use the commutative diagram as in (3.15). The diagram is similar to that in the proof of [37, Theorem 4.2]. Minagawa used a cyclic cover of X branched only on singular points. We use a cyclic cover of X branched along a divisor, but the framework of the proof is almost same.

Proof. Let $p_1, \ldots, p_l \in X$ be the non-rigid singular points of X such that $p_1, \ldots, p_{l'}$ for some $l' \leq l$ are the points which satisfy

$$\phi_{U_i} \neq 0$$

for $i = 1, \ldots, l'$, where U_i is a small Stein neighborhood of p_i.

First we prepare notations to introduce the diagram (3.15). Let m be a sufficiently large integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth member D_m such that $D_m \cap \operatorname{Sing} X = \emptyset$. Let

$$\pi\colon Y := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X(iK_X) \to X$$

be the cyclic cover determined by D_m. There exists a good $\mathbb{Z}_m$-equivariant resolution ([1]) $\nu\colon \tilde{Y} \to Y$ which induces an isomorphism $\nu^{-1}(Y \setminus \pi^{-1}\{p_1, \ldots, p_l\}) \to Y \setminus \pi^{-1}\{p_1, \ldots, p_l\}$ and a birational morphism $\mu\colon \tilde{X} := \tilde{Y}/\mathbb{Z}_m \to X$. These induce the following cartesian diagram;

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{\pi}"] \arrow[d, "\nu"] & \tilde{X} \arrow[d, "\mu"] \\ Y \arrow[r, "\pi"] & X. \end{tikzcd} \tag{3.13}$$

Let $\pi_i\colon V_i := \pi^{-1}(U_i) \to U_i$ and $\nu_i\colon \tilde{V}_i := \nu^{-1}(V_i) \to V_i$ be morphisms induced by the morphisms in the above diagram. Put $\tilde{U}_i := \tilde{V}_i/\mathbb{Z}_m$. Then we get the following cartesian diagram;

$$\begin{tikzcd} \tilde{V}_i \arrow[r, "\tilde{\pi}_i"] \arrow[d, "\nu_i"] & \tilde{U}_i \arrow[d, "\mu_i"] \\ V_i \arrow[r, "\pi_i"] & U_i. \end{tikzcd} \tag{3.14}$$

Put $F := \operatorname{Exc}(\nu), E := \operatorname{Exc}(\mu), D' := \pi^{-1}(D_m)$ and $L' := \mathcal{O}_Y(D') = \mathcal{O}_Y(\pi^*(-K_X))$. Note that F has normal crossing support since ν is good. Also put $F_i := \operatorname{Exc}(\nu_i)$ and $E_i := \operatorname{Exc}(\mu_i)$. Let $\mathcal{F}^{(0)}$ be the $\mathbb{Z}_m$-invariant part of

$\tilde{\pi}_*(\Omega^2_{\tilde{Y}}(\log F)(-F) \otimes \nu^* L')$. Let U be the smooth part of X. Note that $\mathcal{F}^{(0)}|_U \simeq \Omega^2_U \otimes \omega_U^{-1}$. Set $\mathcal{F}_i^{(0)} := \mathcal{F}^{(0)}|_{\tilde{U}_i}$ and $U_i' := U_i \setminus \{p_i\}$. Note that $\mathcal{F}_i^{(0)}|_{U_i'} \simeq \Omega^2_{U_i'} \otimes \omega^{-1}_{U_i'}$.

We have the following commutative diagram;

$$\begin{CD} H^1(U, \Omega^2_U \otimes \omega_U^{-1}) @>{\oplus\psi_i}>> \oplus_{i=1}^{l'} H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)}) @>>> H^2(\tilde{X}, \mathcal{F}^{(0)}) \\ @V{\oplus p_{U_i}}VV @V{\simeq}VV \\ \oplus_{i=1}^{l'} H^1(U_i', \Omega^2_{U_i'} \otimes \omega^{-1}_{U_i'}) @>{\oplus\phi_i}>> \oplus_{i=1}^{l'} H^2_{E_i}(\tilde{U}_i, \mathcal{F}_i^{(0)}). \end{CD} \tag{3.15}$$

We identify $H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)})$ and $H^2_{E_i}(\tilde{U}_i, \mathcal{F}_i^{(0)})$ by the natural homomorphism induced by restriction. Note that $\mathcal{F}_i^{(0)} \simeq \mathcal{F}_{U_i}^{(0)}$, where $\mathcal{F}_{U_i}^{(0)}$ is the sheaf defined in Section 3.3.2. Hence ϕ_i is ϕ_{U_i} in Section 3.3.2.

Next we see that p_{U_i} in the diagram (3.15) is the restriction homomorphism of T^1 as follows. Let $T^1_X, T^1_{V_i}, T^1_{U_i}$ be the tangent spaces of the functors $\mathrm{Def}_X, \mathrm{Def}_{V_i}, \mathrm{Def}_{U_i}$ respectively. By [54, §1 Theorem 2] or the proof of Proposition 3.2.2 in this paper, we can see that the first order deformations of V_i, U_i are bijective to those of the smooth part V_i', U_i'. Similarly we can see the same correspondence for X. So we have

$$T^1_X \simeq H^1(U, \Theta_U) \simeq H^1(U, \Omega^2_U \otimes \omega_U^{-1}),$$

$$T^1_{V_i} \simeq H^1(V_i', \Theta_{V_i'}) \simeq H^1(V_i', \Omega^2_{V_i'} \otimes \omega^{-1}_{V_i'}),$$
$$T^1_{U_i} \simeq H^1(U_i', \Theta_{U_i'}) \simeq H^1(U_i', \Omega^2_{U_i'} \otimes \omega^{-1}_{U_i'}),$$

where $\Theta_U, \Theta_{V_i'}, \Theta_{U_i'}$ are the tangent sheaves of U, V_i', U_i' respectively. Hence p_{U_i} is regarded as the restriction homomorphism $T^1_X \to T^1_{U_i}$.

We want to lift $\eta_i \in H^1(U_i', \Omega^2_{U_i'} \otimes \omega^{-1}_{U_i'}) \simeq T^1_{U_i}$ which induces a non-trivial deformation of U_i to an element of $H^1(U, \Omega^2_U \otimes \omega_U^{-1}) \simeq T^1_X$. In order to do that, we consider $\phi_i(\eta_i) \in H^2_{E_i}(\tilde{U}_i, \mathcal{F}_i^{(0)})$ and lift it by using the diagram (3.15).

Since $\tilde{\pi}$ is finite, $H^2(\tilde{X}, \mathcal{F}^{(0)})$ is a direct summand of

$$H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log F)(-F) \otimes \nu^* L')$$

and this is zero by the vanishing theorem by Guillen-Navarro Aznar-Puerta-Steenbrink ([46] Theorem 7.30 (a)). Hence $\oplus\psi_i$ is surjective.

By the assumption that $\phi_i \neq 0$ for $i = 1, \ldots, l'$, there exists $\eta_i \in H^1(U_i', \Omega^2_{U_i'} \otimes \omega^{-1}_{U_i'}) \setminus \operatorname{Ker} \phi_i$. By the surjectivity of $\oplus\psi_i$, there exists $\eta \in H^1(U, \Omega^2_U \otimes \omega_U^{-1})$ such that $\psi_i(\eta) = \phi_i(\eta_i)$. Then we have

$$p_{U_i}(\eta) \notin \mathrm{Ker}(\phi_i). \tag{3.16}$$

We want to see that $p_{U_i}(\eta)$ induces a non-trivial deformation of a singularity $p_i \in U_i$. For that purpose, we study the deformation of V_i induced by $p_{U_i}(\eta)$ and see that it does not come from a deformation of the resolution of V_i.

Since V_i has only rational singularities, the birational morphism $\nu_i \colon \tilde{V}_i \to V_i$ induces a morphism of the functors $\mathrm{Def}_{\tilde{V}_i} \to \mathrm{Def}_{V_i}$ ([63, Theorem 1.4 (c)]) and the homomorphism $H^1(\tilde{V}_i, \Theta_{\tilde{V}_i}) \to H^1(V_i', \Theta_{V_i'})$ on their tangent spaces. This homomorphism can be rewritten as

$$(\nu_i)_* \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \to H^1(V_i', \Omega^2_{V_i'} \otimes \omega^{-1}_{V_i'})$$

and this is a homomorphism induced by an open immersion $V_i' \hookrightarrow \tilde{V}_i$. Note that infinitesimal deformations of U_i come from $\mathbb{Z}_m$-equivariant deformations of V_i and $H^1(U_i', \Theta_{U_i'}) \simeq H^1(V_i', \Theta_{V_i'})^{\mathbb{Z}_m}$.

Note that ϕ_i is the $\mathbb{Z}_m$-invariant part of the homomorphism

$$\tau_i \colon H^1(V_i', \Omega^2_{V_i'} \otimes \omega^{-1}_{V_i'}) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})).$$

Claim 3.3.7. $\mathrm{Im}(\nu_i)_* \subset \mathrm{Ker}\,\tau_i$.

Proof of Claim. We can write

$$K_{\tilde{V}_i} = \nu_i^* K_{V_i} + \sum_{j=1}^{m_i} a_{i,j} F_{i,j},$$

where $F_i = \bigcup_{j=1}^{m_i} F_{i,j}$ is the irreducible decomposition and $a_{i,j} \geq 1$ are some integers for $j = 1, \ldots, m_i$ since V_i is terminal Gorenstein. We can define a homomorphism

$$\alpha_i \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i}))$$

as a composite of the following homomorphisms;

$$\alpha_i\colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) = H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(-\sum_{j=1}^{m_i} a_{i,j}F_{i,j} - \nu_i^* K_{V_i}))$$
$$\to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-\sum_{j=1}^{m_i} a_{i,j}F_{i,j} - \nu_i^* K_{V_i})) \to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \tag{3.17}$$

since $a_{i,j} \geq 1$.

Note that $\operatorname{Ker} \tau_i = \operatorname{Im} \rho_i$, where we put

$$\rho_i\colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \to H^1(V_i', \Omega^2_{V_i'} \otimes \omega^{-1}_{V_i'}).$$

We can see that $(\nu_i)_*$ factors as

$$(\nu_i)_*\colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \overset{\alpha_i}{\to} H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \overset{\rho_i}{\to} H^1(V_i', \Omega^2_{V_i'} \otimes \omega^{-1}_{V_i'}).$$

Hence $\operatorname{Ker} \tau_i = \operatorname{Im} \rho_i \supset \operatorname{Im}(\nu_i)_*$. □

By Claim 3.3.7 and the relation (3.16), we get $p_{U_i}(\eta) \notin \operatorname{Im}(\nu_i)_*$. This means that a deformation of V_i induced by $p_{U_i}(\eta)$ does not come from that of the resolution $\tilde{V}_i$. In the following, we check that the deformation of V_i goes out from the minimal stratum of the stratification on the Kuranishi space $\operatorname{Def}(V_i)$ introduced in Section 3.3.1.

Let r_i be the Gorenstein index of the singular point p_i and $\pi_i^{-1}(p_i) =: \{q_{i1}, \ldots, q_{ik(i)}\}$, where $k(i) := \frac{m}{r_i}$. Let

$$V_i := \amalg_{j=1}^{k(i)} V_{i,j}$$

be the decomposition into the connected components of V_i. Fix a stratification on each $\operatorname{Def}(V_{i,j})$ for $j = 1, \ldots, k(i)$ as in Section 3.3.1. We see that $p_{U_i}(\eta) \in T^1_{U_i} \subset T^1_{V_{i,1}}$ induces a deformation $g_{i,1}\colon \mathcal{V}_{i,1} \to \Delta^1$. By the property of the Kuranishi space, there exists a holomorphic map $\varphi_{i,1}\colon \Delta^1 \to \operatorname{Def}(V_{i,1})$ which induces the above deformation of $V_{i,1}$. Let $S_{i,k}$ be the minimal stratum of $\operatorname{Def}(V_{i,1})$. Then the image of $\varphi_{i,1}$ is not contained in $S_{i,k}$. and, for general $t \in \Delta^1$, we have $\varphi_{i,1}(t) \in S_{i,k'}$ for some $k' < k$. Let $g\colon \mathcal{X} \to \Delta^1$ be a small deformation of X over a disc induced by $\eta \in H^1(U, \Theta_U)$. Then g induces a deformation of $V_{i,1}$ We can continue this process as long as $\phi_i \neq 0$ and reach a deformation of X whose general fiber has the required condition in the statement of Theorem 3.3.5. □

Lemma 3.3.4 and Theorem 3.3.5 imply the following.

Corollary 3.3.8. *Let X be a $\mathbb{Q}$-Fano 3-fold with only ordinary terminal singularities. Then X has a $\mathbb{Q}$-smoothing.*

Proof. By Lemma 3.3.4, we can continue the process in the proof of Theorem 3.3.5 until we get a $\mathbb{Q}$-smoothing since deformations of ordinary terminal singularities are ordinary. □

Remark 3.3.9. Let (U, p) be a germ of a non-ordinary 3-dimensional terminal singularity. We determine when the coboundary map ϕ_U vanishes in Chapter 5.

3.3.4 Non-smoothable examples

The smoothing problem is delicate. We exhibit an example of a non-smoothable weak Fano 3-fold with terminal Gorenstein singularities. Thus the Fano assumption is necessary in Corollary 3.3.8.

First, we treat a proposition necessary for constructing non-smoothable varieties.

Example 3.3.10. ([44]) Let $p \in X$ be a Stein space with an isolated rational Gorenstein singularity of dimension 3 and $\pi \colon Y \to X$ a crepant partial resolution, that is, π is a proper birational morphism such that Y is normal and $K_Y = \pi^* K_X$. Let $E \subset Y$ be the exceptional locus of π and $\Sigma \subset Y$ the singular locus of Y. By [4], Y has a finite dimensional semi-universal deformation space $\mathrm{Def}(Y)$. The following is an important proposition for checking smoothability.

Proposition 3.3.11. *([44, Proposition 1.3]) Assume that Y has only ordinary double points $p_1, \ldots, p_n$ as singularities. Let $\nu \colon Z \to Y$ be a small resolution of ordinary double points and set $C_i := \nu^{-1}(p_i)$. Let $E_1, \ldots, E_m$ be the irreducible components of $\nu^{-1}(E)$. Let r be the rank of the matrix $(C_i \cdot E_j)_{1 \le i \le n, 1 \le j \le m}$*

Then $\mathrm{Def}(Z)$ and $\mathrm{Def}(Y)$ are smooth and we have

$$\dim \mathrm{Def}(Y) = \dim \mathrm{Def}(Z) + n - r.$$

Moreover, the following are equivalent.

(a) Y has a smoothing.

(b) There exist nonzero rational numbers $\alpha_1, \ldots, \alpha_n$ such that $(\sum_{i=1}^n \alpha_i C_i \cdot E_j) = 0$ for all j.

We use this proposition in the next example.

Example 3.3.12. There is an example of a weak Fano 3-fold with an ordinary double point which does not have a smoothing. It is written in [38, Example 3.7]. We now explain it for the convenience of the reader.

Let $\bar{X}$ be the cone over the del Pezzo surface $S = \mathbb{F}_1$ of degree 8. Then $\bar{X}$ is a Fano 3-fold with an isolated rational Gorenstein singularity $\bar{p} \in \bar{X}$. Let $f\colon Z \to X$ be the blow-up at $\bar{p}$. Then f is a crepant resolution and $Z \simeq \mathrm{Proj}_S(\mathcal{O}_S \oplus \omega_S^{-1})$. The exceptional divisor E of f is isomorphic to $\mathbb{F}_1$. Let $C \subset E$ be the -1-curve on $\mathbb{F}_1$ and $\mu\colon \mathbb{F}_1 \to \mathbb{P}^2$ the contraction of C. Then Z is a weak Fano 3-fold with a $(-1,-1)$-curve C, that is, the normal bundle $\mathcal{N}_{C/Z} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Let $\nu\colon Z \to X$ be a birational contraction of C, that is, a projective birational morphism to a projective variety X. We can construct this by a base point free divisor $E + \pi^*(-K_S) + \pi^*\mu^*\mathcal{O}_{\mathbb{P}^2}(1)$, where $\pi\colon Z \simeq \mathrm{Proj}_S(\mathcal{O}_S \oplus \omega_S^{-1}) \to S$ is the projection. We can check that, for an irreducible curve $\Gamma \subset Z$, we have $D_X \cdot \Gamma = 0$ if and only if $\Gamma = C$.

Then X is a weak Fano 3-fold with an ordinary double point $p = \nu(C)$. The divisor $F := \nu(E)$ is isomorphic to $\mathbb{P}^2$ and passes through $p \in X$. We see that F is not $\mathbb{Q}$-Cartier since the local analytic class group at $p \in X$ is torsion free and a Cartier divisor through a singular point is not smooth. Hence X is not $\mathbb{Q}$-factorial.

Let V be a Stein neighborhood of $\bar{p}$ and $g\colon X \to \bar{X}$ an induced birational morphism. We can apply Proposition 3.3.11 to the birational morphism $g^{-1}(V) \to V$. Hence $g^{-1}(V) \subset X$ is not smoothable. This implies that X is not smoothable.

3.4 A $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold with a Du Val elephant

In this section, we prove the $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold with a Du Val elephant.

3.4.1 Existence of an essential resolution of a pair

We need a resolution of a 3-dimensional hypersurface singularity and its divisor with good properties as follows.

Proposition 3.4.1. *Let Y be a 3-dimensional variety with only hypersurface singularities and D a Cartier divisor on Y with only Du Val singularities. Assume that a finite group G acts on Y and the action preserves D.*

Then there exists a G-equivariant resolution of singularities $f\colon \tilde{Y} \to Y$ of Y with the following properties;

(i) The strict transform $\tilde{D} \subset \tilde{Y}$ of D is smooth,

(ii) We have $K_{\tilde{D}} = f_D^* K_D$, where $f_D\colon \tilde{D} \to D$ is the morphism induced by f.

Proof. Let $f_D\colon \tilde{D} \to D$ be the minimal resolution of D which is a composition of the blow-ups at smooth points. Let $f_1\colon \tilde{Y}_1 \to Y$ be a composition of the blow-ups at the same smooth points as f_D. We can assume that f and f_D are G-equivariant since we can take G-invariant centers of the blow-ups for f_D.

Next, we can take a G-equivariant resolution $f_2\colon \tilde{Y} \to \tilde{Y}_1$ such that f_2 is isomorphism on $\tilde{Y}_1 \setminus \operatorname{Sing} \tilde{Y}_1$. Note that f_2 induces an isomorphism on $\tilde{D}$. We see that the composition $f := f_1 \circ f_2 \colon \tilde{Y} \to Y$ satisfies the required condition. Thus we finish the proof. □

3.4.2 Classification of 3-dimensional terminal singularities

Let $(p \in U)$ be a germ of a 3-dimensional terminal singularity. By Reid's result [52], (U, p) is locally isomorphic to

$$0 \in (f = 0)/\mathbb{Z}_r \subset \mathbb{C}^4/\mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $\mathbb{C}^4$ diagonally and $f \in \mathbb{C}[x, y, z, u]$ and x, y, z, u are $\mathbb{Z}_r$-semi-invariant functions on $\mathbb{C}^4$. By the list in [52](6.4), we have a $\mathbb{Z}_r$-semi-invariant function $h \in \mathbb{C}[x, y, z, u]$ such that

$$D_h := (f = h = 0)/\mathbb{Z}_r \subset (f = 0)/\mathbb{Z}_r =: U_f$$

has only a Du Val singularity at the origin and $D_h \in |-K_{U_f}|$.

3.4.3 Some ingredients for the proof

Let X be an algebraic scheme and D its closed subscheme. For the functor $\operatorname{Def}_{(X,D)}\colon \operatorname{Art}_k \to (Sets)$, let $T^1_{(X,D)} := \operatorname{Def}_{(X,D)}(A_1)$ be the tangent space.

We use the following fact that deformations of a pair of a variety and its divisor.

Lemma 3.4.2. *Let X be a 3-dimensional variety with only terminal singularities and D a $\mathbb{Q}$-Cartier divisor on X. Let $Z \subset X$ be a 0-dimensional subset. Let $\iota\colon U := X \setminus Z \hookrightarrow X$ be an open immersion. Set $D_U := D \cap U$.*

Then the restriction homomorphism $\iota^*\colon T^1_{(X,D)} \to T^1_{(U,D_U)}$ is an isomorphism.

Proof. We can construct the inverse $\iota_*\colon T^1_{(U,D_U)} \to T^1_{(X,D)}$ of ι^* as follows. $\xi \in T^1_{(U,D_U)}$ corresponds to a deformation $U_1 \to \operatorname{Spec} A_1$ and an A_1-flat ideal sheaf $\mathcal{I}_{D_{U_1}}$. We see that $\mathcal{O}_{X_1} := \iota_*\mathcal{O}_{U_1}$ is a sheaf of A_1-flat algebras by a similar argument as in the proof of Proposition 3.2.2. Moreover, we see that $\mathcal{I}_{D_1} := \iota_*\mathcal{I}_{D_{U_1}}$ is an A_1-flat ideal sheaf. Indeed there is an exact sequence $0 \to \mathcal{I}_{D_U} \to \mathcal{I}_{D_{U_1}} \to \mathcal{I}_{D_U} \to 0$ and, by taking its push-forward by ι, we obtain an exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{I}_{D_1} \to \mathcal{I}_D \to 0. \tag{3.18}$$

The surjectivity in (3.18) follows from $R^1\iota_*\mathcal{I}_{D_U} = 0$. We can show that $R^1\iota_*\mathcal{I}_{D_U} = 0$ similarly as Claim 3.2.7 since $\mathcal{I}_D$ can be written locally as an eigenspace of some invertible sheaf with respect to the group action induced by the index one cover. By the sequence (3.18), we see that $\mathcal{I}_{D_1}$ is flat over A_1. Thus $(\mathcal{O}_{X_1}, \mathcal{I}_{D_1})$ defines an element $\iota_*(\eta) \in T^1_{(X,D)}$ and this determines ι_*. □

Let $p \in U$ be a Stein neighborhood of a 3-dimensional terminal singularity p with the Gorenstein index r. By the classification of 3-dimensional terminal singularities, there exists $D \in |-K_U|$ with only Du Val singularity at p. Let m be a positive multiple of r and $\pi_U\colon V \to U$ the $\mathbb{Z}_m$-cyclic cover of U determined by the isomorphism $\mathcal{O}_U(rK_U) \simeq \mathcal{O}_U$ as in Section 3.3.2. Set $\Delta := \pi_U^{-1}(D)$. Then V has terminal Gorenstein singularities at $Q := \pi^{-1}(p)$ and Δ has Du Val singularities at Q. Let $\nu\colon \tilde{V} \to V$ be the $\mathbb{Z}_m$-equivariant resolution of singularities of (V, Δ) constructed in Proposition 3.4.1. Let $\tilde{\Delta} := \nu_*^{-1}(\Delta) \subset \tilde{V}$ be the strict transform of Δ and F the exceptional divisor of ν. Then we have the coboundary map

$$\tau_{(V,\Delta)}\colon H^1(V', \Omega^2_{V'}(\log \Delta')) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log \tilde{\Delta})), \tag{3.19}$$

where $V' := V \setminus Q$ and $\Delta' := V' \cap \Delta$. By Lemma 3.4.2, we see that

$$T^1_{(V,\Delta)} \simeq T^1_{(V',\Delta')} \simeq H^1(V', \Theta_{V'}(-\log \Delta')) \simeq H^1(V', \Omega^2_{V'}(\log \Delta')(-K_{V'} - \Delta')).$$

By fixing a $\mathbb{Z}_m$-equivariant isomorphism $\mathcal{O}_V \simeq \mathcal{O}_V(-K_V - \Delta)$, we finally obtain an isomorphism

$$T^1_{(V,\Delta)} \simeq H^1(V', \Omega^2_{V'}(\log \Delta')).$$

This isomorphism is $\mathbb{Z}_m$-equivariant and the $\mathbb{Z}_m$-invariant parts are

$$T^1_{(U,D)} \simeq H^1(U', \Omega^2_{U'}(\log D')).$$

For deformations of Δ, we have the following.

Lemma 3.4.3. *Let $\iota_\Delta \colon \Delta' \hookrightarrow \Delta$ be the open immersion. Then the restriction homomorphism $\iota_\Delta^* \colon T^1_\Delta \to T^1_{\Delta'}$ is injective.*

Proof. For $\Delta_1 \in \mathrm{Def}_\Delta(A_1)$, we have $(\iota_\Delta)_* \iota_\Delta^* \mathcal{O}_{\Delta_1} \simeq \mathcal{O}_{\Delta_1}$ since Δ is S_2. $\square$

We have the following commutative diagram;

$$\begin{tikzcd}
T^1_{(V,\Delta)} \arrow[r] \arrow[d, "p_\Delta"] & T^1_{(V',\Delta')} \arrow[r] \arrow[d, "p_{\Delta'}"] & H^1(V', \Omega^2_{V'}(\log \Delta')) \arrow[d, "P_{\Delta'}"] \\
T^1_\Delta \arrow[r] & T^1_{\Delta'} \arrow[r] & H^1(\Delta', \Omega^1_{\Delta'}),
\end{tikzcd}$$

where $P_{\Delta'}$ is induced by the residue homomorphism. This implies that the elements of $\mathrm{Im}\, P_{\Delta'}$ is coming from elements of T^1_Δ. We also have the following diagram;

$$\begin{tikzcd}
H^1(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) \arrow[r, "R_\Delta"] & H^1(\Delta', \Omega^1_{\Delta'}) \\
T^1_{\tilde{\Delta}} \arrow[u, "\simeq"] \arrow[r] \arrow[dr, "(\nu_\Delta)_*"'] & T^1_{\Delta'} \arrow[u, "\simeq"'] \\
 & T^1_\Delta \arrow[u, "\iota^*_\Delta"']
\end{tikzcd}$$

The vertical isomorphisms are induced by the isomorphism $\mathcal{O}_\Delta(K_\Delta) \simeq \mathcal{O}_\Delta$ since $\nu_\Delta^* K_\Delta = K_{\tilde{\Delta}}$. The homomorphism $(\nu_\Delta)_*$ is the blow-down morphism by Wahl ([63]). It is well known that $(\nu_\Delta)_* = 0$ since Δ has a Du Val singularity (cf. [8, 2.10]). Hence we see that $R_\Delta = 0$ as well.

We have the following lemma.

Lemma 3.4.4. *Let $R_\Delta \colon H^1(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) \to H^1(\Delta', \Omega^1_{\Delta'})$ be the restriction homomorphism as above.*

Then we have $P_{\Delta'}(\mathrm{Ker}\, \tau_{(V,\Delta)}) \subset \mathrm{Im}\, R_\Delta = 0$. In particular, this implies that $\tau_{(V,\Delta)}(\eta) \neq 0$ for $\eta \in H^1(V', \Omega^2_{V'}(\log \Delta')) \simeq T^1_{(V,\Delta)}$ which induces a smoothing of Δ.

Proof. We have a diagram

$$\begin{tikzcd}
H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log \tilde{\Delta})) \arrow[r, "\alpha_{(V,\Delta)}"] \arrow[d] & H^1(V', \Omega^1_{V'}(\log \Delta')) \arrow[d] \\
H^1(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) \arrow[r, "R_\Delta"] & H^1(\Delta', \Omega^1_{\Delta'})
\end{tikzcd}$$

The vertical homomorphisms are induced by the residue homomorphisms and the horizontal homomorphisms are induced by open immersions. Hence the diagram is commutative. Since $\operatorname{Ker} \tau_{(V,\Delta)} = \operatorname{Im} \alpha_{(V,\Delta)}$, we obtain the claim by the diagram. $\square$

We also need the following Lefschetz type statement.

Proposition 3.4.5. *Let Y be a normal projective 3-fold with only isolated singularities and $\Delta \subset Y$ its ample Cartier divisor with only isolated singularities. Assume that $H^1(Y, \mathcal{O}_Y) = 0$. Let $\nu \colon \tilde{Y} \to Y$ be a resolution of singularities of the pair (Y, Δ) which is isomorphism on $Y \setminus (\operatorname{Sing} Y \cup \operatorname{Sing} \Delta)$ such that the strict transform $\tilde{\Delta}$ of Δ is smooth. Let $r_{\tilde{\Delta}} \colon \operatorname{Pic} \tilde{Y} \to \operatorname{Pic} \tilde{\Delta}$ be the restriction homomorphism.*

Then $\operatorname{Ker} r_{\tilde{\Delta}}$ is generated by ν-exceptional divisors.

Proof. It is enough to show that

$$r_{\Delta} \colon \operatorname{Cl} Y \to \operatorname{Cl} \Delta$$

is injective. Indeed we have a commutative diagram

$$\begin{tikzcd} \operatorname{Cl} \tilde{Y} \arrow[r, "r_{\tilde{\Delta}}"] \arrow[d, "\nu_*"'] & \operatorname{Cl} \tilde{\Delta} \arrow[d, "(\nu_{\Delta})_*"] \\ \operatorname{Cl} Y \arrow[r, "r_{\Delta}"] & \operatorname{Cl} \Delta \end{tikzcd}$$

and, if r_{Δ} is injective, can see that

$$\operatorname{Ker} r_{\tilde{\Delta}} \subset \operatorname{Ker} (\nu_{\Delta})_* \circ r_{\tilde{\Delta}} = \operatorname{Ker} r_{\Delta} \circ \nu_* = \operatorname{Ker} \nu_*$$

and $\operatorname{Ker} \nu_*$ is generated by ν-exceptional divisors.

Let m be a sufficiently large integer such that $m\Delta$ is very ample. By [49, Theorem 1], there exists a very general smooth element $\Delta_m \in |m\Delta|$ which is disjoint with $\operatorname{Sing} \Delta$ and

$$r_{\Delta_m} \colon \operatorname{Cl} Y \to \operatorname{Cl} \Delta_m$$

is an isomorphism. Take $A \in \operatorname{Ker} r_{\Delta}$. Then we have $A \cdot \Delta = 0$ as a rational equivalence class of a cycle on Y. Then we have

$$A \cdot \Delta_m = 0$$

as a rational equivalence class on Y.

We show that $A|_{\Delta_m} = 0 \in \operatorname{Cl} \Delta_m$ as follows. It is enough to show that $A|_{\Delta_m}$ is numerically trivial on Δ_m since $H^1(\Delta_m, \mathcal{O}_{\Delta_m}) = 0$. Let $\Gamma \in \operatorname{Cl} \Delta_m$ be any element. Since r_{Δ_m} is an isomorphism, there exists $F \in \mathrm{Cl}\, Y$ such that $F|_{\Delta_m} = \Gamma$. We have

$$A|_{\Delta_m} \cdot \Gamma = (A \cdot \Delta_m) \cdot F = 0$$

by the intersection theory. Indeed $A \cdot \Delta_m$ is a sum of several curves which are regularly immersed since $\Delta_m \cap \mathrm{Sing}\, Y = \emptyset$. Hence $A|_{\Delta_m} = 0 \in \mathrm{Cl}\, \Delta_m$ and we get $A = 0 \in \mathrm{Cl}\, Y$ since $\mathrm{Cl}\, Y \overset{\simeq}{\to} \mathrm{Cl}\, \Delta_m$. Thus we get r_Δ is injective and we finish the proof. $\square$

3.4.4 Proof of the theorem

Our strategy of the proof of Theorem 3.1.8 is similar to that of [45, Theorem 1.3]. In [45, Theorem 1.3], there are two crucial ingredients. One is the non-vanishing of the coboundary map of local cohomology group ([45, Theorem 1.1]). And another is the vanishing of a composition of homomorphisms between some cohomology groups ([45, Proposition 1.2]). We modify these propositions to our setting of a pair of a variety and its divisor.

Proof of Theorem 3.1.8. By Corollary 3.3.8, we can assume that the singularities on X are non ordinary terminal singularities. We prepare the notations to introduce the diagram (3.22).

Let m be a positive integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth element D_m which satisfies $D_m \cap \mathrm{Sing}\, D = \emptyset$ and intersects transversely with D. Let $\pi \colon Y := \mathrm{Spec} \oplus_{i=0}^{m-1} \mathcal{O}_X(iK_X) \to X$ be the cyclic cover determined by D_m. Note that Y is terminal Gorenstein. Put $\{p_1, \dots, p_l\} := \mathrm{Sing}\, D$. Note that $\mathrm{Sing}\, X \subset \mathrm{Sing}\, D$ since all the singularities on X are non-Gorenstein. Also note that $G := \mathrm{Gal}(Y/X) \simeq \mathbb{Z}_m$ acts on Y and $\Delta := \pi^{-1}(D)$ is G-invariant.

Let U_i be a sufficiently small Stein neighborhood of p_i such that $U_i \setminus \{p_i\}$ is smooth and $K_{V_i} = 0$, where $V_i := \pi^{-1}(U_i)$. Let $\pi_i : V_i \to U_i$ be the morphism induced by π.

By Proposition 3.4.1, we can take a $\mathbb{Z}_m$-equivariant resolution $\nu \colon \tilde{Y} \to Y$ of Y such that $\nu|_{\nu^{-1}(Y \setminus \mathrm{Sing}\, \Delta)}$ is an isomorphism, $\tilde{\Delta} := (\nu^{-1})_* \Delta$ is smooth and

$$\nu_\Delta^* K_\Delta = K_{\tilde{\Delta}},$$

where $\nu_\Delta\colon \tilde{\Delta} \to \Delta$ is induced by ν. Then we have the following diagram;

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{\pi}"] \arrow[d, "\nu"] & \tilde{X} \arrow[d, "\mu"] \\ Y \arrow[r, "\pi"] & X. \end{tikzcd} \tag{3.20}$$

We also have the following diagram induced by the above diagram;

$$\begin{tikzcd} \tilde{V}_i \arrow[r, "\tilde{\pi}_i"] \arrow[d, "\nu_i"] & \tilde{U}_i \arrow[d, "\mu_i"] \\ V_i \arrow[r, "\pi_i"] & U_i. \end{tikzcd} \tag{3.21}$$

Put $F := \operatorname{Exc}(\nu) \subset \tilde{Y}$, $F_i := \operatorname{Exc}(\nu_i)$, $E := \operatorname{Exc}(\mu)$ and $E_i := \operatorname{Exc}(\mu_i)$. Put $\tilde{\Delta}_i := (\nu_i^{-1})_* \Delta_i$, where $\Delta_i := \Delta \cap V_i$.

Let $\mathcal{F}^{(0)}$ be the $\mathbb{Z}_m$-invariant part of $\tilde{\pi}_* \Omega^2_{\tilde{Y}}(\log \tilde{\Delta})$ and set $\mathcal{F}^{(0)}_i := \mathcal{F}^{(0)}|_{\tilde{U}_i}$. Set $U := X \setminus \operatorname{Sing} D$. Note that $\mathcal{F}^{(0)}|_U \simeq \Omega^2_U(\log D_U)$, where $D_U := D \cap U$.

Hence we have the following diagram;

$$\begin{tikzcd} H^1(U, \Omega^2_U(\log D_U)) \arrow[r, "\oplus\psi_i"] \arrow[d, "\oplus p_{U_i}"] & \bigoplus_{i=1}^l H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)}) \arrow[r, "\oplus\beta_i"] \arrow[d, "\simeq"] & H^2(\tilde{X}, \mathcal{F}^{(0)}) \\ \bigoplus_{i=1}^l H^1(U'_i, \Omega^2_{U'_i}(\log D'_i)) \arrow[r, "\oplus\phi_i"] & \bigoplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \mathcal{F}^{(0)}_i), & \end{tikzcd} \tag{3.22}$$

where $U'_i := U_i \setminus \{p_i\}$ and $D'_i := D \cap U'_i$.

We have restriction homomorphisms $\iota^*\colon T^1_{(X,D)} \to T^1_{(U,D_U)}$ and $\iota_i^*\colon T^1_{(U_i,D_i)} \to T^1_{(U'_i,D'_i)}$, where $\iota\colon U \hookrightarrow X$ and $\iota_i\colon U'_i \hookrightarrow U_i$ are open immersions. By Lemma 3.4.2 and the arguments around it, we see that

$$H^1(U, \Omega^2_U(\log D_U)) \simeq T^1_{(X,D)},$$

$$H^1(U'_i, \Omega^2_{U'_i}(\log D'_i)) \simeq T^1_{(U_i,D_i)}.$$

By using the diagram (3.22), we want to lift $\eta_i \in H^1(U'_i, \Omega^2_{U'_i}(\log D'_i)) \simeq T^1_{(U_i,D_i)}$ which induces a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i) to X. For that purpose, we consider $\phi_i(\eta_i)$ and lift it to $H^1(U, \Omega^2_U(\log D_U))$.

Note that ϕ_i is the $\mathbb{Z}_m$-invariant part of the coboundary map $\tau_i\colon H^1(V'_i, \Omega^2_{V'_i}(\log \Delta'_i)) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log \tilde{\Delta}_i))$. We see that τ_i is same as $\tau_{(V_i,\Delta_i)}$ introduced in (3.19). Thus we can use the results in Section 3.4.3. By Lemma 3.4.4, we see that

$$P_{\Delta'_i}(\operatorname{Ker}\tau_i) \subset \operatorname{Im} R_{\Delta_i} = 0, \tag{3.23}$$

where $P_{\Delta'_i}\colon H^1(V'_i, \Omega^2_{V'_i}(\log \Delta'_i)) \to H^1(\Delta'_i, \Omega^1_{\Delta'_i})$ and $R_{\Delta_i}\colon H^1(\tilde{\Delta}_i, \Omega^1_{\tilde{\Delta}_i}) \to H^1(\Delta'_i, \Omega^1_{\Delta'_i})$ are defined as in Section 3.4.3.

There exists $\eta_i \in T^1_{(U_i,D_i)}$ which induces a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i) by the description in Section 3.4.2. Note that $\phi_i(\eta_i) \neq 0$ by the relation (3.23). To lift $\phi_i(\eta_i)$ to $H^1(U, \Omega^2_U(\log D_U))$, we need the following claim.

Claim 3.4.6. $\beta_i \circ \phi_i = 0$.

Proof of Claim. $\beta_i \circ \phi_i$ is the $\mathbb{Z}_m$-invariant part of a composition of the homomorphisms

$$\begin{aligned} H^1(V'_i, \Omega^2_{V'_i}(\log \Delta'_i)) &\to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log \tilde{\Delta}_i)) \\ &\simeq H^2_{F_i}(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log \tilde{\Delta})) \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log \tilde{\Delta})). \end{aligned} \tag{3.24}$$

By considering its dual, it is enough to show that the $\mathbb{Z}_m$-invariant part of the homomorphism

$$\Phi_i\colon H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \to H^1(V'_i, \Omega^1_{V'_i}(\log \Delta'_i)(-\Delta'_i))$$

is zero. We show that $\Phi_i = 0$ in the following.

For a $\mathbb{Z}$-module M, we set $M_{\mathbb{C}} := M \otimes \mathbb{C}$. Let $\mathcal{K}_{(\tilde{Y},\tilde{\Delta})}$ be a sheaf of groups defined by an exact sequence

$$1 \to \mathcal{K}_{(\tilde{Y},\tilde{\Delta})} \to \mathcal{O}^*_{\tilde{Y}} \to \mathcal{O}^*_{\tilde{\Delta}} \to 1.$$

We have a commutative diagram with two horizontal exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \arrow[r] & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}) \arrow[r] & H^1(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) \\ 0 \arrow[r] & H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[r] \arrow[u, "\epsilon"] & H^1(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}})_{\mathbb{C}} \arrow[r] \arrow[u, "\delta_{\tilde{Y}}"] & H^1(\tilde{\Delta}, \mathcal{O}^*_{\tilde{\Delta}})_{\mathbb{C}}, \arrow[u, "\delta_{\tilde{\Delta}}"] \end{tikzcd}$$

where the injectivity follows since we see that $H^0(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) = 0$ and that $H^0(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}}) \to H^0(\tilde{\Delta}, \mathcal{O}^*_{\tilde{\Delta}})$ is surjective. We see that $\delta_{\tilde{Y}}$ is an isomorphism and $\delta_{\tilde{\Delta}}$ is injective since we have $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$ for $i = 1, 2$ and $H^1(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}) = 0$. Hence we see that ϵ is an isomorphism.

Set $\mathcal{K}_{(V_i',\Delta_i')} := \mathcal{K}_{(\tilde{Y},\tilde{\Delta})}|_{V_i'}$. We have a commutative diagram

$$\begin{tikzcd}
H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \arrow[d] & H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[l, "\simeq"'] \arrow[d, "\Phi_i'"] \\
H^1(V_i', \Omega^1_{V_i'}(\log \Delta_i')(-\Delta_i')) & H^1(V_i', \mathcal{K}_{(V_i',\Delta_i')})_{\mathbb{C}}. \arrow[l]
\end{tikzcd}$$

Hence it is enough to show that $\Phi_i' = 0$. Moreover we have a commutative diagram

$$\begin{tikzcd}
H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[r, hook] \arrow[d] & H^1(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}})_{\mathbb{C}} \arrow[d, "\Phi_i''"] \\
H^1(V_i', \mathcal{K}_{(V_i',\Delta_i')})_{\mathbb{C}} \arrow[r, hook] & H^1(V_i', \mathcal{O}^*_{V_i'})_{\mathbb{C}}.
\end{tikzcd}$$

Since ν is an isomorphism outside $\operatorname{Sing} \Delta$, we see that $\Phi_i'' = 0$ by Proposition 3.4.5. Hence we see that $\Phi_i' = 0$ and we finish the proof of Claim 3.4.6. $\square$

By Claim 3.4.6, we have $\beta_i(\phi_i(\eta_i)) = 0$. Thus there exists $\eta \in H^1(U, \Omega^2_U(\log D'))$ such that $\psi_i(\eta) = \phi_i(\eta_i)$ for each i. Then $P_{\Delta_i'}(p_{U_i}(\eta) - \eta_i) \in P_{\Delta_i'}(\operatorname{Ker} \tau_i) \subset \operatorname{Im} R_{\Delta_i} = 0$ by the relation (3.23). Hence we have

$$P_{\Delta_i'}(p_{U_i}(\eta)) = P_{\Delta_i'}(\eta_i) \in H^1(\Delta_i', \Omega^1_{\Delta_i'}). \tag{3.25}$$

Note that this element corresponds to an element of $T^1_{\Delta_i}$ which induces a smoothing of Δ_i by the definition of η_i.

By Theorem 3.2.9, there exists a deformation $f\colon (\mathcal{X}, \mathcal{D}) \to \Delta^1$ of (X, D) induced by η. By the relation (3.25), we see that f induces a smoothing of Δ_i. Note that $\operatorname{Sing} V_i \subset \operatorname{Sing} \Delta_i$ and this relation is preserved by deformation since $\mathcal{D}_t \in |-K_{X_t}|$ contains all non-Gorenstein points of X_t, where $X_t := f^{-1}(t)$ for $t \in \Delta^1$. We see that a deformation of V_i becomes smooth along a deformation of Δ_i which is smooth since a deformation of $\Delta_i \subset V_i$ is still a Cartier divisor, Thus f is a $\mathbb{Q}$-smoothing and we finish the proof of Theorem 3.1.8. $\square$

Remark 3.4.7. There are many examples of $\mathbb{Q}$-Fano 3-folds without Du Val elephants. See Section 6.4 for such examples.

3.4.5 Genus bound for primary $\mathbb{Q}$-Fano 3-folds

Definition 3.4.8. Let X be a $\mathbb{Q}$-Fano 3-fold. Let $\tilde{\mathrm{Cl}}X$ be the quotient of the divisor class group $\mathrm{Cl}\,X$ by its torsion part. X is called *primary* if

$$\tilde{\mathrm{Cl}}X \simeq \mathbb{Z} \cdot [-K_X].$$

Takagi [60] proved the following theorem on the genus bound of certain primary $\mathbb{Q}$-Fano 3-folds.

Theorem 3.4.9. *([60, Theorem 1.5]) Let X be a primary $\mathbb{Q}$-Fano 3-fold with only terminal quotient singularities. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

By combining his result and our results, we get the following genus bound.

Theorem 3.4.10. *Let X be a primary $\mathbb{Q}$-Fano 3-fold. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

Proof. By Theorem 3.1.8, there is a deformation $\mathcal{X} \to \Delta^1$ of X such that $\mathcal{X}_t$ has only quotient singularities and $|-K_{\mathcal{X}_t}|$ contains an element with only Du Val singularities for $t \neq 0$. By Theorem 5.28 of [35], we have $h^0(X, -K_X) = h^0(\mathcal{X}_t, -K_{\mathcal{X}_t})$. By Theorem 3.4.9, we have

$$h^0(X, -K_X) = h^0(\mathcal{X}_t, -K_{\mathcal{X}_t}) \leq 10.$$

$\square$

Chapter 4

Deformations of weak Fano manifolds

4.1 Introduction

We consider algebraic varieties over an algebraically closed field k of characteristic zero.

The Kuranishi space of a smooth projective variety has bad singularities in general. Even in the surface case, Vakil [62] exhibited several examples of smooth projective surfaces of general type with arbitrarily singular Kuranishi spaces.

On the other hand, in some nice situations, the Kuranishi space is smooth. A famous result is that the Kuranishi space of a Calabi–Yau manifold is smooth. The Kuranishi space of a Fano manifold X is also smooth since $H^2(X, \Theta_X) = 0$ by the Kodaira–Nakano vanishing theorem, where Θ_X is the tangent sheaf of X.

In this chapter, we look for several nice projective manifolds with smooth Kuranishi space.

A smooth projective variety X is called a *weak Fano manifold* if the anticanonical divisor $-K_X$ is nef and big. The following is the main theorem of this chapter.

Theorem 4.1.1. *Deformations of a weak Fano manifold are unobstructed.*

Previously, Ran proved the unobstructedness for a weak Fano manifold with a smooth anticanonical element ([47, Corollary 3]). Minagawa's argument in [38] implies the unobstructedness when $|-2K_X|$ contains a smooth element. However these assumptions are not satisfied for a general weak Fano manifold as explained in Example 4.2.9. We prove it for the general case.

We use the T^1-lifting technique developed by Ran, Kawamata, Deligne and Fantechi-Manetti. Another approach is dealt with by Buchweitz–Flenner in [7].

The following more general result implies Theorem 4.1.1.

Theorem 4.1.2. *Let X be a smooth projective variety. Assume that $H^1(X, \mathcal{O}_X) = 0$ and there exists a positive integer m and a smooth divisor $D \in |-mK_X|$ such that $H^1(D, \mathcal{N}_{D/X}) = 0$.*

Then deformations of X are unobstructed.

We sketch the proof of Theorem 4.1.2. Instead of proving the unobstructedness directly, we first prove the unobstructedness for the pair of a weak Fano manifold X and a smooth element D of $|-mK_X|$ for a sufficiently large integer m in Theorem 4.2.2. Next we show that the unobstructedness for (X, D) implies the unobstructedness for X.

We also show that the Kuranishi space of a smooth projective surface is smooth if the Kodaira dimension of the surface is negative or 0 in Theorem 4.3.2. It seems to be known to experts but we give a proof for the convenience of the reader.

4.2 Proof of theorem

Fix an algebraically closed field k of characteristic zero. Let Art_k be the category of Artinian local k-algebras with residue field k and $Sets$ the category of sets. For a proper variety X over k and an effective Cartier divisor D on X, let $\mathrm{Def}_{(X,D)}\colon Art_k \to Sets$ be the functor sending $A \in Art_k$ to the set of equivalence classes of proper flat morphisms $f\colon X_A \to \operatorname{Spec} A$ together with effective Cartier divisors $D_A \subset X_A$ and marking isomorphisms $\phi_0\colon X_A \otimes_A k \to X$ such that $\phi_0(D_A \otimes_A k) = D$. This is the pair version of the deformation functor Def_X defined in [28]. We see that $\mathrm{Def}_{(X,D)}$ is a *deformation functor* in the sense of Fantechi–Manetti ([11, Introduction]).

We need the following lemma.

Lemma 4.2.1. *Let Z be a smooth proper variety over k and $\Delta \subset Z$ a smooth divisor. Set $A_n := k[t]/(t^{n+1})$ for a non-negative integer n. Let $Z_n \to \operatorname{Spec} A_n$ and $\Delta_n \subset Z_n$ be deformations of Z and Δ. Let $\Omega^\bullet_{Z_n/A_n}(\log \Delta_n)$ be the de Rham complex of Z_n/A_n with logarithmic poles along Δ_n (cf. [25, (7.1.1)]). Then we have the following:*

(i) the hypercohomology group $\mathbb{H}^k(Z_n, \Omega^\bullet_{Z_n/A_n}(\log \Delta_n))$ is a free A_n-module for all k;

(ii) the spectral sequence

$$E_1^{p,q} := H^q(Z_n, \Omega^p_{Z_n/A_n}(\log \Delta_n)) \Rightarrow \mathbb{H}^{p+q}(Z_n, \Omega^\bullet_{Z_n/A_n}(\log \Delta_n)) \tag{4.1}$$

degenerates at E_1;

(iii) the cohomology group $H^q(Z_n, \Omega^p_{Z_n/A_n}(\log \Delta_n))$ is a free A_n-module and commutes with base change for any p and q.

Proof. We can prove this by the same argument as in [9, Théorème 5.5]. We give a proof for the convenience of the reader. We can assume that $k = \mathbb{C}$ by the Lefschetz principle.

(i) Set $U := Z \setminus \Delta$. Let $\iota\colon U \hookrightarrow Z$ be the open immersion. We see that the complex $\Omega^\bullet_{Z_n/A_n}(\log \Delta_n)$ is quasi-isomorphic to $\iota_* \Omega^\bullet_{U_n/A_n}$ by a standard argument as in [46, Proposition 4.3], where $U_n \to \operatorname{Spec} A_n$ is a deformation of U which is induced by $Z_n \to \operatorname{Spec} A_n$. We have an isomorphism $\mathbb{H}^k(Z_n, \iota_* \Omega^\bullet_{U_n/A_n}) \simeq \mathbb{H}^k(U_n, \Omega^\bullet_{U_n/A_n})$ since we have $R^i \iota_* \Omega^j_{U_n/A_n} = 0$ for $i > 0$ and all j. Moreover we have $\mathbb{H}^{p+q}(U_n, \Omega^\bullet_{U_n/A_n}) \simeq H^{p+q}(U, A_n)$, where the latter is the singular cohomology on U with coefficient A_n since $\Omega^\bullet_{U_n/A_n}$ is a resolution of the sheaf $A_{n,U}$, where $A_{n,U}$ is a constant sheaf on U associated to A_n (See [9, Lemme 5.3]). Hence we obtain (i) since we have

$$\mathbb{H}^{p+q}(Z_n, \Omega^\bullet_{Z_n/A_n}(\log \Delta_n)) \simeq H^{p+q}(U, A_n) \simeq H^{p+q}(U, \mathbb{C}) \otimes A_n.$$

Moreover we obtain the equality

$$\dim_{\mathbb{C}} \mathbb{H}^{p+q}(Z_n, \Omega^\bullet_{Z_n/A_n}(\log \Delta_n)) = \dim_{\mathbb{C}}(A_n) \cdot \dim_{\mathbb{C}} \mathbb{H}^{p+q}(Z, \Omega^\bullet_Z(\log \Delta)).$$

(ii) By the argument as in [9, (5.5.5)], we see that

$$\dim_{\mathbb{C}} H^q(Z, \Omega^p_{Z_n/A_n}(\log \Delta_n)) \leq \dim_{\mathbb{C}}(A_n) \cdot \dim_{\mathbb{C}} H^q(Z, \Omega^p_Z(\log \Delta)) \tag{4.2}$$

and equality holds if and only if $H^q(Z, \Omega^p_{Z_n/A_n}(\log \Delta_n))$ is a free A_n-module. By the spectral sequence (4.1), we have

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^q(Z_n, \Omega^p_{Z_n/A_n}(\log \Delta_n)) \geq \dim_{\mathbb{C}} \mathbb{H}^k(Z_n, \Omega^\bullet_{Z_n/A_n}(\log \Delta_n)). \tag{4.3}$$

By the two inequalities (4.2), (4.3) and (i), we obtain

$$\dim_{\mathbb{C}}(A_n)\cdot\sum_{p+q=k}\dim_{\mathbb{C}} H^q(Z,\Omega_Z^p(\log\Delta))\geq\dim_{\mathbb{C}}(A_n)\cdot\dim_{\mathbb{C}}\mathbb{H}^k(Z,\Omega_Z^\bullet(\log\Delta)). \tag{4.4}$$

We have equality in the inequality (4.4) since the spectral sequence (4.1) degenerates at E_1 when $n=0$ by [10, Corollaire (3.2.13)(ii)]. Hence we have equality in (4.3) and obtain (ii).

(iii) This follows from (i) and (ii). □

To prove Theorem 4.1.2, we prove the following theorem on unobstructedness of deformations of a pair.

Theorem 4.2.2. *Let X be a smooth proper variety such that $H^1(X,\mathcal{O}_X)=0$. Assume that there exists a smooth divisor $D\in|-mK_X|$ for some positive integer m. Then deformations of (X,D) are unobstructed, that is, $\mathrm{Def}_{(X,D)}$ is a smooth functor.*

Proof. Set $A_n:=k[t]/(t^{n+1})$ and $B_n:=k[x,y]/(x^{n+1},y^2)\simeq A_n\otimes_k A_1$. For $[(X_n,D_n),\phi_0]\in\mathrm{Def}_{(X,D)}(A_n)$, let $T^1((X_n,D_n)/A_n)$ be the set of isomorphism classes of pairs $((Y_n,E_n),\psi_n)$ consisting of deformations (Y_n,E_n) of (X_n,D_n) over B_n and marking isomorphisms $\psi_n\colon Y_n\otimes_{B_n}A_n\to X_n$ such that $\psi_n(E_n\otimes_{B_n}A_n)=D_n$, where we use a ring homomorphism $B_n\to A_n$ given by $x\mapsto t$ and $y\mapsto 0$. Then we see the following.

Claim 4.2.3. We have

$$T^1((X_n,D_n)/A_n)\simeq H^1(X_n,\Theta_{X_n/A_n}(-\log D_n)), \tag{4.5}$$

where $\Theta_{X_n/A_n}(-\log D_n)$ is the dual of $\Omega^1_{X_n/A_n}(\log D_n)$.

Proof. We can prove this by a standard argument (cf. [56, Proposition 3.4.17]) using $B_n=A_n\otimes_k A_1$. □

Hence, by [11, Theorem A], it is enough to show that the natural homomorphism

$$\gamma_n\colon H^1(X_n,\Theta_{X_n/A_n}(-\log D_n))\to H^1(X_{n-1},\Theta_{X_{n-1}/A_{n-1}}(-\log D_{n-1}))$$

is surjective for the above X_n,D_n and for $X_{n-1}:=X_n\otimes_{A_n}A_{n-1}, D_{n-1}:=D_n\otimes_{A_n}A_{n-1}$.

Note that we have a perfect pairing

$$\Omega^1_{X_n/A_n}(\log D_n) \times \Omega^{d-1}_{X_n/A_n}(\log D_n) \to \mathcal{O}_{X_n}(K_{X_n/A_n} + D_n) \simeq \omega^{\otimes 1-m}_{X_n/A_n},$$

where we set $d := \dim X$. We have $\mathcal{O}_{X_n}(K_{X_n/A_n} + D_n) \simeq \omega^{\otimes 1-m}_{X_n/A_n}$ since we have $H^1(X, \mathcal{O}_X) = 0$ (See [20, Theorem 6.4(b)], for example.). Thus we see that

$$H^1(X_n, \Theta_{X_n/A_n}(-\log D_n)) \simeq H^1(X_n, \Omega^{d-1}_{X_n/A_n}(\log D_n) \otimes \omega^{\otimes m-1}_{X_n/A_n}).$$

Let

$$\pi_n \colon Z_n := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_{X_n}(iK_{X_n/A_n}) \to X_n$$

be the ramified covering defined by a section $\sigma_{D_n} \in H^0(X_n, -mK_{X_n/A_n})$ which corresponds to D_n. We have an isomorphism

$$\pi_n^* \Omega^1_{X_n/A_n}(\log D_n) \simeq \Omega^1_{Z_n/A_n}(\log \Delta_n)$$

for some divisor $\Delta_n \in |-\pi_n^* K_{X_n/A_n}|$. Hence we see that

$$(\pi_n)_* \Omega^{d-1}_{Z_n/A_n}(\log \Delta_n) \simeq \bigoplus_{i=0}^{m-1} \Omega^{d-1}_{X_n/A_n}(\log D_n)(iK_{X_n/A_n})$$

and $\Omega^{d-1}_{X_n/A_n}(\log D_n) \otimes \omega^{\otimes m-1}_{X_n/A_n}$ is one of the direct summands.

Hence it is enough to show that the natural restriction homomorphism

$$r_n \colon H^1(Z_n, \Omega^{d-1}_{Z_n/A_n}(\log \Delta_n)) \to H^1(Z_{n-1}, \Omega^{d-1}_{Z_{n-1}/A_{n-1}}(\log \Delta_{n-1}))$$

is surjective, where we set $Z_{n-1} := Z_n \otimes_{A_n} A_{n-1}$ and $\Delta_{n-1} := \Delta_n \otimes_{A_n} A_{n-1}$, since γ_n is an eigenpart of r_n. By Lemma 4.2.1(iii), we see the required surjectivity. This completes the proof of Theorem 4.2.2. □

Remark 4.2.4. Iacono [23] proved Theorem 4.2.2 when $m = 1$ without the assumption $H^1(X, \mathcal{O}_X) = 0$ in [23, Corollary 4.5] as a consequence of the analysis of DGLA.

Remark 4.2.5. We can remove the assumption $H^1(X, \mathcal{O}_X) = 0$ when $m = 1$ by a similar argument as in [47, Corollary 2]. In that case, we see that $\mathcal{O}_{X_n}(K_{X_n/A_n} + D_n) \simeq \mathcal{O}_{X_n}$ since we have $H^0(X_n, K_{X_n/A_n} + D_n) \simeq A_n$ by Claim 4.2.1, with X_n, D_n as in the proof of Theorem 4.2.2.

We do not know whether we can remove the assumption $H^1(X, \mathcal{O}_X) = 0$ in Theorem 4.2.2 when m is arbitrary.

Theorem 4.2.2 implies Theorem 4.1.2 as follows.

Proof of Theorem 4.1.2. Since $H^1(D, \mathcal{N}_{D/X}) = 0$, we see that the forgetful morphism

$$\mathrm{Def}_{(X,D)} \to \mathrm{Def}_X$$

between functors is smooth. Since $\mathrm{Def}_{(X,D)}$ is smooth by Theorem 4.2.2, we see that Def_X is also smooth. $\square$

Theorem 4.1.2 implies Theorem 4.1.1 as follows.

Proof of Theorem 4.1.1. Let X be a weak Fano manifold of dimension d. By the base point free theorem, we can take a sufficiently large integer m such that $-mK_X$ is base point free and contains a smooth element $D \in |-mK_X|$. We have $H^1(D, \mathcal{N}_{D/X}) = 0$ since there is an exact sequence

$$H^1(X, \mathcal{O}_X(D)) \to H^1(D, \mathcal{N}_{D/X}) \to H^2(X, \mathcal{O}_X)$$

and both outer terms are zero by the Kawamata–Viehweg vanishing theorem. Hence Theorem 4.1.2 implies Theorem 4.1.1. $\square$

Remark 4.2.6. We can prove the following theorem by the same argument as Theorem 4.1.1.

Theorem 4.2.7. *Let X be a complex manifold whose anticanonical bundle is nef and big. Then deformations of X are unobstructed.*

Actually we see that such a complex manifold is Moishezon since there is a big divisor on X. Hence we can show Lemma 4.2.1 and the base-point free theorem in this setting. Using these, we can show Theorem 4.2.7 in the same way as Theorem 4.1.1.

Example 4.2.8. We give an example of a weak Fano manifold such that $H^2(X, \Theta_X) \neq 0$, where Θ_X is the tangent sheaf.

Let $f\colon X \to \mathbb{P}(1,1,1,3)$ be the blow-up of the singular point p of the weighted projective space. We can check that $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3))$ and f is the anticanonical morphism of X. Hence $-K_X = f^*(-K_{\mathbb{P}(1,1,1,3)})$ and this is nef and big. Set $\mathcal{E} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$. By a direct calculation using the relative Euler sequence for $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}) \to \mathbb{P}^2$, we see that

$$h^2(X, \Theta_X) = h^2(X, \Theta_{X/\mathbb{P}^2}) = h^2(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{E}^*) = 1.$$

Hence $H^2(X, \Theta_X) \neq 0$.

Thus we need a technique such as T^1-lifting for the proof of Theorem 4.1.2.

Example 4.2.9. We give an example of a Fano manifold such that neither of the linear systems $|-K_X|$ and $|-2K_X|$ contain smooth elements. Our example is a modification of an example in [29, Example 3.2 (3)].

Let $X := X_{5d} \subset \mathbb{P}(1, \dots, 1, 5, d) = \mathbb{P}(1^n, 5, d)$ be a weighted hypersurface of degree $5d$ and dimension n. Assume that $d \not\equiv 0 \mod 5$ and that $5+n-4d = 2$. (For example, $d = 6, n = 21$.) The latter condition implies that $-K_X = \mathcal{O}_X(2)$. We see that the base locus of $|-K_X|$ and $|-2K_X|$ consists of a point $p := H_1 \cap \dots \cap H_n \cap X_{5d}$, where $H_1, \dots, H_n$ are degree 1 hyperplanes of the first n coordinates of $\mathbb{P}(1^n, 5, d)$. We see that every element of $|-K_X|$ has multiplicity 2 at the base point p and hence is singular. We also see that every element of $|-2K_X|$ has multiplicity 4 at the base point p and hence is singular.

Example 4.2.10. We give an example of a smooth projective variety such that Def_X is not smooth and $-K_X$ is big.

Let $C \subset \mathbb{P}^3$ be a smooth curve with an obstructed embedded deformation which lies in a cubic surface as in [20, Theorem 13.1]. Let $\mu \colon X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C. Then X has an obstructed deformation. See [20, Example 13.1.1]. Note that $-K_X = \mu^*\mathcal{O}_{\mathbb{P}^3}(4) - E$ where $E := \mu^{-1}(C)$ and C is contained in a cubic surface $S \subset \mathbb{P}^3$. Let $\tilde{S} \subset X$ be the strict transform of S. Then we see that $-K_X$ is big since $\tilde{S} + |\mu^*\mathcal{O}_{\mathbb{P}^3}(1)| \subset |-K_X|$.

Example 4.2.11. We give an example of X and $D \in |-K_X|$ such that $\mathrm{Def}_{(X,D)}$ is smooth but Def_X is not smooth.

Let $C \subset \mathbb{P}^3$ be a smooth curve in a quartic surface S such that the Hilbert scheme of curves in $\mathbb{P}^3$ is singular at the point corresponding to C (cf. [20, Exercise 13.2]). Let $X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C. Then X has an obstructed deformation. However the strict transform $D := \tilde{S} \in |-K_X|$ of S is smooth and $H^1(X, \mathcal{O}_X) = 0$. Hence $\mathrm{Def}_{(X,D)}$ is smooth by Theorem 4.2.2.

Example 4.2.12. We give an example of X with an obstructed deformation such that $-K_X$ is nef.

Set $X := T^m \times \mathbb{P}^1$ where T^m is a complex torus of dimension $m \geq 2$. Then X has an obstructed deformation ([30, p.436–441]). Note that $-K_X$ is nef. It is actually semiample.

It is natural to ask the following question:

Problem 4.2.13. *Let X be a smooth projective variety such that $-K_X$ is nef and $H^1(X, \mathcal{O}_X) = 0$. Is the Kuranishi space of X smooth?*

4.3 The surface case

The following lemma states that smoothness of the Kuranishi space is preserved under the blow-up at a point.

Lemma 4.3.1. *Let S be a smooth projective variety and $\nu\colon T \to S$ the blow-up at a point $p \in S$. Then the functor Def_S is smooth if and only if the functor Def_T is smooth.*

Proof. Let $\mathrm{Def}_{(S,p)}$ be the functor of deformations of a closed immersion $\{p\} \subset S$ and $\mathrm{Def}_{(T,E)}$ the functor as in Section 2, where $E := \nu^{-1}(p)$. We can define a natural transformation

$$\nu_*\colon \mathrm{Def}_{(T,E)} \to \mathrm{Def}_{(S,p)}$$

as follows: given $A \in Art_k$ and a deformation $(\mathbf{T}, \mathbf{E})$ of (T, E) over A, we see that $\nu_*\mathcal{O}_{\mathbf{T}}$ is a sheaf of flat A-algebras by [63, Corollary 0.4.4] since we have $R^1\nu_*\mathcal{O}_T = 0$. We also see that $\nu_*\mathcal{O}_{\mathbf{T}}(-\mathbf{E})$ is a sheaf of flat A-modules by [63, Corollary 0.4.4] since we have $R^1\nu_*\mathcal{O}_T(-E) = 0$ by a direct calculation. Hence we can define a deformation $(\mathbf{S}, \mathbf{p})$ of (S, p) over A by sheaves $\mathcal{O}_{\mathbf{S}} := \nu_*\mathcal{O}_{\mathbf{T}}, \mathcal{I}_{\mathbf{p}} := \nu_*\mathcal{O}_{\mathbf{T}}(-\mathbf{E})$ and obtain a natural transformation ν_*.

We can also define a natural transformation

$$\nu^*\colon \mathrm{Def}_{(S,p)} \to \mathrm{Def}_{(T,E)}$$

as follows: given a deformation $(\mathbf{S}, \mathbf{p})$ of (S, p) over $A \in Art_k$, we define a deformation $\mathbf{T}$ of T as the blow-up of $\mathbf{S}$ along $\mathbf{p}$. We can also define a deformation $\mathbf{E}$ of E by the inverse image ideal sheaf $\nu^{-1}\mathcal{I}_{\mathbf{p}} \cdot \mathcal{O}_{\mathbf{T}}$, where $\mathcal{I}_{\mathbf{p}}$ is the ideal sheaf of $\mathbf{p} \subset \mathbf{S}$.

We see that ν_* and ν^* are inverse to each other. Hence we have $\mathrm{Def}_{(T,E)} \simeq \mathrm{Def}_{(S,p)}$ as functors.

We have forgetful morphisms of functors $F_T\colon \mathrm{Def}_{(T,E)} \to \mathrm{Def}_T$ and $F_S\colon \mathrm{Def}_{(S,p)} \to \mathrm{Def}_S$. We see that F_T and F_S are smooth since we have $H^1(E, \mathcal{N}_{E/T}) \simeq H^1(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(-1)) = 0$ and $H^1(\mathcal{N}_{p/S}) = 0$, where we set $d := \dim S$.

Thus we have a diagram

$$\begin{tikzcd} \mathrm{Def}_{(T,E)} \arrow[r, "\sim"] \arrow[d, "F_T"] & \mathrm{Def}_{(S,p)} \arrow[d, "F_S"] \\ \mathrm{Def}_T & \mathrm{Def}_S, \end{tikzcd}$$

where F_T and F_S are smooth. Hence we see the required equivalence. □

By this lemma, we see that a smooth projective surface has unobstructed deformations if and only if its relatively minimal model has unobstructed deformations.

Using Lemma 4.3.1, we can prove the following:

Theorem 4.3.2. *Let X be a smooth projective surface with non-positive Kodaira dimension. Then the deformations of X are unobstructed.*

Proof. By Lemma 4.3.1, we can assume that X does not contain a -1-curve.

If the Kodaira dimension of X is negative, it is known that $X \simeq \mathbb{P}^2$ or $X \simeq \mathbb{P}_C(\mathcal{E})$ for some projective curve C and a rank 2 vector bundle $\mathcal{E}$ on C. In these cases, we see that $H^2(X, \Theta_X) = 0$ by the Euler sequence or the argument in [55, p.204].

If the Kodaira dimension of X is zero, it is a $K3$ surface, an Abelian surface or its étale quotient. It is well known that these surfaces have unobstructed deformations. Hence we are done. □

Remark 4.3.3. Kas [26] gave an example of a smooth projective surface of Kodaira dimension 1 with an obstructed deformation.

Chapter 5

Deformations of $\mathbb{Q}$-Calabi-Yau threefolds and $\mathbb{Q}$-Fano threefolds

5.1 Introduction

We consider algebraic varieties over $\mathbb{C}$.

In this chapter, we continue the study of $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold in Chapter 2. We study the coboundary map of local cohomology introduced in Section 3.3.2 more precisely.

The following is a main result of this chapter.

Theorem 5.1.1. *A $\mathbb{Q}$-Fano 3-fold can be deformed to one with only quotient singularities and singularities isomorphic to $(x^2+y^2+z^3+u^2=0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$.*

This is a generalization of Theorem 3.1.5. The key tool in the proof is the coboundary map ϕ_{U_i} associated to some local cohomology group of a Stein neighborhood U_i of a singularity p_i on X. (See (5.2) for the definition of ϕ_{U_i}.) The following purely local statement is the main result of Section 5.2.

Theorem 5.1.2. *Let (U,p) be a germ of a non-ordinary 3-dimensional terminal singularity (See Definition 3.3.2).*

(i) Assume that the index one cover $(V,q) \not\simeq ((x^2+y^2+z^3+u^2=0),0)$. Then we have $\phi_U \neq 0$.

(ii) Assume that $(V,q) \simeq ((x^2+y^2+z^3+u^2=0),0)$. Then $\phi_U = 0$.

This theorem together with Theorem 3.3.5 implies Theorem 5.1.1. However, the map vanishes for the singularity $(x^2+y^2+z^3+u^2=0)/\mathbb{Z}_4$ as shown in Theorem

5.1.2 (ii). Thus we need another method to treat a general $\mathbb{Q}$-Fano 3-fold. However, as an application of Theorem 5.1.2, we obtain the $\mathbb{Q}$-smoothability of some $\mathbb{Q}$-Calabi–Yau 3-fold (Corollary 5.2.6).

We treat an example of a $\mathbb{Q}$-Fano 3-fold X with several terminal singularities isomorphic to $(x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4$ in Section 5.3. This X satisfies that $H^2(X, \Theta_X) \neq 0$. Nevertheless, we shall see that we can deform the terminal singularities on X.

5.2 Coboundary map of local cohomology

Let (U, p) be a germ of a 3-dimensional non-ordinary terminal singularity. By the classification([40], [52]), we have

$$(U, p) \simeq ((x^2 + y^2 + g(z, u) = 0), 0)/\mathbb{Z}_4 \subset (\mathbb{C}^4/\mathbb{Z}_4, 0), \tag{5.1}$$

where $g(z, u) \in \mathfrak{m}^2_{\mathbb{C}^4,0}$ is some $\mathbb{Z}_4$-semi-invariant polynomial in z, u and $\sigma \in \mathbb{Z}_4$ acts on $\mathbb{C}^4$ by $\sigma \cdot (x, y, z, u) \mapsto (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u)$.

We explain the coboundary map of local cohomology which is introduced in Section 3.3.2 to find a $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold (See also [45, Section 1], [37, Section 4]). Let $\nu\colon \tilde{V} \to V$ be a $\mathbb{Z}_4$-equivariant good resolution and $F \subset \tilde{V}$ its exceptional divisor. Let

$$\tau_V\colon H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V))$$

be the coboundary map of the local cohomology. Let $\tilde{\pi}\colon \tilde{V} \to \tilde{U} := \tilde{V}/\mathbb{Z}_4$ be the finite morphism induced by π and $E \subset \tilde{U}$ the exceptional locus of $\mu\colon \tilde{U} \to U$. Let $\mathcal{F}^{(0)}_U$ be the $\mathbb{Z}_4$-invariant part of $\tilde{\pi}_* \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)$. Then we have the coboundary map

$$\phi_U\colon H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_E(\tilde{U}, \mathcal{F}^{(0)}_U) \tag{5.2}$$

which is the $\mathbb{Z}_4$-invariant part of τ_V.

We have $H^1_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the proof of [59, Theorem 4]. We also have $H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the Guillén–Navarro Aznar–Puerta–Steenbrink vanishing theorem. Thus we have an exact sequence

$$\begin{aligned} 0 \to H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)) &\to H^1(V', \Omega^2_{V'}(-K_{V'})) \\ &\xrightarrow{\tau_V} H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)) \to 0 \end{aligned} \tag{5.3}$$

We have the following inequality.

Proposition 5.2.1. *We have*

$$\dim \operatorname{Ker} \tau_V \leq \dim \operatorname{Im} \tau_V. \tag{5.4}$$

Proof. This is proved in Remark after [45, Theorem(1.1)]. Let us recall the proof for the convenience of the reader.

By the exact sequence (5.3), it is enough to show that

$$h^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \leq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)).$$

We have a surjection

$$H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F))$$

since we have $H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F) \otimes \mathcal{O}_F) = \operatorname{Gr}^2_F H^5_{\{q\}}(V, \mathbb{C}) = 0$. By the local duality, we have

$$H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F))^* \simeq H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)).$$

Moreover we see that the differential homomorphism

$$d\colon H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \to H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))$$

is surjective by analysing the spectral sequence

$$H^q(\tilde{V}, \Omega^p_{\tilde{V}}(\log F)(-F)) \Rightarrow \mathbb{H}^{p+q}(\tilde{V}, \Omega^\bullet_{\tilde{V}}(\log F)(-F)) = 0$$

as in the proof of [45, Theorem 1.1]. Thus we obtain relations

$$\begin{aligned} h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \geq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)) &= h^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \\ &\geq h^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \end{aligned} \tag{5.5}$$

and this implies (5.4). □

Let $T^1_{(V,q)}$, $T^1_{(U,p)}$ be the sets of first order deformations of the germs (V, q) and (U, p) respectively. Recall that we have an isomorphism $T^1_{(V,q)} \simeq \mathcal{O}_{V,q}/J_{V,q}$ of $\mathcal{O}_{V,q}$-modules for the Jacobian ideal $J_{V,q} \subset \mathcal{O}_{V,q}$. Hence we have a surjective $\mathcal{O}_{V,q}$-module homomorphism $\varepsilon\colon \mathcal{O}_{V,q} \to T^1_{(V,q)}$ which sends $h \in \mathcal{O}_{V,q}$ to the corresponding deformation $\varepsilon_h \in T^1_{(V,q)}$. Also we have a commutative diagram

$$\begin{array}{ccc} T^1_{(U,p)} & \xrightarrow{\simeq} & H^1(U', \Omega^2_{U'}(-K_{U'})) \\ \Big\downarrow & & \Big\downarrow \\ T^1_{(V,q)} & \xrightarrow{\simeq} & H^1(V', \Omega^2_{V'}(-K_{V'})), \end{array}$$

where the horizontal isomorphisms are restrictions by open immersions and the upper terms inject into the lower terms as $\mathbb{Z}_4$-invariant parts. Thus, in the proof of Theorem 5.2.3, we identify $T^1_{(V,q)}, T^1_{(U,p)}$ and $H^1(V', \Omega^2_{V'}(-K_{V'})), H^1(U', \Omega^2_{U'}(-K_{U'}))$ respectively via these isomorphisms.

We use the following notion of right equivalence ([16, Definition 2.9]).

Definition 5.2.2. Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the convergent power series ring of n variables. Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$.

f is called *right equivalent* to g if there exists an automorphism φ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$. We write this as $f \overset{r}{\sim} g$.

By using these ingredients, we prove the following.

Theorem 5.2.3. *Let (U, p) be a germ of a non-ordinary 3-dimensional terminal singularity.*

(i) Assume that the index one cover $(V, q) \not\simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Then we have $\phi_U \neq 0$.

(ii) Assume that $(V, q) \simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Then $\phi_U = 0$.

Proof. (i) Suppose that $\phi_U = 0$. We show the claim by contradiction. We can write $g(z, u) = \sum a_{i,j} z^i u^j$ for some $a_{i,j} \in \mathbb{C}$ for $i, j \geq 0$ which are zero except finitely many of them. Since the generator $\sigma \in \mathbb{Z}_4$ acts on g by $\sigma \cdot g = -g$ and on $z^i u^j$ by $\sigma \cdot z^i u^j = \sqrt{-1}^{2i+j} z^i u^j$, we see that $a_{i,j} \neq 0$ only if

$$2i + j \equiv 2 \mod 4. \tag{5.6}$$

Let $J_g := (\frac{\partial g}{\partial z}, \frac{\partial g}{\partial u}) \subset \mathbb{C}[z, u]$ be the Jacobian ideal of the polynomial g. Note that we have $T^1_{(V,q)} \simeq \mathbb{C}[z, u]/(g, J_g)$ since $\varepsilon_x = \varepsilon_y = 0 \in T^1_{V,q}$.

(**Case 1**) Assume that $a_{0,2} \neq 0$. We can write

$$g(z, u) = u^2(1 + h_1(z, u)) + h_2(z)$$

for some polynomials $h_1(z,u) \in (z,u) \subset \mathbb{C}[z,u]$ and $h_2(z) \in (z) \subset \mathbb{C}[z]$. Thus $g(z,u) \in \mathcal{O}_{\mathbb{C}^2,0}$ is right equivalent to $u^2 + h_2(z)$. We see that $h_2(z) \in \mathcal{O}_{\mathbb{C},0}$ is right equivalent to z^{2i_0+1} for some positive integer i_0 since $(g = 0)$ has an isolated singularity and by the condition (5.6). Thus we have

$$(V,q) \simeq ((x^2 + y^2 + z^{2i_0+1} + u^2 = 0), 0).$$

If $i_0 = 1$, it contradicts the assumption $(V,q) \not\simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Hence we have $i_0 \geq 2$. By calculating the partial derivatives of $x^2 + y^2 + z^{2i_0+1} + u^2$, we see that $\varepsilon_1, \varepsilon_z, \varepsilon_{z^2} \in T^1_{(V,q)}$ are linearly independent and

$$\dim T^1_{(V,q)} \geq 3.$$

On the other hand, we see that $\tau_V(\varepsilon_z) = 0$ since we assumed $\phi_U = 0$ and $\varepsilon_z \in T^1_{(U,p)}$. By this and the fact that τ_V is an $\mathcal{O}_{V,q}$-module homomorphism, we obtain a surjection $\mathbb{C}[z,u]/(z,u) \to \operatorname{Im}\tau_V$ since $\varepsilon_u = 0$. By this surjection and $\mathbb{C}[z,u]/(z,u) \simeq \mathbb{C}$, we obtain $\dim \operatorname{Im}\tau_V \leq 1$. By this and the inequality (5.4), we obtain an inequality

$$\dim T^1_{(V,q)} = \dim \operatorname{Im}\tau_V + \dim \operatorname{Ker}\tau_V \leq 1 + 1 = 2$$

and it is a contradiction.

(**Case 2**) Assume that $a_{0,2} = 0$. Then we see that $a_{i,j} \neq 0$ only if $2i + j \geq 6$ by (5.6). Note that a monomial $z^i u^j$ with $2i + j \geq 6$ is some multiple of either z^3, z^2u^2, zu^4 or u^6. By computing partial derivatives of these monomials, we see that $(g, J_g) \subset (z^2, zu^2, u^4)$. Thus we see that $\varepsilon_1, \varepsilon_z, \varepsilon_{zu}, \varepsilon_u, \varepsilon_{u^2}, \varepsilon_{u^3} \in T^1_{(V,q)}$ are linearly independent and we obtain

$$\dim T^1_{(V,q)} \geq 6. \tag{5.7}$$

On the other hand, by the assumption $\phi_U = 0$, we have $\tau_V(\varepsilon_z) = 0, \tau_V(\varepsilon_{u^2}) = 0$ since $\varepsilon_z, \varepsilon_{u^2} \in T^1_{(U,p)}$. Thus we have a relation $(z, u^2) \subset \operatorname{Ker}\tau_V \circ \varepsilon \subset \mathcal{O}_{V,q}$ and obtain a surjection $\mathbb{C}[z,u]/(z,u^2) \to \operatorname{Im}\tau_V$. This implies an inequality $\dim \operatorname{Im}\tau_V \leq \dim \mathbb{C}[z,u]/(z,u^2) = 2$. By this inequality and the inequality (5.4), we have an inequality

$$\dim T^1_{(V,q)} = \dim \operatorname{Ker}\tau_V + \dim \operatorname{Im}\tau_V \leq 2 + 2 = 4.$$

This contradicts (5.7).

Hence we obtain $\phi_U \neq 0$ and finish the proof of (i).

(ii) For non-negative integers i, j, we set

$$b^{i,j} := \dim H^j(\tilde{V}, \Omega^i_{\tilde{V}}(\log F)(-F)),$$

$$l^{i,j} := \dim H^j(F, \Omega^i_{\tilde{V}}(\log F) \otimes \mathcal{O}_F).$$

Let $s_k(V, q)$ for $k = 0, 1, 2, 3$ be the Hodge number of the Milnor fiber of (V, q) as in [59, Section 4]. By [59, Theorem 6], we have $s_0 = 0, s_1 = b^{1,1}, s_2 = b^{1,1} + l^{1,1}$ and $s_3 = l^{0,2}$. We see that $l^{0,2} = 0$ by [59, Lemma 2]. Since the sum $\sum_{k=0}^{3} s_k(V, q)$ is the Milnor number of (V, q), we obtain $2b^{1,1} + l^{1,1} = 2$. Since $b^{1,1} \neq 0$ by [45, Theorem 2.2], we obtain

$$b^{1,1} = 1, \quad l^{1,1} = 0. \tag{5.8}$$

There exists an exact sequence

$$\begin{aligned} H^0(F, \Omega^1_{\tilde{V}}(\log F) \otimes \mathcal{O}_F) \to H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \to H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)) \\ \to H^1(F, \Omega^1_{\tilde{V}}(\log F) \otimes \mathcal{O}_F). \end{aligned} \tag{5.9}$$

Since $l^{1,0} = 0$ by [59, Lemma 1], the both outer terms are zero and the homomorphism in the middle is an isomorphism. By this and (5.8), we have

$$\mathbb{C} \simeq H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)) \simeq H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))^*. \tag{5.10}$$

Suppose that $\tau_V(\varepsilon_z) \neq 0$. Then $\varepsilon_z \notin \operatorname{Ker} \tau_V$. This implies that $\operatorname{Ker} \tau_V = 0$ since $T^1_{(V,q)} \simeq \mathbb{C}[z]/(z^2)$ as $\mathbb{C}[z]$-modules. Thus $\mathbb{C}^2 \simeq \operatorname{Im} \tau_V \simeq H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))$. This contradicts (5.10).

Thus we obtain $\tau_V(\varepsilon_z) = 0$. Since $T^1_{(U,p)} \simeq \mathbb{C}$ is generated by ε_z, we see that $\phi_U = 0$. Thus we finish the proof of (ii). □

We have another coboundary map

$$\bar{\tau}_V \colon H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(-\nu^* K_V))$$

and this fits in the commutative diagram

$$\begin{tikzcd} H^1(V', \Omega^2_{V'}(-K_{V'})) \arrow[r, "\bar{\tau}_V"] \arrow[d, "\tau_V"] & H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(-\nu^* K_V)) \\ H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)), \arrow[ur, hook, "\tau'_V"'] & \end{tikzcd} \tag{5.11}$$

where the injectivity of τ'_V is proved in the proof of [45, Theorem 1.1].

Let $\bar{\mathcal{F}}_U^{(0)} := (\tilde{\pi}_* \Omega^2_{\tilde{V}}(-\nu^* K_V))^{\mathbb{Z}_4}$ be the $\mathbb{Z}_4$-invariant part. Let

$$\bar{\phi}_U \colon H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_E(\tilde{U}, \bar{\mathcal{F}}_U^{(0)})$$

be the coboundary map. It is the $\mathbb{Z}_4$-invariant part of $\bar{\tau}_V$. As the $\mathbb{Z}_4$-invariant part of the diagram (5.11), we obtain the following diagram;

$$\begin{tikzcd} H^1(U', \Omega^2_{U'}(-K_{U'})) \arrow[r, "\bar{\phi}_U"] \arrow[d, "\phi_U"] & H^2_E(\tilde{U}, \bar{\mathcal{F}}_U^{(0)}) \\ H^2_E(\tilde{U}, \mathcal{F}_U^{(0)}). \arrow[ur, hook, "\phi'_U"'] & \end{tikzcd}$$

By these arguments, we obtain the following corollary of Theorem 5.2.3.

Corollary 5.2.4. *Let (U, p) be a germ of a non-ordinary 3-dimensional terminal singularity. Assume that $\bar{\phi}_U = 0$.*

Then we have $(U, p) \simeq \big((x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4(1, 3, 2, 1), 0\big)$.

Since V has only rational singularities, we can define the blow-down morphism ([63])

$$\nu_* \colon H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) \to H^1(V', \Omega^2_{V'}(-K_{V'})).$$

We can prove the relation

$$\operatorname{Im} \nu_* \subset \operatorname{Ker} \tau_V = \operatorname{Ker} \bar{\tau}_V \tag{5.12}$$

by the same argument as in Claim 3.3.7.

5.2.1 Application to $\mathbb{Q}$-smoothing problems

By Proposition 5.2.3 and Theorem 3.3.5 we obtain the following. It almost solves the conjecture in [3, 4.8.3].

Corollary 5.2.5. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then X can be deformed to a $\mathbb{Q}$-Fano 3-fold with only quotient singularities and the non-ordinary terminal singularities $(x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4(1, 3, 2, 1)$.*

As another corollary of Theorem 5.2.3, we obtain a similar result for $\mathbb{Q}$-Calabi–Yau 3-fold. Here, a *$\mathbb{Q}$-Calabi–Yau 3-fold* means a normal projective 3-fold with only terminal singularities whose canonical divisor is a torsion class. Let r be the Gorenstein index of X, that is, the minimal positive integer such that

$\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$. The isomorphism $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$ determines the global index one cover $\pi\colon Y := \operatorname{Spec} \oplus_{j=0}^{r-1} \mathcal{O}_X(jK_X) \to X$.

Corollary 5.2.6. *Let X be a $\mathbb{Q}$-Calabi–Yau 3-fold. Assume that the global index one cover $Y \to X$ is $\mathbb{Q}$-factorial.*

Then a $\mathbb{Q}$-Calabi–Yau 3-fold X can be deformed to one with only quotient singularities and singularities isomorphic to $(x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4(1, 3, 2, 1)$.

Proof. The proof is a modification of the proof of [37, Main Theorem 1]. We sketch the proof for the convenience of the reader.

We can assume that X has only quotient singularities and non-ordinary terminal singularities by [37, Main Theorem 1]. First we prepare notations to define the diagram (5.13).

Let $p_1, \ldots, p_l \in X$ be the non-ordinary singularities and $U_1, \ldots, U_l$ their Stein neighborhoods. Let $\nu\colon \tilde{Y} \to Y$ be a good $\mathbb{Z}_r$-equivariant resolution, $\tilde{\pi}\colon \tilde{Y} \to \tilde{X} := \tilde{Y}/\mathbb{Z}_r$ the quotient morphism and $\mu\colon \tilde{X} \to X$ the induced birational morphism.

Let $V_i := \pi^{-1}(U_i)$, $\tilde{V}_i := \nu^{-1}(V_i)$ and $\nu_i := \nu|_{\tilde{V}_i} \colon \tilde{V}_i \to V_i$ be the restrictions. Let $\tilde{U}_i := \mu^{-1}(U_i)$ and $\tilde{\pi}_i := \tilde{\pi}|_{\tilde{V}_i} \colon \tilde{V}_i \to \tilde{U}_i$ the induced finite morphism. Let $\bar{\mathcal{F}}^{(0)} := \left(\tilde{\pi}_* \Omega^2_{\tilde{Y}}(-\nu^* K_V)\right)^{\mathbb{Z}_r}$ be the $\mathbb{Z}_r$-invariant part and $\bar{\mathcal{F}}_i^{(0)} := \bar{\mathcal{F}}^{(0)}|_{\tilde{U}_i}$ its restriction.

Then we have the diagram

$$\begin{tikzcd} H^1(X', \Omega^2_{X'}(-K_{X'})) \arrow[r, "\oplus\psi_i"] \arrow[d, "\oplus p_{U_i}"] & H^2_E(\tilde{X}, \bar{\mathcal{F}}^{(0)}) \arrow[r, "\oplus B_i"] \arrow[d, "\varphi_i"', "\simeq"] & H^2(\tilde{X}, \bar{\mathcal{F}}^{(0)}) \\ H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'})) \arrow[r, "\bar{\phi}_i"] & \oplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \bar{\mathcal{F}}_i^{(0)}) & \end{tikzcd} \tag{5.13}$$

Note that $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i$ is the $\mathbb{Z}_r$-invariant part of the composition

$$\begin{aligned} H^1(V_i', \Omega^2_{V_i'}(-K_{V_i'})) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(-\nu_i^* K_{V_i})) \to H^2_F(\tilde{Y}, \Omega^2_{\tilde{Y}}(-\nu^* K_Y)) \\ \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(-\nu^* K_Y)). \end{aligned} \tag{5.14}$$

We see that this is zero by [45, Proposition 1.2] since we assumed that Y is $\mathbb{Q}$-factorial. Thus we also see that $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i = 0$.

There exists an element $\eta_i \in H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'}))$ such that $\bar{\phi}_i(\eta_i) \neq 0$ by Corollary 5.2.5. Since $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i(\eta_i) = 0$, there exists $\eta \in H^1(X', \Omega^2_{X'}(-K_{X'}))$ such that $\psi_i(\eta) = \varphi_i^{-1}(\bar{\phi}_i(\eta_i))$. By the relation (5.12) and $p_{U_i}(\eta) - \eta_i \in \operatorname{Ker} \bar{\phi}_i$, we see that $p_{U_i}(\eta) \notin \operatorname{Im}(\nu_i)_*$, where we use the inclusion $H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'})) \subset H^1(V_i', \Omega^2_{V_i'}(-K_{U_i'}))$. By arguing as in the end of the proof of Theorem 3.3.5, we can deform singularity $p_i \in U_i$ as long as $\bar{\phi}_i \neq 0$. By Corollary 5.2.4, we obtain a required deformation since the deformations of a $\mathbb{Q}$-Calabi–Yau 3-fold are unobstructed [42].

$\square$

5.3 Examples

We investigate an example of a $\mathbb{Q}$-Fano 3-fold with the singularity $(x^2+y^2+z^3+u^2 = 0)/\mathbb{Z}_4(1,3,2,1)$. The following is a modification of the example given in [43, Example 5].

Example 5.3.1. Let $S := \mathbb{P}^1 \times \mathbb{P}^1$ and $L := \mathcal{O}_S(-1,-1) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$. Set

$$\varphi_1 := s_0 s_1 \prod_{i=0}^{3}(s_0 - \zeta_4^i s_1),\ \varphi_2 := t_0 t_1 \prod_{j=0}^{3}(t_0 - \zeta_4^j t_1) \in H^0(\mathbb{P}^1, \mathcal{O}(6)),$$

where $\zeta_4 := \sqrt{-1}$ and s_0, s_1 and t_0, t_1 are the homogeneous coordinates on $\mathbb{P}^1$. Set $b_0 := \mathrm{pr}_1^* \varphi_1 \otimes \mathrm{pr}_2^* \varphi_2$.

Let $W_0 \subset \mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3})$ be a hypersurface defined by an equation $f_{W_0} := Y^2 Z = X^3 + b_0 Z^3$, where X, Y, Z are sections determined by natural inclusions

$$X \colon L^{\otimes 2} \hookrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3},$$

$$Y \colon L^{\otimes 3} \hookrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3},$$

$$Z \colon \mathcal{O}_S \hookrightarrow \mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}.$$

By [43, Example 5], a small deformation of W_0 is a hypersurface of the form

$$W = (Y^2 Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}),$$

where $a \in H^0(S, L^{\otimes -4}), b \in H^0(S, L^{\otimes -6})$. Set $p_1 := [0:1], p_2 := [1:0], p_3 := [1:1], p_4 := [\zeta_4 : 1], p_5 := [-1:1], p_6 := [-\zeta_4 : 1] \in \mathbb{P}^1$. Let $\pi \colon W_0 \to S$ be the composition $W_0 \hookrightarrow \mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}) \to S$. We see that W_0 has 36 singular points $p_{i,j}$ for $1 \leq i, j \leq 6$ such that $\pi(p_{i,j}) = (p_i, p_j) \in S$. They are all isomorphic to a cDV singularity

$$0 \in A_{1,2} := (st + x^3 + y^2 = 0) \subset \mathbb{C}^4$$

The elliptic fibration $\pi \colon W_0 \to S$ has a section $\Sigma := (X = Z = 0)$. Since W_0 is smooth along Σ and $\mathcal{N}_{\Sigma/W_0} \simeq \mathcal{O}_S(-1,-1)$, we have a contraction morphism $\nu \colon W_0 \to X_0$. X_0 is a Fano 3-fold with only terminal Gorenstein singularities and we have $H^2(X_0, \Theta_{X_0}) \neq 0$ by [43].

Let $\phi := \phi_1 \times \phi_2$ be an automorphism determined by

$$\phi_1 = \phi_2 = \begin{pmatrix} \zeta_4 & 0 \\ 0 & 1 \end{pmatrix} \in \operatorname{Aut} \mathbb{P}^1 \simeq \mathrm{PGL}(2, \mathbb{C}).$$

Note that

$$\phi_1^* s_0 = \zeta_4 s_0, \ \phi_1^* s_1 = s_1, \ \phi_2^* t_0 = \zeta_4 t_0, \ \phi_2^* t_1 = t_1.$$

This induces $\Phi \in \operatorname{Aut}(\mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}))$ by the base change. We also have $\psi \in \operatorname{Aut}(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3})$ induced by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}$$

and this ψ induces $\Psi \in \operatorname{Aut}_S(\mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}))$ such that

$$\Psi^* X = -X, \ \Psi^* Y = -\zeta_4 Y, \ \Psi^* Z = Z.$$

Set $\theta := \Psi \circ \Phi$. Then we see that $\theta^*(f_{W_0}) = -f_{W_0}$ and θ induces $\theta_{W_0} \in \operatorname{Aut} W_0$. Since $\theta_{W_0}^4 = id$, we have the $\mathbb{Z}_4$-action on W_0. Let $V_0 := W_0/\mathbb{Z}_4$ be the quotient. Since the section Σ of π is preserved by θ_{W_0}, we have the contraction $\mu \colon V_0 \to Y_0 := X_0/\mathbb{Z}_4$ of $\Sigma/\mathbb{Z}_4 \subset V_0$. Note that the 4 singularities $p_{i,j}$ such that $1 \leq i, j \leq 2$ are fixed by the automorphism θ and the remaining 32 singularities $p_{i,j}$ are divided into 8 θ-orbits. Thus we see that V_0 has 8 $A_{1,2}$-singularities and 4 singular points which are isomorphic to

$$A_{1,2}/4 := (st + x^3 + y^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4_{s,t,x,y}/\mathbb{Z}_4(1,1,2,3),$$

where $\mathbb{Z}_4$ acts on s, t, x, y with weights $1, 1, 2, 3$. Let $q := \mu(\Sigma/\mathbb{Z}_4) \in Y_0$ be the singular point arises from the contraction μ. We can check that the singular point q is isomorphic to $(xy + zu = 0)/\mathbb{Z}_4(1,1,1,1)$. We see that this singularity is rigid. Let $T^1_{V_0}$ and $T^1_{Y_0}$ be the sets of first order deformations of V_0 and Y_0 respectively. We have a blow-down morphism $\mu_* \colon T^1_{V_0} \to T^1_{Y_0}$ ([63]) since Y_0 has only rational singularities. Thus we see that $\mu_* \colon T^1_{V_0} \to T^1_{Y_0}$ is an isomorphism.

We see that $X_0 \to Y_0$ is étale outside finite points and $-K_{Y_0}$ is ample.

We shall show that $H^2(V_0, \Theta_{V_0}) \neq 0$. It is enough to show that $T^1_{V_0} \to H^0(V_0, \mathcal{T}^1_{V_0})$ is not surjective, where $\mathcal{T}^1_{V_0} := \underline{\mathrm{Ext}}^1(\Omega^1_{V_0}, \mathcal{O}_{V_0})$ is the Ext sheaf and is supported on singularities of V_0.

By the description of deformations of W_0, we have a surjective homomorphism

$$H^0(S, L^{\otimes -4}) \oplus H^0(S, L^{\otimes -6}) \to T^1_{W_0}$$

which sends (a, b) on the L.H.S. to a deformation

$$(Y^2Z = X^3 + \lambda a X Z^2 + (b_0 + \lambda b)Z^3) \subset \mathbb{P}_S(\mathcal{O}_S \oplus L^{\otimes 2} \oplus L^{\otimes 3}) \times \operatorname{Spec} \mathbb{C}[\lambda]/(\lambda^2).$$

Since we have $\Psi^*(XZ^2) = -XZ^2$ and $\Psi^*(Z^3) = Z^3$, by taking the $\mathbb{Z}_4$-equivariant deformations, we obtain a surjective homomorphism

$$H^0(S, L^{\otimes -4})^{\mathbb{Z}_4} \oplus H^0(S, L^{\otimes -6})^{[2]} \to T^1_{V_0},$$

where $H^0(S, L^{\otimes -4})^{\mathbb{Z}_4}$ is the $\mathbb{Z}_4$-invariant part and

$$H^0(S, L^{\otimes -6})^{[2]} := \{s \in H^0(S, L^{\otimes -6}) \mid \theta^*_{W_0}(s) = -s\}.$$

We can compute that $H^0(S, L^{\otimes -4})^{\mathbb{Z}_4}$ has a basis

$$s_1^4 \otimes t_1^4, s_1^4 \otimes t_0^4, s_0 s_1^3 \otimes t_0^3 t_1, s_0^2 s_1^2 \otimes t_0^2 t_1^2, s_0^3 s_1 \otimes t_0 t_1^3, s_0^4 \otimes t_1^4, s_0^4 \otimes t_0^4.$$

Thus we obtain $\dim H^0(S, L^{\otimes -4})^{\mathbb{Z}_4} = 7$.

Let $q_1, \ldots, q_8 \in V_0$ be the $A_{1,2}$-singularities. For $i = 1, \ldots, 8$, the semi-universal family of a $A_{1,2}$-singularity q_i is $(st + x^3 + \sigma_i x + \tau_i + y^2 = 0) \subset \mathbb{C}^4 \times \mathbb{C}^2$ over $\mathbb{C}^2$ with coordinates σ_i, τ_i. Thus we have

$$\mathcal{T}^1_{V_0, q_i} \simeq \mathbb{C}\sigma_i \oplus \mathbb{C}\tau_i.$$

Let $r_1, \ldots, r_4 \in V_0$ be the $A_{1,2}/4$-singularities. The semi-universal family of r_j is given by $(st + x^3 + \rho_j x + y^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4 \times \mathbb{C}$ over $\mathbb{C}$ with a coordinate ρ_j. Thus we have

$$\mathcal{T}^1_{V_0, r_j} \simeq \mathbb{C}\rho_j.$$

We have a commutative diagram

$$\begin{tikzcd}
H^0(S, L^{\otimes -4})^{\mathbb{Z}_4} \oplus H^0(S, L^{\otimes -6})^{[2]} \arrow[r] \arrow[d] & T^1_{V_0} \arrow[d] \\
\bigoplus_{i=1}^8 (\mathbb{C}\sigma_i \oplus \mathbb{C}\tau_i) \oplus \bigoplus_{j=1}^4 \mathbb{C}\rho_j \arrow[r, "\simeq"] & H^0(V_0, \mathcal{T}^1_{V_0}).
\end{tikzcd}$$

We see that the image of $H^0(S, L^{\otimes -4})^{\mathbb{Z}_4}$ is contained in $\bigoplus_{i=1}^8 \mathbb{C}\sigma_i \oplus \bigoplus_{j=1}^4 \mathbb{C}\rho_j$ and the image of $H^0(S, L^{\otimes -6})^{[2]}$ is contained in $\bigoplus_{i=1}^{8} \mathbb{C}\tau_i$. Thus we see that

$$\begin{aligned}
&\dim \operatorname{Coker}(\alpha\colon T^1_{V_0} \to H^0(V_0, \mathcal{T}^1_{V_0})) \\
&\quad \geq \dim \operatorname{Coker}(\alpha'\colon H^0(S, L^{\otimes -4})^{\mathbb{Z}_4} \to \bigoplus_{i=1}^{8} \mathbb{C}\sigma_i \oplus \bigoplus_{j=1}^{4} \mathbb{C}\rho_j) \geq 8 + 4 - 7 = 5. \quad (5.15)
\end{aligned}$$

Thus α is not surjective and $H^2(Y_0, \Theta_{Y_0}) \neq 0$.

By Theorem 3.3.5 and Corollary 5.2.5, we can deform 8 $A_{1,2}$-singularities. The remaining 4 $A_{1,2}/4$-singularities can not be treated by Corollary 5.2.5. Nevertheless, we can deform these singularities. Indeed, the elements $s_0^4 \otimes t_0^4, s_0^4 \otimes t_1^4, s_1^4 \otimes t_0^4, s_1^4 \otimes t_1^4 \in H^0(S, L^{\otimes -4})^{\mathbb{Z}_4}$ induce $\mathbb{Q}$-smoothings of $p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}$ respectively.

Chapter 6

Deforming non-Du Val elephants of $\mathbb{Q}$-Fano 3-folds

6.1 Introduction

In this chapter, we consider Conjecture 3.1.7 on deformations of elephants of a $\mathbb{Q}$-Fano 3-fold. The conjecture states that; *Let X be a $\mathbb{Q}$-Fano 3-fold such that $|-K_X| \neq \emptyset$. Then X has a deformation $f\colon \mathcal{X} \to \Delta^1$ such that $|-K_{\mathcal{X}_t}|$ contains a Du Val elephant for $t \neq 0$.*

We show that, if there is an elephant with only isolated singularities and X has only quotient singularities, we have such a good deformation as follows.

Theorem 6.1.1. *Let X be a $\mathbb{Q}$-Fano 3-fold with only quotient singularities. Assume that there exists $D \in |-K_X|$ with only isolated singularities.*

Then there exists a deformation $(\mathcal{X}, \mathcal{D}) \to \Delta^1$ of (X, D) over a unit disc such that $\mathcal{D}_t$ has only Du Val singularities for $t \neq 0$.

The assumption that X has only quotient singularities is reasonable since it is conjectured that a $\mathbb{Q}$-Fano 3-fold X can be deformed to one with only quotient singularities. Indeed, it is solved in most of the cases (See Theorem 3.1.5 and Theorem 5.1.1).

The statement is not empty since there is an example of a $\mathbb{Q}$-Fano 3-fold with only terminal quotient singularities and with only non-Du Val elephants (Example 6.4.4).

6.1.1 Strategy of the proof

Let $\operatorname{Sing} D := \{p_1, \ldots, p_l\}$, $U_i \subset X$ a Stein neighborhood of p_i for $i = 1, \ldots, l$ and $D_i := D \cap U_i$. Let $T^1_{(X,D)}, T^1_{(U_i,D_i)}$ be the sets of first order deformations of the pair (X, D) and (U_i, D_i) respectively. Since deformations of the pair (X, D) are unobstructed by Theorem 3.1.6, it is enough to find an element $\eta \in T^1_{(X,D)}$ which deforms singularities of D_i. We study the restriction homomorphism $\oplus p_{U_i} \colon T^1_{(X,D)} \to \oplus_{i=1}^l T^1_{(U_i,D_i)}$ and want to lift a local deformation $\eta_i \in T^1_{(U_i,D_i)}$. There exists an exact sequence

$$T^1_{(X,D)} \overset{\oplus p_{U_i}}{\to} \oplus_{i=1}^l T^1_{(U_i,D_i)} \to H^2(X, \Theta_X(-\log D)),$$

where $\Theta_X(-\log D)$ is the sheaf of tangent vectors which vanish along D. One direct approach is to try to prove $H^2(X, \Theta_X(-\log D)) = 0$. However, this strategy does not work well. Thus we should study the map $\oplus p_{U_i}$ more precisely.

For this purpose, we use some local cohomology groups supported on the exceptional divisor of a suitable "V-resolution" $\mu_i \colon \tilde{U}_i \to U_i$ of the pair (U_i, D_i) for $i = 1 \ldots, l$. A V-resolution means a proper birational morphism such that $\tilde{U}_i$ has only quotient singularities and $\mu_i^{-1}(D_i)$ has VNC support. We use the commutative diagram of the form

$$\begin{tikzcd} T^1_{(X,D)} \arrow[r, "\oplus p_{U_i}"] \arrow[dr, "\oplus \psi_i"'] & \oplus_{i=1}^l T^1_{(U_i,D_i)} \arrow[d, "\oplus \phi_i"] \\ & \oplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \Omega^2_{\tilde{U}_i}(\log \tilde{D}_i + E_i)), \end{tikzcd}$$

where $\tilde{D}_i \subset \tilde{U}_i$ is the strict transform of D_i. The two key statements are the non-vanishing of ϕ_i and the surjectivity of ψ_i. In order to show $\phi_i \neq 0$, we should carefully choose a V-resolution $\mu_i \colon \tilde{U}_i \to U_i$. We first choose a suitable weighted blow-up $\mu_{i,1} \colon U_{i,1} \to U_i$ such that $K_{U_{i,1}} + \tilde{D}_i - \mu_{i,1}^*(K_{U_i} + D_i)$ has negative coefficient (Lemma 6.3.1, Lemma 6.3.3 and Lemma 6.3.4). Next we construct a suitable V-resolution $\mu_{i,12} \colon \tilde{U}_{i,2} \to \tilde{U}_{i,1}$ of the pair $(U_{i,1}, \mu_{i,1}^{-1}(D_i))$ (Lemma 6.3.6). By these careful choices, we can achieve $\phi_i \neq 0$ in Lemma 6.3.10. The surjectivity of ψ_i follows from the fact that $X \setminus D$ is affine. Here we need the Fano assumption.

6.2 Preliminaries on deformations of a pair

6.2.1 Preliminaries on weighted blow-up

We prepare several properties of the weighted blow-up. We refer [22, Section 3] for more details.

Let $v := \frac{1}{r}(a_1, \dots, a_n) \in \frac{1}{r}\mathbb{Z}^n$, $N := \mathbb{Z}^n + \mathbb{Z}v$ a lattice and $M := \mathrm{Hom}(N, \mathbb{Z})$. Let $e_1 := (1, 0, \dots, 0), \dots, e_n := (0, \dots, 0, 1)$ be a basis of $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $\sigma := \mathbb{R}^n_{\geq 0} \subset \mathbb{R}^n$ the cone determined by $e_1, \dots, e_n$. Let $U_v := \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$ be the associated toric variety. We know that $U_v \simeq \mathbb{C}^n/\mathbb{Z}_r(a_1, \dots, a_n)$, where the R.H.S. is the quotient of $\mathbb{C}^n$ by the $\mathbb{Z}_r$-action

$$(x_1, \dots, x_n) \mapsto (\zeta_r^{a_1} x_1, \dots, \zeta_r^{a_n} x_n),$$

where $x_1, \dots, x_n$ are the coordinates on $\mathbb{C}^n$ and ζ_r is the primitive r-th root of unity.

Let $v_1 := \frac{1}{r}(b_1, \dots, b_n) \in N$ be a primitive vector such that $b_i > 0$ for all i. Let Σ_1 be a fan which is formed by the cones σ_i generated by $\{e_1, \dots, e_{i-1}, v_1, e_{i+1}, \dots, e_n\}$ for $i = 1, \dots, n$. Let U_1 be the toric variety associated to the fan Σ_1. Let $\mu_1 \colon U_1 \to U$ be the toric morphism associated to the subdivision. It is a birational morphism with an exceptional divisor $E_1 := \mu_1^{-1}(0) \simeq \mathbb{P}(b_1, \dots, b_n)$. We call μ_1 the *weighted blow-up* with weights v_1.

Let $f := \sum f_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[x_1, \dots, x_n]$ be the $\mathbb{Z}_r$-semi-invariant polynomial with respect to the $\mathbb{Z}_r$-action on $\mathbb{C}^n$. Let

$$\mathrm{wt}_{v_1}(f) := \min\{\sum_{j=1}^n \frac{b_j i_j}{r} \mid f_{i_1, \dots, i_n} \neq 0\}$$

be *the v_1-weight* of f. Let $D_f := (f = 0)/\mathbb{Z}_r \subset U$ be the divisor determined by f and $D_{f,1} \subset U_1$ the strict transform of D_f. Then we have the following;

$$K_{U_1} = \mu_1^* K_U + \frac{1}{r}(\sum_{i=1}^n b_i - r) E_1, \tag{6.1}$$

$$D_{f,1} = \mu_1^* D_f - \mathrm{wt}_{v_1}(f) E_1. \tag{6.2}$$

Let $U_{1,i} \subset U_1$ be the affine open subset which corresponds to the cone σ_i. Then we have $U_1 = \bigcup_{i=1}^n U_{1,i}$ and

$$U_{1,i} \simeq \mathbb{C}^n/\mathbb{Z}_{a_i}(-a_1, \dots, \overset{i\text{-th}}{r}, \dots, -a_n).$$

Moreover the morphism $\mu_1|_{U_{1,i}}: U_{1,i} \to U$ is described by

$$(x_1, \dots, x_n) \mapsto (x_1 x_i^{a_1/r}, \dots, x_i^{a_i/r}, \dots, x_n x_i^{a_n/r}).$$

6.2.2 Deformations of a divisor in a terminal 3-fold

We first define discrepancies of a log pair.

Definition 6.2.1. Let U be a normal variety and D its divisor such that $K_U + D$ is $\mathbb{Q}$-Cartier, that is, $m(K_U + D)$ is a Cartier divisor for some positive integer m. Let $\mu\colon \tilde{U} \to U$ be a proper birational morphism from another normal variety and $E_1, \dots, E_l$ its exceptional divisors. Let $\tilde{D} \subset \tilde{U}$ be the strict transform of D.

We define a rational number $a(E_i, U, D)$ as the number such that

$$m(K_{\tilde{U}} + \tilde{D}) = \mu^*(m(K_U + D)) + \sum_{i=1}^{l} ma(E_i, U, D)E_i.$$

We call $a(E_i, U, D)$ the *discrepancy* of E_i with respect to the pair (U, D).

Let U be a Stein neighborhood of a 3-fold terminal singularity of Gorenstein index r and D a $\mathbb{Q}$-Cartier divisor on U. We have the index one cover $\pi_U\colon V := \operatorname{Spec} \oplus_{j=0}^{r-1} \mathcal{O}_U(jK_U) \to U$ determined by an isomorphism $\mathcal{O}_U(rK_U) \simeq \mathcal{O}_U$. Let $G := \operatorname{Gal}(V/U) \simeq \mathbb{Z}_r$ be the Galois group of π_U. This induces a G-action on the pair (V, Δ), where $\Delta := \pi_U^{-1}(D)$. We can define functors of G-equivariant deformations of (V, Δ) as follows.

Definition 6.2.2. Let $\operatorname{Def}^G_{(V,\Delta)}\colon (Art_{\mathbb{C}}) \to (Sets)$ be a functor such that, for $A \in (Art_{\mathbb{C}})$, a set $\operatorname{Def}^G_{(V,\Delta)}(A) \subset \operatorname{Def}_{(V,\Delta)}(A)$ is the set of deformations $(\mathcal{V}, \boldsymbol{\Delta})$ of (V, Δ) over A with a G-action which is compatible with the G-action on (V, Δ).

We can also define the functor $\operatorname{Def}^G_V\colon (Art_{\mathbb{C}}) \to (Sets)$ of G-equivariant deformations of V similarly.

Proposition 6.2.3. *We have isomorphisms of functors*

$$\operatorname{Def}^G_{(V,\Delta)} \simeq \operatorname{Def}_{(U,D)}, \qquad \operatorname{Def}^G_V \simeq \operatorname{Def}_U. \tag{6.3}$$

Moreover, these functors are unobstructed and the forgetful homomorphism $\operatorname{Def}_{(U,D)} \to \operatorname{Def}_U$ *is a smooth morphism of functors.*

Remark 6.2.4. The latter isomorphism $\operatorname{Def}^G_V \simeq \operatorname{Def}_U$ is given in [42, Proposition 3.1].

Proof. For a G-equivariant deformation of (V,Δ), we can construct a deformation of (U,D) by taking its quotient by G. Conversely, given a deformation $(\mathcal{U},\mathcal{D})$ of (U,D). Let $\iota\colon U' := U \setminus \{p\} \hookrightarrow U$ be an open immersion and $\mathcal{U}' \to \operatorname{Spec} A$ a deformation of U' induced by $\mathcal{U}$. Let $\omega^{[i]}_{\mathcal{U}/A} := \iota_*\omega^{\otimes i}_{\mathcal{U}'/A}$. This is flat over A_n by [31, Theorem 12]. Thus we can construct a G-equivariant deformation of (V,Δ) by

$$\pi_{\mathcal{U}}\colon \mathcal{V} := \operatorname{Spec}_{\mathcal{U}} \oplus_{i=0}^{r-1}\omega^{[i]}_{\mathcal{U}/A} \to \mathcal{U}$$

and $\mathbf{\Delta} := \pi^*_{\mathcal{U}}(\mathcal{D})$, where $\pi_{\mathcal{U}}$ is defined by an isomorphism $\varphi_{s_{\mathcal{U}}}\colon \omega^{[r]}_{\mathcal{U}/A} \simeq \mathcal{O}_{\mathcal{U}}$ for some nowhere vanishing section $s_{\mathcal{U}} \in H^0(\mathcal{U},\omega^{[r]}_{\mathcal{U}/A})$. Note that $\pi_{\mathcal{U}}$ is independent of the choice of a section $s_{\mathcal{U}}$. We can check that these constructions are converse to each other. Thus we obtain the required isomorphisms of functors.

Since V has only l.c.i. singularities and Δ is its Cartier divisor, we see the latter statements. Thus we finish the proof of Proposition 6.2.3. □

These arguments imply the following.

Proposition 6.2.5. *Let $U, D, \pi_U\colon V \to U, \Delta$ as above. Then we have*

$$T^1_{(U,D)} \simeq (T^1_{(V,\Delta)})^G, \quad T^1_U \simeq (T^1_V)^G.$$

We check these isomorphisms in the following examples.

Example 6.2.6. Let $U := \mathbb{C}^3/\mathbb{Z}_2(1,1,1)$ and $D := (x^3+y^3+z^3=0)/\mathbb{Z}_2 \subset U$ its divisor. In this case, we can write $V = \mathbb{C}^3$ and $\Delta = (x^3+y^3+z^3=0) \subset V$. We have

$$T^1_{(U,D)} \simeq (T^1_{(V,\Delta)})^{\mathbb{Z}_2} \simeq (\mathcal{O}_{\mathbb{C}^3,0}/(x^2,y^2,z^2))^{[-1]} \simeq \mathbb{C}\eta_x \oplus \mathbb{C}\eta_y \oplus \mathbb{C}\eta_z \simeq \mathbb{C}^3,$$

where $(\mathcal{O}_{\mathbb{C}^3,0}/(x^2,y^2,z^2))^{[-1]} := \{f \in \mathcal{O}_{\mathbb{C}^3,0}/(x^2,y^2,z^2) \mid g\cdot f = -f\}$.

6.2.3 Additional lemma

We need the following lemma due to Professor Angelo Vistoli.

Lemma 6.2.7. *Let $f \in \mathbb{C}[x,y,z]$ be a polynomial which defines an isolated singularity $0 \in D := (f=0) \subset \mathbb{C}^3$. Assume that $\mathcal{D} := (f+tx=0) \subset \mathbb{C}^3 \times \Delta^1$ defines a smoothing of D over a unit disk Δ^1. Let $g \in \mathbb{C}[x,y,z]$ be a polynomial such that $\operatorname{mult}_0 g \geq 2$. Then $\mathcal{D}' := (f+t(x+g)=0) \subset \mathbb{C}^3 \times \Delta^1$ is also a smoothing of D.*

Proof. Consider the linear system

$$\{C_{[s:t]} := (sf + t(x+g) = 0) \subset \mathbb{C}^3 \mid [s:t] \in \mathbb{P}^1\}.$$

By Bertini's theorem $C_{[s:t]}$ is smooth away from the base points of the linear system, for all but finitely many values of $[s:t]$. If $p \in \mathbb{C}^3$ is a base point of the linear system, then either p is the origin, in which case $C_{[0:1]}$ is smooth at p, or is not, and in this case $C_{[1:0]}$ is smooth at p. Since being smooth at a base point is an open condition, we have that C_t is smooth at all points of $\mathbb{C}^3$ for all but finitely many values of t. □

6.2.4 Blow-down morphism of deformations

Let X be an algebraic variety and $\tilde{X} \to X$ its resolution of singularities. Suppose we have a deformation $\tilde{\mathcal{X}} \to \operatorname{Spec} A$ over an Artin ring A. If X has only rational singularities, we can "blow-down" the deformation $\tilde{\mathcal{X}}$ to a deformation of X.

We need the following proposition in general setting.

Proposition 6.2.8. *([63, Section 0]) Let X be an algebraic scheme over k and $A \in \mathrm{Art}_k$. Let $\mathcal{X} \to \operatorname{Spec} A$ be a deformation of X and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{X}$, flat over A, inducing $F := \mathcal{F} \otimes_A k$ on X.*

If $H^1(X, F) = 0$, then ϕ^0 is an isomorphism and $H^0(\mathcal{X}, \mathcal{F})$ is A-flat.

Proposition 6.2.8 implies the following.

Corollary 6.2.9. *Let $X \to Y$ be a proper birational morphism of integral normal k-schemes. Assume that $R^1 f_* \mathcal{O}_Y = 0$.*

Then there exists a morphism of functors

$$f_* \colon \operatorname{Def}_X \to \operatorname{Def}_Y$$

defined as follows: For a deformation $\mathcal{X} \to \operatorname{Spec} A$ of X over $A \in \mathrm{Art}_k$, we define its image by f_* as the scheme $\mathcal{Y} = (Y, f_* \mathcal{O}_{\mathcal{X}})$.

We call this transformation the blow-down morphism.

For a surface with non-rational singularities, Wahl considered "equisingularity" of deformations via the blow-down transformation. Although the blow-down transformation is not always possible, we can still consider the "equisingular deformation functor" as follows.

Definition 6.2.10. Let $U := \operatorname{Spec} R$ be an affine normal surface over k with a singularity at p and $f \colon X \to U$ a resolution of a singularity such that $f^{-1}(p)$ has SNC support. Wahl ([63, (2.4)]) defined an equisingular deformation of the resolution of a singularity as a deformation of (X, E) whose blow-down can be defined. More precisely, he defined a functor $\mathrm{ES}_X \colon \mathrm{Art}_k \to (Sets)$ by setting

$$\mathrm{ES}_X(A) := \{(\mathcal{X}, \mathcal{E}) \in \mathrm{Def}_{(X,E)}(A) \mid H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) : A\text{-flat.}\}$$

There exists a natural transformation $f_* \colon \mathrm{ES}_X \to \mathrm{Def}_U$ and this induces a linear map $f_*(A_1) \colon \mathrm{ES}_X(A_1) \to \mathrm{Def}_U(A_1)$ on the tangent spaces.

Equisingular deformation should preserve some properties of a singularity. For example, it is known that equisingular deformations of an isolated 2-dimensional hypersurface singularity do not change the Milnor number ([63]). In particular, smoothings of a hypersurface singularity can not be equisingular. However, the situation is a bit different in higher codimension case. Although a singularity has high multiplicity in general, an equisingular deformation may be induced by an equation of multiplicity one. This phenomenon does not happen in the hypersurface case as shown in Lemma 6.2.7. In the following, we exhibit such an example due to Wahl ([64]) of a deformation of an isolated complete intersection singularity (ICIS for short).

Example 6.2.11. Let $U := (xy - z^2 = x^4 + y^4 + w^2 = 0) \subset \mathbb{C}^4$ be an ICIS and $\mathcal{U} := (xy - z^2 + tw = x^4 + y^4 + w^2 = 0) \subset \mathbb{C}^4 \times \mathbb{C}$ a deformation of U, where x, y, z, w are coordinates on $\mathbb{C}^4$ and t is a deformation parameter of $\mathbb{C}$. For any value of t, the singularity $\mathcal{U}_t$ is a cone (C_t, K_{C_t}) for a smooth curve C_t of genus 3 and its canonical bundle, that is, $\mathcal{U}_t \simeq \mathrm{Spec} \oplus_{k=0}^{\infty} H^0(C_t, kK_{C_t})$. We see that $C_0 \simeq (xy - z^2 = x^4 + y^4 + w^2 = 0) \subset \mathbb{P}(1,1,1,2)$ is a hyperelliptic curve and C_t for $t \neq 0$ is a smooth quartic curve in $\mathbb{P}^2$. The singularity has a resolution

$$f_t \colon \mathrm{Tot}(\mathcal{O}_{C_t}(K_{C_t})) := \mathrm{Spec} \oplus_{k=0}^{\infty} \mathcal{O}_{C_t}(kK_{C_t}) \to \mathcal{U}_t,$$

where $\mathrm{Tot}(\mathcal{O}_{C_t}(K_{C_t}))$ is the total space of the line bundle $\mathcal{O}_{C_t}(K_{C_t})$. It is actually a contraction of the zero section. Thus we get a family of contractions $\tilde{\mathcal{U}} \to \mathcal{U}$. Let $\eta_w \in T^1_{(U,p)}$ be the element corresponding to the deformation $\mathcal{U}$. By the above description, we see that $\eta_w \in \mathrm{Im}(f_0)_*$. Recall that $T^1_{U,p} \simeq \mathcal{O}_{U,p}^{\oplus 2}/J_p$ for the Jacobian sub-module J_p determined by the partial derivatives of the defining equations of U. Since the order of w is one, we see that $\eta_w \notin \mathfrak{m}_{U,p}^2 T^1_{(U,p)}$.

We use the pair version of the blow-down transformation as follows.

Let X be a normal variety with only rational singularities and $D = \sum_{j \in J} D_j$ a sum of effective Cartier divisors D_j on X. Let $\mu \colon \tilde{X} \to X$ be a resolution of singularities of X. Let $\tilde{D} \subset \tilde{X}$ be the strict transform of D and $E = \sum_{i=1}^{m} E_i$ the exceptional locus of μ. Since X has only rational singularities, we see that $\mu_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X$ and $R^1 \mu_* \mathcal{O}_{\tilde{X}} = 0$.

Proposition 6.2.12. *Let $X, D, \tilde{X}, \tilde{D}, E$ be as above. Then we can define a morphism of functors*

$$\mu_* \colon \mathrm{Def}_{(\tilde{X}, \tilde{D}+E)} \to \mathrm{Def}_{(X,D)}$$

Proof. Consider a deformation $(\tilde{\mathcal{X}}, \sum_{j \in J} \tilde{\mathcal{D}}_j + \sum_{i=1}^{m} \mathcal{E}_i)$ of $(X, \tilde{D} + E)$ over $A \in \mathrm{Art}_k$. We can blow down a deformation $\tilde{\mathcal{X}}$ of $\tilde{X}$ over A as in Corollary 6.2.9 since $R^1 \mu_* \mathcal{O}_{\tilde{X}} = 0$.

Let $\mathbf{I}_{D_j}, \mathbf{I}_{E_i} \subset \mathcal{O}_{\tilde{\mathcal{X}}}$ be the ideal sheaves of given deformations of D_j, E_i respectively. We can write

$$\mu^* D_j = \tilde{D}_j + \sum_{i=1}^{m} a_{i,j} E_i$$

by some non-negative integers $a_{i,j}$. We can define a deformation of $D_j \subset X$ by the ideal

$$\mu_* \left(\mathbf{I}_{\tilde{D}_j} \cdot \prod_{i=1}^{m} \mathbf{I}_{E_i}^{a_{i,j}} \right) \subset \mathcal{O}_{\mathcal{X}}.$$

We can check that this ideal is A-flat by Proposition 6.2.8 (iii) and

$$R^1 \mu_* \mathcal{O}_{\tilde{X}}(\tilde{D}_j + \sum_{i=1}^{m} a_{i,j} E_i) = R^1 \mu_* \mu^* \mathcal{O}_X(D_j) = 0.$$

$\square$

Example 6.2.13. Let $D \subset U$ be a reduced divisor in a smooth 3-fold U. Let $\mu \colon \tilde{U} \to U$ be a proper birational morphism from another smooth variety $\tilde{U}$. Let $\tilde{D} \subset \tilde{U}$ be the strict transform of D and E the μ-exceptional divisor. Then we can define a natural transformation $\mu_* \colon \mathrm{Def}_{(\tilde{U}, \tilde{D}+E)} \to \mathrm{Def}_{(U,D)}$ and this induces a homomorphism $\mu_* \colon T^1_{(\tilde{U}, \tilde{D}+E)} \to T^1_{(U,D)}$ on the tangent spaces. We use this homomorphism in the proof of Lemma 6.3.10. The point is that we can define the blow-down transformation even if some irreducible component of D has non-isolated singularities. When D has only isolated singularities, the definition of the blow-down transformation is easier (See (6.7), for example).

6.3 Deformations of elephants with isolated singularities

In this section, we treat deformations of a pair of a $\mathbb{Q}$-Fano 3-fold and a member of $|-K_X|$ with only isolated singularities.

6.3.1 First blow-up

Consider a $\mathbb{Q}$-Fano 3-fold X and its elephant D with only isolated singularities. Take a non-Du Val singularity p on D and its Stein neighborhood $U \subset X$. We first prepare lemmas on a weighted blow-up of the Stein neighborhood U. We want to construct a weighted blow-up $U_1 \to U$ whose exceptional divisor E_1 has a negative discrepancy $a(E_1, U, D)$. Since U is analytic locally isomorphic to $\mathbb{C}^3$ or $\mathbb{C}^3/\mathbb{Z}_r(1, a, r-a)$ for some coprime integers r and a, we argue for these latter spaces. We use the same symbol 0 for the origin of $\mathbb{C}^3$ and its image on $\mathbb{C}^3/\mathbb{Z}_r$.

The following is the easiest case where a singularity on a divisor is a hypersurface singularity of multiplicity 3 or higher.

Lemma 6.3.1. *Let $U := \mathbb{C}^3$ and $D \subset U$ a divisor with an isolated singularity at* 0*. Assume that $m_D := \operatorname{mult}_0 D \geq 3$. Let $\mu_1 \colon U_1 \to U$ be the blow-up at the origin* 0*.*

Then the discrepancy $a(E_1, U, D)$ satisfies

$$a(E_1, U, D) = 2 - m_D \leq -1. \tag{6.4}$$

Proof. This follows since we have $K_{U_1} = \mu_1^* K_U + 2E_1$ and $D_1 = \mu_1^* D - m_D E_1$. $\square$

We use the following notion of right equivalence ([16, Definition 2.9]).

Definition 6.3.2. Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the convergent power series ring of n variables. Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$.

f is called *right equivalent* to g if there exists an automorphism φ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$. We write this as $f \overset{r}{\sim} g$.

The following double point in a smooth neighborhood is actually the most tricky case.

Lemma 6.3.3. *Let $0 \in D := (f = 0) \subset \mathbb{C}^3 =: U$ be a divisor such that $\operatorname{mult}_0 D = 2$ and $0 \in D$ is not a Du Val singularity.*

Then there exists a birational morphism $\mu_1 \colon U_1 \to U$ which is a weighted blow-up of weights $(3, 2, 1)$ or $(2, 1, 1)$ for a suitable coordinate system on U such *that the discrepancy $a(E_1, U, D)$ of the μ_1-exceptional divisor E_1 satisfies*

$$a(E_1, U, D) \leq -1.$$

Proof. By taking a suitable coordinate change, we can write $f = x^2 + g(y,z)$ for some $g(y,z) \in \mathbb{C}[y,z]$ which defines a reduced curve $(g(y,z) = 0) \subset \mathbb{C}^2$. We see that $\operatorname{mult}_0 g(y,z) \geq 3$ since, if $\operatorname{mult}_0 g(y,z) = 2$, we see that D has a Du Val singularity of type A at 0. We can write $g(y,z) = \sum g_{i,j} y^i z^j$ for $g_{i,j} \in \mathbb{C}$. We divide the argument with respect to the multiplicity $\operatorname{mult}_0 g(y,z)$ of the polynomial $g(y,z)$.

(**Case 1**) Consider the case $\operatorname{mult}_0 g(y,z) \geq 4$. Let $\mu_1 \colon U_1 \to U$ be the weighted blow-up with weights $(2,1,1)$ and $D_1 \subset U_1$ the strict transform of D. Then we have

$$K_{U_1} = \mu_1^* K_U + 3E_1,$$

$$\mu_1^* D = D_1 + m_D E_1,$$

where $m_D = \min\{4, \min\{i + j \mid g_{i,j} \neq 0\}\}$. By the assumption $\operatorname{mult}_0 g(y,z) \geq 4$, we see that $g_{i,j} \neq 0$ only if $i + j \geq 4$. Thus we see that $m_D = 4$. Since we have

$$K_{U_1} + D_1 = \mu_1^*(K_U + D) - E_1,$$

the weighted blow-up μ_1 satisfies the required property.

(**Case 2**) Consider the case $\operatorname{mult}_0 g(y,z) = 3$. Let $g^{(k)} := \sum_{i+j \leq k} g_{i,j} y^i z^j$ be the *k-jet* of g. We divide this into two cases with respect to $g^{(3)}$. The proof uses the arguments in the classification of simple singularities of type D and E ([16, Theorem 2.51, 2.53]).

(2.1) Suppose that $g^{(3)}$ factors into at least two different factors. By [16, Theorem 2.51], we see that $g \overset{r}{\sim} y(z^2 + y^{k-2})$ for some $k \geq 4$. Thus $0 \in D$ is a Du Val singularity of type D_k. This contradicts the assumption.

(2.2) Suppose that $g^{(3)}$ has a unique linear factor. We can write $g^{(3)} = y^3$ by a suitable coordinate change. By the proof of [16, Theorem 2.53], the 4-jet $g^{(4)}$ can be written as

$$g^{(4)} = y^3 + \alpha z^4 + \beta y z^3$$

for some $\alpha, \beta \in \mathbb{C}$.

(i) If $\alpha \neq 0$, we obtain $g \overset{r}{\sim} y^3 + z^4$ by the same argument as [16, Theorem 2.53, Case E_6]. Thus we see that $0 \in D$ is a Du Val singularity of type E_6.

(ii) If $\alpha = 0$ and $\beta \neq 0$, we obtain $g \overset{r}{\sim} y^3 + yz^3$ by the same argument as [16, Theorem 2.53, Case E_7]. Thus we see that $0 \in D$ is a Du Val singularity of type E_7.

(iii) Now assume that $\alpha = \beta = 0$. In this case, the 5-jet $g^{(5)}$ can be written as

$$g^{(5)} = y^3 + \gamma z^5 + \delta y z^4$$

for some $\gamma, \delta \in \mathbb{C}$.

If $\gamma \neq 0$, we obtain $g \overset{r}{\sim} y^3 + z^5$ by the same argument as [16, Theorem 2.53, Case E_8]. Thus we see that $0 \in D$ is a Du Val singularity of type E_8.

If $\gamma = 0$ and $\delta \neq 0$, we can write $g = y^3 + yz^4 + h_6(y, z)$ for some $h_6(y, z) \in \mathbb{C}[y, z]$ such that $\operatorname{mult}_0 h_6(y, z) \geq 6$. Let $\mu_1 \colon U_1 \to U$ be the weighted blow-up with weights $(3, 2, 1)$ on (x, y, z) and E_1 its exceptional divisor. Then we can calculate

$$K_{U_1} = \mu_1^* K_U + 5E_1,$$

$$\mu_1^* D = D_1 + 6E_1$$

by the formula (6.2). Thus we obtain

$$K_{U_1} + D_1 = \mu_1^*(K_U + D) - E_1.$$

Hence μ_1 has the required property.

If $\gamma = \delta = 0$, we can write $g = y^3 + h_6$ for some nonzero h_6 such that $\operatorname{mult}_0 h(y, z) \geq 6$. Let $\mu_1 \colon U_1 \to U$ be the weighted blow-up with weights $(3, 2, 1)$ as above. We can similarly check that this μ_1 has the required property.

□

For $U = \mathbb{C}^3/\mathbb{Z}_r(1, a, r-a)$, we can take $1/r(1, a, r-a)$-weighted blow-up for the first blow-up as follows.

Lemma 6.3.4. *Let $U = \mathbb{C}^3/\mathbb{Z}_r(1, a, r-a)$ be the quotient variety for some coprime integers r and a such that $0 < a < r$ and $D \in |-K_U|$ an anticanonical divisor with only isolated singularity at $0 \in U$. Let $\pi_U \colon V = \mathbb{C}^3 \to U$ be the quotient morphism and $\Delta := \pi_U^{-1}(D)$. Assume that $\operatorname{mult}_0 \Delta \geq 2$. Let $\mu_1 \colon U_1 \to U$ be the weighted blow-up with weights $1/r(1, a, r-a)$ and E_1 its exceptional divisor.*

Then we have an inequality on the discrepancy

$$a(E_1, U, D) \leq -1.$$

Proof. Let $f = \sum f_{i,j,k} x^i y^j z^k$ be the defining equation of $\Delta \subset \mathbb{C}^3$ at $0 \in \mathbb{C}^3$. We have

$$K_{U_1} = \mu_1^* K_U + \frac{1}{r}(1 + a + r - a - r)E_1 = \mu_1^* K_U + \frac{1}{r}E_1,$$

$$\mu_1^* D = D_1 + \frac{m_D}{r}E_1,$$

where $m_D := \min\{i + aj + (r - a)k \mid f_{i,j,k} \neq 0\}$. We see that $m_D \geq 2$ since Δ is singular. Thus we can write

$$K_{U_1} + D_1 = \mu_1^*(K_U + D) + \frac{1}{r}(1 - m_D)E_1$$

and $\frac{1}{r}(1 - m_D) < 0$. Since $K_U + D$ is a Cartier divisor, we see that $\frac{1}{r}(1 - m_D)$ is a negative integer. Thus μ_1 satisfies the required condition. $\square$

6.3.2 Second blow-up

Let $U_1 \to U$ be either one of the weighted blow-ups constructed in Section 6.3.1. We use the same notation as Section 6.3.1.

We define the "VNC"-pair as follows.

Definition 6.3.5. (cf. [58, Definition 1.16]) Let U be a normal variety with only cyclic quotient singularities and $D \subset U$ its reduced divisor. A pair (U, D) is called a *VNC pair* if, for each point p, there exists a Stein neighborhood U_p and a cyclic cover $\pi_p \colon V_p \to U_p$ such that V_p is smooth and $\pi_p^{-1}(D \cap U_p)$ is a normal crossing divisor. We call D a *VNC divisor*. Moreover, if $\pi_p^{-1}(D \cap U_p)$ is a smooth divisor for each point $p \in U$, we call (U, D) a *V-smooth pair*.

Let U be a normal variety and D its divisor. Let $\mu \colon \tilde{U} \to U$ is a proper birational morphism. We say that μ is a *V-resolution* of the pair (U, D) if $(\tilde{U}, \mu^{-1}(D))$ is a VNC pair.

We construct a useful V-resolution $U_2 \to U_1$ of $(U_1, D_1 + E_1)$ as follows.

Lemma 6.3.6. *Let $\mu_1 \colon U_1 \to U, D_1$ and E_1 be those as in Section 6.3.1.*

Then there exists a projective birational morphism $\mu_{12} \colon U_2 \to U_1$ and a finite set $Z \subset U_1$ such that U_2 has only quotient singularities, $\mu_{12}^{-1}(D_1 \cup E_1)$ has VNC support, and $U_1' := U_1 \setminus Z$ satisfies the following conditions; U_1' and $U_2' := \mu_{12}^{-1}(U_1')$ are smooth and the morphism $\mu_{12}' := \mu_{12}|_{U_2'} \colon U_2' \to U_1'$ is a composition of blow-ups of smooth curves in the strict transforms of $D_1' := U_1' \cap D_1$.

As a consequence, the discrepancies satisfy

$$a(E_{2,j}', U_1', D_1') \leq 0$$

for all μ'_{12}-exceptional divisors $E'_{2,j}$.

Proof. By the construction of $\mu_1 \colon U_1 \to U$, we see that U_1 has only terminal quotient singularities and $-K_{U_1}$ is μ_1-ample. Let m be a sufficiently large integer such that $-mK_{U_1}$ is μ_1-free. Take a general member $A_m \in |-mK_{U_1}|$ such that A_m and E_1 intersects transversely, $A_m \cap \operatorname{Sing} U_1 = \emptyset$ and the intersection of $(D_1)^{\mathrm{sm}}$ and A_m is transversal.

Let $\pi_1 \colon V_1 := \operatorname{Spec} \oplus_{j=0}^{m-1} \mathcal{O}_{U_1}(jK_{U_1}) \to U_1$ be the cyclic cover branched along A_m. Set $\Delta_1 := \pi_1^{-1}(D_1)$ and $F_1 := \pi_1^{-1}(E_1)$. Then we see that V_1 is smooth and the non-SNC locus of $\Delta_1 \cup F_1$ is contained in $\Delta_1 \cap F_1$. We can construct a $\mathbb{Z}_m$-equivariant resolution $f_{k,1} \colon V_k \to V_1$ of the pair $(V_1, \Delta_1 + F_1)$ which is a composition

$$f_{k,1} \colon V_k \overset{f_{V_{k-1}}}{\to} V_{k-1} \to \cdots \to V_2 \overset{f_{V_1}}{\to} V_1$$

of blow-ups $f_{V_i} \colon V_{i+1} \to V_i$ of smooth irreducible subvarieties $Z_i \subset V_i$ for $i = 1, \ldots, k-1$ such that $f_{k,1}^{-1}(\Delta_1 \cup F_1)$ has SNC support. Let $\Delta_i \subset V_i$ be the strict transform of Δ_1 and $f_{i,1} := f_{V_1} \circ \cdots \circ f_{V_{i-1}} \colon V_i \to V_1$. We can take these centers Z_i such that $Z_i \subset \Delta_i$ and with the following conditions;

(i) $Z_i \subset \operatorname{Sing} \Delta_i$ if Δ_i is singular,

(ii) Z_i is contained in the non-SNC locus of $f_{i,1}^{-1}(\Delta_1 \cup F_1)$ if Δ_i is smooth.

Let

$$B^0 := \bigcup_{\dim f_{i,1}(Z_i)=0} f_{i,1}(Z_i)$$

be the union of the images of Z_i over points on V_1 and

$$B^1 := \bigcup_{f_{i_1,1}(Z_{i_1}) \neq f_{i_2,1}(Z_{i_2})} (f_{i_1,1}(Z_{i_1}) \cap f_{i_2,1}(Z_{i_2}))$$

the intersections of the different images of 1-dimensional centers. We also set $B^{\mathrm{sing}} := \pi_1^{-1}(\operatorname{Sing} U_1)$ and $B^{\mathrm{ram}} := \pi_1^{-1}(A_m) \cap (\Delta_1 \cap F_1)$. Let

$$B_0 := B^0 \cup B^1 \cup B^{\mathrm{sing}} \cup B^{\mathrm{ram}} \tag{6.5}$$

be the 0-dimensional locus we want to remove and $Z := \pi_1(B_0)$. Since $B^{\mathrm{sing}} \subset B_0$, we see that $U'_1 := U \setminus Z$ is smooth. Since $f_{k,1}$ is $\mathbb{Z}_m$-equivariant, we can take $U_2 := V_k/\mathbb{Z}_m$ and $f_{k,1}$ induces a birational morphism $\mu_{12} \colon U_2 \to U_1$. Let $U'_2 := \mu_{12}^{-1}(U'_1)$ and $\mu'_{12} := \mu_{12}|_{U'_2} \colon U'_2 \to U'_1$. We shall see that μ'_{12} is a composition of blow-ups of smooth curves in the following.

Let $V'_1 := V_1 \setminus B_0$, $V'_i := f_{i,1}^{-1}(V'_1)$, where $f_{i,1} := f_{V_{i-1}} \circ \cdots f_{V_1} \colon V_i \to V_1$. We see that $Z'_i := Z_i \cap V'_i$ is the center of the blow-up $f_{V'_i} := f_{V_i}|_{V'_{i+1}} \colon V'_{i+1} \to V'_i$. Let $\sigma \in \mathbb{Z}_m$ be the generator. We see that $\mathbb{Z}_m$-orbit $\{Z'_1, \sigma \cdot Z'_1, \ldots, \sigma^{m-1} \cdot Z'_1\}$ are disjoint since we have $B^1 \subset B_0$ as in (6.5). Let $f_{W_1} \colon W_2 \to W_1 := V_1$ be the blow-up of the $\mathbb{Z}_m$-invariant center $\bigcup_{j=0}^{m-1} \sigma^j Z'_1$. By repeating this operation, we can construct the morphism $V'_k \to V'_1$ as a composition of blow-ups of $\mathbb{Z}_m$-invariant centers;

$$V'_k = W_l \to W_{l-1} \to \cdots \to W_2 \to W_1 = V'_1,$$

where $f_{W_i} \colon W_{i+1} \to W_i$ is a blow-up of a $\mathbb{Z}_m$-invariant smooth curve Z_{W_i} contained in the strict transform of $\Delta'_1 := \Delta_1 \cap V'_1$. By this construction, the $\mathbb{Z}_m$-action on V_1 induces the action on each W_i and we can take the $\mathbb{Z}_m$-quotient $Q_i := W_i/\mathbb{Z}_m$ and $g_{Q_i} \colon Q_{i+1} \to Q_i$ for $i = 1, \ldots, l-1$. Let $\pi'_1 := \pi_1|_{V'_1} \colon V'_1 \to U'_1$ and $R_{V'_1} \subset V'_1$ its ramification divisor. Let $R_{W_i} \subset W_i$ be the strict transform of $R_{V'_1}$ which is the ramification locus of $\pi'_i \colon W_i \to Q_i$. Let $Z_{Q_i} := \pi'_i(Z_{W_i})$ and $D'_i \subset Q_i$ the strict transform of $D'_1 \subset U'_1 = Q_1$. We see that g_{Q_i} is a blow-up of a smooth curve Z_{Q_i} such that $Z_{Q_i} \subset D'_i$ since R_{W_i} is disjoint with the center $Z_{W_i} \subset W_i$ of $W_{i+1} \to W_i$ and the quotient morphism $W_i \to Q_i$ is étale around Z_{W_i}. Thus we see that μ'_{12} is a composition of blow-ups of smooth curves. By this and the smoothness of U'_1, we see that U'_2 is also smooth.

We can check the inequality $a(E'_{2,j}, U'_1, D'_1) \leq 0$ as follows; We have an equality

$$\begin{aligned}\sum_j a(E'_{2,j}, U'_1, D'_1) E'_{2,j} &= K_{Q_l} + D'_l - (\mu'_{12})^*(K_{Q_1} + D'_1) \\ &= \sum_i (g_{k,i+1})^*(K_{Q_{i+1}} + D'_{i+1} - g^*_{Q_i}(K_{Q_i} + D'_i)). \quad (6.6)\end{aligned}$$

We also have $K_{Q_{i+1}} + D'_{i+1} - g^*_{Q_i}(K_{Q_i} + D'_i) = (1 - \mathrm{mult}_{Z_{Q_i}}(D'_i)) g_{Q_i}^{-1}(Z_{Q_i})$ and $1 - \mathrm{mult}_{Z_{Q_i}}(D'_i) \leq 0$. Thus we see that $a(E'_{2,j}, U'_1, D'_1) \leq 0$ for each j.

We finish the proof of Proposition 6.3.6. □

Remark 6.3.7. Note that U_2 is a normal variety with only quotient singularities, but can have non-isolated singularities.

6.3.3 Lemmas on cohomology groups

Let $U := \mathbb{C}^3/\mathbb{Z}_r(1, a, r-a)$ and $D \in |-K_U|$ with only isolated singularity at $0 \in D$. Let $\pi \colon V := \mathbb{C}^3 \to U$ be the quotient morphism and $\Delta := \pi^{-1}(D)$. We can assume that $\Delta = (f = 0) \subset \mathbb{C}^3$. By Proposition 6.2.5, we have

$$T^1_{(U,D)} \simeq (T^1_{(V,\Delta)})^{\mathbb{Z}_r}$$

and we regard $T^1_{(U,D)}$ as a subspace of $T^1_{(V,\Delta)}$. Since V is smooth, we also have $T^1_{(V,\Delta)} \simeq T^1_{\Delta} \simeq \mathcal{O}_{V,0}/J_{f,0}$ for the Jacobian ideal $J_{f,0} \subset \mathcal{O}_{V,0}$. Thus $T^1_{(V,\Delta)}$ has a $\mathcal{O}_{V,0}$-module structure and we fix an $\mathcal{O}_{V,0}$-module homomorphism

$$\varepsilon \colon \mathcal{O}_{V,0} \to T^1_{(V,\Delta)}$$

such that, for $h \in \mathcal{O}_{V,0}$, an element $\varepsilon(h) \in T^1_{(V,\Delta)}$ is a deformation $(f + th = 0) \subset V \times \operatorname{Spec} \mathbb{C}[t]/(t^2)$ of V. Let $\mathfrak{m}^2 T^1_{(U,D)} := \mathfrak{m}^2_{V,0} T^1_{(V,\Delta)} \cap T^1_{(U,D)}$ be the set of deformations induced by functions with multiplicity 2 or more.

Since D has an isolated singularity at $0 \in U$, we have $T^1_{(U,D)} \simeq T^1_{(U',D')}$, where $U' := U \setminus 0$ and $D' := D \cap U$ (cf. Lemma 3.4.2).

Let $U_1 \to U$ be one of the weighted blow-ups constructed in Section 6.3.1. We can define the *blow-down morphism* $(\mu_1)_* \colon T^1_{(U_1,D_1+E_1)} \to T^1_{(U,D)}$ as a composition

$$(\mu_1)_* \colon T^1_{(U_1,D_1+E_1)} \overset{\iota_1^*}{\to} T^1_{(U',D')} \overset{\simeq}{\to} T^1_{(U,D)}, \tag{6.7}$$

where ι_1^* is the restriction by an open immersion $\iota_1 \colon U' \simeq U_1 \setminus E_1 \hookrightarrow U_1$.

Lemma 6.3.8. *Let $\mu_1 \colon U_1 \to U, D_1$ and E_1 be those as in Section 6.3.1.*

Then we have the following

(i) $T^1_{U_1} = 0$.

(ii) $\operatorname{Im}(\mu_1)_* \subset \mathfrak{m}^2 T^1_{(U,D)}$.

Proof. (i) We shall show $H^1(U_1, \Theta_{U_1}) = 0$ as follows.

Let m_1 be a positive integer such that $|-m_1 K_{U_1}|$ contains a smooth member D_{m_1} such that $D_{m_1} \cap \operatorname{Sing} U_1 = \emptyset$. Let $\pi_1 \colon V_1 \to U_1$ be the degree m_1 cyclic cover branched along D_{m_1}. We have $H^1(U_1, \Theta_{U_1}) \simeq H^1(U_1, (\iota_1)_* \Omega^2_{U_1'}(-K_{U_1'}))$, where $U_1' := U_1 \setminus \operatorname{Sing} U_1$ and $\iota_1 \colon U_1' \hookrightarrow U_1$ is the open immersion. We see that this is a $\mathbb{Z}_{m_1}$-invariant part of $H^1(V_1, \Omega^2_{V_1}(L_1))$, where $L_1 := \pi_1^*(-K_{U_1})$. Note that L_1 is a Cartier divisor and $\mu_1 \circ \pi_1$-ample over U. By the vanishing theorem on a toric variety (cf. [14, Theorem 1.1]), we see that $H^1(V_1, \Omega^2_{V_1}(L_1)) = 0$. Thus, as its subspace, we obtain $H^1(U_1, \Theta_{U_1}) = 0$.

(ii) Take $\eta_1 \in T^1_{(U_1,D_1+E_1)}$. We have an exact sequence

$$H^0(U_1, \mathcal{O}_{U_1}(D_1)) \to H^0(D_1, \mathcal{N}_{D_1/U_1}) \to H^1(U_1, \mathcal{O}_{U_1}) = 0.$$

Hence the deformation of D_1 induced by η_1 comes from some divisor $D'_1 \in |D_1|$. In particular, it can be extended over a unit disc Δ^1. We also obtain $H^0(E_1, \mathcal{N}_{E_1/U_1}) = 0$ since $-E_1$ is μ_1-ample. Hence η_1 induces a trivial deformation of E_1 over a unit disc.

By these arguments and (i), the first order deformation η_1 can be extended to a deformation $(\mathcal{U}_1, \mathcal{D}_1 + \mathcal{E}_1) \to \Delta^1$ of $(U_1, D_1 + E_1)$ over a unit disc Δ^1 such that $\mathcal{U}_1 \simeq U_1 \times \Delta^1$. By taking its image by $\mu_1 \times \mathrm{id}\colon U_1 \times \Delta^1 \to U \times \Delta^1$, we obtain a deformation $(\mathcal{U}, \mathcal{D}) \to \Delta^1$ of (U, D).

If $U = \mathbb{C}^3$, let m_1 be a positive integer such that

$$(\mu_1 \times \mathrm{id})^* \mathcal{D} = \mathcal{D}_1 + m_1 \mathcal{E}_1.$$

If $U = \mathbb{C}^3/\mathbb{Z}_r$, let m_1 be a rational number such that

$$(\mu_1 \times \mathrm{id})^* (r\mathcal{D}) = r\mathcal{D}_1 + r m_1 \mathcal{E}_1.$$

For $t \in \Delta^1$, let $\mathcal{D}_t, \mathcal{D}_{1,t}$ be the fibers of $\mathcal{D}, \mathcal{D}_1$ over t and $m_{1,t}$ a rational number such that $\mu_1^* \mathcal{D}_t = \mathcal{D}_{1,t} + m_{1,t} E_1$. The above relations imply that $m_{1,t}$ is invariant for all $t \in \Delta^1$.

First assume that U is smooth. Recall that $\mu_1\colon U_1 \to U$ is a weighted blow-up of weights (a, b, c), where (a, b, c) is $(1, 1, 1)$, $(3, 2, 1)$ or $(2, 1, 1)$ by Lemmas 6.3.1, 6.3.3. Suppose that there exists $\eta_1 \in T^1_{(U_1, D_1 + E_1)}$ such that

$$(\mu_1)_*(\eta_1) \in T^1_{(U,D)} \setminus \mathfrak{m}^2 T^1_{(U,D)}. \tag{6.8}$$

We use the inclusion $T^1_{(U,D)} \subset T^1_{(V,\Delta)}$ as above. Take $h_1 \in \mathcal{O}_{V,0}$ such that $\varepsilon(h_1) = (\mu_1)_*(\eta_1)$. By the condition (6.8), we obtain $\mathrm{mult}_0\, h_1 \leq 1$. Hence we see that $m_{1,t} \leq \max\{a, b, c\} =: M$ by the formula (6.2). However, we can check that $m_{1,0} > M$. Indeed, if the weight is $(3, 2, 1)$, we see that $M = 3$ and $m_{1,0} \geq 6$ by the calculation in the proof of Lemma 6.3.3. If the weight is $(2, 1, 1)$, the maximum is 2 and $m_{1,0} \geq 4$ by the calculation in the same lemma. This is a contradiction.

Next assume that U has a quotient singularity. Suppose that there is an element $\eta_1 \in T^1_{(U_1, D_1 + E_1)}$ with the condition (6.8). As in the case U is smooth, we can take $h_1 \in \mathcal{O}_{V,0}$ such that $\varepsilon(h_1) = (\mu_1)_*(\eta_1)$ and $\mathrm{mult}_0\, h_1 \leq 1$. Then we see that $m_{1,t} \leq (1/r) \max\{1, a, r - a\}$ for $t \neq 0$ by the formula (6.2). However we see that $m_{1,0} \geq 1 + 1/r$ by the calculation in the proof of Lemma 6.3.4. This is a contradiction.

Hence we finish the proof of (ii). $\square$

We define a reflexive sheaf of differential forms for a VNC pair as follows.

Definition 6.3.9. ((cf. [58, Definition 1.17])) For a VNC pair (U, D), let $\iota\colon U' \hookrightarrow U$ be the smooth locus of U and $D' := D \cap U'$. We define $\tilde{\Omega}^i_U(\log D) := \iota_* \Omega^i_{U'}(\log D')$.

Let U, U_1, U_2, D_2 as in Lemma 6.3.6. Let $\mu_2 := \mu_1 \circ \mu_{12}\colon U_2 \to U$ and $E_2 := \mu_2^{-1}(0)$ the μ_2-exceptional divisor.

Since $U_2 \setminus E_2 \simeq U \setminus 0 = U'$, we have the coboundary map

$$\phi_U\colon H^1(U', \Omega^2_{U'}(\log D')) \to H^2_{E_2}(U_2, \tilde{\Omega}^2_{U_2}(\log D_2 + E_2)).$$

We fix an isomorphism $S_D\colon \mathcal{O}_U(-K_U - D) \simeq \mathcal{O}_U$ and it induces an isomorphism

$$\varphi_{S_D}\colon T^1_{(U,D)} \to H^1(U', \Theta_{U'}(-\log D')) \to H^1(U', \Omega^2_{U'}(\log D')).$$

We have the following lemma.

Lemma 6.3.10. *We have* $\operatorname{Ker} \phi_U \subset \varphi_{S_D}(\mathfrak{m}^2 T^1_{(U,D)})$. *In particular, we have* $\phi_U \neq 0$.

Proof. Let $E_{12} \subset U_2$ be the μ_{12}-exceptional locus. Let $U'_1 := U_1 \setminus Z$, $U'_2 := \mu_{12}^{-1}(U'_1)$ and $U''_1 := U_1 \setminus (\mu_{12}(E_{12}) \cup Z)$. We have the following relation;

$$\begin{tikzcd}
 & U'_2 \arrow[r, hook, "\iota_2"] \arrow[d, "\mu'_{12}"] & U_2 \arrow[d, "\mu_{12}"] \\
U''_1 \arrow[ur, hook, "\iota_{12}"] \arrow[r, hook] & U'_1 \arrow[r, hook] & U_1 \arrow[d, "\mu_1"] \\
 & U' \arrow[ul, hook', "\iota_1"] \arrow[r, hook] & U.
\end{tikzcd} \tag{6.9}$$

Set $D'_j := D_j \cap U'_j, E'_j := E_j \cap U'_j$ for $j = 1, 2$.

Let G'_2 be a divisor on U'_2 supported on E'_2 such that

$$\{-(K_{U_2} + D_2 + E_2) + \mu_2^*(K_U + D)\}|_{U'_2} \sim G'_2.$$

We see that G'_2 is effective since we have

$$\begin{aligned} G'_2 = -E'_2 &+ \left\{-(K_{U'_2} + D'_2) + (\mu'_{12})^*(K_{U'_1} + D'_1)\right\} \\ &+ (\mu'_{12})^* \{-(K_{U_1} + D_1) + \mu_1^*(K_U + D)\} |_{U'_1} \geq -E'_2 + 0 + (\mu'_{12})^* E'_1 \geq 0 \end{aligned} \tag{6.10}$$

by Lemmas in Section 6.3.1 and Lemma 6.3.6. Set $G''_1 := G'_2 \cap U''_1$. Note that we have an open immersion

$$\iota\colon U' = U \setminus 0 \simeq U_2 \setminus \mu_2^{-1}(0) \hookrightarrow U_2.$$

Then we have the following commutative diagram;

$$\begin{tikzcd}
H^1(U_2, \tilde{\Omega}^2_{U_2}(\log D_2 + E_2)) \arrow[rr, "\iota^*"] \arrow[d, "\iota_2^*"] & & H^1(U', \Omega^2_{U'}(\log D')) \\
H^1(U'_2, \Omega^2_{U'_2}(\log D'_2 + E'_2)) \arrow[d, "\phi_{G'_2}"] & & \\
H^1(U'_2, \Omega^2_{U'_2}(\log D'_2 + E'_2)(G'_2)) \arrow[r, "\iota_{12}^*"] & H^1(U''_1, \Omega^2_{U''_1}(\log D''_1 + E''_1)(G''_1)), \arrow[ruu, "\iota_1^*"'] &
\end{tikzcd} \tag{6.11}$$

where $\iota^*, \iota_1^*, \iota_2^*, \iota_{12}^*$ are the restriction by open immersions $\iota, \iota_1, \iota_2, \iota_{12}$ as in the diagram (6.9) and $\phi_{G'_2}$ is induced by an injection $\mathcal{O}_{U'_2} \hookrightarrow \mathcal{O}_{U'_2}(G'_2)$.

Since U'_2 is smooth and $D'_2 + E'_2$ is a SNC divisor by the construction of μ_{12}, we have a natural isomorphism

$$T^1_{(U'_2, D'_2+E'_2)} \simeq H^1(U'_2, \Theta_{U'_2}(-\log D'_2+E'_2)) \simeq H^1(U'_2, \Omega^2_{U'_2}(\log D'_2+E'_2)(-K_{U'_2}-D'_2-E'_2)).$$

The isomorphism S_D induces an isomorphism $\mu_2^*(S_D)\colon \mathcal{O}_{U_2}(\mu_2^*(-K_U - D)) \simeq \mathcal{O}_{U_2}$ and this induces an isomorphism

$$H^1(U'_2, \Omega^2_{U'_2}(\log D'_2 + E'_2)(-K_{U'_2} - D'_2 - E'_2)) \simeq H^1(U'_2, \Omega^2_{U'_2}(\log D'_2 + E'_2)(G'_2)).$$

Thus we have an isomorphism

$$\varphi_{\mu_2^*(S_D)}\colon T^1_{(U'_2, D'_2+E'_2)} \simeq H^1(U'_2, \Omega^2_{U'_2}(\log D'_2 + E'_2)(G'_2)).$$

The homomorphisms $\varphi_{\mu_2^*(S_D)}$ and $\iota_1^* \circ \iota_{12}^*$ fit in the commutative diagram

$$\begin{tikzcd}
H^1(U_2', \Omega^2_{U_2'}(\log D_2' + E_2')(G_2')) \arrow[r, "\iota_1^* \circ \iota_{12}^*"] & H^1(U', \Omega^2_{U'}(\log D')) \\
T^1_{(U_2', D_2'+E_2')} \arrow[u, "\varphi_{\mu_2^*(S_D)}"', "\simeq"] \arrow[r] \arrow[d, "(\mu_{12}')_*"] & T^1_{(U',D')} \arrow[u, "\varphi_{S_D}"', "\simeq"] \\
T^1_{(U_1', D_1'+E_1')} \arrow[d, "\simeq"] & T^1_{(U,D)} \arrow[u, "\simeq"] \\
T^1_{(U_1, D_1+E_1)}, \arrow[ru, "(\mu_1)_*"] &
\end{tikzcd} \tag{6.12}$$

where $(\mu_{12}')_*$ can be defined by the properties of μ_{12}' described in Lemma 6.3.6. We have $\operatorname{Im}(\mu_1)_* \subset \mathfrak{m}^2 T^1_{(U,D)}$ by Lemma 6.3.8. By this and the above diagrams, we see the claim. We finish the proof of Lemma 6.3.10. □

6.3.4 Proof of Theorem

We define "simultaneous $\mathbb{Q}$-smoothing" as follows.

Definition 6.3.11. Let X be a normal 3-dimensional variety with only terminal singularities and $D \in |-K_X|$ an anticanonical element.

We call a deformation $f\colon (\mathcal{X}, \mathcal{D}) \to \Delta^1$ a *simultaneous* $\mathbb{Q}$*-smoothing* if $\mathcal{X}_t$ and $\mathcal{D}_t$ have only quotient singularities and $(\mathcal{X}_t, \mathcal{D}_t)$ is a V-smooth pair for $t \neq 0$ (Definition 6.3.5).

We give the proof of the main theorem in the following.

Theorem 6.3.12. *Let X be a $\mathbb{Q}$-Fano 3-fold with only quotient singularities such that there exists an element $D \in |-K_X|$ with only isolated singularities.*

Then (X, D) has a simultaneous $\mathbb{Q}$-smoothing.

Proof. Let m be a sufficiently large integer such that $|-mK_X|$ contains a smooth element D_m such that $\operatorname{Sing} D \cap D_m = \emptyset$. Let $\pi\colon Y \to X$ be a cyclic cover branched along D_m and $\Delta := \pi^{-1}(D)$. This induces an index one cover around each point of $\operatorname{Sing} X$ and Y is smooth.

Let $p_1, \ldots, p_l \in \operatorname{Sing} D$ be the image of non-Du Val singular points of Δ and $p_{l+1}, \ldots, p_{l+l'}$ the image of Du Val singularities of Δ. Let $U_i \subset X$ be a Stein neighborhood of p_i and $D_i := D \cap U_i$ for $i = 1, \ldots, l+l'$. For $i = 1, \ldots, l$, let $\mu_{i,1}\colon U_{i,1} \to U_i$

be the weighted blow-up constructed in Section 6.3.1 and $\mu_{i,12}\colon U_{i,2} \to U_{i,1}$ the birational morphism constructed in Lemma 6.3.6. Let $\mu_{i,2} := \mu_{i,1} \circ \mu_{i,12}\colon U_{i,2} \to U_i$ be the composition. For $i = l+1, \dots, l+l'$, let $\mu_i\colon \tilde{U}_i \to U_i$ be a projective birational morphism such that $(\tilde{U}_i, \mu_i^{-1}(D_i))$ is a VNC pair.

By patching these $\mu_{i,2}$ for $i = 1, \dots, l$ and μ_i for $i = l+1, \dots, l+l'$, we construct a birational morphism $\mu\colon \tilde{X} \to X$ such that $\mu^{-1}(D) \subset \tilde{X}$ is a VNC divisor. Let $\tilde{D} \subset \tilde{X}$ be the strict transform of D and $E \subset \tilde{X}$ the μ-exceptional divisor. Also let $\tilde{D}_i := \tilde{D} \cap \mu^{-1}(U_i)$ and $E_i := \mu^{-1}(p_i)$ for $i = 1, \dots, l+l'$.

We use the following diagram;

$$\begin{tikzcd}
H^1(X', \Omega^2_{X'}(\log D')) \arrow[r, "\oplus\psi_i"] \arrow[d, "\oplus p_{U_i}"] & H^2_E(\tilde{X}, \tilde{\Omega}^2_{\tilde{X}}(\log \tilde{D} + E)) \arrow[r] \arrow[d, "\oplus\varphi_i"', "\simeq"] & H^2(\tilde{X}, \tilde{\Omega}^2_{\tilde{X}}(\log \tilde{D} + E)) \\
\oplus_{i=1}^{l+l'} H^1(U'_i, \Omega^2_{U'_i}(\log D'_i)) \arrow[r, "\oplus\phi_i"] & \oplus_{i=1}^{l+l'} H^2_{E_i}(\tilde{U}_i, \tilde{\Omega}^2_{\tilde{U}_i}(\log \tilde{D}_i + E_i)),
\end{tikzcd} \tag{6.13}$$

where $X' := X \setminus \{p_1, \dots, p_{l+l'}\}$ and $D' := D \cap X'$.

Let $i \in \{1, \dots, l\}$ be a fixed number and $\eta_i \in H^1(U'_i, \Omega^2_{U'_i}(\log D'_i))$ an element inducing a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i). We see that $H^2(\tilde{X}, \tilde{\Omega}^2_{\tilde{X}}(\log \tilde{D} + E)) = 0$ since $\tilde{X} \setminus (\tilde{D} + E) \simeq X \setminus D$ is a smooth affine variety and $H^2(\tilde{X}, \tilde{\Omega}^2_{\tilde{X}}(\log \tilde{D} + E))$ is a subquotient of $H^4(\tilde{X} \setminus (\tilde{D} + E), \mathbb{C}) = 0$ by the mixed Hodge theory on V-manifolds. Thus there exists $\eta \in H^1(X', \Omega^2_{X'}(\log D'))$ such that $\psi_i(\eta) = (\varphi_i)^{-1}(\phi_i(\eta_i))$. Since $\eta_i - p_{U_i}(\eta) \in \operatorname{Ker} \phi_i$, we see that

$$\varphi^{-1}_{S_{D_i}}(\eta_i - p_{U_i}(\eta)) \in \mathfrak{m}^2 T^1_{(U_i, D_i)} \tag{6.14}$$

by Lemma 6.3.10. Let $\pi_i\colon V_i \to U_i$ be the index one cover and $\Delta_i := \pi_i^{-1}(D_i) \subset V_i$. By (6.14) and Lemma 6.2.7, we see that $p_{U_i}(\eta)$ induces a smoothing of Δ_i. Thus it induces a deformation of (U_i, D_i) to a V-smooth pair. By Theorem 3.2.9, we can lift the first order deformation η to a deformation $f\colon (\mathcal{X}, \mathcal{D}) \to \Delta^1$ of (X, D) over a unit disc Δ^1. This f induces a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i) for $i = 1, \dots, l$. Thus we can deform all non-Du Val singularities of D and obtain a $\mathbb{Q}$-Fano 3-fold with a Du Val elephant as a general fiber of the deformation f. Moreover, by Theorem 3.1.8, there exists a simultaneous $\mathbb{Q}$-smoothing of this $\mathbb{Q}$-Fano 3-fold. Thus we finish the proof of Theorem 6.3.12. □

6.4 Examples

Shokurov and Reid proved the following theorem.

Theorem 6.4.1. *Let X be a Fano 3-fold with only canonical Gorenstein singularities.*

Then a general member $D \in |-K_X|$ has only Du Val singularities.

For non-Gorenstein $\mathbb{Q}$-Fano 3-folds, this statement does not hold. We give several examples of $\mathbb{Q}$-Fano 3-folds without Du Val elephants.

Example 6.4.2. ([24]) Iano-Flethcer gave an examples of a $\mathbb{Q}$-Fano 3-fold without elephants. Let $X := X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ be a weighted complete intersection of degree 12 and 14. Then we have $|-K_X| = \emptyset$ and general X have only terminal quotient singularities.

Iano-Fletcher gave a list of 95 families of $\mathbb{Q}$-Fano 3-fold weighted hypersurfaces. General members of those families have only quotient singularities and they have Du Val elephants. However, by taking special members in those families, we can construct weighted hypersurfaces without Du Val elephants as follows.

Example 6.4.3. Let $X := X_{14} := ((x^{14} + x^2 y_1^6) + w^2 + y_1^3 y_2^4 + y_2^7 + y_1 z^4 = 0) \subset \mathbb{P}(1,2,2,3,7)$ be a weighted hypersurface with coordinates x, y_1, y_2, z, w of weights $1,2,2,3,7$ respectively. This is a modified version of an example in [3, 4.8.3].

We can check that X has only terminal singularities. It has three $1/2(1,1,1)$-singularities on the (y_1, y_2)-axis, a terminal singularity $(x^2+w^2+z^4+y_2^4 = 0)/\mathbb{Z}_2(1,1,1,0)$ and a $1/3(1,2,1)$-singularity at $[0:0:0:1:0]$.

We see that $|-K_X| = \{D\}$ and D has an elliptic singularity $(w^2 + y_2^4 + z^4 = 0)/\mathbb{Z}_2$. In fact, this is log canonical.

Next we give a $\mathbb{Q}$-Fano 3-fold with only quotient singularities and with only non-log canonical elephants. Thus the statement of Theorem 6.1.1 is not empty.

Example 6.4.4. Let $X := (x^{15} + xy^7 + z^5 + w_1^3 + w_2^3 = 0) \subset \mathbb{P}(1,2,3,5,5)$ be a weighted hypersurface, where x, y, z, w_1, w_2 are coordinate functions with degrees $1,2,3,5,5$ respectively. We can check that X has a $1/2(1,1,1)$-singularity and three $1/5(1,2,3)$-singularities. Thus X is a $\mathbb{Q}$-Fano 3-fold with only terminal quotient singularities.

On the other hand, we have $|-K_X| = \{D\}$, where $D := (z^5 + w_1^3 + w_2^3 = 0) \subset \mathbb{P}(2,3,5,5)$. We see that the singularity $p = [1:0:0:0] \in D$ is isomorphic to a singularity $(x_1^5 + x_2^3 + x_3^3 = 0)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$-action is of type $1/2(1,1,1)$. The singularity is not Du Val. In fact, we see that it is even not log canonical by computing a resolution of singularity explicitly.

Next, we give an example of a $\mathbb{Q}$-Fano 3-fold with only non-normal elephants.

Example 6.4.5. Let $X := X_{16} := (x^{16}+x(z^5+zy^6)+yu^2+w^4 = 0) \subset \mathbb{P}(1,2,3,4,7)$ be a weighted hypersurface with coordinates x, y, z, w, u with weights $1, 2, 3, 4, 7$ respectively.

Firstly, we check that X has only terminal singularities. By computing the Jacobian of the defining equation of X, we see that X is quasi-smooth outside the points on an affine piece $y \neq 0$ such that $x = w = u = 0$ and $z(z^4 + y^6 = 0)$. We can describe the singularities as follows; An affine piece $(x \neq 0)$ is smooth. An affine piece $(y \neq 0)$ has two singularities isomorphic to $(xz + w^4 + u^2 = 0) \subset \mathbb{C}^4$ and an singularity $(xz + w^4 + u^2 = 0)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on x, z, w, u with weights $1/2(1, 1, 0, 1)$. They are terminal by the classification ([36, Theorem 6.5]). On a piece $(z \neq 0)$, there exists a $1/3(2, 1, 2)$-singularity. A piece $(w \neq 0)$ is smooth. A piece $(u \neq 0)$ has a $1/7(1, 3, 4)$-singularity.

Next, we check that $|-K_X|$ has only non-normal elements. Indeed, we have $|-K_X| = \{D\}$ with $D = (yu^2 + w^4 = 0) \subset \mathbb{P}(2, 3, 4, 7)$ and the singular locus of D is non-isolated.

Thus it is meaningful to consider Conjecture 3.1.7 when the singularity of an elephant is non-isolated.

On the other hand, we could not find an example of a $\mathbb{Q}$-Fano 3-fold without Du Val elephants such that $h^0(X, -K_X) \geq 2$. Thus the following question is natural.

Problem 6.4.6. *Let X be a $\mathbb{Q}$-Fano 3-fold such that $h^0(X, -K_X) \geq 2$.*

Does there exist a Du Val elephant of X? Or, does there exist a normal elephant of X?

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