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A THEORETICAL AND COMPUTATIONAL STUDY OF THE  
ION BEAM EXPLOSIVE INSTABILITY

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Ph.D.  
1974

TO MY WIFE

## MEMORANDUM

This dissertation is submitted to the University of Warwick in support of my application for admission to the degree of Doctor of Philosophy. It contains an account of my own work performed at the Department of Physics at the University of Warwick during the period October 1969 to September 1972 under the general supervision of Dr G Rowlands. It also contains an account of work carried out using the computing facilities at the U.K.A.E.A. Culham Laboratory under the supervision of Mr B McNamara during the same period.

No part of this dissertation has been used previously in a degree thesis submitted to this or any other University. The work described in this thesis is the result of my own independent research except where specifically acknowledged in the text.

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Raymond Peter Hand  
October 1973

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## ABSTRACT

This thesis contains an account of a theoretical study of the non-linear interaction of three waves in a plasma. Kinetic equations are derived, which describe the interaction for a plasma obeying the fluid equations of magnetohydrodynamics, and for a plasma obeying the Vlasov-Poisson equations. In the magnetohydrodynamic case we find that energy is transferred back and forth between the three waves in a periodic manner.

We study a Vlasov plasma with a diffuse ion beam propagating through it, and find that one wave, a Landau unstable ion sound wave propagating on the beam, behaves as though its energy were negative. The kinetic equations predict that as it transfers energy to two other waves, it grows in amplitude, and that all three waves reach an infinite amplitude in a finite time. This phenomenon is known as an 'explosive instability', and the mechanisms governing the interaction are analysed in order to find stabilisation processes.

We show that the unstable oscillations cause a heating of the beam ions, a reduction in beam velocity and a distortion of the beam distribution function, all of which contribute to stabilising the interaction. We find that the heating is the dominant process, causing heavy Landau damping of the unstable waves.

Two computer programs are described which were used to simulate the beamed plasma configuration in order to test the theory. The results show that heating is the dominant beam effect, and that wave growth saturates as the heating becomes significant. Some of the computational techniques developed during the implementation of the programs are described, and a critical survey is made of the programs themselves.

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## CHAPTER 1

### INTRODUCTION

#### 1.1 Foreword

A difficulty in studying wave propagation in plasmas is that the equations describing the evolution of a plasma are nonlinear, and in general cannot be solved exactly. The most common simplification made, is to assume that the plasma is close to a steady state, and hence that the dependent plasma variables can be expressed in terms of a time independent part and a (much smaller) time dependent part. By using such an expansion in the plasma equations, and by neglecting products of small terms, one gets linear equations for the time dependent quantities, which can be solved to yield the dispersive characteristics of the plasma. The study of such equations is known as the 'linear plasma theory', and a central feature of the theory is that arbitrary (small) perturbations about a steady state are expressible as a superposition of eigenmodes which evolve independently of one another.

Over a period of time, much longer than the typical oscillation period for an eigenmode however, the effects of the neglected nonlinear terms could accumulate and modify the linear solution significantly. The waves therefore, can no longer be considered to evolve independently, but rather to interact weakly with one another.

For small perturbations, the dominant nonlinear term will be that formed from the product of two of the small time dependent quantities. The inclusion of such a term yields plasma equations which describe the modification of the 'linear' wave due to the presence of two other waves in the plasma. This phenomenon is known as a 'three wave interaction' and is responsible for many interesting and unusual plasma phenomena which cannot be described by the linear theory.

In this thesis, we study the interaction of three waves in an electrically neutral plasma. As Kadomtsev, Mikhailovskii and Timofeev (1965) show, it is possible for one of the waves participating in the interaction to behave as though its energy were negative. The consequence of a 'negative energy' wave interacting with two 'normal' waves is that all three wave amplitudes can grow to infinity in a finite time. This phenomenon is known as an 'explosive instability', and clearly describes a result which is unphysical. We therefore study the mechanisms governing three wave interactions, and also the physical properties of negative energy waves, in order to develop a theory describing the stabilisation of the interaction.

## 1.2 History of Work in the Field

The phenomenon of two waves 'beating' together to drive a third is well known in the theory of harmonic oscillators. The mechanism behind the phenomenon is a nonlinear interaction between the waves, and the condition that the coupling is significant is

$$\omega_m + \omega_n = \omega_l, \quad (1.2.1)$$

where the  $\omega$ 's are the frequencies of the three waves  $l$ ,  $m$  and  $n$ . If this condition is not met, a resonance situation does not occur, and the wave  $l$  fails to oscillate in sympathy with the driving waves. In the case of plasma waves, the situation is more complex. Kadomtsev (1965) shows that for plasma waves to interact, they must also satisfy the wavevector condition

$$k_m + k_n = k_l. \quad (1.2.2)$$

Much of the early work on the three wave interaction was done by Kadomtsev, and is described in his book (1965). He derives a general form for the kinetic equation describing the exchange of energy between three coupled waves. It takes the form

$$\frac{\partial \phi_\ell}{\partial t} = \sum_{\substack{k_m, k_n \\ k_m + k_n = k_\ell}} \Gamma_{\ell mn} \phi_m \phi_n, \quad (1.2.3)$$

where the  $\phi$ 's are electrostatic potentials, and  $\Gamma_{\ell mn}$  is a measure of the strength of the interaction between the three waves  $\ell$ ,  $m$  and  $n$ .

The usual approximation used in deriving the kinetic equation is to assume that dissipative effects are negligible. Malfliet (1969) however, has derived a kinetic equation which takes account of unstable or damped waves propagating, and as such is more difficult to analyse and solve. The important point in Malfliet's paper is that the effects of wave growth and damping can be superimposed on a kinetic equation derived assuming zero dissipation. For example, the effects of wave  $\ell$  having a Landau damping rate  $\gamma_\ell$  can be included by simply adding a term  $-\gamma_\ell \phi_\ell$  to the right hand side of equation (1.2.3). More will be said on the subject of dissipative effects later in this Section.

The kinetic three wave interaction equation has been solved for a number of situations. In particular, Galeev and Oraevskii (1963) show that the interaction can cause the decay of large amplitude Alfvén waves. Since Alfvén waves represent exact solutions to the nonlinear plasma equations, it would be expected that a large amplitude Alfvén wave could propagate indefinitely. Galeev and Oraevskii however, demonstrate how a nonlinear coupling between the Alfvén wave and a spectrum of pairs of low amplitude ion acoustic waves results in the one way transfer of energy from the Alfvén wave to the background wave field. A similar result, concerning the decay of high frequency plasma waves has been demonstrated by Bakai (1970). His result applies to a plasma in which low and high frequency waves are propagating.

The first mention of the possibility of 'negative energy' waves participating in a three wave interaction was by Kadomtsev, Mikhailovskii

and Timofeev (1965). They show that for a plasma to support 'negative energy' waves, it must be well away from thermodynamic equilibrium. They describe some of the anomalous properties of such waves, and make the proposition that an 'explosive instability' can occur if energy is transferred from the negative energy wave to two other positive energy waves. The explosive instability is examined in some detail by Coppi, Rosenbluth and Sudan (1969) for the case of a plasma with a loss cone distribution in a strong magnetic field. They calculate the strength of the coupling between the three waves by balancing energy terms derived from the single particle scattering of waves.

Fukai, Krishan and Harris (1969) also examine the explosive instability for a plasma with a loss cone distribution. They calculate the coupling strength by using a Hamiltonian formalism for the derivation of the kinetic equation. They further show how the four wave kinetic equation describes a stabilisation process for the instability. The four wave equation has solutions which predict that the waves suffer an amplitude dependent frequency shift, and hence the frequency condition (1.2.1) for the interaction would be violated. The four wave interaction however, is not the unique stabilising process, and Dum and Sudan (1969) consider the perturbations on the particle orbits caused by the large amplitude waves, as a possible stabilising process. The assertion made is that large amplitude oscillations can cause a significant enough change to the distribution function, and hence also to the dispersion relation, for the negative energy waves to cease to satisfy the requirements for an explosive instability.

As we said earlier, the problem of an explosive instability in a region of dissipation, is more difficult to solve. Not only is the kinetic equation of Malfliet (1969) more complicated due to adding effects such as Landau damping and growth, but the problem of analysing multiple triplets

of waves appears. As Jarmen, Stenflo, Wilhelmsson and Engelmann (1969) show, while phase changes due to either dissipation or the four wave process of Fukai et al can stabilise explosive instabilities, they can also de-stabilise situations which are otherwise stable.

The experimental evidence of the explosive instability is very limited. Although the physical phenomenon of negative energy waves is known in the field of parametric amplifiers (cf. Sturrock, 1960), its observation in plasmas is speculative. Dum and Sudan (1969) comment that the bursts of radiation and associated particle ejection observed in mirror confined plasmas, suggest that an explosive instability is reaching saturation point. The suggestion is only qualitative, but is accepted by many authors as an explanation for the observations.

The limited experimental knowledge of the explosive instability makes it an ideal subject to simulate on a computer. Little or no work has previously been done on the simulation of an explosively unstable plasma, although numerical solutions to the kinetic equation have been found for a large number of initial conditions: see for example Wilhelmsson, Stenflo and Engelmann (1970) and Stenflo, Wilhelmsson and Weiland (1970). This is probably because simulation programs have only recently become sophisticated enough to handle such problems.

The restrictions of speed and core size of the second generation computers forced onto the computational physicist, the need to write highly optimised programs which had to be tailored to a given class of plasma problems. A significant improvement was made however by the use of the 'Fast Fourier Transform' algorithm of Gentleman and Sande (1966) for solving Poisson's equation for the electrostatic potential. This algorithm gives an exact solution to the discrete form of Poisson's equation encountered in simulation programs, as well as providing the physically useful Fourier

potential coefficients in the process. Some of the programming techniques and high speed data dumping and restoring methods are discussed in detail by Boris and Roberts (1969A).

With the advent of the large third generation computers and high speed data transfer channels, it is now possible to implement two and three dimensional simulation programs which are relatively free from the restriction of being tailored to specific needs. The increase in speed of these machines over the previous generation opens up the possibility of employing extremely sophisticated and accurate numerical analysis techniques, as well as simulating a larger number of plasma particles. The larger and cheaper memory available on present day computers also makes the task of simulating more particles easier.

### 1.3 Scope and Plan of Thesis

A difficulty in studying the propagation of plasma waves is that the equations which describe the evolution of a plasma are nonlinear. We can treat the equations by using a perturbation theory, and we find that it is necessary to go to second order in the theory in order to study wave interactions, since the linear theory only describes independent modes of oscillation. If we attempt to do this however using a standard perturbation theory, we find that the second order solutions contains secular terms; that is, terms which are unbounded in time. We can however, deal with this problem of secularity by employing a multiple timescale perturbation theory described by Frieman (1963), which has its origins in the field of nonlinear mechanics (cf. Bogoliubov and Krylov, 1947).

In Chapter 2 we demonstrate this technique for the case of a plasma which obeys the fluid equations of magnetohydrodynamics (cf Montgomery and Tidman (1964), Chapter 13). We examine a plasma configuration in which sound waves and circularly polarised Alfvén waves propagate. We show that

nonlinear kinetic equations can be derived, which describe the exchange of energy between three such waves. The kinetic equations can be solved to give solutions describing a periodic exchange of energy back and forth between the three waves. We further show how the multiple timescale technique can be used to derive kinetic equations describing four wave and higher order interactions.

In Chapter 3, the same technique is applied to the more detailed Vlasov-Poisson plasma equations, and again kinetic equations are derived which describe the interaction between three waves. In this case however, for a certain class of waves, the equations predict that the three waves will all grow to an infinite amplitude in a finite time. This phenomenon, the 'explosive instability', was first investigated by Kadomtsev, Mikhailovski and Timofeev (1965). A necessary condition that three waves can interact in this unstable manner, is that either one or two of the waves must have what is called 'negative energy'.

The concept of 'negative energy' waves is well known in the theory of parametric amplifiers (cf Sturrock, 1960), where it is used for studying the amplification of waves propagating in dispersive media. In Chapter 3 we also examine the physical characteristics of 'negative energy' waves propagating in a plasma through which a diffuse ion beam is travelling. We show that such a beamed plasma can support three waves which satisfy the criteria for an explosive instability. An expression is derived for the time taken by the waves to reach infinite amplitudes.

The infinite solution however is unphysical and violates the approximation in the multiple timescale perturbation theory, that the waves are of small amplitude. The examination of the physical meaning of 'negative energy' waves however, gives us insight into the mechanisms by which the instability must saturate. In Chapter 4 we analyse the behaviour of the

plasma particles when a 'negative energy' wave propagates, and in particular when it interacts with two other positive energy waves. For the plasma configuration we study, we find three stabilisation mechanisms, the most dominant of which is ion heating. This causes the interacting waves to be heavily Landau damped.

In Chapter 5, we study the numerical and physical problems associated with simulating the explosively unstable beamed plasma on a computer. We describe the research and developments carried out on two computer programs in an attempt to simulate the instability. We discuss methods of avoiding numerical 'noise' caused by machine rounding errors and approximations in the numerical analysis. A method is given for calculating initial particle co-ordinates such that the plasma has spatial uniformity and low noise properties. We also give a condition for avoiding spurious particle correlations which can arise in a numerical plasma of such uniformity. The results of the computer simulations are described in Section 5.4, and a comparison is made with the theory, specifically with respect to the proposed saturation mechanisms. The results indicate that Landau damping stabilises the instability.

#### 1.4 Summary of Results

In Chapter 2 we derive kinetic equations for the interaction of two circularly polarised Alfvén waves with an ion acoustic wave. We use the multiple timescale perturbation theory, and the results are the same as given by Galeev and Sagdeev (1969) for the same case. The example is given, not only to demonstrate the multiple timescale nature of the problem, but also to display the technique itself, and in particular we find the frequency and wavevector conditions (1.2.1 and 2) appearing intrinsically in the formal mathematics. Sagdeev and Galeev use a more phenomenological approach to solve the problem. From the kinetic equations, an energy theorem can be derived which has an interesting parallel in the case of positive and 'negative energy' waves interacting.



In Chapter 2 we apply the multiple timescale method to perturbations in a neutral plasma in which the ions are split into two distributions. The majority form a Maxwellian around  $v = 0$ , and the rest form a diffuse beam. We find that a three wave interaction can occur between two ion sound waves propagating on the beam, and an ion sound wave propagating in the 'stationary' ions. One of the beam modes has 'negative energy'. In deriving the kinetic equations, we again find that (1.2.1 and 2) form an intrinsic part of the formal mathematics. We solve the kinetic equations for the three waves and find that their amplitudes grow in a similar manner to the trigonometric tangent function. We show that the function becomes infinite in a time which is inversely proportional to the wave amplitudes at time  $t = 0$ .

The physical characteristics of the waves and medium are analysed, and we find that the ion beam can support a 'negative energy' wave ( $\ell$ ) providing

$$U_b > \omega_\ell / k_\ell,$$

$$\text{and} \quad \left. \frac{\partial f_{0i}}{\partial v} \right|_{v = \omega_\ell / k_\ell} > 0,$$

where  $U_b$  is the average beam velocity, and  $f_{0i}$  is the steady state value of the total ion distribution function. We find that beam particle kinetic energy is transferred to the 'negative energy' wave at its phase velocity in the same manner as is found in inverse Landau damping (cf. Drummond, 1965). The wave loses energy to the two positive energy waves and yet still gains energy from the particles. In this way, all three waves grow, the driving force being the beam kinetic energy.

In Chapter 4 we analyse the behaviour of the ion beam particles when the beam modes are propagating, in order to find a process which might limit the explosive growth of the waves. By studying the behaviour of the

beam ions, we propose three mechanisms which may cause the saturation of the wave growth. The first is that the average beam velocity decreases. The beam mode frequencies are proportional to the beam velocity, and a reduction in beam velocity therefore causes the resonance condition (1.2.1) to be violated. Secondly, the large amplitude oscillations cause the beam ions to heat up and hence cause the waves to be heavily Landau damped. Finally, a distortion of the beam distribution function takes place. This results in severe Landau damping of the positive energy beam wave and a reduction in the Landau growth rate of the negative energy beam wave. The relative importance of the three processes depends on the plasma parameters, but in the case where we use the parameter set chosen for the simulation programs, we find that the heating of the beam ions is the dominant effect. The heating is enough to cause Landau damping of the waves in one or two periods of oscillation.

In Chapter 5, we outline the work done on two plasma simulation programs, GALAXY and NOVA, and give the results of simulations of the beamed plasma configuration. We show that the instability can be simulated, even using the GALAXY program which is prone to a high level of numerical 'noise'. The GALAXY results predict a sharp decrease in the ion beam velocity as the waves grow, but disagree quantitatively with the theory in that the deceleration is much more rapid than the theory predicts. Unfortunately, the high noise level in GALAXY makes a detailed analysis of the beam valueless.

We propose a method for simulating a comparatively noise free plasma by initially placing the particles on closed orbits in a phase space which is spatially periodic. In this way, particles remain on pre-determined trajectories, chosen to minimise fluctuations in space charge. When selected waves are perturbed, their evolution is clearly measurable against

the background spectrum of waves. The method is used in the NOVA program, and gives a background level of wave amplitudes which are  $10^{-6}$  of those found in GALAXY. We also show that this initial choice of particle co-ordinates causes no spurious binary correlation effects, even though the spatial configuration is lattice like.

The results of the NOVA program show that although the beam ions decelerate slightly, they heat up significantly, as do the stationary ions. The results indicate that beam heating is responsible for the saturation of the instability.

## CHAPTER 2

### THE THREE WAVE INTERACTION

#### 2.1 Introduction

We are going to study the behaviour of a plasma for states which are initially close to a steady state. As Bernstein and Trehan (1960) have demonstrated, being close to a steady state implies the existence of some small dimensionless parameter in terms of which one can expand the time dependent quantities in the equations governing the plasma. The expanded equations to lowest order in the small parameter are just those which govern the steady state. To next order one obtains a set of linear equations with constant coefficients, the theory of which is known as linear plasma theory. The literature on linear plasma theory is extensive, and Bernstein and Trehan's comprehensive review article (1960) makes reference to many of the original and most useful publications. A central feature of the theory is that arbitrary (small) perturbations about a steady state are expressible as a superposition of eigenmodes which evolve independently of one another. We can however examine the interaction between the eigenmodes by considering the nonlinear terms which are neglected in the linear theory cf. Coppi, Rosenbluth and Sudan (1969).

Frieman (1963) demonstrates the well known result that to do this by simply equating terms of second order in the expanded plasma equations gives solutions which are secular. That is, second order perturbed quantities which are proportional to time, and hence diverge for large times, even for cases where we know the result remains finite. Frieman further shows that one may avoid the occurrence of such non-physical solutions by taking advantage of the fact that for small perturbations, the effect of the nonlinear terms is only important over a timescale very much longer than the

typical oscillation period of the eigenmodes. The multiple timescale nature of the problem arises naturally since (if we call the dimensionless expansion parameter  $\epsilon$ ), the linear terms in the expanded equations are proportional to  $\epsilon$  whereas the nonlinear terms are proportional to  $\epsilon^2$  and higher. Consideration of the time dependence of the lowest order perturbed quantities  $q$  (say) due to nonlinear terms (of order  $q^n$  say) shows that they will vary on a timescale of order  $1/\epsilon^{n-1}$ . This being the case, we treat the problem using a multiple timescale perturbation theory employed by Frieman, which has a built in formalism for removing secular solutions.

In this Chapter, we demonstrate the technique in the case of a plasma which obeys the fluid equations of magnetohydrodynamics (cf. Montgomery and Tidman (1964), Chapter 13). This particular example has been treated previously by Galeev and Oraevskii (1963) using a different mathematical technique. Their method involves time averaging the expanded nonlinear equations in order to obtain kinetic equations governing the long timescale effects of the nonlinear terms on the first order quantities. These authors show that the resulting kinetic equations describe the transfer of energy back and forth between three waves propagating in the plasma. This process is known as the 'three wave interaction' and is a feature of great importance to the rest of the thesis.

The multiple timescale (MTS) technique is ideally suited to the study of the three wave interaction, due to the linear and nonlinear phenomena occurring on widely separated timescales, and it can be extended with very little difficulty to deal with multi-wave interactions. Moreover, the strength of the coupling between the waves appears explicitly as the formal mathematics evolves, whereas in methods such as the 'detailed balance of microscopic energy' used by Coppi, Rosenbluth and Sudan (1969), and the Hamiltonian formalism used by Fukai, Frisban and Harris (1969), the coupling

strength has to be calculated independently of the derivation of the kinetic equation.

The example is given however, not only to demonstrate the perturbation technique, but also to highlight the nature of the energy transfer. We will see that to assume a fluid description prevents the detailed analysis of particle effects, which is important in a certain category of three wave interactions; namely those associated with negative energy waves. The physical meaning of negative energy waves, and the particle nature of their interaction with other waves is discussed in Chapter 3.

## 2.2 The Multiple Timescale Approach

For a plasma close to a steady state, we write the time dependent plasma variables as a power series in the small parameter  $\epsilon$ ; namely

$$q_n = q_{n0} + \epsilon q_{n1} + \epsilon^2 q_{n2} \text{ etc.},$$

where  $q_{n0}$  is the steady state value of the quantity  $q_n$ , and  $\epsilon$  is a measure of the strength of the perturbation. Now we expect different order non-linear terms to describe different physical processes. Thus we expect the plasma system to behave differently on the various timescales, unity,  $1/\epsilon$ ,  $1/\epsilon^2$  etc., each of which corresponds to different order nonlinear terms becoming important. The method described here exploits this, in that we assume that solutions to the plasma equations can be found with a time dependence of the form

$$q_n = q_{n0}(t, \epsilon t_1, \epsilon^2 t_2 \dots) + \epsilon q_{n1}(t, \epsilon t_1, \epsilon^2 t_2 \dots) \text{ etc.}$$

The variables  $t_1$ ,  $t_2$  etc. are related to the real time  $t$  by the relation

$$\frac{dt_n}{dt} = 1; \quad n = 1, 2, \dots \text{ etc.} \quad (2.2.1)$$

Frieman (1963) shows that the freedom in the solutions of (2.2.1) due to the

choice of initial conditions on the plasma equations allows us to treat  $t$ ,  $\epsilon t_1$ ,  $\epsilon^2 t_2$  etc. as independent variables. Thus the time derivative  $(\partial/\partial t)$  can be formally expanded in powers of  $\epsilon$  as

$$\frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \epsilon t_1} + \epsilon^2 \frac{\partial}{\partial \epsilon^2 t_2} \text{ etc.} \quad (2.2.2)$$

Throughout this thesis we shall use the notation that  $t$ 's subscripts are omitted. As such,  $\epsilon$  is used as a 'book-keeping' parameter for the ordering of not only the time dependent plasma quantities but also the time-scales themselves.

The magnetohydrodynamic (MHD) equations for the velocity  $\underline{v}$  of the plasma 'fluid', the charge density  $n$ , and the magnetic field  $\underline{H}$  are

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = \frac{1}{n} \left( -\nabla(p + H^2/8\pi) + (\underline{H}/4\pi \cdot \nabla) \underline{H} \right),$$

$$\frac{\partial \underline{H}}{\partial t} + (\underline{v} \cdot \nabla) \underline{H} + \underline{H}(\nabla \cdot \underline{v}) - (\underline{H} \cdot \nabla) \underline{v} = 0, \text{ and}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0,$$

where  $p$  is the fluid pressure  $= nc^2$  and  $c$  is the sound speed. The first equation represents the conservation of momentum, the third represents the conservation of charge, and the second derives from Maxwell's equations. The steady state plasma is considered to have a uniform density  $n_0$ , zero fluid velocity, and a uniform magnetic field  $H_0$  in the  $z$  direction. Following the example of Sagdeev and Galeev (1969) we examine perturbations about this steady state in the form of circularly polarised Alfvén waves propagating in the  $z$  direction, and sound waves also propagating along  $z$ .

We write an expansion in the form dictated by (2.2.1) and (2.2.2), namely

$$\underline{H} = \underline{H}_0 + \epsilon H_{p1}(z, t, \epsilon t, \epsilon^2 t, \dots) + \epsilon^2 H_{p2}(z, t, \epsilon t, \epsilon^2 t, \dots) \text{ etc.,}$$

$$n = n_0 + \epsilon n_{z1}(z, t, \epsilon t, \epsilon^2 t, \dots) + \epsilon^2 n_{z2}(z, t, \epsilon t, \epsilon^2 t, \dots) \text{ etc.,}$$

$$\underline{v} = \epsilon v_{p1}(z, t, \epsilon t, \epsilon^2 t, \dots) + \epsilon^2 v_{p2}(z, t, \epsilon t, \epsilon^2 t, \dots) \text{ etc.} +$$

$$\epsilon v_{z1}(z, t, \epsilon t, \epsilon^2 t, \dots) + \epsilon^2 v_{z2}(z, t, \epsilon t, \epsilon^2 t, \dots) \text{ etc.,}$$

where the subscript z refers to perturbations in the z direction, which are associated with the sound waves, and subscript p refers to perturbations perpendicular to the z direction, which are associated with the circularly polarised Alfvén waves.

Substituting these expansions into the expanded MHD equations, we can separate out the z and p components of the momentum equation, and equating terms proportional to  $\epsilon$  gives the four equations

$$\frac{\partial v_{z1}}{\partial t} + \frac{c^2}{n_0} \frac{\partial n_{z1}}{\partial z} = 0, \quad (2.2.3)$$

$$\frac{\partial n_{z1}}{\partial t} + n_0 \frac{\partial v_{z1}}{\partial z} = 0, \quad (2.2.4)$$

$$\frac{\partial v_{p1}}{\partial t} - \frac{H_0}{4\pi n_0} \frac{\partial H_{p1}}{\partial z} = 0, \quad (2.2.5)$$

$$\frac{\partial H_{p1}}{\partial t} - H_0 \frac{\partial v_{p1}}{\partial z} = 0. \quad (2.2.6)$$

These are just the linear MHD equations for the configuration, and we assume solutions of the form

$$v_{z1} = \sum_{\ell} v_{z1}^{\ell} \exp i (k_{\ell} z - \omega_{\ell} t),$$

and similar summations in  $\exp i (k_{\ell} z - \omega_{\ell} t)$  for  $n_{z1}$ ,  $v_{p1}$  and  $H_{p1}$ , where the  $k$ 's are wavevectors and the  $\omega$ 's are frequencies. Using these solutions and looking at the  $\ell$ -th Fourier component, the system of equations (2.2.3, 4, 5 and 6) reduces to the matrix equation



$$\underline{M} \cdot \underline{q} = 0 \quad (2.2.7)$$

where  $\underline{q}$  is a column vector with the four components  $v_{z1}^\ell$ ,  $n_{z1}^\ell$ ,  $v_{p1}^\ell$  and  $H_{p1}^\ell$ . The condition that non trivial solutions of (2.2.7) exist is that the determinant of  $\underline{M}$  vanishes. This gives us the dispersion relation

$$(\omega_\ell^2 - k_\ell^2 v_A^2)(\omega_\ell^2 - k_\ell^2 c^2) = 0, \quad (2.2.8)$$

where  $v_A$  is the 'Alfven velocity' given by

$$v_A^2 = H_o^2 / 4\pi n_o.$$

From the matrix equation we can also show that

$$v_{p1} = -H_o k_\ell H_{p1}^\ell / 4\pi n_o \omega_\ell, \quad (2.2.9)$$

$$\text{and} \quad v_{z1} = c^2 k_\ell n_{z1}^\ell / n_o \omega_\ell. \quad (2.2.10)$$

Equating terms in the expanded MHD equations which are proportional to  $\epsilon^2$  gives the four equations

$$\begin{aligned} \frac{\partial v_{z2}}{\partial t} + \frac{c^2}{n_o} \frac{\partial n_{z2}}{\partial z} &= -\frac{\partial v_{z1}}{\partial \epsilon t} - v_{z1} \frac{\partial v_{z1}}{\partial z} \\ &+ \frac{n_{z1}}{n_o} c^2 \frac{\partial n_{z1}}{\partial z} - \frac{H_{p1}}{4\pi n_o} \frac{\partial H_{p1}}{\partial z}, \end{aligned} \quad (2.2.11)$$

$$\frac{\partial n_{z2}}{\partial t} + n_o \frac{\partial v_{z2}}{\partial z} = -\frac{\partial n_{z1}}{\partial \epsilon t} - n_{z1} \frac{\partial v_{z1}}{\partial z} - v_{z1} \frac{\partial n_{z1}}{\partial z}, \quad (2.2.12)$$

$$\frac{\partial v_{p2}}{\partial t} - \frac{H_o}{4\pi n_o} \frac{\partial H_{p2}}{\partial z} = -\frac{\partial v_{p1}}{\partial \epsilon t} - \frac{n_{z1}}{4\pi n_o} \frac{\partial H_{p1}}{\partial z}, \quad (2.2.13)$$

$$\frac{\partial H_{p2}}{\partial t} - H_o \frac{\partial v_{p2}}{\partial z} = -\frac{\partial H_{p1}}{\partial \epsilon t} - v_{z1} \frac{\partial H_{p1}}{\partial z}. \quad (2.2.14)$$

Equations (2.2.11) and (2.2.12) represent the second order correction to the sound wave perturbation. They are similar to equations (2.2.3) and (2.2.4) respectively, except that there are nonlinear driving terms on the

right hand sides. Likewise, equations (2.2.13) and (2.2.14) represent the second order correction to the Alfvén wave perturbation, and in the same manner are similar to equations (2.2.5) and (2.2.6).

Since the nonlinear terms on the right hand sides of equations (2.2.11, 12, 13 and 14) are products of two plane wave forms, the second order corrections will take on a plane wave form, having frequencies and wavevectors which are sums and differences of those of the eigenmodes. These second order waves are often called 'virtual' or 'beat' waves, and have wavevectors which are members of the same set as the eigenmodes', since the  $k$ 's constitute a quasi-continuum. The set of their frequencies however intersects but is not identical to the set of eigenmode frequencies since a frequency  $\bar{\omega}_\ell$  (given by  $\bar{\omega}_\ell = \omega_m + \omega_n$ ) and a wavevector  $k_\ell$  (given by  $k_\ell = k_m + k_n$ ) are not necessarily related through the dispersion relation (2.2.8). So a second order wave formed from two eigenmodes beating together is not necessarily an eigenmode itself. The case when it is, is a special case and will be treated as such.

Meanwhile, therefore, we write solutions of (2.2.11, 12, 13 and 14) in the form

$$v_{z2} = \sum_{\ell} v_{z2}^{\ell} \exp i(k_{\ell} z - \bar{\omega}_{\ell} t),$$

and similarly for the other second order quantities. We substitute these solutions into (2.2.11, 12, 13 and 14) and look at the  $\ell$ -th Fourier component. From (2.2.11) and (2.2.12) we eliminate  $n_{z2}^{\ell}$ , and from (2.2.13) and (2.2.14) we eliminate  $v_{p2}^{\ell}$ . We substitute the first order quantities  $v_{p1}^{\ell}$  and  $n_{z1}^{\ell}$  from (2.2.9) and (2.2.10) respectively, thus reducing the system of equations to two. The first is for the second order correction to the sound wave, and is

$$\begin{aligned}
(\bar{\omega}_\ell^2 - k_\ell^2 c^2) v_{z2}^\ell &= -i(\bar{\omega}_\ell - \omega_\ell) \times \left\{ \frac{\partial v_{z1}^\ell}{\partial \epsilon t} e^{-i(\omega_\ell - \bar{\omega}_\ell)t} + \right. \\
&+ \sum_{k_m + k_n = k_\ell} (ik_\ell / 4\pi n_0) H_{p1}^m H_{p1}^n e^{-i(\omega_m + \omega_n - \bar{\omega}_\ell)t} \left. \right\}.
\end{aligned}
\tag{2.2.15}$$

The second is for the Alfvén wave, and is

$$\begin{aligned}
(\bar{\omega}_\ell^2 - k_\ell^2 V_A^2) H_{p2}^\ell &= -i(\bar{\omega}_\ell + V_A^2 k_\ell / \omega_\ell) \times \left\{ \frac{\partial H_{p1}^\ell}{\partial \epsilon t} e^{-i(\omega_\ell - \bar{\omega}_\ell)t} + \right. \\
&+ \sum (V_A^2 ik_\ell^2 k_n / \omega_\ell \omega_n) H_{p1}^m v_{z1}^n e^{-i(\omega_m + \omega_n - \bar{\omega}_\ell)t} \left. \right\}.
\end{aligned}
\tag{2.2.16}$$

From (2.2.15) and (2.2.16) we can write the solutions for  $v_{z2}^\ell$  and  $H_{p2}^\ell$ . We notice however that but for the term in  $\partial/\partial \epsilon t$  on the right hand side of (2.2.15) we have a secular solution for  $v_{z2}^\ell$  when  $\bar{\omega}_\ell^2 - k_\ell^2 c^2$  is zero or of order  $\epsilon$ . Similarly, but for the  $\partial/\partial \epsilon t$  term in (2.2.16)  $H_{p2}^\ell$  is secular when  $\bar{\omega}_\ell^2 - k_\ell^2 V_A^2$  is small. The multiple timescale theory however, provides us with a procedure for removing the secular solutions. That is, by equating the right hand sides of (2.2.15) and (2.2.16) to zero when  $(\bar{\omega}_\ell - \omega_\ell)$  approaches zero.

The interesting equations however arise from the condition that secular solutions for  $v_{z2}^\ell$  and  $H_{p2}^\ell$  do not exist. For  $H_{p2}^\ell$  this condition can be written as

$$\frac{\partial H_{p1}^\ell}{\partial \epsilon t} = - \sum_{k_m + k_n = k_\ell} (V_A^2 ik_\ell^2 k_n / \omega_\ell \omega_n) H_{p1}^m v_{z1}^n e^{-i\Omega}, \tag{2.2.17}$$

where  $\Omega = \omega_m + \omega_n - \omega_\ell \lesssim \epsilon$ . Equation (2.2.17) can also take on the form

$$\frac{\partial H_{p1}^m}{\partial \epsilon t} = - \sum_{k_m + k_n = k_\ell} (V_A^2 ik_m^2 k_n / \omega_m \omega_n) H_{p1}^\ell v_{z1}^{n*} e^{-i\Omega}. \tag{2.2.18}$$

The condition that secular solutions for  $v_{z2}^\ell$  do not exist can be written as

$$\frac{\partial v_{z1}^n}{\partial \epsilon t} = - \sum_{k_m + k_n = k_\ell} (i k_n / 4 \pi n_0) H_{p1}^{\ell} H_{p1}^{m*} e^{-i \Omega}, \quad (2.2.19)$$

where the \* denotes the complex conjugate.

Equations (2.2.17, 18 and 19) are kinetic equations for the slow variation ( $\epsilon t$ ) of the first order quantities. As we have said, a condition that they hold, is that  $\Omega$  is close to zero. Thus we can write

$$\begin{aligned} \omega_m + \omega_n - \omega_\ell &\sim \epsilon \text{ or less} \\ \text{and } k_m + k_n - k_\ell &\sim \epsilon \text{ or less} \end{aligned} \quad (2.2.20)$$

With these conditions, the equations describe the slow variation in amplitude of a wave  $\ell$  which is being driven by pairs of waves beating together at the original wave's frequency and wavevector. They are the kinetic equations governing the 'three wave interaction', and the conditions (2.2.20) are known as the 'resonance conditions' for the three wave interaction.

### 2.3 Solution and Discussion of the Kinetic Equations

Equations essentially equivalent to (2.2.17, 18 and 19) were derived by Sagdeev and Galeev (1969) by time averaging the nonlinear expanded MHD equations, and noting that the slow variation of the first order quantities is nonzero providing that the resonance conditions (2.2.20) are satisfied. An objection to this approach is that the three wave interaction equation and the expression for the second order plasma quantities are separated ad hoc, whereas in the MTS approach they separate as a necessary condition for avoiding secular solutions.

Sagdeev and Galeev (1969) have solved equations (2.2.17, 18 and 19) under the condition that the summation is replaced by a single product term. This assumption is valid if all other waves have negligible amplitudes or if the non negligible waves fail to satisfy the resonance conditions.

Sagdeev and Galeev solve the equations for  $\Omega$  being identically zero, in terms of symmetrised wave amplitudes  $C_\ell$  defined by

$$|C_\ell|^2 = \left\{ \frac{n_0 |v_z^\ell|^2}{2} + \frac{|H_p^\ell|^2}{4\pi} + \frac{c^2 |n_z^\ell|^2}{2n_0} \right\} / \omega_\ell.$$

The equations take on the form

$$\frac{i \partial C_\ell}{\partial \epsilon t} = V_1 C_m C_n, \quad (2.3.1)$$

$$\frac{i \partial C_m}{\partial \epsilon t} = V_2 C_\ell C_n^*, \quad (2.3.2)$$

$$\frac{i \partial C_n}{\partial \epsilon t} = V_3 C_\ell C_m^*, \quad (2.3.3)$$

where  $|V_1| = |V_2| = |V_3| = (|\omega_\ell \omega_m \omega_n| / n_0 c^2)^{1/2}$ .

These equations permit two classes of wave interaction illustrated in Figures 2.1(a) and (b). That is, energy transfer from mode  $\ell$  to modes  $m$  and  $n$ , and energy transfer from modes  $m$  and  $n$  to mode  $\ell$ .

As stated in Section 2.2 we now drop the  $\epsilon$ 's in (2.3.1, 2 and 3), and treat the  $C$ 's as 'small'. For the case  $C_\ell = 0$  and  $C_m \gg C_n$  at time  $t = 0$ , the equations can be solved in terms of sines and cosines to give a result of the form shown in Figure 2.2, displaying the energy transfer back and forth between the three waves. Furthermore Sagdeev and Galeev (1969) show that the  $|C|^2$  can be interpreted as the number of 'quasi particles' or 'quanta'  $N$  in a mode, since the  $N$ 's are defined by

$$N_\ell = \omega_\ell^{-1} W_\ell,$$

where the  $W$ 's are the mode energies. From (2.2.1, 2 and 3), using the  $N$ 's we can derive

$$\frac{\partial N_\ell}{\partial t} + \frac{\partial N_m}{\partial t} = 0, \quad (2.2.4)$$

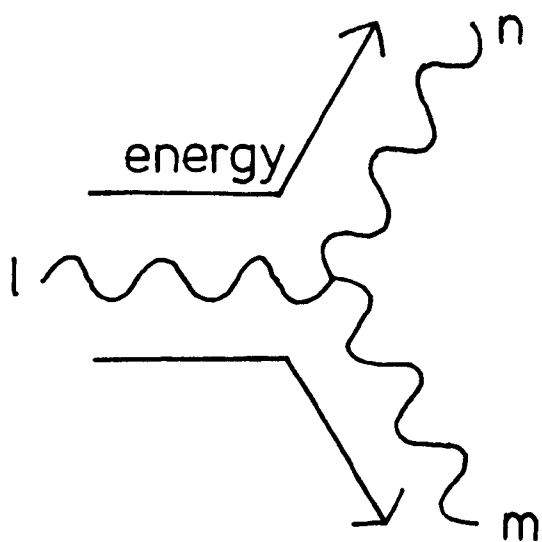


FIGURE 2.1(a)

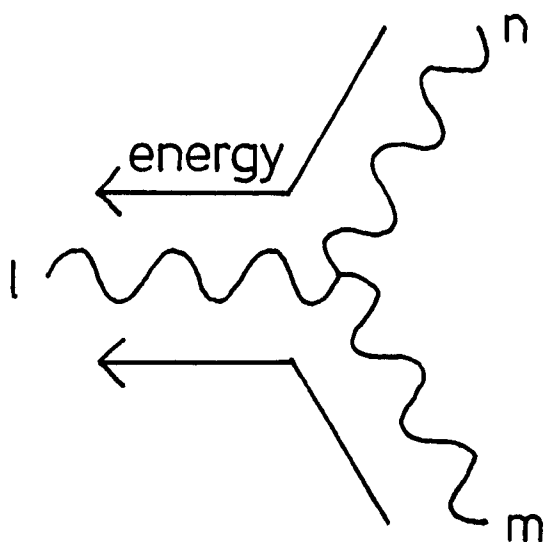


FIGURE 2.1(b)

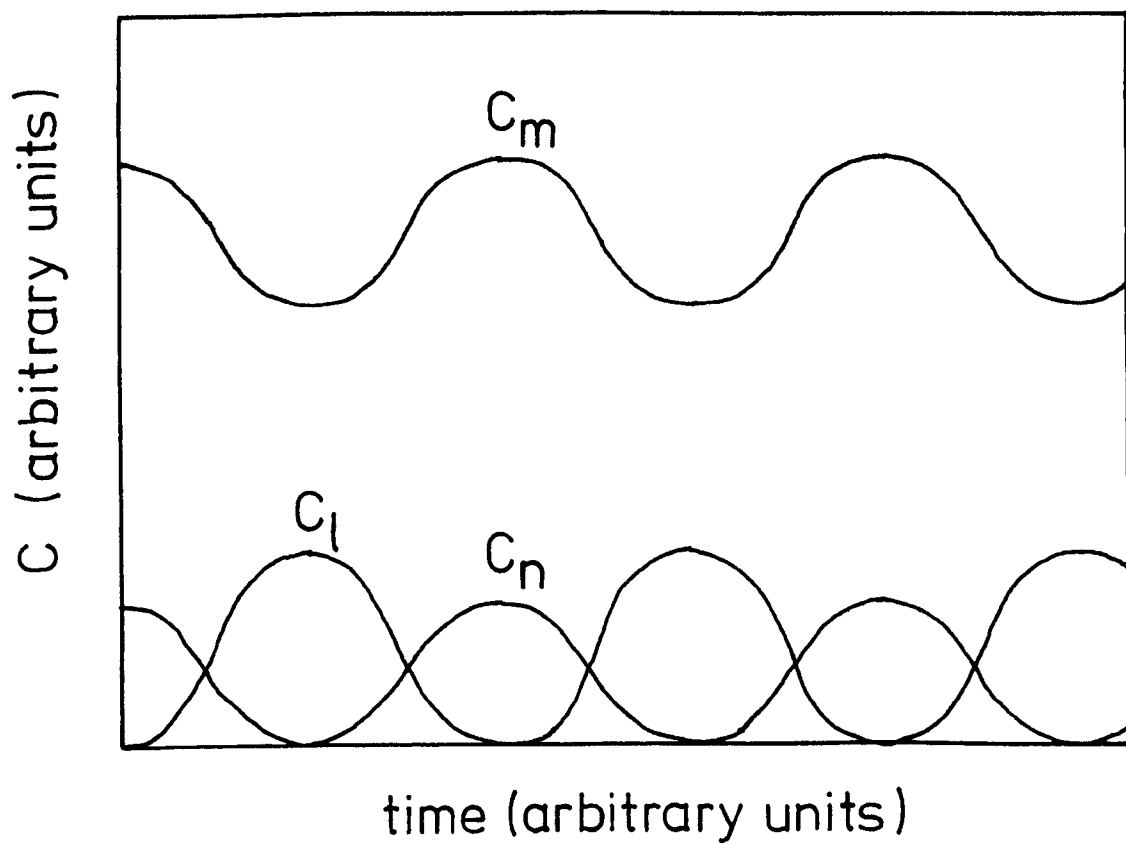


FIGURE 2.2

$$\frac{\partial N_{\ell}}{\partial t} + \frac{\partial N_n}{\partial t} = 0, \quad (2.2.5)$$

$$\frac{\partial N_m}{\partial t} = \frac{\partial N_n}{\partial t} = 0. \quad (2.2.6)$$

The relations (2.2.4, 5 and 6) are called the 'Manley-Rowe' relations and are well known in the theory of parametric amplifiers. They tell us that if a 'quantum' of energy is gained (or lost) by mode  $\ell$ , then one will be lost (or gained respectively) by both mode  $m$  and  $n$ .

The case we have treated is for a plasma in which only three waves propagate. In reality, there will be many pairs  $m$  and  $n$  satisfying the resonance conditions (2.2.20) with a wave  $\ell$ . As shown by Galeev and Oraevskii (1963) and Sagdeev and Galeev (1969), if the amplitude of the Alfvén wave  $\ell$  is larger than the  $m$ 's and  $n$ 's it will decay irreversibly providing that the number of  $(m,n)$  pairs is large enough, and the initial phases of the  $(m,n)$  waves are random. Sagdeev and Galeev treat this case by using a random phase approximation to the waves in the system. It is then found that a large amplitude Alfvén wave is transformed into random oscillations of the medium rather than an ordered pair of waves capable of interacting back.

This example has been given not only to display the MTS technique, but also to highlight the limitations of a fluid description of a plasma when studying the three wave interaction. The important features are twofold. Firstly, as we will see in Chapter 3 there exists a class of three wave interactions for which the single  $(m,n)$  product case and the case of an arbitrarily large summation have qualitatively the same behaviour, the only difference being the timescale of the nonlinear process. Secondly, we will see that by using the approach of solving the Vlasov equation, we are able to analyse the particle physics involved in the interactions of 'negative energy waves', a problem which cannot be analysed using the MHD equations.

## CHAPTER 3

### THREE WAVE INTERACTION IN A VLASOV PLASMA

#### 3.1 Introduction

In this Chapter we derive a kinetic equation for the amplitude of a monochromatic electrostatic wave which is interacting weakly with other similar waves in a one dimensional neutral Vlasov plasma. We use the multiple timescale (MTS) method outlined in the previous Chapter.

The equations for the slow variation of the wave amplitudes are found to take on the general form

$$\frac{\partial \phi_\ell}{\partial t} = \sum_{m,n} M_{\ell mn} \phi_m \phi_n, \quad (3.1.1)$$

where the  $\phi$ 's are the electrostatic potentials of the waves, and the  $M$ 's describe the wave coupling strength and are often called the 'matrix elements' of the kinetic equations. These kinetic equations are similar in form to those derived in the previous Chapter (cf. 2.3.1, 2 and 3).

In order to make the general kinetic equations solvable, we assume that only three waves are participating in the interaction; that is, that the summation in (3.1.1) disappears. As stated in Chapter 2, this is not the general case, but we will discuss the problems and implications of considering the sum over many wave pairs in Section 3.5. The assumption is reasonable when the three waves under study have amplitudes far in excess of the background level (while still of course remaining 'small' compared with the electron temperature, for the perturbation expansion to hold). This assumption also has particular relevance to the computational plasma



simulations, since as we will see in Chapter 5, none of the three chosen waves can interact with any but one another. This is because the simulation wavevectors are discrete, and in all other cases, the wavevector resonance condition on the existence of the kinetic equations is violated.

We show that a consequence of adopting the Vlasov approach is that the matrix elements of the kinetic equations can be written in the form

$$M_{lmn} = F_{lmn} / (\partial \epsilon / \partial \omega) \big|_{\ell},$$

where  $F_{lmn}$  is a function symmetric in the subscripts  $\ell$ ,  $m$  and  $n$ , and  $\epsilon$  is the plasma dispersion function. We further show that if  $\partial \epsilon / \partial \omega \big|_{\ell}$  is negative, the kinetic equations describe waves which grow to an infinite amplitude in a finite time. This phenomenon is called an 'explosive instability' and cannot be described by an MHD calculation, (cf. Chapter 2, where the matrix elements are all the same sign).

We therefore examine those properties of the waves and the medium which make  $\partial \epsilon / \partial \omega$  negative, in order to determine the physical mechanisms driving the instability. The plasma configuration we study consists of a Maxwellian distribution of electrons about velocity  $v = 0$ , the bulk of the ions also in a Maxwellian about  $v = 0$ , and a diffuse Maxwellian ion beam about  $v = U_b$ . In Appendix B we show that this configuration can support ion sound waves with phase velocities around  $v = 0$ , and also 'Doppler shifted' ion sound waves with phase velocities around  $v = U_b$ . We choose to examine the interaction of one of the 'nearly stationary' ion sound waves, one 'ion beam' wave with phase velocity just above  $U_b$ , and one 'ion beam' wave with phase velocity just below  $U_b$ . Now for the 'slow' beam mode we find that  $\partial \epsilon / \partial \omega$  is negative, whereas for the 'fast' beam mode and the 'ordinary' ion mode, it is positive.

Now Stix (1962) has shown that the change in the total energy of a

dispersive medium when a monochromatic electrostatic wave propagates in it, is proportional to  $|E|^2 \partial(\omega\epsilon)/\partial\omega$ , where  $E$  is the amplitude of the wave electric field. In the case of the wave being a normal mode of the system and hence satisfying the dispersion equation  $\epsilon = 0$ , the change in energy as calculated by Stix, is proportional to  $|E|^2 \partial\epsilon/\partial\omega$ . From this, we might conclude that the energy of the 'slow' ion beam mode is negative. The physical meaning of 'a wave with negative energy' is examined in Section 3.2, and an explanation is given for how the energy of a medium can decrease when a wave propagating in it increases in amplitude. Many authors e.g. Kadomtsev, Mikhailovskii and Timofeev (1965), and Dikarov, Rudakov and Ryutov (1965) simply state that the energy of a monochromatic wave can be negative, and hence when it interacts with two ordinary 'positive energy' waves (through a three wave interaction like that discussed in Chapter 2), energy is transferred from it to them resulting in the unbounded growth of all three wave amplitudes. This statement describes a very useful and economical mathematical model for describing the explosive instability. It does however pose the question of what is meant physically by a negative energy wave.

In this Chapter, we examine the energy  $W$  added to a plasma when a monochromatic electrostatic wave propagates, and find for normal modes ( $\epsilon = 0$ ), that it is proportional to  $\partial\epsilon(\omega)/\partial\omega$  which for some media can be negative. However, by deriving  $W$  from the Vlasov equation (Appendix A) it is clearly equal to the sum of the electrostatic wave energy and the change in that part of the charged particle kinetic energy which is associated with the coherent wave motion. We argue moreover, that this sum can be negative in a beamed plasma in which beam particles slow down, and some of their energy is transferred to the wave. It transpires that the greater the amount of kinetic energy extracted from the beam (at the wave's phase velocity), the larger the amplitude of the wave. In this way,

if some of the beam kinetic energy is transferred to two other waves, all three can grow in amplitude. So, rather than regarding one of the waves as having negative energy, we use the more physical picture of treating the beam kinetic energy as the driving force for the instability. This creates a better understanding of explosive instabilities, and also gives us a starting point for examining the saturation of the instability in the following Chapter.

In the course of this Chapter, the symbol  $\epsilon$  is used both as a label for the 'smallness parameter' in the perturbation expansion, and also to signify the dispersion function. When any confusion may arise, the meaning is stated explicitly.

### 3.2 Negative Energy Waves

Stix (1962) has shown that the change in energy of a dispersive medium when a monochromatic electrostatic wave propagates, is given by

$$W = \frac{|E|^2}{16\pi} \frac{\partial}{\partial \omega} \{\omega \epsilon_h\} , \quad (3.2.1)$$

where  $E$  is the amplitude of the wave electric field, and  $\epsilon_h$  is the Hermitian part of the dielectric function for the medium. The result (3.2.1) is not the usual constant multiple of  $|E|^2$  which we expect for a wave, and Stix analyses it by looking at electron plasma oscillations. In this case,  $W$  can be interpreted as the sum of the electrostatic energy ( $|E|^2$ ) and that portion of the charged particle kinetic energy which is associated with the perturbation of the particle distribution function. In Appendix A we derive (3.2.1) explicitly for a plasma using the Vlasov equation and Poisson's equation to describe the system, except for the factor of  $1/16\pi$  in Stix's derivation which arises from his formulation of Poynting's theorem. The Vlasov derivation automatically produces the two separate terms, and the physical interpretation of each is obvious, since one is a constant multiple of  $|E|^2$ , and the other derives from

$$\int v^2 (f - f_0) dv \text{ (averaged over the phase of the oscillation),}$$

where  $f$  is the perturbed particle distribution function and  $f_0$  is the unperturbed distribution function. This term describes the kinetic energy associated with the perturbation on the particles. The advantage of the Vlasov approach is that the physics is clear in the general case (we need not choose a specific example as Stix does). When we add the two terms to get the total energy associated with the wave, we get the result that

$$W \propto |E|^2 \frac{\partial}{\partial \omega} \{ \omega \epsilon \} \quad (\text{cf. A.10})$$

Now it is well known that there exist media which for certain values of  $\omega$  and  $k$ , have  $\partial \epsilon / \partial \omega$  negative. Such a wave will be called a 'negative energy wave'. From the expression for the energy of such a wave (3.2.1) we see that as its amplitude increases, its energy decreases. Thus if it loses energy to two positive energy waves for instance, all three increase in amplitude. The questions now arise; what sort of medium exhibits this anomalous dispersive characteristic, and what happens to a plasma when a negative energy wave propagates and interacts with other waves.

Landau and Lifshitz (1960) show from entropy considerations that a region of transparency in a medium in thermodynamic equilibrium will exhibit normal dispersion i.e.  $\partial(\omega \epsilon) / \partial \omega$  is positive. For media away from such an equilibrium, the thermodynamic theory ceases to hold, and the positiveness of  $\partial(\omega \epsilon) / \partial \omega$  can no longer be assumed. For this reason, we choose to look at a plasma with beams in order to find negative energy dispersion characteristics. Such an example is relevant to the theme of this thesis, and will bring out the physics of negative energy waves.

Sturrock (1960) has demonstrated that if two waves of equal amplitude are excited with phase velocities  $u'$  and  $-u'$  with respect to a medium moving

with the (non-relativistic) velocity  $U_b$ , then the energy of the two waves (as measured in a stationary frame) is given by:-

$$\begin{aligned} W \text{ (forward wave)} &= + \left\{ \frac{U_b + |u'|}{|u'|} \right\} W', \\ W \text{ (backward wave)} &= - \left\{ \frac{U_b - |u'|}{|u'|} \right\} W', \end{aligned} \tag{3.2.2}$$

where  $W'$  is the energy of the waves measured in the moving frame. We observe that the two waves have energies of opposite sign if

$$U_b > |u'|. \tag{3.2.3}$$

We might conclude from (3.2.2) and (3.2.3) that negative energy waves are a classical relativistic phenomenon, but this would only be a half truth since it fails to explain physically why such a wave can increase in amplitude as it decreases in energy.

An important thing to note however, is that the medium needs to be moving faster than a given velocity (3.2.3) i.e. there needs to be a large amount of kinetic energy in the medium itself. Unfortunately, most of the literature on explosive instabilities takes the study of the waves themselves no further than stating 'negative energy waves exist', and little work has been done on the combined physics of the medium and the wave. Such a study is essential since it reveals the full physics of these waves, and also gives a deeper understanding of explosive instabilities, and how they saturate. The approach of analysing the medium has previously been ignored only to yield oversimplified and often misleading results. This point will be taken up in greater detail in Chapter 4, but at present we will concern ourselves with an interpretation of the statement that a negative energy wave increases in amplitude as it decreases in energy.

The important point here is that 'energy' in this sense is a sum of electrostatic energy and particle kinetic energy. In a medium in which

a beam carries a wave, if the excitation of the wave can be achieved by decelerating some of the beam particles (namely by a special kind of perturbation) then the change in energy of the system due to the presence of the wave is proportional to  $|E|^2$  minus the change in kinetic energy of the perturbed particles. Clearly, since a plasma's kinetic energy greatly exceeds its potential energy, we need only decelerate a minute fraction of the particles in order to excite such a wave. Moreover for such a configuration, the larger the wave amplitude we want, the greater the number of particles we need to perturb (decelerate).

A question arising at this point is whether we need to decelerate particles in order to excite a negative energy wave on a beam. Guided by the relativistic argument of Sturrock (3.2.2) we examine the 'slow' beam mode, and if we consider a Maxwellian distribution of beam particles we see that there are more particles travelling faster than the wave's phase velocity than slower, providing  $U_b > |u'|$  (cf. 3.2.3). Hence, the net effect of adjusting local particle velocities to the waves phase velocity is a reduction in kinetic energy, that is a deceleration.

This argument is not dissimilar to the quasilinear theory of the saturation of 'Landau unstable' plasma oscillations. As shown by Drummond (1965), the growth of plasma oscillations with  $\omega$  and  $k$  such that  $\partial f_0 / \partial v$  is positive at  $v = \omega/k$ , occurs at the expense of a reduction in particle kinetic energy. It is also true that the Landau damping factor  $\gamma$  is given by

$$\gamma = \text{Im}(\epsilon) / \partial \epsilon / \partial \omega, \quad (3.2.4)$$

namely that a negative energy wave is Landau unstable if  $\text{Im}(\epsilon)$  is negative. The full circle of equivalence is completed by condition (3.2.3).

We maintain that a negative energy wave propagating on a beam is one

with phase velocity below the beam velocity  $U_b$ . Moreover, its amplitude increases as we increase the perturbation on the particles (as we expect) - the perturbation however being increased by decelerating an increased number of beam particles. This interpretation avoids making the misleading claim that a wave in a medium can, independently of the medium, be assigned an energy which is less than zero.

At this point we can see what is happening when a negative energy wave interacts with two positive energy waves. The positive energy waves have frequencies  $\omega_m$  and  $\omega_n$  such that  $\omega_m + \omega_n = \omega_\ell$  (the frequency of the negative energy wave). Similarly the wavevectors  $k_m + k_n = k_\ell$ , which as we saw in Chapter 2 are the two 'resonance conditions' for the interaction of three waves. Now since the two lower frequency waves beat together to form a wave of phase velocity  $\omega_\ell/k_\ell$ , we have a situation where particle kinetic energy can be transferred to them (since  $\partial f_0/\partial v$  is positive at that velocity). This causes the negative energy wave amplitude to increase, and then we have an explosive instability (since all three waves are driving one another's amplitude up). The details of the explosive instability will be developed in Section 3.4.

### 3.3 Basic Equations

To obtain the equations governing the three wave interaction discussed in Section 3.1 we start with the one dimensional Vlasov equation,

$$\frac{\partial f_j}{\partial t} + v \frac{\partial f_j}{\partial x} - \left(\frac{e}{m}\right)_j \frac{\partial \phi}{\partial x} \frac{\partial f_j}{\partial v} = 0, \quad (3.3.1)$$

where the subscript  $j$  refers to the plasma species, namely ions (i) or electrons (e). The electrostatic potential  $\phi$  is given by Poisson's equation,

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi n_0 e \int_{-\infty}^{\infty} (f_i - f_e) dv, \quad (3.3.2)$$

where  $n_0$  is the total particle number density of one of the species, and  $e$  is the electronic charge. Normalisation of the ion and electron particle distribution functions  $f_i$  and  $f_e$  is such that

$$\int_{-\infty}^{\infty} f_{i,e} dv = n_{i,e}/n_0.$$

Consider the plasma to be neutral with a Maxwellian electron velocity distribution about  $v = 0$ , and an ion distribution made up of the bulk of the ions in a Maxwellian at  $v = 0$ , and the rest forming a Maxwellian beam about  $v = U_b$ . We write the beam density as  $\eta^2$  which is a small quantity. As shown in Appendix B, this ordering gives us a configuration which supports negative energy waves, and also makes possible the quantitative calculations of timescales, amplitudes, etc.

We want to find an expression for the potential  $\phi$  correct to second order in the small parameter  $\epsilon = e\phi_0/k_B T_e$ , where  $k_B$  is Boltzmann's constant,  $T_e$  is the electron temperature, and  $\phi_0$  is some characteristic value of the electrical potential. We introduce the MTS expansion

$$\begin{aligned} f_i = & (1 - \eta^2) f_{oi}^0(v, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \eta^2 f_{oi}^b(v - U_b, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \epsilon f_{1i}(x, v, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \epsilon^2 f_{2i}(x, v, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \text{terms of order } \epsilon^3 \text{ and higher.} \end{aligned}$$

$$\begin{aligned} f_e = & f_{oe}(v, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \epsilon f_{1e}(x, v, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \epsilon^2 f_{2e}(x, v, t, \epsilon t, \epsilon^2 t, \dots) + \text{etc.} \end{aligned}$$

$$\begin{aligned} \phi = & \epsilon \phi_1(x, t, \epsilon t, \epsilon^2 t, \dots) + \\ & + \epsilon^2 \phi_2(x, t, \epsilon t, \epsilon^2 t, \dots) + \text{etc.} \end{aligned}$$



For purposes of treating the total ion distribution, e.g. on the right hand side of Poisson's equation, or the  $\partial/\partial v$  term in the Vlasov equation, we call the unperturbed ion distribution function  $f_{oi}$  where

$$f_{oi} = (1 - \eta^2) f_{oi}^o + \eta^2 f_{oi}^b.$$

We do however treat the 'stationary' and 'beam' ions as separate terms when this is necessary e.g. in time derivatives, where the two terms may have different temporal behaviours.

Making use of the expanded form of the time derivative (2.2.2) we equate terms of order unity in the expanded Vlasov equation for ions, and get

$$\frac{\partial f_{oi}^o}{\partial t} = 0.$$

Similarly, equating spatially independent terms of order  $\epsilon$  gives

$$\frac{\partial f_{oi}^o}{\partial \epsilon t} + \frac{\partial f_{oi}^b}{\partial t} = 0.$$

Since  $f_{oi}^o$  is independent of  $t$ , integration of the above equation with respect to  $t$  gives in general, a secular contribution to  $f_{oi}^b$ . We therefore conclude that

$$\frac{\partial f_{oi}^o}{\partial \epsilon t} = \frac{\partial f_{oi}^b}{\partial t} = 0.$$

In a similar way, if we consider the Vlasov equation for electrons, we find that

$$\frac{\partial f_{oe}}{\partial \epsilon t} = \frac{\partial f_{oe}}{\partial t} = 0.$$

Therefore, equating terms of order  $\epsilon$  in the expanded forms of (3.3.1 and 2) gives

$$\frac{\partial f_{1j}}{\partial t} + \frac{v \partial f_{1j}}{\partial x} - \left(\frac{e}{m}\right)_j \frac{\partial \phi_1}{\partial x} \frac{\partial f_{oj}}{\partial v} = 0, \quad (3.3.3)$$

$$\text{and} \quad \frac{\partial^2 \phi_1}{\partial x^2} = -4\pi n_0 e \int_{-\infty}^{\infty} (f_{1i} - f_{1e}) dv. \quad (3.3.4)$$

It is well known that equations (3.3.3 and 4) should be solved as an initial value problem in order not to lose the significance of the Landau contour. To avoid lengthy algebra however, we will simply assume solutions of (3.3.3 and 4) to be of the form

$$f_{1j} = \sum_{\ell} f_{1j}^{\ell} \exp(ik_{\ell}x - i\omega_{\ell}t),$$

$$\text{and} \quad \phi_1 = \sum_{\ell} \phi_1^{\ell} \exp(ik_{\ell}x - i\omega_{\ell}t), \quad (3.3.5)$$

and when the necessity arises, ensure that the appropriate velocity integrals are done using the Landau contour. In such a case, the  $\omega$ 's will be understood to refer to complex frequencies. For the main part however, they are to be considered as the real part of the frequency unless specifically stated otherwise.

Looking at the  $\ell$ -th Fourier component of (3.3.3) we get

$$f_{1j}^{\ell}(k_{\ell}, \omega_{\ell}) = - \frac{(\frac{e}{m})_j k_{\ell} \partial f_{0j} / \partial v}{(\omega_{\ell} - k_{\ell} v)} \phi_1^{\ell}(k_{\ell}, \omega_{\ell}). \quad (3.3.6)$$

Using Poisson's equation (3.3.4) to eliminate  $\phi_1^{\ell}$ , we get the well known linear dispersion relation

$$\epsilon(\omega_{\ell}, k_{\ell}) = 1 + \sum_j \frac{\omega_{pj}^2}{k_{\ell}^2} \int_c \frac{k_{\ell} \partial f_{0j} / \partial v}{(\omega_{\ell} - k_{\ell} v)} dv = 0, \quad (3.3.7)$$

where  $\omega_{pj}$  is the plasma frequency for the species  $j$ , and  $c$  denotes the Landau contour.

As we saw in Chapter 2, the formal mathematics of the M.T.S. approach

not only gives us the linear solutions, but also describes systematically the second order corrections produced by cross modulations of the first order waves. The description given in Section 2.2 of the characteristics of the second order waves' frequencies and wavevectors, carries over into this example. If we follow a parallel path of analysis to that in Chapter 2, we will again find that the case of the beat wave's frequency and wavevector satisfying the dispersion relation (3.3.7) gives rise to an equation describing the three wave interaction. Accordingly, we equate terms of order  $\epsilon^2$  in the expanded Vlasov and Poisson equations, and get

$$\begin{aligned} \frac{\partial f_{1j}}{\partial \epsilon t} + \frac{\partial f_{0j}^b}{\partial \epsilon t} + \frac{\partial f_{0j}^o}{\partial \epsilon^2 t} + \frac{\partial f_{2j}}{\partial t} + v \frac{\partial f_{2j}}{\partial x} \\ - \left(\frac{e}{m}\right)_j \frac{\partial \phi_2}{\partial x} \frac{\partial f_{0j}}{\partial v} - \left(\frac{e}{m}\right)_j \frac{\partial \phi_1}{\partial x} \frac{\partial f_{1j}}{\partial v} = 0, \end{aligned} \quad (3.3.8)$$

and

$$\frac{\partial^2 \phi_2}{\partial x^2} = -4\pi n_0 e \int (f_{2i} - f_{2e}) dv, \quad (3.3.9)$$

where the beam term in (3.3.8) will be omitted for the case of  $j$  referring to electrons. As we did in Chapter 2, we assume solutions of the form

$$f_{2j} = \sum_{\ell} f_{2j}^{\ell} \exp(ik_{\ell} x - i\bar{\omega}_{\ell} t),$$

and

$$\phi_2 = \sum_{\ell} \phi_2^{\ell} \exp(ik_{\ell} x - i\bar{\omega}_{\ell} t).$$

Before making these substitutions, we can separate the spatially dependent parts of (3.3.8) from the spatially independent parts to give us two equations. The spatially independent terms form an equation describing the quasi-linear effect of the waves on the unperturbed distribution functions, and for the ion species can be written as

$$\frac{\partial f_{0i}^b}{\partial \epsilon t} + \frac{\partial f_{0i}^o}{\partial \epsilon^2 t} - \left(\frac{e}{m}\right)_i \left\{ \frac{\partial \phi_1}{\partial x} \frac{\partial f_{1i}}{\partial v} \right\}_s = 0, \quad (3.3.10)$$

where the subscript s indicates that we only take spatially independent products for the bracket term.

We shall return to the study of (3.3.10) in Section 4.2, but meanwhile making the above substitutions for  $f_{2j}$  and  $\phi_2$  into (3.3.8 and 9), and looking at the  $\ell$ -th Fourier component, we are left with

$$\begin{aligned} & (-i\bar{\omega}_\ell + ik_\ell v) f_{2j}^\ell - \left(\frac{e}{m}\right)_j ik_\ell \phi_2^\ell \frac{\partial f_{0j}}{\partial v} = \\ & - \frac{\partial f_{1j}^\ell}{\partial \epsilon t} e^{-i(\omega_\ell - \bar{\omega}_\ell)t} + \left(\frac{e}{m}\right)_j \sum_{k_n, k_m}^{k_n + k_m = k_\ell} ik_n \phi_1^n \frac{\partial f_{1j}^m}{\partial v} e^{-i(\omega_n + \omega_m - \bar{\omega}_\ell)t}, \end{aligned} \quad (3.3.11)$$

and

$$k_\ell^2 \phi_2^\ell = 4\pi n_0 e \int (f_{2i}^\ell - f_{2e}^\ell) dv. \quad (3.3.12)$$

Equation (3.3.11) gives us an expression for  $f_{2j}^\ell$ . Writing this expression firstly for  $j$  representing ions and then for  $j$  representing electrons, we can substitute for  $f_{2i}^\ell$  and  $f_{2e}^\ell$  in (3.3.12) to give

$$\begin{aligned} & \phi_2^\ell \left\{ 1 + \sum_j \frac{\omega_{pj}^2}{k_\ell^2} \int_c \frac{k_\ell \partial f_{0j} / \partial v}{(\bar{\omega}_\ell - k_\ell v)} dv \right\} = \\ & = \frac{4\pi n_0 e}{k_\ell^2} \int_c \frac{dv}{(\bar{\omega}_\ell - k_\ell v)} \left\{ -i \frac{\partial}{\partial \epsilon t} (f_{1i}^\ell - f_{1e}^\ell) e^{-i(\omega_\ell - \bar{\omega}_\ell)t} \right. \\ & \quad - \left(\frac{e}{m}\right)_i \sum_{k_n, k_m} k_n \phi_1^n \frac{\partial f_{1i}^m}{\partial v} e^{-i(\omega_n + \omega_m - \bar{\omega}_\ell)t} \\ & \quad \left. + \left(\frac{e}{m}\right)_e \sum_{k_n, k_m} k_n \phi_1^n \frac{\partial f_{1e}^m}{\partial v} e^{-i(\omega_n + \omega_m - \bar{\omega}_\ell)t} \right\}. \end{aligned} \quad (3.3.13)$$

Note that the left hand side of (3.3.13) can be written as

$$\phi_2^\ell(k_\ell, \bar{\omega}_\ell) \in (\bar{\omega}_\ell, k_\ell),$$

where  $\epsilon$  is the dispersion function as it appeared in (3.3.7). So we can formally write

$$\phi_2^\ell(k_\ell, \bar{\omega}_\ell) = \frac{\text{R.H.S. of (3.3.13)}}{\epsilon(\bar{\omega}_\ell, k_\ell)},$$

except when  $\epsilon = 0$ , that is when the second order beat wave is a normal mode of the system. To avoid such a secular solution for  $\phi_2^\ell$ , we must insist that the right hand side of (3.3.13) is identically zero when  $\bar{\omega}_\ell = \omega_\ell$ .

So doing, gives us an equation for the 'slow' variation of  $(f_{1i}^\ell - f_{1e}^\ell)$ , and substituting their values from (3.3.6) we see that the first term on the right hand side of (3.3.13) simplifies to

$$i \frac{\partial \phi_1^\ell}{\partial \epsilon t} \bigg|_{\omega_\ell, k_\ell} \frac{\partial \epsilon}{\partial \omega}. \quad (3.3.14)$$

Now for the second two terms on the right hand side of (3.3.13) to be nonzero, the argument in the exponentials must be zero, otherwise the 'fast'  $t$ -timescale oscillations would average out on the  $\epsilon t$ -timescale to zero. Therefore we arrive at the conditions for a non-trivial expression for the slow variation of  $\phi_1^\ell$ , namely that

$$\omega_\ell = \omega_m + \omega_n, \quad (3.3.15)$$

$$k_\ell = k_m + k_n.$$

These are the 'resonance conditions' discussed in Chapter 2. In a similar manner to the derivation of result (3.3.14) we can substitute for  $f_{1i}^\ell$  and  $f_{1e}^\ell$  into the second two terms on the right hand side of (3.3.13). We can therefore write the condition that the right hand side of (3.3.13) is identically zero as

$$i \frac{\partial \phi_1^\ell}{\partial \epsilon t} = \sum_{\substack{k_n + k_m = k_\ell \\ \omega_m + \omega_n = \omega_\ell}} \frac{\Gamma_{\ell mn}}{\alpha_\ell} \phi_1^m \phi_1^n, \quad (3.3.16)$$

where

$$\Gamma_{lmn} = \frac{1}{2} \sum_j \omega_{pj}^2 \left( \frac{e}{m} \right)_j \int_c \frac{k_l k_m k_n \partial f_{oj} / \partial v dv}{(\omega_l - k_l v)(\omega_m - k_m v)(\omega_n - k_n v)},$$

and  $\alpha_l = k_l^2 \left. \frac{\partial \epsilon}{\partial \omega} \right|_l.$  (3.3.17)

Equation (3.3.16) is the equation governing the interaction of three waves  $\phi_1^l$ ,  $\phi_1^m$  and  $\phi_1^n$ . It is of a similar form to those derived in Chapter 2 (cf. 2.3.1, 2 and 3), and the resonance conditions for such an interaction are identical to those derived using the M.H.D. approach. Moreover, the interaction equation (3.3.16) and the matrix element (3.3.17) are structured identically to those described by Kadomtsev (1965).

The factor of 1/2 in  $\Gamma_{lmn}$  arises because the complete sum in (3.3.17) counts all terms twice, i.e. includes the  $\phi_1^m \phi_1^n$  term as well as the  $\phi_1^n \phi_1^m$  term.

For the case  $\bar{\omega}_l \neq \omega_l$ , namely when the second order beat wave is not a normal mode of the system, we can see from (3.3.13) that

$$\left\langle -\frac{i\partial}{\partial \epsilon t} (f_{li}^l - f_{le}^l) e^{-i(\omega_l - \bar{\omega}_l)t} \right\rangle_{\text{time average}} \rightarrow 0,$$

due to the rapidly oscillating exponential factor. Therefore, in a similar manner to the derivation of (3.3.16), we get the solution for  $\phi_2^l$  to be

$$\phi_2^l \epsilon(\bar{\omega}_l, k_l) = \sum_{\substack{k_n + k_m = k_l \\ \omega_n + \omega_m = \omega_l}} \bar{\Gamma}_{lmn} \phi_1^m \phi_1^n, \quad (3.3.18)$$

where  $\bar{\Gamma}_{lmn}$  is of the same form as  $\Gamma_{lmn}$  given by (3.3.17), except that  $\omega_l$  is replaced by  $\bar{\omega}_l$ . In this case of course,  $\epsilon(\bar{\omega}_l, k_l)$  is a nonzero quantity.

Equation (3.3.16) governs the three wave interaction in the Vlasov representation of a plasma, and forms the starting point for the discussion in Section 3.4. Equation (3.3.10) represents a quasi-linear effect of the waves on the ions, which we will study in Section 4.2 in connection with the violation of the resonance conditions (3.3.15).

### 3.4 The Explosive Instability

As we stated in Section 3.2, if the plasma was in thermodynamic equilibrium, that is did not have the ion beam, then  $\partial(\omega\epsilon)/\partial\omega$  would be positive. For those  $\omega$  corresponding to normal modes therefore, since  $\epsilon(\omega, k) = 0$ ,  $\partial\epsilon/\partial\omega$  would be positive. That being the case, all three wave energies would be positive and the analogous equation to (3.3.16) for the three wave interaction would have  $\alpha_\ell$  (cf. 3.3.17) positive. Equation (3.3.16) and the corresponding equations for  $\partial\phi_1^m/\partial\epsilon t$  and  $\partial\phi_1^n/\partial\epsilon t$  could then be solved in an identical manner to those discussed in Chapter 2. That is, the solutions would be of the form shown in Figure 2.2, exhibiting the Manley-Rowe behaviour of energy transfer back and forth between the three waves.

The deviation from thermodynamic equilibrium caused by the presence of the ion beam however, gives rise to the possibility of having negative energy waves, and a corresponding negative  $\alpha$  for some waves in (3.3.16). As shown in Appendix B, for the beam plasma system described in the previous section, we can choose three waves  $\phi_1^\ell$ ,  $\phi_1^m$  and  $\phi_1^n$  which have the following characteristics:

$$\begin{aligned}\omega_\ell &= k_\ell (U_b - \delta n v_{th,e}), \\ \omega_m &= k_m (U_b + \delta n v_{th,e}), \\ \omega_n &= k_n \delta v_{th,e} (1 - n^2)^{1/2},\end{aligned}\tag{3.4.1}$$

where  $n^2$  is the ratio of the number of beam ions to the total number of ions,  $\delta^2$  is the electron to ion mass ratio,  $U_b$  is the ion beam velocity, and  $v_{th,e}$  is the electron thermal speed. In Appendix B, we also show that

$$\left. \frac{\partial(\omega\epsilon)}{\partial\omega} \right|_{m,n} > 0 > \left. \frac{\partial(\omega\epsilon)}{\partial\omega} \right|_{\ell},$$

namely that wave  $\ell$  has 'negative energy' and waves  $m$  and  $n$  have positive energy. Numerical values for all the relevant quantities are given in Appendix B, for use elsewhere, and from them we find that the resonance conditions (3.3.15) can be satisfied with a suitable choice of  $U_b$  and  $\eta$ .

In the following, the time derivative is understood to refer to the 'slow' timescale, and the  $\phi$ 's are understood to be first order quantities. Then the equation for  $\phi_{\ell}$  namely (3.3.16), and the equivalent equations for  $\phi_m$  and  $\phi_n$  for the case when only three waves participate, take the form

$$\begin{aligned} \frac{\partial\phi_{\ell}}{\partial t} &= \frac{i \Gamma_{\ell mn}}{|\alpha_{\ell}|} \phi_m^* \phi_n^*, \\ \frac{\partial\phi_m}{\partial t} &= \frac{i \Gamma_{\ell mn}}{|\alpha_m|} \phi_n^* \phi_{\ell}^*, \\ \frac{\partial\phi_n}{\partial t} &= \frac{i \Gamma_{\ell mn}}{|\alpha_n|} \phi_{\ell}^* \phi_m^*. \end{aligned} \tag{3.4.2}$$

In equations (3.4.2),  $\alpha_m$  and  $\alpha_n$  are positive, but  $\alpha_{\ell}$  is negative. We have transformed to the dependent variable  $\phi_{\ell}^*$ , rather than  $\phi_{\ell}$ , in order to get the equations in symmetric form. This makes no difference to the matrix element  $\Gamma_{\ell mn}$ , which is seen from Appendix B to be real. Because however, the frequency and wavevector associated with  $\phi_{\ell}^*$  are  $-\omega_{\ell}$  and  $-k_{\ell}$ , the resonance conditions also take on a symmetric form, namely

$$\begin{aligned} \omega_{\ell} + \omega_m + \omega_n &= 0, \\ k_{\ell} + k_m + k_n &= 0. \end{aligned}$$

Making the variable changes  $A_{\ell,m,n} = \phi_{\ell,m,n} |\alpha_{\ell,m,n}|^{1/2}$ , and  $M = \Gamma_{\ell mn} / (|\alpha_{\ell}| |\alpha_m| |\alpha_n|)^{1/2}$ , equations (3.4.2) take on the completely symmetrical form of



$$\frac{\partial A_\ell}{\partial t} = i M A_m^* A_n^*,$$

$$\frac{\partial A_m}{\partial t} = i M A_n^* A_\ell^*, \quad (3.4.3)$$

$$\frac{\partial A_n}{\partial t} = i M A_\ell^* A_m^*.$$

Since the A's are complex quantities, and M may be positive or negative, then with no loss of generality we may write

$$A_\ell = B_\ell e^{i\theta_\ell}, \quad (3.4.4)$$

with similar expressions for  $A_m$  and  $A_n$ ,

and  $M = |M| \exp(-i\beta)$ ,

where the B's and  $\theta$ 's are real functions of time, and  $\beta$  is real. Substituting (3.4.4) into (3.4.3) and equating imaginary parts gives

$$\frac{\partial \theta}{\partial t} = |M| \left\{ \frac{B_m B_n}{B_\ell} + \frac{B_\ell B_n}{B_m} + \frac{B_\ell B_m}{B_n} \right\} \cos \theta, \quad (3.4.5)$$

where  $\theta = \theta_\ell + \theta_m + \theta_n - \beta$ . Equating real parts gives

$$\frac{\partial B_\ell}{\partial t} = |M| B_m B_n \sin \theta,$$

$$\frac{\partial B_m}{\partial t} = |M| B_n B_\ell \sin \theta, \quad (3.4.6)$$

$$\frac{\partial B_n}{\partial t} = |M| B_\ell B_m \sin \theta.$$

Digressing slightly, we see that equations (3.4.3) give us a relation similar to the Manley-Rowe relations discussed in Chapter 2. In this case, the relation is

$$\frac{d}{dt} (A_\ell A_\ell^*) = \frac{d}{dt} (A_m A_m^*) = \frac{d}{dt} (A_n A_n^*) \quad (3.4.7)$$

The important point to note about (3.4.7) is that the quantities  $AA^*$  are given by

$$AA^* = k^2 |\phi|^2 \frac{\partial \epsilon}{\partial \omega},$$

whereas the corresponding quantities derived in Chapter 2 were  $n_o v^2/2$  (for sound waves), and  $H^2/4\pi$  (for Alfvén waves). That is, the present relation (3.4.7) refers to energy as defined in Section 3.2. In other words, the conserved quantities are no longer simply the energies of oscillation, but also included is the contribution from changes in particle kinetic energy due to the presence of the wave. From general considerations, we must have conservation of energy, and the relation (3.4.7) describes such a conservation. Furthermore, it reaffirms the physics of negative energy waves as discussed in Section 3.2. That is, it provides a physically meaningful energy theorem in which it is not necessary to classify one of the waves as having negative energy, but simply to highlight the individual energy terms which contribute to the balance.

From the relation (3.4.7) we can easily show that

$$\begin{aligned} B_\ell^2 - B_m^2 &= \text{constant} = B_{\ell o}^2 - B_{m o}^2, \\ B_\ell^2 - B_n^2 &= \text{constant} = B_{\ell o}^2 - B_{n o}^2, \end{aligned} \tag{3.4.8}$$

where the subscript 'o' defines the variable at time  $t = 0$ . Using (3.4.5) and (3.4.6) we see that

$$\frac{\sin \theta}{\cos \theta} d\theta = \frac{dB_\ell}{B_\ell} + \frac{dB_m}{B_m} + \frac{dB_n}{B_n}.$$

Integration gives  $B_\ell B_m B_n \cos \theta = \text{constant} = C$ . If we suppose the amplitude  $B_\ell$  is zero at time  $t = 0$ , then the constant  $C$  is clearly zero. Further, if  $B_\ell$ ,  $B_m$  and  $B_n$  are all nonzero at some later time, then  $\cos \theta$  must be zero for all time. Therefore, since  $\sin \theta = 1$ , we can write the equation for  $B_\ell$  (3.4.6), substituting  $B_m$  and  $B_n$  from (3.4.8), in the form

$$\int_0^{B_\ell} \frac{d\bar{B}_\ell}{(\bar{B}_\ell^2 - B_{mo}^2)^{1/2} (\bar{B}_\ell^2 - B_{no}^2)^{1/2}} = |M|t. \quad (3.4.9)$$

n

For the case  $B_{mo} = B_{no}$ , the integral is simple, and yields

$$B_\ell = B_{mo} \tan (|M|B_{mo}t).$$

It will be noted that the wave amplitude, and indeed all three amplitudes become infinite in a finite time, namely

$$t_\infty = \frac{\pi}{2|M|B_{mo}}. \quad (3.4.10)$$

As stated in Section 3.1, this phenomenon is called the 'explosive instability', and the waves participating in the interaction are said to be 'explosively unstable'.

For the general case of  $B_{mo} \neq B_{no}$ , (3.4.9) is an elliptic integral of the first kind. When  $B_{mo} > B_{no} > 0$ , we can solve it, to give

$$|M|t = \frac{1}{B_{mo}} F(\psi, q), \quad (3.4.11)$$

where  $F$  is the elliptic integral of the first kind with argument  $\psi$  given by

$$\psi = \tan^{-1} (B_\ell/B_{no}),$$

and amplitude  $q$  given by

$$q = \frac{(B_{mo}^2 - B_{no}^2)^{1/2}}{B_{mo}}.$$

As shown in Appendix B, the shortest time in which the three waves can reach an infinite amplitude, is given by the simple expression (3.4.10), namely when  $B_{mo} = B_{no}$ , and for the waves we are interested in, takes the value

$$t_\infty = \frac{128\sqrt{3}}{3|\phi_{mo}|},$$

where the unit of potential ( $\phi$ ) is the electron temperature, and the unit of

time is the inverse electron plasma frequency ( $\omega_{pe}^{-1}$ ). The units adopted for these calculations and for the plasma simulation programs are discussed in Appendix B.

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### 3.5 Discussion

We have examined a one dimensional plasma configuration whose equilibrium can be pictured as in Figure 3.1, where  $v_\ell$ ,  $v_m$  and  $v_n$  are the phase velocities of the three ion sound waves  $\ell$ ,  $m$  and  $n$  having frequency characteristics described by equations (3.4.1). We have seen that the three waves can interact in such a manner that they all gain an infinite amplitude in a finite time. Figure 3.2(a) shows diagrammatically an interpretation of the mechanisms driving the instability. Figure 3.2(b) is the diagram usually adopted, since it closely resembles the three wave interaction discussed in Chapter 2. Unfortunately, albeit an economical picture it is non-physical, whereas the version given in Figure 3.2(a) as well as being a physical picture of events, gives us a lead in looking for processes which stabilise the instability.

Now it was pointed out earlier that only three waves are considered to participate in the interaction. We saw in Chapter 2 that if a large number of wave pairs exchange energy with a given wave, then an energy equalisation occurs between the various modes, and the interaction peters out. If however, we have a negative energy wave present, as Dikarov, Rudakov and Ryutov (1965) have shown, no such equilibrium can be reached so long as the three wave interaction dominates the time evolution of the waves. This is because an increased number of positive energy pairs 'feeding' off a negative energy wave accentuates the growth of the negative energy wave. For this reason, by restricting our attention to the simple case of only three waves, we do not omit any of the basic physics of the ion beam explosive instability.

Clearly the statement that the waves gain an infinite amplitude is

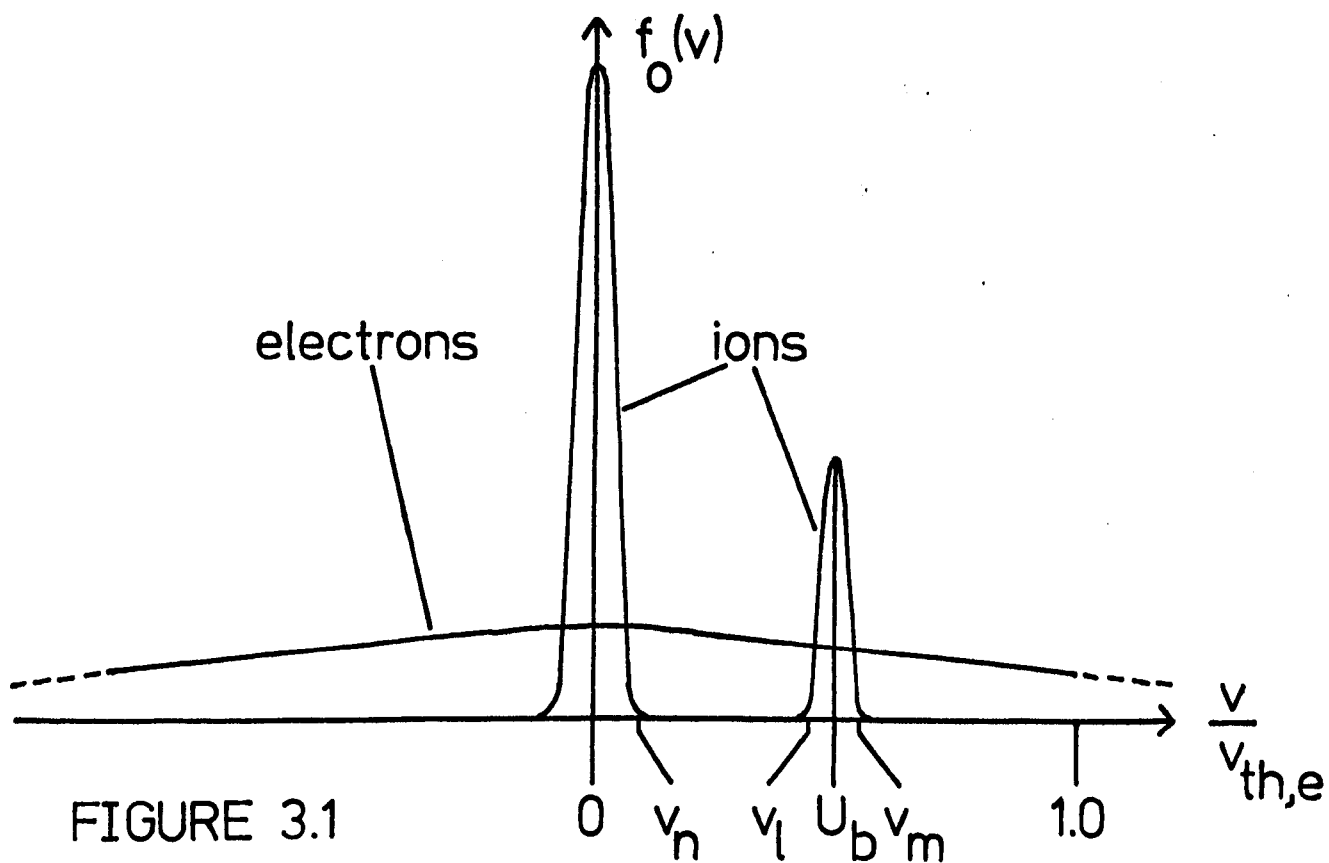


FIGURE 3.1

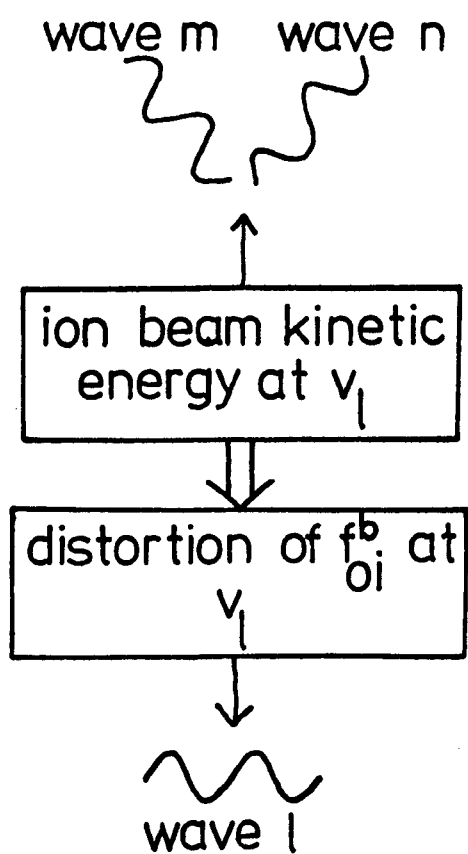


FIGURE 3.2(a)

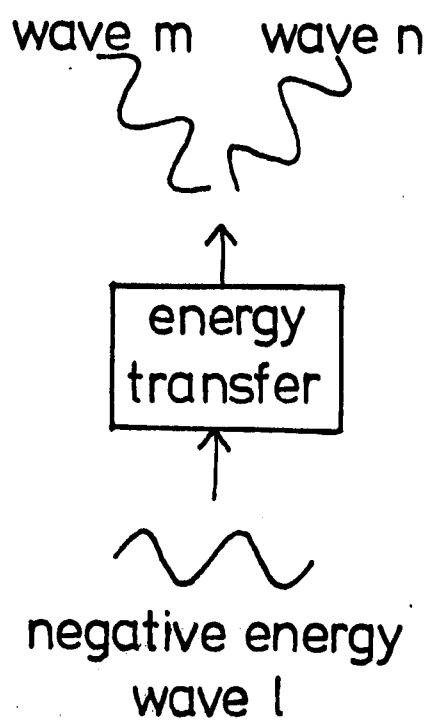


FIGURE 3.2(b)

physically unacceptable. Moreover, the result violates the perturbation expansion used to derive it. In the next Chapter we look at possible stabilisation mechanisms, which must clearly come from considering more than just the effect of the waves on one another. Hence we look in more detail at the medium, in particular at the beam ions.

## CHAPTER 4

### THE SATURATION OF THE INSTABILITY

#### 4.1 Introduction

In this Chapter we analyse the effects on the plasma particles of the three explosively unstable waves discussed in the previous Chapter. By doing this, we show how the beam particle distribution is adjusted by the wave electric field into a configuration in which the three waves can no longer interact explosively.

In Section 4.2 we study the quasi-linear equation (3.3.10) which arose directly from the multiple timescale formalism. From it we find that the beam can slow down and that the beam temperature can increase. The most important consequence of the beam velocity decreasing is that the frequencies of the two beam modes decrease accordingly. This causes a breakdown in the frequency resonance condition (3.3.15) and hence the interaction ceases. In a plasma with many waves propagating however, a pair of positive energy waves previously out of resonance with the negative energy wave, may then interact explosively with it. In this way the beam will slow down due to a succession of interactions until the two ion distributions merge and  $\partial\epsilon/\partial\omega$  is positive for all waves.

An increase in ion beam temperature has the effect of increasing the Landau damping of the positive energy beam mode and decreasing the Landau growth of the negative energy mode. As Fried and Gould (1961) show, if the ion and electron temperatures become comparable, Landau damping will be sufficient to prevent the wave from propagating at all. For the plasma parameters chosen in the computer simulations, the dominant effect is found

to be the increase in ion beam temperature, and we find that only a slight increase is enough to quench the instability.

The approach adopted in this Chapter, namely of analysing the effect of the waves on the medium, has been used previously by Dum and Sudan (1969). They investigate the case of explosively unstable high frequency electrostatic flute modes in a mirror confined plasma. By analysing an expression for the effective transverse diffusion coefficient, they show that it is enhanced considerably by the explosively unstable modes. This results in a plasma configuration in which negative energy waves cannot propagate. In Section 4.3 we derive an analogous result for the present case by studying the Vlasov equation directly. We find that the average beam velocity decreases, that the beam ions heat up, and that the ion beam distribution function distorts in a way which increases Landau damping.

#### 4.2 Quasi-linear Effects on the Ion Beam

Equation (3.3.10) describes the quasi-linear effects of the waves on the beam distribution function and on the 'stationary' ions. Since we know however, that it is the presence of the beam which is allowing negative energy waves to propagate, we restrict the analysis to the study of the quasi-linear effects on the beam.

Equating terms in (3.3.10) which have a velocity dependence like  $(v - U_b)$ , we can write the kinetic equation for  $f_{oi}^b$  (dropping the  $\epsilon$ 's) as

$$\frac{\partial f_{oi}^b}{\partial t} = \left(\frac{e}{m}\right)_i \left\{ \frac{\partial \phi_1}{\partial x} \frac{\partial f_{1i}}{\partial v} \right\}_s,$$

where the subscript 's' denotes a spatially independent bracket term, and  $f_{1i}$  only includes the  $\partial f_{oi}^b / \partial v$  part of  $\partial f_{oi} / \partial v$ . So, in terms of Fourier coefficients, substituting  $f_{1i}^q$  from (3.3.6) we can say that

$$\frac{\partial f_{oi}^b}{\partial t} = \left(\frac{e}{m}\right)_i^2 \sum_{k_q \neq 0} i k_q^2 |\phi_1^q|^2 \frac{\partial}{\partial v} \left\{ \frac{\partial f_{oi}^b / \partial v}{(\omega_q - k_q v)} \right\} \quad (4.2.1)$$



Now we analyse (4.2.1) for a time dependence of both the average beam velocity  $U_b$  and the beam thermal velocity  $v_{th,b}$ . We only give the formal mathematics of the  $U_b$  case however, since the two are very similar.

We define  $U_b$  in the usual way as

$$U_b = \int f_{oi}^b v dv,$$

and we can then integrate (4.2.1) to give

$$-\frac{\partial U_b}{\partial t} = \left(\frac{e}{m}\right)^2 \sum_{k_q \neq 0} i k_q^2 |\phi_1^q|^2 \int \frac{\partial f_{oi}^b / \partial v}{(\omega_q - k_q v)} dv. \quad (4.2.2)$$

To a good approximation we can say that the integrand of equation (4.2.2) only has non-zero values near the phase velocities of the two beam modes. That is, where the absolute value of the integrand's numerator is a maximum, and its denominator goes to zero.

Now  $k_m$  is the wavevector for the fast beam mode at  $v = U_b + \delta v_{th,e}$ , and this mode has positive energy. Let  $\gamma_m$  be the Landau damping rate for this mode (damped since the distribution function has a negative slope at that velocity). Similarly  $k_\ell$  is the wavevector of the slow (negative energy) beam mode at  $v = U_b - \delta v_{th,e}$  and we let the growth rate be  $\gamma_\ell$  (growing since the slope is positive at that velocity). Using this notation, both the  $\gamma$ 's are positive, and since we are treating a configuration in which the Landau rates are small, we can apply the limit rule for the integral in (4.2.2). That is, we use the relation

$$\lim_{\gamma \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{g(v) dv}{(v-a \pm \gamma)} = P \int_{-\infty}^{\infty} \frac{g(v) dv}{(v-a)} \mp i \pi g(a),$$

where  $P$  denotes the principle part integral. Now we sum over positive and

negative wavevectors in equation (4.2.2), since both contribute, and change variables such that all the  $k$ 's and  $\gamma$ 's (for convenience) are positive.

We use the transformations

$$\begin{aligned}\omega(-k) &= -\omega(k), \\ \text{and } \gamma(-k) &= \gamma(k),\end{aligned}$$

which are the reality conditions on the Fourier expansion. So doing, we write equation (4.2.2) in full as

$$\begin{aligned}-\frac{\partial U_b}{\partial t} &= \\ &- \left(\frac{e}{m}\right)_i^2 |k_\ell| |\phi_1^\ell|^2 \left\{ P_\ell + i\pi \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_\ell} \right\} \\ &+ \left(\frac{e}{m}\right)_i^2 |k_\ell| |\phi_1^\ell|^2 \left\{ P_\ell - i\pi \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_\ell} \right\} \\ &- \left(\frac{e}{m}\right)_i^2 |k_m| |\phi_1^m|^2 \left\{ P_m - i\pi \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_m} \right\} \\ &+ \left(\frac{e}{m}\right)_i^2 |k_m| |\phi_1^m|^2 \left\{ P_m + i\pi \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_m} \right\}.\end{aligned}$$

We note that the principle part integrals cancel to leave

$$\begin{aligned}-\frac{\partial U_b}{\partial t} &= 2 \left(\frac{e}{m}\right)_i^2 \pi |k_\ell| |\phi_1^\ell|^2 \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_\ell} \\ &- |k_m| |\phi_1^m|^2 \left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_m} \}. \quad (4.2.3)\end{aligned}$$

Now  $\left. \frac{\partial f_{oi}^b}{\partial v} \right|_{v=v_\ell}$  is positive, and if  $f_{oi}^b$  is Maxwellian with thermal velocity  $v_{th,b}$ , it takes on the value

$$\begin{aligned}\left\{ \frac{\delta n v_{th,e}}{2 v_{th,b}} \right\} \exp - \left\{ \frac{\delta n v_{th,e}}{2 v_{th,b}^2} \right\}^2 \\ = G \text{ (say),}\end{aligned}$$

where the values of the parameters  $\delta$ ,  $n$ ,  $v_{th,e}$  and  $v_{th,b}$  are discussed in

Appendix B. We can therefore write

$$\frac{\partial U_b}{\partial t} = -2\pi \left(\frac{e}{m}\right)_i^2 G (k_\ell |\phi_1^\ell|^2 + k_m |\phi_1^m|^2). \quad (4.2.4)$$

So, as the waves increase in amplitude, the beam velocity decreases. This has the effect of (a) shifting the interaction out of resonance, and (b) eventually changing the negative energy waves into positive energy waves. Equation (4.2.4) will break down as the beam velocity approaches zero, since the assumption of separating the beam and stationary terms of the ion distribution function ceases to hold.

To study the quasi-linear equation (4.2.1) for a time dependence in  $v_{th,b}$ , we make the usual definition

$$v_{th,b}^2 = \int f_{oi}^b (v - U_b)^2 dv,$$

and a similar analysis to that done for the beam velocity yields the kinetic equation

$$\frac{\partial v_{th,b}^2}{\partial t} = 4\pi \left(\frac{e}{m}\right)_i^2 G (k_\ell |\phi_1^\ell|^2 + k_m |\phi_1^m|^2) v_{th,b}^2 / v_{th,e} \quad (4.2.5)$$

There is a close similarity in form between (4.2.4) and (4.2.5), and in Chapter 5 we compare these results with results from computational plasma simulations.

### 4.3 Diffusion and Friction Coefficients

In the last Section we assumed that a time dependence in  $f_{oi}^b$  meant a time dependence in  $U_b$  and  $v_{th,b}^2$ . In this Section we examine the Vlasov equation to see how the effective collision terms arising from the wave field affect a plasma configuration of the form under discussion. In particular we examine in graphical form, the properties of the diffusion and friction operators acting on  $f_{oi}^b$  in order to build up a physical picture of the evolution of the ion beam.

Writing the ion Vlasov equation in terms of the Fourier coefficients  $f_k$  and  $\phi_k$  as

$$(\omega - kv)f_k = -\left(\frac{e}{m}\right)_i k\phi_k \frac{\partial f_o}{\partial v} + \left(\frac{e}{m}\right)_i \sum_{k'} (k-k')\phi_{k-k'} \frac{\partial f_{k'}}{\partial v},$$

we extract from the right hand side those terms diagonal along  $f_k$  and denote them by  $-iv_k f_k$ . Now  $v_k$  has the physical meaning of an effective collision frequency, and in general will be an operator. Following Kadomtsev (1965) we can write

$$(\omega - kv + iv_k)f_k = -\left(\frac{e}{m}\right)_i k\phi_k \frac{\partial f_o}{\partial v} + \left(\frac{e}{m}\right)_i \sum_{k'} (k-k')\phi_{k-k'} \frac{\partial f_{k'}}{\partial v} + iv_k f_k. \quad (4.3.1)$$

Equation (4.3.1) gives us an expression for  $f_k$ , and if we substitute this expression into the nonlinear term on the right hand side of (4.3.1) we get

$$\begin{aligned} (\omega - kv + iv_k)f_k &= -\left(\frac{e}{m}\right)_i k\phi_k \frac{\partial f_o}{\partial v} + \left(\frac{e}{m}\right)_i \sum_{k'} (k-k')\phi_{k-k'} \times \\ &\times \frac{\partial}{\partial v} \left\{ -g_{k'} \left(\frac{e}{m}\right)_i k'\phi_{k'} \frac{\partial f_o}{\partial v} + g_{k'} iv_{k'} f_{k'} + \right. \\ &\left. + g_{k'} \left(\frac{e}{m}\right)_i \sum_{k''} \frac{\partial f_{k''}}{\partial v} (k'-k'')\phi_{k'-k''} \right\} + iv_k f_k, \end{aligned} \quad (4.3.2)$$

where  $g_{k'}$  is an operator with the property

$$g_{k'} (\omega - k'v + iv_{k'})f_{k'} = f_{k'}, \text{ for all } k'.$$

For such an operation to have any meaning, we must be able to extract those terms diagonal along  $f_k$  from equation (4.3.2), in order to get an expression for the collision operator. There is only one such term in the summation over  $k''$ , namely when  $k'' = k$ , and then extracting terms diagonal along  $f_k$  gives

$$\left(\frac{e}{m}\right)_i \sum_{k'} (k-k') \phi_{k-k'} \frac{\partial}{\partial v} \left\{ g_{k'} \left(\frac{e}{m}\right)_i (k'-k) \phi_{k'-k} \frac{\partial f_k}{\partial v} \right\} + i v_k f_k = 0.$$

From this equation we are able to inspect the  $k = 0$  terms. That is, the terms which describe the effect of collisions on the background state.

Putting  $k = 0$ , and changing the summation subscript  $k'$  to  $k$  we see that

$$\begin{aligned} i v_0 f_0 &= \left(\frac{e}{m}\right)_i^2 \sum_k k^2 |\phi_k|^2 \frac{\partial}{\partial v} \left\{ g_k \frac{\partial f_0}{\partial v} \right\} \\ &= \left(\frac{e}{m}\right)_i^2 \sum_k k^2 |\phi_k|^2 \left\{ \frac{\partial g_k}{\partial v} \frac{\partial f_0}{\partial v} + g_k \frac{\partial^2 f_0}{\partial v^2} \right\}. \end{aligned} \quad (4.3.3)$$

In particular, we shall analyse the physical effect of the two terms on the right hand side of (4.3.3) in the case of  $f_0$  being  $f_{oi}^b$ . In this case we can write

$$f_0 = f_{oi}^b \exp\{- (v-U_b)^2 / 2v_{th,b}^2\}.$$

Hence,

$$\begin{aligned} i v_0 f_{oi}^b &= \left(\frac{e}{m}\right)_i^2 \sum_k k^2 |\phi_k|^2 \left\{ -\frac{\partial g_k}{\partial v} \frac{(v-U_b)}{v_{th,b}^2} \right. \\ &\quad \left. + g_k \left[ \frac{(v-U_b)^2 - v_{th,b}^2}{v_{th,b}^4} \right] \right\} f_{oi}^b. \end{aligned} \quad (4.3.4)$$

We shall consider the two terms on the right hand side of (4.3.4) separately.

If we treat  $g_k$  as an algebraic multiplier rather than an operator (this involves assuming that  $v_k$  is a simple number), then the second term on the right hand side of (4.3.4) can be written as

$$\left(\frac{e}{m}\right)_i^2 \sum_k k^2 |\phi_k|^2 \frac{1}{(\omega - kv + i v_k)} \left\{ \frac{(v-U_b)^2 - v_{th,b}^2}{v_{th,b}^4} \right\} f_{oi}^b. \quad (4.3.5)$$

It will be recalled (cf. 3.4.1) that the dispersion equation (3.3.7) has two solutions for waves propagating on the beam, namely

$$\omega = k(U_b \pm \delta n v_{th,e}).$$

Moreover, we chose to consider the case of only two beam modes propagating in the plasma. We signified these by  $\phi_\ell$  (negative energy) and  $\phi_m$  (positive energy). Substituting these two terms into the summation in (4.3.5) gives their contribution to be

$$\begin{aligned} & \left(\frac{e}{m}\right)_i^2 \left\{ \frac{k_\ell^2 |\phi_\ell|^2}{k_\ell(v-U_b) - k_\ell \delta n v_{th,e} + i v_\ell} + \frac{k_m^2 |\phi_m|^2}{k_m(U_b-v) + k_m \delta n v_{th,e} + i v_m} \right\} \times \\ & \times \left\{ \frac{(v-U_b)^2 - v_{th,b}^2}{4 v_{th,b}} \right\} f_{oi}^b. \end{aligned} \quad (4.3.6)$$

Breaking this down even further, we first look at the function

$$\frac{k_\ell^2 |\phi_\ell|^2}{k_\ell(v-U_b) - k_\ell \delta n v_{th,e} + i v_\ell} + \frac{k_m^2 |\phi_m|^2}{k_m(U_b-v) + k_m \delta n v_{th,e} + i v_m}. \quad (4.3.7)$$

Now for the case we are interested in, in order to satisfy the resonance conditions (3.3.15),  $k_\ell$  must be greater than  $k_m$ . As we will explain in the next Chapter, the choice  $k_\ell = 3k_m$  is made for the plasma simulation programs, and so we shall use that value in the present discussion. Furthermore,  $|\phi_\ell|$  will be greater than  $|\phi_m|$  since as we stated in Section 4.2, the mode  $\ell$  suffers Landau growth, whereas mode  $m$  is Landau damped. This being the case, the function (4.3.7) is of the form shown in Figure 4.1, where it is shown just away from the real  $v$  axis, where it will be non-singular. From (4.3.6), this non-symmetrical function multiplies

$$\left\{ \frac{(v-U_b)^2 - v_{th,b}^2}{4 v_{th,b}} \right\} f_{oi}^b. \quad (4.3.8)$$

The effect of the term (4.3.8) is to increase the thermal velocity  $v_{th,b}$  since  $f_{oi}^b$  decreases in the region between  $-v_{th,b}$  and  $+v_{th,b}$ , and increases everywhere else (cf. Figure 4.2).

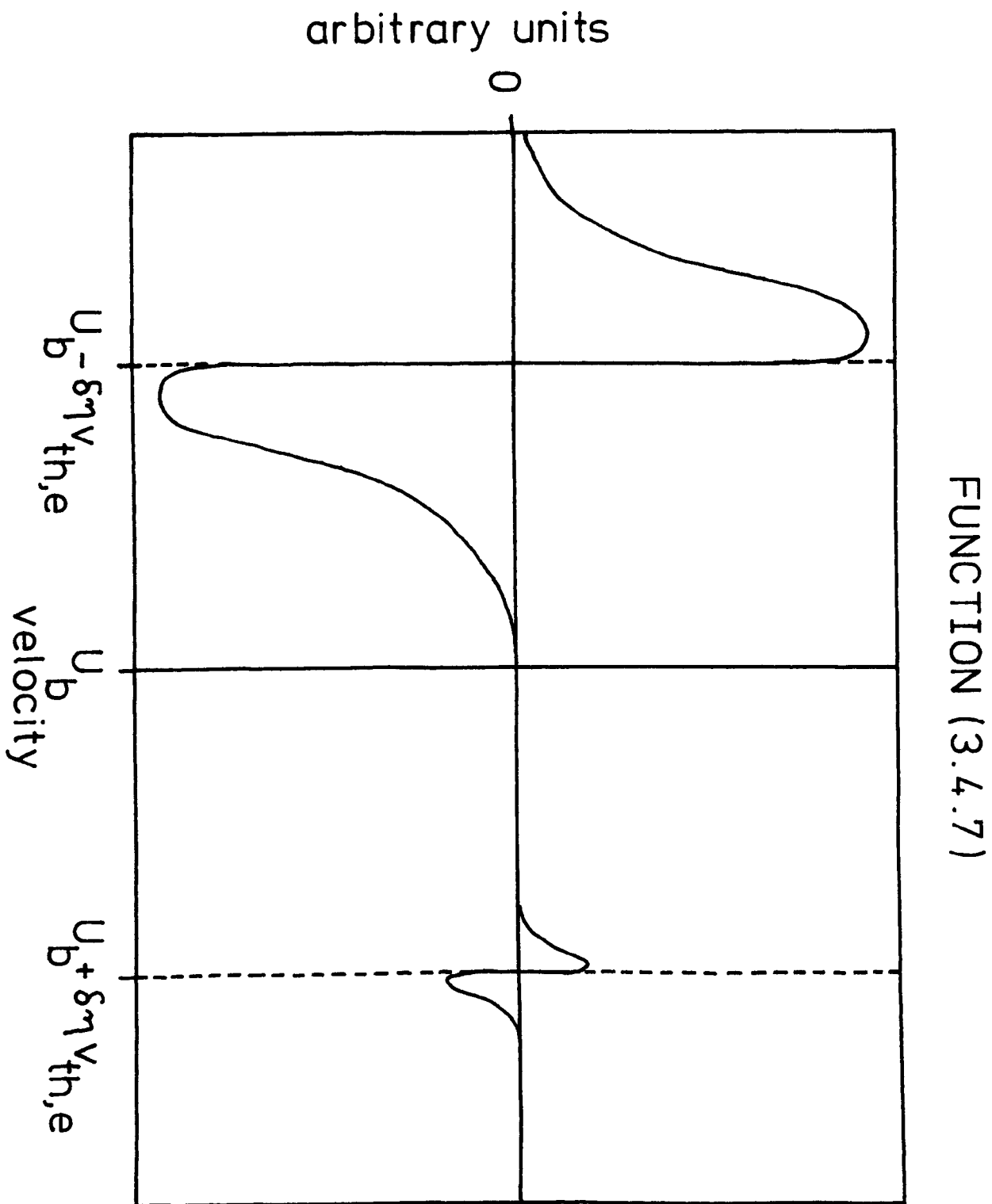


FIGURE 4.1

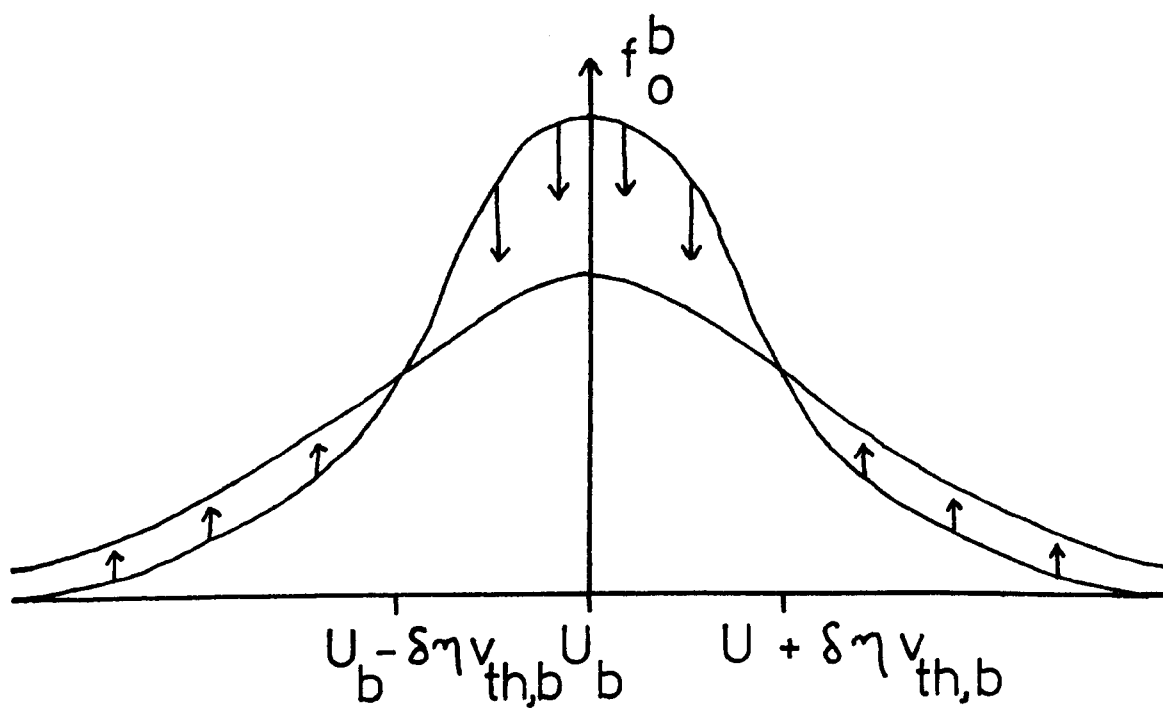


FIGURE 4.2

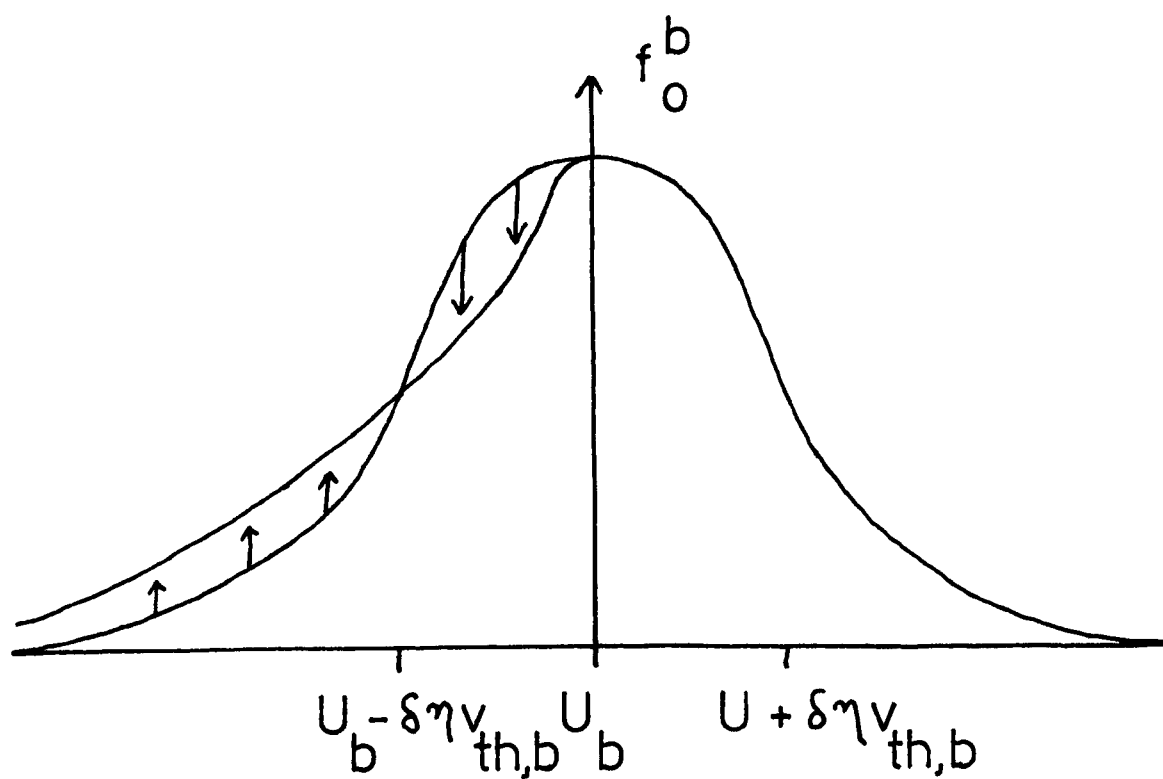


FIGURE 4.3



As well as this heating effect, the beam distribution function is distorted non-symmetrically due to the term (4.3.7). If we consider the dominant part of (4.3.7), namely the part which affects the beam particles near  $v = U_b - \delta n v_{th,e}$ , the beam will distort as shown in Figure 4.3. This distortion, like the heating effect will decrease the Landau growth rate of the negative energy wave by reducing the gradient  $\partial f_{oi}^b / \partial v$  at the wave's phase velocity. If we also consider the remaining part of (4.3.7), namely that which affects the beam particles near  $v = U_b + \delta n v_{th,e}$  we find that the Landau damping rate of the positive energy beam mode increases.

As Fried and Gould (1961) have shown, an increase in ion temperature causes heavy Landau damping of ion sound waves. The effect of the second term in the equation for the effective collision frequency (4.3.4) however, is not only to heat the ions, but also to distort the beam distribution function around the phase velocities of the two beam waves.

Moving on to the first term in (4.3.4), it can be written as

$$- \left( \frac{e}{m} \right)_i^2 \left\{ \frac{k_\ell^3 |\phi_\ell|^2 (v - U_b)}{(k_\ell U_b - k_\ell \delta n v_{th,e} - k_\ell v + i v_\ell)^2} \cdot \frac{1}{v_{th,b}^2} + \frac{k_m^3 |\phi_m|^2 (v - U_b)}{(k_m U_b + k_m \delta n v_{th,e} - k_m v + i v_m)^2} \cdot \frac{1}{v_{th,b}^2} \right\} f_{oi}^b, \quad (4.3.9)$$

where we have substituted the algebraic form of  $g_k$  and the beam modes' wavevectors and frequencies. Since  $k_\ell^3 |\phi_\ell|^2$  is greater than  $k_m^3 |\phi_m|^2$  as discussed earlier, the multiplicand of  $f_{oi}^b$  in (4.3.9) is of the form shown in Figure 4.4. That is, the particle distribution function is distorted such that it decreases in the region  $v > U_b$  and increases in the region  $v < U_b$ . This corresponds to a friction effect in that the average particle velocity decreases. The deceleration moreover, is velocity dependent and hence further contributes to the distortion of the Maxwellian form of the particle distribution.

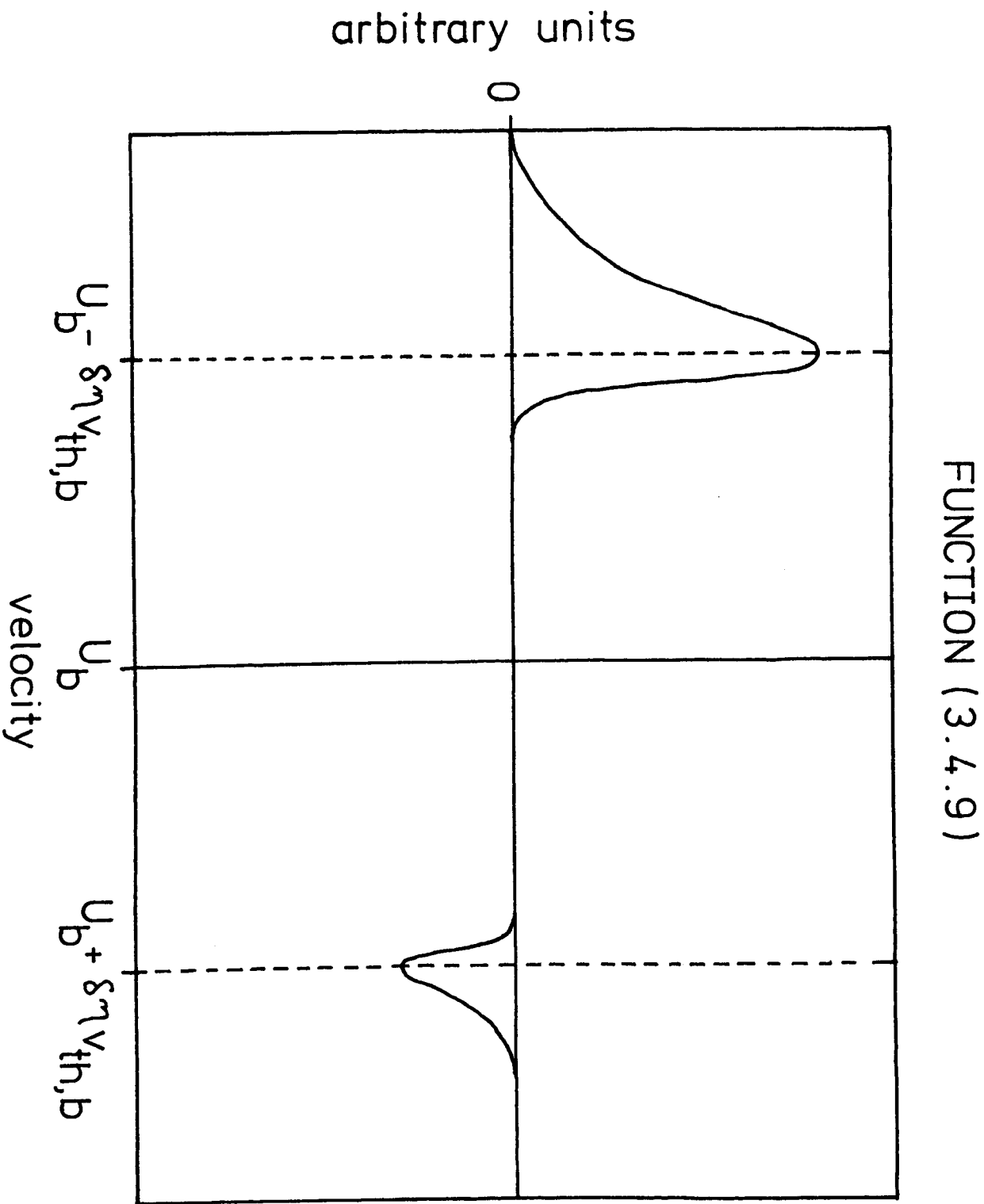


FIGURE 4.4

#### 4.4 Discussion

In this Chapter we have analysed possible stabilising mechanisms for the explosively unstable plasma described in Section 3.4. A quasi-linear analysis of the effects of the waves on the beam ions shows that the beam velocity ( $U_b$ ) can decrease, and the beam temperature ( $T_b$ ) can increase. The equations describing these phenomena (4.2.4 and 5), can be written in the form

$$\frac{dU_b}{dt} = -A, \quad (4.4.1)$$

and

$$\frac{dT_b}{dt} = 2AT_b,$$

where  $A$  is linearly related to the wave quantity  $|\phi|^2$ . In order to compare the relative importance of these two effects, we can make the crude approximation that the waves are growing such that  $A$  is proportional to some power of  $t$ . For convenience we write equations (4.4.1) in the form

$$\frac{dU_b}{dt} = -nBt^{n-1},$$

and

$$\frac{dT_b}{dt} = 2nBt^{n-1}T_b,$$

where  $B$  is a constant. These equations can be solved to give

$$U_b = U_b(t=0) - Bt^n,$$

and

$$T_b = T_b(t=0) + e^{2Bt^n} - 1.$$

Although  $A$  is not proportional to a power of  $t$ , this example shows that while the beam velocity decreases linearly with the quantity  $(t^n)$ , the temperature increases exponentially. As we will see in Chapter 5, the computer plasma simulations also display the much more rapid change in  $T_b$  compared with  $U_b$ , and for this reason we maintain that the heating of the ion beam is more important than the deceleration.

As Fried and Gould (1961) have shown, ion sound waves cannot propagate unless the ions are cold. They show that if the beam ion temperature is of the same order as the electron temperature, then the Landau damping rate of an ion sound wave will be of the same order as its frequency. In other words, the waves will dissipate after a few oscillation periods. Jarmen, Stenflo, Wilhelmsson and Engelmann (1969) show that the Landau damping rate needs only to be greater than a quantity of order  $1/t_{\infty}$  for it to suppress an explosive instability. The heating effect could therefore stabilise the instability through Landau damping.

Since the frequencies of the two beam modes depend on the beam velocity (cf. 3.4.1), the beam deceleration will shift the frequencies  $\omega_l$  and  $\omega_m$ . This causes a breakdown of the frequency resonance conditions (3.3.15), and hence the efficiency of energy transfer between the three waves decreases. This mechanism can therefore also contribute to the stabilisation of the interaction.

The analysis of the ion beam done in Section 4.3 however, shows that as well as a deceleration and heating, the beam ion distribution function distorts. The effects of this distortion are perhaps the most difficult to quantify. It has the effect however of decreasing the Landau growth rate of the negative energy wave, and increasing the damping rate of the positive energy wave. Moreover, since the distortion is non-symmetrical, it causes a perturbation of the average beam velocity (decreasing it), over and above the deceleration discussed earlier.

Other authors have suggested stabilisation mechanisms for the explosive instability. In particular, Fukai, Krishan and Harris (1970) show how the instability can be stabilised by the higher order nonlinear interactions, and in particular by four wave coupling. They show that the four wave interaction equation takes the form

$$\frac{\partial \phi_{\ell}}{\partial t} = \sum_m M |\phi_m|^2 \phi_{\ell},$$

where the matrix element  $M$  is a function of the characteristics of waves  $\ell$  and  $m$ . This equation describes an amplitude dependent shift in the frequency of wave  $\ell$ , which as we stated earlier could stabilise the interaction. It does not however tell us why the frequency should shift. Moreover, as Jarmen, Stenflo, Wilhelmsson and Engölmann (1969) show, while such a shift can stabilise explosive instabilities, it can also destabilise situations which are otherwise stable. For this reason we do not consider this proposal further.

## CHAPTER 5

### THE GALAXY AND NOVA PROGRAMS

#### 5.1 Introduction

In this Chapter we shall discuss the development and results of two computer programs which simulate a plasma. They are 'GALAXY', written by Boris and Roberts (1969 A and B), and 'NOVA' written by McNamara and Langdon (1971). Individual details of each are given in Sections 5.2 and 5.3. Both simulate two dimensional plasmas in which the particles interact through their Coulomb forces, and self consistent magnetic fields are ignored. A constant magnetic field in the third direction however, can be incorporated into either program.

The 'plasma' is considered to be inside a square or rectangular region, which for numerical reasons is divided into a mesh of square cells. The region is periodic in both directions and typically 10,000 to 100,000 particles are used in the simulation. The macroscopic quantities of charge density, potential and electric field are calculated as discrete functions of the x-y cell co-ordinates. The amount of charge in each cell is calculated according to some prescription, and a charge density matrix is built up. A discrete form of Poisson's equation is solved numerically to give a matrix of values of the potential in each of the cells. A simple numerical difference scheme is then used to calculate the electric field vectors in each cell. The positions and velocities of the individual particles are integrated forward through a small time interval, from their equations of motion. Using the updated particle positions, the electric field is recalculated ready for integrating the co-ordinates through the next timestep. This means that full plasma diagnostics can be obtained, either in-line or

by separate programs examining data dumped during the simulation onto some backing store.

At first, it appears that the simulated plasma differs in many characteristics from a real plasma. In particular, the small number of particles used, the discrete nature of the system, and usually an unphysical electron to ion mass ratio, need explanation. The difference arises from computational constraints, but with careful programming the qualitative differences can be minimised. The result is that the programs can only simulate a restricted set of plasma configurations.

As demonstrated by Buneman (1959) a plasma satisfying the Vlasov-Poisson equations can be simulated by a large number of macro-particles which obey the equations of motion

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \alpha,$$

where  $\alpha$  is the acceleration. The macro-particles do not bear a one to one correspondence with real plasma particles, but for reasons of minimising statistical fluctuations in the numerical analysis, a 'large' number of particles is nevertheless desirable. The small number of particles used in practice, is due to the limitations of computer speed and core size. The limitation of core size can be overcome by using a large backing store for dumping and retrieving plasma co-ordinates, but data transfers take time. By multiplexing these input/output operations with central processor activity, this overhead can be considerably reduced. The major considerations therefore are those of computer speed and efficient encoding of the most frequently used routines. Within a given computer configuration, and with a fully optimised program, the choice of the number of particles depends on how many timesteps need executing and how long the computer can be dedicated to that program. In the case of GALAXY and NOVA, as few as 10,000 particles moving for a few hundred plasma periods requires computer activity for a time of the

order of hours. By making the system periodic in both directions, and by solving Poisson's equation for a doubly periodic region, the fact that a small number of particles are used is not critical, since their trajectories are determined by a highly accurate electric field, calculated from a wide spectrum of infinite electrostatic waves.

The discrete nature of the macroscopic quantities arises from the need to solve difference equations rather than differential equations. Since the electric field is a step function of the cell positions, all the particles in a given cell suffer the same acceleration, thus reducing binary collisions between particles in the same cell to zero. It is desirable for the plasma to evolve collision free, and although extremely short range collisions are nonexistent, the particles do suffer from collisions with the grid, that is the discontinuous electric field across cell boundaries. The severity of such collisions depends on the method adopted for calculating the electric field, and in particular on the way the charge density matrix is built up.

In the original version of GALAXY, the charge of a particle was allotted to the centre of the cell in which it resided. The potential was then calculated using a fast Fourier transform technique which we shall discuss later. This method of charge allocation is usually called the nearest grid point (NGP) approximation, and has one severe drawback. In order that the electric field does not fluctuate wildly from cell to cell, an extremely large number of particles must occupy each cell, and as we have said, this is prohibited by its high running cost.

Many methods of overcoming this 'grid noise' are possible, the most obvious being to allocate charge according to a more realistic formula. The best known method of doing this is to adopt the 'cloud in cell' (CIC) method used by Birdsall and Fuss (1969) in which the charge of a particle is



distributed among its nearest neighbour cells according to its position within its cell. Unfortunately, due to the integer nature of the GALAXY co-ordinate buffers and the co-ordinate integration routine, such a modification would have increased the co-ordinate integration time by a factor of four or more.

The method adopted by the author was to calculate the charge density using the NGP approximation, and smooth the electric field using the formula

$$E_{ij} = (E_{i+1,j+1} + E_{i+1,j} + E_{i+1,j-1} + E_{i,j-1} + \\ + E_{i-1,j-1} + E_{i-1,j} + E_{i-1,j+1} + E_{i,j+1})/8,$$

where  $i$  and  $j$  are  $x$  and  $y$  cell numbers. That is, by defining the 'smoothed' electric field to be the average of the values in the eight surrounding cells. This form of smoothing is analogous to calculating the electric field from the potential using a higher order difference scheme to the first order method used in GALAXY. It has been tested by both the author and Hockney (1970), and gives smoothness comparable with the CIC method. It has the advantage over the CIC method of being faster, in that the arithmetic is not only short, but performed only on the electric field matrix rather than each particle's co-ordinates.

In a real plasma, since the ions are heavier than the electrons, they move more slowly unless of course, the ion temperature greatly exceeds the electron temperature. For the present case however, this creates numerical difficulties since the ion motion is highly important. For stability of the numerical integration scheme, the timestep of integration must be small enough to ensure that the fastest particles move only  $1/2$  of a cell in one step. This means that tens of thousands of timesteps are required to simulate a significant amount of ion movement. This being too expensive, the ion to

electron mass ratio is usually decreased in simulation programs to a level which can be afforded. The ratio however, must be maintained high enough for the electrons and not the ions to respond to local changes in space charge, otherwise much of the physics would be lost. It is found from testing various values, that a ratio of about 16:1 is the lowest 'safe' limit for simulating ion motion. For a large range of phenomena, this ratio does not affect results qualitatively, and a simple re-scaling can give accurate quantitative agreement with experiment (cf. McNamara, Boris, Cook and Sykes, 1969). The value of 16:1 was the value used by the author for the experiments discussed in Sections 5.2, 5.3 and 5.4.

In order to simulate the explosively unstable plasma discussed earlier, and get results which can be compared with the theory, the programs must have two important properties. These are, that the program diagnostics include an accurate Fourier spectrum of the waves, and that the plasma uniformity in the second direction be maintained exactly.

The uniformity was maintained in essentially the same way in both GALAXY and NOVA. GALAXY uses a  $64 \times 64$  mesh of square cells, but only one strip of cells in the x-direction was occupied by particles. When the 64 component charge density vector was calculated, it was replicated in each of the other 63 strips parallel to it. This made it possible for the standard two dimensional Poisson solver to be used with only minor re-scaling modifications. More importantly, 64 times as many particles than in the full two dimensional program, could be simulated.

In the NOVA program, the numerical analysis was performed on a rectangular grid of  $128 \times 4$  cells (the reason for these choices of grid sizes will be explained later in this Section). Having only 4 cells in the (uniform) y direction made the replication of the charge density vector eight times faster than in GALAXY. Moreover, with 128 cells covering the

same 'real' length of plasma as in GALAXY, and with floating point co-ordinate arithmetic, numerical errors were considerably reduced in NOVA.

It will be noticed that in both programs, the number of cells along each side of the grid is a power of two. This is because of the numerical technique used to solve Poisson's equation. The charge density matrix is Fourier transformed, and Poisson's equation is solved in k-space. For a neutral plasma, this simply involves setting

$$\phi_k = \frac{4\pi n_k}{k^2}, \quad k \neq 0,$$

$$\phi_k = 0, \quad k = 0,$$

where  $n_k$  is the k-th component of the transformed charge density. In this way, the potential coefficients are available for comparison with theory, and can be Fourier synthesised to give the potential in real space from which the electric field can be calculated. The Poisson solver in GALAXY was written by Boris (1968) and operates on a square grid of side  $2^n$  cells, where n lies between 2 and 6. The NOVA version is more general, operating on a rectangular grid of size  $2^m \times 2^n$  cells, where m and n have a minimum value of 2 and a maximum governed by the size of the computer. Their common denominator and central feature is the 'Fast Fourier Transform' method of Gentleman and Sande (1966).

The 'Fast Fourier Transform' is an algorithm whereby an N point transform is defined in terms of smaller transforms having lengths which are factors of N. This is a recursive formula which may be implemented directly, although by simulating the recursion, one avoids the unnecessary calculation of some complex exponentials. By making N a power of 2, the complete transform is defined in terms of 2-point transforms which can be done using only addition and subtraction. The logic of the algorithm requires a minimum sequence length of 4, but is easily adaptable to cover multi-dimensional

sequences. The reason that the 'Fast Fourier Transform' was implemented in GALAXY and NOVA, rather than using a standard five point difference scheme for solving Poisson's equation, is twofold. Firstly, it provides the exact solution for a discrete periodic system, and secondly it gives the physically invaluable potential coefficients as part of the diagnostic information.

From the nature of these programs it is clear that an extremely large number of plasma parameters are available for analysis at each timestep. For the most part however, this data will be reduced to those quantities which are important to the physical problem in hand. In the present case for instance, we only require tables of wave amplitudes, beam temperature and average beam velocity at each timestep. It is necessary however, to examine the plasma behaviour globally to ensure that no extraneous physical or numerical phenomena are present. This procedure, necessary to determine whether the experiment is functioning as it was designed to do, is impossible to perform manually on account of the size of the data. The task of getting a global picture of particle trajectories and overall plasma evolution must therefore be done with the aid of the computer.

For this reason, the 'MOVIE' package was written by Boris, Hodges, Roberts and Hamilton (1969) for use with the GALAXY program. MOVIE produces frame(s) of film at each timestep, of selected particle positions or views in k-space, using a model 120 Benson-Lehner microfilm recorder. The necessity for an efficient and versatile graphics package for the examination of raw data cannot be overemphasised. The author used the package during the testing of various methods of initialising particle co-ordinates to give an equilibrium plasma, and during the testing of methods to perturb selected waves without introducing extra effects. This saved a considerable amount of time during the setting up of the experiment, and clearly could also aid the physicist in finding the physical processes which govern some plasma phenomena.

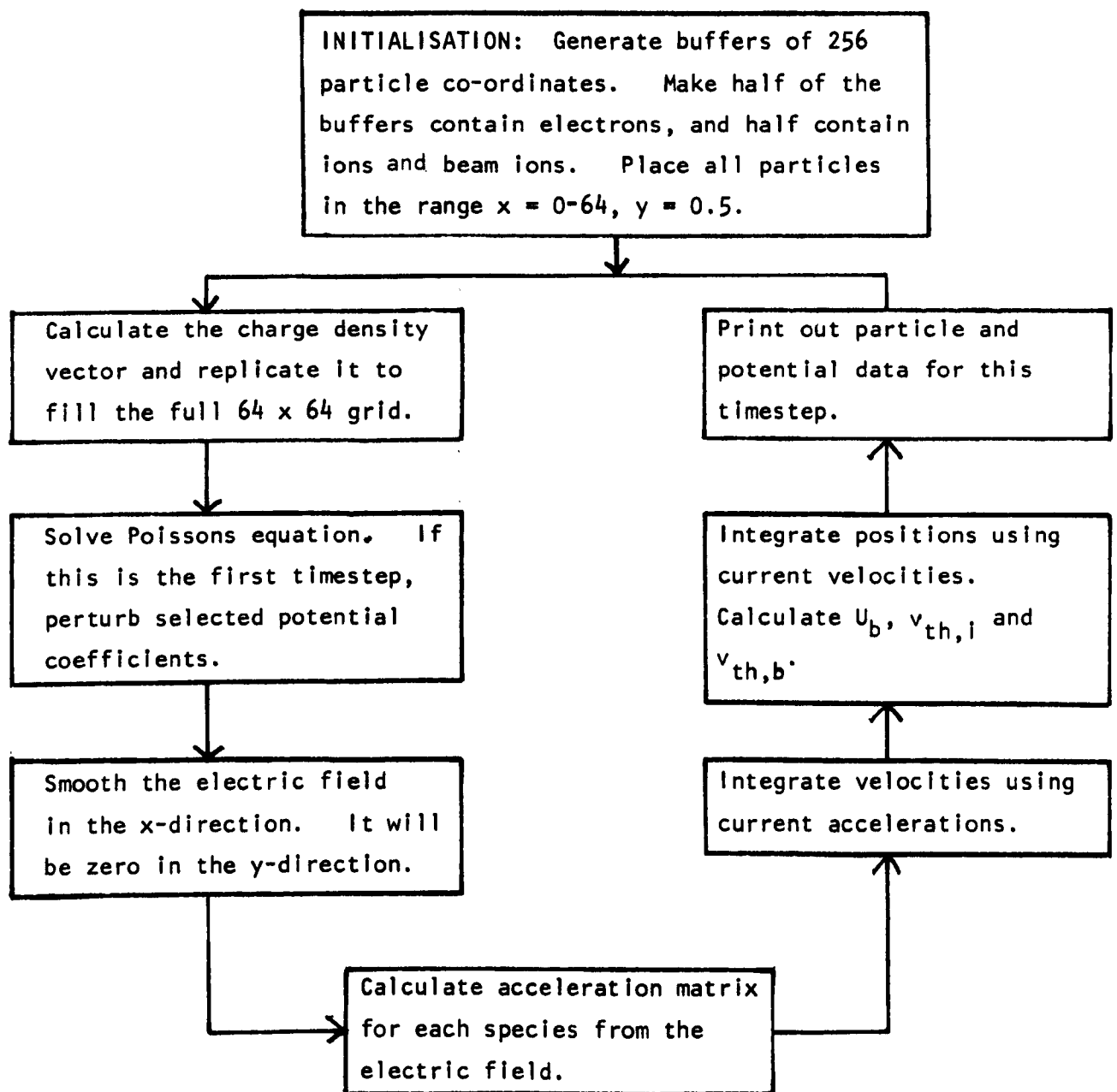
Having outlined the framework of the simulation programs, in Sections 5.2 and 5.3 we shall discuss GALAXY and NOVA individually, in terms of accuracy, speed and versatility. The results of the experiments, and the comparison with theory are given in Section 5.4.

## 5.2 An Appraisal of GALAXY

A two dimensional plasma simulation program, GALAXY, was written by Boris and Roberts (1969 A and B) to run on the ICL KDF-9 computer at the UKAEA Culham Laboratory. It was written to test a number of programming techniques in an environment of repetitive computations involving a large number of variables. It was tailored to the core size, word length, instruction set and logic of the KDF-9, and evolved into a powerful tool for studying nonlinear phenomena in plasmas (cf. McNamara, Boris, Cook and Sykes, 1969). A flowchart of the program in the form used by the author is given in Figure 5.1.

The positions and velocities of a large number of point particles are computed at successive closely spaced timesteps by calculating the electric field acting on each of them from Poisson's equation. An immediate problem arises in the choice of initial particle positions and velocities, such that they form a steady state distribution such as that shown in Figure 3.1.

The simplest way to achieve spatial uniformity is to assign particle positions on a regular lattice within the region of the plasma. Unless particle positions and velocities are carefully chosen however, the net result of such an initialisation is that the positions randomise after a few timesteps, and it is a long time before collective effects are measurable against the background noise. The time taken for the KDF-9 to execute a single timestep of GALAXY with a particle configuration of the form adopted by the author, was typically 10-20 seconds, and so the cost in terms of computing time prohibited the regular use of such a technique. In the next



Flowchart of GALAXY showing variations required to deal with the one dimensional explosive instability. Details of input/output and vector integration are given by Boris and Roberts (1969 A and B).

FIGURE 5.1

Section however, we show how a careful 'lattice-type' choice of positions and velocities can yield a low-noise numerical plasma. The method unfortunately, did not lend itself to GALAXY because of other limitations of the program, which we will discuss later in this Section.

Another common method of particle positioning is to place them completely randomly to begin with, and damp the resulting noise during the solving of Poisson's equation. Apart from being a physically questionable practice, it is easily shown that in a two dimensional plasma, the energy spectrum generated by randomly positioned particles is of the form

$$W(k) \propto 1/k^2,$$

which diverges for small  $k$ . For such a plasma in thermal equilibrium, the spectrum should be of the form

$$W(k) \propto 1/(k_D^2 + k^2),$$

where  $k_D = 2\pi\lambda_D^{-1}$ , the inverse Debye screening length. This concentration of energy into the longer wavelengths causes severe problems when studying the behaviour of low amplitude, long wavelength phenomena. Figure 5.2 shows a plot of  $\phi_k(t)$  from a typical run of GALAXY initialised using random positioning. The value of  $k$  corresponds to the longest wavelength mode in the plasma. No smooth waveform can be detected, and the amplitude, which approaches unity at times cannot be considered 'small' for purposes of relating results to a perturbation theory.

The diverging form of the energy spectrum arises from neglecting binary correlations within the plasma; but calculating particle positions to give the exact Debye distribution would take many hours of computing time in a program like GALAXY. Williamson (1970) however, proposed an algorithm for correlating initial particle positions such that they displayed the correct Debye distribution to a high degree of accuracy. His method, which we shall

GALAXY: using random positions

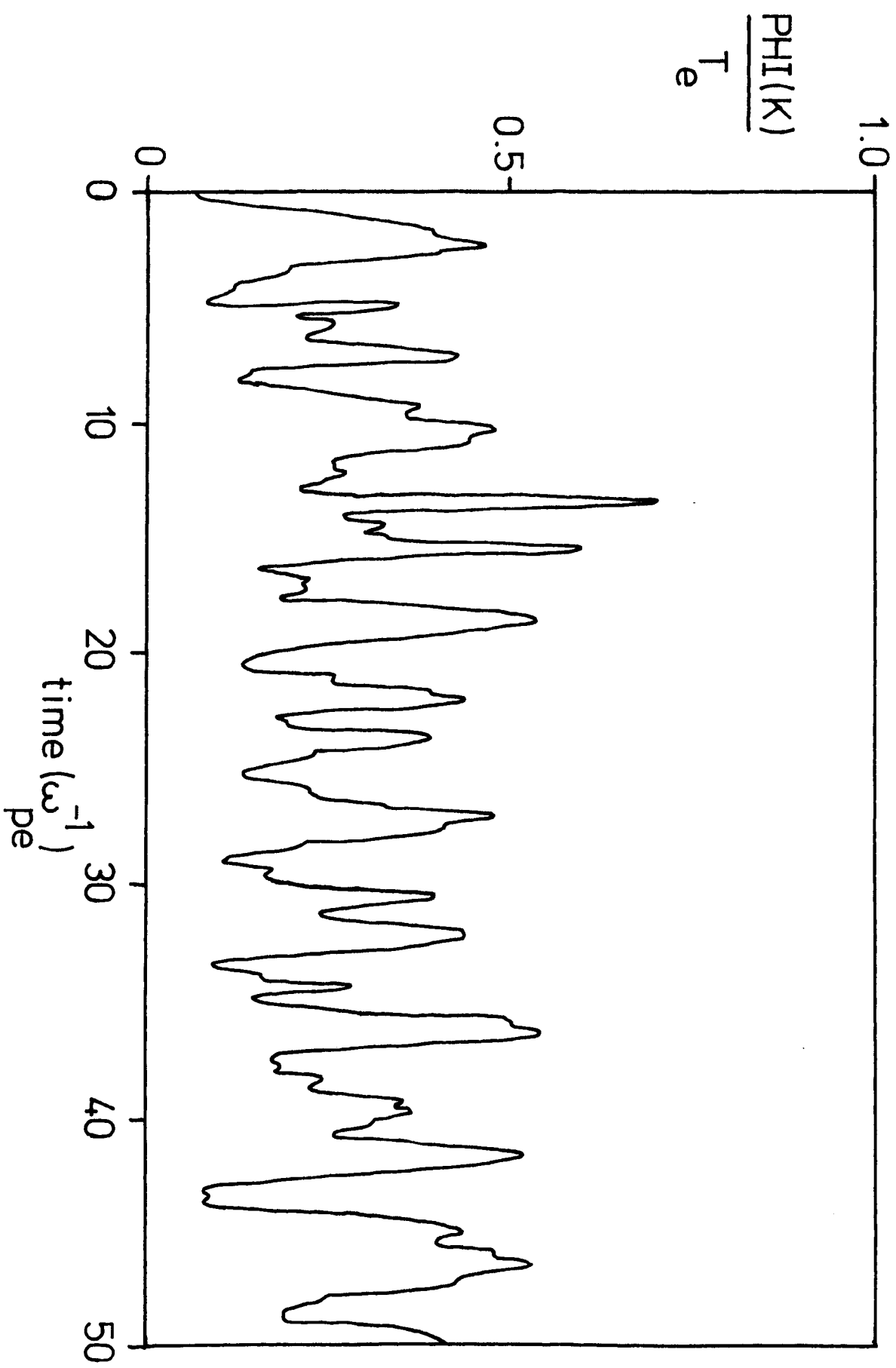


FIGURE 5.2



not discuss in detail here, involves positioning one species, say ions, randomly, and to each ion associating an electron at a distance determined by an appropriate function of a normalised random variable. Using his algorithm, Williamson claims that 99.95% of the electrons are within  $0.001 \lambda_D$  of their correct position, and that less than one electron in  $10^{12}$  is misplaced by more than  $0.01 \lambda_D$ . It was Williamson's method which the author used in production runs of GALAXY in order to help cancel numerical 'noise' generated by approximations and rounding errors elsewhere in the program.

Initialisation of particle velocities is a simpler task, and there are many ways of generating a large number of velocity values which for instance are normally distributed. The method used by the author for runs of GALAXY, was to generate the x-velocity from the relation

$$v_x = v_{th} (-2 \log_e R),$$

where  $v_{th}$  is the thermal velocity of the species, and R is a random number in the range 0-1. The y-velocities were set to zero, since it was desired to simulate a one dimensional plasma.

As we have said, GALAXY uses a  $64 \times 64$  grid for its numerical analysis. In terms of physical units, we use non-dimensional quantities as follows,

Debye screening length	=	$\lambda_D$	=	1 cell,
electron plasma frequency	=	$\omega_{pe}$	=	1,
electron thermal velocity	=	$v_{th,e}$	=	1,
electronic charge	=	e	=	-1,
electronic mass	=	$m_e$	=	1,
electron temperature	=	$T_e$	=	1,

and all plasma quantities are given in terms of these units (potential is given in terms of  $T_e$ ). In these units, the Poisson solver uses the discrete set of k values,

$$k = \frac{\pm 2\pi}{64\lambda_D} \quad (0, 1, 2, \dots, 31),$$

to form the Fourier sum. This gives a spectrum of waves from  $-k_D/2$  to  $+k_D/2$  which may interact. This choice of data ensures that only those waves which would have been heavily Landau damped ( $|k| > k_D/2$ ) are omitted from the sum.

We choose the wavevectors of the three interacting waves to be

$$k_\ell = 3.2\pi/64\lambda_D \quad (\text{slow beam mode}),$$

$$k_m = 1.2\pi/64\lambda_D \quad (\text{fast beam mode}),$$

$$k_n = 2.2\pi/64\lambda_D \quad (\text{ion sound mode}).$$

This choice is made because they are the three longest wavelength modes which satisfy the wavevector resonance conditions (3.3.15), and as such are the three waves least affected by Landau growth or damping. The frequency resonance conditions are satisfied by making the ratio of beam ions to total ions,

$$\eta^2 = 1/4,$$

and by making the ion beam velocity,

$$u_b = \frac{2 + \sqrt{3}}{8} \approx 0.4665.$$

As far as possible we have tried to make the simulated plasma satisfy the limitations imposed in Appendix B for calculating the explosion time  $t_\infty$ . Two of the approximations are not satisfied as well as ideally they might be. The quantity  $\eta^2$  is required to be very much less than unity. Unfortunately, a value much less than  $1/4$  would have made the number of simulated beam ions insufficient to support the collective oscillations required. Also, it is necessary that the beam velocity satisfies

$$v_{th,i} \ll u_b \ll v_{th,e},$$

but in fact it is not much smaller than  $v_{th,e}$ . Its value is determined by

the other parameters of the system, and the frequency resonance conditions. The only way it can be made smaller without upsetting other requirements, is by making  $n^2$  smaller also.  $U_b$  is however  $\gg v_{th,i}$  since the electron to ion temperature ratio is arbitrary, and the choice of 32 satisfies the inequality.

As shown in Appendix B, we expect the explosion time  $t_{\omega}$  to be of the order of many hundreds of plasma periods if the  $\phi$ 's are initially  $\approx 0.1$ . With a timestep as large as  $0.5 (\omega_{pe}^{-1})$ , the constraint of computer time restricts the total number of particles used to less than 5000. Our choice of 4096 seems small, but being only one dimensional, corresponds to 64 plasma 'sheets' per Debye length, which is within the physical and numerical approximations.

GALAXY was tested many times with various data in order to find a satisfactory set of parameters to simulate the explosive instability. It was found that perturbing the three selected Fourier potential coefficients was unnecessary, since the level of thermal fluctuations is large enough to cause the three waves to go unstable.

The GALAXY program employs an exceptionally fast form of vector integration devised by Boris and Roberts (1969 A and B), which is based on packing the x and y parts of a co-ordinate (position, velocity or acceleration) into a single 48 bit KDF-9 word. By suitable packing, both components of a co-ordinate can be integrated using a single integer addition, but unfortunately, the method has only first order accuracy in the timestep  $\Delta t$ . Since our simulation concerns ion motion over many hundreds of plasma periods, the timestep was chosen to be 0.5. This however, causes an accumulation of errors in the electron trajectories and a corresponding electron noise problem over the whole energy spectrum. To modify the vector integration routine to use a higher order scheme would increase the GALAXY execution time by an order of magnitude, which is clearly out of the question.

As we said in the previous Section, by employing the NGP approximation for determining the charge density matrix, and by smoothing the electric field after solving Poisson's equation, we get an electric field matrix which has continuity comparable with CIC calculations. Although this is the case, the accuracy is not as good as that for CIC. Whereas our method takes an inaccurate electric field and smooths it, the CIC method uses an accurate charge density and automatically gets a smooth field from it. The important point to note is that our method, while damping erratic fluctuations in the particle trajectories caused by the vector integration technique, gives Fourier potential coefficients derived from the rough NGP approximation.

The results of the GALAXY runs are therefore open to a certain amount of criticism, although by carefully analysing the results, the ion behaviour can be isolated and examined. We will discuss the actual results and their reliability further in Section 5.4. These inaccuracies and limitations however, indigenous in the GALAXY framework, do not necessarily cause trouble in all plasma problems. Many plasma configurations have been studied accurately using GALAXY, and as a test-bed for programming techniques, it provided a good background for writing the NOVA program.

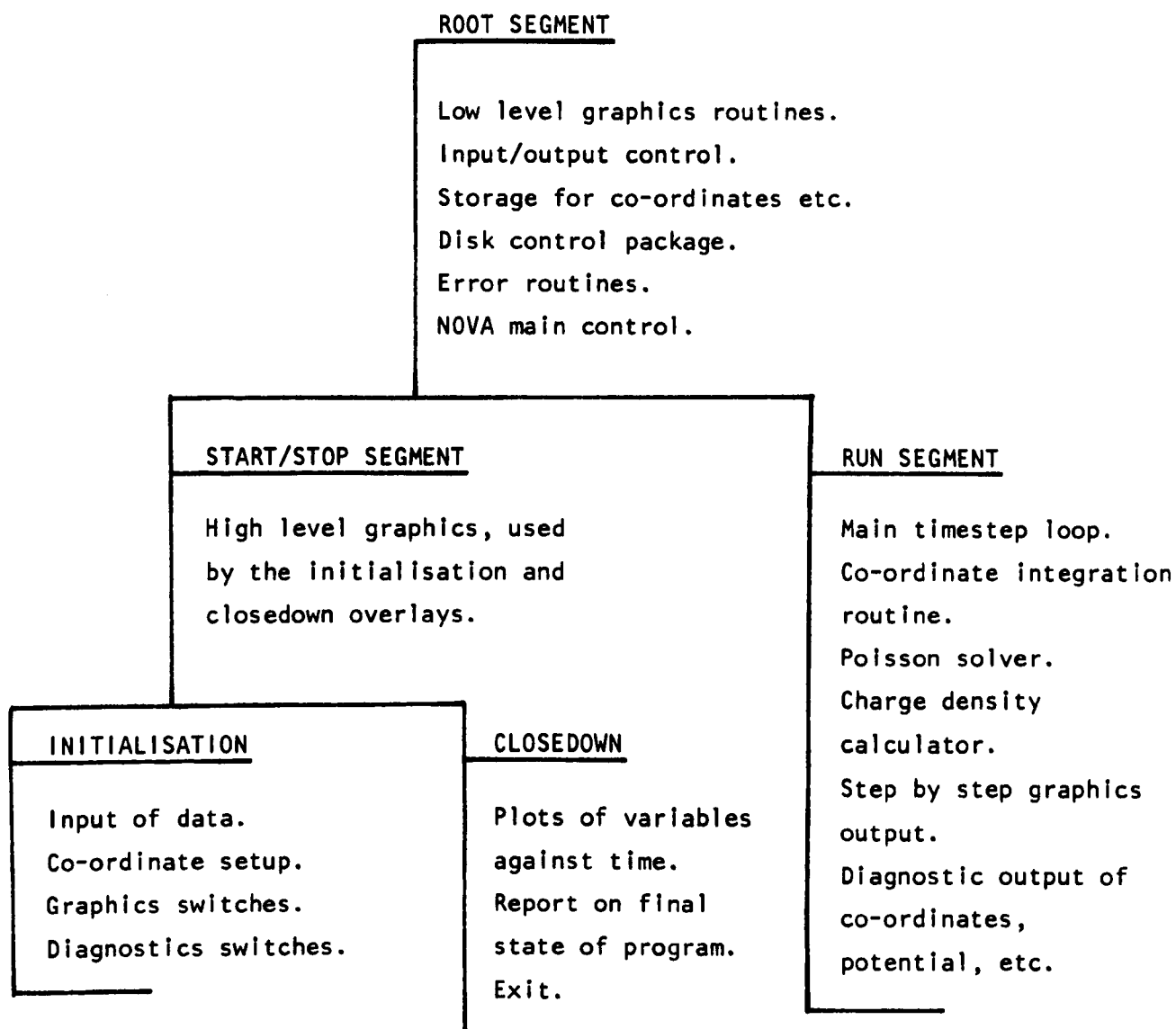
### 5.3 An Appraisal of NOVA

NOVA is a general program for numerically solving multi-timestep problems. It was written by McNamara and Langdon (1971) and was first used as a two dimensional plasma simulation program on the CDC-6600 computer at LRL Livermore, USA. Since all but the Poisson solver was written in Fortran IV, it was easily transported to run on the ICL System 4/70 computer at the UKAEA Culham Laboratory. NOVA has a number of features which make it considerably more versatile and accurate than GALAXY. The added generality and more complicated numerics can be afforded however, since the 4/70 is a much larger and faster machine than the KDF-9.

The data for runs of NOVA only differs from those of GALAXY in that 128 cells are used in the x-direction, and the Debye length is two cells rather than one. This means that we are simulating the same sized plasma as in GALAXY (64 Debye lengths), but in a numerically more accurate fashion. The speed and size of the 4/70 makes it possible to simulate the 4096 particle plasma completely in core, and perform analysis of the system during run time. If more particles are required than can be coped with in core, the data transfers in and out take place to magnetic disk rather than tape. This not only makes the transfers orders of magnitude faster, but also keeps them free from the read/write errors which tape is prone to giving.

The program itself uses a more sophisticated timestep loop than GALAXY. The charge density is calculated according to a generalised form of the CIC method. A given particle can be assumed to have its charge distributed over any range according to any functional dependence, such as an exponential Debye screening function or a simple nearest neighbour distribution. The method used for the runs of NOVA discussed here, is to spread the charge of a particle evenly over half a Debye length centred at the particle's co-ordinate. The fraction of charge which then resides within each cell is allocated to that cell for purposes of building up the charge distribution. The numerical accuracy and stability of using floating point variables in this manner makes the energy spectrum and electric fields almost totally free from numerical noise. It also makes it possible to use co-ordinate integration schemes which are accurate to a high order in the timestep  $\Delta t$ , thus making the choice of  $\Delta t = 0.5$  numerically stable.

The major structural difference between NOVA and GALAXY is that those NOVA packages which are not used at every timestep are overlaid. The overlay structure of NOVA is shown in Figure 5.3, and this structure means that the initialisation and closedown packages do not occupy valuable core space during the main execution of the program.



Overlay structure of NOVA. The Root Segment is always in core. Program starts with Start/Stop and Initialisation Segments in core. These are overwritten by the Run Segment for the main program execution, and this in turn is overwritten by the Start/Stop and Closedown Segments when the main run has finished.

FIGURE 5.3

In the initialisation overlay, we use an algorithm designed to allocate positions and velocities to particles in such a way that the electrostatic energy stays at a level many orders of magnitude below the typical particle energies. This is achieved by placing the particles on a set of closed orbits in phase space, which in a periodic one dimensional plasma is trivial since all orbits are closed. In higher dimensions however, it is a non-trivial problem, and it is important that the orbits are closed and not ergodic since a finite number of particles on an ergodic orbit will produce charge fronts moving along the orbits, generating unwanted potential fluctuations. In general, the closed orbits will have to be discovered numerically.

In the one dimensional case however, we select the closed orbits by firstly assuming that the three Maxwellian distributions (electrons, ions and beam ions) can be approximated by step functions of the form shown in Figure 5.4. The number of particles at any given velocity is the integer part of the normalised Maxwellian distribution, and the total should be as close as possible to the number required in that species. Invariably, since we are truncating real numbers to integers the totals will not tally. In order therefore, to maintain charge neutrality and the correct charge and mass balance between the three species, we simply modify their charges and masses accordingly. The result may be slightly too few ions having slightly too great a mass and charge, but the system acts in the same way as the desired configuration. Having split the distributions into sets of particles in this manner, we then allot them positions on the grid in such a way that the particles within each set are uniformly distributed, and the total plasma is uniform.

In this way, the electric field is zero, and hence particle velocities remain constant. The particles will follow the closed (uniform) orbits formed by the other members of their own set and but for numerical rounding

Step function approximation to a Maxwellian

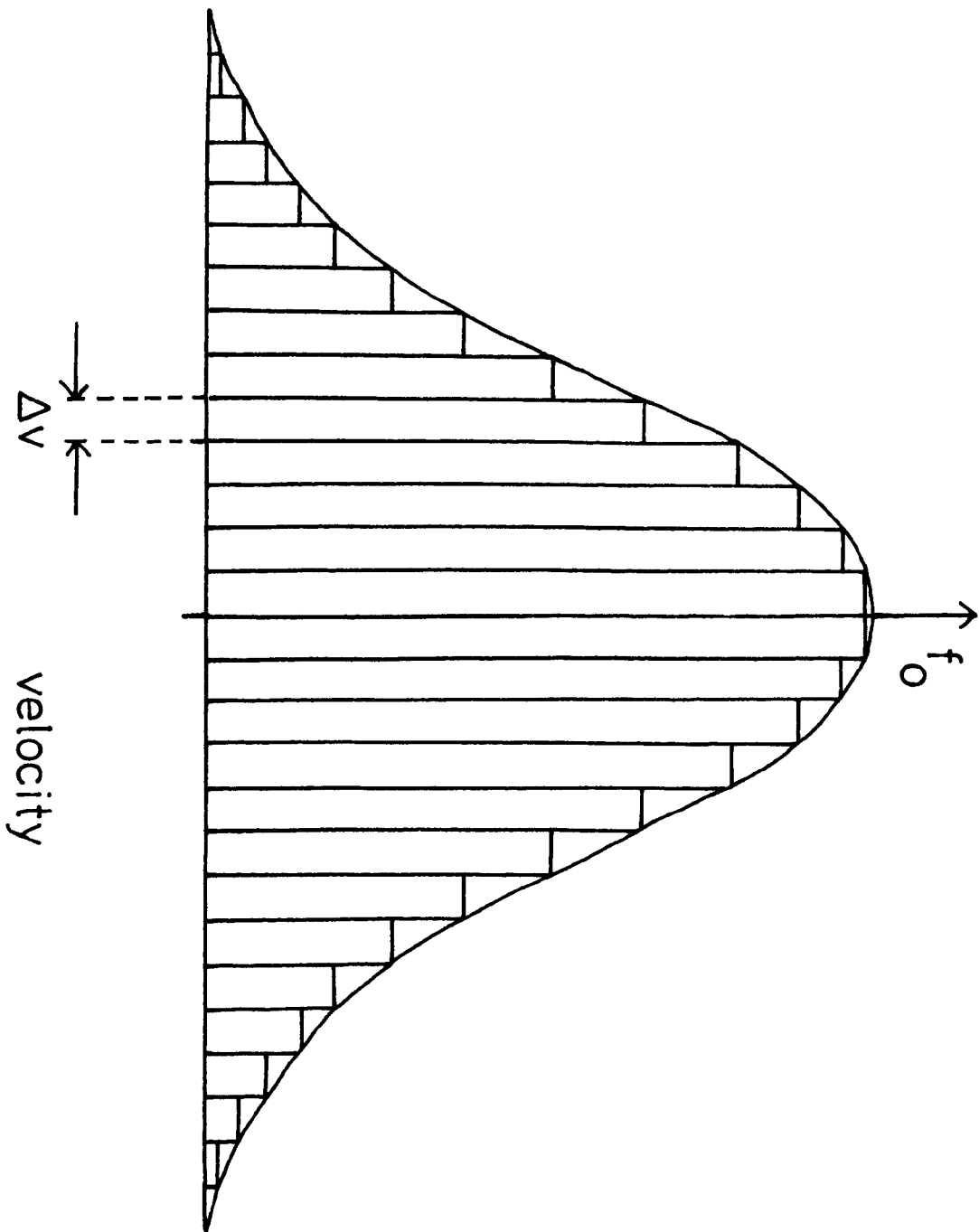


FIGURE 5.4



error, permanently move in zero electric field. For the configuration we are studying, the rounding error at time  $t = 0$  gives potential coefficients of the order of  $10^{-10}$ . The regular lattice nature of each of the sets of particles raises the problem of whether particle correlations are too high for the system to behave like a real plasma. As we show in Appendix C however, the effect on the plasma is small providing the velocity spread of each set of particles ( $\Delta v$  in Figure 5.4) is small enough.

On top of this 'equilibrium', we wish to impose perturbations of the order of  $10^{-7}$  or  $10^{-8}$  so that they are much larger than the background level, but nevertheless much less than unity. In the GALAXY program, the perturbations were applied to the Fourier potential coefficients of the appropriate wavevectors, but this process affects all the possible eigenmodes of the system having those wavevectors. We wish to exclude the possibility of exciting plasma oscillations or normal ion sound waves at the selected ion beam mode wavevectors, for instance. To achieve this, we start with the transform of the linearised Vlasov equation for the electrons,

$$(-i\omega + ikv)f_{1e}^k = \left(\frac{e}{m}\right)_e ik\phi_1^k \frac{\partial f_{oe}}{\partial v}(v).$$

The three waves in question all have phase velocities much less than  $v_{th,e}$ , and so we can write

$$v f_{1e}^k = \left(\frac{e}{m}\right)_e \phi_1^k \frac{\partial f_{oe}}{\partial v}(v),$$

to describe how the electrons behave when a slow wave  $k$  propagates. Since  $f_{oe}$  is a Maxwellian, this takes the form

$$f_{1e}^k = -\left(\frac{e}{m}\right)_e \frac{\phi_1^k f_{oe}(v)}{v_{th,e}^2},$$

and so we can write the perturbation on an electron at  $(x,v)$  with a slow wave  $k$  propagating as

$$p(x,v) = A f_{oe}(v) \sin kx,$$

where A is a measure of the amplitude of the perturbation. Since three slow waves are propagating, we need to perturb the electrons such that

$$p(x,v) = f_{oe}(v) \{ A \sin k_x x + B \sin k_m x + C \sin k_n x \}.$$

Similarly, perturbations on the stationary ions and beam ions are calculated, and we displace each particle from its equilibrium orbit according to these formulae. In this way, we excite only the three waves which will interact through the explosive instability mechanisms.

As well as being general enough to be easily adapted to the requirements of different plasma configurations, NOVA is considerably more sophisticated than GALAXY both in its numerical analysis and in its structure. This sophistication is reflected in the low noise nature of its results, which are discussed in the following Section.

#### 5.4 Results of the Simulation Experiments

The preparation, testing and running of the GALAXY program for this research took place from October 1969 until August 1971 by which time the 4/70 computer had been installed at Culham and the NOVA program was available for development. The NOVA program then took priority, continuing until August 1972.

We give here the results of a run of GALAXY which took place in August 1971. It is one of many using different initial particle co-ordinates calculated using the algorithm of Williamson (1970), all of which gave similar results. Figure 5.5 shows the growth, saturation and decay of the three interacting waves. Only the maximum value of the potential at each wave cycle is used to form the curve, and the plasma oscillations at these wavevectors have been phased out. The average value of the other potential coefficients is also shown in Figure 5.5. Two of the amplitudes reach about

# Results from the main run of GALAXY

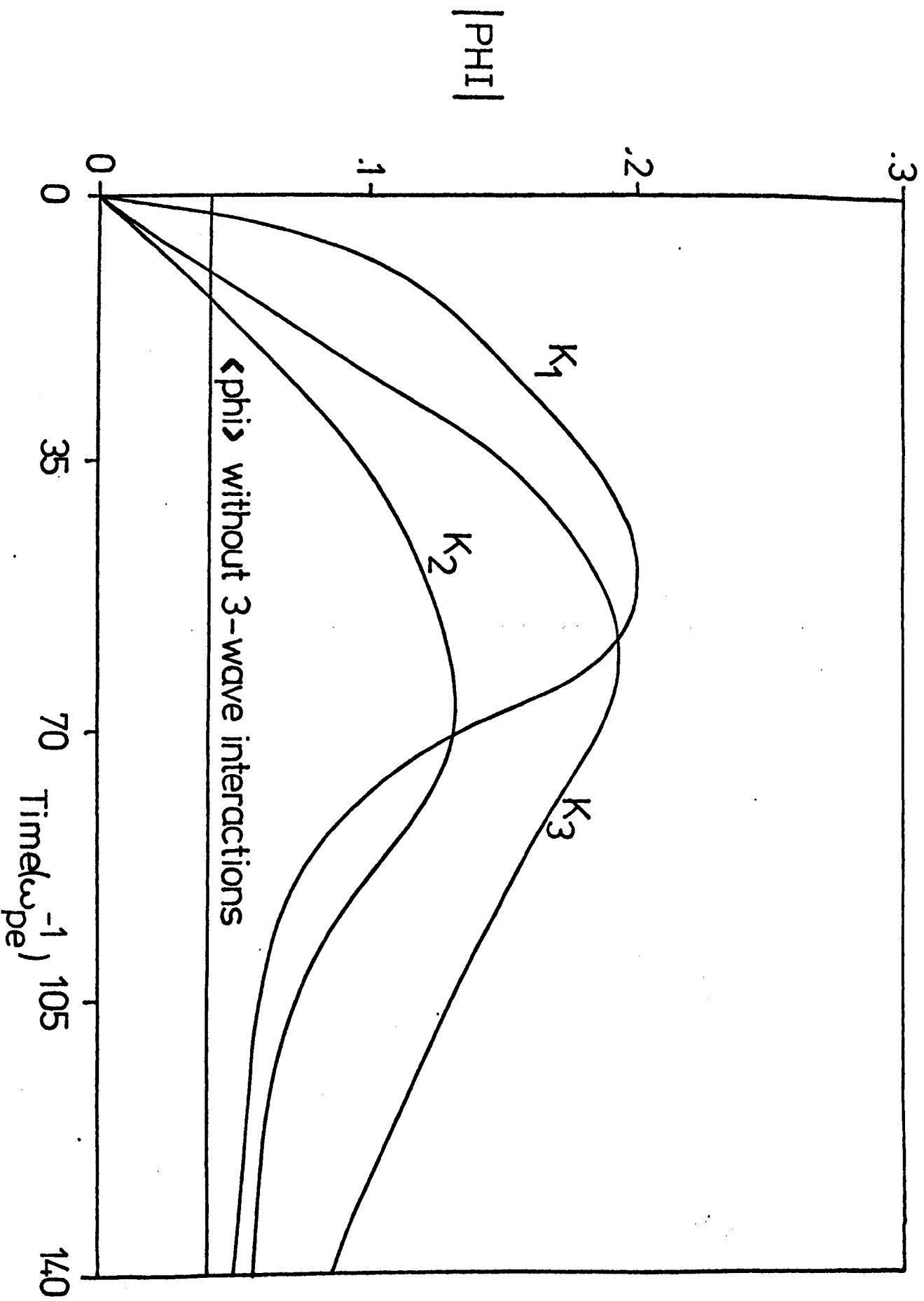


FIGURE 5.5

five times the background level before saturating around  $t = 70 \omega_{pe}^{-1}$ . This corresponds to a growth rate which is higher than the theory predicts for the early stages of growth. As we see from Figure 5.6, this behaviour is accompanied by a drastic reduction in the ion beam velocity. After about  $70 \omega_{pe}^{-1}$  the beam velocity has reduced enough for  $\partial\epsilon/\partial\omega$  to become positive, and hence for the explosive nature of the interaction to cease.

The three wave interaction equation (3.3.16) was solved numerically for purposes of comparing the simulation results with the theory. The numerical solution took into account the beam velocity equation (4.2.4), and the dependence of  $\partial\epsilon/\partial\omega$  in (3.3.16) on the beam velocity. The results are shown in Figures 5.7 and 5.8, and clearly do not agree quantitatively with the simulation results. The theory predicts that the wave amplitudes grow and that the beam velocity slows down at a considerably slower rate than the simulation experiment shows. Although the simulation does give results which behave qualitatively in the way that the theory predicts, the high noise level and the faults in the program outlined in Section 5.2 cast certain doubts on the reliability of the results.

Because of the high noise level and the large amplitude of the ion beam oscillations, no detailed analysis of the ion beam was possible. We give the results here however, since they show that a definite change does take place in the configuration of the ion beam, although the nature of the change is not determined conclusively by GALAXY.

As with GALAXY, many production runs of NOVA were executed once the 'equilibrium' start had been written and tested. They all gave similar results for different perturbations on the waves, and the results of one of the runs are shown in Figures 5.9 and 5.10. These results are more conclusive in that the changes in wave amplitude vary many orders of magnitude compared with the statistical fluctuations. Moreover, the three waves

Result from the main run of GALAXY

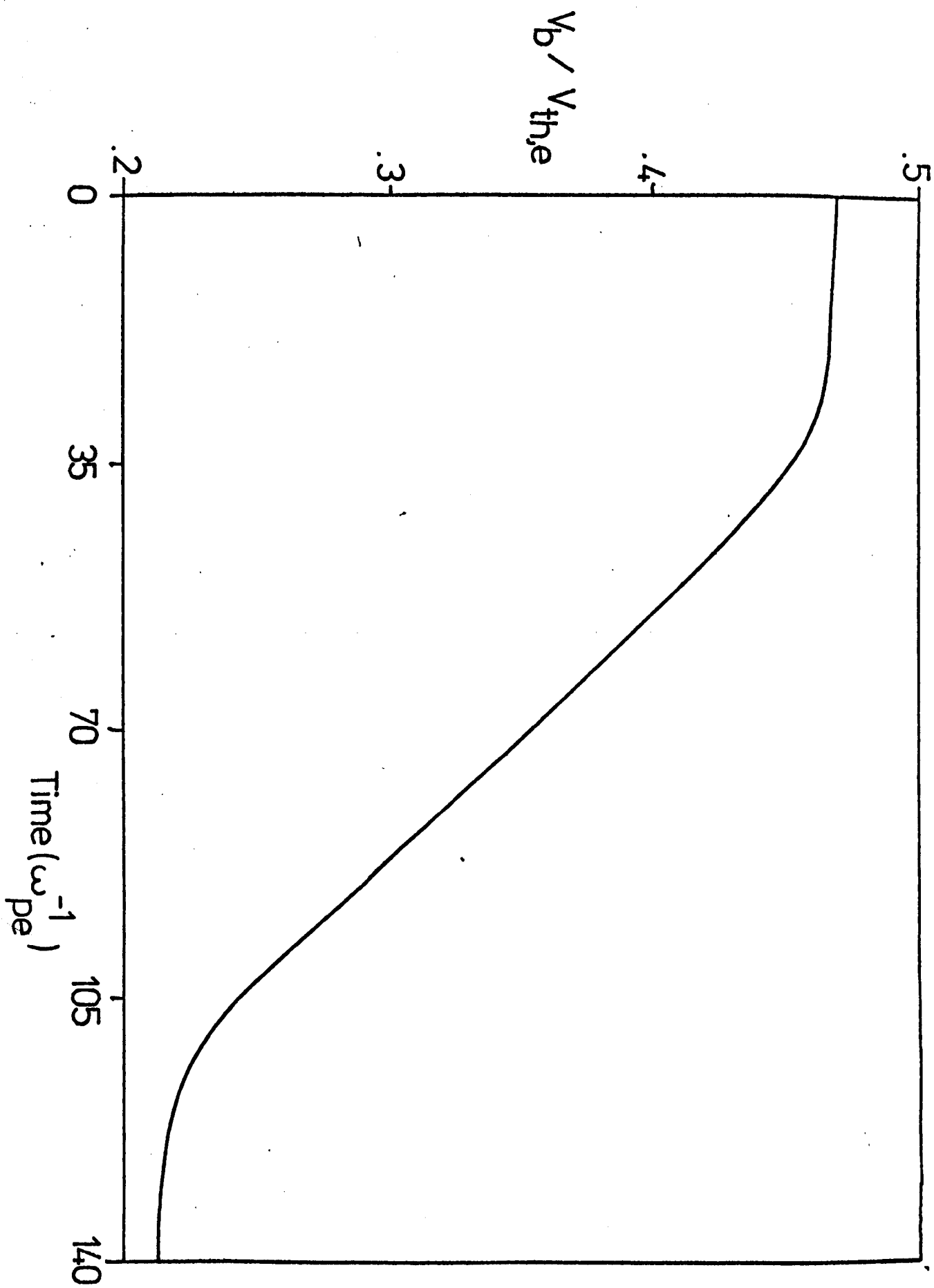


FIGURE 5.6

Numerical solution with no Landau damping

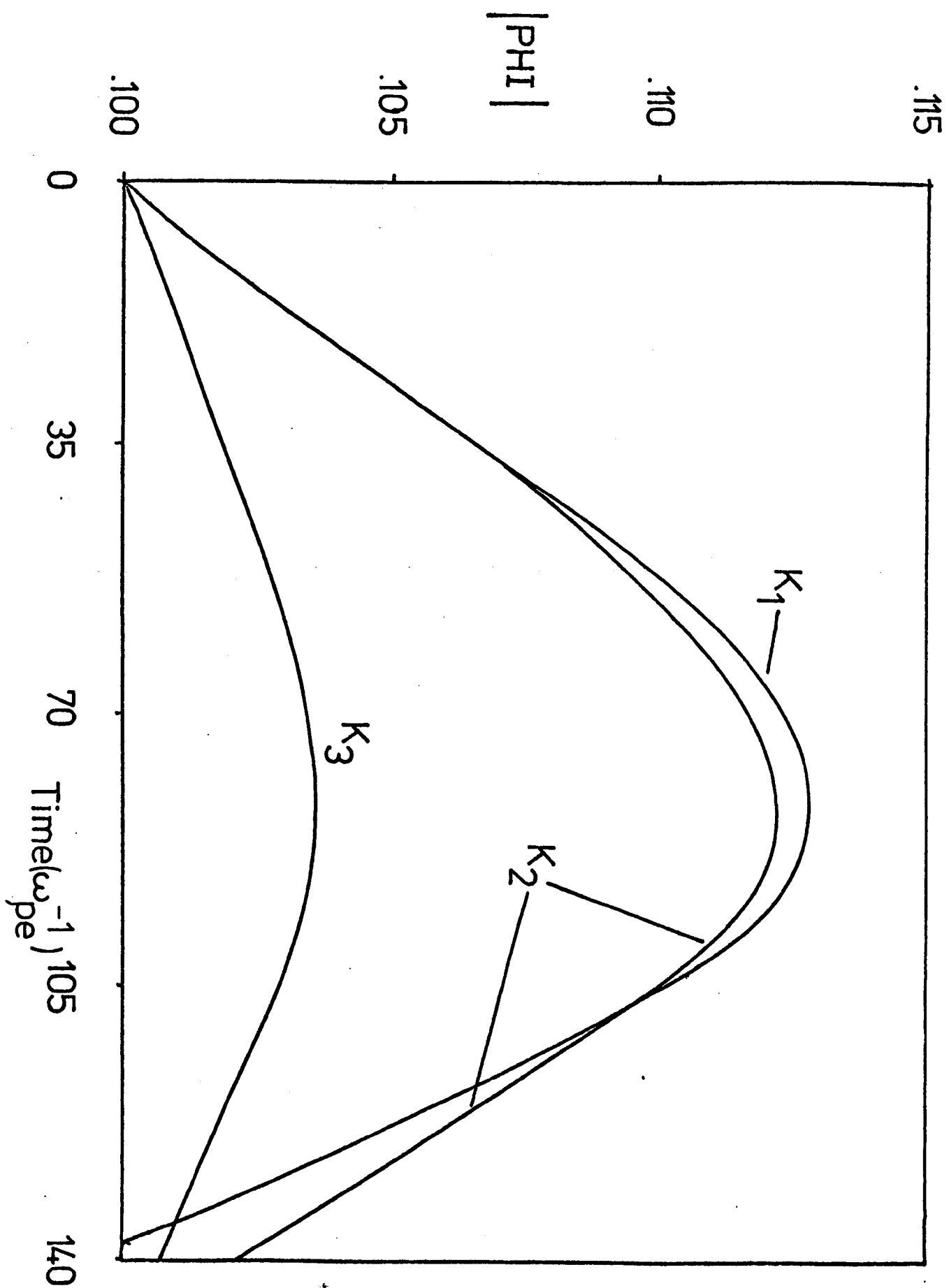


FIGURE 5.7

Numerical solution with no Landau damping

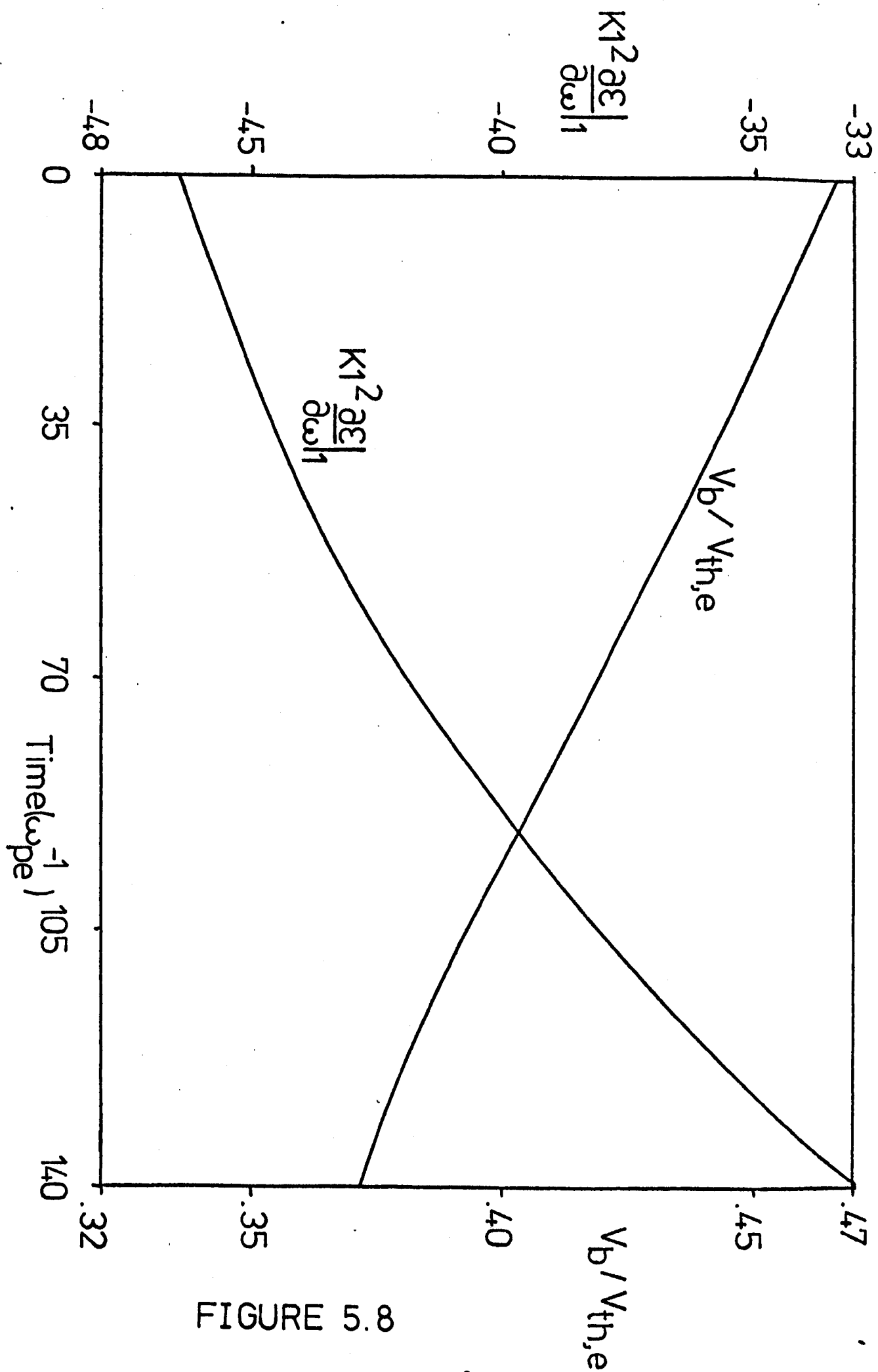


FIGURE 5.8

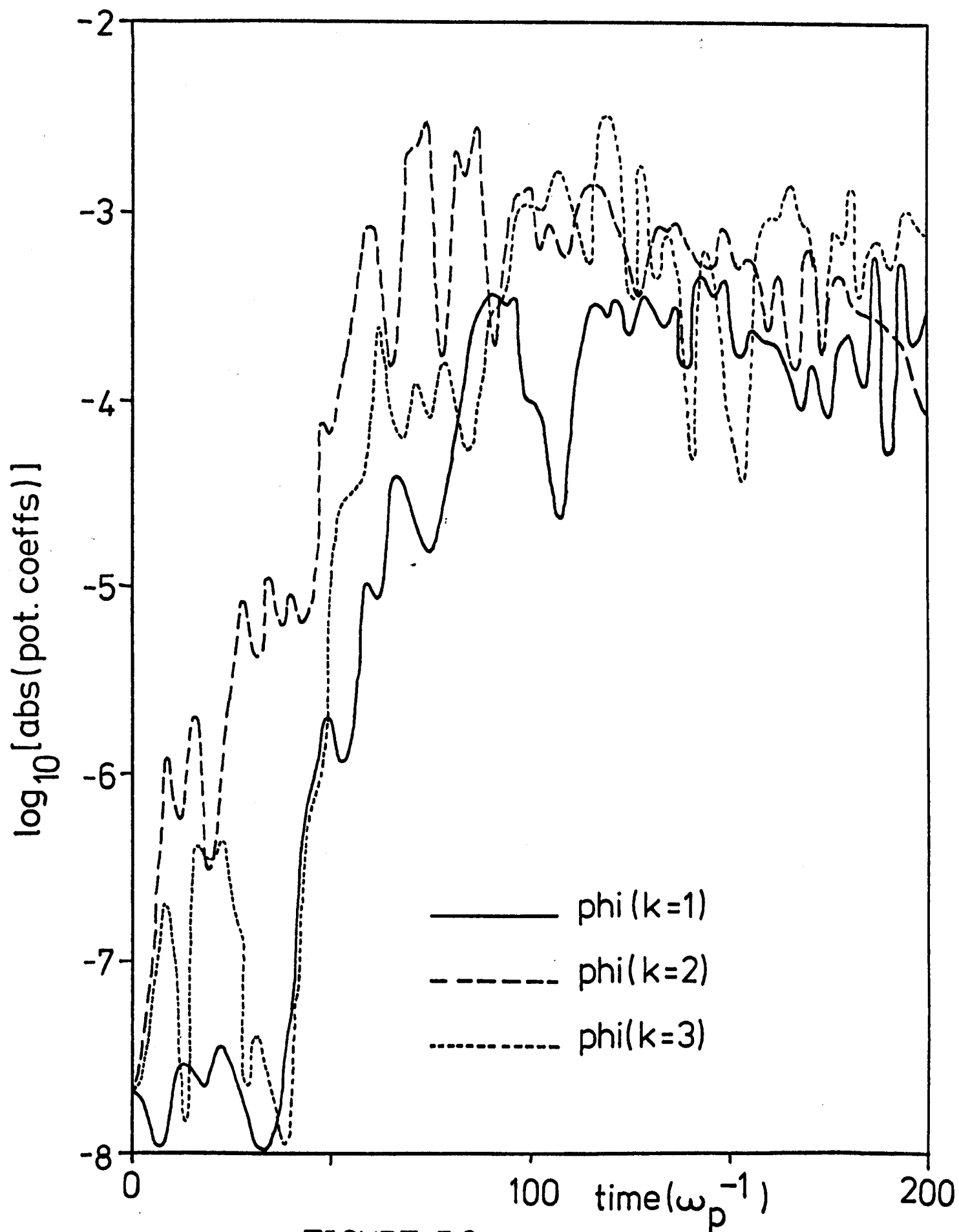


FIGURE 5.9  
NOVA RUN NO. 12



Velocities are plotted as ratios  
of their values at time  $t = 0.0$

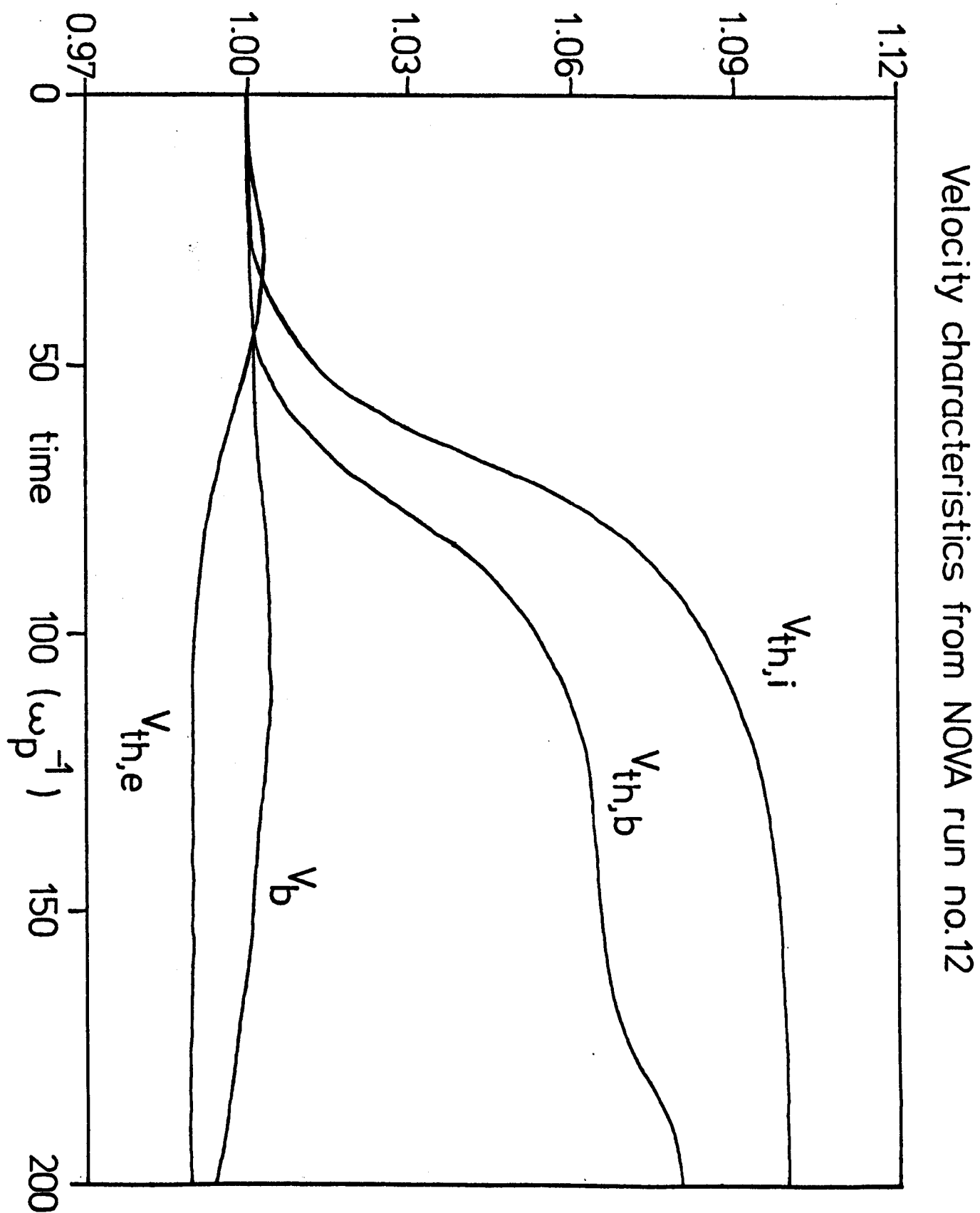


FIGURE 5.10

follow the same growth and saturation paths as one another, while the background level of waves fluctuated randomly with amplitudes of  $10^{-8}$  to  $10^{-6}$ .

The particle diagnostics shown in Figure 5.10 are more detailed than could be evaluated from runs of GALAXY, since the quantities  $v_{th,i}$ ,  $v_{th,b}$  and  $v_{th,e}$  are numerically more stable in the NOVA simulations. The significant difference between the results of GALAXY and NOVA, is that the NOVA simulation yields a much slower deceleration of the beam particles. The saturation mechanisms appear to be more closely associated with beam distortion and heating. It should also be noted that the stationary ions heat up due to the large amplitude sound wave propagating. We see from Figure 5.10 that the beam velocity only changes by about 0.005, which corresponds to a frequency mismatch of about 0.001. Even over a time of order 100-200 plasma periods, this only causes a reduction in the growth rate of the waves of 10% - 20%. This is enough to affect the explosive instability to a small extent, but the importance of the NOVA results can be seen by examining the wave and particle behaviour together. They show that the growth and saturation of the waves is synchronised with an increase in the ion temperatures and a slight reduction in beam velocity. This result is in agreement with the assertion made in Section 4.4 that the heating effect is more important than the beam deceleration.

The NOVA results however, are extremely difficult to test quantitatively, since the function describing the distortion of the beam (3.4.9) changes rapidly. Around the beam mode phase velocities, the function is almost singular, and hence the distortion cannot be numerically reproduced in a satisfactory manner. The NOVA results do however show a qualitative agreement with the assertions made in Sections 4.2 and 4.3. They show the importance of studying the combined behaviour of the waves and medium, when the explosive instability is being examined.

From Figure 5.10, since the wave amplitudes are for the most time  $10^{-5}$  to  $10^{-3}$ , a four wave interaction would be considerably weaker than the explosive instability interaction. For this reason the four wave mechanism for the saturation of the instability given by Fukai, Krishan and Harris (1969 and 1970), is not considered to be reflected in the NOVA results.

## 5.5 Discussion

The experiments performed with GALAXY and NOVA used the full extent of the programs' capabilities and accuracy. The ion to electron mass ratio is reduced to its minimum numerically stable value, since working with the real ratio is too expensive. The effects we wish to observe take place over a long enough timescale for numerical errors to accumulate, and hence numerical stability and accuracy is imperative. A central feature of the study, namely the wave amplitudes, must be calculated with precision, and the particle trajectories must be computed in a way which minimises numerical errors. We have outlined in this Chapter, the numerical and physical considerations for a simulation program to meet these requirements.

In the case of GALAXY, many features which should be regarded as essential are absent, either due to the integer nature of the vector integration and charge allocation, or the slow execution speed of the program. Bearing in mind the limitations of GALAXY two results are important, namely that the explosive instability can be simulated on a computer, and that the saturation mechanism is linked to the behaviour of the medium.

The NOVA program on the other hand is more sophisticated than GALAXY both in its construction and its numerics. Although both programs give every diagnostic detail of the plasma evolution, NOVA's accurate numerics and low noise properties make more of them usefully available. It is possible for instance to study a genuinely 'small' perturbation on the waves without background noise dominating the results (cf. Figure 5.10). The

amplitudes moreover, always remain low enough to be described by a perturbation theory, even though they change many orders of magnitude.

The GALAXY results therefore, although they tell part of the story of the explosive instability, are not as detailed as the NOVA results, which indicate that the saturation occurs because the beam ions heat up, as predicted in Sections 4.2 and 4.3.

## CHAPTER 6

### CONCLUDING REMARKS

#### 6.1 Theory

We have shown in this thesis that a multiple timescale perturbation theory can be used to derive kinetic equations governing the nonlinear coupling between three waves in a plasma. We find that if the plasma exhibits 'normal dispersion', that is if  $\partial\epsilon/\partial\omega$  is everywhere positive, energy will be transferred back and forth between the three waves in a periodic manner.

We show however, that in a simple plasma with a diffuse ion beam travelling through it, a wave can exist for which  $\partial\epsilon/\partial\omega$  is negative. This is called a 'negative energy' wave and when it couples nonlinearly with two 'positive energy' waves, an explosive instability occurs. By studying the properties of the medium when such a wave propagates, we find that the energy for the instability is provided by the kinetic energy of those beam ions with velocity close to the wave's phase velocity. This gives us a starting point for finding a process to stabilise the instability.

We show that there is a quasi-linear effect of the growing waves on the ion beam distribution function. This causes the beam ions to heat up and also to decrease in average velocity. The heating results in heavy Landau damping of the interacting waves, and the beam deceleration causes a frequency change in the waves, and a corresponding loss of efficiency in the energy transfer mechanism between the waves. By studying an expression for the effective collision frequency in the plasma, we find an additional stabilising effect, namely a distortion of the beam distribution function.

This distortion is such that it decreases the Landau growth rate of the negative energy wave, and increases the Landau damping rate of a positive energy wave participating in the interaction. We further show that the most dominant of these effects is the heating of the beam ions.

In the derivation and solution of the three wave kinetic equations, two mathematical and physical problems stand out.

Firstly, since the kinetic equations arise as a condition for avoiding secular terms in a perturbation expansion, their validity is limited to the cases of exact frequency and wavevector resonances. It is clear from physical considerations however, that the resonances must have some finite width, and that this width corresponds to the 'nearly secular' solutions. Within the multiple timescale framework these terms remain part of the expansion and are not eliminated. To build the theory on a physically more realistic basis, and yet provide some mathematically rigorous criterion and procedure for selecting such terms, would be desirable, although at present it remains an outstanding task.

The second problem is to find a method of solving the kinetic equations when there is more than one pair of waves satisfying the resonance conditions with another wave. In the case when all the waves have the same sign of energy, this can be dealt with using the random phase approximation, but with explosive interactions, the effect of multiple pairs is additive. As such, their importance is ~~easy to~~ determine qualitatively, but difficult to quantify.

Although it is possible to find plasma configurations which are explosively unstable, the instability is often accompanied by many others physical processes, some tending to stabilise and other to destabilise the situation. A detailed analysis of the plasma is therefore necessary, in order to ascertain which processes are responsible for saturating the instability.

A central argument presented in this thesis is that the physics of explosive instabilities is closely linked to the behaviour of the plasma medium. There also, lies an indication of the processes which cause the explosive instability to stabilise. The existence of negative energy waves depends on the existence of an energy source, capable of being transferred to the wave, and the nature of the energy source depends on the configuration of the steady state plasma. It is necessary therefore, when studying explosive instabilities, to examine the effects of the large amplitude oscillations on the background plasma.

In this thesis we have demonstrated that this approach is profitable, and that to consider only the equations governing the time development of the wave spectrum, is insufficient. It is believed that this may be true in other situations in which explosive instabilities arise.

## 6.2 Computation

The GALAXY program, described in Section 5.2 has many limitations, and these are reflected in the results of the simulation experiments performed with this program. The first order numerical scheme used to integrate particle co-ordinates, and the integer nature of the charge density matrix, give rise to a high level of 'numerical noise' in the wave energy spectrum. They also give particle diagnostics which are not always reliable, and a great deal of tailoring of the program to specific requirements needs to be done.

Nevertheless, the GALAXY results show the rapid growth of the three explosively unstable waves relative to the background wave field. They also show an (exaggerated) decrease in the velocity of the ion beam, and a saturation of wave growth synchronised with the beam velocity becoming constant. The important aspect of the GALAXY results, is that the saturation of the wave growth is linked with the behaviour of the beam ions.

The NOVA program, described in Section 5.3 however, has more accurate numerics than GALAXY, and with the co-ordinate initialisation process (also described in Section 5.3), gives a level of numerical noise which is orders of magnitude lower than can be achieved using GALAXY. The improved numerical schemes also make the particle diagnostics more reliable, and the results are almost free from numerical noise.

The NOVA results show the three explosively unstable waves growing in amplitude many orders of magnitude, and reaching a maximum when their energies are still much smaller than the electron temperature. Their growth is synchronised with an increase in the temperature of the beam ions, and with a slight decrease in the average beam velocity. Furthermore, the beam temperature reaches its maximum value as the wave growth saturates.

This increase in beam temperature causes the two beam modes to suffer Landau damping, and hence causes the instability to stabilise. These results agree with the theory developed in Chapter 4, in which the dominant saturation effect was found to be due to an increase in beam temperature, with a secondary effect due to the beam decelerating.

The difference in sophistication between the NOVA program and the GALAXY program is due to the fact that NOVA is run on the ICL 4/70 computer, a much larger and faster machine than the ICL KDF-9 which GALAXY operates on.

The computing power available on the large third generation computers, makes it possible to simulate two and three dimensional plasmas using numerical methods of a high order of accuracy. Such programs can now complement experiment by simulating realistic magnetic field configurations. They can be used to examine unresearched situations, and give results which can be relied on. The possibility of numerical analysis functions being performed by hardware, combined with the high speed data transfer channels



available nowadays, enables a plasma to be simulated using a realistic mass ratio and a large number of timesteps.

It is desirable however, to make such programs as transportable as possible from machine to machine, in order to avoid the duplication of programming effort. A program coded heavily in assembly language is not therefore a major aim, although such optimisation is necessary on slower computers. The commonest high level language, Fortran IV, has the disadvantage that the program has little or no control over the central processor during data transfers. An efficient buffering scheme is therefore desirable in order that the central processor may be processing one buffer, as the previous one is being dumped to backing store, and as the next is being brought in from backing store.

Buffering schemes of this nature however, can be highly complicated, especially when some buffers need to be processed differently or more often than others, and when the processing decision needs to be made at run time. McNamara and Langdon (1973) have however, developed a triple buffering package which is both general and simple, and is written in Fortran IV. The routines which need to be recoded in assembly language to permit asynchronous input and output, are few and well documented.

For the more restricted set of plasma problems, for which a one dimensional electron plasma affords a good approximation, a technique developed by Christiansen (1970) may become more widespread than at present. Christiansen's 'bit pushing algorithm' involves mapping phase space  $(x, v_x)$  onto the computer memory, such that each small region of phase space  $(x \text{ to } x + \Delta x, v_x \text{ to } v_x + \Delta v_x)$  is mapped to say six binary bits of the memory. The bit pattern in this region indicates the number of electrons in that region of phase space. Using a six bit unit, up to  $63 (= 2^6 - 1)$  electrons can reside in the region around  $(x, v_x)$ . With an intelligent choice of

core layout, the co-ordinates can be updated each timestep by a series of logical shift instructions and address modifications applied to the words in memory.

Using this algorithm, an extremely large number of particles can be simulated, perhaps of order millions, and all of them can be integrated within a second or two. It is worth noting that for a Fermi system, where the occupation number of any state is either 0 or 1, then as many phase space 'cells' can be dealt with as there are available bits in the computer.

The future of computational plasma simulations therefore, seems not to be static. In Chapter 5, we discussed various problems arising in simulation programs, such as the use of low mass ratios and particle numbers, and the use of low order numerical techniques in an attempt to gain speed. Recent developments both in software and hardware, are providing many of the solutions to these problems. Computational research into the kind of plasma situation discussed in this thesis, in which the programs are used near the limits of their validity, will in future be performed with speed and accuracy far above that found in GALAXY and NOVA.

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## APPENDIX A

### ENERGY CALCULATION FROM THE VLASOV EQUATION

In this Appendix, we calculate the change in the total energy of a plasma when a monochromatic electrostatic wave is excited. We use the Vlasov equation (3.3.1) and Poisson's equation (3.3.2), and perform the calculation for a one dimensional one component plasma. The result however, can be easily modified to apply to more general plasma configurations. We express the electric field of the monochromatic wave as

$$E_1 = E_1^\ell \exp(ik_\ell x - i\omega_\ell t + \gamma_\ell t) + \text{c.c.}, \quad (\text{A.1})$$

where  $k_\ell$  is the wavevector and  $\omega_\ell + i\gamma_\ell$  the (complex) frequency of the wave, and c.c. denotes the complex conjugate of the immediately preceding term. The first order correction to the steady state distribution function is likewise expressed as

$$f_1 = f_1^\ell \exp(ik_\ell x - i\omega_\ell t + \gamma_\ell t) + \text{c.c.} \quad (\text{A.2})$$

We write the Vlasov equation for the second order quantities  $f_2$  and  $E_2$  as

$$\frac{\partial f_2}{\partial t} + v \frac{\partial f_2}{\partial x} - E_2 \frac{\partial f_0}{\partial v} = E_1 \frac{\partial f_1}{\partial x}. \quad (\text{A.3})$$

Using the definitions (A.1 and A.2) and the expression for  $f_1^\ell$  (3.3.6), the right hand side of (A.3) can be written as

$$\begin{aligned} \frac{\partial}{\partial v} \left\{ \frac{i(E_1^\ell)^2 e^{2i(k_\ell x - \omega_\ell t)} e^{2\gamma_\ell t}}{(\omega_\ell + i\gamma_\ell - k_\ell v)} \frac{\partial f_0}{\partial v} + \text{c.c.} + \right. \\ \left. + \frac{2\gamma_\ell |E_1^\ell|^2 e^{2\gamma_\ell t}}{(\omega_\ell + i\gamma_\ell - k_\ell v)(\omega_\ell - i\gamma_\ell - k_\ell v)} \frac{\partial f_0}{\partial v} \right\}, \quad (\text{A.4}) \end{aligned}$$

that is, an oscillatory part and a slowly varying part.

From the form of equation (A.3) it is seen that we can write the second order correction to the distribution function as

$$f_2 = f_{22}^{\ell} \exp(2ik_{\ell}x - 2i\omega_{\ell}t + 2\gamma_{\ell}t) + \text{c.c.} + f_{20}.$$

From Poisson's equation we see that the second order electric field may be written as

$$E_2 = E_{22}^{\ell} \exp(2ik_{\ell}x - 2i\omega_{\ell}t + 2\gamma_{\ell}t) + \text{c.c.}$$

Using these definitions, we can rewrite the Vlasov equation for  $f_2$ , (A.3), as

$$(-2\omega_{\ell} - 2i\gamma_{\ell} + 2k_{\ell}v)f_{22}^{\ell} + iE_{22}^{\ell} \frac{\partial f_0}{\partial v} = \frac{\partial}{\partial v} \left\{ \frac{(E_1^{\ell})^2}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial f_0}{\partial v} \right\}, \quad (\text{A.5})$$

$$\text{and } \frac{\partial f_{20}}{\partial t} = \frac{\partial}{\partial v} \left\{ \frac{2\gamma_{\ell}|E_1^{\ell}|^2 e^{2\gamma_{\ell}t}}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)(\omega_{\ell} - i\gamma_{\ell} - k_{\ell}v)} \frac{\partial f_0}{\partial v} \right\}. \quad (\text{A.6})$$

In obtaining equations (A.5) and (A.6) we have associated the oscillatory part of (A.4) with  $f_{22}^{\ell}$  and the slowly varying part with  $f_{20}$ .

We can solve (A.6) to give

$$f_{20} = \frac{\partial}{\partial v} \left\{ \frac{|E_1^{\ell}|^2 e^{2\gamma_{\ell}t}}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)(\omega_{\ell} - i\gamma_{\ell} - k_{\ell}v)} \frac{\partial f_0}{\partial v} \right\}. \quad (\text{A.7})$$

From (A.5) we can write  $f_{22}^{\ell}$  as

$$f_{22}^{\ell} = \frac{i E_{22}^{\ell} \frac{\partial f_0}{\partial v}}{2(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} - \frac{1}{2(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^{\ell})^2}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial f_0}{\partial v} \right\}. \quad (\text{A.8})$$

Using this expression for  $f_{22}^{\ell}$  in Poisson's equation for  $E_{22}^{\ell}$ , we can write  $E_{22}^{\ell}$  as

$$E_{22}^{\ell} = \frac{-\int \frac{dv}{2(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^{\ell})^2 \partial f_0 / \partial v}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \right\}}{2ik_{\ell} + \frac{i}{2} \int \frac{\partial f_0 / \partial v dv}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)}} .$$

Substituting this into equation (A.8) gives

$$\begin{aligned} f_{22}^{\ell} &= \frac{-\frac{\partial f_0}{\partial v} \int \frac{dv}{2(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^{\ell})^2 \partial f_0 / \partial v}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \right\}}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v) \left\{ 4k_{\ell} - \int \frac{\partial f_0 / \partial v dv}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \right\}} \\ &- \frac{1}{2(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^{\ell})^2 \partial f_0 / \partial v}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \right\}. \end{aligned} \quad (A.9)$$

Now the expression for the total energy in a plasma is

$$\text{Energy} = \int v^2 f dv + E^2,$$

which to second order in the amplitude expansion is

$$\text{Energy} = \int v^2 (f_0 + f_1 + f_2) dv + E_1^2,$$

which is just a kinetic energy term plus a potential energy term. Using the expressions for  $f_1$ ,  $f_2$  and  $E_1$  given by (A.2), (A.7 and A.9) and (A.1) respectively, we obtain

$$\begin{aligned} \text{Energy} &= \int v^2 f_0 dv + \\ &+ \int \frac{iv^2 E_1^{\ell} e^{i(k_{\ell}x - \omega_{\ell}t)} e^{\gamma_{\ell}t} \partial f_0 / \partial v dv}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} + \text{c.c.} \\ &+ \int \frac{v^2 \partial f_0 / \partial v dv}{(\omega_{\ell} + i\gamma_{\ell} - k_{\ell}v)} \times \end{aligned}$$



$$\begin{aligned}
& \times \frac{\int \frac{dv}{2(\omega_\ell + i\gamma_\ell - k_\ell v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^\ell)^2 e^{2i(k_\ell x - \omega_\ell t)} e^{2\gamma_\ell t} \frac{\partial f_o}{\partial v}}{(\omega_\ell + i\gamma_\ell - k_\ell v)} \right\} \frac{\partial f_o}{\partial v}}{\left\{ 4k - \int \frac{\partial f_o / \partial v}{(\omega_\ell + i\gamma_\ell - k_\ell v)} dv \right\}} + c.c. \\
& - \int \frac{v^2 dv}{2(\omega_\ell + i\gamma_\ell - k_\ell v)} \frac{\partial}{\partial v} \left\{ \frac{(E_1^\ell)^2 e^{2i(k_\ell x - \omega_\ell t)} e^{2\gamma_\ell t} \frac{\partial f_o}{\partial v}}{(\omega_\ell + i\gamma_\ell - k_\ell v)} \right\} + c.c. \\
& + \int v^2 \frac{\partial}{\partial v} \left\{ \frac{|E_1^\ell|^2 \frac{\partial f_o}{\partial v} e^{2\gamma_\ell t}}{(\omega_\ell + i\gamma_\ell - k_\ell v)(\omega_\ell - i\gamma_\ell - k_\ell v)} \right\} dv \\
& + (E_1^\ell)^2 e^{2i(k_\ell x - \omega_\ell t)} e^{2\gamma_\ell t} + c.c. + 2|E_1^\ell|^2 e^{2\gamma_\ell t}.
\end{aligned}$$

Many of the terms in this expression are rapidly oscillating, and if we take  $\gamma_\ell \ll \omega_\ell$ , we can calculate the time average of the energy ( $W$ ) by averaging over the phase of the oscillations. The average energy, correct to second order is then seen to be

$$W = KE + \int v^2 \frac{\partial}{\partial v} \left\{ \frac{|E_1^\ell|^2 \frac{\partial f_o}{\partial v}}{(\omega_\ell - k_\ell v)} \right\} dv + 2|E_1^\ell|^2,$$

where KE is the kinetic energy of the unperturbed plasma. The second term describes that part of the particle energy associated with the coherent wave motion, and the final term is the electrostatic energy. These last two terms can be combined, and we can then write

$$W = KE + |E_1^\ell|^2 \frac{\partial}{\partial \omega} \left\{ \omega \epsilon \right\}_{\omega=\omega_\ell, k=k_\ell}, \quad (A.10)$$

where  $\epsilon$  is the dispersion function of the form given by (3.3.7). The second term in (A.10) is what we refer to in the thesis as the energy of a monochromatic wave in a dispersive medium.

We have derived from the Vlasov equation and Poisson's equation, a form

first indicated by von Laue (1905) for the energy of a monochromatic wave in a dispersive medium. The importance of the result lies in the dependence on  $\partial\epsilon/\partial\omega$ , which for certain  $\epsilon$ 's, can be negative.

## APPENDIX B

### EVALUATION OF $t_{\infty}$

In this Appendix, we examine the dispersive characteristics of three explosively unstable ion sound waves propagating in the beamed plasma system described in Section 3.3. We determine the restrictions on the parameters of the system, and make a numerical evaluation of the time taken for the waves to reach an infinite amplitude, using the expression (3.4.10) for  $t_{\infty}$ . We define the fraction of ions which are present in the beam as

$$\eta^2 = \int f_{oi}^b dv / \int (f_{oi}^o + f_{oi}^b) dv.$$

We impose the restrictions that

$$\begin{aligned} \eta^2 &\ll 1, \\ v_{th,i} &\ll (U_b \text{ or } \omega/k) \ll v_{th,e}, \end{aligned} \quad (B.1)$$

and  $k\lambda_D \ll 1,$

where  $U_b$  = average velocity of the beam ions,

$v_{th,i}$  = ion thermal velocity,

$v_{th,e}$  = electron thermal velocity,

$\lambda_D$  = Debye screening length =  $v_{th,e}/\omega_{pe}$ ,

$\omega_{pe}$  = electron plasma frequency,

and  $\omega/k$  refers to the phase velocity of any of the waves we consider.

Following Fried and Gould (1961), with these approximations (B.1), the dispersion relation (3.3.7) can be simplified to

$$\epsilon(\omega, k) \approx \left(\frac{k_D}{k}\right)^2 - \omega_{pi}^2 \left\{ \frac{(1 - \eta^2)}{\omega^2} + \frac{\eta^2}{(\omega - kU_b)^2} \right\} = 0,$$

where  $\omega_{pi}$  is the ion plasma frequency  $= \delta \omega_{pe}$  ( $\delta^2$  is the electron to ion mass ratio), and  $k_D$  is the inverse Debye length  $= 2\pi\lambda_D^{-1}$ .

For waves with phase velocities  $\omega/k \approx U_b$ , we can further approximate the dispersion relation to

$$\epsilon(\omega, k) \approx \left(\frac{k_D}{k}\right)^2 - \frac{\eta^2 \omega_{pi}^2}{(\omega - kU_b)^2} = 0. \quad (B.2)$$

Similarly, for waves with phase velocities near zero (well away from  $U_b$ ) we can write

$$\epsilon(\omega, k) \approx \left(\frac{k_D}{k}\right)^2 - \frac{(1 - \eta^2) \omega_{pi}^2}{\omega^2} = 0. \quad (B.3)$$

It should be pointed out that (B.2) and (B.3) can be derived using the magnetohydrodynamic description of the time dependent plasma quantities. This however, does not detract from our using the Vlasov approach to the three wave problem in Chapter 3, since only in the Vlasov approach can we determine the importance of particle effects for waves with  $\partial\epsilon/\partial\omega$  negative.

Equation (B.2) can be solved to give

$$\omega^\pm = k(U_b \pm \delta \eta v_{th,e}), \quad (B.4)$$

which we associate with ion sound oscillations propagating on the beam, the (+) mode travelling slightly faster than  $U_b$  and the (-) mode travelling slightly slower. Similarly (B.3) can be solved to give

$$\omega = \pm k\delta(1 - \eta^2)^{1/2} v_{th,e}, \quad (B.5)$$

which is the conventional ion sound mode propagating in the bulk of the ions. Three waves participate in the interaction we are studying, and we choose to deal with one from each of the branches of (B.4) and a forward moving wave from the branch (B.5). We call these waves  $\ell$ ,  $m$  and  $n$  such that

$$\begin{aligned}
\omega_l &= k_l(U_b - \delta \eta v_{th,e}), \\
\omega_m &= k_m(U_b + \delta \eta v_{th,e}), \\
\text{and } \omega_n &= k_n \delta (1 - \eta^2)^{\frac{1}{2}} v_{th,e}.
\end{aligned} \tag{B.6}$$

It will be noticed from (B.2) and (B.3) that mode  $l$  has  $\partial \epsilon / \partial \omega$  negative, and modes  $m$  and  $n$  have  $\partial \epsilon / \partial \omega$  positive. Therefore, if we can select all the parameters of the problem to satisfy (B.1) and the resonance conditions (3.3.15), then as demonstrated in Section 3.4, we can expect the three waves to be explosively unstable. In order to get numerical values for all the parameters, we choose to deal in the non dimensional units used in the two plasma simulation programs discussed in Chapter 5. In those units,

$$\begin{aligned}
\text{electron plasma frequency} &= \omega_{pe} = 1, \\
\text{electron thermal velocity} &= v_{th,e} = 1, \\
\text{Debye length} &= \lambda_D = 1, \\
\text{electronic charge} &= e = -1, \\
\text{electronic mass} &= m_e = 1, \\
\text{electron temperature} &= T_e = 1.
\end{aligned}$$

The consequences and practicality of using these units, as well as the justification for using peculiar values for  $\delta$ , the  $k$ 's etc., is discussed in Chapter 5. Meanwhile we simply state that for the plasma simulation programs we choose to use

$$\delta = 1/4 \text{ and } \eta^2 = 1/4. \tag{B.8}$$

Fried and Gould (1961) have shown that ion oscillations are heavily damped unless the ion temperature (in our case  $T_i$  and  $T_b$ ) is much less than the electron temperature (cf. B.1). For this reason we take

$$\begin{aligned}
T_i/T_e &= 1/32, \\
\text{and } T_b/T_e &= 1/32.
\end{aligned} \tag{B.9}$$

Further, we choose the three wavevectors to be

$$\begin{aligned} k_\ell &= 3.2\pi/64\lambda_D, \\ k_m &= 1.2\pi/64\lambda_D, \\ \text{and } k_n &= 2.2\pi/64\lambda_D, \end{aligned} \tag{B.10}$$

thus satisfying  $k\lambda_D \ll 1$ , and the wavevector resonance condition. Since we must also satisfy the frequency resonance condition, we find using (B.6), (B.7), (B.8) and (B.10) that

$$\begin{aligned} U_b &= (2 + \sqrt{3})/8, \\ &\approx 0.4665. \end{aligned} \tag{B.11}$$

It will be noticed that values such as  $1/4$  are being considered very much less than unity. This has to be done however, if we are going to satisfy all the restrictions simultaneously, unless we were in a position to choose a more realistic mass ratio than  $1/16$ . As explained in Chapter 5 however, choosing a more realistic mass ratio is computationally prohibited for practical reasons, and the values of the parameters given above are the most suitable compromise between accurate theory and practical computing.

Turning to the calculation of  $t_\infty$ , we can write (3.4.10) in the form

$$t_\infty = \frac{(|\alpha_\ell| |\alpha_n|)^{\frac{1}{2}} \pi}{2 |\Gamma_{\ell mn}| |\phi_{m0}|}, \tag{B.12}$$

and note that using the above parameter values,

$$\left. \frac{\partial \epsilon}{\partial \omega} \right|_\ell = \frac{-2}{\delta \eta k_\ell^3},$$

$$\text{and } \left. \frac{\partial \epsilon}{\partial \omega} \right|_n = \frac{2}{\delta k_n^3 (1 - \eta^2)^{\frac{1}{2}}} \approx \frac{2}{\delta k_n^3}.$$

$$\text{Therefore } |\alpha_\ell| = \frac{2}{\delta \eta |k_\ell|} = \frac{512}{3\pi},$$

and  $|\alpha_n| \approx \frac{2}{\delta |k_n|} = \frac{128}{\pi}$ .

Therefore  $t_\infty = \frac{128 \sqrt{3}}{|\Gamma_{lmn}| |\phi_{mo}|^3}$ . (B.13)

In order to calculate the matrix element  $\Gamma_{lmn}$ , we use the expression (3.3.17) which can be written as

$$\Gamma_{lmn} = \frac{1}{2} \sum_j \left( \frac{e}{m} \right)_j \left\{ \frac{I_{lj}}{(V_m - V_l)(V_n - V_l)} + \frac{I_{mj}}{(V_l - V_m)(V_n - V_m)} - \frac{I_{nj}}{(V_l - V_n)(V_m - V_n)} \right\}, \quad (B.14)$$

where

$$I_{lj} = \omega_{pj}^2 \int_c \frac{\partial f_{oj} / \partial v \, dv}{(V_l - v)},$$

and  $V_l = \omega_l / k_l$  etc.

As outlined in Section 3.3, the  $f_{oj}$  are Maxwellian, normalised to unity, and so can be expressed as

$$f_{oj} = \frac{1}{\sqrt{2\pi} v_{th,j}} \exp(-v^2 / 2 v_{th,j}^2).$$

Using this form we can then write

$$I_{lj} = \frac{\omega_{pj}^2}{2 v_{th,j}^2} (1 + w Z(w)),$$

where  $w = V_l / \sqrt{2} v_{th,j}$  and  $Z$  is the well known plasma dispersion function defined as

$$Z(w) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dq \exp(-q^2) / (q - w),$$

for  $\text{Im}(w) > 0$  and the analytic continuation of this for  $\text{Im}(w) < 0$ .

For the electron terms we have  $w \ll 1$ , and we can expand  $Z(w)$  in a power series. In this way we can write

$$I_{le} = \frac{1}{\lambda_D^2} \{1 + iw \sqrt{\pi} e^{-w^2} - 2w^2 + \frac{4w^3}{3} - O(w^6)\}.$$

Now the first term in  $I_{le}$  does not contribute to  $\Gamma_{lmn}^e$  since

$$\frac{1}{(V_m - V_\ell)(V_n - V_\ell)} + \frac{1}{(V_\ell - V_m)(V_n - V_m)} + \frac{1}{(V_\ell - V_n)(V_m - V_n)} = 0.$$

For small  $w$ ,  $e^{-w^2} \approx 1$  and substituting this into (B.14) shows that the imaginary contribution to  $\Gamma_{lmn}^e$  from  $I_{le}$  is also identically zero. The first term in  $I_{le}$  which gives any contribution to  $\Gamma_{lmn}^e$  is the  $-2w^2$  term. If we ignore terms of order  $w^4$  and higher, we find after some algebraic manipulation that

$$\begin{aligned} \Gamma_{lmn}^e &= \frac{1}{2} \left(\frac{e}{m}\right) \frac{1}{e \lambda_D^2} \frac{1}{v_{th,e}^2}, \\ &= -\frac{1}{2} \text{ in our units.} \end{aligned} \quad (B.15)$$

For the ion terms,  $w \gg 1$ , since we have made  $v_{th,i}$  much less than  $\delta v_{th,e}$ , and so we may use the asymptotic value of  $Z$ . Using this, we can write

$$I_{li} = \frac{\omega_{pi}^2}{v_{th,i}^2} \left\{ 1 + iw \sqrt{\pi} e^{-w^2} - 1 - \frac{1}{2w^2} - O(w^{-4}) \right\}.$$

For the same reason as for the electron terms, the first term in  $I_{li}$  does not contribute to  $\Gamma_{lmn}^i$ . Further, since  $w \gg 1$ , the imaginary term in  $I_{li}$  will be exponentially small. Therefore, if we neglect terms of order  $w^{-4}$  and higher in  $I_{li}$ , we can approximate  $\Gamma_{lmn}^i$  to

$$\Gamma_{lmn}^i = -\frac{1}{2} \left(\frac{e}{m}\right)_i \omega_{pi}^2 \left\{ \frac{V_\ell V_m + V_\ell V_n + V_m V_n}{v_\ell^2 v_m^2 v_n^2} \right\}.$$



Substituting the numerical values for the parameters, we obtain

$$\begin{aligned}\Gamma_{\ell mn}^i &= -\frac{1}{2} \frac{(28 + 16\sqrt{3})}{(42 + 7\sqrt{3})}, \\ &\approx -\frac{1}{2}.\end{aligned}$$

Combining this with result (B.15) we calculate the total matrix element as

$$|\Gamma_{\ell mn}| \approx 1.$$

Therefore, from (B.13) we see that

$$t_{\infty} = \frac{128\sqrt{3}}{3|\phi_{mo}|}. \quad (\text{B.16})$$

This value for  $t_{\infty}$  is for the case when the potential grows like a tangent function, that is when  $B_{mo} = B_{no}$ . We shall however, briefly examine the general case of  $B_{mo} \neq B_{no}$ . From (3.4.11) we see that

$$B_{\ell} = B_{no} \operatorname{tn}(|M|B_{mo}t, K),$$

where  $\operatorname{tn}$  is the elliptic integral of the first kind with argument  $|M|B_{mo}t$  and amplitude  $K$ .  $K$  is also an elliptic integral whose argument  $q$  is given by

$$q = (B_{mo}^2 - B_{no}^2)^{1/2} / B_{mo}.$$

The function  $\operatorname{tn}$  is similar to the trigonometric tangent function except that instead of becoming infinite at  $\pi/2$ , it does so at  $K(q)$ . If we take for simplicity  $\phi_{mo} = \phi_{no}$ , then  $q$  depends only on the  $\alpha$ 's, namely

$$q = (\alpha_m - \alpha_n)^{1/2} / \alpha_m^{1/2},$$

which for the present case is of order 0.86. From tables of elliptic integrals, we find that

$$K(q = 0.86) \approx 2.139,$$

which is greater than  $\pi/2$ . In this way, we get a value for  $t_{\infty}$  which exceeds

the value given by (B.16) by a factor of order 1.35. Moreover, for  $q$  non-zero, the value of  $K$  always exceeds  $\pi/2$ , and hence gives a value of  $t_\infty$  greater than that given by (B.16).

## APPENDIX C

### TWO PARTICLE CORRELATIONS IN THE SIMULATED PLASMA

The particle co-ordinate initialisation routine used in the NOVA program approximates the Maxwellian particle velocity distribution to a sum of step functions of width  $\Delta v$  and height  $n(v)$  (cf. Figure 5.4). For a given group of  $n(v)$  particles with the same velocity  $v$ , we space them evenly along the  $x$ -axis from  $x = 0$  to  $x = 64\lambda_D$ , and as such their positions are highly correlated. This correlation however, is only between particles in the same group, and inter-group correlations can be ignored since the co-ordinates of one group do not depend on those of another. We wish to determine whether the correlation which does exist is large enough to invalidate the correspondence being made between the simulated plasma and a Vlasov plasma.

Following the approach adopted by Rutherford and Frieman (1963), the BBGKY hierarchy of equations is closed by neglecting three particle correlations, and so we can write the two and three particle distribution functions  $f_2$  and  $f_3$  as

$$\begin{aligned} f_2(1,2,t) &= f(1,t) f(2,t) + g(1,2,t) , \\ \text{and} \quad f_3(1,2,3,t) &= f(1,t) f(2,t) f(3,t) + \\ &+ f(1,t) g(2,3,t) + f(2,t) g(3,1,t) + f(3,t) g(1,2,t) , \end{aligned}$$

where  $f(1,t)$  is the one particle distribution function,  $g(1,2,t)$  is the two particle correlation function and  $1,2, \dots$  denote phase space points  $(\underline{x}_1, \underline{v}_1)$  and  $(\underline{x}_2, \underline{v}_2)$ . We consider the spatially homogeneous case, in which case  $f(1,t)$  is independent of  $\underline{x}_1$  and  $g(1,2,t)$  only depends on  $\underline{x}_1$  and  $\underline{x}_2$  as  $(\underline{x}_1 - \underline{x}_2)$ . The hierarchy reduces to two coupled equations for  $f(1,t)$  and  $g(1,2,t)$  which simplify further with the assumption that  $n\lambda_D^3 \gg 1$ . We can then write

$$\frac{\partial f_i}{\partial t} = \frac{1}{m_i} \sum_j n_j \int \frac{\partial \phi_{ij}}{\partial x_1} (1,2) \frac{\partial g_{ij}}{\partial v} (1,2,t) d^3 x_2 d^3 v_2, \quad (C.1)$$

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) g_{ij}(1,2,t) - \frac{1}{m_i} \frac{\partial f_i}{\partial v_1} (1) \sum_k n_k \int \frac{\partial \phi_{ik}}{\partial x_1} (1,3) \times \\ & \times g_{jk}(2,3,t) d^3 x_3 d^3 v_3 - \frac{1}{m_j} \frac{\partial f_j}{\partial v_2} (2) \sum_k n_k \int \frac{\partial \phi_{jk}}{\partial x_2} (2,3) \times \\ & \times g_{ik}(1,3,t) d^3 x_3 d^3 v_3 = \quad (C.2) \\ & = \frac{\partial \phi_{ij}}{\partial x_1} (1,2) \left\{ \frac{f_j(2)}{m_i} \frac{\partial f_i(1)}{\partial v_1} - \frac{f_i(1)}{m_i} \frac{\partial f_j(2)}{\partial v_2} \right\}. \end{aligned}$$

Here,  $i$  and  $j$  denote particle species, and  $\phi_{ij}(1,2)$  is the interparticle potential energy  $= e_i e_j / |x_1 - x_2|$ .

The assumption that  $n\lambda_D^3 \gg 1$  allows us to solve (C.2) over times for which the one particle distribution function may be considered time independent. To this end, it is convenient to Fourier transform in the variable  $(x_1 - x_2)$ , and we write the transform of  $g_{ij}(1,2,t)$  as

$$g_{ij}(k, v_1, v_2, t) = \frac{1}{(2\pi)^3} \int g_{ij}(1,2,t) e^{-ik(x_1 - x_2)} d^3(x_1 - x_2).$$

Equation (C.2) can then be written

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + L_i(k, v_1) + L_j(-k, v_2) \right) g_{ij}(k, v_1, v_2, t) = \\ & = S_{ij}(k, v_1, v_2), \quad (C.3) \end{aligned}$$

$$\text{where } S_{ij}(k, v_1, v_2) = \frac{4\pi e_i e_j}{(2\pi)^3 k^2} ik \left\{ \frac{f_j(v_2)}{m_i} \frac{\partial f_i}{\partial v_1} - \frac{f_i(v_1)}{m_j} \frac{\partial f_j}{\partial v_2} \right\},$$

and  $L_i$  and  $L_j$  are linear operators defined by

$$L_i(k, v_1) = ikv_1 - \frac{4\pi e_i k}{m_i k^2} \frac{\partial f_i}{\partial v_1} \sum_l n_l e_l \int d^3 v_l,$$

and similarly for  $L_j$ . If we denote the Laplace Transform of  $g_{ij}(k, v_1, v_2, t)$  by

$$\tilde{g}_{ij}(k, v_1, v_2, p) = \int_0^{\infty} g_{ij}(k, v_1, v_2, t) e^{-pt} dt,$$

we can transform expression (C.3) so long as  $g_{ij}$  does not grow faster than  $\exp(at)$ , for some positive  $a$ . So doing, we obtain

$$\tilde{g}_{ij}(p) = \frac{1}{p + L_i + L_j} \left\{ \frac{S_{ij}(p)}{p} + g_{ij}(t=0) \right\}. \quad (C.4)$$

Inverting the transform, the correlation function can be written as

$$g_{ij}(t) = \frac{1}{2\pi i} \int_c \frac{e^{pt} dp}{(p + L_i + L_j)} \left\{ \frac{S_{ij}}{p} + g_{ij}(0) \right\},$$

where  $c$  denotes the Bromwich contour from  $-i\infty$  to  $+i\infty$ , to the right of all the singularities in the integrand.

Now the operator  $(p + L_i + L_j)^{-1}$  is the Laplace transform of the operator  $\exp(-L_i t) \exp(-L_j t)$ . Further, the Laplace transform of  $\exp(-L_i t)$  is  $(p + L_i)^{-1}$ , and therefore

$$e^{-L_i t} = \frac{1}{2\pi i} \int_{c_1} \frac{e^{p_1 t} dp_1}{(p_1 + L_i)},$$

and

$$e^{-L_j t} = \frac{1}{2\pi i} \int_{c_2} \frac{e^{p_2 t} dp_2}{(p_2 + L_j)},$$

where  $c_1$  and  $c_2$  are Bromwich contours in the  $p_1$  and  $p_2$  planes respectively.

We can now write

$$\begin{aligned} \frac{1}{(p + L_i + L_j)} &= \int_0^{\infty} e^{-(p + L_i + L_j)t} dt = \\ &= \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{dp_1 dp_2}{(p - p_1 - p_2)} \frac{1}{(p_1 + L_i)} \frac{1}{(p_2 + L_j)}, \end{aligned} \quad (C.5)$$

where  $\text{Re}(p) > \text{Re}(p_1 + p_2)$ . Using the form (C.5) for the operator  $(p+L_i+L_j)^{-1}$  in the expression for the correlation function (B.4) we obtain

$$\tilde{g}_{ij}(p) = \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{dp_1 dp_2}{(p-p_1-p_2)} \frac{1}{(p_1+L_i)} \frac{1}{(p_2+L_j)} \left\{ \frac{S_{ij}}{p} + g_{ij}(0) \right\}.$$

Inverting the above, we obtain

$$g_{ij}(t) = \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{dp_1}{(p_1+L_i)} \frac{dp_2}{(p_2+L_j)} \times \\ \times \left\{ \frac{(e^{(p_1+p_2)t} - 1) S_{ij}}{(p_1 + p_2)} + e^{(p_1 + p_2)t} g_{ij}(0) \right\}. \quad (C.6)$$

The problem of calculating the correlation function has now become the problem of identifying the operators  $(p_1 + L_i)^{-1}$  and  $(p_2 + L_j)^{-1}$ . We find that they can be identified with the operators which appear in the solution to the linearised Vlasov equation. This initial value problem, first solved by Landau (1946) contains the relationship

$$\frac{1}{p_1 + L_i(k, v_1)} = \frac{1}{p_1 + ikv_1} \left\{ 1 - \frac{\beta_i(k, v_1)}{\epsilon(k, p_1)} \int \frac{\sum n_i e_i d^3 v_1}{(p_1 + ikv_1)} \right\},$$

$$\text{where } \beta_i(k, v_1) = -\frac{4\pi e_i}{m_i k^2} ik \frac{\partial f_i}{\partial v_1},$$

and  $\epsilon(k, p_1)$  is the dispersion function. Substituting this expression for  $(p_1 + L_i)^{-1}$ , and the corresponding expression for  $(p_2 + L_j)^{-1}$  into equation (C.6), we obtain

$$g_{ij}(t) = \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{dp_1}{(p_1 + ikv_1)} \left\{ 1 - \frac{\beta_i(k, v_1)}{\epsilon(k, p_1)} \int \frac{\sum n_i e_i d^3 v_1}{(p_1 + ikv_1)} \right\} \times \\ \times \frac{dp_2}{(p_2 - ikv_2)} \left\{ 1 - \frac{\beta_j(-k, v_2)}{\epsilon(-k, p_2)} \int \frac{\sum n_j e_j d^3 v_2}{(p_2 - ikv_2)} \right\} \times$$

$$\times \left\{ \frac{(e^{(p_1+p_2)t} - 1) s_{ij}}{(p_1 + p_2)} + e^{(p_1+p_2)t} g_{ij}(0) \right\}. \quad (C.7)$$

In order to perform the integrals in (C.7) we note that the operator  $(p_1 + L_1)^{-1}$  has poles at  $p_1 = -ikv_1$  and at the zeros of the dispersion function  $\epsilon(k, p_1)$ . We will first consider the stable cases, namely when all the solutions of  $\epsilon(k, p) = 0$  are in the half plane  $\text{Re}(p) < 0$ . In this case, the operators  $(p_1 + L_1)^{-1}$  and  $(p_2 + L_2)^{-1}$  are analytic in the half planes  $\text{Re}(p_1) > 0$  and  $\text{Re}(p_2) > 0$  respectively, and we can move the contours  $c_1$  and  $c_2$  just to the right of the imaginary axis. It follows that as  $t$  approaches infinity,  $g_{ij}(t)$  remains finite. That is,

$$g_{ij}(\infty) = \lim_{p \rightarrow 0^+} \{ \tilde{p} g_{ij}(p) \},$$

remains finite and is independent of the initial correlations  $g_{ij}(0)$ .

A more rigorous analysis of the time dependence of  $g_{ij}(t)$  shows that initial correlations die away like  $\exp(-2\gamma(k)t)$  where  $\gamma(k)$  is the Landau damping rate of mode  $k$ . These damping terms arise from analytically continuing the integrand of (C.6) into the half planes  $\text{Re}(p_1) < 0$  and  $\text{Re}(p_2) < 0$ , and picking up the first poles associated with the zeros in  $\epsilon(k, p_1)$  and  $\epsilon(-k, p_2)$ . The timescale for the establishment of  $g_{ij}(\infty)$  is therefore of order  $1/\gamma(k)$ .

Before we consider the poles at  $p_1 = -ikv_1$  and  $p_2 = ikv_2$ , we note that in the unstable case, we will have terms in  $g_{ij}(t)$  which grow like  $\exp(2\gamma(k)t)$ . The criterion under discussion for the 'Vlasov plasma' however, is that

$$\frac{g(1,2,t)}{f(1,t)f(2,t)} \ll 1,$$

and the term  $f(1,t)f(2,t)$  has similar Landau growth features to  $g(1,2,t)$ .

This ratio therefore is unaffected by the poles corresponding to the zeros

in the dispersion function. The important poles are those at  $p_1 = -ikv_1$  and  $p_2 = ikv_2$ .

Taking account of these poles in expression (C.7), we obtain

$$\begin{aligned}
 g_{ij}(t) = & \frac{1}{(2\pi i)^2} \left\{ 1 - \frac{\beta_i(k, v_1)}{\epsilon(k, -ikv_1)} I(1, i) \right\} \cdot \left\{ 1 - \frac{\beta_j(-k, v_2)}{\epsilon(-k, ikv_2)} I(2, j) \right\} \times \\
 & \times \left\{ \frac{(e^{-ik(v_1-v_2)t} - 1)}{-ik(v_1-v_2)} \frac{4\pi e_i e_j ik}{(2\pi)^3 k^2} \left\{ \frac{f_j(v_2)}{m_i} \frac{\partial f_i}{\partial v_1} - \right. \right. \\
 & \left. \left. - \frac{f_i(v_1)}{m_j} \frac{\partial f_j}{\partial v_2} \right\} + e^{ik(v_1-v_2)t} g(0) \right\}, \quad (C.8)
 \end{aligned}$$

where the  $I$ 's are the velocity integrals in (C.7).

In the simulation program initialisation, we ensure that particles of velocity  $v_n$  are uncorrelated to particles of velocity  $v_m$ , for  $m \neq n$ . Providing we have a large selection of velocity groups, as Rutherford and Frieman (1963) demonstrate, terms like

$$g(0) e^{-ik(v_1-v_2)t}$$

can be considered to vanish by a phase mixing process.

The case still exists however, of the highly correlated particles of the same velocity, namely the situation when  $m = n$ . It should be noted that such particles in the simulation program belong to the same species, and hence  $i = j$  in expression (C.8). Under these circumstances, the Balescu-Lenard term vanishes since

$$\frac{e^{-ik(v_1-v_2)t} - 1}{-ik(v_1-v_2)} \rightarrow 1 \text{ as } v_1 \rightarrow v_2,$$

and

$$\left\{ \frac{f_j(v_2)}{m_i} \frac{\partial f_i}{\partial v_1} - \frac{f_i(v_1)}{m_j} \frac{\partial f_j}{\partial v_2} \right\} \rightarrow 0,$$

when  $i = j$  and  $v_1 \rightarrow v_2$ . This is physically reasonable, since the collision



effects of two particles of the same species travelling with the same velocity is expected to be zero. We can therefore write down a simplified form of the correlation function as

$$g_{ii}(k, v, t) = \frac{-1}{4\pi^2} \left\{ 1 + \omega_{pi}^2 \frac{ik}{k^2} \frac{\partial f_i}{\partial v} \frac{1}{\epsilon(k, -ikv)} \int \frac{dv}{(p+ikv)} \right\} \times \\ \times \left\{ 1 - \omega_{pi}^2 \frac{ik}{k^2} \frac{\partial f_i}{\partial v} \frac{1}{\epsilon(-k, ikv)} \int \frac{dv}{(p-ikv)} \right\} g_{ii}(t=0).$$

Two further simplifications can be made since

$$\int_{-\infty}^{\infty} \frac{dv}{(p+ikv)} = \int_{-\infty}^{\infty} \frac{dv}{(p-ikv)} = \frac{\pi}{k},$$

and  $\epsilon(k, -ikv) = \epsilon(-k, ikv)$ . We therefore obtain

$$g_{ii}(k, v, t) = \frac{-1}{4\pi^2} \left\{ 1 + \frac{\omega_{pi}^4}{k^4} \left( \frac{\partial f_i}{\partial v} \right)^2 \frac{\pi^2}{\epsilon^2} \right\} g_{ii}(0). \quad (C.9)$$

Since we approximated the distribution function to a sum of step functions, for any group of particles with velocity  $v$ , the term  $(\partial f_i / \partial v)^2$  in (C.9) is zero. We can therefore write

$$g_{ii}(x, v, t) = \frac{-1}{4\pi^2} g_{ii}(0), \quad (C.10)$$

for the steady state correlation, reached after the short timescale effects discussed earlier, have damped away.

In order to determine the effect of this correlation term on the one particle distribution function, we substitute (C.10) into (C.1). In the one dimensional case we are studying, the electric field between two 'particles' is a constant, and  $g(0)$  is a Dirac delta function of space and a step function of velocity. Equation (C.1) can therefore be written as

$$\frac{\partial f_i}{\partial t} = \frac{n_i}{m_i} \int \frac{e_i^2}{4\pi} \frac{\partial}{\partial v} \left\{ \frac{n(v)}{n_i} \frac{\Delta v}{v_{th,i}} L \delta(x_1 - x_2 - d) \right\} dx dv,$$

where  $L$  is the size of the system ( $64\lambda_D$ ), and  $d$  is the distance between two particles in the same step function group of velocity  $v$  and width  $\Delta v$ .

The integrals are trivial and we obtain

$$\frac{\partial f_i}{\partial t} = \frac{\omega_{pe}}{16\pi^3} \left( \frac{n(v)}{n_i} \right) \left( \frac{\Delta v}{v_{th,i}} \right) \frac{64}{v_{th,e}}.$$

Since  $f_i$  has the dimensions of  $v_{th,i}^{-1}$ , we can write the growth rate due to numerical correlation effects as

$$\gamma_c = \frac{64}{16\pi^3} \omega_{pi} \left( \frac{n(v)}{n_i} \right) \left( \frac{\Delta v}{v_{th,i}} \right).$$

Therefore, in order to ensure that spurious correlation effects are small in the simulated plasma, the number of particles in a given group must be small compared with the total in that species, and there must be many velocity groups per thermal velocity.

In the case of the NOVA program discussed in Section 5.3, since the total time of the simulation might be of order  $200 \omega_{pe}^{-1}$ , we require  $\gamma_c$  to be less than  $1/200 \omega_{pe}$ . This is to ensure that wave growth is small compared with Landau effects and small compared with the original wave amplitudes. This was achieved in the simulations by making  $\Delta v/v_{th,i} = 30$ .