On Controllability and Stability of Uncertain Systems

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For
my family
the memory of my mother
ITA turma 85

Declaration

The work in this thesis is, to the best of my knowledge, original, except where explicitly stated otherwise.

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Summary

In the first part of the work, we consider the problem of giving upper bounds for $|x(T) - z(T)|$, the error between the final states of a nominal finite dimensional system $\dot{x} = Ax + Bu$, $x(0) = x_o$, and of the system disturbed by multiple structured perturbations of the form

$$\dot{z}(t) = Az(t) \ + \ \sum_{k=1}^{r} D_k F_k(C_k z(t), t) \ + \ Bu(t) \qquad z(0) = x_o$$

which accounts for the uncertainties on the entries of the matrix A. In approaching the problem we introduce a framework which involves some weight-functions and provides a scaling technique that allows for enlarging the class of perturbations and for getting lower bounds for the error.
In the second part, we contribute towards the problem of robustness of stability of $\dot{x} = Ax$. To account for the uncertainties we consider linear but time-varying structured perturbations yielding the disturbed system

$$\dot{x} = Ax + BD(t)Cx \qquad x(0) = x_o \qquad (*)$$

We determine the real time-varying stability radius

$$r_{R,t} = \{||D||_{L^\infty} \ ; \ \text{the equilibrium of } (*) \text{ is not asymptotically stable}\}$$

for the linear oscillator by means of a special algorithm. Also we study its asymptotic behaviour for small dampings by using an averaging method. Finally we study n-dimensional systems under periodic perturbations and give a result which generalises the characterisation of destabilising perturbation from time-invariant to that of time-varying periodic perturbations.

Contents

INTRODUCTION

There are several situations in which a mathematical model may arise. Some of these situations are essentially mathematical in their nature and lead to theoretical objects in a somewhat abstract and precise way. Others, however, may lead to a model that contains uncertain parameters. This is the case in many applications in control engineering and other areas where usually the uncertainty is a result of some approximation or simplification undertaken on originally more sophisticated models of real phenomena. In this way, order reduction, linearisation around equilibria of nonlinear system, finite dimensional approximation of infinite dimensional problems, and many other practical situations will produce mathematical models bearing some kind of uncertainty on the knowledge of most of their parameters. Ultimately, a model is a representation of the real world and as such it is necessarily an idealisation of the real dynamics. The essence of the robustness approach is to interpret the real system (reality) as a perturbation of the ideal one. Thus, the question of robustness of a model plays a fundamental role on the problem of designing a controller for a plant. [1]

The focus of this dissertation is rather on the robustness issues of conditioning of controllability and stability radii for finite dimensional deterministic systems. The mathematical model, referred to as the

[1]Here, the term robustness is used in the sense that "a controller is said to be *robust* if it works well for a large class of perturbed systems." (Hinrichsen-Pritchard [12])

nominal system, will be the abstract differential equations

$$\dot{x} = Ax + Bu \quad x(0) = x_o \tag{0.1}$$

for the controllability problem (B is a given real $n \times m$ matrix, u the input function and $x_o \in \mathbf{R}^n$ the initial condition) and

$$\dot{x} = Ax \qquad x(0) = x_o \tag{0.2}$$

for the stability study.

The work on conditioning of controllability provides a new approach, inaugurated in Pijnacker-Pritchard-Townley[19], to the problem of designing controllers with a desired performance in face of uncertainties. Contributions to the more theoretical issue of continuous metrics (as opposed to discrete metrics of the type yes/no) and the distance between "controllable" and "uncontrollable" appeared as early as 1986 (v. Eising [20] and [21]), but in this dissertation the problem is posed somewhat in a more practical fashion as that of giving estimates for the error between the final states of a nominal system and a system that accounts for its uncertainties.

Concerning the problem of robustness of stability, Doyle-Stein[25] emphasised the importance of explicit uncertainty models such as multiplicative and additive and towards the beginning of the 1980's there was a renewed interest in frequency domain to address the problem of feedback design to provide perfomance in the face of uncertainties (Doyle-Stein[25], Postelthwaite-Edmunds-McFarlaine[26], Cruz-Freudenberg-Looze[27], for instance). However, most of these results do not exploit information about the structure of a perturbation. Doyle[24] introduced the μ-analysis for systems with structured uncertanties, a general approach to norm-bounded perturbation problems with arbitrary structure, which was based on an extension of singular values techniques. This method has motivated a series of works on robustness where the use of structural information is essential. Most of those works centered on single-input, single-output (scalar) systems and, although the study of multivariable control systems has begun by the same time (v. Safonov[22] and the special issue [23]), one can argue that with state-space methods there is less need for explicitating a distinction between scalar and multivariable systems. In Hinrichsen-Pritchard[8] and [9], state-space technique were brought into the issue of robustness of stability by means of the concept of real and complex stability radii which asserts the problem as that of finding a minimum norm destabilising structured perturbation belonging to certain classes. Following the lines of Hinrichen and Pritchard, the present work attempts to give some contributions towards a characterisation of real stability radius for a class of linear but time-variyng structured perturbations.

The dissertation is organised as follows. Chapter 1 establishes an estimate for the error between the final states of the nominal system and of the system disturbed by multi-structured perturbations of the form

$$\dot{z}(t) = Az(t) \; + \; \sum_{i=1}^{r} D_i F_i(C_i z(t), t) \; + \; Bu \tag{0.3}$$

which accounts for the uncertainties on the entries of the matrix A. The perturbations $F_i : \mathbf{R}^{p_i} \times [O, T] \longrightarrow \mathbf{R}^{q_i}$ are assumed to be Lipschitz maps.

Some weight functions are introduced in the problem, requiring a special topological framework and some weighted operators to be brought in. The result is translated into an inequality of the form

$$|E(T)| \leq \frac{\gamma_o \; ||M_\alpha||}{1 - \gamma_o \; ||L_\alpha||} \; ||C_\alpha x||$$

where the operators M_α, C_α and L_α are properly introduced in terms of the weight $\alpha = (\alpha_1, \cdots, \alpha_r)$. It should be noted that the novelty of introducing the weights allows for a subsequent improvement on the estimates and on the class of perturbation.

The evaluation of the norms $||M_\alpha||$ and $||C_\alpha x||$ does not present much conceptual difficulty, but with respect to $||L_\alpha||$ the situation is different. Chapter 2 provides a characterisation of the norm of the operator L_α by means of introducing a linear quadratic optimal control problem. The characterisaton is then obtained in terms of a parametrised differential Riccati equation and, equivalently, via a Hamiltonian approach. The question of computational algorithms for evaluating and minimising $||L_\alpha||$ is also assessed. In chapter 3 we consider the problem of minimising $||C_\alpha x||$ with respect to the input u. For this it is used the technique from Functional Analysis of reducing a constrained problem into an unconstrained one by means of Lagrange multipliers.

The second part of the work concerns the problem of robustness of stability. Chapter 4 introduces what will be the focus of the following chapters, namely the concept of stability radii. In chapter 5 we use a result from the theory of dynamical systems and implement an algorithm culminating in the exact evaluation of $r_{R,t}$ for the damped linear oscillator. The time-varying real stability radius $r_{R,t}$ is defined as the infimum of all $||D||_{L^\infty}$ such that the equilibrium of $\dot{x} = Ax + BD(t)Cx$ is not asymptotically stable. This result is complemented in chapter 6 with an asymptotic study of $r_{R,t}$ for small dampings by means of a slight modification of the classical averaging method.

An outcome of the study of the oscillator is that $r_{R,t}$ can be made strictly less than the time-invariant real stability radius and at the boundary between the regions of stability and instability we can have the combination of a periodic perturbation yielding a periodic solution of the system. Inspired by this result, in chapter 7 we address the case of periodic perturbations to culminate in a generalisation of the time-invariant characterisation of destabiling perturbations to the case of periodic ones.

Finally, chapter 8 constitutes a conclusion to the work and presents some comments as well as some directions for further research.

In any academic relationship there is always an interplay of power, scopes and interests, even if tacit or unnoticed. My supervisor's interest has basically been on scientific results while mine have sometimes pointed to recognising the cognitive process underlying a research activity. I would gratefully concede that the nature and scope of the present work is a result of the supervisor's sound guidance. And the circumstances plus the turbulence of collateral thoughts, feelings and failures have been so mine! This comment brings about two crucial aspects of a thesis which are usually overlooked by the academic community: that it is ultimately a text, and that it is written by a student (a learner).

It is commonly mentioned that usually you do not finish a work such as a music record or an academic thesis. You abandon it when it is assumed that it has reached a fair status. Basically, you want to evolve to other (or further) enquiries or rather you are led to its completion due to unbearable pressures to finish it off. However, it is at the end of such work that we feel that it would have been then the right moment to start it on. But how many times do we need until we learn to recognise when the time is next to us? Anyway, at this point there is often an awareness of what was going on while it happened and a willingness to carry out some new enterprise that is to some extent the continuation of what was done, even if in a totally different scope and form.

There is a huge distance between the idealisation of a work and its accomplishment. Since my time as an undergraduate student I have felt something very frustrating whenever we undertake some task. Before starting it off we urge ourselves with enthusiasm like that of a child. We make plans and conceive projects which would virtually lead to the creation of a masterpiece, so daring and ambitious and complete are they. But indeed, things start running differently. Some excuses arise to put the work off which hasten it towards the deadline. Unexpected elements in the contents which our naïveté or lack of knowledge did not consider at the beginning. A friend who calls for our attention in a crucial moment (or it is us who call for the friend). Someone who falls ill, the little niece who arrives for silly conversations. The gas which starts leaking. The money which we run out of and the debts which increase.

Then we do what is possible, hurry what we can, alter the project, catch up with time, put off sleeping!... Then, the work is finished at last. It turns out to be exactly of our size, smaller than the idealisation and with the taste of time. If it was not for this rivalry between the mind and the arms, the mind and the heart, and the heart and the ghosts, life would be less painful. But I think I know the end of the story: the heart wins over only to give in afterwards. However, when the work is called finished, our heart is uplifted by the feeling of "mission accomplished". We smile to ourselves and make the vow that next time everything will be different and hopefully better.

Chapter 1

CONDITIONING OF CONTROLLABILITY

This chapter deals with the problem of giving upper bounds for $|x(T) - z(T)|$, the error between the final state of a nominal finite dimensional system $\dot{x} = Ax + Bu$ and that of the one disturbed by multiple structured perturbations of the form

$$\dot{z}(t) = Az(t) + \sum_{j=1}^{r} D_j F_j (C_j z(t), t) + Bu(t)$$

for the same initial condition.

The novelty here is twofold: we consider multiple structured perturbations and the class of perturbations is envisaged by means of some artefact which allows for a tightening of the bounds.[1]

1.1 Introduction

We shall consider a linear multivariable control system with m inputs described in state space form by the following set of first order linear ordinary differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad t \in [0, T] \tag{1.1}$$

[1]A slightly different version of this chapter appeared in Botelho-Pritchard[34].

where $T > 0$ and the matrices $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ (with m and n being positive integers) are given so that the system (1.1), denoted (A, B) for short, is *controllable.*

For a given initial state $x(0) = x_o \in \mathbf{R}^n$ and any time $t \in [0, T]$, the coordinates of $x(t)$ are

$$x(t) = e^{At}x_o + \int_o^t e^{A(t-s)}Bu(s)ds$$

After it was shown that the traditional discrete methods providing yes- no answers for testing the controllability of a system might lead to wrong conclusion, some efforts have been made in order to overcome this drawback by considering the distance between a system and the set of uncontrollable systems (Eising [20],[21]). However, this approach is more of theoretical importance than of practical relevance since in general the purpose of a nominal model is to enable one to design some suitable control to implement on the real system. Thus, once a control is designed in order to steer a system from a given initial state to a desired final state, a number of circumstances like considering simplified models to represent plants with complicated dynamic behaviour, or the lack of precise knowledge of the values of the parameters involved, in short the presence of uncertainties on the system, will imply that it may not result in the desired performance. So it is useful to have an estimate of the error between the final states of the nominal and the perturbed system which accounts for the uncertainties. We can apply the same method introduced in Pijnacker-Pritchard-Townley[19] and consider time-invariant perturbations yielding a perturbed system of the form

$$\dot{z} = Az + D\Delta Cz + Bu$$

Then we can show that

$$||L||\,\gamma_o < 1 \Longrightarrow |x(T) - z(T)| \leq \frac{\gamma_o||M||}{1 - \gamma_o||L||}||Cx||$$

where $\gamma_o = ||\Delta||$ and

$$\begin{aligned} Mv &= \textstyle\int_0^T e^{A(T-s)}Dv(s)ds \\ Lv\,(t) &= \textstyle\int_0^t Ce^{A(t-s)}Dv(s)ds \end{aligned}$$

This interesting result nonetheless presents some shortcomings, namely

1. We have single structured perturbations. So it does not account for disturbances on all entries of A.

2. The class of perturbation is restricted to the time-invariant ones only.

3. There is no flexibility allowed on the sufficient condition. Therefore, the result is inconclusive if $\gamma_o \geq ||L||^{-1}$.

We shall present here a generalisation of this theory by considering multiple-structured, possibly nonlinear, time-varying perturbations. Hence we avoid the first two shortcomings above. To cope with the third, we bring about a novelty in the method of approaching the proof by introducing weight-functions $\alpha = (\alpha_1, \ldots, \alpha_r)$. Thus we shall prove the following conditioning of controllability:

$$||L_\alpha||\gamma_o < 1 \Longrightarrow |x(T) - Z(T)| \leq \frac{\gamma_o||M_\alpha||}{1 - \gamma_o||L_\alpha||}||C_\alpha x||$$

where γ_o is a constant and $M_\alpha, L_\alpha, C_\alpha$ are now maps which depend on the weight-functions.

This result allows the designer some flexibility to seek convenient α_i to reduce the value of $||L_\alpha||$. The significance of this is that one can find $\tilde{\alpha}$ such that $||L_{\tilde{\alpha}}||\gamma_o < 1$ even when $||L||\gamma_o \geq 1$ or, more generally, $||L_\alpha||\gamma_o \geq 1$ for other choices of α. Furthermore, we can improve the estimate (i.e., tighten the upper bound) by first minimising $||L_\alpha||$ with respect to α and then, for the optimal $\tilde{\alpha}$, to minimise $||C_{\tilde{\alpha}}x||$ with respect to the input u.

In order to account for the uncertainties of the system, assume perturbed systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t), t) + Bu(t) \tag{1.2}$$

where $f : \mathbf{R}^n \times [0, T] \longrightarrow \mathbf{R}^n$ is given by multiple nonlinear mappings given by

$$f(x(t), t) = \sum_{i=1}^{r} D_i F_i(C_i x(t), t) \tag{1.3}$$

where each $D_i \in \mathbf{R}^{n \times q_i}$ and $C_i \in \mathbf{R}^{p_i \times n}$, with $p_i, q_i \in \mathbf{N}$, are given matrices.

Each perturbation $F_i : \mathbf{R}^{p_i} \times [0,T] \longrightarrow \mathbf{R}^{q_i}$ is assumed to be Lipschitz in the sense that there exists a positive number γ_i such that, for all $x, y \in \mathbf{R}^{p_i}$,

$$|F_i(x,t) - F_i(y,t)|_{R^{q_i}} \leq \gamma_i |x - y|_{R^{p_i}}$$

and we shall denote

$$\gamma_o = \max_{1 \leq i \leq r} \gamma_i \tag{1.4}$$

In particular, note that F_i constant and $F_i(x,t) = x$ are classic examples of this class of perturbations.

Also we shall illustrate this scaling technique by considering an example for which we can determine analytically the error $|x(T) - z(T)|$ as a function of the magnitute of the perturbation:

Example 1.1.1 (Linear undamped oscillator) *As the nominal system, consider the case of small oscillations of a frictionless pendulum near its equilibrium position. Assume that we have zero initial conditions and the system is subjected to a constant unitary force during the time interval* $[0, \pi]$*:*

$$\begin{array}{l} \ddot{\varphi}(t) + \varphi(t) = 1 \quad t \in [0,\pi] \\ \varphi(0) = \dot{\varphi}(0) = 0 \end{array}$$

whose matrix representation is $\dot{x} = Ax + Bu$ *with* $u(t) \equiv 1$,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

On the other hand, suppose we have uncertainty in the nominal values for the "spring constant" and "damping factor" so that the perturbed system is taken to be of the form

$$\begin{aligned} \dot{z}(t) &= Az(t) + \left(\sum_{k=1}^{2} D_k \delta_k C_k z(t) \right) + Bu(t) \qquad (1.5) \\ z(0) &= 0 \end{aligned}$$

with δ_1, δ_2 being two real numbers and

$$D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 = [1 \quad 0] \quad C_2 = [0 \quad 1]$$

The solution of the nominal system is

$$x(t) = \begin{bmatrix} 1 - \cos t \\ \sin t \end{bmatrix}$$

so that

$$x(\pi) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Concerning the perturbed system, note that it reduces to

$$\begin{array}{l} \ddot{\varphi}(t) - \delta_2\dot{\varphi}(t) + (1-\delta_1)\varphi(t) = 1 \\ \varphi(0) = \dot{\varphi}(0) = 0 \end{array}$$

whose solution will depend on the nature of the roots

$$\frac{\delta_2}{2} \pm \sqrt{\left(\frac{\delta_2}{2}\right)^2 - (1-\delta_1)}$$

of the characteristic equation. Since we expect the perturbations to be "small" and hence $\delta_2^2 < 4(1-\delta_1)$, the oscillatory solutions of the perturbed system are given by

$$z(t) = \begin{bmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} (a^2+w^2)^{-1}(e^{at}(a \ \sin wt - w \ \cos wt) + 1) \\ e^{at} \sin wt \end{bmatrix}$$

where $a = (\delta_2/2)$ and $w = \sqrt{(1-\delta_1) - a^2}$.

1.2 The scaling technique

Let $E : [0,T] \longrightarrow \mathbf{R}^n$ denote the difference $E = z - x$, where x is the solution of the nominal system and z the solution of the perturbed system with $x(0) = z(0) = x_o$, for any given $x_o \in \mathbf{R}^n$. By differentiating, we have the initial value problem

$$\begin{aligned} \dot{E}(t) &= AE(t) + \sum_{i=1}^{r} D_i F_i(C_i z(t), t) \\ E(0) &= 0 \end{aligned} \tag{1.6}$$

Consider the integral form of (1.6), namely

$$E(t) = z(t) - x(t) = \int_0^t e^{A(t-s)} \left(\sum_{j=1}^{r} D_j F_j(C_j z(s), s) \right) ds \tag{1.7}$$

In order to study the problem within a concise formulation, it is convenient to set up a tidy framework to work within.

Let

$$\begin{aligned} v_i(.) &= C_i z(.) \\ y_i(.) &= C_i x(.) \end{aligned}$$

From equation (1.7), we have

$$v_i(t) = y_i(t) + C_i \int_0^t e^{A(t-s)} \left(\sum_{j=1}^{r} D_j F_j(v_j(s), s) \right) ds \tag{1.8}$$

for $t \in [0,T]$ and $i = 1, 2, \ldots, r$.

At this point we artificially introduce some weight functions.

Consider $\alpha_1, \ldots, \alpha_r$ any real valued positive continuous functions defined on the closed interval $[0,T]$

Thus, equation (1.8) can be rewriten as follows:

$$\alpha_i(t) v_i(t) = \alpha_i(t) y_i(t) \; + \tag{1.9}$$

$$+\alpha_i(t)C_i\int_0^t e^{A(t-s)}\sum_{j=1}^{r}\frac{1}{\alpha_j(s)}D_j\alpha_j(s)F_j\left(\frac{1}{\alpha_j(s)}\alpha_j(s)v_j(s),s\right)ds$$

Once the input u is fixed ($u(.)\in L^\infty$, say), each $\alpha_i(.)y_i(.)$ is also fixed in $L^2[0,T;\mathbf{R}^{p_i}]$. Thus, our first concern is to guarantee the existence of $z\in L^2[0,T;\mathbf{R}^n]$ so that

$$v_\alpha=(\alpha_1v_1,\alpha_2v_2,\ldots,\alpha_rv_r)=(\alpha_1C_1z,\alpha_2C_2z,\ldots,\alpha_rC_rz)$$

satisfies the above system of equations.

For any given r-tuples

$$\begin{array}{ll}\alpha=(\alpha_1,\ldots,\alpha_r) & \alpha_i\in C[0,T;a,b]\\ p=(p_1,\ldots,p_r)\in\mathbf{N}^r & \\ q=(q_1,\ldots,q_r)\in\mathbf{N}^r & \end{array}$$

we construct the following linear spaces structures:

1. Let $\mathbf{R}^p=\mathbf{R}^{p_1}\times\cdots\times\mathbf{R}^{p_r}$.

   For convenience, we will write either $a=\begin{bmatrix}a_1\\ \vdots\\ a_r\end{bmatrix}$ or $a=(a_1,\ldots,a_r)$ for an element $a\in\mathbf{R}^p$.

   On $\mathbf{R}^p$, we shall take the usual inner product

$$<(\mu_1,\cdots,\mu_r),(\nu_1,\cdots,\nu_r)>_{R^p}=\sum_{i=1}^{r}<\mu_i,\nu_i>_{R^{p_i}}$$

2. Let $L^{2,p}$ denote the following product of Hilbert spaces

$$L^{2,p}=L^2[0,T;\mathbf{R}^{p_1}]\times\cdots\times L^2[0,T;\mathbf{R}^{p_r}]$$

   with the inner product

$$<(g_1,\cdots,g_r),(h_1,\cdots,h_r)>_{L^{2,p}}=\sum_{i=1}^{r}<g_i,h_i>_{L^{2,p_i}}$$

L^{2,p_i} is a short notation for $L^2[0,T;\mathbf{R}^{p_i}]$,

We have that $L^{2,p}$ is a Hilbert space with

$$||g||^2_{L^{2,p}} = \sum_{i=1}^{r} ||g_i||^2_{L^{2,p_i}}$$

3. We define $\mathbf{R}^q$ and $L^{2,q}$ similarly.

   Also, for consistency of notation we put, say,

$$L^{r,s} = L^r[0,T;\mathbf{R}^s] \quad \text{for } s \in \mathbf{N} \text{ and } 1 \leq r \leq \infty$$

Now, we move on to introduce some operators.

Let $F_\alpha : \mathbf{R}^p \times [0,T] \longrightarrow \mathbf{R}^q$ be given by

$$F_\alpha(\mu,t) = \begin{bmatrix} \alpha_1(t)F_1\left(\alpha_1(t)^{-1}\mu_1,t\right) \\ \vdots \\ \alpha_r(t)F_r\left(\alpha_r(t)^{-1}\mu_r,t\right) \end{bmatrix}$$

for all $\mu = (\mu_1,\ldots,\mu_r) \in \mathbf{R}^p$, and define the operator $\tilde{F}_\alpha$ on $L^{2,p}$ by putting

$$\left(\tilde{F}_\alpha g\right)(t) = F_\alpha(g(t),t) \quad \forall g = (g_1,\cdots,g_r) \in L^{2,p}$$

The Lipschitz assumption on F_i is translated into a nice property for $\tilde{F}_\alpha$:

Proposition 1.2.1

$\tilde{F}_\alpha$ is a map from $L^{2,p}$ into $L^{2,q}$ satisfying the Lipschitz condition

$$||\tilde{F}_\alpha g - \tilde{F}_\alpha h||_{L^{2,q}} \leq \gamma_o||g-h||_{L^{2,p}} \quad \forall g,h \in L^{2,p}$$

Proof:
Given any $g = (g_1, \ldots, g_r)$ and $h = (h_1, \ldots, h_r)$ in $L^{2,p}$, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^{r} \int_0^T \left|\alpha_i(t)F_i\left(\alpha_i(t)^{-1}g_i(t),t\right) - \alpha_i(t)F_i\left(\alpha_i(t)^{-1}h_i(t),t\right)\right|^2 dt \leq \\
&\quad = \textstyle\sum_{i=1}^{r} \int_0^T \alpha_i(t)^2 \left|F_i\left(\alpha_i(t)^{-1}g_i(t),t\right) - F_i\left(\alpha_i(t)^{-1}h_i(t),t\right)\right|^2 dt \\
&\quad \leq \textstyle\sum_{i=1}^{r} \gamma_i^2 \int_0^T |g_i(t) - h_i(t)|^2 dt
\end{aligned}$$

from which the result follows. □

By introducing a few more definitions, the system of equations generated by (1.9) can be written in a concise form as

$$v_\alpha(t) = y_\alpha(t) + C_\alpha(t) \int_0^t e^{A(t-s)} D_\alpha(s)(\tilde{F}_\alpha v_\alpha)(s)ds \tag{1.10}$$

where

$$v_\alpha(t) = \begin{bmatrix} \alpha_1(t)v_1(t) \\ \vdots \\ \alpha_r(t)v_r(t) \end{bmatrix} \qquad y_\alpha(t) = \begin{bmatrix} \alpha_1(t)y_1(t) \\ \vdots \\ \alpha_r(t)y_r(t) \end{bmatrix}$$

For each t, we define

$$C_\alpha(t) : \mathbf{R}^n \longrightarrow \mathbf{R}^p \quad \text{and} \quad D_\alpha(t) : \mathbf{R}^q \longrightarrow \mathbf{R}^n$$

by setting

$$C_\alpha(t)x = \begin{bmatrix} \alpha_1(t)C_1 x \\ \vdots \\ \alpha_r(t)C_r x \end{bmatrix} \qquad \forall x \in \mathbf{R}^n$$

and for all $\mu \in \mathbf{R}^q$,

$$D_\alpha(t)\mu = \begin{bmatrix} \alpha_1(t)^{-1}D_1 & \cdots & \alpha_r(t)^{-1}D_r \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_r \end{bmatrix} = \sum_{i=1}^{r} \alpha_i(t)^{-1} D_i \mu_i$$

It is easy to check that the adjoint operators

$$C_\alpha(t)^* : \mathbf{R}^p \longrightarrow \mathbf{R}^n \quad \text{and} \quad D_\alpha(t)^* : \mathbf{R}^n \longrightarrow \mathbf{R}^q$$

are given respectively by

$$C_\alpha(t)^*\ \nu = \sum_{i=1}^{r} \alpha_i(t) C_i^* \nu_i$$

$$D_\alpha(t)^*\ g = \begin{bmatrix} \alpha_1(t)^{-1} D_1^* g \\ \vdots \\ \alpha_r(t)^{-1} D_r^* g \end{bmatrix}$$

Also,

$$D_\alpha(t) D_\alpha(t)^*\ g = \sum_{i=1}^{r} \alpha_i(t)^{-2} D_i D_i^* g$$

$$C_\alpha(t)^*\ C_\alpha(t)\ x = \sum_{i=1}^{r} \alpha_i(t)^2 C_i^* C_i x$$

Motivated by equation (1.10) we can define the operator L_α from $L^{2,q}$ into $L^{2,p}$ by setting, for all w in $L^{2,q}$,

$$L_\alpha w(t) = C_\alpha(t) \int_0^t e^{A(t-s)} D_\alpha(s) w(s) ds \tag{1.11}$$

or, equivalently,

$$L_\alpha w = \begin{bmatrix} \alpha_1 L_{11} \alpha_1{}^{-1} & \cdots & \alpha_1 L_{1r} \alpha_r{}^{-1} \\ \vdots & & \vdots \\ \alpha_r L_{r1} \alpha_1{}^{-1} & \cdots & \alpha_r L_{rr} \alpha_r{}^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}$$

so that

$$(L_\alpha w)_i = \sum_{j=1}^{r} \alpha_i(.) L_{ij} \alpha_j(.)^{-1} w_j \tag{1.12}$$

with

$$L_{ij} u\,(t) \;=\; C_i \int_0^t e^{A(t-s)} D_j u(s) ds \quad \forall u \in L^{2,q_j}$$

Note that each component $(L_\alpha w)_i$ is the classic input-output operator, which is a bounded linear mapping from L^{2,p_i} into L^{2,q_i}. Therefore, L_α is a bounded linear operator from $L^{2,q}$ into $L^{2,p}$. This follows from Young's theorem and the fact that $g \in L^2[0,T;\mathbf{R}^n]$ when $w \in L^{2,q}$, where

$$g(t) = \sum_{j=1}^{r} \frac{1}{\alpha_j(t)} D_j w_j(t) \;\in\; \mathbf{R}^n \tag{1.13}$$

We take

$$||L_\alpha|| = \sup_{v\neq 0} \frac{||L_\alpha v||_{L^{2,p}}}{||v||_{L^{2,q}}}$$

Note that when $\alpha_1(t) = \ldots = \alpha_r(t) \equiv \mu$, for some positive constant μ, we have that $||L_\alpha||$ is invariant with respect to μ.

Also, since each L_{ij} is compact, this implies that L_α is also compact.

The proposition below provides the expression for L_α^*.

Proposition 1.2.2 *The adjoint $L_\alpha^* : L^{2,p} \longrightarrow L^{2,q}$ is given by*

$$L_\alpha^* y(s) = D_\alpha(s)^* \int_s^T e^{A^*(t-s)} C_\alpha(t)^* y(t) dt \tag{1.14}$$

Proof: For any $w \in L^{2,q}$ and $y \in L^{2,p}$,

$$\begin{aligned} &< y, L_\alpha w >_{L^{2,p}} = \textstyle\sum_{i=1}^r < y_i, (L_\alpha w)_i >_{L^{2,p_i}} \\ &= \textstyle\sum_{i=1}^r \int_0^T < y_i(t), \alpha_i(t) C_i \int_0^t e^{A(t-s)} g(s) ds >_{R^{p_i}} dt \end{aligned}$$

so that

$$\begin{aligned} &< y_i, (L_\alpha w)_i >_{L^{2,p_i}} = \textstyle\int_0^T \int_0^t < e^{A^*(t-s)} \alpha_i(t) C_i^* y_i(t), g(s) >_{R^n} dsdt \\ &= \textstyle\int_0^T \int_s^T < e^{A^*(t-s)} \alpha_i(t) C_i^* y_i(t), g(s) > dtds \\ &= \textstyle\sum_j \int_0^T < \alpha_j(s)^{-1} D_j^* \int_s^T e^{A^*(t-s)} \alpha_i(t) C_i^* y_i(t) dt, w_j(s) >_{R^{q_j}} ds \\ &=< D_\alpha(s)^* \textstyle\int_s^T e^{A^*(t-s)} \alpha_i(t) C_i^* y_i(t) dt, w(s) >_{R^q} \end{aligned}$$

Therefore, we have

$$\begin{aligned} &< y, L_\alpha w >_{L^{2,p}} = \textstyle\sum_{i=1}^r < y_i, (L_\alpha w)_i >_{L^{2,p_i}} \\ &=< D_\alpha(s)^* \textstyle\int_s^T e^{A^*(t-s)} \left(\sum_{i=1}^r \alpha_i(t) C_i^* y_i(t)\right) dt, w(s) >_{R^q} \\ &=< D_\alpha(s)^* \textstyle\int_s^T e^{A^*(t-s)} C_\alpha(t)^* y(t), w(s) >_{R^q} \end{aligned}$$

□

In virtue of (1.14), we have

$$L_\alpha^* L_\alpha w\,(s) = D_\alpha(s)^* \int_s^T e^{A^*(t-s)} C_\alpha(t)^* C_\alpha(t) \left(\int_0^t e^{A(t-s)} D_\alpha(s) w(s) ds \right) dt$$

We can write equation 1.10 in a more convenient way as

$$v_\alpha = y_\alpha + L_\alpha(\tilde{F}_\alpha v_\alpha) \tag{1.15}$$

and this establishes the above equation as an equivalent representation for our previous system 1.6:

$$\begin{aligned} \dot{E}(t) &= AE(t) + \sum_{j=1}^{r} D_j F_j(C_j z(t), t) \\ E(0) &= 0 \end{aligned} \tag{1.16}$$

Finally, from what has been constructed, we have the following expression for $E(T)$:

$$E(T) = \int_0^T e^{A(T-s)} D_\alpha(s) F_\alpha(v_\alpha(s), s) ds$$

and we introduce the operator $M_\alpha : L^{2,q} \longrightarrow R^n$ by

$$M_\alpha w = \int_0^T e^{A(T-s)} D_\alpha(s) w(s) ds \tag{1.17}$$

One easily have that $M_\alpha^* : \mathbf{R}^n \longrightarrow L^{2,q}$ is given by

$$(M_\alpha^* g)(t) = \begin{bmatrix} \alpha_1(t)^{-1} D_1^* e^{A^*(T-t)} g \\ \vdots \\ \alpha_r(t)^{-1} D_r^* e^{A^*(T-t)} g \end{bmatrix}$$

for each $g \in \mathbf{R}^n$.

Such M_α is a bounded linear operator. In fact, for every w in $L^{2,q}$,

$$\begin{aligned} |M_\alpha w|_{R^n}^2 &\leq \left(\int_0^T |e^{A(T-s)} D_\alpha(s) w(s)|_{R^n} ds\right)^2 \\ &\leq e^{2\|A\|T} \left(\int_0^T \left|\sum_{j=1}^r \alpha_j(s)^{-1} D_j w_j(s)\right|_{R^n} ds\right)^2 \\ &\leq e^{2\|A\|T} \left(\sum_{j=1}^r \|D_j\| . (\max_s \alpha_j(s)^{-1}) . \int_0^T |w_j(s)| ds\right)^2 \\ &\leq e^{2|A\|T} . (\max_j \|D_j\|) . (\max\{\max_s \alpha_j(s)^{-1}\}) \, rT^2 . \|w\|_{L^{2,q}} \end{aligned}$$

where we have used the inequalities:

$$\begin{aligned} \left(\sum_{j=1}^r a_j\right)^2 &\leq r \sum_{j=1}^r a_j^2 \\ \left(\int_a^b |f(t)| dt\right)^2 &\leq (b-a) \left(\int_a^b |f(t)|^2 dt\right) \end{aligned}$$

both being consequences of the Cauchy-Schwartz inequality in different measure spaces.

Hence, we take

$$||M_\alpha|| = \sup_{w \neq 0} \frac{|M_\alpha w|_{R^n}}{||w||_{L^{2,q}}}$$

Remark 1.2.1: Eventually, an alternative way of establishing the framework for the theory in dealing with the weight-functions α is to develop the analysis using weighted spaces $L^2_{\alpha_i}$. Since the α_i are taken to be continuous on the compact interval $[0, T]$, the L^2 and $L^2_{\alpha_i}$ norms are equivalent because

$$\left(\min_{t \in [0,T]} \alpha_i(t) \right) ||u||^2_{L^2(0,T)} \leq ||u||^2_{L^2_{\alpha_i}(0,T)} \leq \left(\max_{t \in [0,T]} \alpha_i(t) \right) ||u||^2_{L^2(0,T)}$$

However, incorporating the weights on the operators seems to be more prevalent when the matter in question is to apply some scaling technique to tighten the upper bounds.

1.3 Upper bound for the error

From what we have seen, the existence of solution $E(t)$ for the problem 1.6 is equivalent to the existence of some v_α such that

$$v_\alpha = y_\alpha + L_\alpha(\tilde{F}_\alpha v_\alpha) \tag{1.18}$$

where y_α is fixed in $L^{2,p}$, and the final state $E(T)$ is given by

$$E(T) = M_\alpha(\tilde{F}_\alpha v_\alpha) \tag{1.19}$$

We shall make use of the classical contraction mapping argument in order to estimate the error $|E(T)| = |z(T) - x(T)|$. The following theorem is the main result of the chapter.

Theorem 1.3.1 (Conditioning of controllability)
Suppose

$$\gamma_o < \frac{1}{||L_\alpha||}$$

Then, we have

$$|z(T) - x(T)|_{R^n} \leq \frac{\gamma_o \|M_\alpha\|}{1 - \gamma_o \|L_\alpha\|} \|y_\alpha\|_{L^{2,p}} \tag{1.20}$$

where $y_\alpha = (\alpha_1 C_1 x, \ldots, \alpha_r C_r x) \in L^{2,p}$ *is fixed for each fixed input* u *in* $L^\infty[0, T; \mathbf{R}^m]$.

Proof: For y_α fixed, define the operator H_α from $L^{2,p}$ into itself by putting

$$H_\alpha(w) = y_\alpha + L_\alpha(\tilde{F}_\alpha w) \qquad \forall w \in L^{2,p}$$

We have, for every $g, h \in L^{2,p}$,

$$\begin{aligned} \|H_\alpha(g) - H_\alpha(h)\|_{L^{2,p}} &= \|L_\alpha(\tilde{F}_\alpha g - \tilde{F}_\alpha h)\|_{L^{2,p}} \\ &\leq \|L_\alpha\| . \|\tilde{F}_\alpha g - \tilde{F}_\alpha h\|_{L^{2,q}} \\ &\leq \|L_\alpha\| \gamma_o \|g - h\|_{L^{2,p}} \end{aligned}$$

Since $\|L_\alpha\|\gamma_o < 1$, it follows that H_α is a strict contraction.

Therefore, from the Contraction Mapping Theorem, it follows that there exists a unique $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_r) \in L^{2,p}$ such that $H_\alpha(\bar{w}) = \bar{w}$.

Define $v \in L^{2,p}$ by putting $v_i(t) = \alpha_i(t)^{-1} \bar{w}_i(t)$.

Then, $v_\alpha = y_\alpha + L_\alpha(\tilde{F}_\alpha v_\alpha)$ and

$$\begin{aligned} \|v_\alpha\|_{L^{2,p}} &\leq \|y_\alpha\|_{L^{2,p}} + \|L_\alpha\| \cdot \|\tilde{F}_\alpha v_\alpha\|_{L^{2,q}} \\ &\leq \|y_\alpha\|_{L^{2,p}} + \|L_\alpha\| \, \gamma_o \, \|v_\alpha\|_{L^{2,p}} \end{aligned}$$

and, since $\|L_\alpha\|\gamma_o < 1$, we have $(1 - \|L_\alpha\|\gamma_o)\|v_\alpha\|_{L^{2,p}} \leq \|y_\alpha\|_{L^{2,p}}$. So,

$$\|v_\alpha\|_{L^{2,p}} \leq \frac{1}{1 - \|L_\alpha\|\gamma_o} \|y_\alpha\|_{L^{2,p}}$$

On the other hand, by equation 1.19 we can write

$$|E(T)|_{R^n} \leq \|M_\alpha\| \gamma_o \|v_\alpha\|_{L^{2,p}}$$

and the result follows. □

Remark 1.3.1: This theorem generalises the cases of single structured perturbations and of time-invariant perturbations. In particular, for the case of single time-invariant perturbations, the conditioning result becomes:

$$|z(T) - x(T)| \leq \frac{\gamma_o \, ||M||}{1 - \gamma_o ||L||} ||Cx||$$

with γ_o being the operator norm of the perturbation. We shall refer to the above upper bound as *conditioning number* for this perturbation configuration. Accordingly, for our configuration we define

$$f_\alpha(\gamma_o) = \frac{\gamma_o \, ||M_\alpha||}{1 - \gamma_o ||L_\alpha||} ||C_\alpha x|| \tag{1.21}$$

and shall refer to it as the *weighted conditioning number*. A welcome aspect of theorem 1.3.1 is the presence of the functions $\alpha = (\alpha_1, \ldots, \alpha_r)$ which allows for an improvement both on the condition $\gamma_o ||L_\alpha|| < 1$, which may eventually be achieved even when $\gamma_o ||L|| \geq 1$, and on the weighted conditioning number by conveniently choosing α. This flexibility enables *at least in principle* the following optimisation strategy: first pick the optimal $\tilde{\alpha}$ which renders $||L_{\tilde{\alpha}}||$ minimal; then, for this choice of $\tilde{\alpha}$ find the optimal input such that $||C_{\tilde{\alpha}} x||$ is also minimal.

Remark 1.3.2: Concerning the evaluation of the norms involved,

(i) for $||C_\alpha x||$ we easily have

$$||C_\alpha x||^2 = \sum_{k=1}^{r} \int_0^T |\alpha_k(t) C_k x(t)|^2 dt$$

(ii) for $||M_\alpha||$, first note that

$$M_\alpha M_\alpha^* = \int_0^T \sum_{i=1}^{r} \frac{1}{\alpha_i(s)^2} e^{A(T-s)} D_i D_i^* e^{A^*(T-s)} ds \tag{1.22}$$

Thus, $M_\alpha M_\alpha^*$ is a linear transformation from $\mathbf{R}^n$ into itself and as such can be identified with a $n \times n$ real matrix. So the determination of $||M_\alpha||$ can be reduced to the problem of evaluating the operator norm of a matrix;

(iii) for $||L_\alpha||$, the situation is more complicated and we address this problem in next chapter.

Chapter 2

CHARACTERISATION OF $||L_\alpha||$

We provide a characterisation of the norm of the operator L_α by introducing a linear quadratic optimal control problem. The characterisaton is obtained in terms of a parametrised differential Riccati equation. The new framework derived from the multi-structured character of the perturbation and the presence of the weight-functions adds some new complications to the algebraic developments of the proof. Also it is shown the equivalence of this characterisation and another one in terms of Hamiltonian systems. The study is completed by assessing the question of a computational algorithm for evaluating $||L_\alpha||$. Finally we illustrate the result on the conditioning of controllability problems to the example of the undamped linear oscillator.

2.1 Introduction

To motivate the method pursued here, consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + D_\alpha(t)v(t) \quad x(0) = 0 \\ y(t) &= C_\alpha(t)x(t) \end{aligned} \tag{2.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$, A , C_α and D_α are the ones defined in the conditioning study.

For any given $v \in L^{2,q}$, the output y is given in $L^{2,p}$ by

$$y(t) = C_\alpha(t) \int_0^t e^{A(t-s)} D_\alpha(s) v(s) ds$$

so that

$$\begin{aligned} y &= L_\alpha v \\ \|y\|_{L^{2,p}} &\leq \|L_\alpha\| . \|v\|_{L^{2,q}} \end{aligned}$$

This enables us to write $\|L_\alpha\|^{-2} = \sup \mathcal{A}$ where

$$\mathcal{A} = \{\rho > 0 \, ; \, \|v\|^2_{L^{2,p}} - \rho \|y\|^2_{L^{2,q}} \geq 0 \, , \, \forall v \in L^{2,q} \, \}$$

where $y(t) = C_\alpha(t) x(t)$ and x is the solution of the system 2.1.

Note that y also depends on the input v since the state of the system is a function of the input. So we can introduce the following optimization problem on $L^{2,q}$, which is a Linear Quadratic Optimal Control Problem and we shall refer to it as (OP):

Minimise (properly or not) the cost functional given by

$$J_\rho(v) = \|v\|^2_{L^{2,q}} - \rho \|y\|^2_{L^{2,p}} \tag{2.2}$$

subject to the constraint:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + D_\alpha(t) v(t) \quad x(0) = \varphi \quad t \in [0, T] \\ y(t) &= C_\alpha(t) x(t) \end{aligned} \tag{2.3}$$

We should mention that a more clarifying notation for the cost should read $J_\rho(v; \varphi, [0, T])$, since the initial state and the time interval also determine the value of the cost functional.

The analysis of this problem follows the technique presented by Dietmar Salamon in his course Control Theory given at the University of Warwick in 1987.

We will provide a twofold characterisation of $||L_\alpha||$ because it brings about the resolution of a matrix Riccati equation which can be connected with Hamiltonian systems.

2.2 Solution of the optimal control problem

Let us introduce the operator $\mathcal{F}_\alpha : \mathbf{R}^n \longrightarrow L^{2,p}$ by putting $\forall\varphi \in \mathbf{R}^n$

$$(\mathcal{F}_\alpha\varphi)(t) = C_\alpha(t)e^{At}\varphi \tag{2.4}$$

Thus the output of (2.3) is

$$y = \mathcal{F}_\alpha\varphi + L_\alpha v \tag{2.5}$$

and, along the system (2.3), the functional J_ρ can be expressed as

$$\begin{aligned} J_\rho(v) &= < v, v >_{L^{2,q}} -\rho < \mathcal{F}_\alpha\varphi + L_\alpha v, \mathcal{F}_\alpha\varphi + L_\alpha v >_{L^{2,p}} \\ &= < v, v >_{L^{2,q}} -\rho\{< \mathcal{F}_\alpha\varphi, \mathcal{F}_\alpha\varphi >_{L^{2,p}} + \\ &\qquad +2 < \mathcal{F}_\alpha\varphi, L_\alpha v >_{L^{2,p}} + < L_\alpha v, L_\alpha v >_{L^{2,p}}\} \end{aligned}$$

Therefore, by solving $J_\rho'(v) = 0$ we obtain the necessary condition for the optimal input:

$$\tilde{v} = \rho L_\alpha^* \tilde{y} \tag{2.6}$$

$$\tilde{y} = \mathcal{F}_\alpha\varphi + L_\alpha\tilde{v} \tag{2.7}$$

The question now is to know if $I - \rho L_\alpha^* L_\alpha$ is invertible and positive definite, where I represents the identity map on $L^{2,q}$.In other words, to know if equations 2.6 and 2.7 have a unique solution. The next proposition shows that the answer is affirmative, provided that ρ is restricted.

Proposition 2.2.1 *Suppose $\rho < ||L_\alpha||^{-2}$. Then (OP) admits a unique solution.*

Proof: Given any $v \in L^{2,p}$,

$$||(I - \rho L_\alpha^* L_\alpha)v||_{L^{2,p}} \geq ||v||_{L^{2,p}} - \rho||L_\alpha^* L_\alpha v||_{L^{2,p}} \geq (1 - \rho||L_\alpha||^2)||v||_{L^{2,p}}$$

where $(1 - \rho||L_\alpha||^2) > 0$ does not depend on v. This shows that $I - \rho L_\alpha^* L_\alpha$ admits continuous inverse.

Also, we have that $I - \rho L_\alpha^* L_\alpha$ is positive definite, since

$$< (I - \rho L_\alpha^* L_\alpha)v, v >_{L^{2,p}} \geq (1 - \rho ||L_\alpha||^2)||v||_{L^{2,p}}^2.$$

□

Once we have established the existence and uniqueness of solution for the optimization problem, we might wish to obtain it explicitly. To do this, first notice that the optimal input is given by

$$\tilde{v}(s) = \rho(L_\alpha^* \tilde{y})(s) = -D_\alpha^*(s) \int_T^s e^{A^*(t-s)} \rho C_\alpha^*(t)\tilde{y}(t)dt$$

Now, denote

$$w(s) = \int_T^s e^{A^*(t-s)} \rho C_\alpha^*(t) y(t) dt \tag{2.8}$$

In particular, note that w is absolutely continuous and therefore is in $L^{2,n}$.

Thus, we have that equations 2.6 and 2.7 are equivalent to the following coupling, which we shall refer to as (CS):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + D_\alpha(t)v(t) \qquad x(0) = \varphi \\ y(t) &= C_\alpha(t)x(t) \end{aligned}$$

$$\begin{aligned} \dot{w}(t) &= -A^*w(t) + \rho C_\alpha(t)^* y(t) \qquad w(T) = 0 \\ v(t) &= -D_\alpha(t)^* w(t) \end{aligned}$$

and we can establish the existence and uniqueness of solution for our optimisation problem:

Theorem 2.2.2 *Suppose $\rho < ||L_\alpha||^{-2}$. Then, there exists a unique optimal control $\tilde{v} \in L^{2,q}$ for the problem (OP):*

Minimise $J_\rho(v; \varphi, [0, T]) = ||v||_{L^{2,q}}^2 - \rho||y||_{L^{2,p}}^2$
Subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + D_\alpha(t)v(t) \qquad x(0) = \varphi \in \mathbf{R}^n \\ y(t) &= C_\alpha(t)x(t) \end{aligned}$$

Moreover, $\tilde{v}$ is continuous and given by the feedback law

$$\tilde{v}(t) = -D_\alpha(t)^* P(t)x(t) \quad \forall t \in [0,T] \tag{2.9}$$

where each $n \times n$ matrix $P(t)$ is self-adjoint and defined by

$$< \varphi, P(t)\varphi >_{R^n} = \inf_{v \in L^{2,q}} J_\rho(v; \varphi, [0,T]) \tag{2.10}$$

In particular, $P(T) = 0$.

Proof: For the optimal cost we have (we will leave out the tildes for simplicity of notation):

$$\begin{aligned} J_\rho(v) &= < v, v > -\rho < y, y > \\ &= < v, v > -\rho < \mathcal{F}_\alpha \varphi + L_\alpha v, y > \\ &= < v, v - \rho L_\alpha^* y > -\rho < \mathcal{F}_\alpha \varphi, y > \end{aligned}$$

So,

$$J_\rho(v) = -\rho < \mathcal{F}_\alpha \varphi, y >_{L^{2,p}} \tag{2.11}$$

since $v - \rho L_\alpha^* y = 0$. On the other hand,

$$\begin{aligned} < \mathcal{F}_\alpha \varphi, y >_{L^{2,p}} &= \int_0^T < C_\alpha(t) e^{At}\varphi, y(t) >_{R^p} dt \\ &= < \varphi, \int_0^T e^{A^* t} C_\alpha(t)^* y(t) dt >_{R^n} \end{aligned}$$

which gives

$$\mathcal{F}_\alpha{}^* y = \int_0^T e^{A^* t} C_\alpha(t)^* y(t) dt$$

Using equations 2.8 and 2.11, we can write

$$J_\rho(v) = < \varphi, -\rho \mathcal{F}_\alpha{}^* y > = < \varphi, w(0) >$$

for the optimal cost. At this point, we can introduce the operator $P(0)$ from $\mathbf{R}^n$ into itself (which we shall identify with its matrix) by defining

$$P(0)\varphi = w(0) \qquad \forall \varphi \in \mathbf{R}^n \tag{2.12}$$

To write $P(0)$ explicitly in terms of the others operators involved, we recall that equations 2.6 and 2.7 combined yield $\mathcal{F}_\alpha\varphi = (I - \rho L_\alpha L_\alpha^*)y$. So, provided $\rho < ||L_\alpha||^{-2}$ we have

$$\begin{aligned} P(0)\varphi &= w(0) = -\rho\mathcal{F}_\alpha{}^*y \\ &= -\rho\mathcal{F}_\alpha{}^*(I - \rho L_\alpha L_\alpha^*)^{-1}\mathcal{F}_\alpha\varphi \end{aligned}$$

So, $P(0) = -\rho\mathcal{F}_\alpha{}^*(I - \rho L_\alpha L_\alpha^*)^{-1}\mathcal{F}_\alpha$, which is clearly a bounded linear operator on $\mathbf{R}^n$. Moreover, it is self- adjoint. Indeed, for any $h, g \in L^{2,p}$, put $H = (I - \rho L_\alpha L_\alpha^*)^{-1}h$ and $G = (I - \rho L_\alpha L_\alpha^*)^{-1}g$. Then,

$$\begin{aligned} < h, (I - \rho L_\alpha L_\alpha^*)^{-1}g > &= < (I - \rho L_\alpha L_\alpha^*)H, G > \\ &= < H, G > -\rho < H, L_\alpha L_\alpha^* G > \\ &= < H, (I - \rho L_\alpha L_\alpha^*)G > \\ &= < (I - \rho L_\alpha L_\alpha^*)^{-1}h, g > \end{aligned}$$

To procede to the definition of $P(t)$, for $t \neq 0$, first note that

$$v(0) = D_\alpha(0)^* w(0) = -D_\alpha(0)^* P(0)x(0)$$

Now, take $0 < \xi < T$ arbitrary and consider the problem:
Minimize $J_\rho(v; x(\xi), [\xi, T]) = ||v||^2_{L^{2,q}} - \rho||y||^2_{L^{2,p}}$
Subject to

$$\begin{aligned} &\dot{x} = Ax + D_\alpha v \quad \text{on } [\xi, T] \\ &y = C_\alpha x \end{aligned}$$

It follows from the uniqueness of the solution of (OP) that $v = -D_\alpha w$ restricted to $[\xi, T]$ is optimal for the problem above. Also, we have that the optimal cost is

$$J_\rho(v; x(\xi), [\xi, T]) =< x(\xi), w(\xi) >=< x(\xi), P(\xi)x(\xi) > \tag{2.13}$$

where $P(\xi)$ is defined by:

$$P(\xi)\varphi = w(\xi) = -\int_\xi^T e^{A^*(t-\xi)}\rho C_\alpha(t)^* y(t)dt \quad \forall\varphi \in R^n \tag{2.14}$$

with w being the solution of

$$\begin{aligned} \dot{x} &= Ax + D_\alpha v & x(\xi) = \varphi \\ y &= C_\alpha x \\ \dot{w} &= -A^* w + \rho C_\alpha^* y & w(T) = 0 \\ v &= -D_\alpha^* w \end{aligned}$$

on $[\xi, T]$. Finally, since ξ is arbitrary on the open interval $(0, T)$, we have defined a family of self-adjoint bounded operators $\{P(t)\}_{t\in[0,T]}$ with

$$w(t) = P(t)x(t) \qquad \forall t \in [0, T]$$

□

2.3 Riccati equation

As an important consequence of the last theorem, we can obtain a characterisation of $||L_\alpha||$ in terms of a parametrised differential Riccati equation on $[0, T]$, which we shall refer to as (DRE).

Corollary 2.3.1

(i) *If*

$$\rho < \frac{1}{||L_\alpha||^2}$$

then there exists a family $\{P(t)\}_{t\in[0,T]}$ of self-adjoint bounded linear operators (a posteriori, matrices) such that $P(.) \in C^1[0, T; R^{n\times n}]$ and $P(t)$ is the unique solution of the following (DRE):

$$\dot{P}(t) + A^*P(t) + P(t)A - \rho C_\alpha(t)^*C_\alpha(t) - P(t)D_\alpha(t)D_\alpha(t)^*P(t) = 0$$
$$P(T) = 0$$

(ii) *Conversely, if $P(.) \in C^1[0, T; R^{n\times n}]$ is such that $P(t)$ is a self-adjoint solution of (DRE) on $[0, T]$, then*

$$\rho \leq \frac{1}{||L_\alpha||^2}$$

Proof of (i) For $\rho < ||L_\alpha||^{-2}$, we define $P(.)$ as before by putting, for any fixed $\varphi \in R^n$ and $\xi \in [0, T]$,

$$P(\xi)\varphi = w(\xi)$$

where

$$\begin{aligned} \dot{x}(t) &= Ax(t) - D_\alpha(t)D_\alpha(t)^* w(t) & x(\xi) &= \varphi \\ \dot{w}(t) &= -A^* w(t) + \rho C_\alpha(t)^* C_\alpha(t)x(t) & w(T) &= 0 \end{aligned}$$

or, equivalently,

$$w(\xi) = -\int_\xi^T e^{A^*(t-\xi)} \rho C_\alpha(t)^* C_\alpha(t)x(t)dt$$

(Cf. equation 2.14).

Moreover, x given by $x(t)$ is continuous on $[\xi, T]$, so that w is differentiable on ξ, $\forall \xi \in [0,T]$. From this it follows that $P(.) \in C^1[0,T; R^{n\times n}]$ and

$$\begin{aligned} &\dot{P}(\xi)x(\xi) + P(\xi)\dot{x}(\xi) = \dot{w}(\xi) \\ &\dot{P}(\xi)x(\xi) + P(\xi)\{Ax(\xi) - D_\alpha(\xi)D_\alpha(\xi)^* P(\xi)x(\xi)\} = \\ &\quad = -A^* P(\xi)x(\xi) + \rho C_\alpha(\xi)^* C_\alpha(\xi)x(\xi) \\ &\{\dot{P}(\xi) + A^* P(\xi) + P(\xi)A - \rho C_\alpha(\xi)^* C_\alpha(\xi) - \\ &\quad -P(\xi)D_\alpha(\xi)D_\alpha(\xi)^* P(\xi)\}\ \varphi = 0 \end{aligned}$$

Since φ is arbitrary, (DRE) follows. Also, $P(T) = w(T) = 0$.

The uniqueness is a consequence of the theory of ordinary differential equations with initial (and, by a change of variables, final) conditions. □

Proof of (ii) Equation 2.14 gives the optimal cost:

$$< \varphi, P(0)\varphi >_{R^n} = \inf_{v \in L^{2,q}} J_\rho(v; \varphi[0,T]) \qquad \forall \varphi \in R^n$$

In particular, the case $\varphi = 0$ yields

$$0 = inf_v J_\rho(v; 0, [0,T]) \leq J_\rho(v; 0, [0,T]) \quad \forall v \in L^{2,q}$$

Thus, along

$$\begin{aligned} &\dot{x} = Ax + D_\alpha v \quad x(0) = 0 \\ &y = C_\alpha x \end{aligned}$$

it follows that

$$0 \leq ||v||^2_{L^{2,q}} - \rho||L_\alpha v||^2_{L^{2,p}}$$

$$\left(\frac{||L_\alpha v||_{L^{2,p}}}{||v||_{L^{2,q}}}\right)^2 \leq \frac{1}{\rho}$$

for every $v \neq 0$.

Hence, since $||L_\alpha|| = \sup_{v\neq 0} \frac{||L_\alpha v||_{L^{2,p}}}{||v||_{L^{2,q}}}$, the proof is complete. □

Remark 2.3.1: $||L_\alpha||$ is obtained iteratively by searching for the

smallest value of $\rho > 0$ which breaks down the regularity of the solution of the Riccati equation (more precisely, its boundedness). Of course, one can argue that when it comes to the computations, to detect this breaking down of boundedness brings in itself some pratical and logical difficulties. The approach via Hamiltonian systems, which we consider now, turns out to be more accessible for the numerical computations.

2.4 Hamiltonian systems

Another way to consider the coupling system (CS) is to rewrite it as

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & -D_\alpha(t)D_\alpha(t)^* \\ \rho C_\alpha(t)^* C_\alpha(t) & -A^* \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

This is brought up because a connection between Hamiltonian systems and matrix Riccati equations can be established and this is stated in the proposition below. When it comes to computations, the approach via Hamiltonian systems may turn out easier to tackle.

Recall that a linear Hamiltonian system is one that may be written as $\dot{V}(t) = JH(t)V(t)$ where $H(t)$ is a symmetric matrix which is continuous on an interval and

$$J = \begin{bmatrix} O_n & -I_n \\ I_n & O_n \end{bmatrix} \qquad V = \begin{bmatrix} X \\ W \end{bmatrix}$$

I_n is the identity matrix of order n and O_n is the $n \times n$-matrix with zeroes as entries.

Consider the matrix equation (HS):

$$\begin{bmatrix} \dot{X}(t) \\ \dot{W}(t) \end{bmatrix} = \begin{bmatrix} A & -D_\alpha(t)D_\alpha(t)^* \\ \rho C_\alpha(t)^* C_\alpha(t) & -A^* \end{bmatrix} \begin{bmatrix} X(t) \\ W(t) \end{bmatrix}$$

with final time conditions

$$\begin{bmatrix} X(T) \\ W(T) \end{bmatrix} = \begin{bmatrix} I_n \\ O_n \end{bmatrix}$$

where each $X(t)$ and $W(t)$ are $n \times n$ real matrices.

Theorem 2.4.1 (i) *If $P(.) \in C^1[0,T;R^{n\times n}]$ is such that $P(t)$ is the unique self-adjoint solution of the Riccati equation (DRE), then*

$$\begin{aligned} X(t) &= \Phi(t,T) \\ W(t) &= P(t)\Phi(t,T) \end{aligned}$$

is the unique solution of the Hamiltonian system (HS).

Here, $\Phi(.,.)$ is the evolution operator generated by

$$A - D_\alpha(.)D_\alpha(.)^* P(.)$$

(ii) *Conversely, if $\begin{bmatrix} X(t) \\ W(t) \end{bmatrix}$ is the C^1-solution of (HS) and $X(t)$ is invertible on $[0,T]$, then*

$$P(t) = W(t)X(t)^{-1}$$

is the self-adjoint C^1-solution of (DRE).

Proof of (i) Consider (HS) in the form

$$\begin{array}{ll} \dot{X}(t) = AX(t) - D_\alpha(t)D_\alpha(t)^* W(t) & X(T) = I \\ \dot{W}(t) = -A^* W(t) + \rho C_\alpha(t)^* C_\alpha(t) X(t) & W(T) = 0 \end{array}$$

If we write $\dot{X}(t) = (A - D_\alpha(t)D_\alpha(t)^* P(t))X(t)$, it follows that

$$||A - D_\alpha(t)D_\alpha(t)^* P(t)|| \leq ||A|| + ||D_\alpha(t)||.||P(t)||$$

So, since both $D_\alpha(.)$ and $P(.)$ are continuous on the compact $[0,T]$ and therefore bounded, it follows that

$$A - D_\alpha(t)D_\alpha(t)^* P(t)$$

generates the evolution operator $\Phi(t,s)$ satisfying

$$\dot{\Phi}(t,T) = (A - D_\alpha(t)D_\alpha(t)^* P(t))\Phi(t,T) \tag{2.15}$$
$$\Phi(T,T) = I \tag{2.16}$$

We claim that $X(t) = \Phi(t,T)$ and $W(t) = P(t)\Phi(t,T)$ solve the system (HS).

Indeed,

$$\dot{X}(t) = \dot{\Phi}(t,T) = (A - D_\alpha(t)D_\alpha(t)^* P(t))X(t) \tag{2.17}$$
$$\dot{W}(t) = \dot{P}(t)\Phi(t,T) + P(t)\dot{\Phi}(t,T) \tag{2.18}$$

Substituting the expressions for $\dot{P}(t)$ from (DRE) and $\dot{\Phi}(t,T)$ from (2.15) into equation 2.18, we can see that (HS) is satisfied. The final conditions are easily seen to be verified as well. □

Proof of (ii) Define

$$P(t) = W(t)X^{-1}(t)$$

Then we have that $P(T) = W(T) = 0$ and $P(.) \in C^1[0,T;\mathbf{R}^{n\times n}]$ whenever X and W are continuously differentiable. Therefore,

$$\dot{P}(t)X(t) + P(t)\dot{X}(t) = \dot{W}(t)$$

and, again, substituting the expressions for $\dot{X}(t)$ and $\dot{W}(t)$ from (HS) in the above equation, we have:

$$\begin{aligned}\{\dot{P}(t) + A^*P(t) + P(t)A - \rho C_\alpha(t)^* C_\alpha(t) - \\ -P(t)D_\alpha(t)D_\alpha(t)^* P(t)\}\, X(t) = 0\end{aligned}$$

Since $X(t)$ is invertible, it follows that $P(t)$ satisfies (DRE). To prove that $W(t)^*X(t)^{-1} = (X(t)^{-1})^* W(t)$ (i.e., $P(t)$ is self-adjoint), first note that this is equivalent to

$$X(t)^*W(t) = W(t)^*X(t)$$

Thus, consider h given by

$$h(t) = X(t)^*W(t) - W(t)^*X(t) \qquad \forall t \in [0,T]$$

By differentiating, we have

$$\dot{h}(t) = \left(\dot{X}(t)^* W(t) + X(t)^* \dot{W}(t)\right) - \left(\dot{W}(t)^* X(t) + W(t)^* \dot{X}(t)\right)$$

Now, inserting the expressions for $\dot{X}(t)$ and $\dot{W}(t)$ from (HS) and manipulating the simplifications, we get $\dot{h}(t) = 0$. So, since $h(T) = 0$, it follows that $h(t) = 0$, $\forall t \in [0, T]$. □

Corollary 2.4.2 (i) *If $\rho < ||L_\alpha||^{-2}$, then there exist $X(t)$ and $W(t)$ which solves (HS) uniquely, with each $X(t)$ invertible.*

(ii) *If $X(t)$ and $W(t)$ are the unique solution of (HS), with $X(t)$ invertible, then $\rho \leq ||L_\alpha||^{-2}$.*

Proof: Immediate. □

2.5 Numerical evaluation of $||L_\alpha||$

Corollary 2.4.2 indicates that if, for each fixed α, we solve (HS) backwards in time from the final condition $X(T) = I$, $W(T) = 0$, starting with small values of $\rho > 0$ to obtain $detX(t) \neq 0$, $\forall t \in [0, T]$, and keep on increasing ρ, the value of $||L_\alpha||^{-2}$ will be equal to the first ρ for which the solution of (HS) is such that $detX(\hat{t}) = 0$, for some $\hat{t} \in [0, T]$. Hence the computation of $||L_\alpha||$ can be done by scanning the values of $detX(t)$ along the interval $[0, T]$, for each ρ, until we find the first value of ρ which gives $detX(t) \neq 0$ for some t.

We can improve our insight into this algorithm that allows for the numerical evaluation of the norm of the input-output operator by trying to get a better understanding of it as a function of the time interval. For simplicity of reasoning, let us focus only on the case of single perturbation.

Consider the map $f : [0, T] \longrightarrow [0, \infty)$ defined by $f(\xi) = ||L_\xi||$, where, for each $\xi \in [0, T]$, L_ξ is the bounded linear operator from

$L^2[\xi, T; \mathbf{R}^p]$ into $L^2[\xi, T; \mathbf{R}^q]$ given by

$$L_\xi w\ (t) = \int_\xi^t K(t-s)w(s)ds$$

where $K(t-s) = Ce^{A(t-s)}D$. It is our purpose to study some properties of f so that we could be able to derive an algorithm for the evaluation of $||L|| = f(0)$ by means of iterations on the parameter ρ in either the Riccati equation or Hamiltonian characterisation of $||L||$ (cf. corollary 2.3.1 and theorem 2.4.1). Concerning such an algorithm, it is desirable that it could be useful for implementations on computers.

It is immediate from the definition that each L_ξ is bounded and the function f is non-increasing, i.e., $||L_{\xi_1}|| \le ||L_{\xi_2}||$ for $0 \le \xi_1 \le \xi_2 \le T$.

In Hinrichsen-Ilchmann-Pritchard [18] it was shown, for the case L_ξ from $L^2[\xi, \infty; \mathbf{R}^p]$ into $L^2[\xi, \infty; \mathbf{R}^q]$ defined similarly, that $||L_\xi||$ is constant, for all $\xi \in [0, \infty]$. The next proposition shows that the situation is different in our case.

Proposition 2.5.1
There exists a sequence (T_k) on $[0, T]$ converging to T such that

$$\lim_{k\to\infty} ||L_{T_k}|| = 0$$

Proof:
For each $k \in N$, let

$$T_k = \frac{k}{k+1}T$$

Take $w \in L^2[0, T; \mathbf{R}^p]$. In particular, $w \in L^2[T_k, T; \mathbf{R}^p]$ and we have:

$$||L_{T_k}w||^2 = \int_{T_k}^T || \int_{T_k}^t K(t-s)w(s)ds||^2 dt \le \gamma^2(T-T_k)^2 \int_{T_k}^T ||w(s)||^2 ds$$

where $\gamma = (||C||.||D||Mc_o)^2$.

Therefore, for each $k \in \mathbf{N}$,

$$0 \le ||L_{T_k}|| \le \gamma(T - T_k)$$

and the result follows.

□

Note that, as a straightforward consequence of Lebesgue's theorem of dominated convergence, we have that $\int_{T_k}^{T} ||w(s)||^2 ds \longrightarrow 0$ as $T_k \longrightarrow T$.

Also, proposition 2.5.1 and the fact that f is non-increasing yield that there exists $\delta > 0$ such that $f(\xi) < f(0)$, $\forall \xi \in (\delta, T)$.

So, the set $\mathcal{P}$ defined by

$$\mathcal{P} = \{\delta \in [0,T]\ ;\ ||L_\xi|| < ||L||\ ,\ \forall \xi \in (\delta, T)\}$$

is bounded and non-empty. So, let us denote $\hat{t} = \inf \mathcal{P}$.

It can be shown that $f(\hat{t}) = f(0)$ by using continuity arguments. This yields that the bounded set

$$\mathcal{H} = \{\mu \in [0,T]\ ;\ ||L_\xi|| = ||L||\ \forall \xi \in [0, \mu)\}$$

is non-empty and we have the following result:

Proposition 2.5.2

(i) $\hat{t} = \inf \mathcal{P} = \max \mathcal{H}$

(ii) *Suppose* $\mu \in \mathcal{H}$ *and* $||L_\mu \hat{w}|| = ||L_\mu||.||\hat{w}||$ *for some* $\hat{w} \in L^2[\mu, T; \mathbf{R}^m]$.

Then, w *defined by*

$$w = \begin{cases} 0 & on \quad [0,\mu) \\ \hat{w} & on \quad [\mu, T] \end{cases}$$

is such that $||Lw|| = ||L_\mu||.||\hat{w}||_{L^2[0,T;R^m]}$.

Proof of (i) Denote $\sup \mathcal{H} = \mu_o$.

If $\hat{t} < \mu_o$, then either $\mu_o \in \mathcal{P}$, which contradicts the fact that $\hat{t} \in \mathcal{H}$, or there exists $\bar{\mu} \in (\mu_o, T]$ such that $||L_{\bar{\mu}}|| = ||L||$. But the non-increasing character of f implies that $||L_\xi|| = ||L||$, $\forall \xi \in [0, \bar{\mu}]$. Hence, $\bar{\mu} > \mu_o$ and $\bar{\mu} \in \mathcal{H}$, which is a contradiction.

If $\hat{t} > \mu_o$, take m such that $\mu_o < m < \hat{t}$. Then, $m > \mu_o$ implies that

$$\exists \xi_m \in (\mu_o, m] \quad \text{such that} \quad ||L_{\xi_m}|| < ||L|| \tag{2.19}$$

and $m < \hat{t}$ implies that

$$\exists \theta_m \in (m, \hat{t}] \quad \text{such that} \quad ||L_\xi|| = ||L|| \ \forall \xi \in [0, \theta_m]$$

Therefore $\xi_m \in [0, \theta_m) \subset [0, T]$, which gives $||L_{\xi_m}|| = ||L||$. This contradicts 2.19

Finally, since $\sup \mathcal{H} = \hat{t}$ and $\hat{t} \in \mathcal{H}$, the result follows. □

Proof of (ii)

$$\begin{aligned} ||Lw||^2 &= \int_0^T \left\| \int_0^t K(t-s)w(s)ds \right\|^2 dt = \int_\mu^T \left\| \int_\mu^t K(t-s)\hat{w}(s)ds \right\|^2 dt = \\ &= ||L_\mu \hat{w}||^2 = ||L_\mu||^2 . ||\hat{w}||_{L^2[\mu,T;R^m]} \end{aligned}$$

□

It is important to note that if $\hat{t} = 0$ then f is strictly decreasing. Indeed, if

$$\hat{t} = 0 \quad \text{and} \quad ||L_\xi|| = ||L||, \ \forall \xi \in [0, \bar{\xi}]$$

for some $\bar{\xi}$ in $(0, T]$, then there would exist a $\bar{\delta} > 0$ such that

$$||L_\xi|| < ||L||, \ \forall \xi \in (\bar{\delta}, T]$$

with $0 < \bar{\delta} < \bar{\xi} + 0 \leq T$. This means that $\bar{\xi} \in (\bar{\delta}, T]$. Therefore, $||L_{\bar{\xi}}|| < ||L||$, which is a contradiction.

2.6 Example of application of the theory

Now we address example 1.1.1 in order to study the bounds for the error between the nominal and the perturbed systems. For this example we can obtain analytically the exact value of $E(\pi) = |x(\pi) - z(\pi)|$. The first interesting aspect of this double structured perturbation is that, for the same value of

$$\gamma_o = \max\{|\delta_1|, |\delta_2|\}$$

we have different values of $E(\pi)$ corresponding to different combinations of perturbations δ_1 (uncertainty on the spring factor) and δ_2 (uncertainty on the damping constant) that produce the same γ_o. The table below shows this feature of the response of the system, which we were able to notice ad hoc.

δ_1	δ_2	γ_o	$E(\pi)$
0.1	0.07	0.1	0.33
0.08	0.1	0.1	0.34
-0.1	0.05	0.1	0.18
0.0	-0.1	0.1	0.15
0.1	0.0	0.1	0.22

Figure 2.1: Some magnitudes of perturbations yielding the same $\gamma_o = 0.1$

We shall concentrate on small perturbations, say, $|\delta_i| \leq 0.1$, for which $\gamma_o = 0.1$.

Define

$$\Omega_{\gamma_o} = \{(\delta_1, \delta_2) \in [-0.1, 0.1] \times [-0.1, 0.1] \; ; \; \max_{i=1,2} |\delta_i| = \gamma_o\}$$

and let Ω'_{γ_o} be a subset of Ω_{γ_o} with a finite number of elements. Then we denote

$$E_{\gamma_o}(\pi) = \max\{E(\pi) \; ; \; (\delta_1, \delta_2) \in \Omega'_{\gamma_o}\}$$

and, for the weighted conditioning numbers,

$$K_\alpha = \frac{\gamma_o ||M_\alpha||}{1 - \gamma_o ||L_\alpha||} \, ||C_\alpha x||_{L^{2,2}}$$

With this notation, we have that $E_{\gamma_o}(\pi) = E_{0.1}(\pi) = 0.34$ for the set of values of perturbations of figure 2.6.

In order to compare the upper bounds K_α with this exact error $E_{\gamma_o}(\pi)$ we set

$$r_\alpha = \frac{K_\alpha}{E_{\gamma_o}}$$

We proceed to the relevant computations for the example.

Applying the algorithm for determining $||L_\alpha||$ we have, considering the Hamiltonian approach and after the change of variables $t = \pi - \tau$, the following initial value problem

$$\begin{array}{rcll} \dot{X}(\tau) & = & -AX(\tau) + \sum_{i=1}^{2} \alpha_i(\pi-\tau)^{-2} D_i D_i^* W(\tau) & X(0) = I_2 \\ \dot{W}(\tau) & = & \rho \sum_{i=1}^{2} \alpha_i(\pi-\tau)^2 C_i^* C_i X(\tau) + A^* W(\tau) & W(0) = 0_2 \end{array}$$

This yields the system of equations

$$\begin{array}{ll} \dot{X}_{11} = -X_{21} & X_{11}(0) = 1 \\ \dot{X}_{12} = -X_{22} & X_{12}(0) = 0 \\ \dot{X}_{21} = +X_{11} + (\beta_1^{-2} + \beta_2^{-2}) W_{21} & X_{21}(0) = 0 \\ \dot{X}_{22} = +X_{12} + (\beta_1^{-2} + \beta_2^{-2}) W_{22} & X_{22}(0) = 1 \end{array}$$

$$\begin{array}{ll} \dot{W}_{11} = -W_{21} - \rho\beta_1^2 X_{11} & W_{11}(0) = 0 \\ \dot{W}_{12} = -W_{22} - \rho\beta_1^2 X_{12} & W_{12}(0) = 0 \\ \dot{W}_{21} = +W_{11} - \rho\beta_2^2 X_{21} & W_{21}(0) = 0 \\ \dot{W}_{22} = +W_{12} - \rho\beta_2^2 X_{22} & W_{22}(0) = 0 \end{array}$$

where $\beta_i(\tau) = \alpha_i(\pi - \tau)$.

Proceding in an analogous manner, similar sets of differential equations can be obtained considering the Riccati equation approach:

$$\begin{array}{ll} \dot{p}_1 = -2p_2 - \rho\beta_1^2 - (\beta_1^{-2} + \beta_2^{-2}) p_2^2 & p_1(0) = 0 \\ \dot{p}_2 = -p_3 + p_1 - (\beta_1^{-2} + \beta_2^{-2}) p_2 p_3 & p_2(0) = 0 \\ \dot{p}_3 = 2p_2 - \rho\beta_2^2 - (\beta_1^{-2} + \beta_2^{-2}) & p_3(0) = 0 \end{array}$$

Using Runge-Kutta method, we can integrate either system to obtain $||L_\alpha||$. Obviously, for finite time problems the integration of the

Hamiltonian equation is the natural alternative, since it is logically sound to detect the vanishing of the determinant of a matrix as opposed to the blowing up of some matrix entry. For infinite time problems the right-hand side of the Riccati equation vanishes as t tends to infinity and the differential equation reduces to an algebraic one.

Figure 2.2 shows the behaviour of $detX(t)$ as ρ varies for the case $\alpha_1(t) = 1+0.3t$ and $\alpha_2(t) = 1.5-0.1t$. Note that, as you increase ρ from values near zero, $\rho_2 = 0.115$ is the first one that renders $detX(t) = 0$ for some t. Hence $||L_\alpha|| = (\rho_2)^{-1/2} = 2.95$ for this choice of weight α. A remarkable aspect of all these computations is that always the first time that $\det X(t)$ vanishes is at zero. So the experimentation gives $\hat{t} = 0$ in our theoretical considerations.

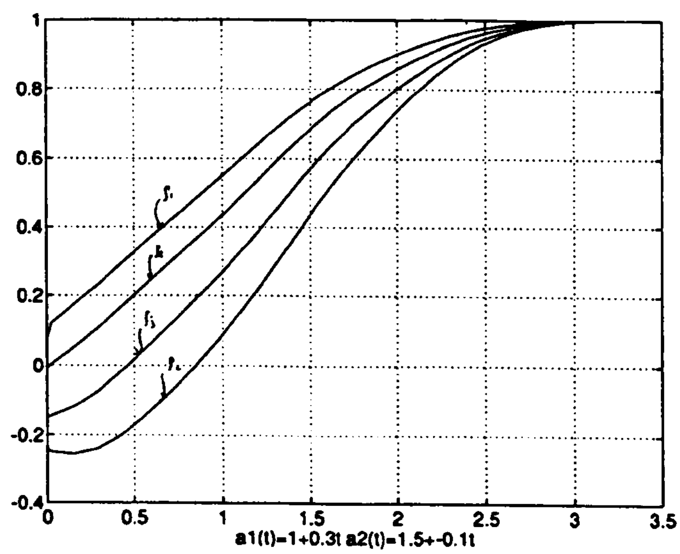

Figure 2.2: The graph $detX(t) \times t$ for successive values of ρ: $\rho_1 = 0.10$, $\rho_2 = 0.115$, $\rho_3 = 0.15$, $\rho_4 = 0.20$.

Case 1: Constant weights $\alpha_1(t) \equiv a$ and $\alpha_2(t) \equiv c$

For the determination of $||C_\alpha x||$ we have

$$||C_\alpha x||^2 = \int_0^\pi \left(\alpha_1(t)^2 x_1(t)^2 \ + \ \alpha_2(t)^2 x_2(t)^2\right) dt$$

where $x = (x_1, x_2) \in L^2[0, \pi; \mathbf{R}^2]$ the solution of the nominal system.

Thus,

$$||C_\alpha x|| = \left(a^2 \int_0^\pi (1-\cos t)^2 dt + c^2 \int_0^\pi \sin^2 t dt\right)^{1/2}$$

and so

$$||C_\alpha x|| = \sqrt{(3a^2+c^2)\frac{\pi}{2}} \tag{2.20}$$

With respect to the evaluation of $||M_\alpha||$, we have

$$M_\alpha M_\alpha^* = \int_0^\pi \left(\frac{1}{\alpha_1(t)^2} + \frac{1}{\alpha_2(t)^2}\right) e^{A(\pi-s)} D_1 D_1^* e^{A^*(\pi-s)} ds$$

We can easily check from

$$x(t) = \begin{bmatrix} 1-\cos t \\ \sin t \end{bmatrix} = \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds$$

and $e^{A(t+s)} = e^{At}e^{As}$ that

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Thus,

$$M_\alpha M_\alpha^* = \begin{bmatrix} \int_0^\pi h(s)\sin^2 s ds & -\int_0^\pi h(s)\sin s\cos s ds \\ -\int_0^\pi h(s)\sin s\cos s ds & \int_0^\pi h(s)\cos^2 s ds \end{bmatrix}$$

where

$$h(s) = \frac{1}{\alpha_1(s)^2} + \frac{1}{\alpha_2(s)^2} \tag{2.21}$$

So,

$$M_\alpha M_\alpha^* = \left(\frac{1}{a^2} + \frac{1}{c^2}\right) \begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$$

For this particular case of constant weights, the evaluation of $||M_\alpha M_\alpha^*||$ can be made rather easily by noticing that

$$\begin{aligned} ||M_\alpha||^2 &= < M_\alpha g, M_\alpha g > = < g, M_\alpha M_\alpha^* g > \\ &= \left(\tfrac{1}{a^2} + \tfrac{1}{c^2}\right) \tfrac{\pi}{2} |g| \end{aligned}$$

Then

$$||M_\alpha|| = \sqrt{\left(\frac{1}{a^2} + \frac{1}{c^2}\right)\frac{\pi}{2}}$$

It is worthwhile to note that for the even more particular case when the constant weights are the same, $\alpha_1(t) = \alpha_2(t) \equiv \mu$, then

$$||M_\alpha||.||C_\alpha x|| = \pi\sqrt{2}$$

independently of the value of μ. Since $||L_\alpha|| = 2.42$ also is invariant with μ in this case, the conditioning number can be promptly evaluated:

$$K_1 = ||M_\alpha||.||C_\alpha x||.\frac{\gamma_o}{1-\gamma_o||L_\alpha||} = \pi\sqrt{2} \times \frac{0.1}{1-0.1\times 2.42} = 0.58$$

Also,

$$r_1 = \frac{K_1}{E_{0.1}(\pi)} = \frac{0.58}{0.34} = 1.72$$

Figure 2.3 summarises the outcome for some choices of constant weights α.

$\alpha_1(t)$	$\alpha_2(t)$	$\|\|C_\alpha x\|\|$	$\|\|M_\alpha\|\|$	$\|\|L_\alpha\|\|$	$r(\alpha)$
μ	μ	$\sqrt{\pi}/\mu$	$\mu\sqrt{2\pi}$	2.42	1.72
a	c	$\sqrt{(3a^2+c^2)\pi/2}$	$\sqrt{(a^{-2}+c^{-2})\pi/2}$		
1.0	0.5	2.26	2.80	3.45	2.80
0.5	1.0	1.66	2.80	2.67	1.86
0.5	2.0	2.73	2.58	4.59	3.84

Figure 2.3: Case of constant weights. $\gamma_o = 0.1$ and $E_{0.1}(\pi) = 0.34$

Case 2: Time-varying weights.

Since we do not have an algorithm for the minimisation of $||L_\alpha||$, we shall consider weights given by

$$\begin{aligned}\alpha_1(t) &= a + bt \\ \alpha_2 &= c + dt\end{aligned}$$

where $a, c \neq 0$ and $b > -a\pi^{-1}$, $d > -c\pi^{-1}$ to avoid singularities. Then, by scanning over the values of admissible a, b, c and d, we can search for a convenient pair of weight-functions that improves our estimate either by allowing a larger class of perturbations — if $||L_\alpha||$ is smaller than the one for time-invariant weights — or by tightening up the bounds.

We have

$$||C_\alpha x||^2 = \int_0^\pi (a + bt)^2(1 - \cos t)^2 dt + \int_0^\pi (c + dt)^2 \sin^2 t dt$$

For the evaluation of $||M_\alpha||$ we can use a computer program[1] to get the norm via

$$\begin{aligned}||M_\alpha||^2 &= ||H|| = \max_{|g|=1} |Hg| \\ &= \max\{\sqrt{\lambda}\,;\ \lambda \in \sigma(H^*H)\}\end{aligned}$$

where $H = M_\alpha M_\alpha^*$.

Naturally, the norm $||L_\alpha||$ is obtained by means of the algorithm from chapter 2. Figure 2.4 illustrates the results obtained for some choices of α. Note that it was possible to get lower values for $||L_\alpha||$ than any of the ones obtained in the case of constant weights.

[1] We have used the Mathematica software on a PC.

$\alpha_1(t)$	$\alpha_2(t)$	$\|\|C_\alpha x\|\|$	$\|\|M_\alpha\|\|$	$\|\|L_\alpha\|\|$	$r(\alpha)$
1	1+0.5t	3.14	2.41	3.00	3.18
1	1.8-0.5t	2.54	4.58	2.19	4.38
1.8-0.5t	1.8-0.5t	1.92	7.18	1.45	4.75
1+0.3t	1.5-0.1t	4.12	1.42	2.95	2.44

Figure 2.4: Case of weights depending on t.

Chapter 3

ON THE OPTIMISATION OF THE CONDITIONING NUMBER

We present a remark on the nature and status of the problem of finding the optimal α which gives the minimal $||L_\alpha||$. Namely, we characterise it as a convex non-differentiable minimisation problem in infinite dimensional spaces for which no algorithm is availabe so far. Also, we give a complete functional analytic abstract approach to the problem of minimising $||C_\alpha x||$ with respect to the input. l/

3.1 Introduction

We recall that with respect to the nominal system $\dot{x} = Ax + Bu$ for which the account for the uncertainties on its structure led to the study of perturbed systems of the form

$$\dot{z}(t) = Az(t) + \sum_{i=1}^{r} D_i F_i(C_i z(t), t) + Bu(t)$$

with Lipschitz perturbations, the error in the final state after a time T — and starting at a given initial state x_o — was estimated as follows

$$|z(T) - x(T)|_{R^n} \leq \frac{\gamma_o ||M_\alpha||}{1 - \gamma_o ||L_\alpha||} ||C_\alpha x||_{L^{2,p}} \tag{3.1}$$

whenever

$$||L_\alpha||.\gamma_o < 1 \tag{3.2}$$

The artificial introduction of the weight α allows one to seek for a convenient α that guarantees the sufficient condition (3.2) to be satisfied. So, even when $||L|| \geq \gamma_o^{-1}$, inequation (3.2) can still be satisfied for a convenient choice of α and this in principle enlarges the class ("magnitude") of perturbations that can be considered. Moreover, one can move a step further and try to find α which gives smaller values of $||L_\alpha||$ and, for this choice of the weight-function, to get a control that gives the minimal $||C_\alpha||$.

3.2 Remark on the minimisation of $||L_\alpha||$

Wee have already seen that merely evaluating $||L_\alpha||$ was on itself a problem far from being trivial (Cf. chapter 2). So, not surprisingly, the question of providing an algorithm for minimising $||L_\alpha||$ with respect to α turns out a very difficult and as yet unsolved one.

We shall prove here that, after a change of variables given by $\alpha_i(t) = e^{\varphi_i(t)}$, the mapping $\varphi \mapsto ||L_\varphi^* L_\varphi||$ is convex. However, no result on its smoothness is available and, more than this, we are somewhat inclined to expect that this mapping is non-differentiable at the point where it achieves its minimum.[1] Therefore, what we have at hand here is an extremely nontrivial non-differentiable convex minimisation problem. Although in the literature one can find simple algorithms for minimizing a convex optimization problem in finite-dimensional spaces, finding general results is a more delicate task. There is a proof of convergence of the subgradient method for constrained convex problems in Hilbert spaces (cf. comments in Shor [18]) but as yet no algorithm is available for finding subdifferential in the context of more general infinite dimensional spaces for non-differentiable functionals.

Thus, at the present stage of the research we have to content ourselves with searching for a satisfactorily small value of $||L_\alpha||$ by trying different choices of α. This was the procedure privileged in last chap-

[1] Because, according to Pritchard's informal comments, this is not so for constant single weights.

ter's treatment of example 1.1.1, where the choice for α's of the form of straight lines was assumed — as opposed to argued — as providing a reasonably wide range of possibilities.

Now, in order to prove the convexity of the mapping $\varphi \mapsto ||L_\varphi^* L_\varphi||$, we first recall the following facts which establish the context for a non-differentiable convex minimisation problem.

If X is a real Banach space and $f : X \longrightarrow \mathbf{R}$ is a convex mapping, then

$$f(\tilde{\alpha}) = \min_X f(\alpha) \qquad \text{iff} \qquad 0 \in \partial f(\tilde{\alpha})$$

Moreover, every local minimum is also a global one.

If Ω is a convex subset of X and $f : \Omega \longrightarrow \mathbf{R}$ convex, then

- $\alpha^* \in X^*$ such that $f(\alpha) \geq f(\tilde{\alpha}) + \alpha^*(\alpha - \tilde{\alpha})$, $\forall \alpha \in \Omega$, is called a *subgradient* of f at $\tilde{\alpha}$.
- $\partial f(\tilde{\alpha}) = \{\alpha^* \in X^* \ ; \ \alpha^* \text{ is a subgradient of } f \text{ at } \tilde{\alpha}\ \}$ is called the *subdifferential* of f at $\tilde{\alpha}$.

The monotone multivalued map $\partial f : \Omega \longrightarrow 2^{X^*}$ is called the *subdifferential* of f.

Since the dual of $X = C[0,T;a,b]$ is the space $X^* = NBV[0,T;a,b]$ of the functions of bounded variations, it is not a straightforward task to find out the $\tilde{\alpha}$ for which $0 \in \partial f(\tilde{\alpha})$ is verified.

On the other hand, since $Q_\alpha = L_\alpha^* L_\alpha$ is a compact self-adjoint operator in $\mathcal{L}(L^{2,q})$, it has some useful properties:

(i) $< Q_\alpha w, w >_{L^{2,q}} \geq 0, \forall w \in L^{2,q}$.

(ii) $||Q_\alpha|| = \sup\{| < Q_\alpha w, w > | \text{ such that } ||w||_{L^{2,q}} = 1\}$

(iii) $||Q_\alpha|| = \max\{\lambda \ ; \ \lambda \text{ is an eigenvalue of } Q_\alpha\}$

Based on a short proof for the convexity of the largest singular value of a matrix given in Sezginer-Overton [15] we show the convexity of the function that associates the largest eigenvalue of $L_\varphi^* L_\varphi$ to each φ defined by the change of variables $\varphi = (\varphi_1, \ldots \varphi_r)$ given by

$$\alpha_i(t) = e^{\varphi_i(t)} \tag{3.3}$$

Proposition 3.2.1

Let Ω be a convex subset of $C[0,T;a,b] \times \ldots \times C[0,T;a,b]$. Then, the mapping $\varphi \in \Omega \longmapsto ||Q_\varphi||$ is convex.

Proof:

Let $S = \{w \in L^{2,q} \; ; \; ||w||_{L^{2,q}} = 1\}$.
We remind that

$$||Q_\varphi|| = \sup_S ||Q_\varphi w||_{L^{2,q}} = \max\{\sqrt{\lambda} \; ; \; \lambda \in \sigma(Q_\varphi)\}$$

and

$$||Q_\varphi w||^2_{L^{2,q}} = \sum_{i=1}^{r} ||\,(Q_\varphi w)_i\,||^2_{L^{2,p_i}}$$

Consider $f(\varphi) = ||Q_\varphi||$. We remind that f is convex on Ω if

$$hf(\varphi) + (1-h)f(\psi) \geq f(h\varphi + (1-h)\psi) \;\; \forall \varphi, \psi \in \Omega \;\; 0 \leq h \leq 1$$

and one can show that if f continuous is such that

$$\frac{1}{2}f(\varphi) + \frac{1}{2}f(\psi) \geq f\left(\frac{\varphi+\psi}{2}\right) \quad \forall\varphi, \varphi \in \Omega$$

then f is convex on Ω.
For any $\varphi, \psi \in \Omega$ the triangle inequality yields

$$\frac{1}{2}f(\varphi) + \frac{1}{2}f(\psi) = \frac{1}{2}||Q_\varphi|| + \frac{1}{2}||Q_\psi|| \geq \frac{1}{2}||Q_\varphi + Q_\varphi||$$

Denote $\Gamma = (\varphi + \psi)/2$. We wish to prove that

$$\frac{1}{2}||Q_\varphi + Q_\psi|| \geq ||Q_\Gamma||$$

For each $\varphi \in \Omega$, consider the operator $E(\varphi) : L^{2,q} \to L^{2,q}$ where

$$(E(\varphi)y)_i\,(t) = e^{\varphi_i(t)} y_i(t)$$

For simplicity of notation we shall write

$$E\left(\frac{\psi-\varphi}{2}\right)=E$$

$$E\left(-\frac{\psi-\varphi}{2}\right)=E^{-1}$$

Thus

$$\begin{aligned} Q_\varphi &= E^{-1}Q_\Gamma E \\ Q_\psi &= EQ_\Gamma E^{-1} \end{aligned}$$

Indeed, for $w \in L^{2,q}$

$$\begin{aligned} (E^{-1}Q_\Gamma Ew)_i &= e^{\frac{\psi_i-\varphi_i}{2}}(Q_\Gamma Ew)_i \\ &= e^{-\frac{\psi_i-\varphi_i}{2}}\sum_{j=1}^r e^{\Gamma_i}Q_{ij}e^{-\Gamma_j}(Ew)_j \\ &= e^{-\frac{\psi_i-\varphi_i}{2}}e^{\frac{\psi_i+\varphi_i}{2}}\sum_{j=1}^r Q_{ij}e^{-\frac{\varphi_j+\psi_j}{2}}e^{\frac{\psi_j-\varphi_j}{2}}w_j \\ &= \sum_{j=1}^r e^{\varphi_i}Q_{ij}e^{-\varphi_j}w_j \\ &= (Q_\varphi w)_i \end{aligned}$$

By the Cauchy-Schwartz inequality we can write

$$\begin{aligned} ||Q_\varphi+Q_\psi|| &= ||E^{-1}Q_\Gamma E+EQ_\Gamma E^{-1}|| \\ &\geq |< u\,,\,(E^{-1}Q_\Gamma E+EQ_\Gamma E^{-1})v >_{L^{2,q}}| \end{aligned}$$

for all $u,v\in S$, $||u||=||v||=1$.

So, in particular, choose

$$u=\frac{Eu_o}{||Eu_o||_{L^{2,q}}}$$

$$v=\frac{Ev_o}{||Ev_o||_{L^{2,q}}}$$

where v_o is an eigenvector corresponding to the largest eigenvalue of $Q_\Gamma^* Q_\Gamma$, that is,

$$Q_\Gamma^* Q_\Gamma v_o = ||Q_\Gamma||^2 v_o$$

and

$$u_o = Q_\Gamma v_o$$

Then, denoting $a = ||Eu_o||_{L^{2,q}}$ and $b = ||Ev_o||_{L^{2,q}}$,

$$\begin{aligned} ||Q_\varphi + Q_\psi|| &\geq (ab)^{-1} |< Eu_o, E^{-1}Q_\Gamma E^2 v_o >_{L^{2,q}} + \\ &\quad + < Eu_o, EQ_\Gamma E^{-1} Ev_o >_{L^{2,q}}| \\ &= (ab)^{-1} |< u_o, Q_\Gamma E^2 v_o > + < Eu_o, EQ_\Gamma v_o >| \\ &= (ab)^{-1} |< Q_\Gamma^* u_o, E^2 v_o > + < Eu_o, Eu_o >| \\ &= (ab)^{-1} |< Q_\Gamma^* Q_\Gamma v_o, E^2 v_o > + a^2| \\ &= (ab)^{-1} |< v_o, E^2 v_o > ||Q_\Gamma||^2 + a^2| \\ &= (ab)^{-1} (b^2 ||Q_\Gamma||^2 + a^2) \end{aligned}$$

so that

$$||Q_\varphi + Q_\psi|| \geq \frac{b}{a} ||Q_\Gamma||^2 + \frac{a}{b}$$

Therefore, writing $\nu = ||Q_\Gamma||(b/a)$, we have

$$\frac{1}{2}||Q_\varphi + Q_\psi|| \geq ||Q_\Gamma|| \frac{1}{2}\left(\nu + \frac{1}{\nu}\right) \geq ||Q_\Gamma||$$

since

$$\frac{1}{2}\left(\nu + \frac{1}{\nu}\right) \geq 1 \qquad \forall \nu > 0$$

□

From what has been proved, and taking f to be given by

$$f(\varphi) = ||Q_\varphi||$$

we have that if

$$0 \in \partial f(\varphi_o) \quad \text{for some } \varphi_o \in C[0,T;a,b] \times \ldots \times C[0,T;a,b]$$

then

$$||L_{\varphi_o}||^2 = ||Q_{\varphi_o}|| = \min\{||Q_\varphi|| \; ; \; \varphi \in C[0,T;a,b] \times \ldots \times C[0,T;a,b]\}$$

3.3 The minimisation of $||C_\alpha x||$

Consider the problem of determining the optimal input $\tilde{u}$ that renders $||C_\alpha x||$ minimal so that the upper bound for the error is tightened.

We will consider the following functional as the one to be minimised:

$$\varphi_\varepsilon(x,u) = \frac{1}{2}\int_0^T \Bigg\{ \sum_{i=1}^{r} < \alpha_i(t) C_i x(t), \alpha_i(t) C_i x(t) >_{R^{p_i}} +$$

$$+ \varepsilon^2 < u(t), u(t) >_{R^m} \Bigg\} dt$$

The introduction of the parameter $\varepsilon > 0$ is to avoid complications from the fact that the extremum problem may have no relevant solutions otherwise. For instance, by plugging this regularity factor in, one can yield convexity of the cost functional.

The problem is formulated under the constraints

$$\dot{x} = Ax + Bu \qquad x(0) = x_o \qquad \text{and} \qquad x(T) = x_T$$

which can be integrated to give

$$x(t) - x_o - \int_0^t (Ax(t) + Bu(t))\,dt = 0$$
$$x(T) - x_T = 0$$

with the additional condition that the system (A, B) is controllable. Thus, we are left with a classical problem of minima under equality constraints where the objective functional and the map defining the constraint are differentiable. The ordinary way to attack this kind of problem is to reduce it to an unconstrained one via Lagrange multipliers method. The precise formulation of the method is summarised in the following result from the literature:

Theorem 3.3.1 (Theorem 26.1 in Deimling [1])
Suppose Z, Y are real Banach spaces, and that

$$\varphi : B_\mu(\tilde{z}) \subseteq Z \longrightarrow \mathbf{R} \;\; \textit{and} \qquad F : B_\mu(\tilde{z}) \longrightarrow Y$$

are continuously (Fréchet-)differentiable.

Also, assume that $Fz_o = 0$ and that the range $R(F'(\tilde{z}))$ is closed.

If

$$\varphi(\tilde{z}) = \min\{\varphi(z) : z \in B_\mu(\tilde{z}) \quad \textit{and} \quad Fz = 0\}$$

then there exist $\lambda \in \mathbf{R}$ and $y^* \in Y^*$, not all zero, such that

$$\lambda\varphi'(\tilde{z}) + (F'(\tilde{z}))^* y^* = 0$$

Moreover, if $F'(\tilde{z})$ is onto, then $\lambda \neq 0$.

Now, we explore this theory in the context of our control problem. Consider the spaces

$$\begin{array}{rcl} Z &=& C[0,T;\mathbf{R}^n] \times L^\infty[0,T;\mathbf{R}^m] \\ Y &=& C[0,T;\mathbf{R}^n] \times \mathbf{R}^n \end{array}$$

and the problem:
Minimise (on Z)

$$\varphi_\varepsilon(z) = \frac{1}{2}\int_0^T \left\{ \sum_{i=1}^r \alpha_i(t)^2 < C_i^* C_i \tilde{x}(t), x(t) >_{R^n} + \varepsilon^2 < \tilde{u}(t), u(t) >_{R^m} \right\} dt$$

subject to

$$(Fz)(t) = \left(x(t) - x_o - \int_0^t (Ax(s) + Bu(s))ds \ , \ x(T) - x_T \right) = (0,0)$$

where $x_o, x_T \in \mathbf{R}^n$ are given.

One can easily check that, for every $z = (x,u) \in Z$,

$$\varphi'(\tilde{z})z = \int_0^T \left\{ \sum_{i=1}^r \alpha_i(t)^2 < C_i^* C_i \tilde{x}(t), x(t) >_{R^n} + \varepsilon^2 < \tilde{u}(t), u(t) >_{R^m} \right\} dt$$

and

$$(F'(\tilde{z})z)(t) = \left(x(t) - \int_0^t (Ax(s) + Bu(s))ds, x(T) \right)$$

Since the system (A, B) is controllable, we have that the range of $F'(\tilde{z})$ is the whole space Y and so it is closed.

Proposition 3.3.2
A necessary condition for $(x,u) \in Z = C[0,T;\mathbf{R}^n] \times L^\infty[0,T;\mathbf{R}^m]$ to be optimal for our minimization problem is that there exist $\lambda \in \mathbf{R}$, $c \in \mathbf{R}^n$ and an absolutely continuous w_o, not all zero, such that

$$\begin{array}{rcll} \dot{w}_o(t) &=& A^* w_o(t) + \lambda \left(\sum_{i=1}^r \alpha_i(t)^2 C_i^* C_i x(t) \right) & w_o(T) = -c \\ \dot{x}(t) &=& Ax(t) + Bu(t) & x(T) = x_T \\ \lambda u(t) &=& \varepsilon^{-2} B^* w_o(t) & x(0) = x_o \end{array}$$

Proof: For convenience, in the environment of the proof we will use the notation $\tilde{z} = (\tilde{x}, \tilde{u})$ for the optimal element. Thus, by theorem 3.3.1, we have that there exist $\lambda \in R$ and $y^* \in Y^*$, not all zero, such that

$$\lambda \varphi_\varepsilon'(\tilde{z})z + ((F'(\tilde{z}))^* y^*)z = 0 \tag{3.4}$$

for all $z = (x, u) \in Z$.

Since $Y^* = NBV[0, T; \mathbf{R}^n] \times \mathbf{R}^n$, we can write $y^* = (g, c)$, for some normalized vector function of bounded variation $g = (g_1, g_2, \ldots, g_n)$ and some vector c in $\mathbf{R}^n$. Also, $\forall z = (x, u)$,

$$\begin{aligned} ((F'(\tilde{z}))^* y^*)z &= \ < (g, c), F'(\tilde{z})z > \\ &= \ < g, x(.) - \textstyle\int_0^{\cdot}(Ax + Bu)ds > + < c, x(T) > \\ &= \ \textstyle\int_0^T \left[x(t) - \int_0^t (Ax + Bu)ds\right] dg(t) + < c, x(T) >_{\mathbf{R}^n} \end{aligned}$$

where the Riemann-Stieltjes integral of the last equality follows from Riezs Representation Theorem for the dual space of $C[0, T; \mathbf{R}^n]$.

So, equation 3.4 becomes

$$\lambda \int_0^T \left[< \sum_{i=1}^r \alpha_i(t)^2 C_i^* C_i \tilde{x}(t), x(t) >_{R^n} + \varepsilon^2 < \tilde{u}(t), u(t) >_{R^m}\right] dt +$$

$$+ \int_0^T \left[x(t) - \int_0^t (Ax(s) + Bu(s))ds\right] dg(t) + < c, x(T) >_{R^n} = 0$$

which is valid for every $z = (x, u) \in Z$. We can obtain representations for g and $\tilde{u}$ by considering the cases $u = 0$ and $x = 0$, respectively. Indeed, for $u = 0$ we have

$$\int_0^T y(t)dg(t) = -\lambda \int_0^T < a(t), x(t) > dt - < c, x(T) > \tag{3.5}$$

where

$$y(t) = x(t) - \int_0^t Ax(s)ds \tag{3.6}$$

and, for simplicity, we are denoting

$$a(t) = \sum_{i=1}^r \alpha_i(t)^2 C_i^* C_i \tilde{x}(t) \tag{3.7}$$

Now, for every $y \in C[0,T;\mathbf{R}^n]$, we know that equation 3.6 has a unique solution $x = Hy$, where H is given by

$$Hy\ (t) \ = \ y(t) + \int_0^t e^{A(t-s)} y(s) ds$$

Hence, equation 3.5 becomes

$$\begin{aligned}
&\textstyle\int_0^T y(t)dg(t) = \\
&\quad = -\lambda \textstyle\int_0^T < a(t), (Hy)(t) >_{R^n} dt - < c, x(T) >_{R^n} = \\
&\quad = - \textstyle\int_0^T < \lambda a(t), y(t) > dt - \int_0^T \lambda a(t), \int_0^t e^{A(t-s)} y(s) ds > dt - \\
&\qquad - < c, y(T) > - \textstyle\int_0^T < c, e^{A(T-s)} y(s) > ds = \\
&\quad = - \textstyle\int_0^T < \lambda a(t), y(t) > dt - \int_0^T \int_t^T < \lambda a(s), e^{A(s-t)} y(t) > dsdt - \\
&\qquad - < c, y(T) > - \textstyle\int_0^T < c, e^{A(T-s)} y(s) > ds = \\
&\quad = \textstyle\int_0^T < y(t), -\lambda a(t) - \lambda \int_t^T e^{A^*(s-t)} a(s) ds + e^{A^*(T-t)} c > dt - \\
&\qquad - < y(T), c >
\end{aligned}$$

Thus

$$\begin{aligned}
\textstyle\int_0^T y(t)dg(t) \ \ &= \ \ - \textstyle\int_0^T < y(t), \lambda a(t) + \\
&\qquad + A^* e^{-A^* t} \left[\lambda \textstyle\int_t^T e^{A^* s} a(s) ds + e^{A^* T} c \right] > dt - < y(T), c >
\end{aligned}$$

Let us introduce

$$w_o(t) = -\lambda \int_t^T e^{A^*(s-t)} \left(\sum_{i=1}^{r} \alpha_i(s)^2 C_i^* C_i \tilde{x}(s) \right) - e^{A^*(T-t)} c \tag{3.8}$$

Such w_o is absolutely continuous and satisfies:

$$\begin{aligned}
\dot{w}_o(t) &= -A^* w_o(t) + \lambda a(t) \\
w_o(T) &= -c
\end{aligned}$$

So, we are left with the following representation for $dg(t)$:

$$\int_0^T y(t)dg(t) = - \int_0^T < y(t), \dot{w}_o(t) >_{R^n} dt - < y(T), c >_{\mathbf{R}^n} \tag{3.9}$$

and

$$w_o(T) = -c \tag{3.10}$$

On the other hand, for $x = 0$ we obtain:

$$\begin{aligned}
\textstyle\int_0^T &< \lambda\varepsilon^2\tilde{u}(t), u(t) >_{R^m} dt = \\
&= \textstyle\int_0^T \left[\int_0^t Bu(s)ds\right] dg(t) \\
&= -\textstyle\int_0^T < \int_0^t Bu(s)ds, \dot{w}_o(t) >_{R^n} dt - \\
&\quad - \textstyle\int_0^T < Bu(s), c >_{R^n} ds \\
&= \textstyle\int_0^T < Bu(s), \int_T^s \dot{w}_o(t)dt >_{R^n} ds - \\
&\quad - \textstyle\int_0^T < Bu(s), c >_{R^n} ds
\end{aligned}$$

Therefore,

$$\int_0 < u(t), \lambda\varepsilon^2\tilde{u}(t) > dt = \int_0^T < u(t), B^*(w_o(t) - w_o(T)) - B^*c > dt$$

so that

$$\lambda\varepsilon^2\tilde{u}(t) - B^*w_o(t) + B^*(w_o(T) + c) = 0$$

from which the result follows. □

Since the controllability of the system (A, B) is an inherent assumption for the whole study we are taking up, the possibility of having $\lambda = 0$, which would render the last proposition inconclusive, is denied because (A, B) controllable is obviously equivalent to $R(F'(\tilde{z})) = Y$. This implies that $\lambda \neq 0$. We remark, in passing, that although we only need for our purposes that the controllability of (A, B) is a sufficient condition for $\lambda \neq 0$, it can be shown that the condition is also necessary (V. Deimling [1]).

We summarise the optimisation result as follows:

Theorem 3.3.3
If $(x, u) \in C[0, T; \mathbf{R}^n] \times L^\infty[0, T; \mathbf{R}^m]$ is optimal for the minimization problem, then there exist $p \in \mathbf{R}^n$ and a absolutely continuous function w, not all zero, such that we have the coupling (CS1):

$$\begin{aligned}
\dot{w}(t) &= -A^*w(t) + \textstyle\sum_{i=1}^r \alpha_i(t)^2 C_i^* C_i x(t) \quad w(T) = p \\
\\
\dot{x}(t) &= Ax(t) + Bu(t) \quad x(0) = x_o \qquad x(T) = x_T \\
u(t) &= \varepsilon^{-2}B^*w(t)
\end{aligned}$$

Proof: Take $w = (1/\lambda)w_o$ and $p = -(1/\lambda)c$. □

The system (CS1) can be transformed into an initial value problem by means of the change of variables:

$$\begin{aligned} w(\tau) &= w(T-\tau) \\ x(\tau) &= x(T-\tau) \end{aligned}$$

resulting

$$\begin{bmatrix} \dot{w} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A^* & -\sum_i(\alpha_i(T-\tau))^2 C_i{}^* C_i \\ \varepsilon^{-2}BB^* & -A \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix}$$

with

$$\begin{bmatrix} w(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} p \\ x_T \end{bmatrix} \quad \text{and} \quad x(T) = x_o$$

Example 1.1.1 revisited: In the case of our example of the frictionless pendulum, the system becomes:

$$\dot{w}_1 = -w_2 - \beta_1^2 x_1 \qquad w_1(0) = p_1 \tag{3.11}$$

$$\dot{w}_2 = w_1 - \beta_2^2 x_2 \qquad w_2(0) = p_2 \tag{3.12}$$

$$\dot{x}_1 = -x_2 \qquad x_1(0) = x_{T1} \quad x_1(\pi) = x_{o1} \tag{3.13}$$

$$\dot{x}_2 = x_1 + \varepsilon^{-2} w_2 \qquad x_2(0) = x_{T2} \quad x_2(\pi) = x_{o2} \tag{3.14}$$

for $\beta_i(\tau) = \alpha_i(T-\tau)$.

The optimal control is given by

$$u(\tau) = \frac{1}{\varepsilon^2} w_2(\tau) \tag{3.15}$$

From (3.14) and (3.13) we get

$$u(\tau) = -(x_1 + \ddot{x}_1) \tag{3.16}$$

so that all we need to do to get the minimising input is to find $x_1(\tau)$. By differentiating (3.14) twice we have

$$\ddot{w}_2 = -\varepsilon^2 (x_1^{(iv)} + \ddot{x}_1) \tag{3.17}$$

On the other hand, equations (3.11) through (3.14) yield

$$\begin{aligned}\ddot{w}_2 &= \dot{w}_1 - (2\beta_2\dot{\beta}_2 x_2 - \beta_2^2\dot{x}_2) \\ &= -\varepsilon^2(\ddot{x}_1 + x_1) - \beta_1^2 x_1 + 2\beta_2\dot{\beta}_2\dot{x}_1 + \beta_2^2\ddot{x}_1\end{aligned}$$

which comparing with (3.17) gives the following fourth-order boundary value problem:

$$x_1^{(iv)} + (2 + \frac{1}{\varepsilon^2}\beta_2^2)\ddot{x}_1 + 2\frac{1}{\varepsilon^2}\beta_2\dot{\beta}_2\dot{x}_1 + (1 - \frac{1}{\varepsilon^2}\beta_1^2)x_1 = 0 \tag{3.18}$$

$$\begin{array}{ll} x_1(0) = 2 & x_1(\pi) = 0 \\ \dot{x}_1(0) = -\dot{x}_2(0) = 0 & \dot{x}_1(\pi) = -\dot{x}_2(\pi) = 0 \end{array}$$

with $u = -(\ddot{x}_1 + x_1)$.

Note that for $\beta_1(\tau)\beta_2(\tau) \equiv 1$, which is the optimal weight that gives the minimal $||L_\alpha||$ in the case of constant weights, the problem to be solved reduces to

$$x_1^{(iv)} + (2 + \frac{1}{\varepsilon^2})\ddot{x}_1 + (1 - \frac{1}{\varepsilon^2})x_1 = 0 \tag{3.19}$$

$$\begin{array}{ll} x_1(0) = 2 & x_1(\pi) = 0 \\ \dot{x}_1(0) = 0 & \dot{x}_1(\pi) = 0 \end{array}$$

$$u = -(\ddot{x}_1 + x_1)$$

For $\varepsilon > 1$, the characteristic roots of the above equation are all complex numbers and equation (3.19) can be solved analytically to yield a solution in terms of sines and cosines.

Figure 3.1 shows the values of $\varphi_\varepsilon(x, u)$ for ε between 1 and 2. Note that for $\varepsilon = 1.1$ we have a minimum. Figure 3.2 shows the graphs of the optimal control u against t and the phase diagram $x_1 \times \dot{x}_1$ for this optimal value of ε.

Remark 3.2.1: The classical state-space approach to our optimisation problem is to formulate the optimum control problem of seeking a linear input

$$u(t) = -B^* P(t)x(t)$$

subject to the dynamics $\dot{x} = Ax + Bu$. The optimal gain matrix P is supposed to minimise the performance criterium

$$J(u) =< x(0), P(0)x(0) > +\varepsilon^2 \int_0^T ||u(t)||^2 dt$$

The closed-loop dynamic behaviour is the one given by

$$\dot{x} = (A - BB^* P)x$$

that is,

$$x(t) = e^{At}x(0) + \int_0^t \Phi(t,s)Bu(s)ds \qquad \text{with} \qquad |x(T) - x_T| \leq k$$

where $\Phi(t,s)$ is the transition matrix of the closed-loop system.

The optimal gain $\tilde{P}$ is then given by $\tilde{P} = B^*M$, where M is the solution of the Riccati equation:

$$\begin{aligned} \dot{M}(t) + M(t)A + A^*M(t) - M(t)BB^*M(t) &= 0 \\ M(T) &= 0 \end{aligned}$$

The reason for privileging a more abstract functional analytic approach is that one idea of further research is to readdress all the topics in this work with an abstract formulation. This could provide an insight to these robustness problems which could facilitate some generalisation to infinite dimensional systems. In particular, it seems that one way to approach the problem of finding algorithms for nondifferentiable convex functional in infinite dimensional spaces is by means of approximating schemes and A-proper mappings techniques, which are again nontrivial abstract formulations.

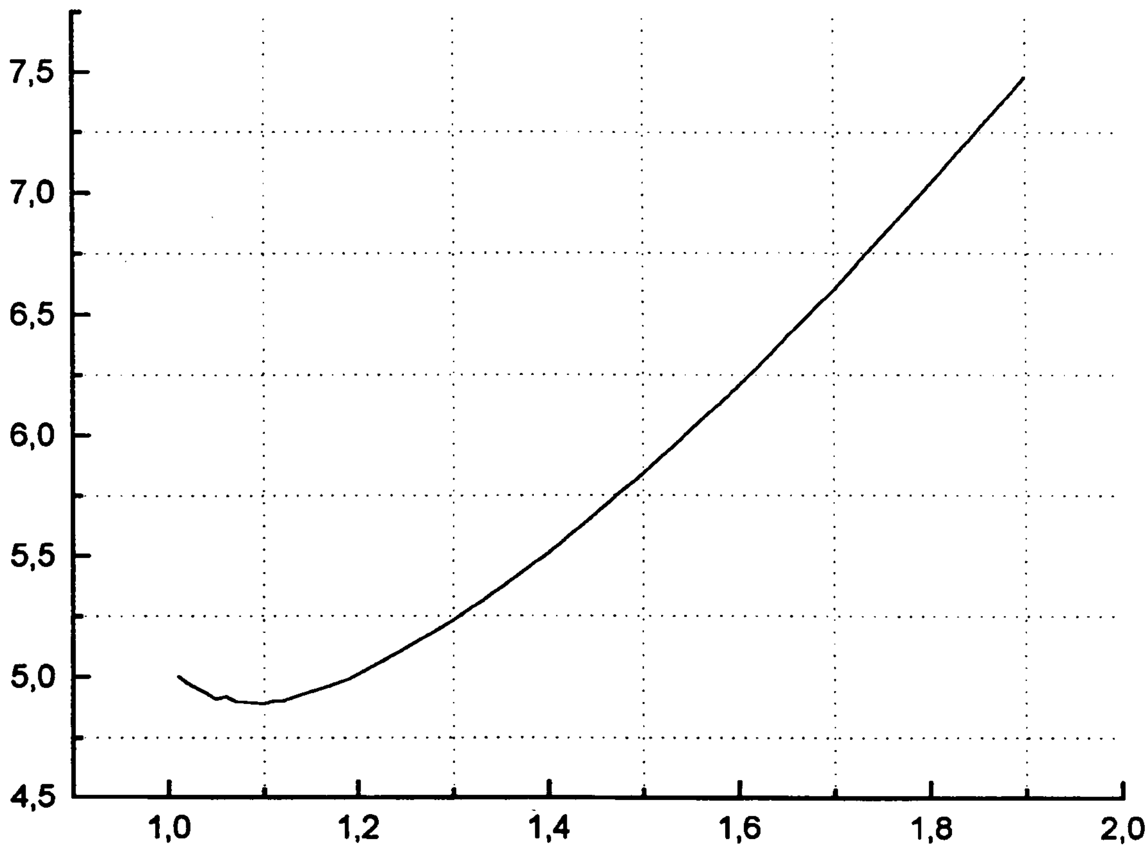

Figure 3.1: Graph of the minimal costs $\varphi_\varepsilon(x,u)$ plotted for different values of the regularity factor ε

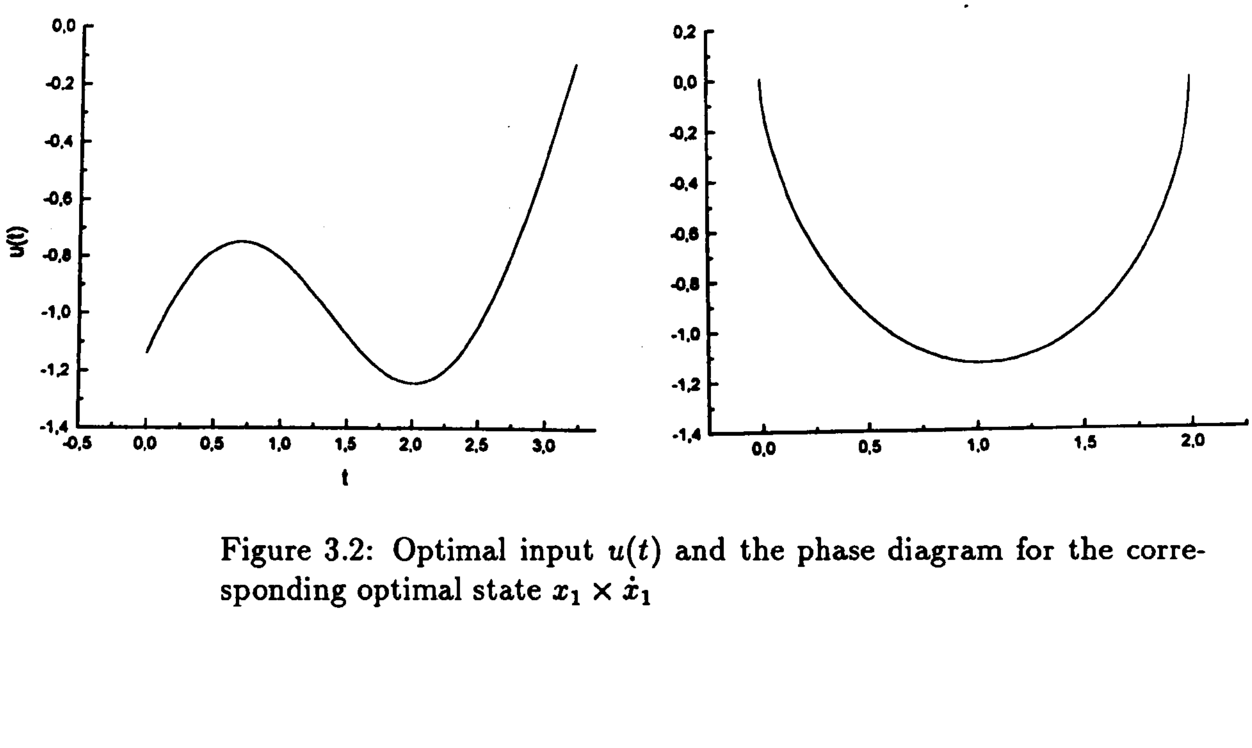

Figure 3.2: Optimal input $u(t)$ and the phase diagram for the corresponding optimal state $x_1 \times \dot{x}_1$

Chapter 4

INTRODUCTION TO THE PART ON STABILITY

We focus on the robustness of stability for linear autonomous systems by considering a class of minimum norm destabilisation problems. The present chapter is an introductory one, where we set up the backgrounds and the context for what is to be the concern of chapters 4 through 6. Nothing here is original work, except the treatment of the well-posedness of the perturbed system, which was adapted from the one in Hinrichsen-Pritchard [13].

4.1 Preliminaries

Consider the nominal system:

$$\dot{x} = Ax \qquad A \in \mathbf{R}^{n \times n} \tag{4.1}$$

where it is assumed that $\sigma(A) \subset \mathbf{C}^-$. This means (in other words, it is equivalent to the fact) that, for any norm $||.||$ in $\mathbf{R}^n$, there exist constants $M, \mu > 0$ such that

$$||e^{At}x_o|| \leq Me^{-\mu t}|x_o| \tag{4.2}$$

for all $t \geq 0, x_0 \in \mathbf{R}^n$. So, the equilibrium point $x = 0$ is asymptotically (more precisely, exponentially) stable and we shall say, for short, that the system is stable.

We recall from the theory of ordinary differential equations the definitions:

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ — for some vector field f with $f(0) = 0$, possibly after a suitable change of variables — is (Lyapunov) *stable* if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ depending only on ε and not on t, such that for every x_o, $|x_o| < \delta$, the solution of $\dot{x} = f(x)$ with initial condition $x(0) = x_o$ can be extended onto the whole half-line $t > 0$ and satisfies $|x(t)| < \varepsilon$, $\forall t > 0$.

The equilibrium point $x = 0$ is said to be *asymptotically stable* if it is (Lyapunov) stable and $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$, for every solution $x(t)$ with initial condition $x(0)$ lying in a sufficiently small neighbourhood of the origin.

To account for the uncertainties on the nominal system we shall consider time-varying perturbations of the class given by disturbed systems of the form:

$$\dot{x} = Ax + B\mathcal{D}(Cx) \tag{4.3}$$

with the matrices $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ giving the structure of the perturbation.

$\mathcal{D} : L^2(0,\infty;\mathbf{R}^p) \longrightarrow L^2(0,\infty);\mathbf{R}^m)$ is given by

$$\mathcal{D}y(t) := D(t)y(t) \qquad \forall t \geq 0$$

for some $D \in L^\infty(0,\infty;\mathbf{R}^{m\times p}) \cap L^1(0,\infty;\mathbf{R}^{m\times p})$. Notice that this implies $D \in L^p$, $\forall p \geq 1$.

Clearly, such $\mathcal{D}$ is a bounded linear operator with

$$||\mathcal{D}|| = ||D||_{L^\infty}$$

(cf. Royden[3]).

Sometimes it is convenient to represent the system (4.3) as the following formal[1] output feedback configuration:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= \mathcal{D}y \end{aligned} \tag{4.4}$$

[1]Formal in the sense that the matrices B and C only account for the structure of the perturbations and do not bear any correspondence with the concepts of controllability and observability of the system. For the present study we do not need any assumptions on controllability of (A, B) at all.

Obviously , a better representation of the structure of the perturbation that accounts for the uncertainties in the original open-loop model would be covered by multiple pairs (B_k, C_k) yielding systems of the form

$$\dot{x} = Ax + \sum_k B_k \mathcal{D}(C_k x)$$

but it seems premature to pursue this level of complexity as yet.

In Hinrichsen-Pritchard[8] and [9] it was introduced a state space approach to the problem of measuring the robustness of stability. Essentially the problem was formalised as a minimum norm destabilising perturbation one. This means that we look for parameters $r_{K,P}$ with the property that

$$||\mathcal{D}|| < r_{K,P} \quad \Longrightarrow \quad (4.3) \text{ is asymptotically stable}$$

and such that

$$r_{K,P} = \inf\{||\mathcal{D}|| \,;\, \mathcal{D} \quad \text{is in } P \text{ and } (4.3) \text{ is not asymptotically stable}\}$$

The symbol P is used to indicate some particular class of perturbation (not necessarily the one we are considering in this work). On the other hand, **K** stands for the scalar field, which is allowed to be either **R** or **C**. This flexibility for **K** might seem somewhat unfounded, since we are dealing with matrix A with real entries, but the reasons for this allowance will become clear in the sequel.

Clearly, in order to be consistent one should expect only real perturbations to be considered. After all, the matrix A is real and so should the perturbations be, since our account for uncertainties should match our perceptions derived from concrete applications. Also, from the definition it follows immediately that

$$0 \leq r_{C,p} \leq r_{R,P}$$

which means that $r_{C,p}$ is a more conservative bound of stability. Actually, in some cases, it is arbitrarily conservative in the sense that for some special choices of A, B, C and the class of perturbations, the quotient $r_{R,p}/r_{C,p}$ can become unbounded (cf. Hinrichsen-Pritchard [12]).

However, the problem is that, contrary to its real counterpart, $r_{C,P}$ presents some robust features in the sense that they are invariant with respect to different classes of perturbations which are relevant in the applications. So, in the case of doubt about the class of perturbation to be considered, the choice for complex stability radii seems more reasonable. Furthermore, it should be noted that for some classes of perturbations we can have $r_{C,P} = r_{R,P}$ (cf. Hinrichsen-Pritchard [13]) .

In Hinrichsen-Pritchard[8] the notion of stability radii was introduced for the class of unstructured time-invariant perturbations, i.e., for systems of the form [2]

$$\dot{x} = (A + \Delta)x \qquad \Delta \in \mathbf{K}^{n \times n} \text{ constant matrix}$$

by essentially measuring the distance of A from the set of unstable matrices. The next step undertaken in Hinrichsen-Pritchard [9] was to consider structured constant perturbations:

$$\dot{x} = (A + B\Delta C)x$$

with $B \in \mathbf{K}^{n \times m}$, $C \in \mathbf{K}^{p \times n}$, $\Delta \in \mathbf{K}^{m \times p}$, and to introduce the stability radii

$$r_K = \inf\{||\Delta|| \, ; \, \sigma(A + B\Delta C) \cap i\mathbf{R} \neq \emptyset\}$$

It is assumed that

$$r_K = \infty \quad \text{if} \quad \sigma(A + B\Delta C) \cap i\mathbf{R} = \emptyset \qquad \forall \, \Delta$$

Since the set of unstable matrices (more properly not asymptotically stable) is closed and its boundary consists of matrices with at least one eigenvalue on the imaginary axis, it follows that for any stable matrix A (that is, such that $\sigma(A) \subset \mathbf{C}^-$), whenever the set

$$\{\Delta \, ; \, \sigma(A + B\Delta C) \cap i\mathbf{R} \neq \emptyset\}$$

is non-empty there exists a matrix $\Delta_o \in \mathbf{K}^{m \times p}$ such that

$$\sigma(A + B\Delta_o C) \cap i\mathbf{R} \neq \emptyset$$

and

$$||\Delta_o|| = \inf\{ \, ||\Delta|| \, ; \, \sigma(A + B\Delta C) \cap i\mathbf{R} \neq \emptyset\}$$

[2]Note that $B = I_m$ and $C = I_p$ in this case.

so that the infimum is attained.

The stability radii r_C (for the case $\mathbf{K} = \mathbf{C}$) and r_R (for $\mathbf{K} = \mathbf{R}$) are in general different and have been characterised:

$$r_C = \frac{1}{\max_{w \in R} ||G(iw)||} = \frac{1}{||L||}$$

$$r_R = [\max\{||G_R(iw)|| \; ; \; w \in \mathbf{R} \quad \text{and} \quad G_I(iw) = 0\}]^{-1} \quad \text{if } m = p = 1$$

$$r_R = \frac{1}{\max_{w \in R} dist(G_R(iw), G_I(iw)\mathbf{R})} \quad \text{if } m = 1$$

$$r_R = \left[\sup_{w \in R} \inf_{\gamma \in (0,1]} \sigma_2 (\begin{bmatrix} G_R(iw) & -\gamma G_I(iw) \\ \gamma^{-1} G_I(iw) & G_R(iw) \end{bmatrix}) \right]^{-1} \quad \text{for } m \neq 1$$

If $G \equiv 0$, the stability radii are said to be ∞ by definition.

Here, $G(s) = G_R(s) + iG_I(s) = C(sI - A)^{-1}B \;\in \mathbf{C}^{p \times m}$ is the transfer function for the system (4.4) and L represents its input-output operator given by

$$Lu\,(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds$$

All the norms in question are the operator norms.

We recall that the real and imaginary parts of the transfer matrix $G(s)$ for $s = \alpha + iw$ is given by

$$\begin{aligned} G_R(\alpha + iw) &= C(w^2 I + (\alpha I - A)^2)^{-1}(\alpha I - A)B \\ G_I(\alpha + iw) &= -wC(w^2 I + (\alpha I - A)^2)^{-1}B \end{aligned}$$

The global maximisation of $||G(iw)||$ may eventually present computational difficulties for some systems . As a way around this problem, one can count on alternative characterisation involving the notions of Hamiltonian matrix and algebraic Riccati equation.

First , for a cost function of the form

$$J_\rho(x_o, v) = \int_0^\infty (||v(t)||^2 - \rho^2 ||y(t)||^2)dt$$

to be minimised subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \quad t \geq 0 \quad x(0) = x_o \\ y(t) &= Cx(t) \end{aligned}$$

we have that $J_\rho(0,v) \geq 0$, for all $v \in L^2(0,\infty;\mathbf{C}^p)$, if and only if $\rho \leq r_C$.

So, the Riccati equation associated with the above minimisation problem is

$$A^*P + PA - \rho^2 C^*C - PBB^*P = 0 \qquad (ARE_\rho)$$

and we have

Theorem 4.1.1

Let $\mathcal{H}_n$ denote the real vector subspace of $\mathbf{R}^{n\times n}$ consisting of all Hermitian $n \times n$ matrices and $\mathcal{H}_n^+$ the convex cone of positive semidefinite matrices in $\mathcal{H}_n$.

Suppose $\sigma(A) \subset \mathbf{C}^-$ and $\rho \geq 0$.

Then, the complex stability radius r_C has the following properties:

(i) *It is the largest value of ρ for which (ARE_ρ) has a solution in $\mathcal{H}_n^+$.*

(ii) *If $\rho \leq r_C$ then there always exist solutions of (ARE_ρ) in $\mathcal{H}_n^+$ and if $\rho < r_C$ and $P(\rho)$ is the smallest such solution, then $P(.)$ is an increasing, analytic function on $(0, r_C)$ and it is stabilising in the sense that $\sigma(A + BB^*P(\rho)) \subset \mathbf{C}^-$.*

(iii) *If $\rho = r_C$ and $P(r_C)$ is the smallest solution in $\mathcal{H}_n^+$, then there exists $(w,x) \in \mathbf{R} \times \mathbf{C}^n$ such that*

$$(A + BB^*P(r_C))x = iwx$$

Moreover,

$$\Delta_o = \frac{B^*P(r_C)x(Cx)^*}{|Cx|^2}$$

is a minimum norm destabilising perturbation.

We have also a characterisation of destabilising time-invariant perturbations

Proposition 4.1.2 *$\Delta \in \mathbf{K}^{m\times p}$ is destabilising*

iff

there exist $w \in \mathbf{R}$ and a nonzero $y \in \mathbf{C}^p$ such that

$$G(iw)\Delta y = y \tag{4.5}$$

Agreeably, r_C can be an acceptable lower estimate for r_R, although there are cases where this estimate is too conservative. One remark concerning pros and cons of these two stability radii is that r_C depends continuously on (A, B, C) whereas in certain cases r_R may be an over-optimistic indicator of robustness since it may jump under slight changes in the structure.

Finally, Hinrichsen-Pritchard [13] investigated the robustness of stability of linear finite-dimensional systems under complex or real perturbations by considering perturbed systems

$$\dot{x}(t) = Ax(t) + B\mathcal{E}(Cx)(t)$$

with the operator $\mathcal{E}$ being one of the following types[3]:

1. $\mathcal{E}(y)(t) = N(y(t))$ where $N : \mathbf{K}^p \longrightarrow \mathbf{K}^m$ is differentiable at 0, is locally Lipschitz and of finite gain, and $N(0) = 0$.
2. $\mathcal{E}(y)(t) = D(t)y(t)$ where $D(.) \in L^\infty(\mathbf{R}^+; \mathbf{K}^{p\times m})$.
3. $\mathcal{E}(y)(t) = N(y(t), t)$ where $N(.,.) : \mathbf{K}^p \times \mathbf{R}^+ \longrightarrow \mathbf{K}^m$ is locally Lipschitz in the first variable, and continuous and of finite gain uniformly in the second variable. Also, $N(0, t) = 0$ for all $t \in \mathbf{R}^+$.
4. $\mathcal{E} : L^2(\mathbf{R}^+; \mathbf{K}^p) \longrightarrow L^2(\mathbf{R}^+; \mathbf{K}^m)$ is causal, weakly Lipschitz continuous and of finite gain. $\mathcal{E}(0) = 0$.

For each of the above classes of perturbation it was defined the real and complex stability radii of the system as the norm of the "smallest" destabilising perturbation. Interestingly, if complex perturbations are considered, the four corresponding complex stability radii coincide, whereas for real perturbations they depend on the specific class.

Our aim in this part of the work on robustness of stability is to give some contributions to the study of the real stability radius for the class of linear but time-varying perturbations given by item 2 above, namely, $\mathcal{E}y\,(t) = \mathcal{D}y\,(y) = D(t)y(t)$. We define and denote this real stability radius as

$$r_{R,t} = \inf\{||D||_{L^\infty} \; ; \; D \in L^\infty \text{ and (4.3) is not asymptotically stable}\}$$

[3] In each case, $\mathbf{K}$ is taken to be either $\mathbf{R}$ or $\mathbf{C}$.

To illustrate and apply our study on robustness of stability we shall always be addressing the following

Example 4.1.3 (The damped linear oscillator) *Consider the equation of small oscillations of a pendulum with friction*

$$\ddot{\varphi} + 2\xi\dot{\varphi} + \varphi = 0$$

which can be put in the form

$$\dot{x} = Ax$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix}$$

For this system we have that

$$\sigma(A) = \left\{ \lambda \in \mathbf{C};\ \ \lambda = -\xi \pm \sqrt{\xi^2 - 1} \right\}$$

so that if the coefficient of friction is positive then $\sigma(A) \subset \mathbf{C}^-$ and the lower equilibrium position of the pendulum ($x_1 = x_2 = 0$) is a stable focus. (As $\xi \to 0$, the focus becomes a center).

The explicit formula for the solution is

$$\begin{aligned} &\varphi(t) = re^{\xi t}cos(wt - \theta) = \alpha e^{\xi t}coswt + \beta e^{\xi t}sinwt \\ &w = \sqrt{1 - \xi^2} \end{aligned}$$

where the coefficients r and θ (or α and β) can be determined from the initial conditions.

For small ξ we have $w \approx 1 - \frac{\xi^2}{2}$. Thus, the friction increases the period only very slightly.

If the coefficient of friction is large ($\xi > 1$), the pendulum does not execute damped oscillations but rather goes directly into its equilibrium position.

Now, we assume a perturbation of the restoring force:

$$\ddot{\varphi}(t) + 2\xi\dot{\varphi}(t) + (1 + d(t))\varphi(t) = 0 \tag{4.6}$$

or equivalently

$$\dot{x}(t) = Ax(t) + Bd(t)Cx(t)$$

with

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0]$$

The transfer function for this system is

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^2 + 2\xi s + 1}$$

so that

$$G(iw) = G_R(iw) + iG_I(iw) = \frac{1 - w^2}{(1 - w^2)^2 + 4\xi^2 w^2} - i\frac{2\xi w}{(1 - w^2)^2 + 4\xi^2 w^2}$$

Simple calculations give the stability radii r_C and r_R. Thus,

$$r_C = \begin{cases} 2\xi\sqrt{1 - \xi^2} & \text{if } 0 < \xi \leq \frac{\sqrt{2}}{2} \\ 1 & \text{if } \frac{\sqrt{2}}{2} \leq \xi < 1 \end{cases}$$

The frequency w_o which yield $|G(iw_o)| = \max_{w \in R} |G(iw)|$ are

$$\begin{array}{ll} w_o = \pm\sqrt{1 - 2\xi^2} & \text{for } 0 < \xi < \frac{\sqrt{2}}{2} \\ w_o = 0 & \text{for } \frac{\sqrt{2}}{2} \leq \xi < 1 \end{array}$$

On the other hand, since $r_R = |G(iw_*)|^{-1}$, for $w_* \in \mathbf{R}$ such that $G_I(iw_*) = 0$, we have that $r_R = 1$.

We wish to compare $r_{R,t}$ with this value of the real stability radius for time-invariant perturbations with those concerning time-varying ones.

We close this introduction with a proof of the existence and uniqueness of (global) solution for the perturbed system (4.3).

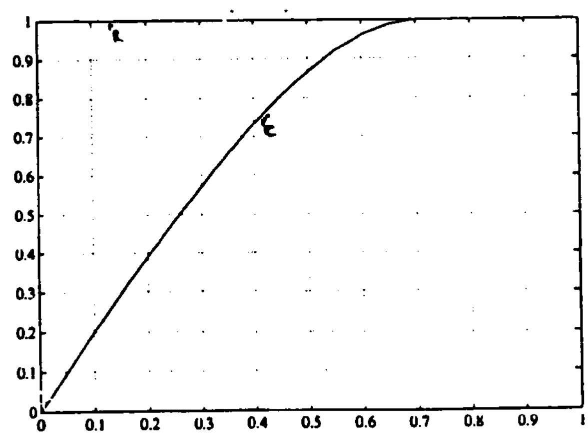

Figure 4.1: r_R and r_C as functions of the parameter ξ.

4.2 Well-posedness of the perturbed system

The question on the existence of solution for the system can be interpreted as a problem of existence of fixed point for linear bounded operators in Hilbert spaces. Indeed, note that for every initial condition $x(0) = x_o \in \mathbf{R}^n$, the formal solution of (4.3) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}B(\mathcal{D}y)(\tau)d\tau \quad \forall t \geq 0 \tag{4.7}$$

where $y = Cx$. Thus, we can consider the functional equation

$$y = \mathcal{F}x_o + L\mathcal{D}y \tag{4.8}$$

by introducing the operator $\mathcal{F} : \mathbf{R}^n \longrightarrow L^2(\mathbf{R}^+; \mathbf{K}^p)$ and the input-output operator $L : L^2(\mathbf{R}^+; \mathbf{K}^m) \longrightarrow L^2(\mathbf{R}^+; \mathbf{K}^p)$, defined respectively by

$$\begin{aligned} \mathcal{F}x_o(t) &= Ce^{At}x_o && (4.9) \\ Lu(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau && (4.10) \end{aligned}$$

The hypothesis on the stability of the matrix A yields that both $\mathcal{F}$ and L are bounded linear operators.

Now, if for each initial condition $x_o \in \mathbf{R}^n$ arbitrarily fixed, we define the operator $\mathcal{H}_{x_o}$ from $L^2(\mathbf{R}^+; \mathbf{K}^p)$ into itself by

$$\mathcal{H}_{x_o}(y) = \mathcal{F}x_o + L\mathcal{D}y \tag{4.11}$$

the question of global solutions of equations becomes equivalent to a problem of existence and uniqueness of fixed points for $\mathcal{H}_{x_o}$.

Proposition 4.2.1

Suppose $||\mathcal{D}|| < ||L||^{-1}$.

Then given any initial condition $x_o \in \mathbf{R}^n$, *the operator* $\mathcal{H}_{x_o}$ *above has a unique fixed point* $y \in L^2(\mathbf{R}^+; \mathbf{K}^p)$. *This means that there exists a unique* $x(.) \in L^2(\mathbf{R}^+; \mathbf{K}^p)$ *such that*

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B\mathcal{D}(Cx)(t) \quad \forall t \geq 0 \\ &x(0) = x_o \end{aligned}$$

Moreover, the equilibrium of $\dot{x} = Ax + B\mathcal{D}(Cx)$ *is asymptotically stable.*

Proof:

The operator $\mathcal{H}_{x_o}$ is easily seen to be a contraction if $||\mathcal{D}|| < ||L||^{-1}$. So, the contraction mapping theorem gives the existence and uniqueness of a fixed point y for $\mathcal{H}_{x_o}$. This L^2-function y obviously vanishes at infinity.

To prove the asymptotic stability we write, given any $t_o > 0$,

$$x(t) = e^{A(t-t_o)}z + \int_{t_o}^{t} e^{A(t-\tau)}B(\mathcal{D}y)(\tau)d\tau \quad \forall t \geq t_o$$

where $z \in \mathbf{R}^n$ is the solution of $y(t_o) = Cz$.

We have from (4.2),

$$||e^{A(t-t_o)}z|| \leq Me^{-\mu(t-t_o)}|z| \quad \forall t \geq t_o$$

So, for $t > t_o$,

$$|x(t)| \leq Me^{-\mu(t-t_o)}|z| + M.||B||.||\mathcal{D}||. \int_{t_o}^{t} ||e^{-\mu(t-\tau)}|y(\tau)|d\tau$$

We have, from Schwarz inequality, that:

$$\begin{aligned} \textstyle\int_{t_o}^{t} e^{-\mu(t-\tau)}|y(\tau)|d\tau &\leq \left(\textstyle\int_{t_o}^{t} e^{-2\mu(t-\tau)}d\tau\right)^{1/2}\left(\textstyle\int_{t_o}^{t}|y(\tau)|^2 d\tau\right)^{1/2} \\ &\leq (2\mu)^{-1/2}\left(1-e^{-2\mu(t-t_o)}2\mu\right)^{1/2}||y||_{L^2(to,t;K^p)} \\ &\leq (2\mu)^{-1/2}||y||_{L^2(t_o,t)} \\ &\leq (2\mu)^{-1/2}||y||_{L^2(t_o,\infty)} \end{aligned}$$

and we can write

$$|x(t)| \leq Me^{-\mu(t-t_o)}|z| + M.||\mathcal{D}||.(2\mu)^{-1/2}||y||_{L^2(t_o,\infty;K^p)}$$

But $||y||_{L^2(t_o,\infty;K^p)} \longrightarrow 0$ as $t_o \longrightarrow \infty$, which means that for any given $\varepsilon > 0$ there corresponds $\alpha > 0$ such that

$$t_o > \alpha \implies ||y||_{L^2(t_o,\infty)} < \frac{\sqrt{2\mu}}{M.||\mathcal{D}||}.\frac{\varepsilon}{2}$$

Now, for this t_o sufficiently large, take

$$\tilde{t} > t_o + \frac{1}{\mu}\log\left(\frac{2M|z|}{\varepsilon}\right)$$

Then,

$$Me^{-\mu(t-t_o)}|z| < \frac{\varepsilon}{2}$$

so that $|x(t)| < \varepsilon$ for all $t > \tilde{t}$. □

The proposition above establishes that $||L||^{-1}$ is a bound for the stability of the perturbed system in the sense that it gives the radius of a circle such that the system is asymptotically stable for perturbations whose "magnitude" is within it.

Chapter 5

EXACT DETERMINATION OF $r_{R,t}$

We use a result from the theory of dynamical systems as a tool for deriving an algorithm yielding the exact evaluation of the time-varying real stability radius $r_{R,t}$ for the second-order linear oscillator parametrised by the damping factor. The interesting outcome is that $r_{R,t}$ can be a less conservative measure of the robustness of stability than r_R as long as one consider linear oscillations with sufficiently small damping factor. This result originally appeared in Hinrichsen-Pritchard[13], where only perturbations on the spring constant were assumed. Here we consider perturbations on both the damping and the spring factors and conclude that in this case $r_{R,t}$ is invariant with respect to the structure of perturbation.

5.1 Preliminaries

In Hinrichsen-Pritchard[13] one can find a characterisation, due to H. Gonzales, of asymptotically stability of the equilibrium of 2-dimensional time-varying system of the form:

$$\dot{x}(t) = M(t)x(t) \tag{5.1}$$

with $M(t) = [m_{ij}(t)]$, or in the extended notation,

$$M(t) = \begin{bmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{bmatrix}$$

and each $m_{ij}(.) : [0,\infty) \longrightarrow \mathbf{R}$, $i,j = 1,2$, is measurable (or piecewise constant) and satisfies

$$m_{ij}^{-} \leq m_{ij}(t) \leq m_{ij}^{+} \qquad t \geq 0 \tag{5.2}$$

where $m_{ij}^{-} \leq m_{ij}^{+}$ are given real numbers.

To state the theorem, first we need to go through some definitions and notations.

For each $k = 1,2,\ldots,16$, denote by

$$V^{(k)} := \begin{bmatrix} v_{11}^{(k)} & v_{12}^{(k)} \\ v_{21}^{(k)} & v_{22}^{(k)} \end{bmatrix}$$

the 2×2 real matrix such that the $(i,j)-$th entry $v_{ij}^{(k)}$ is either m_{ij}^{-} or m_{ij}^{+}. Note that this really comes up to 16 distinct variations at the most.

Denote by $\mathcal{V}$ the set of all these matrices $V^{(k)}$. Also, we are taking trX and $detX$ to mean the trace and the determinant of a given matrix X, respectively.

We shall consider the following optimal control problem, which we shall refer to as (AP):

Find the optimal $\hat{T} > 0$, $\hat{N}(.)$ and $\hat{y}(\hat{T})$ so that $\hat{y}(\hat{T})$ is the maximal element of the set of all $y(T)$ such that $y(.)$ is the solution of the following boundary value problem:

$$\begin{aligned} &\ddot{y}(t) - trN(t)\,\dot{y}(t) + detN(t)\,y(t) = 0 \\ &y(0) = -1 \\ &\dot{y}(T) = 0 \\ &\dot{y}(t) \neq 0 \quad \textit{for}\ 0 < t < T \end{aligned}$$

where $N(.) : [0,T] \longrightarrow \mathcal{V}$ is piecewise constant. (Thus, the aspect of the maps $N(.)$ is that the interval $[0,T]$ is partitioned into r sub-intervals so that, for all t in each sub-interval, $N(t) = V^{(k)}$ for some $V^{(k)} \in \mathcal{V}$.

Finally, let us consider the two following conditions:

G1 : $\max_{V^{(k)} \in \mathcal{V}} trV^{(k)} < 0$ and $\min_{V^{(k)} \in \mathcal{V}} detV^{(k)} > 0$

G2 : at least one of the following conditions is satisfied:
G2.1 : Either

$$(trV^{(k)})^2 - 4 \det V^{(k)} \geq 0 \quad \text{for all} \quad V^{(k)} \in \mathcal{V} \tag{5.3}$$

G2.2 : or $\hat{y}(\hat{T}) < 1$, where $\hat{y}(\hat{T})$ is the solution of (AP).

Then, the theorem that gives the characterisation of asymptotical stability can be stated as follows.

Theorem 5.1.1 (Gonzales's theorem, v. Hinrichsen-Pritchard[13])

Consider the system 5.1 as described above and the optimal control problem (AP).

*The equilibrium of (5.1) is asymptotically stable iff (**G1**) and (**G2**) are satisfied.*

The problem (AP) admits an optimal triple $(\hat{y}(.), \hat{N}(.), \hat{T})$ that can be constructed by means of an algorithm, which we present here:

Step 1: Let $V^{(k_o)}$ be the determinantwise maximal element of $\mathcal{V}$, in the sense that

$$detV^{(k_o)} = \max\{detV^{(k)} \; ; \; V^{(k)} \in \mathcal{V}\}$$

and let $t_1 > 0$ denote the first time at which one of the lines

$$\begin{aligned} L_1 &= \{(y,z) \in \mathbf{R}^2 \; ; \; z = v_{11}^{(k_o)} y\} \\ L_2 &= \{(y,z) \in \mathbf{R}^2 \; ; \; z = v_{22}^{(k_o)} y\} \\ L_3 &= \{(y,z) \in \mathbf{R}^2 \; ; \; y = 0\} \end{aligned}$$

is hit by the orbit $(y_1(t), \dot{y}_1(t))$, $t > 0$, given by the solution of the initial value problem:

$$\begin{aligned}
&\ddot{y}_1(t) - trV^{(k_o)}\dot{y}_1(t) + detV^{(k_o)}y_1(t) = 0 \qquad t > 0\\
&y_1(0) = -1\\
&\dot{y}_1(0) = 0
\end{aligned}$$

Figure 5.1 illustrates such situation when $v_{11}^{(k_o)} < 0 < v_{22}^{(k_o)}$.

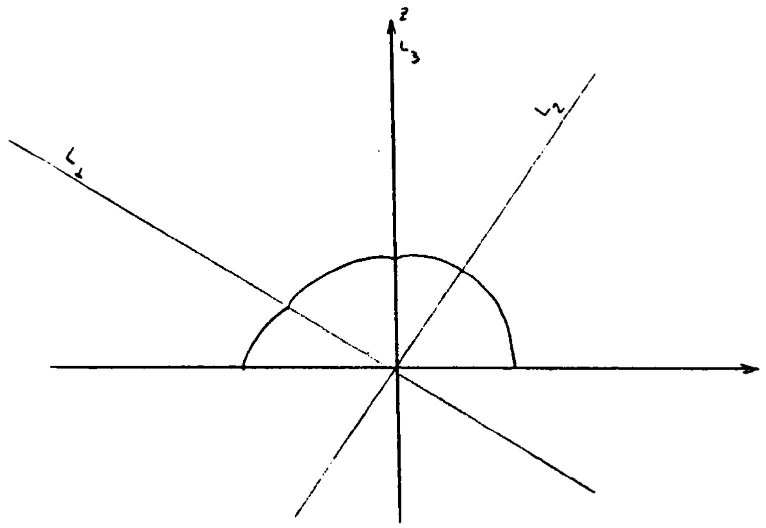

Figure 5.1: The switching lines for the optimal trajectory.

At this point we have the optimal $\hat{N}(.)$ and $\hat{y}(.)$ on the interval $[0, t_1]$. More specifically, let us denote by $\hat{n}_{ij}(t)$ the entries of the optimal $\hat{N}(.)$. Then,

$$\begin{aligned}
\hat{n}_{ij}(t) &= v_{ij}^{(k_o)} \qquad \forall t \in [0, t_1]\\
\hat{y}(t) &= y_1(t) \qquad \forall t \in [0, t_1]
\end{aligned}$$

Now, we move on towards their construction for $t > t_1$.

Step 2: If $(y_1(t_1), \dot{y}_1(t_1)L_1$, that is, L_1 is the first line to be hit by the orbit, we switch $\hat{n}_{22}(t)$ according to the following rule:

if $\lim_{t\to t_1^-} \hat{n}_{22}(t) = m_{22}^+$ then set $\lim_{t\to t_1^+} \hat{n}_{22}(t) = m_{22}^-$;

if $\lim_{t\to t_1^-} \hat{n}_{22}(t) = m_{22}^-$ then set $\lim_{t\to t_1^+} \hat{n}_{22}(t) = m_{22}^+$.

The other entries of $\hat{N}(t)$ remain unchanged for $t \to t_1^+$ (that is, in a right-neighbourhood of t_1).

If $(y_1(t_1), \dot{y}_1(t_1) \in L_2$, we change $\hat{n}_{11}(t)$ in an analogous way:
if $\lim_{t\to t_1^-} \hat{n}_{11}(t) = m_{11}^-$ then set $\lim_{t\to t_1^+} \hat{n}_{11}(t) = m_{11}^+$;
if $\lim_{t\to t_1^-} \hat{n}_{11}(t) = m_{11}^+$ then set $\lim_{t\to t_1^+} \hat{n}_{11}(t) = m_{11}^-$.
The other entries remain the same.

If $(y_1(t_1), \dot{y}_1(t_1)) \in L_3$, we change both $\hat{n}_{12}(t)$ and $\hat{n}_{21}(t)$ in the same way, sticking to the other entries.

Step 3: Consider the matrix $\hat{N}(t_1^+)$, where we are denoting

$$\lim_{t\to t_1^+} \hat{n}_{ij}(t) := \hat{n}_{ij}(t_1^+)$$

and let $t_2 > 0$ denote the first time at which one of the lines L_1, L_2 or L_3 is hit by the orbit $(y_2(t), \dot{y}_2(t))$, $t > 0$, of the new initial value problem:

$$\begin{aligned} &\ddot{y}_2(t) - tr\hat{N}(t_1^+)\dot{y}_2(t) + det\hat{N}(t_1^+)y_2(t) = 0 \qquad t > 0 \\ &y_2(0) = y_1(t_1) \\ &\dot{y}_2(0) = \dot{y}_1(t_1) \end{aligned}$$

Step 4: Set

$$\begin{aligned} \hat{N}(t) &= \hat{N}(t_1^+) \quad \forall t \in [t_1, t_1 + t_2] \\ \hat{y}(t) &= y_2(t) \qquad \forall t \in [t_1, t_1 + t_2] \end{aligned}$$

Step 5: Again, the same rules for changing $\hat{N}(t)$ are applied resulting a new matrix

$$\hat{N}(t) = \hat{N}((t_1 + t_2)^+) \text{ , for } t \in [t_1 + t_2, t_1 + t_2 + t_3]$$

with t_3 being determined by means of the same rules.

The process is continued until the line $\dot{y} = 0$ is reached for the first time, say t_r. This time gives the optimal $\hat{T} = t_1 + t_2 + \ldots + t_r$ and the functions $\hat{y}(.)$ and $\hat{N}(.)$ constructed by means of the algorithm yield the maximal triple

$$(\hat{N}(.), \hat{y}(.), \hat{T})$$

In the case of the linear oscillator, and more generally, 2-dimensional systems, we can use this characterisation to determine $r_{R,t}$ exactly. This is done in the modes of Hinrichsen-Pritchard [13], where the method is founded on a result by H. Gonzales. We shall provide the method here, but this time assuming perturbations on both the "spring constant" and the "damping factor".

5.2 Time-varying real stability radius $r_{R,t}$ for the linear oscillator

Consider the following perturbation of the linear oscillator introduced in example (4.1.3):

$$\ddot{\varphi} + (2\xi + q(t))\dot{\varphi} + (1 + p(t))\varphi = 0 \tag{5.4}$$

where $0 < \xi \leq 1$ is a given parameter.

We assume that, given $\alpha, \beta \geq 0$,

$$p(.) : [0, \infty) \rightarrow [-\alpha, \alpha]$$

and

$$q(.) : [0, \infty) \rightarrow [-\beta, \beta]$$

are two L_{LOC} functions defined a.e.

In particular, this means that $p(.), q(.) \in L^{\infty}(0, \infty)$. Also, we remind that functions which are measurable and essentially bounded belong to L_{LOC}.

The phase-space representation of (5.4) is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u(t) &= d(t)y(t) \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$d(t) = [-p(t) \qquad -q(t)]$$

The case studied in Hinrichsen-Pritchard [13], namely, $d(t) = p(t)$ and $C = [1 \quad 0]$, can be reproduced from our treatment by taking β to be zero and performing the obvious changes in the structure of the perturbation.

We shall apply Gonzales's theorem to our perturbed oscillator (5.4), in which case the corresponding matrix $M(t)$ is given by

$$\begin{bmatrix} 0 & 1 \\ -1-p(t) & -2\xi - q(t) \end{bmatrix}$$

so that

$$m_{ij}^- \le m_{ij}(t) \le m_{ij}^+ \qquad t \ge 0$$

with

$$m_{11}^- = m_{11}^+ = 0$$
$$m_{12}^- = m_{12}^+ = 1$$
$$m_{21}^- = -1 - \alpha \qquad m_{21}^+ = -1 + \alpha$$
$$m_{22}^- = -2\xi - \beta \qquad m_{22}^+ = -2\xi + \beta$$

We have the set $\mathcal{V} = \{V^{(k)} = [v_{ij}^{(k)}]$ with $k = 1, 2, 3, 4\}$ where

$$V^{(1)} = \begin{bmatrix} 0 & 1 \\ -1-\alpha & -2\xi - \beta \end{bmatrix} \qquad V^{(2)} = \begin{bmatrix} 0 & 1 \\ -1-\alpha & -2\xi + \beta \end{bmatrix}$$

$$V^{(3)} = \begin{bmatrix} 0 & 1 \\ -1+\alpha & -2\xi - \beta \end{bmatrix} \qquad V^{(4)} = \begin{bmatrix} 0 & 1 \\ -1+\alpha & -2\xi + \beta \end{bmatrix}$$

and we have the following table:

$V^{(k)}$	$trV^{(k)}$	$detV^{(k)}$	$\left(trV^{(k)}\right)^2 - 4detV^{(k)}$
$V^{(1)}$	$-2\xi - \beta$	$1 + \alpha$	$(2\xi + \beta)^2 - 4(1 + \alpha)$
$V^{(2)}$	$-2\xi + \beta$	$1 + \alpha$	$(2\xi - \beta)^2 - 4(1 + \alpha)$
$V^{(3)}$	$-2\xi - \beta$	$1 - \alpha$	$(2\xi + \beta)^2 - 4(1 - \alpha)$
$V^{(4)}$	$-2\xi + \beta$	$1 - \alpha$	$(2\xi - \beta)^2 - 4(1 - \alpha)$

The way we shall use the theorem is by seeking the first destabilising perturbation. More precisely, we shall start off from a situation of asymptotic stability and seek $d(t)$ with minimum norm $||d(.)||_{L^\infty} = \sup_{t\geq 0}\sqrt{p(t)^2+q(t)^2}$ which renders the system **not** asymptotic stable. Hence, $r_{R,t}(\xi) = ||d(.)||_{L^\infty}$ for this minimum norm destabilising $d(.)$.

For $V^{(k)} \in \mathcal{V}$ we have

$$\begin{array}{lll} \max\, trV^{(k)} = trV^{(2)} = trV^{(4)} = -2\xi+\beta < 0 & \text{iff} & \beta < 2\xi \\ \min\, detV^{(k)} = detV^{(3)} = detV^{(4)} = 1-\alpha > 0 & \text{iff} & \alpha < 1 \end{array}$$

It is natural to start from hypothesis that guarantee (G1) and expect the destabilisation to come from the study of (G2), for otherwise $||d(.)|| \geq 1 = r_R$. Therefore, we shall assume

$$0 \leq \alpha < 1 \tag{5.5}$$

$$0 \leq \beta < 2\xi \tag{5.6}$$

We have

lemma 5.2.1 *Under our assumptions, condition (G2.1) is not satisfied.*

Proof: Just note that

$$\sup_{0\leq\beta<2\xi}\left(\frac{2\xi-\beta}{2}\right) = \xi \leq 1 < 1+\alpha$$

and (G2.1) fails for $V^{(2)}$. □

This lemma means that the destabilising process will be carried out through following the steps of the algorithm for the optimal problem (AP).

Step 1: We have that

$$V_1 = [v_{ij}^{(1)}] = \begin{bmatrix} 0 & 1 \\ -1-\alpha & -2\xi-\beta \end{bmatrix}$$

is maximal in the sense that

$$detV_1 = \max_{V^{(k)}\in\mathcal{V}} detV^{(k)} = 1+\alpha$$

The three switching lines are

$$\begin{aligned} L_1 &= \{(y,z)\ ;\ z=0\} \\ L_2 &= \{(y,z)\ ;\ z=(-2\xi-\beta)y\} \\ L_3 &= \{(y,z)\ ;\ y=0\} \end{aligned}$$

However, the lines L_1 and L_2 can be neglected, since the starting point $(-1,0)$ already belongs to $\dot{y}=0$ and, with respect to the line L_2, there is no change to be carried on once it is hit because $m_{11}^{-} = m_{11}^{+}$. So, the only switching line that is relevant is $y=0$.

Exploring the algorithm further, we need to determine t_1, the first time at which the line $y=0$ is hit by the solution $y_1(.)$ of

$$\begin{aligned} \ddot{y}_1 + (2\xi+\beta)\dot{y}_1 + (1+\alpha)y_1 &= 0 \\ y_1(0) &= -1 \\ \dot{y}_1(0) &= 0 \end{aligned} \tag{5.7}$$

Its characteristic equation is

$$\lambda^2 + (2\xi+\beta)\lambda + (1+\alpha) = 0 \tag{5.8}$$

whose solutions are

$$\lambda = -\left(\frac{2\xi+\beta}{2}\right) \pm \sqrt{\left(\frac{2\xi+\beta}{2}\right)^2 - (1+\alpha)} \tag{5.9}$$

In order to study these roots of the characteristic equation, we ought to analyse three cases. For convenience of notation, let us put

$$a = \frac{2\xi+\beta}{2}$$

and

$$C_1(\xi) = \left\{(t,s)\in[0,1)\times[0,2\xi)\ ;\ 1+t < \left(\frac{2\xi+s}{2}\right)^2\right\}$$

$$C_2(\xi) = \left\{(t,s)\in[0,1)\times[0,2\xi)\ ;\ 1-t < \left(\frac{2\xi+s}{2}\right)^2 < 1+t\right\}$$

$$C_3(\xi) = \left\{(t,s) \in [0,1) \times [0,2\xi) ; \left(\frac{2\xi+s}{2}\right)^2 < 1-t\right\}$$

An initial study of C_1 and C_2 allows us to show that $r_{R,t}(\xi) = 1$ for $\xi \geq \sqrt{2}/2$. First, $(\alpha, \beta) \in C_1$ gives the following lemma

lemma 5.2.2 $\xi \geq \sqrt{2} \implies r_{R,t}(\xi) = 1$

Proof: The characteristic equation 5.8 has two distinct real roots in this case, so that the solution of the initial value problem 5.7 is

$$y_1(t) = -e^{-at}\left(\cosh(dt) + \frac{a}{d}\sinh(dt)\right) \tag{5.10}$$

$d = \sqrt{a^2 - (1+\alpha)}$, which gives that the system goes directly to its equilibrium position without executing any oscillation.

The asymptotic stability can only be broken by putting either $\alpha = 1$ or $\beta = 2\xi$. Therefore, the destabilising $d(.)$ with minimum norm will give $||d(.)||_{L^\infty} = \sqrt{1^2 + 0^2}$. □

Figure 5.2 shows the phase-diagram for $(\alpha, \beta) \in C_1(\xi)$.

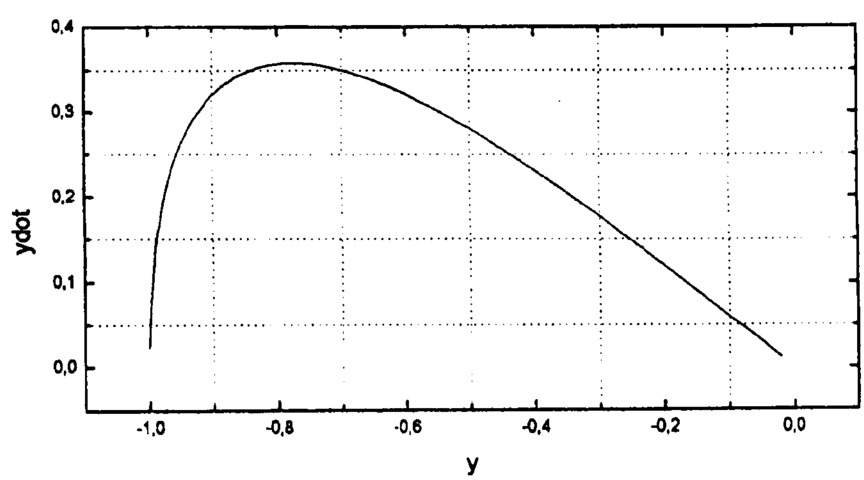

Figure 5.2: Trajectory for $\alpha = 0.20$, $\beta = 1.0$ and $\xi = 0.80$

On the other hand, suppose $(\alpha, \beta) \in C_2(\xi)$. We have that (5.7) yields a characteristic equation with two complex roots. Denoting

$$b = \sqrt{(1+\alpha) - a^2}$$

which is also positive, the roots are $-a \pm ib$. Using the initial conditions, we obtain the solution of (5.7) :

$$y_1(t) = -e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right) \tag{5.11}$$

$$\dot{y}(t) = e^{-at}\left(\frac{1+\alpha}{b}\right)\sin bt \tag{5.12}$$

Since t_1 is the first time at which $y(t_1) = 0$, we have that

$$t_1 = \frac{1}{b}\arctan\left(-\frac{b}{a}\right)$$

is such that

$$\frac{\pi}{2b} < t_1 < \frac{\pi}{b} \qquad \cos bt_1 = -\frac{a}{\sqrt{1+\alpha}} \qquad \sin bt_1 = \frac{b}{\sqrt{1+\alpha}}$$

Now we set, for all $t \in [0, t_1]$,

$$\hat{y}(t) = y_1(t) = -e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right)$$

which yields

$$\dot{y}_1(t_1) = e^{-at_1}\left(\frac{a^2+b^2}{b}\right)\sin bt_1 = \sqrt{1+\alpha}\, e^{-at_1}$$

Within the same approach we can obtain :

$$y_1(t_1) = -e^{-at_1}\left(-\frac{a}{\sqrt{1+\alpha}} + \frac{a}{b}.\frac{b}{\sqrt{1+\alpha}}\right) = 0$$

Therefore, at this time t_1 the solution $y_1(t)$ hits the line $y = 0$ and we are supposed to change $\hat{n}_{12}(t)$ and $\hat{n}_{21}(t)$. The new initial value problem after switching at time t_1, namely

$$\begin{aligned} \ddot{y}_2 + (2\xi + \beta)\dot{y}_2 + (1-\alpha)y_2 &= 0 \\ y_2(0) &= 0 \\ \dot{y}_2(0) &= \dot{y}_1(t_1) \end{aligned} \tag{5.13}$$

has the solution

$$y_2(t) = \mu\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) \tag{5.14}$$

where $\lambda_1 = -a + \sqrt{a^2 - (1-\alpha)}$ and $\lambda_2 = -a - \sqrt{a^2 - (1-\alpha)}$ are the two real roots of the characteristic equation associated to 5.13 and we are denoting

$$\mu = \frac{\sqrt{1+\alpha}}{\lambda_1 - \lambda_2}\, e^{-at_1}$$

The time t_2 for which $\dot{y}_2(t_2) = 0$ is uniquely determined by

$$\frac{\lambda_1}{\lambda_2} = e^{(\lambda_2 - \lambda_1)t_2} \tag{5.15}$$

(Note that $\lambda_1/\lambda_2 > 0$, since $0 < \alpha < 1$, which can be shown by a contradiction argument).

Also,

$$y_2(t_2) = \mu\left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right) e^{\lambda_1 t_2} = \frac{\sqrt{1+\alpha}}{\sqrt{1-\alpha}}\, e^{-a(t_1+t_2)}$$

So we have

Proposition 5.2.3 $r_{R,t}(\xi) = 1$ *for* $\xi \geq \sqrt{2}/2$.

Proof: The system is asymptotic stable whenever $y_2(t_2) < 1$, that is,

$$\frac{\sqrt{1+\alpha}}{a + \sqrt{a^2 - (1-\alpha)}} e^{\lambda_1 t_2} e^{-at_1} < 1$$

So, if $\sqrt{2}\, a > e^{-a\bar{t}_1}$, with $\bar{t}_1$ determined by

$$\cos\sqrt{2-a^2}\,\bar{t}_1 = -\frac{a}{\sqrt{2}} \text{ and } \frac{\pi}{2} \leq \sqrt{2-a^2}\,\bar{t}_1 \leq \pi$$

we have

$$\frac{\sqrt{1+\alpha}}{a + \sqrt{a^2 - (1-\alpha)}} e^{\lambda_1 t_2} e^{-at_1} < \frac{\sqrt{2}}{2a}.1.e^{-a\bar{t}_1} < 1$$

Hence, for $\xi \geq 1/\sqrt{2}$ we have that $\xi + \beta/2 \geq 1/\sqrt{2}$ and

$$\sqrt{2}\, a \geq 1 > e^{-a\bar{t}_1}$$

so that the asymptotic stability is guaranteed. Therefore, the minimum norm destabilising perturbation gives $||d(.)|| = \sqrt{1^2 + 0^2}$.

This together with lemma 5.2.2 complete the proof. □

To study the asymptotic stability for $\xi < \sqrt{2}/2$, we need to analyse the cases $C_2(\xi)$ and $C_3(\xi)$. With respect to the first one, the solution of the initial value problem after switching at time t_1 was determined above. For the case $(\alpha, \beta) \in C_3(\xi)$, we have $a^2 < 1 + \alpha$ so that equation (5.8) has two complex roots, giving the same $y_1(t)$ and t_1 as before. The initial value problem obtained after the switching time is the following:

$$\begin{aligned} \ddot{y}_2(t) + (2\xi + \beta)\dot{y}_2(t) + (1 - \alpha)y_2(t) &= 0 \qquad\qquad t \in [0, t_2] \\ y_2(0) &= 0 \\ \dot{y}_2(0) = \dot{y}_1(t_1) &= e^{-at_1}\sqrt{1+\alpha} \end{aligned}$$

t_2 to be determined in the sequel.

Its solution is given by

$$y_2(t) = \frac{\sqrt{1+\alpha}}{c}\, e^{-a(t_1+t)} \sin ct$$

where $c = \sqrt{(1-\alpha) - a^2}$.

Hence, t_2, the time at which the solution hits $\dot{y} = 0$, is uniquely determined by

$$0 < t_2 < \frac{\pi}{2c} \qquad t_2 = \frac{1}{c} arctan\left(\frac{c}{a}\right) \qquad \cos ct_2 = \frac{a}{\sqrt{1-\alpha}}$$

and the value of $y_2(t_2)$ is

$$y_2(t_2) = \sqrt{\frac{1+\alpha}{1-\alpha}}\, e^{-a(t_1+t_2)}$$

In order to evaluate $r_{R,t}(\xi)$ for each positive $\xi < \sqrt{2}/2$ fixed , we proceed as follows: First, we choose a pair of values for α and β and check if $(\alpha, \beta) \in C_k(\xi)$ with $k = 2, 3$. If the answer is negative, we pick another pair of values. If positive, we input their values in the corresponding computer program that returns the phase-diagram with the switching curves and check whether the trajectory hits the axis $\dot{y} = 0$ at $(1, 0)$. If so, the choice (α, β) gives a destabilising perturbation (in the boundary between stability and instability). If it hits the axis to the left of $(1, 0)$, we have a perturbation giving asymptotically stability. If it hits to the right, the system is unstable. Figure 5.3 shows three choices of (α, β) representing each of the possibilities.

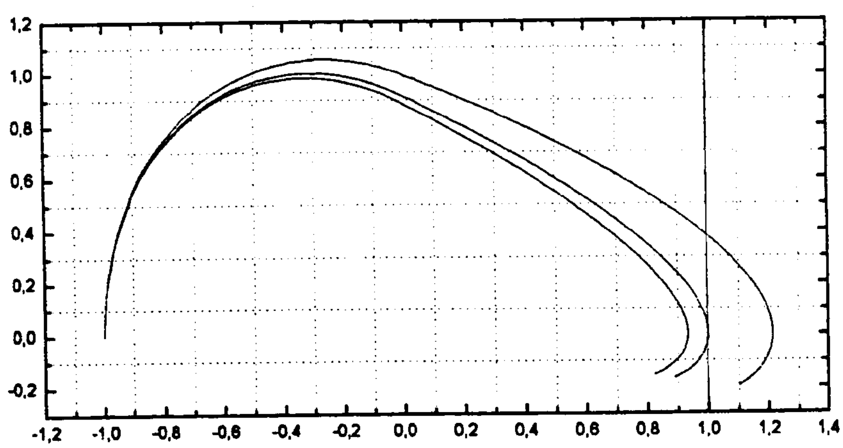

Figure 5.3: Phase-diagrams corresponding to three choices of (α, β) when $\xi = 0.20$. (a) $\alpha = 0.80$, $\beta = 0.20$; (b) $\alpha = 0.80$, $\beta = 0.16$; (c) $\alpha = 0.80$, $\beta = 0.05$

For each value of ξ this procedure is repeated so that we obtain a curve giving all the values of the critical $(\tilde{\alpha}, \tilde{\beta})$ for which the trajectory of the switching systems hits the axis $\dot{y} = 0$ at $(1, 0)$. The value of $r_{R,t}(\xi)$ will be equal to $\min\sqrt{\tilde{\alpha}^2 + \tilde{\beta}^2}$. Figure 5.4 gives this curve (the boundary between stability and instability) for the case $\xi = 0.20$. Note that the minimum norm destabilising $d(.)$ happens to be the one corresponding to $\tilde{\alpha} = 0.60$ and $\tilde{\beta} = 0$, giving $||d(.)|| = \sqrt{0.60^2 + 0^2} = 0.60$. Hence, the minimum norm destabilising perturbation is the one obtained for $\tilde{\beta} = 0$. It turned out that this behaviour was characteristic for all $\xi < \sqrt{2}/2$. So, the structured of perturbation favoured in our study (with disturbances both on the damping and on the spring constant) resulted in the same $r_{R,t}$ obtained in Hinrichsen-Pritchard[13] for perturbations only on the spring constant.

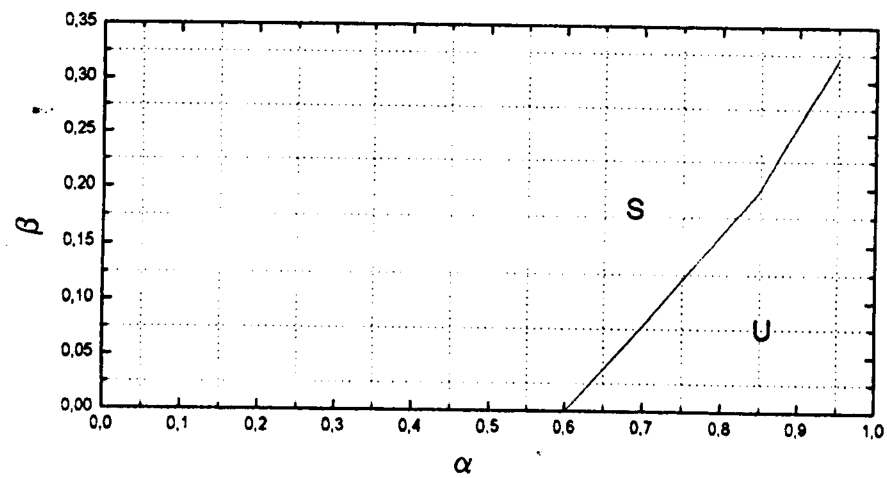

Figure 5.4: Instability region for a damping factor $\xi = 0.20$. (S) represents the stability region whilst (U) represents the instability region.

Figure 5.2 shows the graphics of r_C, $r_{R,t}$ and $r_R \equiv 1$ against ξ, obtained by performing Gonzales's algorithm and studying, for different values of α and β, the points at which the asymptotic stability was broken by having $y_2(t_2) = 1$.

Obviously, the asymptotic analysis for $r_{R,t}$ is made when $\beta = 0$ and hence is the same as the one that appears in [13], yielding $r_{R,t}(\xi) \approx \pi\xi$ near $\xi = 0$. Next chapter will provide in detail an asymptotical analysis by an averaging method.

An important additional conclusion is that for the minimum destabilising $d(.)$ the solution of the oscillator is periodic. Its closed orbit is obtained by mirroring the critical phase-diagram (that is, the one that hits $(1,0)$) of figure 5.2 over the horizontal axis.

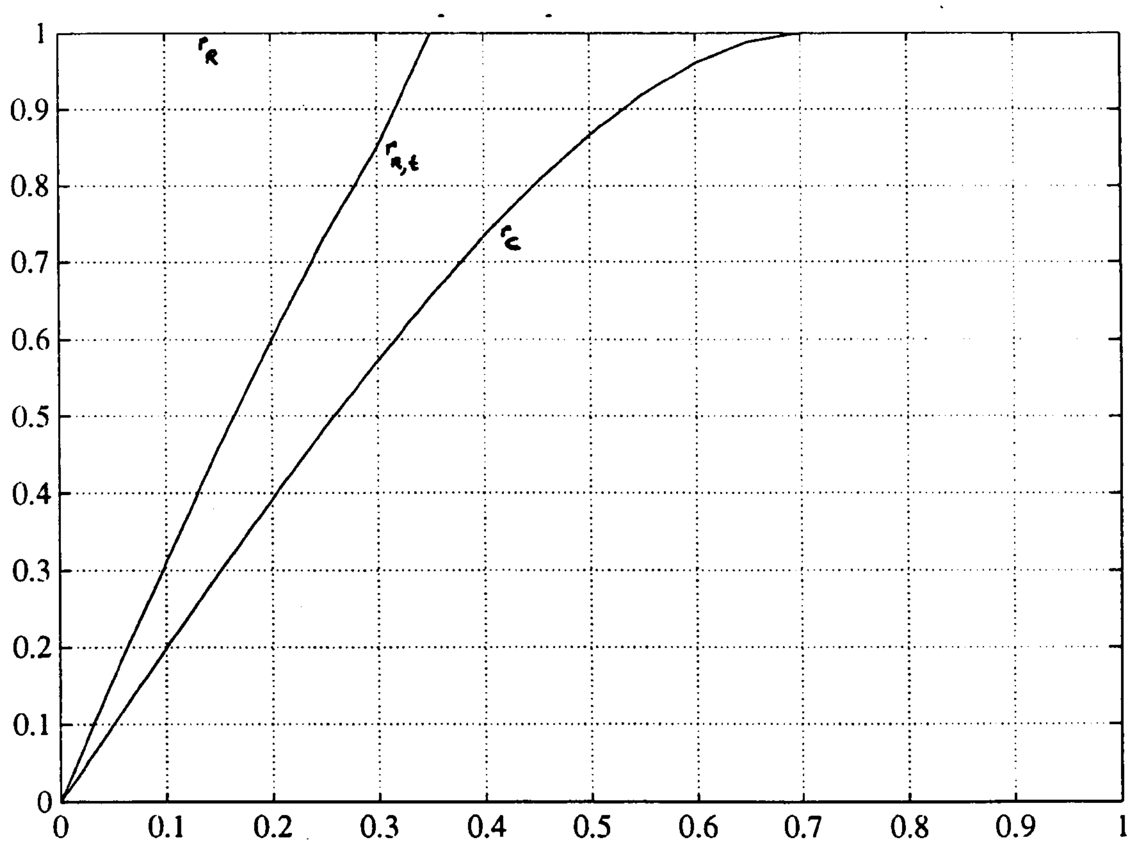

Figure 5.5: Stability radii as functions of the damping parameter ξ.

Chapter 6

ASYMPTOTIC ANALYSIS

We show that, for the 2-dimensional linear oscillator with small damping, there exists a periodic perturbation on the spring factor with L^∞-norm strictly less than r_R. Moreover, the corresponding asymptotic behaviour of the solution turns out to be also periodic.

The relevant point here is that we provide an alternative perturbation method which is conceptually simpler than the one in the previous chapter and can eventually be generalised to systems of any order n.

6.1 Introduction

The motivation for the method we provide here is the observation that, for linear systems subject to periodic perturbations, the boundary between regions of stability and instability will present periodic solutions. To make this statement more precise, we borrow from Verhulst[2] a summary of the Flocquet theory.

By analysing simple examples, we can see that dynamical systems represented by equations with periodic coefficients may have both periodic and non-periodic, even unbounded, solutions. The essential issue here is provided by the result that each fundamental matrix $\Phi(t)$ of equations of the form $\dot{x} = A(t)x$ with $A(t)$ a continuous T-periodic $n \times n$ matrix can be written as

$$\Phi(t) = P(t)e^{MT}$$

with $P(t)$ a T-periodic $n \times n$ matrix and M a constant matrix of the same order. Indeed, by using simple arguments one can show that the fundamental matrices $\Phi(t)$ and $\Phi(t+T)$ are linearly dependent, which implies that there exists a constant $n \times n$ matrix M such that

$$\Phi(t+T) = \Phi(t)e^{MT}$$

and from this one has that $P(t) = \Phi(t)e^{-MT}$ is T-periodic.

The eigenvalues λ of e^{MT} are called *characteristic multipliers.* Each complex number α such that

$$\lambda = e^{\alpha T}$$

is called a *characteristic exponent.* One can choose the exponents α in a way that they coincide with the eigenvalues of the matrix M. Both the stability of the trivial solutions and the existence of periodic solutions of $\dot{x} = A(t)x$ are determined by the eigenvalues of the matrix M. We summarise this comment as follows:

- If there exists a T-periodc solution, then one or more of the characteristic exponents are purely imaginary (and the absolute value of multiplier equals 1).
- The trivial solution is asymptotically stable if and only if all characteristc exponents have negative real part (multipliers have absolute values less than 1).
- The trivial solution is stable if and only if all characteristc exponents have real part ≤ 0 while all the exponents with real part zero have multiplicity 1.

Concerning the problem of calculating the characteristc exponents and multipliers, we have:

Theorem 6.1.1 (v. Verhulst, p. 82) *Suppose λ_i and α_i, $i = 1, 2, ..., n$ are the characteristc multipliers and exponents, respectively, of $\dot{x} = A(t)x$ Then*

$$\lambda_1 \lambda_2 \cdots \lambda_n = exp(\int_0^T trA(t)dt)$$

$$\sum_{i=1}^{n} \alpha_i = \frac{1}{T}\int_0^T trA(t)dt \qquad (mod\frac{2\pi i}{T})$$

where $trA(t)$ is the trace of the matrix $A(t)$.

So, the existence of characteristic exponents α with real part zero (for certain values of the parameter $\xi > 0$) can mean the transition case between unstable and stable solutions. (It can be shown that) in such a transition case the system has periodic solutions (of period either T or $2T$). Thus, by criteriously searching for periodic solutions we are determining the boundaries of the stability domains and, as such, an estimate for $r_{R,t}$.

6.2 Asymptotic analysis of the linear oscillator under periodic perturbations

In the present section we apply a perturbation method to the linear oscillator parametrised by the damping ξ and show that, for small values of ξ, $r_{R,t}$ can be made strictly less than r_R.

Thus, consider the perturbed linear oscillator given by:

$$\dot{x} = (A + Bd(t)C)x \tag{6.1}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0]$$

Since we are interested in the behaviour for small values of the parameter ξ, we assume:

The perturbation near $\xi_o = 0$ is given by

$$d(t;\xi) = a(t)\sum_{k=1}^{\infty} p_k \xi^k \tag{6.2}$$

where $p_k \in \mathbf{R}$, not all zero, and $a(.)$ is a continuous real T-periodic function. Note that this power expansion start at $k = 1$, or equivalently, we are taking p_o. The reason for this will become clear in the sequel.

The periodic solution near $\xi_o = 0$ is of the form

$$x(t;\xi) = \sum_{k=0}^{\infty} x_k(t)\xi^k \tag{6.3}$$

— with each $x_k(t) \in \mathbf{R}^2$ periodic of the same period. We assume that the power series are absolutely convergent.

Furthermore, note that $A = A(\xi) = A_1 + A_2\xi$ with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

Substituting (6.2) and (6.3) into (6.1),

$$\dot{x}_o(t) + \sum_{k=1}^{\infty} \dot{x}_k(t)\xi^k = A_1 x_o(t) + \sum_{k=1}^{\infty} A_1 x_k(t)\xi^k +$$

$$+ \sum_{k=1}^{\infty} A_2 x_{k-1}(t)\xi^k + \sum_{k=1}^{\infty} Ba(t) \left(\sum_{j=0}^{k} p_j C x_{k-j} \right) \xi^k$$

so that

$$\dot{x}_o - A_1 x_o(t) + \sum_{k=1}^{\infty} \left(\dot{x}_k(t) - A_1 x_k(t) - A_2 x_{k-1}(t) - Ba(t) \sum_{j=0}^{\infty} p_j y_{k-j} \right) \xi^k = 0$$

where $y_k = Cx_k$.

Hence we have:

$$\dot{x}_o(t) - A_1 x_o(t) = 0 \tag{6.4}$$

and, for $k \geq 1$,

$$\dot{x}_k(t) - A_1 x_k(t) = A_2 x_{k-1}(t) + \sum_{j=1}^{k} Ba(t) p_j y_{k-j} \tag{6.5}$$

In terms of the state coordinates of the oscillator, the above set of equations give

$$\ddot{\varphi}_o(t) + \varphi_o(t) = 0 \tag{6.6}$$

and

$$\ddot{\varphi}_k(t) + \varphi_k(t) = -2\dot{\varphi}_{k-1}(t) - a(t)\sum_{j=1}^{k} p_j y_{k-j} \qquad k = 1, 2, \cdots \tag{6.7}$$

At this point the reason for assuming $p_o = 0$ becomes clear: equation (6.6) would be

$$\ddot{\varphi}_o(t) + (1 + p_o a(t))\varphi_o(t) = 0$$

otherwise. For $a(t) = \cos t$, the stability considerations lead to the study of a Mathieu equation

$$\ddot{\varphi}_o(t) + (\delta + \varepsilon p \cos t)\varphi_o(t) = 0$$

and to forced Mathieu equation for $k \geq 1$. This would add unnecessary complications to the problem. Furthermore, the classical perturbation analysis for this equation gives $\varepsilon = 0$ when $\delta = 1$ in the boundary between stability and instability (see, for instance, [28]).

Going back to our procedure, the general solution of (6.6) is

$$\varphi_o(t) = c_1 cost + c_2 sint \tag{6.8}$$

which is bounded and, as such, stable.

Inserting (6.8) into $\ddot{\varphi}_1 + \varphi_1 = -2\varphi_o - a(t)p_1\varphi_o$ yields

$$\ddot{\varphi}_1 + \varphi_1 = 2c_1 sint - 2c_2 cost - a(t)p_1(c_1 cost + c_2 sint) \tag{6.9}$$

We observe that a 2π-periodic perturbation $a(t) = d_1 \cos t + e_1 \sin t$, the outcome is that the only stable solution is the trivial one $\varphi \equiv 0$, which can be easily checked. Note that we can embed p_1 into the notation of the coefficients of $a(t)$.

For a π-periodic perturbation of the form

$$a(t) = \frac{d_o}{2} + d_2 \cos 2t + e_2 \sin 2t$$

we have

$$\ddot{\varphi}_1 + \varphi_1 = M_1 \cos t + N_1 \sin t + M_3 \cos 3t + N_3 \sin 3t \tag{6.10}$$

where

$$M_1 = -\left(2c_2 + \frac{d_o}{2}c_1 + \frac{1}{2}d_2c_1 + \frac{1}{2}e_2c_2\right)$$

$$N_1 = 2c_1 - \frac{d_o}{2}c_2 + \frac{1}{2}d_2c_2 - \frac{1}{2}e_2c_1$$

$$M_3 = -\frac{1}{2}\left(d_2c_1 - e_2c_2\right)$$

$$N_3 = -\frac{1}{2}\left(d_2c_2 + e_2c_1\right)$$

Again, according to the theory of stability of forced oscillations we have that the solution φ_1 of (6.10) is stable when the coefficients of $\cos t$ and $\sin t$ all vanish so that forced terms with frequency equal to the natural frequency of the system do not appear and resonance is avoided. Thus,

$$\begin{aligned} c_1d_2 + c_2e_2 &= -d_oc_1 - 4c_2 \\ c_2d_2 - c_1e_2 &= d_oc_2 - 4c_1 \end{aligned}$$

This system of equations yields:

$$d_2 = -\frac{d_o\left(c_1{}^2 - c_2{}^2\right) + 8c_1c_2}{c_1{}^2 + c_2{}^2}$$

$$e_2 = \frac{4\left(c_1{}^2 - c_2{}^2\right) - 2d_oc_1c_2}{c_1{}^2 + c_2{}^2}$$

and we have the following expression for $a(t)$:

$$a(t) = \frac{d_o}{2} + \sqrt{d_2{}^2 + e_2{}^2}\,\cos(2t+\theta) = \frac{d_o}{2} + \sqrt{d_o{}^2 + 16}\,\cos(2t+\theta)$$

with

$$\cos\theta = \frac{d_2}{\sqrt{d_2{}^2 + e_2{}^2}} \quad \text{and} \quad \sin\theta = -\frac{e_2}{\sqrt{d_2{}^2 + e_2{}^2}}$$

so that

$$\tan\theta = -\frac{e_2}{d_2} = \frac{4\left(c_1{}^2 - c_2{}^2\right) - 2d_oc_1c_2}{d_o\left(c_1{}^2 - c_2{}^2\right) + 8c_1c_2}$$

For $\xi > 0$ sufficiently small such that

$$\varphi(t;\xi) = \varphi_o(t) + \mathcal{O}(\xi)$$

we have that

$$||d(\cdot)||_{L^\infty} = \max_{t\in[0,\pi]} |\xi a(t)| = \xi \max_{t\in[0,\pi]} \left| \frac{d_o}{2} + \sqrt{{d_o}^2 + 16}\cos(2t+\theta) \right|$$

We can get the minimal estimate by taking $d_o = 0$, which yields

$$||d(.)||_{L^\infty} = 4\xi \qquad \text{near } \xi = 0$$

To obtain the asymptotic behaviour of the trajectory, we can use an averaging method. For this, we shall be relying on the classical Lagrange method summarised in the following lemma:

lemma 6.2.1 (v. theorem 11.1 in Verhulst [2])
Consider the initial values problems

$$\dot{x} = \xi f(t,x) \qquad x(0) = x_o$$

and

$$\dot{y}_a = \xi f^o(y_a) \qquad y_a = x_o$$

where $x, y_a, x_o \in D \subset \mathbf{R}^n$, $t \in [0,\infty)$ *and* $f : [0.\infty) \times D \longrightarrow \mathbf{R}^n$ *is* T*-periodic in* t*, with* T *a constant independent of* ξ.

$$f^o(y) = \frac{1}{T}\int_0^T f(t,y)dt$$

(y *is kept constant in performing the integration).*
Suppose also that

1. f *and* $\frac{\partial f}{\partial x}$ *are defined, continuous and bounded by a constant* M *(independent of* ξ*) in* $[0,\infty) \times D$.

2. y_a *is contained in an internal subset of* D.

Then, $x(t) - y_a(t) = \mathcal{O}(\xi)$ *on the time-scale* $1/\xi$*, that is, there exist constant* K_1 *and* K_2 *independent of* ξ *such that*

$$|x(t) - y_a(t)|_{R^n} \le K_1\xi \quad \text{for} \quad 0 \le \frac{1}{\xi}t \le K_2$$

as $\xi \longrightarrow 0$.

Consider the perturbed oscillator

$$\begin{aligned}
&\ddot{\varphi}(t) + 2\xi\dot{\varphi}(t) + [1 + \xi(\ \tfrac{d_o}{2} + d_2\cos 2t\ +\ e_2\sin 2t\ ))\varphi(t)\ =\ 0\\
&\varphi(0) = c\\
&\dot{\varphi}(0) = 0
\end{aligned}$$

for small values of the parameter $\xi >\underline{0}$.

We apply the transformation:

$$\varphi(t) = y_1(t)\cos t + y_2(t)\sin t \tag{6.11}$$

$$\dot{\varphi}(t) = -\dot{y}_1(t)\sin t + \dot{y}_2\cos t \tag{6.12}$$

Note that this implies that

$$\dot{y}_1\cos t + \dot{y}_2\sin t = 0 \tag{6.13}$$

By differentiating (6.12) with respect to t and substituting $\varphi, \dot{\varphi}$ and $\ddot{\varphi}$ in the equation for the perturbed oscillator, we get

$$-\dot{y}_1\sin t + \dot{y}_2\cos t =$$

$$-\xi\left[2(-y_1\sin t + y_2\cos t) + (\frac{d_o}{2} + d_2\cos 2t + e_2\sin 2t)(y_1\cos t + y_2\sin t)\right]$$

Hence we can prove the following

Theorem 6.2.2
Consider the system:

$$\begin{aligned}
&\ddot{\varphi}(t) + 2\xi\dot{\varphi} + (1 + d(t))\varphi(t) = 0 \quad t > 0\\
&\varphi(0) = c\\
&\dot{\varphi}(0) = 0
\end{aligned}$$

Then, for sufficiently small values of $\xi > 0$, there exists a periodic perturbation of the form $d(t) = \xi a(t)$ for some continuous function $a(.)$, with $a(t+T) = a(t)$, such that

$$||d(.)||_{L^\infty} = 4\xi \qquad near \qquad \xi_o = 0$$

and the corresponding solution $\varphi(t)$ is periodic of the same period T.

In particular, for the linear oscillator

$$\ddot{\varphi} + \xi\dot{\varphi}(t) + \varphi(t) = 0$$

with uncertainty on the spring constant, the time-varying real stability radius $r_{R,t}$ is strictly less than r_R in some neighbourhood of $\xi_o = 0$.

Proof: We can think of this equation together with (6.13) as an algebraic system for the unknowns $\dot{y}_1$ and $\dot{y}_2$. Solving it and averaging over $[0, \pi]$ yields

$$\dot{y}_{1a} = \frac{\xi}{4}\left[(e_2 - 4)y_{1a} + (d_o - d_2)y_{2a}\right] \qquad y_{1a}(0) = c$$

$$\dot{y}_{2a} = \frac{\xi}{4}\left[(d_o + d_2)y_{1a} + (e_2 + 4)y_{2a}\right] \qquad y_{2a}(0) = 0$$

Solving this system of differential equations for $d_o = d_2 = 0$ and $e_2 = 4$, we have that $\varphi_a(t) = c\ \cos t$, so that

$$\varphi(t) = c\ \cos t + \mathcal{O}(\xi)$$

being the periodic solution for ξ near zero and

$$d(t) = 4\xi \sin 2t$$

Moreover, note that the solution is periodic. Hence $d(t)$ is a destabilizing perturbation and we have:

$$r_{R,t} < ||d(.)||_{L^\infty} = 4\xi < 1 = r_R$$

for ξ sufficiently small. This completes the proof. □

So we can conclude that the norm of the perturbation for small ξ turns out to be a reasonably tight upper bound for $r_{R,t}$ since the asymptotical study derived from Gonzales's approach gave $r_{R,t} \approx \pi\xi$ (cf. Hinrichsen-Pritchard [13]).

An advantage of the present result is the simplicity of its method in comparison with the one undertaken in the previous chapter. Note also that there is nothing special with the order of the system. This method can be applied to parametrised systems of any order n as long as the time-invariant matrix $A(\xi)$ can be written as

$$A(\xi) = A_1 + A_2\xi$$

where

$$A_1 = \begin{bmatrix} 0 & I \\ -Q & 0 \end{bmatrix}$$

for some positive definite matrix Q.

This results also opens up the possibility of having a method for obtaining $r_{R,t}$ approximately by imposing periodic perturbation with a periodic solution with the same period. In a way, the next chapter is related with this approach.

Chapter 7

PERIODIC PERTURBATIONS

This chapter explores the situation where one can have periodic solution as a response to periodic perturbation. The main result is a characterisation of destabilising periodic perturbations which generalises the existing time-invariant version. Another relevant point is that it broadens the field of investigations on time-varying real stability radius: this resonance phenomenum may eventually be explored to provide some algorithm to give approximations for $r_{R,t}$ in the case of general n-dimensional systems.

7.1 Introduction

We consider the usual open-loop asymptotically stable system $\dot{x} = Ax$, with $\sigma(A) \subset \mathbf{C}^-$, and the following feedback configuration to account for its uncertainties:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ u(t) &= D(t)y(t) \end{aligned} \tag{7.1}$$

$$\begin{array}{ll} A \in \mathbf{R}^{n \times n} & \sigma(A) \subset \mathbf{C}^- \\ B \in \mathbf{R}^{n \times m} & C \in \mathbf{R}^{p \times n} \end{array}$$

We assume that $D(t)$ is defined on $[0,\infty)$ as a real $m \times p$ matrix and of period $T = 2\pi/w_o$, for some positive parameter w_o, in the sense that

$$D(t+kT) = D(t) \qquad k = 0 \pm 1, \pm 2, \dots$$

whenever $D(t)$ is defined. Also, $D(.) \in L(\underline{0},T;\mathbf{R}^{m\times p})$, and so integrable over every finite interval and hence its Fourier coefficients are well defined.

Furthermore, we suppose that $D(.) \in L^\infty(0,T;\mathbf{R}^{m\times p})$. Then the extension of $D(t)$ by periodicity to $(-\infty,\infty)$ is in $L^2(\mathbf{R};\mathbf{R}^{m\times p})$ and the Fourier series of $D(.)$ converges to $D(t)$ almost everywhere:

$$D(t) = \frac{1}{2}D_o + \sum_1^\infty (D_k \cos kw_ot \ + \ E_k \sin kw_ot) \qquad \text{a.e.} \tag{7.2}$$

where $D_k, E_k \in \mathbf{R}^{m\times p}$ are the Fourier coefficients of $D(t)$.

Given a initial condition $x(0) = x_o \in \mathbf{R}^n$, we shall impose the solution of (7.1) to be periodic of the same period T and also in $L^2(-\infty,\infty;\mathbf{R}^n)$. Thus, we can write

$$x(t) = \frac{1}{2}a_o + \sum_{k=1}^\infty (a_k \cos kw_ot \ + \ b_k \sin kw_ot) \qquad \text{a.e.} \tag{7.3}$$

where $a_k, b_k \in \mathbf{R}^n$ are the Fourier coefficients of $x(t)$.

In terms of the observation $y = Cx$, equation (7.3) gives

$$y(t) = \frac{1}{2} + \sum_{k=1}^\infty (y_k \cos kw_ot \ + \ \hat{y}_k \sin kw_ot) \qquad \text{a.e.} \tag{7.4}$$

Sometimes it is convenient to treat the problem by means of the complex form of the Fourier series. Thus, we have the following equivalent expressions for $D(t)$ and $y(t)$:

$$D(t) = \sum_{k=-\infty}^\infty \Delta_k e^{ikw_ot} \quad \text{a.e.} \tag{7.5}$$

where $\sum_{-\infty}^\infty$ means the limit as $N \to \infty$ of $\sum_{-N}^N$.

$$y(t) = \sum_{k=-\infty}^\infty Y_k e^{ikw_ot} \qquad \text{a.e.} \tag{7.6}$$

Recall that the coefficients in the trigonometric and complex form of the Fourier series are related as follows:

$$\begin{aligned} D_o &= 2\Delta_o & \\ D_k &= \Delta_k + \Delta_{-k} & \text{for } k = 1, 2, \dots \\ E_k &= i(\Delta_k - \Delta_{-k}) & \text{for } k = 1, 2, \dots \end{aligned}$$

or equivalently, for $k \geq 1$,

$$\begin{aligned} \Delta_k &= \tfrac{1}{2}(D_k - iE_k) \\ \Delta_{-k} &= \tfrac{1}{2}(D_k + iE_k) \end{aligned}$$

Note that, for $k \neq 1$, $\Delta_{-k} = \overline{\Delta_k}$ whenever $D(t)$ is a real perturbation.

Analogous expressions are valid for the Fourier coefficients of $y(t)$.

7.2 Periodic destabilisation

First we derive a necessary condition to be satisfied by the corresponding Fourier coefficients.

lemma 7.2.1 (Compatibility equations) *Consider the closed-loop system (7.1) with the above periodicity assumptions on the perturbation and the solution. Then*

$$G(ikw_o) \sum_{j=-\infty}^{\infty} \Delta_j Y_{k-j} = Y_k \quad \text{for } k = 0, \pm 1, \pm 2, \dots \tag{7.7}$$

where $G(s) = C(sI - A)^{-1}B \in \mathbf{C}^{p \times m}$ *is the transfer matrix of the open-loop system.*

Proof: The equation $\dot{x}(t) - Ax(t) - BD(t)Cx(t) = 0$ implies that the Fourier coefficients satisfy

$$ikw_o X_k - AX_k - B(D(.)Cx)_k = 0$$

where $(D(.)Cx)_k = \sum_{j=-\infty}^{\infty} \Delta_j C X_{k-j} = \sum_{j=-\infty}^{\infty} \Delta_{k-j} C X_j$ is the Fourier coefficient of the product $D(t)Cx(t)$. Then.

$$(ikw_o I_n - A)X_k - B \sum_{j=-\infty}^{\infty} \Delta_j C X_{k-j} = 0$$

Since $\sigma(A) \subset \mathbf{C}^-$, we have that $(ikw_oI - A)$ is nonsingular and (7.7) follows. □

The next theorem is the main result of this chapter. Recall that (Y_k) in l^2 means that $\sum_{k=-\infty}^{\infty} |Y_k|^2 < \infty$. Also, the hypotheses on $D(.)$ guarantee that it is in L^2 and hence the sequence of its Fourier coefficients is in l^2.

Theorem 7.2.2 *Suppose $D(.) \in L^\infty(0,T;\mathbf{K}^{m\times p}) \bigcap C(0,T;\mathbf{K}^{m\times p})$.*
$D(t)$ T-periodic is destabilising

iff

there exists a nonzero sequence $(Y_k)_{k=-\infty}^{\infty}$ in l^2 such that the set of equations (7.7) is satisfied, i.e.,

$$G(ikw_o) \sum_{j=-\infty}^{\infty} \Delta_j Y_{k-j} = Y_k$$

Proof:

For $D(t)$ T-periodic and destabilising whose Fourier coefficients are such that the perturbation is at the boundary between stability and instability, the Flocquet's theory gives the existence of periodic solution $x(.) \in L^2(0,T;\mathbf{K}^n)$ of period either T or 2T. We can assume the period of the solution to be the same, for if T_D is the period of D and $2T_D$ is the period of x, we can consider the Fourier expansion of $D(t)$ as a $2T_D$-periodic function. The necessity follows from lemma 7.2.1.

On the other hand, suppose (7.7) holds. Take $x(t)$ such that

$$X_k = (ikw_oI - A)^{-1} B \sum_{j=-\infty}^{\infty} \Delta_j Y_{k-j}$$

Then the Fourier coefficient of $Ax(t) + BD(t)Cx(t)$ is

$$(Ax(.) + BD(.)Cx(.))_k = AX_k + B \sum_{j=-\infty}^{\infty} \Delta_j Y_{k-j}$$

$$\begin{aligned} &= AX_k + (ikw_oI - A)(ikw_oI - A)^{-1}B\textstyle\sum_{j=-\infty}^{\infty}\Delta_j Y_{k-j} \\ &= AX_k + (ikw_oI - A)X_k \\ &= ikw_oX_k \end{aligned}$$

which is the Fourier coefficient of $\dot{x}(t)$. □

Remark 7.2.1: This result generalises proposition 4.1.2 from the context of time-invariant perturbations to periodic ones. Also, we can regard it as providing a method for obtaining upper bounds for the time-varying stability radii by considering the following problem:

Problem $(*)$: *Find* Δ_k *with* $\sum ||\Delta_k||^2 < \infty$ *and* $D(t) \in \mathbf{K}^{m\times p}$ *of minimum norm*

$$||D(.)||_{L^\infty} = \sup_{t\in(0,T)} ||D(t)||$$

such that there exist Y_k*, not all zero and with* $\sum |Y_k|^2 < \infty$*, so that (7.7) is satisfied, i.e.,*

$$G(ikw_o)\sum_{j=-\infty}^{\infty}\Delta_j Y_{k-j} = Y_k$$

Thus, such $D(t)$ is destabiling and hence $r_{K,t} \leq ||D||_{L^\infty}$.

It is worthy to note at this point that the Fourier coefficients Δ_k of the perturbation, and consequently the norm $||D||_{L^\infty}$, depend on certain parameters. They are the frequency w_o and the nature of the unstable periodic solution imposed by equation (7.6), that is, the initial conditions for the system. So we can say that the variables w_o and $Y_k = CX_k$, for $k \in \mathbf{Z}$ are the entities to play with in order to search for a relevantly small $||D||_{L^\infty}$.

Obviously, to solve this infinite set of solutions is far from being a simple task but the expectation that one can obtain upper bounds which are less conservative than r_R opens a new front of research that can be pursued on. The natural aim, of course, is to culminate in a complete methodology to approach the compatibility equations (7.7) in its full generality (that is, infinite number of Fourier coefficients).

Let us first establish that problem $(*)$ has a solution and, more precisely, that theorem 7.2.2 reproduces the time-invariant complex stability radius:

Proposition 7.2.3 *Suppose $\Delta_o \in \mathbf{C}^{m\times p}$ is the minimum destabilising matrix that gives $||\Delta_o|| = r_C$.*
Then, $D(t) = \Delta_o$ is a solution for problem $(*)$.

Proof: Suppose $\Delta_k = 0$ for all $k \neq 0$ (i.e., time-invariant perturbation) and let k_o and w_o be such that $||G(ik_o w_o)|| = \max_{w\in R} ||G(iw)||$ (without loss of generality, we can assume $k_o > 0$).
Then (7.7) becomes $G(ikw_o)\Delta_o Y_k = Y_k$ for all k.
Take $Y_k = 0$ for all $k \neq k_o$, with $Y_{k_o} \neq 0$ such that

$$\Delta_o G(ik_o w_o)U_{k_o} = U_{k_o}$$

where $U_{k_o} = \Delta_o Y_{k_o}$.
Then, the (destabilising) minimum norm solution of the above equation is

$$\Delta_o = \frac{U_{k_o}[G(ik_o w_o)U_{k_o}]^*}{||G(ik_o w_o)U_{k_o}||^2}$$

where $[G(i_{k_O} w_o)U_{k_o}]^*$ is a linear form in $(\mathbf{C}^p)^*$, the dual space of $\mathbf{C}^p$, whose existence is guaranteed by the Hahn-Banach theorem (v. Hinrichsen-Pritchard[12]).
We have $||\Delta_o|| = r_C$.

□

Also we have the following:

Proposition 7.2.4 *The compatibility equation (7.7) provides an upper bound for r_R.*

Proof:

Take $\Delta_k = 0$ for $k \neq 0$ and, for a nonzero $\Delta_o \in \mathbf{R}^{m\times p}$, equation (7.7) gives

$$G(0)\Delta_o Y_o = Y_o$$

Hence

$$|\Delta_o| = \frac{1}{||G(0)||} \geq r_R$$

for any real $Y_o \neq 0$. □

In the context of real perturbations ($\mathbf{K} = \mathbf{R}$) we have that $\Delta_o \in \mathbf{R}^{m\times p}$ and $\Delta_{-j} = \overline{\Delta_j}$ for $j \neq 0$. Similar relations are valid for Y_k , the Fourier coefficients of $y = Cx$. Therefore, (7.7) can be written as

$$\sum_{j=-\infty}^{-1} G(ikw_o)\Delta_j Y_{k-j} + [G(ikw_o)\Delta_o - I]\, Y_k + \sum_{j=1}^{\infty} G(ikw_o)\Delta_j Y_{k-j} = 0$$

and we have the new expression for the compatibility equation in the case of real perturbations:

$$[G(ikw_o)\Delta_o - I]\, Y_k + \sum_{j=1}^{\infty} G(ikw_o)\left(\overline{\Delta_j} Y_{k+j} + \Delta_j Y_{k-j}\right) = 0$$

Thus we have:

Theorem 7.2.5

Suppose $\Delta_o \in \mathbf{R}^{m\times p}$ and $\Delta_1, \Delta_2 \ldots \Delta_{2k_o} \in \mathbf{C}^{m\times p}$ are the minimum norm solutions of

$$[G(ikw_o)\Delta_o - I]\, Y_k + G(ikw_o)\left[\sum_{j=1}^{k_o-k} \overline{\Delta_j} Y_{k+j} + \sum_{j=1}^{k_o+k} \Delta_j Y_{k-j}\right] = 0 \quad (7.8)$$

for $k = 0, 1, \ldots, k_o - 1$ and

$$[G(ik_o w_o)\Delta_o - I]\, Y_k + G(ik_o w_o)\sum_{j=1}^{2k_o} \Delta_j Y_{k_o-j} = 0 \qquad (7.9)$$

for some vectors $Y_o \in \mathbf{R}^p$ and $Y_1, Y_2, \ldots, Y_{k_o} \in \mathbf{C}^p$.

Let Δ_j for $j \geq 2k_o + 1$ be given by

$$\Delta_{k+k_o} = \left(-\sum_{j=k-k_o}^{k+k_o-1} \Delta_j Y_{k-j}\right)\frac{Y_{-k_o}}{|Y_{k_o}|^2} \qquad k = k_o + 1, k_o + 2, \ldots \quad (7.10)$$

where $Y^*_{k+k_o}$ in the dual space $(\mathbf{C}^p)^*$ is aligned with Y_{-k_o} .

If $\sum_{j=2k_o+1}^{\infty} |\Delta_j|^2 < \infty$ then

$$D(t) = \sum_{j=-\infty}^{\infty} \Delta_k e^{ikw_o t}$$

is destabilising in the sense that the equilibrium of the closed-loop system is not asymptotically stable.

Proof:

Take $y(t) = \sum_{j=-k_o}^{k_o} Y_k e^{ikw_o t}$ with $Y_{-j} = \overline{Y_j}$ for $1 \leq j \leq k_o$.

Then equation (7.7) reduces to (7.8) for $0 \leq k \leq k_o$.

For $k \geq k_o + 1$, we have that $Y_k = 0$ and equation (7.8) reduces to

$$G(ikw_o)\left(\Delta_{k+k_o} Y_{-k_o} + \sum_{j=k-k_o}^{k+k_o-1} \Delta_{k-j}\right) = 0 \qquad (7.11)$$

which is verified "minimum-normwise" when Δ_{k+k_o} is given by (7.10).

Therefore $D(t) = \sum_{k=-\infty}^{\infty} \Delta_k e^{ikw_o t}$ yields a periodic solution of $\dot{x} = Ax + BD(t)Cx$ and the result follows. □

Example 7.2.6 *Suppose $k_o = 1$ in theorem 7.2.5.*

Then (7.8) and (7.9) become

$$(G(0)\Delta_o - I)Y_o + G(0)\left(\overline{\Delta_1}Y_1 + \Delta_1 Y_{-1}\right) = 0 \qquad (7.12)$$

$$(G(iw_o)\Delta_o - I)Y_1 + G(iw_o)(\Delta_1 Y_o + \Delta_2 Y_{-1}) = 0 \qquad (7.13)$$

The task of solving the above equations for $\Delta_o, \Delta_1, \Delta_2$ imply some compromises. First, we should bear in mind that we are aiming for solutions with minimum norm. Second, there are parameters that are open to choice. They are the frequency of the periodic perturbation (w_o) and Y_o, Y_1, which ultimately have to do with the initial conditions to be applied on the closid-loop system. The parameters should be chosen conveniently in a way that the Δ_k's to be determined subsequently are such that $\sum_{k=-\infty}^{\infty} |\Delta_k|^2 < \infty$.

For $k \geq 2$ we take $Y_k = 0$ and

$$\Delta_{k+1} = -(\Delta_{k-1}Y_1 + \Delta_k Y_o)\frac{Y_{-1}^*}{|Y_1|^2} \tag{7.14}$$

is a (minimum-norm) solution of

$$G(ikw_o)(\Delta_{k+1}Y_{-1} + \Delta_{k-1}Y_1 + \Delta_k Y_o) = 0 \tag{7.15}$$

This difference equation can be solved once $Y_o, Y_1, \Delta_o, \Delta_1$ and Δ_2 are already given from (??) and (7.13).

We shall develop this reasoning in detail for the linear oscillator of example 4.1.3. There we had a single-input/single-output system (i.e., m=p=1) so that the Fourier coefficients of D(t) and y(t) are scalars. Also, at this point it is convenient to bring about also the trigonometric representation of the Fourier coefficients since we are interested in real perturbations.

Thus, let us write

$$\begin{aligned}
&\Delta_o = \tfrac{1}{2}d_o \\
&\Delta_k = \tfrac{1}{2}(d_k - ie_k) \quad k \geq 1 \\
&Y_o = \tfrac{1}{2}y_o \\
&Y_1 = \tfrac{1}{2}(y_1 - i\hat{y}_1)
\end{aligned}$$

Recall that we have $Y_{-k} = \overline{Y}_k$ for $k \geq 1$ and $Y_{-1} = \overline{Y}_1$.

Furthermore, $Y_{-1}^* = Y_1$.

Now we can reduce the two-steps difference equation (7.14) to a one-step equation by introducing the trigonometric coefficients in (7.14) to yield:

$$\begin{aligned}
d_{k+1} - ie_{k+1} = &-(y_1 - i\hat{y}_1)\frac{(y_1 - i\hat{y}_1)}{y_1^2 + \hat{y}_1^2}(d_{k-1} - ie_{k-1}) - \\
&- y_o\frac{y_1 + i\hat{y}_1}{y_1^2 + \hat{y}_1^2}(d_k - ie_k)
\end{aligned}$$

which, after carrying out the multiplications, give

$$d_{k+1} - ie_{k+1} = \frac{1}{y_1^2 + \hat{y}_1^2}\Big[(-(y_1^2 - \hat{y}_1^2) + i2y_1\hat{y}_1)d_{k-1} +$$

$$+ (2y_1\hat{y}_1 + i(y_1^2 - \hat{y}_1^2))e_{k-1} - (y_o y_1 + iy_o\hat{y}_1)d_k - (y_o\hat{y}_1 - iy_o y_1)e_{k-1}\Big]$$

Thus

$$d_{k+1} = ad_{k-1} - be_{k-1} + cd_k - de_k$$
$$e_{k+1} = bd_{k-1} + ae_{k-1} + dd_k + ce_k$$

where

$$a = -\frac{y_1^2 - \hat{y}_1^2}{y_1^2 + \hat{y}_1^2} \qquad b = -\frac{2y_1\hat{y}_1}{y_1^2 + \hat{y}_1^2}$$

$$c = -\frac{y_o y_1}{y_1^2 + \hat{y}_1^2} \qquad d = \frac{y_o\hat{y}_1}{y_1^2 + \hat{y}_1^2}$$

Define, for $k \geq 2$,

$$F_k = \begin{bmatrix} d_{k-1} \\ e_{k-1} \\ d_k \\ e_k \end{bmatrix}$$

Then we have the following one-step difference equation

$$F_{k+1} = \mathcal{A}F_k \quad k \geq 2 \tag{7.16}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & -b & c & -d \\ b & a & d & c \end{bmatrix}$$

whose solution is

$$F_{k+1} = \mathcal{A}^{k-1}F_2 \qquad k \geq 1 \tag{7.17}$$

The idea now is that if F_2 is taken to be a linear combination of eigenvectors of $\mathcal{A}$ associated to an eigenvalue λ with $|\lambda| < 1$, then we have that $\sum_{k=-\infty}^{\infty} |F_k|^2 < \infty$ and, consequently, that the Fourier series of $D(t)$ converges in L^2.

So the first point is to ensure that there is a convenient choice of y_o, y_1 and $\hat{y}_1$ for which there exists an eigenvalue of $\mathcal{A}$ with absolute value strictly less than 1:

lemma 7.2.7 *Suppose $m = p = 1$.*

Then, there are values of $y_o, y_1, \hat{y}_1 \in \mathbf{R}$ which yield that there exists at least one eigenvalue λ of the matrix $\mathcal{A}$ with $|\lambda| < 1$.

Proof:

First note that the entries v_k of an eigenvector V in $ker\ (\mathcal{A} - \lambda I)$ are given by the system of equations:

$$\begin{aligned} \lambda v_1 &= v_3 \\ \lambda v_2 &= v_4 \\ (a + (c-\lambda)\lambda)v_1 - (b + d\lambda)v_2 &= 0 \\ (b + d\lambda)v_1 + (a + (c-\lambda)\lambda)v_2 &= 0 \end{aligned}$$

so that $(v_1, v_2) \neq (0,0)$ if and only if

$$(a + (c-\lambda)\lambda)^2 + (b + d\lambda)^2 = 0 \tag{7.18}$$

If $y_o = 0$ then $c = d = 0$ and any eigenvalue has absolute value equal to 1. Indeed, equation (7.18) becomes:

$$(a - \lambda^2)^2 + b^2 = 0$$

so that $\lambda^2 = a \pm ib$ which gives

$$|\lambda^2| = \sqrt{a^2 + b^2} = 1$$

for any values of y_1 and $\hat{y}_1$ not simultaneously equal to zero.

If $\hat{y}_1 = 0$ (with $y_1 \neq 0$), then the choice $y_o/y_1 = \nu + \nu^{-1}$ for any $\nu \in \mathbf{R}$ with $|\nu| > 1$ gives one real eigenvalue $\lambda = \nu^{-1}$ with absolute value less than 1.

Indeed, (7.18) gives

$$\lambda^2 + \frac{y_o}{y_1}\lambda + 1 = 0$$

which yields $\lambda = \pm i$ when $y_o = 0$. If $y_o \neq 0$, it follows that any complex root has absolute value equal to 1, since the product of the roots equal

1 and the coefficients are real. However, if $y_o y_1^{-1} > 2$, which implies that both the roots are real, choose

$$\frac{y_o}{y_1} = \nu + \frac{1}{\nu}$$

for any $\nu \in \mathbf{R}$ with $|\nu| > 1$. Then the roots are

$$\lambda_1 = -\nu \quad \text{and} \quad \lambda_2 = -\frac{1}{\nu}$$

If $y_1 = 0$ (with $\hat{y}_1 \neq 0$, then the choice $y_o/\hat{y}_1 = \nu + \nu^{-1}$ for any $\nu \in \mathbf{R}$ with $|\nu| < 1$ results in one purely imaginary eigenvalue $\lambda = \nu i$ with absolute value less than 1.

In fact, now equation (7.18) is

$$(1 + \lambda^2)^2 = \frac{y_o^2}{y_1^2}\lambda^2$$

This equation is satisfied if

$$\lambda^2 - i\frac{y_o}{\hat{y}_1}\lambda - 1 = 0$$

Again, $y_o = 0$ gives $\lambda = \pm 1$ but the choice

$$\frac{y_o}{\hat{y}_1} = \nu + \frac{1}{\nu}$$

with $-1 < \nu < 1$ gives the result.

□

We observe that the case $y_o y_1 \hat{y}_1 \neq 0$ is not as elegant to treat analytically, but one can easily check computationally that there will be choices of the numbers above giving complex nonreal eigenvalues with moduli less than 1. Figure 7.2 lists some values of $|\lambda|$ obtained in the case of the damped linear oscilllator.

Proceeding with the algorithm, choose $y_o, y_1, \hat{y}_1$ so that $\mathcal{A}$ has eigenvalues with absolute values strictly less than 1 and let λ be one such eigenvalue.

The choice for

$$F_2 = \begin{bmatrix} d_1 \\ e_1 \\ d_2 \\ e_2 \end{bmatrix}$$

is restricted by equations (7.12) and (7.13).

After substituting the complex Fourier coefficients by their trigonometric pairs and equating the real and imaginary parts respectively, those equations result in the following system of compatibility equations which we shall refer to as (SCE):

$$\begin{aligned} \tfrac{1}{2}y_o d_o + y_1 d_1 + \hat{y}_1 e_1 &= G(0)^{-1} y_o \\ G_R(y_o d_1 + y_1 d_o + y_1 d_2 + \hat{y}_1 e_1) + G_I(\hat{y}_1 d_o - \hat{y}_1 d_2 + y_o e_1 + y_1 e_2) &= 2y_1 \\ G_I(y_o d_1 + y_1 d_o + y_1 d_2 + \hat{y}_1 e_2) + G_R(\hat{y}_1 d_o - \hat{y}_1 d_2 + y_o e_1 + y_1 e_2) &= -2\hat{y}_1 \end{aligned}$$

We note that G_R and G_I are simplified notation for $G_R(iw_o)$ and $G_I(iw_o)$, respectively.

Observe that (SCE) consists of three equations and five unknowns, namely, d_o, d_1, d_2, e_1 and e_2. Reminding that the eigenvectors of $\mathcal{A}$ are of the form

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

we can avoid the drawback of having the number of equations less than the number of unknowns by taking F_2 to be a certain linear combinations of eigenvectors of $\mathcal{A}$. Thus, we have to consider two cases.

We first consider the instance when the geometric multiplicity of the eigenvalue λ, which is never greater than its (algebraic) multiplicity, equals 2.

Suppose dim $ker(\mathcal{A} - \lambda I) = 2$, that is, there exist two linearly independent eigenvectors V and U associated to λ.

In this case, set

$$F_2 = \begin{bmatrix} d_1 \\ e_1 \\ d_2 \\ e_2 \end{bmatrix} = \nu V + \mu U$$

Substituting these expressions for d_1, d_2, e_1, e_2 in (SCE) results in a consistent system of three equations and three unknowns: d_o, ν and μ. We shall refer to this modified (SCE) as (modSCE).

It can easily be checked that (modSCE) has a unique solution. Therefore, for ν, μ determined from (modSCE), we have

$$F_3 = \mathcal{A}F_2 = \nu\mathcal{A}V + \mu\mathcal{A}U = \lambda(\nu V + \mu U)$$

$$F_4 = \mathcal{A}F_3 = \lambda^2(\nu V + \mu U)$$

and so on, to yield the solution

$$F_{k+1} = \lambda^{k-1}(\nu V + \mu W) \quad k = 1, 2, \ldots \tag{7.19}$$

More precisely,

$$\begin{bmatrix} d_k \\ e_k \\ d_{k+1} \\ e_{k+1} \end{bmatrix} = \lambda^{k-1}\nu \begin{bmatrix} v_1 \\ v_2 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix} + \lambda^{k-1}\mu \begin{bmatrix} u_1 \\ u_2 \\ \lambda u_1 \\ \lambda u_2 \end{bmatrix}$$

so that, for $k \geq 1$,

$$\Delta_k = \frac{1}{2}(d_k - ie_k) = \frac{1}{2}\lambda^{k-1}\left[(\nu v_1 + \mu u_1) - i(\nu v_2 + \mu u_2)\right] \tag{7.20}$$

On the other hand, if dim $ker(\mathcal{A} - \lambda I) = 1$, take an eigenvector

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix} \in ker(\mathcal{A} - \lambda I)$$

and set

$$F_2 = \nu V + \mu \overline{V}$$

for $\nu, \mu \in \mathbf{R}$ being the solution of (modSCE), the resulting consistent system when F_2 is plugged in (SCE).

Since $\mathcal{A}V = \lambda V$ and $\mathcal{A}$ is a real matrix, we have that $\mathcal{A}\overline{V} = \overline{\lambda V}$,– i.e., $\overline{\lambda}$ also is an eigenvalue of $\mathcal{A}$.

So it follows that

$$F_{k+1} = \lambda^{k-1}\nu V + \overline{\lambda}^{k-1}\mu\overline{V}$$

Then

$$d_k = \lambda^{k-1}\nu v_1 + \overline{\lambda}^{k-1}\mu\overline{v}_1$$

$$e_k = \lambda^{k-1}\nu v_2 + \overline{\lambda}^{k-1}\mu\overline{v}_2$$

Therefore, for the case when the eigenspace of $\mathcal{A}$ has dimension 1, we have for $k \geq 1$,

$$\Delta_k = \frac{1}{2}\left[\lambda^{k-1}\nu(v_1 - iv_2) + \overline{\lambda}^{k-1}\mu(\overline{v_1} - i\overline{v_2})\right] \tag{7.21}$$

The following proposition summarises the algorithm for obtaining the destabilising periodic perturbation $D(t)$.

Proposition 7.2.8 *Let $\lambda = re^{i\theta} \in \sigma(\mathcal{A})$ be such that $|\lambda| < 1$ for some $y_o, y_1, \hat{y}_1$.*

(1) *Suppose dim ker$(\mathcal{A} - \lambda I) \geq 2$ and*

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix} \qquad U = \begin{bmatrix} u_1 \\ u_2 \\ \lambda u_1 \\ \lambda u_2 \end{bmatrix}$$

are two linearly independent eigenvectors of $\mathcal{A}$ associated to λ.

Let

$$D(t) = \frac{1}{2}d_o + \frac{r\ Re\left(Pe^{i(w_o t+\theta)}\right) - r^2 Re\ P}{1 - 2r\cos(w_o t + \theta) + r^2} \tag{7.22}$$

where

$$P = \lambda^{-1}((\nu v_1 + \mu u_1) - i(\nu v_2 + \mu u_2))$$

and d_o, ν, μ are the solutions of (modSCE).

Then $D(t)$ given by (7.22) is a $(2\pi/w_o)$-periodic destabilising perturbation of $\dot{x} = Ax + BD(t)Cx$.

(2) *Alternatively, suppose dim ker$(\mathcal{A} - \lambda I) = 1$ and*

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

is an eigenvector of $\mathcal{A}$.

Let

$$\begin{aligned} D(t) &= \frac{1}{2}d_o + \frac{r\ Re\left(P_\nu e^{i(w_o t+\theta)}\right) - r^2 Re(P_\nu)}{1 - 2r\cos(w_o t + \theta) + r^2} + \\ &\quad + \frac{r\ Re\left(P_\mu e^{i(w_o t-\theta)}\right) - r^2 Re(P_\mu)}{1 - 2r\cos(w_o t - \theta) + r^2} \end{aligned} \tag{7.23}$$

where d_o, ν, μ are the solutions of (modSCE) and

$$P_\nu = \lambda^{-1}\nu(v_1 - iv_2)$$

$$P_\mu = \overline{\lambda}^{-1}\mu(\overline{v_1} - i\overline{v_2})$$

Then $D(t)$ given by (7.23 is a $2\pi/w_o$-periodic destabilising perturbation of $\dot{x} = Ax + BD(t)Cx$.

Proof:

In virtue of (7.20) we can write, for the case (1),

$$\Delta_k = \frac{1}{2}r^k e^{ik\theta}P \qquad k \geq 1$$

Therefore, we have that the perturbation $D(t)$ which solves the compatibility equation (7.7) is

$$D(t) = \frac{1}{2}d_o + \frac{1}{2}\overline{P}\sum_{k=1}^{\infty}\left(re^{-i(w_o t+\theta)}\right)^k + \frac{1}{2}P\sum_{k=1}^{\infty}\left(re^{i(w_o t+\theta)}\right)^k$$

The two geometric series above are convergent since $r < 1$. Also,

$$\sum_{k=1}^{\infty} r^k e^{\pm ik(w_o t+\theta)} = \frac{re^{\pm i(w_o t+\theta)}}{1 - re^{\pm i(w_o t+\theta)}}$$

Substituting and performing the straightforward calculations lead to the result.

The proof for case (2) is analogous. □

Example 4.1.3 revisited: Concerning the example of the damped linear oscillator, we illustrate here the theory for some specific choices of the parameters which allow a flexibility for seeking upper bounds for $r_{R,t}$.

The choice

$$\begin{aligned} y_o &= 2.5 \\ y_1 &= 0.5 \\ \hat{y}_1 &= 0 \end{aligned}$$

gives

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -5 & 0 \\ 0 & -1 & 0 & -5 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = -0.2087$ with algebraic multiplicity 2

$\lambda_2 = -4.7913$ with algebraic multiplicity 2

so that we choose λ_1, since $|\lambda_1| = 0.2087 < 1$.

Hence we have the eigenvectors

$$V = \begin{bmatrix} 1 \\ 0 \\ -0.2087 \\ 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -0.2087 \end{bmatrix}$$

Plugging $F_2 = \nu V + \mu U$ into (modSCE) gives

$$\begin{aligned} d_o &= 2.1822 \\ \nu &= -0.4554 \\ \mu &= -0.8348 \end{aligned}$$

Thus, fixing $\xi = 0.3$, proposition 7.2.8 yields

$$||d(.)||_{L^\infty} = \max|d(t)| = 2.1847 \qquad \text{for } w_o = 1$$

$$||d(.)||_{L^\infty} = \max|d(t)| = 1.0256 \qquad \text{for } w_o = 0.1$$

Figure 7.2 shows the graphs of $d(t)$ for two different values of w_o.

Figure 7.2 illustrates the behaviour of $||d(.)||$ with respect to w_o and y_1 (for $\xi 0.3$, $y_o = -5$, $y_1 = 0$ and $\hat{y}_1 = 1$ in the first case and $\xi = 0.3$, $y_o = 6$, $w_o = 1$ and $\hat{y}_1 = 0$ in the second case). Note that the value of $||d(.)||$ "jumps" at $w_o = 0.7$. This unexpected behaviour happens again for other values of w_o (near $w_o = 1.3$, for instance) and whether this is a consequence of some eventual mistake on the computation or something intrinsic to the nature of the problem it is a question still to be solved.

Remark 7.2.2: Suppose we consider a truncated expansion of the perturbation in the form:

$$D(t) = \frac{1}{2}d_o + d_1 \cos w_o t + e_1 \sin w_o t$$

or equivalently

$$D(t) = \Delta_{-1}e^{-iw_o t} + \Delta_o + \Delta_1 e^{iw_o t}$$

Then the compatibility equation gives

$$G(ikw_o)(\Delta_{-1}Y_{k+1} + \Delta_o Y_k + \Delta_1 Y_{k-1}) = Y_k \tag{7.24}$$

Assuming $\Delta_{-1} \neq 0$, we can write the following homogeneous difference equation:

$$\underline{Y_{k+1}} = -\frac{\Delta_1}{\Delta_{-1}} Y_{k-1} - \frac{1}{\Delta_{-1}} \left(\Delta_o - \frac{1}{G(ikw_o)} \right) Y_k \qquad (7.25)$$

Again, we only need to consider this equation for $k \geq 1$, since we are dealing with real systems and therefore $Y_{-k} = \overline{Y_k}$.

Define

$$Z_k = \begin{bmatrix} Y_{k-1} \\ Y_k \end{bmatrix}$$

Then equation (7.25) can be expressed as

$$Z_{k+1} = \mathcal{A}(k) Z_k \qquad k \geq 1 \qquad (7.26)$$

with

$$\mathcal{A}(k) = \begin{bmatrix} 0 & 1 \\ a & b_k \end{bmatrix}$$

$$a = -\frac{\Delta_1}{\Delta_{-1}} \qquad b_k = -\frac{1}{\Delta_{-1}} \left(\Delta_o - \frac{1}{G(ikw_o)} \right)$$

The transition matrix for this system is

$$\begin{aligned} \Phi(k+1, l) &= \mathcal{A}(k-1)\mathcal{A}(k-2) \ldots \mathcal{A}(l) \quad k > l \\ \Phi(k, k) &= I \end{aligned}$$

and the solution of (7.26) is

$$Z_k = \Phi(k) Z_l \qquad k = 1, 2, \ldots$$

One can check that $\Phi(k, 1)$ has terms of the form $b_1 b_2 \ldots b_j$ in its entries.

Since the transfer matrix $G(ikw_0)$ is "low pass" for most practical systems, i.e., $|G(ikw_o)| \to 0$ as $k \to \infty$, it follows that $b_1 b_2 \ldots b_k \to \infty$ as $k \to \infty$. Then, any sequence of solutions (Y_K) will not be on l^2 unless the coefficients of $D(t)$ are chosen in a way that $\mathcal{A}(k_o) Z_{k_o} = 0$ for some k_o, so that one can have $Z_k = 0$ for $k > k_o$.

But

$$\begin{bmatrix} 0 & 1 \\ a & b_{k_o} \end{bmatrix} \begin{bmatrix} M_{k_o-1} \\ N_{k_o-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if and only if $N_{k_o-1} = 0$ and $a.M_{k_o-1} = 0$, which one can easily check that is not achievable for real perturbations (i.e., $a \neq 0$).

We remark that if we allow for complex perturbations, one such solution would be to choose w_o such that $||G(iw_o)|| = ||G||_{H^\infty}$.

Therefore $\Delta_o = G(iw_o)^{-1}$ gives $b_1 = 0$.

For $\Delta_1 = 0$, we would have

$$\mathcal{A}(1)Z_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for any initial condition of the form

$$Z_1 = \begin{bmatrix} M \\ 0 \end{bmatrix}$$

But then $||D(.)||$ could be made arbitrarily close to r_C by taking Δ_{-1} sufficiently small. This is no upper bound for $r_{R,t}$.

Finally, we observe that if we consider a perturbation with more harmonics

$$D(t) = \sum_{-p}^{p} \Delta_k e^{ikw_o t}$$

the nature of the problem with the transition matrix is still the same. The difference is only that its order would be greater, but one still would have to cope with the unboundedness of some of its entries. Even the fact that then one would have more parameters (that is, more coefficients Δ_k) to fiddle about is hardly promising.

y_o	y_1	$\hat{y}_1$	λ	$\|\lambda\|$
0	1	0	$\pm i$	1
0	0	1	± 1	1
0	1	1	$\mp 0.707 \pm 0.707i$	1
2.5	1	0	-0.5	0.5
5	2	0	-0.5	0.5
5	0	2	$\pm 0.5i$	0.5
± 1	± 1	± 1	$0.5 \pm 0.5i$	0.707
4	0.5	0.3	$0.097 \pm 0.144i$	0.144
1	1	2	$0.790 \pm 0.272i$	0.836

Figure 7.1: Some examples illustrating different values of $|\lambda|$ for different choices of initial condition for the oscillator.

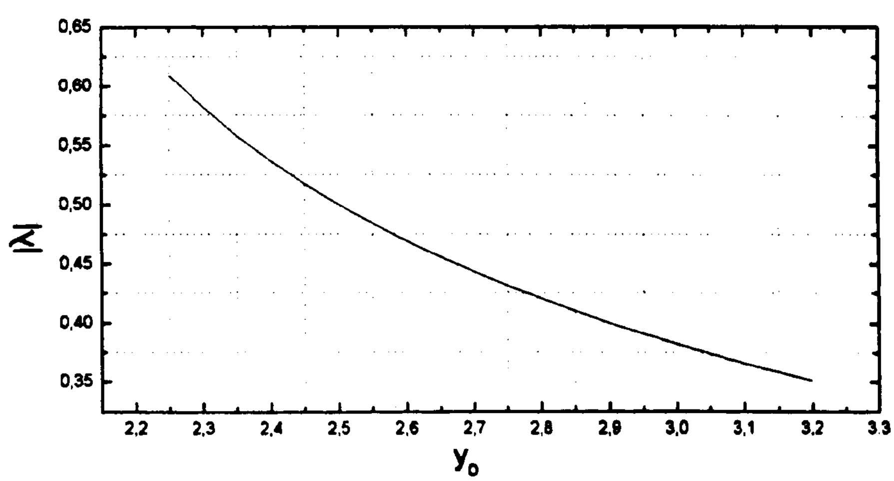

Figure 7.2: Variation of $|\lambda|$ with respect to y_o when $y_1 = 1$ and $\hat{y}_1 = 0$.

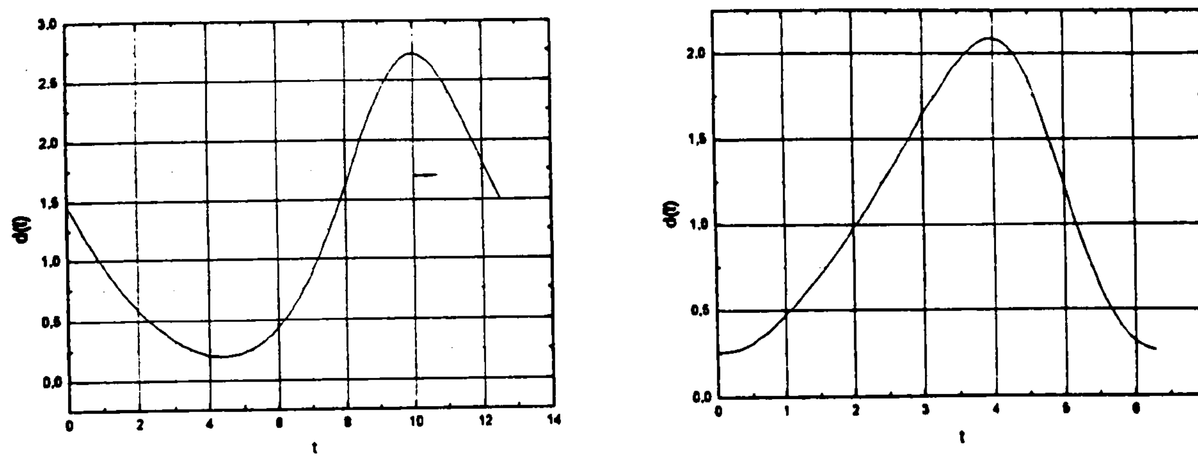

Figure 7.3: The graph to the left shows $d(t) \times t$ for $w_o = 0.5$ while the right one is for $w_o = 1$. The other parameters are $\xi = 0.3$, $y_o = -5$, $y_1 = 0$ and $\hat{y}_1 = 1$

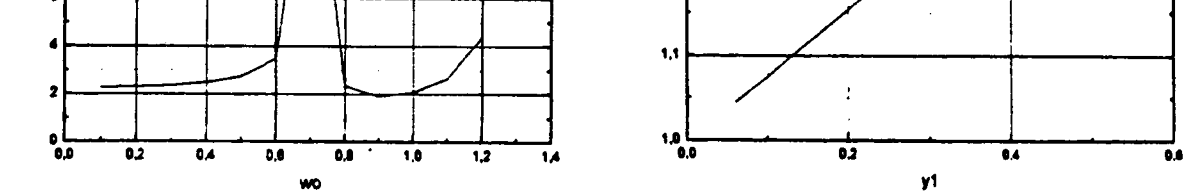

Figure 7.4: Dependence of $||d(.)||$ on w_o and y_1.

Chapter 8

CONCLUSION

This chapter presents a concluding remark on the whole set of the dissertation and the research work herein developed. It adds to a number of such remarks which have already appeared in each chapter when opportune. Also some suggestions for further investigations are here outlined.

The first part of the work has to do fundamentally with a new approach to problems of designing controllers in the face of uncertainties. This conditioning of controllability problem breaks out of the paradigm of giving yes/no answers to the question of controllability to concern with the more practical aim of giving a measure of the robustness of the controllability.

A peculiarity of our approach is that it assumes multiple-structured nonlinear perturbations. Thus, it accounts for a class of perturbations which is very general both in the structural and substantial aspects of the disturbances. In other words, it stretches the reach of concrete systems, plants and situations for which the results can be employed as useful tools for designing.

Another peculiarity is that it introduces a scaling technique which allows one to consider perturbations with "magnitude" bigger than those allowed when no scaling is considered. In this sense, it enlarges the class of perturbations under consideration. On the other hand, this scaling technique opens up the possibility of minimising the conditioning number. However, this optimisation perspective relies on the availability of minimising algorithms for the norms of the operators involved. With respect to $||C_\alpha x||$ there is no setback because the problem can be posed in the context of differentiable functionals and the Lagrange multipliers method can be used. But the minimisation of $||L_\alpha||$ is not as gratifying since its dependence on the weight α is not differentiable and an algorithm for non-differentiable infinite dimensional minimisation problems is not available as yet.

Eventually, this can be an interesting area of investigation and to this respect, it may turn out of some use the theory of approximation schemes (approximation of infinite dimensional problems by a sequence of finite dimensional ones, Galerkin methods, and the like) via A-proper mappings, along the same lines as, say, Botelho [31].

Central to the part of the work concerned with robustness of stability is the contribution towards an approximation of the values of $r_{R,t}$, *the real stability radius in the context of time-varying linear perturbations.* For second order systems, we presented an algorithm culminating in the exact evaluation of $r_{R,t}$ and an asymptotic analysis which can be generalised for parametrised systems of arbitrary order. Both the algorithm and the asymptotic study were applied to the damped linear oscillator so that the results add to the analysis presented in Hinrichsen-Pritchard [13] and re-approach early works on the stability boundaries for the Mathieu equation (Narendra-Taylor [28] and Parks [29] among others).

In order to overcome the drawback that no characterisation of $r_{R,t}$ is as yet available for systems of dimension greater than 2, what one can have at the present stage of research is to count on bounds for it. After proving that, for certain values of the damping, $r_{R,t}$ can lie between r_C and r_R in the case of the linear oscillator, we introduced a new approach to the problem by means of considering periodic perturbations and the theory of Fourier series with the purpose to generate a method that yields upper bounds for $r_{R,t}$ in the general case of n-dimensional systems.

One nice outcome of this approach is a characterisation of destabilising periodic perturbations which generalises Hinrichsen-Pritchard's characterisation for constant perturbations. As such, the new result generalises or reproduces previous propositions in the theory of time-invariant stability radii r_C and r_R. This fact has theoretical relevance on itself. On what concerns computability and practical design of concrete systems, a very welcome result would be to show some instance of the method leading to less conservative upper bounds to $r_{R,t}$ than the time-invariant real stability radius. Unfortunately, this remains an open question still.

We have been able to prove unfounded the original expectation of the supervisor that imposing on the system a periodic perturbation, with infinitely many harmonics and yielding an elliptic orbit as a response, would lead to the desired tighter upper bound. This task demanded a great deal of conceptual meanderings and difficult technical manipulations until we were able to negate the conjecture. The reason for this struggle was the intrinsic compromise between topological and algebraic issues: the need to guarantee L^2-convergence of Fourier series together with compatibility algebraic constraints on the Fourier coefficients. But some very promising insights have come out of this strive which are directing some of my further research activities. They concern some concepts on dynamical systems, control theory and the nature of research activity. One of these directions is on improving the method in order to tackle the problem of tightening the bounds for $r_{R,t}$: it seems crucial to drop the continuity on the resulting destabilising periodic perturbation. This is already being done in Botelho-Guimarães [33], where together with this point we focus on the question of how bad or good (computationally) it is to replace nonlinear terms by linear perturbations on the linearisation of the van der Pol's equation.

We conclude this chapter with some remarks. The first three of them (two other suggestions for further research and one result obtained by means of frequency domain methods) are organised as three separate sections whilst the remaining minor comments were concentrated in one last section.

8.1 The dual observability problem

A natural question which arises, once the problem of conditioning of controllability is tackled, concerns the search for similar results for its dual problem, the observability one. The idea is that, given a mathematical model of the dynamics of a real process, the observation of the state for the model may not lead to the actual initial condition of the process. Again, this is due to a number of uncertainties on parameters or concessions on behalf of some simplification assumed during the modelling of the system.

An approach to be favoured here would be to consider both the model and a perturbation of it and then to deduce an estimate for the error between the initial conditions coming out from each system (that is, the nominal and the perturbed ones).

Thus, suppose that the system:

$$\begin{aligned} \dot{x} &= Ax \qquad x(0) = x_o = My \\ \tilde{y} := FMy &= Cx \end{aligned}$$

is continuously initially observable on $[0, T]$.

Here, we are taking $A \in \mathbf{R}^{n \times n}$, $C \in \mathbf{R}^{p \times n}$ and F is a bounded linear operator from $\mathbf{R}^n$ into $L^2[0, T; \mathbf{R}^p]$ defined, for every $\varphi \in \mathbf{R}^n$, by

$$F\varphi\,(t) = Ce^{At}\varphi$$

M is assumed to be a bounded linear operator from $L^2[0, T; \mathbf{R}^p]$ into $\mathbf{R}^n$ such that the product (in the sense of composition) MF is the identity map I_n. Obviously, the natural candidate for M would be $M = (F^*F)^{-1}F^*$.

The perturbed system is taken to be of the form:

$$\begin{aligned} &\dot{z} = Az + g(z) \quad z(0) = z_o \\ &y = Cz \end{aligned}$$

To avoid unnecessary formal complications at this stage of the analysis, we will consider single nonlinear perturbations. So we assume g to be given by

$$g(z) = EN(Gz)$$

We are given the output y of the nonlinear system and we use the observer to estimate the initial state $x_o = My$ so we can solve for $x(.)$ and Gx. Then the error is given by

$$\begin{array}{ll} \dot{e} = Ae + EN(Gz) & e(0) = e_o = z_o - x_o \\ MCe = 0 & (\text{i.e., } y - \tilde{y} \in ker\ M) \end{array}$$

whose solution is

$$e = e^{A\cdot}e_o + L_E N(Gz) \tag{8.1}$$

Hence, $0 = MCe = e_o + MCL_E N(Gz)$, from which it follows the expression for the auxiliary initial state:

$$e_o = -MCL_E N(Gz) \tag{8.2}$$

So, we can write (8.1) more conveniently as

$$e = (I - e^{A\cdot}MC)L_E N(Gz)$$

or

$$u = G(I - e^{A\cdot}MC)L_E N(u) + Gx \qquad \text{for } u = Gz$$

and we are faced again with the problem of ensuring the existence of a unique fixed point for a map $H(u) = G(I - e^{A\cdot}MC)L_E N(u) + Gx$ on some ball $\overline{B_a}(0) \subset L^2[o, T; \mathbf{R}^p]$, for a convenient value of $a > 0$.

Now, for $a \in S$, it follows that

$$1 - ||G(I - e^{A\cdot}MC)L_E||k(a) > 0$$

Therefore, if $u, v \in \overline{B_a}(0)$ we have

$$||H(u) - H(v)|| \leq ||G(I - e^{A\cdot}MC)L_E||\ k(a)\ ||u - v||$$

which gives that H is a strict contraction from $\overline{B_a}(0)$ into itself, since

$$||H(u)|| \leq ||G(I - e^{A\cdot}MC)L_E||\ k(a)\ ||u||\ +\ ||Gx|| \leq a$$

So, it follows that $H(u) = u$ has a unique solution in $\overline{B_a}(0)$ and the estimate can be performed in the same way as in the proof for the controllability case.

□

From the result above, one could expect a good deal of similarities to be shared with on the treatment of both the controllability problem and its dual. However, the remaining task of characterising the norms of some of the operators involved may turn out to be even more delicate than that for the controllability problem.

8.2 Impulsive perturbations

Consider again the oscillator

$$\ddot{\varphi}(t) + 2\xi\dot{\varphi} + (1 + d(t))\varphi = 0 \tag{8.3}$$

with the following solution and perturbation:

$$\varphi(t) = a_1 \cos w_o t + b_1 \sin w_o t = c \sin(w_o t + \phi) \tag{8.4}$$

$$d(t) = -1 + w_o^2 + \delta(t) \tag{8.5}$$

with $\delta(t)$ periodic of period $T = 2\pi/w_o$.

Substituting (8.4) and (8.5) in (8.3) gives

$$\delta(t) = -2\xi w_o cotan(w_o t + \phi) \tag{8.6}$$

as the necessary condition for compatibility.

Such $d(.)$ is not in L^∞, since $|d(t)| \to \infty$ as $t \to \hat{t} = T, 2T, \ldots$. Hence, one can infer from this observation that one way to get an elliptic orbit is to apply some periodically impulsive forcing on the system. This suggests that an approach in the sense of distributions to the convergence of the Fourier series can eventually be fruitful in some problems of robustness of stability.

Thus, we can consider the disturbances on the nominal system to be represented by periodic tempered distributions, that is, perturbations

$D(.)$ with Fourier coefficients Δ_k satisfying

$$\sum_{k=-\infty}^{\infty} (1+k^2)^{s/2} \, ||\Delta_k|| \; < \; \infty$$

Of course, this approach requires some theoretical issues (existence of global solutions for the system and stability theory) to be considered in this new context. We present here the main points of one such stability theory.

Suppose $\tilde{f}$ is a generalised function of some kind, say, a tempered distribution, that is, $\tilde{f} \in S'$, the space of bounded linear functionals on the Schwarz space S' of rapidly decreasing functions.

We say that

$$\tilde{f} \; \in \; \mathcal{H}_T^s(0,\infty)$$

if for each $p \in \mathbf{N}$, $\tilde{f}$ is identified with a sequence $(F_{pk})_{k \in Z}$ of generalized Fourier coefficients obtained via either one of these procedures:

(i) if $\tilde{f}$ is regular (i.e., $\tilde{f} \; = f$, an ordinary function),

$$F_{pk} = \frac{1}{T} \int_{(p-1)T}^{pT} f(t) e^{-ik(2\pi/T)t} dt$$

(ii) otherwise, we make use of the following theory:

Theorem(Champeney [5])
If $\tilde{f}_p \in S'$ has period t, then the Fourier transform $\tilde{F}_p$ of $\tilde{f}_p$ (exists) and can be written in the form:

$$\tilde{F}_p(w) = \sum_{k=-\infty}^{\infty} F_{pk} \tilde{\delta} \left(w - \frac{k}{T} \right) \qquad \text{in } S'$$

where $F_{pk} \in \mathbf{C}$, $k \in \mathbf{Z}$ and $\tilde{\delta}$ is the Dirac's delta "function".

Moreover, $\exists \; A_p > 0, M_p > 0$ such that for each k

$$|F_{pk}| \leq A_p |k|^{M_p}$$

When $\tilde{f}$ is regular and periodic, the generalized Fourier coefficients F_{pk} are identical to the ordinary ones.

Theorem(Champeney [5])
If $\tilde{f}_p \in S'$ has period T and has Fourier coefficients F_{pk}, then

$$\tilde{f}_p(t) = \sum_{k=-\infty}^{\infty} F_{pk} e^{ik(2\pi/T)t} \qquad \text{in } S'$$

Basically, by taking these F_{pk}, what we are doing is to identify $\tilde{f}$ with a Fourier series $\sum_k F_{pk} e^{ik(2\pi/T)t}$ on each interval $J_p = ((p-1)t, pT)$:

Of course, we need to make explicit sense to the convergence

$$\tilde{f}_p(t) = \sum_{k=-\infty}^{\infty} F_{pk} e^{ik(2\pi/T)t}$$

in the theorem above.

The characterisation can now be formalised by saying that

$$\tilde{f} \in \mathcal{H}_T^s$$

iff

for each $p \in \mathbf{N}$, $\tilde{f}$ is identified with a sequence $(F_{pk})_{k \in Z}$ of generalized Fourier coefficients satisfying

$$||\tilde{f}_p||_{H^s} := \sum_{k=-\infty}^{\infty} \left(1+k^2\right)^s |F_{pk}| < \infty$$

and

$$||\tilde{f}||_{\mathcal{H}_T^s(0,\infty)} := \sup_{p \in N} ||\tilde{f}_p||_{H^s} < \infty$$

We can have the well-posedness of this perturbed system by first establishing the boundedness of the operators D and L.

Assume the periodic perturbation $D(.)$ is such that it has a Fourier series

$$D(t) \sim \sum_{k=-\infty}^{\infty} \Delta_k e^{ik(2\pi/T)t}$$

with

$$\sup_j |\Delta_{k-j}| = m_k < \infty \quad \forall k \in \mathbf{Z}$$

Note that this is the case when, for instance, Δ_k is constant and therefore $D(.)$ is a tempered distribution.

Then

$$\left(1+k^2\right)^{-s/2} |\Delta_{k-j}| \le \left(1+k^2\right)^{-s/2} |m_k|$$

so that $\sum_k (1+K^2)^{-s/2}|\Delta_{k-j}|$ converges uniformly with respect to j if we assume that $\sum_k (1+k^2)^{-s/2}|m_k|$ converges for some $s > 0$.

Now, concerning the operator $\mathcal{D}$ defined by

$$\mathcal{D}y\ (t) = D(t)y(t)$$

we have that, given $y \in \mathcal{H}_T^o(0,\infty)$, then for each $p \in \mathrm{N}$, $\mathcal{D}y$ can be identified with a sequence $(\zeta_{pk})_k$, that is

$$D(t)y(t) \sim \sum_k \zeta_{pk} e^{ik(2\pi/T)t} \qquad \text{on each } J_p$$

where

$$\zeta_{pk} = \sum_{j=-\infty}^{\infty} \Delta_{k-j} y_{pk}$$

Note that for each p and k, the Hölder's inequality yields

$$|\zeta_{pk}| \le \sum_j |\Delta_{k-j}|.|y_{pk}| \le m_k \sum_j |y_{pk}| < \infty$$

Thus, we can write

$$\sum_k (1+k^2)^{-s/2}|\zeta_{pk}| \le \sum_k \sum_j (1+k^2)^{-s/2}|\Delta_{k-j}|.|y_{pk}| = c_s . \sum_{j=-\infty}^{\infty} |y_{pk}|$$

where

$$c_s = \sum_{k=-\infty}^{\infty} (1+k^2)^{-s/2}|\Delta_{k-j}| < \infty$$

independently of $j \in \mathbf{Z}$, so that the order of "summation" can be interchanged. So,

$$\sum_{p\in\mathrm{N}} \sum_{k=-\infty}^{\infty} (1+k^2)^{-s/2}|\zeta_{pk}| \le c_s \sup_{p\in\mathrm{N}} \sum_{j=-\infty}^{\infty} |y_{pk}| < \infty$$

which proves that $\mathcal{D}$ is a bounded linear operator from $\mathcal{H}_T^o(0,\infty)$ into $\mathcal{H}_T^{-s/2}(0,\infty)$ and

$$||\mathcal{D}|| \leq c_s = \sum_{k=-\infty}^{\infty} (1+k^2)^{-s/2}|\Delta_{k-j}| < \infty$$

independently of $j \in \mathbf{Z}$.

On the other hand, in order to have that the input-output operator L is a bounded linear operator from $\mathcal{H}_T^{-s/2}(0,\infty)$ into $\mathcal{H}_T^o(0,\infty)$ we can pursue the following reasoning:

Suppose $u : [0,\infty) \longrightarrow \mathbf{C}$ is an ordinary function. The hypothesis that $u \in \mathcal{H}_T^s(0,\infty)$ implies that, for each $p \in \mathbf{N}$, we would take a T-periodic function u_p such that $u_p = u$ on $J_p = ((p-1)T, pT)$.

So, $Lu = Lu_p$ on J_p and we can say that $(Lu)_p = Lu_p$ is a T-periodic function such that $Lu = (Lu)_p$ on J_p and the system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu_p(t) \qquad t \geq 0 \\ &y(t) = Cx(t) \\ &x(0) = 0 \end{aligned}$$

yields

$$\hat{y}(s) = G(s)\hat{u}_p(s) \quad \text{and} \quad \hat{y} = (Lu_p)^\wedge(s)$$

Hence

$$(Lu_p)^\wedge(s) = G(s)\hat{u}_p(s)$$

so that the Fourier coefficients of (Lu_p) are given by

$$(Lu_p)_k = G\left(i\frac{k}{T}\right)\hat{u}_p\left(i\frac{k}{T}\right) = G\left(i\frac{k}{T}\right)u_{pk}$$

Therefore,

$$(Lu)_{pk} = (Lu_p) = G\left(i\frac{k}{T}\right)$$

where u_{pk} are the Fourier coefficients of u_p and

$$G\left(i\frac{k}{T}\right) = C\left(i\frac{k}{T}I - A\right)^{-1} B$$

so that

$$\sup_{p\in\mathbf{N}} \sum_{k=-\infty}^{\infty} |(Lu)_{pk}| \leq \gamma_s \sup_{p\in\mathbf{N}} \sum_{k=-\infty}^{\infty} (1+k^2)^{-s/2}|u_{pk}| < \infty$$

where

$$\gamma_s = \sup_{k\in\mathbf{Z}}(1+k^2)^{s/2}\left|G\left(i\frac{k}{T}\right)\right|$$

is assumed to be finite.

This gives that L is bounded and $||L|| \leq \gamma_s$.

It should be noted that the construction of the spaces $\mathcal{H}_T^s$, although allowing a simple treatment in the proof of the boundedness of the operator $\mathcal{D}$, resulted somewhat ackward when dealing with the boundedness of the operator L. This feature is inverted if we change from a series approach to a Fourier transform formulation of the problem. In fact, consider the Sobolev space H^s, $s \in \mathbf{R}$, defined as the completion of the Schwartz space $\mathcal{S}$ of the smooth functions f with $f(t) = 0$, $\forall t < 0$ and for which $\sup |t^\alpha f^{(\beta)} f(t)| < \infty$, for all $\alpha, \beta \in \{0, 1, 2, \ldots\}$, with respect to the norm

$$||f||_{H^s} = ||\Lambda^s f||_{L^2}$$

where

$$(\Lambda^s f)^\wedge(w) = (1+|w|^2)^{s/2}\hat{f}(w)$$

(V. Folland[14], for instance.)

Suppose $s > 0$. Then we have

Proposition 8.2.1
Consider $L : H^o = L^2(0,\infty;\mathbf{R}^m \longrightarrow H^s(0,\infty;\mathbf{R}^p)$, *given by*

$$Lu\,(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

If

$$\gamma_s^2 = \sup_{w\in R}(1+|w|^2)^s|G(iw)|^2 < \infty$$

then L *is bounded and* $||L|| < \gamma_s$.

Proof:
The result follows easily, since

$$||Lu||^2_{H^s} = ||(\Lambda^s Lu)^\wedge||^2_{L^2} = \int_{-\infty}^{\infty} (1+|w|^2)^s |G(iw)|^2 . |\hat{u}(w)|^2 dw$$

□

Proposition 8.2.2
Suppose $\mathcal{D}$ is the operator from $H^s(0,\infty;\mathbf{R}^p)$ into $L^2(0,\infty;\mathbf{R}^m)$ defined by

$$\mathcal{D}y\ (t) = D(t)y(t)$$

where $D(.) \in H^{-s}(0,\infty;\mathbf{R}^p)$ is such that $\hat{D} \in K^{-s}$, the space of all locally integrable $\hat{f}$ with

$$\int_{-\infty}^{\infty} (1+|w|^2)^{-s/2} |\hat{f}(w)| dw \ < \ \infty$$

Then, $\mathcal{D}$ is bounded and

$$||\mathcal{D}|| \leq \int_{-\infty}^{\infty} (1+|w|^2)^{-s/2} |\hat{D}(w)| dw$$

Proof:
Consider the map $y \in H^s \mapsto D_{-s}y_s$ where

$$D_{-s} = \Lambda^{-s} D \qquad \text{and } y_s = \Lambda^s y$$

Note that

$$\begin{array}{lll} D \in H^{-s} & \Longleftrightarrow & \Lambda^{-s} D \in L^2 \\ y \in H^s & \Longleftrightarrow & \Lambda^s y \in L^2 \end{array}$$

We have

$$(D_{-s}.y_s)^\wedge(w) = \int_{-\infty}^{\infty} D_{-s}(t) y_s(t) e^{-iwt} dt = \hat{D}_{-s} * \hat{y}_s\ (w)$$

since $y_s, \hat{D}_{-s} \in L^2$. Hence,

$$(D_{-s}.y_s)^\wedge(w) = \int_{-\infty}^{\infty} h(w-\eta).f(\eta)d\eta$$

where

$$h(w) = (1+|w|^2)^{-s/2}\hat{D}(w)$$
$$f(w) = (1+|w|^2)^{s/2}\hat{y}(w)$$

Now, using Hölder's inequality, we can write

$$|(D_{-s}y_s)^\wedge(w)| = \left|\int_{-\infty}^{\infty} h(w-\eta)^{1/2}h(w-\eta)^{1/2}f(\eta)d\eta\right| \leq$$

$$\leq \left(\int_{-\infty}^{\infty} |h(w-\eta)|d\eta\right)^{1/2}\left(\int_{-\infty}^{\infty} |h(w-\eta)|.|f(\eta)|^2 d\eta\right)^{1/2}$$

because the translation invariance property of Lebesgue measure gives

$$\int_{-\infty}^{\infty}(1+|w-\eta|^2)^{-s/2}||\hat{D}(w-\eta)|d\eta =$$

$$= \int_{-\infty}^{\infty}(1+|\xi|^2)^{-s/2}|\hat{D}(\xi)|d\xi = c$$

and also

$$\int_{-\infty}^{\infty} |h(w-\eta)|.|f(\eta)|^2 d\eta \ < \ \infty$$

since $h, |f|^2 \in L^1$.

So, by the Plancherel's and Fubini-Tornelli's theorems

$$\begin{aligned}
||D_{-s}y_s||_{L^2}^2 &= \textstyle\int_{-\infty}^{\infty} |(D_{-s}y_s)^\wedge(w)|^2 dw \\
&\leq c\textstyle\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |h(w-\eta)|.|f(\eta)|^2 d\eta\right) dw \\
&= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(w-\eta)|.|f(\eta)|^2 dw d\eta \\
&= c||y||_{H^s}^2.
\end{aligned}$$

□

The following result shows that this theory generalises the case of constant perturbations.

Proposition 8.2.3
If $\mathcal{D} : L^2 \longrightarrow H^{-s}$, $s \geq 0$, is given by

$$\mathcal{D}y\ (t) \equiv \Delta y(t)$$

for $\Delta \in \mathbf{R}^{m \times p}$ constant, then

$$||\mathcal{D}|| = ||\Delta||$$

Proof:
Consider $y \in L^2(0, \infty; \mathbf{R}^p)$. Then

$$\begin{array}{rcl} ||\mathcal{D}y||^2_{H^{-s}} & = & ||(\Lambda_{-s}\Delta y)^\wedge||^2_{L^2} \\ & = & \int_{-\infty}^{\infty}(1+|w|^2)^{-s}|\Delta\hat{y}(w)|^2 dw \\ & = & ||\Delta||^2 \int_{-\infty}^{\infty}(1+|w|^2)^{-s}|\hat{y}(w)|^2 \\ & \leq & ||\Delta||.||y||_{L^2} \end{array}$$

since $(1+|w|^2)^{-s} \leq 1$.

This shows that $||\mathcal{D}|| \leq ||\Delta||$.

On the other hand, for any $\varepsilon > 0$, take $y_\varepsilon \in L^2(0, \infty; \mathbf{R}^p)$ such that

$$\hat{y}_\varepsilon(w) = \sqrt{\varepsilon f(\varepsilon w)}\ v$$

where $f : \mathbf{R} \longrightarrow \mathbf{R}^+$ is an integrable function with $\int_{-\infty}^{\infty} f = 1$ and $v \in \mathbf{R}^p$ is such that $|v| = v^* v = 1$.

Then $|\hat{y}_\varepsilon(w)|^2 = \varepsilon f(\varepsilon w)$ and

$$||y_\varepsilon||^2_{L^2} = \int_{-\infty}^{\infty} f = 1$$

Also, $\lim_{\varepsilon \to 0} \varepsilon f(\varepsilon w) = \delta(w)$ in $\mathcal{S}'$.

Therefore,

$$||\mathcal{D}y_\varepsilon||^2_{H^{-s}} = ||\Delta||^2 \int_{-\infty}^{\infty}(1+|w|^2)^{-s}|\hat{y}_\varepsilon(w)|^2 dw$$

But

$$\int_{-\infty}^{\infty}(1+|w|^2)^{-s}|\hat{y}_\varepsilon(w)|^2 dw = \int_{-\infty}^{\infty}(1+|w|^2)^{-s}\varepsilon f(\varepsilon w) dw \to 1$$

as $\varepsilon \to 0$. Since y_ε is in the boundary of the unit ball and tends to some y_o in $\mathcal{S}'$ as $\varepsilon \to \infty$, it follows that $||y_o|| = 1$ and

$$\lim_{\varepsilon \to 0} ||Dy_\varepsilon||_{H^{-s}} = ||Dy_o||_{H^{-s}}$$

We have

$$\begin{aligned} \lim_{\varepsilon\to 0} ||Dy_\varepsilon||^2_{H^{-s}} &= \lim_{\varepsilon\to\infty} ||\Delta||^2 \textstyle\int_{-\infty}^{\infty}(1+|w|^2)^{-s}\varepsilon f(\varepsilon w)dw \\ &= ||\Delta||^2 \textstyle\int_{-\infty}^{\infty}(1+|w|^2)^{-s}\delta(w)dw \\ &= ||\Delta||^2 \end{aligned}$$

from which it follows that $||\mathcal{D}|| \geq ||\Delta||$. □

The theory for evaluation of $||D||_{L^\infty}$ can be constructed following exactly the same steps developed in last chapter. Naturally, there still remains a number of questions concerning the topology that has been constructed. Some of interesting ones are:

1. Is $\mathcal{H}^s_T(0,\infty)$ a Banach space?

2. Is S dense in each $\mathcal{H}^s_T(0,\infty)$ with respect to the norm defined above?

3. Denote $\mathcal{P}$ the space of functions from $[0,\infty)$ into $\mathbf{C}$ that are T-periodic and $\mathbf{C}^\infty$.

   For
   $$f \in \mathcal{P} \longmapsto f_p = \sum_k F_{pk} e^{ik(2\pi/T)t}$$
   where the convergence is in the normal sense, define $\Lambda_{p,s}$ on $\mathcal{P}$ setting,
   $$(\Lambda_{p,s})_k = \left(1+k^2\right)^s F_{pk}$$
   We should have that $\Lambda_{p,s} f \in \mathcal{P}$.

4. Do we have the crucial decaying property that $f(t) \to 0$ as $t \to \infty$ for all $f \in \mathcal{H}^s_T(0,\infty)$?

5. Can we have any result of the form $|| \cdot ||_{\mathcal{H}^s_T} \leq || \cdot ||_{\mathcal{H}^r_T}$?

6. Can we have some kind of Sobolev lemma in this context?

7. What happens when $\tilde{f}$ has period T?

8. Note that for $s = 0$, the characterisation gives

$$\sum_k |F_{pk}| < \infty \qquad \text{and} \qquad \sup_p \sum_k |F_{pk}| < \infty$$

   Can we characterise $\mathcal{H}^s_T(0,\infty)$ for $s > 0$ and $s < 0$?

8.3 Convolutive perturbations

The approach to robustness analysis favoured in the whole body of the present dissertation is the state space one. Also relevant is the H^∞-approach to robustness of stability, which uses frequency domain techniques and started to flourish at the beginning of the 80's (see Zames[35] and Francis[36]). Regardless of any eventual reasoning either to differentiate the scope of each approach or to point out their merits and drawbacks, some class of perturbations can be more naturally dealt with in one or the other approach preferentiably. In particular, if we consider convolutive perturbations, the natural technique for studying the real stability radius turns out to be the frequency domain one. In this remark we provide a proof, by means of a frequency domain method, that the real stability radius for the class of convolutive perturbations is equal to the time-invariant complex stability radius.

Let $K(.) \in H^\infty$ be given. We recall that

$$||K||_{H^\infty} := ess \sup_{w \in \mathbf{R}} |\hat{K}(iw)|$$

where $\hat{K}(iw)$ is the Fourier-Plancherel transform of K.

We consider a perturbation of the form

$$\mathcal{N}y \ \ (t) = (K * y)(t) = \int_0^t K(t-\tau)y(\tau)d\tau$$

for every $y(.) \in L^2(0,\infty;\mathbf{R}^p)$.

Clearly, we have that $\mathcal{N}y \in L^2(0,\infty;\mathbf{R}^m)$ and

$$||\mathcal{N}y||_{L^2} \leq ||K||_{H^\infty}||y||_{L^2}$$

so that we can define the robustness measure for this case as being

$$r_{R,*} = \inf\{||K||_{H^\infty} \ ; \ K(t) \in \mathbf{R}^{m\times p} \quad \text{and}$$

$$\dot{x} = Ax + B(K * Cx) \quad \text{is not asymptotically stable}\}$$

We have the following result:

Proposition 8.3.1

$$r_{R,*} = r_C$$

Proof: Take $\Delta_o \in \mathbf{C}^{m\times p}$ such that

$$||\Delta_o|| = r_C = \inf\{||\Delta|| \ ; \ \sigma(A + B\Delta C) \cap i\mathbf{R} \neq \emptyset\}$$

Clearly, $r_{R,\star} \geq r_C$.
We have that $\Delta_o = \Delta_1 + i\Delta_2$ for $\Delta_1, \Delta_2 \in \mathbf{R}^{m\times p}$.
Also,

$$||\Delta_o|| = r_C = \frac{1}{\max_{w\in\mathbf{R}} ||G(iw)||} = \frac{1}{||G||_{H^\infty}}$$

where $||G||_{H^\infty} = \max_{w\in\mathbf{R}} ||G(iw)|| = ||G(iw_o)||$ for some $w_o \in \mathbf{R}$.

Since we are focussing on *real* perturbations, we can consider only $w \geq 0$. So we introduce $K \in H^\infty$ by putting $K(0) = 0$ and

$$\hat{K}(iw) = \frac{\Delta_1 + i\frac{w}{w_o}\Delta_2}{1 + i\left(\frac{w}{w_o} - \frac{w_o}{w}\right)q} \quad \forall w > 0$$

for some $q > 0$.

Note that, since $\hat{K}(iw_o) = \Delta_o$ is destabilizing, there exists a non-zero $y \in \mathbf{C}^p$ such that

$$G(iw_o)\hat{K}(iw_o)y = y$$

or, for the non-zero vector $u = \hat{K}(iw_o)y$,

$$(I - \hat{K}(iw_o)G(iw_o))u = 0$$

From this we conclude that

$$\det(I - \hat{K}(iw_o)G(iw_o)) = 0$$

and consequently

$$\inf_{\Re e\, s \geq 0} |\det(I - \hat{K}(s)G(s))| = 0$$

This finally implies that K is destabilising.
On the other hand,

$$\begin{aligned} |\Delta_o z|^2 &= < \Delta_1 z + i\Delta_2 z, \Delta_1 z + i\Delta_2 z > \\ &= |\Delta_1 z|^2 + |\Delta_2 z|^2 + 2\Im m\ (< \Delta_1 z, \Delta_2 z >) \end{aligned}$$

Since $|\Delta_o z|^2 = |\Delta_1 z|^2 + |\Delta_2|^2$ when $z \in \mathbf{R}^p$, we have that

$$\begin{aligned} |\Delta_1 z|^2 \leq |\Delta_o z|^2 \leq r_C^2 |z|^2 \\ |\Delta_2 z|^2 \leq |\Delta_o z|^2 \leq r_C^2 |z|^2 \end{aligned}$$

and

$$2\Im m(< \Delta_1 z, \Delta_2 z >) \ \leq \ r_C^2 |z|^2 - \left(|\Delta_1 z|^2 + |\Delta_2 z|^2\right) \qquad (8.7)$$

This enables us to write

$$|\hat{K}(iw)z|^2 = \left| \frac{1}{1 + \left(\alpha - \frac{1}{\alpha}\right) qi} \right|^2 < \Delta_1 z + i\alpha\Delta_2 z, \Delta_2 z + i\alpha\Delta_2 z >$$

where $\alpha > 0$ denotes the ratio $\alpha = w/w_o$.
Hence, by using (8.7), we have

$$|\hat{K}(iw)z|^2 \leq \frac{1}{1 + \left(\alpha - \frac{1}{\alpha}\right)^2 q^2} \left[(1-\alpha)|\Delta_1 z|^2 + (\alpha^2 - \alpha)|\Delta_2 z|^2 + \alpha r_C^2 |z|^2\right]$$

Note that $||\hat{K}(iw)|| \rightarrow 0$ as $w \rightarrow 0$, that is, $\alpha \rightarrow 0$.

Now, for $0 < \alpha \leq 1$ we have

$$|\hat{K}(iw)z|^2 \leq \frac{1}{1+\left(\alpha-\frac{1}{\alpha}\right)^2 q^2} r_C^2 |z|^2 \leq r_C^2 |z|^2$$

and for $\alpha > 1$,

$$|\hat{K}(iw)z|^2 \leq \frac{\alpha^2}{1+\left(\alpha-\frac{1}{\alpha^2}\right)^2 q^2} r_C^2 |z|^2 \leq \frac{4q^2}{4q^2-1} r_C^2 |z|^2$$

where

$$\frac{4q^2}{4q^2-1} = \max_{\alpha>1} \frac{\alpha^2}{1+\left(\alpha-\frac{1}{\alpha}\right)^2 q^2}$$

Therefore,

$$r_C = ||\hat{K}(iw_o)|| \leq ||\hat{K}||_{H^\infty} \leq r_C \quad \text{if } 0 < w \leq w_o$$

and

$$r_C \leq ||K||_{H^\infty} \leq \sqrt{\frac{4q^2}{4q^2-1}}\, r_C \quad \text{if } w > w_o$$

Since

$$\lim_{q \leftarrow \infty} \sqrt{\frac{4q^2}{4q^2-1}} = 1$$

we have that

$$r_{R,*} \leq ||K||_{H^\infty} = r_C$$

and we have the result. □

8.4 Other remarks

Remark 8.4.1: (Minimisation of $||L_\alpha||$ revisited)

We have already commented on the problem of non-availability of an algorithm for the minimisation of $||L_\alpha||$ elsewhere on the dissertation. One of the major obstacles to this purpose is that presently there is no result showing any smoothness property on the dependence of the norm on the weights α and, more than this, the differentiability of $||L_\alpha^* L_\alpha||$ seems unlikely. Although for the time being we are not prepared to venture on computability issues to the respect of minimisation algorithms and we are not supposed to expect any differentiability on the problem, we present here an optimality condition for the case when the functional is differentiable at the point where the extremum is achieved.

For simplicity, we shall assume $r = 1$ (i.e., single structured perturbations). Note that once more the condition is expressed in terms of a Hamiltonian system.

Proposition 8.4.1

Suppose $\tilde{\lambda} = \min_\alpha \max\{\lambda \in \mathbf{R}\ ;\ \lambda \textit{ is an eigenvalue of } L_\alpha^* L_\alpha\}$
Furthermore, suppose that

$$f(\alpha) := \max\{\lambda\ ;\ \lambda \textit{ is an eigenvalue of } L_\alpha^* L_\alpha\}$$

is differentiable at $\tilde{\alpha}$, *where* $f(\tilde{\alpha}) = \min_\alpha f(\alpha)$.
Then, λ *can be expressed by the following coupling*

$$\begin{array}{rcll} \dot{z}(t) & = & Az(t) + (\lambda\beta(t))^{-1}DD^* w(t) & z(0) = 0 \\ \dot{w}(t) & = & -A^* w(t) - \beta(t)C^* C z(t) & w(T) = 0 \end{array}$$

with $\beta(t) = \alpha(t)^2$.
Moreover, suppose $Z(t), W(t) \in \mathbf{R}^{n\times n}$ *solve*

$$\begin{bmatrix} \dot{Z} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} A & (\lambda\beta)^{-1}DD^* \\ -\beta C^* C & -A^* \end{bmatrix} \begin{bmatrix} Z \\ W \end{bmatrix} \qquad \begin{array}{l} Z(T) = I_n \\ W(T) = O_n \end{array}$$

Then, for any $\eta \in ker Z(0)$,

$$\begin{array}{l} z(t) = Z(t)\eta \\ w(t) = W(t)\eta \end{array}$$

solve the above coupling.

Proof:
Let u be an eigenvector of $L_\alpha^* L_\alpha$ associated with $\tilde{\lambda}$ (which we shall denote merely as λ for simplicity). Then $L_\alpha^* L_\alpha u = \lambda u$. Putting

$$v(t) = \frac{1}{\alpha(t)} u(t)$$

we can write the expression that gives implicitly the functional $\lambda = ||L_\alpha^* L_\alpha||$ to be minimised:

$$\int_t^T D^* e^{A^*(s-t)} \beta(s) C^* C \int_o^s e^{A(s-\rho)} D v(\rho) d\rho ds = \lambda \beta(t) v(t)$$

Here, λ and the eigenvector v depend on β.

$$\int_t^T D^* e^{A^*(s-t)} C^* \beta(s) \left\{ \int_o^s C e^{A(s-\rho)} D v(\rho) d\rho \right\} ds = \lambda \beta(t) v(t)$$

$$\begin{aligned} L^*(\beta L v)(t) &= \lambda \beta(t) v(t) \\ < v(t), L^*(\beta L v)(t) > &= < v(t), \lambda \beta(t) v(t) > \end{aligned}$$

where L is the ordinary input-output operator which is obtained for $\alpha \equiv 1$.

Without loss of generality, assume that v is normalized. Then, $< v, \beta v >= 1$ and

$$< d\beta v, v > +2 < \beta v, dv >= 0$$

We have

$$L^*(\beta L v) = \lambda \beta v$$

so that

$$L^*(d\beta L v) + L^*(\beta L dv) = d\lambda \beta v + \lambda \beta dv$$

Take inner product on the right with v to yield

$$< Lv, d\beta L v >= d\lambda + \lambda < v, d\beta v >$$

So the equations are

$$\begin{aligned} < Lv, d\beta L v > -\lambda < v, d\beta v > &= d\lambda & (8.8) \\ < \beta v, v > &= 1 & (8.9) \end{aligned}$$

Thus, equation 8.8 above is

$$\int_o^T d\beta(t)\{|Lv(t)|^2 - \lambda|v(t)|^2\}dt = d\lambda \tag{8.10}$$

Therefore, the optimal β will be the one that renders

$$|Lv(t)|^2 = \lambda|v(t)|^2 \qquad \text{a. e.} \tag{8.11}$$

so that the right-hand side is zero.

On the other hand, denote

$$z(t) = \int_o^t e^{A(t-s)}Dv(s)ds$$

Then

$$\dot{z} = Az + Dv \quad , \quad z(0) = 0 \quad \text{and} \quad Cz = Lv$$

Also, $\lambda\beta v = L^*(\beta Lv)$ and $Lv = Cz$ yield

$$\lambda\beta(t)v(t) = \int_t^T D^* e^{A^*(s-t)}\beta(s)C^*Cz(s)ds$$

Denote

$$w(t) = \int_t^T e^{A^*(s-t)}\beta(s)C^*Cz(t)dt$$

For the second part of the proof, we have

$$\begin{aligned}
\dot{z}(t) &= \dot{Z}(t)\eta = [AZ(t) + (\lambda\beta(t))^{-1}DD^*W(t)]\eta \\
&= Az(t) + (\lambda\beta(t))^{-1}DD^*w(t) \\
\dot{w}(t) &= \dot{W}(t)\eta = [-A^*W(t) - \beta(t)C^*CZ(t)]\eta \\
&= -A^*w(t) - \beta(t)C^*Cz(t)
\end{aligned}$$

and

$$\begin{aligned}
&z(0) = Z(0)\eta = 0 \qquad (\text{since } \eta \in ker Z(0)) \\
&w(T) = W(T)\eta = 0\eta = 0
\end{aligned}$$

□

Particularly interesting is equation (8.11) to the extent that it may suggest one direction for investigation on what concerns problems of stability. One can think of introducing the weight-functions and use the scaling technique to get (de-)stabilisation.

Remark 8.4.2: (Formalisation at a higher degree of abstraction)

The whole body of the work done here can be put in a more abstract functional analytic approach. We have already done this in the section on the minimisation of $||C_\alpha x||$ in chapter 3. The existence of T-periodic solution for $\dot{x} = Ax + BD(t)Cx$ with T-periodic continuous perturbation $D(.)$ can be set as a problem of existence of solution for an operator equation $Px = Qx$, with P being a strictly-γ-contraction and Q completely continuous after a certain projection technique is applied. This is being done in detail in Botelho-Gonçalves [32]. The concept of controllability may eventually be addressed under some approximation scheme formalisation, via the theory of A-proper mappings and encompassing infinite dimensional systems as well. In the context of reflexive Banach spaces, the exact controllability of a system $\dot{x} = Ax + Bu$ with L^2-inputs is equivalent to a map $\mathcal{B}$ being A-proper with respect to a projection scheme π. Here, $\mathcal{B}$ is defined by

$$\mathcal{B}u = \int_0^T S(T-s)Bu(s)ds$$

where $(S(t))_{t\geq 0}$ is a nonexpansive semigroup generated by A (which is the case when A is strongly dissipative). Cf. Deimling [1].

Remark 8.4.3: (Cognitive processes and "discourse analysis")

A question about robustness is one about the nature of a model, and the affinity of the notions of model and interpretation is self-evident. As an exercise of thought, we could move from the notion of robustness of dynamical systems to a broader one which could encompass thought, consciousness and cognitive processes as its objects. Or else, we could use the concept of dynamical systems to represent and study cognitive processes. In any case, for these representations the notion of robustness gets some distinct meanings: whilst for the usual representation of plants in engineering applications it is desirable that the system is robust, for cognitive processes the more robust the system the less efficient it. After all, the role of life is to introduce as much indetermination as possible on the natural tendency towards stability and geometrisation. Thus, in some examples a measure of robustness could be understood in the sense of establishing the degree of openness (or maybe randomness) of the system in order to operate efficiently: it needs to be opened to all kinds of disturbances that can eventually redirect the fate of its performance and undermine its aims. In this sense, robustness would have to do with the lack of comprehension of the way language works (and the way it works out).

However, in order to engage in such line of questionings, some basic issues in an alien domain should be tackled preliminarily to ground further discussions in the domain of science. And my point is that this calls for a new linguistic approach to Mathematics and to the mathematical thought in the process of doing research. A more extensive analysis of these claims are to be found in Botelho [30].

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Last words

— ··· *and at the end of the World, what is left for the last man?*

— *The last word!*