Collective Rationality and Monotone Path Division Rules*

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14 July 2014

Abstract

We impose the axiom Independence of Irrelevant Alternatives on division rules for the conflicting claims problem. With the addition of Consistency and Resource Monotonicity, this characterizes a family of rules which can be described in three different but intuitive ways. First, a rule is identified with a fixed monotone path in the space of awards, and for a given claims vector, the path of awards for that claims vector is simply the monotone path truncated by the claims vector. Second, a rule is identified with a set of parametric functions indexed by the claimants, and for a given claims problem, each claimant receives the value of his parametric function at a common parameter value, but truncated by his claim. Third, a rule is identified with an additively separable, strictly concave social welfare function, and for a given claims problem, the amount awarded is the maximizer of the social welfare function subject to the constraint of choosing a feasible award.

1 Introduction

A conflicting claims problem is a situation in which a divisible homogeneous good must be distributed among a group, each individual in the group having an objective claim on the good, but where the amount of the good is insufficient to satisfy all the claims.[1] An example is dividing the liquidated value of a bankrupt firm among its creditors. How should the good be divided among the claimants? We seek a rule which chooses, for any problem, a feasible allocation or award. An award is feasible

*I thank Atila Abdulkadiroğlu, Bhaskar Dutta, Albin Erlanson, Peter Hammond, Hervé Moulin, and Karol Szwagrzak for useful discussions. I am grateful for seminar participants at the University of Warwick, Southern Denmark University, and the 12th Meeting of the Society for Social Choice and Welfare. I also thank an associate editor and referees for comments.

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[1]Though such problems are as old as civilization (examples are found in the Talmud), formal study of such problems began with O'Neill (1982). See Thomson (2003, 2013a) for surveys of the literature stemming from this seminal paper.

if it completely exhausts the good to be divided and if every individual receives an amount between 0 and his respective claim.

1.1 Overview of Results

We impose the axiom Independence of Irrelevant Alternatives (IIA) on rules. This axiom states that if the chosen award for a problem is also feasible for a second problem whose feasible set is a subset of the original problem, then that award is also chosen for the second problem. This is the same axiom introduced by Nash (1950) in the domain of bargaining problems. In the context of individual choice, this axiom is sometimes known as Chernoff's condition (Chernoff, 1954) or Sen's α (Sen, 1969).

We also impose two axioms that are common in the literature: Consistency and Resource Monotonicity. Consistency states that if a division rule chooses an award for a group of claimants, then it should not choose to reallocate the awards of any subgroup when considered as a separate problem. Resource Monotonicity states that if the amount to be divided increases, then no claimant's award should decrease.

Theorem 1 shows that IIA, Consistency, and Resource Monotonicity characterize a family of rules which can be described in three different but intuitive ways:

- Consider a fixed, weakly monotone path in the space of awards. For any group of claimants and any vector of claims for that group, the path of awards is simply the fixed path truncated by the claims vector. We refer to all such rules as monotone path rules.

- Consider a set of parametric functions, one for each individual. Each parametric function depends only on a single parameter, in which it is weakly increasing. For any problem, each parametric function is truncated by the individual's claim, and a common parameter is found so that the sum of the truncated parametric functions evaluated at that parameter equals the amount to be divided. We refer to all such rules as claims independent parametric rules.

- Consider an additively separable, strictly concave social welfare function. For any problem, the amount awarded is the maximizer of the social welfare function subject to the constraint of choosing a feasible award. We refer to all such rules as collectively rational additively separable (CRAS) rules.

We also consider a property which is dual to IIA. Rather than taking the awards as what matters to the individuals, as IIA does, this dual property takes the losses (the difference between an individuals claim and his award) as what matters. Theorem 2 shows that IIA and its dual are effectively incompatible: the queueing rule (which is generally considered to be normatively unappealing) is the only rule to satisfy Consistency, Resource Monotonicity, IIA, and the dual of IIA.

If there is no a priori reason to treat the claimants differently, then one would want the rule to give the same award to individuals with the same claim, a property known as Symmetry. Theorem 3 shows that the constrained equal awards rule is the only rule in our family which satisfies Symmetry.

We conclude with several results relating IIA to other well-known axioms, such as Claims Truncation Invariance (Dagan and Volij, 1993) and Upper Composition (Moulin, 2000).

1.2 Related Literature

To our knowledge, the only other work to consider IIA in the domain of claims problems are a pair of papers by Kıbrıs (2012, 2013). From these papers, the result closest to ours (Kıbrıs, 2012, Theorem 3) is one which characterizes the family of rules that maximize some social welfare function (not necessarily additively separable). The axioms imposed by Kıbrıs are IIA, Continuity, and an axiom called Others-oriented Claims Monotonicity. In general, these rules are not Resource Monotonic. Another way in which this result differs from ours is that the population of claimants is fixed, and thus Consistency does not apply.

Recently, Stovall (2014) characterized the family of (possibly asymmetric) parametric rules, of which the claims independent parametric rules are a special case. The family of symmetric parametric rules (Young, 1987) is also a special case of the family of parametric rules. The only overlap between the family of symmetric parametric rules and the family of claims independent parametric rules is the constrained equal awards rule (see Theorem 3). Additionally, Stovall (2014, Theorem 3) shows that a parametric rule maximizes an additively separable and claims dependent social welfare function. This differs from our result as Theorem 1 characterizes a rule which maximizes a social welfare function which does not depend on the claims.

Considering other domains, the monotone path rules and the CRAS rules each have analogues in the literature on the bargaining problem. Monotone path rules are similar to the solutions given by Thomson and Myerson (1980), though they consider only strictly monotone paths and a fixed population. CRAS rules are analogous to the family of rules characterized by Lensberg (1987). Since the domain of bargaining problems is much richer than the domain of claims problems, it should come as no surprise that these families of rules are *not* equivalent in the domain of bargaining problems. The main axiom imposed by Thomson and Myerson is a strong monotonicity axiom, which would be equivalent to a strict resource monotonicity axiom here. The main axiom imposed by Lensberg is a consistency axiom similar to the one we impose. Interestingly, in the domain of bargaining problems and in conjunction with Lensberg's other axioms, this implies IIA.

In the literature on fair division under single-peaked preferences, Moulin (1999) characterizes the family of monotone path rules. The key axioms are a consistency axiom and resource monotonicity axiom (similar to the ones we impose), and, because of strategic considerations for an individual reporting his peak, a strategy-proofness axiom. In recent work, Erlanson and Szwagrzak (2014) use these same axioms to characterize the family of CRAS rules. Thus similar to our result, in this domain the monotone path rules and CRAS rules are equivalent.[2]

[2] The domain used by Erlanson and Szwagrzak is more general than the domain used by Moulin. The latter assumes a single divisible resource to divide, while the former allows for multiple resources

This work joins a growing literature studying asymmetric rules for the claims problem. In addition to the work by Stovall and Kıbrıs discussed above, Moulin (2000), Naumova (2002), Chambers (2006), and Hokari and Thomson (2003) all consider rules which are not symmetric. We discuss these papers more in Section 6 when we relate IIA to other well-known axioms.

2 Definitions

We adopt the following notation. Let $\mathcal{N}$ denote the set of finite subsets of the natural numbers, $\mathbb{N}$. Let $\mathbb{R}_+$ denote the non-negative real numbers, $\mathbb{R}_{++}$ the positive real numbers, and $\overline{\mathbb{R}}$ the extended real numbers. Let $\mathbf{0}$ denote a vector of zeros and Ω a vector in which all coordinates are $+\infty$. For $x, y \in \mathbb{R}^N$, we use the vector inequalities $x \geqq y$ if $x_i \geq y_i$ for all $i \in N$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for every $i \in N$. For $N' \subset N$, let $x_{N'}$ denote the projection of x onto the subspace $\mathbb{R}^{N'}$.

A *claims problem* is a tuple (N, c, E), where $N \in \mathcal{N}$, $c \in \mathbb{R}^N_{++}$, and $E \in \mathbb{R}_+$, all satisfying $E \leq \sum_{i \in N} c_i$. Let $X(N, c, E)$ denote the set of efficient feasible awards vectors for the problem (N, c, E), i.e.

$$X(N,c,E) \equiv \left\{ x \in \mathbb{R}^N_+ : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i = E \right\}.$$

A *division rule* is a function S such that for every problem (N, c, E), we have $S(N, c, E) \in X(N, c, E)$.

Some well-known rules include the proportional rule, the constrained equal awards rule, the constrained equal losses rule, the Talmud rule (Aumann and Maschler, 1985), and the queueing rule. The constrained equal awards and queueing rules are members of the family that we characterize here, while the others are not.

A convenient way of graphically representing a rule is by the path of awards it generates. For a fixed $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the path of awards generated by S is the set of all awards $S(N, c, E)$ as E varies from 0 to the sum of claims $\sum_N c_i$. See Figure 1. Thus a rule can be identified with a collection of paths, one for every $N \in \mathcal{N}$ and every $c \in \mathbb{R}^N_{++}$.

3 Main Results

In this section we introduce the axioms imposed and the family of rules which are characterized by those axioms.

to divide, some of which are divisible and some of which are indivisible. Also, while Moulin imposes an efficiency axiom, Erlanson and Szwagrzak use a weaker unanimity axiom. However, in the special case of a single divisible good, Erlanson and Szwagrzak (2014, Lemma 3) show that this (in conjunction with the other axioms) implies efficiency. Thus Erlanson and Szwagrzak's main result and Moulin's main result together imply that monotone path rules and CRAS rules are equivalent when allocating a single divisible resource under single peaked preferences.

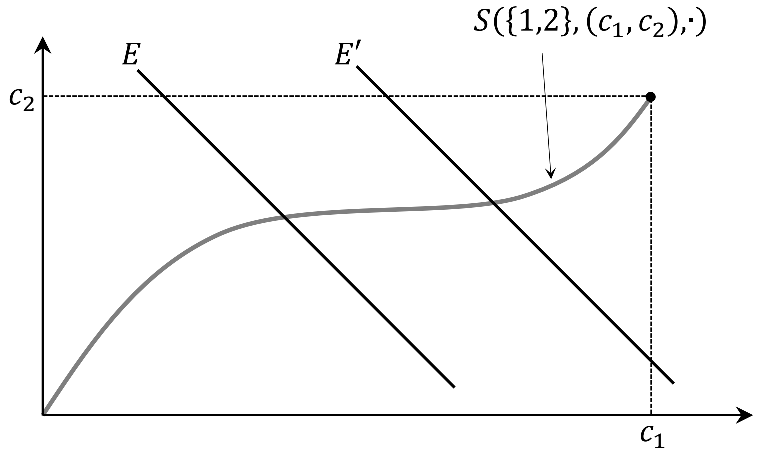

Figure 1: ***Path of awards.*** *The path of awards is the set of all awards as the endowment varies from zero to the sum of claims.*

3.1 Axioms

We impose three axioms. The first states that decreasing the amount to divide should not cause any claimant's award to increase.

Resource Monotonicity. *For every* (N, c, E) *and* $E' < E$*, we have* $S(N, c, E') \leq S(N, c, E)$.

A stronger version of this axiom is Strict Resource Monotonicity, which holds when the latter inequality is changed to a strict inequality. Strict Resource Monotonicity is satisfied by the "equal sacrifice" rules (Young, 1988; Naumova, 2002), as we discuss in Section 6. However, it is violated by the constrained equal awards rule, the constrained equal losses rule, the Talmud rule, and the queueing rule.

The next axiom states that how a rule divides between two claimants does not change if all other claimants are removed from the problem.

Bilateral Consistency. *For every* (N, c, E) *and* $\{i, j\} \subset N$*, if* $x = S(N, c, E)$*, then*

$$(x_i, x_j) = S(\{i, j\}, (c_i, c_j), x_i + x_j).$$

A stronger version of this axiom, called Consistency, holds for any $N' \subset N$. Consistency is arguably the most prominent axiom in the literature on fair allocation. See Thomson (2013b) for a survey.

Our final axiom is an adaptation of one introduced by Nash (1950) in the domain of bargaining problems. IIA states that if the solution to a problem is feasible for a different problem (of the same group of claimants) with a smaller feasible set, then it will be the solution to this second problem.

Independence of Irrelevant Alternatives. *For every* (N, c, E) *and* (N, c', E)*, if*

$S(N,c,E) \in X(N,c',E) \subset X(N,c,E)$, *then* $S(N,c',E) = S(N,c,E)$.

Observe that $S(N,c,E) \leqq c'$ is necessary and sufficient to have $S(N,c,E) \in X(N,c',E)$. Observe also that $c' \leqq c$ is sufficient to have $X(N,c',E) \subset X(N,c,E)$.[3] Thus every (N,c,E) and c' satisfying $S(N,c,E) \leqq c' \leqq c$ will also satisfy the conditions of IIA.

In the context of individual choice, IIA is considered to be a standard rationality assumption. Thus one can think of IIA as imposing some amount of "rationality" on the social choice function S.

IIA is undoubtedly a strong assumption.[4] It imposes on the division rule a certain amount of independence from the claims vector. For example, suppose that for rule S, problem (N,c,E), and award vector $x \equiv S(N,c,E)$, there exists $i \in N$ such that $x_i < c_i$. Now suppose that it is discovered that there was a mistake in determining the claim of individual i, and that his claim is actually c'_i, where $x_i < c'_i < c_i$. Then, setting $c'_j = c_j$ for all $j \neq i$, (N,c',E) is a claims problem. IIA thus implies that $x = S(N,c',E)$. In essence, IIA says that since the division rule deemed x a just award for the problem (N,c,E), then it must deem x a just award for (N,c',E). The fact that the claim of individual i decreased does not alter this assessment since x is still feasible for the problem (N,c',E).

Note that IIA takes for granted the idea that it is the awards that matter. Thus IIA would not be as compelling in applications where the award is actually a bad, e.g. fair taxation. (Taxation problems are formally identical to claims problems: c is interpreted as the vector of incomes, E is interpreted as the revenue that must be raised via taxation, and $S(N,c,E)$ is the assignment of taxes.) We will return to this point in Section 4.

In Section 6, we collect a number of results which relate IIA to other well-known axioms in the literature.

3.2 Monotone Path Rules

Consider the following solution for a claims problem: Fix a monotone path in the space of awards for all possible claimants. For any group of claimants N, take the projection of that fixed path onto the subspace of awards for that group N. For a vector of claims c for that group N, truncate the projected path by c. This defines a path of awards for this particular N and c. Following this procedure for any N and c defines a collection of paths of awards, which defines a division rule. We now formalize this way of dividing.

For $M \subset \mathbb{N}$ and for $x, y \in \overline{\mathbb{R}}^M$, a path from x to y is a continuous function $p : [0,1] \to \overline{\mathbb{R}}^M$ such that $p(0) = x$ and $p(1) = y$. A path is weakly monotone if $t > t'$ implies $p(t) \geqq p(t')$. Let $\mathcal{P}$ denote the family of weakly monotone paths in $\overline{\mathbb{R}}^{\mathbb{N}}_+$ from $\mathbf{0}$ to Ω. Call $p \in \mathcal{P}$ a *monotone path*.

[3] But it is not necessary. For example, if $N = \{1,2\}$, $c = (2,2)$, $c' = (1,3)$, and $E = 2$, then $X(N,c',E) \subset X(N,c,E)$ yet $c' \not\leqq c$.

[4] Of course, one could equally argue that IIA is a strong assumption in the context of bargaining problems.

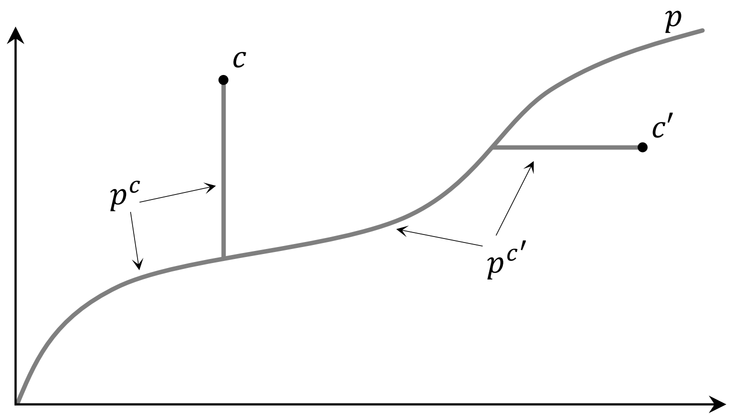

Figure 2: ***Monotone Path Rule.*** *For two claimants, a monotone path p, as well as the truncated paths for claims c and c′.*

For a given $p \in \mathcal{P}$, $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, let p^c denote the path from $\mathbf{0}$ to c obtained by taking, for every t, the meet of c and the projection of $p(t)$ onto $\overline{\mathbb{R}}^N_+$, i.e.

$$p^c(t) \equiv p_N(t) \wedge c.$$

See Figure 2.

Note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} p^c_i(t)$ is continuous, weakly increasing, and satisfies $\sum_{i \in N} p^c_i(0) = 0$ and $\sum_{i \in N} p^c_i(1) = \sum_{i \in N} c_i$. Hence, for any problem (N, c, E), there exists $t \in [0, 1]$ such that $\sum_{i \in N} p^c_i(t) = E$. Moreover, if t and t' are such that $\sum_{i \in N} p^c_i(t) = \sum_{i \in N} p^c_i(t') = E$, then $p^c_i(t) = p^c_i(t')$ for every $i \in N$. Hence for any $p \in \mathcal{P}$, we can define a division rule S^p as:

$$S^p(N, c, E) = p^c(t),$$

where t is chosen such that $\sum_{i \in N} p^c_i(t) = E$. We say a rule S is a *monotone path rule* if there exists $p \in \mathcal{P}$ such that $S = S^p$.

3.3 Claims Independent Parametric Rules

Let $\mathcal{F}$ denote the family of functions $f : \mathbb{N} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}_+$ such that, for any $i \in \mathbb{N}$, $f(i, \cdot)$ is continuous, weakly increasing, and satisfies $f(i, -\infty) = 0$ and $f(i, +\infty) = +\infty$. From now on we write $f(i, \cdot)$ as f_i. We refer to $f \in \mathcal{F}$ as a *claims independent parametric function.*

Note that for any $N \in \mathcal{N}$ and $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous and weakly monotonic in λ, and that $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$ and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any claims problem (N, c, E), there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. Furthermore, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that

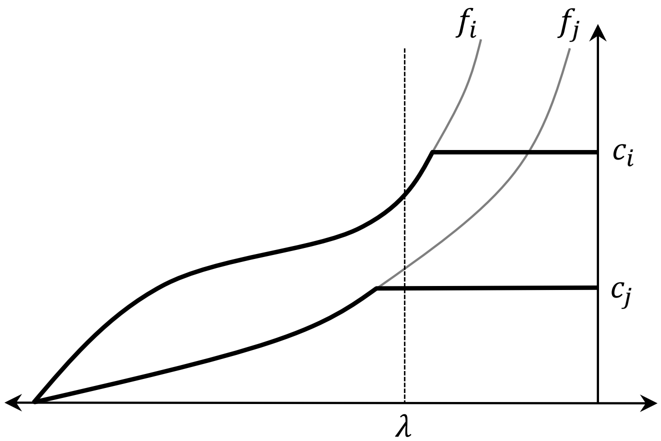

Figure 3: ***Claims Independent Parametric Rule.*** *The parametric functions for two claimants.*

$\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$. Hence for any $f \in \mathcal{F}$, we can define a division rule S^f as:

$$S^f(N, c, E) = (\min\{f_i(\lambda), c_i\})_{i \in N} ,$$

where λ is chosen such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. We say a rule S is a *claims independent parametric rule* if there exists $f \in \mathcal{F}$ such that $S = S^f$. See Figure 3.

The claims independent parametric rules are a special case of the (asymmetric) parametric rules characterized by Stovall (2014). In that paper, a parametric function was a continuous function $g : \mathbb{N} \times \mathbb{R}_{++} \times \overline{\mathbb{R}} \to \mathbb{R}_+$ which was weakly increasing in the third argument and satisfying $g_i(c_i, -\infty) = 0$, $g_i(c_i, +\infty) = c_i$. Let $\mathcal{G}$ denote the family of parametric functions. A division rule could be defined from $g \in \mathcal{G}$ as was done above for claims independent parametric functions:

$$S^g(N, c, E) = (g_i(c_i, \lambda))_{i \in N} ,$$

where λ is chosen such that $\sum_{i \in N} g_i(c_i, \lambda) = E$. Thus for $f \in \mathcal{F}$, the claims independent parametric rule S^f is also a parametric rule S^g, where $g_i(c_i, \lambda) \equiv \min\{f_i(\lambda), c_i\}$.

As we show later in Theorem 1, monotone path rules and claims independent parametric rules are in fact the same family of rules. Indeed, this is easy to see now. For $f \in \mathcal{F}$, define the monotone path

$$p(t) = (f_i(h(t)))_{i \in \mathbb{N}}$$

where h is any strictly increasing bijection from $[0, 1]$ to $\overline{\mathbb{R}}$. Showing the converse is similar.

3.4 Collectively Rational Additively Separable Rules

A social welfare function (SWF) is a real-valued function of awards vectors. We say a SWF is additively separable if there exists $U : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ such that for any $N \in \mathcal{N}$

and $x \in \mathbb{R}_+^N$, the SWF can be written in the form $\sum_{i \in N} U(i, x_i)$. Let $\mathcal{U}$ denote the family of functions $U : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ that are continuous and strictly concave in the second variable. From now on we write $U(i, \cdot)$ as U_i.

Note that for any $U \in \mathcal{U}$, $\arg\max_{x \in X(N,c,E)} \sum_{i \in N} U_i(x_i)$ is single-valued. Hence for any $U \in \mathcal{U}$, we can define a division rule S^U as:

$$S^U(N, c, E) = \underset{x \in X(N,c,E)}{\arg\max} \sum_{i \in N} U_i(x_i).$$

We say a rule S is a *collectively rational additively separable (CRAS) rule* if there exists $U \in \mathcal{U}$ such that $S = S^U$.

The family of CRAS rules are similar to the family of rules characterized by Lensberg (1987) for bargaining problems.[5] Indeed, any claims problem (N, c, E) can be viewed as a bargaining problem by considering the set $\left\{x \in \mathbb{R}_+^N : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i \leq E\right\}$.[6] However Lensberg's result does not imply ours as the class of all bargaining problems is much larger than the class of bargaining problems derived from claims problems.

3.5 Main Theorem

Our main theorem states that these three axioms characterize each of these families of rules, and thus these families are in fact one and the same.

Theorem 1 *The following are equivalent:*

1. *S satisfies Resource Monotonicity, Bilateral Consistency, and Independence of Irrelevant Alternatives.*
2. *S is a monotone path rule.*
3. *S is a claims independent parametric rule.*
4. *S is a collectively rational additively separable rule.*

We note a few things concerning this theorem. First, it shows the appeal of this family of rules as it can be described in three different but intuitive ways. Monotone path rules are geometrically appealing. Claims independent parametric rules are easy to relate to the family of parametric rules. CRAS rules are a natural method of division, one that has been proposed in many other social choice problems.

A second thing to note is that continuity of the rule is not assumed, though it is implied by the axioms. It is a well-known result that Resource Monotonicity implies continuity in the endowment, and this fact is used in the proof. The other part of continuity, that of continuity in the claims vector, is not needed in the proof (though obviously IIA would play a key role in establishing this property).

[5]Lensberg requires that each U_i be continuous and strictly increasing, and that the SWF $\sum_N U_i$ be strictly quasi-concave for each $N \in \mathcal{N}$. Here we require each U_i to be strictly concave to guarantee Resource Monotonicity.

[6]A bargaining problem for $N \in \mathcal{N}$ is a compact, convex subset A of $\mathbb{R}_+^N$ which satisfies the following property: if $x \in A$ and $0 \leqq y \leqq x$, then $y \in A$.

The following examples demonstrate that the axioms in Theorem 1 are independent.

- **Bilateral Consistency.** Let $p, p' \in \mathcal{P}$ be two different monotone paths. Consider the rule which divides according to S^p for all two-person claims problems and $S^{p'}$ for all claims problems with more than two claimants. Such a rule would satisfy Resource Monotonicity and IIA but not Bilateral Consistency.

- **IIA.** The proportional rule

$$P(N, c, E) = \frac{E}{\sum_N c_i} c$$

satisfies Bilateral Consistency and Resource Monotonicity, but not IIA.

- **Resource Monotonicity.** We sketch an example here, but a complete example is given in the appendix. Consider a rule for only two claimants. This rule divides according to a fixed path, which is not weakly monotone. Specifically, this path is always increasing in the first claimant's coordinate, but does sometimes decrease in the second claimant's coordinate. However it never has a slope less than -1, thus guaranteeing that it intersects with any endowment line only once. Given this fixed path, we now describe the path of awards for a given claims vector. For $(c_1, c_2) \in \mathbb{R}^2_{++}$, the path of awards is the intersection of the the fixed path with the rectangle defined by the origin and (c_1, c_2). If the fixed path exits and then subsequently re-enters the rectangle, then the path of awards travels along the border of the rectangle from the point of exit to the point of entry. Once the fixed path leaves the rectangle permanently, the path of awards travels along the border of the rectangle from the point of exit to the point (c_1, c_2). See Figure 4. We have thus described the path of awards for any claims vector, and thus completely described the rule for two claimants. Such a rule satisfies IIA but obviously not Resource Monotonicity. In the appendix, we describe how this rule can be extended to any problem so as to satisfy Bilateral Consistency.

The proof of Theorem 1 is given in the appendix, but a brief sketch here can shed some light on the relation between the three families of rules in the theorem. First, from the rule S we define a binary relation R over the space of possible claimant-award pairs as follows. We write $(i, x_i)R(j, x_j)$ if a smaller endowment is needed for S to award x_i to claimant i than is needed for S to award x_j to claimant j. (Of course, the endowment needed for S to award x_i to claimant i could depend on the claims of i and j. However IIA guarantees that this is not the case.) The key step in the proof is using the axioms to show that R has a numerical representation r.

Now consider when $r(i, x_i) = r(j, x_j)$. If this is the case, then the point (x_i, x_j) must lie on the path of awards for problems with $N = \{i, j\}$. In this way we can trace out the path used to define a monotone path rule, namely $x \in \mathbb{R}^{\mathbb{N}}_+$ is in the

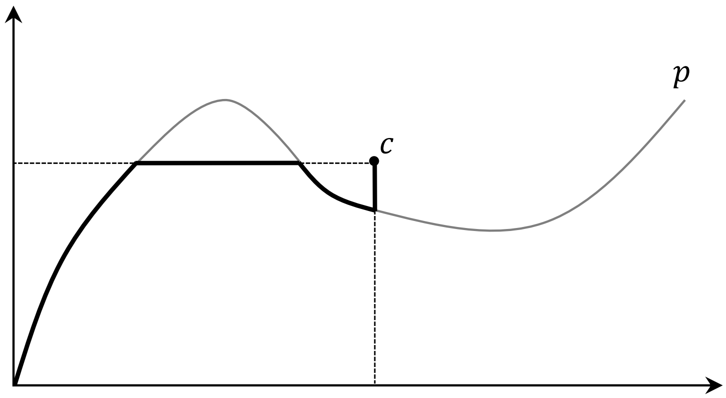

Figure 4: ***Violation of Resource Monotonicity.*** *The path is not monotone, and so Resource Monotonicity is violated. A path of awards is shown in bold for the claims vector c.*

path if $r(i, x_i) = r(j, x_j)$ for every $\{i, j\} \subset \mathbb{N}$.[7] Relatedly, taking the inverse of r with respect to the award yields the claims independent parametric function. Finally, note that for a CRAS rule, the first order conditions for the maximization problem require that the solution equalize marginal utilities across the claimants (for those claimants whose award is not constrained by their respective claim). Thus we can take r to represent the marginal utility function. We get the SWF by integrating r and summing across the claimants.

4 Duality

An alternative way of viewing a claims problem is how to divide up the losses. Since the losses for the problem (N, c, E) is $E - \sum_N c_i$, we define the *dual of* (N, c, E) as the problem $(N, c, \sum_N c_i - E)$. Let $L(N, c, E)$ denote the set of efficient feasible loss vectors for the problem (N, c, E), i.e.

$$L(N, c, E) \equiv \left\{ x \in \mathbb{R}^N_+ : \mathbf{0} \leqq x \leqq c \text{ and } \sum_{i \in N} x_i = \sum_{i \in N} c_i - E \right\}.$$

Note that for every (N, c, E), we have $X(N, c, E) = L(N, c, \sum_N c_i - E)$, which is to say that the set of feasible awards for a problem is equal to the set of feasible losses for the dual of that problem. The *dual of the rule* S is

$$S^d(N, c, E) \equiv c - S(N, c, \sum_{i \in N} c_i - E).$$

[7] A natural alternative to this is to derive the path directly by defining $p_N(t) = S(N, (t, \dots, t), t)$. However, this can only define the path for some finite N. Since the group of possible claimants is infinite, extending this path to infinite dimensions would take considerable more work.

Thus S^d allocates losses the same way that S allocates gains, and vice versa. The *dual of the axiom* A is the axiom A^d such that

$$S \text{ satisfies } A \text{ if and only if } S^d \text{ satisfies } A^d.$$

One can show that the dual of Consistency is Consistency itself. Similarly, the dual of Resource Monotonicity is itself. This is not true of IIA.

IIA-Dual. *For every (N, c, E) and (N, c', E'), if $L(N, c', E') \subset L(N, c, E)$ and $c - S(N, c, E) \in L(N, c', E')$, then $c' - S(N, c', E') = c - S(N, c, E)$.*

Thus IIA-Dual treats losses as what matters, and so would be more normatively appealing in applications in which the resource to be divided is actually a bad.

It is easy to see the implications of replacing IIA with IIA-Dual in Theorem 1. Namely, the result would be a rule which divides losses according to a monotone path rule. However what would be the result if IIA-Dual was added to the axioms in Theorem 1? Since the intuition behind IIA and IIA-Dual are somewhat incompatible, the result is a rule which is usually considered to be normatively unappealing.

The queueing rule divides the endowment by lining up the claimants in a queue and then awarding the first person in the queue his full claim, then the second person his full claim, and continuing until the endowment is exhausted. Let $\mathcal{L}$ denote the set of strict linear orders over $\mathcal{N}$. For $\succ \in \mathcal{L}$, define the rule $S^{\succ}$ as:

$$S^{\succ}(N, c, E) = \left(\min \left\{ c_i, \max \left\{ 0, E - \sum_{j \in N : j \succ i} c_j \right\} \right\} \right)_{i \in N}$$

We say a rule S is a queueing rule if there exists $\succ \in \mathcal{L}$ such that $S = S^{\succ}$.

Theorem 2 *The rule S satisfies Resource Monotonicity, Bilateral Consistency, IIA, and IIA-Dual if and only if S is a queueing rule.*

The proof is given in the appendix.

5 Symmetry

If there is no a priori reason to treat the claimants differently, then one would want the rule to give the same award to individuals with the same claim. This is captured in the following axiom.

Symmetry. *For every problem (N, c, E) and $\{i, j\} \in N$, if $c_i = c_j$, then $S_i(N, c, E) = S_j(N, c, E)$.*

The constrained equal awards rule is the rule that gives everyone the same award, unless that award is more than a claimant's claim in which case that claimant gets his full claim:

$$CEA(N, c, E) = (\min\{c_i, \lambda\})_{i \in N}$$

where λ is chosen such that $\sum_N CEA_i(N,c,E) = E$. It should be obvious that the constrained equal awards rule is a claims independent parametric rule and that it satisfies Symmetry. In fact it is the only claims independent parametric rule to satisfy Symmetry.

Theorem 3 *The constrained equal awards rule is the only rule to satisfy Resource Monotonicity, Bilateral Consistency, IIA, and Symmetry.*

The proof is straightforward given Theorem 1, and so is omitted. We note that adding IIA to the set of axioms in Young (1987, Theorem 1) would give the same result. Thus we could replace Resource Monotonicity with the assumption that S is continuous in Theorem 3 and the result would still hold.

6 Relating IIA to Other Axioms

We conclude with some results relating IIA to other well-known axioms in the literature.

Firstly, one may wonder what the implications would be of assuming another well-known rationality axiom: the Weak Axiom of Revealed Preference (WARP).

WARP. *For every (N,c,E) and (N,c',E), if $S(N,c,E), S(N,c',E) \in X(N,c,E) \cap X(N,c',E)$, then $S(N,c,E) = S(N,c',E)$.*

Given the structure here, it is not difficult to show that IIA and WARP are in fact equivalent. (For this and subsequent results, it will be useful to recall that if (N,c,E) and c' satisfy $S(N,c,E) \leqq c' \leqq c$, then the conditions of IIA are satisfied.)

Lemma 1 *A rule satisfies IIA if and only if it satisfies WARP.*

Proof. It is obvious that WARP implies IIA. To show the other direction, let S satisfy IIA. Set $x \equiv S(N,c,E)$, $x' \equiv S(N,c',E)$, and $y \equiv x \vee x'$. Hence we have $x \leqq y$ and $x' \leqq y$. Also, since $x, x' \in X(N,c,E) \cap X(N,c',E)$, we must have $y \leqq c$ and $y \leqq c'$. Thus IIA implies both $S(N,y,E) = x$ and $S(N,y,E) = x'$, which implies $x = x'$. ■

We impose Resource Monotonicity rather than Strict Resource Monotonicity for normative reasons. However the next result gives another reason to prefer the former over the latter: IIA is incompatible with Strict Resource Monotonicity.

Lemma 2 *A rule cannot satisfy both IIA and Strict Resource Monotonicity.*

Proof. Suppose S satisfies both IIA and Strict Resource Monotonicity. Choose (N,c,E) such that $E < \sum_N c_i$. Then Strict Resource Monotonicity implies $x \equiv S(N,c,E) < c$. Choose any $i \in N$. Set $c'_i = x_i$ and $c'_j = c_j$ for every $j \neq i$. Then IIA implies $x = S(N,c',E)$. But then for any $E' \in (E, \sum_N c'_i)$, we must have $S_i(N,c',E') = x_i$, violating Strict Resource Monotonicity. ■

IIA is similar to two invariance properties prominent in the literature: Claims Truncation Invariance (Dagan and Volij, 1993) and Upper Composition (Moulin, 2000).[8] Both of these axioms (like IIA) impose an invariance on the award due to a specific change in the claims.

Claims Truncation Invariance. *Set* $t(N, c, E) \equiv (\min\{c_i, E\})_{i \in N}$. *For every* (N, c, E), *we have* $S(N, c, E) = S(N, t(N, c, E), E)$.

Upper Composition. *For every* (N, c, E) *and* $E' < E$, *we have* $S(N, c, E') = S(N, S(N, c, E), E')$.

Claims Truncation Invariance says that claims in excess of the endowment should be ignored. Upper Composition says that if the claims vector is replaced by the awards from another problem with the same group, same claims vector, but larger endowment, then the award should not change.

The next result states that IIA implies Claims Truncation Invariance.

Lemma 3 *If a rule satisfies IIA, then it satisfies Claims Truncation Invariance.*

Proof. Let S satisfy IIA and fix (N, c, E). Note that $S(N, c, E) \leqq t(N, c, E) \leqq c$. Hence IIA implies $S(N, c, E) = S(N, t(N, c, E), E)$. ■

The converse is not true as the Talmud rule (Aumann and Maschler, 1985) satisfies Claims Truncation Invariance, but not IIA.

It is not true that IIA implies Upper Composition. This is because Upper Composition implies Resource Monotonicity whereas IIA does not. (Thus the example given in B satisfies IIA but not Upper Composition.) However, IIA and Resource Monotonicity jointly imply Upper Composition.

Lemma 4 *If a rule satisfies IIA and Resource Monotonicity, then it satisfies Upper Composition.*

Proof. Let S satisfy IIA and Resource Monotonicity. Fix (N, c, E) and $E' < E$. By Resource Monotonicity, $S(N, c, E') \leqq S(N, c, E)$. Also $S(N, c, E) \leqq c$ by definition. Thus IIA implies $S(N, c, E') = S(N, S(N, c, E), E')$. ■

The converse is not true as the Proportional rule satisfies Upper Composition but not IIA.

A number of papers consider consistent and resource monotonic rules satisfying either Claims Truncation Invariance or Upper Composition. Moulin (2000) derives a rich family of rules satisfying Consistency, Upper Composition, the dual of Upper Composition, and a scale invariance property called Homogeneity. Chambers (2006) studies a similar family, though without Homogeneity. The overlap between these rules and the monotone path rules is non-trivial, the constrained equal awards rule being the most prominent member of both. Hokari and Thomson (2003) introduce an asymmetric generalization of the Talmud rule. The axioms they impose are Consistency, Claims Truncation Invariance, the dual of Claims Truncation Invariance, and Homogeneity. The only overlap between this family and the monotone path rules is

[8] Dagan and Volij (1993) call their axiom "Independence of Irrelevant Claims." Some authors alternately use "Composition Down" for Upper Composition.

the queueing rule. Young (1988) introduces the family of equal sacrifice rules, which is characterized by Consistency, Symmetry, Upper Composition, and Strict Resource Monotonicity. Naumova (2002) characterizes an asymmetric version of this family. However, given Lemma 2, there is no overlap between the equal sacrifice rules and the monotone path rules.

Appendix

Throughout the appendix, we shorten notation by writing the problem (N, c, E) as (c, E), as the group of claimants N is implicit in the claims vector c.

A Proof of Theorem 1

It is a straightforward exercise to show that each of these families of rules satisfies the axioms.

So we show that the axioms are sufficient to get each family of rules, i.e. we show that statement 1 implies 2, 3, and 4. Let S satisfy Bilateral Consistency, Resource Monotonicity, and IIA. It is straightforward to show that Resource Monotonicity implies the following continuity axiom.

Resource Continuity. *For every (c, E), for every sequence $(E^k)_{k\in\mathbb{N}}$ where $E^k \to E$, we have $S(c, E^k) \to S(c, E)$.*

We follow the general proof strategy of Kaminski (2000, 2006) and Stovall (2014) by defining a binary relation over a suitable outcome space. The key step in the proof is showing that this binary relation has a numerical representation.

For $i \neq j$ and $x_i, x_j > 0$, define:

$$G(i, x_i, j, x_j) \equiv \inf\{E : x_i = S_i((x_i, x_j), E)\}$$

Resource Continuity implies that we can replace inf with min. Note that we always have $G(i, x_i, j, x_j) \leq x_i + x_j$. Set

$$Y \equiv \mathbb{N} \times \mathbb{R}_+$$

and define the binary relation R_1 over Y as follows:

$$(i, x_i)R_1(j, x_j) \text{ if } i \neq j \text{ and either } x_i = 0 \text{ or } G(i, x_i, j, x_j) \leq G(j, x_j, i, x_i).$$

Note that if $(i, x_i)R_1(j, x_j)$ and $x_j = 0$, then it must be that $x_i = 0$. Define the binary relation R_2 over Y as follows:

$$(i, x_i)R_2(j, x_j) \text{ if } i = j \text{ and } x_i \leq x_j.$$

Set $R \equiv R_1 \cup R_2$.

A.1 R Has a Numerical Representation

We show that R is complete, transitive, and that there exists a countable R-dense subset of Y. Thus, by Cantor's classic result, R has a numerical representation.

By definition, it is obvious that R is complete. The following series of lemmas prove the other properties.

Lemma 5 *Suppose $i \neq j$ and $x_i, x_j > 0$. Then $(i, x_i)R_1(j, x_j)$ if and only if $x_i + x_j = G(j, x_j, i, x_i)$.*

Proof. Let $E^i = G(i, x_i, j, x_j)$ and $E^j = G(j, x_j, i, x_i)$. Note that $S_i((x_i, x_j), E^i) = x_i$ and $S_j((x_i, x_j), E^j) = x_j$. Suppose $E^i \leq E^j$. Then Resource Monotonicity implies $S_i((x_i, x_j), E^j) = x_i$, which implies $x_i + x_j = E^j$. Going the other direction, if $E^i > E^j$, then Resource Monotonicity implies $S_j((x_i, x_j), E^i) = x_j$. Hence $E^i = x_i + x_j > E^j$. ■

Thus for $i \neq j$ and $x_i, x_j > 0$, $(i, x_i)P_1(j, x_j)$ if and only if $G(i, x_i, j, x_j) < x_i + x_j$.

Lemma 6 *For every $(i, x_i), (j, x_j), (k, x_k) \in Y$ where i, j, k are distinct and $x_i, x_j, x_k > 0$, there exists E such that*

$$S((x_i, x_j, x_k), E) = (G(k, x_k, i, x_i) - x_k, G(k, x_k, j, x_j) - x_k, x_k).$$

Proof. Set

$$E \equiv G(k, x_k, i, x_i) + G(k, x_k, j, x_j) - x_k$$

and

$$(y_i, y_j, y_k) \equiv S((x_i, x_j, x_k), E).$$

We show that we must have $y_i + y_k = G(k, x_k, i, x_i)$ and $y_j + y_k = G(k, x_k, j, x_j)$. By way of contradiction, assume without loss of generality that $y_i + y_k < G(k, x_k, i, x_i)$. Then since $y_i + y_j + y_k = E$ it must be that $y_j + y_k > G(k, x_k, j, x_j)$. But then Bilateral Consistency and Resource Monotonicity would imply both $y_k < x_k$ and $y_k = x_k$.

Since $y_i + y_k = G(k, x_k, i, x_i)$, Bilateral Consistency implies that $y_k = x_k$. This implies $y_i = G(k, x_k, i, x_i) - x_k$. Similarly, we have $y_j = G(k, x_k, j, x_j) - x_k$. ■

Lemma 7 *Suppose $i \neq j$, $x_i, x_j > 0$, and $(j, x_j)R_1(i, x_i)$. Set $E^* \equiv G(j, x_j, i, x_i)$ and $x_i^* \equiv E^* - x_j$. Then for any $(c_i, c_j) \geq (x_i^*, x_j)$, we have $S((c_i, c_j), E^*) = (x_i^*, x_j)$.*

Proof. First consider the case when $(c_i, c_j) \geq (x_i, x_j)$. Then $X((x_i, x_j), E) \subset X((c_i, c_j), E)$ for every $E \leq x_i + x_j$. Also note that Lemma 5 implies $E^* \leq x_i + x_j$, which implies $x_i^* \leq x_i$.

Note that for every $E' \leq \min\{x_i, x_j\}$, we have $X((x_i, x_j), E') = X((c_i, c_j), E')$, and so IIA implies $S((x_i, x_j), E') = S((c_i, c_j), E')$. Thus if $E^* \leq \min\{x_i, x_j\}$, then we are done since $S((x_i, x_j), E^*) = (x_i^*, x_j)$. So assume $E^* > \min\{x_i, x_j\}$. Note also that for every $E'' < E^*$, we have $S_j((x_i, x_j), E'') < x_j$ (by definition of E^*) and $S_i((x_i, x_j), E'') < x_i$ (since $(j, x_j)R_1(i, x_i)$). Thus for every $E' \leq \min\{x_i, x_j\} < E^*$, we have $S_j((c_i, c_j), E') < x_j$ and $S_i((c_i, c_j), E') < x_i$.

So set $E^1 \equiv \min\{x_i, x_j\}$ and $(y_i^1, y_j^1) \equiv S((c_i, c_j), E^1) = S((x_i, x_j), E^1)$. Thus $y_i^1 < x_i$ and $y_j^1 < x_j$.

Set $E^2 \equiv \min\{E^*, E^1 + \min\{x_i - y_i^1, x_j - y_j^1\}\}$ and $(y_i^2, y_j^2) \equiv S((c_i, c_j), E^2)$. Resource Monotonicity implies that $(y_i^2, y_j^2) \geq (y_i^1, y_j^1)$. Since $y_i^2 \geq y_i^1$, we have $y_j^2 = E^2 - y_i^2 \leq E^2 - y_i^1$. But by definition, $E^2 \leq E^1 + x_j - y_j^1$, and thus $y_j^2 \leq E^1 + x_j - y_j^1 - y_i^1$. But since $y_j^1 + y_i^1 = E^1$, this implies $y_j^2 \leq x_j$. Similarly, we can show $y_i^2 \leq x_i$. Thus we have $(0, 0) \leqq (y_i^2, y_j^2) \leqq (x_i, x_j)$ and $y_i^2 + y_j^2 = E^2$, which means

$(y_i^2, y_j^2) \in X((x_i, x_j), E^2)$. IIA then implies $(y_i^2, y_j^2) = S((c_i, c_j), E^2) = S((x_i, x_j), E^2)$. If $E^2 = E^*$, then we have proved the result since $S((x_i, x_j), E^*) = (x_i^*, x_j)$. So assume $E^2 < E^*$. Thus we have $y_i^2 < x_i$ and $y_j^2 < x_j$.

Similarly, for $n = 3, 4, \ldots$, set $E^n \equiv \min\{E^*, E^{n-1} + \min\{x_i - y_i^{n-1}, x_j - y_j^{n-1}\}\}$ and $(y_i^n, y_j^n) \equiv S((c_i, c_j), E^n)$. As before, we can show $(y_i^n, y_j^n) = S((c_i, c_j), E^n) = S((x_i, x_j), E^n)$. Again, if $E^n = E^*$ for any n, then we are done. So we assume $E^n < E^*$ for every n. Thus $y_i^n < x_i$ and $y_j^n < x_j$.

Now we show $E^n \to E^*$. So suppose not, i.e. $E^n \to \hat{E} < E^*$. Then since $E^n < E^*$ by assumption, we must have $\min\{x_i - y_i^n, x_j - y_j^n\} \to 0$. So either $y_i^n \to x_i$ or $y_j^n \to x_j$. If the former, then $S((x_i, x_j), E^n) = (y_i^n, y_j^n) \to (x_i, \hat{E} - x_i)$. Resource Continuity implies then that $S((x_i, x_j), \hat{E}) = (x_i, \hat{E} - x_i)$. But then by definition of E^* and the fact that $(j, x_j) R_1 (i, x_i)$, we must have $\hat{E} \geq E^*$, which is a contradiction. Similarly, we get a contradiction if $y_j^n \to x_j$.

Since $E^n \to E^*$, Resource Continuity implies $S((c_i, c_j), E^n) \to S((c_i, c_j), E^*)$ and $S((x_i, x_j), E^n) \to S((x_i, x_j), E^*) = (x_i^*, x_j)$. But since $S((c_i, c_j), E^n) = S((x_i, x_j), E^n)$ for every n, we must have $S((c_i, c_j), E^*) = (x_i^*, x_j)$. This proves the result for $(c_i, c_j) \geq (x_i, x_j)$.

Now take any $(c_i, c_j) \geq (x_i^*, x_j)$. Set $(c_i', c_j') = (\max\{c_i, x_i\}, c_j)$. Then by the above result, we have $S((c_i', c_j'), E^*) = (x_i^*, x_j)$. But $(x_i^*, x_j) \in X((c_i, c_j), E^*)$. Hence IIA implies $S((c_i, c_j), E^*) = (x_i^*, x_j)$. ■

Lemma 8 *R is transitive.*

Proof. Let $(i, x_i) R (j, x_j) R (k, x_k)$. The proof is straightforward if either x_i, x_j, or x_k is 0. Hence assume $x_i, x_j, x_k > 0$.

Case 1: i, j, k distinct.

By way of contradiction, suppose $(k, x_k) P_1 (i, x_i)$. Then Lemma 5 implies

$$G(k, x_k, i, x_i) < x_i + x_k. \tag{1}$$

By Lemma 6, there exists E such that

$$S((x_1, x_2, x_3), E) = (G(k, x_k, i, x_i) - x_k, G(k, x_k, j, x_j) - x_k, x_k).$$

For brevity, set $y_i \equiv G(k, x_k, i, x_i) - x_k$ and $y_j \equiv G(k, x_k, j, x_j) - x_k$. Lemma 5 implies $G(k, x_k, j, x_j) = x_j + x_k$. Hence $y_j = x_j$. Bilateral Consistency implies $S((x_i, x_j), y_i + x_j) = (y_i, x_j)$. However since $G(i, x_i, j, x_j) \leq G(j, x_j, i, x_i)$, Resource Monotonicity implies $y_i = x_i$. Hence $G(k, x_k, i, x_i) = x_i + x_k$, which contradicts (1).

Case 2: $i = k \neq j$.

So let $x_i' = x_k$. We need to show that $x_i \leq x_i'$. By way of contradiction, suppose $x_i > x_i'$. Let $E^1 = x_i' + x_j$ and $E^2 = x_i + x_j$. Then $E^1 < E^2$. Set $E^* = G(j, x_j, i, x_i')$. By Lemma 5 we have $E^1 = G(i, x_i', j, x_j)$. Hence $(j, x_j) R_1 (i, x_i')$ implies $E^* \leq E^1$. Lemma 5 also implies $E^2 = G(j, x_j, i, x_i)$. But Lemma 7 implies $S((x_i, x_j), E^*) = (x_i^*, x_j)$, which means we must have $E^2 \leq E^*$. But then $E^2 \leq E^1$, which is a contradiction.

Case 3: $i = j \neq k$.

So let $x'_i = x_j$. Then $x_i \leq x'_i$ by definition of R_2. By way of contradiction, suppose $(k, x_k)P_1(i, x_i)$. Since $(i, x'_i)R_1(k, x_k)$, Case 2 above implies $x'_i \leq x_i$. Hence $x'_i = x_i$. But then we have $(k, x_k)P_1(i, x_i)$ and $(i, x_i)R_1(k, x_k)$, a contradiction.

Case 4: $i \neq j = k$.

Similar to Case 3.

Case 5: $i = j = k$.

Then $x_i \leq x_j \leq x_k$. ■

Lemma 9 $\mathbb{N} \times \mathbb{Q}_{++}$ *is an* R*-dense subset of* Y*.*

Proof. Let $(i, x_i)P(j, x_j)$. Then we must have $x_j > 0$. (If $x_j = 0$, then $(j, x_j)R(i, x_i)$ which contradicts $(i, x_i)P(j, x_j)$.)

Case 1: $i = j$.

Then $x_i < x_j$, so the result follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Case 2: $i \neq j$ and $x_i = 0$.

Then for any $x'_j \in (0, x_j)$, we must have $(i, x_i)P(j, x'_j)P(j, x_j)$. The result follows from Case 1 and transitivity.

Case 3: $i \neq j$ and $x_i > 0$.

Set $E^* \equiv G(i, x_i, j, x_j)$ and $x^*_j \equiv E^* - x_i$. Obviously $S((x_i, x_j), E^*) = (x_i, x^*_j)$. Lemma 5 implies $G(i, x_i, j, x_j) < x_i + x_j$, which implies $x_j > x^*_j$. Choose $x'_j \in (x^*_j, x_j) \cap \mathbb{Q}$. Then $(j, x'_j)P(j, x_j)$ by definition. By way of contradiction, suppose $(j, x'_j)P(i, x_i)$. Set $E' \equiv G(j, x'_j, i, x_i)$ and $x'_i \equiv E' - x'_j$. Lemma 5 implies $G(j, x'_j, i, x_i) < x_i + x'_j$, which implies $x_i > x'_i$. Lemma 7 implies that $S((x_i, x_j), E') = (x'_i, x'_j)$. But as we already noted above $S((x_i, x_j), E^*) = (x_i, x^*_j)$, and since $x_i > x'_i$, Resource Monotonicity implies that $x^*_j \geq x'_j$, which is a contradiction. Hence it must be that $(i, x_i)R(j, x'_j)$. ■

Hence R is complete, transitive, and there exists a countable R-dense subset of Y. Thus there exists $r : Y \to \mathbb{R}$ such that

$$(i, x_i)R(j, x_j) \Leftrightarrow r(i, x_i) \leq r(j, x_j).$$

Note that for every i, $r(i, \cdot)$ is strictly increasing.

Before continuing, we prove the following lemma, which will be useful in what follows.

Lemma 10 *Fix* (c, E) *and set* $x \equiv S(c, E)$*. Set* $\lambda^* \equiv \max_{i \in N}\{r(i, x_i)\}$*. Then for every* $i \in N$ *where* $x_i < c_i$*,*

$$r(i, x_i) \leq \lambda^* \leq \lim_{x'_i \downarrow x_i} r(i, x'_i).$$

Proof. Obviously $r(i, x_i) \leq \lambda^*$. If $r(i, x_i) = \lambda^*$, then the result obviously holds since $r(i, \cdot)$ is strictly increasing. So suppose $r(i, x_i) < \lambda^* = r(j, x_j)$ for some $j \in N$ such that $j \neq i$.

We show that for every $x_i' \in (x_i, c_i)$, we have $\lambda^* < r(i, x_i')$. By way of contradiction, suppose $\lambda^* = r(j, x_j) \geq r(i, x_i')$. Since $r(i, \cdot)$ is strictly increasing, we can assume $r(j, x_j) > r(i, x_i')$ without loss of generality. Note that $x_i' > x_i \geq 0$. Thus $\lambda^* \geq r(i, x_i')$ implies that $x_j > 0$. So set $E^* \equiv G(i, x_i', j, x_j)$ and $x_j^* \equiv E^* - x_i'$. Lemma 5 implies $E^* = x_i' + x_j^* < x_i' + x_j$, which implies $x_j^* < x_j$. By Lemma 7, $S((c_i, c_j), E^*) = (x_i', x_j^*)$. However Bilateral Consistency implies $S((c_i, c_j), x_i + x_j) = (x_i, x_j)$. Since $x_i < x_i'$ and $x_j > x_j^*$, this must violate Resource Monotonicity.

Thus for $x_i' \in (x_i, c_i)$, we have $\lambda^* < r(i, x_i')$. Hence $\lambda^* \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$. ■

A.2 Claims Independent Parametric Rule

Here we show that S must be a claims independent parametric rule.

For $\lambda \in \overline{\mathbb{R}}$, define

$$f_i(\lambda) \equiv \begin{cases} 0 & \text{if } \lambda < r(i,0) \\ \sup\{x_i : r(i, x_i) \leq \lambda\} & \text{if } \lambda \geq r(i,0) \end{cases}.$$

Lemma 11 *For every $(i, x_i) \in Y$, we have $f_i(r(i, x_i)) = x_i$.*

Proof. Fix $(i, x_i) \in Y$. Since $r(i, \cdot)$ is increasing, we have

$$\{x_i' : r(i, x_i') \leq r(i, x_i)\} = \{x_i' : x_i' \leq x_i\}.$$

Hence $f_i(r(i, x_i)) = \sup\{x_i' : x_i' \leq x_i\} = x_i$. ■

Lemma 12 *$f \equiv \{f_i\}_{\mathbb{N}} \in \mathcal{F}$.*

Proof. Fix $i \in \mathcal{N}$.

First we show f_i is weakly increasing. Let $\lambda, \lambda' \in \overline{\mathbb{R}}$ be given where $\lambda < \lambda'$. If $\lambda < r(i, 0)$, then $f_i(\lambda) = 0$. Since the definition of f_i implies $f_i(\lambda') \geq 0$, we have $f_i(\lambda) \leq f_i(\lambda')$. If $\lambda \geq r(i, 0)$, then observe that $\emptyset \neq \{x_i : r(i, x_i) \leq \lambda\} \subset \{x_i : r(i, x_i) \leq \lambda'\}$. Hence $f_i(\lambda) \leq f_i(\lambda')$.

Next we show f_i is continuous. Suppose f_i were discontinuous. Then since f_i is weakly increasing, there must exist $x_0 \in \mathbb{R}_+$ such that x_0 is not in the image of f_i. But this is a contradiction since $\lambda = r(i, x_0) \in \mathbb{R}$ and $f_i(\lambda) = x_0$ by Lemma 11.

Finally we show $f_i(-\infty) = 0$ and $f_i(+\infty) = +\infty$. Note that $-\infty < r(i, 0)$. Hence $f_i(-\infty) = 0$ by definition of f_i. Note that $+\infty \geq r(i, x_i)$ for every $x_i \in \mathbb{R}_+$. Hence $f_i(+\infty) = \sup \mathbb{R}_+ = +\infty$. ■

Lemma 13 *$S = S^f$.*

Proof. Fix (c, E) and set $x \equiv S(c, E)$. Set $\lambda^* \equiv \max_{i \in N}\{r(i, x_i)\}$. Fix $i \in N$.

Note that $\lambda^* \geq r(i, x_i)$, which means $f_i(\lambda^*) \geq f_i(r(i, x_i))$ since f_i is weakly increasing. Lemma 11 then implies $f_i(\lambda^*) \geq x_i$.

If $x_i = c_i$, then $\min\{f_i(\lambda^*), c_i\} = c_i$, which proves the result. So assume $x_i < c_i$. By Lemma 10, $\lambda^* \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$. Thus $f_i(\lambda^*) \leq f_i(\lim_{x_i' \downarrow x_i} r(i, x_i'))$ since f_i is weakly increasing. But then Lemma 11 and continuity of f_i imply $f_i(\lambda^*) \leq x_i$. We have thus established that for the case where $x_i < c_i$, we must have $f_i(\lambda^*) = x_i$. Thus $\min\{f_i(\lambda^*), c_i\} = x_i$. ■

A.3 Monotone Path Rule

Here we show that S must be a monotone path rule.

Set

$$p(t) = (f_i(h(t)))_{i \in \mathbb{N}}$$

where h is any strictly increasing bijection from $[0,1]$ to $\overline{\mathbb{R}}$. For example: $h(t) = 1 - \frac{1}{2t}$ for $0 \leq t \leq \frac{1}{2}$ and $h(t) = \frac{t}{1-t} - 1$ for $\frac{1}{2} < t \leq 1$.

Lemma 14 *$p \in \mathcal{P}$.*

Proof. Since h is a strictly increasing bijection from $[0,1]$ to $\overline{\mathbb{R}}$, we must have $h(0) = -\infty$ and $h(1) = +\infty$. Also, Lemma 12 establishes that $f_i(-\infty) = 0$ and $f_i(+\infty) = +\infty$ for every $i \in \mathbb{N}$. Hence $p(0) = \mathbf{0}$ and $p(1) = \Omega$. Also, p must be weakly monotone and continuous since h is increasing and continuous and each f_i is weakly increasing and continuous. ■

Lemma 15 *$S = S^p$.*

Proof. Follows from Lemma 13. ■

A.4 CRAS Rule

Here we show that S must be a CRAS rule.

Since $r(i,a)$ is strictly increasing, it is Riemann integrable. Define

$$U_i(x_0) \equiv -\int_0^{x_0} r(i,a)\,\mathrm{d}a.$$

Lemma 16 *$U \equiv \{U_i\}_{\mathbb{N}} \in \mathcal{U}$.*

Proof. Fix $i \in \mathbb{N}$. First we show U_i is strictly concave. Fix $x, x' \in \mathbb{R}_+$ such that $x < x'$. Fix $\alpha \in (0,1)$. Set $\hat{x} \equiv \alpha x + (1-\alpha)x'$. Note that

$$U_i(\hat{x}) = U_i(x) - \int_x^{\hat{x}} r(i,a)\,\mathrm{d}a$$

and

$$\begin{aligned}\alpha U_i(x) + (1-\alpha)U_i(x') &= U_i(x) - (1-\alpha)\int_x^{x'} r(i,a)\,\mathrm{d}a \\ &= U_i(x) - \int_x^{\hat{x}} r(i,a)\,\mathrm{d}a + \alpha\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a - (1-\alpha)\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a.\end{aligned}$$

Hence, U_i is strictly concave if and only if

$$\alpha\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a - (1-\alpha)\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a < 0.$$

But since $x < \hat{x} < x'$ and since $r(i,a)$ is strictly increasing, we have

$$\int_x^{\hat{x}} r(i,a)\,\mathrm{d}a < \int_x^{\hat{x}} r(i,\hat{x})\,\mathrm{d}a = (1-\alpha)(x'-x)r(i,\hat{x})$$

and

$$\int_{\hat{x}}^{x'} r(i,a)\,\mathrm{d}a > \int_{\hat{x}}^{x'} r(i,\hat{x})\,\mathrm{d}a = \alpha(x'-x)r(i,\hat{x}),$$

which, inputting into the above equation, gives the result.

The continuity of U_i follows from the Fundamental Theorem of Calculus. ■

Lemma 17 *Suppose $i \neq j$, $x_i, x_j > 0$ and $(j,x_j)P_1(i,x_i)$. Set $x_i^* \equiv G(j,x_j,i,x_i) - x_j$. Then for any $x_i' \in (x_i^*, x_i)$, we have $(j,x_j)P_1(i,x_i')$.*

Proof. Lemma 5 implies $G(j,x_j,i,x_i) < x_i + x_j = G(i,x_i,j,x_j)$. Thus $x_i^* < x_i$ so (x_i^*, x_i) is not empty. Fix $x_i' \in (x_i^*, x_i)$. Lemma 7 implies $S((x_i', x_j), G(j,x_j,i,x_i)) = (x_i^*, x_j)$. Thus $G(j,x_j,i,x_i') \leq G(j,x_j,i,x_i)$. Since $x_i' > x_i^* = G(j,x_j,i,x_i) - x_j$, we have $G(j,x_j,i,x_i') < x_i' + x_j$. Lemma 5 then implies $(j,x_j)P_1(i,x_i')$. ■

Lemma 18 *For every $i \in \mathbb{N}$ and $x_i \in \mathbb{R}_+$, it is without loss of generality to assume $\lim_{x_i' \uparrow x_i} r(i,x_i') = r(i,x_i)$.*

Proof. If $x_i = 0$, then the only sequence in $\mathbb{R}_+$ below 0 is the sequence of zeros. Hence $\lim_{x_i' \uparrow 0} r(i,x_i') = r(i,0)$.

So assume $x_i > 0$. Since $r(i,\cdot)$ is strictly increasing, we obviously have $\lim_{x_i' \uparrow x_i} r(i,x_i') \leq r(i,x_i)$. By way of contradiction, suppose $a \equiv \lim_{x_i' \uparrow x_i} r(i,x_i') < r(i,x_i)$. Note that without loss of generality, we can assume that there exists $(j,x_j) \in Y$ where $j \neq i$ such that $r(j,x_j) \in (a, r(i,x_i))$. (If no such (j,x_j) existed, then we could "cut out" the discontinuity at (i,x_i) without changing the ordering of r, and thus have $a = r(i,x_i)$.) Then for every $x_i' \in (0,x_i)$, we have $r(i,x_i') < r(j,x_j) < r(i,x_i)$. But then $(j,x_j)P_1(i,x_i)$, so Lemma 17 implies that $r(j,x_j) < r(i,x_i')$ for every $x_i' \in (G(j,x_j,i,x_i) - x_j, x_i)$, which is a contradiction. ■

Lemma 19 *Fix (c,E) and set $x \equiv S^U(c,E)$. Suppose $0 < E < \sum_N c_i$. Then there exists λ^* such that for every $i \in N$ where $x_i < c_i$,*

$$r(i,x_i) \leq \lambda^* \leq \lim_{x_i' \downarrow x_i} r(i,x_i').$$

Proof. Set $C \equiv \{y \in \mathbb{R}^N : 0 \leqq y \leqq c\}$. Note that x is the unique solution to the problem

$$\min_{y \in C} \sum_{i \in N} -U_i(y_i) \;\text{ subject to }\; E - \sum_{i \in N} y_i = 0.$$

Also note that there exists y in the interior of C such that $E = \sum_N y_i$ and $\sum_N -U_i(y_i)$ is finite. Hence (Rockafellar, 1970, Corollary 28.2.2), there exists λ^* such that x is the solution to

$$\min_{y \in C} \left[\sum_{i \in N} -U_i(y_i) + \lambda^* \left(E - \sum_{i \in N} y_i \right) \right] = \lambda^* E + \sum_{i \in N} \min_{y_i \in [0,c_i]} \left[-U_i(y_i) - \lambda^* y_i \right].$$

Thus for every i and $y_i \in [0, c_i]$, we have $-U_i(y_i) - \lambda^* y_i \geq -U_i(x_i) - \lambda^* x_i$. So if $x_i \in [0, c_i)$, then $-U_i(y_i) - \lambda^* y_i \geq -U_i(x_i) - \lambda^* x_i$ for every $y_i \in \mathbb{R}_+$ since $-U_i(y_i)$ is strictly convex (by Lemma 16). Thus for every i where $x_i < c_i$, λ^* is a subderivative of $-U_i(\cdot)$ at x_i. Thus by Lemma 18 and (Rockafellar, 1970, Theorem 24.2), we have $r(i, x_i) \leq \lambda^* \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$. ■

Lemma 20 $S = S^U$.

Proof. Fix (c, E) and set $x \equiv S(c, E)$ and $x^U \equiv S^U(c, E)$. Obviously if $E = 0$ then $x = x^U = \mathbf{0}$. Also, if $E = \sum_N c_i$ then $x = x^U = c$. So assume $0 < E < \sum_N c_i$. By Lemma 10, we have (setting $\lambda \equiv \max_{i \in N}\{r(i, x_i)\}$)

$$r(i, x_i) \leq \lambda \leq \lim_{x_i' \downarrow x_i} r(i, x_i')$$

for every $i \in N$ where $x_i < c_i$. By Lemma 19, there exists λ^U such that

$$r(i, x_i^U) \leq \lambda^U \leq \lim_{x_i' \downarrow x_i^U} r(i, x_i')$$

for every $i \in N$ where $x_i^U < c_i$.

By way of contradiction, suppose $x \neq x^U$. Then there exists $i \in N$ such that $x_i \neq x_i^U$. If $x_i^U < x_i \leq c_i$, then $\lim_{x_i' \downarrow x_i^U} r(i, x_i') < r(i, x_i)$ since $r(i, \cdot)$ is strictly increasing. Note that $r(i, x_i) \leq \lambda$ (even if $x_i = c_i$) by definition of λ. Hence $\lambda^U < \lambda$. But this implies $x_j^U \leq x_j$ for every $j \in N$, which implies $\sum_N x_j^U < \sum_N x_j$. But this is a contradiction since both equal E. If $x_i < x_i^U \leq c_i$, then by similar reasoning we get a contradiction. Hence we must have $x = x^U$. ■

B Example Violating Resource Monotonicity

Instead of describing the paths of awards for the rule as we did in Section 3, we will define a function that is almost parametric. This function will be parametric for every $i \in \mathbb{N}$ (meaning f_i satisfies the conditions of a parametric function) except for $i = 2$, which will have f_2 *not* weakly increasing.

Let $\hat{f} \in \mathcal{F}$ satisfy $\hat{f}_1(0) = +\infty$ and $\hat{f}_i(0) = 0$ for all $i \neq 1$. Thus $\hat{f}$ gives priority to claimant 1 over all the other claimants. Also assume $\hat{f}_1$ is strictly increasing and concave on the interval $(-\infty, 0)$ (i.e. $\hat{f}_1'(\lambda) > 0$ and $\hat{f}_1''(\lambda) \geq 0$) and that $\hat{f}_i$ is strictly increasing on the interval $(0, +\infty)$ for every $i \neq 1$. Define the function f from $\hat{f}$ as follows: $f_i = \hat{f}_i$ for every $i \neq 2$. For $i = 2$, let

$$f_2(\lambda) = \begin{cases} f_1(\lambda) & \lambda < a \\ \frac{-\epsilon}{b-a}(\lambda - a) + f_1(a) & a \leq \lambda < b \\ f_1(\lambda) - f_1(b) + f_1(a) - \epsilon & \lambda \geq b \end{cases}$$

for some fixed a, b, and ϵ satisfying $-\infty < a < b < 0$, $0 < \epsilon < f_1(a)$, and $\frac{\epsilon}{b-a} < f_1'(a)$. Thus we have the following: $f_2(-\infty) = 0$, $f_2(0) = +\infty$, and $f_1 + f_2$ is strictly increasing on the interval $(-\infty, 0)$.

For $N \in \mathcal{N}$ where either $1, 2 \in N$ or $2 \notin N$, and for any $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous and increasing in λ, $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$, and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any E satisfying $0 \leq E \leq \sum c_i$, there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. Furthermore, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that $\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$.

For $N \in \mathcal{N}$ where $2 \in N$ and $1 \notin N$, and for $c \in \mathbb{R}^N_{++}$, the function $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ is continuous, $\sum_{i \in N} \min\{f_i(-\infty), c_i\} = 0$, and $\sum_{i \in N} \min\{f_i(+\infty), c_i\} = \sum_{i \in N} c_i$. Hence, for any E satisfying $0 \leq E \leq \sum c_i$, there exists $\lambda \in \overline{\mathbb{R}}$ such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$. However note that $\sum_{i \in N} \min\{f_i(\lambda), c_i\}$ may not be increasing. However here it does not matter because individual 2 is given priority over every individual $i \neq 1$. Thus as before, if λ and λ' are such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = \sum_{i \in N} \min\{f_i(\lambda'), c_i\} = E$, then it must be that $\min\{f_i(\lambda), c_i\} = \min\{f_i(\lambda'), c_i\}$ for every $i \in N$.

So we can define a division rule S^f as:

$$S^f(N, c, E) = \left(\min\{f_i(\lambda), c_i\}\right)_{i \in N},$$

where λ is chosen such that $\sum_{i \in N} \min\{f_i(\lambda), c_i\} = E$.

Obviously S^f does not satisfy Resource Monotonicity.

To see that S^f satisfies Consistency, let (N, c, E) and $N' \subset N$ be given. Set $x = S(N, c, E)$. Then there exists λ such that $x_i = \min\{f_i(\lambda), c_i\}$ for every $i \in N$. But then λ also satisfies $\sum_{N'} \min\{f_i(\lambda), c_i\} = \sum_{N'} x_i$. Thus by the definition of S^f, we have $S(N', c_{N'}, \sum_{N'} x_i) = x_{N'}$.

To see that S^f satisfies IIA, let (N, c, E) and c' satisfy the conditions of IIA. Thus there exists λ such that for every $i \in N$, we have $\min\{f_i(\lambda), c_i\} \leq c'_i$. If $\min\{f_i(\lambda), c_i\} = f_i(\lambda)$, then we have $f_i(\lambda) \leq c'_i$. If $\min\{f_i(\lambda), c_i\} = c_i$, then since $X(N, c', E) \subset X(N, c, E)$, we have $c_i = c'_i$. In either case, we must have $\min\{f_i(\lambda), c'_i\} = \min\{f_i(\lambda), c_i\}$. Note then that $\sum_N \min\{f_i(\lambda), c'_i\} = \sum_N \min\{f_i(\lambda), c_i\} = E$. Thus by definition of S^f, we must have $S(N, c', E) = (\min\{f_i(\lambda), c'_i\})_N$, which means $S(N, c', E) = S(N, c, E)$.

C Proof of Theorem 2

Let S satisfy Bilateral Consistency, Resource Monotonicity, IIA, and IIA-Dual.

Define the binary relation $\succ$ over $\mathbb{N}$ as follows: $i \succ j$ if $i \neq j$ and there exists $(c_i, c_j) \in \mathbb{R}^{\{i,j\}}_{++}$ such that $S_i((c_i, c_j), c_i) = c_i$. The following lemmas demonstrate that $\succ$ is a strict linear order over $\mathbb{N}$.

Lemma 21 *It is never the case that for $N = \{i, j\} \subset \mathbb{N}$, $c \in \mathbb{R}^{\{i,j\}}_{++}$, and $E \in (0, c_i + c_j)$, that we have $0 < S(c, E) < c$.*

Proof. By way of contradiction, suppose there exists $N = \{i, j\} \subset \mathbb{N}$, $c \in \mathbb{R}^{\{i,j\}}_{++}$, and $E \in (0, c_i + c_j)$ such that $0 < S(c, E) < c$. Without loss of generality, assume $c_i \leq c_j$. There exists $\epsilon > 0$ such that $2\epsilon < S_i(c, E) < c_i - 2\epsilon$. Set $c' \equiv (c_i - \epsilon, c_j + \epsilon)$. Note that for $E \leq c_j$, we have $X(c', E) \subset X(c, E)$ and $S(c, E) \in X(c', E)$. Thus

IIA implies $S(c,E) = S(c',E)$. Similarly, for $E \geq c_i$, we have $L(c',E) \subset L(c,E)$ and $c - S(c,E) \in L(c',E)$. Thus IIA-Dual implies $c - S(c,E) = c' - S(c',E)$.

Suppose $E \in [c_i, c_j]$. Then we have $S(c,E) = S(c',E)$ and $c - S(c,E) = c' - S(c',E)$. But this implies $c = c'$, which is a contradiction.

Suppose $E > c_j$. Then we have $c - S(c,E) = c' - S(c',E)$, which implies $S(c,E) = c - c' + S(c',E) = S(c',E) + (\epsilon, -\epsilon)$. Thus $S(c,E) \in X(c',E)$ and $S(c',E) \in X(c,E)$. Consider the claim $c'' = (c_i, c_j + \epsilon)$. Note that $X(c,E) \subset X(c'',E)$ and $X(c',E) \subset X(c'',E)$ and either $S(c'',E) \in X(c,E)$ or $S(c'',E) \in X(c',E)$. If $S(c'',E) \in X(c,E)$, then IIA implies $S(c'',E) = S(c,E)$. But since $S(c,E) \in X(c',E)$, IIA implies $S(c'',E) = S(c',E)$. Thus $S(c,E) = S(c',E)$. But then we get $c = c'$ which is a contradiction. Similarly, we get a contradiction if $S(c'',E) \in X(c',E)$.

Similarly, if $E < c_i$, we get a contradiction. But that exhausts all possible values of E. ∎

Lemma 22 *For every $\{i,j\} \subset \mathbb{N}$, either $i \succ j$ or $j \succ i$, but not both.*

Proof. Fix $\{i,j\} \subset \mathbb{N}$. Lemma 21 implies that either $i \succ j$ or $j \succ i$. We now show that both cannot be true. By way of contradiction, suppose both $i \succ j$ and $j \succ i$. Thus there exists $(c_i, c_j) \in \mathbb{R}_{++}^{\{i,j\}}$ and $(c_i', c_j') \in \mathbb{R}_{++}^{\{i,j\}}$ such that $S((c_i,c_j),c_i) = (c_i,0)$ and $S((c_i',c_j'),c_j') = (0,c_j')$. But then Lemma 7 implies that both $S((c_i,c_j'),c_i) = (c_i,0)$ and $S((c_i,c_j'),c_j') = (0,c_j')$, which must contradict Resource Monotonicity. ∎

Lemma 23 *$\succ$ is transitive.*

Proof. Let $i \succ j$ and $j \succ k$. By Lemma 22, for any $c_i, c_j, c_k > 0$, we have $S_i((c_i,c_j),c_i) = c_i$ and $S_j((c_j,c_k),c_j) = c_j$. Note then that we must have $G(j,c_j,i,c_i) = c_i + c_j$ and $G(j,c_j,k,c_k) = c_j$. Lemma 6 implies there exists E such that

$$S((c_i,c_j,c_k),E) = (G(j,c_j,i,c_i) - c_j, c_j, G(j,c_j,k,c_k) - c_j) = (c_i,c_j,0).$$

Bilateral Consistency then implies $S((c_i,c_k),c_i) = (c_i,0)$. Thus $i \succ k$ by definition. ∎

We now complete the proof by showing that S must be the queueing rule.

Lemma 24 *$S = S^{\succ}$.*

Proof. Fix (c,E) and set $x \equiv S(c,E)$. Let $\{i,j\} \in N$. We show $i \succ j$ if and only if $x_j = c_j$ implies $x_i = c_i$. Bilateral Consistency implies $(x_i,x_j) = S((c_i,c_j),x_i+x_j)$. Suppose $i \succ j$ and $x_j = c_j$. Then Lemmas 21 and 22 imply $S((c_i,c_j),c_i) = (c_i,0)$. Resource Monotonicity then implies $x_i = c_i$. Going the other direction, suppose that $x_j = c_j$ implies $x_i = c_i$. Then it cannot be that $S((c_i,c_j),c_j) = (0,c_j)$. Hence $j \not\succ i$. By Lemma 22 we must have $i \succ j$. ∎

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