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WEAK SOLUTIONS TO A MULTI-PHASE FIELD SYSTEM OF PARABOLIC EQUATIONS RELATED TO ALLOY SOLIDIFICATION *

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Abstract. Existence of weak solutions to a phase field model for solidification of alloys is studied. The model consists of balance equations for the energy and the concentrations of the alloy components which are coupled to a system of Allen-Cahn equations describing the motion of phase and grain boundaries. The system is stated in terms of thermodynamic potentials corresponding to (inverse) temperature and chemical potentials divided by the temperature and phase field variables describing the presence of the possible phases. The fields of the conserved quantities are functions of these variables, and difficulties arise from the growth properties. The existence proof is based on a perturbation method. The differential equations are solved for functions with nicer growth properties. After, appropriate estimates are derived in order to let the perturbation vanish.

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1 Introduction

The existence of weak solution to a system of nonlinear parabolic differential equations of the structure

$$\partial_t \psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}) = \nabla \cdot \boldsymbol{L}\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi}\big)\nabla \boldsymbol{u} \tag{1}$$

$$\omega(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\partial_t \boldsymbol{\phi} = \nabla \cdot a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - a_{,\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u},\boldsymbol{\phi}) \tag{2}$$

for vector valued functions $\boldsymbol{u}$ and $\boldsymbol{\phi}$ on a bounded domain Ω in a finite time interval $I = (0, \mathcal{T})$ will be shown. The equations result from a phase field model for the solidification process of multi-component alloys that has been developed in [6].

The first set (1) is a system of balance equations for the internal energy and the concentrations of the alloy components. The function ψ is convex in $\boldsymbol{u}$ so that its partial derivative $\psi_{,\boldsymbol{u}}$ with respect to $\boldsymbol{u}$ is monotone in $\boldsymbol{u}$. In $\boldsymbol{\phi}$, the function ψ and its derivatives with respect to $\boldsymbol{u}$ and $\boldsymbol{\phi}$ are bounded. The right hand side is a diffusion term with a positive semi-definite coefficient matrix $\boldsymbol{L} = (L_{ij})_{i,j}$ of real-valued functions of $\psi_{,\boldsymbol{u}}$ and $\boldsymbol{\phi}$.

The second set (2) of the equations is of a gradient flow structure based on an entropy functional involving a portion of Ginzburg-Landau type [12]. The real-valued function ω is uniformly positive in its arguments. The function a is two-homogeneous and convex in $\nabla\boldsymbol{\phi}$ so that the partial derivative $a_{,\nabla\boldsymbol{\phi}}$ is monotone in $\nabla\boldsymbol{\phi}$. Finally, w is a multi-well potential that could be split into a convex part and a bounded non-convex part.

When analyzing the system (1), (2), the main difficulties arise from the growth properties of the function ψ which are due to thermodynamically motivated choices for certain potentials. Writing $\boldsymbol{u} = (u_0, \dots, u_N)$, first, ψ can contain an additive term of the form $g(u_0) = -\log(-u_0)$ so that $\psi \nearrow \infty$ if $u_0 \nearrow 0$. Second, ψ can be of linear growth in $\tilde{\boldsymbol{u}} := (u_1, \dots, u_N)$ so that $\psi_{,\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}} \to 0$ as $\|\tilde{\boldsymbol{u}}\| \to \infty$. To precise the problems arising from these growth properties, suppose that the existence of solutions to approximating problems can be shown (indeed, here, a perturbation method will be used). In order to obtain a solution to the original problem from the approximations, in general, convergence in certain L^p spaces is necessary, i.e., estimates of differences of the form $f(t+s, x+h) - f(t,x)$ for small (s,h) are needed. In the case of parabolic differential equations the term with the time derivative yields a control of terms involving time differences, but in the present case only for $\psi_{,\boldsymbol{u}}$, and the above stated growth properties make it difficult to deduce a control of time differences for $\boldsymbol{u}$ itself.

Not only the time differences impose difficulties. Standard estimates gained by testing (1) with $\boldsymbol{u}$ and (2) with $\partial_t\boldsymbol{\phi}$ yield a bound for $\nabla\boldsymbol{u}$ in L^2 from the diffusion term. But the weak growth of ψ in $\boldsymbol{u}$ provides no estimate of $\boldsymbol{u}$ itself. In order to overcome this problem, suitable boundary conditions are imposed, namely Robin boundary conditions of the form

$$-\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi})\nabla\boldsymbol{u} \cdot \nu_{ext} = \boldsymbol{B}(\boldsymbol{u} - \boldsymbol{u}_{bc}) \tag{3}$$

where ν_{ext} is the external unit normal on $\partial\Omega$ and $\boldsymbol{B} = (\beta_{ij})_{i,j}$ is a positive semi-definite coefficient matrix of real-valued functions. Such boundary conditions can provide an L^2 estimate of $\boldsymbol{u}$ on the boundary of the domain, whence the Poincaré inequality gives the desired control.

The procedure applied in the present work is as follows. First, a function ψ of quadratic growth in $\boldsymbol{u}$ is considered. Existence of weak solutions is shown using a Galerkin method.

Thanks to the quadratic growth, the above stated difficulties do not arise, and generically derived estimates are sufficient for the limiting procedure.

The idea to solve the equations for a potential ψ of linear growth is then to approximate it by potentials of the form $\psi^{(\nu)}(\boldsymbol{u}) := \psi(\boldsymbol{u}) + \nu|\boldsymbol{u}|^2$ and let $\nu \searrow 0$. Applying methods of Alt and Luckhaus [2], this procedure successfully delivers a weak solution to the limiting problem. A related work is the one of Eck [4] where existence, uniqueness, and regularity of weak solutions to a model for two alloy components, i.e., $N = 2$, is shown. There, the model is formulated in terms of $\boldsymbol{c} := \psi_{,\boldsymbol{u}}$ and $\boldsymbol{\phi}$, and the nonlinearities are located in $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{c}, \boldsymbol{\phi})$. Degenerate diffusion coefficients L_{ij} then are used that simplify the treatment of (1).

Under a strong assumption on the diffusion matrix, namely, the exclusion of certain cross effects in the diffusion, it is also possible to manage the terms of the structure $g^{(0)}(u_0) := -\log(-u_0)$ appearing additively in ψ. Here, an approximation $g^{(\eta)}(u_0)$ of quadratic growth is used, and ideas of Alt and Pawlow [3] are applied when letting $\eta \searrow 0$. Unfortunately, the last limiting procedure is only possible for potentials $\psi^{(\eta)}$ of quadratic growth in the other variables $\tilde{\boldsymbol{u}}$, the combined problem is still open. The reason is that mixed terms of the form $|u_0(t+h) - u_0(t)||\tilde{\boldsymbol{u}}(t+h) - \tilde{\boldsymbol{u}}(t)|$ appear and cannot be appropriately estimated. It should be remarked that Luckhaus and Visintin [13] can show existence of a weak solution in this case, but without coupling to a system of equations as (2). Their work is based on [2], and they use an approximation of $g^{(0)}$ with a function of linear growth. A strong assumption on the energy flux is used to obtain $u_0 < 0$ in the limit of the approximation.

The focus of this work lies on handling $\boldsymbol{u}$ and ψ. The functions for the system (2), ω, a, and w, are chosen such that the managing of $\boldsymbol{\phi}$ is kept simple. The boundary conditions for $\boldsymbol{\phi}$ read

$$a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \cdot \nu_{ext} = 0. \tag{4}$$

Special difficulties do not appear except perhaps in the coupling term $\psi_{,\boldsymbol{\phi}}$. In works of Colli, Gajewski, Krejči, Sprekels, Zheng et al. (for instance, cf. [11], but see also the references therein), non-local models for $\boldsymbol{\phi}$ are considered where again the difficulties due to the logarithmic term in u_0 appear, but multiple conserved quantities are not taken into account so that Moser type iterations can be applied. Concerning the famous Penrose-Fife model [14], which is the simplest model involving the above stated difficulties, the articles of Horn et al. and Klein [9, 10] are worth to be mentioned.

Given initial data

$$\boldsymbol{u}(t=0) = \boldsymbol{u}_{ic}, \quad \boldsymbol{\phi}(t=0) = \boldsymbol{\phi}_{ic}. \tag{5}$$

clearly must fulfill consistency conditions. For example, when considering the problem involving $g^{(0)} = -\log(-u_0)$ the initial value for u_0 should satisfy $u_{0,ic} < 0$.

The article is organized as follows. In the following section a brief description of the model leading to equations of the form (1) and (2) is presented, and also the mentioned growth properties of ψ are motivated. The existence results are stated in the section after in form of three theorems, including precise statements on the notation and the assumption on the data. In the subsequent sections, the theorems then are proved.

2 Phase field modeling of alloy solidification

2.1 Notation

Throughout this article, $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is an open bounded domain with Lipschitz boundary, and $I = (0,\mathcal{T}) \subset \mathbb{R}$ is a time interval. For a number $K \in \mathbb{N}$ define the sets $\mathrm{H}\Sigma^K := \{\boldsymbol{v} = (v_1,\dots,v_K) \in \mathbb{R}^K : \sum_{i=1}^K v_i = 1\}$ and $\Sigma^K := \{\boldsymbol{v} \in \mathrm{H}\Sigma^K : v_i \geq 0 \quad \forall i\}$. The tangent space $\mathrm{T}_{\boldsymbol{v}}\mathrm{H}\Sigma^K$ on $\mathrm{H}\Sigma^K$ in some point $\boldsymbol{v} \in \mathrm{H}\Sigma^K$ can naturally be identified with the subspace $\mathrm{T}\Sigma^K := \{\boldsymbol{w} \in \mathbb{R}^K : \sum_{i=1}^K w_i = 0\}$. Hence, if f is a real-valued function defined on $\mathrm{H}\Sigma^K$ then its gradient in some point on $\mathrm{H}\Sigma^K$ will be identified with an element of $\mathrm{T}\Sigma^K$. And if $g : \mathbb{R} \to \mathrm{H}\Sigma^K$ is a differentiable function then $g'(r) \in \mathrm{T}\Sigma^K$, $r \in \mathbb{R}$. Let $Y^N := \mathbb{R} \times \mathrm{T}\Sigma^N$. Elements $\boldsymbol{y} \in Y^N$ sometimes will be written in the form $\boldsymbol{y} = (y_0, \tilde{\boldsymbol{y}})$ with $y_0 \in \mathbb{R}$ and $\tilde{\boldsymbol{y}} = (y_1,\dots,y_N) \in \mathrm{T}\Sigma^N$. The tangent space of Y^N in some point $\boldsymbol{y} \in Y^N$ can be identified with Y^N again. With $\mathrm{Bilin}(Y^N, Y^N)$ the bilinear forms on Y^N are denoted.

To integrate functions on $\partial\Omega$ the notation $\mathrm{d}\mathcal{H}^{d-1}$ is used for the surface area element, $\mathcal{H}^{d-1}$ being the Hausdorff measure of dimension $d-1$. Integration with respect to the bulk uses $\mathrm{d}x = \mathrm{d}\mathcal{L}^d$ with the d-dimensional Lebesgue measure $\mathcal{L}^d$. Analogously, $\mathrm{d}t = \mathrm{d}\mathcal{L}^1$ appears when integrating with respect to time. With L^p, $1 \leq p \leq \infty$, the space of measurable functions with the pth moment being Lebesgue integrable is denoted. The space $W^{m,p}$ then denotes the measurable functions such that weak derivatives up to order m exit with their pth moment being integrable. In the case $p = 2$ the notation $H^m = H^{m,2} = W^{m,2}$ will be used. Moreover, the isometric isomorphisms $L^p(I \times \Omega) \cong L^p(I; L^p(\Omega))$, $1 \leq p \leq \infty$, often are implicitly applied. The notation $C^{m,\alpha}$ stands for functions with continuous derivative up to order m that are Hölder continuous of order α. The index $\alpha = 0$ sometimes is omitted, i.e., $C^m = C^{m,0}$.

Numerous estimates will appear involving constants independent of variables but dependent only on given data as the considered domain Ω, the time interval $I = (0,\mathcal{T})$ etc. In spite of the fact that they may change from line to line they remain denoted by C. When applying compactness methods, convergence results in general only hold for subsequences. For shorter presentation, this is usually not explicitly stated, and it was abstained from an indication on the indexes.

Several theorems and results common to specialists in partial differential equations are used without explicit reference. For instance, concerning results on Dirac sequences, the Picard-Lindelöf theorem, the Riesz compactness theorem in L^p spaces, the Lebesgue dominated convergence theorem, the Vitali convergence theorem, the Fatou lemma, Rellich and Sobolev embeddings, the trace theorem for Sobolev functions, the Poincaré inequality, and the Gronwall lemma the book of Alt [1] is an appropriate reference. Concerning compactness results involving spaces of functions mapping real intervals to Banach spaces confer the books of Zeidler [16] (in particular, book II), and the article of Simon [15].

As has already been done in the introduction, partial derivatives are denoted by subscripts after a comma except with respect to space and time. For example, $s_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi})$ is the derivative of the function $s = s(\boldsymbol{c}, \boldsymbol{\phi})$ in a point $(\boldsymbol{c}, \boldsymbol{\phi})$ with respect to the variables represented by $\boldsymbol{\phi}$.

2.2 Phase field model

A general framework based on the phase field approach to model the microstructure formation during alloy solidification has been developed in [6] and results in a system of nonlinear parabolic partial differential equations of the form (1), (2). A derivation of the equations governing the evolution is briefly sketched since this motivates the growth assumptions mentioned in the introduction and helps to interpret several terms in the estimates that will be derived later on.

Introducing phase field variables and defining a Ginzburg-Landau type entropy, the evolution of the phase fields is defined by an L^2 gradient flow. That system is coupled to a set of balance equations for the internal energy and concentrations of the alloy components. To take kinetic anisotropy of the phase interfaces into account, a deviation from the gradient flow structure is allowed by introducing a positive kinetic coefficient depending on the phase fields and their gradients.

A system with M possible phases and N components is considered. The entropy functions reads

$$S(\boldsymbol{c},\boldsymbol{\phi}) = \int_\Omega \Big(s(\boldsymbol{c},\boldsymbol{\phi}) - \big(a(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) + w(\boldsymbol{\phi}) \big) \Big).$$

The vector $\boldsymbol{\phi} = (\phi_\alpha)_{\alpha=1}^M$ consists of phase field variables. Each variable ϕ_α describes the local fraction of the corresponding phase α. They are required to fulfill the constraint

$$\sum_{\alpha=1}^M \phi_\alpha = 1 \quad \Leftrightarrow \quad \boldsymbol{\phi} \in \mathrm{H}\Sigma^M. \tag{6}$$

By e or c_0 the internal energy density and by c_i, $1 \leq i \leq N$, the concentration of component i is denoted. Also the concentrations fulfill a constraint:

$$\boldsymbol{c} := (c_0, c_1, \dots c_N) \in \mathbb{R} \times \Sigma^N. \tag{7}$$

The bulk entropy contribution $s(\boldsymbol{c},\boldsymbol{\phi})$ is concave in $\boldsymbol{c}$. The function $a : \mathrm{H}\Sigma^M \times (\mathrm{T}\Sigma^M)^d \to \mathbb{R}$ is a gradient energy density which is non-negative and homogeneous of degree two in the second variable, i.e.,

$$a(\boldsymbol{\phi},X) \geq 0 \qquad \text{and} \quad a(\boldsymbol{\phi},rX) = r^2 a(\boldsymbol{\phi},X) \quad \forall\,(\boldsymbol{\phi},X) \in \Sigma^M \times (\mathrm{T}\Sigma^M)^d,\ r \in \mathbb{R}^+. \tag{8}$$

The multi-well potential $w : \mathrm{H}\Sigma^M \to \mathbb{R}$ is a non-negative function with exactly M global minima at the points $\boldsymbol{e}_\beta = (\delta_{\alpha\beta})_{\alpha=1}^M$, $1 \leq \beta \leq M$, with $w(\boldsymbol{e}_\beta) = 0$, i.e.,

$$w(\boldsymbol{\phi}) \geq 0, \quad \text{and} \quad w(\boldsymbol{\phi}) = 0 \Leftrightarrow \boldsymbol{\phi} = \boldsymbol{e}_\beta \text{ for some } \beta \in \{1,\dots,M\}. \tag{9}$$

In the original formulation of the model in [6] the ϕ_α are demanded to be non-negative, i.e., $\boldsymbol{\phi} \in \Sigma^M$. But since they have no physical meaning and only are a mathematical device to describe the phase interface motion this assumption can be dropped. From an analytical point of view, the smooth potentials w used in the following cannot guarantee non-negativity in general, in contrast to to potentials of obstacle type (e.g., see [7]).

To define the evolution of the phase field variables a weighted L^2 product is defined. Given a sufficiently smooth field $\boldsymbol{\phi} : \Omega \to \mathrm{H}\Sigma^M$ let

$$(\boldsymbol{w},\boldsymbol{v})_{\omega,\boldsymbol{\phi}} := \int_\Omega \omega(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\,\boldsymbol{w}\cdot\boldsymbol{v} \quad \forall \boldsymbol{w},\boldsymbol{v} \in L^2(\Omega;\mathrm{T}\Sigma^M).$$

The function ω is positive and homogeneous of degree zero in the second variable away from zero, i.e.,

$$\omega(\boldsymbol{\phi}, X) \geq 0 \qquad \text{and} \quad \omega(\boldsymbol{\phi}, rX) = \omega(\boldsymbol{\phi}, X) \quad \forall\, (\boldsymbol{\phi}, X) \in \Sigma^M \times \mathbb{R}^{d\times M},\ r > r_0 \tag{10}$$

with some small $r_0 > 0$. For possible choices of a, ω, and w see [7]. The above structural assumptions will later on be supplemented with growth and regularity assumptions.

The evolution of the system with respect to the phase field variables is defined by

$$(\partial_t \boldsymbol{\phi}, \boldsymbol{v})_{\omega,\boldsymbol{\phi}} = \frac{\mathrm{d}}{\mathrm{d}\delta} S(\boldsymbol{c}, \boldsymbol{\phi} + \delta \boldsymbol{v})\Big|_{\delta=0} \quad \forall \boldsymbol{v} \in C^\infty(\Omega, \mathrm{T}\Sigma^M).$$

Applying the boundary conditions (4) on $\partial\Omega$ the definition yields for all sufficiently smooth functions $v : \Omega \to \mathrm{T}\Sigma^M$ that

$$\int_\Omega \omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \partial_t \boldsymbol{\phi} \cdot \boldsymbol{v} = \int_\Omega \Big(\nabla \cdot a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) + s_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}) \Big) \cdot \boldsymbol{v}. \tag{11}$$

The balance equations for the conserved quantities read

$$\partial_t c_i = -\nabla \cdot \boldsymbol{j}_i(\boldsymbol{c}, \boldsymbol{\phi}, \nabla \boldsymbol{u}(\boldsymbol{c}, \boldsymbol{\phi})), \quad 0 \leq i \leq N, \tag{12}$$

with fluxes of the form

$$\boldsymbol{j}_i(\boldsymbol{c}, \boldsymbol{\phi}, \nabla \boldsymbol{u}(\boldsymbol{c}, \boldsymbol{\phi})) = \sum_{j=0}^{N} L_{ij}(\boldsymbol{c}, \boldsymbol{\phi}) \nabla(-u_j(\boldsymbol{c}, \boldsymbol{\phi})).$$

The thermodynamic potentials $\boldsymbol{u}$ are defined to be

$$\boldsymbol{u} = (u_0, u_1, \ldots, u_N) := -s_{,\boldsymbol{c}}(\boldsymbol{c}, \boldsymbol{\phi}) \in Y^N$$

The matrix $\boldsymbol{L}(\boldsymbol{c}, \boldsymbol{\phi}) = (L_{ij}(\boldsymbol{c}, \boldsymbol{\phi}))_{i,j=0}^N$ of diffusion coefficients is symmetric and positive semi-definite. It can be shown that this together with the gradient flow structure for the evolution of the $\boldsymbol{\phi}$ yields thermodynamic consistency in the sense that the entropy production is non-negative. In order to maintain (7) during the evolution there is the condition

$$\sum_{i=1}^{N} L_{ij}(\boldsymbol{c}, \boldsymbol{\phi}) = 0, \quad 1 \leq j \leq N. \tag{13}$$

It is worth to remark that the balance equations (12) can be interpreted as a gradient flow of the entropy with respect to a weighted H^{-1}-product.

On the boundary $\partial\Omega$ of the domain the fluxes are assumed to fulfill (3), i.e.,

$$\boldsymbol{j}_i(\boldsymbol{c}, \boldsymbol{\phi}, \nabla u(\boldsymbol{c}, \boldsymbol{\phi})) \cdot \nu_{ext} = -\sum_{j=0}^{N} L_{ij}(\boldsymbol{c}, \boldsymbol{\phi}) \nabla u_j(\boldsymbol{c}, \boldsymbol{\phi}) \cdot \nu_{ext} = \sum_{j=0}^{N} \beta_{ij}(u_j - u_{bc,j}), \tag{14}$$

where the coefficient matrix $\boldsymbol{B} = (\beta_{ij})_{i,j}$ is positive semi-definite and, similarly to (13), satisfies $\sum_{i=1}^N \beta_{ij} = 0$, $1 \leq j \leq N$, and $\boldsymbol{u}_{bc} = (u_{bc,0}, \ldots, u_{bc,N})$ is an appropriate function mapping to Y^N.

In the following strong formulation of the problem a Lagrange multiplier λ appears which is due to the constraint (6) and results from (11) when replacing $\boldsymbol{v}$ by test functions mapping to $\mathbb{R}^M$. The notation of partial derivatives with respect to a single phase field variable ϕ_α is used which, in view of (6), does not exist. But taking λ into account one observes that, effectively, the derivative in direction $\boldsymbol{e}_\alpha - \frac{1}{M}\boldsymbol{l} \in \mathrm{T}\Sigma^M$ where $\boldsymbol{e}_\alpha = (\delta_{\beta\alpha})_{\beta=1}^M$ and $\boldsymbol{l} = (1, \dots, 1) \in \mathbb{R}^M$ is on hand. Analogous facts hold for the partial derivatives with respect to $\nabla\phi_\alpha$.

Definition 2.1 *Find functions*

$$\boldsymbol{c} : I \times \Omega \to \mathbb{R} \times \Sigma^N, \quad \boldsymbol{\phi} : I \times \Omega \to \mathrm{H}\Sigma^M$$

that solve on $I \times \Omega$ the system

$$\begin{aligned}\partial_t c_i &= -\nabla \cdot \boldsymbol{j}_i(\boldsymbol{c}, \boldsymbol{\phi}, \nabla \boldsymbol{u}(\boldsymbol{c}, \boldsymbol{\phi})) = \nabla \cdot \left(\sum_{j=0}^N L_{ij}(\boldsymbol{c}, \boldsymbol{\phi}) \nabla u_j(\boldsymbol{c}, \boldsymbol{\phi}) \right), \\ \omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\, \partial_t \phi_\alpha &= \nabla \cdot a_{,\nabla\phi_\alpha}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - a_{,\phi_\alpha}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - w_{,\phi_\alpha}(\boldsymbol{\phi}) - s_{,\phi_\alpha}(\boldsymbol{c}, \boldsymbol{\phi}) - \lambda,\end{aligned}$$

where $0 \le i \le N$ and $1 \le \alpha \le M$ and λ is given by

$$\lambda = \frac{1}{M} \sum_{\beta=1}^M \left(\nabla \cdot a_{,\nabla\phi_\beta}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - a_{,\phi_\beta}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - w_{,\phi_\beta}(\boldsymbol{\phi}) - s_{,\phi_\beta}(\boldsymbol{c}, \boldsymbol{\phi}) \right),$$

subject to initial conditions (5) *and boundary conditions* (14) *and*

$$\left(a_{,\nabla\phi_\alpha}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) - \frac{1}{M} \sum_{\beta=1}^M a_{,\nabla\phi_\beta}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \right) \cdot \nu_{ext} = 0.$$

2.3 Reduced grand canonical potential

Instead of using densities of the conserved quantities as variables the thermodynamic potentials $\boldsymbol{u}$ can be used. The good thermodynamic quantity to reformulate the diffusion equations is the reduced grand canonical potential ψ, defined to be the Legendre transform (cf. [5]) of $-s$ with respect to internal energy and concentrations. This transform is carefully carried out in the following.

Assume that

R1 the function $(-s) : C \to \mathbb{R}$ is of the class C^2 on the convex open set $C \subset \mathbb{R} \times \mathrm{H}\Sigma^N$ and strictly convex,

R2 its derivative $-s_{,\boldsymbol{c}} = D(-s) : C \to U$ is a C^1-diffeomorphism into a convex open set $U \subset \mathbb{R} \times \mathrm{T}\Sigma^N$.

Assumption R1 implies that $D^2(-s)(\boldsymbol{c})$, acting on $(\mathbb{R} \times \mathrm{T}\Sigma^N)^2$, is positive and has full rank so that, locally, assumption R2 is already satisfied.

In the following, the sets C and U are considered as subsets of $\mathbb{R}^{N+1}$, and $\boldsymbol{c} \cdot \boldsymbol{u}$ is the standard scalar product on $\mathbb{R}^{N+1}$ for elements $\boldsymbol{c} \in C$ and $\boldsymbol{u} \in U$.

Lemma 2.1 *With the assumptions R1 and R2 the Legendre transform of the entropy density*

$$(-s)^*(\boldsymbol{u}) := \sup_{\boldsymbol{c}\in C}\{\boldsymbol{c}\cdot\boldsymbol{u} + s(\boldsymbol{c})\}, \quad \boldsymbol{u}\in U,$$

is a well-defined real valued function $(-s)^* : U \to \mathbb{R}$. *Besides*

$$(-s)^*_{,\boldsymbol{u}}(\boldsymbol{u}) = D(-s)^*(\boldsymbol{u}) = \boldsymbol{c}.$$

Proof: For a given $\boldsymbol{u}$ the quantity $\boldsymbol{c}(\boldsymbol{u}) := (D(-s))^{-1}(\boldsymbol{u})$ exists by assumption R2. From the convexity of $-s$ in assumption R1 it follows that this is the only critical point of $\boldsymbol{c}\mapsto\boldsymbol{c}\cdot\boldsymbol{u}+s(\boldsymbol{c})$, and that this is the global maximum. Hence $(-s)^*(\boldsymbol{u}) := \boldsymbol{c}(\boldsymbol{u})\cdot\boldsymbol{u}+s(\boldsymbol{c}(\boldsymbol{u}))$ is well-defined. The identity for the derivative follows easily using $D(-s)(\boldsymbol{c}) = \boldsymbol{u}$. □

Definition 2.2 *If the entropy density s satisfies the assumptions R1 and R2 then the density of the reduced grand canonical potential is defined by*

$$\psi : U \to \mathbb{R}, \quad \psi(\boldsymbol{u}) := (-s)^*(\boldsymbol{u}) = \boldsymbol{c}(\boldsymbol{u})\cdot\boldsymbol{u} + s(\boldsymbol{c}(\boldsymbol{u})) \tag{15}$$

where $\boldsymbol{c}$ *as a function of* $\boldsymbol{u}$ *is given as the unique solution to* $D(-s)(\boldsymbol{c}) = \boldsymbol{u}$.

Sufficient conditions to obtain $-s$ from a given ψ read similar to the assumptions R1 and R2 and lead to the notation

$$-s(\boldsymbol{c}) = \psi^*(\boldsymbol{c}) = \boldsymbol{c}\cdot\boldsymbol{u}(\boldsymbol{c}) - \psi(\boldsymbol{u}(\boldsymbol{c})) \quad \text{with } \boldsymbol{u}(\boldsymbol{c}) \text{ solution to } \boldsymbol{c} = D\psi(\boldsymbol{u}). \tag{16}$$

Here and throughout this article, the object $D\psi$ will always be considered as mapping to the space $\mathbb{R}\times \mathrm{H}\Sigma^N$. Naturally, one would identify the tangential space on Y^N in some $\boldsymbol{u}\in Y^N$ with the linear space $Y^N \subset \mathbb{R}^{N+1}$ again, endowed with the standard scalar product induced from $\mathbb{R}^{N+1}$, and think of $D\psi(\boldsymbol{u})$ or, more precisely, the gradient $\mathrm{grad}(\psi)(\boldsymbol{u})$ being an element of Y^N. The object $\boldsymbol{c}(\boldsymbol{u})$ is then obtained by adding the vector $\boldsymbol{l} := \frac{1}{N}(0,1,\ldots,1) \in \mathbb{R}\times\mathrm{H}\Sigma^N \subset \mathbb{R}^{N+1}$, i.e., $\boldsymbol{c}(\boldsymbol{u}) = \mathrm{grad}(\psi)(\boldsymbol{u}) + \boldsymbol{l} \in \mathbb{R}\times\mathrm{H}\Sigma^N$. That this is the right object can be seen when looking at the derivative of ψ in a direction $\boldsymbol{v}\in Y^N$. Indeed, since $\boldsymbol{l}\perp Y^N$ it holds that

$$\langle D\psi(\boldsymbol{u}),\boldsymbol{v}\rangle = \mathrm{grad}\psi(\boldsymbol{u})\cdot\boldsymbol{v} = (\mathrm{grad}\psi(\boldsymbol{u}) + \boldsymbol{l})\cdot\boldsymbol{v} = \boldsymbol{c}(\boldsymbol{u})\cdot\boldsymbol{v}.$$

To motivate the challenges by the growth of ψ mentioned in the introduction an example for a binary alloy of two components A and B is considered. The free energy density

$$f(T,c_1,c_2) = L_A\frac{T-T_A}{T_A}c_1 + L_B\frac{T-T_B}{T_B} + \tilde{R}T(c_1\log(c_1)+c_2\log(c_2)) - c_vT\Big(\log(\frac{T}{T_{ref}})-1\Big)$$

corresponds to the model of an ideal solution (cf. [8]) which is widely used in thermodynamics and materials science. The L_i, T_i, and $\tilde{R}$, c_v, and T_{ref} are material constants related, among others, to latent heats and melting temperatures. Using the relation $s = -f_{,T}$ the internal energy is linear in the temperature T,

$$e = f + Ts = -L_Ac_1 - L_Bc_2 + c_vT.$$

Inverting this relation enables to write $-s$ as a function in $\boldsymbol{c} = (e, c_1, c_2)$,

$$-s(\boldsymbol{c}) = \Big(\frac{L_A}{T_A}c_1 + \frac{L_B}{T_B}c_2\Big) + \tilde{R}(c_1\log(c_1) + c_2\log(c_2)) - c_v\log\Big(\frac{1}{c_vT_{ref}}(e + L_Ac_1 + L_Bc_2)\Big).$$

Since $\boldsymbol{u} \in Y^3$ it is clear that $u_1 = -u_2$. Computing the derivative of s gives

$$\begin{aligned} u_0 &= \partial_{c_0}(-s)(\boldsymbol{c}) = -\frac{c_v}{e + L_Ac_1 + L_Bc_2} \quad = \quad -\frac{1}{T}, \\ 2u_1 &= L_A\Big(\frac{1}{T_A} - \frac{c_v}{e + L_Ac_1 + L_Bc_2}\Big) - L_B\Big(\frac{1}{T_B} - \frac{c_v}{e + L_Ac_1 + L_Bc_2}\Big) + \tilde{R}\log(\frac{c_1}{c_2}). \end{aligned}$$

Using $c_2 = 1 - c_1$ the above functions can be inverted, and $\boldsymbol{c}$ can be written as a function in $\boldsymbol{u}$. A short calculation yields

$$\begin{aligned} e(\boldsymbol{u}) &= -\frac{c_v}{u_0} - L_A\frac{1}{1 + e^{v_1(\boldsymbol{u})}} - L_B\frac{1}{1 + e^{v_2(\boldsymbol{u})}}, \\ c_1(\boldsymbol{u}) &= \frac{1}{1 + e^{v_1(\boldsymbol{u})}}, \quad c_2(\boldsymbol{u}) = \frac{1}{1 + e^{v_2(\boldsymbol{u})}} \end{aligned}$$

where

$$-v_2(\boldsymbol{u}) = v_1(\boldsymbol{u}) = \frac{1}{\tilde{R}}\Big(L_A(u_0 - u_A) - L_B(u_0 - u_B) - 2u_1\Big)$$

with $u_A := \frac{-1}{T_A}$ and $u_B := \frac{-1}{T_B}$. The entropy density becomes

$$s = \Big(\frac{L_Au_A}{1 + e^{v_1(\boldsymbol{u})}} + \frac{L_Bu_B}{1 + e^{v_2(\boldsymbol{u})}}\Big) + \tilde{R}\Big(\frac{\log(1 + e^{v_1(\boldsymbol{u})})}{1 + e^{v_1(\boldsymbol{u})}} + \frac{\log(1 + e^{v_2(\boldsymbol{u})})}{1 + e^{v_2(\boldsymbol{u})}}\Big) - c_v\log(-u_0T_{ref}).$$

Inserting this and $\boldsymbol{c}(\boldsymbol{u})$ into (15) gives the reduced grand canonical potential density

$$\begin{aligned} \psi(\boldsymbol{u}) = &\Big(\frac{L_A(u_A - u_0)}{1 + e^{v_1(\boldsymbol{u})}} + \frac{L_B(u_B - u_0)}{1 + e^{v_2(\boldsymbol{u})}}\Big) \\ &+ \Big(\frac{u_1 + \tilde{R}\log(1 + e^{v_1(\boldsymbol{u})})}{1 + e^{v_1(\boldsymbol{u})}} + \frac{u_2 + \tilde{R}\log(1 + e^{v_2(\boldsymbol{u})})}{1 + e^{v_2(\boldsymbol{u})}}\Big) - c_v\big(1 + \log(-u_0T_{ref})\big). \end{aligned} \tag{17}$$

Up to the last term which tends to infinity as $u_0 \nearrow 0$ the growth in $\boldsymbol{u}$ is linear. It is worth to remark that, once existence of a (weak) solution was established for the above ψ, automatically values for the concentrations c_i between zero and one and positivity of the temperature would be ensured.

2.4 Strong formulation of the differential equations

The aim is now to write down the equations governing the evolution in terms of $(\boldsymbol{u}, \boldsymbol{\phi})$ instead of $(\boldsymbol{c}, \boldsymbol{\phi})$ as in Definition 2.1. For this purpose, the density of the reduced grand canonical potential ψ including its derivatives is used.

In the preceding subsection it is shown how the reduced grand canonical potential density of a phase can be computed given the free energy density of the phase provided

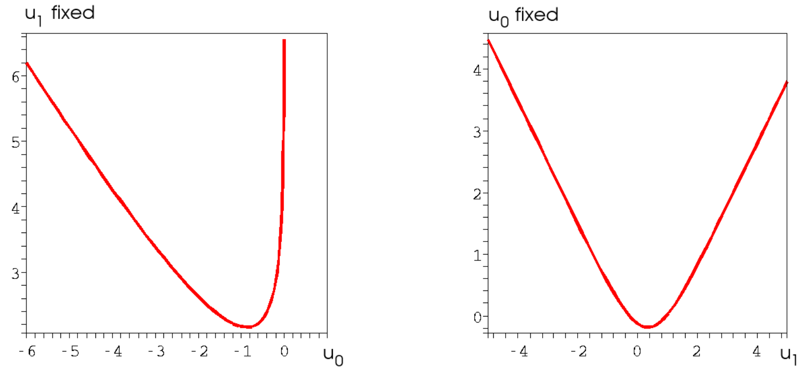

Figure 1: Reduced grand canonical potential ψ given by (17) with the values $L_A = 1$, $L_B = 1.2$, $u_A = 0.8$, $u_B = 1.4$, $\tilde{R} = 1$, $c_v = 1$ and $T_{ref} = 1$ as a function of u_0 and on the line $u_1 + u_2 = 0$ respectively. The function ψ is strictly convex. On the right picture, the slope is $\psi_{,u1} - \psi_{,u2} = c_1 - c_2 \in (-1, 1) \subset \mathbb{R}$.

some structural conditions are satisfied. In a multi-phase system, assume the existence of densities $\psi^\alpha : U_\alpha \to \mathbb{R}$, $1 \le \alpha \le M$, for the possible phases with $U_\alpha \subset \mathbb{R} \times \mathrm{T}\Sigma^N$ defined in assumption R2 in Section 2.3. Assume further that $U = \bigcap_{\alpha=1}^M U_\alpha$ is non-empty. The function $\psi : U \times \mathrm{H}\Sigma^M \to \mathbb{R}$ is obtained as a suitable interpolation of the ψ^α such that $\psi(\boldsymbol{u}, \boldsymbol{e}_\alpha) = \psi^\alpha(\boldsymbol{u})$,

$$\psi : U \times \Sigma^M \to \mathbb{R}, \quad \psi(\boldsymbol{u}, \boldsymbol{\phi}) = \sum_{\alpha=1}^{M} \psi^\alpha(\boldsymbol{u}) h(\phi_\alpha)$$

with a function $h : [0, 1] \to [0, 1]$ satisfying $h(\varphi) = 0$ if $\varphi \le 0$ and $h(\varphi) = 1$ if $\varphi \ge 1$.

In view of (15) one may write

$$s(\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) = \psi(\boldsymbol{u}, \boldsymbol{\phi}) - \boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{u} \tag{18}$$

where $\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}) = \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ or, equivalently, $-s_{,\boldsymbol{c}}(\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) = \boldsymbol{u}$ (see Definition 2.2, considering the $\boldsymbol{\phi}$ just as parameters). If the dependence of ψ on $\boldsymbol{\phi}$ is smooth enough (C^2, as assumed in the following section, is sufficient) then varying $\boldsymbol{\phi}$ is possible in (18). The derivative of the left hand side of (18) in a direction $\boldsymbol{\zeta} \in \mathrm{T}\Sigma^M$ is

$$\begin{aligned}\sum_{\alpha=1}^{M} \frac{\mathrm{d}}{\mathrm{d}\phi_\alpha} \Big(s\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \cdot), \cdot\big) \Big)\Big|_{\boldsymbol{\phi}} \zeta_\alpha &= s_{,\boldsymbol{c}}(\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \cdot \psi_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi})\, \boldsymbol{\zeta} + s_{,\boldsymbol{\phi}}(\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} \\ &= -\boldsymbol{u} \cdot \psi_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{\zeta} + s_{,\boldsymbol{\phi}}(\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \cdot \boldsymbol{\zeta}\end{aligned}$$

but the right hand side yields $\psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} - \boldsymbol{u} \cdot \psi_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{\zeta}$. Comparing finally furnishes the relation (in the sense of gradients belonging to the space $\mathrm{T}\Sigma^M$)

$$s_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}) = -\psi^*_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}) = \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{where } \boldsymbol{c} = \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}). \tag{19}$$

Definition 2.3 *The evolution is governed by the partial differential equations*

$$\partial_t \psi_{,u_i}(\boldsymbol{u},\boldsymbol{\phi}) = -\nabla \cdot \boldsymbol{j}_i(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi},\nabla \boldsymbol{u}) = \nabla \cdot \left(\sum_{j=0}^{N} L_{ij}(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi}) \nabla u_j \right), \quad (20)$$

$$\omega(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\partial_t \phi_\alpha = \nabla \cdot a_{,\nabla\phi_\alpha}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - a_{,\phi_\alpha}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - w_{,\phi_\alpha}(\boldsymbol{\phi}) + \psi_{,\phi_\alpha}(\boldsymbol{u},\boldsymbol{\phi}) - \lambda \quad (21)$$

where $0 \le i \le N$ *and* $1 \le \alpha \le M$ *with* λ *given by*

$$\lambda = \frac{1}{M} \sum_{\beta=1}^{M} \big(\nabla \cdot a_{,\nabla\phi_\beta}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - a_{,\phi_\beta}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - w_{,\phi_\beta}(\boldsymbol{\phi}) + \psi_{,\phi_\beta}(\boldsymbol{u},\boldsymbol{\phi}) \big) .$$

The differential equations are subject to initial conditions

$$\boldsymbol{u}(t=0) = \boldsymbol{u}_{ic}, \quad \boldsymbol{\phi}(t=0) = \boldsymbol{\phi}_{ic}$$

and boundary conditions

$$\boldsymbol{j}_i(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi},\nabla\boldsymbol{u}) \cdot \nu_{ext} = \sum_{j=0}^{N} \beta_{ij}(u_j - u_{bc,j}), \qquad 0 \le i \le N,$$

$$\Big(a_{,\nabla\phi_\alpha}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) - \frac{1}{M}\sum_{\beta=1}^{M} a_{,\nabla\phi_\beta}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) \Big) \cdot \nu_{ext} = 0, \qquad 1 \le \alpha \le M.$$

3 Existence results

3.1 General assumptions

First, some assumptions are stated that are imposed for all following theorems. They concern the nonlinearities in the phase field equations (21).

G1 In addition to the structural assumptions (9) $w \in C^{1,1}(\mathrm{H}\Sigma^M)$ satisfies

$$|w(\boldsymbol{\phi})| \le w_0(1+|\boldsymbol{\phi}|^p), \quad |w_{,\boldsymbol{\phi}}(\boldsymbol{\phi})| \le w_1(1+|\boldsymbol{\phi}|^{p-1}), \quad w(\boldsymbol{\phi}) \ge w_2|\boldsymbol{\phi}|^p - w_3, \quad (22)$$

for all $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$ where the w_i are positive constants. Here, $p > 2$ is such that $1 - \frac{d}{2} > -\frac{d}{p}$, hence, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Observe that if $\boldsymbol{\phi} \in L^p(I \times \Omega; \mathrm{H}\Sigma^M)$ then $w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \in L^{p^*}(I \times \Omega; \mathrm{T}\Sigma^M)$ with the dual exponent $p^* := \frac{p}{p-1}$ to p. The restriction to the growth of w is necessary in order to obtain the strong convergence of the gradients of the phase field variables in Subsection 4.4. But it is not essential for the properties of the phase field model since for the asymptotic analysis shown in [6] only the structure of w on and close to Σ^M is of interest.

G2 In addition to (8) the gradient term $a \in C^{1,1}(\mathrm{H}\Sigma^M \times (\mathrm{T}\Sigma^M)^d)$ fulfills

$$a_0|\boldsymbol{X}|^2 \le a(\boldsymbol{\phi},\boldsymbol{X}) \le a_1(|\boldsymbol{\phi}|^2 + |\boldsymbol{X}|^2), \quad (23)$$

$$a_{,\boldsymbol{\phi}}(\boldsymbol{\phi},\boldsymbol{X}) \le a_2\big(|\boldsymbol{\phi}| + |\boldsymbol{X}|\big), \quad a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\boldsymbol{X}) \le a_3\big(|\boldsymbol{\phi}| + |\boldsymbol{X}|\big), \quad (24)$$

$$\big(a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\boldsymbol{X}) - a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\hat{\boldsymbol{X}})\big) : (\boldsymbol{X} - \hat{\boldsymbol{X}}) \ge a_4|\boldsymbol{X} - \hat{\boldsymbol{X}}|^2, \quad (25)$$

for all $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$ and $\boldsymbol{X}, \hat{\boldsymbol{X}} \in (\mathrm{T}\Sigma^M)^d$ where the a_i are positive constants.

G3 In addition to (10) the kinetic coefficient $\omega \in C^{0,1}(\mathrm{H}\Sigma^M \times (\mathrm{T}\Sigma^M)^d)$ satisfies

$$\omega_0 \le \omega(\boldsymbol{\phi}, \boldsymbol{X}) \le \omega_1, \tag{26}$$

for all $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$ and $\boldsymbol{X} \in (\mathrm{T}\Sigma^M)^d$ where the ω_i are positive constants. Observe that $\frac{1}{\omega} \in C^{0,1}(\mathrm{H}\Sigma^M \times (\mathrm{T}\Sigma^M)^d)$.

3.2 Reduced grand canonical potential of quadratic growth

Concerning the balance equations (20) and initial and boundary conditions assume the following.

Q1 The reduced canonical potential $\psi \in C^{2,1}(Y^N \times \mathrm{H}\Sigma^M)$ satisfies

$$|\psi(\boldsymbol{u}, \boldsymbol{\phi})| \le k_6(1 + |\boldsymbol{u}|^2), \quad |\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{v}| \le k_5(1 + |\boldsymbol{u}|)|\boldsymbol{v}|, \tag{27}$$

$$\boldsymbol{v} \cdot \psi_{,\boldsymbol{uu}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{v} \ge k_0|\boldsymbol{v}|^2, \quad |\boldsymbol{w} \cdot \psi_{,\boldsymbol{uu}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{v}| \le k_1|\boldsymbol{w}||\boldsymbol{v}|, \tag{28}$$

$$|\psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{\zeta}| \le k_2(1 + |\boldsymbol{u}|), \quad |\boldsymbol{v} \cdot \psi_{,\boldsymbol{u\phi}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{\zeta}| \le k_3|\boldsymbol{v}||\boldsymbol{\zeta}|, \quad |\psi(0, \boldsymbol{\phi})| \le k_4, \tag{29}$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}) \in Y^N \times \mathrm{H}\Sigma^M$, $\boldsymbol{v}, \boldsymbol{w} \in Y^N$, and $\boldsymbol{\zeta} \in \mathrm{T}\Sigma^M$ where the k_i are positive constants. The assumption (28) implies that

$$(\psi_{,\boldsymbol{uu}}(\cdot))^{-1} \in C^{0,1}(Y^N \times \mathrm{H}\Sigma^M, \mathrm{Bilin}(Y^N, Y^N)).$$

Q2 The matrix $\boldsymbol{L} = (L_{ij})_{i,j=0}^N$ with coefficients

$$L_{ij} \in C^{0,1}(\mathbb{R} \times \mathrm{H}\Sigma^N \times \mathrm{H}\Sigma^M) \cap L^\infty(\mathbb{R} \times \mathrm{H}\Sigma^N \times \mathrm{H}\Sigma^M)$$

uniformly in its arguments fulfills

$$\boldsymbol{L} \text{ is symmetric and positive semi-definite,} \tag{30}$$

$$\ker(\boldsymbol{L}) = \mathrm{span}\{(0, 1, \dots, 1) \in \mathbb{R}^{N+1}\} = (Y^N)^\perp, \tag{31}$$

$$\boldsymbol{v} \cdot \boldsymbol{L}(\boldsymbol{c}, \boldsymbol{\phi})\boldsymbol{v} \ge l_0|\boldsymbol{v}|^2, \quad \boldsymbol{w} \cdot \boldsymbol{L}(\boldsymbol{c}, \boldsymbol{\phi})\boldsymbol{v} \le L_0|\boldsymbol{w}||\boldsymbol{v}| \tag{32}$$

for all $\boldsymbol{w}, \boldsymbol{v} \in Y^N$, $\boldsymbol{c} \in \mathbb{R} \times \mathrm{H}\Sigma^N$ and $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$ where $0 < l_0 \le L_0$ are constants.

Q3 The initial data $\boldsymbol{u}_{ic} \in L^2(\Omega; Y^N)$, $\boldsymbol{\phi}_{ic} \in H^1(\Omega; \Sigma^M)$ are such that

$$\int_\Omega \Big[\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \cdot \boldsymbol{u}_{ic} - \psi(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) + w(\boldsymbol{\phi}_{ic}) + |\nabla \boldsymbol{\phi}_{ic}|^2\Big] dx \le C. \tag{33}$$

Observe that $\boldsymbol{\phi}_{ic} \in L^\infty(\Omega; \Sigma^M)$, that $w(\boldsymbol{\phi}_{ic}) \in L^1(\Omega)$ thanks to (22), and that $|\nabla \boldsymbol{\phi}_{ic}|^2 \in L^1(\Omega)$, whence the two last terms could have been dropped.

Q4 The boundary data for $\boldsymbol{u}$ fulfill $\boldsymbol{u}_{bc} \in C^0(\overline{I \times \partial\Omega}; Y^N)$. Furthermore, the coefficient matrix $\boldsymbol{B} = (\beta_{ij})_{i,j=0}^N \in C^0(\overline{I \times \partial\Omega}; \mathrm{Bilin}(Y^N, Y^N))$ is symmetric and satisfies

$$\begin{aligned} &\ker(\boldsymbol{B}) \supset \mathrm{span}\{(0, 1, \dots, 1) \in \mathbb{R}^{N+1}\} = (Y^N)^\perp, \\ &|\boldsymbol{w} \cdot \boldsymbol{B}(t, x)\boldsymbol{v}| \le \beta_1|\boldsymbol{w}||\boldsymbol{v}|, \quad \boldsymbol{v} \cdot \boldsymbol{B}(t, x)\boldsymbol{v} \ge \beta_0|\boldsymbol{v}|^2 \end{aligned} \tag{34}$$

for all $(t, x) \in \overline{I \times \partial\Omega}$ and $\boldsymbol{w}, \boldsymbol{v} \in Y^N$ where $0 \le \beta_0 \le \beta_1$ are constants.

Theorem 3.1 *If the assumptions G1–G3 and Q1–Q4 are fulfilled then there are functions*

$$\boldsymbol{u} \in L^2(I; H^1(\Omega; Y^N)), \quad \boldsymbol{\phi} \in H^1(I \times \Omega; \mathrm{H}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{H}\Sigma^M)$$

such that

$$\boldsymbol{\phi}(t, \cdot) \to \boldsymbol{\phi}_{ic} \quad \text{in } L^2(\Omega; \mathrm{H}\Sigma^M) \text{ as } t \searrow 0 \tag{35}$$

and such that

$$\begin{aligned} 0 = \quad & \int_I \int_\Omega \Big[-\partial_t \boldsymbol{v} \cdot (\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})) + \nabla \boldsymbol{v} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \nabla \boldsymbol{u} \Big] \,\mathrm{d}x\mathrm{d}t \\ + & \int_I \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{B}(\boldsymbol{u} - \boldsymbol{u}_{bc}) \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\ + & \int_I \int_\Omega \Big[\omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \partial_t \boldsymbol{\phi} \cdot \boldsymbol{\zeta} + a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) : \nabla\boldsymbol{\zeta} \Big] \,\mathrm{d}x\mathrm{d}t \\ + & \int_I \int_\Omega \Big[a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \cdot \boldsymbol{\zeta} + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \cdot \boldsymbol{\zeta} - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} \Big] \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{36}$$

for all $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$ *with* $\boldsymbol{v}(\mathcal{T}) = 0$ *and* $\boldsymbol{\zeta} \in H^1(I \times \Omega; \mathrm{T}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{T}\Sigma^M)$.

Proof: The proof of the theorem is given in several steps, each one corresponding to one of the subsections in Section 4.

4.1 For a Galerkin approximation, the existence of solutions $(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})_{n \in \mathbb{N}}$ mapping the time interval into finite dimensional subspaces $Y^{(n)} \times X^{(n)}$ of $H^1(\Omega; Y^N) \times H^1(\Omega; \mathrm{T}\Sigma^M)$ is shown.

4.2 Uniform estimates in n are derived. It is shown that for $m \le n$ and some C independent of m, n

$$\begin{aligned} \|\boldsymbol{u}^{(n)}\|_{L^\infty(I; L^2(\Omega; Y^N))} + \|\nabla \boldsymbol{u}^{(n)}\|_{L^2(I; L^2(\Omega; (Y^N)^d))} + \|\partial_t \boldsymbol{u}^{(n)}\|_{L^2(I; (Y^{(m)}))^*} &\le C, \\ \|\boldsymbol{\phi}^{(n)}\|_{L^\infty(I; L^p(\Omega; \mathrm{H}\Sigma^M))} + \|\nabla \boldsymbol{\phi}^{(n)}\|_{L^\infty(I; L^2(\Omega; (\mathrm{T}\Sigma^M)^d))} + \|\partial_t \boldsymbol{\phi}^{(n)}\|_{L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M))} &\le C. \end{aligned}$$

4.3 The imposed regularity and growth assumptions enable to go to the limit as $n \to \infty$ in most of the terms in the weak formulation of the Galerkin problem.

4.4 Strong convergence of $\nabla \boldsymbol{\phi}^{(n)} \to \nabla \boldsymbol{\phi}$ in L^2 has to be shown in order to handle the terms involving ω, $a_{,\boldsymbol{\phi}}$, and $a_{,\nabla\boldsymbol{\phi}}$. The idea is to use $\boldsymbol{\zeta}^{(n)} = \boldsymbol{\phi}^{(n)} - \boldsymbol{\phi}$ as test function for the Galerkin system and to use (25) to get $|\nabla\boldsymbol{\phi}^{(n)} - \nabla\boldsymbol{\phi}|$ under control. The fact that $\boldsymbol{\zeta}^{(n)}$ is no admissible test function makes it necessary to construct an approximation appropriately converging strongly to $\boldsymbol{\phi}$.

4.5 To conclude the proof, assertion (35) is shown.

$\square$

3.3 Reduced grand canonical potential of linear growth

Consider now a reduced grand canonical potentials of the form

$$\psi(\boldsymbol{u},\boldsymbol{\phi}) = g(u_0) + \sum_{\alpha=1}^{M} h(\phi_\alpha)\lambda^{(\alpha)}(\boldsymbol{u})$$

where $h:\mathbb{R}\to[0,1]$, the functions $\lambda^{(\alpha)}$ are convex but only of linear growth in $\boldsymbol{u}$, and g is of quadratic growth. Because of the special structure of ψ it makes sense to split the variable $\boldsymbol{u}$. Recall the notation $\boldsymbol{u} = (u_0, \tilde{\boldsymbol{u}})$ with $u_0 \in \mathbb{R}$ and $\tilde{\boldsymbol{u}} \in \mathrm{T}\Sigma^N$.

The idea of solving the problem in Definition 2.3 is to approximate the above ψ with potentials satisfying the assumption Q1, namely

$$\psi^{(\nu)}(\boldsymbol{u},\boldsymbol{\phi}) := \nu|\tilde{\boldsymbol{u}}|^2 + \psi(\boldsymbol{u},\boldsymbol{\phi}).$$

After, compactness arguments are applied to the solutions in order to deduce a limiting function which solves the differential equations with the original ψ. The arguments follow the lines of [2] for the potentials $\boldsymbol{u}$. The challenge is to tackle the problems due to the coupling to the phase field variables $\boldsymbol{\phi}$.

Given some small $\overline{\nu} > 0$ assume the following:

L1 The functions $g \in C^{2,1}(\mathbb{R})$, $\lambda^{(\alpha)} \in C^{2,1}(Y^N)$, and $h \in W^{3,\infty}(\mathbb{R})$ fulfill

$$\begin{aligned}
&|g(u_0)| \le g_0(1+u_0^2), \quad |g'(u_0)| \le g_1(1+|u_0|), \quad |g''(u_0)| \le g_2,\\
&\boldsymbol{v}\cdot\psi_{,\boldsymbol{uu}}(\boldsymbol{u},\boldsymbol{\phi})\boldsymbol{v} \ge k_0|v_0|^2, \quad |\boldsymbol{w}\cdot\lambda^{(\alpha)}_{,\boldsymbol{uu}}(\boldsymbol{u})\boldsymbol{v}| \le \hat{k}_1|\boldsymbol{w}||\boldsymbol{v}|,\\
&|\lambda^{(\alpha)}(\boldsymbol{u})| \le \hat{k}_2(1+|\boldsymbol{u}|), \quad |\lambda^{(\alpha)}_{,\boldsymbol{u}}(\boldsymbol{u})\cdot\boldsymbol{v}| \le \hat{k}_3|\boldsymbol{v}|, \quad |\lambda^{(\alpha)}(0)| \le \hat{k}_4,\\
&h(r) = 0 \text{ if } r \le 0, \quad h(r) = 1 \text{ if } r \ge 1, \quad 0 \le h'(r) \le k_7
\end{aligned} \tag{37}$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in Y^N$, $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$, $\alpha \in \{1,\dots,M\}$, and $r \in \mathbb{R}$, where the g_i, the $\hat{k}_i$ and the k_i are positive constants.

L2 The assumptions in Q2 remain fulfilled.

L3 For initial data $(\boldsymbol{u}_{ic},\boldsymbol{\phi}_{ic})$ as in Q3

$$\int_\Omega \Big[\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic},\boldsymbol{\phi}_{ic})\cdot\boldsymbol{u}_{ic} - \psi^{(\nu)}(\boldsymbol{u}_{ic},\boldsymbol{\phi}_{ic}) + w(\boldsymbol{\phi}_{ic}) + |\nabla\boldsymbol{\phi}_{ic}|^2\Big]\,dx \le C.$$

holds with a constant C independent of ν as long as $\nu \in [0,\overline{\nu}]$.

L4 The assumptions in Q4 are fulfilled. In addition it holds that $\beta_0 > 0$, and the boundary data $\boldsymbol{u}_{bc}$ are such that for some $C > 0$

$$\|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}_{bc},\boldsymbol{\phi})\|_{L^2(I;L^2(\partial\Omega;Y^N))} \le C \quad \text{for all } \nu \in [0,\overline{\nu}],\ \boldsymbol{\phi} \in H^{1,2}(I\times\Omega;\mathrm{H}\Sigma^M).$$

In Theorem 3.1 one can allow for $\beta = 0$ which corresponds to no-flux or homogeneous Neumann boundary conditions for $\boldsymbol{u}$. In the proof a control of an approximating Galerkin solution $\boldsymbol{u}^{(n)}$ in L^2 is obtained from the quadratic growth of ψ. But in the present situation that estimate is not available any more in the limiting case $\nu = 0$ (more precisely, the estimate (55) is not valid any more), whence the above stated Robin boundary conditions with $\beta_0 > 0$ are essential to get a control of $\tilde{\boldsymbol{u}}$. For u_0 the condition could be relaxed since, by assumption (37), the situation for u_0 is as in Theorem 3.1.

Theorem 3.2 *If the assumptions G1–G3 and L1–L4 are fulfilled then there are functions*

$$\boldsymbol{u} \in L^2(I; H^1(\Omega; Y^N)), \quad \boldsymbol{\phi} \in H^1(I \times \Omega; \mathrm{H}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{H}\Sigma^M)$$

such that

$$\boldsymbol{\phi}(t, \cdot) \to \boldsymbol{\phi}_{ic} \quad \text{in } L^2(\Omega; \mathrm{H}\Sigma^M) \text{ as } t \searrow 0 \tag{38}$$

and such that

$$\begin{aligned} 0 = \quad & \int_I \int_\Omega \Big[-\partial_t \boldsymbol{v} \cdot (\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})) + \nabla \boldsymbol{v} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \nabla \boldsymbol{u} \Big] \, \mathrm{d}x \mathrm{d}t \\ & + \int_I \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{B}(\boldsymbol{u} - \boldsymbol{u}_{bc}) \, \mathrm{d}\mathcal{H}^{d-1} \mathrm{d}t \\ & + \int_I \int_\Omega \Big[\omega(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) \partial_t \boldsymbol{\phi} \cdot \boldsymbol{\zeta} + a_{,\nabla \boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) : \nabla \boldsymbol{\zeta} \Big] \, \mathrm{d}x \mathrm{d}t \\ & + \int_I \int_\Omega \Big[a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \cdot \boldsymbol{\zeta} - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} \Big] \, \mathrm{d}x \mathrm{d}t \end{aligned} \tag{39}$$

for all $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$ *with* $\boldsymbol{v}(\mathcal{T}) = 0$ *and* $\boldsymbol{\zeta} \in H^1(I \times \Omega; \mathrm{T}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{T}\Sigma^M)$.

Proof: The proof of the theorem is given in several steps, each one corresponding to one of the subsections in Section 5.

5.1 The perturbed reduced grand canonical potential $\psi^{(\nu)}$ fulfills the assumptions of Theorem 3.1 yielding a solution $(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)})$ and providing a set of useful estimates. By functional analytical facts on the considered spaces candidates $(\boldsymbol{u}, \boldsymbol{\phi})$ for a solution to the weak problem are obtained. It remains to handle the nonlinearities.

5.2 Several preparatory facts on $\psi^{(\nu)}$ and its Legendre transform are shown which are of technical nature.

5.3 The core of the proof is to show that the set of functions $\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)})\}_\nu$ is precompact in L^1.

5.4 The results are sufficient to go to the limit in the weak formulation of the problem for the perturbed potential $\psi^{(\nu)}$ as $\nu \to 0$.

□

3.4 Reduced grand canonical potential with logarithmic term

In this subsection, a reduced grand canonical potential of the form

$$\psi : (-\infty, 1) \times \mathrm{T}\Sigma^N \times \mathrm{H}\Sigma^M \to \mathbb{R},$$

$$\psi(\boldsymbol{u}, \boldsymbol{\phi}) = \underbrace{-c_v\big(1 + \log(T_{ref}(1-u_0))\big)}_{=:g(u_0)} + \nu|\tilde{\boldsymbol{u}}|^2 + \sum_{\alpha=1}^{M} h(\phi_\alpha)\lambda^{(\alpha)}(\boldsymbol{u})$$

is considered. The functions $\lambda^{(\alpha)}$ are convex and of linear growth in $\boldsymbol{u}$. Observe that, in contrast to the potential in the example in Subsection 2.3, there is a shift by 1 in u_0. This is done only for technical reasons, namely, to have a well defined value at $\boldsymbol{u} = 0$.

Again, the idea is to approximate ψ with potentials satisfying the conditions in assumption Q1 in order to apply Theorem 3.1. After, apply compactness arguments to the solutions in order to deduce a limiting function. To obtain convergence in u_0, truncation techniques as in [3] are used.

The approximation of the function g and, hence, ψ is done as follows. For $\eta \in [0, \overline{\eta}]$ with some small $\overline{\eta} > 0$ let y_η and z_η be the points such that $g'(y_\eta) = \frac{1}{\eta}$ and $g'(z_\eta) = \eta$. The points exist if $\overline{\eta}$ is small enough since g' is continuous, $g'(u_0) \to \infty$ as $u_0 \nearrow 1$ and $g'(u_0) \to 0$ as $u_0 \searrow -\infty$. Clearly $y_\eta \to 1$ and $z_\eta \to \infty$ as $\eta \to 0$. Uniqueness follows from the fact that g is strictly convex, hence, g' is strictly monotone increasing.

Let $g_\eta^+ : \mathbb{R} \to \mathbb{R}$ be the unique polynomial of degree 2 such that $g_\eta^+(y_\eta) = g(y_\eta)$, $(g_\eta^+)'(y_\eta) = g'(y_\eta)$, and $(g_\eta^+)''(y_\eta) = g''(y_\eta)$. Analogously, let $g_\eta^- : \mathbb{R} \to \mathbb{R}$ be the unique quadratic polynomial such that $g_\eta^-(z_\eta) = g(z_\eta)$, $(g_\eta^-)'(z_\eta) = g'(z_\eta)$ and $(g_\eta^-)''(z_\eta) = g''(z_\eta)$. Define

$$g^{(\eta)}(u_0) := \begin{cases} g_\eta^+(u_0), & y_\eta \le u_0, \\ g(u_0), & z_\eta \le u_0 \le y_\eta, \\ g_\eta^-(u_0), & u_0 \le z_\eta, \end{cases}$$

and then $\psi^{(\eta)} \in C^{2,1}(Y^N \times \mathrm{H}\Sigma^M)$ by

$$\psi^{(\eta)}(\boldsymbol{u}, \boldsymbol{\phi}) = g^{(\eta)}(u_0) + \nu|\tilde{\boldsymbol{u}}|^2 + \sum_{\alpha=1}^{M} h(\phi_\alpha)\lambda^{(\alpha)}(u).$$

Observe that, in this part of the article, η varies but ν is a fixed positive constant. Letting $\eta \to 0$ it must be shown that a solution $(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ to the perturbed problem converges to a function $(\boldsymbol{u}, \boldsymbol{\phi})$ with $u_0 < 1$ almost everywhere. For this purpose, an estimate of the form

$$\|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})\|_{L^2} \le C$$

will be derived. Since $g'(u_0) = -c_v \frac{1}{u_0 - 1}$ this enables to get the desired result. Unfortunately, in order to obtain that estimate, additional assumptions on the coefficients L_{ij} and the boundary conditions have to be imposed. Cross effects between mass and energy diffusion have to be neglected, and Robin boundary conditions are only imposed for the flux of u_0 while it is assumed that there is no flux of the u_i, $i \ge 1$, across the external boundary.

The precise assumptions are:

B1 The functions $\psi^{(\eta)} \in C^{2,1}(Y^N)$, $\lambda^{(\alpha)} \in C^{2,1}(Y^N)$, and $h \in W^{3,\infty}(\mathbb{R})$ fulfill

$$\begin{aligned}
&\boldsymbol{v} \cdot \psi^{(\eta)}_{,\boldsymbol{uu}}(\boldsymbol{u}, \boldsymbol{\phi})\boldsymbol{v} \geq \hat{k}_0|\tilde{\boldsymbol{v}}|^2, \quad |\boldsymbol{w} \cdot \lambda^{(\alpha)}_{,\boldsymbol{uu}}(\boldsymbol{u})\boldsymbol{v}| \leq \hat{k}_1|\boldsymbol{w}||\boldsymbol{v}|, \\
&|\lambda^{(\alpha)}(\boldsymbol{u})| \leq \hat{k}_2(1 + |\boldsymbol{u}|), \quad |\lambda^{(\alpha)}_{,\boldsymbol{u}}(\boldsymbol{u}) \cdot \boldsymbol{v}| \leq \hat{k}_3|\boldsymbol{v}|, \quad |\lambda^{(\alpha)}(0)| \leq \hat{k}_4, \\
&h(r) = 0 \text{ if } r \leq 0, \quad h(r) = 1 \text{ if } r \geq 1, \quad 0 \leq h'(r) \leq k_7
\end{aligned} \tag{40}$$

for all $\eta \in [0, \overline{\eta}]$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in Y^N$, $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$, $\alpha \in \{1, \ldots, M\}$, and $r \in \mathbb{R}$, where the $\hat{k}_i$ and k_i are positive constants. Moreover, there is a small $\delta_0 > 0$ and a constant k_8 such that

$$\psi^{(\eta)}_{,u_0}(\boldsymbol{u}, \boldsymbol{\phi}) \geq K_\eta(u_0 - 1) - k_8 \quad \text{whenever } u_0 > 1 - \delta_0 \tag{41}$$

with $0 < K_\eta \to \infty$ as $\eta \to 0$.

B2 The coefficients L_{ij} are as in assumption Q2 but, in addition, fulfill

$$L_{i0} = L_{0i} = 0 \quad \forall i \in \{1, \ldots, N\}. \tag{42}$$

B3 The initial data $(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})$ are as in assumption Q3 and, in addition, such that

$$\begin{aligned}
&\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) = \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \text{ and } \|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\|_{L^2(\Omega)} \leq C \quad \text{for all } \eta \in [0, \overline{\eta}], \\
&\int_\Omega \Big[\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \cdot \boldsymbol{u}_{ic} - \psi^{(\eta)}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) + w(\boldsymbol{\phi}_{ic}) + |\nabla \boldsymbol{\phi}_{ic}|^2\Big] dx \leq C
\end{aligned} \tag{43}$$

for all $\eta \in [0, \overline{\eta}]$ with C independent of η. Observe that the first assumption means that $u_{ic,0} \in L^\infty(\Omega)$ is bounded away from $-\infty$ and 1.

B4 For the energy flux the boundary condition

$$\boldsymbol{j}_0 \cdot \nu_{ext} = \beta_{00}(u_0 - u_{bc,0})$$

is imposed with a continuous function $\beta_{00} : I \times \Omega \to \mathbb{R}$ satisfying

$$0 < \beta_0 \leq \beta_{00}(t, x) < \beta_1$$

and a function $u_{bc,0} \in C(\overline{I \times \partial\Omega}; Y^N) \cap L^2(I; L^2(\partial\Omega; Y^N))$ such that

$$\|\psi^{(\eta)}_{,u_0}(u_{bc,0}, \tilde{\boldsymbol{u}}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})\|_{L^2(I; L^2(\partial\Omega))} \leq C$$

for all sets $\{\tilde{\boldsymbol{u}}^{(\eta)}\}_{\eta \in [0, \overline{\eta}]} \subset \mathrm{T}\Sigma^N$, $\{\boldsymbol{\phi}^{(\eta)}\}_{\eta \in [0, \overline{\eta}]} \subset \mathrm{H}\Sigma^M$ with

$$\sup_{\eta \in [0, \overline{\eta}]} \Big(\|\boldsymbol{\phi}^{(\eta)}\|_{L^2(I; L^2(\partial\Omega; \mathrm{H}\Sigma^M))} + \|\tilde{\boldsymbol{u}}^{(\eta)}\|_{L^2(I; L^2(\partial\Omega; \mathrm{H}\Sigma^N))}\Big) \leq C. \tag{44}$$

For the mass no-flux boundary conditions are imposed:

$$\boldsymbol{j}_i \cdot \nu_{ext} = 0, \quad i = 1, \ldots, N.$$

Theorem 3.3 *If the assumptions G1–G3 and B1–B4 are fulfilled then there are functions*

$$\boldsymbol{u} \in L^2(I; H^1(\Omega; Y^N)), \quad \boldsymbol{\phi} \in H^1(I \times \Omega; \mathrm{H}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{H}\Sigma^M)$$

such that

$$\begin{aligned} &u_0 < 1 \text{ almost everywhere}, \\ &\boldsymbol{\phi}(t, \cdot) \to \boldsymbol{\phi}_{ic} \quad \text{in } L^2(\Omega; \mathrm{H}\Sigma^M) \text{ as } t \searrow 0, \end{aligned} \tag{45}$$

and such that

$$\begin{aligned} 0 = \quad & \int_I \int_\Omega \Big[-\partial_t \boldsymbol{v} \cdot (\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})) + \nabla \boldsymbol{v} : L(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \nabla \boldsymbol{u} \Big] \,\mathrm{d}x\mathrm{d}t \\ + & \int_I \int_{\partial\Omega} v_0 \cdot \beta_{00}(u_0 - u_{bc,0}) \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\ + & \int_I \int_\Omega \Big[\omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \partial_t \boldsymbol{\phi} \cdot \boldsymbol{\zeta} + a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) : \nabla\boldsymbol{\zeta} \Big] \,\mathrm{d}x\mathrm{d}t \\ + & \int_I \int_\Omega \Big[a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) \cdot \boldsymbol{\zeta} + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \cdot \boldsymbol{\zeta} - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \cdot \boldsymbol{\zeta} \Big] \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{46}$$

for all $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$ *with* $\boldsymbol{v}(\mathcal{T}) = 0$ *and* $\boldsymbol{\zeta} \in H^1(I \times \Omega; \mathrm{T}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{T}\Sigma^M)$.

Proof: The proof of the theorem is given in several steps, each one corresponding to one of the subsections in Section 6.

6.1 The perturbed potential $\psi^{(\eta)}$ fulfills the assumptions of Theorem 3.1. Since the other assumptions are satisfied, too, there is a weak solution $(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ to the perturbed problem with additional estimates independent of η.

6.2 An estimate for the $\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ is derived. Together with the other estimates, a candidate $(\boldsymbol{u}, \boldsymbol{\phi})$ for a solution to (46) can be obtained, and it can be shown that the candidate satisfies $u_0 \leq 1$. A subsequence of the $\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ converges weakly to some limiting function $\boldsymbol{b}$ in L^2.

6.3 The function $\boldsymbol{b}$ has to be identified with $\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$. In order to go to the limit in the coupling term in the phase field equation $\psi^{(\eta)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ strong convergence of the $\boldsymbol{u}^{(\eta)}$ to $\boldsymbol{u}$ will be shown. The main task is to get a control of time differences of the form $|u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t)|$. The images of the functions $u_0^{(\eta)}$ are projected to a compact interval where the second derivatives of the $\psi^{(\eta)}$ with respect to u_0 are bounded from below by a positive constant independent of η. A control of time differences of the truncated functions is obtained from the standard estimates. Moreover, the error due to the truncation, measured in the norm of the space $L^1(I \times \Omega)$, can be made arbitrarily small.

6.4 Collecting the obtained convergence results it is possible to let $\eta \to 0$ in the weak formulation of the perturbed problem and to show that the candidate $(\boldsymbol{u}, \boldsymbol{\phi})$ in fact is a solution to (46). In particular, it is shown that the solution fulfills $u_0 < 1$ almost everywhere.

□

4 Proof of Theorem 3.1

4.1 Galerkin approximation

Let the set $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega;Y^N)$ be a Schauder basis of $H^1(\Omega;Y^N)$ such that the matrix $\big((\boldsymbol{e}_i,\boldsymbol{e}_j)_{L^2(\Omega;Y^N)}\big)_{i,j=0}^n$ is regular for each $n\in\mathbb{N}$. Similarly, let $\{\boldsymbol{b}_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega;\mathrm{T}\Sigma^M)$ be a Schauder basis of $H^1(\Omega;\mathrm{T}\Sigma^M)$ such that $\big((\boldsymbol{b}_i,\boldsymbol{b}_j)_{L^2(\Omega;\mathrm{T}\Sigma^M)}\big)_{i,j=0}^n$ is regular, $n\in\mathbb{N}$. Given some $n\in\mathbb{N}$ define the finite dimensional Galerkin spaces

$$Y^{(n)} := \mathrm{span}\{\boldsymbol{e}_m, 0\le m\le n\}, \quad X^{(n)} := \mathrm{span}\{\boldsymbol{b}_m, 0\le m\le n\}.$$

The Galerkin ansatz reads

$$\boldsymbol{u}^{(n)}(t,x) = \sum_{k=0}^n u^{(k,n)}(t)\boldsymbol{e}_k(x), \quad \boldsymbol{\phi}^{(n)}(t,x) = \boldsymbol{1}^M + \sum_{l=0}^n \phi^{(l,n)}(t)\boldsymbol{b}_l(x)$$

with functions $u^{(k,n)}\in C^1(I)$, $\phi^{(l,n)}\in C^1(I)$. The aim is to solve the following problem: *Find* $(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}) \in C^1(I;Y^{(n)})\times C^1(I;X^{(n)})$ *such that*

$$\boldsymbol{u}^{(n)}(t=0) = \boldsymbol{u}_{ic}^{(n)} := \sum_{k=0}^n \big(\boldsymbol{u}_{ic},\boldsymbol{e}_k\big)_{L^2(\Omega;Y^N)}\boldsymbol{e}_k, \tag{47}$$

$$\boldsymbol{\phi}^{(n)}(t=0) = \boldsymbol{\phi}_{ic}^{(n)} := \sum_{l=0}^n \big(\boldsymbol{\phi}_{ic},\boldsymbol{b}_l\big)_{L^2(\Omega;\mathrm{T}\Sigma^M)}\boldsymbol{b}_l, \tag{48}$$

and such that for each $t\in I$

$$\begin{aligned}
0 = &\int_\Omega \Big[\boldsymbol{v}^{(n)}\cdot\big(\psi_{,\boldsymbol{uu}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\partial_t\boldsymbol{u}^{(n)} + \psi_{,\boldsymbol{u\phi}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\partial_t\boldsymbol{\phi}^{(n)}\big)\Big]\,\mathrm{d}x \\
&+ \int_\Omega \Big[\nabla\boldsymbol{v}^{(n)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}),\boldsymbol{\phi}^{(n)})\nabla\boldsymbol{u}^{(n)}\Big]\,\mathrm{d}x + \int_{\partial\Omega}\Big[\boldsymbol{v}^{(n)}\cdot\boldsymbol{B}(\boldsymbol{u}^{(n)}-\boldsymbol{u}_{bc})\Big]\,\mathrm{d}\mathcal{H}^{d-1} \\
&+ \int_\Omega \Big[\omega(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\boldsymbol{\zeta}^{(n)}\cdot\partial_t\boldsymbol{\phi}^{(n)} + \nabla\boldsymbol{\zeta}^{(n)} : a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\Big]\,\mathrm{d}x \\
&+ \int_\Omega \Big[\boldsymbol{\zeta}^{(n)}\cdot\big(a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)}) + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}) - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\big)\Big]\,\mathrm{d}x
\end{aligned} \tag{49}$$

for all test functions of the form

$$\boldsymbol{v}^{(n)} = \sum_{k=0}^n v^{(k,n)}\boldsymbol{e}_k, \quad \boldsymbol{\zeta}^{(n)} = \sum_{l=0}^n \zeta^{(l,n)}\boldsymbol{b}_l \tag{50}$$

with real coefficients $v^{(k,n)}$ *and* $\zeta^{(l,n)}$.

By assumption (28) and the properties of the basis functions $\{\boldsymbol{e}_k\}_k$ it holds that

$$\int_\Omega \sum_{m_1,k=0}^{n,n} \boldsymbol{e}_{m_1}\cdot\psi_{,\boldsymbol{uu}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\boldsymbol{e}_k\,\mathrm{d}x \ge k_0\int_\Omega \Big|\sum_{k=0}^n \boldsymbol{e}_k\Big|^2\,\mathrm{d}x = k_0\sum_{i,j=0}^n\int_\Omega \boldsymbol{e}_i\cdot\boldsymbol{e}_j\,\mathrm{d}x > 0.$$

Similarly, assumption (26) and the properties of the $\{\boldsymbol{b}_l\}_l$ imply

$$\int_\Omega \sum_{m_2,l=0}^{n,n} \omega(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)})\boldsymbol{b}_{m_2}\cdot\boldsymbol{b}_l \,\mathrm{d}x \geq \omega_0 \int_\Omega \left|\sum_{l=0}^{n} \boldsymbol{b}_l\right|^2 \mathrm{d}x > 0.$$

Therefore, choosing $\boldsymbol{v}^{(n)} = \boldsymbol{e}_k$, $k = 0,\ldots,n$, and $\boldsymbol{\zeta}^{(n)} = \boldsymbol{b}_l$, $l = 0,\ldots,n$, in (49) yields a system for the coefficients functions $u^{(k,n)}$, $\phi^{(l,n)}$ with matrices before the vectors $\partial_t(u^{(k,n)})_k$ and $\partial_t(\phi^{(l,n)})_l$ that can be inverted. By the regularity assumptions on the occurring functions, namely in G1, G2, G3, Q1, Q2, and Q4 all terms in (49) are Lipschitz continuous with respect to the coefficient functions $u^{(k,n)}(t)$ and $\phi^{(l,n)}(t)$, and continuous with respect to t. Applying standard results for ordinary differential equations (e.g., the theorem of Picard-Lindelöf) there is a unique solution $(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \in C^1(I; Y^{(n)} \times X^{(n)})$ to (49) subject to the initial data $(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)})$ given in (52).

Using test functions $(\boldsymbol{v}^{(m)}, \boldsymbol{\zeta}^{(m)})$ of the form (50) with n replaced by m and coefficient functions $v^{(k,m)} \in C^1(I)$ fulfilling $v^{(k,m)}(\mathcal{T}) = 0$ and $\zeta^{(l,m)} \in C^0(I)$, equation (49) becomes after partially integrating with respect to t over I for $n \geq m$

$$\begin{aligned}
0 = &- \int_I\int_\Omega \partial_t\boldsymbol{v}^{(m)}\cdot(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)}))\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_\Omega \nabla\boldsymbol{v}^{(m)}\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}), \boldsymbol{\phi}^{(n)}) : \nabla\boldsymbol{u}^{(n)}\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_{\partial\Omega} \boldsymbol{v}^{(m)}\cdot\boldsymbol{B}(\boldsymbol{u}^{(n)} - \boldsymbol{u}_{bc})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
&+ \int_I\int_\Omega \omega(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)})\boldsymbol{\zeta}^{(m)}\cdot\partial_t\boldsymbol{\phi}^{(n)}\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_\Omega \Big[\nabla\boldsymbol{\zeta}^{(m)} : a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)}) + \boldsymbol{\zeta}^{(m)}\cdot a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)})\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_\Omega \Big[\boldsymbol{\zeta}^{(m)}\cdot w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}) - \boldsymbol{\zeta}^{(m)}\cdot\psi_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})\Big]\,\mathrm{d}x\mathrm{d}t. \qquad (51)
\end{aligned}$$

4.2 Uniform estimates

The goal is now to derive appropriate estimates to let $n \to \infty$ in (51). For this purpose, test (49) with $\boldsymbol{v}^{(n)} = \boldsymbol{u}^{(n)}$ and $\boldsymbol{\zeta}^{(n)} = \partial_t\boldsymbol{\phi}^{(n)}$ and integrate with respect to t over some time interval $\tilde{I} = (0, \tilde{t})$, $\tilde{t} < \mathcal{T}$ to find

$$\begin{aligned}
0 = &\int_{\tilde{I}}\int_\Omega \Big[\partial_t\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})\cdot\boldsymbol{u}^{(n)} - \psi(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})\big)\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{\tilde{I}}\int_\Omega \Big[\partial_t\big(a(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)}) + w(\boldsymbol{\phi}^{(n)})\big)\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{\tilde{I}}\int_\Omega \Big[\omega(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)})|\partial_t\boldsymbol{\phi}^{(n)}|^2 + \nabla\boldsymbol{u}^{(n)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}), \boldsymbol{\phi}^{(n)})\nabla\boldsymbol{u}^{(n)}\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{\tilde{I}}\int_{\partial\Omega} \Big[\boldsymbol{u}^{(n)}\cdot\boldsymbol{B}(\boldsymbol{u}^{(n)} - \boldsymbol{u}_{bc})\Big]\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t.
\end{aligned}$$

Here, the regularity assumptions on w, a, and ψ were used again.

Thanks to properties of the basis functions $\{\boldsymbol{e}_k\}_k$ and $\{\boldsymbol{b}_l\}_l$ clearly as $n \to \infty$

$$\begin{aligned} \boldsymbol{u}_{ic}^{(n)} &\to \boldsymbol{u}_{ic} \quad \text{almost everywhere and in } L^2(\Omega; Y^N), \\ \boldsymbol{\phi}_{ic}^{(n)} &\to \boldsymbol{\phi}_{ic} \quad \text{almost everywhere, in } H^1(\Omega; \mathrm{H}\Sigma^M) \text{ and in } L^p(\Omega; \mathrm{H}\Sigma^M). \end{aligned} \tag{52}$$

This yields, using the Lebesgue convergence theorem and the growth properties (27), (22), and (23), that

$$\begin{aligned} &\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \text{ in } L^2(\Omega; Y^N), \quad \psi(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)}) \to \psi(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \text{ in } L^1(\Omega) \\ &w(\boldsymbol{\phi}_{ic}^{(n)}) \to w(\boldsymbol{\phi}_{ic}) \text{ in } L^1(\Omega) \quad a(\boldsymbol{\phi}_{ic}^{(n)}, \nabla \boldsymbol{\phi}_{ic}^{(n)}) \to a(\boldsymbol{\phi}_{ic}, \nabla \boldsymbol{\phi}_{ic}) \text{ in } L^1(\Omega). \end{aligned} \tag{53}$$

By (47), (48), and assumption (33) it follows that

$$\begin{aligned} &\int_\Omega \left[\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}(\tilde{t}), \boldsymbol{\phi}^{(n)}(\tilde{t})) \cdot \boldsymbol{u}^{(n)}(\tilde{t}) - \psi(\boldsymbol{u}^{(n)}(\tilde{t}), \boldsymbol{\phi}^{(n)}(\tilde{t}))\right] \mathrm{d}x \\ &+ \int_\Omega \left[w(\boldsymbol{\phi}^{(n)}(\tilde{t})) + a(\boldsymbol{\phi}^{(n)}(\tilde{t}), \nabla \boldsymbol{\phi}^{(n)}(\tilde{t}))\right] \mathrm{d}x \\ &+ \int_{\tilde{I}} \int_\Omega \left[\omega(\boldsymbol{\phi}^{(n)}, \nabla \boldsymbol{\phi}^{(n)}) |\partial_t \boldsymbol{\phi}^{(n)}|^2 + \nabla \boldsymbol{u}^{(n)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}), \boldsymbol{\phi}^{(n)}) \nabla \boldsymbol{u}^{(n)}\right] \mathrm{d}x\mathrm{d}t \\ &+ \int_{\tilde{I}} \int_\Omega \left[\boldsymbol{u}^{(n)} \cdot \beta\big(\boldsymbol{u}^{(n)} - \boldsymbol{u}_{bc}\big)\right] \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\ \leq \quad &\int_\Omega \left[\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)}) \cdot \boldsymbol{u}_{ic}^{(n)} - \psi(\boldsymbol{u}_{ic}^{(n)}, \boldsymbol{\phi}_{ic}^{(n)}) + w(\boldsymbol{\phi}_{ic}^{(n)}) + a(\boldsymbol{\phi}_{ic}^{(n)}, \nabla \boldsymbol{\phi}_{ic}^{(n)})\right] \mathrm{d}x \\ \to \quad &\int_\Omega \left[\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) \cdot \boldsymbol{u}_{ic} - \psi(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}) + w(\boldsymbol{\phi}_{ic}) + a(\boldsymbol{\phi}_{ic}, \nabla \boldsymbol{\phi}_{ic})\right] \mathrm{d}x \quad \leq \quad C. \end{aligned} \tag{54}$$

Assumption (28) gives

$$\begin{aligned} &\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \cdot \boldsymbol{u}^{(n)} - \psi(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \\ = \quad &\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\big(\psi_{,\boldsymbol{u}}(\theta \boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \cdot \theta \boldsymbol{u}^{(n)} - \psi(\theta \boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})\big)\, \mathrm{d}\theta - \psi(0, \boldsymbol{\phi}^{(n)}) \\ = \quad &\int_0^1 \big(\theta \boldsymbol{u}^{(n)} \cdot (\psi_{,\boldsymbol{u}\boldsymbol{u}}(\theta \boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \boldsymbol{u}^{(n)})\big)\, \mathrm{d}\theta - \psi(0, \boldsymbol{\phi}^{(n)}) \\ \geq \quad &\frac{k_0}{2} |\boldsymbol{u}^{(n)}|^2 - k_4. \end{aligned} \tag{55}$$

By assumption (34) and using Young's inequality with a small δ (later specified)

$$\begin{aligned} &\int_{\tilde{I}} \int_{\partial\Omega} \left[\boldsymbol{u}^{(n)} \cdot \boldsymbol{B} \boldsymbol{u}^{(n)} - \boldsymbol{u}^{(n)} \cdot \boldsymbol{B} \boldsymbol{u}_{bc}\right] \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\ \geq \quad &\beta_0 \int_{\tilde{I}} \int_{\partial\Omega} |\boldsymbol{u}^{(n)}|^2 \, \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t - \beta_1 \int_{\tilde{I}} \int_{\partial\Omega} |\boldsymbol{u}^{(n)}| |\boldsymbol{u}_{bc}| \, \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\ \geq \quad &(\beta_0 - \beta_1 \delta) \int_{\tilde{I}} \|\boldsymbol{u}^{(n)}\|_{L^2(\partial\Omega)}^2 \, \mathrm{d}t - C(\beta_1, \delta) \int_{\tilde{I}} \|\boldsymbol{u}_{bc}\|_{L^2(\partial\Omega)}^2 \, \mathrm{d}t. \end{aligned} \tag{56}$$

Now, the estimate (54) yields thanks to the assumptions (32), (22), (23), and (26)

$$\int_\Omega \left[\tfrac{k_0}{2}|\boldsymbol{u}^{(n)}(\tilde{t})|^2 + w_2|\boldsymbol{\phi}^{(n)}(\tilde{t})|^p + a_0|\nabla\boldsymbol{\phi}^{(n)}(\tilde{t})|^2\right] \mathrm{d}x \\ + \int_{\tilde{I}}\int_\Omega \left[\omega_0|\partial_t\boldsymbol{\phi}^{(n]}|^2 + l_0|\nabla\boldsymbol{u}^{(n)}|^2\right] \mathrm{d}x\mathrm{d}t - \int_{\tilde{I}}\int_{\partial\Omega} \delta\beta_1|\boldsymbol{u}^{(n)}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \quad \leq \quad C. \quad (57)$$

By the trace theorem for Sobolev functions there is a constant C_{tr} such that

$$-\delta\beta_1 \int_{\tilde{I}}\int_{\partial\Omega} |\boldsymbol{u}^{(n)}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \geq -\delta\beta_1 C_{Tr}\int_{\tilde{I}}\int_\Omega |\boldsymbol{u}^{(n)}|^2 + |\nabla\boldsymbol{u}^{(n)}|^2 \,\mathrm{d}x\mathrm{d}t.$$

Choose $\delta > 0$ so small such that $l_0 - \delta\beta_1 C_{Tr} > 0$. Then (57) gives

$$\int_\Omega \tfrac{k_0}{2}|\boldsymbol{u}^{(n)}(\tilde{t},x)|^2 \,\mathrm{d}x \leq C + \int_0^{\tilde{t}}\int_\Omega \delta\beta_1 C_{Tr}|\boldsymbol{u}^{(n)}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t.$$

Applying the Gronwall Lemma to $t \mapsto \int_\Omega |\boldsymbol{u}^{(n)}(t,x)|^2 \mathrm{d}x$ then yields with (57)

$$\|\boldsymbol{u}^{(n)}\|_{L^\infty(I;L^2(\Omega;Y^N)} + \|\boldsymbol{\phi}^{(n)}\|_{L^\infty(I;L^p(\Omega;\mathrm{H}\Sigma^M))} + \|\nabla\boldsymbol{\phi}^{(n)}\|_{L^\infty(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d))} \\ + \|\partial_t\boldsymbol{\phi}^{(n)}\|_{L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))} + \|\nabla\boldsymbol{u}^{(n)}\|_{L^2(I;L^2(\Omega;(Y^N)^d))} \quad \leq \quad C. \quad (58)$$

Choose now time dependent coefficients $v^{(k,n)}(t)$ in (49) and integrate with respect to t over I. With the assumptions (32) and (34) and with estimate (58) it follows that

$$\begin{aligned}
& \Big|\int_I\int_\Omega \boldsymbol{v}^{(n)} \cdot \partial_t\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\Big| \\
= & \Big|\int_I\int_\Omega \nabla\boldsymbol{v}^{(n)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}),\boldsymbol{\phi}^{(n)})\nabla\boldsymbol{u}^{(n)} \,\mathrm{d}x\mathrm{d}t \\
& \quad + \int_I\int_{\partial\Omega} \boldsymbol{v}^{(n)} \cdot \boldsymbol{B}(\boldsymbol{u}^{(n)} - \boldsymbol{u}_{bc}) \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t\Big| \\
\leq & \; L_0\|\nabla\boldsymbol{v}^{(n)}\|_{L^2(I;L^2(\Omega;(Y^N)^d))}\|\nabla\boldsymbol{u}^{(n)}\|_{L^2(I;L^2(\Omega;(Y^N)^d))} \\
& + \beta_1\|\boldsymbol{v}^{(n)}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\big(\|\boldsymbol{u}^{(n)}\|_{L^2(I;L^2(\partial\Omega;Y^N))} + \|\boldsymbol{u}_{bc}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\big) \\
\leq & \; C\|\boldsymbol{v}^{(n)}\|_{L^2(I;H^1(\Omega;Y^N))}
\end{aligned}$$

so that for all natural numbers $n \geq m$ with some constant $C(m)$ independent of n

$$\|\partial_t\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\|_{L^2(I,(Y^{(m)})^*)} \quad \leq \quad C(m). \quad (59)$$

By (28) and (29) $|\partial_t\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})| \geq k_0|\partial_t\boldsymbol{u}^{(n)}| - k_3|\partial_t\boldsymbol{\phi}^{(n)}|$, hence from (58) and (59) for $n \geq m$ with some $C(m)$ independent of n

$$\|\partial_t\boldsymbol{u}^{(n)}\|_{L^2(I,(Y^{(m)})^*)} \quad \leq \quad C(m). \quad (60)$$

4.3 First convergence results

Since the Hilbert spaces $L^2(I;H^1(\Omega;Y^N))$, $L^2(I;L^2(\partial\Omega;Y^N))$, and $H^1(I\times\Omega;\mathrm{H}\Sigma^M)$ are reflexive, in view of (58), there are functions $\boldsymbol{u}$ and $\boldsymbol{\phi}$ such that for a subsequence as $n\to\infty$ (as mentioned previously already, whenever there are convergence statements in the following, in general, they are only valid for subsequences which are relabeled with n again)

$$\boldsymbol{\phi}^{(n)} \rightharpoonup \boldsymbol{\phi} \qquad \text{in } H^1(I\times\Omega;\mathrm{H}\Sigma^M), \tag{61}$$

$$\boldsymbol{u}^{(n)} \rightharpoonup \boldsymbol{u} \qquad \text{in } L^2(I;H^1(\Omega;Y^N)) \text{ and in } L^2(I;L^2(\partial\Omega;Y^N)). \tag{62}$$

By the compactness of the embedding

$$\Big\{\boldsymbol{\zeta}\in L^p(I;H^1(\Omega;\mathrm{H}\Sigma^M))\ :\ \partial_t\boldsymbol{\zeta}\in L^2(I;L^2(\Omega;\mathrm{H}\Sigma^M))\Big\}\hookrightarrow L^p(I;L^p(\Omega;\mathrm{H}\Sigma^M))$$

with p as in assumption G(1) the results (58) and (61) lead to

$$\boldsymbol{\phi}^{(n)}\to\boldsymbol{\phi} \text{ almost everywhere and in } L^q(I\times\Omega;\mathrm{H}\Sigma^M) \tag{63}$$

for $q=2$ and $q=p$. Also the embedding

$$\Big\{\xi\in L^2(I;H^1(\Omega;Y^N)),\partial_t\xi\in L^2(I;(Y^{(m)})^*)\Big\}\hookrightarrow L^2(I;L^2(\Omega;Y^N)) \tag{64}$$

exists and is compact. The estimates (58) and (60) therefore imply that there is some $\hat{\boldsymbol{u}}\in L^2(I;L^2(\Omega;Y^N))$ such that $\boldsymbol{u}^{(n)}\to\hat{\boldsymbol{u}}$ almost everywhere and in $L^2(I;L^2(\Omega;Y^N))$. By (62) (the weak limit is unique) $\hat{\boldsymbol{u}}=\boldsymbol{u}$, hence

$$\boldsymbol{u}^{(n)}\to\boldsymbol{u} \text{ almost everywhere and in } L^2(I;L^2(\Omega;Y^N)). \tag{65}$$

By assumptions (28) and (29) and using (58) the functions $\nabla\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})$ are uniformly bounded in $L^2(I;L^2(\Omega;(Y^N)^d))$, and thanks to assumption (27) also the functions $\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\in L^2(I;L^2(\Omega;Y^N))$. The estimate (59) and (64) yield precompactness of the $\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})$ in $L^2(I;L^2(\Omega;Y^N))$. Thanks to (63) and (65) this furnishes

$$\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\to\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}) \text{ almost everywhere and in } L^2(I;L^2(\Omega;Y^N)). \tag{66}$$

In the preceding subsection it was already demonstrated that

$$\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}^{(n)},\boldsymbol{\phi}_{ic}^{(n)})\to\psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic},\boldsymbol{\phi}_{ic}) \text{ almost everywhere and in } L^2(\Omega;Y^N). \tag{67}$$

By the assumptions on $\boldsymbol{L}$ in Q2 the functions $\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}),\boldsymbol{\phi}^{(n)})\nabla\boldsymbol{v}^{(m)}$ converge to $\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi})\nabla\boldsymbol{v}^{(m)}$ almost everywhere and strongly in $L^2(I;L^2(\Omega;(Y^N)^d))$. With (62) this implies

$$\nabla\boldsymbol{v}^{(m)}:\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}),\boldsymbol{\phi}^{(n)})\nabla\boldsymbol{u}^{(n)} \\ \to\nabla\boldsymbol{v}^{(m)}:\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi}),\boldsymbol{\phi})\nabla\boldsymbol{u} \text{ in } L^1(I;L^1(\Omega)). \tag{68}$$

Using (63), (65), the first growth assumption in (29), and (58) it holds that

$$\psi_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}) \to \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \text{ a.e. and in } L^2(I; L^2(\Omega; Y^N)). \tag{69}$$

Similarly, by (63) $w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}) \to w_{,\boldsymbol{\phi}}(\boldsymbol{\phi})$ almost everywhere. By (22) $|w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)})|^{p^*} \leq C(w_1)(1 + |\boldsymbol{\phi}^{(n)}|^p)$. With (63) and the theorem of dominated convergence

$$w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)}) \to w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \text{ a.e. and in } L^{p^*}(I \times \Omega; \mathrm{T}\Sigma^M). \tag{70}$$

To go to the limit in the terms involving a and ω strong convergence of $\boldsymbol{\phi}^{(n)} \to \boldsymbol{\phi}$ is necessary.

4.4 Strong convergence of the gradients of the phase fields

The first goal is to construct functions strongly converging to $\boldsymbol{\phi}$ in $H^1(I \times \Omega; \mathrm{H}\Sigma^M)$ and in $L^p(I \times \Omega; \mathrm{H}\Sigma^M)$ which are admissible test functions in (51). After, the strong monotonicity (25) of a is used to obtain the desired result.

Let $\mathcal{P}(\overline{I}; H^1(\Omega; \mathrm{H}\Sigma^M))$ be the set of polynomials $\boldsymbol{q} : [0, \mathcal{T}] \to H^1(\Omega; \mathrm{H}\Sigma^M)$. By standard density results these polynomials are dense in $H^1(I \times \Omega; \mathrm{H}\Sigma^M)$ and in $L^p(I \times \Omega; \mathrm{H}\Sigma^M)$ with p as in assumption (22). Let $\{\boldsymbol{q}_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials in $\mathcal{P}(\overline{I}; H^1(\Omega; \mathrm{H}\Sigma^M))$ with

$$\boldsymbol{q}_n \to \boldsymbol{\phi} \quad \text{in } H^1(I \times \Omega; \mathrm{H}\Sigma^M) \text{ and in } L^p(I \times \Omega; \mathrm{H}\Sigma^M) \quad \text{as } n \to \infty.$$

The union of the Galerkin spaces $X^{(\infty)} := \bigcup_{m\in\mathbb{N}} X^{(m)}$ is dense in $H^1(\Omega; \mathrm{H}\Sigma^M)$ and $L^p(\Omega; \mathrm{H}\Sigma^M)$. By projection of the coefficients of the polynomials $\boldsymbol{q}_n$ to the spaces $X^{(m)}$, for each $n \in \mathbb{N}$ there are polynomials $\{\boldsymbol{q}_n^{(m)}\}_{m\in\mathbb{N}} \subset \mathcal{P}(\overline{I}; X^{(m)})$ with

$$\boldsymbol{q}_n^{(m)} \to \boldsymbol{q}_n \quad \text{in } \mathcal{P}(\overline{I}; H^1(\Omega; \mathrm{H}\Sigma^M)) \text{ and in } \mathcal{P}(\overline{I}; L^p(\Omega; \mathrm{H}\Sigma^M)) \quad \text{as } m \to \infty.$$

Taking an appropriate diagonal sequence $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}} := \{\boldsymbol{q}_n^{(n)}\}_{n\in\mathbb{N}}$ this means that there are functions $\boldsymbol{f}^{(n)} \in C^1(I; X^{(n)})$ with

$$\boldsymbol{f}^{(n)} \to \boldsymbol{\phi} \quad \text{a.e., in } H^1(I \times \Omega; \mathrm{H}\Sigma^M) \text{ and in } L^p(I \times \Omega; \mathrm{H}\Sigma^M) \quad \text{as } n \to \infty \tag{71}$$

and, in addition, thanks to (63), for $q = 2$ and $q = p$

$$\|\boldsymbol{\phi}^{(n)} - \boldsymbol{f}^{(n)}\|_{L^q(I\times\Omega;\mathrm{T}\Sigma^M)} \to 0 \quad \text{as } n \to \infty. \tag{72}$$

Now, let $m = n$ in (51) and take $\boldsymbol{v}^{(n)} = 0$ and $\boldsymbol{\zeta}^{(n)} = (\boldsymbol{\phi}^{(n)} - \boldsymbol{f}^{(n)})$ as test function. The functions $w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)})$ are bounded in $L^{p^*}(I \times \Omega; \mathrm{T}\Sigma^M)$ (cf. the remark in assumption G1). Then by (58) and using the growth assumptions (29), (24), and (26), the convergence in

(72) implies

$$
\begin{aligned}
&\left|\int_I\int_\Omega a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)}):(\nabla\boldsymbol{\phi}^{(n)}-\nabla\boldsymbol{f}^{(n)})\,\mathrm{d}x\mathrm{d}t\right|\\
\leq\;&\left|\int_I\int_\Omega\big(\omega(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\partial_t\boldsymbol{\phi}^{(n)}+a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\big)\cdot\boldsymbol{\zeta}^{(n)}\,\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\int_I\int_\Omega\big(w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)})-\psi_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)})\big)\cdot\boldsymbol{\zeta}^{(n)}\,\mathrm{d}x\mathrm{d}t\right|\\
\leq\;&\omega_1\|\partial_t\boldsymbol{\phi}^{(n)}\|_{L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))}\|\boldsymbol{\zeta}^{(n)}\|_{L^2(I\times\Omega;\mathrm{T}\Sigma^M)}\\
&+a_2\big(\|\boldsymbol{\phi}^{(n)}\|_{L^2(I;L^2(\Omega;\mathrm{H}\Sigma^M))}+\|\nabla\boldsymbol{\phi}^{(n)}\|_{L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d))}\big)\|\boldsymbol{\zeta}^{(n)}\|_{L^2(I\times\Omega;\mathrm{T}\Sigma^M)}\\
&+\|w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)})\|_{L^{p^*}(I\times\Omega;\mathrm{T}\Sigma^M)}\|\boldsymbol{\zeta}^{(n)}\|_{L^p(I\times\Omega;\mathrm{T}\Sigma^M)}\\
&+k_2C\big(1+\|\boldsymbol{u}^{(n)}\|_{L^2(I;L^2(\Omega;Y^N))}\big)\|\boldsymbol{\zeta}^{(n)}\|_{L^2(I\times\Omega;\mathrm{T}\Sigma^M)}\\
\leq\;&C(\|\boldsymbol{\zeta}^{(n)}\|_{L^p(I\times\Omega;\mathrm{T}\Sigma^M)}+\|\boldsymbol{\zeta}^{(n)}\|_{L^2(I\times\Omega;\mathrm{T}\Sigma^M)})\quad\to\quad 0\quad\text{as } n\to\infty.
\end{aligned}
\tag{73}
$$

By (71), (63) for $q=2$, and by assumption (24) $a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{f}^{(n)})\to a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi})$ in $L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d))$. Since in addition $\nabla\boldsymbol{\zeta}^{(n)}\rightharpoonup 0$ in $L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d))$ by (71) and (61) is follows that

$$
\int_I\int_\Omega a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{f}^{(n)}):\nabla\boldsymbol{\zeta}^{(n)}\,\mathrm{d}x\mathrm{d}t\to 0\quad\text{as } n\to\infty.
\tag{74}
$$

The left hand side of (73) can be computed to

$$
\begin{aligned}
&\int_I\int_\Omega a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)}):(\nabla\boldsymbol{\phi}^{(n)}-\nabla\boldsymbol{f}^{(n)})\,\mathrm{d}x\mathrm{d}t\\
=\;&\int_I\int_\Omega\big(a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})-a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{f}^{(n)})\big):(\nabla\boldsymbol{\phi}^{(n)}-\nabla\boldsymbol{f}^{(n)})\,\mathrm{d}x\mathrm{d}t\\
&+\int_I\int_\Omega a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{f}^{(n)}):(\nabla\boldsymbol{\phi}^{(n)}-\nabla\boldsymbol{f}^{(n)})\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Assumption (25) applied on the first term on the right hand side now furnishes together with the convergence results in (73) and (74) that

$$
\int_I\int_\Omega|\nabla\boldsymbol{\phi}^{(n)}-\nabla\boldsymbol{f}^{(n)}|^2\,\mathrm{d}x\mathrm{d}t\to 0\text{ as } n\to\infty
$$

which, in view of (61) and (71), means that

$$
\boldsymbol{\phi}^{(n)}\to\boldsymbol{\phi}\text{ in } L^2(I;H^1(\Omega;\mathrm{H}\Sigma^M))\text{ and }\nabla\boldsymbol{\phi}^{(n)}\to\nabla\boldsymbol{\phi}\text{ almost everywhere.}
\tag{75}
$$

Thanks to the growth and regularity assumptions in G2 this gives

$$
\begin{aligned}
&a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\to a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\text{ in } L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d)),\\
&a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})\to a_{,\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\text{ in } L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M)).
\end{aligned}
\tag{76}
$$

Moreover, for arbitrary test functions $\boldsymbol{\zeta}^{(m)}$, by the assumptions in G3

$$\omega(\boldsymbol{\phi}^{(n)}, \nabla\boldsymbol{\phi}^{(n)})\boldsymbol{\zeta}^{(m)} \to \omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\boldsymbol{\zeta}^{(m)} \text{ a.e. and in } L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M)). \tag{77}$$

Letting $n \to \infty$ in (51), the convergence results (66), (67), (68), (62), (77), (76), (70), and (69) yield that $(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)})$ can be replaced by $(\boldsymbol{u}, \boldsymbol{\phi})$:

$$\begin{aligned}
0 = &- \int_I \int_\Omega \Big[\partial_t \boldsymbol{v}^{(m)} \cdot (\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}))\Big] \,\mathrm{d}x\mathrm{d}t \\
&+ \int_I \int_\Omega \Big[\nabla \boldsymbol{v}^{(m)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi})\nabla \boldsymbol{u}\Big] \,\mathrm{d}x\mathrm{d}t \\
&+ \int_I \int_{\partial\Omega} \Big[\boldsymbol{v}^{(m)} \cdot \boldsymbol{B}(\boldsymbol{u} - \boldsymbol{u}_{bc})\Big] \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
&+ \int_I \int_\Omega \Big[\boldsymbol{\zeta}^{(m)} \cdot \omega(\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\partial_t \boldsymbol{\phi} + \nabla\boldsymbol{\zeta}^{(m)} : a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\Big] \,\mathrm{d}x\mathrm{d}t \\
&+ \int_I \int_\Omega \Big[\boldsymbol{\zeta}^{(m)} \cdot \big(a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) - \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi})\big)\Big] \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{78}$$

Arbitrary test functions $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$ with $\boldsymbol{v}(\mathcal{T}) = 0$ and $\boldsymbol{\zeta} \in H^1(I \times \Omega; \mathrm{T}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{T}\Sigma^M)$ can be approximated by test functions $(\boldsymbol{v}^{(m)}, \boldsymbol{\zeta}^{(m)})$ that are admissible in (78) by a similar procedure as the definition of the $\boldsymbol{f}^{(n)}$. From (78) it then follows that $(\boldsymbol{u}, \boldsymbol{\phi})$ is a solution to (36). To conclude the proof of Theorem 3.1, (35) must be proved.

4.5 Initial values for the phase fields

The embedding $H^1(I \times \Omega; \mathrm{H}\Sigma^M) \hookrightarrow C^0(\overline{I}; L^2(\Omega; \mathrm{H}\Sigma^M))$ is compact. The convergence result (61) implies that

$$\boldsymbol{\phi}^{(n)} \to \boldsymbol{\phi} \quad \text{in } C^0(\overline{I}; L^2(\Omega; \mathrm{H}\Sigma^M)).$$

In particular, at $t = 0$ recalling (48) and (52)

$$\begin{aligned}
\|\boldsymbol{\phi}(0,\cdot) - \boldsymbol{\phi}_{ic}\|_{L^2(\Omega;\mathrm{T}\Sigma^M)} &\leq \|\boldsymbol{\phi}(0,\cdot) - \boldsymbol{\phi}^{(n)}(0,\cdot)\|_{L^2(\Omega;\mathrm{T}\Sigma^M)} + \|\boldsymbol{\phi}^{(n)}(0,\cdot) - \boldsymbol{\phi}_{ic}\|_{L^2(\Omega;\mathrm{T}\Sigma^M)} \\
&\leq \|\boldsymbol{\phi} - \boldsymbol{\phi}^{(n)}\|_{C^0(\overline{I};L^2(\Omega;\mathrm{T}\Sigma^M))} + \|\boldsymbol{\phi}_{ic}^{(n)} - \boldsymbol{\phi}_{ic}\|_{L^2(\Omega;\mathrm{T}\Sigma^M)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This proves assertion (35) and, hence, Theorem 3.1.

4.6 Additional estimates

In addition to proving Theorem 3.1, the convergence results in the previous subsections allow to deduce estimates for the solution $(\boldsymbol{u}, \boldsymbol{\phi})$ which will turn out to be useful in the coming sections, namely, the so-called entropy estimate (79) (since, there, the entropy $\psi - \psi_{,\boldsymbol{u}} \cdot \boldsymbol{u}$ appears, cf. Definition 2.2), and the estimate (80) for time-differences.

Lemma 4.1 *Assume that* $\beta_0 > 0$ *in assumption* (34) *and let* $(\boldsymbol{u},\boldsymbol{\phi})$ *be a weak solution as in Theorem 3.1 that has been constructed with the Galerkin method presented in the previous subsections. Then the following two estimates hold:*

$$\begin{aligned} \mathrm{esssup}_{\tilde t\in I}\int_\Omega \Big[\psi_{,\boldsymbol{u}}(\boldsymbol{u}(\tilde t),\boldsymbol{\phi}(\tilde t))\cdot\boldsymbol{u}(\tilde t)-\psi(\boldsymbol{u}(\tilde t),\boldsymbol{\phi}(\tilde t))+w_2|\boldsymbol{\phi}(\tilde t)|^p+a_0|\nabla\boldsymbol{\phi}(\tilde t)|^2\Big]\,\mathrm{d}x \\ +\int_I\int_\Omega\Big[\omega_0|\partial_t\boldsymbol{\phi}|^2+l_0|\nabla\boldsymbol{u}|^2\Big]\,\mathrm{d}x\mathrm{d}t+\beta_2\int_I\int_{\partial\Omega}|\boldsymbol{u}|^2\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \quad \le \quad C, \end{aligned} \tag{79}$$

$$\int_0^{\mathcal{T}-s}\int_\Omega(\boldsymbol{u}(t+s)-\boldsymbol{u}(t))\cdot\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t+s),\boldsymbol{\phi}(t))-\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t),\boldsymbol{\phi}(t))\big)\,\mathrm{d}x\mathrm{d}t \ \le\ s\,C. \tag{80}$$

Proof: Replacing $\tilde I$ by I in (54) there is already the estimate

$$\begin{aligned} \mathrm{esssup}_{\tilde t\in I}\bigg(\int_\Omega\Big[\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}(\tilde t),\boldsymbol{\phi}^{(n)}(\tilde t))\cdot\boldsymbol{u}^{(n)}(\tilde t)-\psi(\boldsymbol{u}^{(n)}(\tilde t),\boldsymbol{\phi}^{(n)}(\tilde t))\Big]\,\mathrm{d}x \\ +\int_\Omega\Big[w(\boldsymbol{\phi}^{(n)}(\tilde t))+a(\boldsymbol{\phi}^{(n)}(\tilde t),\nabla\boldsymbol{\phi}^{(n)}(\tilde t))\Big]\,\mathrm{d}x\bigg) \\ +\int_I\int_\Omega\Big[\omega(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})|\partial_t\boldsymbol{\phi}^{(n)}|^2+\nabla\boldsymbol{u}^{(n)}:\boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)},\boldsymbol{\phi}^{(n)}),\boldsymbol{\phi}^{(n)})\nabla\boldsymbol{u}^{(n)}\Big]\,\mathrm{d}x dt \\ +\int_I\int_{\partial\Omega}\Big[\boldsymbol{u}^{(n)}\cdot\boldsymbol{B}\big(\boldsymbol{u}^{(n)}-\boldsymbol{u}_{bc}\big)\Big]\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \quad\le\quad C. \end{aligned} \tag{81}$$

By (65) and (66)

$$\int_\Omega\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}(\tilde t),\boldsymbol{\phi}^{(n)}(\tilde t))\cdot\boldsymbol{u}^{(n)}(\tilde t)\,\mathrm{d}x\to\int_\Omega\psi_{,\boldsymbol{u}}(\boldsymbol{u}(\tilde t),\boldsymbol{\phi}(\tilde t))\cdot\boldsymbol{u}(\tilde t)\,\mathrm{d}x \tag{82}$$

for almost every $\tilde t\in I$. By the last growth assumption in (27), the convergence results (63) and (65) imply for almost every $\tilde t\in I$ that

$$\int_\Omega\psi(\boldsymbol{u}^{(n)}(\tilde t),\boldsymbol{\phi}^{(n)}(\tilde t))\,\mathrm{d}x\to\int_\Omega\psi(\boldsymbol{u}(\tilde t),\boldsymbol{\phi}(\tilde t))\,\mathrm{d}x. \tag{83}$$

By (63) for $q=p$ and assumption (22) it holds for almost every $\tilde t\in I$ that

$$\liminf_{n\to\infty}\int_\Omega w(\boldsymbol{\phi}^{(n)}(\tilde t))\,\mathrm{d}x\ge\int_\Omega w_2|\boldsymbol{\phi}(\tilde t)|^p\,\mathrm{d}x-C. \tag{84}$$

Similarly, by (75) and assumption (23)

$$\liminf_{n\to\infty}\int_\Omega a(\boldsymbol{\phi}^{(n)}(\tilde t),\nabla\boldsymbol{\phi}^{(n)}(\tilde t))\,\mathrm{d}x\ge\int_I\int_\Omega a_0|\nabla\boldsymbol{\phi}(\tilde t)|^2\,\mathrm{d}x, \tag{85}$$

the convergence result (61) with assumption (26) gives

$$\liminf_{n\to\infty}\int_I\int_\Omega\omega(\boldsymbol{\phi}^{(n)},\nabla\boldsymbol{\phi}^{(n)})|\partial_t\boldsymbol{\phi}^{(n)}|^2\,\mathrm{d}x\mathrm{d}t\ge\int_I\int_\Omega\omega_0|\partial_t\boldsymbol{\phi}|^2\,\mathrm{d}x\mathrm{d}t, \tag{86}$$

and (62) with assumption (32) yields

$$\liminf_{n\to\infty} \int_I \int_\Omega \nabla \boldsymbol{u}^{(n)} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}^{(n)}, \boldsymbol{\phi}^{(n)}), \boldsymbol{\phi}^{(n)}) \nabla \boldsymbol{u}^{(n)} \,\mathrm{d}x\mathrm{d}t \geq \int_I \int_\Omega l_0 |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}t. \tag{87}$$

Finally, recalling (56) for $\tilde{I} = I$, by (62) and for δ small enough (such that $\beta_2 := \beta_0 - \delta\beta_1 > 0$, remember that $\beta_0 > 0$ is assumed for this subsection)

$$\liminf_{n\to\infty} \int_I \int_{\partial\Omega} \boldsymbol{u}^{(n)} \cdot \boldsymbol{B}(\boldsymbol{u}^{(n)} - \boldsymbol{u}_{bc}) \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \geq \beta_2 \int_I \int_{\partial\Omega} |\boldsymbol{u}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t - C. \tag{88}$$

Due to (82)–(88), in the limit as $n \to \infty$ the estimate (81) yields the estimate (79).

Define now at times $0 < t_1 < t_2 < \mathcal{T} - \delta$ and small $\delta > 0$

$$\chi_\delta(t) := \begin{cases} 0, & t \notin [t_1, t_2 + \delta], \\ \frac{1}{\delta}(t - t_1), & t \in [t_1, t_1 + \delta], \\ 1, & t \in (t_1 + \delta, t_2), \\ -\frac{1}{\delta}(t - (t_2 + \delta)), & t \in [t_2, t_2 + \delta]. \end{cases}$$

Since $\boldsymbol{u} \in L^2(H^{1,2}(\Omega; Y^N))$ and

$$\chi'_\delta(t) = \begin{cases} \frac{1}{\delta}, & t \in (t_1, t_1 + \delta), \\ -\frac{1}{\delta}, & t \in (t_2, t_2 + \delta), \\ 0, & t \in (-\infty, t_1) \cup (t_1 + \delta, t_2) \cup (t_2 + \delta, \infty), \end{cases}$$

it is clear that $\boldsymbol{v}(t, x) = \chi_\delta(t)(\boldsymbol{u}(t_2, x) - \boldsymbol{u}(t_1, x)) \in H^{1,2}(I \times \Omega; Y^N)$ for almost every t_1, t_2. The properties of the convolution (the functions $\zeta_\delta(t) = \frac{1}{\delta}\chi_{(\tilde{t},\tilde{t}+\delta)}(t)$ where $\chi_{(\tilde{t},\tilde{t}+\delta)}$ is the characteristic function of the interval $(\tilde{t}, \tilde{t} + \delta)$ constitute a Dirac sequence) and the fact that $\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \in L^2(I; L^2(\Omega; Y^N))$ by (66) give

$$⨍_{\tilde{t}}^{\tilde{t}+\delta} \int_\Omega \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t), \boldsymbol{\phi}(t)) \,\mathrm{d}x\mathrm{d}t \to \int_\Omega \psi_{,\boldsymbol{u}}(\boldsymbol{u}(\tilde{t}), \boldsymbol{\phi}(\tilde{t})) \,\mathrm{d}x$$

for almost every $\tilde{t} \in I$. Inserting $\boldsymbol{v}$ and $\boldsymbol{\zeta} = 0$ in (36) yields for almost every t_1, t_2 in the limit as $\delta \searrow 0$ (the dependence on x is dropped and $\boldsymbol{L}(t) := \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t), \boldsymbol{\phi}(t)), \boldsymbol{\phi}(t))$ was set for shorter presentation)

$$\begin{aligned} 0 = \;& \int_{t_1}^{t_1+\delta} \int_\Omega -\tfrac{1}{\delta}(\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t), \boldsymbol{\phi}(t)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big) \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_2}^{t_2+\delta} \int_\Omega \tfrac{1}{\delta}(\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t), \boldsymbol{\phi}(t)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big) \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2+\delta} \int_\Omega \chi_\delta(t) \nabla(\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \boldsymbol{L}(t) \nabla \boldsymbol{u}(t) \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2+\delta} \int_{\partial\Omega} \chi_\delta(t)(\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \boldsymbol{B}(t)(\boldsymbol{u}(t) - \boldsymbol{u}_{bc}(t)) \,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
\to \quad & \int_\Omega (\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_2), \boldsymbol{\phi}(t_2)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1), \boldsymbol{\phi}(t_1))\big)\,\mathrm{d}x \\
& + \int_{t_1}^{t_2} \int_\Omega \nabla(\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \boldsymbol{L}(t)\nabla\boldsymbol{u}(t)\,\mathrm{d}x\mathrm{d}t \\
& + \int_{t_1}^{t_2} \int_{\partial\Omega} (\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)) \cdot \boldsymbol{B}(t)(\boldsymbol{u}(t) - \boldsymbol{u}_{bc}(t))\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t.
\end{aligned}
$$

For a small $s > 0$ such that $\mathcal{T} - s > 0$ let $t_2 = t_1 + s$ and integrate the above identity with respect to t_1 from $t_1 = 0$ to $t_1 = \mathcal{T} - s$. By the convolution estimates

$$
\int_0^{\mathcal{T}-s} \fint_{t_1}^{t_1+s} \|\boldsymbol{L}(t)\nabla\boldsymbol{u}(t)\|_{L^2(\Omega)}\,\mathrm{d}t\mathrm{d}t_1 \le \int_I \|\boldsymbol{L}(t_1)\nabla\boldsymbol{u}(t_1)\|_{L^2(\Omega)}\,\mathrm{d}t_1,
$$

$$
\int_0^{\mathcal{T}-s} \fint_{t_1}^{t_1+s} \|\boldsymbol{B}(t)(\boldsymbol{u}(t) - \boldsymbol{u}_{bc}(t))\|_{L^2(\partial\Omega)}\,\mathrm{d}t\mathrm{d}t_1 \le \int_I \|\boldsymbol{B}(t_1)(\boldsymbol{u}(t_1) - \boldsymbol{u}_{bc}(t_1))\|_{L^2(\partial\Omega)}\,\mathrm{d}t_1
$$

and by (79) it follows that

$$
\begin{aligned}
0 \le \quad & \Big| \int_0^{\mathcal{T}-s}\int_\Omega (\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)) \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s), \boldsymbol{\phi}(t_1+s)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1), \boldsymbol{\phi}(t_1))\big)\,\mathrm{d}x\mathrm{d}t_1 \Big| \\
\le \quad & s \int_0^{\mathcal{T}-s} \Big(\|\nabla\boldsymbol{u}(t_1+s)\|_{L^2(\Omega)} + \|\nabla\boldsymbol{u}(t_1)\|_{L^2(\Omega)}\Big) \fint_{t_1}^{t_1+s} \|\boldsymbol{L}(t)\nabla\boldsymbol{u}(t)\|_{L^2(\Omega)}\,\mathrm{d}t\mathrm{d}t_1 \\
& + s \int_0^{\mathcal{T}-s} \big(\|\boldsymbol{u}(t_1+s)\|_{L^2(\partial\Omega)} + \|\boldsymbol{u}(t_1)\|_{L^2(\partial\Omega)}\big) \fint_{t_1}^{t_1+s} \|\boldsymbol{B}(t)(\boldsymbol{u}(t) - \boldsymbol{u}_{bc}(t))\|_{L^2(\partial\Omega)}\,\mathrm{d}t\mathrm{d}t_1 \\
\le \quad & s \int_I \Big(2L_0\|\nabla\boldsymbol{u}(t_1)\|^2_{L^2(\Omega)} + 2\beta_1\|\boldsymbol{u}(t_1)\|_{L^2(\partial\Omega)}\|\boldsymbol{u}(t_1) - \boldsymbol{u}_{bc}(t_1)\|_{L^2(\partial\Omega)}\Big)\mathrm{d}t_1 \qquad (89)
\end{aligned}
$$

where the last inequality holds thanks to (32) and the assumption on $\boldsymbol{u}_{bc}$ in Q4. Using (29), the first term on the right hand side can be estimated by

$$
\begin{aligned}
& \big|(\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)) \cdot (\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s), \boldsymbol{\phi}(t_1+s)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s), \boldsymbol{\phi}(t_1)))\big| \\
= \quad & \Big|(\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)) \cdot \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s), \theta\boldsymbol{\phi}(t_1+s) + (1-\theta)\boldsymbol{\phi}(t_1))d\theta\Big| \\
= \quad & \Big|(\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)) \\
& \qquad \cdot \int_0^1 \psi_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u}(t_1+s), \theta\boldsymbol{\phi}(t_1+s) + (1-\theta)\boldsymbol{\phi}(t_1))d\theta \cdot (\boldsymbol{\phi}(t_1+s) - \boldsymbol{\phi}(t_1))\Big| \\
\le \quad & \tfrac{1}{2}k_3|\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)||\boldsymbol{\phi}(t_1+s) - \boldsymbol{\phi}(t_1)|.
\end{aligned}
$$

Assumption (28) implies that $\psi_{,\boldsymbol{u}}$ is monotone in $\boldsymbol{u}$ uniformly in $\boldsymbol{\phi}$, hence from (89) and the above computations the estimate (80) is obtained:

$$
0 \le \int_0^{\mathcal{T}-s} \int_\Omega (\boldsymbol{u}(t_1+s) - \boldsymbol{u}(t_1)) \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s), \boldsymbol{\phi}(t_1)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1), \boldsymbol{\phi}(t_1))\big)\,\mathrm{d}x\mathrm{d}t_1
$$

$$
\begin{aligned}
\leq\quad & \Big| \int_0^{\mathcal{T}-s} \int_\Omega (\boldsymbol{u}(t_1+s)-\boldsymbol{u}(t_1)) \\
&\qquad \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s),\boldsymbol{\phi}(t_1+s)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1),\boldsymbol{\phi}(t_1))\big)\,\mathrm{d}x\mathrm{d}t_1 \Big| \\
&+ \Big| \int_0^{\mathcal{T}-s} \int_\Omega (\boldsymbol{u}(t_1+s)-\boldsymbol{u}(t_1)) \\
&\qquad \cdot \big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s),\boldsymbol{\phi}(t_1+s)) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}(t_1+s),\boldsymbol{\phi}(t_1))\big)\,\mathrm{d}x\mathrm{d}t_1 \Big| \\
\leq\quad & s\Big(2L_0\|\nabla\boldsymbol{u}\|^2_{L^2(I;L^2(\Omega;(Y^N)^d))} + 2\beta_1\|\boldsymbol{u}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\|\boldsymbol{u}-\boldsymbol{u}_{bc}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\Big) \\
&+ s\Big(k_3\|\boldsymbol{u}\|_{L^2(I;L^2(\Omega;Y^N))}\|\partial_t\boldsymbol{\phi}\|_{L^2(I;L^2(\Omega;Y^N))}\Big) \\
\leq\quad & s\,C\big(\|\boldsymbol{u}\|_{L^2(I;H^{1,2}(\Omega;Y^N))}, \|\boldsymbol{u}\|_{L^2(I;L^2(\partial\Omega;Y^N))}, \|\partial_t\boldsymbol{\phi}\|_{L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))}\big).
\end{aligned}
$$

For the second last inequality it was used that

$$
\int_\Omega \int_0^{\mathcal{T}-s} \left| \frac{\boldsymbol{\phi}(t+s)-\boldsymbol{\phi}(t)}{s} \right|^2 \mathrm{d}t\mathrm{d}x \leq C\,\|\partial_t\boldsymbol{\phi}\|^2_{L^2(I\times\Omega;\mathrm{T}\Sigma^M)}. \tag{90}
$$

5 Proof of Theorem 3.2

5.1 Solution to the perturbed problem

By the assumptions on the functions g, h, and the $\lambda^{(\alpha)}$ the perturbed potential $\psi^{(\nu)}$ is of the class $C^{2,1}$ (observe that $W^{3,\infty}(\mathbb{R}) \hookrightarrow C^{2,1}(\mathbb{R})$). The assumptions in L1 furthermore imply that the $\psi^{(\nu)}$ also fulfill the growth assumption stated in Q1. In particular, it holds that

$$
|\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u},\boldsymbol{\phi})\cdot\boldsymbol{\zeta}| \leq \hat{k}_2(1+|\boldsymbol{u}|)Mk_7|\boldsymbol{\zeta}|, \tag{91}
$$

$$
|\boldsymbol{v}\cdot\psi^{(\nu)}_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u},\boldsymbol{\phi})\boldsymbol{\zeta}| \leq |\boldsymbol{v}|\hat{k}_3Mk_7|\boldsymbol{\zeta}|, \tag{92}
$$

$$
|\psi^{(\nu)}(0,\boldsymbol{\phi})| \leq M\hat{k}_4, \tag{93}
$$

$$
|\psi^{(\nu)}(\boldsymbol{u},\boldsymbol{\phi})| \leq g_0(1+u_0^2) + \nu|\tilde{\boldsymbol{u}}|^2 + M\hat{k}_2C(1+|\boldsymbol{u}|^2). \tag{94}
$$

With L3–L2, Theorem 3.1 furnishes functions

$$
\boldsymbol{u}^{(\nu)} \in L^2(I;H^1(\Omega;Y^N)), \quad \boldsymbol{\phi}^{(\nu)}H^1(I\times\Omega;\mathrm{H}\Sigma^M)
$$

such that

$$
\boldsymbol{\phi}^{(\nu)}(t,\cdot) \to \boldsymbol{\phi}_{ic} \quad \text{in } L^2(\Omega;\mathrm{H}\Sigma^M) \text{ as } t \searrow 0
$$

and such that

$$
\begin{aligned}
0 = \quad & \int_I\int_\Omega \Big[-\partial_t\boldsymbol{v}\cdot(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic},\boldsymbol{\phi}_{ic}))\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_\Omega \Big[\nabla\boldsymbol{v} : \boldsymbol{L}(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)}),\boldsymbol{\phi}^{(\nu)})\nabla\boldsymbol{u}^{(\nu)}\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_I\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{B}(\boldsymbol{u}^{(\nu)}-\boldsymbol{u}_{bc})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t
\end{aligned}
$$

$$+\int_I\int_\Omega \Big[\omega(\boldsymbol{\phi}^{(\nu)},\nabla\boldsymbol{\phi}^{(\nu)})\partial_t\boldsymbol{\phi}^{(\nu)}\cdot\boldsymbol{\zeta}+a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\nu)},\nabla\boldsymbol{\phi}^{(\nu)}):\nabla\boldsymbol{\zeta}\Big]\,\mathrm{d}x\mathrm{d}t$$
$$+\int_I\int_\Omega \Big[a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\nu)},\nabla\boldsymbol{\phi}^{(\nu)})\cdot\boldsymbol{\zeta}+w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\nu)})\cdot\boldsymbol{\zeta}-\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\cdot\boldsymbol{\zeta}\Big]\,\mathrm{d}x\mathrm{d}t \qquad (95)$$

for all $\boldsymbol{v}\in H^1(I\times\Omega;Y^N)$ with $\boldsymbol{v}(\mathcal{T})=0$ and $\boldsymbol{\zeta}\in H^1(I\times\Omega;\mathrm{T}\Sigma^M)\cap L^p(I\times\Omega;\mathrm{T}\Sigma^M)$. In addition, the following estimates resulting from (79) and (80) are fulfilled (recall that $\beta_0>0$ in consistence with the additional assumption in Lemma 4.1):

$$\begin{aligned}\operatorname{esssup}_{\tilde t\in I}\int_\Omega \Big[&\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(\tilde t),\boldsymbol{\phi}^{(\nu)}(\tilde t))\cdot\boldsymbol{u}^{(\nu)}(\tilde t)-\psi^{(\nu)}(\boldsymbol{u}^{(\nu)}(\tilde t),\boldsymbol{\phi}^{(\nu)}(\tilde t))\\ &+w_2|\boldsymbol{\phi}^{(\nu)}(\tilde t)|^p+a_0|\nabla\boldsymbol{\phi}^{(\nu)}(\tilde t)|^2\Big]\,\mathrm{d}x\\ +\int_I\int_\Omega\Big[\omega_0|\partial_t\boldsymbol{\phi}^{(\nu)}|^2+l_0|\nabla\boldsymbol{u}^{(\nu)}|^2\Big]\,\mathrm{d}x\mathrm{d}t&+\beta_2\int_I\int_{\partial\Omega}|\boldsymbol{u}^{(\nu)}|^2\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \quad\le\quad C, \qquad (96)\end{aligned}$$

$$\begin{aligned}\int_0^{\mathcal{T}-s}\int_\Omega&(\boldsymbol{u}^{(\nu)}(t+s)-\boldsymbol{u}^{(\nu)}(t))\\ &\cdot\big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t))-\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\big)\,\mathrm{d}x\mathrm{d}t \quad\le\quad C\,s. \qquad (97)\end{aligned}$$

5.2 Properties of the Legendre transform

For shorter presentation define the function

$$b^{(\nu)}:Y^N\times\mathrm{H}\Sigma^M\to\mathbb{R},\quad b^{(\nu)}(\boldsymbol{u},\boldsymbol{\phi}):=\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi})\cdot\boldsymbol{u}-\psi^{(\nu)}(\boldsymbol{u},\boldsymbol{\phi})$$

for every $\nu\in[0,\overline{\nu}]$. The following two lemmas were proved in [2] for functions $\psi^{(\nu)}$ not depending on additional parameters $\boldsymbol{\phi}\in\mathrm{H}\Sigma^M$.

Lemma 5.1 *For every $\tilde\delta>0$ there is a constant $C_{\tilde\delta}>0$ independent of ν such that*

$$|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})|\le\tilde\delta b^{(\nu)}(\boldsymbol{z},\boldsymbol{\xi})+C_{\tilde\delta}$$

for all $(\boldsymbol{z},\boldsymbol{\xi})\in Y^N\times\mathrm{H}\Sigma^M$.

Proof: For arbitrary points $\boldsymbol{z},\tilde{\boldsymbol{z}}\in Y^N$ and $\boldsymbol{\xi}\in\mathrm{H}\Sigma^M$ the convexity of $\psi^{(\nu)}$ implies

$$b^{(\nu)}(\boldsymbol{z},\boldsymbol{\xi})-b^{(\nu)}(\tilde{\boldsymbol{z}},\boldsymbol{\xi})\ge(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})-\psi^{(\nu)}_{,\boldsymbol{u}}(\tilde{\boldsymbol{z}},\boldsymbol{\xi}))\cdot\tilde{\boldsymbol{z}}$$

Let $e=\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})/|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})|\in Y^N$. Then

$$\begin{aligned}|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})| &= \tilde\delta\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})\cdot\frac{e}{\tilde\delta}\\ &= \tilde\delta\psi^{(\nu)}_{,\boldsymbol{u}}(\frac{e}{\tilde\delta},\boldsymbol{\xi})\cdot\frac{e}{\tilde\delta}+\tilde\delta\big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z},\boldsymbol{\xi})-\psi^{(\nu)}_{,\boldsymbol{u}}(\frac{e}{\tilde\delta},\boldsymbol{\xi})\big)\cdot\frac{e}{\tilde\delta}\\ &\le \tilde\delta\psi^{(\nu)}_{,\boldsymbol{u}}(\frac{e}{\tilde\delta},\boldsymbol{\xi})\cdot\frac{e}{\tilde\delta}+\tilde\delta\big(b^{(\nu)}(\boldsymbol{z},\boldsymbol{\xi})-b^{(\nu)}(\frac{e}{\tilde\delta},\boldsymbol{\xi})\big)\\ &\le \tilde\delta b^{(\nu)}(\boldsymbol{z},\boldsymbol{\xi})+\tilde\delta\max_{|\tilde{\boldsymbol{z}}|=\frac{1}{\tilde\delta}}\psi^{(\nu)}(\tilde{\boldsymbol{z}},\boldsymbol{\xi}).\end{aligned}$$

In view of (94), the assertion holds with $C_{\tilde\delta} := C\max\{g_0, M\hat k_2, \overline{\nu}\}(1+\frac{1}{\tilde\delta^2})$. $\square$

Lemma 5.2 *For all $\Xi > 0$ there is a function $\omega_\Xi : [0,\infty) \to [0,\infty)$ continuous in zero with $\omega_\Xi(0) = 0$ so that for all $\nu \in [0,\overline{\nu}]$ and all functions $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{\xi} \in H^1(\Omega)$ with $\|\boldsymbol{z}_1\|_{H^1}, \|\boldsymbol{z}_2\|_{H^1}, \|\boldsymbol{\xi}\|_{H^1} \le \Xi$, $\|b^{(\nu)}(\boldsymbol{z}_i,\boldsymbol{\xi})\|_{L^1(\Omega)} \le \Xi$, $i = 1, 2$, and*

$$\int_\Omega \big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_1,\boldsymbol{\xi}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_2,\boldsymbol{\xi})\big)\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)\,\mathrm{d}x \le \delta$$

it holds that

$$\int_\Omega \big|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_1,\boldsymbol{\xi}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_2,\boldsymbol{\xi})\big|\,\mathrm{d}x \le \omega_\Xi(\delta).$$

Proof: Suppose the contrary, i.e., there are $\Xi, \varepsilon > 0$ such that for all $\delta > 0$ there are functions $\boldsymbol{z}_i^{(\delta)} \in H^1(\Omega; Y^N)$, $\boldsymbol{\xi}^{(\delta)} \in H^1(\Omega; \mathrm{H}\Sigma^M)$ and values $\nu_\delta \in [0,\overline{\nu}]$ such that

$$\|\boldsymbol{z}_i^{(\delta)}\|_{H^1} \le \Xi, \quad \|\boldsymbol{\xi}^{(\delta)}\|_{H^1} \le \Xi, \quad \|b^{(\nu_\delta)}(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)})\|_{L^1(\Omega)} \le \Xi, \quad i = 1,2,$$
$$\int_\Omega \big(\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_1^{(\delta)},\boldsymbol{\xi}^{(\delta)}) - \psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_2^{(\delta)},\boldsymbol{\xi}^{(\delta)})\big)\cdot(\boldsymbol{z}_1^{(\delta)}-\boldsymbol{z}_2^{(\delta)})\,\mathrm{d}x \le \delta$$

but

$$\int_\Omega \big|\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_1^{(\delta)},\boldsymbol{\xi}^{(\delta)}) - \psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_2^{(\delta)},\boldsymbol{\xi}^{(\delta)})\big|\,\mathrm{d}x > \varepsilon.$$

There are functions $\boldsymbol{z}_i, \boldsymbol{\xi} \in H^1(\Omega)$ and there is $\nu \in [0,\overline{\nu}]$ such that, for a subsequence as $\delta \to 0$ (still indexed with δ), it holds that $\nu_\delta \to \nu$, $\boldsymbol{z}_i^{(\delta)} \rightharpoonup \boldsymbol{z}_i$ in H^1, and $\boldsymbol{\xi}^{(\delta)} \rightharpoonup \boldsymbol{\xi}$ in H^1. After eventually restricting again on a subsequence, it follows that $(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)}) \to (\boldsymbol{z}_i,\boldsymbol{\xi})$ in L^2 and almost everywhere. Hence $\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)}) \to \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_i,\boldsymbol{\xi})$ almost everywhere. By the preceding lemma

$$\begin{aligned}\int_E |\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)})|\,\mathrm{d}x &\le \tilde\delta\int_E b^{(\nu_\delta)}(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)})\,\mathrm{d}x + \int_E C_{\tilde\delta}\,\mathrm{d}x \\ &\le \tilde\delta\Xi + C_{\tilde\delta}\mathcal{L}^d(E)\end{aligned}$$

for every $\tilde\delta > 0$ and every Borel set $E \subset \Omega$. Choosing first $\tilde\delta$ small and then E such that $\mathcal{L}^d(E)$ becomes sufficiently small the Vitali convergence theorem yields $\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_i^{(\delta)},\boldsymbol{\xi}^{(\delta)}) \to \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_i,\boldsymbol{\xi})$ in $L^1(\Omega)$ whence

$$\int_\Omega \big|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_1,\boldsymbol{\xi}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_2,\boldsymbol{\xi})\big|\,\mathrm{d}x \ge \varepsilon. \tag{98}$$

Using the Fatou lemma and the monotonicity of $\psi^{(\nu_\delta)}_{,\boldsymbol{u}}$ in $\boldsymbol{u}$ one first obtains

$$\begin{aligned}0 &= \liminf_{\delta\to 0}\int_\Omega \big(\psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_1^{(\delta)},\boldsymbol{\xi}^{(\delta)}) - \psi^{(\nu_\delta)}_{,\boldsymbol{u}}(\boldsymbol{z}_2^{(\delta)},\boldsymbol{\xi}^{(\delta)})\big)\cdot(\boldsymbol{z}_1^{(\delta)}-\boldsymbol{z}_2^{(\delta)})\,\mathrm{d}x \\ &\ge \int_\Omega (\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_1,\boldsymbol{\xi}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_2,\boldsymbol{\xi}))\cdot(\boldsymbol{z}_1-\boldsymbol{z}_2)\,\mathrm{d}x\end{aligned}$$

and from this $\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_1,\boldsymbol{\xi}) = \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{z}_2,\boldsymbol{\xi})$ almost everywhere in contradiction to (98). $\square$

The following lemma concerns the dependence of the $\psi^{(\nu)}$ on $\boldsymbol{\phi}$.

Lemma 5.3 *Consider series* $\{\boldsymbol{u}^{(m)}\}_{m\in\mathbb{N}} \subset Y^N$, $\{\boldsymbol{\phi}^{(m)}\}_{m\in\mathbb{N}} \subset \mathrm{H}\Sigma^M$, *and* $\{\nu_m\}_{m\in\mathbb{N}} \subset [0,\overline{\nu}]$ *such that* $\boldsymbol{\phi}^{(m)} \to \boldsymbol{\phi}$ *in* $\mathrm{H}\Sigma^M$, $\nu_m \searrow 0$, *and there is* $\boldsymbol{u} \in Y^N$ *such that*

$$\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$$

as $m \to \infty$. *Then*

$$\psi^{(\nu_m)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}) \to \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{as } m \to \infty.$$

Proof: By (19) it must be shown that

$$(\psi^{(\nu_m)})^*_{,\boldsymbol{\phi}}(\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}), \boldsymbol{\phi}^{(m)}) \to \psi^*_{,\boldsymbol{\phi}}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \quad \text{as } m \to \infty. \tag{99}$$

The regularity assumptions on ψ in L1 provide that, for a given $\boldsymbol{\phi} \in \mathrm{H}\Sigma^M$, the function $\psi_{,\boldsymbol{u}}(\cdot, \boldsymbol{\phi})$ is a C^1-diffeomorphism mapping an open set $U_{\boldsymbol{\phi}} \subset Y^N$ to an open set $C_{\boldsymbol{\phi}} \subset \mathbb{R} \times \mathrm{H}\Sigma^N$. These sets may be proper subsets in contrast to the situation for $\psi^{(\nu_m)}_{,\boldsymbol{u}}(\cdot, \boldsymbol{\phi})$ which, thanks to the quadratic growth in u, is defined on the total space Y^N and maps onto the total space $\mathbb{R} \times \mathrm{H}\Sigma^N$.

Let $q^{(\nu_m)}(\boldsymbol{u}) := \nu_m|\tilde{\boldsymbol{u}}|^2$. The special structure of $\psi^{(\nu_m)}(\boldsymbol{u}, \boldsymbol{\phi}) = q^{(\nu_m)}(\boldsymbol{u}) + \psi(\boldsymbol{u}, \boldsymbol{\phi})$ yields for all $\boldsymbol{c} \in C_{\boldsymbol{\phi}}$ that

$$(\psi^{(\nu_m)})^*(\boldsymbol{c}, \boldsymbol{\phi}) = q^{(\nu_m)}(\boldsymbol{c}) + \psi^*(\boldsymbol{c}, \boldsymbol{\phi}).$$

Furthermore, the regularity of ψ in $\boldsymbol{\phi}$ implies that if $\boldsymbol{c} \in C_{\boldsymbol{\phi}}$ then also $\boldsymbol{c} \in C_{\tilde{\boldsymbol{\phi}}}$ for all $\tilde{\boldsymbol{\phi}}$ in a small ball around $\boldsymbol{\phi}$. Hence, fixing $\boldsymbol{c}$, variations with respect to $\boldsymbol{\phi}$ are possible and give

$$(\psi^{(\nu_m)})^*_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}) = \psi^*_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}).$$

Let now $\boldsymbol{c} := \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$. Since $\boldsymbol{\phi}^{(m)} \to \boldsymbol{\phi}$ there are a small $\varepsilon > 0$ and $m_1 \in \mathbb{N}$ such that

$$B_\varepsilon(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})) \subset C_{\boldsymbol{\phi}^{(m)}} \subset \mathbb{R} \times \mathrm{H}\Sigma^N \quad \text{for all } m \geq m_1$$

where $B_\varepsilon(\boldsymbol{v})$ stands the ball of radius ε centered in $\boldsymbol{v} \in Y^N$. Therefore $(\psi^{(\nu_m)})^*_{,\boldsymbol{\phi}}(\boldsymbol{c}, \boldsymbol{\phi}^{(m)}) = \psi^*_{,\boldsymbol{\phi}}(c, \boldsymbol{\phi}^{(m)})$ for all $\boldsymbol{c} \in B_\varepsilon(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}))$ as long as $m \geq m_1$.

Since $\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \in C_{\boldsymbol{\phi}}$ there is some $m_2 \in \mathbb{N}$, $m_2 \geq m_1$, with

$$\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}) \in B_\varepsilon(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})) \quad \text{for all } m \geq m_2$$

whence

$$(\psi^{(\nu_m)})^*_{,\boldsymbol{\phi}}(\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}), \boldsymbol{\phi}^{(m)}) = \psi^*_{,\boldsymbol{\phi}}(\psi^{(\nu_m)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(m)}, \boldsymbol{\phi}^{(m)}), \boldsymbol{\phi}^{(m)}) \quad \text{for all } m \geq m_2.$$

Since $\psi^*_{,\boldsymbol{\phi}}$ is continuous this gives the desired result (99). $\square$

5.3 Compactness of the conserved quantities

As a first step to show precompactness of the set $\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\}_{\nu\in[0,\overline{\nu}]}$ in $L^1(I\times\Omega;Y^N)$ the convergence result (101) involving time differences $\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))$ will be proved. Define the set

$$E^{(\nu)}_{s,\Xi} := \Big\{t\in[0,\mathcal{T}-s] : \tilde{e}^{(\nu)}_{s,\Xi}(t)\le\Xi\Big\} \tag{100}$$

where

$$\begin{aligned}\tilde{e}^{(\nu)}_{s,\Xi}(t) \;:=\;& \|\boldsymbol{u}^{(\nu)}(t)\|_{H^1(\Omega;Y^N)} + \|\boldsymbol{u}^{(\nu)}(t+s)\|_{H^1(\Omega;Y^N)} + \|\boldsymbol{\phi}^{(\nu)}(t)\|_{H^1(\Omega;\mathrm{H}\Sigma^M)}\\ &+\frac{1}{s}\int_\Omega(\boldsymbol{u}^{(\nu)}(t+s)-\boldsymbol{u}^{(\nu)}(t))\\ &\qquad\cdot\big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t))-\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\big)\,\mathrm{d}x\\ &+\Big\|\frac{\boldsymbol{\phi}^{(\nu)}(t+s)-\boldsymbol{\phi}^{(\nu)}(t)}{s}\Big\|_{L^2(\Omega;\mathrm{T}\Sigma^M)}\\ &+\|b^{(\nu)}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s))\|_{L^1(\Omega)}+\|b^{(\nu)}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\|_{L^1(\Omega)}.\end{aligned}$$

By (96) and (97) and using (90) there is a constant $C>0$ such that

$$C\ge\int_0^{\mathcal{T}-s}\tilde{e}^{(\nu)}_{s,\Xi}(t)\,\mathrm{d}t=\int_{E^{(\nu)}_{s,\Xi}}\tilde{e}^{(\nu)}_{s,\Xi}(t)\,\mathrm{d}t+\int_{[0,\mathcal{T}]\setminus E^{(\nu)}_{s,\Xi}}\tilde{e}^{(\nu)}_{s,\Xi}(t)\,\mathrm{d}t\ge\Xi\,\mathcal{L}^1(E^{(\nu)}_{s,\Xi})$$

whence $\mathcal{L}^1(E^{(\nu)}_{s,\Xi})$ becomes arbitrarily small when choosing Ξ sufficiently large. Applying Lemma 5.2 of the previous subsection with $\delta=s\Xi$ gives

$$\int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega\big|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t))-\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\big|\,\mathrm{d}x\mathrm{d}t\le\mathcal{T}\omega_\Xi(s\Xi).$$

Thanks to (92)

$$\begin{aligned}&\int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega\big|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s))-\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t))\big|\,\mathrm{d}x\mathrm{d}t\\ =&\int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega\Big|\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\underbrace{\theta\boldsymbol{\phi}^{(\nu)}(t+s)+(1-\theta)\boldsymbol{\phi}^{(\nu)}(t)}_{=:\phi_\theta})\,\mathrm{d}\theta\Big|\,\mathrm{d}x\mathrm{d}t\\ =&\int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega\Big|\int_0^1\psi^{(\nu)}_{,\boldsymbol{u}\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)}(t+s),\phi_\theta)\,\mathrm{d}\theta\cdot\big(\boldsymbol{\phi}^{(\nu)}(t+s)-\boldsymbol{\phi}^{(\nu)}(t)\big)\Big|\,\mathrm{d}x\mathrm{d}t\\ \le\;& s\int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega\hat{k}_3Mk_7\Big|\frac{\boldsymbol{\phi}^{(\nu)}(t+s)-\boldsymbol{\phi}^{(\nu)}(t)}{s}\Big|\,\mathrm{d}x\mathrm{d}t\quad\le\quad s\,C\,\Xi.\end{aligned}$$

For the last inequality it was used that, on bounded domains, the L^1 norm can be estimated by the L^2 norm, and estimate (90) was applied. Altogether, using (96) and Lemma

5.1 with $\tilde{\delta} = 1$:

$$\begin{aligned}
&\int_0^{\mathcal{T}-s}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\right| \mathrm{d}x\mathrm{d}t \\
= \;& \int_{E^{(\nu)}_{s,\Xi}}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\right| \mathrm{d}x\mathrm{d}t \\
& + \int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\right| \mathrm{d}x\mathrm{d}t \\
\leq \;& 2\mathrm{esssup}_{t\in I}\int_\Omega |\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))|\,\mathrm{d}x\mathcal{L}^1(E^{(\nu)}_{s,\Xi}) \\
& + \int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t+s)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t))\right| \mathrm{d}x\mathrm{d}t \\
& + \int_{[0,\mathcal{T}-s]\setminus E^{(\nu)}_{s,\Xi}}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t+s),\boldsymbol{\phi}^{(\nu)}(t)) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\right| \mathrm{d}x\mathrm{d}t \\
\leq \;& 2\Big(\mathrm{esssup}_{t\in I}\int_\Omega b^{(\nu)}(\boldsymbol{u}^{(\nu)}(t),\boldsymbol{\phi}^{(\nu)}(t))\,\mathrm{d}x + \mathcal{L}^d(\Omega)C_1\Big)\,\mathcal{L}^1(E^{(\nu)}_{s,\Xi})) + s\,C\,\Xi + \mathcal{T}\omega_\Xi(s\Xi) \\
\leq \;& C\mathcal{L}^1(E^{(\nu)}_{s,\Xi})) + s\,C\,\Xi + \mathcal{T}\omega_\Xi(s\Xi).
\end{aligned}$$

Choosing first Ξ sufficiently large and, after, s sufficiently small, the right hand side becomes arbitrarily small, independently of $\nu \in [0,\overline{\nu}]$, hence as $\Xi \to \infty$, $s \to 0$

$$\sup_{\nu\in[0,\overline{\nu}]}\int_0^{\mathcal{T}-s}\int_\Omega \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(t+s) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(t)\right| \mathrm{d}x\mathrm{d}t \to 0. \tag{101}$$

In order to show precompactness of the $\psi^{(\nu)}_{,\boldsymbol{u}}$ in $L^1(I\times\Omega;Y^N)$, to each $\kappa > 0$ a finite number of functions $\{\boldsymbol{f}_k\}_k$ has to be found such that the $\psi^{(\nu)}_{,\boldsymbol{u}}$ lie in the union of the balls with radius κ around the $\boldsymbol{f}_k$. The second step consists in showing that it is sufficient if the set $\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\}_{\nu\in[0,\overline{\nu}]}$ is precompact in $L^1(D;Y^N)$ for every $D \subset\subset I\times\Omega$. To see this, let $\kappa > 0$ be given. Observe that for each $\boldsymbol{f} \in L^1(D;Y^N)$ by Lemma 5.1

$$\begin{aligned}
&\|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)}) - \chi_D\boldsymbol{f}\|_{L^1(I\times\Omega;Y^N)} \\
= \;& \int_{(I\times\Omega)\setminus D} |\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})|\,\mathrm{d}x\mathrm{d}t + \int_D \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)}) - \boldsymbol{f}\right|\mathrm{d}x\mathrm{d}t \\
\leq \;& \tilde{\delta}\int_{I\times\Omega} b^{(\nu)}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\,\mathrm{d}x\mathrm{d}t + C_{\tilde{\delta}}\mathcal{L}^d((I\times\Omega)\setminus D) \\
& + \int_D \left|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)}) - \boldsymbol{f}\right|\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{102}$$

Choosing $\tilde{\delta}$ small, thanks to (96) the first term becomes smaller than $\kappa/3$. After, choose D appropriately so that the second term becomes smaller than $\kappa/3$, i.e., choose D such that $\mathcal{L}^d((I\times\Omega) - D) < \kappa/(3C_{\tilde{\delta}})$. Finally, use the assumption that there are functions $\boldsymbol{f}_1,\ldots,\boldsymbol{f}_{k(\kappa,D)} \in L^1(D;Y^N)$ such that

$$\left\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\right\}_{\nu\in[0,\overline{\nu}]} \subset \bigcup_{i=1}^{k(\kappa,D)} B_{\kappa/3}(\boldsymbol{f}_i)$$

where $B_\varepsilon(\boldsymbol{f}) = \{\boldsymbol{g} \in L^1(D;Y^N) : \|\boldsymbol{g}-\boldsymbol{f}\|_{L^1(D;Y^N)} < \varepsilon\}$ to find a suitable $\boldsymbol{f} = \boldsymbol{f}_i \in L^1(D;Y^N)$ such that the last term in (102) becomes smaller than $\kappa/3$, too.

The next step to show precompactness of the $\psi^{(\nu)}_{,\boldsymbol{u}}$ in $L^1(D;Y^N)$ for each $D \subset\subset I \times \Omega$ is to construct approximating step functions. Let $K \in \mathbb{N}$ and $s = \mathcal{T}/K$, and define the functions

$$\boldsymbol{v}^{(\nu)}(t,x) := \begin{cases} \boldsymbol{u}^{(\nu)}(t,x), & \text{if } t \notin E^{(\nu)}_{s,\Xi}, \\ 0, & \text{elsewhere,} \end{cases}$$

$$\boldsymbol{\zeta}^{(\nu)}(t,x) := \begin{cases} \boldsymbol{\phi}^{(\nu)}(t,x), & \text{if } t \notin E^{(\nu)}_{s,\Xi}, \\ 0, & \text{elsewhere.} \end{cases}$$

The step functions (with steps in time, not in space) are defined by

$$T^{(\nu)}_{r,s,\Xi}(t,x) := \sum_{i=1}^{K} \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})((i-1)s+r,x)\chi_{(i-1)s,is]}(t)$$

where $r \in [0,s)$ will later be chosen appropriately. The following calculation is essential for a control of the error between the original function and the step function. For times $t_1 = j_1 s$ and $t_2 = j_2 s$ with $j_1, j_2 \in \{0,\dots,K\}$

$$\begin{aligned}
&\fint_0^s \int_{t_1}^{t_2} \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(t) - T^{(\nu)}_{r,s,\Xi}(t) \right\|_{L^1(\Omega;Y^N)} \mathrm{d}t\mathrm{d}r \\
= \quad & \sum_{i=j_1+1}^{j_2} \fint_0^s \int_{(i-1)s}^{is} \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(t) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})((i-1)s+r) \right\|_{L^1} \mathrm{d}t\mathrm{d}r \\
= \quad & \sum_{i=j_1+1}^{j_2} \fint_0^s \int_0^s \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})((i-1)s+\tilde{r}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})((i-1)s+r) \right\|_{L^1} \mathrm{d}\tilde{r}\mathrm{d}r \\
= \quad & \sum_{i=j_1+1}^{j_2} \fint_0^s \int_{(i-1)s}^{is} \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})((i-1)s+\tilde{r}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})(\tilde{t}) \right\|_{L^1} \mathrm{d}\tilde{t}\mathrm{d}\tilde{r} \\
= \quad & \frac{1}{s} \int_{t_1}^{t_2} \int_0^s \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})((i-1)s+\tilde{r}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})(\tilde{t}) \right\|_{L^1} \mathrm{d}\tilde{r}\mathrm{d}\tilde{t};
\end{aligned}$$

inserting $q = \tilde{r} + (i-1)s - \tilde{t} \in ((i-1)s - \tilde{t}, is - \tilde{t})$ this is estimated by

$$\begin{aligned}
\leq \quad & \frac{1}{s} \int_{t_1}^{t_2} \int_{-s}^s \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}+q) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})(\tilde{t}) \right\|_{L^1} \mathrm{d}q\mathrm{d}\tilde{t} \\
\leq \quad & \frac{1}{s} \int_{t_1}^{t_2} \int_{-s}^s \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}+q) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}) \right\|_{L^1} \mathrm{d}q\mathrm{d}\tilde{t} \\
& + \frac{1}{s} \int_{t_1}^{t_2} \int_{-s}^s \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})(\tilde{t}) \right\|_{L^1} \mathrm{d}q\mathrm{d}\tilde{t} \\
\leq \quad & 2 \sup_{|q|\leq s} \int_{t_1}^{t_2} \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}+q) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}) \right\|_{L^1(\Omega;Y^N)} \mathrm{d}\tilde{t} \\
& + 2 \int_{E^{(\nu)}_{s,\Xi}} \left\| \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(\tilde{t}) - \psi^{(\nu)}_{,\boldsymbol{u}}(0,0)(\tilde{t}) \right\|_{L^1(\Omega;Y^N)} \mathrm{d}\tilde{t}.
\end{aligned}$$

The result (101) states that the first term on the right hand side tends to zero as $s \to 0$. Using Lemma 5.1 with $\tilde{\delta} = 1$, (93), and (96) the second term is estimated by

$$2\mathcal{L}^1(E^{(\nu)}_{s,\Xi})\Big(\mathrm{esssup}_{\tilde{t}\in I}\int_\Omega b^{(\nu)}(\boldsymbol{u}^{(\nu)}(\tilde{t},x),\boldsymbol{\phi}^{(\nu)}(\tilde{t},x))\,\mathrm{d}x + C\Big) \quad \le \quad C\,\mathcal{L}^1(E^{(\nu)}_{s,\Xi})$$

and becomes arbitrarily small when choosing Ξ sufficiently large. Therefore, if a small $\kappa > 0$ is given then it is possible to choose some large Ξ, some small s (by choosing K big), and some $r_\nu \in [0,s]$ for every $\nu \in [0,\overline{\nu}]$ such that

$$\int_{t_1}^{t_2} \|\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})(t) - T^{(\nu)}_{r_\nu,s,\Xi}(t)\|_{L^1(\Omega;Y^N)}\,\mathrm{d}t \le \kappa.$$

Hence, if the set of step functions $\{T^{(\nu)}_{r_\nu,s,\Xi}\}_{\nu\in[0,\overline{\nu}]}$ is precompact in $L^1(D)$ for every $D \subset\subset I\times\Omega$ and every s,Ξ, then choose s small enough such that $D \subset\subset [0,\mathcal{T}-s]\times\Omega$ and apply the above result to get that the set $\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\}_{\nu\in[0,\overline{\nu}]}$ is precompact in $L^1(D;Y^N)$.

Finally, consider the set $\{T^{(\nu)}_{r_\nu,s,\Xi}\}_{\nu\in[0,\overline{\nu}]}$ as a subset of $L^1(D;Y^N)$ for some $D\subset\subset I\times\Omega$. It remains to demonstrate that there is a function $\tilde{T}\in L^1(D;Y^N)$ and a subsequence $(\nu_k)_{k\in\mathbb{N}}$ such that $T^{(\nu_k)}_{r_{\nu_k},s,\Xi} \to \tilde{T}$ in $L^1(D;Y^N)$. Since K, s, and Ξ are fixed now it remains to examine whether the sets $\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu)},\boldsymbol{\zeta}^{(\nu)})((i-1)s+r_\nu)\}_{\nu\in[0,\overline{\nu}]}$ are precompact in $L^1(D_x;Y^N)$ for every $D_x\subset\subset\Omega$, $i=1,\dots,K$. It holds that

$$\begin{aligned} \bar{t}^{(\nu)} := (i-1)s + r_\nu \in E^{(\nu)}_{s,\Xi} \quad &\Rightarrow \quad \boldsymbol{v}^{(\nu)}(\bar{t}^{(\nu)}) = 0,\\ \bar{t}^{(\nu)} \notin E^{(\nu)}_{s,\Xi} \quad &\Rightarrow \quad \|\boldsymbol{v}^{(\nu)}(\bar{t}^{(\nu)})\|_{H^1(D_x;Y^N)} \le \Xi, \end{aligned}$$

and analogously for $\boldsymbol{\zeta}^{(\nu)}$. It follows that for every sequence $(\nu_k)_{k\in\mathbb{N}} \subset [0,\overline{\nu}]$ there is a subsequence, still denoted by $(\nu_k)_k$, there is $\tilde{\nu}\in[0,\overline{\nu}]$, and there are functions $\bar{\boldsymbol{v}}\in H^1(D_x;Y^N)$ and $\bar{\boldsymbol{\zeta}}\in H^1(D_x;\mathrm{H}\Sigma^M)$ such that $\nu_k\to\tilde{\nu}$ and

$$\begin{aligned} &\boldsymbol{v}^{(\nu_k)}(\bar{t}^{(\nu_k)}) \to \bar{\boldsymbol{v}} \text{ weakly in } H^1(D_x;Y^N), && \text{strongly in } L^2(D_x;Y^N), && \text{and a.e.},\\ &\boldsymbol{\zeta}^{(\nu_k)}(\bar{t}^{(\nu_k)}) \to \bar{\boldsymbol{\zeta}} \text{ weakly in } H^1(D_x;\mathrm{T}\Sigma^M), && \text{strongly in } L^2(D_x;\mathrm{T}\Sigma^M), && \text{and a.e.} \end{aligned}$$

as $k\to\infty$. The same arguments as in the proof of Lemma 5.2 in the previous subsection yield the assertion:

$$\psi^{(\nu_k)}_{,\boldsymbol{u}}(\boldsymbol{v}^{(\nu_k)},\boldsymbol{\zeta}^{(\nu_k)})(\bar{t}^{(\nu_k)}) \to \psi^{(\tilde{\nu})}_{,\boldsymbol{u}}(\bar{\boldsymbol{v}},\bar{\boldsymbol{\zeta}}) \text{ in } L^1(D_x;Y^N).$$

Altogether, it was proved that

$$\Big\{\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)},\boldsymbol{\phi}^{(\nu)})\Big\}_{\nu\in[0,\overline{\nu}]} \subset L^1(I\times\Omega;Y^N) \text{ is precompact.} \tag{103}$$

5.4 Convergence statements

The aim of this section is to let $\nu\to 0$ in (95) in order to obtain (39).

Since the set of functions $\{\|\boldsymbol{u}^{(\nu)}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\}_{\nu\in[0,\overline{\nu}]}$ is bounded the Poincaré inequality yields

$$\|\boldsymbol{u}^{(\nu)}\|_{L^2(I;L^2(\Omega;Y^N))} \leq C.$$

By this, the other estimates in (96), and (103) there are functions $\boldsymbol{u} \in L^2(I; H^1(\Omega; Y^N))$, $\boldsymbol{b} \in L^1(I \times \Omega; Y^N)$, and $\boldsymbol{\phi} \in H^1(I \times \Omega; \mathrm{H}\Sigma^M)$ so that for a subsequence as $\nu \to 0$

$$\begin{aligned} \boldsymbol{\phi}^{(\nu)} &\rightharpoonup \boldsymbol{\phi} && \text{in } H^1(I \times \Omega; \mathrm{H}\Sigma^M), \\ \boldsymbol{u}^{(\nu)} &\rightharpoonup \boldsymbol{u} && \text{in } L^2(I; H^1(\Omega; Y^N)) \text{ and in } L^2(I; L^2(\partial\Omega; Y^N)), && (104) \\ \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) &\to \boldsymbol{b} && \text{in } L^1(I \times \Omega; Y^N). && (105) \end{aligned}$$

The second convergence result is already sufficient to obtain the second line of (39) from the third line of (95) as long as the test function fulfills $\boldsymbol{v} \in L^2(I; L^2(\partial\Omega; Y^N))$.

With the same arguments as in Subsection 4.3

$$\boldsymbol{\phi}^{(\nu)} \to \boldsymbol{\phi} \text{ in } L^q(I \times \Omega; \mathrm{H}\Sigma^M) \text{ and almost everywhere,} \tag{106}$$

for $q = 2$ and $q = p$ the value in (22).

To identify $\boldsymbol{b}$ with $\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ the monotonicity trick is applied. But the fact that $\psi_{,\boldsymbol{u}}$ only converges in L^1 must be faced. Let for $R > 0$

$$\mathcal{P}_R : Y^N \to B_R(0) \subset Y^N, \quad \mathcal{P}_R(\boldsymbol{v}) := \begin{cases} \boldsymbol{v}, & \text{if } |\boldsymbol{v}| \leq R, \\ \frac{R}{|\boldsymbol{v}|}\boldsymbol{v}, & \text{if } |\boldsymbol{v}| > R. \end{cases}$$

Bu the convexity of $\psi^{(\nu)}$, the $\psi^{(\nu)}_{,\boldsymbol{u}}$ are monotone in $\boldsymbol{u}$, hence for all $\boldsymbol{v} \in L^2(I \times \Omega; Y^N)$

$$0 \leq \int_I \int_\Omega \mathcal{P}_R\big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{\phi}^{(\nu)}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)})\big) \cdot (\boldsymbol{v} - \boldsymbol{u}^{(\nu)}) \,\mathrm{d}x\mathrm{d}t.$$

The convergence in (106), (104), and (105) yields, thanks to the assumptions in L1 and the Lebesgue convergence theorem,

$$0 \leq \int_I \int_\Omega \mathcal{P}_R\big(\psi_{,\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{\phi}) - \boldsymbol{b}\big) \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t.$$

Insert $\boldsymbol{v} = \boldsymbol{u} + \varepsilon\bar{\boldsymbol{v}}$ with some $\bar{\boldsymbol{v}} \in L^2(I \times \Omega; Y^N)$, multiply by ε, and let $\varepsilon \to 0$ to obtain

$$0 \leq \int_I \int_\Omega \mathcal{P}_R\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \boldsymbol{b}\big) \cdot \bar{\boldsymbol{v}} \,\mathrm{d}x\mathrm{d}t.$$

Since $R > 0$ and $\bar{\boldsymbol{v}}$ are arbitrary one can conclude that $\boldsymbol{b} = \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ almost everywhere, whence from (105)

$$\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \text{ in } L^1(I \times \Omega; Y^N) \text{ and almost everywhere.} \tag{107}$$

Therefore, for every test function $\boldsymbol{v} : I \times \Omega \to Y^N$ such that $\partial_t \boldsymbol{v} \in L^\infty(I \times \Omega; Y^N)$

$$\partial_t\boldsymbol{v}\cdot\big(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) - \psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big) \to \partial_t\boldsymbol{v}\cdot\big(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) - \psi_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big) \text{ in } L^1(I\times\Omega). \tag{108}$$

Similar arguments as have been used to obtain (68) give

$$\nabla \boldsymbol{v} : \boldsymbol{L}(\psi^{(\nu)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}), \boldsymbol{\phi}^{(\nu)}) \nabla \boldsymbol{u}^{(\nu)} \to \nabla \boldsymbol{v} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi}) \nabla \boldsymbol{u} \text{ in } L^1(I \times \Omega) \tag{109}$$

if the test function fulfills $\boldsymbol{v} \in L^2(I; H^1(\Omega; Y^N))$. Taking (108) and (109) together, the limit as $\nu \to 0$ in the first and second line of (95) in fact is the first line of (39).

The growth assumption on $\boldsymbol{v}$, namely, $\partial_t \boldsymbol{v} \in L^\infty(I \times \Omega; Y^N)$, can now be relaxed to the assumption in Theorem 3.2. Indeed, once the limit (108) is established for sufficient smooth test functions, the growth assumptions on ψ as stated in L1 resulting in $\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \in L^2(I \times \Omega; Y^N)$ enable to approximate functions $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$.

Also the terms involving the functions ω, a, and w in the fourth and fifth line of (95) can be handled as previously in Subsection 4.3 and 4.4. Observe that $\boldsymbol{\zeta} = \boldsymbol{\phi}^{(\nu)} - \boldsymbol{\phi}$ is an admissible test function in (95). The following arguments of that subsection can be applied again to show strong convergence of $\nabla \boldsymbol{\phi}^{(\nu)}$ to $\nabla \boldsymbol{\phi}$ in $L^2(I; L^2(\Omega; (\mathrm{T}\Sigma^M)^d))$ and, therefore, to let $\nu \to 0$ in the terms involving ω and a. For handling the w term, the arguments around the result (70) can be applied again in view of (106) and (106). In particular, the limiting terms are exactly those appearing in (39).

It remains to consider the last term in (95). The growth assumptions on $\psi^{(\nu)}_{,\boldsymbol{\phi}}$ in L1, more precisely (91), give, thanks to (96),

$$\|\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)})\|_{L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M))} \le C\big(1 + \|\boldsymbol{u}^{(\nu)}\|_{L^2(I; L^2(\Omega; Y^N))}\big) \le C,$$

whence there is some $\bar{\boldsymbol{\zeta}} \in L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M))$ such that

$$\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) \rightharpoonup \bar{\boldsymbol{\zeta}} \text{ in } L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M)).$$

By (106) and (107) the assumptions in Lemma 5.3 are fulfilled almost everywhere, hence

$$\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) \to \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{almost everywhere as } \nu \to 0.$$

Together this means that

$$\psi^{(\nu)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\nu)}, \boldsymbol{\phi}^{(\nu)}) \rightharpoonup \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \text{ in } L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M))$$

which, as long as $\boldsymbol{\zeta} \in L^2(I; L^2(\Omega; \mathrm{T}\Sigma^M))$, is sufficient to go to the limit in the last term of (95) and to obtain the last term of (39).

Assertion (38) can be derived with similar arguments as in Subsection 4.5 which concludes the proof of Theorem 3.2.

6 Proof of Theorem 3.3

6.1 Solution to the perturbed problem

From the approximation of $g = g^{(0)}$ by the $g^{(\eta)}$ it is obvious that there are functions $\tilde{k}_1(\eta)$ and $\tilde{k}_0(\eta)$ with $\tilde{k}_1(\eta) \ge g^{(\eta)}_{,u_0 u_0}(u_0) \ge \tilde{k}_0(\eta) > 0$ for $\eta > 0$ where $\tilde{k}_1(\eta) \to \infty$ and $\tilde{k}_0(\eta) \to 0$ as $\eta \to 0$. The assumptions in B1 furthermore imply that the perturbed potentials $\psi^{(\eta)}$

fulfill the properties stated in Q1. In view of B2–B4, the assumptions of Theorem 3.1 are satisfied. Thus, there are functions

$$\boldsymbol{u}^{(\eta)} \in L^2(I; H^1(\Omega; Y^N)), \quad \boldsymbol{\phi}^{(\eta)} \in H^1(I \times \Omega; \mathrm{H}\Sigma^M)$$

such that

$$\boldsymbol{\phi}^{(\eta)}(t, \cdot) \to \boldsymbol{\phi}_{ic} \quad \text{in } L^2(\Omega; \mathrm{H}\Sigma^M) \text{ as } t \searrow 0$$

and such that

$$\begin{aligned}
0 = \quad & \int_I \int_\Omega -\partial_t \boldsymbol{v} \cdot \big(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big)\, \mathrm{d}x\mathrm{d}t \\
& + \int_I \int_\Omega \nabla \boldsymbol{v} : \boldsymbol{L}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)}) \nabla \boldsymbol{u}\, \mathrm{d}x\mathrm{d}t \\
& + \int_I \int_{\partial\Omega} v_0 \cdot \beta_{00}(u_0^{(\eta)} - u_{bc,0})\, \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
& + \int_I \int_\Omega \Big[\omega(\boldsymbol{\phi}^{(\eta)}, \nabla \boldsymbol{\phi}^{(\eta)}) \partial_t \boldsymbol{\phi}^{(\eta)} \cdot \boldsymbol{\zeta} + a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)}, \nabla \boldsymbol{\phi}^{(\eta)}) : \nabla \boldsymbol{\zeta}\Big]\, \mathrm{d}x\mathrm{d}t \\
& + \int_I \int_\Omega \Big[a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)}, \nabla \boldsymbol{\phi}^{(\eta)}) \cdot \boldsymbol{\zeta} + w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)}) \cdot \boldsymbol{\zeta} - \psi^{(\eta)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \cdot \boldsymbol{\zeta}\Big]\, \mathrm{d}x\mathrm{d}t \qquad (110)
\end{aligned}$$

for all test functions $\boldsymbol{v} \in H^1(I \times \Omega; Y^N)$ with $\boldsymbol{v}(\mathcal{T}) = 0$ and $\boldsymbol{\zeta} \in H^1(I \times \Omega; \mathrm{T}\Sigma^M) \cap L^p(I \times \Omega; \mathrm{T}\Sigma^M)$. Estimate (79) for the solution $(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ looks slightly different with respect to the boundary term, namely

$$\begin{aligned}
\operatorname{esssup}_{\tilde{t} \in I} \int_\Omega \Big[&\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\tilde{t}), \boldsymbol{\phi}^{(\eta)}(\tilde{t})) \cdot \boldsymbol{u}^{(\eta)}(\tilde{t}) - \psi^{(\eta)}(\boldsymbol{u}^{(\eta)}(\tilde{t}), \boldsymbol{\phi}^{(\eta)}(\tilde{t})) \\
&+ w_2 |\boldsymbol{\phi}^{(\eta)}(\tilde{t})|^p + a_0 |\nabla \boldsymbol{\phi}^{(\eta)}(\tilde{t})|^2\Big]\, \mathrm{d}x \\
+ \int_I \int_\Omega \Big[\omega_0 |\partial_t \boldsymbol{\phi}^{(\eta)}|^2 &+ l_0 |\nabla \boldsymbol{u}^{(\eta)}|^2\Big]\, \mathrm{d}x\mathrm{d}t + \beta_2 \int_I \int_{\partial\Omega} |u_0^{(\eta)}|^2\, \mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \quad \le \quad C. \qquad (111)
\end{aligned}$$

Thanks to assumption (40) a computation similarly to (55) shows that

$$\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \cdot \boldsymbol{u}^{(\eta)} - \psi^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \ge \frac{\hat{k}_0}{2} |\tilde{\boldsymbol{u}}^{(\eta)}|^2 - k_4.$$

With (111) and applying the Poincaré inequality to $u_0^{(\eta)}$ furnishes the estimate

$$\|\boldsymbol{u}^{(\eta)}\|_{L^2(I; L^2(\Omega))} \le C \quad \text{for all } \eta \in (0, \overline{\eta}]. \qquad (112)$$

Estimate (80) reads in the present situation

$$\begin{aligned}
\int_0^{\mathcal{T}-s} \int_\Omega &(\boldsymbol{u}^{(\eta)}(t+s) - \boldsymbol{u}^{(\eta)}(t)) \\
&\cdot \big(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t+s), \boldsymbol{\phi}^{(\eta)}(t)) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\big)\, \mathrm{d}x\mathrm{d}t \quad \le \quad C\, s. \qquad (113)
\end{aligned}$$

6.2 Estimate of the conserved quantities

Let $\chi(t) := \chi_{(0,\tilde{t})}(t)$ be the characteristic function of the interval $\tilde{I} = (0, \tilde{t})$, and define

$$\boldsymbol{v}_\delta(t, x) := \fint_t^{t+\delta} \chi(s)\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(s, x), \boldsymbol{\phi}^{(\eta)}(s, x))\,\mathrm{d}s.$$

The functions $\varphi_\delta(s) = \frac{1}{\delta}\chi_{(-\delta,0)}(s)$ constitute a Dirac sequence. By the growth assumptions in B1 $\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \in L^2(I; L^2(\Omega; (Y^N)^d))$, and the properties of Dirac sequences yield

$$\varphi_\delta * \chi\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \to \chi\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \text{ in } L^2(I; L^2(\Omega; (Y^N)^d))$$

in the limit as $\delta \searrow 0$. Since

$$\begin{aligned} &\big(\varphi_\delta(\cdot) * \chi(\cdot)\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\cdot, x), \boldsymbol{\phi}^{(\eta)}(\cdot, x))\big)(t) \\ = &\int_{\mathbb{R}} \varphi_\delta(t-s)\chi(s)\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(s, x), \boldsymbol{\phi}^{(\eta)}(s, x))\,\mathrm{d}s \\ = &\fint_t^{t+\delta} \chi(s)\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(s, x), \boldsymbol{\phi}^{(\eta)}(s, x))\,\mathrm{d}s \quad = \quad \nabla\boldsymbol{v}_\delta \end{aligned}$$

it is clear that $\nabla\boldsymbol{v}_\delta \in L^2(I; L^2(\Omega; (Y^N)^d))$ and, hence,

$$\nabla\boldsymbol{v}_\delta \to \chi\nabla\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \text{ in } L^2(I; L^2(\Omega; (Y^N)^d)).$$

Analogously $\boldsymbol{v}_\delta \to \chi\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ in $L^2(I; L^2(\Omega; Y^N))$, thus it holds that

$$\boldsymbol{v}_\delta \to \chi\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \text{ in } L^2(I; H^1(\Omega; Y^N)). \tag{114}$$

Define the forward and backward discrete time derivatives by

$$\partial_t^\delta f(t) := \frac{1}{\delta}\Big(f(t+\delta) - f(t)\Big), \quad \partial_t^{-\delta} f(t) := \frac{1}{\delta}\Big(f(t) - f(t-\delta)\Big)$$

for a function $f : \mathbb{R} \to Z$ mapping into some Banach space Z. Then

$$\partial_t\boldsymbol{v}_\delta(t, x) \quad = \quad \partial_t^\delta\big(\chi(\cdot)\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\cdot, x), \boldsymbol{\phi}^{(\eta)}(\cdot, x))\big)(t),$$

hence $\boldsymbol{v}_\delta \in H^1(\tilde{I} \times \Omega; Y^N)$ if $\delta < \mathcal{T} - \tilde{t}$.

Let $\boldsymbol{\zeta} = 0$ and $\boldsymbol{v} = \boldsymbol{v}_\delta$ in (110) and suppose that $\delta < \mathcal{T} - \tilde{t}$. Then

$$\begin{aligned} 0 = &\int_I\int_\Omega -\partial_t\boldsymbol{v}_\delta \cdot \big(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\big)\,\mathrm{d}x\mathrm{d}t \\ &+ \int_I\int_\Omega \nabla\boldsymbol{v}_\delta : \boldsymbol{L}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)})\nabla\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \\ &+ \int_I\int_{\partial\Omega} v_{\delta,0} \cdot \beta_{00}(u_0^{(\eta)} - u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t. \end{aligned} \tag{115}$$

Extend $(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})$ for $t \in (-\delta, 0)$ by $(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})$. Using that $\boldsymbol{y} \cdot (\boldsymbol{y} - \boldsymbol{z}) \geq \frac{1}{2}(|\boldsymbol{y}|^2 - |\boldsymbol{z}|^2)$ for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^{N+1}$ a short calculation shows that

$$
\begin{aligned}
& -\int_I \int_\Omega \partial_t \boldsymbol{v}_\delta \cdot (\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}))\, \mathrm{d}x\mathrm{d}t \\
= & -\int_0^{\mathcal{T}} \int_\Omega \partial_t^\delta \big(\chi \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})\big) \cdot (\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}))\, \mathrm{d}x\mathrm{d}t \\
\geq & \int_0^{\tilde{t}} \frac{1}{2\delta} \int_\Omega \Big(|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))|^2 - |\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t-\delta), \boldsymbol{\phi}^{(\eta)}(t-\delta))|^2 \Big)\, \mathrm{d}x\mathrm{d}t \\
= & \frac{1}{2} \fint_{\tilde{t}-\delta}^{\tilde{t}} \|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\|^2_{L^2(\Omega;Y^N)}\, \mathrm{d}t - \frac{1}{2}\|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\|^2_{L^2(\Omega;Y^N)}.
\end{aligned}
$$

Using again the properties of a convolution with a Dirac sequence it holds that

$$
\fint_{\tilde{t}-\delta}^{\tilde{t}} \|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\|^2_{L^2(\Omega;Y^N)}\, \mathrm{d}t \quad \to \quad \|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\tilde{t}), \boldsymbol{\phi}^{(\eta)}(\tilde{t}))\|^2_{L^2(\Omega;Y^N)}
$$

for almost every $\tilde{t} \in I$, whence, thanks to (43), the first term in (115) can estimated

$$
\begin{aligned}
& -\int_I \int_\Omega \partial_t \boldsymbol{v}_\delta \cdot (\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) - \psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic}))\, \mathrm{d}x\mathrm{d}t \\
\geq \quad & \frac{1}{2} \fint_{\tilde{t}-\delta}^{\tilde{t}} \|\psi_{,\boldsymbol{u}}(\boldsymbol{u}(t), \boldsymbol{\phi}(t))\|^2_{L^2(\Omega;Y^N)}\, \mathrm{d}t - \frac{1}{2}\|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}_{ic}, \boldsymbol{\phi}_{ic})\|^2_{L^2(\Omega;Y^N)} \\
\to \quad & \frac{1}{2}\|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\tilde{t}), \boldsymbol{\phi}^{(\eta)}(\tilde{t}))\|^2_{L^2(\Omega;Y^N)} - C
\end{aligned}
\tag{116}
$$

for almost every $\tilde{t} \in I$ as $\delta \to 0$.

Now, the second term of (115) will be estimated. Thanks to assumption (42)

$$
\begin{aligned}
& \nabla \boldsymbol{u}^{(\eta)} : \psi^{(\eta)}_{,\boldsymbol{uu}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \boldsymbol{L}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)}) \nabla \boldsymbol{u}^{(\eta)} \\
= \quad & |\nabla u_0^{(\eta)}|^2 g^{(\eta)}_{,u_0u_0}(u_0^{(\eta)}) L_{00}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)}) \\
& + 2\nu \sum_{i,j=1}^N \nabla u_i^{(\eta)} \cdot L_{ij}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)}) \nabla u_j^{(\eta)} \\
& + \sum_\alpha h(\phi_\alpha) \nabla \boldsymbol{u}^{(\eta)} : \lambda^{(\alpha)}_{,\boldsymbol{uu}}(\boldsymbol{u}^{(\eta)}) \boldsymbol{L}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)}) \nabla \boldsymbol{u}^{(\eta)}.
\end{aligned}
$$

The positivity of $\boldsymbol{L}$ (see assumption (30)) implies that $L_{00} \geq 0$, therefore for the integral of the first term

$$
\int_{\tilde{I}} \int_\Omega |\nabla u_0^{(\eta)}|^2 g^{(\eta)}_{,u_0u_0}(u_0^{(\eta)}) L_{00}(\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)})\, \mathrm{d}x\mathrm{d}t \geq 0.
$$

The integral of the second and third term can be estimated using (40), (32), and (111):

$$
\begin{aligned}
&\Big|\int_{\tilde I}\int_\Omega 2\nu \sum_{i,j=1}^N \nabla u_i^{(\eta)}\cdot L_{ij}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla u_j^{(\eta)}\,\mathrm{d}x\mathrm{d}t\Big| \\
&+\Big|\int_{\tilde I}\int_\Omega \sum_\alpha h(\phi_\alpha)\nabla \boldsymbol{u}^{(\eta)} : \lambda_{,\boldsymbol{uu}}^{(\alpha)}(\boldsymbol{u}^{(\eta)})\boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla \boldsymbol{u}^{(\eta)}\,\mathrm{d}x\mathrm{d}t\Big| \\
&\leq (2\nu + M\hat{k}_1)L_0 \int_I\int_\Omega |\nabla \boldsymbol{u}^{(\eta)}|^2\,\mathrm{d}x\mathrm{d}t \quad \leq \quad C.
\end{aligned}
$$

Moreover, using the estimate (111)

$$
\begin{aligned}
&\Big|\int_{\tilde I}\int_\Omega \nabla\boldsymbol{\phi}^{(\eta)} : \psi_{,\boldsymbol{\phi}u}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla \boldsymbol{u}^{(\eta)}\,\mathrm{d}x\mathrm{d}t\Big| \\
&\qquad\qquad \leq \quad k_3L_0\int_0^{\tilde t}\int_\Omega \tfrac{1}{2}(|\nabla\boldsymbol{\phi}^{(\eta)}|^2 + |\nabla\boldsymbol{u}^{(\eta)}|^2)\,\mathrm{d}x\mathrm{d}t \quad \leq \quad C.
\end{aligned}
$$

Altogether, the second term of (115) is estimated as

$$
\begin{aligned}
&\int_I\int_\Omega \nabla\boldsymbol{v}_\delta : \boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla\boldsymbol{u}^{(\eta)}\,\mathrm{d}x\mathrm{d}t \\
\to\quad &\int_I\int_\Omega \chi\nabla\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}) : \boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla\boldsymbol{u}^{(\eta)}\,\mathrm{d}x\mathrm{d}t \\
=\quad &\int_{\tilde I}\int_\Omega \Big(\nabla\boldsymbol{u}^{(\eta)} : \psi_{,\boldsymbol{uu}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla\boldsymbol{u}^{(\eta)}\Big)\,\mathrm{d}x\mathrm{d}t \\
+&\int_{\tilde I}\int_\Omega \Big(\nabla\boldsymbol{\phi}^{(\eta)} : \psi_{,\boldsymbol{\phi}u}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\nabla\boldsymbol{u}^{(\eta)}\Big)\,\mathrm{d}x\mathrm{d}t \\
\geq\quad &\int_{\tilde I}\int_\Omega |\nabla u_0^{(\eta)}|^2 g_{,u_0u_0}^{(\eta)}(u_0^{(\eta)})L_{00}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}),\boldsymbol{\phi}^{(\eta)})\,\mathrm{d}x\mathrm{d}t - C.
\end{aligned}
\tag{117}
$$

Considering the third term of the right hand side of (115) observe first that by (114) and the trace theorem for Sobolev functions it holds that

$$
v_{\delta,0} \to \chi\psi_{,u_0}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}) \text{ in } L^2(I;L^2(\partial\Omega)).
$$

This yields with the assumptions in B4 and since, by (111), condition (44) is satisfied

$$
\begin{aligned}
&\int_I\int_{\partial\Omega} v_{\delta,0}\beta_{00}(u_0^{(\eta)} - u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
\to\quad &\int_I\int_{\partial\Omega} \chi\psi_{,u_0}^{(\eta)}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\beta_{00}(u_0^{(\eta)} - u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
=\quad &\int_0^{\tilde t}\int_{\partial\Omega} \big(\psi_{,u_0}^{(\eta)}(u_0^{(\eta)},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)}) - \psi_{,u_0}^{(\eta)}(u_{bc,0},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\big)\beta_{00}(u_0^{(\eta)} - u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t \\
+&\int_0^{\tilde t}\int_{\partial\Omega} \psi_{,u_0}^{(\eta)}(u_{bc,0},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\beta_{00}(u_0^{(\eta)} - u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\geq &\quad \int_0^{\tilde t}\int_{\partial\Omega}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\psi^{(\eta)}_{,u_0}(\underbrace{\theta u_0^{(\eta)}+(1-\theta)u_{bc,0}}_{=:v_{bc,\theta}},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\,\mathrm{d}\theta\cdot\beta_{00}(u_0^{(\eta)}-u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t\\
&-\beta_1\int_0^{\tilde t}\int_{\partial\Omega}|\psi^{(\eta)}_{,u_0}(u_{bc,0},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)})||u_0^{(\eta)}-u_{bc,0}|\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t\\
\geq &\quad \int_0^{\tilde t}\int_{\partial\Omega}(u_0^{(\eta)}-u_{bc,0})\cdot\Big(\int_0^1\psi^{(\eta)}_{,u_0u_0}(v_{bc,\theta},\tilde{\boldsymbol{u}}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\,\mathrm{d}\theta\Big)\beta_{00}(u_0^{(\eta)}-u_{bc,0})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t\\
&-\beta_1\|\psi^{(\eta)}_{,\boldsymbol{u}}(u_{bc,0},\boldsymbol{\phi}^{(\eta)})\|_{L^2(I;L^2(\partial\Omega;Y^N))}\Big(\|u_0^{(\eta)}\|_{L^2(I;L^2(\partial\Omega;Y^N))}+\|u_{bc,0}\|_{L^2(I;L^2(\partial\Omega;Y^N))}\Big)\\
\geq &\quad -C.
\end{aligned}
\tag{118}
$$

To conclude, choosing $(\boldsymbol{v},\boldsymbol{\zeta})=(\boldsymbol{v}_\delta,0)$ in (110) yields as $\delta\to0$ thanks to the estimates (116), (117), and (118) that

$$
\operatorname{esssup}_{\tilde t\in I}\|\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)}(\tilde t),\boldsymbol{\phi}^{(\eta)}(\tilde t))\|^2_{L^2(\Omega;Y^N)}\leq C \quad \text{for all } \eta\in(0,\overline{\eta}]. \tag{119}
$$

As a conclusion from (111), (112), and (119) there are functions $\boldsymbol{u}\in L^2(I;H^1(\Omega;Y^N))$, $\boldsymbol{b}\in L^2(I;L^2(\Omega;Y^N))$ and $\boldsymbol{\phi}\in H^1(I\times\Omega;\mathrm{H}\Sigma^M)\cap L^p(I\times\Omega;\mathrm{H}\Sigma^M)$ such that for a subsequence as $\eta\to0$

$$
\begin{aligned}
\boldsymbol{u}^{(\eta)} &\rightharpoonup \boldsymbol{u} && \text{in } L^2(I;H^1(\Omega;Y^N)) \text{ and in } L^2(I;L^2(\partial\Omega;Y^N)), && (120)\\
\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)}) &\rightharpoonup \boldsymbol{b} && \text{in } L^2(I;L^2(\Omega;Y^N)), && (121)\\
\boldsymbol{\phi}^{(\eta)} &\rightharpoonup \boldsymbol{\phi} && \text{in } H^1(I\times\Omega;\mathrm{H}\Sigma^M), &&\\
\boldsymbol{\phi}^{(\eta)} &\to \boldsymbol{\phi} && \text{in } L^q(I\times\Omega;\mathrm{H}\Sigma^M) \text{ and almost everywhere, } q=2,p. && (122)
\end{aligned}
$$

The goal is to show that $(\boldsymbol{u},\boldsymbol{\phi})$ is a solution to (46) by considering the limit of (110) as $\eta\to0$. Strong convergence of $\nabla\boldsymbol{\phi}(\eta)$ to $\nabla\boldsymbol{\phi}$ in $L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d))$ can be shown as in Subsection 4.4. As a consequence, (70), (76), and (77) hold true with $\boldsymbol{\phi}^{(n)}$ replaced by $\boldsymbol{\phi}^{(\eta)}$ and $\boldsymbol{\zeta}^{(m)}$ by $\boldsymbol{\zeta}$:

$$
\begin{aligned}
&w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)})\to w_{,\boldsymbol{\phi}}(\boldsymbol{\phi}) \text{ in } L^{p^*}(I\times\Omega;\mathrm{T}\Sigma^M),\\
&a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)},\nabla\boldsymbol{\phi}^{(n)})\to a_{,\nabla\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) \text{ in } L^2(I;L^2(\Omega;(\mathrm{T}\Sigma^M)^d)),\\
&a_{,\boldsymbol{\phi}}(\boldsymbol{\phi}^{(\eta)},\nabla\boldsymbol{\phi}^{(n)})\to a_{,\boldsymbol{\phi}}(\boldsymbol{\phi},\nabla\boldsymbol{\phi}) \text{ in } L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M)),\\
&\omega(\boldsymbol{\phi}^{(\eta)},\nabla\boldsymbol{\phi}^{(n)})\boldsymbol{\zeta}\cdot\partial_t\boldsymbol{\phi}^{(n)}\to\omega(\boldsymbol{\phi},\nabla\boldsymbol{\phi})\boldsymbol{\zeta}\cdot\partial_t\boldsymbol{\phi} \text{ in } L^1(I;L^1(\Omega)).
\end{aligned}
\tag{123}
$$

It remains to identify $\boldsymbol{b}$ with $\psi_{,\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{\phi})$ and to show $\psi^{(\eta)}_{,\boldsymbol{\phi}}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})\rightharpoonup\psi_{,\boldsymbol{\phi}}(\boldsymbol{u},\boldsymbol{\phi})$ in $L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))$ and almost everywhere.

As a first step it is shown that $u_0\leq1$ almost everywhere. Define

$$
W_1:=\big\{(t,x)\in I\times\Omega: u_0(t,x)>1\big\},\quad |W_1|:=\mathcal{L}^{d+1}(W_1)
$$

It follows from (119) and assumption (41) that

$$\begin{aligned} C &\geq \int_{I\times\Omega} |\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})|\,\mathrm{d}x\mathrm{d}t && \geq \int_{I\times\Omega} |\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})|\,\mathrm{d}x\mathrm{d}t \\ &\geq \int_{W_1} |\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})|\,\mathrm{d}x\mathrm{d}t && \geq \int_{W_1} \Big|K_\eta(u_0^{(\eta)}-1)-k_8\Big|\,\mathrm{d}x\mathrm{d}t \\ &\geq K_\eta \int_{W_1} |u_0^{(\eta)}-1|\,\mathrm{d}x\mathrm{d}t - k_8|W_1|. \end{aligned}$$

The weak lower semi-continuity of norms implies

$$\int_{W_1} (u_0-1)\,\mathrm{d}x\mathrm{d}t \leq \liminf_{\eta\to 0} \int_{W_1} |u_0^{(\eta)}-1|\,\mathrm{d}x\mathrm{d}t \leq \liminf_{\eta\to 0} \frac{C+k_8|W_1|}{K_\eta} = 0,$$

hence $|W_1| = 0$ and

$$u_0 \leq 1 \text{ almost everywhere.} \tag{124}$$

6.3 Strong convergence of temperature and chemical potentials

The goal of this subsection is to show strong convergence of the $\boldsymbol{u}^{(\eta)}$ to $\boldsymbol{u}$ in L^1. Since the phase field variables are not of interest here, the value $\boldsymbol{\phi}^{(\eta)}(t,x)$ at which $\psi^{(\eta)}$ and its derivatives are evaluated is dropped for shorter presentation.

Set $\boldsymbol{v}_\theta := \theta\boldsymbol{u}^{(\eta)}(t+s) + (1-\theta)\boldsymbol{u}^{(\eta)}(t)$. Using (40), it follows from (113) that

$$\begin{aligned} s\,C &\geq \int_0^{\mathcal{T}-s}\int_\Omega (\boldsymbol{u}^{(\eta)}(t+s)-\boldsymbol{u}^{(\eta)}(t))\cdot\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\psi^{(\eta)}_{,\boldsymbol{u}}(\boldsymbol{v}_\theta)\,\mathrm{d}\theta\,\mathrm{d}x\mathrm{d}t \\ &\geq \int_0^{\mathcal{T}-s}\int_\Omega \hat{k}_0|\tilde{\boldsymbol{u}}^{(\eta)}(t+s)-\tilde{\boldsymbol{u}}^{(\eta)}(t)|^2\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{125}$$

Extending $\tilde{\boldsymbol{u}}^{(\eta)}$ by zero if $t\in\mathbb{R}\backslash(0,\mathcal{T})$ or if $x\in\mathbb{R}^d\backslash\Omega$, (125) and (112) yield

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} |\tilde{\boldsymbol{u}}^{(\eta)}(t+s,x)-\tilde{\boldsymbol{u}}^{(\eta)}(t,x)|^2\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } s\to 0.$$

To obtain an analogous result for differences in space consider

$$\Omega_h := \Big\{x\in\mathbb{R}^d \,:\, x+\theta h\in\Omega \quad \forall\theta\in[0,1]\Big\}$$

for some $h\in\mathbb{R}^d$. By the assumptions on Ω

$$\mathcal{L}^d(\Omega\backslash\Omega_h)\to 0 \text{ and } \mathcal{L}^d((\Omega-h)\backslash\Omega_h)\to 0 \qquad \text{as } |h|\to 0$$

where $\Omega - h = \{x-h \,:\, x\in\Omega\}$ and $\mathcal{L}^d$ is the Lebesgue measure of dimension d. By (111) there is a upper bound for $\{\|\nabla\tilde{\boldsymbol{u}}^{(\eta)}\|_{L^2(I;L^2(\Omega;(\mathrm{H}\Sigma^N)^d))}\}_{\eta\in(0,\overline{\eta}]}$, hence

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}^d} |\tilde{\boldsymbol{u}}^{(\eta)}(t,x+h)-\tilde{\boldsymbol{u}}^{(\eta)}(t,x)|^2\,\mathrm{d}x\mathrm{d}t \\ =& \int_0^{\mathcal{T}}\int_{\Omega_h} \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{\boldsymbol{u}}^{(\eta)}(t,x+\theta h)\,\mathrm{d}\theta\Big|^2\,\mathrm{d}x\mathrm{d}t \\ &+\int_0^{\mathcal{T}}\int_{\mathbb{R}^d\backslash\Omega_h} |\tilde{\boldsymbol{u}}^{(\eta)}(t,x+h)-\tilde{\boldsymbol{u}}^{(\eta)}(t,x)|^2\,\mathrm{d}x\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^{\mathcal{T}} \int_{\Omega_h} \Big| \int_0^1 \nabla \tilde{\boldsymbol{u}}^{(\eta)}(t, x+\theta h)\cdot h \,\mathrm{d}\theta \Big|^2 \,\mathrm{d}x\mathrm{d}t \\
& + \int_0^{\mathcal{T}} \int_{(\Omega-h)\backslash\Omega_h} |\tilde{\boldsymbol{u}}^{(\eta)}(t,x+h)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^{\mathcal{T}} \int_{\Omega\backslash\Omega_h} |\tilde{\boldsymbol{u}}^{(\eta)}(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \\
\leq\ & \int_0^{\mathcal{T}} \int_{\Omega_h} \int_0^1 |\nabla \tilde{\boldsymbol{u}}^{(\eta)}(t,x+\theta h)|^2 \,\mathrm{d}\theta\, |h|^2 \,\mathrm{d}x\mathrm{d}t \\
& + C\big(\mathcal{L}^d((\Omega-h)\backslash\Omega_h) + \mathcal{L}^d(\Omega\backslash\Omega_h)\big) \\
\to\ & 0 \qquad \text{as } |h| \to 0.
\end{aligned}
$$

Thus, the set $\{\tilde{\boldsymbol{u}}^{(\eta)}\}_\eta$ is precompact in $L^2(I; L^2(\Omega; \mathrm{T}\Sigma^N))$. By (120)

$$
\tilde{\boldsymbol{u}}^{(\eta)} \to \tilde{\boldsymbol{u}} \quad \text{almost everywhere and in } L^2(I; L^2(\Omega; \mathrm{T}\Sigma^N)). \tag{126}
$$

Next, an appropriate convergence result $u_0^{(\eta)} \to u_0$ will be shown. Clearly

$$
\begin{aligned}
& (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t))\big(\psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s), \tilde{\boldsymbol{u}}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s), \tilde{\boldsymbol{u}}^{(\eta)}(t))\big) \\
=\ & (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t)) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s), \underbrace{\theta\tilde{\boldsymbol{u}}^{(\eta)}(t+s) + (1-\theta)\tilde{\boldsymbol{u}}^{(\eta)}(t)}_{=:\tilde{v}_\theta}) \,\mathrm{d}\theta \\
=\ & (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t)) \int_0^1 \psi^{(\eta)}_{,u_0\tilde{\boldsymbol{u}}}(u_0^{(\eta)}(t+s), \tilde{v}_\theta) \,\mathrm{d}\theta \cdot (\tilde{\boldsymbol{u}}^{(\eta)}(t+s) - \tilde{\boldsymbol{u}}^{(\eta)}(t)).
\end{aligned} \tag{127}
$$

Analogously

$$
\begin{aligned}
& (\tilde{\boldsymbol{u}}^{(\eta)}(t+s) - \tilde{\boldsymbol{u}}^{(\eta)}(t)) \cdot \big(\psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t))\big) \\
=\ & (\tilde{\boldsymbol{u}}^{(\eta)}(t+s) - \tilde{\boldsymbol{u}}^{(\eta)}(t)) \cdot \int_0^1 \psi^{(\eta)}_{,\tilde{\boldsymbol{u}}\boldsymbol{u}}(\boldsymbol{v}_\theta) \,\mathrm{d}\theta \ (\boldsymbol{u}^{(\eta)}(t+s) - \boldsymbol{u}^{(\eta)}(t)).
\end{aligned} \tag{128}
$$

Estimate (113) means that

$$
\begin{aligned}
s\,C \ \geq\ & \int_0^{\mathcal{T}-s} \int_\Omega (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t)) \cdot \big(\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)}(t))\big) \,\mathrm{d}x\mathrm{d}t \\
& + \int_0^{\mathcal{T}-s} \int_\Omega (\tilde{\boldsymbol{u}}^{(\eta)}(t+s) - \tilde{\boldsymbol{u}}^{(\eta)}(t)) \cdot \big(\psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t))\big) \,\mathrm{d}x\mathrm{d}t \\
=\ & \int_0^{\mathcal{T}-s} \int_\Omega s\partial_t^s u_0^{(\eta)}(t) \big(\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s), \tilde{\boldsymbol{u}}^{(\eta)}(t))\big) \,\mathrm{d}x\mathrm{d}t \\
& + \int_0^{\mathcal{T}-s} \int_\Omega s\partial_t^s u_0^{(\eta)}(t) \big(\psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s), \tilde{\boldsymbol{u}}^{(\eta)}(t)) - \psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)}(t))\big) \,\mathrm{d}x\mathrm{d}t \\
& + \int_0^{\mathcal{T}-s} \int_\Omega s\partial_t^s \tilde{\boldsymbol{u}}^{(\eta)}(t) \cdot \big(\psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t+s)) - \psi^{(\eta)}_{,\tilde{\boldsymbol{u}}}(\boldsymbol{u}^{(\eta)}(t))\big) \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Plugging the first and the last term of the right hand side to the other side yields with

(127) and (128) that

$$\begin{aligned}
&\int_0^{\mathcal{T}-s}\int_\Omega (u_0^{(\eta)}(t+s)-u_0^{(\eta)}(t))\big(\psi^{(\eta)}_{,u_0}(u_0^{(\eta)}(t+s),\tilde{\boldsymbol{u}}^{(\eta)}(t))-\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)}(t))\big)\,\mathrm{d}x\mathrm{d}t\\
\le\;& \int_0^{\mathcal{T}-s}\int_\Omega \Big|s\partial_t^s u_0^{(\eta)}(t)\int_0^1 \psi^{(\eta)}_{,u_0\tilde{\boldsymbol{u}}}(u_0^{(\eta)}(t+s),\tilde{v}_\theta)\,\mathrm{d}\theta\cdot s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\Big|\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^{\mathcal{T}-s}\int_\Omega \Big|s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\cdot\int_0^1 \psi^{(\eta)}_{,\tilde{\boldsymbol{u}}\boldsymbol{u}}(\boldsymbol{v}_\theta)\,\mathrm{d}\theta\, s\partial_t^s\boldsymbol{u}^{(\eta)}(t)\Big|\,\mathrm{d}x\mathrm{d}t \quad +s\,C\\
\le\;& \int_0^{\mathcal{T}-s}\int_\Omega \Big|s\partial_t^s u_0^{(\eta)}(t)\int_0^1 \sum_{\alpha=1}^M h(\phi_\alpha)\lambda^{(\alpha)}_{,u_0\tilde{\boldsymbol{u}}}(u_0^{(\eta)}(t+s),\tilde{v}_\theta)\,\mathrm{d}\theta\cdot s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\Big|\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^{\mathcal{T}-s}\int_\Omega \Big|s\partial_t^s u_0^{(\eta)}(t)\int_0^1 \sum_{\alpha=1}^M h(\phi_\alpha)\lambda^{(\alpha)}_{,u_0\tilde{\boldsymbol{u}}}(\boldsymbol{v}_\theta)\,\mathrm{d}\theta\cdot s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\Big|\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^{\mathcal{T}-s}\int_\Omega \Big|s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\cdot\int_0^1 2\nu\mathrm{Id}_N+\sum_{\alpha=1}^M h(\phi_\alpha)\lambda^{(\alpha)}_{,\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}}(\boldsymbol{v}_\theta)\,\mathrm{d}\theta\, s\partial_t^s\tilde{\boldsymbol{u}}^{(\eta)}(t)\Big|\,\mathrm{d}x\mathrm{d}t \quad +s\,C\\
\le\;& 2\int_0^{\mathcal{T}-s}\int_\Omega \frac{M}{2}\hat{k}_1|u_0^{(\eta)}(t+s)-u_0^{(\eta)}(t)||\tilde{\boldsymbol{u}}^{(\eta)}(t+s)-\tilde{\boldsymbol{u}}^{(\eta)}(t))|\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^{\mathcal{T}-s}\int_\Omega \frac{M}{2}\hat{k}_1|\tilde{\boldsymbol{u}}^{(\eta)}(t+s)-\tilde{\boldsymbol{u}}^{(\eta)}(t)|^2\,\mathrm{d}x\mathrm{d}t \quad +s\,C.
\end{aligned}$$

In view of (125) and (112) this is for $s\le 1$

$$\begin{aligned}
\le\;& C\Big(\int_0^{\mathcal{T}-s}\int_\Omega |u_0^{(\eta)}(t+s)-u_0^{(\eta)}(t)|^2\,\mathrm{d}x\mathrm{d}t\Big)^{1/2}\\
&\qquad\cdot\sqrt{s}\,\Big(\frac{1}{s}\int_0^{\mathcal{T}-s}\int_\Omega |\tilde{\boldsymbol{u}}^{(\eta)}(t+s)-\tilde{\boldsymbol{u}}^{(\eta)}(t))|^2\,\mathrm{d}x\mathrm{d}t\Big)^{1/2} \quad +s\,C\\
\le\;& \sqrt{s}\,C\big(2\|u_0^{(\eta)}\|_{L^2(I;L^2(\Omega))}\big)+s\,C \quad\le\quad \sqrt{s}\,C.
\end{aligned}\tag{129}$$

For $\delta\in(0,\delta_0)$ define

$$u_{0,\delta}^{(\eta)}:=\max\Big(-\frac{1}{\delta},\min\big(1-\delta,u_0^{(\eta)}\big)\Big)=\kappa_\delta\circ u_0^{(\eta)},\tag{130}$$

i.e., $u_0^{(\eta)}$ is projected to the interval $[-\frac{1}{\delta},1-\delta]$ by the truncation function κ_δ. Let

$$W^+(\delta,\eta):=\big\{(t,x)\in I\times\Omega : u_0^{(\eta)}(t,x)>1-\delta\big\},\quad |W^+(\delta,\eta)|:=\mathcal{L}^{d+1}(W^+(\delta,\eta)),$$

which means that $u_{0,\delta}^{(\eta)}=1-\delta$ on $W^+(\delta,\eta)$. With (119) and (41)

$$\begin{aligned}
C\;\ge\;& \int_{I\times\Omega}|\psi^{(\eta)}_{,u_0}(\boldsymbol{u}^{(\eta)},\boldsymbol{\phi}^{(\eta)})|\,\mathrm{d}x\mathrm{d}t\\
\ge\;& \int_{W^+(\delta,\eta)} K_\eta(u_0^{(\eta)}-1+\delta-\delta)\,\mathrm{d}x\mathrm{d}t-k_7|W^+(\delta,\eta)|\\
=\;& K_\eta\int_{W^+(\delta,\eta)}|u_0^{(\eta)}-u_{0,\delta}^{(\eta)}|\,\mathrm{d}x\mathrm{d}t-(K_\eta\delta+k_7)|W^+(\delta,\eta)|.
\end{aligned}$$

Since $K_\eta \to \infty$ as $\eta \to 0$ and as $|W^+(\delta,\eta)|$ is bounded by $\mathcal{L}^{d+1}(I \times \Omega)$ for all δ and η there exists $\overline{\eta}(\delta)$ and $C > 0$ independent of δ such that for all $\eta \leq \overline{\eta}(\delta)$

$$\int_{W^+(\delta,\eta)} |u_0^{(\eta)} - u_{0,\delta}^{(\eta)}| \,\mathrm{d}x\mathrm{d}t \leq \frac{C}{K_\eta} + \Big(\delta + \frac{k_7}{K_\eta}\Big)|W^+(\delta,\eta)| \leq C\,\delta.$$

On the set

$$W^-(\delta,\eta) := \Big\{(t,x) \in I \times \Omega : u_0^{(\eta)}(t,x) < -\frac{1}{\delta}\Big\}$$

it holds that $|u_0^{(\eta)} - u_{0,\delta}^{(\eta)}|\frac{1}{\delta} \leq (-u_0^{(\eta)})^2$. As by (112) $\|u_0^{(\eta)}\|_{L^2(I;L^2(\Omega))}$ is bounded by a constant independent of η

$$\int_{W^-(\delta,\eta)} |u_0^{(\eta)} - u_{0,\delta}^{(\eta)}| \,\mathrm{d}x\mathrm{d}t \leq C\,\delta,$$

and since $u_0^{(\eta)}$ and $u_{0,\delta}^{(\eta)}$ agree on $I \times \Omega \backslash (W^+(\delta,\eta) \cup W^-(\delta,\eta))$, altogether the following convergence result is obtained (for an appropriate diagonal sequence):

$$\int_I \int_\Omega |u_0^{(\eta)} - u_{0,\delta}^{(\eta)}| \,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \eta, \delta \to 0. \tag{131}$$

Observe that

$$\begin{aligned}
&\psi_{,u_0}^{(\eta)}(u_0^{(\eta)}(t+s), \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t)) - \psi_{,u_0}^{(\eta)}(u_0^{(\eta)}(t), \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t)) \\
=\;& \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\psi_{,u_0}^{(\eta)}(\underbrace{\theta u_0^{(\eta)}(t+s) + (1-\theta)u_0^{(\eta)}(t)}_{=:v_{0,\theta}}, \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\,\mathrm{d}\theta \\
=\;& \int_0^1 \psi_{,u_0u_0}^{(\eta)}(v_{0,\theta}, \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t)) \cdot (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t))\,\mathrm{d}\theta \\
=\;& \int_{u_0^{(\eta)}(t)}^{u_0^{(\eta)}(t+s)} \psi_{,u_0u_0}^{(\eta)}(v_{0,\theta}, \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\,\mathrm{d}v_{0,\theta}.
\end{aligned}$$

Thus the estimate (129) reads

$$C\sqrt{s} \geq \int_0^{T-s} \int_\Omega \int_{u_0^{(\eta)}(t)}^{u_0^{(\eta)}(t+s)} \psi_{,u_0u_0}^{(\eta)}(v_{0,\theta}, \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\mathrm{d}v_{0,\theta} \cdot (u_0^{(\eta)}(t+s) - u_0^{(\eta)}(t))\,\mathrm{d}x\mathrm{d}t.$$

By the convexity of ψ clearly $\psi_{,u_0u_0}^{(\eta)} \geq 0$. Replacing $u_0^{(\eta)}$ by $u_{0,\delta}^{(\eta)}$ can therefore only lower the right side of the above inequality which leads to

$$C\sqrt{s} \geq \int_0^{T-s} \int_\Omega \int_{u_{0,\delta}^{(\eta)}(t)}^{u_{0,\delta}^{(\eta)}(t+s)} \psi_{,u_0u_0}^{(\eta)}(v_{0,\theta}, \tilde{\boldsymbol{u}}^{(\eta)}(t), \boldsymbol{\phi}^{(\eta)}(t))\mathrm{d}v_{0,\theta} \cdot (u_{0,\delta}^{(\eta)}(t+s) - u_{0,\delta}^{(\eta)}(t))\,\mathrm{d}x\mathrm{d}t.$$

But then $v_{0,\theta} \in [-\frac{1}{\delta}, 1-\delta]$ where, for η small enough, $\psi^{(\eta)}$ coincides with ψ. In particular, there is a constant $c_0(\delta) > 0$ such that $\psi_{,u_0u_0}^{(\eta)}(v, \boldsymbol{\phi}^{(\eta)}(t)) \geq c_0(\delta)$. Therefore

$$C\sqrt{s} \quad \geq \quad \int_0^{T-s} \int_\Omega c_0(\delta)|u_{0,\delta}^{(\eta)}(t+s) - u_{0,\delta}^{(\eta)}(t)|^2\,\mathrm{d}x\mathrm{d}t.$$

Since $|u_{0,\delta}^{(\eta)}| \le |u_0^{(\eta)}|$, by (112) there is an upper bound for $\|u_{0,\delta}^{(\eta)}\|_{L^2(I;L^2(\Omega))}$ independent of η and δ. Since by (130) $u_{0,\delta}^{(\eta)} = \kappa_\delta \circ u_0^{(\eta)}$ where $\kappa_\delta \in W^{1,\infty}(\mathbb{R})$, the chain rule for Sobolev functions and (111) gives that there is also an upper bound for the set $\{\|\nabla u_{0,\delta}^{(\eta)}\|_{L^2(I;L^2(\Omega;\mathbb{R}^d))}\}_{\eta,\delta}$. Applying analogous arguments as above for $\tilde{\boldsymbol{u}}^{(\eta)}$, for a given δ, the set $\{u_{0,\delta}^{(\eta)}\}_\eta$ is precompact in $L^2(I;L^2(\Omega))$, whence in $L^1(I;L^1(\Omega))$, too.

The convergence result (131) together with an argument involving diagonal sequences (choose first δ sufficient small and after choose an appropriate η) implies with (120)

$$u_0^{(\eta)} \to u_0 \quad \text{almost everywhere and in } L^1(I;L^1(\Omega)). \tag{132}$$

6.4 Convergence statements

Consider the set

$$W_0 := \{(t,x) \in I \times \Omega : u_0(t,x) = 1\}, \quad |W_0| := \mathcal{L}^{d+1}(W_0).$$

By (126), (132), and (122)

$$\psi_{,u_0}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \to \psi_{,u_0}(\boldsymbol{u}, \boldsymbol{\phi}) = \infty \text{ almost everywhere in } W_0.$$

But the estimate (119) gives in view of (121)

$$\|\psi_{,u_0}(\boldsymbol{u}, \boldsymbol{\phi})\|_{L^2(W_0;Y^N)} \le \liminf_{\eta\to 0} \|\psi_{,u_0}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})\|_{L^2(W_0;Y^N)} \le C,$$

therefore $|W_0| = 0$. As a conclusion, taking (124) into account, $u_0^{(\eta)} \to u_0 < 1$ almost everywhere which proves the first assertion in (45).

If $u_0 < 1$ the kind of way $\psi^{(\eta)}$ approximates ψ implies that $\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}, \boldsymbol{\phi}) = \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ as long as η is small enough. Therefore by (126), (132), and (122) $\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ almost everywhere. Recalling (121) $\boldsymbol{b} = \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$, i.e.,

$$\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \rightharpoonup \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{in } L^2(I;L^2(\Omega)). \tag{133}$$

Analogously as done in Subsection 4.4 for $\boldsymbol{u}^{(n)}$ (cf. the result (68)) it can be derived that

$$\nabla \boldsymbol{v} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}), \boldsymbol{\phi}^{(\eta)})\nabla \boldsymbol{u}^{(\eta)} \to \nabla \boldsymbol{v} : \boldsymbol{L}(\psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi}), \boldsymbol{\phi})\nabla \boldsymbol{u} \quad \text{in } L^1(I;L^1(\Omega)). \tag{134}$$

The assumptions B1 together with (112) yield that the $\|\psi_{,\boldsymbol{\phi}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)})\|_{L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))}$ are bounded by a constant independent of η so that there is $\boldsymbol{f} \in L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M))$ with

$$\psi_{,\boldsymbol{\phi}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \rightharpoonup \boldsymbol{f} \text{ in } L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M)).$$

The special structure of $\psi^{(\eta)}$ implies that $\psi_{,\boldsymbol{\phi}}^{(\eta)} = \psi_{,\boldsymbol{\phi}}$. Since $\psi_{,\boldsymbol{u}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \to \psi_{,\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\phi})$ and $\boldsymbol{\phi}^{(\eta)} \to \boldsymbol{\phi}$ almost everywhere Lemma 5.3 in Subsection 5.2 can be applied to show that $\psi_{,\boldsymbol{\phi}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \to \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi})$ almost everywhere, whence

$$\psi_{,\boldsymbol{\phi}}^{(\eta)}(\boldsymbol{u}^{(\eta)}, \boldsymbol{\phi}^{(\eta)}) \rightharpoonup \psi_{,\boldsymbol{\phi}}(\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{in } L^2(I;L^2(\Omega;\mathrm{T}\Sigma^M)). \tag{135}$$

The convergence results (123) and (43) together with (133)–(135) complete the list necessary to let $\eta \to 0$ in (110), i.e., $(\boldsymbol{u}, \boldsymbol{\phi})$ indeed solves (46). Since the second assertion in (45) can be derived as in Subsection 4.5 the proof of Theorem 3.3 is complete.

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