Convergence of Renormalisation and Rigidity of Dynamical Systems

by

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To António Adrego Pinto,
my parents and my grandmother.

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Declaration

Chapter 2 and 3 are taken from papers
jointly written with David Rand.
All the results of this theses are original,
except when explicitely stated to the contrary.

Summary

Motivated by problems in the theory of renormalisation of dynamical systems, we study the properties of Markov families and fractals defined by embedded trees. Our main results concern the classification of $C^{k+\alpha}$ structures. Two topologically equivalent Markov families are $C^{k+\alpha}$ conjugate if they converge together rapidly enough. This result implies that the attractors of two systems at the accumulation point of periodic doubling are $C^{2.11}$ conjugate. We also introduce and study the limit set of an exponential determined Markov family.

Contents

Chapter 1

Introduction.

Our main goal is the study of Markov families and the respective fractal partitions. We obtain results on convergence and smoothness. Some important applications of our work is the existence of smooth conjugacies for circle maps and for quadratic foldings in the frontier of chaos, which shows the rigidity of these systems.

We start our discussion by considering a number of examples of Cantor sets and fractal partitions generated by dynamical systems and some prototypical rigidity and smooth conjugacy results in section 1.1. Firstly, we give some information on the route to chaos through period-doubling. We discuss the universality properties and their understanding using renormalisation in section 1.2. We prove a rigidity result on the $C^{2+.11}$ smoothness of the conjugacy between quadratic foldings with the Feigenbaum order in section 1.2.11. Our result on smoothness extends to any analytic quadratic foldings infinitely renormalisable and topologically conjugated. Secondly, we give a short survey on circle maps in section 1.3. We prove a general theorem on smooth structures which will have applications for the case of critical circle maps in chapter 2. Using the results on smoothness between Markov families, we describe how to obtain C^∞ smoothness of the conjugacies between analytic diffeomorphisms of the circle with the same periodic rotation number in chapter 3. Our future aim is to generalise this proof to analytic diffeomorphisms of the circle with the same diophantine rotation number.

The study of the problems above is strongly associated with the understanding of Markov families. In chapter 3, we prove some general results on smoothness and convergence of Markov families. We define some geometric properties which imply their smoothness and vice-versa. We prove the existence and the degree of smooth conjugacies between convergent Markov families topologically conjugated.

In chapter 4, we define and prove the existence of limit sets for Markov families consisting of two-sided Markov families. We prove the exponential convergence to them under some geometric assumptions. We give applications to diffeomorphisms of the circle, critical circle maps and quadratic foldings. This will allow us to understand better the horseshoe picture for critical circle maps, which we will describe later.

For two-sided Markov families, we study in chapter 5 the C^{1+} self-similarities in the blown-up of small intervals in their domain. We obtain a strong rigidity result in the smoothness of conjugacies between two-sided Markov families.

1.1 Cantor sets.

The middle-third Cantor set is a well known example of a binary Cantor set. A slightly more general construction of a Cantor set C is the following. Let $I = [0,1]$, $I_0 = [0,a]$ and $I_1 = [b,1]$ where $a < b$. Construct the Cantor set C deleting intervals of various lengths. The intervals obtained in the n-induction step are called the *n-cylinders* . We index them by the the finite *words* $\varepsilon_0 \ldots \varepsilon_{n-1}$ of 0s and 1s in such a way that the n-cylinder indexed by $\varepsilon_0 \ldots \varepsilon_{n-2}0$ lies to the left of the n-cylinder indexed by $\varepsilon_0 \ldots \varepsilon_{n-2}1$ and both are contained in the $n-1$-cylinder indexed by $\varepsilon_0 \ldots \varepsilon_{n-2}$. Thus to each finite word $\varepsilon_0 \ldots \varepsilon_{n-1}$ of 0s and 1s we will associate an interval $I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$, such that

$$I_{\varepsilon_0 \ldots \varepsilon_{n-1}} = I_{\varepsilon_0 \ldots \varepsilon_{n-1}0} \bigcup G_{\varepsilon_0 \ldots \varepsilon_{n-1}} \bigcup I_{\varepsilon_0 \ldots \varepsilon_{n-1}1}$$

where the *gap* $G_{\varepsilon_0 \ldots \varepsilon_{n-1}}$ is the open interval between $I_{\varepsilon_0 \ldots \varepsilon_{n-1}0}$ and $I_{\varepsilon_0 \ldots \varepsilon_{n-1}1}$. Thus the Cantor set is constructed inductively by deleting the gaps. We assume that the ratios $|G_t|/|I_t|$ and $|I_{\varepsilon_0 \ldots \varepsilon_{n-1}}|/|I_{\varepsilon_0 \ldots \varepsilon_{n-2}}|$ are bounded away from 0, i.e. the Cantor set C has *bounded geometry.* It is given by

$$C = \bigcap_{n \geq 0} \bigcup_{\varepsilon_0 \ldots \varepsilon_{n-1}} I_{\varepsilon_0 \ldots \varepsilon_{n-1}}.$$

Then there exists $0 < \nu < \mu < 1$ and constants c, d such that $c\nu^n < I_{\varepsilon_0 \ldots \varepsilon_{n-1}} < d\mu^n$.

Let $\Sigma = \{0,1\}^{\mathbb{Z}_{\geq 0}}$ denote the set of infinite *right-handed words* $\varepsilon_0\varepsilon_1 \ldots$ in Σ. We endow it with the product topology. The mapping $i : \Sigma \to \mathbb{R}$ defined by

$$i(\varepsilon_0\varepsilon_1 \ldots) = \bigcap_{n \geq 0} I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$$

gives an embedding of Σ into $\mathbf{R}$.

Very often the set $C = i(\Sigma)$ will be an invariant set of a hyperbolic dynamical system. For example, there is a map σ defined on Σ above by

$$\sigma(\varepsilon_0\varepsilon_1\ldots) = \varepsilon_1\varepsilon_2\ldots.$$

This induces a map σ' on $C = i(\Sigma)$ which is a candidate for a hyperbolic system. Thus we can ask when does there exist a $C^{1+\beta}$ mapping $f : \mathbf{R} \to \mathbf{R}$ such that $f|_C = \sigma'$.

1.1.1 Cookie-cutters.

Suppose that I_0 and I_1 are two disjoint closed subintervals of I containing the end-points of $I = [-1,1]$. A cookie-cutter is a $C^{1+\alpha}$ map $F : I_0 \cup I_1 \to I$ such that $|dF| > \lambda > 1$ and $F(I_0) = F(I_1) = I$. If

$$\Lambda_n = \{x \in I : F^j x \in I_0 \cup I_1, j = 0, \ldots, n-1\}$$

then Λ_n consists of 2^n disjoint closed subintervals. The intervals

$$I_{\varepsilon_0\ldots\varepsilon_{n-1}} = \{x \in I : F^j x \in I_{\varepsilon_j}, 0 \le j \le n\}$$

and the gaps between them are the n-cylinders of the Cantor set C

$$C = \cap_{n\ge 0}\Lambda_n = \{x \in I : F^j x \in I_0 \cup I_1, \text{ for all } j \ge 0\}.$$

To each infinite right-handed word $\underline{\varepsilon} = \varepsilon_0\varepsilon_1\ldots$ we associate the point $i(\underline{\varepsilon}) = \cap_{n\ge 0} I_{\varepsilon_0,\ldots,\varepsilon_{n-1}}$. If $dF > 0$ this agrees with the coding in section 1.1.

1.1.2 Scaling trees.

A *tree* T consists of a set of vertices of the form $V_T = \bigcup_{n\ge 0} T_n$, where each T_n is a finite set, together with a directed graph on these vertices such that each $t \in T_n$, $n \ge 1$, has a unique edge leaving it. This edge joins t (the *daughter*) to $m(t) \in T_{n-1}$ (its *mother*).

Given such a tree T we define the *limit set* or *set of ends* L_T as the set of all sequences $\underline{t} = t_0t_1\ldots$ such that $m(t_{i+1}) = t_i$ for all $i \ge 0$. We endow L_T with the metric d where

$$d(s_0s_1\ldots, t_0t_1\ldots) = 2^{-n}$$

if $s_i = t_i$ for $0 \leq i \leq n-1$ and $s_n \neq t_n$.

To a binary Cantor set C we associate the abstract tree $T = T_C$ whose n-vertex set T_n contains both the symbols $\varepsilon_0 \ldots \varepsilon_{n-1}$ and $g_{\varepsilon_0 \ldots \varepsilon_{n-2}}$ corresponding respectively to the intervals $I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$ and $G_{\varepsilon_0 \ldots \varepsilon_{n-2}}$ and whose edges connect them to their *mother* $\varepsilon_0 \ldots \varepsilon_{n-2}$. Moreover, there is an ordering $\leq_n$ on the n-vertices induced by the ordering of the corresponding subintervals on the real line. This ordering satisfies the compatibility condition $m(J) <_n m(K)$ implies $J <_{n+1} K$, for all vertices J and K.

The binary Cantor set defines a mapping $\sigma_T : \cup_{n \in \mathbb{N}} T_n \to (0,1)$ by $\sigma_T(t) = |I_t|/|I_{m(t)}|$ and $\sigma_T(g_t) = |G_t|/|I_t|$. Clearly σ_T has to satisfy the condition $\sum_t \sigma_T(t) = 1$ where the sum is over all the vertices with the same mother.

Clearly such a map $\sigma_T : T \to (0,1)$ completely defines the binary Cantor set.

Definition 1 A map $\sigma_T : T \to (0,1)$ as above is called a scaling tree.

Notation. If f and g are functions of a variable x with domain Δ, then we write $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$ with constant d if

$$d^{-1} < \frac{|f(x)|}{|g(x)|} < d$$

for all $x \in \Delta$. Often we will drop the reference to d. Thus if a_n and b_n are sequences then $\mathcal{O}(a_n) = \mathcal{O}(b_n)$ means a_n/b_n and b_n/a_n are bounded away from 0 independently of n. The notation $f(x) = \mathcal{O}(g(x))$ means the same thing as $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$.

Similarly, $f(x) \leq \mathcal{O}(g(x))$ with constant d means $|f(x)/g(x)| < d$ for all $x \in \Delta$.

Definition 2 The scaling tree σ_T has *β-scale determination* if and only if

$$\left|\frac{\sigma_T(\varepsilon_0 \ldots \varepsilon_{n-1})}{\sigma_T(\varepsilon_1 \ldots \varepsilon_{n-1})} - 1\right| \leq \mathcal{O}(|I_{\varepsilon_0 \ldots \varepsilon_{n-1}}|^\beta)$$

and

$$\left|\frac{\sigma_T(g_{\varepsilon_0 \ldots \varepsilon_{n-1}})}{\sigma_T(g_{\varepsilon_1 \ldots \varepsilon_{n-1}})} - 1\right| \leq \mathcal{O}(|G_{\varepsilon_0 \ldots \varepsilon_{n-1}}|^\beta)$$

for all the vertices of T, where β lies between 0 and 1.

Definition 3 (i) A map $s : M \longrightarrow N$ is α-Holder continuous, where α lies between 0 and 1, if and only if there is a constant $c > 0$ such that for all $x, y \in M$, $||s(x) - s(y)|| \leq c||x - y||^{\alpha}$.

(ii) A map s is Holder continuous if and only if there is some $\alpha > 0$ such that the map s is α-Holder continuous. If $\alpha = 1$ then the map s is Lipschitz.

(iii) A map s is C^{1+} smooth if and only if there is some $\alpha > 0$ such that the derivative ds is α-Holder continuous.

(iv) A map s is $C^{1+\beta^-}$ smooth if and only if for all α between 0 and β the derivative ds is α-Holder continuous.

Theorem 1 The map σ' on $C = i(\Sigma)$ has a $C^{1+\beta^-}$ extension to the reals if and only if the scaling tree σ_T has β-scale determination.

Let σ_T and $\sigma_{T'}$ be two scaling trees corresponding to different binary Cantor sets C and D.

Definition 4 The scaling tree σ_T and $\sigma_{T'}$ are *β-scale equivalent* if and only if $\sigma_T(t) \in \sigma_{T'}(t)(1 \pm c|I_t|^{\beta})$ and $\sigma_T(g_t) \in \sigma_{T'}(g_t)(1 \pm c|G_t|^{\beta})$ for all the vertices in T, where β lies between 0 and 1 and c is some constant in $\mathbf{R}^+$.

Define the homeomorphism $h : C \longrightarrow D$ which sends the extreme points of all the n-cylinders of C in the extreme points of the n-cylinders of D preserving their order, for all $n \in \mathbf{N}$.

Theorem 2 The homeomorphism $h : C \longrightarrow D$ has a $C^{1+\beta^-}$ extension to the reals if and only if the scaling trees σ_T and $\sigma_{T'}$ are β-scale equivalent.

The theorems above were proved by Sullivan [28] for the case of C^{1+} extensions. Rand and Pinto [17] generalised it for more general scaling trees than the ones generated by binary Cantor sets and got $C^{1+\beta^-}$ differentiability.

The construction of the binary Cantor set is the simplest non-trivial example of a *scaling tree.* We shall be interested in scaling trees such as this where the analogue of the Cantor set C is generated in one way or another by a dynamical system.

1.1.3 Scaling function

The definition of scaling function and the results below are due to a previous work of Sullivan [28]. These results are corollaries of the theorems in the previous section.

We introduce the *dual* Σ^* of Σ. This is the set Σ^* of all *left infinite words* $\ldots\varepsilon_i\ldots\varepsilon_0$ where $\varepsilon_i = 0$ or 1 for all $i \in \mathbf{Z}_{\leq 0}$. We endow it with the product topology.

Definition 5 The *scaling function* $\sigma : \Sigma^* \to \mathbf{R}$ is defined by $\sigma(\ldots\varepsilon_i\ldots\varepsilon_0) = \lim_{i\to\infty} \sigma_T(\varepsilon_i\ldots\varepsilon_0)$.

Theorem 3 *Sullivan.* A bounded geometry Cantor set C is generated by a cookie-cutter if and only if the scaling function exists and is Holder continuous.

Two Cantor sets C and D are in the same C^{1+} equivalence class if and only if the map $h : C \to D$ which sends the extreme points of the n-cylinders of C to the extreme points of the n-cylinders of D keeping their order has a C^{1+} extension to the reals.

Theorem 4 *Sullivan.* The scaling function is a complete invariant for each C^{1+} equivalence class of Cantor sets with Holder scaling function.

1.2 Feigenbaum period-doubling.

Feigenbaum period-doubling is one of the most common and well-known routes to chaos. There are a lot of experiments in different areas which confirm this phenomenon. One of the most amazing properties is the existence of universal quantitative properties which are independent of the experiments. To analyse and discuss these properties we introduce the concepts of renormalisation and Markov families. Excellent recent work in the proof of the rigidity conjecture is due to Sullivan [30]. He proves that the stable manifold contains all quadratic foldings with the Feigenbaum order. Previous relevant work are due to Feigenbaum, Collet, Tresser, Lanford, Rand and other mathematicians and physicists. Our main result is the proof of the $C^{2.11}$ smoothness between analytic quadratic foldings with the Feigenbaum order. Our results on smoothness extend to any analytic quadratic folding in the frontier of chaos. The smoothness of the conjugacy is given in terms of a balance between the speed of convergence of the respective Markov families and the scaling structures of their cylinders.

1.2.1 Feigenbaum ordering of the interval.

We say that a sequence of points x_i, $i = 0, 1, \dots$ in the interval $[x_1, x_0]$ has the Feigenbaum ordering if for $0 \leq i < 2^{n-1}$, x_{i+2^n} and $x_{i+3.2^{n-1}}$ lie between x_i and $x_{i+2^{n-1}}$ and are ordered so that $x_i - x_{i+2^{n-1}}$ and $x_{i+2^n} - x_{i+3.2^{n-1}}$ have the same sign.

1.2.2 C^2 families of quadratic foldings.

A quadratic folding of the interval $I = [-1, 1]$ is a $C^{1+Lipschitz}$ mapping $f : I \to I$ with $Df > 0$ (resp. $Df < 0$) on $[-1, 0)$ (resp, $(0, 1]$) and such that in some neighbourhood of 0 there is a $C^{1+Lipschitz}$ coordinate system x in which $f(x) = -x^2 + f(0)$. Given such a mapping f let $x_i = f^{i+1}(0)$.

A C^2 family is a 1-parameter family f_μ in $C^2(I,I)$, $\mu \in (\alpha, \beta)$, which is continuous in the C^2 topology. It is full on (α, β) if $f_\mu(1) \to 1$ (resp. $\to -1$) as $\mu \to \alpha^+$ (resp. $\mu \to \beta^-$). For example, $1 - \mu x^2$ is full on (0,2).

Definition 6 (a) f is superstable if 0 is in a p-cycle of f. (b)f is 1-filling if $f(I) = I$. f is p-filling if there exist p disjoint closed sub-intervals $I_1, \dots, I_p$ such that f is a homeomorphism from I_j to I_{j+1} for $1 \leq j < p$, $f(I_p) \subset I_1$ and $g = f^p_{|I_1}$ is such that $g(I_1) = I_1$ (i.e. with respect to I_1, g is 1-filling).

Theorem 5 If f_μ is a full C^2-family on (α, β), then there exists $\alpha < \alpha_1 < \alpha_2 < \dots < \beta_2 < \beta_1 < \beta$ such that 1. f_{α_i} is 2^i -superstable, 2. f_{β_i} is 2^i-filling, and 3. $\gamma = \lim_{i\to\infty} \alpha_i = \lim_{i\to\infty} \beta_i$.

See proof in Rand [21].

Define $(f_\gamma)^{i+1}(0) = x_i$. The sequence of points x_i has the Feigenbaum ordering. These define a scaling tree in which the n-cylinders are the closed intervals $J_{i,n}$ between x_i and x_{i+2^n}, $0 \leq i < 2^n$. Moreover, the Cantor set

$$C_f = \cap_{n\geq 1} \cup_{i=0}^{2^n-1} J_{i,n}$$

defined by this scaling tree is the attractor of f in the sense that every orbit is either eventually periodic or else converges to C_f.

1.2.3 Examples of universality.

We give a heuristic introduction to the Feigenbaum conjectures. Consider the 1-parameter family $f_\mu = 1 - \mu x^2$, $0 < \mu \leq 2$, discussed in the previous section. Recall the meaning of the parameter values α_i, β_i of theorem above. Using a pocket calculator one finds

$$\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_{n+1} - \alpha_n} = \lim_{n\to\infty} \frac{\beta_n - \beta_{n-1}}{\beta_{n+1} - \beta_n} = \delta = 4.669\ldots$$

and using something a bit more powerful, it appears that there exists $\lambda = -.3995\ldots$ such that if $\alpha_\infty = \lim_{n\to\infty} \alpha_n$.

$$\psi = \lim_{n\to\infty} \lambda^{-n} f^{2^n}_{\alpha_\infty} \lambda^n$$

exists and is an analytic function of x^2. Moreover, if one takes any other 1-parameter family one gets the same experimental values for δ and λ and, up to a scale change, the same function ψ. This is an example of universality. The Feigenbaum conjectures are developed from the renormalisation operator.

1.2.4 Renormalisation.

Let f be a quadratic folding and $a = a(f) = -f(1)$, $b = b(f) = f(a)$. Let $D(R)$ denote the set of f's such that (i) $a > 0$ (ii) $b > a$ and (iii) $f(b) \leq a$. For $f \in D(R)$ define the renormalisation Rf of f by

$$Rf(x) = a^{-1} f^2(ax). \tag{1.1}$$

1.2.5 Feigenbaum conjectures.

The explanation (essentially proposed by Feigenbaum [6] and [7] and independently by Collet and Tresser [4] goes as follows: Consider the renormalisation operator defined on some suitable subspace of $D(R)$ consisting of analytic functions. Assume that the following facts are true.

Conjecture 1. The renormalisation operator R has a fixed point f_* with the property that $df_*(0) = 0$ and $d^2 f_*(0) \neq 0$.

Conjecture 2. The only element of the spectrum of $dR(f_*)$ outside the disk $|z| < 1$ is a single eigenvalue $\delta = 4.669\ldots$. The rest of the spectrum is contained in a disk of radius strictly less than 1.

Conjecture 3. The unstable manifold of f_* intersects and is transverse to the submanifolds Σ_n and Λ_n of bifurcation and superstable maps defined as

$$\Sigma_n = \{f : \text{for some p in a } 2^n-\text{cycle of f}, df^{2^n}(p) = -1 \text{ and } d^3f^{2^n}(p)+3d^2f^{2^n}(p) \neq 0\}$$

$$\Lambda_n = \{f : f^{2^n}(0) = 0 \text{ and } f^m(0) \neq 0 \text{ for } 0 < m < 2^n\}.$$

These imply:

1. Since $Rf_* = f_*$, the map f_* satisfies the Cvitanovic-Feigenbaum equation

$$f_*^2(\lambda x) = \lambda f_*(x).$$

2. $R^n f \to f_*$ as $n \to \infty$ implies there exists $\beta > 0$ such that

$$\lambda^{-n} f^{2^n}(\lambda^n x) \to \beta^{-1}(f_*(\beta x))$$

uniformly in x as $n \to \infty$.

3. Conjecture 2 implies that, with respect to R, f_* is a saddle point with a 1-parameter dimensional unstable manifold W^u and a stable manifold W^s of codimension one. W^u defines a universal 1-parameter family of maps $f_{*,\mu}$. For a 1-parameter family f_μ near f_* with $f_0 \in W^s$ one has

$$\lambda^{-n} f_{\mu\delta^{-n}}^{2^n} \circ \lambda^n \to \beta^{-1} f_{*,\mu} \circ \beta$$

for some $\beta > 0$.

4. Obviously, $R(\Sigma_n) \subset \Sigma_{n-1}$ and $R(\Lambda_n) \subset \Lambda_{n-1}$. Thus, conjecture 3 implies that the Σ_n and Λ_n accumulate on W^s exponentially fast with the distances from W^s decreasing like δ^{-n} (to prove this one one uses the fact that R can be linearised along the unstable direction). If f_μ is a 1-parameter family near $f_{*,\mu}$ and transverse to W^s with say $f_0 \in W^s$ then $f_{\alpha_i} \in \Lambda_i$ and therefore

$$\lim_{i\to\infty} \frac{\alpha_i - \alpha_\infty}{\alpha_{i+1} - \alpha_\infty} = \delta$$

if $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. In the same fashion, if μ_n denotes the parameter value at which a $2^n \to 2^{n+1}$ period-doubling occurs then

$$\lim_{i\to\infty} \frac{\mu_i - \mu_\infty}{\mu_{i+1} - \mu_\infty} = \delta$$

where $\mu_\infty = \lim_{n\to\infty} \mu_n = \alpha_\infty$.

This explains where the simpler universal quantities δ, λ and ψ come from.

Lanford [12] gives a proof of conjectures 1 and 2. His proof makes essential use of rigorous computer-generated estimates. Sullivan [30] proves that the stable manifold contains all quadratic foldings with the Feigenbaum order.

1.2.6 Lanford's theorem.

Let Ω denote the unit disc $|z| < 1$ in C and let $\mathcal{L}$ denote the real Banach space of continuous $h : \Omega \to$ C which are holomorphic on Ω, take real values at real points and, if $h(z) = \sum_{n\geq 0} a_n z^n$, then $||h|| = \sum_{n\geq 0} |a_n| < \infty$. Let $\mathcal{A}$ denote the set of maps of the form

$$f(z) = 1 - z^2 h((z^2 - 1)/2.5)$$

where $h \in \mathcal{L}$. By identification with $\mathcal{L}$, $\mathcal{A}$ may be regarded as a real Banach space.

Theorem 6 *Lanford.* There is a polynomial $f_{\mathbf{approx}}$ which is very nearly a fixed point g of R. If $\mathcal{V}$ denotes the ball $||f - f_{\mathbf{approx}}|| < .01$, then

(i) $R_{|\mathcal{V}}$ is well defined and C^∞.

(ii) For $f \in \mathcal{V}$, $dR(f)$ is a compact operator.

(iii) R has a unique fixed point g in $\mathcal{V}$.

(iv) The spectrum of dR(g) consists of a simple real eigenvalue $\delta > 1$ and a countable set of eigenvalues contained strictly inside the circle $|z| = \tau$ for some $0 < \tau < 1$.

As a consequence g has a 1-dimensional unstable manifold $W^u(g)$ and a 1-codimensional stable manifold $W^s(g)$ and if $f \in W^s(g)$ then there exists a constant c depending only upon f such that $||R^n f - g|| < c\tau^n$.

In fact, it is known that τ is determined by the eigenvector which is tangent to the one-parameter family of coordinate changes given by $x \to x + tx^2$. However, this eigenvalue can be removed by replacing the scale change in (1) by a transformation of the form $x \to ax + bx^2$ and using two normalisations to fix a and b. Then τ is determined by the smaller eigenvalue which is approximately 0.13 which generically gives the rate at which the slope at the fixed point of $R^n f$ approaches that of g as $n \to \infty$. We use the value of this eigenvalue to prove theorem 11.

1.2.7 Global rigidity conjecture for period-doubling.

The global rigidity conjecture states that if f and g are quadratic foldings as above with the Feigenbaum ordering then there is a $C^{1+\alpha}$ diffeomorphism $h : \mathbf{R} \to \mathbf{R}$ such that $h(C_f) = C_g$ and $h \circ f = g \circ h$ on C_f. As we shall see, this is essentially equivalent to the conjecture that under repeated renormalisation any such quadratic folding converges to the Feigenbaum fixed point. This is a quadratic folding f which satisfies the functional equations

$$f = a^{-1} f^2 \circ a \tag{1.2}$$

where $a = f^2(0)/f(0) = x_1/x_0$.

Moreover, we prove in chapter 4 that the global rigidity conjecture is equivalent to the conjecture that the scaling tree t_{C_f} corresponding to C_f has the $1+\alpha$-scale property for some $\alpha > 0$ or equivalently that there is a cookie-cutter which has C_f as its maximal invariant set C and has the same scaling function as the one corresponding to the Feigenbaum-Cvitanovic fixed point. To see that this is reasonable note that if f is the Feigenbaum fixed point then it follows immediately from (1.2) that C_f is the maximal invariant set of the cookie-cutter.

$$x \to \begin{cases} a^{-1}x & \text{if } x \in [x_1, x_3] \\ a^{-1}f(x) & \text{if } x \in [x_2, x_0] \end{cases}$$

The global rigidity conjecture is proved in the important recent paper [30] by Sullivan.

Theorem 7 *Sullivan.* The stable manifold contains all quadratic foldings with the Feigenbaum order.

Using the results of Lanford and Sullivan, Rand and independently Sullivan proved that

Theorem 8 Any two analytic quadratic foldings with the Feigenbaum order are C^{1+} conjugated.

The prove of theorem 8 relies in the existence and exponential convergence of Markov families corresponding to these quadratic foldings.

1.2.8 Markov families.

Topological Markov families

A *topological Markov family* F is a family of mappings $F_{n,a}$ with either $n = 0, 1, \dots$ or $n \in \mathbb{Z}$ and a in a finite set S_n which satisfy the following conditions.

(i) For each n and $a \in S_n$, $F_{n,a}$ is a homeomorphism of the closed interval I_a^n into $\mathbb{R}$.

(ii) I_a^n contains in its interior a closed interval C_a^n with following properties.

- $\mathrm{int} C_a^n \bigcap \mathrm{int} C_b^n = \emptyset$ if $a \neq b$.
- If $x \in C_a^n$ and $F_n(x) \in C_b^{n+1}$ then $F_n(C_a^n)$ contains C_b^{n+1}.
- If $b \in S_{n+1}$, there exists $a \in S_n$ such that $F_n(C_a^n)$ contains C_b^{n+1}.

We regard the $F_{n,a}$ as defining a single mapping F_n on C^n, where C^n is the smallest interval containing $\bigcup_{a \in S_n} C_a^n$.

$C^{k+\alpha}$ Markov families

A $C^{k+\alpha}$ *Markov family* F satisfies the following conditions in addition.

(i) $F_{n,a} = F_n | I_a^n$ is a $C^{k+\alpha}$ diffeomorphism of I_a^n into $\mathbb{R}$.

(ii) $|F_n'(x)| > 1$ for all $x \in I^n$ and all n in some norm on $\mathbb{R}$.

Bounded and boundedly extended Markov families

A $C^{k+\alpha}$ *Markov family* F is said to be *bounded* if

(i) $|I^n|/|I^0|$, $|C^n|/|C^0|$, $|I_a^n|/|I_a^0|$ and $|C_a^n|/|C_a^0|$ is bounded away from 0 and ∞ where C^n is the smallest closed interval containing $\bigcup_{a \in S_n} C_a^n$;

(ii) for all n and all $a \in S_n$ the $C^{k+\alpha}$ norm of $F_{n,a} = F_n | I_a^n$ on I_a^n is bounded independently of n and a; and

(iii) There exists $\lambda > 1$ such that $|F_n'(x)| > \lambda$ for all $x \in I^n$ and all n.

A point $x \in C^n$ is *captured* if for all $m > n$, $F_{m-1} \circ \cdots \circ F_n(x) \in C^m$. The set of all captured points is denoted by $\Lambda^n = \Lambda^n(F)$.

Let Σ^n denote the set of infinite right-handed words $\underline{\varepsilon} = \varepsilon_n \varepsilon_{n+1} \ldots$ such that (i) $\varepsilon_j \in S_j$ and (ii) there exists $x \in C^n$ with the property that

$$F_{m-1} \circ \cdots \circ F_n(x) \in C^m_{\varepsilon_m}$$

for all $m > n$. We call these words *admissible*. If $\underline{\varepsilon} = \varepsilon_n \varepsilon_{n+1} \ldots \in \prod_{p \geq n} S_p$ let $\underline{\varepsilon}|p$ denote the finite word $\varepsilon_n \ldots \varepsilon_{n+p-1}$ of length p. Let Σ^n_p denote the set of finite words $\underline{\varepsilon}|p$ where $\underline{\varepsilon} \in \Sigma^n$. We denote by σ and m the mappings $\sigma : \Sigma^n_p \to \Sigma^{n+1}_{p-1}$ and $m : \Sigma^n_p \to \Sigma^n_{p-1}$ given by

$$\begin{aligned} \sigma(\varepsilon_n \ldots \varepsilon_{n+p-1}) &= \varepsilon_{n+1} \ldots \varepsilon_{n+p-1} \\ m(\varepsilon_n \ldots \varepsilon_{n+p-1}) &= \varepsilon_n \ldots \varepsilon_{n+p-2}. \end{aligned}$$

If $\underline{\varepsilon} = \varepsilon_n \varepsilon_{n+1} \ldots \in \Sigma^n$ then we denote by $C_{\varepsilon_n \ldots \varepsilon_m}$ (resp. $I_{\varepsilon_n \ldots \varepsilon_m}$) the closed interval consisting of all $x \in C^m$ such that for all $n \leq j < m$,

$$F_j \circ \cdots \circ F_n(x) \in C^{j+1}_{\varepsilon_{j+1}} \quad (\text{resp. } I^{j+1}_{\varepsilon_{j+1}}).$$

By $\Lambda_{\varepsilon_n \ldots \varepsilon_m}$ we denote the intersection of Λ^n with $C_{\varepsilon_n \ldots \varepsilon_m}$ and by $\tilde{C}_{\varepsilon_n \ldots \varepsilon_m}$ the smallest closed interval containing $\Lambda_{\varepsilon_n \ldots \varepsilon_m}$. Note that if each interval C^n_a is replaced by the subinterval $\tilde{C}^n_a$ in the definition of Λ^n then one obtains the same set Λ^n of captured points.

We therefore assume henceforth that $C^n_a = \tilde{C}^n_a$.

Suppose that J is a closed set contained in the interior of an interval I and let $\tilde{J}$ denote the smallest closed interval containing J. Then $I - \tilde{J}$ consists of two intervals. The interval to the right (resp. left) of J is denoted by $R(J, I)$ (resp. $L(J, I)$).

Definition. A $C^{k+\alpha}$ Markov family F is *boundedly extended* if there exists $\delta_1, \delta_2 > 0$ such that, for all n and all $a \in S_n$, the intervals I^n_a on which F_n is defined and $C^{k+\alpha}$ are such that

$$\delta_1 < \frac{|R(C^n_a, I^n_a)|}{|I^n_a|}, \frac{|L(C^n_a, I^n_a)|}{|I^n_a|} < \delta_2. \tag{1.3}$$

Definition. If F and G are two topological Markov families then we say that they are *topologically conjugated* if for all n there exists a homeomorphism $h_n : \Lambda^n(F) \to \Lambda^n(G)$ such that $G_n \circ h_n = h_{n+1} \circ F_n$ on $\Lambda^n(F)$.

In such a case we call the family $\underline{h} = (h_n)$ the *conjugacy.* The major result of this paper is the derivation of natural necessary conditions for the h_n to be $C^{r+\beta}$ or to have a $C^{r+\beta}$ extension to $\mathbf{R}$. Without loss of generality, we will restrict to the case where the homeomorphisms preserve the order of the real line.

1.2.9 Rand's theorem.

Let F and G be two $C^{1+\gamma}$ Markov families topologically conjugated. We will impose the following condition on the pairs of families F and G that we consider.

Condition A. There exists $c > 0$ and $0 < \tau < 1$ such that for all n and all $\varepsilon \in S_n$,

$$||F_{n,\varepsilon} - G_{n,\varepsilon}||_{C^{1+\gamma}(I_t^n)} \leq c\tau^n.$$

Condition B. There exists $c > 0$ and $0 < \nu < 1$ such that for all $n \in \mathbf{Z}_{\geq 0}$ and all contact words $t, t' \in \Sigma_n$

$$1 - c\nu^n < \frac{|C_t^n(F)|}{|C_t^n(G)|}\frac{|C_{t'}^n(G)|}{|C_{t'}^n(F)|} < 1 + c\nu^n.$$

Theorem 9 *Rand.* Suppose that the bounded and boundedly extended $C^{s+\gamma}$ Markov families F and G are topologically conjugate and satisfy Conditions A and B . Then the conjugacy $\underline{h} = (h_n)$ is $C^{1+\beta}$ smooth for some $\beta \in [0, 1)$.

1.2.10 $C^{r+\beta}$ conjugacy between Markov families.

Let F and G be two $C^{s+\gamma}$ Markov families topologically conjugated. We will impose the following condition on the pairs of families F and G that we consider. It involves the positive function $g(n)$.

Condition A(g). For all n and all $\varepsilon \in S_n$,

$$||F_{n,\varepsilon} - G_{n,\varepsilon}||_{C^{s+\gamma}(I_t^n)} \leq g(n+1).$$

By I^n, I_a^n, I_t^n, C^n, C_a^n and C_t^n we denote the intervals and cylinders $I^n(F)$, $I_a^n(F)$, $I_t^n(F)$, $C^n(F)$, $C_a^n(F)$ and $C_t^n(F)$ for F. We denote the corresponding intervals and cylinders for G_n by J^n, J_a^n, J_t^n, D^n, D_a^n and D_t^n.

If $\varepsilon \in S_n$, let $A_{n,\varepsilon}$ denote the affine map which sends C^n_ε onto D^n_ε preserving orientation. We regard $A_{n,\varepsilon}$ as having domain I^n_ε. If t is the word $\varepsilon_0 \dots \varepsilon_n \in \Sigma^0_{n+1}$ define

$$\begin{aligned} K_t &= G^{-1}_{0,\varepsilon_0} \circ \cdots \circ G^{-1}_{n-1,\varepsilon_{n-1}} : J^n_{\varepsilon_n} \to J^0_t, \\ E_t &= F_{n-1,\varepsilon_{n-1}} \circ \cdots \circ F_{0,\varepsilon_0} : I^0_t \to I^n_{\varepsilon_n}, \quad \text{and} \\ L_t &= K_t \circ A_{n,\varepsilon_n} \circ E_t : I^0_t \to J^0_t. \end{aligned}$$

Now we formulate a condition that controls the behaviour at *contact points.* Let $t = \varepsilon_0 \dots \varepsilon_{n-1}$ and $t' = \varepsilon'_0 \dots \varepsilon'_{n-1}$ be in *contact* i.e. such that C_t and $C_{t'}$ meet in a point. Let $m > 0$ be minimal such that $t|m = t'|m$ and $t|(m+1) \neq t'|(m+1)$. In this case, let $e_{t,t'}$ denote

$$e_{t,t'} = \max_{x \in I_t \bigcap I_{t'}} \{|dE_t(x)|, |dE_{t'}(x)|\}.$$

Then we impose the following condition on all such pairs t, t'

Condition B(g). For all such t and t' and all $0 \leq k \leq s$,

$$||L_{\sigma^m t} - L_{\sigma^m t'}||_{C^k} \leq g(n) e^{k-1}_{\sigma^m(t),\sigma^m(t')}$$

on $I_{\sigma^m t} \bigcap I_{\sigma^m t'}$.

It is not difficult to see that condition B(g) is satisfied, for appropriate g, by those Markov maps arising from renormalisation structures with contact points such as those for diffeomorphism of the circle and cubic critical circle maps.

Theorem 10 *(= theorem 20) Rand and Pinto.* Suppose that the bounded and boundedly extended $C^{s+\gamma}$ Markov families F and G are topologically conjugate and satisfy Conditions A(g) and B(g). Let $e(n) = \max_{t \in \Sigma^0_n} ||dE_t||$. Then the conjugacy $\underline{h} = (h_n)$ is $C^{r+\beta}$ with $\beta \in [0,1)$ such that $r + \beta \leq s$ if the function f given by

$$f(n) = \max_{t \in \Sigma^0_n} e(n)^{r+\beta-1} g(n)$$

is such that $\sum_{j=0}^{\infty} f(j) < \infty$.

Remark. Suppose that F and G satisfy the hypotheses of theorem 10. Then, by boundedness, there exist constants $d_1, d_2 > 0$ and $\mu, \lambda \in (0,1)$ such that for all $t \in \Sigma^0_n$,

$$d_1 \mu^{-n} < |dE_t| < d_2 \lambda^{-n}$$

Thus $g(n)/f(n) \leq c\mu^{(r+\beta-1)n}$ and, in particular, $g(n)$ is exponentially decreasing. If $g(n) < c\tau^n$ then, by theorem 10, the condition $\tau/\lambda^{r+\beta-1} < 1$ is sufficient for the conjugacy to be $C^{r+\beta}$.

1.2.11 Global $C^{2+.11}$ rigidity for period-doubling.

Let f be a quadratic folding with the Feigenbaum order. The family F_f of Markov maps F_n corresponding to the renormalisation $f_n = R^n f$ of f is

$$F_n(x) = \begin{cases} (f_n^2(0))^{-1}x & x \in [f_n^2(0), f_n^4(0)] \\ (f_n^2(0))^{-1}f_n(x) & x \in [f_n^3(0), f_n(0)] \end{cases}$$

The family of Markov maps F_n define the same scaling tree in I as the one induced by the orbit of the critical point of f.

Theorem 11 *(= theorem 21) Rand and Pinto.* Suppose that f and g are real analytic quadratic foldings with the Feigenbaum ordering. Then the canonical homeomorphism $h : \Lambda_f \to \Lambda_g$ has a $C^{2+.11}$ extension to the real line.

1.3 Circle maps.

Our study of the circle maps follows the same lines as the study of the Feigenbaum period-doubling. It has applications to the study of the quantitative universal behaviour of the bifurcation from a quasi-periodic flow to a chaotic or turbulent state with two sharp incomensurate frequencies. It is related to the understanding of the break down of invariant curves in families of invertible analytic maps of the annulus. See Ostlund et al [15] and Feigenbaum et al [8]. For diffeomorphisms of the circle an excellent work is due to Denjoy, Arnol'd, Herman, Yoccoz. They prove that analytic diffeomorphisms of the circle with the same diophantine rotation number are C^∞ conjugated. Using the results on smoothness between Markov families, we describe how to obtain C^∞ conjugacies between analytic diffeomorphisms of the circle with the same periodic rotation number. Our future aim is to give an alternative proof of the result due to Denjoy, Arnol'd, Herman, Yoccoz. Recent work for critical circle maps is due to Rand, Lanford, Mestel and others. We prove a general theorem on smooth structures which will have applications for the case of critical circle maps.

1.3.1 Definition.

A continuous map of the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ lifts to a map f of the universal cover $\mathbf{R}$ of $\mathbf{T}$ into itself such that $f(x+1) = f(x) + 1$. This map f is only unique up to addition of an integer; to enforce uniqueness we demand that $0 \leq f(0) < 1$. If the original circle map is C^r, $0 \leq r \leq \omega$, the lift f is C^r. The set of such lifts is denoted $\mathbf{D}^r$.

If $x \in \mathbf{R}$ and $f \in \mathbf{D}^0$ the rotation number of (f, x) is defined to be

$$\rho(f, x) = \lim_{n \to \infty} \inf n^{-1}(f^n(x) - x).$$

In general the limit does not exist and $\rho(f, x)$ is independent of x (Arnold [1]). The number $\rho(f)$ obtained is called the rotation number of f. It depends continuously upon f in the C^0-topology.

1.3.2 Two parameter family of circle maps.

Now to bring out some important aspects of the circle maps, consider the prototypical 2-parameter family

$$f_{\mu,\nu} = x + \nu - (\mu/2\pi) \sin 2\pi x.$$

If $|\mu| \leq 1$ then $f_{\mu,\nu}$ is a homeomorphism; it is a diffeomorphism if $|\mu| < 1$. If $\mu = 0$ then $f_{\mu,\nu}$ is the rotation R_ν so $\rho(f_{0,\nu}) = \nu$ and the dependence of ρ upon ν is trivial. This is not the case if $\mu \neq 0$. To see this fix $0 < |\mu| < 1$ and let f_ν denote $f_{\mu,\nu}$. Let p/q be a rational number expressed in lowest order terms and

$$I_{p,q} = \{\nu : f_\nu^q(x) = x + p \text{ for some } x\}$$

If $\nu \in I_{p/q}$, f_ν has a periodic orbit of period q (a q-cycle) and $\rho(f_\nu) = p/q$. If $\mu = 0$, $I_{p/q}$ is a point. Otherwise, $I_{p/q}$ is a closed interval.

Consequently, $\rho(f_\nu)$ is constant upon the countably infinite set of intervals $I_{p/q}$ and irrational elsewhere. To see how the intervals $I_{p/q}$ vary as μ changes consider the so-called Arnold tongues:

$$A_{p/q} = \{(\mu, \nu) : f_{\mu,\nu}^q(x) = x + p \text{ for some } x\}.$$

Notice how fast they taper off as $\mu \to 0$. Arnold proved that as $\mu \to 0$ the Lebesgue measure of the union of the $I_{p/q}$ converges to 0, even being an open dense set. Moreover as $\mu \uparrow 1$, they fill more and more of the line and it is conjectured that the union of the $I_{p/q}$ has full Lebesgue measure on $\mu = 1$ and its complement has Hausdorff dimension approximately equal to .87. It is conjectured that this properties are universal for families of circle maps.

1.3.3 Rotation number.

If the rotation number ω is an irrational in $(0,1)$, its rational approximations p_n/q_n are defined inductively by setting $p_0 = 0$ and $q_0 = 1$, and requiring that q_n is the smallest positive integer such that $|q_n\omega - p_n| < |q_{n-1}\omega - p_{n-1}|$. These are the rational numbers obtained by truncation the continued fraction expansion $\omega = 1/(a_1 + 1/(a_2 + \ldots)) = [a_1, a_2, \ldots]$ of ω as follows:

$$p_k/q_k = [a_1, \ldots, a_k, a_{k+1} = \infty].$$

Then p_k and q_k satisfy the recursion relations

$$p_{k+1} = a_k p_k + p_{k-1},\ q_{k+1} = a_k q_k + q_{k-1}.$$

If ω is equal to the golden mean $(\sqrt{5}-1)/2$ then $a_k = 1$ for all $k \geq 1$. Thus the rational approximates of the golden number are the ratios of Fibonacci numbers.

If $\beta \geq 0$, the rotation number ω satisfies a Diophantine condition of order β if there is a constant $c > 0$ such that $|\omega - p/q| \geq c/q^{2+\beta}$ for all $p/q \in \mathbf{Q}$. Define C_β the set of all irrational rotation numbers ω satisfying a Diophantine condition of order β.

The rotation numbers ω in the sets C_0, $\cap_{\beta>0} C_\beta$, $\cup_{\beta\geq 0} C_\beta$, $\mathbf{R} \setminus (\mathbf{Q} \cup (\cup_{\beta\geq 0} C_\beta))$ are called, respectively of constant type, Roth type, Diophantine, Liouville.

1.3.4 Critical circle maps.

A critical circle map g is a circle map with a single critical point 0 which is cubic.

Let p_n/q_n be the nth rational approximant of the rotation number ω of the critical circle map g. Consider the orbit $g^n(0), n \in \mathbf{Z}_{\geq 0}$. This partitions the interval $[\alpha - 1, \alpha]$ into $q_n + 1$ closed intervals, where $\alpha = g(0)$. Let T_n denote the set of such intervals and let T be the tree whose vertex set is $\cup_{n\geq 0} T_n$ and such that the mother of $v \in T_n$ is the interval in T_{n-1} which contains v. Thus T is defined by the cylinder structure. The vertices $t_n, s_n \in T_n$ are in contact if $t_n \cap s_n \neq \emptyset$. If $t_0 t_1 \ldots \in L_T$ then $i(t_0 t_1 \ldots) = \cap_{n\geq 0} t_n$ defines an embedding of T with contact points (see section 1.3.5).

Of course, any map which is topologically conjugate to g, i.e. with the some rotation number, generates the tree T but a different embedding. The question of determining whether two such mappings are smoothly conjugate boils down to showing that these embeddings determine the same smooth structure on L_T. It is conjectured based in renormalisation analysis of critical circle maps that the differentiability of the conjugacy between critical circle maps with the same diophantine rotation number is generally between one and two. Therefore, we can use the following results on smooth structures on embedded trees to obtain computer estimates for the differentiability of the conjugacy between critical circle maps with the same rotation number.

1.3.5 Smooth structures on embedded trees

A *tree* T consists of a set of vertices of the form $V_T = \bigcup_{n\geq 0} T_n$, where each T_n is a finite set, together with a directed graph on these vertices such that each $t \in T_n$, $n \geq 1$, has a unique edge leaving it. Given such a tree T we define the *limit set* or *set of ends* L_T as the set of all sequences $\underline{t} = t_0 t_1 \ldots$ such that $m(t_{i+1}) = t_i$ for all $i \geq 0$.

If $\underline{t} = t_0 t_1 \ldots \in L_T$ then by $\underline{t}|n$ we denote the finite word $t_0 \ldots t_{n-1}$. Let $L_{\underline{t}|n}$ denote the set of $\underline{s} \in L_T$ such that $\underline{s}|n = \underline{t}|n$. This is called a *n-cylinder* of the tree. If L is an open subset of L_T containing $L_{\underline{t}|n}$ and $i : L \rightarrow \mathbb{R}$ a continuous mapping, then we denote by $C_{\underline{t}|n,i}$ the smallest closed interval in $\mathbb{R}$ which contains $i(L_{\underline{t}|n})$. This is also called a *n-cylinder*. Note that both $L_{\underline{t}|n}$ and $C_{\underline{t}|n,i}$ are determined by t_{n-1}. Therefore we shall often write these as $L_{t_{n-1}}$ and $C_{t_{n-1},i}$. Say that $\underline{s} \sim \underline{t}$ if $i(\underline{s}) = i(\underline{t})$.

We shall only be interested in mappings i which respect the cylinder structure of L_T in the following way. We demand that if $\underline{s}|n \neq \underline{t}|n$ then

$$\mathrm{int} C_{\underline{s}|n,i} \bigcap \mathrm{int} C_{\underline{t}|n,i} = \emptyset.$$

Clearly, the mapping $i : L \rightarrow \mathbb{R}$ induces a mapping $L/\sim \rightarrow \mathbb{R}$ which we also denote by i.

Definition 7 *(= Definition 13)* Such a pair (i, L) is a chart of L_T if L is an open set of L_T with respect to the metric d and the induced map $i : L/\sim \rightarrow \mathbb{R}$ is an embedding.

Two charts (i, L) and (j, K) are *compatible* if the equivalence relation $\sim$ corresponding to i agrees with that of j on $L \bigcap K$. They are $C^{1+\alpha}$ *compatible* if they are compatible and the mapping $j \circ i^{-1}$ from $i(L \bigcap K)$ to $j(L \bigcap K)$ has a $C^{1+\alpha}$ extension to a neighbourhood of $i(L \bigcap K)$ in $\mathbb{R}$.

Definition 8 *(= Definition 14)* A $C^{1+\alpha}$ *structure* on L_T is a maximal set of $C^{1+\alpha}$ compatible charts which cover L_T. A $C^{1+\alpha-}$ structure is a maximal set of charts covering L_T which are $C^{1+\beta}$ compatible for all $0 < \beta < \alpha$.

Obviously, a finite set of $C^{1+\alpha}$ compatible charts which cover L_T defines a $C^{1+\alpha}$ structure on L_T. A mapping $h : L_T \to L_T$ is smooth if its representatives in local charts are smooth in the following sense: if $\underline{t} \in L$ and $h(\underline{t}) \in L'$ where (i, L) and (i', L') are charts in the structure then $i' \circ h \circ i^{-1}$ has a smooth extension to a neighbourhood of $i(\underline{t})$ in $\mathbf{R}$. Similarly, we define smooth maps between different spaces.

We shall mostly be concerned with situations where either (i) the smooth structure is defined by a single chart or (ii) the structure is defined by a single embedding of $L_T/\sim$ into the circle $\mathbf{T}^1$.

If $\mathcal{S}$ is a $C^{1+\alpha}$ structure on L_T and i is a chart of $\mathcal{S}$ then we have that $\underline{s}|n$ and $\underline{t}|n$ are *adjacent* if there is no $\underline{u} \in L_T$ such that $C_{\underline{u},i}$ lies between $C_{\underline{s}|n,i}$ and $C_{\underline{t}|n,i}$ and that they are *in contact* if $C_{\underline{s}|n,i} \bigcap C_{\underline{t}|n,i} \neq \emptyset$. Note that these conditions are independent of the choice of the chart i of $\mathcal{S}$ which contains $L_{\underline{s}|n}$ and $L_{\underline{t}|n}$ in its domain. It does however depend upon $\mathcal{S}$ so we only use this terminology when we have a specific structure in mind. If $\underline{s}|n = s_0 \dots s_{n-1}$ and $\underline{t}|n = t_0 \dots t_{n-1}$ then we say that s_{n-1} and t_{n-1} are *adjacent* (resp. *in contact*) if $\underline{s}|n$ and $\underline{t}|n$ are.

Definition 9 *(= Definition 15)* Two $C^{1+\alpha}$ structures $\mathcal{S}$ and $\mathcal{T}$ on L_T are $(1+\alpha)$-*equivalent* if the identity is a $C^{1+\alpha}$-diffeomorphism when it is considered as a map from L_T with one structure to L_T with the other. They are $(1+\alpha^-)$-*equivalent* if the identity is a $C^{1+\beta}$-diffeomorphism for all $0 < \beta < \alpha$.

The scaling tree.

Gaps.

Fix a $C^{1+\alpha}$ structure $\mathcal{S}$ on L_T. If s and t are adjacent but not in contact then there is a *gap* between $i(L_s)$ and $i(L_t)$. We will add a symbol $g_{s,t} = g_{t,s}$ to T_n to stand for this gap if $m(s) = m(t)$. For the chart (i, L) we let $G_{s,t,i}$ denote the smallest closed interval containing the gap. Let $\tilde{T}_n$ denote the set T_n with all the gap symbols $g_{s,t}$ adjoined. Let $\tilde{V}_T = \bigcup_{n\geq 1} \tilde{T}_n$. If $m^p(s) = m^p(t)$ then $G_{s,t,i} = G_{m^{p-1}(s),m^{p-1}(t),i}$.

Primary atlas.

Suppose that $\mathcal{S}$ is a $C^{1+\alpha}$ structure on L_T. Then clearly there exists $N \geq 0$ such that if $T_N = \{t_1, \ldots, t_q\}$, then there are charts (i_j, U_j) of $\mathcal{S}$, $1 \leq j \leq q$, such that the open subset U_j contains the N-cylinder L_{t_j}. We call such a system of charts a *primary N-atlas.*

Scaling tree.

Fix such a primary N-atlas $\mathcal{I} = \{(i_j, U_j)\}_{j=1,\ldots,q}$. To each $s, t \in \tilde{T}_n$, $n \geq N$, we associate the following intervals in $\mathbf{R}$ (see figure 3(a), (b) and (c)).

- $C_{t,\mathcal{I}}$ and $G_{s,t,\mathcal{I}}$: $C_{t,\mathcal{I}}$ is the interval C_{t,i_j} where j is such that $m^r(t) = t_j$ for some $r \geq 1$. Similarly, $G_{s,t,\mathcal{I}}$ is the gap G_{s,t,i_j} if s and t are non-contact adjacent points with $m(s) = m(t)$.

- $C_{s,t,\mathcal{I}}$, $C_{t,s,\mathcal{I}}$ and $D_{t,s,\mathcal{I}}$: If $t, s \in T_n$, are adjacent and in contact, define $P_{t,s,\mathcal{I}} = P_{s,t,\mathcal{I}}$ as the common point between the closed sets $C_{t,\mathcal{I}}$ and $C_{s,\mathcal{I}}$. Define the closed sets $C_{t,s,\mathcal{I}}$ and $C_{s,t,\mathcal{I}}$, respectively, as the sets obtained from $C_{t,\mathcal{I}}$ and from $C_{s,\mathcal{I}}$, by rescaling them by the factor $1/2$, keeping the points $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$ fixed. Define $D_{t,s,\mathcal{I}} = C_{t,s,\mathcal{I}} \bigcup C_{s,t,\mathcal{I}}$ If $t, s \in T_n$ are adjacent but not in contact, define $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$, respectively as the common points of the closed sets $C_{t,\mathcal{I}}$ and $C_{s,\mathcal{I}}$ with the gap $G_{t,s,\mathcal{I}}$. Define the closed sets $C_{t,s,\mathcal{I}}$ and $C_{s,t,\mathcal{I}}$, respectively, as the sets obtained from $C_{t,\mathcal{I}}$ and from $C_{s,\mathcal{I}}$, by rescaling them by the factors $\delta_t/2$, $\delta_s/2$, and flipping them into the gap $G_{t,s,\mathcal{I}}$, keeping the points $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$ fixed. Here $\delta_t = |G_{t,s,\mathcal{I}}|/|C_{m^{p-1}(t),\mathcal{I}}|$ and $\delta_s = |G_{t,s,\mathcal{I}}|/|C_{m^{p-1}(s),\mathcal{I}}|$ where $p \in \mathbf{N}$ is such that $m^{p-1}(t) \neq m^{p-1}(s)$ and $m^p(t) = m^p(s)$.

- $E_{t,s,\mathcal{I}}$: Let $t_1, s_1 \in T_{n+1}$ be the adjacent vertices such that $G_{t_1,s_1,i} = G_{t,s,i}$. Define $E_{t,s,\mathcal{I}} = C_{t,s,\mathcal{I}} \setminus C_{t_1,s_1,\mathcal{I}}$.

$(1+\alpha)$-equivalence.

Now suppose, that in addition to the structure $\mathcal{S}$ and its primary atlas $\mathcal{I}$, we have another structure $\mathcal{T}$ and a primary N_1-atlas $\mathcal{J}$ for it. Redefine $N = \max(N_1, N)$. To each $t \in \tilde{T}_n$, $n > N$, we associate the following numbers.

- *the scaling tree $\sigma_{\mathcal{I}}(t)$:*

$$\sigma_{\mathcal{I}}(t) = \frac{|C_{t,\mathcal{I}}|}{|C_{m(t),\mathcal{I}}|} \quad \text{and} \quad \sigma_{\mathcal{I}}(g_{t,s}) = \frac{|G_{t,s,\mathcal{I}}|}{|C_{m(t),\mathcal{I}}|}$$

This defines a function

$$\sigma_{\mathcal{I}} : \bigcup_{n>N} \tilde{T}_n \to [0,1].$$

The fact that it is undefined for small n does not matter.

- ν_t:

$$\nu_t = \left| 1 - \frac{\sigma_{\mathcal{J}}(t)}{\sigma_{\mathcal{I}}(t)} \right|.$$

- A_t: If $t \in \tilde{T}_n$, let $t_1 < \cdots < t_p$ be the elements of T_n with the same mother as t. Between these there may be gaps represented by symbols of the form $g_{t_{m_n}, t_{m_n+1}}$. Denote these gap symbols by $g_1, \ldots, g_q$. Let

$$A_t = \sum_{j=1}^{p} \nu_{t_j} |C_{t_j,\mathcal{I}}| + \sum_{j=1}^{q} \nu_{g_j} |G_{g_j,\mathcal{I}}|$$

- $\nu_{s,t}$: If $s, t \in T_n$ are in contact,

$$\nu_{s,t} = \left| 1 - \frac{|C_{t,\mathcal{I}}|}{|C_{s,\mathcal{I}}|} \frac{|C_{s,\mathcal{J}}|}{|C_{t,\mathcal{J}}|} \right|$$

- $e_{s,t}$: If $s, t \in T_n$ are adjacent but not in contact,

$$e_{s,t} = |E_{s,t,\mathcal{I}}| = \frac{|G_{s,t,\mathcal{I}}|}{2|C_{m^{p-1}(s),\mathcal{I}}|} \left\{ |C_{t,\mathcal{I}}| - |C_{t_1,\mathcal{I}}| \right\}$$

  where $p \in \mathrm{N}$ is such that $m^{p-1}(t) \neq m^{p-1}(s)$ and $m^p(t) = m^p(s)$ and $t_1, s_1 \in T_{n+1}$ are the adjacent vertices such that $G_{t_1,s_1,\mathcal{I}} = G_{t,s,\mathcal{I}}$.

We use the following notation: if f and g are functions of a variable x with domain Δ, then we write $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$ with constant d if

$$d^{-1} < \frac{|f(x)|}{|g(x)|} < d$$

for all $x \in \Delta$. Often we will drop the reference to d.

Thus if a_n and b_n are sequences then $\mathcal{O}(a_n) = \mathcal{O}(b_n)$ means a_n/b_n and b_n/a_n are bounded away from 0 independently of n. The notation $f(x) = \mathcal{O}(g(x))$ means the same thing as $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$.

Similarly, $f(x) \leq \mathcal{O}(g(x))$ with constant d means $|f(x)/g(x)| < d$ for all $x \in \Delta$.

Scale equivalence.

We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$–*scale equivalent* if for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $f = f_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} f(n) \leq \mathcal{O}(f(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \tilde{T}_n$, $\nu_t \leq f(n)$ and $A_t \leq f(n)$;

(iii) for all $s \in T_n$ adjacent to t but not in contact with it, if $m(s) = m(t)$,
$$A_t e_{t,s,\mathcal{I}}^{-(1+\varepsilon)} + \nu_t e_{t,s,\mathcal{I}}^{-\varepsilon} \leq f(n)$$
while if $m(s) \neq m(t)$ then
$$\nu_t e_{t,s,\mathcal{I}}^{-\varepsilon} \leq f(n).$$

Contact equivalence.

We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$-*contact equivalent* if for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $f_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=0}^{\infty} f_\varepsilon(n) < \infty$;

(ii) for all $s, t \in T_n$, $n \geq N$ such that s and t are in contact,
$$\frac{\nu_{s,t}}{|D_{t,s,\mathcal{I}}|^{\varepsilon}} < f_\varepsilon(n).$$

The definitions of scale equivalence and contact equivalence do not at first sight appear to be symmetric in $\mathcal{I}$ and $\mathcal{J}$. However, from theorem 12, it follows that $\mathcal{I}$ and $\mathcal{J}$ define equivalent $C^{1+\alpha^-}$ structures and this implies that $\mathcal{O}(|C_{t,\mathcal{I}}|) = \mathcal{O}(|C_{t,\mathcal{J}}|)$. Therefore, if we exchange $\mathcal{I}$ and $\mathcal{J}$ in the definitions we have that the definitions are verified for the same α.

Definition 10 *(= Definition 18)* We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$-*equivalent* ($\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$) if they are $(1+\alpha)$-scale equivalent and $(1+\alpha)$-contact equivalent.

Theorems.

Theorem 12 *(= Theorem 17) Rand and Pinto.* Let $\mathcal{S}$ and $\mathcal{T}$ be $C^{1+\alpha}$ structures on L_T and let $\mathcal{I}$ (resp. $\mathcal{J}$) be a primary atlas for $\mathcal{S}$ (resp. $\mathcal{T}$). A sufficient condition for $\mathcal{S}$ and $\mathcal{T}$ to be $C^{1+\alpha^-}$-equivalent is that $\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$.

Theorem 13 *(= Theorem 18) Rand and Pinto.* Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ be as in theorem 12 and suppose that $\mathcal{S}$ and $\mathcal{T}$ are $C^{1+\alpha^-}$ equivalent. Then $\mathcal{I} \overset{\gamma}{\sim} \mathcal{J}$ if for all ε such that $0 < \varepsilon < \gamma$ there exists β such that $0 < \varepsilon < \beta < \gamma \leq \alpha$ and there exists a function $g = g_{\beta,\varepsilon} : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} g(n) \leq \mathcal{O}(g(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \tilde{T}_n$, $|C_{m(t),\mathcal{I}}|^{\beta} < g(n)$;

(iii) for all $t_1, t_2 \in T_n$, which are adjacent but not in contact, if $m(t_1) = m(t_2)$ then

$$\frac{|C_{m(t_1),\mathcal{I}}|^{1+\beta}}{|E_{t_1,t_2,\mathcal{I}}|^{1+\varepsilon}} < g(n),$$

while if $m(t_1) \neq m(t_2)$ then

$$\frac{|C_{m(t_1),\mathcal{I}}|^{\beta}}{|E_{t_1,t_2,\mathcal{I}}|^{\varepsilon}} < g(n);$$

(iv) for all $t_1, t_2 \in T_n$, which are in contact we have that

$$|D_{t_1,t_2,\mathcal{I}}|^{\beta-\varepsilon} < g(n).$$

These conditions hold for some of the most interesting problems. In this case theorem 12 and theorem 13 give a necessary and sufficient condition for $(1+\alpha)$-equivalence.

1.3.6 Universality of circle maps.

We give a critical and non-critical example of universality as motivation for the renormalisation analysis.

Golden diffeomorphisms.

Fix $|\mu| < 1$ and let $f_\nu = f_{\mu,\nu}$, so that f_ν is a diffeomorphism. Let ν_* be the value of ν such that $\rho(f_\nu) = \theta$ is the golden mean and $\nu_n = p_n/q_n$ be the nth rational approximant of ν. By the theory of Herman [9] and Yoccoz [31],

(i) If $f = f_\nu$ then $f^{q_n}(0) - p_n$ decreases as a^n where $a = -\theta$.

(ii) $a^{-n}(f^{q_n}(a^n x) - p_n)$ converges, up to a scale change, to $x \to x + \theta$.

(iii) $\lim_{n\to\infty}(\nu_n - \nu_{n-1})/(\nu_{n+1} - \nu_n) = \delta$ where $\delta = -\theta^{-2}$.

Golden critical circle maps.

A golden critical circle map is a critical circle map with the rotation number equal to the golden mean.

Experimental work indicates the following facts.

(i) if $f = f_\nu$ is golden then $f^{q_n}(0) - p_n$ decreases as a^n where $a = -\theta^{0.527\ldots}$.

(ii) $a^{-n}(f^{q_n}(a^n x) - p_n)$ converges to an analytic function ψ of x^3 as $n \to \infty$

(iii) If ν^n is as above with $\mu = 1$ then $\lim_{n\to\infty}(\nu_n - \nu_{n-1})/(\nu_{n+1} - \nu_n) = \delta$ where $\delta = -\theta^{-2.164\ldots}$.

(iv) If $\nu_\infty = \lim_{n\to\infty} \nu_n$ there is a neighbourhood U of $(1, \nu_\infty)$ such that if $(\mu, \nu) \in U$ then $\rho(f_{\mu,\nu}) = \theta$ if and only if $\nu = \nu_*(\mu)$ and $\mu \leq 1$ where the function ν_* is C^∞ on $\mu \neq 1$ and C^2 at $\mu = 1$. If $\mu < 1$ then $f_{\nu_*(\mu)}$ is analytically conjugate to the rotation R_θ, while if $\mu = 1$ then it is C^{1+}-conjugate to R_θ.

Any two parameter family which satisfies these conditions, perhaps after a change of coordinates in the phase and/or parameter space, is in the golden mean class. Numerical studies show that there are many families in this universality class.

1.3.7 Renormalisation analysis.

Notation. We introduce the following notation.

(i) The set Ω is relatively compact open subset of $\mathbf{C}$.

(ii) The set B_r is the domain $2\pi|\mathrm{Im}(z)| < \ln r$ in $\mathbf{C}$.

(iii) $\mathcal{A}(\Omega)$ is the real Banach space (with sup norm) of continuous functions $f : \overline{B}_r \to \mathbf{C}$ which are holomorphic on B_r, take real values at real points, and satisfy $f(x+1) = f(x) + 1$.

(iv) $\mathcal{D}_r$: the real Banach space (with supremum norm) of continuous functions $f : B_r \to \mathbf{C}$ which are holomorphic on B_r, take real values at real points, and satisfy $f(x+1) = f(x) + 1$.

Fix $\nu = .563$. Let Ω_1 and Ω_2 be respectively small neighbourhoods in $\mathbf{C}$ of the real segments $[\nu - 1, 0]$ and $[0, \nu]$ such that $a\overline{\Omega}_1 \subset \Omega_2$ and $a\overline{\Omega}_2 \subset \Omega_1$, for all $a \in [-0.78, -0.77]$. Consider the set of commuting pairs $(\psi, \eta) \in \mathcal{A}(\Omega_1) \times \mathcal{A}(\Omega_2)$ such that

(i) the closure of $\psi(-a\Omega_1)$ is contained in Ω_2 for all $a \in [-0.78, -0.77]$;

(ii) $\eta(0) < 0 < \psi(0) < 1$;

(iii) $d^2\psi(0) = 0$, $d\psi(\eta(0)) \neq 0 \neq d\eta(\psi(0))$ and $d^2\psi(\eta(0)) \neq 0 \neq d^2\eta(\psi(0))$.

(iv) $d^i(\psi(\eta) - \eta(\psi))(0) = 0$ for $0 \leq i \leq 3$.

A pair (ψ, η) is said to be cubic critical if it also satisfies $d\psi(0) = 0$ and $d^3\psi(0) \neq 0$. This implies $d\eta(0) = 0 = d^2\eta(0) = d^2\psi(0)$ and $d^3\eta(0) \neq 0$.

To see the circle map structure consider ψ and η restricted to $\mathbf{R}$. Each pair (ψ, η) satisfying the conditions above and such that ψ and η are monotone determines a mapping $f = f_{\psi,\eta}$ of the interval $I_{\psi,\eta} = [\eta(0), \psi(0)]$ to itself in the following way: $f(x)$ is defined to be $\psi(x)$ if $x \in [\eta(0), 0]$ and $\eta(x)$ if $x \in [0, \psi(0)]$. This can be regarded as a mapping of the circle $S_{\psi,\eta}$ obtained by identifying the end-points of $I_{\psi,\eta}$ to itself since $f(\eta(0)) = \psi(\eta(0)) = \eta(\psi(0)) = f(\psi(0))$. Moreover, the monotone condition can be dropped if the identification to obtain the circle is defined on neighbourhoods of the end-points. Finally, although the circle mapping defined by f is clearly not necessarily analytic in the standard structure on $S_{\psi,\eta}$, if (ψ, η) satisfies $\psi \circ$

$\eta = \eta \circ \psi$ on a neighbourhood of 0, and $g = \psi \circ \eta^{-1}$ is well defined and a homeomorphism near $\eta(0)$ then by glueing the circle with g one obtains an analytic structure in which f is analytic. This corresponds to letting the dynamics determine the glueing, as explained in Rand [24].

If $f = f_{\psi,\eta}$, let $\tilde{f}$ denote $b^{-1} \cdot f \circ b$ where $b = \psi(0) - \eta(0)$. Let $\overline{f}$ be the lift of $\tilde{f}$ to the universal cover R with the property that $0 \leq \overline{f}(0) < 1$. If $f = f_{\psi,\eta}$ is an homeomorphism, the rotation number $\rho(\psi,\eta)$ of (ψ,η) is defined to be the rotation number of $\tilde{f}$. Then $0 \leq \rho(\psi,\eta) < 1$.

Let f be in $\mathcal{D}_r$, for some $r > 1$ and $(\psi,\eta) = (f, f-1)$. If $\rho(f)$ lies strictly between the rational approximates p_n/q_n and p_{n+1}/q_{n+1} obtained by truncating the continued fraction expansion of $\rho(f)$, then define the renormalisation transformation R^n as follows.

$$(\psi_n, \eta_n) = R^n(\psi,\eta) = (a_n^{-1} \cdot f_n \circ a_n, a_n^{-1} \cdot f_{n+1} \circ a_n)$$

where $f_n = f^{q_n} - p_n$, $a_n = f_n(0) - f_{n+1}(0)$. By Rand [21], we have the following equality for the rotation number, where [] means the characteristic of a number.

$$\rho(R(\psi,\eta)) = (\rho(R(\psi,\eta)))^{-1} - [\rho(R(\psi,\eta))]^{-1}.$$

If (ψ,η) is cubic critical then so is $R^n(\psi,\eta)$. If $\rho(\psi,\eta)$ is equal to the golden-mean, then

$$R(\psi,\eta) = a^{-1} \cdot (\psi, \psi \circ \eta) \circ a$$

where $a = \eta(0) - \psi(\eta(0))$. By Jonker [10], $(\psi,\eta) = (x+\gamma, x+\gamma-1)$, where γ is equal to the golden-mean, is a hyperbolic fixed point of R with a 1-dimensional unstable manifold with associated eigenvalue $-\gamma^{-2}$.

1.3.8 Markov families

Let f be a circle map. Let (ψ,η) be the corresponding pair of maps and $(\psi_m, \eta_m) = R^m(\psi,\eta)$. Define the corresponding Markov family $F_f = (F_m)_{m\in \mathbb{Z}_{\geq 0}}$ by:

$$F_m = \begin{cases} a_m^{-1}\psi_m^j(x) & \text{if } x \in I_j^m \text{ for some } j = 0,\dots,n-1 \\ a_m^{-1}\psi_m^{n-1}\eta_m(x) & x \in I_n^m \end{cases}$$

where $I_j^m = [\psi_m^{n-j-1}\eta_m(0), \psi_m^{n-j-1}\eta_m\psi_m(0)]$ for $j = 0,\dots,n-1$, $I_n^m = [\psi_m^{n-1}\eta_m(0), \psi_m(0)]$, $a_m = -|I_0^m|$ and $1/(n+1) \leq \rho(\eta_m, \psi_m) < 1/n$.

1.3.9 Diffeomorphisms of the circle.

Let f be a homeomorphism of the circle. The map f has rational rotation number $\rho(f) = p/q$ if and only if there is $x \in \Pi^1$ such that $f^q(x) = x + p$. Therefore, the class of homeomorphisms of the circle with rotation number $\rho(f) = p/q$ conjugated to the rotation map $R(x) = x + p/q$ is of infinite codimension.

Denjoy proved that a diffeomorphism of the circle $f \in C^2$ with irrational rotation number $\rho(f) = \alpha$ is topologically conjugated to the rotation map $R(x) = x + \alpha$.

Denjoy constructed some examples of diffeomorphism of the circle $f \in C^1$ with irrational rotation number $\rho(f) = \alpha$ which are not topologically conjugated to the rotation map $R(x) = x + \alpha$.

The question of for which irrational rotation numbers the conjugacy maps are smooth arise.

Arnol'd showed the existence of a diffeomorphism of the circle $f \in C^\omega$ with irrational rotation number $\rho(f) = \alpha$ which are not absolutely continuously conjugated to the rotation map $R(x) = x + \alpha$. This construction is based on the existence of small denominators. This leads to the study of the diophantine properties of the rotation number.

1.3.10 Arnol'd,Herman,Yoccoz theorem.

A local theorem is due to Arnol'd. Arnol'd [2] proved the existence of a C^∞ conjugacy for diffeomorphisms of the circle $f \in C^\infty$ with rotation number of diophantine type sufficiently close in the C^∞ topology to the rotation map with the same rotation number.

Herman [9] proved for the global case, that the set $\mathcal{A}$ of rotation numbers for which the diffeomorphisms of the circle $f \in C^\infty$ are C^∞ conjugated has full Lebesgue measure. The set of rotation numbers $\mathcal{A}$ that he proved the theorem is contained in the set of all Roth rotation numbers. He showed that the biggest set $\mathcal{A}$ should be contained in the set of all diophantine rotation numbers.

Theorem 14 *Yoccoz.* Let $f \in C^k$ be a diffeomorphism of the circle with rotation number $\rho \in C_\beta$, where $\beta \geq 0$ and $k \geq 3$. If $k > 2\beta + 1$ then there exist a conjugacy $h \in C^{k-1-\beta-\varepsilon}$ for all $\varepsilon > 0$ between f and the rotation map $R_\rho(x) = x + \rho$.

Corollary 1 Under the same hypotheses for f and ρ as above, if $f \in C^\infty$ (respectively $f \in C^\omega$) then $h \in C^\infty$ (respectively $h \in C^\omega$).

Yoccoz [31] proved that $\mathcal{A}$ contains the set of all diophantine rotation numbers. He has recently completely characterised the degree of smoothness between circle maps with the same rotation number.

By Jonker [10], if f is an analytic diffeomorphism whose rotation number $\rho(f)$ is the golden-mean, then the speed of convergence of the renormalisation of f can be made arbitrarily fast in the analytic norm, by replacing the renormalisation

$$R(\psi, \eta) = a^{-1}(\psi, \psi \circ \eta) \circ a$$

where $a = \eta(0) - \psi(\eta(0))$, by

$$\tilde{R}(\psi, \eta) = p_{\psi,\eta}^{-1}(\psi, \psi \circ \eta) \circ p_{\psi,\eta}$$

where $\mathcal{A}(\psi, \eta) = p_{\psi,\eta}$ is a bounded affine map and $p_{\psi,\eta} : \mathbf{R} \to \mathbf{R}$ is a polynomial. The affine map $\mathcal{A}$ is chosen in such away that we get rid of the biggest eigenvalues of the stable manifold of the renormalisation operator R_1.

Using the developed analytic tools in the formalism of Markov families to determine the differentiability of the conjugacy between two Markov families F_n and G_n topologically conjugated, we get us a corollary that two analytic diffeomorphisms with golden-mean rotation number are C^∞ conjugated. The proof also works for analytic diffeomorphisms with the same rotation number of periodic type. We conjecture that it also works for any rotation number of constant type.

1.4 Symbolic dynamics and renormalisation.

In a number of cases the renormalisations $R^n f$ of a dynamical system f are conjectured to converge to a horseshoe Λ of R as n goes to infinity. Examples of systems to which these ideas apply include diffeomorphisms and cubic critical maps of the circle, quadratic foldings with kneading invariants of constant type, the boundary of Seigel domains and KAM and critical invariant circles in area-preserving and dissipative twist maps. Such an horseshoe picture for critical circle maps is described in Lanford [11] and for quadratic unfoldings is described in Rand [26]. Let us briefly consider this.

1.4.1 Critical circle maps.

Let f be the lift of a cubic critical map of the circle with rotation number $\rho = \rho(f) \in (0,1)$. To ρ we associate the sequence $\rho(f) = \rho_0\rho_1\ldots$ of positive integers which define the continued fraction of ρ. To f we associate the commuting pair $(\psi_f, \eta_f) = (f, f-1)$. Let R be the renormalisation transformation as defined in section 1.3.7. Then, $\rho(R(\psi,\eta)) = \rho(\psi,\eta)^{-1} - [\rho(\psi,\eta)^{-1}]$. For all $\rho = \rho_0\rho_1\ldots$, define $\sigma^n(\rho) = \rho_n\rho_{n+1}\ldots$. Then, $\rho(R^n(\psi,\eta)) = \sigma^n(\rho)$. The conjecture is that in the space of such pairs there exists a set Λ and a homeomorphism $\mathcal{Q} : (\mathbf{Z}_{\geq 0})^{\mathbf{Z}} \to \Lambda$ such that

(i) $\rho(\mathcal{Q}(\ldots\rho_{-1}\rho_0\rho_1\ldots)) = \rho_0\rho_1\ldots$.

(ii) If $\rho(f) = \rho_0\rho_1\ldots$ and $\ldots\rho_{-2}\rho_{-1}$ is an arbitrary sequence then

$$||R^n(f) - \mathcal{Q}(\sigma^n(\rho(f)))||$$

converges to 0 as n tends to infinity. If $\rho(f)$ has bounded entries then this convergence is exponentially fast.

(iii) $R(\Lambda) = \Lambda$.

(iv) If Λ_N is the image under $\mathcal{Q}$ of $\{1,\ldots,N\}^{\mathbf{Z}} \subset (\mathbf{Z}_{\geq 0})^{\mathbf{Z}}$ then Λ_N is a hyperbolic set for R with 1-dimensional unstable manifolds and 1-codimensional stable manifolds.

1.4.2 Quadratic foldings with kneading invariants of constant type.

Let f be an analytic quadratic folding of the interval $I = [-1,1]$. The dynamics of f is even largely determined by its *kneading invariant* ν_f which is defined in the following way. If $x \in I$ let $\theta_n(x)$ be 1, -1 or 0 according as $f^n(x)$ is orientation preserving at x, reversing or $f^j(x) = 0$ for some $0 \leq j < n$. Let $\underline{\theta}(x) = \theta_0(x)\theta_1(x)\ldots \in \{-1,0,1\}^{\mathbf{N}}$. By Rand [26], then the map $x \to \underline{\theta}(x)$ is monotone and the limit $\nu_f = \lim_{x\to 0^-} \underline{\theta}(x)$ exists in the product topology on $\{-1,0,1\}^{\mathbf{N}}$. The sequence ν_f is called the *kneading invariant* of f. We analyze the subset S of those f of *infinite depth.* If $f \in$ S then there exists $n > 0$ and a subinterval J of I such that $f^n|J$ is conjugate to a element of S. The best known examples of maps f of infinite depth are those corresponding to the accumulation point of period-doubling. For these there is an interval J containing 0 such that $f^2|J$ is conjugate to f.

For n odd or equal to 2, we define the renormalisation operator R_n on $\mathcal{D}_n$ by

$$R_n(f) = a^{-1} \cdot f^n \circ a,$$

where $a = 1/f^n(c)$ and $\mathcal{D}_n$ consists of analytic quadratic foldings such that $R_n(f)$ is a quadratic folding. By Rand [26], the kneading invariant ν_f determines if f belongs or not to $\mathcal{D}_n$. Define the *renormalisation sequence* $\underline{a} = a_1 a_2 \ldots$ of $f_0 = f$ if $f_i = R_{a_i}(f_{i-1})$, for all $i > 0$ and $\sigma^n(\underline{a}) = a_{n+1}a_{n+2}\ldots$. Define the operator R on $\mathcal{D} = \cup_{n\geq 2}\mathcal{D}_n$ by $R|\mathcal{D}_n = R_n$. We conjecture that one has horizontal and vertical strips and a similar picture to that for critical circle maps because the operator R acts on kneading invariants in much the same way as the corresponding transformation there acted on rotation numbers. Moreover, we conjecture that the stable manifolds $H_{\underline{a}}$, $\underline{a} = a_1 a_2 \ldots$, $a_i \geq 2$, will consist of those f whose nonwandering set consists of an infinite number of hyperbolic repellors and a minimal attractor A whose dynamics are described in Jonker [10]. In particular, it follows from the renormalisation that A can be described as follows: There exists a decreasing sequence of closed intervals J_m, $m \geq 0$, with $J_0 = I$ and such that if $l_m = a_0 \ldots a_{m-1}$ then $f^{l_m}|J_m$ is unimodal map and if $J_{m,i} = f^i J_m$ for $i = 0, \ldots, l_m - 1$ then $A = \cap_{m\geq 1}(\cup_i J_{m,i})$.

1.4.3 Limit set.

We reinterpret these pictures in terms of Markov families. We define the $(1+\alpha)$-determination condition for a Markov family. To this $(1+\alpha)$-determined Markov family F_n we associate its limit set $\mathcal{M}$ which is essentially the set of Markov families which are limit points of the sequence

$$\underline{F}_{n_i} = (F_{n_i} F_{n_i+1} \ldots)$$

of Markov families. There is a bi-Lipschitz map $\mathcal{F} : \Omega \to \mathcal{M}$ for an appropriate symbol space Ω.

Let f be a cubic critical map of the circle with $\rho(f) = \rho_0 \rho_1 \ldots$ and F the corresponding Markov family as defined in section 1.3.8. Let $\mathcal{M} = \mathcal{M}_F$. In this case Ω is the set of accumulation points of the sequence $(\sigma^n(\rho(f)))_{n\geq 0}$ so that $\Omega \subset (\mathbf{Z}_{\geq 0})^{\mathbf{Z}}$ and we conjecture that

$$F_{\mathcal{Q}(f)} = \mathcal{F}(\rho(f)).$$

If f is an analytic folding map with *renormalisation sequence* $\underline{a} = a_1 a_2 \ldots$ and F is the corresponding Markov family. Let $\mathcal{M} = \mathcal{M}_F$. In this case Ω is the set of accumulation points of the sequence $(\sigma^n(\underline{a}))_{n\geq 0}$ so that $\Omega \subset (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ and we conjecture that

$$F_{\mathcal{Q}(f)} = \mathcal{F}(\rho(f)).$$

1.4.4 $(1+\alpha)$-determination.

Let F be a topological Markov family. Say $F_i \sim F_j$ if there are orientation preserving homeomorphisms $h : C^i \to C^j$ and $h' : C^{i+1} \to C^{j+1}$ such that $h(C^i_a) = C^j_a$ for all $a \in S_i$, $h'(C^{i+1}_a) = C^{j+1}_a$ for all $a \in S_{i+1}$ and $h' \circ F_i = F_j \circ h$.

We can always choose the S_i such that $S_i \cap S_j = \emptyset$ or $S_i = S_j$ and such that $S_i = S_j$ is equivalent to $F_i \sim F_j$. We always assume that the labelling S_i has this property.

We say that $j \overset{n}{\sim} k$ if and only if $F_{j+q} \sim F_{k+q}$, for all $0 \leq q < n$ and $j < k$.

The Markov family is *adapted* if whenever $S_i = S_j$ then $I^i_a = I^j_a$, for all $a \in S_i$. In the following, we always consider that the Markov family F is adapted.

For all $m, n \geq 0$ and $t \in \Sigma^m_n$ we denote C^m_t by C_t since the dependence upon m is determined by t, whenever it will not be confusing. If there is a gap $G_{t,t'}$ between C_t and $C_{t'}$ we introduce a symbol $g_{t,t'} = g_{t',t}$ and denote by $\hat{\Sigma}^m_n$ the set consisting of these new symbols together with Σ^m_n. When we say that a statement is valid for all $t \in \hat{\Sigma}^m_n$, we mean that it is valid for all t and $g_{t,t'}$ in $\hat{\Sigma}^m_n$.

We denote by J and m the mappings $J : \hat{\Sigma}^l_n \to \hat{\Sigma}^{l+1}_{n-1}$ and $m : \hat{\Sigma}^l_n \to \hat{\Sigma}^l_{n-1}$ given by

$$J(t_0 \dots t_{n-1}) = t_1 \dots t_{n-1} \quad \text{and} \quad J(g_{t,t'}) = g_{J(t),J(t')},$$
$$m(t_0 \dots t_{n-1}) = t_0 \dots t_{n-2} \quad \text{and} \quad m(g_{t,t'}) = m(t).$$

Define the *scaling tree* $\sigma_m = \sigma_{F_m} : \cup_{n\geq 1} \hat{\Sigma}^m_n \to \mathbb{R}$ by

$$\sigma_m(t) = \frac{|C_t|}{|C_{m(t)}|}.$$

For all $j \overset{n}{\sim} k$ and all $t \in \hat{\Sigma}^j_i$ and $t \in \hat{\Sigma}^k_i$ and all $0 \leq i \leq n$, define

$$\mu_t = |1 - \frac{\sigma_j(t)}{\sigma_k(t)}| \quad \text{and} \quad \Lambda_t = \sum_{\{t' \in \hat{\Sigma}^j_i : m(t') = m(t)\}} (\mu_{t'} |C^j_{t'}|).$$

(iv) For all $j \overset{n}{\sim} k$ and all contact words $t, s \in \Sigma_i^j$ and $t, s \in \Sigma_i^k$ and all $0 \leq i \leq n$ define

$$\mu_{t,s} = |1 - \frac{|C_t^j|}{|C_s^j|}\frac{|C_s^k|}{|C_t^k|}|.$$

Definition 11 *(= Definition 22)* A topological Markov family F is $(1+\alpha)$-*scale determined* if and only if it possesses the $(1+\alpha)$-scale property and for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{q=m}^{\infty} g(q) < \mathcal{O}(g(m))$, for all $m \geq 0$.

(ii) For all $j \overset{n}{\sim} k$, let $u = \min\{j, k\}$. For all $a \in S_j$,

$$\frac{|C_a^k|}{|C_a^j|} \in 1 \pm g(u) \quad \text{and} \quad \frac{|I_a^k|}{|I_a^j|} \in 1 \pm g(u).$$

(iii) For all $0 \leq i \leq n$ and all $t \in \hat{\Sigma}_i^j$,

$$\mu_t < g(u+i).$$

If $s, t \in \Sigma_i^j$ are not in contact and $m(s) \neq m(t)$ then

$$\mu_t |E_{t,s}|^{-\varepsilon} < g(u+i)$$

while if $m(s) = m(t)$ then

$$|E_{t,s}|^{-(1+\varepsilon)} A_t + |E_{t,s}|^{-\varepsilon} \mu_t < g(u+i).$$

Definition 12 *(= Definition 23)* A topological Markov family F is $(1+\alpha)$-*contact determined* if it possesses the $(1+\alpha)$-contact property and for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{q=m}^{\infty} g(q) < \mathcal{O}(g(m))$, for all $m \geq 0$.

(ii) For all $j \overset{n}{\sim} k$, let $u = \min\{j, k\}$. For all $0 \leq i \leq n$ and $t, s \in \Sigma_i^j$ are in contact, then

$$\frac{\mu_{t,s}}{|D_{t,s}|^\varepsilon} < g(u+i).$$

1.4.5 The symbolic set.

We define the *symbolic set* Ω which indexes the set of topological Markov maps in the limit of the Markov family F. Let $\mathcal{S} = \{S_i\}_{i=0}^{\infty}$. Let $\Omega \subset \mathcal{S}^{\mathbf{Z}}$ denote the set of all bi-infinite sequences $\underline{s} = \ldots s_{-1}s_0s_1\ldots$ such that for all $s_i \in \mathcal{S}$ and $n \in \mathbf{Z}$ and all $m \geq n$ there exists a sequence $j_i \to \infty$ such that $s_n \ldots s_m$ is the index sequence corresponding to the sequence of Markov maps $F_{j_i} \ldots F_{j_i+m-n}$, i. e. $s_{n+k} = S_{j_i+k}$ for $0 \leq k \leq m-n$. Define the map $\sigma : \Omega \to \Omega$ by $\sigma(\underline{s}) = \underline{z}$, where $z_i = s_{i+1}$, for all $i \in \mathbf{Z}$.

In chapter 4, we prove the existence of a bi-Lipschitz map $\mathcal{F} : \Omega \to \mathcal{M}_F$, where $\mathcal{M}_F$ is essentially the set of limit points of the sequence

$$\underline{F}_{n_i} = (F_{n_i}F_{n_i+1}\ldots)$$

of Markov families. The elements $\mathcal{F}(\underline{s}) = F^{\underline{s}} = (F_{\sigma^n(\underline{s})})_{n\in\mathbf{Z}}$ in $\mathcal{M}_F$ are two-sided Markov families with associated scaling functions $\sigma_{\sigma^n(\underline{s})} : \cup_{l>0}\Sigma_l^{\sigma^n(\underline{s})} \to \mathbf{R}$.

1.4.6 The scaling function.

Let Λ^- denote the set of all $\overline{\tau} = \ldots \tau_{-2}\tau_{-1}$ with the following property. There is $\underline{s} \in \Omega$ such that $\tau_{-n} \in s_{-n}$, for all $n > 0$. Denote $\tau_{-n}\ldots\tau_{-1}$ by $\overline{\tau}|n$. Define $\overline{\Lambda}_{\underline{s}} = \{\tau \in \Lambda^- : \tau_n \in s_n\}$. Define $\overline{\Lambda}_{g(\underline{s})}$ as the set of all $g_{\overline{\tau},\overline{\tau}'}$ with the following property. $\overline{\tau},\overline{\tau}' \in \overline{\Lambda}_{\underline{s}}$, $g_{\tau_{-1},\tau'_{-1}} \in \hat{\Sigma}_1^{\sigma^{-1}(\underline{s})}$ and $\tau_{-i} = \tau'_{-i}$, for all $i > 1$. Let $\Lambda_{\underline{s}} = \overline{\Lambda}_{\underline{s}} \cup \overline{\Lambda}_{g(\underline{s})}$.

The *scaling function* $s_{\underline{s}} = s_{F,\underline{s}} : \Lambda_{\underline{s}} \to \mathbf{R}$ is given by

$$s_{\underline{s}}(\tau) = \lim_{n\to\infty} \sigma_{\sigma^{-n}(\underline{s})}(\overline{\tau}|n) \quad \text{and} \quad s_{\underline{s}}(g_{\tau,\tau'}) = \lim_{n\to\infty} \sigma_{\sigma^{-n}(\underline{s})}(g_{\overline{\tau}|n,\overline{\tau}'|n}).$$

Let the map $g : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ be as in the definition of $(1+\alpha)$-scale determination of $(F_m)_{m\geq 0}$. Define the metric in $\Lambda_{\underline{s}}$ as follows.

$$d(\overline{\tau},\overline{\psi}) = g(n+1) \quad \text{and} \quad d(g_{\tau,\tau'},g_{\psi,\psi'}) = g(n+1)$$

if $\overline{\tau}|n = \overline{\psi}|n$ and $\tau_{-(n+1)} \neq \psi_{-(n+1)}$. Moreover, $\tau_{-1} = \psi_{-1}$ and $\tau'_{-1} = \psi'_{-1}$. If necessary, interchange τ_{-1} and τ'_{-1}. Otherwise, the distance is $g(1)$.

Lemma 1 *(= Lemma 29)* The scaling function $s_{\underline{s}}$ is well-defined and it is Lipschitz with respect to the metric d in $\Lambda_{\underline{s}}$.

Lemma 2 *(= Lemma 30)* Let F and G be two $(1+\alpha)$-determined Markov families topologically conjugated. Let $F^{\underline{s}}$ and $G^{\underline{s}}$ be two limit Markov families corresponding to F and G respectively.

(i) If $F^{\underline{s}}$ and $G^{\underline{s}}$ are $(1+\alpha)$-conjugated then the scaling functions $s_{F,\sigma^m(\underline{s})}$ and $s_{G,\sigma^m(\underline{s})}$ are equal, for all $m \in \mathbf{Z}$.

(ii) Let $F^{\underline{s}}$ and $G^{\underline{s}}$ have bounded geometry. If, for all $m \in \mathbf{Z}$, the scaling functions $s_{F,\sigma^m(\underline{s})}$ and $s_{G,\sigma^m(\underline{s})}$ are equal, then $F^{\underline{s}}$ and $G^{\underline{s}}$ are C^{1+} conjugated.

1.4.7 Convergence of the Markov family F to its ω-limit set $\mathcal{M}_F$.

Define the map $f_{\underline{s}} : \mathbf{N} \to \mathbf{R}^+$ by

$$f_{\underline{s}}(l) = \max\{|C_t| : t \in \Sigma_l^{\underline{s}} \text{ or } t \in \Sigma_l^j \text{ and } S_j \dots S_{j+l-1} = s_0 \dots s_{l-1}\}.$$

Define the map $r_{\varepsilon,\underline{s}} : \mathbf{N} \times \mathbf{N} \to \mathbf{R}^+$ by

$$r_{\varepsilon,\underline{s}}(j,l) = (g_{\varepsilon''}(u))^{(\varepsilon''-\varepsilon')(\varepsilon'-\varepsilon)} + (f_{\underline{s}}(l))^{\varepsilon'-\varepsilon}$$

where $u = \min\{j,l\}$, $\varepsilon < \varepsilon' < \varepsilon'' < \alpha$ and the map $g_{\varepsilon''}$ is defined in $(1+\alpha)$-scale determination.

We suppose the following uniformity condition over the map $r_{\varepsilon,\sigma^n(\underline{s})}$. This is true, if for all $t \in \Sigma_n^m$ and all $m \geq 0$ the length of the intervals $|C_t|$ and $g_{\varepsilon'}(n)$ decrease exponential fast to zero, when n tends to infinity.

Condition U: There is ν_ε between 0 and 1 such that $r_{\varepsilon,\sigma^n(\underline{s})}(j,l) \leq \mathcal{O}(\nu_\varepsilon^{2l})$, for all $j \geq l > 0$ and all $n \in \mathbf{Z}$.

We prove the following theorem on convergence of the Markov family F to its ω-limit set $\mathcal{M}$.

Theorem 15 *(= Theorem 22)* Let F be a bounded Markov family which is $(1+\alpha)$-scale determined and $(1+\alpha)$-contact determined. For all $n \in \mathbf{Z}$, let $r_{\varepsilon,\sigma^n(\underline{s})}$ be the function as defined above. For all $j, l > 0$, such that $S_j \dots S_{j+l-1} = s_n \dots s_{n+l-1}$ and all $0 < \varepsilon < \alpha$

$$\|F_j - F_{\sigma^n(\underline{s})}\|_{C^{1+\varepsilon}(K^j \cup K^{\sigma^n(\underline{s})})} \leq \mathcal{O}_\varepsilon(r_{\varepsilon,\sigma^n(\underline{s})}(j,l)).$$

1.4.8 Application to diffeomorphisms of the circle.

Let f be a diffeomorphism of the circle with constant rotation number and F the associated Markov family to f as defined in section 1.3.7. Suppose that F is $(1+\alpha)$-determined. The symbolic sequence of the Markov family F is given by the continued fraction expansion of the rotation number $\rho = \rho_1 \ldots$ of f. Define $\sigma^m(\rho) = \psi$, where $\psi_i = \rho_{m+i}$, for all $i > -m$ and ψ_i is arbitrary for $i \leq -m$. Endow the set $\mathcal{H} = \{0, \ldots, N\}^{\mathbf{Z}}$, for some large $N > 0$, with the product of the discrete topologies. Define the symbolic set Ω_f as the set of $\theta \in \mathcal{H}$ such that there is a converging subsequence of $(\sigma^m(\rho))_{m>0}$. There is a bi-Lipschitz map $\mathcal{F} : \Omega_f \to \mathcal{M}_f$, where $\mathcal{M}_f$ is the limit set of f consisting of two-sided Markov families. By the bi-Lipschitz map $\mathcal{F}$, the symbolic dynamics in Ω_f are carried on to the limit set $\mathcal{M}_f$. The Markov family F converges to $\mathcal{M}_f$ as proven in theorem 15. Stark [27] proves that if f is a $C^{2+\epsilon}$ diffeomorphism of the circle whose rotation number is of constant type then the renormalisation of f converges in the C^2 norm to the line of the rotations of the circle. By this fact and by theorem 15, the set $\mathcal{M}_f$ just depends upon the rotation number of f. Moreover, as the map $\mathcal{F}$ is bi-Lipschitz then the symbolic set Ω_f just depends upon the rotation number of f.

Similar applications are given in chapter 4 to critical circle maps and quadratic foldings with infinite depth.

1.4.9 Two-sided Markov families.

The ω-limit set of Markov family consists of two-sided Markov families. In chapter 5, we study C^{1+} self-similarities in the blown-up of small intervals in the domains of a two-sided Markov family F. We prove that if two $C^{k+\delta}$ two-sided Markov families F and G are C^{1+} conjugated then they are $C^{k+\delta}$ conjugated. This result opposes to the difficulty in getting higher smoothness in one-sided Markov families. In that case a balance between the speed of convergence of the Markov families and the scaling structure of their cylinders is needed.

Let $F = (F_m)_{m\in\mathbf{Z}}$ and $G = (G_m)_{m\in\mathbf{Z}}$ be $C^{k+\delta}$ weakly bounded two-sided Markov families, where $\delta \in (0,1]$ and $k > 0$.

A Markov family F is *weakly bounded* if there are constants b and e, such that, $|dF_m| > e > 1$ and $||F_m||_{C^{k+\delta}} \leq b$, for all $m \in \mathbf{Z}$.

Let $h = (h_m)_{m\in\mathbf{Z}}$ be a topological conjugacy between F and G.

The conjugacy h has the *uniformity property* if it satisfies the following conditions.

(i) There is a sequence of points $x_m \in C^{F_m}$ such that F_m and h_m are smooth at x_m, $F_m(x_m) = x_{m+1}$ and $|dh_m(x_m)| > M_1 > 0$, for all $m < 0$.

(ii) Moreover, there is a continuous function ε such that $\varepsilon(0) = 0$ and for all $m < 0$,

$$|\frac{h_m(z_m) - h_m(y_m)}{z_m - y_m} - dh_m(x_m)| \leq \varepsilon(\max\{|y_m - x_m|, |z_m - x_m|\}).$$

Theorem 16 *(= Theorem 27)* If h is a topological conjugacy between F and G with the uniformity property then there is a $C^{k+\delta}$ conjugacy $r = (r_m)_{m\in Z}$ between F and G.

Corollary 2 *(= Corollary 18)* Let F and G be $C^{k+\delta}$ constant Markov families. Let the map h be a topological conjugacy between F and G. Let x be a periodic point of F, such that F is smooth at x. If h satisfies the uniformity property at x then there is a $C^{k+\delta}$ conjugacy between F and G.

Chapter 2

A Classification of $C^{1+\alpha}$ Structures on Embedded Trees.

A CLASSIFICATION OF $C^{1+\alpha}$ STRUCTURES ON EMBEDDED TREES.

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Abstract

We classify the $C^{1+\alpha}$ structures on embedded trees. This extends the results of Sullivan [6] on embeddings of the binary tree to trees with arbitrary topology and to embeddings without bounded geometry and with contact points. Such an extension is needed, for example, for applications to the smooth conjugacy and renormalisation problems for circle maps with Diophantine rotation number.

2.1 Introduction.

Although they have more general application, the results proved in this paper are mainly motivated by problems concerning the existence of smooth conjugacies in dynamical systems, and particularly, the newly discovered phenomenon of rigidity of certain infinitely renormalisable dynamical systems. It is often possible to determine classes $\mathcal{C}$ of smooth mappings $f : M \to M$ such that (i) if $f \in \mathcal{C}$ then f possesses in a natural fashion an invariant set Λ_f such that (ii) if $f, g \in \mathcal{C}$ then there exists a natural homeomorphism $h : \Lambda_f \to \Lambda_g$ which conjugates the dynamics of f on Λ_f to those of g on Λ_g (i.e. on Λ_f, $g \circ h = h \circ f$). The question then arises of whether or not this conjugacy h is smooth in the sense that it has an extension to a smooth diffeomorphism of the manifold M which contains Λ_f and Λ_g.

In general, such smoothness is very rare: in some sense, it is usually of infinite codimension. However, the theory of renormalisation has revealed the remarkable fact that many critical systems (and some non-critical ones) are rigid in the sense that, whenever it is possible, then h is smooth. This theory covers both classical examples such as the Arnol'd-Herman-Yoccoz theorem for diffeomorphisms of the circle and the Kolmogorov-Arnol'd-Moser theorem

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for invariant tori which concern non-critical behaviour, and more recent discoveries concerning critical behaviour such as the universality of the Feigenbaum period-doubling attractor, the structure of critical circle mappings and the breakdown of critical invariant circles.

For the Arnol'd-Herman-Yoccoz theorem, one takes for $\mathcal{C}$ the set of smooth (say C^∞) diffeomorphisms of the circle $M = S^1$ whose rotation number is a given Diophantine irrational number ρ. In this case, $\Lambda_f = M$ and the existence of the conjugacy h follows from the relatively easy Denjoy's theorem. The Arnol'd-Herman-Yoccoz theorem states that these conjugacies h are, in fact, C^∞; the main step in the proof being to show that they are $C^{1+\alpha}$ for some $\alpha > 0$.

The case of the Feigenbaum period-doubling attractor appears somewhat differently. For this one takes for $\mathcal{C}$ the set of quadratic folding maps of the interval at the so-called accumulation point of period-doubling. Such a mapping $f \in \mathcal{C}$ has an invariant attracting Cantor set Λ_f for which there is a canonical labelling given by a homeomorphism $i_f : X = \{0,1\}^{\mathbf{Z}_{\geq 0}} \to \Lambda_f$. Thus, if $f, g \in \mathcal{C}$, there is a natural conjugacy $h = i_g \circ i_f^{-1} : \Lambda_f \to \Lambda_g$. Recently Sullivan has proved in [8] that if $f \in \mathcal{C}^2$ then the successive renormalisations $R^n f$ converge exponentially fast to a fixed point of R. It follows from this that the conjugacy h is always $C^{1+\alpha}$ for analytic systems ([6],[4]). Moreover, if f is sufficiently smooth and the rate of convergence is taken into account, then it can be shown that, for period-doubling, the conjugacy is $C^{2+.11}$. This is a corollary of a unified theory for smooth conjugacies for critical and non-critical systems in terms of rapid convergence of renormalisation ([1]).

The embedding $i = i_f : X = \{0,1\}^{\mathbf{Z}_{\geq 0}} \to \mathbf{R}$ of the metric space X induces a *smooth structure* on X in the following sense. A smooth function on X is a function $f : X \to \mathbf{R}$ such that the function $f \circ i^{-1} : i(X) \to \mathbf{R}$ has a smooth extension to $\mathbf{R}$. This definition can clearly be generalised and localised by considering localised embeddings as charts on X. The definition makes sense even though X does not have a manifold structure – it is a fractal. We say that two such structures are *equivalent* if the charts of one are smooth functions in the other. In the case that we are considering, the structures determined by i_f and i_g are equivalent if, and only if, h is $C^{1+\alpha}$. Thus we are lead to the problem of classifying the smooth structures on $X = \{0,1\}^{\mathbf{Z}_{\geq 0}}$.

The case of diffeomorphisms of the circle does not immediately fit into this scheme of things because, in this case, $X = S^1$ and, as is well-known, S^1 possesses a unique smooth structure. However, this ignores the fact that, from our point of view, X has a richer structure since it is *marked* by the orbits of the diffeomorphism. The orbit segment $\{f^j x\}_{j=0}^{q_n-1}$ partitions the circle into $q_n - 1$ segments, and this partition must be respected by our conjugacies.

It turns out that it is most convenient to formalise this in terms of *trees* in the following way. We use the number theory of the rotation number ρ to choose the q_n. Then we regard the segments of the partition by $\{f^j x\}_{j=0}^{q_n-1}$ as the vertices of the tree T at level n. Each of these is connected by an edge to the vertex at level $n-1$ corresponding to the segment that contains it. The way in which these segments sit in S^1 determines a smooth structure on the tree T as described in the next section.

Such a tree also exists in the case of the Feigenbaum period-doubling attractor and is determined by the orbit segments $\{f^j c\}_{j=0}^{2^n-1}$ of the critical point c. The analogue of the segments at level n are the 2^n intervals $I_{\tau_0 \ldots \tau_{n-1}}$ which are defined as the smallest intervals containing the sets $i_f(J_{\tau_0 \ldots \tau_{n-1}})$ where

$$J_{\tau_0 \ldots \tau_{n-1}} = \{\tau_0{}'\tau_1{}' \ldots \in X = \{0,1\}^{\mathbf{Z}_{\geq 0}} : \tau_i{}' = \tau_i \ \text{ if } \ i < n\}.$$

Thus in this case, the vertex corresponding to $I_{\tau_0 \ldots \tau_n}$ is connected by an edge to $I_{\tau_0 \ldots \tau_{n-1}}$.

We will therefore define the notion of a $C^{1+\alpha}$ structure on a tree and prove necessary and sufficient conditions for two structures to be equivalent. For the case of the binary Cantor set (as in the Feigenbaum period-doubling attractor above) this was already done by Sullivan under the assumption of bounded geometry. Our results extend his in a number of directions, including the following:

(i) The topological structure of our trees are much more general than the binary tree implicit in his work.

(ii) We drop the condition of *bounded geometry* and thus allow for trees with unbounded branching such as that involved for typical Diophantine irrational rotation numbers and typical infinitely renormalisable kneading sequences of quadratic foldings.

(iii) We include the case where the intervals corresponding to the vertices of the tree do not have gaps between them. This is the situation for circle mappings.

2.1.1 Smooth structures on embedded trees

A *tree* T consists of a set of vertices of the form $V_T = \bigcup_{n \geq 0} T_n$, where each T_n is a finite set, together with a directed graph on these vertices such that each $t \in T_n$, $n \geq 1$, has a unique edge leaving it. This edge joins t (the *daughter*) to $m(t) \in T_{n-1}$ (its *mother*). We inductively define $m^p(t)$ to be the mother of $m^{p-1}(t)$. Using this notation, t is a *descendant* of $m^p(t)$ and $m^p(t)$ is the *p-ancestor* of t.

Given such a tree T we define the *limit set* or *set of ends* L_T as the set of all sequences $\underline{t} = t_0 t_1 \ldots$ such that $m(t_{i+1}) = t_i$ for all $i \geq 0$. We endow L_T with the metric d where

$$d(s_0 s_1 \ldots, t_0 t_1 \ldots) = 2^{-n}$$

if $s_i = t_i$ for $0 \leq i \leq n-1$ and $s_n \neq t_n$.

If $\underline{t} = t_0 t_1 \ldots \in L_T$ then by $\underline{t}|n$ we denote the finite word $t_0 \ldots t_{n-1}$. Let $L_{\underline{t}|n}$ denote the set of $\underline{s} \in L_T$ such that $\underline{s}|n = \underline{t}|n$. This is called a *n-cylinder* of the tree. If L is an open subset of L_T containing $L_{\underline{t}|n}$ and $i : L \to \mathbf{R}$ a continuous mapping, then we denote by $C_{\underline{t}|n,i}$ the smallest closed interval in $\mathbf{R}$ which contains $i(L_{\underline{t}|n})$. This is also called a *n-cylinder*. Note that both $L_{\underline{t}|n}$ and $C_{\underline{t}|n,i}$ are determined by t_{n-1}. Therefore we shall often write these as $L_{t_{n-1}}$ and $C_{t_{n-1},i}$. Say that $\underline{s} \sim \underline{t}$ if $i(\underline{s}) = i(\underline{t})$.

We shall only be interested in mappings i which respect the cylinder structure of L_T in the following way. We demand that if $\underline{s}|n \neq \underline{t}|n$ then

$$\mathrm{int} C_{\underline{s}|n,i} \bigcap \mathrm{int} C_{\underline{t}|n,i} = \emptyset.$$

Clearly, the mapping $i : L \to \mathbf{R}$ induces a mapping $L/\sim \to \mathbf{R}$ which we also denote by i.

Definition 13 Such a pair (i, L) is a chart of L_T if L is an open set of L_T with respect to the metric d and the induced map $i : L/\sim \to \mathbf{R}$ is an embedding.

Two charts (i, L) and (j, K) are *compatible* if the equivalence relation $\sim$ corresponding to i agrees with that of j on $L \bigcap K$. They are $C^{1+\alpha}$ *compatible* if they are compatible and the mapping $j \circ i^{-1}$ from $i(L \bigcap K)$ to $j(L \bigcap K)$ has a $C^{1+\alpha}$ extension to a neighbourhood of $i(L \bigcap K)$ in $\mathbf{R}$.

Definition 14 A $C^{1+\alpha}$ *structure* on L_T is a maximal set of $C^{1+\alpha}$ compatible charts which cover L_T. A $C^{1+\alpha-}$ structure is a maximal set of charts covering L_T which are $C^{1+\beta}$ compatible for all $0 < \beta < \alpha$.

Obviously, a finite set of $C^{1+\alpha}$ compatible charts which cover L_T defines a $C^{1+\alpha}$ structure on L_T. A mapping $h : L_T \to L_T$ is smooth if its representatives in local charts are smooth in the following sense: if $\underline{t} \in L$ and $h(\underline{t}) \in L'$ where (i, L) and (i', L') are charts in the structure then $i' \circ h \circ i^{-1}$ has a smooth extension to a neighbourhood of $i(\underline{t})$ in $\mathbf{R}$. Similarly, we define smooth maps between different spaces.

We shall mostly be concerned with situations where either (i) the smooth structure is defined by a single chart or (ii) the structure is defined by a single embedding of $L_T/\sim$ into the circle $\mathbf{T}^1$.

If $\mathcal{S}$ is a $C^{1+\alpha}$ structure on L_T and i is a chart of $\mathcal{S}$ then we have that $\underline{s}|n$ and $\underline{t}|n$ are *adjacent* if there is no $\underline{u} \in L_T$ such that $C_{\underline{u}|n,i}$ lies between $C_{\underline{s}|n,i}$ and $C_{\underline{t}|n,i}$ and that they are *in contact* if $C_{\underline{s}|n,i} \bigcap C_{\underline{t}|n,i} \neq \emptyset$. Note that these conditions are independent of the choice of the chart i of $\mathcal{S}$ which contains $L_{\underline{s}|n}$ and $L_{\underline{t}|n}$ in its domain. It does however depend upon $\mathcal{S}$ so we only use this terminology when we have a specific structure in mind. If $\underline{s}|n = s_0 \ldots s_{n-1}$ and $\underline{t}|n = t_0 \ldots t_{n-1}$ then we say that s_{n-1} and t_{n-1} are *adjacent* (resp. *in contact*) if $\underline{s}|n$ and $\underline{t}|n$ are.

Definition 15 Two $C^{1+\alpha}$ structures $\mathcal{S}$ and $\mathcal{T}$ on L_T are $(1+\alpha)$-*equivalent* if the identity is a $C^{1+\alpha}$-diffeomorphism when it is considered as a map from L_T with one structure to L_T with the other. They are $(1+\alpha^-)$-*equivalent* if the identity is a $C^{1+\beta}$-diffeomorphism for all $0 < \beta < \alpha$.

Example 1. Standard binary Cantor set.

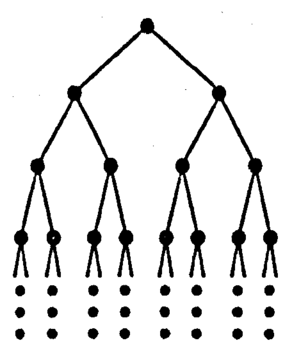

Figure 1.

Consider the binary tree T shown in figure 1. We can index the vertices of the tree by the finite words $\varepsilon_0 \ldots \varepsilon_{n-1}$ of 0s and 1s in such a way that the mother of the vertex $t = \varepsilon_0 \ldots \varepsilon_n$ is $m(t) = \varepsilon_0 \ldots \varepsilon_{n-1}$ and so that $\varepsilon_0 \ldots \varepsilon_{n-1}0$ lies to the left of $\varepsilon_0 \ldots \varepsilon_{n-1}1$. Now to each vertex $t = \varepsilon_0 \ldots \varepsilon_{n-1}$ associate a closed interval I_t so that $I_t \subset I_{m(t)}$, $I_{\varepsilon_0 \ldots \varepsilon_{n-1}0}$ is to the left of $I_{\varepsilon_0 \ldots \varepsilon_{n-1}1}$ and

$$I_{\varepsilon_0 \ldots \varepsilon_{n-1}} = I_{\varepsilon_0 \ldots \varepsilon_{n-1}0} \bigcup G_{\varepsilon_0 \ldots \varepsilon_{n-1}} \bigcup I_{\varepsilon_0 \ldots \varepsilon_{n-1}1}$$

where $G_{\varepsilon_0 \ldots \varepsilon_{n-1}}$ is an open interval between $I_{\varepsilon_0 \ldots \varepsilon_{n-1}0}$ and $I_{\varepsilon_0 \ldots \varepsilon_{n-1}1}$. We assume that the ratios $|G_t|/|I_t|$ are bounded away from 0. Then the lengths of the intervals $I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$ go to 0 exponentially fast as $n \to \infty$ and therefore

$$C = \bigcap_{n \geq 0} \bigcup_{\varepsilon_0 \ldots \varepsilon_{n-1}} I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$$

is a Cantor set.

Let $\Sigma = \{0, 1\}^{\mathbf{Z}_{\geq 0}}$ denote the set of infinite right-handed words $\varepsilon_0\varepsilon_1 \ldots$ of 0s and 1s. Clearly, L_T can be identified with Σ since each $\underline{t} = t_0 t_1 \ldots \in L_T$ can be identified with a word $\varepsilon_0\varepsilon_1 \ldots$ in Σ. The mapping $i : \Sigma \to \mathbf{R}$ defined by

$$i(\varepsilon_0\varepsilon_1 \ldots) = \bigcap_{n \geq 0} I_{\varepsilon_0 \ldots \varepsilon_{n-1}}$$

gives an embedding of L_T into $\mathbf{R}$. This is the simplest non-trivial example of an embedded tree. We shall be interested in embedded trees such as this where the analogue of the Cantor set C is generated in one way or another by a dynamical system.

Very often the set $C = i(L_T)$ will be an invariant set of a hyperbolic dynamical system. For example, there is a map σ defined on L_T above by

$$\sigma(\varepsilon_0\varepsilon_1 \ldots) = \varepsilon_1\varepsilon_2 \ldots.$$

This induces a map σ' on $C = i(L_T)$ which is a candidate for a hyperbolic system. Using our results it is easy to give necessary and sufficient conditions for this map to be smooth in the sense that it has a $C^{1+\alpha}$ extension to $\mathbf{R}$ as a Markov map such as that shown in figure 2 (see [2]).

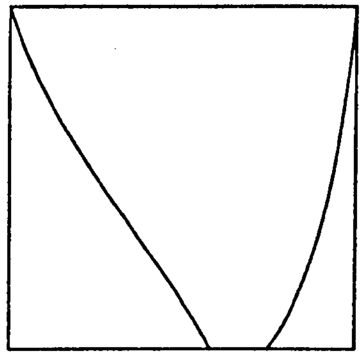

Figure 2.

In the above case the equivalence relation $\sim$ is trivial and there are no contact points. But now consider the case where the tree is embedded in this way but where the gaps G_t are empty. In this case i maps L_T onto an interval but is not an embedding because it is not injective. The equivalence relation $\sim$ on L_T is non-trivial: it identifies the points $\varepsilon_0 \dots \varepsilon_n 1000\dots$ and $\varepsilon_0 \dots \varepsilon_n 0111\dots$. Thus h is injective on all but a countable set. The space $L_T/\sim$ is homeomorphic to an interval. However note that L_T has much more structure than an interval because of the points marked by the cylinder structure. In particular, there are uncountably many smooth structures on L_T, but only one on the interval.

We could regard the vertex set of T as $\bigcup_{n\geq 0} T_n$ where T_n is the set of intervals $I_{\varepsilon_0\dots\varepsilon_{n-1}}$ and the edge relation of T is inclusion. In such a case, we say that T is defined by the cylinder structure.

Rotations of the circle.

This is another example with contact points. Consider the rotation $R_\alpha(x) = x + \alpha$ where α is an irrational number such that $0 < \alpha < 1$, represented as the discontinuous mapping

$$R_\alpha = \begin{cases} x+\alpha & x \in [\alpha-1, 0] \\ x+\alpha-1 & x \in [0,\alpha] \end{cases}$$

Let p_n/q_n be the nth rational approximant of α. Consider the orbit $R_\alpha(0), \dots, R_{(q_n-1)\alpha}(0)$. This partitions the interval $[\alpha-1,\alpha]$ into q_n closed intervals. Let T_n denote the set of such intervals and let T be the tree whose vertex set is $\bigcup_{n\geq 0} T_n$ and such that the mother of $v \in T_n$ is the interval in T_{n-1} which contains v. Thus T is again defined by the cylinder structure. If $t_0 t_1 \dots \in L_T$ then $i(t_0 t_1 \dots) = \bigcap_{n\geq 0} t_n$ defines an embedding of T with contact points.

Of course, any map which is topologically conjugate to R_α would generate the tree T but a different embedding. The question of determining whether two such mappings are smoothly conjugate boils down to showing that these embeddings determine the same smooth structure on L_T. The approach used in the theory of renormalisation is to show that this tree T can be generated by a Markov family $(F_n)_{n\in \mathbb{Z}_{\geq 0}}$ as defined in [4]. This Markov family and its convergence properties determine the $C^{k+\alpha}$ structure on L_T as is proved in [1].

2.1.2 The scaling tree

Gaps.

Fix a $C^{1+\alpha}$ structure $\mathcal{S}$ on L_T. If s and t are adjacent but not in contact then there is a *gap* between $i(L_s)$ and $i(L_t)$. We will add a symbol $g_{s,t} = g_{t,s}$ to T_n to stand for this gap if $m(s) = m(t)$. For the chart (i, L) we let $G_{s,t,i}$ denote the smallest closed interval containing the gap. Let $\tilde{T}_n$ denote the set T_n with all the gap symbols $g_{s,t}$ adjoined. Let $\tilde{V}_T = \bigcup_{n\geq 1} \tilde{T}_n$. If $m^p(s) = m^p(t)$ then $G_{s,t,i} = G_{m^{p-1}(s),m^{p-1}(t),i}$.

Primary atlas.

Suppose that $\mathcal{S}$ is a $C^{1+\alpha}$ structure on L_T. Then clearly there exists $N \geq 0$ such that if $T_N = \{t_1, \ldots, t_q\}$, then there are charts (i_j, U_j) of $\mathcal{S}$, $1 \leq j \leq q$, such that the open subset U_j contains the N-cylinder L_{t_j}. We call such a system of charts a *primary N-atlas.*

Scaling tree.

Fix such a primary N-atlas $\mathcal{I} = \{(i_j, U_j)\}_{j=1,\ldots,q}$. To each $s, t \in \tilde{T}_n$, $n \geq N$, we associate the following intervals in $\mathbf{R}$ (see figure 3(a), (b) and (c)).

- $C_{t,\mathcal{I}}$ and $G_{s,t,\mathcal{I}}$: $C_{t,\mathcal{I}}$ is the interval C_{t,i_j} where j is such that $m^r(t) = t_j$ for some $r \geq 1$. Similarly, $G_{s,t,\mathcal{I}}$ is the gap G_{s,t,i_j} if s and t are non-contact adjacent points with $m(s) = m(t)$.

- $C_{s,t,\mathcal{I}}$, $C_{t,s,\mathcal{I}}$ and $D_{t,s,\mathcal{I}}$: If $t, s \in T_n$, are adjacent and in contact, define $P_{t,s,\mathcal{I}} = P_{s,t,\mathcal{I}}$ as the common point between the closed sets $C_{t,\mathcal{I}}$ and $C_{s,\mathcal{I}}$. Define the closed sets $C_{t,s,\mathcal{I}}$ and $C_{s,t,\mathcal{I}}$, respectively, as the sets obtained from $C_{t,\mathcal{I}}$ and from $C_{s,\mathcal{I}}$, by rescaling them by the factor $1/2$, keeping the points $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$ fixed. Define $D_{t,s,\mathcal{I}} = C_{t,s,\mathcal{I}} \bigcup C_{s,t,\mathcal{I}}$ If $t, s \in T_n$, are adjacent but not in contact, define $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$, respectively as the common points of the closed sets $C_{t,\mathcal{I}}$ and $C_{s,\mathcal{I}}$ with the gap $G_{t,s,\mathcal{I}}$. Define the closed sets $C_{t,s,\mathcal{I}}$ and $C_{s,t,\mathcal{I}}$, respectively, as the sets obtained from $C_{t,\mathcal{I}}$ and from $C_{s,\mathcal{I}}$, by rescaling them by the factors $\delta_t/2$, $\delta_s/2$, and flipping them into the gap $G_{t,s,\mathcal{I}}$, keeping the points $P_{t,s,\mathcal{I}}$ and $P_{s,t,\mathcal{I}}$ fixed. Here $\delta_t = |G_{t,s,\mathcal{I}}|/|C_{m^{p-1}(t),\mathcal{I}}|$ and $\delta_s = |G_{t,s,\mathcal{I}}|/|C_{m^{p-1}(s),\mathcal{I}}|$ where $p \in \mathbb{N}$ is such that $m^{p-1}(t) \neq m^{p-1}(s)$ and $m^p(t) = m^p(s)$.

- $E_{t,s,\mathcal{I}}$: Let $t_1, s_1 \in T_{n+1}$ be the adjacent vertices such that $G_{t_1,s_1,i} = G_{t,s,i}$. Define $E_{t,s,\mathcal{I}} = C_{t,s,\mathcal{I}} \setminus C_{t_1,s_1,\mathcal{I}}$.

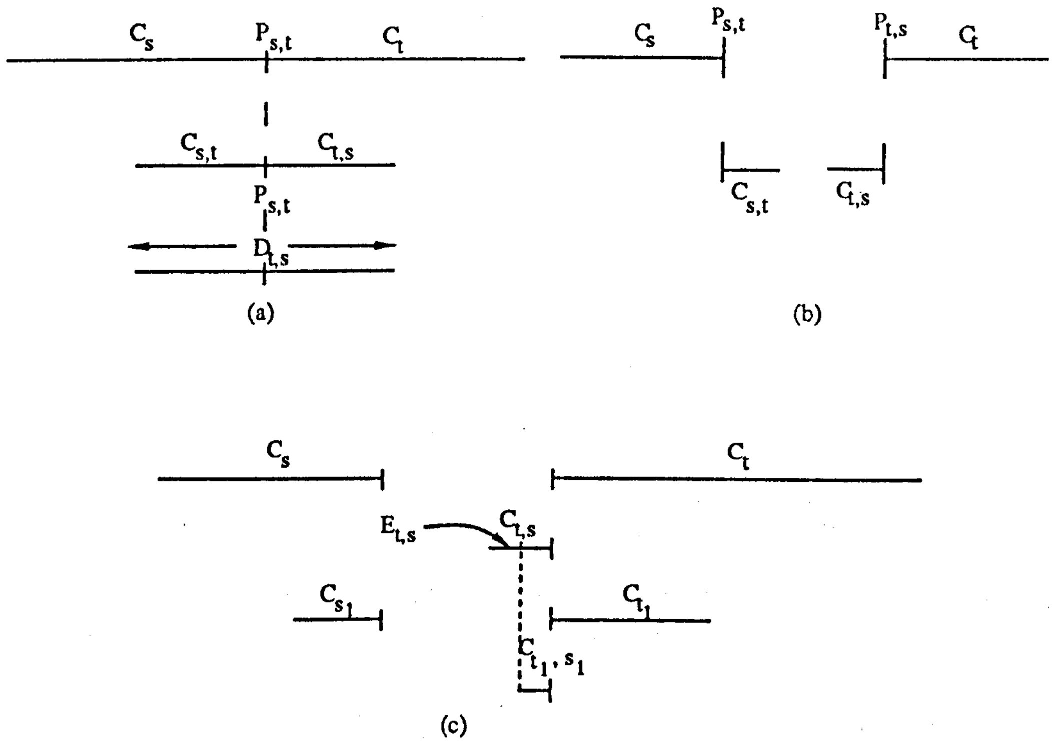

Figure 3.

$(1+\alpha)$-equivalence.

Now suppose, that in addition to the structure $\mathcal{S}$ and its primary atlas $\mathcal{I}$, we have another structure $\mathcal{T}$ and a primary N_1-atlas $\mathcal{J}$ for it. Redefine $N = \max(N_1, N)$. To each $t \in \tilde{T}_n$, $n > N$, we associate the following numbers.

- *the scaling tree* $\sigma_{\mathcal{I}}(t)$:

$$\sigma_{\mathcal{I}}(t) = \frac{|C_{t,\mathcal{I}}|}{|C_{m(t),\mathcal{I}}|} \quad \text{and} \quad \sigma_{\mathcal{I}}(g_{t,s}) = \frac{|G_{t,s,\mathcal{I}}|}{|C_{m(t),\mathcal{I}}|}$$

This defines a function

$$\sigma_{\mathcal{I}} : \bigcup_{n>N} \tilde{T}_n \to [0,1].$$

The fact that it is undefined for small n does not matter.

- ν_t:

$$\nu_t = \left| 1 - \frac{\sigma_{\mathcal{J}}(t)}{\sigma_{\mathcal{I}}(t)} \right|.$$

- A_t: If $t \in \tilde{T}_n$, let $t_1 < \cdots < t_p$ be the elements of T_n with the same mother as t. Between these there may be gaps represented by symbols of the form $g_{t_{m_n}, t_{m_n+1}}$. Denote these gap symbols by $g_1, \ldots, g_q$. Let

$$A_t = \sum_{j=1}^{p} \nu_{t_j} |C_{t_j,\mathcal{I}}| + \sum_{j=1}^{q} \nu_{g_j} |G_{g_j,\mathcal{I}}|$$

- $\nu_{s,t}$: If $s, t \in T_n$ are in contact,

$$\nu_{s,t} = \left| 1 - \frac{|C_{t,\mathcal{I}}|}{|C_{s,\mathcal{I}}|} \frac{|C_{s,\mathcal{J}}|}{|C_{t,\mathcal{J}}|} \right|$$

- $e_{s,t}$: If $s, t \in T_n$ are adjacent but not in contact,

$$e_{s,t} = |E_{s,t,\mathcal{I}}| = \frac{|G_{s,t,\mathcal{I}}|}{2|C_{m^{p-1}(s),\mathcal{I}}|} \{|C_{t,\mathcal{I}}| - |C_{t_1,\mathcal{I}}|\}$$

  where $p \in \mathbb{N}$ is such that $m^{p-1}(t) \neq m^{p-1}(s)$ and $m^p(t) = m^p(s)$ and $t_1, s_1 \in T_{n+1}$ are the adjacent vertices such that $G_{t_1,s_1,\mathcal{I}} = G_{t,s,\mathcal{I}}$.

Throughout the paper we use the following notation: if f and g are functions of a variable x with domain Δ, then we write $\mathcal{O}_x(f(x)) = \mathcal{O}_x(g(x))$ with constant d if

$$d^{-1} < \frac{|f(x)|}{|g(x)|} < d$$

for all $x \in \Delta$. Often we will drop the reference to d.

If it is obvious which variable x is involved then we use the notation $\mathcal{O}(f(x))$ instead. Thus if a_n and b_n are sequences then $\mathcal{O}(a_n) = \mathcal{O}(b_n)$ means a_n/b_n and b_n/a_n are bounded away from 0 independently of n. The notation $f(x) = \mathcal{O}(g(x))$ means the same thing as $\mathcal{O}(f(x)) = \mathcal{O}(g(x))$.

Similarly, $f(x) \leq \mathcal{O}(g(x))$ with constant d means $|f(x)/g(x)| < d$ for all $x \in \Delta$.

Definition 16 We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$-*scale equivalent* if for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $f = f_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} f(n) \leq \mathcal{O}(f(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \tilde{T}_n$, $\nu_t \leq f(n)$;

(iii) for all $s \in T_n$ adjacent to t but not in contact with it, if $m(s) = m(t)$,

$$A_t e_{t,s,\mathcal{I}}^{-(1+\varepsilon)} + \nu_t e_{t,s,\mathcal{I}}^{-\varepsilon} \leq f(n)$$

while if $m(s) \neq m(t)$ then

$$\nu_t e_{t,s,\mathcal{I}}^{-\varepsilon} \leq f(n).$$

Definition 17 We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$-*contact equivalent* if for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $f_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=0}^{\infty} f_\varepsilon(n) < \infty$;

(ii) for all $s, t \in T_n$, $n \geq N$ such that s and t are in contact,

$$\frac{\nu_{s,t}}{|D_{t,s,\mathcal{I}}|^{\varepsilon}} < f_\varepsilon(n).$$

Definitions 16 and 17 do not at first sight appear to be symmetric in $\mathcal{I}$ and $\mathcal{J}$. However, from theorem 17, it follows that $\mathcal{I}$ and $\mathcal{J}$ define equivalent $C^{1+\alpha^-}$ structures and this implies that $\mathcal{O}_t(|C_{t,\mathcal{I}}|) = \mathcal{O}_t(|C_{t,\mathcal{J}}|)$. Therefore, if we exchange $\mathcal{I}$ and $\mathcal{J}$ in the definitions we have that the definitions are verified for the same α.

Definition 18 We say that two such primary atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$-*equivalent* ($\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$) if they are $(1+\alpha)$-scale equivalent and $(1+\alpha)$-contact equivalent.

The main theorem that we prove in this paper is

Theorem 17 Let $\mathcal{S}$ and $\mathcal{T}$ be $C^{1+\alpha}$ structures on L_T and let $\mathcal{I}$ (resp. $\mathcal{J}$) be a primary atlas for $\mathcal{S}$ (resp. $\mathcal{T}$). A sufficient condition for $\mathcal{S}$ and $\mathcal{T}$ to be $C^{1+\alpha^-}$-equivalent is that $\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$.

Theorem 18 Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{I}$ and $\mathcal{J}$ be as in theorem 17 and suppose that $\mathcal{S}$ and $\mathcal{T}$ are $C^{1+\alpha^-}$ equivalent. Then $\mathcal{I} \overset{\gamma}{\sim} \mathcal{J}$ if for all ε such that $0 < \varepsilon < \gamma$ there exists β such that $0 < \varepsilon < \beta < \gamma \leq \alpha$ and there exists a function $g = g_{\beta,\varepsilon} : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} g(n) \leq \mathcal{O}(g(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \tilde{T}_n$, $|C_{m(t),\mathcal{I}}|^{\beta} < g(n)$;

(iii) for all $t_1, t_2 \in T_n$, which are adjacent but not in contact, if $m(t_1) = m(t_2)$ then

$$\frac{|C_{m(t_1),\mathcal{I}}|^{1+\beta}}{|E_{t_1,t_2,\mathcal{I}}|^{1+\varepsilon}} < g(n),$$

while if $m(t_1) \neq m(t_2)$ then

$$\frac{|C_{m(t_1),\mathcal{I}}|^{\beta}}{|E_{t_1,t_2,\mathcal{I}}|^{\varepsilon}} < g(n);$$

(iv) for all $t_1, t_2 \in T_n$, which are in contact we have that

$$|D_{t_1,t_2,\mathcal{I}}|^{\beta-\varepsilon} < g(n).$$

These conditions hold for some of the most interesting problems. In this case theorem 17 and theorem 18 give a necessary and sufficient condition for $(1+\alpha)$-equivalence.

2.1.3 Example: Cylinder structures with bounded geometry

Definition 19 A structure $\mathcal{S}$ has *bounded geometry* if for some primary atlas $\mathcal{I}$, $\sigma_{\mathcal{I}}(t)$ is bounded away from 0 i.e. there exist δ such that $\sigma_{\mathcal{I}}(t) > \delta$ for all $t \in \tilde{T}_n$, $n > N$. Recall that $\sigma_{\mathcal{I}}(t) = |C_{t,\mathcal{I}}|/|C_{m(t),\mathcal{I}}|$ and $\sigma_{\mathcal{I}}(g_{t,s}) = |G_{t,s,\mathcal{I}}|/|C_{m(t),\mathcal{I}}|$

Therefore, bounded geometry implies that for all $t \in \tilde{T}_n$ and all $n > N$, we have that $\sigma_{\mathcal{I}}(t) < 1 - \delta$. Clearly, for bounded geometry, there exists $0 < \lambda < \mu < 1$ and $c, d > 0$ such that for all $n \geq 0$ and all $t \in T_n$, $c\lambda^n < |C_{t,\mathcal{I}}| < d\mu^n$

We introduce a new simpler definition of scale equivalence for a primary atlas $\mathcal{I}$ with bounded geometry.

Definition 20 We say that two such primary atlases with bounded geometry $\mathcal{I}$ and $\mathcal{J}$ are $(1+\alpha)$*-scale equivalent* if for all ε such that $0 \leq \varepsilon < \alpha < 1$, there exists a function $f = f_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} f(n) \leq \mathcal{O}(f(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \tilde{T}_n$, $\nu_t \leq f(n)$;

(iii) if $s \in T_n$ adjacent to t but not in contact with it, then

$$\nu_t |C_{t,\mathcal{I}}|^{-\varepsilon} \leq f(n).$$

Lemma 3 For systems with bounded geometry the two definition of $(1+\alpha)$-scale equivalence are equivalent.

With bounded geometry, theorems 17 and 18 combine to give a simple necessary and sufficient condition for $(1+\alpha)$-equivalence.

Theorem 19 Let $\mathcal{S}$ and $\mathcal{T}$ be $C^{1+\alpha}$ structures on L_T and let $\mathcal{I}$ (resp. $\mathcal{J}$) be a primary atlas with bounded geometry for $\mathcal{S}$ (resp. $\mathcal{T}$) then $\mathcal{S}$ and $\mathcal{T}$ are $C^{1+\alpha^-}$-equivalent if and only if $\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$.

Definition 21 (i) $\mathcal{S}$ is a C^{1+} structure on L_T if and only if $\mathcal{S}$ is a $C^{1+\varepsilon}$ structure for some $\varepsilon > 0$.

(ii) The structures $\mathcal{S}$ and $\mathcal{T}$ are C^{1+}-equivalent if and only if they are $C^{1+\varepsilon}$-equivalent for some $\varepsilon > 0$.

Corollary For bounded geometry, a necessary and sufficient condition for $\mathcal{S}$ to be C^{1+} equivalent to $\mathcal{T}$ is that there is $\lambda \in (0,1)$ such that for all $t \in \tilde{T}_n$ $\nu_t \leq \mathcal{O}(\lambda^n)$ and if s is in contact with t then $\nu_{t,s} \leq \mathcal{O}(\lambda^n)$.

Proof of lemma 3. For all $t \in \tilde{T}_n$, we have that

$$A_t \leq l \max\{\nu_s : m(s) = m(t)\} \leq l f_\varepsilon(n),$$

where l is the maximum length of an N-cylinder.

For all $s \in T_n$ adjacent to t but not in contact with it, we have by definition of $|E_{t,s,\mathcal{I}}|$ and bounded geometry that there is a constant c such that

$$e_{t,s,\mathcal{I}} = |E_{t,s,\mathcal{I}}| > c|C_{t,\mathcal{I}}| > c\delta|C_{m(t),\mathcal{I}}|. \tag{2.1}$$

Then, if $m(s) = m(t)$,

$$\begin{aligned} A_t e_{t,s,\mathcal{I}}^{-(1+\varepsilon)} &+ \nu_t e_{t,s,\mathcal{I}}^{-\varepsilon} \\ &\leq \max\{\nu_s : m(s) = m(t)\}\, |C_{t,\mathcal{I}}| c^{-(1+\varepsilon)} |C_{t,\mathcal{I}}|^{-(1+\varepsilon)} \\ &\quad + \max\{\nu_s : m(s) = m(t)\}\, c^{-\varepsilon} |C_{t,\mathcal{I}}|^{-\varepsilon} \\ &\leq c^{-(1+\varepsilon)} \max\{\nu_s : m(s) = m(t)\}\, |C_{t,\mathcal{I}}|^{-\varepsilon} \leq c^{-(1+\varepsilon)} f_\varepsilon(n); \end{aligned}$$

This proves that definition 20 implies definition 16. The other implication is straightforward. ∎

Proof of theorem 19. By theorem 17 and lemma 3, $\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$ is a sufficient condition for the atlas $\mathcal{S}$ and $\mathcal{T}$ to be $C^{1+\alpha^-}$-equivalent. We now prove that it is also necessary.

For all $t \in \tilde{T}_n$, $|C_{m(t),\mathcal{I}}|^\alpha < c_\alpha (1-\delta)^{n\alpha}$, where c_α is determined by the maximum length of the N-cylinders of the atlas $\mathcal{I}$. Thus, for all adjacent $t_1, t_2 \in T_n$ which are not in contact, if $m(t_1) = m(t_2)$ then by equation 2.1,

$$\frac{|C_{m(t_1),\mathcal{I}}|^{1+\alpha}}{|E_{t_1,t_2,\mathcal{I}}|^{1+\varepsilon}} \leq \mathcal{O}(|C_{m(t_1),\mathcal{I}}|^{\alpha-\varepsilon}) \leq \mathcal{O}((1-\delta)^{n(\alpha-\varepsilon)})$$

where the constants depend only upon ε and $\alpha - \varepsilon$. If $m(t_1) \neq m(t_2)$ then

$$\frac{|C_{m(t_1),\mathcal{I}}|^{\alpha}}{|E_{t_1,t_2,\mathcal{I}}|^{\varepsilon}} \leq \mathcal{O}(|C_{m(t_1),\mathcal{I}}|^{\alpha-\varepsilon}) \leq \mathcal{O}((1-\delta)^{n(\alpha-\varepsilon)})$$

where again the constants depend only upon ε and $\alpha - \varepsilon$.

For all $t_1, t_2 \in T_n$ which are in contact we have that

$$|D_{t_1,t_2,\mathcal{I}}|^{\alpha-\varepsilon} < \mathcal{O}((1-\delta)^{n(\alpha-\varepsilon)}).$$

Therefore, by theorem 18, $\mathcal{I} \overset{\alpha}{\sim} \mathcal{J}$ is also a necessary condition for the atlas $\mathcal{S}$ and $\mathcal{T}$ to be $C^{1+\alpha^-}$-equivalent. ∎

2.2 Proof of theorem 17.

It is sufficient to prove the theorem locally at each point $\underline{t} \in L_T$. Let $i : U_0 \to \mathbb{R}$ be a chart in $\mathcal{I}$ and $j : V_0 \to \mathbb{R}$ be a chart in $\mathcal{J}$ with $\underline{t} \in U_0 \cap V_0$. Then it suffices to show that for some open subsets U and V of $U_0 \cap V_0$ containing $\underline{t}$ the mapping $j \circ i^{-1} : i(U) \to j(V)$ has a $C^{1+\alpha^-}$ extension to $\mathbb{R}$. If this is the case for all such $\underline{t}$ then the result holds globally. Clearly, we can restrict to the case where

(i) the smallest closed interval $\mathbf{I}$ containing $i(U)$ is a cylinder $C_{t,i}$ for some $t \in T_{N_0}$ where $N_0 > N$ or else is the union of two adjacent cylinders of this form which are in contact and

(ii) where the smallest closed interval $\mathbf{J}$ containing $j(V)$ consists of the corresponding cylinders for j.

Now let $\mathbf{I}^n$ (resp. $\mathbf{J}^n$) be the set of end-points of the cylinders $C_{t,i}$ (resp. $C_{t,j}$) where $t \in T_n$, $n \geq N_0$ and $C_{t,i} \subset \mathbf{I}$ (resp. $C_{t,j} \subset \mathbf{J}$). Then $j \circ i^{-1}$ maps $\mathbf{I}^n$ onto $\mathbf{J}^n$ and is a homeomorphism of the closure $\mathbf{I}^\infty$ of $\bigcup_{n \geq N_0} \mathbf{I}^n$ onto the closure $\mathbf{J}^\infty$ of $\bigcup_{n \geq N_0} \mathbf{J}^n$. We will construct a sequence of C^∞ mappings L_n such that

(i) L_n agrees with $j \circ i^{-1}$ on $\bigcup_{N_0 \leq j \leq n} \mathbf{I}^j$,

(ii) L_n is a Cauchy sequence in the space of $C^{1+\beta}$ functions on $\mathbf{I}$ for all $\beta < \alpha$ and therefore converges to a $C^{1+\alpha^-}$ function L_∞ on $\mathbf{I}$.

Then the mapping L_∞ gives the required smooth extension of $j \circ i^{-1}$ and proves the theorem.

The rest of this section consists of the construction of the mappings $L_n : \mathbf{I} \to \mathbf{J}$ and the proof that they converge to a smooth diffeomorphism. We use extensively the fact that for each $n \geq N_0$, $\mathbf{I}$ is the union of cylinder sets of the form $C_{t,i}$ and $G_{s,t,i}$ where $s, t \in T_n$.

2.2.1 A refinement of the $(1+\alpha)$-equivalence property.

Lemma 4 $|C_{t,\mathcal{I}}|/|C_{t,\mathcal{J}}|$ is bounded away from 0 and ∞ i.e. $|C_{t,\mathcal{I}}|/|C_{t,\mathcal{J}}| = \mathcal{O}_t(1)$.

Proof. For all $n \geq N$ and all $t \in T_n$, define $Q(t) = \ln(|C_{t,\mathcal{I}}|/|C_{t,\mathcal{J}}|)$. By the $(1+\alpha)$-scale equivalence

$$|Q(m(t)) - Q(t)| \leq \mathcal{O}(\nu_t) \leq \mathcal{O}(f(n)).$$

Therefore, for all $\underline{t} = t_0 t_1 \ldots \in L_T$, $|Q(t_N) - Q(t_n)| \leq \mathcal{O}(f(N))$. As the set T_N is finite, $|Q(t_n)|$ is bounded above independently of n and t_n. ∎

Corollary. If $t \in T_n$, $n \geq N$,

$$\left| \frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|} - \frac{|C_{m(t),\mathcal{J}}|}{|C_{m(t),\mathcal{I}}|} \right| \leq \mathcal{O}(\nu_t). \tag{2.2}$$

If $s, t \in T_n$ are adjacent but not in contact and $m(s) = m(t)$ then

$$\left| \frac{|G_{t,s,\mathcal{J}}|}{|G_{t,s,\mathcal{I}}|} - \frac{|C_{m(t),\mathcal{J}}|}{|C_{m(t),\mathcal{I}}|} \right| \leq \mathcal{O}(\nu_{g_{t,s}}). \tag{2.3}$$

If they are in contact then

$$\left| \frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|} - \frac{|C_{s,\mathcal{J}}|}{|C_{s,\mathcal{I}}|} \right| \leq \mathcal{O}(\nu_{t,s}). \tag{2.4}$$

Proof. This follows directly from the definition of ν_t, $\nu_{g_{t,s}}$ and $\nu_{t,s}$ and the boundedness of $|C_{t,\mathcal{I}}|/|C_{t,\mathcal{J}}|$. ∎

2.2.2 The map L_t.

For all $n \geq N$ and all $t \in T_n$ define the affine map L_t as follows:

$$L_t(x) = \frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|}(x - P_{t,s,\mathcal{I}}) + P_{t,s,\mathcal{J}},$$

where s is a vertex adjacent to t. The definition of the map L_t is independent of the adjacent vertex s considered because it is an affine map.

Lemma 5 (i) For k equal to 0 and 1 and for all $n > N$ and all pairs of contact vertices $t, s \in T_n$ which are in contact

$$||L_t - L_s||_{C^k} \leq \mathcal{O}(\nu_{t,s}|D_{t,s,\mathcal{I}}|^{1-k}) \tag{2.5}$$

in the domain $D_{t,s,\mathcal{I}}$.

(ii) For all vertices $t \in T_n$ and all $n > N$,

$$||L_t - L_{m(t)}||_{C^0} \leq \mathcal{O}(A_t) \tag{2.6}$$

in the domain $C_{t,\mathcal{I}}$. For all adjacent vertices s and t not in contact, if $m(s) = m(t)$ then

$$||L_t - L_{m(t)}||_{C^0} \leq \mathcal{O}(A_t) \tag{2.7}$$

in the domain $E_{t,s,\mathcal{I}}$. Otherwise

$$||L_t - L_{m(t)}||_{C^0} \leq \mathcal{O}(\nu_t|E_{t,s,\mathcal{I}}|) \tag{2.8}$$

in the domain $E_{t,s,\mathcal{I}}$. Moreover,

$$||dL_t - dL_{m(t)}||_{C^0} \leq \mathcal{O}(\nu_t) \tag{2.9}$$

in the domains $C_{t,\mathcal{I}}$ and $E_{t,s,\mathcal{I}}$.

Proof. Firstly we prove (i). By the corollary to lemma 4 of section 2.2.1 and since $L_t(P_{t,s,\mathcal{I}}) = L_s(P_{s,t,\mathcal{I}}) = P_{t,s,\mathcal{J}} = P_{s,t,\mathcal{J}}$ and $|x - P_{t,s,\mathcal{I}}| \leq \mathcal{O}(|D_{t,s,\mathcal{I}}|)$,

$$|L_t(x) - L_s(x)| = \left|\frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|} - \frac{|C_{s,\mathcal{J}}|}{|C_{s,\mathcal{I}}|}\right| |x - P_{t,s,\mathcal{I}}| \leq \mathcal{O}(\nu_{t,s}|D_{t,s,\mathcal{I}}|) \tag{2.10}$$

and

$$|dL_t - dL_s| = \left|\frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|} - \frac{|C_{s,\mathcal{J}}|}{|C_{s,\mathcal{I}}|}\right| \leq \mathcal{O}(\nu_{t,s}).$$

This proves part (i).

Let $t_1, \ldots, t_p$ denote the vertices in T_n with the same mother as t ordered so that $C_{t_j,\mathcal{I}}$ lies to the left of $C_{t_{j+1},\mathcal{I}}$. Then $t = t_l$ for some l such that $1 \leq l \leq p$. Let $s \in T_n$ be an adjacent vertex to t_1 such that $m(s) \neq m(t)$.

Since $L_{t_1}(P_{t_1,s,\mathcal{I}}) = L_{m(t)}(P_{m(t),m(s),\mathcal{I}}) = P_{t_1,s,\mathcal{J}} = P_{m(t),m(s),\mathcal{J}}$, by the corollary to lemma 4,

$$\begin{aligned} |L_{t_1}(x) - L_{m(t)}(x)| &= \left|\frac{|C_{t_1,\mathcal{J}}|}{|C_{t_1,\mathcal{I}}|} - \frac{|C_{m(t),\mathcal{J}}|}{|C_{m(t),\mathcal{I}}|}\right| |x - P_{t_1,s,\mathcal{I}}| \\ &\leq \mathcal{O}(\nu_{t_1}|C_{t_1,\mathcal{I}}|), \end{aligned}$$

for $x \in C_{t,\mathcal{I}}$ and if s and t are not in contact then

$$|L_{t_1}(x) - L_s(x)| \leq \mathcal{O}(\nu_{t_1}|E_{t_1,s,\mathcal{I}}|),$$

for $x \in E_{t_1,s,\mathcal{I}}$. Therefore, $|P_{t_1,t_2,\mathcal{J}} - L_{m(t)}(P_{t_1,t_2,\mathcal{I}})| \leq \mathcal{O}(\nu_{t_1}|C_{t_1,\mathcal{I}}|)$. If there is a gap g_{t_1,t_2} between the vertices t_1 and t_2 define the map $L_{g_{t_1,t_2}}$ at $G_{t_1,t_2,\mathcal{I}}$ in the following way:

$$L_{g_{t_1,t_2}}(x) = \frac{|G_{t_1,t_2,\mathcal{J}}|}{|G_{t_1,t_2,\mathcal{I}}|}(x - P_{t_1,t_2,\mathcal{I}}) + P_{t_1,t_2,\mathcal{J}}.$$

By the corollary to lemma 4

$$\begin{aligned}\left|L_{g_{t_1,t_2}}(x) - L_{m(t)}(x)\right| &= \left|\frac{|G_{t_1,t_2,\mathcal{J}}|}{|G_{t_1,t_2,\mathcal{I}}|} - \frac{|C_{m(t),\mathcal{J}}|}{|C_{m(t),\mathcal{I}}|}\right| |x - P_{t_1,t_2,\mathcal{I}}| \\ &\quad + |L_{t_1}(P_{t_1,t_2,\mathcal{I}}) - L_{m(t)}(P_{t_1,t_2,\mathcal{I}})| \\ &\leq \mathcal{O}(\nu_{t_1}|C_{t_1,\mathcal{I}}| + \nu_{g_{t_1,t_2}}|G_{t_1,t_2,\mathcal{I}}|).\end{aligned}$$

Therefore,

$$|P_{t_2,t_1,\mathcal{J}} - L_{m(t)}(P_{t_2,t_1,\mathcal{I}})| \leq \mathcal{O}(\nu_{t_1}|C_{t_1,\mathcal{I}}| + \nu_{g_{t_1,t_2}}|G_{t_1,t_2,\mathcal{I}}|).$$

By induction,

$$\begin{aligned}|P_{t_l,t_{l-1},\mathcal{J}} &- L_{m(t)}(P_{t_l,t_{l-1},\mathcal{I}})| \\ &\leq \mathcal{O}(\sum_{m=1}^{l-1} \nu_{t_m}|C_{t_m,\mathcal{I}}| + \sum_{m=1}^{l_1} \nu_{g_{t_{i_m},t_{i_m+1}}}|G_{t_{i_m},t_{i_m+1},\mathcal{I}}|),\end{aligned}$$

where l_1 is the number of gaps between t_1 and t_l. Therefore, if $x \in C_{t,\mathcal{I}}$ or $x \in E_{t,s,\mathcal{I}}$ where s is an adjacent non-contact vertex of t with the same mother as t, then

$$|L_t(x) - L_{m(t)}(x)| \leq \mathcal{O}(A_t).$$

This proves inequalities (2.6), (2.7) and (2.8). Moreover, inequality (2.9) follows because

$$|dL_t - dL_{m(t)}| = \left|\frac{|C_{t,\mathcal{J}}|}{|C_{t,\mathcal{I}}|} - \frac{|C_{m(t),\mathcal{J}}|}{|C_{m(t),\mathcal{I}}|}\right| \leq \mathcal{O}(\nu_t).$$

by the corollary to lemma 4. ∎

2.2.3 The definition of the contact and gap maps.

Lemma 6 For all $\delta \geq 0$ there exists a C^∞ map $\phi : [0, \delta] \to [0, 1]$ such that $\phi = 0$ on $[0, \delta/3]$, $\phi = 1$ on $[2\delta/3, 1]$ and $||\phi||_{C^{k+\alpha}} \leq c_k\delta^{-(k+\alpha)}$, where c_k depends only upon $k \in \mathbb{Z}_{\geq 0}$ and not on $\alpha \in (0, 1]$ or δ.

The proof of this lemma is very simple. Find such a function ϕ_0 for the case $\delta = 1$ and then deduce the general case by letting $\phi(x) = \phi_0(\delta^{-1}x)$. ∎

If s and t are adjacent vertices in T_n we use lemma 6 to choose functions $\phi_{t,s}$ on $G_{t,s,\mathcal{I}}$ and $\psi_{s,t} = \psi_{t,s}$ on $D_{t,s,\mathcal{I}}$ with the following properties.

(i) $\phi_{t,s} = 0$ (resp. $\psi_{t,s} = 0$) on the left-hand third of $E_{t,s,\mathcal{I}}$ (resp. $D_{t,s,\mathcal{I}}$) and $\phi_{t,s} = 1$ (resp. $\psi_{t,s} = 1$) on the right-hand third of $E_{t,s,\mathcal{I}}$ (resp. $D_{t,s,\mathcal{I}}$)

(ii)
$$||\phi_{t,s}||_{C^p} \leq \mathcal{O}(|E_{t,s,\mathcal{I}}|^{-p}) \tag{2.11}$$
and
$$||\psi_{t,s}||_{C^p} \leq \mathcal{O}(|D_{t,s,\mathcal{I}}|^{-p}), \tag{2.12}$$
for all reals p between 0 and 2 and where the constants are independent of all the data.

Extend $\phi_{t,s}$ to all of the gap $G_{t,s,\mathcal{I}}$ as a smooth map by taking it as constant outside $E_{t,s,\mathcal{I}}$. We call the $\phi_{t,s}$ *gap maps* and the $\psi_{t,s}$ *contact maps.*

Note that, for all $n, m > N$ and all non-contact adjacent vertices $t_1, s_1 \in T_n$ and $t_2, s_2 \in T_m$, such that $\{s_1, t_1\} \neq \{s_2, t_2\}$ the domains of the gap maps where they are different from 0 or 1 do not overlap. For all $n \geq N$ and all contact adjacent vertices $t_3, s_3 \in T_n$ and $t_4, s_4 \in T_n$, such that $\{s_3, t_3\} \neq \{s_4, t_4\}$ the domains of the contact maps do not overlap. Moreover, they do not overlap with any domain of any gap map ϕ_{t_2,s_2}, where $t_2, s_2 \in T_m$ and $m \leq n$.

2.2.4 The map $L_n : \mathbf{I} \to \mathbf{J}$.

Construction of L_n on cylinders $C_{t,\mathcal{I}}$ in I

For all $n \geq N_0$ and all vertices $t \in T_n$, define the map L_n on $C_{t,\mathcal{I}} \subset \mathbf{I}$ as follows. For all vertices s_i in contact with t, $L_n = L_t$ on $C_{t,\mathcal{I}} \setminus \cup_i C_{t,s_i,\mathcal{I}}$. If s is in contact with t and s is on the left of t then define L_n on $C_{t,s,\mathcal{I}}$ by
$$L_n = \psi_{t,s} L_t + (1 - \psi_{t,s}) L_s.$$
If s is on the right of t then define L_n on $C_{t,s,\mathcal{I}}$ by
$$L_n = \psi_{t,s} L_s + (1 - \psi_{t,s}) L_t.$$

Extension of L_n to the gaps $G_{t,s,\mathcal{I}}$ in I

For all $n > N_0$ and all non-contact adjacent vertices $t, s \in T_n$, suppose that $C_{t,\mathcal{I}}$ is on the left of $C_{s,\mathcal{I}}$. Define the map L_n on $E_{t,s,\mathcal{I}}$, by
$$L_n|_{E_{t,s,\mathcal{I}}} = L_{m(t)} \phi_{t,s} + L_t (1 - \phi_{t,s}).$$

Define the map L_n on $E_{s,t,\mathcal{I}}$, by

$$L_n|_{E_{s,t,\mathcal{I}}} = L_{m(s)}(1-\phi_{s,t}) + L_s\phi_{s,t}.$$

Finally, in $G_{t,s,\mathcal{I}} \setminus (E_{t,s,\mathcal{I}} \cup E_{s,t,\mathcal{I}})$ define $L_n = L_{n-1}$.

This construction builds an infinitely differentiable map L_n which is defined on the closed interval **I** and which maps **I** diffeomorphically onto **J**.

2.2.5 The sequence of maps $(L_n)_{n>N_0}$ converge in the $C^{1+\varepsilon}$ norm.

The space of $C^{1+\epsilon}$ maps on **I** with the $C^{1+\epsilon}$ norm is a Banach space. In this section we prove that the sequence $(L_n)_{n>N_0}$ is a Cauchy sequence in this space and therefore converges. First we prove the following lemma.

Lemma 7 Suppose $t \in T_n$ and $n > N_0$. Then in the three subsets $C_{t,\mathcal{I}} \setminus \cup_s C_{t,s,\mathcal{I}}$, $D_{t,s,\mathcal{I}}$ and $G_{t,s,\mathcal{I}}$,

$$||L_n - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(f_\epsilon(n-1)).$$

The constants of the inequality only depend upon $\mathcal{I}$ and $\mathcal{J}$.

Proof. We break the proof down into 3 cases corresponding to behaviour in the three subsets $C_{t,\mathcal{I}} \setminus \cup_s C_{t,s,\mathcal{I}}$, $D_{t,s,\mathcal{I}}$, and $G_{t,s,\mathcal{I}}$.

(i) *For $C_{t,\mathcal{I}} \setminus \cup_s C_{t,s,\mathcal{I}}$ where s is in contact with t.* By lemma 5

$$\begin{aligned} ||L_n - L_{n-1}||_{C^{1+\epsilon}} &= ||L_t - L_{m(t)}||_{C^{1+\epsilon}} \\ &\leq \mathcal{O}(\max(A_t, \nu_t)) \leq \mathcal{O}(f_\epsilon(n)) \leq \mathcal{O}(f_\epsilon(n-1)) \end{aligned}$$

(ii) *For $D_{t,s,\mathcal{I}}$.* Suppose s is on the left of t.

$$L_n - L_t = \psi_{t,s}L_t + (1-\psi_{t,s})L_s - L_t = (1-\psi_{t,s})(L_s - L_t)$$

in $C_{t,s,\mathcal{I}}$. Therefore, by inequality (2.5) of lemma 5 and inequality (2.12) we have that in $C_{t,s,\mathcal{I}}$,

$$|L_n - L_t| \leq |1-\psi_{t,s}||L_s - L_t| \leq \mathcal{O}(\nu_{t,s}|D_{t,s,\mathcal{I}}|).$$

Moreover, by lemma 5 and inequality (2.12),

$$|dL_n - dL_t| \leq |d\psi_{t,s}||L_s - L_t| + |\psi_{t,s}||dL_s - dL_t| \leq \mathcal{O}(\nu_{t,s})$$

and

$$\begin{aligned}||dL_n - dL_t||_{C^\epsilon} &\leq ||d\psi_{t,s}||_{C^\epsilon}||L_s - L_t||_{C^0} \\ &+ ||d\psi_{t,s}||_{C^0}||L_s - L_t||_{C^\epsilon} + ||\psi_{t,s}||_{C^\epsilon}||dL_s - dL_t||_{C^0} \\ &\leq \mathcal{O}(\nu_{t,s}|D_{t,s,\mathcal{I}}|^{-\varepsilon})\end{aligned}$$

Therefore,

$$||L_n - L_t||_{C^{1+\epsilon}} \leq \mathcal{O}(\nu_{t,s}|D_{t,s,\mathcal{I}}|^{-\epsilon})$$

in $C_{t,s,\mathcal{I}}$.

If $m(s) \neq m(t)$ then by lemma 5 and the last inequality

$$\begin{aligned}||L_n - L_{n-1}||_{C^{1+\epsilon}} &\leq ||L_n - L_t||_{C^{1+\epsilon}} + ||L_t - L_{m(t)}||_{C^{1+\epsilon}} \\ &+ ||L_{m(t)} - L_{n-1}||_{C^{1+\epsilon}} \\ &\leq \mathcal{O}\left(\nu_{t,s}|D_{t,s,\mathcal{I}}|^{-\epsilon}\right) + \mathcal{O}\left(\max(A_t, \nu_t)\right) \\ &+\mathcal{O}\left(\nu_{m(t),m(s)}|D_{m(t),m(s),\mathcal{I}}|^{-\epsilon}\right) \\ &\leq \mathcal{O}\left(f_\epsilon(n-1)\right),\end{aligned}$$

in $C_{t,s,\mathcal{I}}$.

If $m(s) = m(t)$ then $L_{m(t)} = L_{n-1}$ or

$$||L_{m(t)} - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(\nu_{m(t),z}|D_{m(t),z,\mathcal{I}}|^{-\epsilon}) \leq \mathcal{O}(f_\epsilon(n-1)),$$

where z is a contact vertex of $m(t)$. Therefore, in $C_{t,s,\mathcal{I}}$,

$$||L_n - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(f_\epsilon(n-1)).$$

Moreover, a similar argument to that used for $C_{s,t,\mathcal{I}}$, shows that in $D_{t,s,\mathcal{I}}$,

$$||L_n - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(f_\epsilon(n-1))$$

(iii) *For* $G_{t,s,\mathcal{I}}$. Suppose that $C_{t,\mathcal{I}}$ is on the right of $C_{s,\mathcal{I}}$. By definition of the domains of the gap maps $L_n = L_{n-1}$ in the gap $G_{t,s,\mathcal{I}}$ except in the extended intervals $E_{t,s,\mathcal{I}}$ and $E_{s,t,\mathcal{I}}$. Therefore, in $E_{t,s,\mathcal{I}}$,

$$\begin{aligned}L_n - L_{n-1} &= L_{m(t)}(\phi_{t,s} - 1) + L_t(1 - \phi_{t,s}) \\ &= (L_t - L_{m(t)})(1 - \phi_{t,s})\end{aligned}$$

If $m(t) = m(s)$, by lemma 5 and equation (2.11)

$$||L_n - L_{n-1}||_{C^0} \leq |L_t - L_{m(t)}||1 - \phi_t| \leq \mathcal{O}(A_t),$$

$$\begin{aligned}||dL_n - dL_{n-1}||_{C^0} &\leq |L_t - L_{m(t)}||d\phi_t| + |dL_t - dL_{m(t)}||1 - \phi_t| \\ &\leq \mathcal{O}\left(A_t|E_{t,s,\mathcal{I}}|^{-1}\right) + \mathcal{O}(\nu_t)\end{aligned}$$

and

$$\begin{aligned}||dL_n &- dL_{n-1}||_{C^\epsilon} \\ &\leq ||L_t - L_{m(t)}||_{C^\epsilon}||d\phi_t||_{C^0} + ||L_t - L_{m(t)}||_{C^0}||d\phi_t||_{C^\epsilon} \\ &\quad + ||dL_t - dL_{m(t)}||_{C^0}||1 - \phi_t||_{C^\epsilon} \\ &\leq \mathcal{O}\left(\nu_t|E_{t,s,\mathcal{I}}|^{1-\epsilon-1}\right) + \mathcal{O}\left(A_t|E_{t,s,\mathcal{I}}|^{-(1+\epsilon)}\right) + \mathcal{O}\left(\nu_t|E_{t,s,\mathcal{I}}|^{-\epsilon}\right) \\ &\leq \mathcal{O}\left(A_t|E_{t,s,\mathcal{I}}|^{-(1+\epsilon)}\right) + \mathcal{O}\left(\nu_t|E_{t,s,\mathcal{I}}|^{-\epsilon}\right)\end{aligned}$$

Similarly, in $E_{s,t,\mathcal{I}}$,

$$||L_n - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}\left(A_t|E_{t,s,\mathcal{I}}|^{-(1+\epsilon)}\right) + \mathcal{O}\left(\nu_t|E_{t,s,\mathcal{I}}|^{-\epsilon}\right)$$

If $m(t) \neq m(s)$, by lemma 5 and equation (2.11)

$$||L_n - L_{n-1}||_{C^0} \leq |L_t - L_{m(t)}||1 - \phi_t| \leq \mathcal{O}(\nu_t|E_{t,s,\mathcal{I}}|),$$

$$\begin{aligned}||dL_n - dL_{n-1}||_{C^0} &\leq |L_t - L_{m(t)}||d\phi_t| + |dL_t - dL_{m(t)}||1 - \phi_t| \\ &\leq \mathcal{O}(\nu_t)\end{aligned}$$

and

$$\begin{aligned}||dL_n - dL_{n-1}||_{C^\epsilon} &\leq ||L_t - L_{m(t)}||_{C^\epsilon}||d\phi_t||_{C^0} \\ &\quad + ||L_t - L_{m(t)}||_{C^0}||d\phi_t||_{C^\epsilon} \\ &\quad + ||dL_t - dL_{m(t)}||_{C^0}||1 - \phi_t||_{C^\epsilon} \\ &\leq \mathcal{O}(\nu_t|E_{s,t,\mathcal{I}}|^{-\epsilon})\end{aligned}$$

the constant of the last inequality depending upon ε. ∎

Lemma 8 The sequence of maps $(L_n)_{n>N_0}$ is a Cauchy sequence in the domain I with respect to the $C^{1+\epsilon}$ norm. In fact, $||L_n - L_{n-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(f_\epsilon(n-1))$.

Proof. For all vertices $t \in T_n$, define P_t as the middle point of $C_{t,\mathcal{I}}$ and for all non-contact vertices $t, s \in T_n$, define $Q_{t,s}$ as the extreme point which is common to $E_{t,s,\mathcal{I}}$ and $E_{m(t),m(s),\mathcal{I}}$. Then

$$|(dL_n - dL_{n-1})(P_t)| \leq \mathcal{O}(\nu_t) \quad \text{and} \quad |(dL_n - dL_{n-1})(Q_{t,s})| = 0.$$

For all $x, y \in \mathbf{I}$, if the closed interval between x and y is contained in the union of two of the domains of the form $C_{t,\mathcal{I}}$ or $C_{g_{t,s},\mathcal{I}}$ then denoting $dL_n - dL_{n-1}$ by B_n,

$$\frac{|B_n(y) - B_n(x)|}{|y-x|^\varepsilon} \leq \mathcal{O}(f_\varepsilon(n-1)).$$

Otherwise, take P_x (resp. P_y) to be the nearest point of the form P_t or $Q_{t,s}$ to x (resp. y) in the closed interval between x and y. Let us consider the case that $P_x = P_t$ and $P_y = P_s$. If B_n again denotes $dL_n - dL_{n-1}$,

$$\begin{aligned}
\frac{|B_n(y) - B_n(x)|}{|y-x|^\varepsilon} &\leq \frac{|B_n(y) - B_n(P_y)|}{|y-P_y|^\varepsilon} + \frac{|B_n(P_y)|}{|C_{s,\mathcal{I}}|^\varepsilon} + \\
&\quad + \frac{|B_n(P_x)|}{|C_{t,\mathcal{I}}|^\varepsilon} + \frac{|B_n(P_x) - B_n(x)|}{|P_x - x|^\varepsilon} \\
&\leq \mathcal{O}(f_\varepsilon(n-1)) + \mathcal{O}(\nu_s |C_{s,\mathcal{I}}|^{-\varepsilon}) \\
&\quad + \mathcal{O}(\nu_t |C_{t,\mathcal{I}}|^{-\varepsilon}) + \mathcal{O}(f_\varepsilon(n-1)) \\
&\leq \mathcal{O}(f_\varepsilon(n-1)).
\end{aligned}$$

Similarly, for the other cases. Therefore, $||L_n - L_{n-1}||_{C^{1+\varepsilon}} \leq \mathcal{O}(f_\varepsilon(n-1))$ and L_n is a Cauchy sequence since $\sum_{n=M}^{\infty} f(n) \leq \mathcal{O}(f(M))$, for any $M \geq N_0$. ∎

2.2.6 The conjugacy map L_∞.

Since the sequence $(L_n)_{n \geq N_0}$ is a Cauchy sequence in $C^{1+\varepsilon}(\mathbf{I})$, it converges to a function $L_\infty \in C^{1+\varepsilon}$.

Lemma 9 The map L_∞ is a $C^{1+\alpha^-}$ diffeomorphism of $\mathbf{I}$ onto $\mathbf{J}$ which extends $i^{-1} \circ j$.

Proof. By lemma 4, for all $t \in T_n$ $|C_{t,\mathcal{J}}|/|C_{t,\mathcal{I}}|$ is bounded away from 0 and ∞ and by the hypotheses of $(1+\alpha)$-scale equivalence, and $(1+\alpha)$-contact equivalence if $s, t \in T_n$ are adjacent, (i) $A_t e_{t,s,\mathcal{I}}^{-1} \to 0$, (ii) $\nu_t \to 0$ as $n \to \infty$ and (iii) $\nu_{s,t} \to 0$ depending if s is in contact with t or not and if they have the same mother. Thus there exists $\varepsilon_1 > 0$, $0 < \varepsilon < \varepsilon_1$ and $N_1 > 0$ such that if $n \geq N_1$ then for all $s, t \in T_n$,

$$\varepsilon_1 < |C_{m(t),\mathcal{J}}|/|C_{m(t),\mathcal{I}}|, \ \mathcal{O}(A_t |E_{t,s,\mathcal{I}}|^{-1} + \nu_t) < \varepsilon \text{ and } \mathcal{O}(\nu_{t,s}) < \varepsilon.$$

We break down the proof into four parts corresponding to the sets $C_{t,\mathcal{I}} \setminus C_{t,s,\mathcal{I}}$, $D_{t,s,\mathcal{I}}$, $E_{t,s,\mathcal{I}}$ and $G_{t,s,\mathcal{I}} \setminus E_{t,s,\mathcal{I}}$.

(i) In $C_{t,\mathcal{I}} \setminus C_{t,s,\mathcal{I}}$. $dL_t = |C_{t,\mathcal{J}}|/|C_{t,\mathcal{I}}| > \varepsilon_1$.

(ii) In $D_{t,s,\mathcal{I}}$. Suppose that s is on the left of t. Then, in $D_{t,s,\mathcal{I}}$,

$$\begin{aligned} |dL_n| &= |\psi_{t,s} dL_t + d\psi_{t,s} L_t + (1 - \psi_{t,s}) dL_s - d\psi_{t,s} L_s| \\ &\geq |dL_s| - |d\psi_{t,s}(L_t - L_s) + \psi_{t,s}(dL_t - dL_s)| \\ &\geq |C_{t,\mathcal{J}}|/|C_{t,\mathcal{I}}| - O(\nu_{t,s}) > \varepsilon_1 - \varepsilon > 0, \end{aligned}$$

(iii) In $E_{t,s,\mathcal{I}}$. Suppose t is on the left of s. Then, in $E_{t,s,\mathcal{I}}$,

$$\begin{aligned} |dL_n| &= |\phi_{t,s} dL_t + d\phi_{t,s} L_t + (1 - \phi_{t,s}) dL_{m(t)} - d\phi_{t,s} L_{m(t)}| \\ &\geq |dL_{m(t)}| - |d\phi_{t,s}(L_t - L_{m(t)}) + \phi_{t,s}(dL_t - dL_{m(t)})| \\ &\geq |C_{m(t),\mathcal{J}}|/|C_{m(t),\mathcal{I}}| - O(A_t |E_{t,s,\mathcal{I}}|^{-1} + \nu_t) > \varepsilon_1 - \varepsilon > 0, \end{aligned}$$

(iv) In $G_{t,s,\mathcal{I}} \setminus E_{t,s,\mathcal{I}}$. In different subsets of this set, the map $L_n = L_{n-j}$ for some $j \in \mathbf{N}$. We suppose by induction that $L_{n-j} > \varepsilon_1 - \varepsilon > 0$. For that take $N_0 = \max\{N_0, N_1\}$.

Therefore, $|dL_n| > \varepsilon_1 - \varepsilon > 0$ in $\mathbf{I}$ for all $n > N_0$ which implies that $|L_\infty| \geq \varepsilon_1 - \varepsilon > 0$.

By construction, $L_n(C_{t,\mathcal{I}}) = C_{t,\mathcal{J}}$ for all $t \in T_m$, $N_0 \leq m \leq n$, and therefore L_∞ equals $i^{-1} \circ j$ on the closure of $\bigcup_{n \geq N_0} \mathbf{I}^n$.

As $L_\infty(C_{t,\mathcal{I}}) = C_{t,\mathcal{J}}$, for all vertices $t \in T_n$ and all $n > N_0$, then L_∞ is a $C^{1+\alpha}$ conjugacy between the charts i and j. ■

2.3 Proof of theorem 2.

Suppose that the structures $\mathcal{S}$ and $\mathcal{T}$ are $C^{1+\beta}$-equivalent and are given respectively by the primary atlases $\mathcal{I}$ and $\mathcal{J}$. This equivalence means that the identity is a $C^{1+\beta}$ diffeomorphism between the two structures. Thus, if (i, U) is a chart of $\mathcal{I}$ and $C_{m(z),\mathcal{I}} \subset U$ then there exists a $C^{1+\beta}$ diffeomorphism $h : \mathbf{R} \to \mathbf{R}$ such that $h(C_{m(z),\mathcal{I}}) = C_{m(z),\mathcal{J}}$ and $h(C_{t,\mathcal{I}}) = C_{t,\mathcal{J}}$ for all descendents t of $m(z)$.

By the mean value theorem, there are points $u, v \in C_{m(t),\mathcal{I}}$ such that

$$|dh(u)| = |C_{m(t),\mathcal{J}}|/|C_{m(t),\mathcal{I}}| \quad \text{and} \quad |dh(v)| = |C_{t,\mathcal{J}}|/|C_{t,\mathcal{I}}|.$$

Moreover, since h is $C^{1+\beta}$, we have that, $|dh(u) - dh(v)| \leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta})$. Therefore,

$$\nu_t = 1 - \frac{|C_{t,\mathcal{J}}|}{|C_{m(t),\mathcal{J}}|}\frac{|C_{m(t),\mathcal{I}}|}{|C_{t,\mathcal{I}}|} \leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)).$$

By a similar argument,

$$\nu_{g_{t,s}} = 1 - \frac{|G_{t,s,\mathcal{J}}|}{|C_{m(t),\mathcal{J}}|}\frac{|C_{m(t),\mathcal{I}}|}{|G_{t,s,\mathcal{I}}|} \leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)).$$

Therefore,

$$\begin{aligned} A_t &\leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta}(\sum_{i=1}^{p}|C_{t_i,\mathcal{I}}| + \sum_{n=1}^{q}|G_{t_{i_n},t_{i_n+1},\mathcal{I}}|)) \\ &\leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{1+\beta}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)). \end{aligned}$$

where $t_1, \ldots, t_p$ are all the vertices with the same mother as t and $g_{t_{i_n},t_{i_n+1}}$, with $1 \leq n \leq q$, are the respective gaps between them. By the hypotheses of theorem 2, if $m(t) = m(s)$, then

$$\begin{aligned} A_t|E_{t,s,\mathcal{I}}|^{-(1+\epsilon)} &+ \nu_t|E_{t,s,\mathcal{I}}|^{-\epsilon} \leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{1+\beta}|E_{t,s,\mathcal{I}}|^{-(1+\epsilon)}) \\ &+ \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta}|E_{t,s,\mathcal{I}}|^{-\epsilon}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)). \end{aligned}$$

If $m(t) \neq m(s)$ then

$$\nu_t|E_{t,s,\mathcal{I}}|^{-\epsilon} \leq \mathcal{O}(|C_{m(t),\mathcal{I}}|^{\beta}|E_{t,s,\mathcal{I}}|^{-\epsilon}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)).$$

Thus, the conditions of definition 4 are verified if for $f_\epsilon(n)$ one takes $cg_{\beta,\epsilon}(n)$ where $c > 0$ is some constant. Therefore, the atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\gamma)$-scale equivalent.

If s and t are in contact then, by the mean value theorem, there exists $u \in C_{s,\mathcal{I}}$ and $v \in C_{t,\mathcal{I}}$ such that

$$|dh(u)| = |C_{s,\mathcal{J}}|/|C_{s,\mathcal{I}}| \text{ and } |dh(v)| = |C_{t,\mathcal{J}}|/|C_{t,\mathcal{I}}|.$$

Since the map h is $C^{1+\beta}$,

$$|dh(z) - dh(v)| \leq \mathcal{O}((|C_{t,\mathcal{I}}| + |C_{s,\mathcal{I}}|)^{\beta}) \leq \mathcal{O}(|D_{t,s,\mathcal{I}}|^{\beta}).$$

Therefore,

$$\nu_{g_{t,s}} = 1 - \frac{|C_{t,\mathcal{J}}|}{|C_{s,\mathcal{J}}|}\frac{|C_{s,\mathcal{I}}|}{|C_{t,\mathcal{I}}|} \leq \mathcal{O}(|D_{t,s,\mathcal{I}}|^{\beta})$$

and

$$\frac{\nu_{g_{t,s}}}{|D_{t,s,\mathcal{I}}|^{\epsilon}} \leq \mathcal{O}(|D_{t,s,\mathcal{I}}|^{\beta-\epsilon}) \leq \mathcal{O}(g_{\beta,\epsilon}(n)).$$

The last inequality follows from the hypotheses of the theorem.

Thus, taking $f_\epsilon(n) = cg_{\beta,\epsilon}(n)$, the conditions of definition 5 are verified. Therefore, the atlases $\mathcal{I}$ and $\mathcal{J}$ are $(1+\gamma)$-contact equivalent. This completes the proof that $\mathcal{I}$ and $\mathcal{J}$ are $(1+\gamma)$-equivalent. ∎

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Bibliography

[1] R. de la Llave, A. Pinto and D. A. Rand, Global phase space universality, smooth conjugacies and renormalisation: 2. The $C^{k+\alpha}$ case using rapid convergence of Markov families. In preparation.

[2] A.Pinto and D. A. Rand, Convergence of renormalisation, Markov families and rigidity of dynamical systems: A unification of smooth conjugacy and universality results. In preparation.

[3] D. A. Rand, Universality and renormalisation in dynamical systems. In *New Directions in Dynamical Systems* (ed. T. Bedford and J. Swift) pp. 1-56, (1988), CUP

[4] D A Rand, Global phase space universality, smooth conjugacies and renormalisation: 1. The $C^{1+\alpha}$ case. *Nonlinearity* **1** (1988) 181-202.

[5] D A Rand, The global rigidity conjecture and convergence of renormalisation. In *Workshop on dynamical systems. Papers in honour of Rene Thom.* (eds. Z Coelho and E Shiels) pp. 147-155 Longman, 1988.

[6] D. Sullivan, Differentiable structures on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets. In *Nonlinear evolution and chaotic phenomena.* (eds. G. Gallavotti and P. Zweifel), Plenum, New York, 1988.

[7] D. Sullivan, Quasiconformal homeomorphisms in dynamics, topology and geometry. Preprint 1987

[8] D. Sullivan, On the structure of infinitely many dynamical systems nested inside or outside a given one. To appear in the AMS volume celebrating the Centenary of the American Mathematical Society, 1990.

Chapter 3

Global phase space universality, smooth conjugacies and renormalisation: 2. The $C^{k+\alpha}$ case using rapid convergence of Markov families.

Global phase space universality, smooth conjugacies and renormalisation: 2. The $C^{k+\alpha}$ case using rapid convergence of Markov families.

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Abstract

We prove that the speed of convergence of two Markov families determines the smoothness of the conjugacy between them. One of the applications of this result that we give is that the attractors of any two quadratic foldings at the Feigenbaum accumulation point of period doubling are $C^{2+.11}$ conjugate. Our main result provides the basis for a complete unification of renormalisation and smooth conjugacy results which includes both the classical theorems and more recent results about critical systems.

3.1 Introduction.

In [7] the notion of a Markov family was used to prove that exponentially fast convergence under renormalisation of two dynamical systems with bounded geometry implies that their limit sets are $C^{1+\alpha}$ conjugated. In fact, in a number of cases, such as diffeomorphisms of the circle and quadratic foldings at the accumulation point of period doubling, these conjugacies are actually smoother. In this paper we prove a general theorem for Markov families which gives this extra smoothness in terms of a balance between the speed of convergence of the two Markov families and the scaling structure of their cylinders. In renormalisation problems, these are given by the speed of convergence of the renormalisation and the scaling structure of the critical orbits of the dynamical system. A simple corollary of this theorem (theorem 2) is that the conjugacies between the limit sets of quadratic foldings at the accumulation point of period doubling are $C^{2+.11}$. For diffeomorphisms of the circle of constant type, the fact that the exponential rate of the convergence of the renormalisation can be made arbitrarily large by using a nearly linear high order polynomial diffeomorphism instead of a linear rescaling in the

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renormalisation transformation, allows one to deduce C^∞ conjugacies as a corollary of theorem 1. This result shows that Markov families are a considerably more powerful tool than Feigenbaum-Sullivan scaling function ([8]). This scaling function is a complete invariant of the C^{1+} structure, but by its nature is incapable of detecting extra smoothness. On the other hand, our results show that Markov families determines the extra smoothness. Moreover, this result provides a unification of renormalisation and classical smooth conjugacy results. The general principle is that two infinitely renormalisable systems with bounded geometry are $C^{r+\beta}$ conjugated with $0 \le r \le \infty$ and $0 \le \beta < 1$ if for all $0 < \delta \le r + \beta$ there exists a polynomial renormalisation in which the speed on convergence dominates the $(\delta - 1)^{\text{th}}$ power of the rate at which the smallest geometrical scale goes to zero. An open problem is to replace the condition of bounded geometry by a weaker condition as done in [4] so that problems of non-constant type can be handled.

The observation that the faster speed of convergence implied a C^{2+} conjugacy for period doubling first arose in discussions with Rafael de la Llave in 1987.

3.1.1 Markov families.

Topological Markov families

A *topological Markov family* F is a family of mappings $F_{n,a}$, with either $n = 0, 1, \ldots$ or $n \in \mathbf{Z}$ and a in a finite set S_n, which satisfy the following conditions.

(i) For each n and $a \in S_n$, $F_{n,a}$ is a homeomorphism of the closed interval I_a^n into $\mathbf{R}$.

(ii) I_a^n contains in its interior a closed interval C_a^n with following properties.

- $\text{int}C_a^n \bigcap \text{int}C_b^n = \emptyset$ if $a \neq b$.
- If $x \in C_a^n$ and $F_n(x) \in C_b^{n+1}$ then $F_n(C_a^n)$ contains C_b^{n+1}.
- If $b \in S_{n+1}$, there exists $a \in S_n$ such that $F_n(C_a^n)$ contains C_b^{n+1}.

We regard the $F_{n,a}$ as defining a single mapping F_n on $C^n = \bigcup_{a \in S_n} C_a^n$.

$C^{k+\alpha}$ Markov families

A $C^{k+\alpha}$ *Markov family* F satisfies the following conditions in addition.

(iv) $F_{n,a} = F_n|I_a^n$ is a $C^{k+\alpha}$ diffeomorphism of I_a^n into $\mathbf{R}$.

(v) $|F_n'(x)| > 1$ for all $x \in I^n$ and all n in some norm on $\mathbf{R}$.

Bounded and boundedly extended Markov families

A $C^{k+\alpha}$ *Markov family* F is said to be *bounded* if

(iv) $|I^n|/|I^0|$, $|C^n|/|C^0|$, $|I_a^n|/|I_a^0|$ and $|C_a^n|/|C_a^0|$ is bounded away from 0 and ∞, where C^n is the smallest closed interval containing $\bigcup_{a\in S_n} C_a^n$;

(v) for all n and all $a \in S_n$ the $C^{k+\alpha}$ norm of $F_{n,a} = F_n|I_a^n$ on I_a^n is bounded independently of n and a; and

(vi) there exists $\lambda > 1$ such that $|F_n'(x)| > \lambda$ for all $x \in I^n$ and all n.

A point $x \in C^n$ is *captured* if for all $m > n$, $F_{m-1} \circ \cdots \circ F_n(x) \in C^m$. The set of all captured points in C^n is denoted by $\Lambda^n = \Lambda^n(F)$.

Let Σ^n denote the set of infinite right-handed words $\underline{\varepsilon} = \varepsilon_n\varepsilon_{n+1}\ldots$ such that (i) $\varepsilon_j \in S_j$ and (ii) there exists $x \in C^n$ with the property that

$$F_{m-1} \circ \cdots \circ F_n(x) \in C_{\varepsilon_m}^m$$

for all $m > n$. We call these words *admissible.* If $\underline{\varepsilon} = \varepsilon_n\varepsilon_{n+1}\ldots \in \prod_{p\geq n} S_p$ let $\underline{\varepsilon}|p$ denote the finite word $\varepsilon_n\ldots\varepsilon_{n+p-1}$ of length p. Let Σ_p^n denote the set of finite words $\underline{\varepsilon}|p$ where $\underline{\varepsilon} \in \Sigma^n$. We denote by σ and m the mappings $\sigma : \Sigma_p^n \to \Sigma_{p-1}^{n+1}$ and $m : \Sigma_p^n \to \Sigma_{p-1}^n$ given by

$$\begin{aligned} \sigma(\varepsilon_n\ldots\varepsilon_{n+p-1}) &= \varepsilon_{n+1}\ldots\varepsilon_{n+p-1} \\ m(\varepsilon_n\ldots\varepsilon_{n+p-1}) &= \varepsilon_n\ldots\varepsilon_{n+p-2}. \end{aligned}$$

If $\underline{\varepsilon} = \varepsilon_n\varepsilon_{n+1}\ldots \in \Sigma^n$ then we denote by $C_{\varepsilon_n\ldots\varepsilon_m}$ (resp. $I_{\varepsilon_n\ldots\varepsilon_m}$) the closed interval consisting of all $x \in C^m$ such that for all $n \leq j < m$,

$$F_j \circ \cdots \circ F_n(x) \in C_{\varepsilon_{j+1}}^j \quad (\text{resp. } I_{\varepsilon_{j+1}}^j).$$

By $\Lambda_{\varepsilon_n\ldots\varepsilon_m}$ we denote the intersection of Λ^n with $C_{\varepsilon_n\ldots\varepsilon_m}$ and by $\tilde{C}_{\varepsilon_n\ldots\varepsilon_m}$ the smallest closed interval containing $\Lambda_{\varepsilon_n\ldots\varepsilon_m}$. Note that if each interval C_a^n is replaced by the subinterval $\tilde{C}_a^n$ in the definition of Λ^n then one obtains the same set Λ^n of captured points.

We therefore assume henceforth that $C_a^n = \tilde{C}_a^n$.

Suppose that J is a closed set contained in the interior of an interval I and let $\tilde{J}$ denote the smallest closed interval containing J. Then $I - \tilde{J}$ consists of two intervals. The interval to the right (resp. left) of J is denoted by $R(J, I)$ (resp. $L(J, I)$).

Definition. A $C^{k+\alpha}$ Markov family F is *boundedly extended* if there exists $\delta_1, \delta_2 > 0$ such that, for all n and all $a \in S_n$, the intervals I_a^n on which F_n is defined and $C^{k+\alpha}$ are such that

$$\delta_1 < \frac{|R(C_a^n, I_a^n)|}{|I_a^n|}, \frac{|L(C_a^n, I_a^n)|}{|I_a^n|} < \delta_2. \tag{3.1}$$

All the Markov families of this paper are assumed to have this property.

Definition. If F and G are two topological Markov families then we say that they are *topologically conjugated* if for all n there exists a homeomorphism $h_n : \Lambda^n(F) \to \Lambda^n(G)$ such that $G_n \circ h_n = h_{n+1} \circ F_n$ on $\Lambda^n(F)$.

In such a case we call the family $\underline{h} = (h_n)$ the *conjugacy.* The major result of this paper is the derivation of natural necessary conditions for the h_n to be $C^{r+\beta}$ or to have a $C^{r+\beta}$ extension to $\mathbf{R}$. Without loss of generality, we will restrict to the case where the homeomorphisms preserve the order of the real line.

Conditions A(g) and B(g).

Let G be a $C^{s+\gamma}$ Markov family which is topologically conjugate to F. We will impose the following condition on the pairs of families F and G that we consider. It involves the positive function $g(n)$.

Condition A(g). For all n and all $\varepsilon \in S_n$,

$$||F_{n,\varepsilon} - G_{n,\varepsilon}||_{C^{s+\gamma}(I_\varepsilon^n)} \leq g(n+1).$$

By I^n, I_a^n, I_t^n, C^n, C_a^n and C_t^n we denote the intervals and cylinders $I^n(F)$, $I_a^n(F)$, $I_t^n(F)$, $C^n(F)$, $C_a^n(F)$ and $C_t^n(F)$ for F. We denote the corresponding intervals and cylinders for G_n by J^n, J_a^n, J_t^n, D^n, D_a^n and D_t^n.

If $\varepsilon \in S_n$, let $A_{n,\varepsilon}$ denote the affine map which sends C_ε^n onto D_ε^n preserving orientation. We regard $A_{n,\varepsilon}$ as having domain I_ε^n. If t is the word $\varepsilon_0 \ldots \varepsilon_n \in \Sigma_{n+1}^0$ define

$$\begin{aligned}
K_t &= G_{0,\varepsilon_0}^{-1} \circ \cdots \circ G_{n-1,\varepsilon_{n-1}}^{-1} : J_{\varepsilon_n}^n \to J_t^0, \\
E_t &= F_{n-1,\varepsilon_{n-1}} \circ \cdots \circ F_{0,\varepsilon_0} : I_t^0 \to I_{\varepsilon_n}^n, \quad \text{and} \\
L_t &= K_t \circ A_{n,\varepsilon_n} \circ E_t : I_t^0 \to J_t^0.
\end{aligned}$$

Now we formulate a condition that controls the behaviour at *contact points*. Let $t = \varepsilon_0 \ldots \varepsilon_{n-1}$ and $t' = \varepsilon'_0 \ldots \varepsilon'_{n-1}$ be in *contact* i.e. such that C_t and $C_{t'}$ meet in a point. Let $m > 0$ be minimal such that $t|m = t'|m$ and $t|(m+1) \neq t'|(m+1)$. In this case, let $e_{t,t'}$ denote $\max_{x \in I_t \bigcap I_{t'}} \{|dE_t(x)|, |dE_{t'}(x)|\}$. Then we impose the following condition on all such pairs t, t'

Condition B(g). For all such t and t' and all $0 \leq k \leq s$,

$$||L_{\sigma^m t} - L_{\sigma^m t'}||_{C^k} \leq g(n) e^{k-1}_{\sigma^m(t),\sigma^m(t')}$$

on $I_{\sigma^m t} \bigcap I_{\sigma^m t'}$.

It is not difficult to see that condition B(g) is satisfied, for appropriate g, by those Markov maps arising from renormalisation structures with contact points such as those for diffeomorphism of the circle and cubic critical circle maps.

Theorem 20 Suppose that the bounded and boundedly extended $C^{s+\gamma}$ Markov families F and G are topologically conjugate and satisfy Conditions A(g) and B(g). Let $e(n) = \max_{t \in \Sigma^0_n} ||dE_t||$. Then the conjugacy $\underline{h} = (h_n)$ is $C^{r+\beta}$ with $\beta \in [0,1)$ such that $r + \beta \leq s$, if the function f given by

$$f(n) = e(n)^{r+\beta-1} g(n)$$

is such that $\sum_{j=0}^{\infty} f(j) < \infty$.

Remark. Suppose that F and G satisfy the hypotheses of theorem 20. Then, by boundedness, there exist constants $d_1, d_2 > 0$ and $\lambda, \mu \in (0,1)$ such that for all $t \in \Sigma^0_n$,

$$d_1 \lambda^{-n} < |dE_t| < d_2 \mu^{-n}$$

Thus $g(n)/f(n) \leq c\lambda^{(r+\beta-1)n}$ and, in particular, $g(n)$ is exponentially decreasing. If $g(n) < c\tau^n$ then, by theorem 1, the condition $\tau/\mu^{r+\beta-1} < 1$ is sufficient for the conjugacy to be $C^{r+\beta}$.

3.1.2 Global C^2 rigidity for period-doubling.

We say that a sequence of points x_i $i = 0, 1, \ldots$ in the interval $[x_1, x_0]$ has the *Feigenbaum ordering* if for $0 \leq i < 2^{n-1}$, $x_{i+2^{n-1}}$ and $x_{i+3.2^{n-1}}$ lie between x_i and $x_{i+2^{n-1}}$ and are ordered so that $x_i - x_{i+2^{n-1}}$ and $x_{i+2^n} - x_{i+3.2^n}$ have the same sign. The inductive construction of such a sequence is illustrated in figure 1.

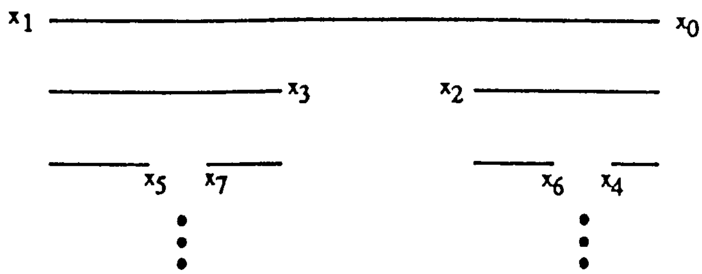

Figure 1.

A quadratic folding of the interval $I = [-1,1]$ is a $C^{1+Lipschitz}$ mapping $f : I \to I$ with $f' > 0$ (resp. $f' < 0$) on $[-1,0]$ (resp, $[0,1]$) and such that in some neighbourhood of 0 there is a $C^{1+Lipschitz}$ coordinate system x in which $f(x) = x^2 + f(0)$. Given such a mapping f let $x_i = f^{i+1}(0)$. Suppose that the x_i have the Feigenbaum ordering. Let $J_{i,n}$ denote the closed interval between x_i and x_{i+2^n}, $0 \leq i \leq 2^n$. The Cantor set

$$\Lambda_f = \bigcap_{n \geq 1} \bigcup_{i=0}^{2n-1} J_{i,n}$$

is the attractor of f in the sense that every orbit is either eventually periodic or else converges to Λ_f.

Theorem 21 Suppose that f and g are real analytic quadratic foldings with the Feigenbaum ordering. Then the canonical homeomorphism $h : \Lambda_f \to \Lambda_g$ has a $C^{2+.11}$ extension to the real line.

Proof. Let Ω denote the unit disk $|z| < 1$ in $\mathbf{C}$ and let $\mathcal{L}$ denote the real Banach space of continuous $h: \bar{\Omega} \to \mathbf{C}$ which are holomorphic on Ω, take real values at real points and are such that, if $h(z) = \sum_{n \geq 0} a_n z^n$ then $||h|| = \sum_{n \geq 0} |a_n| < \infty$. Let $\mathcal{A}$ denote the set of maps of the form

$$f(z) = 1 - z^2 h((z^2 - 1)/2.5)$$

where $h \in \mathcal{L}$. By identification with $\mathcal{L}$, $\mathcal{A}$ may be regarded as a real Banach space.

The *doubling operator*

$$T(f) = a^{-1} \cdot f^2 \circ a \tag{3.2}$$

where $a = a(f) = f(1)$ is well-defined on the open subset $\mathcal{D}(T)$ consisting of those $f \in \mathcal{A}$ such that, if $a = a(f)$ and $b = f(a)$ then $a < 0$, $b > -a$ and $f(b) \leq -a$.

Lanford [1] and [2] has found a polynomial f_{approx} which is very nearly a fixed point of T. Using computer-assisted estimates he then shows that if $\mathcal{V}$ denotes the ball $||f - f_{approx}|| < .01$, then

1. $T|\mathcal{V}$ is well-defined and C^∞;
2. for $f \in \mathcal{V}$, $dT(f)$ is a compact operator; and
3. T has a unique fixed point g in $\mathcal{V}$.

Moreover, using his computer assisted proof, it is easy to show that (Mestel [3]): 4. the spectrum of $dT(g)$ consists of a simple real eigenvalue $\delta > 1$, the eigenvalue $\lambda_0 = a \approx 0.3995$ corresponding to quadratic coordinate changes (see below) and a countable set of eigenvalues contained strictly inside the circle $|z| = \sigma$ for some $0 < \sigma < 0.13$.

If $\tau : \mathbf{C} \to \mathbf{C}$ is holomorphic on a neighbourhood of 0 in $\mathbf{C}$ let ψ_τ denote the infinitesimal coordinate change given by

$$\psi_\tau = \left.\frac{d}{dt}\right|_{t=0} (id + t\tau)^{-1} \circ g \circ (id + t\tau).$$

It is easy to check that if $a^{-1} \cdot \tau \circ a = a^m \tau$ then ψ_τ is an eigenvector of dT with eigenvalue a^m.

We are interested in the eigenvector $\psi_0 = \psi_{\tau_0}$ corresponding to $\tau_0(x) = x^2$. This has eigenvalue $\lambda_0 = a$. Let E be the finite-dimensional spectral projection in $\mathcal{L}$ of dT associated with the eigenvalue λ_0. Then for $f \in \mathcal{A}$ near g, the equation

$$\psi_{\tau(f)} = E(f - g) \tag{3.3}$$

has a unique solution with $\tau(f)$ a monomial of degree 2.

Now, $\psi_{\tau(f)}$ is an infinitesimal coordinate change corresponding to projection of f into $E\mathcal{L}$. Define

$$\alpha(f) = (id + \tau(f))^{-1} \circ f \circ (id + \tau(f))).$$

Then

$$\alpha(f) = f - \psi_{\tau(f)} + \mathcal{O}(|f - g|^2). \tag{3.4}$$

One can regard $\alpha(f)$ as the result of factoring out from $f - g$ the quadratic coordinate transformations in $E\mathcal{L}$ corresponding to deviations from $f = g$. Therefore, consider the transformation

$$T_1(f) = \alpha(T(f)) = (id + \tau(Tf))^{-1} \circ (T(f)) \circ (id + \tau(T(f))).$$

By (3.3) and (3.4) this has derivative

$$dT_1(f) = d\alpha(T(f)) \cdot dT(f) = dT(f) - \psi_{\tau(dT(f))} = (id - E)dT(f)$$

Thus the spectrum of $dT_1(g)$ consists of the simple real eigenvalue $\delta > 1$, and a countable set of eigenvalues contained strictly inside the circle $|z| = \sigma$ for some $0 < \sigma < .13$.

The associated coordinate transformations $(id + \tau(T(f))$ are nonlinear functions of f. On the other hand, if

$$S(f) = B^{-1} \circ T(f) \circ B.$$

where $B = B_f = id + \tau(dT(g) \cdot (f - g))$ then $dS(f) = dT_1(f)$ and the associated coordinate transformations B are bounded affine in f.

Clearly the stable manifolds of S and T are equal. Therefore we have deduced that if $f \in W^s_T(g)$ and $f_n = S^n f$ then there exists a constant c depending only upon f such that $||f_n - g|| < c\sigma^n \leq c(0.13)^n$.

To $f \in W^s_T(g)$ we associate the Markov family

$$G_n(x) = \begin{cases} B^{-1}_{f_n}(x) & \text{if } x \in I_0 = [f_n^2(0), f_n^4(0)] \\ B^{-1}_{f_n}(f_n(x)) & \text{if } x \in I_1 = [f_n^3(0), f_n(0)]. \end{cases}$$

Let F be the corresponding Markov family for the fixed point g. Then F and G satisfy condition $A(g)$ with $g(n) = c(0.13)^n$ for some constant c. Condition $B(g)$ is trivial in this case as there are no contact points. Moreover, since G_n is independent of n, if $t \in \Sigma^0_n$ then $|dE_t| \leq a^{-2n}$ because $|dG_n| \leq a^2$. Therefore, by theorem 1, the homeomorphism $h : \Lambda_f \to \Lambda_g$ has a $C^{r+\beta}$ extension to $\mathbf{R}$ for all r and β such that

$$r + \beta \leq \frac{\ln \sigma}{2 \ln a} + 1 \leq \frac{\ln .13}{2 \ln .3995} + 1 \leq 2.11.$$

This proves the theorem under the assumption that $f \in \mathcal{V}$. The general result follows from Sullivan's theorem ([10]) which says that if f satisfies the hypotheses of the theorem then $T^n(f) \in W^s_T(g)$ for some $n > 0$. ∎

3.2 Proof of theorem 1.

In this section we reduce the proof of theorem 1 to the two main propositions 2 and 3.

Notation

Throughout the paper we use the following notation:

1. If f and g are functions of a variable x with domain Δ, then we write $f(x) = \mathcal{O}_y(g(x))$ with constant d if, for all $x \in \Delta$,

$$d^{-1} < \frac{|f(x)|}{|g(x)|} < d$$

and if the constant d depends only upon the variables y. Often we will drop the reference to d. In many cases the constant d will depend upon the Markov families F and G of theorem 1, but, in this context, these are fixed and this dependence is never explicitly mentioned.

Thus if a_n and b_n are sequences then $a_n = \mathcal{O}(b_n)$ means a_n/b_n and b_n/a_n are bounded away from 0 independently of n.

2. Similarly, $f(x) \leq \mathcal{O}_y(g(x))$ with constant d means $|f(x)/g(x)| < d$ for all $x \in \Delta$.

3. We also use the notation of interval arithmetic for some inequalities where:

   - if I and J are intervals then $I + J$, $I.J$ and I/J have the obvious meaning as intervals,
   - if $I = \{x\}$ then we often denote I by x, and
   - $I \pm \varepsilon$ denotes the interval consisting of those x such that $|x - y| < \varepsilon$ for all $y \in I$.

   Thus $\phi(n) \in 1 \pm \mathcal{O}(\nu^n)$ means that there exists a constant $c > 0$ depending only upon the families F_n and G_n such that for all $n \geq 0$, $1 - c\nu^n < \phi(n) < 1 + c\nu^n$.

Proof of theorem 1

Let the Markov families F and G be as in the statement of theorem 20. As above, if $\varepsilon \in S_n$, let $A_{n,\varepsilon}$ denote the affine map which sends I^n_ε onto J^n_ε. If $\varepsilon\varepsilon' \in \Sigma^n_2$ let

$$K_{n,\varepsilon,\varepsilon'} = G^{-1}_{n,\varepsilon} \circ A_{n+1,\varepsilon'} \circ F_{n,\varepsilon} : I^n_{\varepsilon\varepsilon'} \to J^n_{\varepsilon\varepsilon'}.$$

Then

$$L_t = C_{m(t)} \circ K_{n-1,\varepsilon_{n-1}\varepsilon_n} \circ E_{m(t)}.$$

3.2.1 Gaps

If $s,t \in \Sigma_n^0$ we say that s and t are *adjacent* if there is no $t' \in \Sigma_n^0$ such that $C_{t'}$ lies between C_s and C_t. In this case we say that t and t' and C_s^0 and C_t^0 are in *contact* if $C_s \bigcap C_t \neq \emptyset$. If they are not in contact then there is a *gap* between them. We denote this gap by $G_{s,t}$ or $G_{t,s}$. If $m(s) = m(t)$ then $G_{t,s}$ is called an *n-gap*.

3.2.2 Definitions of the closed sets $C_{t,s}$, $D_{t,s}$ and $E_{t,s}$.

To each $s,t \in \Sigma_n^0$, we associate the following intervals in $\mathbb{R}$ (see figure 2(a), (b) and (c)).

- $C_{s,t}$, $C_{t,s}$ and $D_{t,s}$: If $t,s \in \Sigma_n^0$, are adjacent and in contact, define $P_{t,s} = P_{s,t}$ as the common point between the closed sets C_t and C_s. Let the closed interval $C_{t,s}$ denote the closed interval in C_t of length $\delta|C_t|$ which contains $P_{t,s}$ where, using lemma 15, δ is chosen independently of t so that $C_{t,s} \subset I_t \bigcap I_s$. Define $D_{t,s} = C_{t,s} \bigcup C_{s,t}$. If $t,s \in \Sigma_n^0$ are adjacent but not in contact, define $P_{t,s}$ as the common point of C_t and the gap $G_{t,s}$. There is $\varepsilon > 0$ such that $|G_{t,s}| > \varepsilon|C_t|$, for all $s,t \in \Sigma_n^0$ and all n. Moreover, there is $\delta_1 > 0$ such that if $|x - P_{t,s}| < \delta_1|C_t|$ then $x \in I_t$. Let $0 < \delta < \min\{\varepsilon, \delta_1\}$ and $\delta_t = \delta|C_t|/3$. Define the closed sets $C_{t,s}$ as
$$C_{t,s} = \{x \in G_{t,s} : |x - P_{t,s}| \leq \delta_t\}.$$
Then $C_{t,s} \cap C_{s,t} = \emptyset$, $|C_{t,s}| = \mathcal{O}(|C_t|)$ and $C_{t,s} \subset I_t$.

- $E_{t,s}$: Let $t_1, s_1 \in \Sigma_{n+1}^0$ be the adjacent vertices such that $G_{t_1,s_1} = G_{t,s}$. Define $E_{t,s} = C_{t,s} \setminus C_{t_1,s_1}$. Clearly, $E_{t,s} \subset I_t$. By the choice of δ_t in the definition of C_t, $|E_{t,s}| = \mathcal{O}(|C_t|)$.

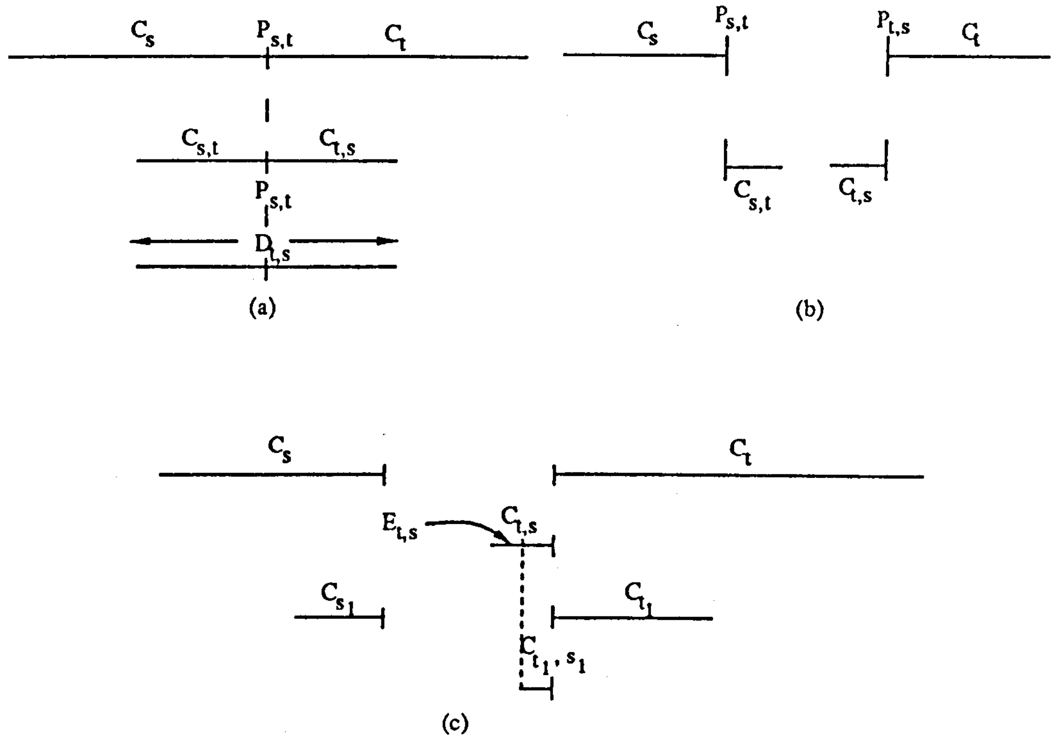

Figure 2.

3.2.3 The definition of the contact and gap maps.

Lemma 10 For all $\delta \geq 0$ there exists a C^∞ map $\phi : [0, \delta] \to [0, 1]$ such that $\phi = 0$ on $[0, \delta/3]$, $\phi = 1$ on $[2\delta/3, \delta]$ and $||\phi||_{C^{k+\alpha}} \leq c_k \delta^{-(k+\alpha)}$ where c_k depends only upon $k \in \mathbf{Z}_{\geq 0}$ and not on $\alpha \in (0, 1]$ or δ.

The proof of this lemma is very simple. Find such a function ϕ_0 for the case $\delta = 1$ and then deduce the general case by letting $\phi(x) = \phi_0(\delta^{-1}x)$. ∎

If s and t are adjacent in Σ_n^0 we use lemma 10 to choose functions $\phi_{t,s}$ on $G_{t,s}$ and $\psi_{s,t} = \psi_{t,s}$ on $D_{t,s}$ with the following properties.

(i) $\phi_{t,s} = 0$ (resp. $\psi_{t,s} = 0$) on the left-hand third of $E_{t,s}$ (resp. $D_{t,s}$) and $\phi_{t,s} = 1$ (resp. $\psi_{t,s} = 1$) on the right-hand third of $E_{t,s}$ (resp. $D_{t,s}$)

(ii) $||\phi_{t,s}||_{C^{k+\alpha}} \leq \mathcal{O}(|E_{t,s}|^{-k-\alpha})$ and $||\psi_{t,s}||_{C^{k+\alpha}} \leq \mathcal{O}(|D_{t,s}|^{-k-\alpha})$, for all integers $0 \leq k \leq s$ and all $0 < \alpha < 1$ such that $k+\alpha < s+\gamma$ and where the constants are independent of all the data.

By lemma 15, $|E_{t,s}| = \mathcal{O}(|dE_t|^{-1}) = \mathcal{O}(|dE_{m(t)}|^{-1})$ and $|D_{t,s}| = \mathcal{O}(e_{t,s}^{-1})$. Therefore,

$$||\psi_{t,s}||_{C^{k+\alpha}} \leq \mathcal{O}(e_{t,s}^{k+\alpha}) \text{ and } ||\phi_{t,s}||_{C^{k+\alpha}} \leq \mathcal{O}(|dE_t|^{k+\alpha}). \tag{3.5}$$

for all integers $0 \leq k \leq s$ and all $0 \leq \alpha < 1$ such that $k+\alpha < s+\gamma$.

Extend $\phi_{t,s}$ to all of the gap $G_{t,s}$ as a smooth map by taking it as constant outside $E_{t,s}$. We call the $\phi_{t,s}$ *gap maps* and the $\psi_{t,s}$ *contact maps.*

Note that, for all $n, m > N$ and all non-contact adjacent vertices $t_1, s_1 \in \Sigma_n^0$ and $t_2, s_2 \in T_m$ such that $\{s_1, t_1\} \neq \{s_2, t_2\}$ the domains of the gap maps where they are different from 0 or 1 do not overlap. For all $n \geq N$ and all contact adjacent vertices $t_3, s_3 \in \Sigma_n^0$ and $t_4, s_4 \in \Sigma_n^0$ such that $\{s_3, t_3\} \neq \{s_4, t_4\}$ the domains of the contact maps do not overlap. Moreover, they do not overlap with any domain of any gap map ϕ_{t_2,s_2} where $t_2, s_2 \in T_m$ and $m \leq n$.

3.2.4 The map $h_n : C^0 \to D^0$.

Construction of h_n on cylinders C_t in C^0.

Lef h_0 be the affine map that sends C^0 onto D^0. For all $n > 0$ and all vertices $t \in \Sigma_n^0$, define the map h_n on $C_t \subset C^0$ as follows. For all words s_i in contact with t, define $h_n = L_t$ on $C_t \subset \cup C_{t,s_i}$. If s is in contact with t and s is on the right of t then

$$h_n(x) = \psi_{t,s}(x)L_t(x) + (1 - \psi_{t,s}(x))L_s(x)$$

for all $x \in C_{t,s}$. If s is on the left of t then

$$h_n(x) = \psi_{t,s}(x)L_s(x) + (1 - \psi_{t,s}(x))L_t(x)$$

for all $x \in C_{t,s}$.

Extension of h_n to the gaps $G_{t,s}$ in C^0.

Suppose that $t, s \in \Sigma_n^0$ are adjacent but not in contact. Moreover, suppose that C_t is on the left of C_s. Define the map h_n on $E_{t,s}$ by

$$h_n|_{E_{t,s}} = \phi_{t,s} \cdot L_{m(t)} + (1 - \phi_{t,s}) \cdot L_t.$$

Define the map h_n on $E_{s,t}$ by

$$h_n|_{E_{s,t}} = (1 - \phi_{s,t}) \cdot L_{m(s)} + \phi_{s,t} \cdot L_s.$$

Finally, in $G_{t,s} \setminus (E_{t,s} \cup E_{s,t})$ define $h_n = h_{n-1}$.

Note that if $C_t = C_{m(t)}$ then on $C_t \cup E_{t,s}$, $h_n = h_{n-1}$. This construction builds an infinitely differentiable map h_n which is defined on the closed interval C^0 and which maps C^0 diffeomorphically onto D^0.

3.2.5 The sequence of maps (h_n) converges in the $C^{r+\beta}$ norm.

The space of $C^{r+\beta}$ maps on the interval C^0 with the $C^{r+\beta}$ norm is a Banach space. In this section we prove that the sequence (h_n) is a Cauchy sequence in this space and therefore converges. Firstly, we prove the following lemma.

Lemma 11 Suppose $t \in \Sigma^0_{n+1}$ and $n > N_0$. Let M be one of the three subsets $D_{t,s}$, $G_{t,s}$ and $C_t \setminus \cup_s C_{t,s}$ where the union is over all those vertices in contact with t. Then, if r and β satisfy the conditions of theorem 1 and $0 \le \delta \le \beta$,

$$||h_{n+1} - h_n||_{C^{r+\delta}(M)} \le \mathcal{O}(g(n)e(n)^{r+\delta-1}).$$

The constants of the inequality only depend upon F and G.

Proof. We break the proof down into 3 cases corresponding to behaviour in the three subsets $C_t \setminus \cup_s C_{t,s}$, $D_{t,s}$ and $G_{t,s}$.

(i) *For $M = C_t \setminus \cup_s C_{t,s}$ where s runs over all those vertices in contact with t.* By proposition 2,

$$\begin{aligned} ||h_{n+1} - h_n||_{C^{r+\delta}} &= ||L_t - L_{m(t)}||_{C^{r+\delta}} \\ &\le \mathcal{O}_{r+\delta}(g(n)|dE_{m(t)}|^{r+\delta-1}) \end{aligned}$$

(ii) *For $M = D_{t,s}$.* Suppose s is on the left of t. Then

$$h_{n+1} - L_t = \psi_{t,s}L_t + (1 - \psi_{t,s})L_s - L_t = (1 - \psi_{t,s})(L_s - L_t)$$

in $C_{t,s}$. Therefore, by proposition 3 and inequality (3.5) we have that in $C_{t,s}$,

$$\begin{aligned} |d^k h_{n+1} - d^k L_t| &\le \sum_{l=0}^{k} a_l |d^l(1 - \psi_{t,s})||d^{k-l}(L_s - L_t)| \\ &\le \mathcal{O}(g(n+1)e_{t,s}^{k-1}) \end{aligned}$$

for all $0 \leq k \leq r+1$. The $a_l \geq 0$ are bounded independently of l and n. Moreover, by proposition 3 and inequality (3.5),

$$\begin{aligned}||d^r h_{n+1} - d^r L_t||_{C^\delta} &\leq \sum_{l=0}^{r} a_l |d^l(1-\psi_{t,s})||_{C^\delta}||d^{r-l}(L_s - L_t)||_{C^0} \\ &\quad + \sum_{l=0}^{r} a_l |d^l(1-\psi_{t,s})||_{C^0}||d^{r-l}(L_s - L_t)||_{C^\delta} \\ &\leq O_{r+\delta}(g(n+1)e_{t,s}^{r+\delta-1}).\end{aligned}$$

Therefore,

$$||h_{n+1} - L_t||_{C^{r+\delta}} \leq O(g(n+1)e_{t,s}^{r+\delta-1})$$

in $C_{t,s}$.

If $m(s) \neq m(t)$ then

$$h_n - L_{m(t)} = (1 - \psi_{m(t),m(s)})(L_{m(s)} - L_{m(t)}). \leq O(g(n)e_{m(t),m(s)}^{r+\delta-1})$$

by lemma 2 and the last inequality. Thus

$$\begin{aligned}||h_{n+1} - h_n||_{C^{r+\delta}} &\leq ||h_{n+1} - L_t||_{C^{r+\delta}} + ||L_t - L_{m(t)}||_{C^{k+\delta}} \\ &\quad + ||L_{m(t)} - h_n||_{C^{r+\delta}} \\ &\leq O\left(g(n)e(n)^{r+\delta-1}\right),\end{aligned}$$

in $C_{t,s}$.

If $m(s) = m(t)$ then $L_{m(t)} = h_n$ or there is a z in contact with $m(t)$ such that $h_n = \psi_{m(t),z}L_{m(t)} + (1 - \psi_{m(t),z})L_z$. In this case,

$$||L_{m(t)} - h_n||_{C^{r+\delta}} \leq O_{r+\delta}(g(n)e_{m(t),z}^{r+\delta-1}).$$

Therefore, in $C_{t,s}$,

$$||h_{n+1} - h_n||_{C^{r+\delta}} \leq O_{r+\delta}(g(n)e(n)^{r+\delta-1}).$$

A similar argument to that used for $C_{s,t}$ gives the same bounds for $D_{t,s}$.

(iii) *For* $M = G_{t,s}$. Suppose that C_t is on the right of C_s. By definition of the domains of the gap maps $h_{n+1} = h_n$ in the gap $G_{t,s}$ except in the extended intervals $E_{t,s}$ and $E_{s,t}$. In $E_{t,s}$,

$$\begin{aligned}h_{n+1} - h_n &= L_{m(t)}(\phi_{t,s} - 1) + L_t(1 - \phi_{t,s}) \\ &= (L_t - L_{m(t)})(1 - \phi_{t,s}).\end{aligned}$$

By proposition 2 and inequality (3.5),

$$\begin{aligned} \|h_{n+1} - h_n\|_{C^k} &\leq \sum_{l=0}^{k} a_l |d^l(L_t - L_{m(t)})||d^{k-l}(1-\phi_t)| \\ &\leq \mathcal{O}_k(g(n)|dE_{m(t)}|^{k-1}) \end{aligned}$$

for all $k = 0, \ldots, r+1$ and the $a_l > 0$ are bounded independently of l and n. Also,

$$\begin{aligned} \|d^r h_{n+1} - d^r h_n\|_{C^\delta} &\leq \sum_{l=0}^{r} a_l \|d^l(L_t - L_{m(t)})\|_{C^\delta} \|d^{k-l}(1-\phi_t)\|_{C^0} \\ &\quad + \sum_{l=0}^{r} a_l \|d^l(L_t - L_{m(t)})\|_{C^0} \|d^{k-l}(1-\phi_t)\|_{C^\delta} \\ &\leq \mathcal{O}_{r+\delta}(g(n)|dE_{m(t)}|^{r+\delta-1}). \end{aligned}$$

Similarly, in $E_{s,t}$,

$$\|h_{n+1} - h_n\|_{C^{r+\delta}} \leq \mathcal{O}_{r+\delta}(g(n)|dE_{m(t)}|^{r+\delta-1}). \blacksquare$$

Lemma 12 $(h_n)_{n>N}$ is a Cauchy sequence in the domain C^0 with respect to the $C^{r+\beta}$ norm.

Proof. Since C^0 is the union of the sets M of the form $D_{t,s}$, $G_{t,s}$ and $C_t \setminus \cup_s C_{t,s}$ where $s, t \in \Sigma_n^0$ it follows that

$$\|h_{n+1} - h_n\|_{C^r(C^0)} \leq \sup_M \|h_{n+1} - h_n\|_{C^r(M)} \leq \mathcal{O}(g(n)e(n)^{r-1})$$

by the previous lemma. It therefore remains to prove that, if $H_n = h_{n+1} - h_n$, then there is $c > 0$ such that for all $x, y \in C^0$, $|d^r H_n(y) - d^r H_n(x)| \leq c|y-x|^\beta$.

Assume without lose of generality that $x < y$. If the interval $[x, y]$ is contained in the union of three or less intervals of the form $D_{t,s}$, $G_{t,s}$ and $C_t \setminus \cup_s C_{t,s}$ where $s, t \in \Sigma_n^0$ then

$$|d^r H_n(y) - d^r H_n(x)| < \mathcal{O}(g(n)e(n)^{r+\beta-1})|y-x|^\beta$$

by lemma 11.

Therefore, suppose that this is not the case. Then $[x, y]$ contains a cylinder C_t in its interior for some $t \in \Sigma_n^0$. Let $t_1, t_2 \in \Sigma_n^0$ be such that C_{t_1} (resp. C_{t_2}) is the leftmost (resp. rightmost) cylinder of this form which is contained in

$[x,y]$ and does not contain x (resp. y) in its interior. Let p_x (resp. p_y) be the left-hand (resp. right-hand) endpoint of C_{t_1} (resp. C_{t_2}). Then

$$\begin{aligned}\frac{|d^r H_n(y) - d^r H_n(x)|}{|y-x|^\beta} &\le \frac{|d^r H_n(y) - d^r H_n(p_y)| + |d^r H_n(p_y)|}{|y-x|^\beta} \\ &+ \frac{|d^r H_n(p_x)| + |d^r H_n(p_x) - d^r H_n(x)|}{|y-x|^\beta} \\ &\le \frac{|d^r H_n(y) - d^r H_n(p_y)|}{|y-p_y|^\beta} + \frac{|d^r H_n(p_y)|}{|C_t|^\beta} \\ &+ \frac{|d^r H_n(p_x)|}{|C_t|^\beta} + \frac{|d^r H_n(x) - d^r H_n(p_x)|}{|x-p_x|^\beta}.\end{aligned}$$

The first and last terms are bounded by $\mathcal{O}(g(n)e(n)^{r+\beta-1}$ by lemma 11. Moreover, by lemma 11 $|d^r H_n(p_x)| \le g(n)e(n)^{r-1}$. Therefore,

$$\frac{|d^r H_n(p_x)|}{|C_t|^\beta} \le \mathcal{O}\left(g(n)e(n)^{r+\beta-1}\right).$$

Thus,

$$|d^r H_n(y) - d^r H_n(x)| \le \mathcal{O}\left(g(n)e(n)^{r+\beta-1}\right)|y-x|^\beta.$$

Consequently, if $m \ge n$ then $||h_m - h_n||_{C^{r+\beta}(C^0)} \le \mathcal{O}(\sum_{j=n}^m f(j))$. Therefore h_m is a Cauchy sequence in the $C^{r+\beta}$ norm. This completes the proof of the lemma. ∎

3.2.6 The map h_∞ is a $C^{r+\beta}$ diffeomorphism.

Since, by lemma 12, the sequence h_n is a Cauchy sequence in $C^{r+\beta}(C^0)$, it converges to a map $\tilde{h}_\infty \in C^{r+\beta}(C^0)$.

Lemma 13 There exists a $C^{r+\beta}$ diffeomorphism h_∞ of C^0 onto D^0 which for all $\underline{\varepsilon} \in \Sigma^0$ and all $n \ge 0$ maps $C_{\underline{\varepsilon}|n}$ onto $D_{\underline{\varepsilon}|n}$.

Proof. By lemma 16, there exists $\varepsilon > 0$ such that for all $t \in \Sigma_n^0$, $|D_t|/|C_t| > \varepsilon$.

We consider separately the four sets $C_t \setminus C_{t,s}$, $D_{t,s}$, $E_{t,s}$ and $G_{t,s} \setminus E_{t,s}$.

(i) In $C_t \setminus C_{t,s}$. $dL_t = |D_t|/|C_t| > \varepsilon$.

(ii) In $D_{t,s}$. Suppose that s is on the left of t. Then by proposition 3 and inequality 3.5, in $D_{t,s}$,

$$\begin{aligned}
|dh_n| &= |\psi_{t,s}dL_t + d\psi_{t,s}L_t + (1-\psi_{t,s})dL_s - d\psi_{t,s}L_s| \\
&\geq |dL_s| - |d\psi_{t,s}(L_t - L_s) + \psi_{t,s}(dL_t - dL_s)| \\
&\geq |D_t|/|C_t| - O(g(n)) > \varepsilon - O(g(n)).
\end{aligned}$$

(iii) In $E_{t,s}$. Suppose t is on the left of s. Then by proposition 2 and inequality 3.5, in $E_{t,s}$,

$$\begin{aligned}
|dh_n| &= |\phi_{t,s}dL_t + d\phi_{t,s}L_t + (1-\phi_{t,s})dL_{m(t)} - d\phi_{t,s}L_{m(t)}| \\
&\geq |dL_{m(t)}| - |d\phi_{t,s}(L_t - L_{m(t)}) + \phi_{t,s}(dL_t - dL_{m(t)})| \\
&\geq |D_{m(t)}|/|C_{m(t)}| - O(g(n)) > \varepsilon - O(g(n)) > 0.
\end{aligned}$$

(iv) In $G_{t,s} \setminus E_{t,s}$. For each $x \in G_{t,s} \setminus E_{t,s}$, $dh_n(x) = dh_{n-j}(x)$ for some $j > 0$.

Therefore there exists $p \geq 0$ and $s_i, t_i \in \Sigma^0_{n_i}$, $1 \leq i \leq p$ such that $\tilde{h}_\infty$ is a $C^{r+\beta}$ diffeomorphism outside of $G = \cup_{i=1}^{p} G_{s_i,t_i}$. If $G_{s_i,t_i} = [a_i, b_i]$ then $\tilde{h}_\infty(a_i) < \tilde{h}_\infty(b_i)$. Therefore, there exists a $C^{r+\beta}$ diffeomorphism h_∞ such that $h_\infty = \tilde{h}_\infty$ outside G.

By construction, $h_n(C_t) = D_t$ for all $t \in \Sigma^0_m$, $m \leq n$. Therefore $h_\infty(C_t) = \tilde{h}_\infty(C_t) = D_t$ for all $t \in \Sigma^0_n$ and all $n \geq 0$.

If $x \in C^n$ let $y \in C^0$ be such that $F_{n-1} \circ \cdots \circ F_0(y) = x$. Define $h^n_\infty(x) = G_{n-1} \circ \cdots \circ G_0 \circ h_\infty(y)$. Then, since h_∞ is $C^{r+\beta}$ then so is h^n_∞. Moreover, $h^n_\infty(\Lambda^n(F)) = \Lambda^n(G)$ since this is true for $n = 0$, and $G_n \circ h^n_\infty = h^{n+1}_\infty \circ F_n$ on $\Lambda^n(F)$. ∎

This completes the proof of theorem 1. It remains to prove propositions 2 and 3.

3.3 Preliminaries.

Lemma 14 If for each n M_n is an interval contained in $F_{n,\epsilon_n}(I^n) \cap G_{n,\epsilon_n}(J^n)$ then

$$||F^{-1}_{n,\epsilon_n} - G^{-1}_{n,\epsilon_n}||_{C^{s+\gamma}(M_{n+1})} \leq O(g(n))$$

Proof. This follows from the fact that on the appropriate domains the mapping $f \to f^{-1}$ is smooth. ∎

Lemma 15 Suppose that F is a bounded and boundedly extended Markov family.

(a) There exists $\lambda \in (0,1)$ such that, if $t \in \Sigma_n^0$, then $|I_t^0| \leq \mathcal{O}(\lambda^n)$.

(b) $|dE_t(y)|/|dE_t(x)|$ is bounded away from 0 and ∞ independently of t and $x, y \in I_t^0$.

(c) $|I_t^0| = \mathcal{O}(|dE_t(x)|^{-1})$; $|C_t|, |R(C_t, I_t)|, |L(C_t, I_t)|, |C_{t,s}|, |E_{t,s}| = \mathcal{O}(|I_t^0|)$; and $|D_{t,s}| = \mathcal{O}(e_{t,s}^{-1})$.

Proof. The proof of (a) follows directly from the boundedness of F.

Next we prove (b). If $t \in \Sigma_{n+1}^0$,

$$\ln \frac{|dE_t(y)|}{|dE_t(x)|} = \sum_{j=0}^{n-1} \log |dF_{j,e_j}(E_{m^{n-j}(t)}(y))| - \ln |dF_{j,e_j}(E_{m^{n-j}(t)}(x))|$$

where $E_{m^n(t)}$ denote the identity map. But $E_{m^{n-j}(t)}(x)$ and $E_{m^{n-j}(t)}(y)$ are in $I_{\sigma^j(t)}^{n-j}$. Therefore, $d(E_{m^{n-j}(t)}(y), E_{m^{n-j}(t)}(y)) < c\lambda^{-j}$ where c is a constant independent of n and t. Let

$$D = \sup_{j \geq 0} \frac{|d^2 F_{j,e_j}|}{|dF_{j,e_j}|}.$$

Then $D < \infty$ and

$$\log \frac{|dE_t(y)|}{|dE_t(x)|} \leq \sum_{j=0}^{n-1} cD\lambda^{-j}.$$

which is therefore bounded away from ∞. By symmetry it is bounded away from $-\infty$.

Part (c) follows directly from (b) because $E_t(C_t^0) = C_{e_n}^n$, $E_t(I_t^0) = I_{e_n}^n$ and $E_t(R(\Lambda_t^0, I_t^0)) = R(\Lambda_{e_n}^n, I_{e_n}^n)$. Therefore, by the mean value theorem, there exists $x, y \in I_t^0$ such that $|I_t^0| = |dE_t(x)|^{-1} |I_{e_n}^n|$ and

$$|R(\Lambda_t^0, I_t^0)| = |dE_t(y)|^{-1} |R(\Lambda_{e_n}^n, I_{e_n}^n)|$$

∎

Lemma 16 (a) $|D_t^0|/|C_t^0| = \mathcal{O}(1)$. (b) $|dL_t| = \mathcal{O}(1)$.

Proof. Since F and G are bounded there exists $c, d > 0$ and $\mu, \lambda \in (0,1)$ such that for all $t \in \Sigma_n^0$

$$c\mu^n < |I_t^0| < d\lambda^n.$$

Moreover, as noted above, $g(n) \leq \mathcal{O}(\tau^n)$ for some $\tau \in (0,1)$. Therefore, (a) follows directly from the results of [7].

Moreover, if $x, y \in I_t^0$ then $|dL_t(y)|/|dL_t(x)| = \mathcal{O}(1)$ by lemma 15 and the definition of L_t. But $L_t(I_t^0) = J_t^0$. Therefore, by the mean value theorem, there exists $x \in I_t^0$ such that $|dL_t(x)| = |J_t^0|/|I_t^0|$. Combining these results we deduce that $|dL_t(x)| = \mathcal{O}(1)$. ∎

Lemma 17 Suppose H_n is a sequence of C^{k+1} local diffeomorphisms such that $H^{(n)} = H_n \circ \cdots \circ H_1$ is well-defined. Then

$$\begin{aligned} d^k \ln dH^{(n)} &= \sum_{l=0}^{k-1}\sum_{i=0}^{n-1} d^{k-l} \ln dH_{i+1}(H^{(i)}) \cdot (dH^{(i)})^{k-l} \\ &\quad \cdot E_l^k(d \ln dH^{(i)}, \ldots d^l \ln dH^{(i)}) \end{aligned}$$

where $H^{(0)}$ denotes the identity map and $E_l^k = E_l^k(x_1, \ldots, x_l)$ is a polynomial of order l with coefficients which are independent of n and i and which satisfies the following conditions

(i) $E_0^k = 1$, $E_k^k = 0$ and $E_1^2(x_1) = x_1$.

(ii) For $l = 0, \ldots, k-1$,

$$\begin{aligned} E_{l+1}^{k+1}(x_1, \ldots, x_{l+1}) &= E_{l+1}^k(x_1, \ldots, x_{l+1}) \\ &\quad + (k-l)x_1 E_l^k(x_1, \ldots, x_l) + F_l^k(x_1, \ldots, x_{l+1}). \end{aligned}$$

(iii) $E_{l+1}^k(x_1, \ldots, x_{l+1})$ is a sum of monomials of the form

$$b_{i_1 \ldots i_j} x_{i_1}^{a_{i_1}} \ldots x_{i_j}^{a_{i_j}}$$

such that $a_{i_1} i_1 + \ldots + a_{i_j} i_j \leq l+1$ and $a_{i_1} + \ldots + a_{i_j} < k$ for all $l = 0, \ldots, k-2$;

(iv) $(k-l)x_1 E_l^k(x_1, \ldots, x_l)$ is a sum of monomials of the form

$$b_{i_1 \ldots i_j} x_1 x_{i_1}^{a_{i_1}} \ldots x_{i_j}^{a_{i_j}}$$

such that $1 + a_{i_1} i_1 + \ldots + a_{i_j} i_j \leq l+1$ and $1 + a_{i_1} + \ldots + a_{i_j} < k+1$ for all $l = 1, \ldots, k-1$.

(v) $F_l^k(x_1,\ldots,x_{l+1})$ is a sum of monomials of the form

$$b_{i_1\ldots i_j} i_m x_{i_1}^{a_{i_1}} \ldots x_{i_m}^{a_{i_m}-1} \ldots x_{i_j}^{a_{i_j}} x_{i_m+1}$$

such that, $a_{i_1} i_1 + \ldots + (a_{i_m}-1) i_m + \ldots + a_{i_j} i_j + i_m + 1 \le l+1$ and $a_{i_1} + \ldots + a_{i_j} + 1 < k+1$ for all $l = 1,\ldots,k-1$. Moreover, $F_0^k = 0$ and $E_{l+1}^{k+1}(x_1,\ldots,x_{l+1})$ is a sum of monomials of the form

$$b_{i_1\ldots i_j} x_{i_1}^{a_{i_1}} \ldots x_{i_j}^{a_{i_j}}$$

such that $a_{i_1} i_1 + \ldots + a_{i_j} i_j \le l+1$ and $a_{i_1} + \ldots + a_{i_j} < k+1$ for all $l = 0,\ldots,k-1$.

Proof. The proof which is by induction on k is omitted because it is straightforward. ∎

Lemma 18 Let G be bounded and boundedly extended $C^{s+\gamma}$ Markov families as above. For all $\underline{\varepsilon} \in \Sigma^0$ and all $0 \le j < n$ define the map $G_{\underline{\varepsilon}}^{j,n} : J_{\varepsilon_n}^n \to J_{\varepsilon_j\ldots\varepsilon_n}^j$ by

$$G_{\underline{\varepsilon}}^{j,n} = G_{j,\varepsilon_j}^{-1} \circ \ldots \circ G_{n-1,\varepsilon_{n-1}}^{-1}.$$

Let $G_{\underline{\varepsilon}}^{n,n}$ denote the identity map.

Then for all $x, y \in J^0$,

$$|\ln \frac{dG_{\underline{\varepsilon}}^{j,n}(y)}{dG_{\underline{\varepsilon}}^{j,n}(x)}| \le c|x-y|^\beta$$

where $\beta = \gamma$ if $s = 1$, or $\beta = 1$ if $s > 1$ and the constant c does not depend upon j, n and $\underline{\varepsilon}$.

Moreover,

$$dG_{\underline{\varepsilon}}^{j,n}(y) \in \exp(\pm c_3) dG_{\underline{\varepsilon}}^{j,n}(x).$$

Proof of lemma 18. By boundedness of G, by the medium value theorem and as $|dG_m^{-1}| < \lambda < 1$, for all $m \ge 0$. Then, for all $x, y \in J_{\varepsilon_n}^n$, there is $z_{x,y}^i \in J_{\varepsilon_n}^n$, such that

$$\begin{aligned}
|\ln \frac{dG_{\underline{\varepsilon}}^{j,n}(y)}{dG_{\underline{\varepsilon}}^{j,n}(x)}| &\le \sum_{i=j+1}^{n} (|\ln |dG_{i-1,\varepsilon_{i-1}}^{-1} \circ G_{\underline{\varepsilon}}^{i,n}(y)| - \ln |dG_{i-1,\varepsilon_{i-1}}^{-1} \circ G_{\underline{\varepsilon}}^{i,n}(x)||) \\
&\le c_1 \sum_{i=j+1}^{n} |G_{\underline{\varepsilon}}^{i,n}(y) - G_{\underline{\varepsilon}}^{i,n}(x)|^\beta \\
&\le c_1 \sum_{i=j+1}^{n} (dG_{\underline{\varepsilon}}^{i,n}(z_{x,y}^i))^\beta |y-x|^\beta \\
&\le c|x-y|^\beta \le c_3,
\end{aligned}$$

where the constant c_3 does not depend of j, n and $\underline{\varepsilon}$. Therefore,

$$dG_{\underline{\varepsilon}}^{j,n}(y) \in \exp(\pm c_3) dG_{\underline{\varepsilon}}^{j,n}(x). \blacksquare$$

Lemma 19 Let G be bounded and boundedly extended $C^{s+\gamma}$ Markov families as above. Then the norm $||\ln dG_{\underline{\varepsilon}}^{j,n}||_{C^k}$ of the map $\ln dG_{\underline{\varepsilon}}^{j,n}$ is bounded independently of j, n and $\underline{\varepsilon}$, for all $k = 0, \ldots, s-1$.

Proof of lemma 19. The case $k = 0$, it is proved by lemma 18. For $k \geq 1$, we will prove by induction in k that $d^k \ln dG_{\underline{\varepsilon}}^{j,n}$ is bounded in the C^0 norm independent of j, n and $\underline{\varepsilon}$.

Case $k = 1$. By lemma 18 and as $k \geq 2$,

$$|\ln \frac{dG_{\underline{\varepsilon}}^{j,n}(y)}{dG_{\underline{\varepsilon}}^{j,n}(x)}| \leq c|x-y|.$$

Therefore, $d \ln dG_{\underline{\varepsilon}}^{j,n}$ is bounded in the C^0 norm independent of j, n and $\underline{\varepsilon}$.

Induction step. By induction hypotheses, we suppose that the following maps $d \ln dG_{\underline{\varepsilon}}^{i,n}, \ldots, d^{k-1} \ln dG_{\underline{\varepsilon}}^{i,n}$ are bounded in the C^0 norm independent of i, n and $\underline{\varepsilon}$. We will prove that $d^k \ln dG_{\underline{\varepsilon}}^{j,n}$ is bounded in the C^0 norm independent of j, n and $\underline{\varepsilon}$.

By lemma 17

$$\begin{aligned} d^k \ln dG_{\underline{\varepsilon}}^{j,n} &= \sum_{l=0}^{k-1} \sum_{i=j+1}^{n} ((d^{k-l} \ln dG_{i-1,\varepsilon_{i-1}}^{-1} \circ G_{\underline{\varepsilon}}^{i,n}) \\ &\quad (dG_{\underline{\varepsilon}}^{i,n})^{k-l} E_l^k (d \ln dG_{\underline{\varepsilon}}^{i,n}, \ldots, d^l \ln dG_{\underline{\varepsilon}}^{i,n})) \end{aligned}$$

where the coefficients of the polynomial E_l^r are independent of i, n and $\underline{\varepsilon}$, for all $k \in \{1, \ldots, s-1\}$.

As the Markov family G is bounded then the first $k+1$-derivatives of the map $G_{i-1,\varepsilon_{i-1}}^{-1}$ are bounded independent of i and $|dG_{i-1,\varepsilon_{i-1}}^{-1}| > b^{-1} > 0$. Therefore,

$$|d^{k-l} \ln dG_{i-1,\varepsilon_{i-1}}^{-1} \circ G_{\underline{\varepsilon}}^{i,n}| \leq b_{r,l}, \tag{3.6}$$

for all $l = 0, \ldots, k-1$, $i = j+1, \ldots, n$ and all $0 \leq j < n$.

As the Markov family G is bounded then $|dG_{i,\varepsilon_i}^{-1}| < \lambda < 1$ and

$$|\sum_{i=j+1}^{n} (dG_{\underline{\varepsilon}}^{i,n})^{k-l}| \leq (\frac{1}{1-\lambda})^{k-l} \leq b_{k,l}, \tag{3.7}$$

for all $l = 0, \ldots, k-1$, $i = j+1, \ldots, n$ and all $0 \leq j < n$.

The induction hypotheses implies

$$|E_l^k(d \ln dG_{\underline{\varepsilon}}^{i,n}, \ldots, d^l \ln dG_{\underline{\varepsilon}}^{i,n})| \leq b_{k,l}, \tag{3.8}$$

for all $l = 0, \ldots, k-1$, $i = j+1, \ldots, n$ and all $0 \leq j < n$.

By lemma 17 and equations (3.6), (3.7) and (3.8)

$$|d^k \ln DG_{\underline{\varepsilon}}^{j,n}| \leq \sum_{l=0}^{k-1} b_{k,l} (\frac{1}{1-\lambda})^{k-l} \leq b_k. \blacksquare$$

Lemma 20 Let G be bounded and boundedly extended $C^{s+\gamma}$ Markov families as above. Then $||\ln dK_t||_{C^{s-1}}$ is bounded independently of t.

Proof of lemma 20. By definition $K_t = G_{\underline{\varepsilon}}^{0,n}$. Therefore, lemma 20 follows by lemma 19.

Lemma 21 For all $t = \varepsilon_0 \ldots \varepsilon_n \in \Sigma_{n+1}^0$ and $1 \leq k < s$,

$$(\mathrm{i})_k \quad ||d \ln dE_t||_{C^{k-1}} \leq \mathcal{O}_k(|dE_t|^k)$$

and

$$(\mathrm{ii})_k \quad ||dE_t||_{C^k} \leq \mathcal{O}_k(|dE_t|^{k+1}).$$

Proof. We firstly prove $(\mathrm{i})_k$ by induction on k. Consider the case $k = 1$. Since $E_0^1 = 1$, $E_1^1 = 0$ and

$$\ln dE_t = \sum_{i=0}^{n-1} \ln dF_{i,\varepsilon_i}(E_{m^{n-i}(t)}),$$

$$|d \ln dE_t| \leq \sum_{i=0}^{n-1} |d \ln dF_{i,\varepsilon_i}(E_{m^{n-i}(t)})||dE_{m^{n-i}(t)}|.$$

But $|dF_{i,\varepsilon_i}| > 1$ and $|d^2 F_{i,\varepsilon_i}|$ is bounded above. Therefore, $|d \ln dF_{i,\varepsilon_i}|$ is bounded above. Thus,

$$\begin{aligned}
|d \ln dE_t| &\leq c_1 \sum_{i=0}^{n-1} |dE_{m^{n-i}(t)}| \\
&\leq c_1 |dE_t| (\sum_{i=0}^{n-1} \left| d(F_{n-1,\varepsilon_{n-1}} \circ \ldots \circ F_{i,\varepsilon_i})(E_{m^{n-i}(t)}) \right|^{-1})
\end{aligned}$$

for some constant c_1. However, there exists $d > 0$ and $0 < \lambda < 1$ such that $|d(F_{n-1,\varepsilon_{n-1}} \circ \ldots \circ F_{i,\varepsilon_i})|^{-1} < d\lambda^{n-i}$. Consequently,

$$|d \ln dE_t| \leq \mathcal{O}(|dE_t|). \tag{3.9}$$

We now consider the case $k = 2$. Since $E_0^2 = 1$ and $E_2^2 = 0$,

$$\begin{aligned} |d^2 \ln dE_t| &\leq \sum_{l=0}^{1}\sum_{i=0}^{n-1} |d \ln dF_{i,\varepsilon_i}(E_{m^{n-i}(t)})| \cdot |dE_{m^{n-i}(t)}|^{2-l} \\ &\quad \cdot |E_l^2(d \ln dE_{m^{n-i}(t)}, \ldots, d^l \ln dE_{m^{n-i}(t)})|. \end{aligned}$$

Moreover, since $|d \ln dF_{i,\varepsilon_i}|$ is bounded above and

$$|E_1^2(d \ln dE_{m^{n-i}(t)})| = |d \ln dE_{m^{n-i}(t)}| \leq \mathcal{O}(|dE_{m^{n-i}(t)}|),$$

it follows that

$$\begin{aligned} |d^2 \ln dE_t| &\leq \sum_{l=0}^{1}\sum_{i=0}^{n-1} \mathcal{O}(|dE_{m^{n-i}(t)}|^{2-l})\mathcal{O}(|dE_{m^{n-i}(t)}|^l) \\ &\leq \sum_{i=1}^{n} \mathcal{O}(|dE_{m^{n-i}(t)}|^2) \\ &\leq \mathcal{O}(|dE_t|^2)\sum_{i=1}^{n} \mathcal{O}(|d(F_{n-1,\varepsilon_{n-1}} \circ \ldots \circ F_{i,\varepsilon_i})|^{-2}) \\ &\leq \mathcal{O}(|dE_t|^2). \end{aligned}$$

Now, as inductive hypothesis assume $(i)_l$ for $1 \leq l \leq k$ we prove that this implies $(i)_{k+1}$. We prove that $(i)_k$ implies $(i)_{k+1}$. By lemma 17, $a_{i_1}i_1 + \ldots + a_{i_j}i_j \leq l$ for all $l = 1, \ldots, k$. Moreover, by $(i)_k$ and lemma 17,

$$\begin{aligned} &E_l^{k+1}(d \ln dE_{m^{n-i}(t)}, \ldots, d^l \ln dE_{m^{n-i}(t)}) \\ &\leq \mathcal{O}_k(|dE_{m^{n-i}(t)}|^{a_{i_1}i_1+\ldots+a_{i_j}i_j}) \leq \mathcal{O}_k(|dE_{m^{n-i}(t)}|^l) \end{aligned}$$

where $0 < l \leq k$. Therefore,

$$\begin{aligned} &|d^{k+1} \ln dE_t| \\ &\leq \sum_{l=0}^{k}\sum_{i=0}^{n-1} |d \ln dF_{i,\varepsilon_i}(E_{m^{n-i}(t)})||dE_{m^{n-i}(t)}|^{k+1-l} \\ &\quad |E_l^{k+1}(d \ln dE_{m^{n-i}(t)}, \ldots, d^l \ln dE_{m^{n-i}(t)})| \\ &\leq \sum_{l=0}^{k}\sum_{i=0}^{n-1} \mathcal{O}(|dE_{m^{n-i}(t)}|^{k+1-l})\mathcal{O}(|dE_{m^{n-i}(t)}|^l) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \mathcal{O}(|dE_{m^{n-i}(t)}|^{k+1}) \\
&\le \mathcal{O}(|dE_t|^{k+1}) \sum_{i=1}^{n} \mathcal{O}(|d(F_{n-1,\epsilon_{n-1}} \circ \ldots \circ F_{i,\epsilon_i})|^{-(k+1)}) \\
&\le \mathcal{O}(|dE_t|^{k+1}).
\end{aligned}
$$

This completes the proof of (i). We now prove (ii) by induction in k. The case $k = 1$ follows from

$$|d^2 E_t| = |dE_t||d \ln dE_t| \le \mathcal{O}(|dE_t|^2).$$

As inductive hypotheses assume that $(\text{ii})_l$ holds for $1 \le l < k$. We then deduce $(\text{ii})_k$. By differentiation,

$$
\begin{aligned}
|d^{k+1} E_t| &= |\sum_{i=1}^{k} c_i d^i E_t d^{k+1-i} \ln dE_t| \\
&\le \sum_{i=1}^{k} \mathcal{O}(|dE_t|^i |dE_t|^{k+1-i}) \\
&\le \mathcal{O}(|dE_t|^{k+1})
\end{aligned}
$$

where the c_i are constants.

Thus,

$$||dE_t||_{C^k} \le \mathcal{O}_k(|dE_t|^{k+1}).$$

∎

Corollary 3 $d(\prod_{i=0}^{j} |d^{m_i} E_t|^{n_i}) \le \mathcal{O}(|dE_t|^{1+\sum_{i=0}^{j} m_i n_i})$.

Proof. By lemma 21, we have that $|d^{m_i} E_t|^{n_i} = \mathcal{O}(|dE_t|^{m_i n_i})$. Thus,

$$\prod_{i=0}^{j} |d^{m_i} E_t|^{n_i} \le \mathcal{O}(|dE_t|^{\sum_{i=0}^{j} m_i n_i}).$$

By differentiation,

$$
\begin{aligned}
d(\prod_{i=0}^{j} |d^{m_i} E_t|^{n_i}) &\le \mathcal{O}(\sum_{k=0}^{j} n_k |d^{m_k+1} E_t||d^{m_k} E_t|^{-1} (\prod_{i=0}^{j} |d^{m_i} E_t|^{n_i}) \\
&\le \sum_{k=0}^{j} \mathcal{O}(|dE_t|^{(m_k+1)-m_k+\sum_{i=0}^{j} m_i n_i}) \\
&\le \mathcal{O}(|dE_t|^{1+\sum_{i=0}^{j} m_i n_i}). \blacksquare
\end{aligned}
$$

3.4 Proposition 1

Before proceeding with proposition 1, we introduce some auxiliary notation that will be of great use throughout the proof of theorem 1.

1. Given a function S and integers $a_0, \dots, a_l \ge 0$ define $S_{a_0 \dots a_l}$ to be the function defined in the following way: Let $T_1 = S^{a_l}$, and for $0 < m \le l$ inductively define $T_{m+1} = S^{a_{l-m}} \cdot dT_m$. Let $S_{a_0 \dots a_l} = T_{l+1}$.
2. Given a function S as above and constants $A_{a_0 \dots a_l}$, for all integers $k, l > 0$ define
$$\chi_{l,k}(S) = \sum_{a_0 + \dots + a_l = k} A_{a_0 \dots a_l} S_{a_0 \dots a_l}.$$
3. Many of the functions that we encounter are of the form $\chi_{k,l}(S)$ and we will often use the notation without explicitly mentioning the coefficients $A_{a_0 \dots a_l}$.

Proposition 1 For all $t = \varepsilon_0 \dots \varepsilon_{n-1} \in \Sigma_n^0$

$$(i)\ ||A_{n-1,\varepsilon_{n-1}} - id||_{C^s(I^{n-1}_{\varepsilon_{n-1}})} \le \mathcal{O}(g(n))$$

$$(ii)\ ||K_{n-1,\varepsilon_{n-1}\varepsilon_n} - id||_{C^s(I^{n-1}_{\varepsilon_{n-1}\varepsilon_n})} \le \mathcal{O}(g(n))$$

$$(iii)\ ||d \ln dK_{n-1,\varepsilon_{n-1}\varepsilon_n}||_{C^{s-1}(I^n_{\varepsilon_{n-1}\varepsilon_n})} \le \mathcal{O}(g(n)).$$

In each case the constants of the inequality depend only upon s and the Markov families F and G.

Proof. Let A, K, F and G denote respectively $A_{n-1,\varepsilon_{n-1}}$, $K_{n1,\varepsilon_{n-1}\varepsilon_n}$, F_{n,ε_n} and G_{n,ε_n}. To prove part (i) we show

$$||A - id||_{C^1(I^{n-1}_{\varepsilon_{n-1}})} \le \mathcal{O}(g(n)) \tag{3.10}$$

and

$$d^k A = 0 \text{ for all } k \ge 2 \tag{3.11}$$

Equation (3.11) follows immediately since A is affine and equation (3.10) follows directly from the the fact that $C^{n-1}_{\varepsilon_{n-1}} = D^{n-1}_{\varepsilon_{n-1}} \pm \mathcal{O}(g(n))$.

We now prove part (ii) of the lemma in four steps.

Step 1. $||K - id||_{C^0(I^{n-1}_{\varepsilon_{n-1}\varepsilon_n})} \le \mathcal{O}(g(n))$.

Since $A = id + \psi$ where $|\psi| \leq \mathcal{O}(g(n))$ and $F = G + \phi$ where $|\phi| \leq \mathcal{O}(g(n))$,

$$K(x) - x = G^{-1}(G(x) + \phi(x) + \psi(G(x) + \phi(x)) - x$$

and therefore

$$\begin{aligned} ||K - id||_{C^0(M)} &\leq ||dG^{-1}||_{C^0(M)}||\phi + \psi(G + \phi)||_{C^0(M)} \\ &\leq \mathcal{O}(g(n)) \end{aligned} \tag{3.12}$$

where $M = I^{n-1}_{\epsilon_{n-1}\epsilon_n}$ since $||dG^{-1}||$ is bounded independently of n.

Note that, by equation (3.12),

$$||d^kG(K) - d^kG||_{C^0(I^{n-1}_{\epsilon_{n-1}\epsilon_n} \bigcap J^{n-1}_{\epsilon_{n-1}\epsilon_n})} \leq \mathcal{O}(g(n)) \tag{3.13}$$

since d^kG is bounded independently of n for $0 \leq k \leq s$.

Step 2. $||K - id||_{C^1(I^{n-1}_{\epsilon_{n-1}\epsilon_n})} \leq \mathcal{O}(g(n))$.

Firstly, note that

$$|\ln \frac{dG(x)}{dG(K(x))}| \leq |\frac{d^2G(z)}{dG(z)}||x - K(x)| \leq \mathcal{O}(g(n)) \tag{3.14}$$

by step 1. But, using $dG^{-1}(A(F)) = dG(K)^{-1}$,

$$dK = \frac{dA(F) \cdot dF}{dG(K)} \in \frac{dF}{dG(K)}(1 \pm \mathcal{O}(g(n))) \subset \frac{dG}{dG(K)}(1 \pm \mathcal{O}(g(n))).$$

Thus, by equation (3.14), $|dK - 1| = \mathcal{O}(g(n))$ and step 2 follows.

Step 3. $||K - id||_{C^2(I^{n-1}_{\epsilon_{n-1}\epsilon_n})} \leq \mathcal{O}(g(n))$.

By the hypotheses of the theorem $|dG(K)| > 1$ and d^2G is bounded independently of n. Therefore using equations (3.12), (3.13) and (3.14)

$$|d^2K| = |\frac{d^2F \cdot dG(K) - d^2G(K) \cdot dF \cdot dK}{(dG(K))^2} dA| \leq \mathcal{O}(g(n)).$$

Let A_1 and A_2 denote the following functions:

$$A_1 = d^2G(K) \cdot dF \cdot dK \text{ and } A_2 = d^2F \cdot dG(K).$$

Step 4. $||K - id||_{C^{k+1}} \leq \mathcal{O}_k(g(n))$ and $||A_2 - A_1||_{C^{k-1}} \leq \mathcal{O}_k(g(n))$.

We use the auxiliary definitions which were introduced at the beginning of this section. Step 4 is proved by induction on $k = 1, \ldots, s$. The case $k = 1$ was proved in steps 1, 2 and 3. The inductive hypothesis is:

$$||A_2 - A_1||_{C^{k-1}} \leq c_k g(n) \quad \text{and} \quad ||K - id||_{C^{k+1}} \leq d_k g(n) \tag{15_k}$$

where the constants c_k and d_k only depend upon k and the Markov families F and G. We prove (15_{k+1}).

The kth derivative $d^k A_1$ of A_1 is of the following form:

$$\sum_{l_1+l_2+l_3=k} a_{l_1 l_2 l_3} d^{l_1+2} G(K) \cdot d^{l_2+1} F \cdot \chi_{l_3+1,l_1}(dK) \tag{16}$$

where $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, $a_{l_1 l_2 0} = 1$ and $a_{l_1 l_2 l_3} > 0$. Moreover, $d^k A_2$ is of the form

$$\sum_{\Lambda} a_{l_1 l_2 l_3} d^{l_1+2} F \cdot d^{l_2+1} G(K) \cdot \chi_{l_3,l_2}(dK_{n,\varepsilon_n \varepsilon_{n+1}}) \tag{17}$$

where the sum is over all the set Λ consisting of those $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ such that $l_1 + l_2 + l_3 = k$ and where $l_3 = 0$ if $l_2 = 0$. By the inductive hypotheses, $||K - id||_{C^{k+1}} \leq d_{k+1} g(n)$. As $\chi_{l_3,l_2}(dK)$ and $\chi_{l_3+1,l_1}(dK)$ are polynomials in $dK \in 1 \pm \mathcal{O}(g(n))$ and $d^2 K, \ldots, d^{l_3+1} K \in \pm\mathcal{O}(g(n))$ and $l_3 \leq k$, then

$$\chi_{l_3,l_2}(dK) \leq 1 \pm \mathcal{O}(g(n)) \quad \text{and} \quad \chi_{l_3+1,l_1}(dK) \leq 1 \pm \mathcal{O}(g(n))$$

and, as $a_{l_1 l_2 0} = 1$,

$$\begin{aligned} |d^k A_2 - d^k A_1| \leq \; & | \sum_{l_1+l_2=k} d^{l_1+2} F \cdot d^{l_2+1} G(K) \cdot (1 \pm \mathcal{O}(g(n))) \\ & - \sum_{l_1+l_2=k} d^{l_1+2} G(K) \cdot d^{l_2+1} F \cdot (1 \pm \mathcal{O}(g(n)))|. \end{aligned}$$

Using the fact that $|d^l G|$ and $|d^l F|$ are bounded for $0 \leq l \leq s$ and equation (3.13) we deduce

$$\begin{aligned} |d^k A_2 - d^k A_1| \leq & \sum_{l_1+l_2=k} | \left(d^{l_1+2} G \pm \mathcal{O}(g(n))\right) \cdot d^{l_2+1} G(K) \cdot (1 \pm \mathcal{O}(g(n))) \\ & - d^{l_1+2} G(K) \cdot (d^{l_2+1} G \pm \mathcal{O}(g(n))) \cdot (1 \pm \mathcal{O}(g(n)))| \\ \leq & \sum_{l_1+l_2=k} |d^{l_1+2} G \cdot d^{l_2+1} G(K) - d^{l_1+2} G(K) \cdot d^{l_2+1} G| + \mathcal{O}(g(n)) \\ \leq & \; \mathcal{O}(g(n)). \end{aligned}$$

If $B = dG(K)$ then by (15_k), $|d^j B|$ is uniformly bounded for all $j \leq k$ independently of n and t. If $A = A_2 - A_1$ then $d^2 K = A/B^2$ whence $d^{k+2} K$ is a sum of terms of the form

$$\prod_{j=1}^{k+1} d^{i_j} S_j / B^{k+2}$$

where $S_j = A$ or B, $i_j \geq 0$, $\sum_{j=1}^{k+1} i_j = k$ and for some j, $S_j = A$. Since $|d^i A| = \mathcal{O}(g(n))$ for $0 \leq i \leq k$ this means that each term is $\leq \mathcal{O}(n)$. Therefore,

$$|d^{k+2}K| \leq \mathcal{O}_k(g(n))$$

which proves part (ii).

Let us prove, for all $k = 1, \ldots, s-1$

$$|d^k \ln dK| \leq \mathcal{O}_k(g(n)). \tag{18}$$

By part (ii), $|d \ln dK| = |d^2K/dK| \leq \mathcal{O}(g(n))$. The map $d^k \ln dK$ is a sum of terms of the form

$$\prod_{j=1}^{k} d^{i_j} S_j/(dK)^k$$

where $S_j = d^2K$ or dK, $i_j \geq 0$, $\sum_{j=1}^{k+1} i_j = k-1$ and for some j, $S_j = d^2K$. Since by part (ii) $|d^i d^2 K| = \mathcal{O}(g(n))$ for $0 \leq i \leq s-2$ this means that each term is $\leq \mathcal{O}(n)$. Therefore,

$$|d^k \ln K| \leq \mathcal{O}_k(g(n))$$

which completes the proof of the proposition. ■

Lemma 22 For all $t \in \Sigma_{n+1}^0$ and $1 \leq l+k < s$,

$$\chi_{l,k}(dA_{n-1,\epsilon_{n-1}}(E_t)dE_t) \leq \mathcal{O}(|dE_t|^{l+k}) \text{ and } \chi_{l,k}(dE_t) \leq \mathcal{O}(|dE_t|^{l+k}).$$

Proof. The map $dA_{n-1,\epsilon_{n-1}}$ is a constant map. Therefore, the proof follows from repeated applications of corollary 3. ■

Lemma 23 Let $S = S_n$ and $S' = S'_n$ be maps such that

$$||S - S'||_{C^i} \leq \mathcal{O}_i(e(n)^{i+1} g(n)) \text{ and } ||S||_{C^i} \leq \mathcal{O}_i(e(n)^{i+1}) \tag{19}$$

for all $0 \leq i < s$. Then,

$$||\chi_{l,k}(S') - \chi_{l,k}(S)||_{C^0} \leq \mathcal{O}_{l,k}(e(n)^{l+k}).$$

for all $1 \leq l+k < s$.

Proof. First, we note that the map $\chi_{l,k}(S)$ is equal to

$$\chi_{l,k}(S) = \sum_{\Gamma} B_{j_0\ldots j_k} \prod_{i=0}^{k} d^{j_i}(S^{a_i}) \tag{20}$$

where Γ is the set of $j_0 \ldots j_k$ such that $j_i \leq i$, $\sum_{i=0}^{k} j_i = k$, $\sum_{i=0}^{k} a_i = l$, $k + l < s$ and the constants $B_{j_0\ldots j_k}$ are bounded independent of n.

For all $j \geq 1$ and $b + j < s$

$$d^j(S^b) = d^j S \cdot S^{b-1} + \sum_{\Gamma} C_{b_0\ldots b_{j-1}} \prod_{i=0}^{j-1} (d^i S)^{b_i} \tag{21}$$

where Γ is the set of all $b_0 \ldots b_{j-1}$ such that $\sum_{i=0}^{j-1} b_i = b$ and $\sum_{i=0}^{j-1}(i+1)b_i = b + j$, $b_i \geq 0$ for all $i = 0, \ldots, k-1$ and there is $i > 0$ such that $b_i > 0$ and the constants $C_{b_0\ldots b_{j-1}}$ are bounded independently of n.

By equation (19),

$$\left(d^i S \pm \mathcal{O}_i(e(n)^{i+1} g(n))\right)^{b_i} \in (d^i S)^{b_i} \pm \mathcal{O}_{b_i,i}(e(n)^{b_i(i+1)} g(n)). \tag{22}$$

In equation (21), $\sum_{i=0}^{j-1}(i+1)b_i = b + j$ and by equation (19) and (22)

$$\begin{aligned}
d^j((S')^b) &= \left((d^j S) \pm \mathcal{O}_j(e(n)^{(j+1)} g(n))\right) \cdot \left(S^{b-1} \pm \mathcal{O}_b(e(n)^{(b-1)} g(n))\right) \\
&\quad + \sum_{\Gamma} C_{b_1\ldots b_{j-1}} \prod_{i=0}^{j-1} \left((d^i S)^{b_i} \pm \mathcal{O}_{b_i,i}(e(n)^{b_i(i+1)} g(n))\right) \\
&\in d^j S \cdot S^{b-1} \pm \mathcal{O}_{b,j}(e(n)^{j+1+b-1} g(n)) \\
&\quad + \sum_{\Gamma} C_{b_1\ldots b_{j-1}} \left(\prod_{i=0}^{j-1} (d^i S)^{b_i} \pm \mathcal{O}_{b,j}(e(n)^{\sum_{i=0}^{j-1} b_i(i+1)} g(n))\right) \\
&\in d^j(S^b) \pm \mathcal{O}_{b,j}(e(n)^{b+j} g(n)).
\end{aligned}$$

In equation (20), $\sum_{i=0}^{k} j_i + a_i = k + l$ and by equation (19) and (23)

$$\begin{aligned}
\chi_{l,k}(S') &\in \sum_{\Gamma} B_{j_0\ldots j_k} \prod_{i=0}^{k} \left(d^{j_i}(S^{a_i}) \pm \mathcal{O}_{a_i,j_i}(e(n)^{a_i+j_i} g(n))\right) \\
&\in \chi_{l,k}(S) \pm \mathcal{O}_{l,k}\left(e(n)^{\sum_{i=0}^{k} a_i+j_i} g(n)\right) \\
&\in \chi_{l,k}(S) \pm \mathcal{O}_{l,k}(e(n)^{l+k} g(n))). \blacksquare
\end{aligned}$$

Lemma 24 For all $t \in \Sigma^0_{n+1}$, define the maps $dE = dE_{m(t)}$, $K = K_{n-1,\epsilon_{n-1}\epsilon_n}$, $A = A_{n-1,\epsilon_{n-1}\epsilon_n}$, $S' = S'_n = dK(E)dE$ and $S = S_n = dA(E)dE$. Then

$$||\chi_{l,k}(S') - \chi_{l,k}(S)||_{C^0} \leq \mathcal{O}_{l,k}(dE^{l+k}).$$

for all $1 \leq l + k < s$.

Proof. For all $0 \leq i < s$, the map d^iS' is equal to

$$d^iS' = \sum_{l=0}^{i} c_l d^{1+l}K(E) \cdot \chi_{l+1,i-l}(dE),$$

where the constants c_l are bounded independently of n.

By lemma 22, $|\chi_{l+1,i-l}(dE)| \leq \mathcal{O}_i(|dE|^{i+1})$ and by proposition 1,

$$d^{1+l}K(E) \in d^{1+l}A(E) \pm \mathcal{O}_l(g(n)).$$

Moreover, $|dA - 1| \leq \mathcal{O}(e(n)^{i+1})$ and $|d^{l+1}A| = 0$, for $l > 0$.

Therefore, $|d^iS|\mathcal{O}_i(|dE|^{i+1})$ and

$$\begin{aligned} d^iS' &\in \sum_{l=0}^{i} c_l \left(d^{1+l}A(E) \cdot \chi_{l+1,i-l}(dE) \pm \mathcal{O}_i(|dE|^{i+1}g(n))\right) \\ &\in d^iS \pm \mathcal{O}_i(|dE|^{i+1}g(n)). \end{aligned}$$

By lemma 23 we obtain the result. ∎

3.5 First main proposition.

Proposition 2 For all $t = \epsilon_0 \ldots \epsilon_n \in \Sigma^0_{n+1}$, $n \geq 0$ and all $0 \leq k \leq r + 1$,

$$||L_t - L_{m(t)}||_{C^k(I^0_t)} \leq \mathcal{O}_k(|dE_{m(t)}|^{k-1}g(n)). \tag{23}$$

Moreover, for all $0 \leq \delta \leq \beta$

$$||L_t - L_{m(t)}||_{C^{r+\delta}(I^0_t)} \leq \mathcal{O}_k(|dE_{m(t)}|^{r+\delta-1}g(n)). \tag{24}$$

Proof. The proof is by induction on k. Firstly we consider the case $k = 0$.

Let A, K, E and C denote respectively $A_{n-1,\epsilon_{n-1}}$, $K_{n-1,\epsilon_{n-1}\epsilon_n}$, $E_{m(t)}$ and $K_{m(t)}$. By lemma 16 the product

$$dC(K(E)) \cdot dK(E) \cdot dE = dL_t$$

is bounded independently of n and t. Moreover, $|dK|$ is bounded by proposition 1. Therefore,

$$|dC(K(E)))| = \mathcal{O}(|dE|^{-1}). \tag{25}$$

For all $x \in I_t^0$,

$$|L_t(x) - L_{m(t)}(x)| = |C(K(y)) - C(A(y))|$$

where $y = E(x) \in I^{n-1}_{\epsilon_{n-1}\epsilon_n}$. But, by proposition 1, $|K(y) - A(y)| \leq \mathcal{O}(g(n))$, for all $y \in I^{n-1}_{\epsilon_{n-1}\epsilon_n}$. Thus by the mean value theorem and equation (25),

$$\begin{aligned} |L_t(x) - L_{m(t)}(x)| &\leq |dC(z)||K(y) - A(y)| \\ &\leq \mathcal{O}((|dE|^{-1}g(n))) \end{aligned}$$

where $z \in J_{\epsilon_{n-1}\epsilon_n}$. This proves the lemma for $k = 0$.

Now consider the case $k = 1$. By lemma 20, $d \ln dC \leq \mathcal{O}(1)$ and by equation (25)

$$|d^2C| = |dC||d \ln dC| \leq \mathcal{O}(|dE|^{-1}) \tag{26}$$

By proposition 1, equation (26) and (25)

$$\begin{aligned} |dL_t(x) - dL_{m(t)}(x)| &\leq |dC(K(E(x))) \cdot dK(E(x))) \cdot dE(x) \\ &\quad -dC(A(E(x))))dA(y)dE(x)| \\ &\leq |dC(K(y)) \cdot (1 \pm \mathcal{O}(g(n))) - dC(A)(y)| \\ &\quad \cdot|dA(y)| \cdot |dE(x)| \\ &\leq \big(|d^2C(z)| \cdot |K(y) - A(y))| + |dC(K(y))|\mathcal{O}(g(n))\big) \\ &\quad \cdot|dA(y)| \cdot |dE(x)| \\ &\leq \mathcal{O}(g(n)) \end{aligned} \tag{27}$$

where $y = E(x)$.

Now we prove the case $k = 2$. By differentiation,

$$|d^2L_t - d^2L_{m(t)}| \leq R|dE| + S|dE| + T$$

where

$$R = |dL_t \cdot d \ln dC(K(E)) \cdot dK(E) - dL_{m(t)} \cdot d \ln dC(A(E)) \cdot dA(E)|,$$

$$S = |d \ln dK(E)||dL_t| \quad \text{and} \quad T = |dL_t - dL_{m(t)}| \cdot |d \ln dE|.$$

Thus,

$$|R(x)| \leq |dL_t(x)| \cdot |d^2 \ln dC| \cdot |K(y) - A(y)| \cdot dK(E) + \mathcal{O}(g(n))$$

since $|dL_t - dL_{m(t)}(x)| \le \mathcal{O}(g(n))$ by equation (27) and $dK(y) = dA(y)(1 + \mathcal{O}(g(n)))$ by proposition 1 and because by lemma 16 and 20 $dL_{m(t)}$, $d\ln dC$ and dA are bounded. But $|d^2 \ln dC|$ is bounded independently of t and, by proposition 1, $|K(y) - A(y)| < \mathcal{O}(g(n))$. Thus, $|R(x)| \le \mathcal{O}(g(n))$.

Furthermore,

$$|S(x)| \le |d\ln dK(y)| \cdot |dL_t| \le \mathcal{O}(g(n))$$

by proposition 1 and lemma 16. Finally,

$$|T(x)| \le |d\ln dE(x)| \cdot |dL_t(x) - dL_{m(t)}(x)| \le |dE| \cdot \mathcal{O}(g(n))$$

by lemma 21 and equation (27). Therefore,

$$|d^2 L_t - d^2 L_{m(t)}| \le \mathcal{O}(|dE| g(n)).$$

We now complete the proof of the proposition by induction. As inductive hypothesis assume that

$$||d^{i+2} L_t - d^{i+2} L_{m(t)}||_{C^0} \le \mathcal{O}(|dE|^{i+1} g(n)) \qquad (28_k)$$

for $i = 0, \ldots, k-1$ and where the constant of the inequality depends only upon i. We prove it for $i = k$ and $k \le r - 1$.

A straightforward calculation gives that

$$\begin{aligned} d^{k+2} L_t - d^{k+2} L_{m(t)} &= \sum_{l_1+l_2+l_3=k} (a_{l_1 l_2 l_3} R_{l_1 l_2 l_3} + b_{l_1 l_2 l_3} S_{l_1 l_2 l_3}) \\ &\quad + \sum_{l_1+l_2=k} c_{l_1 l_2} T_{l_1 l_2} \end{aligned}$$

where the constants $a_{l_1 l_2 l_3}$, $b_{l_1 l_2 l_3}$ and $c_{l_1 l_2}$ are bounded and

$$\begin{aligned} R_{l_1 l_2 l_3} &= d^{l_1+1} L_t \cdot d^{l_2+1} \ln dC(K(E)) \cdot \chi_{l_3+1, l_2}(dK(E) \cdot dE) \\ &\quad - d^{l_1+1} L_{m(t)} \cdot d^{l_2+1} \ln dC(A(E)) \cdot \chi_{l_3+1, l_2}(dA(E) dE) \\ S_{l_1 l_2 l_3} &= d^{l_1+1} L_t \cdot d^{l_2+1} \ln dK(E) \cdot \chi_{l_3+1, l_2}(dE) \end{aligned}$$

and

$$T_{l_1 l_2} = (d^{l_1+1} L_t - d^{l_1+1} L_{m(t)}) \cdot d^{l_2+1} \ln dE.$$

By lemma 24

$$|\chi_{l_3+1, l_2}(dK(E) \cdot dE) - \chi_{l_3+1, l_2}(dA(E) dE)| \le \mathcal{O}_r(|dE|^{l_2+l_3+1} g(n)) \qquad (29)$$

and, since $d^{2+l_2}\ln dC$ is bounded by lemma 20,

$$|d^{l_2+1}\ln dC((K(E))) - d^{l_2+1}\ln dC((A(E)))| \leq \mathcal{O}(K(E) - A(E)) \leq \mathcal{O}(g(n)) \tag{30}$$

by proposition 1. Moreover, by the inductive hypothesis (28_k),

$$|d^{l_1+1}L_t - d^{l_1+1}L_{m(t)}| \leq \mathcal{O}(|dE|^{l_1+1}g(n)). \tag{31}$$

Since

$$\begin{aligned} R_{l_1l_2l_3} \;=\; & d^{l_1+1}L_t \cdot d^{l_2+1}\ln dC(K(E)) \cdot (\chi_{l_3+1,l_2}(dK(E)\cdot dE) \\ & -\chi_{l_3+1,l_2}(dA(E)\cdot dE)) + \chi_{l_3+1,l_2}(dA(E)\cdot dE) \\ & \cdot \Big[\Big(d^{l_2+1}\ln dC(K(E)) - d^{l_2+1}\ln dC(A(E))\Big)\cdot d^{l_1+1}L_t \\ & + d^{l_2+1}\ln dC(A(E))\cdot\Big(d^{l_1+1}L_t - d^{l_1+1}L_{m(t)}\Big)\Big] \end{aligned}$$

it follows immediately from equation (29), (30) and (31) and by lemma 20 and lemma 22, that

$$|R_{l_1l_2l_3}| \leq \mathcal{O}(|dE|^{l_1+l_2+l_3+1}g(n)). \tag{32}$$

To bound $S_{l_1l_2l_3}$, by lemma 22,

$$\chi_{l_3+1,l_2}(dE) \leq \mathcal{O}(|dE|^{l_2+l_3+1})$$

and by proposition 1,

$$d^{1+l_1}\ln dK \leq \mathcal{O}(g(n)).$$

It follows that

$$|S_{l_1l_2l_3}| \leq |d^{l_1+1}L_t|\cdot|d^{l_2+1}\ln dK(E)|\cdot|\chi_{l_3+1,l_2}(dE)| \leq \mathcal{O}(|dE|^{l_2+l_3+1}g(n)).$$

Moreover,

$$|T_{l_1l_2}| \leq |d^{l_1+1}L_{m(t)} - d^{l_1+1}L_t|\cdot|d^{l_2+1}\ln dE| \leq \mathcal{O}(g(n)|dE|^{1+l_1+l_2})$$

by the inductive hypothesis since $|d^{l_2+1}\ln dE| \leq \mathcal{O}(|dE|^{l_2+1})$ by lemma 21.

Putting this together gives

$$|d^{k+2}L_{m(t)} - d^{k+2}L_t| \leq \mathcal{O}(|dE|^{k+1}g(n)).$$

Therefore, we proved equation (23). Let us prove equation (24).

For all $x, y \in I_t^0$, by lemma 15 $|x-y| \leq \mathcal{O}(|dE|^{-1})$. Define the map $U = L_t - L_{m(t)}$. By equation (23) and medium value theorem, there is $z \in I_t^0$ such that

$$\begin{aligned} |d^r U(x) - d^r U(y)| &\leq |d^{r+1}U(z)||x-y|^{1-\delta}|x-y|^\delta \\ &\leq \mathcal{O}(|dE|^{r+\delta-1}g(n))|x-y|^\delta. \blacksquare \end{aligned}$$

Corollary 4 For all $t = \varepsilon_j \ldots \varepsilon_{j+n} \in \Sigma_{n+1}^j$, $n \geq 0$ and all $0 \leq k \leq r+1$,

$$||L_t - L_{m(t)}||_{C^k(I_t^j)} \leq \mathcal{O}_k(|dE_{m(t)}|^{k-1}g(n+j)).$$

Proof of corollary 4. The proof follows in the same way as the proof of proposition 2. $\blacksquare$

3.6 Second main proposition.

Proposition 3 If $t, t' \in \Sigma_n^0$ are in contact and $0 \leq k \leq r+1$ then

$$||L_t - L_{t'}||_{C^k} \leq \mathcal{O}(e_{t,t'}^{k-1}g(n)) \tag{33$_k$}$$

on $I_t \bigcap I_{t'}$ where the constant of the inequality depends only upon k and the Markov families F and G.

Proof. Let us suppose that $t|m = t'|m$ and $t(m+1) \neq t'(m+1)$. Then, the map $L_t = C_{t|m+1} \circ L_{\sigma^m(t)} \circ E_{t|m+1}$ and $L_{t'} = C_{t'|m+1} \circ L_{\sigma^m(t')} \circ E_{t'|m+1}$. Let C, L, L' and E denote respectively $C_{t|m+1}$, $L_{\sigma^m t}$, $L_{\sigma^m t'}$ and $E_{t|m+1} = E_{t'|m+1}$.

The proof of the lemma for $k = 0, 1$ and 2 follows directly from the following facts.

(i) By lemma 16, $|dL| = \mathcal{O}(1)$ and $|dL_t| = \mathcal{O}(1)$ i.e. is bounded from 0 and ∞ independently of t.

(ii) From this and by lemma 15, it follows that $|dC(L(E(x)))| = \mathcal{O}(|dE(x)|^{-1})$, for all $x \in I_{t|m}$.

(iii) By lemma 20, $|d \ln dC| \leq \mathcal{O}(1)$,

$$|d^2C| = |d \ln dC| \cdot |dC| = \mathcal{O}(|dC|)$$

(iv) By condition B(g), for $\delta = 0, 1, 2$ $||L - L'||_{C^\delta} \leq \mathcal{O}(e_{\sigma^m(t),\sigma^m(t')}^{\delta-1}g(n))$. Moreover, $|e_{\sigma^m(t),\sigma^m(t')}dE| \leq \mathcal{O}(e(n))$. By lemma 21, $|d \ln dE| \leq \mathcal{O}(dE)$.

We now prove (33_k) by induction on k for $2 \leq k \leq r+1$. The inductive hypothesis is

$$||L_t - L_{t'}||_{C^i} \leq \mathcal{O}(e_{t,t'}^{i-1} g(n)) \tag{34_k}$$

for $i = 0, \ldots, k-1$.

The derivative $d^k L_t$ has the following form.

$$\begin{aligned}
&\sum_{l_1+l_2+l_3=k-2} a_{l_1 l_2 l_3} \; d^{l_1+1} L_t \cdot d^{l_2+1} \ln dC(L(E)) \cdot \chi_{l_3+1,l_2}(dL(E) \cdot dE) \\
&\quad + \sum_{l_1+l_2+l_3=k-2} b_{l_1 l_2 l_3} d^{l_1+1} L_t \cdot d^{l_2+1} \ln dL(E) \cdot \chi_{l_3+1,l_2}(dE) \\
&\quad + \sum_{l_1+l_2=k-2} c_{l_1 l_2} d^{l_1+1} L_t \cdot d^{l_2+1} \ln dE,
\end{aligned}$$

where the constants $a_{l_1 l_2 l_3}$, $b_{l_1 l_2 l_3}$ and $c_{l_1 l_2}$ are bounded. The derivative $d^k L_{t'}$ can be represented similarly.

Thus,

$$\begin{aligned}
||d^k L_t - d^k L_{t'}|| \;\leq\; & \sum_{l_1+l_2+l_3=k-2} (a_{l_1 l_2 l_3} R_{l_1 l_2 l_3} + b_{l_1 l_2 l_3} S_{l_1 l_2 l_3}) \\
& + \sum_{l_1+l_2=k-2} c_{l_1 l_2} T_{l_1 l_2}
\end{aligned}$$

where

$$\begin{aligned}
R_{l_1 l_2 l_3} &= d^{l_1+1} L_t \cdot d^{l_2+1} \ln dC(L(E)) \cdot \chi_{l_3+1,l_2}(dL(E)dE) \\
&\quad - d^{l_1+1} L_{t'} \cdot d^{l_2+1} \ln dC(L'(E)) \cdot \chi^{l_3+1,l_2}(dL'(E)dE),
\end{aligned}$$

$$\begin{aligned}
S_{l_1 l_2 l_3} &= d^{l_1+1} L_t \cdot d^{l_2+1} \ln dL(E) \cdot \chi_{l_3+1,l_2}(dE) \\
&\quad - d^{l_1+1} L_{t'} \cdot d^{l_2+1} \ln dL'(E) \cdot \chi_{l_3+1,l_2}(dE)
\end{aligned}$$

and

$$T_{l_1 l_2} = d^{l_1+1} L_t \cdot d^{l_2+1} \ln dE - d^{l_1+1} L_{t'} \cdot d^{l_2+1} \ln dE.$$

The fact that

$$|R_{l_1 l_2 l_3}| \leq \mathcal{O}(e_{t,t'}^{l_1+l_2+l_3+1} g(n))$$

with the constant of the inequality only depending upon l_1, l_2 and l_3 follows from the following facts.

(i) By proposition 2, the maps $d^{l_1+1} L_t$ and $d^{l_1+1} L_{t'}$ are bounded above. Moreover, by induction hypotheses (34_k),

$$|d^{l_1+1} L_t - d^{l_1+1} L_{t'}| \leq \mathcal{O}(e(n)^{l_2} g(n)).$$

(ii) By lemma 20, the maps $d^{l_2+1}\ln dC$ and $d^{l_2+2}\ln dC$ are bounded above. Moreover, by condition B(g)

$$\begin{aligned} |d^{l_2+1}\ln dC(L(E)) - d^{l_2+1}\ln dC(L'(E))| &\leq |d^{l_2+2}\ln dC||(L(E)) - (L'(E))| \\ &\leq \mathcal{O}(e^{-1}_{\sigma^m(t),\sigma^m(t')}g(n)). \end{aligned}$$

(iii) Since by proposition 2, $d^i L_{t'}$ is bounded for $i = 1, \ldots, l_3$ and by lemma 22,

$$\chi_{l_3+1,l_2}(dL_{t'}(E)\cdot dE) \leq \mathcal{O}(e^{l_3+l_2+1}_{t,t'}).$$

By lemma 26,

$$|\chi_{l_3+1,l_2}(dL\circ EdE) - \chi_{l_3+1,l_2}(dL'\circ EdE)| \leq \mathcal{O}(e^{l_2+l_3+1}_{t,t'}g(n)).$$

Moreover, it easily follows from the following facts that

$$|S_{l_1 l_2 l_3}| \leq \mathcal{O}(e^{l_1+l_2+l_3+1}_{t,t'}g(n))$$

with the constant of the inequality only depending upon l_1, l_2 and l_3.

(i) By fact (i) above.

(ii) By lemma 25, $|d^{l_2+1}\ln dL'| \leq \mathcal{O}(e^{l_2+1}_{t,t',m})$ is bounded. Moreover,

$$|d^{l_2+1}\ln dL - d^{l_2+1}\ln dL'| \leq \mathcal{O}(e^{l_2+1}_{t,t',m}g(n)).$$

(iii) By lemma 22,

$$|\chi_{l_3+1,l_2}(dE)| \leq \mathcal{O}_{l_3,l_3}(|dE|^{l_2+l_3+1}_{-}).$$

Finally, the bound

$$T \leq \mathcal{O}_{l_1+1,l_2+1}(e^{l_1+l_2+1}_{\sigma^m(t),\sigma^m(t')}g(n)).$$

with the constant of the inequality only dependent upon l_1 and l_2, follows from the following facts.

(i) By fact (i) above.

(ii) By lemma 21,

$$|d^{l_2+1}\ln dE| \leq \mathcal{O}(|dE|^{l_2+2}).$$

To complete the proof of proposition 3, we prove bellow lemma 25 and 26.

∎

Lemma 25 With the notation of the previous proposition, if $L = L_{\sigma^m(t)}$ and $L' = L_{\sigma^m(t')}$ then

$$|d^l \ln dL - d^l \ln dL'| \le \mathcal{O}_l(e^l_{\sigma^m(t),\sigma^m(t')} g(n))$$

for all $1 \le l < k$ and $2 \le k \le r$. Moreover, $||d \ln dL'||_{C^{r-2}}$ is bounded.

Proof. The map $d^l \ln dL'$ is a sum of terms of the form

$$\prod_{j=1}^{l} d^{i_j} S_j / (dL')^l$$

where $S_j = d^2 L'$ or dL', $i_j \ge 0$, $\sum_{j=1}^{l+1} i_j = l - 1$ and for some j, $S_j = d^2 L'$.

By corollary 4, the maps $d^i L$ are bounded for all $i = 1, \dots, r$ and by lemma 16, $\mathcal{O}((dL)^l) = \mathcal{O}(1)$. Therefore, $d^l \ln dL'$ is bounded.

By condition $\mathrm{B}(g)$,

$$\frac{\prod_{j=1}^{l} d^{i_j} S_j}{(dL')^l} \in \frac{\prod_{j=1}^{l} d^{i_j} T_j \pm \mathcal{O}(g(n) e^l_{\sigma^m(t),\sigma^m(t')})}{(dL)^l \pm \mathcal{O}(g(n) e_{\sigma^m(t),\sigma^m(t')})}$$

where $T_j = d^2 L$ if $S_j = d^2 L'$, otherwise $T_j = dL$. By lemma 16, $\mathcal{O}((dL)^l) = \mathcal{O}(1)$. Therefore,

$$|d^l \ln L' - d^l \ln L| \le \mathcal{O}_k(g(n) e^l_{\sigma^m(t),\sigma^m(t')})$$

which completes the proof of the lemma. ∎

Lemma 26 With the notation of the proposition 3, let $S' = S'_n = dL'(E)dE$ and $S = S_n = dL(E)dE$. Then

$$||\chi_{l_3+1,l_1}(S') - \chi_{l_3+1,l_1}(S)||_{C^0} \le \mathcal{O}_{l_3+1,l_1}(e(n)^{l_3+l_1+1}).$$

Proof. Denote $l_3 + 1$ by l and l_1 by k. The map $d^i S'$ is equal to

$$d^i S' = \sum_{l=0}^{i} d_l d^{1+l} L'(E) \cdot \chi_{l+1,i-l}(dE),$$

where the constants d_l are bounded independently of n.

By lemma 22, $|\chi_{l+1,i-l}(dE)| \le \mathcal{O}_i(|dE|^{i+1})$. By condition $\mathrm{B}(g)$,

$$d^{1+l} L'(E) \in d^{1+l} L(E) \pm \mathcal{O}_l(e^l_{\sigma^m(t),\sigma^m(s)} g(n)).$$

As $e_{\sigma^m(t),\sigma^m(s)}|dE| \le e(n)$ and by the results above,

$$\begin{aligned} d^i S' &\in \sum_{l=0}^{i} d_l \left(d^{1+l} L(E) \cdot \chi_{l+1,i-l}(dE) \pm \mathcal{O}_i(e(n)^{i+1} g(n)) \right) \\ &\in d^i S \pm \mathcal{O}_i(e(n)^{i+1} g(n)). \end{aligned}$$

By lemma 22 and corollary 4, $|d^i S| \le \mathcal{O}_i(e(n)^{i+1})$. Therefore, by lemma 23 we obtain the result. ∎

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Bibliography

[1] O. Lanford, A numerical study of the likelihood of phase locking. In *Physica* **14D** 403-408.

[2] O. E. Lanford, A computer-assisted proof of the Feigenbaum conjectures. In *Bull. Am. Math. Soc.* **6** 427-434.

[3] B. D. Mestel, Computer assisted proof of universality for cubic critical maps of the circle with golden mean rotation number. In *Ph.D. Thesis Warwick University.*

[4] A. A. Pinto and D. A. Rand, A classification of $C^{1+\alpha}$ structures on embedded trees. Warwick Preprint.

[5] A.Pinto and D. A. Rand, Convergence of renormalisation, Markov families and rigidity of dynamical systems: A unification of smooth conjugacy and universality results. In preparation.

[6] D. A. Rand, Universality and renormalisation in dynamical systems. In *New Directions in Dynamical Systems* (ed. T. Bedford and J. Swift) pp. 1-56, (1988), CUP

[7] D A Rand, Global phase space universality, smooth conjugacies and renormalisation: 1. The $C^{1+\alpha}$ case. *Nonlinearity* **1** (1988) 181-202.

[8] D. Sullivan, Differentiable structures on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets. In *Nonlinear evolution and chaotic phenomena.* (eds. G. Gallavotti and P. Zweifel), Plenum, New York, 1988.

[9] D. Sullivan, Quasiconformal homeomorphisms in dynamics, topology and geometry. Preprint 1987

[10] D. Sullivan, On the structure of infinitely many dynamical systems nested inside or outside a given one. To appear in the AMS volume celebrating the Centenary of the American Mathematical Society, 1990.

Chapter 4

Symbolic Dynamics and Renormalisation.

Let F be a topological Markov family. Say $F_i \sim F_j$ if there are orientation preserving homeomorphisms $h : C^i \to C^j$ and $h' : C^{i+1} \to C^{j+1}$ such that $h(C_a^i) = C_a^j$ for all $a \in S_i$, $h'(C_a^{i+1}) = C_a^{j+1}$ for all $a \in S_{i+1}$ and $h' \circ F_i = F_j \circ h$.

We can always choose the S_i such that $S_i \cap S_j = \emptyset$ or $S_i = S_j$ and such that $S_i = S_j$ is equivalent to $F_i \sim F_j$. We always assume that the labelling S_i has this property.

We say that $j \overset{n}{\sim} k$ if and only if $F_{j+q} \sim F_{k+q}$, for all $0 \leq q < n$ and $j < k$.

The Markov family is *adapted* if whenever $S_i = S_j$ then $I_a^i = I_a^j$, for all $a \in S_i$.

In this chapter, we always consider that the Markov family F is adapted.

For all $m, n \geq 0$ and $t \in \Sigma_n^m$ we denote C_t^m by C_t since the dependence upon m is determined by t, whenever it will not be confusing. If there is a gap $G_{t,t'}$ between C_t and $C_{t'}$ we introduce a symbol $g_{t,t'} = g_{t',t}$ and denote by $\hat{\Sigma}_n^m$ the set consisting of these new symbols together with Σ_n^m. When we say that a statement is valid for all $t \in \hat{\Sigma}_n^m$, we mean that it is valid for all t and $g_{t,t'}$ in $\hat{\Sigma}_n^m$.

We denote by J and m the mappings $J : \hat{\Sigma}_n^l \to \hat{\Sigma}_{n-1}^{l+1}$ and $m : \hat{\Sigma}_n^l \to \hat{\Sigma}_{n-1}^l$ given by

$$J(t_0 \ldots t_{n-1}) = t_1 \ldots t_{n-1} \quad \text{and} \quad J(g_{t,t'}) = g_{J(t),J(t')},$$
$$m(t_0 \ldots t_{n-1}) = t_0 \ldots t_{n-2} \quad \text{and} \quad m(g_{t,t'}) = m(t).$$

Define the *scaling tree* $\sigma_m = \sigma_{F_m} : \cup_{n\geq 1}\hat{\Sigma}^m_n \to \mathbf{R}$ by

$$\sigma_m(t) = \frac{|C_t|}{|C_{m(t)}|}.$$

For all $j \overset{n}{\sim} k$ and all $t \in \hat{\Sigma}^j_i$ and $t \in \hat{\Sigma}^k_i$ and all $0 \leq i \leq n$, define

$$\mu_t = |1 - \frac{\sigma_j(t)}{\sigma_k(t)}| \text{ and } A_t = \sum_{\{t'\in\hat{\Sigma}^j_i : m(t')=m(t)\}} (\mu_{t'}|C^j_{t'}|).$$

(iv) For all $j \overset{n}{\sim} k$ and all contact words $t, s \in \Sigma^j_i$ and $t, s \in \Sigma^k_i$ and all $0 \leq i \leq n$ define

$$\mu_{t,s} = |1 - \frac{|C^j_t|}{|C^j_s|}\frac{|C^k_s|}{|C^k_t|}|.$$

4.1 Scale and contact determination.

Definition 22 A topological Markov family F is $(1+\alpha)$-*scale determined* if and only if it possesses the $(1+\alpha)$-scale property and for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{q=m}^{\infty} g(q) < \mathcal{O}(g(m))$, for all $m \geq 0$.

(ii) For all $j \overset{n}{\sim} k$, let $u = \min\{j, n\}$. For all $a \in S_j$,

$$\frac{|C^k_a|}{|C^j_a|} \in 1 \pm g(u) \quad \text{and} \quad \frac{|I^k_a|}{|I^j_a|} \in 1 \pm g(u).$$

(iii) For all $0 \leq i \leq n$ and all $t \in \hat{\Sigma}^j_i$,

$$\mu_t < g(u+i).$$

If $s, t \in \Sigma^j_i$ are not in contact and $m(s) \neq m(t)$ then

$$\mu_t |E_{t,s}|^{-\varepsilon} < g(u+i)$$

while if $m(s) = m(t)$ then

$$|E_{t,s}|^{-(1+\varepsilon)} A_t + |E_{t,s}|^{-\varepsilon}\mu_t < g(u+i).$$

Definition 23 A topological Markov family F is $(1+\alpha)$-*contact determined* if it possesses the $(1+\alpha)$-contact property and for all ε such that $0 \leq \varepsilon < \alpha < 1$ there exists a function $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{q=m}^{\infty} g(q) < \mathcal{O}(g(m))$, for all $m \geq 0$.

(ii) For all $j \overset{n}{\sim} k$, let $u = \min\{j, n\}$. For all $0 \leq i \leq n$ and $t, s \in \Sigma_i^j$ are in contact, then

$$\frac{\mu_{t,s}}{|D_{t,s}|^\varepsilon} < g(u+i).$$

Proposition 4 If F is a topological Markov family which is $(1+\alpha)$-scale determined and $(1+\alpha)$-contact determined then there is a $C^{1+\alpha^-}$ Markov family G with the following properties:

(i) $G_m = F_m$ in $K_m = K_m(F)$.

(ii) For all $j \overset{n}{\sim} k$, let $u = \min\{j, n\}$. Then there exist $C^{1+\alpha^-}$ diffeomorphisms $h : C^j \to C^k$ and $h' : C^{j+1} \to C^{k+1}$ such that

$$G_j \circ h = h' \circ G_k$$

in the set $C_n^j = \cup_{t \in \Sigma_n^j} C_t^j$. Moreover,

$$||h - id||_{C^{1+\varepsilon}(C_n^j)} \leq \mathcal{O}_\varepsilon(g(u)) \text{ and } ||h' - id||_{C^{1+\varepsilon}(C_n^j)} \leq \mathcal{O}_\varepsilon(g(u))$$

for all $0 < \varepsilon < \alpha$.

We always assume that the topological Markov family F is $(1+\alpha)$-scale determined and $(1+\alpha)$-contact determined in the following sections of this chapter.

4.2 The symbolic set.

We define the *symbolic set* Ω which indexes the set of topological Markov maps in the limit of the Markov family F. Let $\mathcal{S} = \{S_i\}_{i=0}^{\infty}$. Let $\Omega \subset \mathcal{S}^{\mathbf{Z}}$ denote the set of all bi-infinite sequences $\underline{s} = \ldots s_{-1}s_0s_1\ldots$ such that for all $s_i \in \mathcal{S}$ and $n \in \mathbf{Z}$ and all $m \geq n$ there exists a sequence $i \to \infty$ such that $s_n \ldots s_m$ is the index sequence corresponding to the sequence of Markov maps $F_{j_i} \ldots F_{j_i+m-n}$, i. e. $s_{n+k} = S_{j_i+k}$ for $0 \leq k \leq m-n$.

Fix $\underline{s} = \ldots s_{-1}s_0s_1\ldots \in \Omega$. Two sequences $j_i, n_i \to \infty$, when $i \to \infty$, are called *limiting* if $S_{j_i} \ldots S_{j_i+n_i-1} = s_0 \ldots s_{n_i-1}$.

4.3 The limit Markov family $F^{\underline{s}}$.

Lemma 27 Fix $\underline{s} \in \Omega$. Let j_i, n_i be limit sequences and $u_i = \min\{j_i, n_i\}$.

(i) The scaling tree $\sigma_{\underline{s}} : \cup_{n>0} \hat{\Sigma}^{\underline{s}}_{n} \to \mathbb{R}$ is well defined by

$$\sigma_{\underline{s}}(t) = \lim_{i\to\infty} \frac{|C_t^{j_i}|}{|C_{m(t)}^{j_i}|}$$

and it is independent of the limit sequence.

(ii) For all $t \in \hat{\Sigma}^{\underline{s}}_{l}$,

$$\frac{\sigma_{\underline{s}}(t)}{\sigma_{j_i}(t)} \in 1 \pm g(u_i + l).$$

Fix $\underline{s} \in \Omega$. Let j_i, n_i be limit sequences. Define $C^{\underline{s}}$ to be the limit of C^{j_i} in the sense that the extreme points of C^{j_i} converge to the extreme points of $C^{\underline{s}}$, when i tends to infinity. For all $n > 0$, let $\Sigma^{\underline{s}}_{n}$ denote the set of words $t = t_0 \dots t_{n-1}$ such that $t_i \in s_i$, for all $0 \le i < n$. Let j_i, n_i be limit sequences. Let i be large enough such that $n_i \ge n$. Define $C^{\underline{s}}_{t}$ by the limit of $C_t^{j_i}$ in the sense that the extreme points of $C_t^{j_i}$ converge to the extreme points of $C^{\underline{s}}_{t}$, when i tends to infinity. Define $I^{\underline{s}} = I^{j_i}$ and $I^{\underline{s}}_{a} = I^{j_i}_{a}$, for all $a \in S_{j_i}$. Let $g_{t,t'} \in \hat{\Sigma}^{\underline{s}}_{n}$ if $g_{t,t'} \in \hat{\Sigma}^{j_i}_{n}$. For all $t', t'' \in \Sigma^{\underline{s}}_{n}$ such that $m(t') = m(t'')$, $C^{\underline{s}}_{t'}$ is on the left of $C^{\underline{s}}_{t''}$ if and only if $C^{j_i}_{t'}$ is on the left of $C^{j_i}_{t''}$. Define $K^{\underline{s}} = \cap_{n>0} \cup_{t\in\Sigma^{\underline{s}}_{n}} C_t$ and $K^{\underline{s}}_{t} = K^{\underline{s}} \cap C_t$, for all $t \in \Sigma^{\underline{s}}_{n}$.

Lemma 28 For all $t \in \Sigma^{\underline{s}}_{n}$, the intervals $C^{\underline{s}}_{t}$ are well defined and are independent of the limit sequences $j_i, n_i \to \infty$. Moreover,

$$\frac{C^{\underline{s}}_{t}}{C^{j_i}_{t}} \in 1 \pm g(u_i)$$

Define the map $\sigma : \Omega \to \Omega$ such that $\sigma(\underline{s}) = \underline{v}$, where $v_i = s_{i+1}$, for all $i \in \mathbb{Z}$. The map σ^n is the composition of n maps σ.

Definition 24 The map $F_{\underline{s}} : K^{\underline{s}} \to K^{\sigma(\underline{s})}$ is defined by $F_{\underline{s}}(K^{\underline{s}}_{t}) = K^{\sigma(\underline{s})}_{J(t)}$, for all $t \in \Sigma^{\underline{s}}_{n}$ and all $n > 0$. In the same way, the map $F_{\sigma^m(\underline{s})} : K^{\sigma^m(\underline{s})} \to K^{\sigma^{m+1}(\underline{s})}$ is defined by $F_{\sigma^m(\underline{s})}(K_t^{\sigma^m(\underline{s})}) = K_{J(t)}^{\sigma^{m+1}(\underline{s})}$, for all $t \in \Sigma_n^{\sigma^m(\underline{s})}$, all $n > 0$ and all $m \in \mathbb{Z}$.

Proposition 5 Each map $F_{\sigma^m(\underline{s})}$ has a $C^{1+\alpha^-}$ extension to $I^{\sigma^m(\underline{s})}$ and $\sigma_{\underline{s}} : \cup_{n>0}\hat{\Sigma}_n^{\underline{s}} \to \mathbf{R}$ is the respective scaling function. We also denote the $C^{1+\alpha^-}$ extension by $F_{\sigma^m(\underline{s})}$. These extensions form a $C^{1+\alpha^-}$ Markov family $F^{\underline{s}} = (F_{\sigma^m(\underline{s})})_{m\in\mathbf{Z}}$.

Define the map $f_{\underline{s}} : \mathbf{N} \to \mathbf{R}^+$ by

$$f_{\underline{s}}(l) = \max\{|C_t| : t \in \Sigma_l^{\underline{s}} \text{ or } t \in \Sigma_l^j \text{ and } S_j \ldots S_{j+l-1} = s_0 \ldots s_{l-1}\}.$$

Define the map $r_{\varepsilon,\underline{s}} : \mathbf{N} \times \mathbf{N} \to \mathbf{R}^+$ by

$$r_{\varepsilon,\underline{s}}(j,l) = (g_{\varepsilon''}(u))^{(\varepsilon''-\varepsilon')(\varepsilon'-\varepsilon)} + (f_{\underline{s}}(l))^{\varepsilon'-\varepsilon}$$

where $u = \min\{j,l\}$, $\varepsilon < \varepsilon' < \varepsilon'' < \alpha$ and the map $g_{\varepsilon''}$ is defined in $(1+\alpha)$-scale determination.

Let F and G be two $C^{1+\epsilon}$ Markov families. If $F_i \sim G_j$ and $I_a^{F_i} = I_a^{G_j}$, for all $a \in S_{F_i} = S_{G_j}$, define the map $f_a = F_{i,a} - G_{j,a}$ in $I_a^{F_i}$. For all set $M \subset I_a^{F_i}$, define

$$||f_a||_{C^{1+\epsilon}(M)} = \max_{x,y\in M}\{|f_a(x)| + |df_a(x)| + \frac{|df_a(x) - df_a(y)|}{|x-y|^{\epsilon}}\}.$$

For all set $N \subset I^{F_i}$, define the norm

$$||F_i - G_j||_{C^{1+\epsilon}(N)} = \max_{a\in S_{F_i}} ||f_a||_{C^{1+\epsilon}(N\cap I_a^{F_i})}.$$

Define

$$d_\epsilon(F_i,G_j) = \begin{cases} 1 & \text{if } F_i \overset{Not}{\sim} G_j. \\ ||F_i - G_j||_{C^{1+\epsilon}(I_a^{F_i})} & \text{otherwise.} \end{cases}$$

Theorem 22 Let F be a bounded Markov family which is $(1+\alpha)$-scale determined and $(1+\alpha)$-contact determined. For all $n \in \mathbf{Z}$, let $r_{\varepsilon,\sigma^n(\underline{s})}$ be the function as defined above. For all $j,l > 0$, such that $S_j \ldots S_{j+l-1} = s_n \ldots s_{n+l-1}$ and all $0 < \varepsilon < \alpha$

$$||F_j - F_{\sigma^n(\underline{s})}||_{C^{1+\epsilon}(K^j \cup K^{\sigma^n(\underline{s})})} \leq O_\epsilon(r_{\varepsilon,\sigma^n(\underline{s})}(j,l))$$

where c_ϵ is some constant which only depends upon ε.

We are going to suppose in what follows the following uniformity condition over the map $r_{\varepsilon,\sigma^n(\underline{s})}$. This is true, if for all $t \in \Sigma_n^m$ and all $m \geq 0$ the length of the intervals $|C_t|$ and $g_{\epsilon'}(n)$ decrease exponential fast to zero, when n tends to infinity.

Condition U: There is ν_ϵ between 0 and 1 such that $r_{\epsilon,\sigma^n(\underline{s})}(j,l) \leq O(\nu_\epsilon^{2l})$, for all $j \geq l > 0$ and all $n \in \mathbf{Z}$.

Let $F = (F_n)_{n\geq 0}$ and $G = (G_n)_{n\geq 0}$ be two $C^{1+\epsilon}$ Markov families.

Define the distance between F and G by

$$d_\epsilon(F,G) = \sum_{i=0}^{\infty} \nu_\epsilon^{|i|} d_\epsilon(F_i, G_i).$$

Corollary 5 Let F be a Markov family which is $(1+\alpha)$-scale determined and $(1+\alpha)$-contact determined and such that the maps $r_{\epsilon,\sigma^n(\underline{s})}$ satisfy condition U. For all $j \geq l > 0$, such that $S_j \ldots S_{j+l-1} = s_n \ldots s_{n+l-1}$ consider the Markov families $F^{(j)}$ and $\tilde{F}^{(j)}$ given respectively by $F^{(j)}_m = F_{j+m}$ and $\tilde{F}^{(j)}_m = F_{\sigma^{m+n}(\underline{s})}$. Then

$$d_\epsilon(F^{(j)}, \tilde{F}^{(j)}) \leq c_\epsilon \nu_\epsilon^l$$

if the Markov families are regarded as indexed by $m \geq 0$.

4.4 The scaling function.

Let Λ^- denote the set of all $\overline{\tau} = \ldots \tau_{-2}\tau_{-1}$ with the following property. There is $\underline{s} \in \Omega$ such that $\tau_{-n} \in s_{-n}$, for all $n > 0$. Denote $\tau_{-n} \ldots \tau_{-1}$ by $\overline{\tau}|n$. Define $\overline{\Lambda}_{\underline{s}} = \{\tau \in \Lambda^- : \tau_n \in s_n\}$. Define $\overline{\Lambda}_{g(\underline{s})}$ as the set of all $g_{\overline{\tau},\overline{\tau}'}$ with the following property. $\overline{\tau}, \overline{\tau}' \in \overline{\Lambda}_{\underline{s}}$, $g_{\tau_{-1},\tau'_{-1}} \in \hat{\Sigma}^{\sigma(\underline{s})}_{-1}$ and $\tau_{-i} = \tau'_{-i}$, for all $i > 1$. Let $\Lambda_{\underline{s}} = \overline{\Lambda}_{\underline{s}} \cup \overline{\Lambda}_{g(\underline{s})}$.

The *scaling function* $s_{\underline{s}} = s_{F,\underline{s}} : \Lambda_{\underline{s}} \to \mathbf{R}$ is given by

$$s_{\underline{s}}(\tau) = \lim_{n\to\infty} \sigma_{\sigma^{-n}(\underline{s})}(\overline{\tau}|n) \quad \text{and} \quad s_{\underline{s}}(g_{\tau,\tau'}) = \lim_{n\to\infty} \sigma_{\sigma^{-n}(\underline{s})}(g_{\overline{\tau}|n,\overline{\tau}'|n}).$$

Let $(F_m)_{m\geq 0}$ be $(1+\alpha)$-scale determined and the map $g : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ as in that definition. Define the metric in $\Lambda_{\underline{s}}$ as follows.

$$d(\overline{\tau}, \overline{\psi}) = g(n+1) \qquad \text{and} \qquad d(g_{\tau,\tau'}, g_{\psi,\psi'}) = g(n+1)$$

if $\overline{\tau}|n = \overline{\psi}|n$ and $\tau_{-(n+1)} \neq \psi_{-(n+1)}$. Moreover, $\tau_{-1} = \psi_{-1}$ and $\tau'_{-1} = \psi'_{-1}$. If necessary, interchange τ_{-1} and τ'_{-1}. Otherwise, the distance is $g(1)$.

Lemma 29 The scaling function $s_{\underline{s}}$ is well-defined and it is Lipschitz with respect to the metric d in $\Lambda_{\underline{s}}$.

Lemma 30 Let F and G be two $(1+\alpha)$-determined Markov families topologically conjugated. Let $F^{\underline{s}}$ and $G^{\underline{s}}$ be two limit Markov families corresponding to F and G respectively.

(i) If $F^{\underline{s}}$ and $G^{\underline{s}}$ are $(1+\alpha)$-equivalent then the scaling functions $s_{F,\sigma^m(\underline{s})}$ and $s_{G,\sigma^m(\underline{s})}$ are equal, for all $m \in \mathbb{Z}$.

(ii) Let $F^{\underline{s}}$ and $G^{\underline{s}}$ have bounded geometry. If, for all $m \in \mathbb{Z}$, the scaling functions $s_{F,\sigma^m(\underline{s})}$ and $s_{G,\sigma^m(\underline{s})}$ are equal, then $F^{\underline{s}}$ and $G^{\underline{s}}$ are C^{1+} conjugated.

4.5 The ω-limit set of a Markov family F

On Ω we put the metric $d_\Omega = d_{\Omega,\varepsilon}$, defined by

$$d_{\Omega,\varepsilon}(\underline{s},\underline{r}) = \sum_{i=-\infty}^{+\infty} \nu_\varepsilon^{|i|}\delta(s_i,r_i),$$

where $\delta(s_i,r_i)=0$ if $s_i=r_i$ or $\delta(s_i,r_i)=1$ if $s_i \neq r_i$.

Definition 25 The ω-limit space $\mathcal{M} = \mathcal{M}_{F,\varepsilon}$ of the Markov family F is defined as the set of all Markov families $F^{\underline{s}}$ as defined in proposition 5, i.e. $\mathcal{M} = \{F^{\underline{s}} : \underline{s} \in \Omega\}$. Define the metric $d_{\mathcal{M}} = d_{\mathcal{M},\varepsilon}$ on $\mathcal{M}$ by

$$d_{\mathcal{M},\varepsilon}(F^{\underline{s}},F^{\underline{r}}) = \sum_{i=-\infty}^{+\infty} \nu_\varepsilon^{|i|} d_\varepsilon(F_{\sigma^i(\underline{s})},F_{\sigma^i(\underline{r})}).$$

Corollary 6 The map $\mathcal{F}:\Omega \to \mathcal{M}$ defined by $\mathcal{F}(\underline{s}) = F^{\underline{s}}$ is bi-Lipschitz.

4.5.1 Periodic Markov families

Corollary 7 Let F be a p-periodic bounded Markov family which is $1+\alpha$-scale determined and $1+\alpha$-contact determined. Then there exists a unique limit family $F_{\underline{s}}$ which is topologically conjugated to F. Moreover, for all $\varepsilon \in (0,\alpha)$ there is a function $g = g_\varepsilon : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that $\sum_{n=q}^{\infty} g(n) \leq O_\varepsilon(g(q))$ and

$$||F_m - F_{\sigma^m(\underline{s})}||_{C^{1+\varepsilon}(K_m \cup K_m^{\sigma^m(\underline{s})})} \leq O_\varepsilon(g(m)).$$

Moreover, The set $\mathcal{M}$ is equal to $\mathcal{M} = \{F^{\sigma^j(\underline{s})} : j = 0,\dots,p-1\}$.

4.5.2 Affine branched limit Markov families.

A Markov family F is *affine branched (resp. polynomial branched)* if and only if all the maps F_m have an affine (resp. polynomial) branch $F_{m,\gamma_m} : C^m_{\gamma_m} \to C^{m+1}$ such that $F_{m,\gamma_m}(C^m_{\gamma_m}) = C^{m+1}$. For renormalisations of unimodal maps and circle maps the respective Markov families are affine, or polynomial to increase the velocity of convergence. In both cases the limit Markov families will be affine branched.

Lemma 31 If $F^{\underline{s}}$ is an affine branched Markov family then it is completely determined by the scaling functions $s_{\sigma^m(s)}$, for all $m \in \mathbb{Z}$.

Corollary 8 The Markov family $F^{\underline{s}}$ is the unique element of its $(1+\alpha)$-equivalence class of limit affine branched Markov families.

4.6 Applications.

It is conjectured a horseshoe picture for several dynamical systems. The points of the horseshoe are maps or families of maps and the operation is the renormalisation. Some examples of dynamical systems with these features are the diffeomorphisms of the circle, critical circle maps and quadratic foldings. The horseshoe picture is stated in Lanford [11] for quadratic circle maps and in Rand [26] for quadratic foldings. This picture gives us a better understanding of the renormalisation operator in these maps and their respective universal properties as mentioned in chapter 1. In this chapter, the tool that we use for the study of this phenomena are Markov families with $(1+\alpha)$-determination property associated to these dynamical systems. We prove convergence to a $\mathcal{M}$-limit set consisting of two-sided Markov families. We define a bi-Lipschitz map $\mathcal{F} : \Omega \to \mathcal{M}$, from a symbolic set Ω to $\mathcal{M}$. The set Ω is a subset of the set $\mathcal{H} = \{0, \ldots, N\}^{\mathbb{Z}}$, where $N \in \mathbb{N}$. Let $\sigma : \mathcal{H} \to \mathcal{H}$ be the shift operation defined by $\sigma(\phi) = \eta$, where $\eta_i = \phi_{i+1}$, for all $i \in \mathbb{Z}$. A horseshoe picture is clear in the set $\mathcal{H}$ with the shift operation. The set Ω is the ω-limit set of an orbit in $\mathcal{H}$. The bi-Lipschitz map $\mathcal{F}$ has the important property to carry the dynamics from the symbolic set Ω to the $\mathcal{M}$-limit set of two sided Markov families.

4.6.1 Diffeomorphism of the circle.

Let f be a diffeomorphism of the circle with constant rotation number and F the associated Markov family to f as defined in chapter 1. Suppose that F is $(1+\alpha)$-determined. The symbolic sequence of the Markov family F is given by the continued fraction expansion of the rotation number $\rho = \rho_1 \ldots$ of f. Define $\sigma^m(\rho) = \psi$, where $\psi_i = \rho_{m+i}$, for all $i \geq -m$ and ψ_i is arbitrary for $i < -m$. Endow the set $\mathcal{H}$ with the product of the discrete topologies. Define the symbolic set Ω_f as the set of $\theta \in \mathcal{H}$ such that there is a converging subsequence of $(\sigma^m(\rho))_{m>0}$. By corollary 6, there is a bi-Lipschitz map $\mathcal{F} : \Omega_f \to \mathcal{M}_f$, where $\mathcal{M}_f$ is the limit set of f consisting of two-sided Markov families. By the bi-Lipschitz map $\mathcal{F}$, the symbolic dynamics in Ω_f are carried on to the limit set $\mathcal{M}_f$. The Markov family F converges to $\mathcal{M}_f$ as proven in corollary 5. Stark [27] proves that if f is a $C^{2+\epsilon}$ diffeomorphism of the circle whose rotation number is of constant type then the renormalisation of f converges in the C^2 norm to the line of the rotations of the circle. By this fact and by theorem 22, the set $\mathcal{M}_f$ just depends upon the rotation number of f. Moreover, as the map $\mathcal{F}$ is bi-Lipschitz then the symbolic set Ω_f just depends upon the rotation number of f.

4.6.2 Critical circle maps.

Let f be a critical circle map and F the associated Markov family with $(1+\alpha)$-determination as defined in chapter 1. The symbolic sequence of the Markov family F is given by the continued fraction expansion of the rotation number $\rho = \rho_1 \ldots$. In the same way, as in the section on diffeomorphisms of the circle, we define the symbolic set Ω_f and by corollary 6, there is a bi-Lipschitz map $\mathcal{F} : \Omega_f \to \mathcal{M}_f$, where $\mathcal{M}_f$ is the limit set of f consisting of two-sided Markov families. By the bi-Lipschitz map $\mathcal{F}$, the symbolic dynamics on Ω_f are carried on to the limit set $\mathcal{M}_f$. The Markov family F converges to $\mathcal{M}_f$ in the sense of corollary 5. Let us assume that two Markov families F and G with the same rotation number converge exponentially fast to each other in a $C^{1+\epsilon}$ norm. This assumption, by Jonker [10], is verified for critical circle maps with periodic rotation number, in some open neighbourhood of analytic functions of $x|x|^\epsilon$, and $\varepsilon > 0$ small. By the assumption above and by theorem 22 the set $\mathcal{M}_f$ just depends upon the rotation number of f. Moreover, as the map $\mathcal{F}$ is bi-Lipschtiz then the symbolic set Ω_f just depends upon the rotation number of f.

4.6.3 Quadratic foldings.

There is a similar application for quadratic foldings of the interval, when we use sequences of different renormalisation operators. The renormalisation operator $R_n : \mathcal{D}_n \to \mathcal{D}$ is defined by $R_n(f) = a^{-1} f^n \circ a$, where $a = f^n(0)$ and the set $\mathcal{D}_n$ consists of all quadratic folding maps such that $R_n(f)$ is a quadratic folding map. Let f be an infinitely renormalisable quadratic folding with renormalisation sequence $\underline{a} = a_1 a_2 \ldots$. The symbolic sequence of the associated Markov family F is completely determined by the renormalisation sequence $\underline{a}$. Suppose that F is $(1+\alpha)$-determined. Define $\sigma^m(\underline{a}) = \underline{b}$, where $b_i = a_{m+i}$, for all $i \geq -m$ and b_i is arbitrary, for all $i < -m$. Define the symbolic set Ω_f as the set of $\theta \in \mathcal{H}$ such that there is a converging subsequence $(\sigma^m(\underline{a}))_{m>0}$. By corollary 6, there is a bi-Lipschitz map $\mathcal{F} : \Omega_f \to \mathcal{M}_f$. By the bi-Lipschitz map $\mathcal{F}$, the symbolic dynamics on Ω_f are carried on to the limit set $\mathcal{M}_f$. The Markov family F converges to $\mathcal{M}_f$ as proven in corollary 5. We suppose that two Markov families F and G with the same renormalisation sequence converge exponentially fast to each other in a $C^{1+\epsilon}$ norm. In this case, by theorem 22, the set $\mathcal{M}_f$ just depends on the renormalisation sequence of f. Moreover, as the map $\mathcal{F}$ is bi-Lipschtiz then the symbolic set Ω_f just depends on the renormalisation sequence of f. Let f have renormalisation sequence $22\ldots$. Then, by Sullivan [30], the set $\mathcal{M}_f$ just depends on the renormalisation sequence of f.

4.7 Proofs.

Proof of proposition 4. Let $T^{(j)} = \cup_{m \geq 1} T^{(j)}_m$ be the tree such that $T^{(j)}_m$ is the set of m-cylinders and m-gaps in the domain of the map F_j and such that if $I \in T^{(j)}_m$ then $m(I)$ is the cylinder such that $I \subset m(I)$.

Consider the finite tree $\hat{T}^{(j)}_n = \cup_{1 \leq m \leq n} T^{(j)}_m$. If $j \overset{n}{\sim} k$ then the embeddings $\hat{T}^{(j)}_n$ and $\hat{T}^{(k)}_n$ have the same topological structure. We can define the map $L_m : \mathrm{I}(T^{(j)}) \to \mathrm{I}(T^{(k)})$ as in the section 2.4 of chapter 1, for $m = 0, \ldots, n$ where $\mathrm{I}(T^{(j)}) = I^j$ and $\mathrm{I}(T^{(k)}) = I^k$.

Let $j \overset{n}{\sim} k$ and $u = \min\{j, n\}$. Then by definition 2, the map f_ϵ of lemma 6 of chapter 1 is $f_\epsilon(m) = g(m+u)$. Then, by lemma 6 of chapter 1,

$$||L_m - L_{m-1}||_{C^{1+\epsilon}} \leq \mathcal{O}(f_\epsilon(m-1)) = \mathcal{O}(g(u+m-1))$$

for $m < n$. Moreover, by condition (ii) of definition 2

$$||L_0 - id||_{C^{1+\epsilon}} \leq \mathcal{O}(g(u)).$$

Therefore,

$$||L_n - id||_{C^{1+\epsilon}} \leq O(g(u))$$

where the constants of the inequality only depend upon ε. Define $h = L_n$. Moreover, by a similarly construction we obtain h'. ∎

Notation. We introduce the following notation, with respect to the maps $F^{\underline{s}}$.

(i) For all $t \in \hat{\Sigma}_n^{\underline{s}}$ and all $n > 1$, define

$$\nu_t^{\underline{s}} = |1 - \frac{\sigma_{\underline{s}}(t)}{\sigma_{\sigma(\underline{s})}(J(t))}|.$$

(ii) For all $s, t \in \Sigma_n^{\underline{s}}$ and all $n > 1$ such that s and t are in contact, define

$$\nu_{s,t}^{\underline{s}} = |1 - \frac{|C_t^{\underline{s}}|}{|C_s^{\underline{s}}|}\frac{|C_{J(s)}^{\underline{s}}|}{|C_{J(t)}^{\underline{s}}|}|.$$

(iii) For all $t \in \Sigma_n^{\underline{s}}$ and all $n > 1$, define

$$A_t^{\underline{s}} = \sum_{\{t' \in \hat{\Sigma}_n^{\underline{s}} : m(t') = m(t)\}} \nu_{t'} |C_{t'}|.$$

Proof of lemma 27. Let j_i, n_i be limit sequences and define $u_i = \min\{j_i, n_i\}$. For all $t \in \hat{\Sigma}_n^{\underline{s}}$ and $n \geq 0$, let i be large enough, such that $n_i \geq n$. By condition (ii) of $(1+\alpha)$-scale determination, for all $q \geq p > 0$,

$$\sigma_{j_q}(t) \in \sigma_{j_p}(t)(1 \pm g(u_p + m)).$$

Therefore, the sequence $(\sigma_{j_i}(t))_{i>0}$ converges and by definition, the limit is $\sigma_{\underline{s}}(t)$. Moreover,

$$\frac{\sigma_{\underline{s}}(t)}{\sigma_{j_i}(t)} \in 1 \pm g(u_i + m).$$

Let l_i, m_i be another limit sequences and define $v_i = \min\{l_i, m_i, j_i, n_i\}$. For all $t \in \hat{\Sigma}_n^{\underline{s}}$ and $n \geq 0$, let i be large enough, such that $m_i \geq n$. By condition (ii) of $(1+\alpha)$-scale determination, for all $q \geq p > 0$,

$$\sigma_{j_i}(t) \in \sigma_{l_i}(t)(1 \pm g(v_i + m)).$$

Therefore, the sequence $(\sigma_{l_i}(t))_{i>0}$ converges to the same limit $\sigma_{\underline{s}}(t)$. ∎

Proof of lemma 28. Let j_i, n_i be limit sequences and define $u_i = \min\{j_i, n_i\}$. For all $\underline{t} \in \Sigma^{\underline{s}}$ and $n \geq 0$, let i be large enough, such that $n_i \geq n$. By condition (ii) of $(1+\alpha)$-scale determination, for all $q \geq p > 0$,

$$\begin{aligned} |C^{j_i}_{\underline{t}|n}| &= |C^{j_i}| \prod_{l=1}^{n} \sigma_{j_i}(\underline{t}|l) \\ &\in |C^{\underline{s}}|(1 \pm g(u_i)) \prod_{l=1}^{n} \left(\sigma_{\underline{s}}(\underline{t}|l)(1 \pm g(u_i + l))\right). \subset |C^{\underline{s}}_{\underline{t}|n}|(1 \pm cg(u_i)). \end{aligned}$$

For some constant $c > 0$. Therefore, $\lim_{i \to \infty} |C^{j_i}_{\underline{t}|n}| = |C^{\underline{s}}_{\underline{t}|n}|$. Moreover,

$$\frac{|C^{\underline{s}}_{\underline{t}|n}|}{|C^{j_i}_{\underline{t}|n}|} \in 1 \pm g(u_i). \blacksquare \tag{1}$$

Proof of proposition 5. We prove that $F^{\underline{s}} = (F_{\sigma^m(\underline{s})})_{m \in \mathbb{Z}}$ has $(1+\alpha)$-scale property and $(1+\alpha)$-contact property. By corollary 9, this proves that $F^{\underline{s}}$ has a $C^{1+\alpha^-}$ smooth extension. We use the fact that the Markov family F has $(1+\alpha)$-scale property and $(1+\alpha)$-contact property. We will verify the $(1+\alpha)$-scale property and $(1+\alpha)$-contact property for all $t \in \hat{\Sigma}^{\underline{s}}_n$ and all $n \geq 0$. In the same way, they are verified for all $t \in \hat{\Sigma}^{\sigma^m(\underline{s})}_n$, all $n \geq 0$ and all $m \in \mathbb{Z}$. By definition 24 and by lemma 27 and 28, $\sigma_{\sigma^m(\underline{s})} : \cup_{n>0} \hat{\Sigma}^{\sigma^m(\underline{s})}_n \to \mathbb{R}$ is the scaling function corresponding to $F_{\sigma^m(\underline{s})}$.

Let us prove that the Markov family $F^{\underline{s}}$ has $(1+\alpha)$-scale property.

Let j_i, n_i be limit sequences and $u_i = \min\{j_i, n_i\}$. For all $n > 0$ and all $t \in \hat{\Sigma}^{\underline{s}}_n$, let i be large enough, such that $n_i \geq n$. By lemma 27, $\sigma_{\underline{s}}(t) \in \sigma_{j_i}(t)(1 \pm g(u_i + n - 1))$. Therefore,

$$\frac{\sigma_{\underline{s}}(t)}{\sigma_{\sigma(\underline{s})}(J(t))} \in \frac{\sigma_{j_i}(t)}{\sigma_{j_i}(J(t))}(1 \pm cg(u_i + n - 1))$$

for some constant c. By the equation above and since the Markov family F has $(1+\alpha)$-scale property,

$$\begin{aligned} \nu^{\underline{s}}_t = |1 - \frac{\sigma_{\underline{s}}(t)}{\sigma_{\sigma(\underline{s})}(J(t))}| &\leq |1 - \frac{\sigma_{j_i}(t)}{\sigma_{j_i}(J(t))}| + c\frac{\sigma_{j_i}(t)}{\sigma_{j_i}(J(t))} g(u_i + n - 1) \\ &\leq \nu_{t_i} + c|1 + \nu_{t_i}| g(u_i + n - 1) \leq cg(n) \end{aligned} \tag{2}$$

where c is some constant.

For all $t, t' \in \Sigma^{\underline{s}}_n$ adjacent but not in contact, by condition (ii) of lemma 28 and by the construction of the sets $E^{\underline{s}}_{t,t'}$ and $E^{j_i}_{t,t'}$, then

$$|E^{\underline{s}}_{t,t'}| \in |E^{j_i}_{t,t'}|(1 \pm cg(u_i)), \tag{3}$$

for some constant c. By equation (3), for $\delta = \varepsilon$ or $\delta = 1 + \varepsilon$

$$|E^{\underline{s}}_{t,t'}|^{-\delta} \in |E^{j_i}_{t,t'}|^{-\delta}(1 \pm cg(u_i))^{-\delta} \subset |E^{j_i}_{t,t'}|^{-\delta}(1 \pm cg(u_i)), \tag{4}$$

where the constant c depends only upon ε.

By equations (2) and (4) and since the Markov family F has $(1+\alpha)$-scale property, for i large enough,

$$\begin{aligned} \nu^{\underline{s}}_t |E^{\underline{s}}_{t,t'}|^{-\varepsilon} &\leq (\nu_t + cg(u_i))|E^{j_i}_{t,t'}|^{-\varepsilon}(1 + cg(u_i)) \\ &\leq \nu_t |E^{j_i}_{t,t'}|^{-\varepsilon} + c|E^{j_i}_{t,t'}|^{-\varepsilon} g(u_i) \leq cg(n) \end{aligned}$$

where the constant c depends only upon ε.

For all $t, t' \in \Sigma^{\underline{s}}_n$ adjacent but not in contact, with $m(t) = m(t')$, define $\mathcal{B}^{\underline{s}}_t = \{r \in \hat{\Sigma}^{\underline{s}}_n : m(r) = m(t)\}$ which is equal to $\mathcal{B}^{j_i}_t = \{r \in \hat{\Sigma}^{j_i}_n : m(r) = m(t)\}$. By equations (1), (2) and (4)

$$\begin{aligned} |E^{\underline{s}}_{t,t'}|^{-(1+\varepsilon)} A^{\underline{s}}_t &= |E^{\underline{s}}_{t,t'}|^{-(1+\varepsilon)} \cdot \left(\sum_{\mathcal{B}^{\underline{s}}_t} \nu^{\underline{s}}_r |C^{\underline{s}}_r| \right) \\ &\leq |E^{j_i}_{t,t'}|^{-(1+\varepsilon)} (1 + c|E^{j_i}_{t,t'}|^{-(1+\varepsilon)} g(u_i)) \\ &\quad \cdot \left(\sum_{\mathcal{B}^{j_i}_t} \left(\nu_r (1 + cg(u_i)) |C^{j_i}_r| (1 + cg(u_i)) \right) \right) \\ &\leq |E^{j_i}_{t,t'}|^{-(1+\varepsilon)} A_t + c_1 g(u_i) \end{aligned}$$

where the constant c_1 depends only upon the cardinal of $\mathcal{B}^{\underline{s}}_t$ and ε. Moreover, as the Markov family F has $(1+\alpha)$-scale property, for i large enough

$$|E^{\underline{s}}_{t,t'}|^{-(1+\varepsilon)} A^{\underline{s}}_t \leq |E^{j_i}_{t,t'}|^{-(1+\varepsilon)} A_t + c_1 |E^{j_i}_{t,t'}|^{-(1+\varepsilon)} g(u_i) \leq cg(n),$$

where the constant c depends only upon ε.

Therefore, the Markov family $F^{\underline{s}}$ has $(1+\alpha)$-scale property. Let us prove that the Markov family $F^{\underline{s}}$ has $(1+\alpha)$-contact property.

For all contact words $t, t' \in \Sigma^{\underline{s}}_n$, by equation (1) and by the construction of the sets $C^{\underline{s}}_{t,t'}$ and $C^{j_i}_{t,t'}$,

$$|C^{\underline{s}}_{t,t'}| \in |C^{j_i}_{t,t'}|(1 \pm cg(u_i))$$

for some constant c. Therefore,

$$|C^{\underline{s}}_{t,t'}|^{-\varepsilon} \in |C^{j_i}_{t,t'}|^{-\varepsilon}(1 \pm cg(u_i)) \tag{5}$$

for some constant c which only depends upon ε.

By equations (1) and (5)

$$\begin{aligned}
|C^{\underline{s}}_{t,t'}|^{-\varepsilon}\nu^{\underline{s}}_{t,t'} &\leq |C^{\underline{s}}_{t,t'}|^{-\varepsilon}\frac{|C^{\underline{s}}_{t}|}{|C^{\underline{s}}_{t'}|}\frac{|C^{\sigma(\underline{s})}_{J(t')}|}{|C^{\sigma(\underline{s})}_{J(t)}|} \\
&\leq |C^{j_i}_{t,t'}|^{-\varepsilon}\cdot(1+cg(u_i))\cdot\frac{|C^{j_i}_{t}|}{|C^{j_i}_{t'}|}\frac{|C^{j_i}_{J(t')}|}{|C^{j_i}_{J(t)}|}\cdot(1+cg(u_i)) \\
&\leq |C^{j_i}_{t,t'}|^{-\varepsilon}\nu_{t,t'}(1+cg(u_i))
\end{aligned}$$

where the constant c depends only upon ε.

Moreover, as the Markov family F has $(1+\alpha)$-contact property, for i large enough

$$|C^{\underline{s}}_{t,t'}|^{-\varepsilon}\nu^{\underline{s}}_{t,t'} \leq |C^{j_i}_{t,t'}|^{-\varepsilon}\nu_{t,t'}(1+cg(u_i)) \leq cg(n)$$

where the constant c depends only upon ε.

Therefore, the Markov family $F^{\underline{s}}$ has $(1+\alpha)$-contact property. ∎

Proof of theorem 22. Take two sequences $j_i, l_i \to \infty$ when $i \to \infty$ with the following properties. Denote $u_i = \min\{j_i, l_i\}$ and $u = u_0$ and suppose that $u_{i+1} > u_i$, $j_0 = j$ and $l_0 = l$. Moreover, $S_{j_i}\ldots S_{j_i+l_i-1} = s_n \ldots s_{n+l_i-1}$. As in proposition 4, define the map $h_{j_i} : \mathrm{I}(T^{(j_i)}) \to \mathrm{I}(T^{(j_{i+1})})$ and the map $h'_{j_i} : \mathrm{I}(T^{(j_i+1)}) \to \mathrm{I}(T^{(j_{i+1}+1)})$. Define the map $H : \mathrm{I}(T^{(j)}) \to \mathrm{I}(T^{(\sigma^n(\underline{s})})$ and the map $H' : \mathrm{I}(T^{(j+1)}) \to \mathrm{I}(T^{(\sigma^{n+1}(\underline{s})})$ by

$$H = \lim_{i\to\infty} h_{j_i} \circ \ldots \circ h_{j_0} \quad\text{and}\quad H' = \lim_{i\to\infty} h'_{j_i} \circ \ldots \circ h'_{j_0}.$$

Take ε' and ε'', such that $0 < \varepsilon < \varepsilon' < \varepsilon'' < \alpha$. The two sets of $C^{1+\varepsilon''}$ functions defined on $\mathrm{I}(T^{(j)})$ and on $\mathrm{I}(T^{(j+1)})$, with the respect to the $C^{1+\varepsilon''}$ norm are Banach spaces. Therefore, by proposition 4, the maps H and H' are well defined and

$$||H - id||_{C^{1+\varepsilon''}} \leq c_{\varepsilon'}(g_{\varepsilon''}(u))^{\varepsilon''-\varepsilon'} \quad\text{and}\quad ||H' - id||_{C^{1+\varepsilon'}} \leq c_{\varepsilon'}(g_{\varepsilon''}(u))^{\varepsilon''-\varepsilon'}. \quad (6)$$

Denote $g = (g_{\varepsilon''})^{\varepsilon''-\varepsilon'}$. Define the map $F^a : \mathrm{I}(T^{(\sigma^n(\underline{s}))}) \to \mathrm{I}(T^{(\sigma^{n+1}(\underline{s}))})$ by $F^a = H'F_jH^{-1}$. Therefore, $F_{\sigma^n(\underline{s})} = F^a$ in the set $K^{\sigma^n(\underline{s})}_l$. By equation (6), $||F^a||_{C^{1+\varepsilon'}} \leq c_{\varepsilon'}$. Define $C^{\sigma^n(\underline{s})}_l = \cup_{t\in\Sigma^{\sigma^n(\underline{s})}_l} C^{\sigma^n(\underline{s})}_t$. We prove in two parts that,

$$||F_{\sigma^n(\underline{s})} - F_j||_{C^{1+\varepsilon}(C^{\sigma^n(\underline{s})}_l)} \leq c_\varepsilon r_{\varepsilon,\sigma^n(\underline{s})}(j,l).$$

In the first part, we prove that

$$||F^a - F_j||_{C^{1+\epsilon}(I^{\sigma^n(\underline{s})})} \le c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}.$$

In the second part, we prove that

$$||F_{\sigma^n(\underline{s})} - F^a||_{C^{1+\epsilon}(C_l^{\sigma^n(\underline{s})})} \le c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}.$$

Part one. $||F^a - F_j||_{C^{1+\epsilon}(I^{\sigma^n(\underline{s})})} \le c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}$.

By definition, $DF^a = DH'(F_jH^{-1})DF_j(H^{-1})DH^{-1}$. By equation (6), $|DH - 1| < c_{\epsilon'}g(u)$ and $|DH' - 1| < c_{\epsilon'}g(u)$. Therefore,

$$\begin{aligned} \left|\frac{DF^a}{DF_j}\right| &= \left|\frac{DH' \circ (F_jH^{-1})DF_j \circ H^{-1}DH^{-1}}{DF_j}\right| \\ &\le \left|\frac{DF_j \circ H^{-1}}{DF_j}\right| |1 + c_{\epsilon'}g(u)|. \end{aligned}$$

By hypotheses $||DF_j||_{C^{\epsilon'}} \le c_{\epsilon'}$ and $|DF_j|$ is bounded from zero. Thus,

$$\left|\frac{DF_j \circ H^{-1}}{DF_j}\right| \le 1 \pm c_{\epsilon'}(g(u))^{\epsilon'}.$$

Therefore,

$$|DF^a - DF_j| \le c_{\epsilon'}(g(u))^{\epsilon'}. \tag{7}$$

For all $x \in I_a^{\sigma^n(\underline{s})}$ and all $a \in s_n$, define e as one of the extreme points of the interval $I_a^{\sigma^n(\underline{s})}$. As the Markov family F is adapted, then $F^a(e) = F_j(e)$. Therefore, by equation (7)

$$|F^a(x) - F_j(x)| = \left|\int_e^x (DF^a(y) - DF_j(y))dy\right| \le c_{\epsilon'}(g(u))^{\epsilon'}.$$

Define the map $B : I^{\sigma^n(\underline{s})} \to I^{\sigma^{n+1}(\underline{s})}$ by $B = DF^a - DF_j$. For all $x, y \in I^{\sigma^n(\underline{s})}$, if $|x - y| \le g(u)$ then

$$\begin{aligned} \frac{|B(x) - B(y)|}{|x - y|^\epsilon} &\le \frac{|DF^a(x) - DF^a(y)| + |DF_j(x) - DF_j(y)|}{|x - y|^\epsilon} \\ &\le c_{\epsilon'}|x - y|^{\epsilon'-\epsilon} \le c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}. \end{aligned}$$

If $|x - y| \ge g(u)$, then by equation (7)

$$\frac{|B(x) - B(y)|}{|x - y|^\epsilon} \le \frac{|B(x)| + |B(y)|}{|x - y|^\epsilon} \le c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}.$$

Therefore, $||F^a - F_j||_{C^{1+\epsilon}(I^{\sigma^n(\underline{s})})} \leq c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}$. Similarly, we do the same calculations in the set I^k with the map $F^b = H'^{-1}F_{\sigma^n(\underline{s})}H$. We obtain, $||F^b - F_{\sigma^n(\underline{s})}||_{C^{1+\epsilon}(I^k)} \leq c_{\epsilon'}(g(u))^{\epsilon'-\epsilon}$.

Part two. $||F_{\sigma^n(\underline{s})} - F^a||_{C^{1+\epsilon}(C_l^{\sigma^n(\underline{s})})} \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}$.

The map F^a and $F_{\sigma^n(\underline{s})}$ coincide in the extreme points of the cylinders $C_t^{\sigma^n(\underline{s})}$, for all $t \in \Sigma_l^{\sigma^n(\underline{s})}$. By the medium value theorem, there is $x', x'' \in C_t^{\sigma^n(\underline{s})}$, such that $DF_{\sigma^n(\underline{s})}(x') = DF^a(x'')$. Therefore, for all $x \in C_t^{\sigma^n(\underline{s})}$,

$$\begin{aligned}
&|DF_{\sigma^n(\underline{s})}(x) - DF^a(x)| \\
&\leq |DF_{\sigma^n(\underline{s})}(x) - DF_{\sigma^n(\underline{s})}(x')| + |DF_{\sigma^n(\underline{s})}(x') - DF^a(x'')| \\
&\quad + |DF^a(x'') - DF^a(x)| \\
&\leq c_{\epsilon'}|x - x'|^{\epsilon'} + c_{\epsilon'}|x'' - x|^{\epsilon'} \\
&\leq c_{\epsilon'}|C_t^{\sigma^n(\underline{s})}|^{\epsilon'} \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'}.
\end{aligned}$$

For all $x \in C_t^{\sigma^n(\underline{s})}$ and all $t \in \Sigma_l^{\sigma^n(\underline{s})}$, define e as one of the extreme points of the interval $C_t^{\sigma^n(\underline{s})}$. Thus, $F_{\sigma^n(\underline{s})}(e) = F^a(e)$ and

$$|F_{\sigma^n(\underline{s})}(x) - F^a(x)| = |\int_e^x (DF_{\sigma^n(\underline{s})}(y) - DF^a(y))dy| \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'}.$$

Define the map $E : C^{\sigma^n(\underline{s})} \to C^{\sigma^{n+1}(\underline{s})}$ by $E = DF_{\sigma^n(\underline{s})} - DF^a$. For all $x, y \in C_l^{\sigma^n(\underline{s})}$, if $|x - y| \leq f_{\sigma^n(\underline{s})}(l)$ then

$$\begin{aligned}
\frac{|E(x) - E(y)|}{|x - y|^\epsilon} &\leq \frac{|DF_{\sigma^n(\underline{s})}(x) - DF_{\sigma^n(\underline{s})}(y)| + |DF^a(x) - DF^a(y)|}{|x - y|^\epsilon} \\
&\leq c_{\epsilon'}|x - y|^{\epsilon'-\epsilon} \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}.
\end{aligned}$$

If $|x - y| \geq f_{\sigma^n(\underline{s})}(l)$, then

$$\frac{|E(x) - E(y)|}{|x - y|^\epsilon} \leq \frac{|E(x)| + |E(y)|}{|x - y|^\epsilon} \leq c'_\epsilon(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}.$$

Therefore, $||F_{\sigma^n(\underline{s})} - F^a||_{C^{1+\epsilon}(C_l^{\sigma^n(\underline{s})})} \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}$. Similarly, we do the same calculations with the map $F^b = H'^{-1}F_{\sigma^n(\underline{s})}H$ in $C_l^k = \cup_{t\in\Sigma_l^k}C_t$. We obtain $||F_j - F^b||_{C^{1+\epsilon}(C_l^k)} \leq c_{\epsilon'}(f_{\sigma^n(\underline{s})}(l))^{\epsilon'-\epsilon}$.

Therefore,

$$||F_{\sigma^n(\underline{s})} - F_j||_{C^{1+\epsilon}(C_l^{\sigma^n(\underline{s})}\cup C_l^k)} \leq c_\epsilon r_{\epsilon,\sigma^n(\underline{s})}(j, l). \blacksquare$$

Proof of corollary 5. By theorem 22 and condition U,

$$d_\epsilon(F_{j+m}, F_{\sigma^{m+n}(\underline{s})}) \le c_\epsilon r_{\epsilon,\sigma^{m+n}(\underline{s})}(j+m, l-m) \le c_\epsilon \nu_\epsilon^{2(l-m)},$$

for all $m = 0, \ldots, l-1$. For $m \ge l$,

$$d_\epsilon(F_{j+m}, F_{\sigma^{m+n}(\underline{s})}) \le 1$$

Therefore,

$$d_\epsilon(F^{(j)}, \tilde{F}^{(j)}) \le c_\epsilon \sum_{m=0}^{l-1} \nu_\epsilon^m \nu_\epsilon^{2(l-m)} + \sum_{m=l}^{\infty} \nu_\epsilon^m \le c_\epsilon \nu_\epsilon^l. \blacksquare$$

Proof of lemma 29. For all $0 < n \le m$ and all $\overline{\tau}, g_{\overline{\tau},\overline{\tau}'} \in \Lambda_{\underline{s}}$, by $(1+\alpha)$-scale property of the Markov family F,

$$\begin{aligned} \frac{\sigma_{\sigma^{-m}(\underline{s})}(\overline{\tau}|m)}{\sigma_{\sigma^{-n}(\underline{s})}(\overline{\tau}|n)} &= \prod_{i=n}^{m+1} \frac{\sigma_{\sigma^{-(i+1)}(\underline{s})}(\overline{\tau}|(i+1))}{\sigma_{\sigma^{-i}(\underline{s})}(\overline{\tau}|i)} \\ &\in \prod_{i=n}^{m+1} (1 \pm \nu_{\overline{\tau}|(i+1)}) \\ &\subset 1 \pm \sum_{i=n}^{m+1} cg(i+1) \subset 1 \pm cg(n+1). \end{aligned}$$

Therefore, $s_{\underline{s}}(\overline{\tau})$ is well defined and

$$\frac{s_{\underline{s}}(\overline{\tau})}{\sigma_{\sigma^{-n}(\underline{s})}(\overline{\tau}|n)} \in 1 \pm cg(n+1) \tag{8}$$

Similarly, $s_{\underline{s}}(g_{\overline{\tau},\overline{\tau}'})$ is well defined and

$$\frac{s_{\underline{s}}(g_{\overline{\tau},\overline{\tau}'})}{\sigma_{\sigma^{-n}(\underline{s})}(g_{\overline{\tau}|n,\overline{\tau}'|n})} \in 1 \pm cg(n+1). \tag{9}$$

For all $\overline{\tau}, \overline{\psi} \in \Lambda_{\underline{s}}$, by equation (8) if $\overline{\tau}|n = \overline{\psi}|n$ and $\tau_{n+1} \ne \psi_{n+1}$ then

$$\begin{aligned} s_{\underline{s}}(\overline{\tau}) - s_{\underline{s}}(\overline{\psi}) &\in \sigma_{\sigma^{-n}(\underline{s})}(\overline{\tau}|n)(1 \pm cg(n+1)) \\ &\quad -\sigma_{\sigma^{-n}(\underline{s})}(\overline{\psi}|n)(1 \pm cg(n+1)) \\ &\subset \pm cg(n+1) \subset cd(\tau, \psi). \end{aligned}$$

Similarly, by equation (9), for all $g_{\overline{\tau},\overline{\tau}'}, g_{\overline{\psi},\overline{\psi}'} \in \Lambda_{\underline{s}}$, such that $\overline{\tau}|n = \overline{\psi}|n$ or $\overline{\tau}|n = \overline{\psi}'|n$,

$$|s_{\underline{s}}(g_{\overline{\tau},\overline{\tau}'}) - s_{\underline{s}}(g_{\overline{\psi},\overline{\psi}'})| \le cg(n+1) \le cd(\overline{\tau}, \overline{\psi}).$$

Moreover, for all $g_{\overline{\tau},\overline{\tau}'}, \overline{\psi} \in \Lambda_{\underline{s}}$

$$|s_{\underline{s}}(g_{\overline{\tau},\overline{\tau}'}) - s_{\underline{s}}(\overline{\psi})| \leq 2 \leq cd(g_{\overline{\tau},\overline{\tau}'}, \overline{\psi}). \blacksquare$$

Proof of lemma 30. Let us prove condition (i). By the definition of $(1+\alpha)$-scale conjugacy, for all $\overline{\tau} \in \Lambda_{\underline{s}}$,

$$\frac{\sigma_{F,\sigma^{-n}(\underline{s})}(\overline{\tau}|n)}{\sigma_{G,\sigma^{-n}(\underline{s})}(\overline{\tau}|n)} \in 1 \pm g(n).$$

Therefore, by equation (8) and (9) $s_{F,\underline{s}}(\overline{\tau}) = s_{G,\underline{s}}(\overline{\tau})$. In the same way, $s_{F,\sigma^m(\underline{s})}(\overline{\tau}) = s_{G,\sigma^m(\underline{s})}(\overline{\tau})$, for all $m \in \mathbf{Z}$.

Condition (ii) is proved in theorem 3 of C^{1+} Self-similarities and invariants in Markov partitions. $\blacksquare$

Proof of corollary 6. Let $\underline{s}, \underline{r} \in \Omega$. If

$$s_{-(n-1)} \dots s_0 \dots s_{m-1} = r_{-(n-1)} \dots r_0 \dots r_{m-1},$$

$s_{-n} \neq r_{-n}$ and $s_m \neq r_m$ then by definition of $\underline{s}$ and $\underline{r}$ there is j such that $S_{j+k} = s_{-(n-1)+k} = r_{-(n-1)+k}$ for all $k = 0, \dots, m+n-2$ and $j \geq m+n-2$. By theorem 22 and condition U,

$$\begin{aligned} & d_\epsilon(F_{\sigma^{-(n-1)+k}(\underline{s})}, F_{\sigma^{-(n-1)+k}(\underline{r})}) \\ \leq\ & d_\epsilon(F_{\sigma^{-(n-1)+k}(\underline{s})}, F_{j+k}) + d_\epsilon(F_{j+k}, F_{\sigma^{-(n-1)+k}(\underline{r})}) \\ \leq\ & c_\epsilon \nu_\epsilon^{2(m+n-2-k)}. \end{aligned}$$

Therefore, by definition of $d_{\mathcal{M},\epsilon}$ and $d_{\Omega,\epsilon}$

$$\begin{aligned} d_{\mathcal{M},\epsilon}(F^{\underline{s}}, F^{\underline{r}}) &\leq c_\epsilon(\sum_{i=-(n-1)}^{m-1} \nu_\epsilon^{|i|} \nu_\epsilon^{2(m-1-i)} + \nu_\epsilon^n + \nu_\epsilon^m \\ &\leq c_\epsilon(\nu_\epsilon^n + \nu_\epsilon^m) \leq c_\epsilon d_{\Omega,\epsilon}(\underline{s}, \underline{r}). \end{aligned}$$

On the other hand, by definition of $d_{\mathcal{M},\epsilon}$, $d_{\Omega,\epsilon}$ and d_ϵ,

$$\begin{aligned} d_{\Omega,\epsilon}(\underline{s}, \underline{r}) &\leq c_\epsilon(\nu_\epsilon^n + \nu_\epsilon^m) \\ &\leq c_\epsilon(\nu_\epsilon^n d_\epsilon(F_{\sigma^n(\underline{s})}, F_{\sigma^n(\underline{r})}) + \nu_\epsilon^m d_\epsilon(F_{\sigma^m(\underline{s})}, F_{\sigma^m(\underline{r})})) \\ &\leq c_\epsilon d_{\mathcal{M},\epsilon}(F^{\underline{s}}, F^{\underline{r}}). \blacksquare \end{aligned}$$

Proof of corollary 7. Define the function $g : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ as in $(1+\alpha)$-scale determination and $(1+\alpha)$-contact determination of the Markov family F.

By theorem 22 and as the Markov families F and $F^{\underline{s}}$ have the same sequence of indexes, we obtain corollary 7. ∎

Proof of lemma 31. As $F^{\underline{s}}$ is an affine Markov family then by definition of the scaling function, for all $t \in \hat{\Sigma}_n^{\sigma^m(\underline{s})}$,

$$\sigma_{\sigma^m(\underline{s})}(t) = s_{\sigma^{m+n}(\underline{s})}(\ldots \gamma_{m-1} t). \blacksquare$$

Proof of corollary 8. By condition (i) of lemma 30 and by lemma 31. ∎

4.8 Scale and contact properties.

Let F be a topological Markov family $(F_m)_{m\in\mathcal{R}}$, where $\mathcal{R} = \mathrm{Z}$ or $\mathcal{R} = \mathrm{Z}_{\geq 0}$. We introduce the following notation.

(i) For all $t \in \hat{\Sigma}_n^m$, all $n > 1$ and all $m \in \mathcal{R}$ define

$$\nu_t = |1 - \frac{\sigma_m(t)}{\sigma_{m+1}(J(t))}|.$$

(ii) For all $s, t \in \Sigma_n^m$, all $n > 1$ and all $m \in \mathcal{R}$ such that s and t are in contact define

$$\nu_{s,t} = |1 - \frac{|C_t^m|}{|C_s^m|}\frac{|C_{J(s)}^{m+1}|}{|C_{J(t)}^{m+1}|}|.$$

(iii) For all $t \in \Sigma_n^m$, all $n > 1$ and all $m \in \mathcal{R}$ define

$$\mathcal{B}_t^m = \{t' \in \hat{\Sigma}_n^m : m(t') = m(t)\} \quad \text{and} \quad A_t = \sum_{\mathcal{B}_t^m} \nu_{t'}|C_{t'}^m|.$$

Definition 26 A topological Markov family F has the $(1+\alpha)$-scale property, if for all ε such that $0 \leq \varepsilon < \alpha < 1$ and all $m \in \mathcal{R}$ there exist a function $g = g_\varepsilon : \mathrm{Z}_{\geq 0} \to \mathrm{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} g < \mathcal{O}(g(q))$, for all $q > 1$;

(ii) $|I_a^m|/|I_{J(a)}^{m+1}|$ and $|C_a^m|/|C_{J(a)}^{m+1}|$ are bounded from zero and infinity for all $a \in \hat{\Sigma}_1^m$.

(iii) for all $t \in \hat{\Sigma}_n^m$ and all $n > 1$, $\nu_t \leq g(n)$;

(iv) for all $t' \in \Sigma_n^m$ adjacent to t but not in contact with it, if $m(t') = m(t)$, then

$$A_t |E_{t,t'}^m|^{-(1+\varepsilon)} + \nu_t |E_{t,t'}^m|^{-\varepsilon} \leq g(n),$$

while if $m(t') \neq m(t)$, then

$$\nu_t |E_{t,t'}^m|^{-\varepsilon} \leq g(n).$$

Definition 27 A topological Markov family F has the $(1+\alpha)$-contact property if for all ε such that $0 \leq \varepsilon < \alpha < 1$ and all $m \in \mathcal{R}$ there exist a function $g = g_\varepsilon : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ with the following properties:

(i) $\sum_{n>1} g(n) < \infty$;

(ii) for all $t, t' \in \Sigma_n^m$ and all $n > 1$ such that t and t' are in contact,

$$\frac{\nu_{t,t'}}{|D_{t,t'}^m|^\varepsilon} < g(n).$$

Definition 28 A topological Markov family $(F_m)_{m \in \mathcal{R}}$ has the $(1+\alpha)$-property if and only if it has the $(1+\alpha)$-scale property and $(1+\alpha)$-contact property.

A $C^{1+\alpha^-}$ Markov family G is *weakly bounded*, if for all $0 < \varepsilon < \alpha$, there is a constant $b > 0$ such that,

$$||G_m||_{C^{1+\varepsilon}} < b,$$

for all $m \in \mathcal{R}$.

Corollary 9 A topological Markov family F with the $(1+\alpha)$-property defines a $C^{1+\alpha^-}$ weakly bounded Markov family G such that, $F_m = G_m$ in $K^m = K^m(F_m)$, for all $m \in \mathcal{R}$.

Proof of corollary 9. By the $(1+\alpha)$-property and by theorem 1 of chapter 1, there are $C^{1+\alpha^-}$ maps $G_t : C_t^m \to C_{J(t)}^{m+1}$, for all $t \in \Sigma_2^m$ such that $G_t = F_m$ in K_t^m. Define $G_m = G_t$ in C_t^m.

Corollary 10 Let F be a weakly bounded $C^{1+\alpha^-}$ Markov family. Then it has the $(1+\beta_1)$-property, if for all ε such that $0 < \varepsilon < \beta_1$, there exists β such that $0 < \varepsilon < \beta < \beta_1 \leq \alpha$ and for all $m \in \mathcal{R}$ there exist functions $g = g_{\beta,\varepsilon} : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} g(n) < O(g(q))$, for all $q \in \mathbf{Z}_{\geq 0}$;

(ii) for all $t \in \hat{\Sigma}_n^m$ and all $n > 1$, $|C_t^m|^\beta < g(n+1)$;

(iii) for all non-contact adjacent words $t, t' \in \Sigma_n^m$, if $m(t) = m(t')$ then

$$\frac{|C_t^m|^{1+\beta}}{|E_{t,t'}^m|^{1+\varepsilon}} < g(n),$$

while if $m(t) \neq m(t')$ then

$$\frac{|C_t^m|^{\beta}}{|E_{t,t'}^m|^{\varepsilon}} < g(n),$$

(iv) for all contact words $t, t' \in \Sigma_n^m$, $|D_{t,t'}^m|^{\beta-\varepsilon} < g(n)$.

Proof of 10. By theorem 2 of chapter 1.

More generally, we can consider families of functions g_m for each $m \in \mathcal{R}$, in the definitions 26 and 27 and in corollary 10. In this case we lose the weakly bounded condition in the Markov family F in corollary 9 and 10.

4.9 Scale and contact equivalence.

Let F and G be $C^{1+\alpha}$ Markov families $(F_m)_{m\in\mathcal{R}}$ and $(G_m)_{m\in\mathcal{R}}$ where $\mathcal{R} = \mathbf{Z}$ or $\mathcal{R} = \mathbf{Z}_{\geq 0}$. Let F and G be topologically conjugated. We introduce the following notation.

(i) For all $t \in \hat{\Sigma}_n^{F_m}$ and $t \in \hat{\Sigma}_n^{G_m}$, all $n > 1$ and all $m \in \mathcal{R}$ define

$$\eta_t = |1 - \frac{\sigma_{F_m}(t)}{\sigma_{G_m}(t)}|.$$

(ii) For all $t, s \in \Sigma_n^{F_m}$ and $t, s \in \Sigma_n^{G_m}$, all $n > 1$ and all $m \in \mathcal{R}$ such that t and s are in contact define

$$\eta_{s,t} = |1 - \frac{|C_t^{F_m}|}{|C_s^{F_m}|}\frac{|C_s^{G_m}|}{|C_t^{G_m}|}|.$$

(iii) For all $t \in \Sigma_n^{F_m}$, all $n > 1$ and all $m \in \mathcal{R}$ define

$$A_t = \sum_{t' \in \hat{\Sigma}_n^{F_m} : m(t') = m(t)} \eta_{t'} |C_{t'}^{F_m}|.$$

Definition 29 The Markov families F and G are $(1+\alpha)$-scale equivalent, if for all ε such that $0 \leq \varepsilon < \alpha < 1$ and all $m \in \mathcal{R}$ there exist functions $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n=q}^{\infty} g < \mathcal{O}(g(q))$, for all $q > 1$;

(ii) $|C_a^{F_m}|/|C_a^{G_m}|$ and $|I_a^{F_m}|/|I_a^{G_m}|$ are bounded from zero and infinity, for all $a \in \hat{\Sigma}_n^m$.

(iii) For all $t \in \hat{\Sigma}_n^{F_m}$ and all $n > 1$, then $\eta_t \leq g(n)$;

(iv) for all $t' \in \Sigma_n^{F_m}$ adjacent to t but not in contact with it, if $m(t') = m(t)$, then

$$A_t |E_{t,t'}^{F_m}|^{-(1+\varepsilon)} + \eta_t |E_{t,t'}^{F_m}|^{-\varepsilon} \leq g(n),$$

while if $m(t') \neq m(t)$, then

$$\eta_t |E_{t,t'}^{F_m}|^{-\varepsilon} \leq g(n).$$

Definition 30 The Markov families F and G are $(1+\alpha)$-contact equivalent if for all ε such that $0 \leq \varepsilon < \alpha < 1$ and all $m \in \mathcal{R}$ there exist functions $g = g_\varepsilon : \mathbf{Z}_{\geq 0} \to \mathbf{R}$ with the following properties:

(i) $\sum_{n>1} g(n) < \infty$;

(ii) For all $t, t' \in \Sigma_n^{F_m}$ and all $n > 1$ such that t and t' are in contact,

$$\frac{\eta_{t,t'}}{|D_{t,t'}^{F_m}|^\varepsilon} < g(n).$$

Definition 31 The Markov families F and G are $(1+\alpha)$-equivalent if and only if they are $(1+\alpha)$-scale equivalent and $(1+\alpha)$-contact equivalent.

Let F and G be Markov families. They are $C^{1+\alpha^-}$ conjugated, if for all $0 < \varepsilon < \alpha$, h_m is a $C^{1+\varepsilon}$ diffeomorphism and $F_m h_m = h_{m+1} G_m$ in K^{G_m}, for all $m \in \mathcal{R}$. Moreover, the conjugacy $(h_m)_{\mathcal{R}}$ is bounded, if for all $0 < \varepsilon < \alpha$, there is a constant $b_\varepsilon > 0$ such that $||h_m||_{C^{1+\varepsilon}} < b_\varepsilon$.

Corollary 11 Let F and G be Markov families. If they are $(1+\alpha)$-equivalent then they are $C^{1+\alpha^-}$ bounded conjugated.

Proof of corollary 11. By the definition of $(1+\alpha)$-equivalence and by theorem 1 of chapter 1.

Corollary 12 Let F and G be Markov families $C^{1+\alpha^-}$ conjugated. Then they are $(1+\beta_1)$-equivalent, if for all ε such that $0 < \varepsilon < \beta_1$, there exists β such that $0 < \varepsilon < \beta < \beta_1 \leq \alpha$ and there exist a function $g = g_{\beta,\varepsilon} : Z_{\geq 0} \to R$ with the following properties:

(i) $\sum_{n=q}^{\infty} g(n) < \mathcal{O}(g(q))$, for all $q \in Z_{\geq 0}$;

(ii) for all $t \in \hat{\Sigma}_n^{F_m}$ and all $n > 1$, $|C_t^{F_m}|^\beta < g(n+1)$;

(iii) for all non-contact adjacent words $t, t' \in \Sigma_n^{F_m}$, if $m(t) = m(t')$ then

$$\frac{|C_t^{F_m}|^{1+\beta}}{|E_{t,t'}^{F_m}|^{1+\varepsilon}} < g(n),$$

while if $m(t) \neq m(t')$ then

$$\frac{|C_t^{F_m}|^{\beta}}{|E_{t,t'}^{F_m}|^{\varepsilon}} < g(n),$$

(iv) for all contact words $t, t' \in \Sigma_n^{F_m}$, $|C_{t,t'}^{F_m}|^{\beta-\varepsilon} < g(n)$.

Proof of corollary 12. By theorem 2 of chapter 1.

More generally, we can consider families of functions g_m for each $m \in \mathcal{R}$, in the definitions 29 and 30 and in corollary 12. In this case we lose the bounded condition in the conjugacy h in corollary 11 and 12.

Chapter 5

Two-sided Markov Families

5.1 C^{1+} Self-similarities and invariants in Markov Partitions.

5.1.1 Notation.

Let F be a C^{1+} two-sided bounded Markov family, i.e. there is small $\varepsilon > 0$ such that F is a $C^{1+\varepsilon}$ bounded Markov family. For all $i \in \mathbb{Z}$, *the scaling tree* $\sigma_i = \sigma_{F_i} : \cup_{l\geq 1}\hat{\Sigma}_l^i \to \mathbb{R}$ is defined by

$$\sigma_i(t) = \frac{|C_t^i|}{|C_{m(t)}^i|}.$$

The Markov family F has bounded geometry if there is $\delta > 0$ such that $\sigma_i(t) > \delta$, for all $t \in \Sigma_n^i$, all $n \in \mathbb{N}$ and all $i \in \mathbb{Z}$.

5.1.2 Scale and contact properties.

Definition 32 (i) The Markov family F has the *scale property* if and only if there is $0 < \nu < 1$ and $c > 0$, such that for all word $t \in \hat{\Sigma}_n^m$, all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$

$$1 - c\nu^n < \frac{\sigma_m(t)}{\sigma_{m+1}(J(t))} < 1 + c\nu^n.$$

(ii) The Markov family F has *contact property* if and only if there is $0 < \nu < 1$ and $c > 0$ such that for all contact words $t, s \in \Sigma_n^m$, all $m \in Z$ and all $n \in N$

$$1 - c\nu^n < \frac{|C_t^m|}{|C_{J(t)}^{m+1}|} \frac{|C_{J(s)}^{m+1}|}{|C_s^m|} < 1 + c\nu^n.$$

Theorem 23 Let F be a topological Markov family with bounded geometry. F has scale property and contact property if and only if there is a C^{1+} bounded Markov family G such that $F_m = G_m$ in $K_m = K_m(F_m)$.

5.1.3 C^{1+} conjugacies between Markov families.

Let F and G be two C^{1+} bounded Markov families and topologically conjugated. Therefore, $\hat{\Sigma}_n^m = \hat{\Sigma}_n^{F_m} = \hat{\Sigma}_n^{G_m}$. A C^{1+} conjugacy $h = (h_m)_{m \in Z}$ between F and G is *bounded* if there is small $\varepsilon > 0$ and some constant $b > 0$, such that for all $m \in Z$,

$$||h_m||_{C^{1+\varepsilon}(C^m)} < b.$$

Definition 33 (i) The Markov families F and G are *scale equivalent* if and only if there is $0 < \nu < 1$ and $c > 0$, such that for all $t \in \hat{\Sigma}_n^m$, all $m \in Z$ and all $n \in N$

$$1 - c\nu^n < \frac{\sigma_{G_m}(t)}{\sigma_{F_m}(t)} < 1 + c\nu^n.$$

(ii) The Markov families F and G are *contact equivalent* if and only if there is $0 < \nu < 1$ and $c > 0$ such that for all contact words $t, s \in \Sigma_n^m$, all $m \in Z$ and all $n \in N$

$$1 - c\nu^n < \frac{|C_t^{G_m}|}{|C_t^{F_m}|} \frac{|C_s^{F_m}|}{|C_s^{G_m}|} < 1 + c\nu^n.$$

Theorem 24 Let F and G be two C^{1+} bounded Markov families topologically conjugated. They are scale equivalent and contact equivalent if and only if there is a C^{1+} bounded conjugacy between them.

5.1.4 The scaling family s of a two-sided Markov family F.

Definition 34 (i) $\overline{w} = \ldots \varepsilon_1 \varepsilon_0$ is a backward path of the Markov map F_m if and only if $w|n = \varepsilon_{n-1} \ldots \varepsilon_0 \in \Sigma_n^{m-n}$. The dual Σ_*^m of Σ^m is the set of all backward paths $\overline{w}$ of the Markov map F_m.

(ii) The scaling function $s_m : \Sigma_*^m \to \mathbf{R}$ is defined by

$$s_m(\overline{w}) = \lim_{n\to\infty} \sigma_{m-n}(w|n).$$

Denote by s the scaling family $s = (s_m)_{m\in\mathbf{Z}}$.

Theorem 25 Let F and G be two C^{1+} bounded Markov families topologicaly conjugated. They have the same scaling family s if and only if they are C^{1+} bounded conjugated.

Corollary 13 Let F and G be two $C^{k+\delta}$ bounded Markov families topologically conjugated. They have the same scaling family s if and only if they are $C^{k+\delta}$ bounded conjugated.

5.1.5 Existence and exponential convergence to renormalisation limit Markov families.

Definition 35 (i) For all $m \in \mathbf{Z}$ and $\overline{w}_m \in \Sigma_*^m$, let B^m and $B^{m,n}$ be intervals with the following properties. Their length is bounded away from zero and infinity. The interval $B^{m,0}$ is equal to C^m. B^m is the limit of $B^{m,n}$, when n tends to infinity. In the sense that the extreme points of the interval $B^{m,n}$ converge exponentially fast to the extreme points of B^m.

(ii) For all $\overline{w}_m \in \Sigma_*^m$, denote $\overline{w}_m|n$ by $w_{m,n}$ and $C^{m-n}_{w_{m,n}}$ by $C^{m,n}$. Define the map $A_{m,n} : B^{m,n} \to C^{m,n}$ as the affine map that sends $B^{m,n}$ onto $C^{m,n}$. The map $F_{m,n} : C^{m,n} \to C^m$ is defined by $F_{m,n} = F_{m-1} \circ \ldots \circ F_{m-n}$.

(iii) For all $m \in \mathbf{Z}$ and $w_m \in \Sigma_*^m$, define the n^{th}-*renormalisation Markov map* $R_m^n = R^n_{\overline{w}_m} F : B^{m,n} \to B^{m+1,n}$ by

$$R_m^n = A^{-1}_{m+1,n} \circ F^{-1}_{m+1,n} \circ F_m \circ F_{m,n} \circ A_{m,n}.$$

Let $R^n = (R_m^n)_{m\in\mathbf{Z}}$. By construction, the Markov families F and R^n are topologically conjugated. For all $t \in \hat{\Sigma}_l^m = \hat{\Sigma}_l^{R_m^n}$, denote $C_t^{R_m^n}$ by $B_t^{m,n}$, $K^{R_m^n}$ by $Q^{m,n}$ and $K_t^{R_m^n}$ by $Q_t^{m,n}$.

For all $i \in \mathbf{Z}$, define the scaling tree $\eta_{i,n} = \sigma_{R_i^n} : \cup_{l\geq 1} \hat{\Sigma}_l^i \to \mathbf{R}$ by

$$\eta_{i,n}(t) = \frac{|B_t^{i,n}|}{|B_{m(t)}^{i,n}|}.$$

By definition, $\eta_{i,n}(t) = \sigma_{i-n}(w_{i,n}t)$.

Lemma 32 (i) The scaling tree $\eta_m : \cup_{l\geq 1}\hat{\Sigma}_l^m \to \mathbf{R}$ is well defined by

$$\eta_m(t) = \lim_{n\to\infty} \eta_{m,n}(t).$$

(ii) For all $t \in \hat{\Sigma}_l^m$, there exist $d > 0$ and $0 < \nu < 1$ such that

$$\frac{\eta_m(t)}{\eta_{m,n}(t)} \in 1 \pm d\nu^l\nu^n.$$

Lemma 33 For all $t \in \hat{\Sigma}_l^m$, the sequence of intervals $B_t^{m,n}$ converge to an interval B_t^m.

For all $\underline{t} \in \Sigma^m$, define $Q_{\underline{t}}^m = \cap_{l\geq O} B_{\underline{t}|l}^m$, $Q^m = \cup_{\underline{t}\in\Sigma^m} Q_{\underline{t}}^m$ and $Q = (Q^m)_{m\in\mathbf{Z}}$. Define the renormalisation limit map $R_m^\infty = R_{\underline{w}_m}^\infty : Q^m \to Q^{m+1}$ by $R_m^\infty(Q_{\underline{t}}^m) = Q_{\underline{J(t)}}^{m+1}$. Let $R^\infty = (R_m^\infty)_{m\in\mathbf{Z}}$.

Lemma 34 The scaling function $\eta_i : \cup_{l\geq 1}\hat{\Sigma}_l^i \to \mathbf{R}$ is equal to the scaling function $\sigma_{R_i^\infty} : \cup_{l\geq 1}\hat{\Sigma}_l^i \to \mathbf{R}$, defined by

$$\sigma_{R_i^\infty}(t) = \frac{|B_t^i|}{|B_{m(t)}^i|}.$$

Theorem 26 The renormalisation limit map R^∞ is a C^{1+} bounded Markov family. The Markov family R^∞ is C^{1+} bounded conjugated to R^n by the conjugacy $h^n = (h_{m,n} : B^m \to B^{m,n})_{m\in\mathbf{Z}}$. Moreover, there is $0 < \nu < 1$ and a constant c such that

$$||h_{m,n} - Id||_{C^{1+}(B^m)} < c\nu^n.$$

Corollary 14 The sets $Q^{m,n}$ are C^{1+} self-similar in the sense that there is a C^{1+} diffeomorphism $h_{m,n}$ from Q^m into $Q^{m,n}$ which converges exponentially fast to the identity map in the C^{1+} norm when n tends to infinity.

Definition 36 A C^{1+} class of Markov families is the set of all C^{1+} bounded Markov families $(F_m)_{m\in\mathbf{Z}}$, C^{1+} bounded conjugated between them.

Corollary 15 The renormalisation limit Markov families $(R_m^\infty)_{m\in\mathbf{Z}}$ are complete invariants of their C^{1+} class. We assume that the intervals B^m are the same independent of the element of the C^{1+} class.

5.1.6 Constant Markov families.

A constant Markov family G is a Markov family $(G_m)_{m\in Z}$ such that $G_m = G$ for all $m \in Z$. Then, $\Sigma = \Sigma^m$.

Corollary 16 Let F be a C^{1+} bounded Markov family. F is C^{1+} bounded conjugated to a C^{1+} constant Markov family if and only if there is one renormalisation limit constant Markov family $(R^{\infty}_{\overline{w}_m} F)_{m\in Z}$. The renormalisation limit Markov families $(R^{\infty}_{\overline{w}_m} F)_{m\in Z}$ are constant Markov families, if $\overline{w}_p = \overline{w}_q$ and $B^{F_p} = B^{F_q}$ for all $p, q \in Z$.

Let F be a C^{1+} constant Markov family. Let $\underline{t} \in \Sigma$ be such that there is $\overline{w} \in \Sigma_*$ with the property that $\underline{t}|n_k = \overline{w}|n_k$. Define the n_k^{th} renormalisation constant Markov family at t by $R^{n_k}_{\underline{t}} = R^{n_k}_{\overline{w}}$, for all $k \geq 0$. The C^{1+} conjugacy family $(h_{n_k})_{k\geq 0}$ between $R^{\infty}_{\overline{w}}$ and $R^{n_k}_{\underline{t}}$ converge exponentially fast to the identity in B with respect to the C^{1+} norm, when k tends to infinity.

For all $\underline{t} \in \Sigma$, let $\overline{w} \in \Sigma_*$ and $(n_k)_{k\geq 0}$ be a sequence such that $\underline{t}|n_k = \overline{w}|n_k$. Define the renormalisation limit set at $\underline{t}$ as the set of all renormalisation limit constant Markov families $R^{\infty}_{\overline{w}}$.

For all p-periodic words $\underline{t} = t_1 \dots t_p t_1 \dots \in \Sigma$ there is $\overline{w}_l \in \Sigma_*$ such that $\underline{t}|pm + l = \overline{w}_l|pm + l$, for all $m > 0$ and $l = 0, \dots, p-1$. Therefore, the renormalisation limit set at $\underline{t}$ has a maximum of p different elements:

$$\{R^{\infty}_{\overline{w}_l} : l = 0, \dots, p-1\}.$$

5.1.7 Affine Markov families.

An affine Markov family $(G_m)_{m\in Z}$ is a Markov family such that G_m is a union of affine maps, for all $m \in Z$.

Corollary 17 Let F be a C^{1+} bounded Markov family and G a C^{1+} bounded affine Markov family. Assume that $C^{G_m} = B^{G_m} = B^{F_m}$. Then, F and G are C^{1+} bounded conjugated, if and only if one (all) renormalisation limit Markov families $(R^{\infty}_{\overline{w}_m} F)_{m\in Z}$ are equal to the Markov family G.

This implies that two different affine constant Markov families are not C^{1+} conjugated.

If the constant Markov family F is C^{1+} bounded conjugated to an affine Markov family G then the renormalisation limit set, for all $t \in \Sigma^F$ is equal to $\{G\}$.

5.1.8 Proof of theorem 23 and 24.

Proof of theorem 23. By a simple application of corollary of theorem 3 in chapter 1, there are C^{1+} maps $G_t^m : C_t^m \to C_{J(t)}^{m+1}$, for all $t \in \Sigma_?^m$, such that $G_t^m = F_m$ in $K_t^m = K_t^m(F_m)$. Define $G_m = G_t^m$ in $C_t^m = C_t^m(F_m)$. ∎

Proof of theorem 24. By a simple application of corollary of theorem 3 in chapter 1. ∎

5.1.9 Proof of theorem 25 and 26.

Proof of lemma 32. By definition, $\eta_{m,n}(t) = \sigma_{m-n}(w_{m,n}t)$. By theorem 23, the Markov family F has scale property. Therefore, for all words $t \in \hat{\Sigma}_l^m$ and $l \geq 1$

$$\frac{\sigma_{m-n-1}(w_{m,n+1}t)}{\sigma_{m-n}(w_{m,n}t)} \in 1 \pm c\nu^l \nu^n.$$

Thus, for all $p, q > M$ and $M > 0$

$$\frac{\sigma_{m-n-p}(w_{m,n+p}t)}{\sigma_{m-n-q}(w_{m,n+q}t)} \in 1 \pm d\nu^l \nu^M. \tag{1}$$

By bounded geometry, there is δ such that $\sigma_{m-n-p}(w_{m,n+p}t) > \delta > 0$ and by equation (1), the sequence $(\sigma_{m-n}(w_{m,n}t))_{n\geq 0}$ converges exponentially fast. Therefore the limit $\eta_m(t)$ is well defined, which verifies condition (i). Moreover, condition (ii) is verified by equation 1 and condition (i). ∎

Proof of lemma 33. For all $\underline{t} \in \Sigma^m$, by lemma 32

$$|B_{\underline{t}|l}^m| = \lim_{n\to\infty} |B_{\underline{t}|l}^{m,n}| = \lim_{n\to\infty} |B^{m,n}| \prod_{i=1}^{l} \eta_{m,n}(\underline{t}|i) = |B^m| \prod_{i=1}^{l} \eta_m(\underline{t}|i).$$

∎

Proof of lemma 34. By lemma 33 and lemma 32. ∎

Lemma 35 Let F be a topological Markov family with bounded geometry. Then F is a C^{1+} bounded Markov family if and only if for all $\overline{w}_m \in \Sigma_*^m$ and all $m \in Z$ - or equivalently there exist $\overline{w}_m^1, \ldots, \overline{w}_m^{l_m}$ such that $C^{m-1}_{\overline{w}_m^1|1} \cup \ldots \cup C^{m-1}_{\overline{w}_m^{l_m}|1} = \cup_{\varepsilon \in \Sigma_1^{m-1}} C_\varepsilon^{m-1}$ - the following conditions are verified.

(i) For all $t \in \hat{\Sigma}_l^m$, there exist $d > 0$ and $0 < \nu < 1$,

$$\frac{\eta_m(t)}{\eta_{m,n}(t)} \in 1 \pm d\nu^l \nu^n.$$

(ii) For all contact words $t, s \in \Sigma_l^m$, there exist $d > 0$ and $0 < \nu < 1$

$$\frac{|B_t^{m,n}|}{|B_t^m|} \frac{|B_s^m|}{|B_s^{m,n}|} \in 1 \pm d\nu^l \nu^n.$$

Proof of lemma 35. We will prove that if F is a C^{1+} bounded Markov family then the two conditions of lemma 35 are verified.

Condition (i) is proved by lemma 32.

Let us prove condition (ii). By lemma 33, for all contact words $t, s \in \Sigma_l^m$,

$$\left(\frac{|B_s^{m,n+p}|}{|B_t^{m,n+p}|}\right)_{p \geq 0} \text{ converges to } \frac{|B_s^m|}{|B_t^m|}.$$

For all $t \in \Sigma_l^m$, denote $C^{m-n}_{w_{m,n}t}$ by $C_t^{m,n}$. As the Markov family F has contact property, for all contact words $t, s \in \Sigma_l^m$,

$$\frac{|C_t^{m,n}|}{|C_t^{m,n+1}|} \frac{|C_s^{m,n+1}|}{|C_s^{m,n}|} \in 1 \pm c\nu^l \nu^n.$$

Therefore,

$$\begin{aligned} \frac{|B_t^{m,n}|}{|B_t^{m,n+1}|} \frac{|B_s^{m,n+1}|}{|B_s^{m,n}|} &= \frac{|C_t^{m,n}|}{|C^{m,n}|} \frac{|C_s^{m,n+1}|}{|C^{m,n+1}|} \frac{|C^{m,n}|}{|C_s^{m,n}|} \frac{|C^{m,n+1}|}{|C_t^{m,n+1}|} \\ &= \frac{|C_t^{m,n}|}{|C_t^{m,n+1}|} \frac{|C_s^{m,n+1}|}{|C_s^{m,n}|} \in 1 \pm c\nu^l \nu^n. \end{aligned}$$

Thus, for all $p, q > M$ and $M \geq 1$

$$\frac{|B_t^{m,n+q}|}{|B_t^{m,n+p}|} \frac{|B_s^{m,n+p}|}{|B_s^{m,n+q}|} \in 1 \pm d\nu^l \nu^M.$$

This proves condition (ii).

Let us prove the other implication.

By condition (i), for all words $t \in \hat{\Sigma}_l^m$,

$$\frac{\sigma_{m-1}(w_{m,1}t)}{\sigma_m(t)} = \frac{\eta_{m,1}(t)}{\eta_m(t)} \frac{\eta_m(t)}{\eta_{m,0}(t)} \in (1 \pm d\nu^{l+1})(1 \pm d\nu^l) \subset 1 \pm e\nu^l,$$

for some constant $e > 0$. Thus, the Markov family F has scale property.

By condition (ii), for all contact words $t, s \in \Sigma_l^m$,

$$\frac{|C^{m-1}_{w_{m,1}t}|}{|C_t^m|} \frac{|C_s^m|}{|C^{m-1}_{w_{m,1}s}|} = \frac{|B_t^{m,1}|}{|B_t^m|} \frac{|B_s^m|}{|B_s^{m,1}|} \frac{|B_t^m|}{|B_t^{m,0}|} \frac{|B_s^{m,0}|}{|B_s^m|} \in (1 \pm d\nu^{l+1})(1 \pm d\nu^l) \subset 1 \pm e\nu^l,$$

Thus, the Markov family F has contact property.

Therefore, by theorem 23, F is a C^{1+} bounded Markov family. ∎

Lemma 36 The family of maps R^n and R^∞ are C^{1+} bounded Markov families.

Proof of lemma 36. Let us prove that R^n and R^∞ have scale property. By definition of scale property of the Markov family F and by condition (i) of lemma 35, for all $t \in \hat{\Sigma}_l^m$,

$$\frac{\eta_m(t)}{\eta_{m+1}(J(t))} = \frac{\eta_m(t)}{\eta_{m,0}(t)} \frac{\sigma_m(t)}{\sigma_{m+1}(J(t))} \frac{\eta_{m+1,0}(J(t))}{\eta_{m+1}(J(t))} \in (1 \pm d\nu^l)(1 \pm c\nu^{l-1})(1 \pm d\nu^{l-1}) \subset 1 \pm e_1\nu^l,$$

for some constant $e_1 > 0$. Moreover,

$$\frac{\eta_{m,n}(t)}{\eta_{m+1,n}(J(t))} = \frac{\eta_{m,n}(t)}{\eta_m(t)} \frac{\eta_m(t)}{\eta_{m+1}(J(t))} \frac{\eta_{m+1}(J(t))}{\eta_{m+1,n}(J(t))} \in (1 \pm d\nu^{l+n})(1 \pm e_1\nu^{l-1})(1 \pm d\nu^{l-1+n}) \subset 1 \pm e_2\nu^l,$$

for some constant $e_2 > 0$.

Let us prove that R^n and R^∞ have contact property. The Markov family F has contact property and by condition (ii) of lemma 35, for all contact words $t, s \in \Sigma_l^m$,

$$\frac{|B_t^m|}{|B_{J(t)}^{m+1}|} \frac{|B_{J(s)}^{m+1}|}{|B_s^m|} = \frac{|B_t^m|}{|B_t^{m,0}|} \frac{|B_s^{m,0}|}{|B_s^m|} \frac{|C_t^m|}{|C_{J(t)}^{m+1}|} \frac{|C_{J(s)}^{m+1}|}{|C_s^m|} \frac{|B_{J(t)}^{m+1,0}|}{|B_{J(t)}^{m+1}|} \frac{|B_{J(s)}^{m+1}|}{|B_{J(s)}^{m+1,0}|} \in (1 \pm d\nu^l)(1 \pm c\nu^l)(1 \pm d\nu^{l-1}) \subset 1 \pm e_1\nu^l.$$

Moreover,

$$\begin{aligned}\frac{|B_t^{m,n}|}{|B_{J(t)}^{m+1,n}|}\frac{|B_{J(s)}^{m+1,n}|}{|B_s^{m,n}|} &= \frac{|B_t^{m,n}|}{|B_t^{m}|}\frac{|B_{J(t)}^{m+1}|}{|B_{J(t)}^{m+1,n}|}\frac{|B_t^{m}|}{|B_{J(t)}^{m+1}|}\frac{|B_{J(s)}^{m+1}|}{|B_s^{m}|}\frac{|B_{J(s)}^{m+1,n}|}{|B_{J(s)}^{m+1}|}\frac{|B_s^{m}|}{|B_s^{m,n}|}\\ &\in (1\pm d\nu^{l+n})(1\pm e_1\nu^l)(1\pm d\nu^{l+n-1}) \subset 1\pm e_2\nu^l.\end{aligned}$$

As the Markov family F has bounded geometry, there is $\delta > 0$, such that, for all $t \in \Sigma_l^m$,

$$\eta_{m,n}(t) = \sigma_{m-n}(w_{m,n}t) > \delta.$$

By condition (i) of lemma 32,

$$\eta_m(t) > (1 - d\nu^l\nu^n)\sigma_{m-n}(w_{m,n}t) > (1 - d\nu^l\nu^n)\delta.$$

Therefore, $\eta_m(t) \geq \delta$.

Finally, by theorem 23 we proved the lemma. ∎

Proof of theorem 25. By condition (i) of lemma 32, for all word $t \in \hat{\Sigma}_l^m$, all $m \in \mathbb{Z}$ and all $l \in \mathbb{N}$, $\eta_{F_m}(t)$ and $\eta_{G_m}(t)$ are defined and by hypotheses of the theorem,

$$\eta_{F_m}(t) = \lim_{n\to\infty} \sigma_{F_{m-n}}(w_{m,n}t) = \lim_{n\to\infty} \sigma_{G_{m-n}}(w_{m,n}t) = \eta_{G_m}(t).$$

By condition (ii) of lemma 32,

$$\frac{\sigma_{F_m}(t)}{\eta_{F_m}(t)} \in 1 \pm c_1\nu^l \quad\text{and}\quad \frac{\sigma_{G_m}(t)}{\eta_{G_m}(t)} \in 1 \pm c_2\nu^l,$$

for some constants $c_1, c_2 > 0$. Therefore,

$$\frac{\sigma_{F_m}(t)}{\sigma_{G_m}(t)} \in 1 \pm c\nu^l.$$

Thus, the the Markov families F and G are scale equivalent.

Suppose $B^{F_m} = B^{G_m}$. For all $\underline{t} \in \Sigma^m$ and $m \in \mathbb{Z}$,

$$|B_{t|l}^{F_m}| = |B^{F_m}| \prod_{i=1}^{l} \eta_{F_m}(t|i) = |B^{G_m}| \prod_{i=1}^{l} \eta_{G_m}(t|i) = |B_{t|l}^{G_m}|.$$

By condition (ii) of lemma 35, for all contact words $t, s \in \Sigma_l^m$,

$$\frac{|C_t^{F_m}|}{|B_t^{F_m}|}\frac{|B_s^{F_m}|}{|C_s^{F_m}|} \in 1 \pm d\nu^l \quad\text{and}\quad \frac{|C_t^{G_m}|}{|B_t^{G_m}|}\frac{|B_s^{G_m}|}{|C_s^{G_m}|} \in 1 \pm d\nu^l.$$

Therefore,

$$\frac{|C_t^{F_m}|}{|C_t^{G_m}|}\frac{|C_s^{G_m}|}{|C_s^{F_m}|} \in 1 \pm c\nu^l.$$

Thus, the Markov families F and G are contact equivalent. Finally, by theorem 24, F and G are C^{1+} bounded conjugated.

If F and G are C^{1+} bounded conjugated, then by theorem 24, for all $\overline{w} \in \Sigma_*^m$ and all $m \in \mathbb{Z}$,

$$\frac{\sigma_{F_{m-n}}(\overline{w}|n)}{\sigma_{G_{m-n}}(\overline{w}|n)} \in 1 \pm c\nu^n.$$

Therefore, $s_{F_m}(\overline{w}) = s_{G_m}(\overline{w})$. ∎

Proof of corollary 13. By theorem 25, the family of conjugacies satisfy the unformity hypotheses of the theorem in the section in $C^{k+\delta}$ conjugacy between backward Markov families. ∎

Proof of theorem 26. By lemma 36, the renormalisation limit R^∞ is a C^{1+} bounded Markov family.

We will use the results of chapter 1 to prove the existence of the conjugacy h^n. First, we define a map $g_\varepsilon(n,l)$ that we will use in the proof.

By bounded geometry, there is $\delta > 0$ such that for all word $t \in \hat{\Sigma}_l^m$, all $m \in \mathbb{Z}$ and all $l \in \mathbb{N}$,

$$B_t^m < c\delta^l \tag{2}$$

where the constant c is equal to the maximum length of the intervals B^m, for all $m \in \mathbb{Z}$. Take $\varepsilon > 0$, such that $\nu\delta^{-\varepsilon} = \beta$ for some $0 < \beta < 1$. By equation (2), for all $t \in \Sigma_l^m$, all $m \in \mathbb{Z}$ and all $l \in \mathbb{N}$,

$$\frac{\nu^l}{|B_t^m|^\varepsilon} < c\nu^l(\delta^l)^{-\varepsilon} < c\beta^l. \tag{3}$$

Define $g_\varepsilon(n,l) = c\beta^l\nu^n$.

Let $T^{(m)} = \cup_{l\geq 1} T_l^{(m)}$ be the tree such that $T_l^{(m)}$ is the set of l-cylinders and l-gaps in the domain of the Markov map R_m^∞ and such that if $I \in T_l^{(j)}$ then $m(I)$ is the cylinder such that $I \subset m(I)$. Let $T^{(m,n)} = \cup_{l\geq 1} T_l^{(m,n)}$ be the tree such that $T_l^{(m,n)}$ is the set of l-cylinders and l-gaps in the domain of the Markov map R_m^n and such that if $I \in T_l^{(j)}$ then $m(I)$ is the cylinder such that $I \subset m(I)$.

By construction of the Markov families R^∞ and R^n, the embeddings $T^{(m)}$ and $T^{(m,n)}$ have the same topological structure. We define the map L_l :

$\mathrm{I}(T^{(m)}) \to \mathrm{I}(T^{(m,n)})$ as in the section 2.4 of chapter 1, where $\mathrm{I}(T^{(m)}) = B^m$ and $\mathrm{I}(T^{(m,n)}) = B^{m,n}$.

In chapter 1, take in definition 8 and 5

$$f_\epsilon(l) = g_\epsilon(n,l) = c\beta^l\nu^n, \tag{4}$$

for all $l \geq 0$. By theorem 3 and lemma 6 in chapter 1 and lemma 35 and equation (3) and (4),

$$||L_{l+1} - L_l||_{C^{1+\epsilon}} \leq c\beta^l\nu^n.$$

By condition (i) of definition 35, $||L_0 - \mathrm{id}||_{C^{1+\epsilon}} \leq c\nu^n$. Therefore,

$$||L_\infty - \mathrm{id}||_{C^{1+\epsilon}} \leq c\nu^n.$$

Define the map $h_{m,n} = L_\infty$. ∎

Proof of corollary 14. It is an immediate consequence of theorem 26. ∎

Proof of corollary 15. Let us prove that the Markov family $R^\infty F$ is a complete invariant of its C^{1+} equivalence class. If the Markov family G is C^{1+} bounded conjugated to the Markov family F, then by condition (i) of lemma 35 and theorem 24, for all $t \in \hat{\Sigma}_l^m$, all $l \in \mathrm{N}$ and all $m \in \mathrm{Z}$

$$\begin{aligned}\frac{\eta_{G_m}(t)}{\eta_{F_m}(t)} &= \frac{\eta_{G_m}(t)}{\eta_{G_{m,n}}(t)}\frac{\sigma_{G_{m-n}}(w_{m,n}t)}{\sigma_{F_{m-n}}(w_{m,n}t)}\frac{\eta_{F_{m,n}}(t)}{\eta_{F_m}(t)} \\ &\in 1 \pm 3d\nu^l\nu^n.\end{aligned}$$

Therefore, $\eta_{G_m}(t) = \eta_{F_m}(t)$. As by hypotheses, $B^{G_m} = B^{F_m}$ then $R^\infty F = R^\infty G$. If the Markov families G and F are not C^{1+} bounded conjugated then $(R^\infty G)_{m\in\mathrm{Z}}$ and $(R^\infty F)_{m\in\mathrm{Z}}$ are not C^{1+} bounded conjugated, otherwise, we obtain a contradiction by theorem 26. ∎

Proof of corollary 16. Let F be C^{1+} bounded conjugated to a constant Markov family G. For all $m \in \mathrm{Z}$, take $B^{F_m} = B^{G_m}$. Then, by corollary 15 they have the same renormalisation limit Markov families, i.e. $R^\infty_{\overline{w}_m}F = R^\infty_{\overline{w}_m}G$, for all $m \in \mathrm{Z}$ and $\overline{w}_m \in \Sigma^m_*$. As G is a C^{1+} constant Markov family then $(R^\infty_{\overline{w}_m}G)_{m\in\mathrm{Z}}$ is a C^{1+} constant Markov family when $\overline{w}_p = \overline{w}_q$ and $B^{G_p} = B^{G_q}$, for all $p, q \in \mathrm{Z}$. Therefore, $(R^\infty_{\overline{w}_m}F)_{m\in\mathrm{Z}}$ is a a C^{1+} constant Markov family when $\overline{w}_p = \overline{w}_q$ and $B^{F_p} = B^{F_q}$, for all $p, q \in \mathrm{Z}$.

If $(R^\infty_{\overline{w}_m}F)_{m\in\mathrm{Z}}$ is a C^{1+} constant Markov family then by theorem 26, F is C^{1+} bounded conjugated to it. ∎

Proof of corollary 17. As G_m is a union of affine maps and $C^{G_m} = B^{G_m}$ for all $m \in Z$, then all the renormalisation limit Markov families $R^n G$ coincide with G. As for all $m \in Z$, $B^{G_m} = B^{F_m}$ and F is C^{1+} bounded conjugated to G, then by corollary 15, $R^\infty G = R^\infty F$.

If $R^\infty G = R^\infty F$ then by theorem 26, F is C^{1+} bounded conjugated to G.

■

5.2 $C^{k+\delta}$ conjugacy between two-sided Markov families.

5.2.1 Introduction.

The ω-limit set of Markov family consists of two-sided Markov families. Let F and G be $C^{k+\delta}$ two-sided Markov families. In this section, we prove that if they are C^{1+} conjugated then they are $C^{k+\delta}$ conjugated. This result opposes to the difficulty in getting higher smoothness in one-sided Markov families. In that case a balance between the speed of convergence of the Markov families and the scaling structure of their cylinders is needed.

Let $F = (F_m)_{m \in Z}$ and $G = (G_m)_{m \in Z}$ be $C^{k+\delta}$ weakly bounded two-sided Markov families, where $\delta \in (0, 1]$ and $k > 0$.

A Markov family F is *weakly bounded* if there are constants b and e, such that, $|dF_m| > e > 1$ and $||F_m||_{C^{k+\delta}} \leq b$, for all $m \in Z$.

Let $h = (h_m)_{m \in Z}$ be a topological conjugacy between F and G.

The conjugacy h has the *uniformity property* if it satisfies the following conditions.

(i) There is a sequence of points $x_m \in C^{F_m}$ such that F_m and h_m are smooth at x_m, $F_m(x_m) = x_{m+1}$ and $|dh_m(x_m)| > M_1 > 0$, for all $m < 0$.

(ii) Moreover, there is a continuous function ε such that $\varepsilon(0) = 0$ and for all $m < 0$,

$$\left|\frac{h_m(z_m) - h_m(y_m)}{z_m - y_m} - dh_m(x_m)\right| \leq \varepsilon(\max\{|z_m - x_m|, |y_m - x_m|\}).$$

Theorem 27 If h is a topological conjugacy between F and G with the uniformity property then there is a $C^{k+\delta}$ conjugacy $r = (r_m)_{m \in \mathbb{Z}}$ between F and G.

Corollary 18 Let F and G be $C^{k+\delta}$ constant Markov families. Let the map h be a topological conjugacy between F and G. Let x be a periodic point of F, such that F is smooth at x. If h satisfies the uniformity property at x then there is a $C^{k+\delta}$ conjugacy between F and G.

5.2.2 Proof of theorem 27.

Proof of theorem 27. We will prove in two parts that there is a $C^{k+\delta}$ diffeomorphism $r_0 = s$ from C^{F_0} into C^{G_0} which sends K^{F_0} onto K^{G_0}. In the first part, we prove that there is a sequence of maps s_n converging in the C^0 norm to h_0 in the set K^{F_0}. In the second part, we prove that there is a subsequence of maps $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ converging to a $C^{k+\delta}$ map s, in some $C^{k+\epsilon}$ norm.

By the same argument, there are $C^{k+\delta}$ diffeomorphisms $r_m : C^{F_m} \to C^{G_m}$ which sends K^{F_m} onto K^{G_m}, for all $m < 0$. For all $m \geq 0$, choose a word $v \in \Sigma^0_m$. Define the map $F_{0,m} : C^{F_0}_v \to C^{F_m}$ by $F_{0,m} = F_{m-1} \circ \ldots \circ F_0$ and the map $F^{-1}_{0,m} : C^{F_m} \to C^{F_0}_v$ as the inverse map of $F_{0,m}$. Define the map $r_m : C^{F_m} \to C^{G_m}$ by $r_m = F_{0,m} \circ s \circ F^{-1}_{0,m}$.

First part. Choose the word $\overline{w} \in \Sigma^0_*$ such that $x_n \in C^{F_{-n}}_{\overline{w}|n}$. Denote $C^{F_{-n}}_{\overline{w}|n}$ by A_n and $C^{G_{-n}}_{\overline{w}|n}$ by B_n. Define the map $F^{-1}_{-n} : A_{n-1} \to A_n$ (resp. $G^{-1}_{-n} : B_{n-1} \to B_n$) as the inverse map of $F_{-n} : A_n \to A_{n-1}$ (resp. $G_{-n} : B_n \to B_{n-1}$). Define the maps $f_n : A_n \to C^{F_0}$ and $g_n : B_n \to C^{G_0}$ by

$$f_n = F_{-1} \circ \ldots \circ F_{-n} \quad \text{and} \quad g_n = G_{-1} \circ \ldots \circ G_{-n}.$$

Moreover, define $f_n^{-1} : C^{F_0} \to A_n$ as the inverse map of f_n and $g_n^{-1} : C^{G_0} \to B_n$ as the inverse map of g_n.

Define the affine map $L_n : A_n \to B_n$ by $L_n(A_n) = B_n$. Denote $L_n(x) = \eta_n x + b_n$. By the uniformity property,

$$\eta_n \in dh_{-n}(x_n) \pm \varepsilon(|A_n|). \tag{5}$$

Define the sequence of maps $s_n : C^{F_0} \to C^{G_0}$ by

$$s_n = g_n L_n f_n^{-1},$$

for all $n \geq 0$. By definition, $h_0 = g_n \circ h_{-n} \circ f_n^{-1}$ in K^{F_0}.

For all $y \in K^{F_0}$, let $y_n \in A_n$ be such that $y_n = f_n^{-1}(y)$. By the mean value theorem, there is $z_n \in A_n$, such that,

$$|s_n(y) - h_0(y)| \leq |d(g_n)(z_n)||L_n(y_n) - h_{-n}(y_n)|. \tag{6}$$

By definition of the maps h_{-n} and L_n, $h_{-n}(A_n) = L_n(A_n)$. Thus, there is a point $t_n \in A_n$ such that $h_{-n}(t_n) = L_n(t_n)$. By the uniformity property,

$$\frac{h_{-n}(y_n) - h_{-n}(t_n)}{y_n - t_n} = d_n \in dh_{-n}(x_n) \pm \varepsilon(|A_n|). \tag{7}$$

By equation (5) and (7) and definition of the map L_n,

$$\begin{aligned} |h_{-n}(y_n) - L_n(y_n)| &= |h_{-n}(y_n) - h_{-n}(t_n) - \eta_n(y_n - t_n)| \\ &= |1 - \frac{\eta_n}{d_n}||h_{-n}(y_n) - h_{-n}(t_n)| \\ &\leq |1 - \frac{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}||B_n|. \end{aligned} \tag{8}$$

By the mean value theorem, there is $u_n \in B_n$, such that,

$$|d(g_n)(u_n)||B_n| = |C^{G_0}|. \tag{9}$$

By lemma 38,

$$|d(g_n)(z_n)| \leq \exp(c_3)|d(g_n)(u_n)|. \tag{10}$$

By equation (9) and (10),

$$|d(g_n)(z_n)||B_n| \leq \exp(c_3)|C^{G_0}|. \tag{11}$$

By the uniformity property, $|dh_{-n}(x_n)| > M_1 > 0$. Therefore, by equation (6), (8) and (11),

$$\begin{aligned} |s_n(y) - h_0(y)| &\leq |d(g_n)(z_n)||B_n||1 - \frac{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}| \\ &\leq \exp(c_3)|C^{G_0}||1 - \frac{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}{dh_{-n}(x_n) \pm \varepsilon(|A_n|)}|. \end{aligned}$$

Thus, $|s_n(y) - h_0(y)|$ tends to zero when n tends to infinity. Therefore, the sequence of maps $(s_n)_{n\geq 0}$ converges in the C^0 norm to h_0 in K^{F_0}.

Second part. By the compact embedding lemma to prove the second part it is equivalent to prove by induction in the degree of smoothness r that $||s_n||_{C^{r+s}} \leq b_r$, for all $r = 1, 2, \ldots, k$.

Case $r = 1$. First, we prove that $|ds_n|$ is bounded and then that it is bounded in the δ-Hölder norm, independent of $n \geq 0$.

By the mean value theorem, there is $\theta_n \in C^{F_0}$ and $\psi_n \in C^{G_0}$ such that,

$$|d(f_n^{-1})(\theta_n)||C^{F_0}| = |A_n| \quad \text{and} \quad |d(g_n^{-1})(\psi_n)||C^{G_0}| = |B_n|.$$

For all $y \in C^{F_0}$, by lemma 38 and the last two equalities,

$$\begin{aligned} |ds_n(y)| &= |\eta_n \frac{d(f_n^{-1})(y)}{d(g_n^{-1})(s_n y)}| \leq |\eta_n \exp(2c_3) \frac{d(f_n^{-1})(\theta_n)}{d(g_n^{-1})(\psi_n)}| \\ &\leq \exp(2c_3) \frac{\eta_n}{\eta_n} \frac{|C^{G_0}|}{|C^{F_0}|} \leq c_0 \exp(2c_3), \end{aligned}$$

for some constant $c_0 > 0$. Therefore, ds_n is bounded independently of $n \geq 0$.

Let us prove that the map ds_n is bounded in the δ-Hölder norm. By the equation above, by lemma 38 and the mean value theorem, for all $x, y \in C^{F_0}$, there is $z_{x,y} \in C^{F_0}$,

$$\begin{aligned} \ln |\frac{ds_n(x)}{ds_n(y)}| &= \ln |\frac{d(g_n^{-1})(s_n(y))}{d(g_n^{-1})(s_n(x))} \frac{d(f_n^{-1})(x)}{d(f_n^{-1})(y)} \frac{\eta_n}{\eta_n}| \\ &\leq c(|s_n(x) - s_n(y)|^\delta + |x - y|^\delta) \\ &\leq c|ds_n(z_{x,y})|^\delta |x - y|^\delta + c|x - y|^\delta \\ &\leq c(1 + c_0 \exp(2c_3\delta))|x - y|^\delta. \end{aligned}$$

Thus,

$$\frac{ds_n(x)}{ds_n(y)} \in 1 \pm c_1 |x - y|^\delta,$$

for some constant $c_1 > 0$. Therefore,

$$|ds_n(x) - ds_n(y)| = |ds_n(y)||\frac{ds_n(x)}{ds_n(y)} - 1| \leq c_0 \exp(2c_3) c_1 |x - y|^\delta.$$

We proved that there is a constant $b_1 > 0$, such that

$$||s_n||_{C^{1+\delta}} \leq b_1.$$

Induction step. The induction hypotheses is that $||s_n||_{C^{j+\delta}} \leq b_j$, for all $j = 1, \ldots, r$. Let us prove the case $r + 1$.

For all $y \in C^{F_0}$,

$$d^2 s_n = (d \ln df_n^{-1}) ds_n - (d \ln dg_n^{-1} \circ s_n)(ds_n)^2.$$

Therefore, by induction in r

$$\begin{aligned} d^{r+1}s_n &= L_{r+2}(d\ln df_n^{-1},\ldots,d^r\ln df_n^{-1}, \\ &\quad d\ln dg_n^{-1},\ldots,d^r\ln dg_n^{-1},ds_n,\ldots,d^r s_n) \end{aligned}$$

where L_{r+2} is a polinomial of order $r+2$ with coefficients independent of n.

By lemma 39 and by induction hypotheses, the variables in the equation above are bounded and are δ-Hölder continuous with constants independent of n. Thus, $d^{r+1}s_n$ is bounded and it is δ-Hölder continuous with constant independent of n. ∎

Lemma 37 Let F be a $C^{k+\delta}$ weakly bounded Markov family. Then, for all $r\in\{1,\ldots,k-1\}$,

$$\begin{aligned} d^r\ln df_n^{-1} &= \sum_{l=0}^{r-1}\sum_{i=0}^{n-1}((d^{r-l}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1}) \\ &\quad (df_i^{-1})^{r-l}E_l^r(d\ln df_i^{-1},\ldots,d^l\ln df_i^{-1})) \end{aligned}$$

where E_l^r is a polinomial of order l and the coefficients are independent of $n,i\geq 0$. For $i=0$, we define the map f_i^{-1} equal to the identity map.

Proof of lemma 37. We will prove it by induction in the degree of smoothness r.

Case $r=1$. By differentiation,

$$\ln df_n^{-1}=\sum_{i=0}^{n-1}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1}.$$

Therefore,

$$d\ln df_n^{-1}=\sum_{i=0}^{n-1}(d\ln dF_{-(i+1)}^{-1}\circ f_i^{-1})df_i^{-1}.$$

Thus, the formula is valid for $r=1$, with $E_0^1=1$.

Induction step. Let us suppose by induction hypothesis that the formula is valid for r and let us prove that it is valid for $r+1$. First, we differentiate separately the three terms of the formula in lemma 37.

The derivative of the first term is

$$d(d^{r-l}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1})=(d^{r+1-l}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1})df_i^{-1}.$$

The derivative of the second term is

$$d((df_i^{-1})^{r-l}) = (r-l)(df_i^{-1})^{r-l}(d\ln df_i^{-1}).$$

The derivative of the third term is

$$dE_l^r(d\ln df_i^{-1},\ldots,d^l\ln df_i^{-1}) = F_l^r(d\ln df_i^{-1},\ldots,d^{l+1}\ln df_i^{-1}),$$

where F_l^r has degree l and coefficients independent of i and n. We define the polinomial

$$G_{l+1}^r(x_1,\ldots,x_{l+1}) = F_l^r(x_1,\ldots,x_{l+1}) + (r-l)x_1E_l^r(x_1,\ldots,x_l).$$

The polinomial G_{l+1}^r has degree $l+1$ and the coefficients are independent of i and n. Therefore,

$$\begin{aligned} d^{r+1}\ln df_n^{-1} &= \sum_{l=0}^{r-1}\sum_{i=0}^{n-1}((d^{r+1-l}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1}) \\ &\quad (df_i^{-1})^{r+1-l}E_l^r(d\ln df_i^{-1},\ldots,d^l\ln df_i^{-1})) \\ &\quad +\sum_{l=0}^{r-1}\sum_{i=0}^{n-1}((d^{r-l}\ln dF_{-(i+1)}^{-1}\circ f_i^{-1}) \\ &\quad (df_i^{-1})^{r-l}G_{l+1}^r(d\ln df_i^{-1},\ldots,d^{l+1}\ln df_i^{-1})). \end{aligned}$$

Replacing $l+1$ by l in the second term, we have

$$E_0^{r+1}(x_1,\ldots,x_{l+1}) = E_0^r(x_1,\ldots,x_l) = 1.$$

define $E_r^r = 0$. For $l = \{1,\ldots,r\}$, $E_l^{r+1}(x_1,\ldots,x_l)$ is equal to

$$F_{l-1}^r(x_1,\ldots,x_l) + (r-l+1)x_1E_{l-1}^r(x_1,\ldots,x_{l-1}) + E_l^r(x_1,\ldots,x_l). \blacksquare$$

Lemma 38 Let F be a $C^{k+\delta}$ weakly bounded Markov family. Then, for all $x,y\in C^{F_0}$,

$$|\ln\frac{df_n^{-1}(y)}{df_n^{-1}(x)}| \le c|x-y|^\beta$$

where $\beta=\delta$ if $k=1$, or $\beta=1$ if $k>1$. Moreover,

$$df_n^{-1}(y)\in\exp(\pm c_3)df_n^{-1}(x).$$

Proof of lemma 38. As the Markov family F is weakly bounded and by the medium value theorem and as $|dF_m|>e>1$, for all $x,y\in C^{F_0}$, there is

$z_{x,y} \in C^{F_0}$,

$$\begin{aligned}
|\ln \frac{df_n^{-1}(y)}{df_n^{-1}(x)}| &\leq \sum_{i=0}^{n-1}(|\ln |dF_{-(i+1)}^{-1} \circ f_i^{-1}(y)| - \ln |dF_{-(i+1)}^{-1} \circ f_i^{-1}(x)||) \\
&\leq c_1 \sum_{i=0}^{n-1} |f_i^{-1}(y) - f_i^{-1}(x)|^\beta \leq c_1 \sum_{i=0}^{n-1} (df_i^{-1}(z_{x,y}))^\beta |y-x|^\beta \\
&\leq c|x-y|^\beta \leq c_3,
\end{aligned}$$

for some constant $c_3 > 0$. Therefore,

$$df_n^{-1}(y) \in \exp(\pm c_3) df_n^{-1}(x). \blacksquare$$

Lemma 39 Let F be a $C^{k+\delta}$ weakly bounded Markov family. Then,

$$||\ln df_n^{-1}||_{C^{k-1+\delta}} \leq b_k.$$

Proof of lemma 39. The case $k = 1$, it is proved by lemma 38. For $k \geq 2$, we will prove by induction in r that $d^r \ln df_n^{-1}$ is bounded in the C^0 norm independent of n, for all $r = 1, \dots, k-1$. After, we prove that $d^{k-1} \ln df_n^{-1}$ is δ-Hölder continuous with constant independent of n.

Case $r = 1$. By lemma 38 and as $k \geq 2$,

$$|\ln \frac{df_n^{-1}(y)}{df_n^{-1}(x)}| \leq c|x-y|.$$

Therefore, $d \ln df_n^{-1}$ is bounded in the C^0 norm independent of n.

Induction step. By induction hypotheses, we suppose that the maps $d \ln df_n^{-1}$, $\dots$, $d^{r-1} \ln df_n^{-1}$ are bounded in the C^0 norm independent of n. We will prove that $d^r \ln df_n^{-1}$ is bounded in the C^0 norm independent of n.

By lemma 37,

$$\begin{aligned}
d^r \ln df_n^{-1} &= \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} ((d^{r-l} \ln dF_{-(i+1)}^{-1} \circ f_i^{-1}) \\
&\quad (df_i^{-1})^{r-l} E_l^r (d \ln df_i^{-1}, \dots, d^l \ln df_i^{-1}))
\end{aligned}$$

where the coefficients of the polinomial E_l^r are independent of n and i, for all $r \in \{1, \dots, k-1\}$.

As the Markov family F is weakly bounded, $|dF_{-(i+1)}^{-1}| > b^{-1} > 0$ and because the first $r+1$-derivatives of the map $F_{-(i+1)}^{-1}$ are bounded independent of i

$$|d^{r-l} \ln dF_{-(i+1)}^{-1} \circ f_i^{-1}| \leq b_{r,l}, \tag{12}$$

for all $l = 0, \ldots, r-1$, $i = 0, \ldots, n-1$ and $n \in \mathbb{N}$.

As the Markov family F is weakly bounded, $|dF_{-i}^{-1}| < e^{-1} < 1$. Therefore,

$$|\sum_{i=0}^{n-1}(df_i^{-1})^{r-l}| \le (\frac{1}{1-e^{-1}})^{r-l} \le b_r, \tag{13}$$

for all $l = 0, \ldots, r-1$, $i = 0, \ldots, n-1$ and $n \in \mathbb{N}$.

The induction hypotheses implies

$$|E_l^r(d \ln df_i^{-1}, \ldots, d^l \ln df_i^{-1})| \le b_{r,l}, \tag{14}$$

for all $l = 0, \ldots, r-1$, $i = 0, \ldots, n-1$ and $n \in \mathbb{N}$.

By lemma 37 and equations (12), (13) and (14),

$$|d^r \ln df_n^{-1}| \le \sum_{l=0}^{r-1} b_{r,l}(\frac{1}{1-e^{-1}})^{r-l} \le b_r.$$

Let us prove that the map $d^{k-1} \ln df_n^{-1}$ is δ-Hölder continuous with Hölder constant independent of n. The map $d^{k-1-l} \ln dF_{-(i+1)}^{-1}$ is δ-Hölder continuous for $l = 0$ and it is Lipschitz for $l = 1, \ldots, k-2$. As the Markov family F is weakly bounded, the δ-Hölder (resp. Lipschitz) constant is independent of $i \ge 0$, i.e.

$$||d^{k-1-l} \ln dF_{-(i+1)}^{-1}||_{C^\delta or C^{Lipschitz}} \le c,$$

for some constant $c > 0$. Thus, the map $d^{k-1-l} \ln F_{-(i+1)}^{-1} \circ f_i^{-1}$ is Lipschitz if $l > 0$ or δ-Hölder continuous if $l = 0$. As the Markov family F is weakly bounded, $|df_{-i}^{-1}| < (e^{-1})^i < 1$. Therefore, the Lipschitz (resp. δ-Hölder) constant of the map $d^{k-1-l} \ln F_{-(i+1)}^{-1} \circ f_i^{-1}$ converges exponentially fast to zero, when i tends to infinity.

The map $(df_i^{-1})^{(k-1-l)}$ is Lipschitz where the Lipschitz constant converges exponentially fast to zero, when i tends to infinity because it has bounded nonlinearity and it is exponentially contracting.

The map

$$E_l^{k-1}(d \ln df_i^{-1}, \ldots, d^l \ln df_i^{-1})$$

is Lipschitz with constant independent of i because it is a l-product of maps bounded in the C^1 norm independently of i as proved by induction.

Therefore, the map

$$(d^{k-1-l} \ln dF_{-(i+1)}^{-1} \circ f_i^{-1})(df_i^{-1})^{k-1-l} E_l^{k-1}(d \ln df_i^{-1}, \ldots, d^l \ln df_i^{-1})$$

is a product of δ-Hõlder and Lipschitz maps with constants bounded independent of $i = 0, \ldots, n$ and $n \in \mathbb{N}$. The map $(df_i^{-1})^{k-1-l}$ converges exponentially fast to zero in the $C^{Lipschitz}$ norm when i tends to infinity. Therefore, the product of the three maps above is δ-Hõlder continuous where the δ-Hõlder constant converges exponentially fast to zero, when i tends to infinity. Therefore, the map $d^{k-1} \ln df_n^{-1}$ is δ-Hõlder continuous. ∎

Lemma 40 *Compact embedding lemma.* Let $(f_n)_{n\geq 0}$ be a sequence of $C^{k+\alpha}$ smooth functions f_n defined in an interval $I = [a, c]$, where $k > 0$ and $\alpha \in (0, 1]$. If $||f_n||_{C^{k+\alpha}} \leq b$, for all $n \geq 0$, then there is a subsequence $(f_{n_i})_{i\geq 0}$ converging to a $C^{k+\alpha}$ smooth function f in the $C^{k+\alpha^-}$ norm.

The map f is bounded in the $C^{k+\alpha^-}$ norm, if for all $0 < \varepsilon < \alpha$, f is bounded in the $C^{k+\varepsilon}$ norm.

Corollary 19 The set of all functions $f \in C^{k+\alpha}$ defined in an interval I such that $||f||_{C^{k+\alpha}} \leq b$ is a compact set with respect to the norm $C^{k+\alpha-\varepsilon}$ for all small $\varepsilon > 0$.

Definition 37 A subset of a topological space is called conditionally compact if its closure is compact in its relative topology.

Theorem 28 *Arzela-Ascoli.* If S is compact set then a set in the space of continuous functions with domain S is conditionally compact if and only if it is bounded and equicontinuous.

Proof of the compact embedding lemma. *Likely not original.* As the sequence of maps f_n is bounded in the $C^{k+\alpha}$ norm, then

$$|d^k f_n(x) - d^k f_n(y)| \leq b|x - y|^\alpha,$$

for all $n \geq 0$. Therefore, $(d^k f_n)_{n\geq 0}$ is an equicontinuous family of functions. By the Arzela-Ascoli theorem, there is a subsequence $(d^k f_{n_i})_{i\geq 0}$ converging to a function h in the C^0 norm. In other words, there is a sequence $(l_i)_{i\geq 0}$ converging to zero, such that,

$$|d^k f_{n_i} - h| \leq l_i.$$

As the function h is continuous, then it is integrable. Let us show that the sequence $(d^{k-m} f_{n_i})_{i\geq 0}$ converges to m-times the integral of h in the C^0 norm, for all $m \in \{1, \ldots, k\}$.

$$|d^{k-m} f_{n_i} - \int_a^x \cdots \int_a^x h| \leq |\int_a^x \cdots \int_a^x (d^k f_{n_i} - h)| \leq l_i |c - a|^m.$$

Therefore, the sequence $(f_{n_i})_{i\geq 0}$ converges to k-times the integral of h in the C^k norm.

Let us prove that the sub-sequence $(f_{n_i})_{i\geq 0}$ converges in the $C^{k+\epsilon}$ norm to k-times the integral of h, for all $\varepsilon < \alpha$. Define the map $H = H_{mj} = d^k f_{n_m} - d^k f_{n_j}$. As the sub-sequence $(f_{n_i})_{i\geq 0}$ is contained in a Banach space with respect to the $C^{k+\epsilon}$ norm, we have to prove that

$$\frac{|H(y) - H(x)|}{|y - x|^{\epsilon}}$$

tends to zero when j tends to infinity, for all $m \geq j$.

If $|x - y| > l_j$,

$$\frac{|H(y) - H(x)|}{|y - x|^{\epsilon}} \leq \frac{|H(y)|}{|y - x|^{\epsilon}} + \frac{|H(x)|}{|y - x|^{\epsilon}} \leq \frac{4l_j}{(l_j)^{\epsilon}} \leq 4(l_j)^{1-\epsilon}.$$

If $|x - y| \leq l_j$,

$$\begin{aligned} \frac{|H(y) - H(x)|}{|y - x|^{\epsilon}} &\leq \frac{|d^k f_{n_m}(x) - d^k f_{n_m}(y)| + |d^k f_{n_j}(x) - d^k f_{n_j}(y)|}{|y - x|^{\epsilon}} \\ &\leq \frac{2b|y - x|^{\alpha}}{|y - x|^{\epsilon}} \leq 2b(l_j)^{\alpha-\epsilon} \end{aligned}$$

Therefore, the sequence of functions $(f_{n_i})_{i\geq 0}$ converge to a function f in the $C^{k+\epsilon}$ norm.

The function f is $C^{k+\alpha}$ smooth, because

$$\begin{aligned} |d^k f(x) - d^k f(y)| &\leq |d^k f(x) - d^k f_{n_i}(x)| + |d^k f_{n_i}(x) - d^k f_{n_i}(y)| \\ &\quad + |d^k f_{n_i}(y) - d^k f(y)| \\ &\leq 2l_i + c|x - y|^{\alpha} \end{aligned}$$

and as the sequence $(l_i)_{i\in\mathbb{N}}$ tends to zero when i tends to infinity, we obtain that $|d^k f(x) - d^k f(y)| \leq c|x - y|^{\alpha}$. ∎

5.3 The existence of a Markov map and exponential convergence to the Feigenbaum-Cvitanovic fixed point of period doubling.

5.3.1 Introduction.

Lanford [11] proved the existence of a fixed point g of the renormalisation operator R and the existence of a stable manifold with codimension one in some analytic space. Rand [22] and Sullivan [28] proved independently that two maps in the stable manifold are C^{1+} conjugated. This easily implies that, if f is in the stable manifold then f has the Feigenbaum order and there is a C^{1+} Markov map F with the following property.

$$\overline{\cap_{i \geq 0} F^{-i}(I)} = \overline{\{f^i(0), i \in Z_{\geq 0}\}}. \tag{15}$$

We give a simple proof of the convergence of the renormalisation $R^n f$ of f to a fixed point g of the renormalisation operator. For that, we assume that the map f has the Feigenbaum order and the existence of a C^{1+} Markov map F with the property of equation (15). This means that if f is C^{1+} conjugated to the renormalisation fixed point, then it converges to it. This is the converse of the result of Rand and Sullivan. Sullivan [30] proves a much more general result that $R^n f$ of f convergences to the renormalisation fixed point, just under the assumption that f has the Feigenbaum order. This excellent result relies in the use of a lot of machinery from complexes analysis and one dimensional dynamics.

5.3.2 Theorem 30.

Let $f \in C^2$ be a quadratic fold map of the interval $I = [-1, 1]$. Let $\mathcal{D}$ be the set of all folding maps. The renormalisation operator $R(f) = a^{-1} f^2 \circ a$, where $a = a(f) = f(1)$ is well defined on an open subset $\mathcal{D}(R)$ consisting of those $f \in \mathcal{D}$ such that, if $a = a(f)$ and $b = f(a)$, then $a < 0$, $b > -a$ and $f(b) \leq a$.

Let F be a C^{1+} Markov map with the following property.

$$K = K^0 = \overline{\cap_{i \geq 0} F^{-i}(I)} = \overline{\{f^i(0), i \in Z_{\geq 0}\}}.$$

The Markov map F defines a cylinder partition in I. Denote the n-cylinder containing 0 by $C_n = C^0_{\underline{1}|n}$, where we denote by $\underline{1}|n \in \Sigma^0_n$ the corresponding word, for all $n > 0$. Moreover, define $K_n = C_n \cap K$ and CC as the smallest interval containing K. Define the inverse map of $F^n : C_n \to CC$ by $F^{-n} : CC \to C_n$.

Lemma 41 The map $f^{2^n} : C_n \to C_n$ has the property that in the set K_n,

$$f^{2^n} = F^{-n} f F^n$$

Lemma 41 was proved in Sullivan [28]. Define the set

$$K^n = \overline{\{(R^n f)^i(0) : i \in \mathrm{N}\}}.$$

Define C^n as the smallest interval containing K^n. Define a sequence of linear maps $\alpha_n : \mathrm{R} \to \mathrm{R}$ by $\alpha_n(0) = 0$ and $\alpha_n(f^{2^n}(0)) = f^{2^{n-1}}(0)$. Define the map $p_n : CC \to \mathrm{R}$ by $p_n = \alpha_1 \ldots \alpha_n F^{-n}$. The map p_n is a diffeomorphism from K onto K^n.

Recall that the C^{1+} norm of a map f in a domain M is defined by

$$||f||_{C^{1+}(M)} = \max\{|f(x)|, |df(x)|, + \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in M\},$$

for some $\alpha \in (0, 1]$.

Lemma 42 If f has the Feigenbaum order and the Markov map F is C^{1+} smooth then the sequence of maps $(p_n)_{n>0}$ converge exponentially fast to a diffeomorphism $p : CC \to \mathrm{R}$ in the C^{1+} norm.

Define the set K^∞ by

$$K^\infty = \overline{\{(pfp^{-1})^i(0) : i > 0\}}.$$

Let C^∞ be the smallest interval containing K^∞. As the maps p and p_n are C^{1+} diffeomorphism, then the map $h_n : C^\infty \to \mathrm{R}$ defined by $p_n p^{-1}$ is a C^{1+} diffeomorphism. The map h_n is a C^{1+} conjugacy between the map $pfp^{-1} : K^\infty \to K^\infty$ and $f_n = R^n f : K^n \to K^n$.

Theorem 29 The conjugacies h_n converge exponentially fast to the identity map in the C^{1+} norm, when n tends to infinity.

Theorem 30 Let f be a C^2 fold map with the Feigenbaum order. Let F be a C^{1+} Markov map verifying equation 15. Let the map $d(R^n f)$ be bounded in the C^{1+} norm, for all $n \geq 0$. Then $R^n f$ converges exponentially fast to pfp^{-1} in K^∞ with respect to the C^{1+} norm, when n tends to infinity.

Corollary 20 The renormalisation limit map $R^\infty f : K^\infty \to K^\infty$ is equal to $pfp^{-1} : K^\infty \to K^\infty$.

Define the Markov family $(G_n)_{n \geq 0}$

$$G_n(x) = \begin{cases} bx & x \in [b_2, b_4] \\ bf_n(x) & x \in [b_3, b_1] \end{cases}$$

where $b_i = (f_n)^i(0)$, for all $i = 1, \ldots, 4$ and $b = (b_2)^{-1}$.

The family of Markov maps G_n define the same cylinder partition in CC as the Markov map F. Define the Markov map $F_\infty : C^\infty \to C^\infty$ by

$$G_n(x) = \begin{cases} ax & x \in [a_2, a_4] \\ a(pfp^{-1})(x) & x \in [a_3, a_1] \end{cases}$$

where $a_i = (pfp^{-1})^i(0)$, for all $i = 1, \ldots, 4$ and $a = (a_2)^{-1}$.

Corollary 21 The sequence of Markov maps G_n converges exponentially fast to the Markov map F_∞ in K^∞ with respect to the C^{1+} norm.

Theorem 31 The renormalisation limit map $R^\infty f : K^\infty \to K^\infty$ is a fixed point of the renormalisation operator, i.e. satisfies the Feigenbaum-Cvitanovic equation, $R(g) = g$.

Theorem 32 The fixed point map $R^\infty f : K^\infty \to K^\infty$ is completely determined by the scaling function of the Markov map F and the extreme points of K^∞.

Proof of lemma 41. We represent symbolically $w = \varepsilon_0 \varepsilon_1 \ldots \in K$ by $\sum_{i=0}^\infty \varepsilon_i 2^i$, where ε_i is 1 or 0, depending if $0 \in C^0_{\varepsilon_i}$ or not. In this symbolic representation $f(w) = w + 1$, $F(w) = [w/2]$ and $F^{-1}(w) = 2w + 1$ where []

means the characteristic of a number. Thus,

$$\begin{aligned}
F^n f^{2^n} F^{-n}(w) &= F^n f^{2^n} F^{-n}(\sum_{i=0}^{\infty} \varepsilon_i 2^i) \\
&= F^n f^{2^n}(\sum_{i=0}^{n-1} 2^i + 2^n \sum_{i=0}^{\infty} \varepsilon_i 2^i) \\
&= F^n(2^n - 1 + 2^n \sum_{i=0}^{\infty} \varepsilon_i 2^i + 2^n) \\
&= [(2^n - 1 + 2^n \sum_{i=0}^{\infty} \varepsilon_i 2^i + 2^n)/2^n] \\
&= \sum_{i=0}^{\infty} \varepsilon_i 2^i + 1 = w + 1 = f(w). \blacksquare
\end{aligned}$$

Proof of lemma 42. Define the map $p_{n,m} = \alpha_n \circ \ldots \circ \alpha_m \circ F^{-(m-n+1)}$, where $n < m$. Note that $p_n = p_{1,n}$ We prove this lemma in three parts. We do not distinguish between different constants c.

First part. The sequence of maps $(p_n)_{n>0}$ converges exponentially fast in the C^0 norm.

By the mean value theorem, there is $t_n \in C_n$, such that:

$$dF(t_n)(f^{2^n}(0) - 0) = F(f^{2^n}(0)) - F(0) = f^{2^{n-1}}(0) - 0.$$

Therefore, $\alpha_n(x) = dF(t_n)x$, for all $x \in \mathbb{R}$.

For all $x \in C_{i-1}$, there is $x_i' \in C_{i-1}$, such that $dF^{-1}(x_i').(x - 0) = F^{-1}(x) - F^{-1}(0) = F^{-1}(x)$. By the inverse function theorem there is $x_i \in C_i$ such that $dF^{-1}(x_i') = 1/dF(x_i)$. Thus $F^{-1}(x) = x/dF(x_i)$ and $F^{-(m-n+1)}(x) = x/\prod_{i=n}^{m} dF(x_i)$. Therefore,

$$p_{n,m}(x) = \prod_{i=n}^{m} \frac{dF(t_i)}{dF(x_i)} x.$$

As the points t_i and x_i are in C_i and dF has the same sign for all x in C_1 and it is α -Holder continuous,

$$\begin{aligned}
|\ln|dF(t_i)| - \ln|dF(x_i)|| &\leq c|dF(t_i) - dF(x_i)| \\
&\leq c|t_i - x_i|^\alpha \leq c|C_i|^\alpha
\end{aligned}$$

for all $i > 0$. By bounded geometry, there is μ between 0 and 1 such that $|C_n| \leq c\mu^n$. Thus, by last paragraph,

$$|\ln|\prod_{i=n}^{m} \frac{dF(t_i)}{dF(x_i)}|| \leq c\sum_{i=n}^{m} |C_i|^\alpha \leq c|C_n|^\alpha \leq c(\mu^n)^\alpha.$$

By the inequality above and as $dF(t_i)$ has the same sign as $dF(x_i)$,

$$\prod_{i=n}^{m} \frac{dF(t_i)}{dF(x_i)} \in 1 \pm c|C_n|^\alpha.$$

Thus,

$$p_{n,m}(x) - x = (\prod_{i=n}^{m} \frac{dF(t_i)}{dF(x_i)} - 1)x \in \pm c|C_n|^\alpha [C_{n-1}].$$

Therefore, for all $y \in CC$, there is $x \in C_{n-1}$ such that

$$\begin{aligned} p_m(y) - p_{n-1}(y) &= dF(t_1)dF(t_2)\ldots dF(t_{n-1}).(p_{n,m}(x) - x) \\ &\in c|C_n|^\alpha [I] \in c(\mu^\alpha)^n. \end{aligned}$$

Therefore, the sequence p_n converges in the C^0 norm, when n tends to infinity.

Second part. Let us prove that the sequence $(dp_n)_{n>0}$ converges in the C^0 norm. Define $x^i = F^{-i}(x)$. Thus,

$$dp_{n-1}(x) = \prod_{i=1}^{n-1} \frac{dF(t_i)}{dF(x^i)} \quad \text{and} \quad dp_m(x) = \prod_{i=1}^{m} \frac{dF(t_i)}{dF(x^i)}.$$

By the same argument as in part 1,

$$\prod_{i=n}^{m} \frac{dF(t_i)}{dF(x^i)} \in 1 \pm c|C_n|^\alpha$$

Therefore,

$$\begin{aligned} dp_m(x) - dp_{n-1}(x) &= \left(\prod_{i=1}^{n-1} \frac{dF(t_i)}{dF(x^i)}\right).\left(\prod_{i=n}^{m} \frac{dF(t_i)}{dF(x^i)} - 1\right) \\ &\in (1 \pm c|C_1|^\alpha)(\pm c|C_n|^\alpha) \in c(\mu^\alpha)^n. \end{aligned}$$

Thus, the sequence dp_n converges uniformly in the C^0 norm, when n tends to infinity. In the same way,

$$dp_m^{-1}(x) - dp_{n-1}^{-1}(x) \in c(\mu^\alpha)^n.$$

Therefore, dp_n is bounded away from zero, independent of n.

We will use that

$$\frac{dp_m(x)}{dp_n(x)} = \prod_{i=n+1}^{m} \frac{dF(t_i)}{dF(x^i)} \in 1 \pm c(\mu^\alpha)^n \tag{16}$$

in the proof of theorem 30.

Third part. Define the map $H = H_{m,n} : CC \to \mathbb{R}$ by $H = dp_m - dp_{n-1}$. Let us prove that

$$\frac{|H(x) - H(y))|}{|x-y|^\epsilon}$$

tends exponentially fast to zero, when n tends to infinity, where $\epsilon < \alpha$ and $m > n$. This is a consequence of the exponential convergence in part 2.

First, we prove that $|dp_{n-1}(x) - dp_{n-1}(y)| < c|x-y|^\alpha$. By the same argument as in part one and because F is an expanding map,

$$\frac{dp_{n-1}(x)}{dp_{n-1}(y)} = \prod_{i=1}^{n-1} \frac{dF(y^i)}{dF(x^i)} \in 1 \pm c|x-y|^\alpha. \tag{17}$$

As dp_{n-1} converges exponentially fast to dp,

$$|dp_{n-1}(x) - dp_{n-1}(y)| < |dp_{n-1}(y)|| \frac{dp_{n-1}(x)}{dp_{n-1}(y)} - 1| \leq c|x-y|^\alpha.$$

Denote $\nu = \mu^\alpha$. If $|x-y| > \nu^n$,

$$\frac{|H(x) - H(y))|}{|x-y|^\epsilon} \leq |\frac{|H(x)| + |H(y)|}{|x-y|^\epsilon} \leq \frac{c\nu^n}{(\nu^n)^\epsilon} \leq c(\nu^{1-\epsilon})^n.$$

If $|x-y| < \nu^n$,

$$\begin{aligned} \frac{|H(x) - H(y))|}{|x-y|^\epsilon} &\leq |\frac{|dp_m(x) - dp_m(y)| + |dp_{n-1}(x) - dp_{n-1}(y)|}{|x-y|^\epsilon} \\ &\leq \frac{2c|x-y|^\alpha}{|x-y|^\epsilon} \leq 2c(\nu^{\alpha-\epsilon})^n. \end{aligned}$$

Therefore, the sequence of maps $(p_n)_{n>0}$ converge exponentially fast to a map p in the C^{1+} norm. As the maps dp_n are bounded away from zero, the map p is a C^{1+} diffeomorphism. ∎

Proof of theorem 29. Similarly to lemma 42. ∎

Proof of theorem 30. By definition, in the set CC,

$$f_n = R^n f_{|CC} = \alpha_1 \dots \alpha_n f^{2^n} \alpha_n^{-1} \dots \alpha_{1|CC}^{-1}.$$

Moreover, by lemma 41, in the set K^n,

$$\alpha_1 \dots \alpha_n f^{2^n} \alpha_n^{-1} \dots \alpha_{1|K^n}^{-1} = \alpha_1 \dots \alpha_n F^{-n} f_{|K} F^n \alpha_n^{-1} \dots \alpha_{1|K^n}^{-1}.$$

The last map has a C^{1+} extension to CC but the last equality may not be true in this extension. For all $x \in K^\infty$, define $y = p^{-1}(x)$ and $x_n = p_n(y) \in K^n$. Denote, pfp^{-1} by g and $p_n f p_n^{-1}$ by g_n. Thus, $f_n(x_n) = g_n(x_n)$.

We will prove that f_n converges exponentially fast to g in K^∞, with respect to the C^{1+} norm. We prove it in three parts.

First part. Let us prove $|f_n(x) - g(x)| < e_1 \nu^n$. As all the points $x_n \in K^n$ are accumulation points in K^n and $f_n(x_n) = g_n(x_n)$ then $df_n(x_n) = dg_n(x_n)$. Therefore, there is a constant $c > 0$,

$$|df_n(z)| \leq |df_n(1)| = |dg_n(1)| \leq c.$$

For all $x \in K^\infty$,

$$\begin{aligned} |f_n(x) - g(x)| &\leq |f_n(x) - f_n(x_n)| + |f_n(x_n) - g(x)| \\ &\leq |df_n(z)||x - x_n| + |g_n(x_n) - g(x)| \end{aligned}$$

The first term is less than $c\nu^n$ because $|df_n(z)| \leq c$ and by first part of lemma 42, there is $y \in C$, such that, $|x - x_n| = |p(y) - p_n(y)| \leq c\nu^n$.

The second term is less than $c\nu^n$ because by first part of lemma 42,

$$|g_n(x_n) - g(x)| \leq |p_n f(y) - pf(y)| \leq c\nu^n.$$

Therefore, $|f_n(x) - g(x)| \leq c\nu^n$.

Second part. Let us prove that $|df_n(x) - dg(x)| \leq c(\nu^\alpha)^n$. We have that,

$$\begin{aligned} &|df_n(x) - dg(x)| \\ &\leq |df_n(x) - df_n(x_n)| + |df_n(x_n) - dg(x)| \\ &\leq |df_n(x) - df_n(x_n)| + |dg_n(x_n) - dg(x)| \\ &\leq |df_n(x) - df_n(x_n)| \\ &\quad + |dp(f(y)) \cdot df(y) \cdot dp^{-1}(x)| \left| \frac{dp_n(f(y)) \cdot dp_n^{-1}(x_n)}{dp(f(y)) \cdot dp^{-1}(x)} - 1 \right|. \end{aligned}$$

The first term is less than $c(\nu^\alpha)^n$, because by hypothesis and by the first part of lemma 42,

$$|df_n(x) - df_n(x_n)| \leq c|x - x_n|^\alpha \leq c|p(y) - p_n(y)|^\alpha \leq c\nu^n.$$

Let us prove that the second term is less than $c\nu^n$. As $|df(y)| \leq c$ and p is a diffeomorphism onto its image,

$$|dp(f(y))dF(y)dp^{-1}(x)| \leq c.$$

By equations 16 and 17 of lemma 42,

$$\frac{dp_n(f(y))}{dp(f(y))}\frac{dp_n^{-1}(x_n)}{dp_n^{-1}(x)}\frac{dp_n^{-1}(x)}{dp^{-1}(x)} \in 1 \pm c\nu^n.$$

Therefore, $|df_n(x) - dg(x)| \leq c(\nu^\alpha)^n$.

Third part. Define the map $H = df_n - dg$ and $\gamma = \nu^\alpha$. Let us prove that,

$$\frac{|H(x) - H(z))|}{|x - z|^\epsilon} \leq c(\gamma^{\alpha-\epsilon})^n$$

where $\varepsilon < \alpha$. This is a consequence of the exponential convergence in part 2.

If $|x - z| \geq \gamma^n$,

$$\frac{|H(x) - H(z))|}{|x - z|^\epsilon} \leq \frac{|H(x)| + |H(z)|}{|x - z|^\epsilon} \leq \frac{2c\gamma^n}{(\gamma^\epsilon)^n} \leq c(\gamma^{1-\epsilon})^n$$

If $|x - z| < \gamma^n$ then

$$\begin{aligned}\frac{|H(x) - H(z))|}{|x - z|^\epsilon} &\leq \frac{|df_n(x) - df_n(z)| + |dg(x) - dg(z)|}{|x - z|^\epsilon}\\ &\leq \frac{c|x - z|^\alpha}{|x - z|^\epsilon} \leq c(\gamma^{\alpha-\epsilon})^n.\end{aligned}$$

Therefore, $||f_n - g||_{C^{1+}(K^\infty)} \leq c(\gamma^{\alpha-\epsilon})^n$, for all $n > 0$. ∎

Proof of corollary 20. By theorem 30, the maps $R^n f$ converge exponentially fast to pfp^{-1} in K^∞ with respect to the C^{1+} norm. Therefore,

$$R^\infty f_{|K^\infty} = pfp^{-1}_{|K^\infty}. \blacksquare$$

Proof of corollary 21. Similarly to the proof of theorem 30. ∎

Proof of theorem 31. Similarly to the proof of lemma 42, we have that $p_n F^{-1} : CC \to \mathbb{R}$ is equal to $\alpha_1 \ldots \alpha_n \ldots F{-}1 \ldots F{-}n$ converges exponentially fast to $(dF(0))^{-1}p$ in CC with respect to the C^{1+} norm.

By lemma 41, in set K^n,

$$(R^n f)^2_{|K^n} = p_n f^2 p^{-1}_{n|K^n} = p_n F^{-1} f F p^{-1}_{n|K^n}.$$

Similarly to the proof of theorem 30, we obtain that $(R^n f)^2_{|K^\infty}$ converges exponentially fast to $(dF(0))^{-1} p f p^{-1} dF(0)_{|K^\infty}$, when n tends to infinity.

As the limit of $(R^n f)^2_{|K^\infty}$ is unique and by corollary 20,

$$(R^\infty f)^2_{|K^\infty} = (dF(0))^{-1} R^\infty f dF(0)_{|K^\infty}. \blacksquare$$

Proof of theorem 32. The Markov map F_∞ and F are C^{1+} conjugated. Therefore, they define the same scaling function by theorem 25. The Markov map F_∞ has an affine branch, with the fixed point 0 contained in its domain. Let $\bar{1} \in \Sigma_*^{F_\infty}$ be such that $0 \in C_{\bar{1}|n}$. Therefore, For all $t \in \Sigma_n^{F_\infty}$ and all $n > 0$, $\sigma_{F_\infty}(t) = \lim_{m \to \infty} t(\bar{1}|nt) = \sigma_F(\bar{1}t)$. $\blacksquare$

Bibliography

[1] V. I. Arnol'd, Geometrical methods in the theory of ordinary differential equations. *Springer-Verlag.*

[2] V. I. Arnol'd, Small denominators I; on the Mapping of a circle into itself. *Investijia Akad. Nauk* Math, 25, 1, 1961, 21-96. *Transl.* A.M.S., $2^n d$ series, 46, 213-284.

[3] P. Collet and J. P. Eckmann, Iterated maps on the interval as dynamical systems. *Birkhauser* (1980).

[4] P. Collet and C. Tresser, Iterations d'endomorphismes et groupe de renormalisation. *J. Phys.* C5 (1978) 25-28.

[5] A. Denjoy, Sur les courbes définies par les equations différentielles à la surface du tore. *J. de Math. Pures et Appl.* (9), 11, 1932, 333-375.

[6] M. J. Feigenbaum, Quantitative universality for a class of non-linear transformations. *J. Stat. Phys.* **19** (1978) 25-52.

[7] M. J. Feigenbaum, Universal metric properties of non-linear transformations. *J. Stat. Phys.* **21** (1979) 669-706.

[8] M. J. Feigenbaum, L. P. Kadanoff and S. J. Shenker, Quasi-periodicity in dissipative systems: a renormalisation group analysis. *Physica* 5D (1982) 370-386.

[9] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. In *Publ. IHES* n^{o}49, 1979, 5-233.

[10] L. Jonker and D. A. Rand, Universal properties of maps of the circle with ε-singularities. In *Commun. Math. Phys.* **90** (1983) 273-292.

[11] O. Lanford, A numerical study of the likelihood of phase locking. In *Physica* **14D** 403-408.

[12] O. E. Lanford, A computer-assisted proof of the Feigenbaum conjectures. In *Bull. Am. Math. Soc.* **6** 427-434.

[13] B. D. Mestel, Computer assisted proof of universality for cubic critical maps of the circle with golden mean rotation number. In *Ph.D. Thesis Warwick University.*

[14] J. K. Moser, A rapidly convergent iteration method, part II. In *Ann. scient. Norm. Sup. di Pisa* series III, 20, 1966, 449-535.

[15] S. Ostlund, D. A. Rand, J. Sethna and E. Siggia, Universal properties of the transition from quasi-periodicity to chaos in dissipative systems. In *Physica D* 8 (1983) 303-342.

[16] B. Parry, Lecture notes on numbers and dynamics, Warwick University.

[17] A. A. Pinto and D. A. Rand, A classification of $C^{1+\alpha}$ structures on embedded trees. Warwick preprint: 1/1991.

[18] A. A. Pinto and D. A. Rand, Global phase space universality, smooth conjugacies and renormalisation: 2. The $C^{k+\alpha}$ case using rapid convergence of Markov families. Warwick preprint.

[19] A. A. Pinto and D. A. Rand, Convergence of renormalisation, Markov families and rigidity of dynamical systems: A unification of smooth conjugacy and universality results. In preparation.

[20] A. A. Pinto, M.Sc. Dissertation, University of Warwick, 1988.

[21] D. A. Rand, Universality and renormalisation in dynamical systems. In *New Directions in Dynamical Systems* (ed. T. Bedford and J. Swift) 1-56, (1988), CUP

[22] D. A. Rand, Global phase space universality, smooth conjugacies and renormalisation: 1. The $C^{1+\alpha}$ case. *Nonlinearity* **1** (1988) 181-202.

[23] D. A. Rand, The global rigidity conjecture and convergence of renormalisation. Papers in honour of Rene Thom. Macmillan, 1989.

[24] D. A. Rand, Existence, non-existence and universal breakdown of dissipative golden invariant tori. I. Golden critical circle maps. IHES Preprint, 1989.

[25] D. A. Rand, Existence, non-existence and universal breakdown of dissipative golden invariant tori. II. Convergence of renormalisation for mappings of the annulus. IHES Preprint, 1989.

[26] D. A. Rand, Fractal bifurcation sets, renormalisation strange sets and their universal invariants. Proceedings of the Royal Society Discussion Meeting on Dynamical Chaos.

[27] J. Stark, Smooth conjugacy and renormalisation for diffeomorphisms of the circle. In *Nonlinearity* **1** (1988) 541-575.

[28] D. Sullivan, Differentiable structures on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets. In *Nonlinear evolution and chaotic phenomena.* (eds. G. Gallavotti and P. Zweifel), Plenum, New York, 1988.

[29] D. Sullivan, Quasiconformal homeomorphisms in dynamics, topology and geometry. Preprint 1987

[30] D. Sullivan, Bounds, quadratic differentials, and renormalisation conjectures. To appear in the AMS volume celebrating the Centenary of the American Mathematical Society, 1990.

[31] J. C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. In *Ann. scient. Éc. Norm. Sup.* 4 série, t. 17, 1984, 333-359.