Price of Anarchy for Non-atomic Congestion Games with Stochastic Demands

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Abstract

We generalize the notions of user equilibrium, system optimum and price of anarchy to non-atomic congestion games with stochastic demands. In this generalized model, we extend the two bounding methods from Roughgarden and Tardos (2004) and Correa et al. (2008) to bound the price of anarchy, and compare the upper bounds we have obtained. Our results show that the price of anarchy depends not only on the class of cost functions but also demand distributions and, to some extent, the network topology. The upper bounds are tight in some special cases, including the case of deterministic demands.

Keywords: price of anarchy, user equilibrium, system optimum, stochastic demand

1. Introduction

Non-atomic congestion games (Rosenthal, 1973; Schmeidler, 1973) illustrate non-cooperative situations involving large populations of players competing for a finite set of resources. Routing problem in transportation networks (Wardrop, 1952; Beckmann et al., 1956) is a very important application of non-atomic congestion games. The price of anarchy (PoA), first introduced by Koutsoupias and Papadimitriou (1999) on a load-balancing game, is one of the main measures of system degradation due to lack of coor-

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dination. In non-atomic congestion games, as studied by Roughgarden and Tardos (2004), the PoA is the worst-case system performance at a user equilibrium (UE) compared with the system performance at a system optimum (SO), where a UE (Wardrop, 1952) is a steady state of travelers' selfish routing while an SO represents an optimal usage of traffic resources as a result of a well-coordinated collective action on the whole network.

Quantitative study on the PoA enables us to deem certain outcomes of a game optimal or approximately optimal and to make known the factor influencing the inefficiency of the UE, and further contributes to mechanism design for congestion games. Roughgarden and Tardos (2002, 2004) bounded the PoA when the link cost functions are separable, semi-convex and differentiable. The PoA was proved to be dependent only on the class of the cost functions, independent of the network topology (Roughgarden, 2003). In particular, the PoA with affine cost functions is tightly bounded by 4/3.

The main developments in the research on PoA were extensions to networks with a broader range of cost functions. Chau and Sim (2003) generalized Roughgarden and Tardos' results to the cases with symmetric cost functions. Correa et al. (2004, 2008) gave a geometric proof of the upper bound of the PoA with cost functions that are non-convex, non-differentiable, and even discontinuous. Perakis (2007) extended the work to asymmetric cost functions and bounded the PoA by two parameters of asymmetry and nonlinearity. Sheffi (1985) introduced the notion of stochastic user equilibrium (SUE), which describes the travelers' selfish routing decisions on their subjective perceived travel costs by involving stochastic cost functions. The PoA on logit-based SUE was bounded by Guo et al. (2010) on the basis of Sheffi's model.

Another line of development in the PoA study is to improve the setting of the traffic demand to better reflect reality. Chau and Sim (2003) presented a weaker upper bound on the PoA with elastic demands. Although study on the PoA with stochastic demands is still quite new, efforts have been spent on modeling UE and SO involving demand uncertainty. It was assumed that the objective of selfish travelers was to choose the path that minimizes the mean travel cost (Sumalee and Xu, 2011) or weighted sum of the mean and the variance of the travel cost (Sumalee and Xu, 2011; Bell and Cassir, 2002) with risk-neutral and risk-averse travelers, respectively. A travel time budget was also considered in the equilibrium condition on the basis of reliability (Lo et al., 2006; Shao et al., 2006). However, to deduce the distributions of the path and link flows, all of these studies relied on some assumptions, such as that all the path flows follow the same type of distribution as the demand and have the same variance (or standard deviation) to mean ratio (Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008), and that all the path flows are independent (Clark and Watling, 2005; Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008). Apparently, these assumptions are open to questions for the relationship between the path flows and demands, not only because of lack of empirical data support but also they violate the demand feasibility constraint even in simple networks. In order to have a better reliable result on the PoA, we need to relax the aforementioned assumptions and establish a new equilibrium condition.

In this paper we present an analytical method to determine distributions of path and link flows under given demand distributions and, from a practical perspective, describe travelers' behaviors by path choice probabilities. We generalize the deterministic UE condition to a stochastic version with risk-neutral travelers. For our generalized model we establish upper bounds on the PoA, which are found to depend on cost functions and demand distributions. These upper bounds are shown to be tight in some special cases, including the case of deterministic demands.

The remainder of the paper is organized as follows. Section 2 introduces generalized notions of user equilibrium (UE) and system optimum (SO) under demand uncertainty, formulates the equilibrium condition as a variational inequality problem and discusses existence and uniqueness of an equilibrium. Section 3 extends to our stochastic model the two methods introduced respectively by Roughgarden and Tardos (2004) and Correa et al. (2008) to bound the PoA. We establish upper bounds on PoA with polynomial cost functions and specific demand distributions, namely, general positive-valued distributions and normal distributions. Section 4 discusses connections with existing results in the literature and makes some concluding remarks.

2. The model with stochastic demands

2.1. Preliminaries and notation

Consider a general network $G = (N, E)$, where N and E denote the set of nodes and links, respectively. To each link $e \in E$, we associate a (link) cost function $c_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, which is assumed to be nondecreasing in its argument, the link flow. A subset of nodes form a set of origin-destination (O-D) pairs, denoted by I. We call an O-D pair $i \in I$ a *commodity*. Parallel links are allowed and a node can be in multiple O-D pairs. Denote P_i as the set of all possible paths connecting an O-D pair $i \in I$.

Day-to-day variability of traffic demands is considered as the source of uncertainty in this study. We assume that the demand distributions are given and publicly known, which is based on the fact that a traveler, especially a commuter, has knowledge of the probabilities of possible demand levels from his or her own experiences, although the actual current demand level is unknowable. A similar assumption can be found in the model with deterministic demands, which states that travelers have perfect knowledge of the fixed demand in the network (Wardrop, 1952). The demands of different O-D pairs are assumed to be independent. We adopt the following notation in our study, where capital and lower-case letters are used to express random variables and, if applicable, the corresponding mean values, respectively. For convenience, we divide them into two groups: input and output. The input information provided in the network is as follows.

- $\mathbf{D}$: vector of random traffic demands with component D_i as the random demand between O-D pair $i \in I$;
- $\mathbf{d}$: vector of mean traffic demands with component $d_i > 0$ as the mean demand between O-D pair $i \in I$;
- σ_i^2: variance of D_i, $i \in I$;
- ϵ_i: coefficient of demand variation, i.e., $\epsilon_i = \sigma_i/d_i$, $i \in I$;
- $\bar{\epsilon}$: maximum coefficient of demand variation, i.e., $\bar{\epsilon} = \max_{i \in I}\{\epsilon_i\}$;
- $\underline{\epsilon}$: minimum coefficient of demand variation, i.e., $\underline{\epsilon} = \min_{i \in I}\{\epsilon_i\}$;
- $\delta_{k,e}^i$: link-path incidence indicator, which is 1 if link e is included in path $k \in P_i$ and 0 otherwise, $e \in E$, $i \in I$;
- δ_e^i: link-commodity incidence indicator, i.e., $\delta_e^i = \max_{k \in P_i} \delta_{k,e}^i$, $e \in E$, $i \in I$;
- n_e: number of O-D pairs that use link $e \in E$ in their paths, i.e., $n_e = \sum_{i \in I} \delta_e^i$;
- n: $n = \max_{e \in E}\{n_e\}$. Hence $n \leq |I|$.

On parameter n defined above, we note that, if $n = 1$, then every link is used by only a single O-D pair, which implies that the whole network can be separated into $|I|$ single-commodity sub-networks. Therefore, as far as our problem is concerned for system stability and optimality (to be defined more precisely in Sections 2.3 and 2.4), our problem is reduced to the problem with a single commodity when $n = 1$.

After all travelers make their routing choices, the output in the network is the resulting traffic flows.

F_k^i: random traffic flow on path $k \in P_i$, $i \in I$;
f_k^i: mean traffic flow on path $k \in P_i$, $i \in I$;
$\mathbf{F}$: vector of random path flows, i.e., $\mathbf{F} = (F_k^i : k \in P_i, i \in I)$;
$\mathbf{f}$: vector of mean path flows, i.e., $\mathbf{f} = (f_k^i : k \in P_i, i \in I)$;
V_e: random traffic flow on link $e \in E$;
v_e: mean traffic flow on link $e \in E$;
$\mathbf{V}$: vector of random link flows, i.e., $\mathbf{V} = (V_e : e \in E)$;
$\mathbf{v}$: vector of mean link flows, i.e., $\mathbf{v} = (v_e : e \in E)$;

We denote any instance of a non-atomic congestion game by a triple $(G, \mathbf{D}, \mathbf{c})$, where G is the underlying network, $\mathbf{D}$ and $\mathbf{c}$ are the vectors of random demands and (link) cost functions, respectively.

2.2. Routing strategies

Under the deterministic setting, i.e., $D_i = d_i$ for all $i \in I$, the continuum of players of each O-D pair $i \in I$ is represented by the interval $[0, d_i]$ endowed with the Lebesgue measure. The set of mixed strategies of each player from O-D pair $i \in I$ is

$$\Omega_i = \left\{ \mathbf{p}^i = (p_k^i \geq 0 : k \in P_i) : \textstyle\sum_{k \in P_i} p_k^i = 1 \right\},$$

where p_k^i is the probability that path $k \in P_i$ is chosen. According to (Schmeidler, 1973), a *strategy profile* is a (Lebesgue) measurable function q^i from $[0, d_i]$ to Ω_i, i.e, for each player $x \in [0, d_i]$, $q^i(x) \in \Omega_i$ is his/her mixed strategy. A strategy profile q^i induces the vector $\mathbf{f}^i$ of path flows, $\mathbf{f}^i = (f_k^i : k \in P_i)$, which is called an *action distribution* in (Roughgarden and Tardos, 2002), as follows:

$$f_k^i = \int_0^{d_i} q_k^i(x) dx, \quad \forall\, k \in P_i,$$

where $q_k^i(x)$ is the probability that path $k \in P_i$ is chosen by the player x from O-D pair $i \in I$. Clearly, $\sum_{k \in P_i} f_k^i = d_i$ since $q^i(x) \in \Omega_i$ for all $x \in [0, d_i]$, $i \in I$. (Roughgarden and Tardos, 2002) focused on flow assignments, i.e., action distributions, instead of strategy profiles with the argument that every flow assignment can be induced by some strategy profile and the costs depend only on the flow assignment of a strategy profile. Under the stochastic setting, realized path flows depend on not only the chosen strategy profile but also the realized demand. Therefore, it is necessary for us to work with strategy profiles as primary variables instead of flow assignments. Given that the demands are stochastic, it is reasonable to assume that all the players of a same O-D pair play the same strategy at an equilibrium in such an environment with incomplete information (Myerson, 1998; Ashlagi et al., 2006). Indeed it is unrealistic for a player to know the routing choices of all other players or to distinguish players from a same O-D pair when the demand is uncertain. According to (Myerson, 1998), players can only form perceptions about how other players make routing decisions solely depending on the information of which O-D pairs these players belong to. Mathematically, we assume that for any two different players x and x' in the (random) interval $[0, D_i]$ of the same O-D pair $i \in I$,

$$q_k^i(x) = q_k^i(x') = p_k^i, \quad \forall\, k \in P_i,$$

where $\mathbf{p}^i$ is some mixed strategy in Ω_i. Under this assumption, each strategy profile for players from O-D pair $i \in I$ is now represented by a single mixed strategy $\mathbf{p}^i \in \Omega_i$. Let $\Omega = \prod_{i \in I} \Omega_i$. Then each vector $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$ represents a strategy profile of players from all O-D pairs.

Now let us define random path flows and link flows for our stochastic model. Given a strategy profile represented by $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$, the random path flows can be calculated as follows:

$$F_k^i = \int_0^{D_i} q_k^i(x)dx = \int_0^{D_i} p_k^i dx = p_k^i \cdot D_i, \quad \forall\ k \in P_i,\, i \in I. \tag{1}$$

Since $\mathbf{p}^i \in \Omega_i$, we have:

$$\sum_{k \in P_i} F_k^i = D_i, \quad \forall\ i \in I. \tag{2}$$

It is clear that the flow on each link is the sum of flows on all the paths that include the link:

$$V_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i F_k^i, \quad \forall\ e \in E.$$

Applying (1), we obtain the following formulation for random link flows:

$$V_e = \sum_{i \in I} p_e^i \cdot D_i, \quad \forall\ e \in E, \tag{3}$$

where $p_e^i = \sum_{k \in P_i} \delta_{k,e}^i p_k^i$ is the (link) choice probability of link $e \in E$ for the players from $i \in I$.

Given the link cost functions, the random path cost is simply the sum of the costs of those links that constitute the path, i.e.,

$$c^i_k(\mathbf{F}) = \sum_{e\in E}\delta^i_{k,e}c_e(V_e), \quad \forall\ k \in P_i,\ \forall\ i \in I. \tag{4}$$

We can also compute the total (social) cost as follows:

$$C(\mathbf{F}) = \sum_{e\in E} c_e(V_e)V_e. \tag{5}$$

Remark 1. It is commonly assumed in the literature (Clark and Watling, 2005; Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008) that all path flows $\{F^i_k : k \in P_i, i \in I\}$ are independent, which apparently violates the flow constraints (2). In our study, dependent path flows from a same O-D pair are considered as they should be according to (2) and we only assume that demands $\{D_i : i \in I\}$ are independent. From (1) we can see that the path flows from different O-D pairs are independent, i.e., for any $i, i' \in I$, $i \neq i'$ and any $k \in P_i, k' \in P_{i'}$, path flows F^i_k and $F^{i'}_{k'}$ are independent of each other.

2.3. Equilibrium under stochastic demands (UE-SD)

As discussed in the previous subsection, under stochastic traffic demands, we assume that risk-neutral travelers between a same O-D pair will use the same strategy at a steady state. We define our equilibrium condition such that travelers cannot improve their expected travel costs by unilaterally changing their routing choice strategies.

Definition 1 (UE-SD condition). Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, strategy profile $\mathbf{p} \in \Omega$ is said to be a user equilibrium (UE-SD) if and only if

$$\mathbb{E}[c^i_k(\mathbf{F})] \leq \mathbb{E}[c^i_\ell(\mathbf{F})], \quad \forall\ k, \ell \in P_i, i \in I \text{ with } p^i_k > 0. \tag{6}$$

From the definition we see that, at any UE-SD, all the paths with positive probabilities for the same O-D pair have the equal and minimum expected travel cost. When all travelers play mixed strategies according to the UE-SD condition, the expected travel costs are guaranteed to be at minimum. To solve the equilibrium problem, let us reformulate the UE-SD condition as a variational inequality (VI).

Proposition 1. *Given a transportation game* $(G, \mathbf{D}, \mathbf{c})$ *with stochastic demands, let* $\bar{\mathbf{p}} \in \Omega$ *be a strategy profile. Then* $\bar{\mathbf{p}}$ *is a UE-SD if and only if it satisfies the following VI condition: for any strategy profile* $\mathbf{p} \in \Omega$,

$$(\mathbf{f} - \bar{\mathbf{f}})^T \mathbb{E}\left[c(\bar{\mathbf{F}})\right] \geq 0, \tag{7}$$

where $\bar{\mathbf{F}}$ *is the vector of path flows corresponding to* $\bar{\mathbf{p}}$, *and* $\bar{\mathbf{f}}$ *and* $\mathbf{f}$ *are, respectively, the vector of the mean path flows corresponding to* $\bar{\mathbf{p}}$ *and* $\mathbf{p}$.

Proof. Taking the expectation in (1), we have $f_k^i = p_k^i d_i$. Since demand $d_i > 0$ for every $i \in I$, we can write the UE-SD condition (6) as follows:

$$\mathbb{E}[c_k^i(\mathbf{F})] \leq \mathbb{E}[c_\ell^i(\mathbf{F})], \quad \forall\, k, \ell \in P_i, i \in I \text{ with } f_k^i > 0. \tag{8}$$

Let $\pi_i = \min_{\ell \in P_i} \mathbb{E}[c_\ell^i(\mathbf{F})]$ for any $i \in I$, then (8) is equivalent to

$$\begin{cases} f_k^i(\mathbb{E}[c_k^i(\mathbf{F})] - \pi_i) = 0, \\ f_k^i \geq 0, \end{cases} \qquad \forall\, k \in P_i,\ \forall\, i \in I.$$

Let $\bar{\mathbf{p}}$, $\bar{\mathbf{F}}$ and $\bar{\mathbf{f}}$ be the vectors of path choice probabilities and the corresponding path flows, mean path flows at a UE-SD, respectively. Then

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) = 0,$$

where $\bar{\pi}_i = \min_{\ell \in P_i} \mathbb{E}[c_\ell^i(\bar{\mathbf{F}})]$. For any $\mathbf{f} = (f_k^i \geq 0 : k \in P_i, i \in I)$, we also have

$$\sum_{i \in I} \sum_{k \in P_i} f_k^i(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) \geq 0.$$

Thus

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) \leq \sum_{i \in I} \sum_{k \in P_i} f_k^i(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i). \tag{9}$$

From condition (2) we have $\sum_{k \in P_i} f_k^i = \sum_{k \in P_i} (\bar{f}_k^i) = d_i$ for every $i \in I$. Hence

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)\bar{\pi}_i = \sum_{i \in I} \sum_{k \in P_i} f_k^i \bar{\pi}_i,$$

which together with (9) implies (7):

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)\mathbb{E}[c_k^i(\bar{\mathbf{F}})] \leq \sum_{i \in I} \sum_{k \in P_i} f_k^i \mathbb{E}[c_k^i(\bar{\mathbf{F}})].$$

On the other hand, observe that as the first order optimality condition, the solution of VI problem (7) also solves the following LP problem:

$$\begin{array}{ll} \min & \mathbf{f}^T\mathbb{E}[\mathbf{c}(\bar{\mathbf{F}})] \\ \text{s.t.} & \sum_{k\in P_i} f_k^i = d_i, \quad i \in I, \\ & f_k^i \geq 0, \qquad\quad k \in P_i,\, i \in I, \end{array}$$

the duality of which is

$$\begin{array}{ll} \max & \lambda^T\mathbf{d} \\ \text{s.t.} & \lambda_i \leq \mathbb{E}[c_k^i(\bar{\mathbf{F}})], \quad k \in P_i,\, i \in I. \end{array}$$

Therefore, we have the following complementary slackness conditions:

$$(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \lambda_i)f_k^i = 0,\ k \in P_i,\, i \in I,$$

which imply (6). □

Remark 2. With equations (3) and (4) we can rewrite the VI condition (7) in terms of link flows: $\bar{\mathbf{p}} \in \Omega$ is a UE-SD if and only if it satisfies the condition that, for any vector $\mathbf{p} \in \Omega$ of path choice probabilities,

$$\sum_{e\in E}(v_e - \bar{v}_e)\mathbb{E}[c_e(\bar{V}_e)] \geq 0, \tag{10}$$

where $\bar{V}_e$ is the link flow on link $e \in E$ corresponding to $\bar{\mathbf{p}}$, and v_e and $\bar{v}_e$ are, respectively, the mean link flows on link $e \in E$ corresponding to $\mathbf{p}$ and $\bar{\mathbf{p}}$.

An equivalence between the UE-SD condition and a minimization problem can also be established if the link cost functions are affine, which is stated in the following proposition.

Proposition 2. *Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands and affine cost functions, let $\bar{\mathbf{p}} \in \Omega$ be a vector of path choice probabilities. Then $\bar{\mathbf{p}}$ is a UE-SD if and only if it solves the following minimization problem*

$$\min_{\mathbf{p}\in\Omega} Z(\mathbf{p}) \equiv \sum_{e\in E}\int_0^{v_e} c_e(x)dx, \tag{11}$$

where, as we recall, $v_e = \sum_{i\in I}\sum_{k\in P_i}\delta_{k,e}^i p_k^i d_i$.

Proof. We prove this proposition by verifying the equivalence between VI problem (7) and minimization problem (11). Note that, since the link cost function $c_e(x)$ is continuously differentiable and non-decreasing, function $\int_0^{v_e} c_e(x)dx$ is convex (with respect to v_e) for any $e \in E$. Convexity is invariant under affine maps; therefore, the objective function $Z(\mathbf{p})$ in (11) is convex (with respect to $\mathbf{p}$). In addition, feasible region Ω is convex and compact. Thus minimization problem (11) is a convex optimization problem. It is then necessary and sufficient for $\bar{\mathbf{p}}$ to satisfy the first order optimality condition of (11) (Bertsekas, 1999, Proposition 2.1.2):

$$(\mathbf{p} - \bar{\mathbf{p}})^T \nabla Z(\bar{\mathbf{p}}) \geq 0. \tag{12}$$

We have:

$$\frac{\partial Z(\mathbf{p})}{\partial p_k^i} = \sum_{e \in E} c_e(v_e) \frac{\partial v_e}{\partial p_k^i} = \sum_{e \in E} c_e(v_e)(\delta_{k,e}^i d_i) = c_k^i(f) d_i.$$

In addition, we have $\bar{f}_k^i = \bar{p}_k^i d_i$ by taking the expectation in (1). Thus, condition (12) is equivalent to

$$(\mathbf{f} - \bar{\mathbf{f}})^T \mathbf{c}(\bar{\mathbf{f}}) \geq 0,$$

which in turn is equivalent to (7) when the link cost functions are affine. □

Proposition 2 establishes that the VI condition for a UE-SD is just a restatement of the first order necessary and sufficient condition of a minimization problem, if the cost functions $\mathbf{c}$ are affine. For general link cost functions, we can rewrite condition (7) in the following form by substituting $f_k^i = p_k^i d_i$ and $\bar{f}_k^i = \bar{p}_k^i d_i$:

$$(\mathbf{p} - \bar{\mathbf{p}})^T \mathbf{S}(\bar{\mathbf{p}}) \geq 0, \ \mathbf{p} \in \Omega, \tag{13}$$

where $\mathbf{S}(\mathbf{p})$ is a vector with the same dimension as $\mathbb{E}[\mathbf{c}(\mathbf{F})]$, obtained by replacing element $\mathbb{E}[c_k^i(\mathbf{F})]$ in vector $\mathbb{E}[\mathbf{c}(\mathbf{F})]$ with $\mathbb{E}[c_k^i(\mathbf{F})]d_i$ for every $k \in P_i, \ i \in I$. When link cost functions are continuous, the game admits at least one UE-SD. This is due to the fact that existence of a solution $\bar{\mathbf{p}} \in \Omega$ to VI problem (13) is guaranteed by the continuity of $\mathbf{S}(\mathbf{p})$ and the compactness of Ω.

Let us conclude this subsection with a discussion on non-uniqueness of user equilibria in transportation games with stochastic demands. In deterministic models, the user equilibrium is unique with respect to link flows under the assumption of separable and strictly increasing link cost functions (Beckmann et al., 1956; Dafermos and Sparrow, 1969). As one link flow can correspond to many path flows in general networks, the path flow of a deterministic user equilibrium is not unique. In our stochastic model, path and link flows are random and determined by path and link choice probabilities respectively. The following example shows that multiple UE-SDs may exist even under the assumption of separable and strictly increasing link cost functions.

Example 1. Consider the network in Figure 1. There are two O-D pairs in the network, (s_1, t) and (s_2, t). Each O-D pair is connected by two paths, paths 1 and 2 from s_1 to t and paths 3 and 4 from s_2 to t, where Path 1 consists of links 1 and 3, Path 2 of links 1 and 4, Path 3 of links 2 and 3, and Path 4 of links 2 and 4. The cost function on each link is also indicated in the figure.

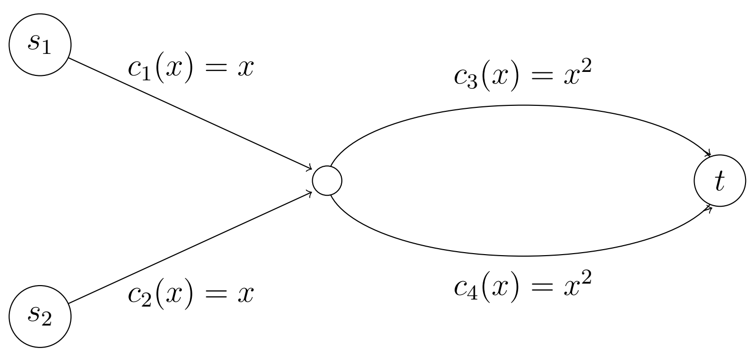


Figure 1: Network with multiple UE-SDs

The demand D_1 from s_1 to t follows a distribution with a mean $d_1 = 1$ and variance $\sigma_1^2 = 1$, while the demand D_2 from s_2 to t follows a different distribution with a mean $d_2 = 1$ and variance $\sigma_2^2 = 4$. Given the definition of UE-SD, a feasible strategy profile is clearly a UE-SD when both paths from each O-D pair have the same expected travel cost, i.e., when

$$\begin{cases} \mathbb{E}[c_1(V_1)] + \mathbb{E}[c_3(V_3)] = \mathbb{E}[c_1(V_1)] + \mathbb{E}[c_4(V_4)], \\ \mathbb{E}[c_2(V_2)] + \mathbb{E}[c_3(V_3)] = \mathbb{E}[c_2(V_2)] + \mathbb{E}[c_4(V_4)], \end{cases}$$

which are equivalent to

$$\mathbb{E}[c_3(V_3)] = \mathbb{E}[c_4(V_4)]. \tag{14}$$

Based on condition (14), we can find many UE-SDs. Here we present two of them for comparison: $\bar{\mathbf{p}}_\mathbf{1} = (1, 0, 0.25, 0.75)$ and $\bar{\mathbf{p}}_\mathbf{2} = (0.5, 0.5, 0.5, 0.5)$. From (3) we can calculate means and variances of link flows and expected link costs for each of the two strategy profiles as shown in Table 1, from which satisfaction of condition (14) at each strategy profile confirms that they are both UE-SDs.

At $\bar{\mathbf{p}}_\mathbf{1}$	v_e	$\mathrm{Var}[V_e]$	$\mathbb{E}[c_e(V_e)]$	At $\bar{\mathbf{p}}_\mathbf{2}$	v_e	$\mathrm{Var}[V_e]$	$\mathbb{E}[c_e(V_e)]$
Link 1	1	1	1	Link 1	1	1	1
Link 2	1	4	1	Link 2	1	4	1
Link 3	1.25	1.25	2.8125	Link 3	1	1.25	2.25
Link 4	0.75	2.25	2.8125	Link 4	1	1.25	2.25

Table 1: Means and variances of link flows, expected link costs at $\bar{\mathbf{p}}_\mathbf{1} = (1, 0, 0.25, 0.75)$ and $\bar{\mathbf{p}}_\mathbf{2} = (0.25, 0.25, 0.25, 0.25)$

It is easy to see that at the two UE-SDs the mean link flows on links 3 and 4 are different and the link choice probabilities are also different. For example, the choice probability of link 3 is 1 for travelers from s_1 to t and 0.25 from s_2 to t in the first UE-SD, while it becomes 0.5 for both O-D pairs in the second UE-SD. Furthermore, in terms of the expected total cost $\mathbb{E}[C(\mathbf{F})]$ (see definition (5)), they are also different at the two UE-SDs as shown in the following calculations (assuming that $\mathbb{E}[D_i^3]$ is finite for $i = 1, 2$), where $\bar{\mathbf{F}}_\mathbf{1}$ and $\bar{\mathbf{F}}_\mathbf{2}$ are path flows resulted from $\bar{\mathbf{p}}_\mathbf{1}$ and $\bar{\mathbf{p}}_\mathbf{2}$, respectively:

$$\begin{aligned}
\mathbb{E}[C(\bar{\mathbf{F}}_\mathbf{1})] &= \mathbb{E}[V_1^2] + \mathbb{E}[V_2^2] + \mathbb{E}[V_3^3] + \mathbb{E}[V_4^3] \\
&= \mathbb{E}[D_1^2] + \mathbb{E}[D_2^2] + \mathbb{E}[(p_1 D_1 + p_3 D_2)^3] + \mathbb{E}[(p_2 D_1 + p_4 D_2)^3] \\
&= 9.4375 + \mathbb{E}[D_1^3] + 0.4375\ \mathbb{E}[D_2^3], \\
\mathbb{E}[C(\bar{\mathbf{F}}_\mathbf{2})] &= \mathbb{E}[V_1^2] + \mathbb{E}[V_2^2] + \mathbb{E}[V_3^3] + \mathbb{E}[V_4^3] \\
&= 12.25 + 0.25\mathbb{E}[D_1^3] + 0.25\ \mathbb{E}[D_2^3].
\end{aligned}$$

Clearly, $\mathbb{E}[C(\bar{\mathbf{F}}_\mathbf{1})] \neq \mathbb{E}[C(\bar{\mathbf{F}}_\mathbf{2})]$ when $\mathbb{E}[D_1^3] + 0.25\ \mathbb{E}[D_2^3] \neq 3.75$. $\square$

Example 1 shows that multiple UE-SDs with different mean link flows, link choice probabilities, and expected total costs can exist. If cost functions are further restricted to being affine, as addressed in Proposition 2, the UE-SD condition can be reformulated as a minimization problem, which is actually in the same form as deterministic user equilibrium condition (Beckmann et al., 1956) with respect to mean link flows. Thus under the same condition of separable and strictly increasing cost functions, the mean link flows of UE-SD are unique. However, link choice probabilities and expected total cost are still non-unique in general, which can be shown by modifying Example 1 with all settings remaining the same except that all link cost functions are affine, $c_e(x) = x$ for all $e = 1, \ldots, 4$.

By solving the minimization problem (11), a strategy profile is a UE-SD if and only if $v_1 = v_2 = v_3 = v_4 = 1$, which can be expressed as follows:

$$p_1 + p_2 = p_3 + p_4 = p_1 + p_3 = p_2 + p_4 = 1.$$

We can find multiple UE-SD strategy profiles from this system of equations, such as $(1, 0, 0, 1)$ and $(0.5, 0.5, 0.5, 0.5)$. The choice probability of link 3 is 1 from s_1 to t and 0 from s_2 to t in the former UE-SD, while both become 0.5 in the latter UE-SD. The expected total cost in the whole network can be calculated as 14 and 11.5 for the two UE-SDs respectively.

2.4. System optimum under stochastic demand (SO-SD)

At a system optimum (SO-SD), traffic is coordinated by a central authority according to mixed strategies. It should be noted in the case of coordination that traffic is assigned according to path choice probabilities rather than by traffic proportions. This is due to the fact that demand is cumulative over the time period concerned, while traffic allocation needs to be made once a traffic flow arrives at a route entrance. The central authority has to implement traffic coordination without full knowledge of the actual demand. The objective of the coordinator is to minimize the expectation of the total travel cost at an SO-SD. This gives rise to our following definition.

Definition 2 (SO-SD condition). Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, a vector $\mathbf{p} \in \Omega$ of path choice probabilities is said to be an SO-SD strategy if it solves the following minimization problem:

$$\min_{\mathbf{p} \in \Omega} T(\mathbf{p}) \equiv \mathbb{E}\left[C(\mathbf{F})\right] = \mathbb{E}\left[\sum_{e \in E} c_e(V_e) V_e\right], \tag{15}$$

where V_e is computed from $\mathbf{p}$ according to (3).

Generally, an SO-SD may not be unique as the optimization problem (15) may have more than one optimal solution, all of which, however, must yield the same expected total cost in the whole network.

3. Price of anarchy

In this section we investigate the price of anarchy (PoA) to be defined below based on the model presented in the preceding section with the expected total cost $T(\cdot)$ defined in the network by (15) as the social (system) objective function. Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, the corresponding PoA is defined as the worst-case ratio between expected total costs at a UE-SD and at an SO-SD:

$$\text{PoA}(G, \mathbf{D}, \mathbf{c}) := \max\left\{\frac{T(\mathbf{p})}{T(\mathbf{q})} : \mathbf{p}, \mathbf{q} \in \Omega, \mathbf{p} \text{ is a UE-SD and } \mathbf{q} \text{ is an SO-SD}\right\}.$$

Here and in the remainder of the paper, it is understood that the corresponding ratio is infinity whenever the denominator is zero.

Let $\mathcal{I}$ be any given set of instances $(G, \mathbf{D}, \mathbf{c})$ of transportation games with stochastic demands, then the PoA with respect to $\mathcal{I}$ is defined as

$$\text{PoA}(\mathcal{I}) := \max_{(G,\mathbf{D},\mathbf{c})\in\mathcal{I}} \text{PoA}(G, \mathbf{D}, \mathbf{c}).$$

Note that even for deterministic demands (i.e., $\mathbf{D}$ is particularly deterministic), the PoA is already unbounded if the link cost functions $\mathbf{c}$ are unrestricted (Roughgarden and Tardos, 2002). In this paper, we will establish upper bounds on the PoA for a fixed set $\mathcal{C}_m$ of link cost functions, the class of polynomial cost functions with degree at most m. For deterministic models, the tight upper bound of the PoA is proved to be $(1 - m(m+1)^{-(m+1)/m})^{-1}$ for the class $\mathcal{C}_m$ of link cost functions. Roughgarden and Tardos (2004) first proved the result by using the fact that the link cost functions are semi-convex and differentiable. Correa et al. (2008) extended the work of Roughgarden and Tardos (2004) by removing the assumption of semi-convex and differentiable cost functions, and bounded the PoA by a geometric method, which resulted in the same tight upper bound for polynomial cost functions. For convenience of comparison in the paper, we refer these two bounding techniques as *convexity* and *geometry method*, respectively. We extend both methods to our stochastic model.

3.1. Price of anarchy under general stochastic demands

Both convexity and geometry method we mentioned above for deterministic models require general bounds on the total cost function $\sum_{e\in E} c_e(v_e)v_e$. In our stochastic model, the expected total cost function is $\mathbb{E}\left[\sum_{e\in E} c_e(V_e)V_e\right]$, which in general is not solely a function of the mean link flows v_e, $e \in E$. In order to extend the bounding techniques to our stochastic model, we make the following general assumption. Denote by $\mathcal{C}(\mathcal{I})$ the class of all link cost functions $\{c_e(\cdot) : e \in E\}$ used in game instances $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$.

Assumption 1. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, there exist non-decreasing functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\overline{t}_e(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$, such that $\underline{s}_e(0) = \overline{s}_e(0) = 0$ and $\underline{t}_e(0) = \overline{t}_e(0) = c_e(0)$; and for any random link flows V_e ($e \in E$) as defined in (3) with $v_e > 0$,

$$0 < \underline{s}_e(v_e) \le \mathbb{E}[c_e(V_e)V_e] \le \overline{s}_e(v_e), \tag{16}$$

$$0 < \underline{t}_e(v_e) \le \mathbb{E}[c_e(V_e)] \le \overline{t}_e(v_e). \tag{17}$$

Note that when $v_e = 0$, we can derive $V_e = 0$ from (3) and the fact that $d_i > 0$, $i \in I$. Hence with $\mathbb{E}[c_e(V_e)] = c_e(0)$ and $\mathbb{E}[c_e(V_e)V_e] = 0$, we can still use $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$ $\underline{t}_e(\cdot)$ and $\overline{t}_e(\cdot)$ to bound $\mathbb{E}[c_e(V_e)V_e]$ and $\mathbb{E}[c_e(V_e)]$ when $v_e = 0$.

The above assumption is satisfied under some mild conditions, which we will discuss in detail later. Based on Correa et al. (2008), we make the following definitions.

Definition 3. Under Assumption 1, for each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define

$$\beta(c_e, \mathcal{I}) = \sup_{x \ge 0,\, y > 0} \frac{x\,(\overline{t}_e(y) - \underline{t}_e(x))}{y\,\overline{t}_e(y)},$$

and

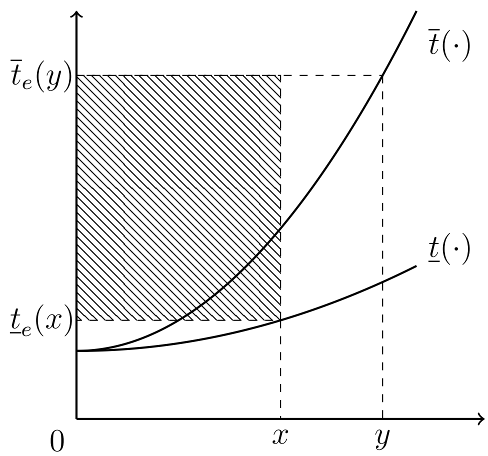
$$\beta(\mathcal{I}) = \sup_{c \in \mathcal{C}(\mathcal{I})} \beta(c, \mathcal{I}).$$

Note that particularly when demands are deterministic, we can choose $\underline{t}_e(\cdot) = \overline{t}_e(\cdot) = c_e(\cdot)$ and obtain

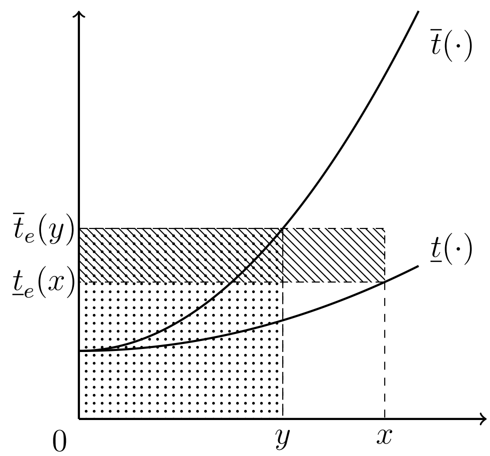
$$\beta(c_e, \mathcal{I}) = \sup_{x \ge 0,\, y > 0} \frac{x\,(c_e(y) - c_e(x))}{y\,c_e(y)} = \sup_{y > x \ge 0} \frac{x\,(c_e(y) - c_e(x))}{y\,c_e(y)},$$

which is proved geometrically to be less than 1 in (Correa et al., 2008). However with stochastic demands, we can no longer guarantee that $\beta(\mathcal{I})$ in Definition 3 is always less than 1. The supremum in the definition of $\beta(c_e, \mathcal{I})$ can be attained under the condition $x > y$. We demonstrate this point later in our study of the PoA. We now use Figure 2 to show how

$(x\,(\bar{t}_e(y) - \underline{t}_e(x)))/(y\,\bar{t}_e(y))$ can be interpreted geometrically under both conditions, $x \geq y$ and $x < y$. As shown in panel (a), when $x \leq y$, the shaded rectangle of area $x\,(\bar{t}_e(y) - \underline{t}_e(x))$ is within the big rectangle of area $y\,\bar{t}_e(y)$. However, in panel (b), the shaded rectangle of area $x\,(\bar{t}_e(y) - \underline{t}_e(x))$ is not completely within the dotted rectangle of area $y\,\bar{t}_e(y)$ due to the possibility that $\bar{t}_e(y) > \underline{t}_e(x)$, which implies $\beta(c_e, \mathcal{I})$ could be more than 1. It shows that the geometric meaning of this ratio is not as clear as its counterpart in the deterministic setting. However, we still use the word "geometry" to refer to the bounding technique motivated from (Correa et al., 2008) to indicate the significance of the motivating work.



(a) $x \leq y$



(b) $x > y$

Figure 2: Geometric interpretation of $\dfrac{x\,(\bar{t}_e(y) - \underline{t}_e(x))}{y\,\bar{t}_e(y)}$ in the definition of $\beta(c_e, \mathcal{I})$

Definition 4. Under Assumption 1, for each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define for each $e \in E$ functions $\phi_e(\cdot)$ and $\eta_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\phi_e(x) = \frac{x\,\underline{t}_e(x)}{\bar{s}_e(x)}, \quad \eta_e(x) = \frac{x\,\bar{t}_e(x)}{\underline{s}_e(x)}.$$

Let

$$\underline{\alpha}(c_e, \mathcal{I}) = \inf_{x>0} \phi_e(x), \quad \overline{\alpha}(c_e, \mathcal{I}) = \sup_{x>0} \eta_e(x),$$

and

$$\underline{\alpha}(\mathcal{I}) = \inf_{c \in \mathcal{C}(\mathcal{I})} \underline{\alpha}(c, \mathcal{I}), \quad \overline{\alpha}(\mathcal{I}) = \sup_{c \in \mathcal{C}(\mathcal{I})} \overline{\alpha}(c, \mathcal{I}).$$

Now we are ready to show our first bound by the geometry method.

Proposition 3. *[General geometry bound] Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ *be any game instance. Under Assumption 1, if* $\beta(\mathcal{I}) < 1$*, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq (1 - \beta(\mathcal{I}))^{-1} \cdot \frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ be a UE-SD with $\bar{\mathbf{V}}, \bar{\mathbf{v}}$ as the corresponding link flows and mean link flows. Let $\mathbf{p}^*$ be an SO-SD with $\mathbf{V}^*, \mathbf{v}^*$ as the corresponding link flows and mean link flows. Given that $\mathbf{d} > 0$, we have $\bar{\mathbf{v}}$, $\mathbf{v}^* \geq 0$. From UE-SD condition (10), we have

$$\begin{aligned}\sum_{e \in E} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)] &\leq \sum_{e \in E} v_e^* \mathbb{E}[c_e(\bar{V}_e)] \\ &= \sum_{e \in E} v_e^* \mathbb{E}[c_e(V_e^*)] + \sum_{e \in E} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right]\right),\end{aligned}$$

which can be rearranged as

$$(1 - R) \sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right] \leq \sum_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right],$$

where, with $\{e \in E : \bar{v}_e > 0\} \neq \emptyset$,

$$\begin{aligned} R &= \frac{\sum_{e \in E} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right]\right)}{\sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \\ &\leq \frac{\sum_{e \in E: \bar{v}_e > 0} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right]\right)}{\sum_{e \in E: \bar{v}_e > 0} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \\ &\leq \max_{e \in E: \bar{v}_e > 0} \frac{v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right]\right)}{\bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]}. \end{aligned}$$

The first inequality above is due to $v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right]\right) \leq 0$ when $\bar{v}_e = 0$ as can be seen from the facts that $\bar{V}_e = 0$ when $\bar{v}_e = 0$ and $\underline{t}_e(\cdot)$ is non-decreasing (Assumption 1). Now we have:

$$\begin{aligned}
R &\leq \max_{e \in E: \bar{v}_e > 0} \left\{ \frac{v_e^*}{\bar{v}_e} - \frac{v_e^* \mathbb{E}[c_e(V_e^*)]}{\bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \right\} \leq \max_{e \in E: \bar{v}_e > 0} \left\{ \frac{v_e^*}{\bar{v}_e} - \frac{v_e^* \underline{t}_e(v_e^*)}{\bar{v}_e \bar{t}_e(\bar{v}_e)} \right\} \\
&= \max_{e \in E: \bar{v}_e > 0} \frac{v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right)}{\bar{v}_e \bar{t}_e(\bar{v}_e)} \leq \beta(\mathcal{I}).
\end{aligned}$$

Hence

$$(1 - \beta(\mathcal{I})) \sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right] \leq \sum_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right]. \tag{18}$$

We have

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} = \frac{\sum_{e \in E} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]}{\sum_{e \in E} \mathbb{E}[c_e(V_e^*)V_e^*]} = R_1 \cdot {R_2}^{-1} \cdot R_3,$$

where

$$R_1 = \frac{\sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]}{\sum_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right]} \leq (1 - \beta(\mathcal{I}))^{-1},$$

according to inequality (18), and

$$\begin{aligned}
R_2 &= \frac{\sum_{e \in E} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\sum_{e \in E} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} = \frac{\sum_{e \in E: \bar{v}_e > 0} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\sum_{e \in E: \bar{v}_e > 0} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} \\
&\geq \min_{e \in E: \bar{v}_e > 0} \frac{\bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} \geq \underline{\alpha}(\mathcal{I}), \qquad (19) \\
R_3 &= \frac{\sum_{e \in E} v_e^* \mathbb{E}[c_e(V_e^*)]}{\sum_{e \in E} \mathbb{E}[c_e(V_e^*)V_e^*]} = \frac{\sum_{e \in E: v_e^* > 0} v_e^* \mathbb{E}[c_e(V_e^*)]}{\sum_{e \in E: v_e^* > 0} \mathbb{E}[c_e(V_e^*)V_e^*]} \\
&\leq \max_{e \in E: v_e^* > 0} \frac{v_e^* \mathbb{E}[c_e(V_e^*)]}{\mathbb{E}[c_e(V_e^*)V_e^*]} \leq \overline{\alpha}(\mathcal{I}). \qquad (20)
\end{aligned}$$

The second equations in (19) and (20) hold because $\bar{V}_e = 0$ and $V_e^* = 0$ when $\bar{v}_e = 0$ and $v_e^* = 0$, respectively. Therefore,

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq (1 - \beta(\mathcal{I}))^{-1} \cdot \frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})},$$

for any pair $\bar{\mathbf{p}}, \mathbf{p}^* \in \Omega$ of a UE-SD and an SO-SD. □

In considering polynomial link cost functions, Roughgarden and Tardos (2004) used the fact that link cost functions are differentiable and semi-convex in their bounding techniques (or more exactly, the convexity of the function $x\, c_e(x)$). In extending their method to our stochastic model, we make the following assumption, which we will show later is satisfied for polynomial link cost functions.

Assumption 2. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, the function $\underline{s}_e(\cdot)$ in Assumption 1 is convex and differentiable. In addition, there exists a function $\lambda_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\underline{s}_e'(\lambda_e(x)x) = \bar{t}_e(x)$ for all $x \geq 0$, where $\underline{s}_e'(\cdot)$ is the derivative of $\underline{s}_e(\cdot)$.

Definition 5. Under Assumptions 1 and 2, for each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define for $e \in E$ functions $\psi_e(\cdot)$ and $\mu_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\psi_e(x) = \frac{x\, \bar{t}_e(x)}{\bar{s}_e(x)}, \quad \mu_e(x) = \frac{\underline{s}_e(\lambda_e(x)\, x)}{\bar{s}_e(x)}.$$

Using $\phi_e(\cdot)$ defined in Definition 4 in addition to $\psi_e(\cdot)$ and $\mu_e(\cdot)$, we define

$$\gamma(c_e, \mathcal{I}) = \inf_{x>0} \left\{ \mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x) \right\},$$

Let

$$\gamma(\mathcal{I}) = \inf_{c \in \mathcal{C}(\mathcal{I})} \gamma(c, \mathcal{I}).$$

Now let us present a bound by the convexity method in the following proposition.

Proposition 4. *[General convexity bound] Let $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ be any game instance. Under Assumptions 1 and 2, if $\gamma(\mathcal{I}) > 0$, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \frac{1}{\gamma(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ and $\mathbf{p}^* \in \Omega$ be respectively a UE-SD and an SO-SD, with

$\bar{\mathbf{v}}, \mathbf{v}^* \geq 0$ as the corresponding mean link flows. Then

$$\begin{aligned}
T(\mathbf{p}^*) &= \sum_{e\in E} \mathbb{E}\left[c_e(V_e^*)V_e^*\right] \geq \sum_{e\in E} \underline{s}_e(v_e^*) \\
&\geq \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\underline{s}_e'(\lambda_e(\bar{v}_e)\bar{v}_e)\right) \\
&= \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\bar{t}_e(\bar{v}_e)\right) \\
&= \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + v_e^*\bar{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right),
\end{aligned}$$

where the first inequality follows from (16), and the second last equation follows from Assumption 2.

Applying UE-SD condition (10) and inequalities (17) in the last line above leads to

$$\begin{aligned}
T(\mathbf{p}^*) &\geq \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e\underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right) \\
&= \sum_{e\in E:\bar{v}_e>0} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e\underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right) \\
&= \sum_{e\in E:\bar{v}_e>0} \left(\mu_e(\bar{v}_e) + \phi_e(\bar{v}_e) - \psi_e(\bar{v}_e)\lambda_e(\bar{v}_e)\right)\bar{s}_e(\bar{v}_e) \\
&\geq \gamma(\mathcal{I}) \sum_{e\in E:\bar{v}_e>0} \bar{s}_e(\bar{v}_e) = \gamma(\mathcal{I})\sum_{e\in E}\bar{s}_e(\bar{v}_e) \geq \gamma(\mathcal{I})\, T(\bar{\mathbf{p}}).
\end{aligned}$$

where the first equation follows from $\underline{s}_e(0) = 0$ (Assumption 1) and the second equation follows from Definitions 4 and 5. Given that $\gamma(\mathcal{I}) > 0$, we have

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq \frac{1}{\gamma(\mathcal{I})},$$

for any pairs $\bar{\mathbf{p}}, \mathbf{p}^* \in \Omega$ of a UE-SD and an SO-SD. □

Remark 3. When demands are deterministic, Propositions 3 and 4 yield the PoA bounds in Correa et al. (2008) and Roughgarden and Tardos (2004), respectively, by choosing $\underline{s}_e(v_e) = \bar{s}_e(v_e) = c_e(v_e)v_e$ and $\underline{t}_e(v_e) = \bar{t}_e(v_e) = c_e(v_e)$ for $e \in E$, which implies $\underline{\alpha}(c_e, \mathcal{I}) = \overline{\alpha}(c_e, \mathcal{I}) = 1$. As we have mentioned before, $\beta(c_e, \mathcal{I}) < 1$ always holds for nondecreasing cost functions according to Correa et al. (2008). Similarly, condition $\gamma(c_e, \mathcal{I}) > 0$ is satisfied since

$$\mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x) = \mu_e(x) + 1 - \lambda_e(x) > 0,$$

due to the fact that $\mu_e(x) > 0$ and $\lambda_e(x) \leq 1$ for $x > 0$ for nonzero cost functions (see Roughgarden and Tardos (2004) for details).

3.2. Price of anarchy with polynomial cost functions

As mentioned previously, for deterministic models, both convexity and geometry method lead to the same PoA upper bound with polynomial link cost functions. For our stochastic model, this is no longer true. After establishing two general PoA upper bounds in this subsection, we will show respectively in the next two subsections that for polynomial link cost functions, the geometry bound on the PoA is better (but not tight) in general, while the convexity bound is better and indeed tight in some special cases.

We consider the set $\mathcal{I}_m$ of game instances for any fixed $m \in \mathbb{Z}_+$ $(m \geq 1)$ with (non-zero) polynomial link cost functions in the form of

$$c_e(x) = \sum_{j=0}^{m} b_{ej} x^j,\ b_{ej} \geq 0,\ j = 0, 1, \ldots, m \text{ and } \sum_{j=0}^{m} b_{ej} > 0;\ e \in E.$$

In other words, $\mathcal{C}(\mathcal{I}_m) = \mathcal{C}_m$, the set of (non-zero) polynomial functions with nonnegative coefficients and degree at most m. Let $\widetilde{\mathcal{C}}_m$ be the subset of $\mathcal{C}_m$ consisting of only one term, namely $\widetilde{\mathcal{C}}_m = \cup_{0\leq j\leq m} \widetilde{\mathcal{C}}_m^j$, where $\widetilde{\mathcal{C}}_m^j = \{bx^j : b > 0\}$ for all $j = 0, 1, \ldots, m$. Let $\widetilde{\mathcal{I}}_m$ be the subset of game instances in $\mathcal{I}_m$ with link cost functions in $\widetilde{\mathcal{C}}_m$. The following lemma shows we can focus on the subset $\widetilde{\mathcal{I}}_m$ when bounding the PoA for instances in $\mathcal{I}_m$.

Lemma 5. *For any instance $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$, we have*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq PoA(\widetilde{\mathcal{I}}_m).$$

Proof. Any instance $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$ can be transformed into an equivalent instance with link cost functions in $\widetilde{\mathcal{C}}_m$ by replacing any link $e \in E$ of cost $c_e(x) = \sum_{j=0}^{m} b_{ej} x^j$ with a directed path consisting of no more than $m + 1$ links of costs $\widetilde{c}_{e,j}(x) = b_{ej} x^j$ $(0 \leq j \leq m)$ such that $b_{ej} > 0$. This equivalent instance clearly belongs to $\widetilde{\mathcal{I}}_m$. The result then follows immediately. □

A similar lemma can be found in (Roughgarden, 2005) for calculating the anarchy value of polynomial cost functions in deterministic models.

We now consider monomial link cost functions in $\widetilde{\mathcal{C}}_m$. Given link cost function $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^j$, i.e., $c_e(x) = b_{ej} x^j$ with $b_{ej} > 0$ for a fixed $j \leq m$, we have:

$$\mathbb{E}\left[c_e(V_e)V_e\right] = b_{ej}\mathbb{E}\left[V_e^{j+1}\right].$$

If $j = 0$, then $\mathbb{E}[c_e(V_e)V_e] = b_{ej}v_e$, which is a function of the mean link flow v_e. For $j \geq 1$, in order to compute $\mathbb{E}[c_e(V_e)V_e]$, we need the moment $\mathbb{E}[V_e^{j+1}]$ of V_e to be finite. Given that $V_e = \sum_{i \in I} p_e^i \cdot D_i$ and $\{D_i : i \in I\}$ are independent, we then need the first $j+1$ moments of D_i to be finite. In addition, in order to construct functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, and $\overline{t}_e(\cdot)$ in Assumption 1, we make the following assumption.

Assumption 3. The first $m+1$ moments of random demands D_i $(i \in I)$ are all finite and positive. In addition, for $j = 2, \ldots, m+1$, there exist $0 < l_j \leq h_j$ such that

$$0 \leq l_j v_e^j \leq \mathbb{E}\left[V_e^j\right] \leq h_j v_e^j, \quad \forall\, e \in E.$$

Positivity of moments is satisfied in general if we consider positive-valued demand distributions, which is reasonable to assume. We also consider normal distributions later since they are widely used in the literature to simulate traffic demands, especially the ones with large (positive) means or relatively small variances, although negative tails are contained (Clark and Watling, 2005; Asakura and Kashiwadani, 1991). Positivity of higher moments for normal distributions is again satisfied easily under the assumption of positive means. With respect to the parameters l_j and h_j, for consistency we define $l_0 = h_0 = l_1 = h_1 = 1$ since $\mathbb{E}[V_e^j] = v_e^j$ for $j = 0, 1$. We will show later how to compute l_j and h_j for $j > 1$ for both positive-valued demand distributions and normal distributions (with positive means).

Under Assumption 3, we can now show that there exist functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\overline{t}_e(\cdot)$, and $\lambda_e(\cdot)$ with which both Assumptions 1 and 2 are satisfied for monomial link cost functions.

Definition 6. For a fixed j $(0 \leq j \leq m)$, let $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^j$. Let

$$\begin{cases} \overline{t}_e(x) = h_j b_{ej} x^j, \\ \underline{t}_e(x) = l_j b_{ej} x^j, \end{cases} \quad \begin{cases} \overline{s}_e(x) = h_{j+1} b_{ej} x^{j+1}, \\ \underline{s}_e(x) = l_{j+1} b_{ej} x^{j+1}, \end{cases}$$

where h_j and l_j are taken from Assumption 3. In addition, let

$$\lambda_e(x) = \begin{cases} \left(\dfrac{h_j}{(j+1)l_{j+1}}\right)^{1/j}, & j > 0, \\ 1, & j = 0. \end{cases}$$

With the functions defined in Definition 6, it is easy to show that Assumption 1 is satisfied. As for Assumption 2, it is indeed that $\underline{s}_e(\cdot)$ is convex and differentiable. We can also check easily that $\underline{s}_e'(\lambda_e(x)x) = \overline{t}_e(x)$ for all $x \geq 0$. We are now ready to compute all necessary parameters to provide specific upper bounds on the PoA.

Lemma 6. *Under Assumption 3, we have*

$$\begin{aligned}
\underline{\alpha}(\widetilde{\mathcal{I}}_m) &= \min_{0\leq j\leq m} \frac{l_j}{h_{j+1}}, \quad \overline{\alpha}(\widetilde{\mathcal{I}}_m) = \max_{0\leq j\leq m} \frac{h_j}{l_{j+1}}, && (21)\\
\beta(\widetilde{\mathcal{I}}_m) &= \max_{1\leq j\leq m} \left\{ \frac{j}{j+1} \cdot \left(\frac{h_j}{(j+1)l_j} \right)^{1/j} \right\}, && (22)\\
\gamma(\widetilde{\mathcal{I}}_m) &= \min_{1\leq j\leq m} \left\{ \frac{l_j}{h_{j+1}} - \frac{j}{j+1} \cdot \frac{h_j}{h_{j+1}} \cdot \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j} \right\}. && (23)
\end{aligned}$$

Proof. We have

$$\phi_e(x) = \frac{x\,\underline{t}_e(x)}{\overline{s}_e(x)} = \frac{l_j}{h_{j+1}}, \quad \eta_e(x) = \frac{x\,\overline{t}_e(x)}{\underline{s}_e(x)} = \frac{h_j}{l_{j+1}}.$$

Hence

$$\underline{\alpha}(c_e, \widetilde{\mathcal{I}}_m) = \frac{l_j}{h_{j+1}}, \quad \overline{\alpha}(c_e, \widetilde{\mathcal{I}}_m) = \frac{h_j}{l_{j+1}}.$$

Since $\mathcal{C}(\widetilde{\mathcal{I}}_m) = \widetilde{\mathcal{C}}_m = \cup_{0\leq j\leq m} \widetilde{\mathcal{C}}_m^j$, we have

$$\begin{aligned}
\underline{\alpha}(\widetilde{\mathcal{I}}_m) &= \min_{0\leq j\leq m} \inf_{c\in\widetilde{\mathcal{C}}_m^j} \underline{\alpha}(c, \widetilde{\mathcal{I}}_m) = \min_{0\leq j\leq m} \frac{l_j}{h_{j+1}},\\
\overline{\alpha}(\widetilde{\mathcal{I}}_m) &= \max_{0\leq j\leq m} \sup_{c\in\widetilde{\mathcal{C}}_m^j} \overline{\alpha}(c, \widetilde{\mathcal{I}}_m) = \max_{0\leq j\leq m} \frac{h_j}{l_{j+1}}.
\end{aligned}$$

For parameter $\beta(c_e, \widetilde{\mathcal{I}}_m)$, we have

$$\frac{x\,(\overline{t}_e(y) - \underline{t}_e(x))}{y\,\overline{t}_e(y)} = \frac{x}{y}\left(1 - \frac{l_j}{h_j}\left(\frac{x}{y}\right)^j\right) \equiv f_j(z),$$

where $z = x/y$, which implies that

$$\beta(\widetilde{\mathcal{I}}_m) = \max_{0\leq j\leq m} \sup_{c\in\widetilde{\mathcal{C}}_m^j} \beta(c, \widetilde{\mathcal{I}}_m) = \max_{0\leq j\leq m} \sup_{z>0} f_j(z).$$

For $1 \le j \le m$, elementary calculus gives

$$\sup_{z>0} f_j(z) = \frac{j}{j+1} \cdot \left(\frac{h_j}{(j+1)l_j} \right)^{1/j},$$

which together with the fact that $f_0(\cdot) \equiv 0$ implies equation (22).

We now consider $\gamma(c_e, \widetilde{\mathcal{I}}_m)$. If $j = 0$, we have: $\lambda_e(x) = \mu_e(x) = 1 = \phi_e(x) = \eta_e(x)$. Thus we have $\gamma(c_e, \widetilde{\mathcal{I}}_m) = 1$ for $c_e(\cdot) \in \widetilde{\mathcal{C}}^0_m$. When $j > 0$ we have

$$\lambda_e(x) = \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j}.$$

Thus

$$\mu_e(x) = \frac{\underline{s}_e(\lambda_e(x)x)}{\overline{s}_e(x)} = \frac{l_{j+1}}{h_{j+1}} \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j+1}.$$

Similar to $\phi_e(\cdot)$ and $\eta_e(\cdot)$, we have $\psi_e(x) = h_j/h_{j+1}$. All three functions, $\lambda_e(\cdot)$, $\mu_e(\cdot)$, and $\psi_e(\cdot)$ are constants, leading to the following:

$$\begin{aligned}
\gamma(c_e, \widetilde{\mathcal{I}}_m) &= \mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x) \\
&= \frac{l_{j+1}}{h_{j+1}} \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j+1} + \frac{l_j}{h_{j+1}} - \frac{h_j}{h_{j+1}} \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j} \\
&= \frac{l_j}{h_{j+1}} - \frac{j}{j+1} \cdot \frac{h_j}{h_{j+1}} \cdot \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j}.
\end{aligned}$$

Since specifically for $j = 1$, we have $l_1/h_2 = 1/h_2 \le 1$, which implies $\gamma(c_e, \widetilde{\mathcal{I}}_m) \le 1$ for $c_e(\cdot) \in \widetilde{\mathcal{C}}^1_m$, according to the definition of $\gamma(\widetilde{\mathcal{I}}_m)$, we obtain equation (23). □

In the following two theorems we present specific geometry and convexity bound on the PoA by applying Proposition 3 and 4 to game instances in $\mathcal{I}_m$.

Theorem 7. *[Geometry upper bound] Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$. *Under Assumption 3, if*

$$\frac{h_j}{l_j} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots, m, \tag{24}$$

then

$$PoA(G, \mathbf{D}, \mathbf{c}) \le \left(1 - \max_{1\le j\le m} \left\{ \frac{j}{j+1} \cdot \left(\frac{h_j}{(j+1)l_j} \right)^{1/j} \right\} \right)^{-1} \cdot \frac{\max_{0\le j\le m} h_j/l_{j+1}}{\min_{0\le j\le m} l_j/h_{j+1}}.$$

Proof. The proof of the theorem is straightforward by applying Proposition 3 for $\widetilde{\mathcal{I}}_m$ combined with Lemma 6. Note that Assumption 1 is satisfied for functions defined in Definition 6 under Assumption 3. Lemma 5 is then used to bound the PoA for game instances in $\mathcal{I}_m$. $\square$

Theorem 8. *[Convexity upper bound] Let $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$. Under Assumption 3, if*

$$\frac{h_j}{l_{j+1}}\left(\frac{h_j}{l_j}\right)^j < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\ j = 1, \ldots, m, \tag{25}$$

then

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \max_{1 \leq j \leq m} \left(\frac{l_j}{h_{j+1}} - \frac{j}{j+1} \cdot \frac{h_j}{h_{j+1}} \cdot \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j} \right)^{-1}.$$

Remark 4. Theorems 7 and 8 both generalize the PoA bounds provided by Roughgarden and Tardos (2002, 2004) and Correa et al. (2008) for deterministic models.

When the traffic demands return to being deterministic, we can choose $l_j = h_j = 1$ for all integer $j \leq m+1$, so that Assumption 3 is satisfied. Both conditions (24) and (25) clearly hold since $(j+1)^{j+1}/j^j > 1$ for $1 \leq j \leq m$. Both the geometry and convexity bound become $\left(1 - m(m+1)^{-(m+1)/m}\right)^{-1}$, which matches the tight upper bound of the PoA in deterministic models.

On the other hand, unlike in deterministic models, conditions (24) and (25) are necessary for our stochastic model. It is due to the fact that parameters we use to bound the PoA in our stochastic model now depend on not only the cost functions but also demand distributions and to some extent, the network structure. Both conditions are constructed based on functional approximations of $\mathbb{E}\left[c_e(V_e)\right]$ and $\mathbb{E}\left[c_e(V_e)V_e\right]$. In general it is difficult to determine what set $\mathcal{I}$ of instances of transportation games with stochastic demands for which these conditions are always satisfied. However, we will derive these two conditions in the next subsection with specific demand distributions.

3.3. Price of anarchy under specific demand distributions

3.3.1. General positive-valued demand distributions

It is natural to consider general positive-valued distributions for demands $\{D_i : i \in I\}$. It was clear that we need to assume the finiteness of the first $m+1$ moments of D_i $(i \in I)$ when considering game instances in $\mathcal{I}_m$. These moments are non-negative when demands follow positive-valued distributions. Let

$$\theta_i^{(j)} = \frac{\mathbb{E}[D_i^j]}{d_i^j} > 0, \quad \forall\, i \in I,\ \forall\, j = 0, 1, \ldots, m+1. \tag{26}$$

Lemma 9. *For any $s, t \in \mathbb{Z}_+$ and any $i \in I$,*

$$\theta_i^{(s+t)} \geq \theta_i^{(s)} \cdot \theta_i^{(t)}. \tag{27}$$

Proof. We have

$$\mathbb{E}[D_i^{s+t}] = \mathbb{E}[D_i^s D_i^t] = \mathbb{E}[D_i^s] \cdot \mathbb{E}[D_i^t] + \mathrm{Cov}(D_i^s, D_i^t), \quad i \in I.$$

Since D_i is a positive random variable, $\mathrm{Cov}(D_i^s, D_i^t) \geq 0$ (see, e.g., Schmidt (2014)). Thus

$$\mathbb{E}[D_i^{s+t}] \geq \mathbb{E}[D_i^s] \cdot \mathbb{E}[D_i^t], \quad \forall\, i \in I,$$

which leads to

$$\frac{\mathbb{E}\left[D_i^{s+t}\right]}{\mathbb{E}[D_i]^{s+t}} \geq \frac{\mathbb{E}[D_i^s]}{\mathbb{E}[D_i]^s} \cdot \frac{\mathbb{E}[D_i^t]}{\mathbb{E}[D_i]^t}, \quad \forall\, i \in I.$$

We then have $\theta_i^{(s+t)} \geq \theta_i^{(s)} \cdot \theta_i^{(t)}$ for all $i \in I$. $\square$

We will need Minkowski's inequality, which is stated in the following lemma.

Lemma 10 (Minkowski's Inequality)**.** *Let X and Y be random variables. Then for $1 \leq q < \infty$,*

$$\left(\mathbb{E}[|X+Y|^q]\right)^{1/q} \leq \left(\mathbb{E}[|X|^q]\right)^{1/q} + \left(\mathbb{E}[|Y|^q]\right)^{1/q}.$$

Denote $\overline{\theta}^{(j)} = \max_{i \in I}\{\theta_i^{(j)}\}$ for $j = 0, 1, \ldots, m$. The following lemma shows positive-valued distributions do satisfy Assumption 3 with $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 2, \ldots, m+1$. Note that for $j = 0, 1$, we also have $l_j = h_j = \overline{\theta}^{(j)} = 1$.

Lemma 11. *For any transportation game $(G, \mathbf{D}, \mathbf{c})$ in which random demands $\{D_i : i \in I\}$ follow positive-valued distributions, the moments of link flows can be bounded as follows:*

$$0 \leq v_e^j \leq \mathbb{E}[V_e^j] \leq \overline{\theta}^{(j)} v_e^j, \quad \forall\, j = 2, \ldots, m+1,\ e \in E.$$

Proof. According to (3), V_e is a non-negative random variable since $\{D_i : i \in I\}$ follow positive-valued distributions. Due to the convexity of x^j on $[0, \infty)$ for $j \geq 2$, the middle inequality follows directly from Jensen's inequality. For the last inequality in the lemma, we have

$$\begin{aligned}\left(\mathbb{E}[V_e^j]\right)^{1/j} &= \left(\mathbb{E}\left[\left(\sum_{i\in I}\delta_e^i p_e^i D_i\right)^j\right]\right)^{1/j} \leq \sum_{i\in I}\delta_e^i\left(\mathbb{E}\left[\left(p_e^i D_i\right)^j\right]\right)^{1/j} \\ &= \sum_{i\in I}\delta_e^i p_e^i\left(\mathbb{E}[D_i^j]\right)^{1/j} = \sum_{i\in I}\delta_e^i p_e^i d_i\left(\theta_i^{(j)}\right)^{1/j} \leq \left(\overline{\theta}^{(j)}\right)^{1/j} v_e,\end{aligned}$$

where the first inequality follows Minkowski's inequality. We then have

$$\mathbb{E}[V_e^j] \leq \overline{\theta}^{(j)} v_e^j, \quad \forall\, e \in E,$$

which completes our proof. □

Substituting $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 0, 1, \ldots, m$ in Theorem 7, we obtain the following specific geometry bound on the PoA.

Proposition 12. *Let $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$. If $\{D_i : i \in I\}$ follow positive-valued distributions with finite first $m+1$ moments and*

$$\overline{\theta}^{(j)} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots m, \tag{28}$$

then

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \max_{1\leq j\leq m}\left(1 - \frac{j}{j+1}\cdot\left(\frac{\overline{\theta}^{(j)}}{j+1}\right)^{1/j}\right)^{-1}\cdot\overline{\theta}^{(m)}\overline{\theta}^{(m+1)}.$$

Proof. We only need to show that $\underline{\alpha}(\widetilde{\mathcal{I}}_m) = 1/\overline{\theta}^{(m+1)}$ and $\overline{\alpha}(\widetilde{\mathcal{I}}_m) = \overline{\theta}^{(m)}$. By setting $s = t+1$ in (27), it is easy to prove that $\overline{\theta}^{(j)}$ is nondecreasing in j. Then

$$\begin{aligned}\underline{\alpha}(\widetilde{\mathcal{I}}_m) &= \min_{0\leq j\leq m}\frac{1}{\overline{\theta}^{(j+1)}} = \frac{1}{\overline{\theta}^{(m+1)}}, \\ \overline{\alpha}(\widetilde{\mathcal{I}}_m) &= \max_{0\leq j\leq m}\overline{\theta}^{(j)} = \overline{\theta}^{(m)}.\end{aligned}$$

□

Similarly, we can substitute $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 0, 1, \ldots, m$ in Theorem 8 to obtain the following specific convexity bound on the PoA.

Proposition 13. *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$. *If* $\{D_i : i \in I\}$ *follow positive-valued distributions with finite first* $m+1$ *moments and*

$$\overline{\theta}^{(m)} < \frac{m+1}{m^{m/(m+1)}}, \tag{29}$$

then

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \max_{1 \leq j \leq m} \left(\frac{1}{\overline{\theta}^{(j+1)}} - \frac{j}{j+1} \cdot \frac{\overline{\theta}^{(j)}}{\overline{\theta}^{(j+1)}} \cdot \left(\frac{\overline{\theta}^{(j)}}{j+1} \right)^{1/j} \right)^{-1}.$$

Remark 5. Since $(j+1)/j^{j/(j+1)}$ is decreasing in j, condition (29) implies condition (28) for $m \geq 1$. When $m = 1$ both conditions are always satisfied and we will discuss this special case in Section 3.4. So let us consider the two conditions for $m = 2, 3, 4$, as the highest power of a link cost function is seldom greater than 4 in practice (Clark and Watling, 2005; Sumalee and Xu, 2011). Table 2 shows applicable ranges of moments up to degree 4 for two conditions (28) and (29). The results indicate that condition (29) for the specific convexity bound in Proposition 13 is much less applicable for polynomial cost functions with higher degrees.

Degree	$j = 2$	$j = 3$	$j = 4$
Geometry condition (28):	$\overline{\theta}^{(2)} < 6.75$	$\overline{\theta}^{(3)} < 9.48$	$\overline{\theta}^{(4)} < 12.21$
Convexity condition (29):	$\overline{\theta}^{(2)} < 1.89$	$\overline{\theta}^{(3)} < 1.75$	$\overline{\theta}^{(4)} < 1.65$

Table 2: Applicable ranges of moments for the two PoA bounds

Example 2. We provide an example with log-normal distributions for the comparison in the above remark. Assume for $i \in I$ that $D_i \sim \ln N(\mu_i, \omega_i)$, i.e., D_i follows a log-normal distribution with mean $d_i = e^{\mu_i + \omega_i^2/2}$ and variance $\sigma_i^2 = (e^{\omega_i^2} - 1)d_i^2$, which means that the coefficient of demand variation $\epsilon_i = \sigma_i/d_i = (e^{\omega_i^2} - 1)^{1/2}$. The moments of D_i are $\mathbb{E}\left[D_i^j\right] = e^{j\mu_i + j^2\omega_i^2/2}$. Thus

$$\theta_i^{(j)} = \frac{\mathbb{E}\left[D_i^j\right]}{d_i^j} = e^{j(j-1)\omega_i^2/2} = (\epsilon_i^2 + 1)^{j(j-1)/2}, \quad \forall\ i \in I,\ j \in \mathbb{Z}^+.$$

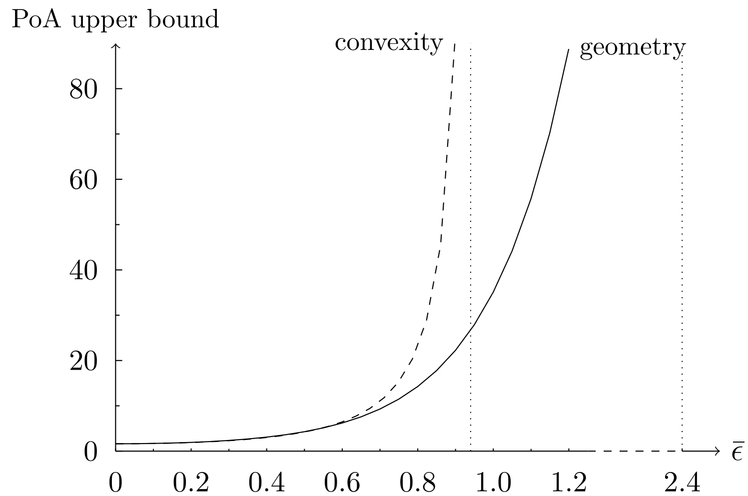


Figure 3: The two PoA upper bounds with quadratic cost functions ($m = 2$) and log-normal distributions

We then have $\overline{\theta}^{(j)} = (\bar{\epsilon}^2 + 1)^{j(j-1)/2}$ for all $j \in \mathbb{Z}^+$. Conditions (28) and (29) can now be expressed in terms of applicable ranges of the maximum coefficient of variation $\bar{\epsilon}$ for different classes of polynomial link cost functions. Table 3 shows maximum values of $\bar{\epsilon}$ for $\mathcal{C}_m$ with $m = 2, 3$, and 4.

Class of cost functions	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Geometry condition (28):	2.40	1.06	0.72
Convexity condition (29):	0.94	0.45	0.29

Table 3: Maximum values of coefficient of variation $\bar{\epsilon}$ of log-normal distributions for the two upper bounds

In Figure 3 we also plot the two specific PoA bounds presented in Propositions 12 and 13 when demands follow log-normal distributions for quadratic cost functions ($m = 2$).

As can be seen, the geometry bound is better and applicable for a wider range of game instances. The vertical dotted (asymptotical) lines help to show that the convexity and geometry bound approach infinity when $\bar{\epsilon} \to 0.94$ and 2.40 respectively. Note that when the demand variation is very small ($\bar{\epsilon} \leq 0.54$ in this case), the convexity bound can be slightly better than the geometry bound although the overall improvement is insignificant.

Similar results can be obtained for $m = 3$ and $m = 4$. $\square$

3.3.2. Normal distributions

As previously mentioned, normal distributions can be used to approximate traffic demands, especially those with large positive means or relatively small variances. As the second class of specific demand distributions, let us consider $D_i \sim N(d_i, \sigma_i^2)$ for $i \in I$, i.e., D_i follows a normal distribution with mean d_i and variance σ_i^2 for $i \in I$. We assume $d_i > 0$ for $i \in I$. Note that this assumption guarantees the non-negativity of mean link flows, which is needed to derive general upper bounds on the PoA in Propositions 3 and 4.

Given that demands $\{D_i : i \in I\}$ are independent, clearly V_e also follows a normal distribution for $e \in E$. The mean v_e and variance σ_e^2 of V_e can be derived from (3) as follows, which is applicable for any independent demand distributions:

$$v_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i p_k^i d_i,$$

and

$$\sigma_e^2 = \mathrm{Var}\left[\sum_{i \in I,\, k \in P_i} \delta_{k,e}^i p_k^i D_i\right] = \mathrm{Var}\left[\sum_{i \in I} \delta_e^i p_e^i D_i\right] = \sum_{i \in I} \delta_e^i (p_e^i)^2 \sigma_i^2.$$

Since $V_e \sim N(v_e, \sigma_e^2)$, the jth moment of the link flow V_e can be written as follows:

$$\mathbb{E}[V_e^j] = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\sigma_e)^r (v_e)^{j-r} (r-1)!!, \quad \forall\, e \in E, \tag{30}$$

where $j \in \mathbb{N}$ is the power degree, $(r-1)!!$ is the double factorial of $r-1$, i.e., $(r-1)!! = (r-1)(r-3)\cdots 1$ (if r is even) with the understanding that $(-1)!! = 1$, and $\binom{j}{r} = j!/((j-r)!r!)$ is a binomial coefficient. (Note that moment formula (30) for the normal distribution can be found in standard texts, e.g., in (Patel and Read, 1996; Ross, 2002, p. 396 (47)).)

In order to bound moments of V_e, we first bound its variance σ_e^2 using the following lemma.

Lemma 14. *The mean v_e and variance σ_e^2 of random link flow V_e ($e \in E$) satisfy the following inequalities:*

$$\frac{\underline{\epsilon}^2}{n} v_e^2 \le \sigma_e^2 \le \overline{\epsilon}^2 v_e^2,$$

where n, $\underline{\epsilon}$, and $\overline{\epsilon}$ are defined in Section 2.1.

Proof. By definition $\epsilon_i = \sigma_i/d_i$, we can bound σ_e^2 from above:

$$\begin{aligned}\sigma_e^2 &= \sum_{i\in I}\delta_e^i(p_e^i)^2\epsilon_i^2 d_i^2 \le \left(\max_{i\in I}\{\epsilon_i\}\right)^2\sum_{i\in I}\delta_e^i(p_e^i d_i)^2\\ &\le \overline{\epsilon}^2\left(\sum_{i\in I,\, k\in P_i}\delta_{k,e}^i p_k^i d_i\right)^2 = \overline{\epsilon}^2 v_e^2,\end{aligned}$$

and bound σ_e^2 from below:

$$\begin{aligned}\sigma_e^2 &\ge \left(\min_{i\in I}\{\epsilon_i\}\right)^2\sum_{i\in I}\delta_e^i(p_e^i d_i)^2 \ge \frac{\underline{\epsilon}^2}{n_e}\left(\sum_{i\in I}\delta_e^i p_e^i d_i\right)^2\\ &= \frac{\underline{\epsilon}^2}{n_e}\left(\sum_{i\in I,\, k\in P_i}\delta_{k,e}^i p_k^i d_i\right)^2 \ge \frac{\underline{\epsilon}^2}{n}v_e^2,\end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality with $n_e = \sum_{i\in I}\delta_e^i$ as defined in Section 2.1. $\square$

We are now ready to bound moments of link flows and show that Assumption 3 is satisfied.

Lemma 15. *For any transportation game $(G, \mathbf{D}, \mathbf{c})$ in which $\{D_i : i \in I\}$ follow normal distributions with positive mean demands, Assumption 3 is satisfied with*

$$l_j = \sum_{r=0,\, r=even}^{j}\binom{j}{r}\left(\frac{\underline{\epsilon}^2}{n}\right)^{r/2}(r-1)!!, \quad h_j = \overline{\theta}^{(j)}, \quad \forall\, j = 2,\ldots,m+1. \tag{31}$$

Proof. Since $D_i \sim N(d_i, \sigma_i^2)$ with $d_i > 0$ and finite σ_i for all $i \in I$, it is clear that all moments of D_i are finite and positive according to (30). We need to show that

$$l_j v_e^j \le \mathbb{E}[V_e^j] \le h_j v_e^j,\ \forall\, j = 2,\ldots,m+1,\ e \in E,$$

where l_j and h_j are defined in (31). Applying Lemma 14 in (30), we obtain

$$\mathbb{E}[V_e^j] \ge \sum_{r=0,\, r=\text{even}}^{j}\binom{j}{r}\left(\frac{\underline{\epsilon}^2}{n}\right)^{r/2}(v_e)^j(r-1)!!,\ e \in E.$$

On the other hand, observe that

$$\theta_i^{(j)} = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\epsilon_i)^r (r-1)!!,\ \forall\ i \in I,$$

which implies

$$\overline{\theta}^{(j)} = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\overline{\epsilon})^r (r-1)!!,$$

which together with Lemma 14 implies the upper bound. □

Lemma 15 indicates that we can apply Theorems 7 and 8 for transportation games $(G, \mathbf{D}, \mathbf{c})$ in which $\{D_i : i \in I\}$ follow normal distributions with positive means by using values of l_j and h_j in (31). Since l_j depends on $\underline{\epsilon}$ and $\underline{\epsilon} \to 0$ implies $l_j \to 1$, without any restriction on $\underline{\epsilon}$ (i.e., no positive lower bound), we would have the same PoA upper bounds as with any positive-valued distributions (with the same moments as those of normal distributions). This is due to the fact that $h_j = \overline{\theta}^{(j)}$ in both settings.

Additionally l_j depends also on $n = \max_{e \in E} n_e$, where n_e as defined in Section 2.1 is the number of O-D pairs that use link e. Clearly, n is a network-related parameter, which means for normal distributions, the two upper bounds (and conditions of their applicability) are not network independent as in deterministic models in general. However the effect of n is limited, as we can also derive upper bounds independent of n by setting $n \to \infty$. From (31), $n \to \infty$ also implies $l_j \to 1$. Thus in such an extreme case, the PoA upper bounds (and conditions of their applicability) for normal distributions would return to the same as those for positive-valued distributions (with the same moments as those of normal distributions). Then from Table 2 we can derive maximum applicable $\overline{\epsilon}$ of normal distributions for both geometry and convexity upper bound, as shown in Table 4. For polynomial cost functions with degree no more than 4, we can bound the PoA using the geometry method for arbitrary network as long as $\overline{\epsilon} < 1.08$. This condition is actually not restrictive, as only normal distributions with relatively small variance are usually used in practice to simulate traffic demands.

In order to demonstrate the effect of small n, we use the case of $\underline{\epsilon} = \overline{\epsilon} = \epsilon$ for simplicity. Figure 4 shows the maximum applicable values of ϵ when $m = 2$ for our geometry and convexity bound. We can see that condition (24) is always satisfied for $n < 7$ and so is condition (25) for $n < 3$. When n is large

Class of cost functions	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Geometry Condition (24):	2.40	1.68	1.08
Convexity Condition (25):	0.94	0.50	0.32

Table 4: Maximum applicable $\overline{\epsilon}$ of normal distributions for the two upper bounds when $n \to \infty$

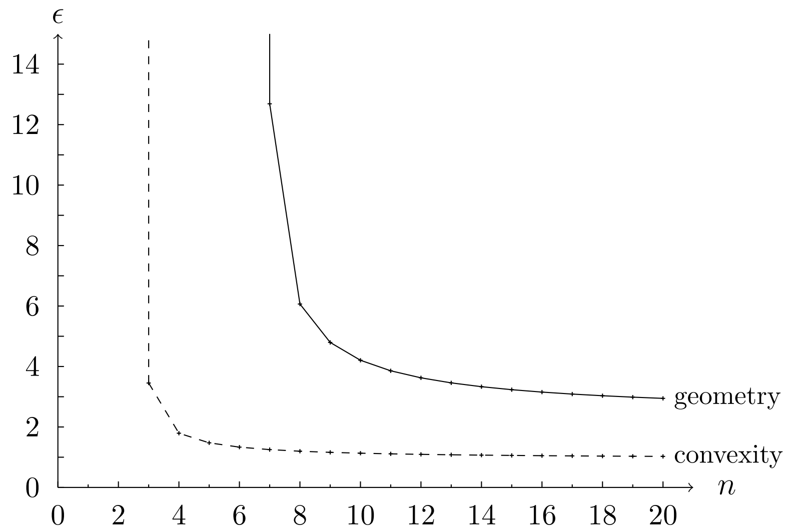


Figure 4: Maximum values of coefficient of variation, ϵ, of normal distributions for the two PoA upper bounds with quadratic cost functions ($m = 2$)

enough, the applicable ranges of ϵ remain almost constant and as discussed above, the upper bounds of these ranges converge to the corresponding values reported in Table 4. Similar results can be found for $m = 3$ and $m = 4$. As noted in Section 2.1, the case of $n = 1$ is equivalent to the case of single commodity, which we will treat as a special simple case later in Section 3.4, as we will do for the case of $m = 1$.

We also compare the two upper bounds for different values of ϵ, which is illustrated in Figure 5 for $m = 2$ and $n = 5$ with clearly better a quality of the geometry bound.

3.4. Two special cases

We have provided two general upper bounds on the PoA for transportation games with general networks and general polynomial cost functions un-

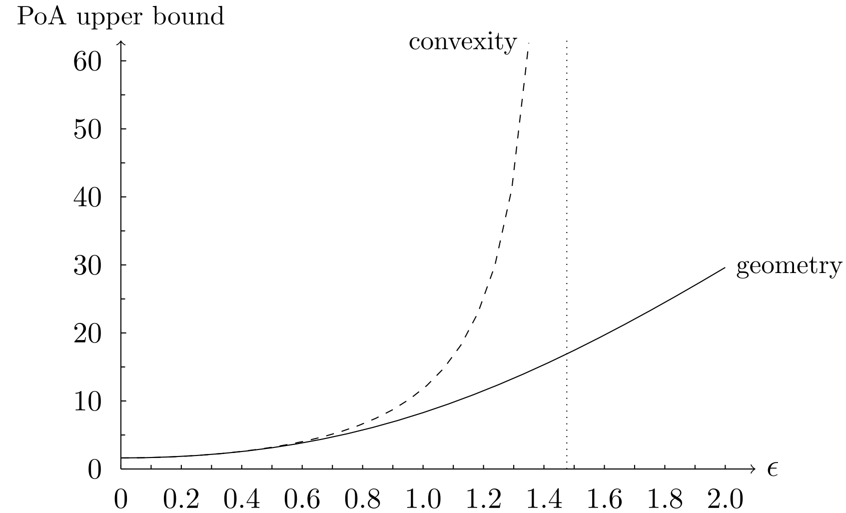


Figure 5: The two PoA upper bounds with quadratic cost functions ($m = 2$) and normal distributions and $n = 5$

der two specific classes of demand distributions, namely, general positive-valued distributions and normal distributions. With these settings, the geometry method leads to a better upper bound under less stringent conditions. In this subsection we will investigate two special cases in which the convexity method will lead to a better (and in fact tight) upper bound.

The first special case is with single-commodity networks ($|I| = 1$), or equivalently $n = 1$ as noted before, while the second special case is of affine cost functions. Interestingly, both conditions (24) and (25) in these two special cases are satisfied automatically as in deterministic models.

3.4.1. Single commodity networks

Consider any transportation game $(G, D, \mathbf{c}) \in \mathcal{I}_m$ such that G has a single O-D pair. Since $|I| = 1$, we will drop the subscript i in writing relevant parameters, such as writing D instead of D_i. In order to satisfy Assumption 3, we assume the first $m+1$ moments of D are finite and positive. Then $V_e = p_e \cdot D$. Thus $\mathbb{E}\left[V_e^j\right] = \theta^{(j)} v_e^j$, where $\theta^{(j)} = \mathbb{E}\left[D^j\right]/d^j$. We can then select $l_j = h_j = \theta^{(j)}$ for all $j = 0, 1, \ldots, m+1$ and hence Assumption 3 is satisfied.

Condition (24) is satisfied since $h_j/l_j = 1$ for all $j = 1, \ldots, m$. The geometry bound in Theorem 7 can be calculated as follows:

$$\begin{aligned}\mathrm{PoA}(G, D, \mathbf{c}) &\leq (1 - \max_{1\leq j\leq m} j(j+1)^{-(j+1)/j})^{-1} \cdot \frac{\max_{0\leq j\leq m} \theta^{(j)}/\theta^{(j+1)}}{\min_{0\leq j\leq m} \theta^{(j)}/\theta^{(j+1)}} \\ &= (1 - m(m+1)^{-(m+1)/m})^{-1} \cdot \frac{\max_{0\leq j\leq m} \theta^{(j)}/\theta^{(j+1)}}{\min_{0\leq j\leq m} \theta^{(j)}/\theta^{(j+1)}}.\end{aligned}$$

We claim that $\theta^{(j+1)} \geq \theta^{(j)} \geq 1$ for all j in both settings for the demand distributions, general positive-valued distributions and normal distributions. For positive-valued distributions, it follows directly from (27). For normal distributions, we can use (30) to derive the result. Thus $\max_{0\leq j\leq m} \theta^{(j)}/\theta^{(j+1)} = \theta^{(0)}/\theta^{(1)} = 1$. The geometry bound can then be simplified further as follows:

$$\mathrm{PoA}(G, D, \mathbf{c}) \leq (1 - m(m+1)^{-(m+1)/m})^{-1} \cdot \max_{0\leq j\leq m} \frac{\theta^{(j+1)}}{\theta^{(j)}}. \tag{32}$$

Condition (25) becomes

$$\frac{\theta^{(j)}}{\theta^{(j+1)}} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots, m,$$

which is also satisfied given that $\theta^{(j+1)} \geq \theta^{(j)} \geq 1$ for all j. The convexity bound in Theorem 8 becomes:

$$\mathrm{PoA}(G, D, \mathbf{c}) \leq \max_{1\leq j\leq m} \left\{ \left(1 - \frac{j}{j+1} \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1} \frac{\theta^{(j+1)}}{\theta^{(j)}} \right\}. \tag{33}$$

We claim that the convexity bound in (33) is better than the geometry

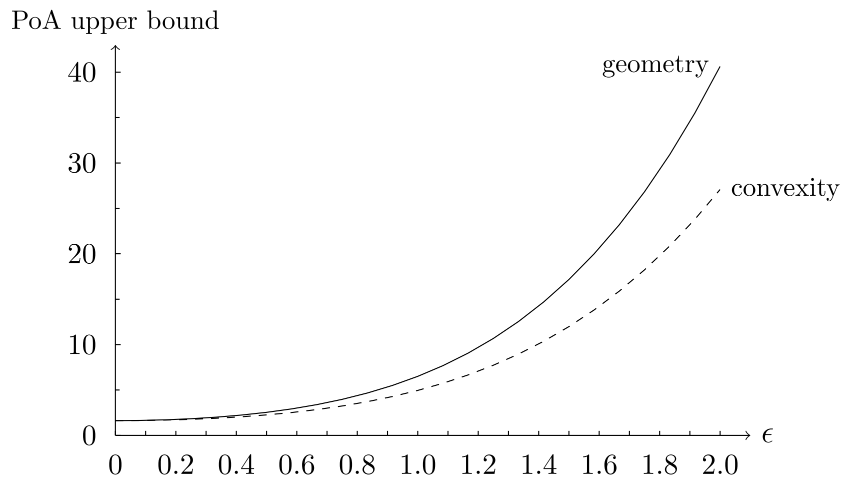


Figure 6: The two PoA upper bounds for single commodity networks with log-normal distributions and quadratic cost functions ($m = 2$)

bound in (32). In fact,

$$\begin{aligned}
&\max_{1\le j\le m}\left\{\left(1-\frac{j}{j+1}\cdot\left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1}\cdot\frac{\theta^{(j+1)}}{\theta^{(j)}}\right\}\\
&\quad\le \max_{1\le j\le m}\left(1-\frac{j}{j+1}\cdot\left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1}\cdot\max_{1\le j\le m}\frac{\theta^{(j+1)}}{\theta^{(j)}}\\
&\quad\le \max_{1\le j\le m}\left(1-\frac{j}{j+1}\left(\frac{1}{j+1}\right)^{1/j}\right)^{-1}\cdot\max_{0\le j\le m}\frac{\theta^{(j+1)}}{\theta^{(j)}}\\
&\quad= (1-m(m+1)^{-(1+m)/m})^{-1}\cdot\max_{0\le j\le m}\frac{\theta^{(j+1)}}{\theta^{(j)}},
\end{aligned}$$

where the second inequality is due to $\theta^{(j+1)}/\theta^{(j)} \ge 1$ and $\theta^{(0)} = \theta^{(1)} = 1$.

Figure 6 shows these two upper bounds for the log-normal distributions discussed in Section 3.3.1 for quadratic link cost functions ($m = 2$), in which the convexity bound is strictly better than the geometry bound. In what follows we provide an example to show that the convexity bound in (33) is actually tight.

Example 3. Consider a two-link network in Figure 7. The cost function on the upper link is a constant, $c_1(x) = \mathbb{E}[D^j]$, and that on the lower link is a polynomial function, $c_2(x) = x^j$ for a fixed j, $1 \leq j \leq m$.

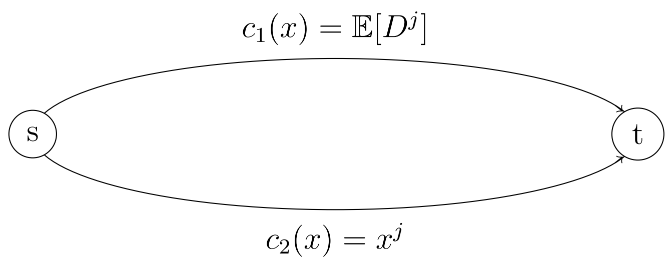


Figure 7: Two-link network with polynomial cost functions

As the expected total cost on the lower link is never greater than that on the upper link, strategy $\bar{\mathbf{p}} = (0, 1)$ is a UE-SD. We have

$$T(\bar{\mathbf{p}}) = \mathbb{E}[D^{j+1}] = \theta^{(j+1)} d^{j+1}.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*)$ be an SO-SD strategy, which minimizes the expected total cost

$$T(\mathbf{p}) = p_1 \theta^{(j)} d^{j+1} + (p_2)^{j+1}\, \theta^{(j+1)} d^{j+1}.$$

Hence

$$p_1^* = 1 - \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \quad \text{and} \quad p_2^* = \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j},$$

which lead to

$$T(\mathbf{p}^*) = \left(1 - \frac{j}{j+1} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \right) \theta^{(j)} d^{j+1}.$$

Thus, for this instance,

$$\text{PoA} \geq \left(1 - \frac{j}{(j+1)} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \right)^{-1} \cdot \frac{\theta^{(j+1)}}{\theta^{(j)}}, \tag{34}$$

which shows that the convexity bound in (33) is tight. $\square$

3.4.2. Affine cost functions

We now consider a transportation game $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_1$, i.e., all link cost functions belong to $\mathcal{C}_1$, the set of all non-zero affine functions with non-negative coefficients:

$$c_e(x) = a_e x + b_e, \text{ where } a_e, b_e \geq 0 \text{ and } a_e + b_e > 0, \ \forall\, e \in E.$$

Assume that $\{D_i : i \in I\}$ have positive means and finite second moments. From Lemma 14, we can choose $h_2 = 1 + \overline{\epsilon}^2$ and $l_2 = 1 + \underline{\epsilon}^2/n$. Hence Assumption 3 is satisfied.

Condition (24) is satisfied since $h_1 = l_1 = 1$. With the chosen values of h_2 and l_2, the geometry bound in Theorem 7 can be simplified:

$$\mathrm{PoA}(G, \mathbf{D}, \mathbf{c}) \leq \frac{4}{3}(1 + \overline{\epsilon}^2). \tag{35}$$

Condition (25) reduces to $\left(1 + \underline{\epsilon}^2/n\right)^{-1} < 4$, which is always satisfied. The convexity bound in Theorem 8 is simplified as follows:

$$\begin{aligned} \mathrm{PoA}(G, \mathbf{D}, \mathbf{c}) &\leq \left(\frac{1}{1 + \overline{\epsilon}^2} - \frac{1}{2} \cdot \frac{1}{1 + \overline{\epsilon}^2} \cdot \frac{1}{2(1 + \underline{\epsilon}^2/n)} \right)^{-1} \\ &= \frac{4}{3}\left(1 + \overline{\epsilon}^2\right) \left(\frac{1 + \underline{\epsilon}^2/n}{1 + (4/3) \cdot \underline{\epsilon}^2/n} \right). \end{aligned} \tag{36}$$

It is apparent that the convexity bound in (36) is better than the geometry bound in (35). In addition, the bound in (36) indicates that it is network dependent in general.

Figure 8 shows the two bounds with different values of n for affine cost functions when $\underline{\epsilon} = \overline{\epsilon} = \epsilon$. We can see that the geometry bound is the limiting convexity bound when n tends to infinity.

We conclude our consideration of the special case of affine cost functions by noting that the convexity bound in (36) is actually tight when $n = 1$ as can be easily verified by direct computation with the special case $j = 1$ of Example 3.

4. Discussion and concluding remarks

In this study, we have presented a general model for transportation games, taking variation of the traffic demands into account. The notion of mixed

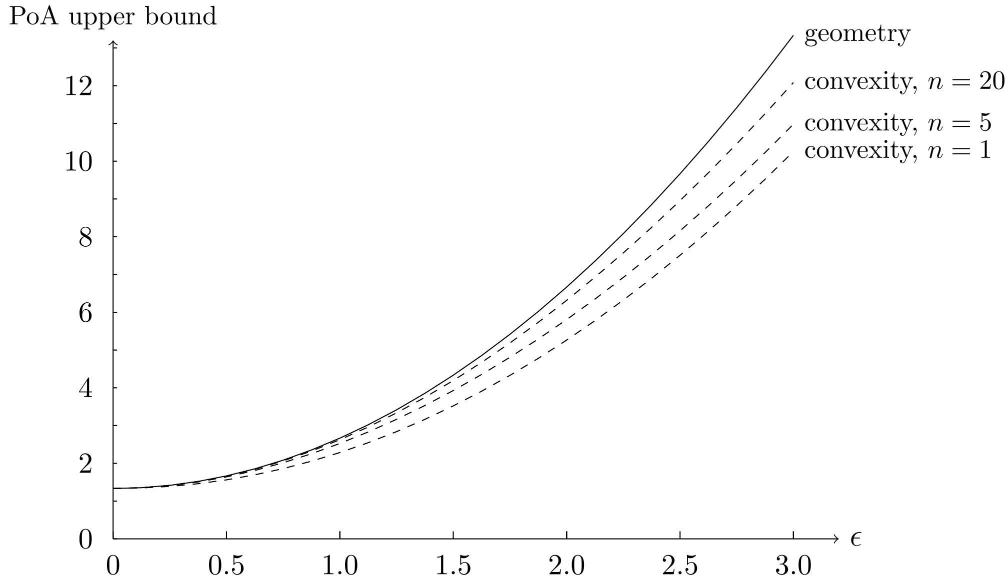


Figure 8: The two PoA upper bounds with affine cost functions for different values of n

strategies is adopted in our models of user equilibrium and system optimum with stochastic demands to describe travelers' and coordinator's behaviors in a stochastic environment. The user equilibrium condition is reformulated as a VI problem for general cost functions. Unlike in the deterministic models, multiple equilibria can exist in our stochastic model.

We have extended two bounding techniques from Roughgarden and Tardos (2004) and Correa et al. (2008) and established two different upper bounds on the PoA for our stochastic model, namely, the convexity and geometry bound, respectively. Unlike in the deterministic models, the two upper bounds are applicable in general only under certain conditions. In our opinion, these conditions are technical limitations of the bounding techniques we have used. We believe that in general, if these conditions are not met, the PoA can still be bounded even though we are not able to prove it at present.

We have derived two specific PoA upper bounds for the class of polynomial link cost functions with positive-valued demand distributions as well as normal distributions, which are commonly used to approximate demand distributions. Numerical results show that in general the geometry bound is better and more applicable than the convexity bound. However, for single-commodity networks, the convexity bound is tight (and hence better than the geometry bound). Similarly, when only affine link cost functions are considered, the convexity bound is again better than the geometry one. One possible explanation is that the convexity method relies on convex under-approximation, which is less effective with highly non-linear (convex) cost functions. On the other hand, it seems this method is more effective if we have a good approximation of $\mathbb{E}\left[c_e(V_e)V_e\right]$, which is indeed case when the network is of a single commodity (and hence no approximation is needed) or when the demand variation is very small. In general, both upper bounds can be improved if we can have better approximation of $\mathbb{E}\left[c_e(V_e)V_e\right]$ for some specific types of demand distributions. For the class of polynomial link cost functions, better approximation of $\mathbb{E}\left[c_e(V_e)V_e\right]$ means larger l_j and smaller h_j for $j \in \mathbb{Z}^+$ in Assumption 3.

All upper bounds obtained for our stochastic model under various specific settings generalize the corresponding upper bounds obtained by Roughgarden and Tardos (2002, 2004) and Correa et al. (2008) for deterministic demands. The stochasticity of demands plays an important role in the formulation of these upper bounds in our model. Unlike in the deterministic models, these upper bounds can goes to infinity when the demand variation tends to infinity. It shows that travelers' selfish routing can cause serious system degradation with stochastic demands. In addition, while the upper bounds in the deterministic models are network independent, those in our stochastic model can be network dependent (through the number n of O-D pairs whose paths share a particular link in the network).

Finally, all upper bounds obtained in this paper can be reached under the assumption of separable cost functions. Extension of this study with non-separable cost functions is a future research direction. In addition, it would be interesting to see if novel methods can be found to bound the PoA when necessary conditions proposed in this study are relaxed.

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