Kinematic Fast Dynamo Problem

by

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Declarations

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. This work has not been submitted for any other degree.

An original scheme of the fluid flow that is presented in Fig. 1.1 has been suggested by Dr. Kozlovski. He also has helped me with technical results, which contained in Subsection 3.3.1 “Discretization and the Weierstrass transform toolbox”.

Abstract

In the present work we develop an approach to the classical kinematic fast dynamo problem for flows [32] in the real 3-dimensional space. We suggest a fluid flow that may possibly generate a magnetic field which energy grows exponentially fast with time in the present of slow diffusivity. In order to verify the construction we study a discrete system and prove that an analogous statement holds true for the Poincaré map of the provisional flow and vector fields in the plane.

This problem falls into the framework of open dynamical systems with random holes.

1 Introduction

1.1 A problem of magnetohydrodynamics

The subject of magnetohydrodynamics is evolution and interaction of motions of an electrically conducting fluid and an electromagnetic field. Typical examples of electrically conducting fluids that dynamo theory is dealing with are the liquid layer of the core of the Earth or convection zones of stars, although we will be studying very simplified models. Dynamo theory studies the mechanism of generation of magnetic fields in electrically conducting fluids as a phenomenon of magnetohydrodynamics [25]. The classical kinematic fast dynamo problem [32], [36] is dating back to 1970-s and concerns the evolution of a magnetic field in a conducting fluid flow in the presence of small diffusion, or, in other words, when the magnetic Reynolds number is large. The magnetic Reynolds number R_s is a dimensionless parameter that is used to describe the relative balance of magnetic advection to magnetic diffusion. It is proportional to the electric conductivity and the velocity of the fluid and to the length of a characteristic fluid structure. The kinematic dynamo equations read

$$\begin{cases} \dfrac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B + \varepsilon \Delta B \\ \nabla \cdot v = \nabla \cdot B = 0, \end{cases} \tag{1.1}$$

where v is the known velocity field of the conducting fluid filling a certain compact domain M. We will be assuming that the vector field v is tangent to the boundary ∂M; B is the magnetic field, and $\varepsilon = \frac{1}{R_s}$ is a parameter corresponding to the speed of diffusion through the boundary of M. The case of slow diffusion corresponds to an almost perfectly conducting fluid.

Definition 1. The action of the velocity field v on the magnetic field B described by the system (1.1) is called *dynamo action*. A divergence-free C^1 vector field v with compact support is called a *kinematic fast dynamo* if the magnetic field grows stronger exponentially fast with time.

Dynamo action and chaotic motion turn out to be closely related. It has been shown by Klapper and Young [19] that the growth rate of the magnetic field is bounded by topological entropy of the fluid flow. Kozlovski [21] has shown that the growth rate is related to the topological entropy, Lyapunov exponents, and topological pressure. The limit chaotic motion, corresponding to the perfectly conducting liquid ($\varepsilon = 0$), causes the magnetic field B to inherit the complexity of the Lagrangian chaos.

It turns out that in dynamo theory the magnetic field reflects closely the motions of the fluid, just as the swirls of cream in a cup of coffee reveal the pattern of eddies stirred by spoon. In other words, the changes of magnetic field keep the track of the movements of the fluid, and one can reconstruct the geometry of the flow from the magnetic field. If we consider a magnetic field as a collection of magnetic lines, the fast dynamo corresponds to the growth of an average line length in a flow and thus stretching and folding properties of the flow.

The Lorenz force causes a feedback action of the magnetic field on the velocity field. When the magnetic field is small, one can neglect this action. Whence the full nonlinear system of magnetohydrodynamics may be reduced [10] to the system (1.1) in the case of an incompressible fluid.

The full pre-Maxwell system of magnetohydrodynamics may be written as

$$\text{Ampere's Law} \qquad \nabla \times B = \mu J, \tag{1.2}$$

$$\text{Faraday's Law} \qquad \nabla \times E = -\frac{\partial B}{\partial t}, \tag{1.3}$$

$$\text{Ohm's Law} \qquad J = \sigma(E + v \times B). \tag{1.4}$$

The magnetic field is divergence-free:

$$\nabla \cdot B = 0. \tag{1.5}$$

In the equations above $B(\bar{x}, t)$ is the magnetic field, $E(\bar{x}, t)$ is the electric field, $J(\bar{x}, t)$ is the current, μ is the magnetic permeability in the vacuum, σ is the electrical conductivity, and v is the velocity field of the fluid.

We can substitute (1.4) into (1.2) and apply the curl operator to both sides. Then we get the induction equation

$$\frac{\partial B}{\partial t} - \nabla \times (v \times B) - \varepsilon \nabla^2 B = 0, \tag{1.6}$$

where

$$\varepsilon = \frac{1}{\mu\sigma} = \text{ magnetic density.}$$

We may expand

$$\nabla \times (v \times B) = B \cdot \nabla v - v \cdot \nabla B + (\nabla \cdot B)v - (\nabla \cdot v)B,$$

and recall the incompressibility condition $\nabla \cdot v = 0$. Together with (1.5) we get

$$\nabla \times (v \times B) = (B \cdot \nabla)v - (v \cdot \nabla)B.$$

Finally, we substitute it to (1.6) and obtain (1.1):

$$\frac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B + \varepsilon \Delta B = 0.$$

The following question is well-known as "The Kinematic Fast Dynamo Problem".

Problem 1. Whether or not there exist a divergence-free velocity field v with a compact support $\operatorname{supp} v = M$ such that the energy $E(t) = \|B(t)\|^2_{\mathcal{L}^1(M)}$ of the magnetic field $B(t)$ grows exponentially with time for some initial condition $B(0) = B_0$ with $\operatorname{supp} B_0 = M$, and for arbitrary small diffusivity ε? In other words [1], does kinematic fast dynamo exist?

The exponential growth of the magnetic energy is equivalent to

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\ln\int_{\mathbb{R}^d}|B(z,t,\varepsilon)|\mathrm{d}z > 0 \tag{1.7}$$

The main interest is related to stationary velocity fields v in two- and three-dimensional domains M.

Looking at the heat equation one may deduce [34] that the exponent of the Laplace operator is acting on vector fields by convolution with the heat kernel:

$$(\exp(\varepsilon\Delta)v)(z) = \int_{\mathbb{R}^d}\frac{1}{(\sqrt{2\pi}\varepsilon)^d}\exp\Big(-\frac{|z-t|^2}{2\varepsilon^2}\Big)v(t)\mathrm{d}t$$

1.1.1 The main result

We suggest a fluid flow on a 3-dimensional manifold immersed in $\mathbb{R}^3$, that may possibly generate a magnetic field which energy grows exponentially fast with time in the present of slow diffusivity; and therefore give a positive answer to a long standing Problem 1. The flow is chaotic and structurally stable. In order to verify the example we show that an analogous statement holds true for the Poincaré map of the provisional flow and vector fields in the plane. The main result is the following

Theorem 9. There exists a volume preserving piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\|(\exp(\varepsilon\Delta)F_*)^n\,B_0\|_{\mathcal{L}_1} > 0.$$

The map F may be realised as a Poincaré map of an incompressible fluid flow filling a compact domain in $\mathbb{R}^3$ (an immersed 3-dimensional manifold with a boundary).

1.1.2 Discrete problem

The Problem 1 has a discrete analogue, where the flow action is replaced by a diffeomorphism, and dissipation is represented by action of $\exp(\varepsilon\Delta)$. The Kinematic Fast Dynamo problem for diffeomorphisms has been stated by Arnold [1] in the following form.

Problem 2. Does there exist a volume-preserving diffeomorphism $g\colon M \to M$ of a compact manifold M such that the energy of the magnetic field B grows exponentially with the number of iterations of the map

$$B \mapsto \exp(\varepsilon\Delta)(g_* B) \tag{1.8}$$

for some initial vector field B_0 and for arbitrary small diffusivity ε?

In other words,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\int_{\mathbb{R}^d}|(w_\varepsilon * g_*)^n B_0(z)|\mathrm{d}z > 0, \tag{1.9}$$

where w_ε is the d-dimensional Gaussian density with isotropic variance ε:

$$w_\varepsilon(z) \stackrel{\text{def}}{=} \frac{1}{(\sqrt{2\pi\varepsilon})^d}\exp\Big(-\frac{|z|^2}{2\varepsilon^2}\Big); \tag{1.10}$$

where g_* is induced action on vector fields and $*$ stands for convolution. Nowadays the discrete analogue is a problem of particular interest itself and maps have become a popular model for fast dynamos [6], [13], [14], [30].

1.2 Brief history

While the realistic dynamo problem is still open, the non-dissipative case, corresponding to perfectly conducting fluid ($\varepsilon = 0$ in the equation (1.1)), is easy. It is well known [33], [14] that non-dissipative kinematic fast dynamos exist on all manifolds.

Theorem. *On an arbitrary n-dimensional manifold any divergence-free vector field with a stagnation point with a unique positive eigenvalue is a non-dissipative kinematic fast dynamo.*

The case of realistic dynamo action $\varepsilon > 0$ is not so simple. There is numerical evidence of dynamo action in helical flows [6], ABC flows [16], and Möbius flows [31]. Yet, there is no rigorous mathematical argument for these examples nor for flows in $\mathbb{R}^3$ in general. In particular, there is no continuity of the spectrum of the corresponding operator as $\varepsilon \to 0$.

The only constructions known are discrete dynamos in two dimensional surfaces with non-trivial first homology group $H_1(M, \mathbb{R})$.

Main features of these examples are coming from the cat map on the torus [3]. Consider

$$g\colon T^2 \to T^2, \qquad g\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1.$$

The expanding direction at all points is given by eigenvector $B_0 = \left(\frac{1+\sqrt{5}}{2}\right)$ with eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}$. Therefore, the constant magnetic field $B \equiv B_0$ grows exponentially with number of iterations of the map g:

$$B_n = (g_*)^n B_0 = \lambda^n B_0; \qquad \|B_n\| = \lambda^n \|B_0\|.$$

Added diffusion doesn't spoil the example, since an average of a constant field is the same constant field.

This example has been generalised in [24] to arbitrary diffeomorphisms of the torus. Later, a more general result has been established [1].

Theorem. *Let $g\colon M \to M$ be an area-preserving diffeomorphism of the two-dimensional compact Riemannian manifold M. Then g is a dissipative fast dynamo if and only if the* *induced linear operator g_{*1} on the first homology group has an eigenvector λ with $|\lambda| > 1$. The dynamo growth rate is independent of ε:*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|B_n\| = \ln |\lambda|$$

for almost any initial vector field B_0. (Here $B_{n+1} = \exp(\varepsilon\Delta) g_* B_n$.)

The argument exploits duality between vectors and one-forms on the surfaces and commutativity between the Laplace-Beltrami operator and the exterior derivative. Therefore, it is not possible to extend it to higher dimensions.

On the negative side, there are antidynamo theorems, specifying geometric properties of the manifold M where flows with fast dynamo action are impossible. A very early result [11] states that "A steady magnetic field in $\mathbb{R}^3$ that is symmetric with respect to rotations about a given axis cannot be maintained by a steady velocity field that is also symmetric with respect to rotations about the same axis". This result has been generalised [22], [26] and it is now understood that the symmetry of the magnetic field alone is not compatible with exponential growth.

Theorem. *A transitionally, helically, or axially symmetric magnetic field in $\mathbb{R}^3$ cannot be maintained by a dissipative dynamo action.*

Our goal is to construct a 3-dimensional flow, that will resolve Problem 1 positively. A possible model is discussed below. In order to study the flow, we begin with Poincaré map. Theorem 9 (p. 162) shows that the inequality (1.9) holds true with g chosen to be a simplified Poincaré map of the flow. Although simpler than the flow itself, the Poincaré map is still difficult to study. Therefore we begin with a simple one-dimensional map, which would be a reduction of the Poincaré map, and show in Theorem 6 (p. 94) that the inequality (1.9) holds true for this one-dimensional case.

1.3 Provisional fluid flow

The following model for the fluid flow on a 3-dimensional manifold, displayed in Figure 1.1, has been suggested by Dr. O. Kozlovski. Topologically, the manifold is equivalent to a solid 3-dimensional body whose boundary is a sphere with three handles. The vector field has two lines of saddles ℓ_1 and ℓ_2, which are orthogonal to each other and do not intersect. Light blue two-dimensional surfaces consist of separatrices of the saddles. Blue dashed lines with arrows represent solid tubes $\tau_{1,\ldots,4}$ with cylindrical boundaries that connect two surfaces. Dark blue arrows stand for the velocity field of the fluid flow, and red arrows is the stationary initial induction field B_0. We assume that the fluid flow is stationary outside of a neighbourhood of the manifold and its velocity tends to zero rapidly near the boundary. Blue boundaries mark "the dynamo manifold", where the exponential growth of the initial induction field takes place.

The induced mapping between the sections $\{\pi, \sigma_{1,\ldots,4}\}$, is shown in Figure 1.2. In particular, we see that any point that leaves the dynamo manifold due to diffusion is being attracted to the unstable manifolds of the saddles S_1 and S_2. In addition, we see two heteroclinic connections clearly. To complete the construction one has to define gluing between the green surfaces $\sigma_{1,\ldots,4}$ by tubes, and to make sure that unstable separatrices of two periodic saddle points S_1 and S_2 eventually enter the tube τ_3. This will guarantee that all trajectories, that leave the manifold due to diffusion, either return back shortly, and the frozen into the fluid[1] magnetic field doesn't change much, or go into a long tube τ_3, which causes large return time. An alternative would be to make unstable separatrices to be attracted to periodic cycles of

[1] We say that a vector v field is frozen into a moving fluid if $E + v \times B = 0$, which corresponds $\sigma \gg 1$ in the Ohm's law (1.4). In practice, it means that when a surface consisting of magnetic field lines is moved by the flow, it changes, but none of the field lines become orthogonal to the surface.

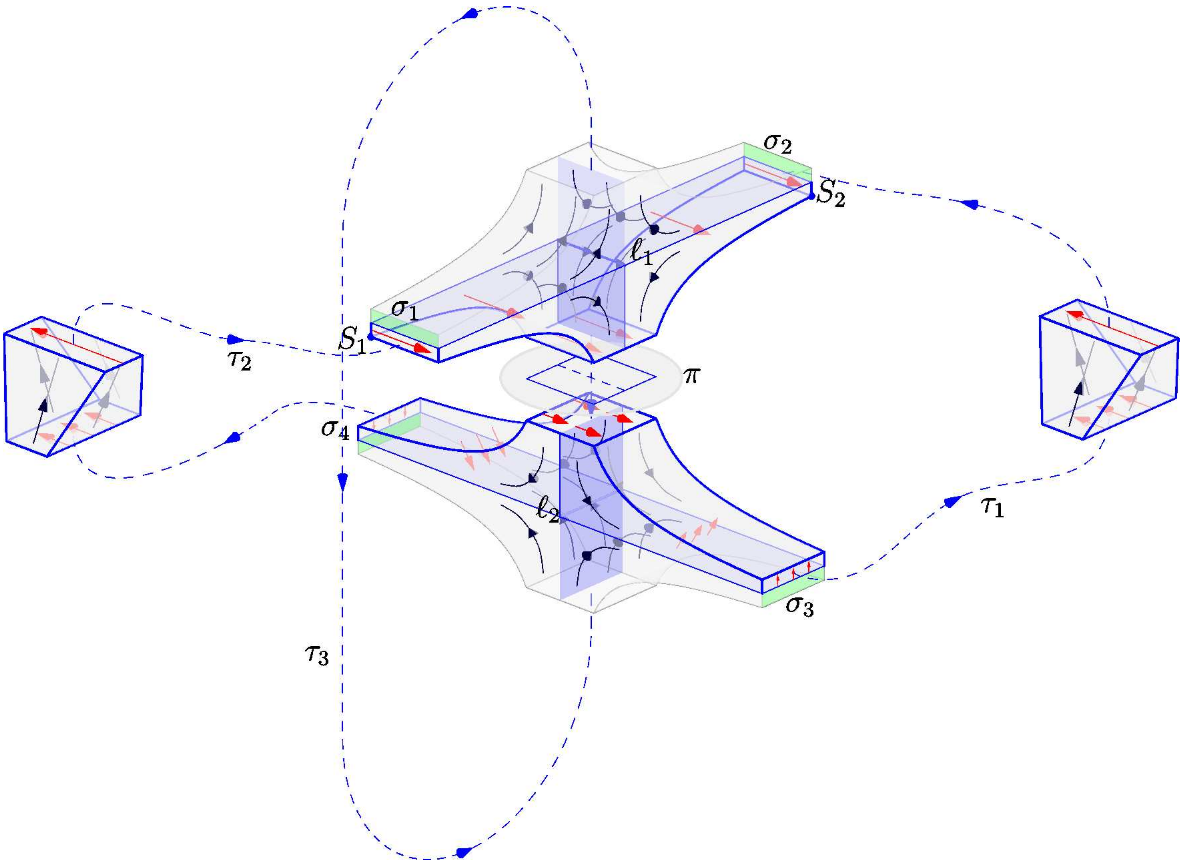

Figure 1.1: Dynamo manifold with the fluid flow (blue) and magnetic induction field (red). The labels S_1 and S_2 mark periodic saddle points.

a huge period. This seems to be possible, although we are still working on the details.

We also would like to point out, that any small perturbation of the presented 3-dimensional flow possess fast dynamo action as well. Therefore, once this example is verified, we will be able to show that dynamo flows are generic.

1.4 Poincaré map

In order to study the flow, one can consider a global Poincaré section π, and the first return map F. The intersection between the plane π and the dynamo manifold has four connected components. Three of them are intersections with the tubes τ_1, τ_2, τ_3 and another one is "a square" which is shown in Figure 1.1. The restriction of the Poincaré map onto the square is representative for studying the flow action; and deserves a special consideration. In particular, it is an unfolded[1] Baker's map and demonstrates chaotic properties. Since near the intersection with the separatrices of the saddles ℓ_2 the first return time is huge, a proper 2-dimensional model for the Poincaré map would be a map with a Z-shaped hole, as shown in Figure 1.3.

Outside of the square the first return map F has the following properties.

1. It is piecewise continuous and bijective.
2. It is area preserving.
3. The Euclidean norm of the differential is uniformly bounded $\|\mathrm{d}F\| \leq 1 + \mu$ for a small $\mu > 0$.
4. The Hessian is small $\|\mathrm{d}^2 F\| \leq \mu_2$ for a small $\mu_2 > 0$.

In addition, we shall impose an artificial condition in order to guarantee that the map outside of the unit square doesn't "bend" too much. This condition in principle should be replaced by a statement similar to Yomdin's Lemma on volume growth [35].

Consequently, as a first step we may try to show that the *unfolded Baker's map* itself is a fast dynamo in the presence of slow diffusion through the boundary. This is the main result of the present work (Theorem 9 p. 162).

[1] In literature two different maps are being referred to as "Baker's map". By unfolded we mean the one that doesn't change orientation of the vector field. A precise definition is given by (1.11).

1.5 Principal obstacles and general strategy

In the absence of diffusion ($\varepsilon = 0$) we may choose the initial magnetic field B_0 to be collinear with the stretching direction of the Baker's map on the square and then transfer it out all over the dynamo manifold using the fluid flow. The added diffusion makes the energy of the vector field to dissipate through the boundary as a solution of the heat equation. Baker's maps were suggested as a model for kinematic fast dynamo long ago ([13], for example), and a numerical evidence was found for the exponential growth of magnetic energy [12]. However, there was no rigorous analytical argument in the presence of diffusion.

In order to be more specific, let us introduce a shorthand notation for the unit square

$$\square\colon = \{(x, y) \in \mathbb{R}^2 \mid |x| < 1,\ |y| < 1\}.$$

and consider the unfolded Baker's map

$$P(x,y) = \begin{cases} \left(\frac{x-1}{2}, 2y+1\right), & \text{if } x \in \square,\ -1 < y < 0, \\ \left(\frac{x+1}{2}, 2y-1\right), & \text{if } x \in \square,\ 0 < y < 1, \\ F(x,y), & \text{if } (x,y) \in \mathbb{R}^2 \setminus \square; \end{cases} \tag{1.11}$$

where $F\colon \mathbb{R}^2 \setminus \square \to \mathbb{R}^2 \setminus \square$ is an area-preserving piecewise diffeomorphism with uniformly bounded Jacobian $\|\partial F\| < 1 + \mu$ (as the Euclidean norm of a linear operator) and such that any point has not more than $d \ll M$ preimages with respect to F^M for some large M. Our goal is to show that there exists a vector field B_0 such that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \int_{\mathbb{R}^2} \left|(\exp(\varepsilon\Delta)P_*)^n B_0\right| > 0. \tag{1.12}$$

It is sufficient to construct two cones C_1 and C_2 in the space of essentially bounded vector fields with finite $\mathcal{L}_1$ norm such that for some $\delta > 0$ and any sufficiently large m

$$(\exp(\varepsilon\Delta)P_*)^m(\overline{C_1}) \subsetneq C_2 \subsetneq C_1 \quad \text{and} \quad \|(\exp(\varepsilon\Delta)P_*)^m|_{C_1}\| \geq (1+\delta)^m$$

The argument is based on the following ideas.

Noise instead of diffusion. The idea to replace the diffusion by noise added to the system has been used by Klapper and Young in [19]. One can introduce a "small perturbation" of the original map

$$P_t \colon = P + t$$

and associate a composition of small perturbations to any sequence $\bar{t} \in \ell_\infty(\mathbb{R}^2)$ by

$$P^m_{\bar{t}} \colon = P_{t_m} \circ P_{t_{m-1}} \circ \ldots \circ P_{t_0}.$$

Then by the Noise Lemma 2.2.1 with $\widehat{t} = 0, t_{m-1}, \ldots, t_1$:

$$(\exp(\varepsilon\Delta)P_*)^m v(z) = \int_{\mathbb{R}^{2(n-1)}} w_\varepsilon(t_1) w_\varepsilon(t_2) \ldots w_\varepsilon(t_{m-1}) (\exp(\varepsilon\Delta) P^m_{\widehat{t}*} v)(z) \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_{m-1}, \tag{1.13}$$

where w_ε is the two-dimensional Gaussian kernel with isotropic variance ε, defined by (1.10). It follows that it is enough to construct a pair of cones C_1 and C_2 such that for arbitrary sequence of small vectors $\widehat{t}$

$$\exp(\varepsilon\Delta) P^m_{\widehat{t}*}(\overline{C_1}) \subsetneq C_2 \subsetneq C_1 \quad \text{and} \quad \| \exp(\varepsilon\Delta) P^m_{\widehat{t}*} |_{C_1} \| \geq (1+\delta)^m$$

The choice of the norm. By definition, a cone is a convex subset which is invariant with respect to multiplication by a non-negative real number. The cones we will be dealing with have a general form

$$\mathrm{Cone}\,(v_k, \alpha_k) \colon = \{ d v_k + w \mid \|w\| \leq d 2^{-\alpha_k} \|v_k\|, d \in \mathbb{R}^+ \}.$$

We say that the cone $\mathrm{Cone}\,(v_1, \alpha_1)$ is smaller than the cone $\mathrm{Cone}\,(v_2, \alpha_2)$, if $\alpha_1 > \alpha_2 > 0$. We do not require here that $\mathrm{Cone}\,(v_1, \alpha_1) \cap \mathrm{Cone}\,(v_2, \alpha_2) \neq \varnothing$.

In order to construct a pair of cones, it helps to choose the norm in a proper way. The diffusion represented by convolution with the Gaussian kernel means that the energy of a vector field that changes direction rapidly cannot grow very fast due to cancellations [14].

It is not unreasonable to suggest therefore that *piecewise constant vector fields* will grow rapidly. Following this idea, we introduce a class $\mathcal{G}(m,\delta)$ of partitions of the real plane with the following properties.

1. The unit square $\square$ contains at most 4^m and at least 4^{m-1} elements of the partition; the interior of an element of the partition does not intersect the boundary of the square.

2. Any element of the partition contains a square with side length $\frac{1}{m}2^m$ and is contained in a square with side length 2^{m+1}.

3. Any square with a side δ may be covered by at most $N_\delta = 2^{2m+1}\delta^2$ elements of the partition.

To any partition Ω of the class $\mathcal{G}(m,\delta)$ we associate a weighted $\mathcal{L}_1$ norm on the space of vector fields by (cf. Subsection 4.2.2):

$$\|v\|_{\Omega,\mathcal{L}_1} \stackrel{\text{def}}{=} \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v|,$$

where π_y represents orthogonal projection onto the expanding direction of the Baker's map. The supremum norm of a vector field v we denote by $\|v\|_\infty \stackrel{\text{def}}{=} \sup|v|$. Finally, on the space of essentially bounded vector fields with finite $\mathcal{L}_1$ norm, we introduce a new norm, combining the two

$$\|v\| \stackrel{\text{def}}{=} \max\Big(\|v\|_{\Omega,\mathcal{L}_1}, 2^{-m/4}\sup|v|\Big).$$

Canonical partitions. We would like to approximate the operator $P^m_{\hat{t}*}$ by a linear operator between two suitable subspaces of piecewise constant vector fields with a simple-looking matrix. In order to do that we construct a pair of so called *canonical partitions* Ω^1 and Ω^2 of the class $\mathcal{G}(m,\delta)$, associated to a sequence of perturbations $\widehat{t}$ (Subsection 4.2.3) and introduce two subspaces $\mathfrak{X}_{\Omega^1}$ and $\mathfrak{X}_{\Omega^2}$ of piecewise constant vector fields associated to partitions Ω^1 and Ω^2, respectively. On every subspace of piecewise constant vector fields we choose a normalised basis

$$\Big\{\chi^s_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega_{ij}}; \quad \chi^u_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega_{ij}}\Big\}_{i,j\in\mathbb{Z}}, \tag{1.14}$$

where π_x represents the orthogonal projection onto the contracting direction of the Baker's map. The construction of canonical partitions rely on the study of small perturbations of the doubling map. It is easy to observe that the Baker's map and the doubling map are closely related, and the former is just an extension of the latter. The canonical partition for the sequence $\widehat{t}$ is set to be a direct product of two canonical partitions associated to suitably chosen perturbations of the doubling map.

The first approximation. Once two partitions are chosen, we define a linear operator $\mathcal{A}_t\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ by its matrix elements so that

$$\int_{\Omega^2_{kl}} P^m_{\widehat{t}*} v = \int_{\Omega^2_{kl}} \mathcal{A}_t v \qquad \text{for all } \Omega^2_{kl} \in \Omega^2 \text{ and any } v \in \mathfrak{X}_{\Omega^1}.$$

The choice of partitions allows us to establish the following facts about the matrix of the operator $\mathcal{A}_t$ in canonical bases (1.14).

1. There exists a small number $0 < \gamma_1 < 0.01$ such that $\sup|a^{kl}_{ij}| < 2^{\gamma_1 m}$. (Proposition 4.3.2).

2. There exists an $\frac{15}{16} < \alpha < 1$ such that for all $|t| \le 2^{-m\alpha}$ we have a decomposition $\mathcal{A}_t = \mathcal{B}_t \oplus \mathcal{C}_t$. The matrix elements of the operator $\mathcal{B}_t$ satisfy (Proposition 4.3.1)

$$\#\big\{(i,j,k,l) \mid (i,j) \in \Box,\ (k,l) \in \Box,\ b^{kl}_{ij} \neq 1\big\} \le 2^{(4\frac{1}{2}-\alpha)m}$$

and the matrix of the operator $\mathcal{C}_t$ is small

$$\sum_{\square}\sum_{\square}|c_{ij}^{kl}| \leq 100m(1+\mu)^{2m}2^{-m\alpha},$$

where μ comes from the upper bound on the Jacobian of $F|_{\mathbb{R}^2\setminus\square}$.

Using the inequalities above, we deduce that for all sufficiently small $|\bar{t}| \leq 2^{-m\alpha}$ we have (Lemma 4.3.19)

$$\|\mathcal{A}_t - \mathcal{A}_0\| \leq 2^{(2\frac{3}{4}+\gamma_1-\alpha)m};$$

where $\mathcal{A}_0$ corresponds to the zero sequence $t = 0$. Afterwards, we establish the following facts

1. There exist two cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}_t(C_1) \subset C_2$ and C_2 is much smaller than C_1 (Theorem 8 p. 141).

2. The operator $\mathcal{A}_t$ is a good approximation to $P_{\widehat{t*}}$ (Corollary 2 of Proposition 4.3.3):

$$\|\exp(\delta\Delta)(P_{\widehat{t*}}^m - \mathcal{A}_t)\nu\|_{\Omega^2} \leq 2^{2+(2+\alpha)m}\sup\operatorname{diam}(\Omega_{kl}^2)\|\nu\|_{\Omega^1}, \qquad \text{where } \delta = 2^{-m\alpha}. \tag{1.15}$$

The second approximation. The goal is to get rid of dependence of partitions Ω^1 and Ω^2 on $\bar{t}$ and to show that for any partition Ω^3 of the class $\mathcal{G}(m,\delta)$ there exists a linear operator $D\colon \mathfrak{X} \to \mathfrak{X}_{\Omega^3}$ such that for any $\delta = 2^{-m\alpha}$ and any $|t| \leq \delta$ the following properties hold true:

1. There exists a cone $C_3 \in \mathfrak{X}_{\Omega^3}$, smaller than the cone C_1, such that (Proposition 4.4.4):

$$D\exp(\delta\Delta)\mathcal{A}_t(C_1) \subset C_3.$$

2. The norm of the operator $D\exp(\delta\Delta)\mathcal{A}_t$ grows exponentially with m: for any $v \in \mathfrak{X}_{\Omega^1}$ we have (Lemma 4.4.4):

$$\|D\exp(\delta\Delta)\mathcal{A}_t v\|_{\Omega^3} \geq \left(2^{2m} - 2^{\frac{3}{2}m}\right)\|v\|_{\Omega^1}.$$

3. The operators $D\exp(\delta\Delta)\mathcal{A}_t$ and $\exp(\delta\Delta)\mathcal{A}_t$ are close. There exists a small $\gamma_2 > 0$ such that for any $v \in \mathfrak{X}_{\Omega^1}$ we have (Lemma 4.4.1):

$$\|(D\exp(\delta\Delta)\mathcal{A}_t - \exp(\delta\Delta)\mathcal{A}_t)v\|_{\Omega^3} \leq 2^{(2-\gamma_2)m}\|v\|_{\Omega^1}. \tag{1.16}$$

Combining the first (1.15) and the second (1.16) approximations, we get an invariant cone for the operator $\exp(\delta\Delta)P_{\widehat{t}*}^{2m}$ and derive from it an invariant cone for the operator $(\exp(\delta\Delta)P_*)^{2m}$.

It may seem at first sight that the examples chosen are too simple since they are linear. However, it appears that they are sufficiently complicated to analyse and the same approach is applicable to non-trivial perturbations, since most estimates are based on distortion properties and the distortion is easy to control for perturbations of hyperbolic maps.

1.6 Outline

The work presented has three chapters. In chapter 2 "A proof of the fast dynamo theorem" we give sufficient conditions (Invariant Cone Hypothesis 1) for a piecewise C^2 transformation of $\mathbb{R}^n$ to be a fast dynamo. In the following Chapters 3 and 4 we construct measure-preserving piecewise-C^2 transformations $\ell \colon \mathbb{R} \to \mathbb{R}$ and $T \colon \mathbb{R}^2 \to \mathbb{R}^2$, respectively, that satisfy the Hypothesis. As mentioned above, the arguments in the two-dimensional case rely on some parts of the analysis of the one-dimensional system.

We begin the Chapter 2 with a few general constructions; we give a definition to *small random perturbations* (Subsection 2.1.1), introduce a norm in the space of vector fields, and fix the type of cones we are interested in. Then we explain how to reduce the system with diffusion to a system generated by a small random perturbation of a certain map. Finally, in the Section 2.2, we prove the fast dynamo theorem for maps satisfying the Invariant Cone

Hypothesis.

The Chapter 3 we start with a few definitions, that introduce the central elements of the construction. In particular, we define a *class of partitions* $\mathcal{G}(m)$ of the real line with a certain "uniform" property (Definition 4, p. 34); and a *norm* in the space of essentially bounded integrable functions we will be using throughout (Definition 3, p. 26). In addition, we introduce a *transfer operator*, that we will use to define an induced action of a piecewise diffeomorphism on functions (Definition 5 p. 35).

In Section 3.2.1 we consider a so-called *toy dynamo operator* between two linear spaces $A\colon X_1 \to X_2$ in the most abstract way, i.e. in terms of its matrix coefficients. We show that for any toy dynamo operator there exist two cones $C_1 \subset X_1$ and $C_2 \subset X_2$ such that $A(C_1) \subsetneq C_2$; and C_2 is much smaller than C_1. This is the content of Theorem 3 p. 45.

In Section 3.2 we show that a *toy dynamo operator* approximates a transfer operator, induced by a large iteration m of a small random perturbation of the so-called *dynamo map* (Subsection 3.1). The dynamo map is an expanding map on the unit interval complemented by reflection outside. More precisely, given $1 < s_2 < 2 < s_1 < 3$, we define (3.3)

$$\ell(x) = \begin{cases} s_1 x + s_1 - 1, & \text{if } -1 < x < \frac{2}{s_1} - 1; \\ s_2 x + 1 - s_2, & \text{if } \frac{2}{s_1} < x < 1; \\ -x, & \text{otherwise}; \end{cases}$$

and associate a small random perturbation to any sequence $\xi \in \ell_\infty(\mathbb{R})$. Essentially, the toy dynamo operator is given by the transition matrix between two partitions of the class $\mathcal{G}(m)$ associated a small perturbation ℓ_ξ^m of the map ℓ. Namely, we see that $a_{ij} = 1$ if $\ell_\xi^m(\Omega_i^1) \cap \Omega_j^2 = \Omega_j^2$ and $\ell_\xi^m |_{\Omega_i^1}$ is increasing. Figure 1.4 shows a few iterations of the map without perturbation ($\xi \equiv 0$) and with the largest possible perturbation ($\xi \equiv \delta$). We see that transition matrices should coincide in many places for $\Omega_i^1, \Omega_j^2 \subset [-1, 1]$.

In Subsection 3.2.2 we introduce a *canonical partition* Ω^ξ of the class $\mathcal{G}(m)$, associated to a small perturbation ξ. The partition has the following property. For any interval I with $\ell_\xi^k(I) \subset [-1,1]$ for all $0 \le k < m$ there exists an element of the partition Ω_i^ξ such that $I \subseteq \Omega_i^\xi$. In addition, the partition is "uniform": any interval of the length δ contains not more than N_δ elements of the partition.

In Subsection 3.2.3 we show that the operator $\ell_{\xi *}^m$ may be very well approximated (Theorem 4 on p. 62) by a *toy dynamo operator* $\mathcal{T}$ defined on the space of piecewise constant functions associated to Ω^ξ. For any essentially bounded and absolutely integrable function f,

$$\|(\ell_{\xi *}^m - \mathcal{T}) \exp(\delta\Delta) f\|_2 \le \left(\frac{s_1^3 \delta}{2^{1/2} s_2}\right)^m \cdot m\|f\|_1.$$

Here we can choose parameters s_1, s_2 of the map ℓ, and a constant α such that the approximation is good enough: namely $\|\mathcal{T}\| = 2^m$ and

$$\frac{s_1^3}{2^{1/2+\alpha} s_2} < 2.$$

In Section 3.3 we construct an invariant cone for the operator $\exp(\delta\Delta)\ell_{\xi *}^m \exp(\delta\Delta)$ in the space of essentially bounded absolutely integrable functions. In order to do that, we show that the image of the Weierstrass transform with Gaussian kernel with isotropic variance $\delta = 2^{-m\alpha} \gg \sup|\Omega_j^1|$ may be well approximated by step functions on a partition Ω^1 of the class $\mathcal{G}(m)$. Namely, for any partition Ω^2 of the class $\mathcal{G}(m)$ we have (Lemma 3.3.2):

$$\|D_{\Omega^1} W_\delta f - W_\delta f\|_1 \le \frac{\max(\sup|\Omega_j^1|, \sup|\Omega_j^2|)}{\delta}\|f\|_2 \le \frac{1}{s_2^m \delta}\|f\|_2.$$

Based on this simple idea, we construct an invariant cone in the space of essentially bounded integrable functions "around" the cone in the space of piecewise constant functions.

In chapter 4 we construct a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies Invariant Cone Hypothesis 1. Informally speaking, we take a certain iteration of an unfolded Baker's map on the square and complement it by a non-expanding area preserving map outside. The argument runs in a similar way to the one-dimensional case; and according to the general strategy described above in Subsection 1.5.

In the beginning we fix notation related to mappings of $\mathbb{R}^2$ and vector fields. In particular, we define (4.3) the Gaussian kernel W_δ and the Weierstrass transform operator on vector fields.

In Section 4.2, we define the dynamical system we will be working with. It is easy to see that the energy of the vector fields that change direction rapidly does not grow exponentially fast. We are going, as before, to replace diffusion by small random perturbations, and we have almost no control on the map outside the square. Therefore we need to introduce a delay in return time artificially. One of possible solutions is to use a *tower construction.*

In Subsection 4.2.1 we define the phase space X to be a *tower of M floors*, which is a union of the real plane $\mathbb{R}^2$ and $M-1$ copies of it with the central square cut off:

$$X \stackrel{\text{def}}{=} \big(\{0\} \times \mathbb{R}^2\big) \cup \big(\{1, \ldots, M-1\} \times (\mathbb{R}^2 \setminus \Box)\big),$$

where $\Box = [-1,1]^2$. We also define a map $F\colon X \to X$, to be, generally speaking, Baker's map on the square $\Box$ and some area-preserving map transferring points outside of the square to a different floor.

$$F(z,n) \stackrel{\text{def}}{=} \begin{cases} (F_0(z), 0), & \text{if } n = 0 \text{ and } z \in \Box; \\ (F_{n+1}(z), (n+1) \operatorname{mod} (M-1)), & \text{otherwise.} \end{cases} \tag{4.5}$$

(See p. 97 for definition of the maps F_n, $n = 1, \ldots, M-1$.) We also introduce small perturbations F_ξ of the map F. Afterwards, we define the map $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ we will be dealing with as a large iteration of the map F_ξ.

In Subsection 4.2.2, we introduce (4.11) a *norm* on the space of vector fields we will be using to construct invariant cones. It is similar to the norm we were using in one dimensional case, but we "weight" $\mathcal{L}_1$ norm only in expanding direction. A linear operator we define to be an action induced by P^2_ξ in ordinary way (4.10).

In Subsection 4.2.3 we exploit similarities between Baker's map, its inverse, and the doubling map, and construct a *canonical partition* for the Baker's map as a direct product of two partitions for suitably chosen small random perturbations of the doubling map.

In Section 4.3 we introduce a subspace $\mathfrak{X}_{\Omega^1}$ of piecewise constant vector fields associated to the canonical partition Ω^1. We define the basis on $\mathfrak{X}_{\Omega^1}$ to be $V^s_{\Omega^1} \cup V^u_{\Omega^1}$, where

$$V^s_{\Omega^1} \overset{\text{def}}{=} \left\{ \frac{1}{|\pi_x(\Omega^1_{ij})|} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \chi_{\Omega^1_{ij}} \right\}_{i\in\mathbb{Z}, j\in\mathbb{Z}} \quad \text{and} \quad V^u_{\Omega^1} \overset{\text{def}}{=} \left\{ \frac{1}{|\pi_x(\Omega^1_{ij})|} \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \chi_{\Omega^1_{ij}} \right\}_{i\in\mathbb{Z}, j\in\mathbb{Z}}.$$

The vectors that have only $\chi_{\Omega^s_{ij}}$ components, are parallel to the contracting direction of the Baker's map and vectors that have only $\chi_{\Omega^u_{ij}}$ components, are parallel to the expanding direction of the Baker's map. Using the operator $P_{\xi*}$, we define an associated linear operator $\mathcal{A}$ between $\mathfrak{X}_{\Omega^1}$ and a suitable subspace of piecewise constant vector fields $\mathfrak{X}_{\Omega^2}$ and such that

$$\int_{\Omega^2_{kl}} P^2_{\xi*}\nu = \int_{\Omega^2_{kl}} A\nu \tag{4.16}$$

via its matrix elements. It is natural to write the operator $\mathcal{A}$ as a direct sum of four linear operators $\mathcal{A} = SS \oplus SU \oplus US \oplus UU$, where

$$SS\colon \langle V^s_{\Omega^1}\rangle \to \langle V^s_{\Omega^2}\rangle; \quad SU\colon \langle V^s_{\Omega^1}\rangle \to \langle V^u_{\Omega^2}\rangle; \quad US\colon \langle V^u_{\Omega^1}\rangle \to \langle V^s_{\Omega^2}\rangle; \quad UU\colon \langle V^u_{\Omega^1}\rangle \to \langle V^u_{\Omega^2}\rangle.$$

The growth of the energy is guaranteed by the operator UU, and we will study it separately in the next section. We conclude this section with construction of a pair of cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$, such that $\mathcal{A}(C_1) \subsetneq C_2$ and the cone C_1 is much smaller than C_2.

In Subsection 4.3.1 we establish that the matrix of the operator UU demonstrates properties similar to the ones of "toy dynamo operator" we studied in the Chapter 3. Namely, its central part, corresponding to the elements from the unit square, has a plenty of 1's, and the absolute value of elements is majorated by a small power of 2 (Propositions 4.3.1 and 4.3.2).

In Subsection 4.3.2 we justify the choice of the operator $\mathcal{A}$, and show that operators $W_\delta \mathcal{A}$ and $W_\delta P^2_\xi$ are close on the subspace of piecewise constant vector fields $\mathfrak{X}_{\Omega^1}$. The estimations are based on the fact that δ is chosen so that

$$\max(\sup|\pi_x(\Omega^2_{ij})|, \sup|\pi_y(\Omega^2_{ij})|) \ll \delta$$

and the construction of canonical partitions.

In Subsection 4.3.3 we construct a pair of cones for the operator $\mathcal{A}$, the larger cone $C_1 \subset \mathfrak{X}_{\Omega^1}$ and a much smaller cone $\mathcal{A}(C_1) \subset C_2 \subset \mathfrak{X}_{\Omega^2}$. We use the decomposition $\mathcal{A} = SS \oplus SU \oplus US \oplus UU$, and exploit simplicity of the matrix UU along with upper bounds on other operators.

The Subsection 4.4.1 repeats the Section 3.3 of the one-dimensional Chapter 3 with obvious modifications adjusting the arguments to dimension two. In particular, the length of the intervals of the partitions in the upper bounds is replaced by the diameter of the elements.

In Subsection 4.4.2 we construct of an invariant cone for the operator $W_{\frac{\delta}{2m}} P^2_{\xi *} W_{\frac{\delta}{2m}}$.

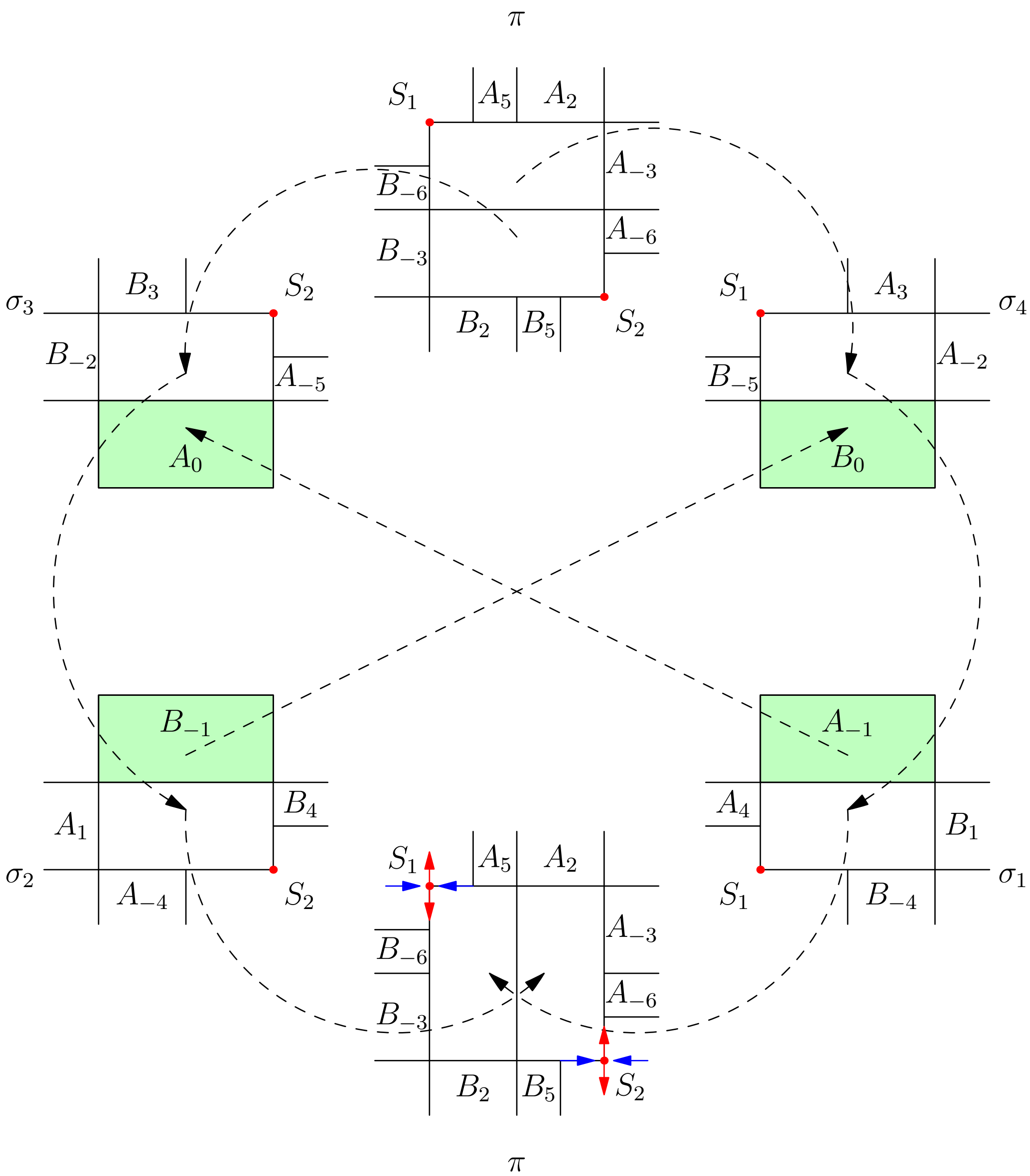

Figure 1.2: The mapping between the sections $\pi, \sigma_{1,\ldots,4}$, induced by the fluid flow; $A_k = \Phi'(A_{k-1})$, $B_k = \Phi''(B_{k-1})$. The points S_1 and S_2 are periodic saddles; blue and red arrows show stable and unstable manifolds, respectively.

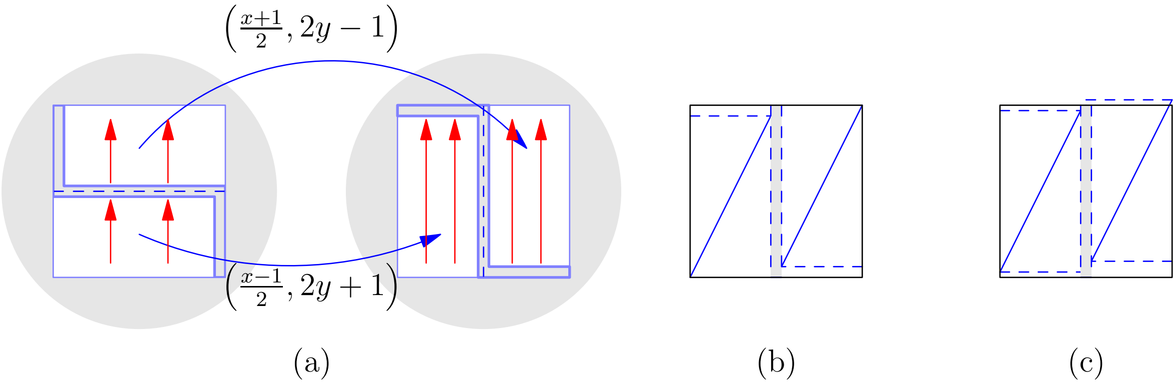

Figure 1.3: (a) Unfolded Baker's map, that appears as the first return map to the section π; (b) Doubling map with a hole; and (c) a small perturbation of the doubling map with a hole.

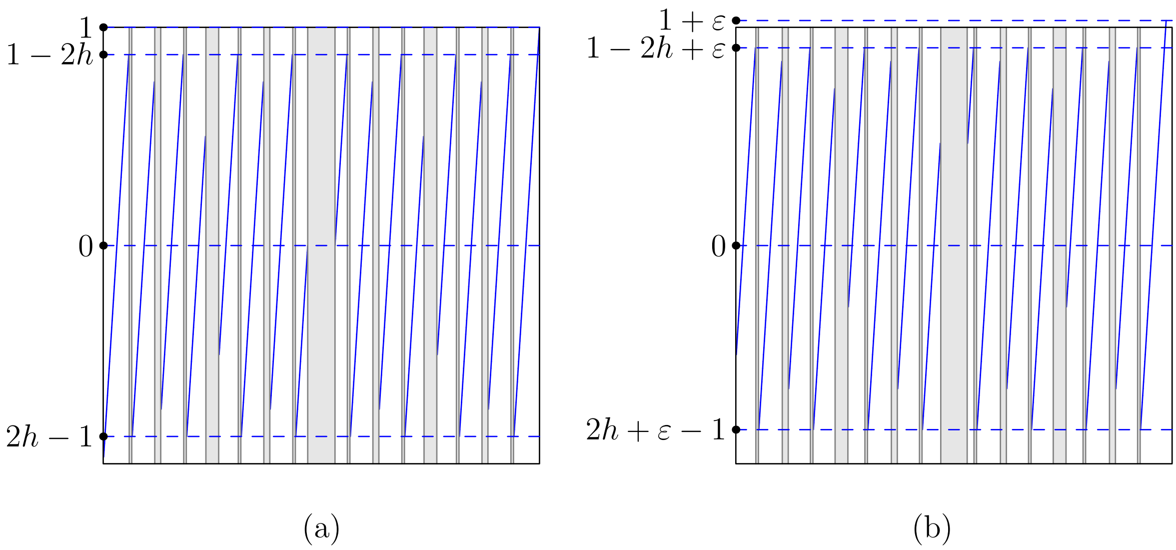

Figure 1.4: First four iterations of a small perturbation of the doubling map with a hole of the width h: (a) the case of the zero sequence; and (b) the case of the constant sequence $\xi_k \equiv \varepsilon$.

2 A proof of the fast dynamo theorem

In this Chapter we give a proof for the fast dynamo theorem for maps satisfying certain hypothesis. Later, in the Chapter 3 we construct a one-dimensional map satisfying this hypothesis and in the Chapter 4 we prove that its two-dimensional extension also satisfies these conditions. The two-dimensional map may be realised as a Poincaré map of a smooth stationary vector field in $\mathbb{R}^3$.

2.1 Basic constructions

In this Section we introduce objects central for our investigations: small random perturbations of a dynamical system and a norm in the space of vector fields. We also specify the type of cones in the space vector fields we are interested in.

2.1.1 Small random perturbations

We construct a random dynamical system using skew-products. Let X be a real manifold and let $f\colon X \to X$ be a transformation. We consider its extension

$$\widehat{f}\colon X \times \mathbb{R}^n \to X \qquad \widehat{f}(x,\xi) \stackrel{\text{def}}{=} f(x) + \xi(1). \tag{2.1}$$

Let $\Sigma \subset \ell_\infty(\mathbb{R}^n)$ be a shift-invariant subset of two-sided bounded sequences of vectors in $\mathbb{R}^n$. We introduce a skew product over the Bernoulli shift

$$\sigma \times \widehat{f} \colon \Sigma \times X \to \Sigma \times X \qquad (\sigma \times \widehat{f})(\xi, z) \stackrel{\text{def}}{=} (\sigma(\xi), \widehat{f}(z, \xi(1))). \tag{2.2}$$

The induced transformation on fibers we denote by

$$f_\xi \colon X \to X, \qquad f_\xi(z) \stackrel{\text{def}}{=} \widehat{f}(z, \xi(1)). \tag{2.3}$$

Its iterations are given by

$$f_\xi^k(z) \stackrel{\text{def}}{=} \widehat{f}(f_\xi^{k-1}(z), \xi(k)). \tag{2.4}$$

Remark 1. The following identities follow from the definition of the map f_ξ.

$$f_\xi^k = f_{\xi(k)} \circ f_{\xi(k-1)} \circ \ldots \circ f_{\xi(1)}; \tag{2.5}$$

$$f_\xi^{-k} = (f_\xi^k)^{-1} = f_{\xi(1)}^{-1} \circ f_{\xi(2)}^{-1} \circ \ldots \circ f_{\xi(k)}^{-1}; \tag{2.6}$$

$$f_\xi^{n-k} = f_\xi^n \circ f_\xi^{-k} = f_\xi^{-k} \circ f_\xi^n = \begin{cases} f_{\sigma^n(\xi)}^{n-k}, & \text{if } n < k; \\ f_{\sigma^k(\xi)}^{n-k}, & \text{if } n > k. \end{cases} \tag{2.7}$$

Definition 2. We call the map f_ξ *a random perturbation* of the map f associated to the sequence $\xi \in \Sigma$.

2.1.2 Norm in the space of vector fields

Piecewise constant vector fields are proved to be very useful to us. We define a norm in the space of essentially bounded and absolutely integrable vector fields Φ, using partitions.

The norm we are about to introduce is related to the map f. Since topological entropy is an upper bound for the growth rate of the energy, the system has to be chaotic. We shall assume therefore that the map is hyperbolic and choose an n_u-dimensional unstable manifold.

We let π_u to be projection along the unstable foliation onto the stable manifold. We denote by λ_{n_u} the n_u-dimensional Lebeague measure on the unstable manifold.

Let us fix a large number $m \gg 1$. Its role will be clarified in the next Subsection.

Definition 3. A norm in the space of essentially bounded and absolutely integrable functions, associated to a partition $\Omega(m) = \bigcup_{j=1}^{\infty} \Omega_j$ of $\mathbb{R}^n$ is given by

$$\|f\|_{\Omega} = \max\Big(\sum_{j\in\mathbb{Z}} \frac{2^{-n_u m}}{\lambda_{n_u}(\pi_u(\Omega_j))} \int_{\Omega_j} |f(x)|\mathrm{d}x, 2^{-\alpha m} \sup|f|\Big); \tag{2.8}$$

where the choice of α depends on n.

The first term we refer to as the weighted $\mathcal{L}_1$-norm and write

$$\|f\|_{\Omega,\mathcal{L}_1} : = \sum_{j\in\mathbb{Z}} \frac{2^{-n_u m}}{\lambda_{n_u}(\pi_u(\Omega_j))} \int_{\Omega_j} |f(x)|\mathrm{d}x,$$

it depends, of course, on the partition chosen.

We denote by Φ_Ω the subspace of Φ, consisting of piecewise constant vector fields associated to the partition Ω.

2.1.3 Cones in vector fields on $\mathbb{R}^n$

We reserve a notation for a cone of the radius r with the main axis v_0 in the space Φ_Ω:

$$\mathrm{Cone}\,(v_0, r, \Omega) \stackrel{\mathrm{def}}{=} \Big\{\eta = dv_0 + \varphi \mid \varphi \in \Phi_\Omega, \int (f_*^m \varphi)v_0 = 0;\ \|\varphi\|_\Omega \le dr\|v_0\|\Big\}. \tag{2.9}$$

We extend the cone $\mathrm{Cone}\,(v_0, r, \Omega)$ to include general functions from the main space:

$$\widehat{\mathrm{Cone}}\,(v_0, r, \varepsilon, \Omega) \stackrel{\mathrm{def}}{=} \Big\{f = \eta + g, \mid \eta \in \mathrm{Cone}\,(v_0, r, \Omega)\,,\ \|g\|_\Omega \le \varepsilon\|\eta\|_\Omega\Big\}. \tag{2.10}$$

We say that the cone $\widehat{\mathrm{Cone}}\left(v_0, r_1, \varepsilon_1, \Omega^1\right)$ is smaller than the cone $\widehat{\mathrm{Cone}}\left(v_0, r_2, \varepsilon_2, \Omega^2\right)$ and write $\widehat{\mathrm{Cone}}\left(v_0, r_1, \varepsilon_1, \Omega^1\right) \ll \widehat{\mathrm{Cone}}\left(v_0, r_2, \varepsilon_2, \Omega^2\right)$, if $r_1 > r_2$ and $\varepsilon_1 > \varepsilon_2$; we do not assume here that $\widehat{\mathrm{Cone}}\left(v_0, r_1, \varepsilon_1, \Omega^1\right) \cap \widehat{\mathrm{Cone}}\left(v_0, r_2, \varepsilon_2, \Omega^2\right) \ne \varnothing$.

2.2 Fast dynamo theorem

In this Section we set the hypothesis and give a proof of the fast dynamo theorem 1.

The first step the Noise Lemma 2.2.1, which suggests to replace the operator $(\exp(\delta\Delta)f_*)^n$ in our considerations with the operator $\exp(\delta\Delta)f^n_{t*}$ for some sequence t.

We begin with a simple observation that the exponent of the Laplacian operator[1] in $\mathbb{R}^n$, is the convolution with the Gaussian kernel, in particular

$$\exp(\delta\Delta)v = w_\delta * v, \text{ where } w_\delta(x) = \frac{1}{\sqrt{2\pi}\delta}\exp\big(-\frac{x^2}{2\delta^2}\big).$$

The latter operator is also known as *the Weiertstrass transform* $W_\delta(v) \stackrel{\text{def}}{=} (w_\delta * v)$; this notation we use throughout.

The following statement is generally known, but we give a proof for completeness.

Lemma 2.2.1 (Noise Lemma). *For any map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and for any vector field v in $\mathbb{R}^n$ we have*

$$W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-1} v(x) = \int_{\mathbb{R}^{n(m-1)}} w_\delta(t_1) w_\delta(t_2) \dots w_\delta(t_{m-1}) (W_{\frac{\delta}{2}} f^m_{\overline{0t}*} v)(x) \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_{m-1}, \tag{2.11}$$

where $\overline{0t} = (0, t_1, t_2, \dots, t_{m-1}) \in \mathbb{R}^m$.

Proof. Observe that $f^{-1}(x - t) = f_t^{-1}(x)$, because $f_t(x) = f(x) + t$. By straightforward

[1] $\Delta\colon v \to \mathrm{d}^2 v$ in the case of the real line.

calculation,

$$\begin{aligned}W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-1} v(x) &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} W_\delta f_* v(x) = \\ &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} \int_{\mathbb{R}^n} w_\delta(t)(f_*v)(x-t)\mathrm{d}t = \\ &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} \int_{\mathbb{R}^n} w_\delta(t_1)(f_{t_1 *}v)(x)\mathrm{d}t_1 = \ldots = \\ &= W_{\frac{\delta}{2}} \int_{\mathbb{R}^{n(m-1)}} w_\delta(t_1)\ldots w_\delta(t_{m-1})(f_* f_{t_1 *}\ldots f_{t_{m-1}*}v)(x)\mathrm{d}t_1\ldots\mathrm{d}t_{m-1} = \\ &= \int_{\mathbb{R}^{n(m-1)}} w_\delta(t_1)\ldots w_\delta(t_{m-1})(W_{\frac{\delta}{2}} f^m_{0t*} v)(x)\mathrm{d}t_1\ldots\mathrm{d}t_{m-1}.\end{aligned}$$

■

Corollary 1. *For any map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, any vector field v_0 in $\mathbb{R}^n$, and for any $k = k_0 m \gg m$*

$$\begin{aligned}&W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{k-1} W_{\frac{\delta}{2}} v_0(x) = \\ &\quad= \int_{\mathbb{R}^{k-k_0}} w_\delta(t_1)w_\delta(t_2)\ldots w_\delta(t_{k-k_0}) \left(W_{\frac{\delta}{2}} f^m_{0t*} W_{\frac{\delta}{2}}\right)^{k_0} v_0(x)\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_{k-k_0}. \quad (2.12)\end{aligned}$$

We shall put the following conditions on the map f.

Hypothesis 1 (Invariant Cone)**.** *There exist an $m \gg 1$, a partition $\Omega(m)$, a vector field v_0, and four numbers $r_2(m) \ll r_1(m)$, $\varepsilon_2(m) \ll \varepsilon_1(m) \ll 1$ such that for any sequence ξ with $\|\xi\|_\infty \le \delta$*

$$W_{\frac{\delta}{2}} f^m_{0t*} W_{\frac{\delta}{2}} \colon \widehat{\mathrm{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega) \to \widehat{\mathrm{Cone}}\,(v_0, r_2, \varepsilon_2, \Omega) \subset \widehat{\mathrm{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega) \quad (2.13)$$

Moreover, there exists $0 < \gamma < 0.01$ such that for any field $v \in \widehat{\mathrm{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega)$

$$2^{m-2}\|v\|_\Omega \le \|W_{\frac{\delta}{2}} f^m_{0t*} W_{\frac{\delta}{2}} v\|_\Omega \le 2^{(1+\gamma)m}\|v\|_\Omega. \quad (2.14)$$

We construct a map $f\colon \mathbb{R} \to \mathbb{R}$ satisfying this hypothesis in the Chapter 3 and a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying this condition in the Chapter 4.

We choose $\delta = 2^{-m\alpha}$, a partition $\Omega = \Omega(m) = \bigcup \Omega_j$, the vector field $v_0 \ge 0$, and fix four dimension parameters of two cones r_1, r_2, ε_1, and ε_2 such that the Hypothesis holds true.

Lemma 2.2.2. *In the notations introduced above, for any* $v \in \overline{\widehat{\text{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega)}$

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1})(W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v)\mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \in \widehat{\text{Cone}}\left(v_0, e^2 r_2, e^2 \varepsilon_2, \Omega\right). \tag{2.15}$$

Proof. By the Hypothesis assumption, we know that for any $|t| \in [-\delta, \delta]^m$ and any vector field $v \in \overline{\widehat{\text{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega)}$

$$W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v = dv_0 + \psi_t + g_t \in \widehat{\text{Cone}}\,(v_0, r_2, \varepsilon_2, \Omega)\,,$$

where $\psi_t \in \Phi_\Omega$, $\|\psi_t\|_\Omega \le dr_2 \|v_0\|_\Omega$ and $\|g_t\|_\Omega \le d\varepsilon_2 \|v_0\|_\Omega$. Observe that Ω is independent on t. Therefore,

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) v_0 \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} =$$
$$= v_0 \Big(\int_{-\delta}^{\delta} w_{\frac{\delta}{m}}(t)\mathrm{d}t \Big)^{m-1} = v_0 \Big(1 - \frac{2}{m}\Big)^{m-1} \ge e^{-2} v_0, \tag{2.16}$$

for m large enough. Since $\psi_t \in \Phi_\Omega$ for any $t \in [-\delta, \delta]^m$,

$$\int_{[-\delta,\delta]^m} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \psi_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \in \Phi_\Omega,$$

and we calculate Ω-norm.

$$\Big\| \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \psi_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \Big\|_\Omega \le$$
$$\le \sum_{j \in \mathbb{Z}} \frac{2^{-n_u m}}{\lambda_{n_u}(\pi_u(\Omega_j))} \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \Big(\int_{\Omega_j} |\psi_t(x)| \mathrm{d}x \Big) \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \le$$
$$\le \sup_t \|\psi_t\|_\Omega \le dr_2 \|v_0\|_\Omega.$$

Similarly,

$$\Big\| \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) g_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \Big\|_\Omega \le d\varepsilon_2 \|v_0\|_\Omega.$$

Observe that

$$\int_{-1}^{1}\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\psi_t(x)\mathrm{d}t_1\dots\mathrm{d}t_{m-1}\mathrm{d}x =$$

$$= \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\mathrm{d}t_1\dots\mathrm{d}t_{m-1}\cdot\int_{-1}^{1}\psi_t(x)\mathrm{d}x = 0.$$

Summing up, for any $v \in \overline{\widehat{\mathrm{Cone}}\,(v_0,\varepsilon_1,r_1,\Omega)}$

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})W_{\frac{\delta}{2m}}f_{t*}^{m}W_{\frac{\delta}{2m}}v\mathrm{d}t_1\dots\mathrm{d}t_{m-1} \in \widehat{\mathrm{Cone}}\left(v_0,e^2r_2,e^2\varepsilon_2,\Omega\right).$$

■

Lemma 2.2.3. *In the notations introduced above, assume in addition that* $2^{\gamma m}e^{-m} \ll \varepsilon_2(m)$*.*

Then

$$W_{\frac{\delta}{2m}}f_*(W_\delta f_*)^{m-1}W_{\frac{\delta}{2m}} : \overline{\widehat{\mathrm{Cone}}\,(v_0,r_1,\varepsilon_1,\Omega)} \to \widehat{\mathrm{Cone}}\left(v_0,e^2r_2,e^2\varepsilon_2,\Omega\right) \subsetneq \widehat{\mathrm{Cone}}\,(v_0,r_1,\varepsilon_1,\Omega)\,;$$

Moreover, there exists $0<\gamma<0.01$ *such that for any field* $v \in \widehat{\mathrm{Cone}}\,(v_0,r_1,\varepsilon_1,\Omega)$

$$2^{m-5}\|v\|_\Omega \le \|W_{\frac{\delta}{2m}}f_*(W_\delta f_*)^{m-1}W_{\frac{\delta}{2m}}v\|_\Omega \le 2^{(1+\gamma)m}\|v\|_\Omega$$

Proof. By Lemma 2.2.1 for any $v \in \overline{\mathrm{Cone}\,(v_0,r_1,\Omega)}$

$$\begin{aligned} W_{\frac{\delta}{2m}}f_*(W_{\frac{\delta}{m}}f_*)^{m-1}W_{\frac{\delta}{2m}}v &= \\ &= \int_{\mathbb{R}^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})(W_{\frac{\delta}{2m}}f_{t*}^{m}W_{\frac{\delta}{2m}}v)\mathrm{d}t_1\mathrm{d}t_2\dots\mathrm{d}t_{m-1} = \\ &= \left(\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}} + \int_{[-\delta,\delta]^{m-1}}\right)\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}f_{t*}^{m}W_{\frac{\delta}{2m}}v)\mathrm{d}\bar{t}. \end{aligned} \tag{2.17}$$

By Lemma 2.2.2 we know that for any $v \in \overline{\widehat{\mathrm{Cone}}\,(v_0,r_1,\varepsilon_1,\Omega)}$

$$\int_{[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}f_{t*}^{m}W_{\frac{\delta}{2m}}v)\mathrm{d}\bar{t} \in \widehat{\mathrm{Cone}}\left(v_0,e^2r_2,e^2\varepsilon_2,\Omega\right).$$

We estimate the first term

$$\Big\|\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v)\mathrm{d}\bar{t}\Big\|_\Omega \leq$$
$$\leq \sum_{j\in\mathbb{Z}}\frac{2^{-n_u m}}{\lambda_{n_u}(\pi_u(\Omega_j))}\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\Big(\int_{\Omega_j}\Big|W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v(x)\Big|\mathrm{d}x\Big)\mathrm{d}\bar{t} \leq$$
$$\leq \sup_t \|W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v\|_\Omega \int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t}.$$

We shall find an upper bound for the integral:

$$\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t} \leq$$
$$\leq 2^m\Big(\int_\delta^{+\infty} w_{\frac{\delta}{m}}(t)\mathrm{d}t\Big)^m + 2m\int_\delta^{+\infty}\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\mathrm{d}t_1\dots\mathrm{d}t_{m-1} \leq$$
$$\leq 2^m e^{-m^2} + 2me^{-m}.$$

We may also recall that there exists $0<\gamma<0.01$ such that for any $v \in \overline{\widehat{\mathrm{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega)}$, we have $\sup_t \|W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v\|_\Omega \leq 2^{(1+\gamma)m}\|v\|_\Omega$. Therefore

$$\sup_t \|W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v\|_\Omega \int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t} \leq 2^{(1+\gamma)m}\cdot 2me^{-m}\|f\|_\Omega.$$

We need to verify $2^{(1+\gamma)m}\cdot 2me^{-m} \lll 2^{m-5}\varepsilon_2$, which is equivalent to $2^{\gamma m+6}e^{-m} \lll \varepsilon_2$.

For the second inequality we recall the second condition of the Hypothesis

$$\forall v \in \overline{\widehat{\mathrm{Cone}}\,(v_0, r_1, \varepsilon_1, \Omega)}\colon \|W_{\frac{\delta}{2m}} f_{\xi*}^m W_{\frac{\delta}{2m}} v\|_\Omega \geq 2^{m-2}\|v\|_\Omega.$$

Then

$$\Big\|\int_{[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v)\mathrm{d}\bar{t}\Big\|_\Omega \geq$$
$$\geq \inf_{t\in[-\delta,\delta]^m}\|W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v\|_\Omega\cdot\int_{[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j) \geq 2^{m-2}e^{-2}\|v\|_\Omega. \quad (2.18)$$

Taking into account

$$\Big\|\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} f_{t*}^m W_{\frac{\delta}{2m}} v)\mathrm{d}\bar{t}\Big\|_\Omega \leq 2^m\varepsilon_2\|v\|_\Omega,$$

we get the result. ∎

Theorem 1 (Fast dynamo theorem)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise-C^2 transformation satisfying the Invariant Cone Hypothesis and an additional condition $\left(\frac{2\gamma}{e}\right)^m \ll \varepsilon_2$. Then there exists an essentially bounded vector field v, with absolutely integrable components such that*

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\left\|(\exp(\delta\Delta)f_*)^n v\right\|_{\mathcal{L}_1} > 0,$$

Proof. It follows by straightforward calculation that $W_\delta W_\delta = W_{2\delta}$ for any number $\delta > 0$. The Theorem follows from Corollorary 1 of Lemma 2.2.1 and Lemma 2.2.3 with $v = W_{\frac{\delta}{2m}} v_0$. ∎

3 Fast dynamo on the real line

This Chapter is dedicated to the construction of a transformation $\ell\colon \mathbb{R} \to \mathbb{R}$ satisfying the Invariant Cone Hypothesis 1, p. 28. In perspective, the operator ℓ_* corresponds to the induced action on vector fields on the unstable manifold of the Poincaré map of the provisional flow. The unstable manifold is one dimensional and the settings are the following. Vector fields on a one-dimensional real manifold may be identified with functions $\mathbb{R} \to \mathbb{R}$; and an induced action on vector fields on $\mathbb{R}$ is given by a transfer operator $(\ell_* v)(y) = \sum\limits_{x \in \ell^{-1}(y)} \mathrm{d}\ell(x) v(x)$, where $\ell\colon \mathbb{R} \to \mathbb{R}$ is a piecewise-differentiable function.

The main result is the following

Theorem 2 (Invariant cone)**.** *There exist a measure preserving piecewise-smooth transformation $\ell\colon \mathbb{R} \to \mathbb{R}$, a cone C in the space Φ of essentially bounded absolutely integrable vector fields on $\mathbb{R}$, and a norm $\|\cdot\|$ in Φ such that for an $m \gg 1$ large enough and any sequence $\|\xi\|_\infty \le \delta$ with $\delta = 2^{-m\alpha}$ for $\frac{15}{16} < \alpha < 1$ we have*

$$W_{\frac{\delta}{2m}} \ell^m_{\xi *} W_{\frac{\delta}{2m}} \colon \overline{C} \to C; \qquad \forall f \in \overline{C}\colon \|W_{\frac{\delta}{2m}} \ell^m_{\xi *} W_{\frac{\delta}{2m}} f\| \ge \frac{1}{4} \|\ell^m_*\| \cdot \|f\|. \tag{3.1}$$

3.1 Notation

In this Section we fix notation we use throught the proof of Invariant Cone Theorem 2.

The following letters are reserved for constants: α, β, γ, γ_1, $\varkappa$, s_1, s_2. The admissible range of values will be specified later.

Given a subset $I \subset \mathbb{R}^n$ we denote by $|I|$ its Lebesgue measure. We say that two sets I_1 and I_2 are δ-close and write $|I_1 - I_2| < \delta$ if I_1 belongs to the δ-neighbourhood of I_2 or I_2 belongs to the δ-neighbourhood of I_1. Otherwise, we write $|I_1 - I_2| > \delta$. The indicator function of a set I we denote by χ_I.

Let δ_{ij} be the Dirac delta function:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

The supremum norm of a sequence of real numbers $\xi \in \ell_\infty(\mathbb{R})$ we denote by $\|\xi\| = \sup_{k \in \mathbb{N}} |\xi_k|$. Whenever supremum or infimum are taken along the whole range of values, we omit the range.

We write $x \ll y$ when x is *exponentially small* compared to y, namely, there exist a small number $0 < \varepsilon < 1$ such that $x < 2^{-\varepsilon m} y$.

Let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16} < \alpha < 1$.

Definition 4. We say that a collection of intervals $\Omega = \{\Omega_j\}_{j \in \mathbb{Z}}$ makes *a partition of the class* $\mathcal{G}(m, \delta, s_1, s_2)$, if $\bigcup \overline{\Omega_j} = \mathbb{R}$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$, and the following conditions hold true.

1. The interval $[-1, 1]$ contains at least 2^{m-1} and at most 2^m intervals of the partition, and $\{\pm 1\}$ are the end points of some intervals of the partition.

2. The length of intervals Ω_j is bounded away from zero and from infinity

$$\frac{1}{m s_1^m} \leq |\Omega_j| \leq 2\Big(\frac{1}{s_1^m} + \frac{1}{s_2^m}\Big).$$

3. Any interval $I \subset \mathbb{R}$ of the length $|I| = \delta$ contains not more than

$$N_\delta = 2^{m+1}\delta^{\log_{s_1} 2} = 2^{m(1-\alpha\log_{s_1} 2)+1}$$

intervals of the partition.

4. Any interval of the partition $\Omega_j \subset \mathbb{R} \setminus [-1 - m\delta, 1 + m\delta]$ has length $|\Omega_j| = 2^{-m}$.

We write $\mathcal{G}(m, \delta, s_1, s_2)$ to indicate dependence on m, δ, s_1, and s_2; we will abuse notations and omit m, δ, s_1, or s_2, when it leads to no confusion and the dependence is of no importance.

We number intervals of a partition Ω in the natural order, starting from $\Omega_0 \ni 0$. We set Ω_{N_l} to be the most left interval of Ω inside $[-1, 1]$, and Ω_{N_r} to be the most right interval of Ω inside $[-1, 1]$.

Here we deal with essentially bounded absolutely integrable functions on the real line. We refer to the space $\Phi \stackrel{\text{def}}{=} \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R})$ as the main space. "Any function" refers to a function from the main space always.

Given a partition $\Omega = \{\Omega_j\}_{j\in\mathbb{Z}}$ of the class $\mathcal{G}$, we denote the associated space of step functions by Φ_Ω and address the basis $\{\chi_{\Omega_j}\}_{j\in\mathbb{Z}}$ as the canonical basis of Φ_Ω.

Definition 5. We associate *a weighted transfer operator* f_*, acting on the main space, to a map f on the real line by[1]

$$(f_*\phi)(x)\colon = \sum_{y\in f^{-1}(x)} \operatorname{sgn} \mathrm{d}f(y)\phi(y). \tag{3.2}$$

3.1.1 The dynamical system

Here we define the system we will be studying. We have specified the phase space to be the space of essentially bounded and absolutely intgrable vector fields on $\mathbb{R}$. Now we define a

[1] Transfer operator is a bounded linear operator. In this case, it is chosen to be one dimensional analogue of induced action on vector fields by area-preserving transformations. Transfer operators with negative coefficients have been considered, for instance, in [15].

transformation and a norm. We also fix the type of cones we will be dealing with.

The transformation of Φ. Let $s_2 \leq 2 \leq s_1$, be two real numbers such that $\log \frac{s_1}{s_2} = \varkappa \ll 1$. and let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16} \leq \alpha < 1$. Consider the map $\ell \colon \mathbb{R} \to \mathbb{R}$

$$\ell(x) = \begin{cases} s_1 x + s_1 - 1, & \text{if } -1 < x < \frac{2}{s_1} - 1; \\ s_2 x + 1 - s_2, & \text{if } \frac{2}{s_1} - 1 < x < 1; \\ -x, & \text{otherwise.} \end{cases} \tag{3.3}$$

and define its extension $\widehat{\ell} \colon \mathbb{R}^2 \to \mathbb{R}$ by $\widehat{\ell}(x, y) = \ell(x) + y$. We associate a small perturbation ℓ_ξ to any sequence $\xi \in \ell_\infty(\mathbb{R})$ and $\|\xi\|_\infty \leq \delta$.

The map ℓ outside the unit interval is not important to us and we chose a simple map that changes direction of the vector field, to make it non-trivial. The exact form is not relevant here. We associate a transfer operator to a map ℓ_ξ^m according to (3.2). We will be studying the action $\ell_{\xi *}^m \colon \Phi \to \Phi$.

Norm in the space of vector fields. Piecewise constant vector fields have proved to be very useful to us. We define a norm in the space Φ of essentially bounded and absolutely integrable vector fields on $\mathbb{R}$, using partitions.

Definition 6. A norm in the space Φ_Ω of essentially bounded and absolutely integrable functions, associated to a partition $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ of $\mathbb{R}$, is given by

$$\|f\|_\Omega = \max\Big(\sum_{j \in \mathbb{Z}} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} |f(x)| \mathrm{d}x, 2^{-m/2} \sup |f| \Big). \tag{3.4}$$

The first term we refer to as the weighted $\mathcal{L}_1$-norm and write

$$\|f\|_{\Omega, \mathcal{L}_1} := \sum_{j \in \mathbb{Z}} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} |f(x)| \mathrm{d}x,$$

it depends, of course, on the partition chosen.

This definition agrees with general definition in Subsection 2.1.2 with $\alpha = 1/2$.

The subspace of Φ, consisting of piecewise constant vector fields associated to the partition Ω we denote by Φ_Ω. Observe that for any step function $\phi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j} \in \Phi_\Omega$ we have that

$$\|\phi\|_\Omega = \max\Big(2^{-m} \sum_{j\in\mathbb{Z}} |c_j|, 2^{-m/2} \sup |c_j|\Big). \tag{3.5}$$

Cones in vector fields on $\mathbb{R}$. We reserve a notation for a cone of radius r with the main axis $\chi_{[-1,1]}$ in the space Φ_Ω of piecewise constant functions, associated to a partition Ω:

$$\mathrm{Cone}\,(r, \Omega) \stackrel{\text{def}}{=} \Big\{\eta = d\chi_{[-1,1]} + \varphi \mid \varphi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\varphi\|_\Omega \le dr\Big\}. \tag{3.6}$$

We extend the cone $\mathrm{Cone}\,(r,\Omega)$ to include general functions from the main space:

$$\widehat{\mathrm{Cone}}\,(r, \varepsilon, \Omega) \stackrel{\text{def}}{=} \Big\{f = \eta + g, \mid \eta \in \mathrm{Cone}\,(r,\Omega)\,,\ \|g\|_\Omega \le \varepsilon \|\eta\|_\Omega\Big\}. \tag{3.7}$$

This definition agrees with general definition in Subsection 2.1.3.

3.2 Transfer operator as a dynamo operator

The plan is to choose suitable subspaces of Φ and approximate the operator $\ell^m_{\xi*}$ by an operator with a simple matrix. The latter we call a *generalised toy dynamo operator.*

Afterwards, we prove that there exists a map $\ell\colon \mathbb{R} \to \mathbb{R}$ such that for any small perturbation ℓ^m_ξ with $\|\xi\|_\infty \le \delta$ we can find a generalised toy dynamo operator $\mathcal{A}\colon \Phi \to \Phi$ and two partitions Ω^1 and Ω^2 of $\mathbb{R}$ such that $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ and $\|(\ell^m_{\xi*} - \mathcal{A})W_\delta\| \le 2^{-\gamma m}(\|\ell^m_{\xi*}\| + \|\mathcal{A}\|)$ for some $\gamma > 0$.

3.2.1 Generalised toy dynamo operators

Here we give a definition and show that any generalised toy dynamo operator $\mathcal{A}$ possess a pair of cones $C_1, C_2 \subset \Phi$ such that $C_2 \ll C_1$ and $\mathcal{A}(\overline{C_1}) \subset C_2$ (but $C_2 \not\subset C_1$).

Let $\mathcal{A}$ be a linear operator acting on the main space. Assume that there exists two partitions Ω^1, Ω^2 of the class $\mathcal{G}$ such that $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$. Here and below we denote by N_l^1 and N_r^1 the indices of the first and the last intervals of the partition Ω^1 inside $[-1,1]$, respectively; and let N_l^2 and N_r^2 be the indices of the first and the last intervals of the partition Ω^2 inside $[-1,1]$, respectively. In other words, the sets $\Omega_i^2 \times \Omega_j^1$ with $N_l^2 \le i \le N_r^2$, and $N_l^1 \le j \le N_r^1$ make a partition of the unit square.

We define several sets of indices in order to describe the properties of the operator $\mathcal{A}$ important to us. Let a_{ij} be coefficients of the matrix of $\mathcal{A}$ in the canonical bases of the subspaces Φ_{Ω^1} and Φ_{Ω^2}.

Accelerator:

$$\mathrm{Ar}\colon = \big\{ j \in \{N_l^1, \dots, N_r^1\} \mid \#\{ i \in \{N_l^2, \dots, N_r^2\} \mid a_{ij} = 1\} \ge 2^m - N_\delta \big\}. \tag{3.8}$$

Inflow diffusion:

$$\mathrm{D_{in}}\colon = \big\{ (i,j) \in \{N_l^2, \dots, N_r^2\} \times \{N_l^1, \dots, N_r^1\} \mid a_{ij} \ne 1 \big\}. \tag{3.9}$$

Outflow diffusion:

$$\begin{aligned} \mathrm{D_{out}}\colon = \{N_l^2 - mN_\delta, \dots, N_r^2 + mN_\delta\} \times \{N_l^1 - mN_\delta, \dots, N_r^1 + mN_\delta\} - \\ - \{N_l^2, \dots, N_r^2\} \times \{N_l^1, \dots, N_r^1\}. \end{aligned} \tag{3.10}$$

Indifferent subspace:

$$\mathrm{Sp}\colon = \mathbb{Z}^2 \setminus \{N_l^2 - mN_\delta, \dots, N_r^2 + mN_\delta\} \times \{N_l^1 - mN_\delta, \dots, N_r^1 + mN_\delta\}. \tag{3.11}$$

We are interested in linear operators $\mathcal{A}$ such that the following conditions hold true for the matrix coefficients in the canonical bases.

(D1) $\max |a_{ij}| + 1 \le m^2 \big(\frac{s_1}{s_2}\big)^m$;

(D2) $\#\mathrm{D_{in}} \leq m\delta s_1^{2m}$;

(D3) for any pair $(i,j) \in \mathrm{Sp}$ we have $a_{ij} = 0$ whenever $|i-j| > mN_\delta$;

(D4) $\#\mathrm{Ar} \geq 2^{m-2}$.

Definition 7. We say that a linear operator $\mathcal{A}\colon \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R}) \to \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R})$ is a *generalised toy dynamo* if there exist two partitions Ω^1 and Ω^2 of the class $\mathcal{G}$ such that $\mathcal{A}(\Phi_{\Omega^1}) \subset \Phi_{\Omega^2}$ and the conditions (D1)–(D4) hold true in the settings introduced above.

Remark 2. All theorems and the main result hold true for an operator $\mathcal{A}$ that satisfies conditions (D1)–(D4) with right parts of the inequalities multiplied by polynomials in m.

When we have several partitions, e.g. Ω^1, Ω^2, and Ω^3 of the class $\mathcal{G}$ we refer to the norms associated to the partitions by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, respectively.

We will need the following fact.

Remark 3. For any $s_1 \leq 2 \leq s_2$, satisfying $\big(\log s_1 - \log s_2\big) \ll 1$, and $\delta = 2^{-\alpha m}$ there exists a number $0 < \gamma_1 = 2(1-\alpha) < 1/4$ such that

$$m^2\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}. \tag{3.12}$$

for m large enough.

Lemma 3.2.1. *Let $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a generalised toy dynamo and let $\phi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega^1_j}$ be a step function. Then*

$$\sum_{i=N_l^2}^{N_r^2} \sum_{j=N_l^1}^{N_r^1} |c_j| \cdot |1-a_{ij}| \leq 2^{m(3/2+\gamma_1)} \|\phi\|_1.$$

Proof. By straightforward calculation,

$$\sum_{i=N_l^2}^{N_r^2} \sum_{j=N_l^1}^{N_r^1} |c_j| \cdot |1-a_{ij}| = \sum_{(i,j)\in\mathrm{D_{in}}} |c_j| \cdot |1-a_{ij}| \leq \sup|1-a_{ij}| \cdot \#\mathrm{D_{in}} \cdot \sup|c_j| \leq$$

$$\leq \frac{s_1^m}{s_2^m} \cdot m^2\delta s_1^{2m} \cdot 2^{m/2}\|\phi\|_1 \leq 2^{m(3/2+\gamma_1)}\|\phi\|_1.$$

■

Definition 8. Let Ω^1, Ω^2 be two partitions of the class $\mathcal{G}(m)$. We define *the kernel* of $\mathcal{A}^*\chi_{[-1,1]}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ to be the set

$$\operatorname{Ker} \mathcal{A}^*\chi_{[-1,1]}\colon = \left\{\phi \in \Phi_{\Omega^1} \mid \int_{-1}^{1} \mathcal{A}\phi(x)\mathrm{d}x = 0\right\} =$$

$$= \left\{\phi = \sum_{j\in\mathbb{Z}} c_j|\Omega^1_j|\colon \sum_{i=N^2_l}^{N^2_r}\sum_{j\in\mathbb{Z}} a_{ij}c_j|\Omega^2_i| = 0\right\}. \quad (3.13)$$

Proposition 3.2.1. *Let* $2^{\gamma_1} < s_2 < 2$. *Then for any two partitions of the class* $\mathcal{G}$ *and a generalised toy dynamo* $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$

$$\Phi_{\Omega^1} = \chi_{[-1,1]} \oplus \operatorname{Ker} \mathcal{A}^*\chi_{[-1,1]}.$$

In other words, for any $\phi \in \Phi_{\Omega^1}$ *there exist* $\psi \in \operatorname{Ker} \mathcal{A}^*\chi_{[-1,1]}$ *and* $d \in \mathbb{R}$ *such that*

$$\phi = d\chi_{[-1,1]} + \psi. \quad (3.14)$$

Proof. Let $\chi_{[-1,1]} = \sum_{j\in\mathbb{Z}} u_j\chi_{\Omega^1_j}$, where $u_j = 1$ for $N^1_l \le j \le N^1_r$ and $u_j = 0$ otherwise. Let $\phi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j} \in \Phi_{\Omega^1}$ be a step function. We want to find a function $\psi \in \Phi_{\Omega^1}$ such that $\psi \in \operatorname{Ker}\mathcal{A}^*$. By definition of the kernel 8, using (3.14) we write

$$\int_{-1}^{1} \mathcal{A}\psi(x)\mathrm{d}x = \int_{-1}^{1} \mathcal{A}(\varphi - d\chi_{[-1,1]})(x)\mathrm{d}x = \int_{-1}^{1} \sum_{i,j\in\mathbb{Z}} a_{ij}(c_j - du_j)\chi_{\Omega^2_i}(x)\mathrm{d}x =$$

$$= \sum_{i=N^2_l}^{N^2_r}\sum_{j\in\mathbb{Z}} a_{ij}(c_j - du_j)|\Omega^2_i| = 0.$$

We want to solve the last equality for d. It is sufficient to show that for any generalised toy dynamo $\mathcal{A}$ we have that

$$\sum_{i=N^2_l}^{N^2_r}\sum_{j\in\mathbb{Z}} u_ja_{ij}|\Omega^2_i| \neq 0.$$

By straightforward calculation,

$$\sum_{i=N^2_l}^{N^2_r}\sum_{j\in\mathbb{Z}} u_ja_{ij}|\Omega^2_i| = \sum_{j=N^1_l}^{N^1_r}\sum_{i=N^2_l}^{N^2_r} a_{ij}|\Omega^2_i| = 2(N^1_r - N^1_l) + \sum_{j=N^1_l}^{N^1_r}\sum_{i=N^2_l}^{N^2_r} (a_{ij} - 1)|\Omega^2_i|.$$

Using conditions (D1)–(D4), we estimate the last term as follows

$$\sum_{j=N_l^1}^{N_r^1}\sum_{i=N_l^2}^{N_r^2}(a_{ij}-1)|\Omega_i^2| = \sum_{\mathrm{D_{in}}}(a_{ij}-1)|\Omega_i^2| \le \#\mathrm{D_{in}}\cdot \sup|a_{ij}-1|\cdot \sup|\Omega_i^2| \le$$

$$\le s_1^{2m}m^2\delta\cdot\frac{s_1^m}{s_2^m}\cdot 2(s_1^{-m}+s_2^{-m}).$$

We see that $\sum_{i=N_l^2}^{N_r^2}\sum_{j\in\mathbb{Z}} u_j a_{ij}|\Omega_i|\neq 0$ under condition that

$$s_1^{2m}m^2\delta\cdot\frac{s_1^m}{s_2^m}\cdot 2(s_1^{-m}+s_2^{-m}) < 2(N_r^1-N_l^1). \tag{3.15}$$

Recall that, since Ω^1 is of the class $\mathcal{G}$, we have $N_r^1-N_l^1>2^{m-1}$. We also know from (3.12) that there exists $\gamma_1<1/4$ such that

$$m^2\delta\cdot\frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}.$$

Therefore (3.15) holds true under condition that $2^{\gamma_1}<s_2<2$. ∎

Lemma 3.2.2. *Let $\eta = d\chi_{[-1,1]}+\psi\in\Phi_\Omega$ be a step function such that $\|\psi\|\le dr$ for some $r\ll 1$. Then $\eta\in\mathrm{Cone}\left(\frac{2r}{1-2r},\Omega\right)$.*

Proof. We would like to write $\psi=\beta\chi_{[-1,1]}+\widetilde{\psi}$, where $\widetilde{\psi}=\sum_{j\in\mathbb{Z}}\widetilde{c_j}\chi_{\Omega_j}$ and $\sum_{j=N_l}^{N_r}\widetilde{c_j}=0$. Let us assume that $\psi=\sum_{j\in\mathbb{Z}}c_j\chi_{\Omega_j}$, then

$$\sum_{j=N_l}^{N_r}c_j\chi_{\Omega_j}=\beta\sum_{j=N_l}^{N_r}\chi_{\Omega_j}+\sum_{j=N_l}^{N_r}\widetilde{c_j}\chi_{\Omega_j}.$$

implies $\widetilde{c_j}=c_j-\beta$ and consequently $\sum_{j=N_l}^{N_r}(c_j-\beta)=0$. Thus we have an upper bound for $|\beta|$:

$$|\beta|=\left|\frac{1}{N_r-N_l}\sum_{j=N_l}^{N_r}c_j\right|\le\frac{1}{2^{m-1}}\sum_{j\in\mathbb{Z}}|c_j|=2\|\psi\|\le 2dr.$$

Therefore we deduce that

$$\eta=(d+\beta)\chi_{[-1,1]}+\widetilde{\psi}\in C\Big(\frac{|\beta|}{|d|-|\beta|},\Omega\Big)\subset C\Big(\frac{2r}{1-2r},\Omega\Big).$$

∎

Definition 9. Given Ω^1 and Ω^2, two partitions of the class $\mathcal{G}$, we define *a linear operator* $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ by the matrix

$$E_{ij} = \begin{cases} 1, & \text{if } N_l^2 \le i \le N_r^2 \text{ and } N_l^1 \le j \le N_r^1, \\ \\ \delta_{ij}, & \text{otherwise.} \end{cases} \tag{3.16}$$

Remark 4. The operator $\mathcal{E}$ is a generalised toy dynamo.

Lemma 3.2.3. *Consider a function* $\varphi \in \operatorname{Ker} \mathcal{E}^* \chi_{[-1,1]}$. *Then* $\|\mathcal{E}\varphi\|_2 \le \|\varphi\|_1$.

Proof. Let $\varphi \in \Phi_{\Omega^1}$ be a step function. We may write $\varphi = \sum\limits_{j\in\mathbb{Z}} c_j \chi_{\Omega_j^1}$, then

$$\mathcal{E}\varphi = \sum_{j=N_l^1}^{N_r^1} c_j \chi_{[-1,1]} + \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) c_j \chi_{\Omega_j^2};$$

and the condition $\varphi \in \operatorname{Ker} \mathcal{E}^* \chi_{[-1,1]}$ implies $\sum\limits_{j=N_l^1}^{N_r^1} c_j = 0$. Therefore

$$\|\mathcal{E}\varphi\|_2 = \Big\| \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) c_j \chi_{\Omega_j^2} \Big\|_2 = \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) |c_j| \le \|\varphi\|_1.$$

■

Proposition 3.2.2. *Let* s_1 *be small enough so that* $\log_2 s_1 \le 64/63$. *Let* Ω^1 *and* Ω^2 *be partitions of the class* $\mathcal{G}$. *Consider a generalised toy dynamo operator* $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$. *Then for any* $\phi \in \Phi_{\Omega^1}$

$$\|(\mathcal{A} - \mathcal{E})\phi\|_2 \le 2^{m(1/2+\gamma_1)} \|\phi\|_1,$$

where γ_1 *satisfies the inequality* (3.12).

Proof. Let $\phi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j^1} \in \Phi_{\Omega^1}$ be a step function with the unit norm

$$\|\phi\|_1 = \max\big(2^{-m} \sum_{j\in\mathbb{Z}} |c_j|, 2^{-m/2}, \sup |c_j|\big) = 1,$$

which implies $\sum_{j\in\mathbb{Z}}|c_j| \le 2^m$ and $\sup|c_j| \le 2^{m/2}$. By straightforward calculation,

$$(\mathcal{A}-\mathcal{E})\phi = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} c_j(a_{ij}-E_{ij})\chi_{\Omega_i^2} =$$

$$= \sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1} c_j(a_{ij}-1)\chi_{\Omega_i^2} + \sum_{\mathrm{D_{out}}} c_j(a_{ij}-\delta_{ij})\chi_{\Omega_i^2} + \sum_{\mathrm{Sp}} c_j(a_{ij}-\delta_{ij})\chi_{\Omega_i^2}.$$

Observe that

$$\#\mathrm{D_{out}} \le \left(4m^2N_\delta^2 + 2mN_\delta(N_r^2-N_l^2) + 2mN_\delta(N_r^1-N_l^1)\right) =$$

$$= 2mN_\delta(2mN_\delta + N_r^2 - N_l^2 + N_r^1 - N_l^1).$$

Therefore, using $\|\phi\|_1 \le 1$, Lemma 3.2.1, definition of the set $\mathrm{D_{out}}$, and condition (D3),

$$\|(\mathcal{A}-\mathcal{E})\phi\|_{\mathcal{L}_1,\Omega^2} \le$$

$$\le 2^{-m}\Big(\sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1}|c_j|\cdot|a_{ij}-1| + \sum_{\mathrm{D_{out}}}|c_j|\cdot|a_{ij}-\delta_{ij}| + \sum_{\mathrm{Sp}}|c_j|\cdot|a_{ij}-\delta_{ij}|\Big) \le$$

$$\le 2^{-m}\big(2^{m(3/2+\gamma_1)} + \sup|c_j|\cdot\sup|a_{ij}|\cdot\#\mathrm{D_{out}} + \sum_{\mathbb{Z}}|c_j|\cdot mN_\delta\cdot\sup|a_{ij}|\big) \le$$

$$\le 2^{m(1/2+\gamma_1)} + 2^{-m/2}\frac{s_1^m}{s_2^m}\cdot 2mN_\delta(2mN_\delta + N_r^2 - N_l^2 + N_r^1 - N_l^1) + mN_\delta\frac{s_1^m}{s_2^m}.$$

By straightforward calculation we see that for s_1 small enough so that $\log_2 s_1 \le 64/63$

$$\frac{s_1^m}{s_2^m}\cdot mN_\delta < \frac{s_1^m}{s_2^m}\cdot m2^{m(1-\alpha\log_{s_1}2)} \le \frac{s_1^m}{s_2^m}\cdot\frac{\delta s_1^{2m}}{2^m} = 2^{m\gamma_1}.$$

Therefore, under the same condition, since $N_\delta \ll 2^m$,

$$2^{-m/2}\cdot\frac{s_1^m}{s_2^m}\cdot m^2N_\delta^2 < 2^{m\gamma_1}\cdot mN_\delta\cdot 2^{-m/2} < 2^{m(1/2+\gamma_1)}.$$

Finally,

$$2^{1-m/2}mN_\delta\cdot\frac{s_1^m}{s_2^m}\cdot\left(N_r^2-N_l^2+N_r^1-N_l^1\right) \le \frac{s_1^m}{s_2^m}\cdot 2^{m/2}\cdot 4mN_\delta < 2^{m(1/2+\gamma_1)}.$$

Summing up,

$$\|(\mathcal{A}-\mathcal{E})\phi\|_2 \le 3\cdot 2^{m(1/2+\gamma_1)}.$$

Now, for the maximum norm, we have that

$$\|(\mathcal{A}-\mathcal{E})\phi\|_\infty \le \max_{x\in\mathbb{R}} \sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} |c_j|\cdot|a_{ij}-E_{ij}|\chi_{\Omega_i^2}(x) \le \sum_{j\in\mathbb{Z}} |c_j|\frac{s_1^m}{s_2^m} \le 2^m\frac{s_1^m}{s_2^m}.$$

Thus $2^{-m/2}\|(\mathcal{A}-\mathcal{E})\phi\|_\infty \le 2^{m(1/2+\gamma_1)}$. ■

Lemma 3.2.4. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}$. Let $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a linear operator with the matrix defined by (3.16) in the canonical basis. Then for any function $\phi \in \Phi_{\Omega^1}$*

$$\|\mathcal{E}\phi\|_2 \le 2^m\|\phi\|_1 \quad \textit{and} \quad \|\mathcal{E}\chi_{[-1,1]}\|_2 \ge 2^{m-2}.$$

Proof. Let $\phi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega_j^1} \in \Phi_{\Omega^1}$ be a step function of the unit norm. Then, by straightforward calculation,

$$\mathcal{E}\phi = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} c_jE_{ij}\chi_{\Omega_i^2} = \sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1} c_j\chi_{\Omega_i^2} + \sum_{\mathrm{D_{out}\cup D_{in}}} c_j\delta_{ij}\chi_{\Omega_i} =$$

$$= \sum_{j=N_l^1}^{N_r^1} c_j\chi_{[-1,1]} + \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big)c_j\chi_{\Omega_j};$$

so the weighted $\mathcal{L}_1$-norm is

$$\|\mathcal{E}\phi\|_{2,\mathcal{L}_1} = 2^{-m}\cdot\Big|\sum_{j=N_l^1}^{N_r^1} c_j\Big|\cdot(N_r^2-N_l^2) + 2^{-m}\Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big)|c_j| \le 2^m+1.$$

The upper estimate for the supremum norm is easy:

$$\|\mathcal{E}\phi\|_\infty = \max_{x\in\mathbb{R}}\Big(\sum_{j=N_l^1}^{N_r^1} c_j\chi_{[-1,1]}(x) + \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big)c_j\chi_{\Omega_j}(x)\Big) \le 2^m.$$

Hence $\|\mathcal{E}\phi\|_2 \le 2^m\|\phi\|$. Obviously,

$$\|\mathcal{E}\chi_{[-1,1]}\|_2 \ge \|\mathcal{E}\phi\|_{2,\mathcal{L}_1} = 2^{-m}(N_r^1-N_l^1)(N_r^2-N_l^2) \ge 2^{m-2}.$$

■

Let us consider two cones $\mathrm{Cone}\left(1,\Omega^1\right) \subset \Phi_{\Omega^1}$ and $\mathrm{Cone}\left(2^{(\gamma_1-1/2)m},\Omega^2\right) \subset \Phi_{\Omega^2}$ in correspondence with general definition p. 37:

$$\mathrm{Cone}\left(1,\Omega^1\right) \stackrel{\text{def}}{=} \Big\{\phi = d\chi_{[-1,1]} + \psi \mid \psi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\psi\|_1 \le d\Big\}; \tag{3.17}$$

$$\mathrm{Cone}\left(2^{(\gamma_1-1/2)m},\Omega^2\right) \stackrel{\text{def}}{=}$$
$$\Big\{\phi = d\chi_{[-1,1]} + \psi \mid \psi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^2_j};\ \sum_{j=N_l}^{N_r} c_j = 0; \|\psi\|_2 \le d2^{m(\gamma_1-1/2)}\Big\}. \tag{3.18}$$

Theorem 3. *Assume that m is large enough so that the inequality (3.12) holds true for some $0<\gamma_1<1/4$ and all sufficiently small $\varkappa$. Additionally, assume that $\log_2 s_1 \le 64/63$. Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}$. Then for any generalised toy dynamo $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ we have $\mathcal{A}\colon \overline{\mathrm{Cone}\left(1,\Omega^1\right)} \to \mathrm{Cone}\left(2^{m(\gamma_1-1/2)},\Omega^2\right)$; Moreover, for any $\eta \in \mathrm{Cone}\left(1,\Omega^1\right)$ we have $\|\mathcal{A}\eta\|_2 \ge (N_r^2 - N_l^1)\|\eta\| \ge 2^{m-1}\|\eta\|$.*

Proof. Let $\phi \in \mathrm{Cone}\left(1,\Omega^1\right)$ be a step function, $\phi = d\chi_{[-1,1]} + \psi$, where $\psi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j}$, with $\|\psi\|_1 \le d$ and $\sum_{j=N_l^1}^{N_r^1} c_j = 0$. We may write

$$\mathcal{A}\phi = (\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\phi = d\mathcal{E}\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\psi.$$

Obviously, $\|\phi\|_1 \le 2d$, thus by Proposition 3.2.2

$$\|(\mathcal{A}-\mathcal{E})\phi\|_2 \le \|\mathcal{A}-\mathcal{E}\|\cdot\|\phi\|_1 \le d2^{m(1/2+\gamma_1)+1}.$$

By Lemma 3.2.3, $\|\mathcal{E}\psi\|_2 \le \|\psi\|_1 = d$. Therefore $\|(\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\psi\|_2 \le d2^{m(1/2+\gamma_1)+1} + d$, so we conclude

$$\mathcal{A}\phi = \tilde{d}\chi_{[-1,1]} + d(\mathcal{A}-\mathcal{E})\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\psi + \mathcal{E}\psi,$$

where $\tilde{d} \ge d2^{m-2}$ and $\|d(\mathcal{A}-\mathcal{E})\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\psi + \mathcal{E}\psi\|_2 \le d(2^{m(1/2+\gamma_1)+1}+1)$. Theorem now follows from Lemma 3.2.2.

■

Now we proceed to approximation. First of all we shall show that with any sequence ξ we can associate a pair of canonical partitions of the class $\mathcal{G}$. Then we approximate the operator $\ell^m_{\xi*}$ by a generalised toy dynamo (Theorem 4, p. 62).

3.2.2 Canonical partition for the perturbation ℓ^m_ξ

In this section we construct a partition of the class $\mathcal{G}(m)$ associated to the sequence ξ. Later we will refer to it as the canonical partition of the map ℓ^m_ξ.

Recall Definition 4 of the partition $\mathcal{G}$:

Definition 4. We say that a collection of intervals $\Omega = \{\Omega_j\}_{j\in\mathbb{Z}}$ makes *a partition of the class* $\mathcal{G}(m, \delta, s_1, s_2)$, if $\bigcup \overline{\Omega_j} = \mathbb{R}$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$, and the following conditions hold true.

1. The interval $[-1, 1]$ contains at least 2^{m-1} and at most 2^m intervals of the partition, and $\{\pm 1\}$ are the end points of some intervals of the partition.

2. The length of intervals Ω_j is bounded away from zero and from infinity

$$\frac{1}{ms_1^m} \leq |\Omega_j| \leq 2\Big(\frac{1}{s_1^m} + \frac{1}{s_2^m}\Big).$$

3. Any interval $I \subset \mathbb{R}$ of the length $|I| = \delta$ contains not more than

$$N_\delta = 2^{m+1}\delta^{\log_{s_1} 2} = 2^{m(1-\alpha\log_{s_1} 2)+1}$$

intervals of the partition.

4. Any interval of the partition $\Omega_j \subset \mathbb{R} \setminus [-1 - m\delta, 1 + m\delta]$ has length $|\Omega_j| = 2^{-m}$.

We fix s_1 and s_2 in the definition of the map ℓ (3.3) and a sequence $\xi \in \ell_\infty(\mathbb{R})$ with a norm $\|\ell\| \leq \delta = 2^{-m\alpha}$.

The map ℓ_ξ is piecewise linear and so any iteration is a such. Let

$$-\infty = a_0^{(k)} < a_1^{(k)} < \ldots < a_{N_k+1}^{(k)} = +\infty \tag{3.19}$$

be all the points of discontinuity of the map ℓ_ξ^k. Define the corresponding partition of the real line $\mathfrak{a}^{(k)} = \bigcup_{j=0}^{N_k} \overline{\mathfrak{a}_j^{(k)}}$, where $\mathfrak{a}_j^{(k)} = (a_j^{(k)}, a_{j+1}^{(k)})$ are the partition intervals. Observe that for any k we have that $\{\pm 1\}$ are the endpoints of some intervals of the partition. Let $a_{l_k}^{(k)} = -1$ and $a_{r_k}^{(k)} = 1$.

We shall modify the partition $\mathfrak{a}^{(m)}$ and obtain the canonical partition for the map ℓ_ξ^m.

Definition 10. We call a branch $\ell_\xi^n(\mathfrak{a}_j^{(n)})$ of the map ℓ_ξ^n *main*, if for any $0 < k < n$ we have that $\ell_\xi^k(\mathfrak{a}_j^{(n)}) \subset [-1, 1]$.

Definition 11. We call a main branch $\ell_\xi^k(\mathfrak{a}_j^{(k)})$ of the map ℓ_ξ^k *long*, if

$$\left|\ell_\xi^k(\mathfrak{a}_j^{(k)}) \cap [-1, 1]\right| > \frac{2}{s_2}.$$

Lemma 3.2.5. *The map ℓ_ξ^m has at most $2^{m(1-\alpha_1)+1}$ main branches that are not long, where $\alpha_1 < \frac{\alpha}{\log_2 s_1}$ is chosen such that $s_1^{\alpha_1} < 2^\alpha$.*

Proof. Let $\mathfrak{a}_j^{(m)}$ be a domain of a main branch which is not long, that is $|\ell_\xi^m(\mathfrak{a}_j^{(m)}) \cap [-1, 1]| < \frac{2}{s_2}$. Since $\ell_\xi^{(m)}(\mathfrak{a}_j^{(m)})$ is an interval, a connected subset of $\mathbb{R}$, we conclude $|\ell_\xi^m({a_j}^{(m)}) + 1| > 1 - \frac{1}{s_2}$ or $|\ell_\xi^m({a_{j+1}}^{(m)}) - 1| > 1 - \frac{1}{s_2}$. Without loss of generality we may assume that the first holds true. By definition, $a_j^{(m)}$ is a point of discontinuity. Therefore, for some $k < m$ we have that $\ell_\xi^k(a_j^{(m)}) = -1 + \xi(k)$; hence we deduce that $\ell_\xi^{(m)}(a_j^{(m)}) = \ell_\xi^{m-k}(-1 + \xi(k))$. So we conclude $|\ell_\xi^{m-k}(-1 + \xi(k)) + 1| > 1 - \frac{1}{s_2}$, and, consequently, $k < m(1 - \alpha_1) + 1$. Indeed, if $k > m(1 - \alpha_1) + 1$, then $m - k < m\alpha_1 - 1$, and it follows that

$$|\ell_\xi^{m-k}(-1 + \xi(k)) + 1| < s_1^{m\alpha_1 - 1}\delta < \frac{2}{s_1} = 2 - \frac{2}{s_2}.$$

Since the map ℓ_ξ^k has at most 2^k main branches, we conclude that there are at most $2^{m(1-\alpha_1)+1}$ points $a_j^{(m)}$ such that $\ell_\xi^k(a_j^{(m)}) = -1 + \xi(k)$.

Summing up, the map ℓ_ξ^m has at most $2^{m(1-\alpha_1)+1}$ main branches that are not long. ■

Lemma 3.2.6. *Let $1 \le k \le m\alpha \log_{s_1} 2$ and let (a, b) be the domain of a main branch of the map ℓ_ξ^k. Then*

$$|\ell_\xi^k(a) + 1| < \delta \frac{s_1^k - 1}{s_1 - 1} < \frac{2}{s_1} - \delta;$$
$$|\ell_\xi^k(b) - 1| < \delta \frac{s_2^k - 1}{s_2 - 1} < \frac{2}{s_1} - \delta.$$

Proof. By induction in k. The case $k = 1$ is obvious. Recall that $\mathfrak{a}^{(k)} \subset \mathfrak{a}^{(k+1)}$ and

$$\mathfrak{a}^{(k+1)} \setminus \mathfrak{a}^{(k)} = \{\ell_\xi^{-k}(-1), \ell_\xi^{-k}(1), \ell_\xi^{-k}(2/s_1 - 1)\}.$$

Therefore for $x \in \mathfrak{a}^{(k)}$ we have

$$\ell_\xi^{k+1}(x) = \ell_{\sigma^k(\xi)} \ell_\xi^k(x) = s_1 \ell_\xi^k(x) + s_1 - 1 - \xi(k+1), \quad \text{if } |\ell_\xi^k(x) + 1| < 2/s_1 - \delta$$

$$\ell_\xi^{k+1}(x) = \ell_{\sigma^k(\xi)} \ell_\xi^k(x) = s_2 \ell_\xi^k(x) - s_2 + 1 - \xi(k+1), \quad \text{if } |\ell_\xi^k(x) - 1| < 2/s_1 - \delta$$

In the first case we know that, by induction assumption,

$$|\ell_\xi^{k+1}(x) + 1| \le s_1 |\ell_\xi^k(x) + 1| + |\xi(k+1)| + 1 \le \frac{s_1^{k+1} - 1}{s_1 - 1} \delta.$$

In the second case,

$$|\ell_\xi^{k+1}(x) - 1| \le s_2 |\ell_\xi^k(x) - 1| + |\xi(k+1)| + 1 \le \frac{s_2^{k+1} - 1}{s_2 - 1} \delta.$$

■

Corollary 1. *Let $1 \le k \le m\alpha \log_{s_1} 2$. Then for any domain (a, b) of a main branch of the map ℓ_ξ^k we have that $\frac{2}{s_1} - 1 = 1 - \frac{2}{s_2} \in \ell_\xi^k(a, b)$.*

Proof. By assumption, $\ell_\xi^k(a) < \ell_\xi^k(b)$ and from Lemma 3.2.6 it follows that

$$\ell_\xi^k(a) < \frac{2}{s_1} - \delta - 1 < \frac{2}{s_1} - 1 < 1 - \frac{2}{s_2} + \delta < \ell_\xi^k(b).$$

■

Corollary 2. *Let $1 \le n \le m\alpha \log_{s_1} 2$. Then any main branch of the map ℓ_ξ^n is δ-close to one of the ends of the interval $[-1,1]$: in other words, either $1-\delta \in \ell_\xi^n(a,b)$ or $\delta - 1 \in \ell_\xi^n(a,b)$, or both.*

Proof. By induction in n. The case $n = 1$ is obvious. Observe that (a,b) cannot be an interval of continuity of the map ℓ_ξ^n for any $k < n$. Therefore ℓ_ξ^{n-1} is either continuous at a, or at b, or at both end points. In any case (a,b) belongs to an interval of continuity of ℓ_ξ^{n-1} satisfying conditions of Corollary 1 of Lemma 3.2.6. By definition of ℓ_ξ, we see that either $\ell_\xi^{n-1}(a) = \frac{2}{s_1} - 1$ or $\ell_\xi^{n-1}(b) = \frac{2}{s_1} - 1$. Without loss of generality assume that $\ell_\xi^{n-1}(b) = \frac{2}{s_1} - 1$. Then we see that $\ell_\xi^n(a,b) \supset (\xi(n), 1+\xi(n)) \ni 1-\delta$. Similarly, $\ell_\xi^{n-1}(a) = \frac{2}{s_1} - 1$ implies that $\ell_\xi^n(a,b) \supset (-1+\xi(n), \xi(n)) \ni \delta - 1$. ■

Lemma 3.2.7. *The map ℓ_ξ^k for any $1 \le k \le m\alpha \log_{s_1} 2$ has exactly 2^k long branches.*

Proof. By induction in k. The case $k = 1$ is trivial. It follows from Lemma 3.2.6 and Corollary 1 of Lemma 3.2.6 that any long branch of the map ℓ_ξ^k contains at least two long branches of the map ℓ_ξ^{k-1}. ■

Corollary 1. *The map ℓ_ξ^m has at least 2^{m-2} long branches, provided $2\alpha \log_{s_1} 2 > 1$.*

Proof. If $2\alpha \log_{s_1} 2 > 1$, then $m - m\alpha \log_{s_1} 2 < m\alpha \log_{s_1} 2$ and therefore the map $P_\eta^{m - m\alpha \log_{s_1} 2}$ has at least $2^{m - m\alpha \log_{s_1} 2}$ long branches for any $\eta \in \ell_\infty(\mathbb{R})$ with $\|\eta\| \le \delta$. Let $\eta = \sigma^{m\alpha \log_{s_1} 2}\xi$. Then we can decompose $\ell_\xi^m = \ell_\eta^{m(1-\alpha \log_{s_1} 2)} \ell_\xi^{m\alpha \log_{s_1} 2}$. According to Lemma 3.2.7 the map $\ell_\xi^{m\alpha \log_{s_1} 2}$ has at $2^{m\alpha \log_{s_1} 2}$ long branches. By definition of a long branch, its image is at least $\frac{2}{s_2}$ long; using Corollaries 1 and 2 of Lemma 3.2.6, we deduce that for any domain (a,b) of a long branch we have that either $(-1+\delta,-1+\frac{2}{s_2}) \subset \ell_\xi^{m\log_{s_1}2}(a,b)$ or $(1-\frac{2}{s_2};1-\delta) \subset \ell_\xi^{m\log_{s_1}2}(a,b)$. Moreover, any of two intervals $(-1,\frac{2}{s_1}-1)$ and $(\frac{2}{s_1}-1,1)$ contains exactly $2^{m(1-\alpha\log_{s_1}2)-1}$ long branches of the map $\ell_\xi^{m(1-\alpha\log_{s_1}2)}$.

We can find an upper bound for the length of a domain of a long branch of the map $\ell_\eta^{m(1-\alpha\log_{s_1}2)}$: it easy to show by induction in number of iterations that any long branch (a,b) has a domain of the length at least

$$|b-a| = (2 - s_1^{m(1-\alpha)\log_{s_1}2}\delta)s_1^{-m(1-\alpha\log_{s_1}2)} = 2s_1^{-m(1-\alpha\log_{s_1}2)} - \delta \geq s_1^{-m(1-\alpha\log_{s_1}2)}$$

Therefore any of the intervals $(-1+\delta,\frac{2}{s_1}-1)$ and $(\frac{2}{s_1}-1,1-\delta)$ contains at least

$$2^{m(1-\alpha\log_{s_1}2)-1} - s_1^{m(1-\alpha\log_{s_1}2)}\delta = 2^{m(1-\alpha\log_{s_1}2)-1} - 2^{m(\log_2 s_1-2\alpha)} \geq 2^{m(1-\alpha\log_{s_1}2)-1} - 2$$

long branches of the map $\ell_\xi^{m(1-\alpha\log_{s_1}2)}$.

Therefore, the composition has at least $2^{m\alpha\log_{s_1}2}(2^{m(1-\alpha\log_{s_1}2)-1}-2)$ long branches, which comes as $2^{m-1} - 2^{m\alpha\log_{s_1}2} > 2^{m-2}$, as promised. ■

Canonical partition construction. Let us consider the set of end points of domains of long branches

$$D_l \colon= \{x \mid x \text{ is an endpoint of a domain of a long branch of the map } \ell_\xi^m\} \cup \{\pm 1\} =$$
$$= \{-1 = d_1 < d_2 < \ldots < d_N = 1\},$$

and define a partition $\Omega = \bigcup_{j=1}^{N} \Omega_j$ of the interval $[-1,1]$ by $\Omega_j = (d_j, d_{j+1})$; $j = 1,\ldots,N$.

Let us denote by $U_\varepsilon(\Omega_j)$ a neighbourhood of Ω_j of the size ε.

We shall set $\varepsilon = (2s_1^m)^{-1}$. If for some $\Omega_j = (d_j, d_{j+1})$, containing a long branch of the map ℓ_ξ^m, there exist points of discontinuity of the map ℓ_ξ^m in a neighbourhood $U_\varepsilon(\Omega_j) \cap [-1,1]$, then we extend the interval Ω_j to include all these points.

Let $\Omega'_j = (d'_j, d'_{j+1})$, $j = 1, \ldots, N$ be a new collection of intervals. If there exist two intervals (d'_j, d'_{j+1}) and (d'_{j+2}, d'_{j+3}) containing long branches of the map ℓ_ξ^m, and such that $d_{j+2} - d_{j+1} < (ms_1^m)^{-1}$, then we replace the interval (d_j, d_{j+1}) in Ω' with the interval (d_j, d_{j+2}).

Now the length of any interval of the partition Ω', containing a long branch, is not more than $2(s_1^{-m} + s_2^{-m})$. Assume that there exist two intervals (d'_j, d'_{j+1}) and (d'_{j+2}, d'_{j+3}) containing long branches of the map ℓ_ξ^m, such that $d'_{j+2} - d'_{j+1} > s_2^{-m}$, then we split the interval (d'_{j+1}, d'_{j+2}) into intervals of the size s_2^{-m}, allowing one of them to be longer, or smaller, if necessary. More precisely, let $\Omega'_j = (d'_j, d'_{j+1})$ be an interval of Ω' that doesn't contain a long branch. Let $n := \left[s_2^m(d'_{j+1} - d'_j)\right]$ be the number of "whole" intervals of the length s_2^{-m} that could fit inside (d'_j, d'_{j+1}). If $(d'_{j+1} - d'_j) - ns_2^m < s_1^{-m}$, we split the interval (d'_j, d'_{j+1}) into n intervals; adding the intervals $(d'_j + ks_2^{-m}, d'_j + (k+1)s_2^{-m})$, $0 \le k < n$ to the partition Ω'. Otherwise, we split the interval (d'_j, d'_{j+1}) into $n+1$ intervals, adding $(d'_j + ks_2^{-m}, d'_j + (k+1)s_2^{-m})$, $0 \le k \le n$ to Ω'.

The intervals $(a_0^{(m)}, -1)$ and $(1, a_{N_m}^{(m)})$, do not contain any long branches, and we define the partition there as described above. Finally, we define the partition on $(-\infty, a_1^{(m)})$ and $(a_{N_m}^{(m)}, +\infty)$ splitting them into equal intervals of the length 2^{-m}.

We have obtained a partition of the real line, that satisfies Conditions (D2) and (D4) of Definition 4. We have to check other conditions of Definition 4.

Lemma 3.2.8. *The partition constructed satisfies Condition (D3). Any interval $I \subset \mathbb{R}$ of the length δ contains at most $N_\delta < 2^{m(1-\alpha \log_{s_1} 2)+1}$ intervals of the partition.*

Proof. The statement holds true for any interval $I \subset \mathbb{R} \setminus [a_1^{(m)}, a_{N_m}^{(m)}]$ of the length δ. Assume that $I \subset [a_1^{(m)}, a_{N_m}^{(m)}]$, and $|I| = \delta$. Then there are two possibilities:

1. the interval I contains a long branch;

2. the interval I doesn't contain a long branch.

Consider the first case. Observe that for any $k_0 \le m$ and for any interval $I_0 \subset [-1,1]$ of the length $|I_0| < s_1^{-k_0}$ such that $\ell_\xi^k(I_0) \subset [-1,1]$ for all $k < k_0$, the map $\ell_\xi^{k_0}$ is one-to-one on I_0. (Easy to check by induction). Since for any long branch $\mathfrak{a}_j^{(m)}$ we have $1 - \frac{2}{s_1} \in \ell_\xi(\mathfrak{a}_j^{(m)})$, we conclude that $k_0 : = [-\log_2 \delta \log_{s_1} 2] = [\alpha m \log_{s_1} 2]$ and then we see that the map $\ell_\xi^{k_0}$ is one-to-one on any interval I_0 of the length less than δ such that $\ell_\xi^k(I_0) \subset [-1,1]$ for all $k < k_0$. Thus any interval of the length δ contains at most 2^{m-k_0} long branches of the map ℓ_ξ^m. Consequently, any interval I with $|I| \le \delta$ contains at most $2^{m-k_0} < N_\delta$ intervals of the partition with a long branch inside.

Assume now that the interval I of the length $|I| = \delta$ contains some intervals of the partition that do not contain a long branch inside. Let $I_0 \subset I$ be a maximal by inclusion subinterval not containing a long branch. Then by construction of the partition, it contains at most one interval of the partition Ω of the length less than s_2^{-m}. Since the interval I contains at most 2^{m-k_0} long branches, it may contain not more than $2^{m-k_0} + 2$ intervals I_0 without a long branch inside. Therefore, the interval I contains not more than $\delta s_2^m + 2^{m-k_0+1} < N_\delta$ intervals of the partition.

In the second case, an argument similar to the one above shows that an interval I of the length $|I| = \delta$ and without a long branch inside contains not more than $\delta s_2^m + 1 < N_\delta$ intervals of the partition.

■

Lemma 3.2.9. *The partition constructed satisfies Condition (D1) of Definition 4. The interval $[-1,1]$ contains at least 2^{m-1} and at most $2^m - 2^{m\alpha \log_{s_1} 2} + m\delta s_1^m$ intervals of the partition.*

Proof. By Corollary 1 of Lemma 3.2.7, the map ℓ_ξ^m has at least 2^{m-2} long branches, provided

s_1 is chosen such that $2\alpha \log_{s_1} 2 > 1$. Every long branch belongs to exactly one of intervals of the partition, and the escaping set of measure $m\delta$ contains at most $m\delta s_1^m$ intervals. ∎

Summing up, we conclude that the construction leads to a partition of the class $\mathcal{G}$, as desired.

We shall refer to the resulting partition Ω as the canonical partition of the map ℓ_ξ^m.

Lemma 3.2.10. *Any interval of a canonical partition Ω has at most two main branches of the map ℓ_ξ^m.*

Proof. If an interval Ω_j of the partition contains more than one main branch, one of the main branches is not long. Let it be $\mathfrak{a}_k^{(m)}$. Then by Definition 11 $\left|\ell_\xi^m(\mathfrak{a}_k^{(m)}) \cap [-1,1]\right| < \frac{2}{s_2}$.

Now we repeat the calculation of Lemma 3.2.5. The end points of the interval $\mathfrak{a}_k^{(m)}$ are the points of discontinuity of the map ℓ_ξ^m. Then there exists two numbers $n_1 < m$ and $n_2 < m$ such that $\ell_\xi^{n_1}(a_k^{(m)}) = -1 + \xi(n_1)$ and $\ell_\xi^{n_2}(a_{k+1}^{(m)}) = 1 - \xi(n_2)$. Therefore,

$$|\ell_\xi^m(a_k^{(m)}) + 1| = |\ell_{\sigma^{n_1}\xi}^{m-n_1}(-1+\xi(n_1)) + 1| \leq s_1^{m-n_1}\delta;$$

$$|\ell_\xi^m(a_{k+1}^{(m)}) - 1| = |\ell_{\sigma^{n_2}\xi}^{m-n_2}(1-\xi(n_2)) - 1| \leq s_2^{m-n_2}\delta.$$

Since by assumption $\left|\ell_\xi^m(\mathfrak{a}_k^{(m)})\right| < \frac{2}{s_2}$, we deduce $2 - \delta(s_2^{m-n_2} + s_1^{m-n_1}) \leq \frac{2}{s_2}$. The latter is equivalent to $\delta(s_2^{m-n_2} + s_1^{m-n_1}) \geq \frac{2}{s_1}$, which implies that either $\delta s_2^{m-n_2} \geq \frac{1}{s_1}$, or $\delta s_1^{m-n_1} \geq \frac{1}{s_1}$, or both. Hence we get an upper bound on n_1 or n_2, respectively:

$$n_{1,2} < m_0 \colon = m\Big(1 - \frac{\alpha}{\log_2 s_1}\Big) + 10.$$

Therefore one of the end points of $\mathfrak{a}_k^{(m)}$ is an end point of the main branch of the map ℓ_ξ^n with $n < m_0$. Observe that all main branches of the map ℓ_ξ^m are long. Any interval of the length $s_1^{-m_0}$ contains at not more than one main branch of the map $\ell_\xi^{m_0}$. Therefore the distance between short main branches of the map ℓ_ξ^m is at least $s_1^{-m_0} \gg 2(s_1^{-m} + s_2^{-m}) = \sup|\Omega_j|$, and any interval of the partition contains not more than one short main branch of the map ℓ_ξ^m. Therefore, any interval of the partition contains at most two main branches. ■

3.2.3 Approximating $\ell_{\xi*}^m$ by a generalised toy dynamo operator

Here we prove the main result of this Section, Theorem 4, which establishes the existence of a generalised toy dynamo operator, a close approximation of $\ell_{\xi*}^m$ for arbitrary $\|\xi\|_\infty \le \delta$.

Construction. Let a partition Ω^2 of the class $\mathcal{G}$ be given. Let ℓ_ξ^m be as above, and let $\mathfrak{a}^{(m)}$ be a partition of the real line by its points of discontinuity and let Ω^1 be the canonical partition of the map ℓ_ξ^m. Introduce the joint partition: $\mathfrak{a}^{(m)} \cup \Omega^1 = \{d_j\}_{j\in\mathbb{Z}}$. We assume the natural numbering: $[d_0; d_1] \ni 0$ and $d_j < d_{j+1}$ for any $j \in \mathbb{Z}$. Define the image of the joint partition by

$$\{b_j^\pm \colon = \lim_{y\to d_j\pm 0} \ell_\xi^m(y)\}_{j\in\mathbb{Z}}.$$

Then on the interval (d_j, d_{j+1}) the map ℓ_ξ^m is given by

$$\ell_\xi^m(x) \colon = \frac{b_{j+1}^- - b_j^+}{d_{j+1} - d_j} x + \frac{b_j^+ d_{j+1} - b_{j+1}^- d_j}{d_{j+1} - d_j}, \quad d_j < x < d_{j+1}.$$

We define an approximating map $\widehat{\ell}_\xi^m$ to be

$$\widehat{\ell}_\xi^m(x) \colon = \frac{\lfloor b_{j+1}^- \rfloor - \lceil b_j^+ \rceil}{d_{j+1} - d_j} x + \frac{\lceil b_j^+ \rceil d_{j+1} - \lfloor b_{j+1}^- \rfloor d_j}{d_{j+1} - d_j}, \quad d_j < x < d_{j+1};$$

where $\lfloor x \rfloor$ stands for the closest to x point of the partition Ω^2, which is smaller than x; and $\lceil x \rceil$ stands for the closest to x point of the partition Ω^2, which is larger than x. In particular, branches of the map $\widehat{\ell}_\xi^m$ are not longer than branches of the map ℓ_ξ^m.

We define an operator $\mathcal{T} \colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ by

$$(\mathcal{T}\phi)(x) \colon = \sum_{y \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y) \phi(y). \tag{3.20}$$

Lemma 3.2.11. *The operator $\mathcal{T}$ is a linear operator between two subspaces of step functions associated to the partitions Ω^1 and Ω^2 (see p. 26 for definition): $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$.*

Proof. Linearity is obvious. It is sufficient to show that for any interval $\Omega^1_j \in \Omega^1$ of the first partition,

$$(\mathcal{T}\chi_{\Omega^1_j})(x)\colon = \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y)\chi_{\Omega^1_j}(y) \in \Phi_{\Omega^2}.$$

By definition of $\widehat{\ell}_\xi^m$, we see $\lim\limits_{y\to d_j-0} \widehat{\ell}_\xi^m(y) = [b_j^-]$ and $\lim\limits_{y\to d_j+0} \widehat{\ell}_\xi^m(y) = [b_j^+]$, therefore all points of $\Omega^2_k \subset \Omega^2$ have the same number of preimages with respect to ℓ_ξ^m for any interval Ω^2_k. Moreover, $\widehat{\ell}_\xi^{-m}(\Omega^2_k)$ does not contain any point of Ω^1 inside, as it is piecewise monotone on a subpartition $\Omega^1 \cup \mathfrak{a}^{(m)}$. ■

Definition 12. We introduce *the k-escaping set*

$$E_k\colon = \{x \in [-1,1] \mid \exists n < k\ \ell_\xi^n(x) \notin [-1,1]\}. \tag{3.21}$$

Lemma 3.2.12. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2}*

$$\sup_{y\in\Omega^2_i} \#\{x \in \Omega^1_j \mid \ell_\xi^m(x) = y\} \le m^2 \frac{s_1^m}{s_2^m}.$$

Proof. Observe that the map is one-to-one on any interval $I \subset [-1,1]\setminus E_m$ of the length $|I| \le 2s_1^{-m}$.

Given an element Ω^1_j, consider a maximal by inclusion interval $I \subset E_m \cap \Omega^1_j$, such that $|I| \le s_1^{-m}$. We shall show that

$$\max_{y\in\mathbb{R}} \#\big\{x \in I \mid \ell_\xi^m(x) = y\big\} \le 3ms_1^3. \tag{3.22}$$

There are two possibilities:

1. the map ℓ_ξ^m is continuous on $I \subset E_m \cap \Omega^1_j$;

2. the map ℓ_ξ^m is not continuous on $I \subset E_m \cap \Omega^1_j$.

In the first case the map $\ell_\xi|_I$ is a bijection and (3.22) holds true.

Now consider the second case: the map ℓ_ξ^m is not continuous on $I \subset E_m \cap \Omega_j^1$. We may find the smallest k_0 such that $\ell_\xi^{k_0}(I) \not\subset [-1, 1]$. Then

$$\ell_\xi^{k_0}(I) \cap [-1, 1] \subset (-1, -1 + s_1^{k_0 - m}) \sqcup (1 - s_1^{k_0 - m}, 1).$$

Let $m_0 \stackrel{\text{def}}{=} \frac{1+m\alpha}{\log_2 s_1} - 2$. It follows by induction in k that for any $k_0 \le k < m_0$ the image $\ell_\xi^k(I) \cap [-1, 1]$ may be covered by two disjoint intervals in particular,

$$\ell_\xi^k(I) \cap [-1, 1] \subset (-1, -1 + \delta_k^1 + s_1^{k-m}) \sqcup (1 + \delta_k^2 - s_1^{k-m}, 1),$$

where $\delta_k^1 = \sum_{j=k_0}^{k} s_1^{k-j}\xi_j$ and $\delta_k^2 = \sum_{j=k_0}^{k} s_2^{k-j}\xi_j$ with $|\delta_k^{1,2}| \le s_1^{k-k_0+1}\delta$, and $-1 + \frac{2}{s_1} \notin \ell_\xi^k(I)$ for all $k_0 \le k < m_0$. In particular, for any $x_1, x_2 \in I$ such that $\ell_\xi^{k_0}(x_1) \in (-1, -1 + s_1^{k_0-m})$ and $\ell_\xi^{k_0}(x_2) \in (1 - s_1^{k_0-m}, 1)$ we have for all $k < m_0$:

$$\left|\ell_\xi^k(x_1) - \ell_\xi^k(x_2)\right| \ge (1 + \delta_k^2 - s_1^{k-m}) - (-1 + \delta_k^1 + s_1^{k-m}) = 2 - 2s_1^{k-m} + (\delta_k^2 - \delta_k^1) \ge 1.$$

The map $\ell_{\sigma^{k_0}\xi}^{m_0-k_0}$ is a bijection on any of the intervals $(-1, -1 + s_1^{k_0-m})$ and $(1 - s_1^{k_0-m}, 1)$.

Therefore, we deduce that the map $\ell_\xi^{m_0}$ is a bijection on I. It follows that the image $\ell_\xi^{m_0}(I)$ consists of not more than $3m_0$ intervals each of which is not longer than $s_1^{m_0-m}$. Let $\eta = \sigma^{m_0}\xi$ and consider the map $\ell_\eta^{m-m_0}$. We claim that it is a bijection on any interval $I \subset \mathbb{R}$ of the length $|I| \le s_1^{m_0-m-3}$. Indeed, if $\ell_\eta^{m-m_0}$ is continuous on I, then it is a bijection. Assume that for some $k_0 \le m - m_0$ the map $\ell_\eta^{k_0}$ is not continuous on I. Then

$$\ell_\eta^{k_0}(I) \cap [-1, 1] \subset (-1; -1 + s_1^{m_0+k_0-m-3} + \delta) \sqcup (1 - s_1^{m_0+k_0-m-3} - \delta; 1),$$

and for any $k_0 < k \le m - m_0$

$$\ell_\eta^k \cap [-1, 1] \subset (-1; -1 + s_1^{k+1}\delta + s_1^{m_0+k-m-3}) \sqcup (1 - s_1^{k+1} - s_1^{m_0+k-m-3}; 1).$$

By straightforward calculation we see that provided $s_1 \le 2^{2\alpha}$

$$s_1^{m_0+k-m-3} + s_1^{k+1}\delta \le \frac{1}{s_1}.$$

Therefore for any interval I of the length $|I| \leq s_1^{m_0-m-3}$ and for any two points $x_1, x_2 \in I$ with $x_1 \neq x_2$ we have that $\ell_\eta^k(x_1) \neq \ell_\eta^k(x_2)$ for all $1 \leq k \leq m - m_0$. We see that the image $\ell_\xi^{m_0}$ may be covered by not more than $3m_0 s_1^3$ intervals of the length $s_1^{m_0-m-3}$. Hence we conclude that for any interval $|I| \leq s_1^{-m}$

$$\sup_{y\in\mathbb{R}} \#\{x \in I \mid \ell_\xi^m(x) = y\} \leq 3m_0 s_1^3 < 3m s_1^3 + 3.$$

Since by Lemma 3.2.10 any interval of the partition contains at most two main branches, the set $\Omega \cap E_m$ is a union of not more than two intervals, which may be covered by $2 + 2\frac{s_1^m}{s_2^m}$ disjoint intervals of the length s_1^{-m}. Therefore

$$\max_{y\in\mathbb{R}} \#\{x \in \Omega_i^1 \mid \ell_\xi^m(x) = y\} \leq 3m s_1^3 \left(\frac{s_1}{s_2}\right)^m < m^2 \left(\frac{s_1}{s_2}\right)^m.$$

■

Corollary 1. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2} the matrix of the operator $\mathcal{T}$ satisfies condition* (D1)*:*

$$\max |\tau_{ij}| + 1 \leq m^2 \left(\frac{s_1}{s_2}\right)^m.$$

Proof. Recall the definition of the operator $\mathcal{T}$:

$$(\mathcal{T}\phi)(x) \colon = \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y)\phi(y). \tag{3.20}$$

Then for $\phi = \chi_{\Omega_j^1}$ we have

$$(\mathcal{T}\phi)(x) \colon = \sum_{i\in\mathbb{Z}} \tau_{ij}\chi_{\Omega_i^2}(x) = \sum_{i\in\mathbb{Z}} \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y)\chi_{\Omega_j^1}(y)\chi_{\Omega_i^2}(x) =$$

$$= \sum_{i\in\mathbb{Z}} \sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_j^1} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y)\chi_{\Omega_i^2}(x); \tag{3.23}$$

therefore

$$\tau_{ij} = \sum_{y\in\widehat{\ell}_\xi^{-m}(\Omega_i^2)\cap\Omega_j^1} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y).$$

The definition of the map $\widehat{\ell}_\xi$ guarantees that τ_{ij} are well-defined; in particular

$$|\tau_{ij}| \le \#\{x \in \Omega^1_j \mid \widehat{\ell}^m_\xi(x) = y \in \Omega^2_i\},$$

and the right hand side is independent on the choice of y. Obviously,

$$\sup_x \#\{x \in \Omega^1_j \mid \widehat{\ell}^m_\xi(x) = y \in \Omega^2_i\} \le \sup_x \#\{x \in \Omega^1_j \mid \ell^m_\xi(x) = y \in \Omega^2_i\},$$

■

Corollary 2. *We have the following upper bound for a total number of preimages of a point $x \in \mathbb{R}$:*

$$\sup_{x\in\mathbb{R}} \big\{y \in \mathbb{R} \mid \ell^m_\xi(y) = x\big\} \le 2m^2\Big(\frac{2s_1}{s_2}\Big)^m; \tag{3.24}$$

$$\sup_{x\in\mathbb{R}} \big\{y \in \mathbb{R} \mid \widehat{\ell}^m_\xi(y) = x\big\} \le 2m^2\Big(\frac{2s_1}{s_2}\Big)^m. \tag{3.25}$$

Proof. By definition of a partition of the class $\mathcal{G}$, the interval $[-1, 1]$ contains not more than 2^m intervals of the partition; and intervals $[-1-m\delta; -1]$ and $[1; 1+m\delta]$ contain not more than mN_δ intervals of the partition each. Finally, both maps are bijections on the complement to $[-1 - m\delta, 1 + m\delta]$. ■

Lemma 3.2.13. *Let Ω^1 be the canonical partition of the map ℓ^m_ξ. Let Ω^2 be another partition of the class $\mathcal{G}$. Then*

$$\#\big\{(i, j) \in \Box \mid \Omega^2_i \subset [-1, 1] \cap \ell^m_\xi(\Omega^1_j \cap E_m)\big\} \le m^2\delta s_1^{2m}$$

Proof. We shall prove that

$$\sum_{\mathrm{D_{in}}^2} |\Omega^2_i| \le \sum_{\mathfrak{a}^{(m)}_j \subset E_m} |\ell^m_\xi(\mathfrak{a}^{(m)}_j)| \le s_1^m\delta;$$

then the Lemma will from from the lower bound on the size of the elements of partition.

Indeed, by induction one can show that

$$\sum_{\mathfrak{a}_j^{(k)} \subset E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \leq \frac{s_1^k - 2^k}{s_1 - 2}\delta.$$

The case $k = 1$ is trivial. Then we proceed

$$\sum_{\mathfrak{a}_j^{(k)} \subset E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \leq \sum_{\mathfrak{a}_j^{(k-1)} \subset E_{k-1}} |\ell_\xi^k(\mathfrak{a}_j^{(k-1)})| + \sum_{\mathfrak{a}_j^{(k)} \subset E_k \setminus E_{k-1}} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \leq$$

$$\leq s_1\delta \cdot \frac{s_1^{k-1} - 2^{k-1}}{s_1 - 2} + 2^k\delta = \frac{s_1^k - 2^k}{s_1 - 2}\delta.$$

■

Corollary 1. *Let Ω^1 be the canonical partition of the map ℓ_ξ^m. Let Ω^2 be another partition of the class $\mathcal{G}$; and let $\widehat{\ell}_\xi^m$ be a map defined as above on p. 54. Then*

$$\#\{(i,j) \in \Box \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_j^1 \cap E_m)\} \leq m^2\delta s_1^{2m}$$

Proof. The inequality for the map $\widehat{\ell}_\xi^m$ follows from the fact that images of all branches under adjusted map $\widehat{\ell}_\xi^m$ are shorter than the images of the same branches under the original map ℓ_ξ^m. ■

Proposition 3.2.3. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2} the operator $\mathcal{T}$ defined by (3.20) is a generalised toy dynamo.*

Proof. We have checked the condition (D1) already. We should verify the following conditions.

(D2) $\#\mathrm{D_{in}} \leq 3m^2\delta s_1^{2m}$;

(D3) for any pair $(i,j) \in \mathrm{Sp}$ we have that $\tau_{ij} = 0$ whenever $|i-j| > mN_\delta$;

(D4) $\#\mathrm{Ar} \geq 2^{m-2}$.

where

$$\mathrm{Ar}\colon = \{j \in \{N_l^1 \ldots N_r^1\} \mid \#\{i \in \{N_l^2 \ldots N_r^2\} \mid \tau_{ij} = 1\} \geq 2^m - N_\delta\};$$

$$\mathrm{D_{in}}\colon = \{(i,j) \in \{N_l^2 \ldots N_r^2\} \times \{N_l^1 \ldots N_r^1\} \mid \tau_{ij} \neq 1\}.$$

To verify the condition (D2): $\#\mathrm{D_{in}} \leq 3s_1^{2m}m^2\delta$ we shall show that $\sum_{\mathrm{D_{in}}} |\Omega_i^2| \leq 3s_1^m m\delta$ and then taking into account $|\Omega_i^2| \geq \frac{s_1^{-m}}{m}$ we get the result. Let E_m be the m-escaping set as defined by (3.21) above.

We introduce three subsets of the set $\mathrm{D_{in}}$.

$$\mathrm{D_{in}}^1 \colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \setminus \widehat{\ell}_\xi^m(\Omega_j^1 \setminus E_m)\}$$

— complement to the images of the main branches;

$$\mathrm{D_{in}}^2 \colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_j^1 \cap E_m)\}$$

— image of the points that were mapped outside $[-1,1]$ and back;

$$\mathrm{D_{in}}^3 \colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_j^1 \setminus E_m)\}$$

— image of the points that were inside $[-1,1]$ in first m iterations.

We claim that $\mathrm{D_{in}} = \mathrm{D_{in}}^1 \cup \mathrm{D_{in}}^2 \cup \mathrm{D_{in}}^3$: indeed, for any pair of indices $(i,j) \in \mathrm{D_{in}}$ we have that $\Omega_j^1 \cap E_m \neq \varnothing$. We shall show that $\sum_{\mathrm{D_{in}}^1} |\Omega_i^2| \leq s_1^m m\delta$, $\sum_{\mathrm{D_{in}}^2} |\Omega_i^2| \leq s_1^m m\delta$, and $\#\mathrm{D_{in}}^3 \leq s_1^{2m}m^2\delta$.

We start with $\mathrm{D_{in}}^1$ and recall the original partition $\mathfrak{a}^{(m)}$ by the points of discontinuity of the map ℓ_ξ^m. Let $J(\mathrm{D_{in}}^1)$ be the union of intervals with indices corresponding to $\mathrm{D_{in}}^1$:

$$J(\mathrm{D_{in}}^1) \colon = \bigcup_{j \colon (i,j) \in \mathrm{D_{in}}^1} (\Omega_j^1 \setminus E_m).$$

We may write then

$$\sum_{\mathrm{D_{in}}^1} |\Omega_i^2| \leq 2 \cdot \#\{\mathfrak{a}_j^{(m)} \subset J(\mathrm{D_{in}}^1)\} - \sum_{\mathfrak{a}_j^{(m)} \subset J(\mathrm{D_{in}}^1)} |\widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)})| \leq 2^{m+1} - \sum_{\mathfrak{a}_j^{(m)} \subset [-1,1] \setminus E_m} |\widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)})|;$$

and we shall show by induction in k that

$$2^{k+1} - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| < s_1^k \cdot k 2^{-k\alpha}, \text{ where } \|\xi\|_\infty \leq 2^{-k\alpha}.$$

The case $k = 1$ is trivial. Let $\mathfrak{b}^{(k)}$ be the canonical partition of the map ℓ_0^k. This partition has 2^k elements in $[-1,1]$. There exists a correspondence between the sets of indices $\tau\colon \{i \in \mathbb{Z} \mid \mathfrak{a}_i^{(k)} \subset (-1,1)\} \to \{-2^k, \dots, 2^k - 1\}$ that satisfies $\mathrm{d}\ell_\xi^k \mid_{\mathfrak{a}_j^{(k)}} = \mathrm{d}\ell_0^k \mid_{\mathfrak{b}_{\tau(j)}^{(k)}}$ and $\tau(j_1) \neq \tau(j_2)$ for all $j_1 \neq j_2$. In particular, $\operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^k \mid_{\mathfrak{a}_j^{(k)}} = \operatorname{sgn} \mathrm{d}\widehat{\ell}_0^k \mid_{\mathfrak{b}_{\tau(j)}^{(k)}}$.

We split the intervals $\mathfrak{a}_j^{(k)}$ into two groups:

$$B_1^k\colon = \{j \in \{-2^k, \dots, 2^k - 1\} \mid j = \tau(i) \text{ for some } i \in \mathbb{Z}\};$$

$$B_2^k\colon = \{-2^k, \dots, 2^k - 1\} \setminus B_1^k.$$

We also see that $\ell_0^k(\mathfrak{b}_j^{(k)}) = [-1,1]$ for any interval of the partition $\mathfrak{b}^{(k)}$.

$$\begin{aligned}
2^{k+1} - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| &= \Big(\sum_{j \in B_1^k} + \sum_{j \in B_2^k}\Big) |\ell_0^k(\mathfrak{b}_{\tau(j)}^{(k)})| - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| = \\
&= \sum_{j \in B_1^k} |\ell_0^k(\mathfrak{b}_j^{(k)}) \setminus \ell_\xi^k(\mathfrak{a}_j^k)| + \sum_{j \in B_2^k} |\ell_0^k(\mathfrak{b}_j^{(k)})| \leq \\
&\leq s_1 \sum_{j \in B_1^{k-1}} |\ell_0^{k-1}(\mathfrak{b}_j^{(k-1)}) \setminus \ell_\xi^{k-1}(\mathfrak{a}_{\tau(j)}^{k-1})| + 2^k\delta + 2 \sum_{j \in B_2^{k-1}} |\ell_0^{k-1}(\mathfrak{b}_j^{(k-1)})| \leq \\
&\leq s_1 \Big(2^{k-1} - \sum_{B_1^{k-1}} |\ell_\xi^{k-1}(\mathfrak{a}_{\tau(j)}^{k-1})|\Big) + 2^k\delta \leq \\
&\leq s_1^k(k-1)\delta + 2^k\delta \leq s_1^k k\delta.
\end{aligned}$$

Therefore we deduce that

$$2^{m+1} - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| < s_1^m \cdot m2^{-m\alpha}, \text{ where } \|\xi\|_\infty \leq 2^{-k\alpha}.$$

Since there are not more than 2^m main branches, and the length of intervals of the partition Ω^2 is bounded $|\Omega_i^2| \leq 2(s_2^{-m} + s_1^{-m})$, we get

$$2^{m+1} - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\widehat{\ell}_\xi^k(\mathfrak{a}_j^{(k)})| < s_1^m \cdot m2^{-m\alpha} + 2^m(s_2^{-m} + s_1^{-m}) \leq 2m\delta s_1^m;$$

provided $s_2 < 2 < s_1$ are chosen such that $s_1 s_2 > 2^{1+\alpha}$, which is possible.

The inequality for the set ${\mathrm{D_{in}}}^2$ follows from 1 of Lemma 3.2.13.

Finally, for the set ${\mathrm{D_{in}}}^3$ we observe that $(i,j) \in {\mathrm{D_{in}}}^3$ if and only if there exist two main branches $\mathfrak{a}_{j_1}^{(m)}, \mathfrak{a}_{j_2}^{(m)} \subset \Omega_j^1$ such that for any $k < m$ we have $\widehat{\ell}_\xi^k(\mathfrak{a}_{j_1}^{(m)}) \subset [-1,1]$ and $\widehat{\ell}_\xi^k(\mathfrak{a}_{j_2}^{(m)}) \subset [-1,1]$, and $\widehat{\ell}_\xi^m(\mathfrak{a}_{j_1}^{(m)}) \cap \widehat{\ell}_\xi^m(\mathfrak{a}_{j_2}^{(m)}) \cap \Omega_i^2 \neq \varnothing$. Since both $\mathfrak{a}_{j_1}^{(m)}$ and $\mathfrak{a}_{j_2}^{(m)}$ are belong to the same element of the partition we conclude that either $|\ell_\xi^m(\mathfrak{a}_{j_1}^{(m)})| \leq \frac{2}{s_2}$ or $|\ell_\xi^m(\mathfrak{a}_{j_2}^{(m)})| \leq \frac{2}{s_2}$. By Lemma 3.2.5 there are at most $2^{m(1-\alpha_1)}$ main branches with this property. Without loss of generality we assume the latter. Then by definition of $\widehat{\ell}_\xi^m$ we have $|\widehat{\ell}_\xi^m(\mathfrak{a}_{j_2}^{(m)})| \leq \frac{2}{s_2} + 2({s_2}^{-m} + {s_1}^{-m})$. Hence $\#{\mathrm{D_{in}}}^3 \leq 2^{m(1-\alpha_1)} \frac{2N_\delta}{\delta}$. It follows that

$$\#\mathrm{D_{in}} = \#{\mathrm{D_{in}}}^1 + \#{\mathrm{D_{in}}}^2 + \#{\mathrm{D_{in}}}^3 \leq 2s_1^{2m} m\delta + 2^{m(1-\alpha)}\frac{2N_\delta}{\delta} \leq 3s_1^{2m} m\delta,$$

as required.

The condition (D3) follows from the fact that the map $\widehat{\ell}_\xi^m$ is linear and on the compliment $\mathbb{R} \setminus [-1-m\delta, 1+m\delta]$ (in other words, the complement consists of two pieces of continuity), and, moreover, it is given by $\widehat{\ell}_\xi^m(x) = x + b$ on these set. Therefore, $\tau_{ij} = 0$ whenever $|i-j| > b \cdot N_\delta \delta^{-1}$. Obviously, $|b| \leq m\delta$, so we get $\tau_{ij} = 0$ whenever $|i-j| > mN_\delta$.

■

Now it only remains to show that the generalised toy dynamo, constructed from the map $\widehat{\ell}_\xi^m$, is a good approximation to the operator $\ell_{\xi*}^m$.

Theorem 4. *Let Ω^2 be a partition of the class $\mathcal{G}$. Consider a sequence $\xi \in \ell_\infty(\mathbb{R})$ with $\|\xi\|_\infty \leq 2^{-m\alpha}$ and let Ω^1 be the canonical partition of the map ℓ_ξ^m. Then for the operator $\mathcal{T} = \widehat{\ell}_{\xi*}^m \colon \Phi \to \Phi$ defined by (3.20) and for any essentially bounded integrable function $g \in \mathcal{L}_1(\mathbb{R})$ we have*

$$\|(\ell_{\xi*}^m - \mathcal{T})W_\delta g\|_2 \leq \left(\frac{s_1^3}{2^{1/2+\alpha}s_2}\right)^m \cdot m\|g\|_1.$$

Proof. Let $\|g\|_{\Omega_1} = 1$ and let $f = W_\delta g$. Then $\|f\|_\infty \le \|g\|_\infty \le 2^{m/2}$, since

$$\|g\|_1 = \max\Big(2^{-m}\sum_{j\in\mathbb{Z}}\frac{1}{|\Omega_j^1|}\int_{\Omega_j^1}|g(x)|\mathrm{d}x, 2^{-m/2}\sup_{x\in\mathbb{R}}|g(x)|\Big)\le 1.$$

By definition, we write

$$\mathcal{T}f(x) = \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \mathrm{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y); \tag{3.26}$$

$$\ell_{\xi*}^m f(x) = \sum_{y\in\ell_\xi^{-m}(x)} \mathrm{sgn}\big(\ell_\xi^m\big)'(y)f(y). \tag{3.27}$$

We begin with weighted $\mathcal{L}_1$-norm.

$$\|\ell_{\xi*}^m f - \widehat{\ell}_{\xi*}^m f\|_2 = \sum_{j\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_j^2|}\int_{\Omega_j^2}\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x \le$$

$$\le \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big)\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x + \tag{3.28}$$

$$+ \Big(\frac{s_1}{2}\Big)^m\Big(\int_{-1-m\delta}^{-1} + \int_1^{1+m\delta}\Big)\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x + \tag{3.29}$$

$$+ 2^{-m}\sum_{j=N_l^2}^{N_r^2}\frac{1}{|\Omega_j^2|}\int_{\Omega_j^2}\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x. \tag{3.30}$$

We estimate all three terms separately. By the very definition, on the infinite intervals $(1+m\delta, +\infty)$ and $(-\infty, -1-m\delta)$ the map ℓ_ξ^m is given by $\ell_\xi^m(x) = (-1)^m\big(x + \sum_{j=1}^m \xi(j)\big)$. Therefore, the map $\widehat{\ell}_\xi^m$ is one to one on each of the intervals $(-\infty, -1-m\delta)$ and $(1+\delta, +\infty)$; moreover,

$$\ell_\xi^{-m}((-\infty, -1-m\delta)\cup(1+m\delta, +\infty)) \subset (-\infty, -1)\cup(1, +\infty).$$

Observe that for the last point $a_N \in \mathbb{R}$ of the last point of discontinuity of the map ℓ_ξ^m we have, using Lemma 3.3.3:

$$\int_{a_N}^{+\infty}|f(x)|\mathrm{d}x = \int_{a_N}^{+\infty}|(W_\delta g)(x)|\mathrm{d}x = \sum_{j=N_2}^{+\infty}\frac{2^{-m}}{|\Omega_j^2|}\int_{\Omega_j^2}|W_\delta g(x)|\mathrm{d}x \le \|W_\delta g\|_1 \le \frac{mN_\delta}{s_2^m\delta}.$$

The first difference we estimate by the sum of absolute values.

$$\int_{1+m\delta}^{+\infty}\Big|\sum_{y\in\widehat{\ell}_\xi^{-m}(x)}\mathrm{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y)-\sum_{y\in\ell_\xi^{-m}(x)}\mathrm{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big|\mathrm{d}x=$$

$$=\int_{1+m\delta}^{+\infty}\Big|f(\widehat{\ell}_\xi^{-m}(x))-f(\ell_\xi^{-m}(x))\Big|\mathrm{d}x\le 2\Big(\int_{-\infty}^{-1}+\int_1^{+\infty}\Big)|f(x)|\mathrm{d}x=$$

$$=2\Big(\int_{-\infty}^{a_1}+\int_{a_1}^{-1}+\int_1^{a_N}+\int_{a_N}^{+\infty}\Big)|f(x)|\mathrm{d}x,$$

where a_1 and a_N are the first and the last points of discontinuity of the map ℓ_ξ^m. Summing up,

$$\int_{1+m\delta}^{+\infty}|\ell_{\xi*}^m f(x)-\mathcal{T}f(x)|\mathrm{d}x\le 4m\delta\sup|f|+4\|f\|_1\le 4\Big(m\delta 2^{m/2}+\frac{mN_\delta}{s_2^m\delta}\Big)\|g\|_1\le 8\frac{mN_\delta}{s_2^m\delta}. \quad (3.31)$$

Similarly,

$$\int_{-\infty}^{-1-m\delta}|\ell_{\xi*}^m f(x)-\mathcal{T}f(x)|\mathrm{d}x\le 8\frac{mN_\delta}{s_2^m\delta}. \quad (3.32)$$

Summing up (3.31) and (3.32), and taking into account that $\|f\|_1=1$, we get an upper bound for the first term (3.28):

$$\Big(\int_{-\infty}^{-1-m\delta}+\int_{1+m\delta}^{+\infty}\Big)|\ell_{\xi*}^m f(x)-\mathcal{T}f(x)|\mathrm{d}x\le 16\frac{mN_\delta}{s_2^m\delta}. \quad (3.33)$$

Now we use a rough upper bound to estimate the second term. Since by Corollary 2 of Lemma 3.2.12 any point has at most $2m^2\big(\frac{2s_1}{s_2}\big)^m$ preimages with respect to $\widehat{\ell}_\xi^m$ or ℓ_ξ^m; and taking into account $\|f\|_\infty\le 2^{m/2}$.

$$\int_1^{1+m\delta}|(\ell_{\xi*}^m-\mathcal{T})f(x)|\mathrm{d}x\le\int_1^{1+m\delta}|\ell_{\xi*}^m f(x)|+|\mathcal{T}f(x)|\mathrm{d}x\le$$

$$\le\sup_x\big(|\ell_{\xi*}^m f(x)|+|\mathcal{T}f(x)|\big)m\delta\le\Big(\frac{s_1}{s_2}\Big)^m m^3 2^{m+1}\delta\|f\|_\infty\le$$

$$\le m2^{m(3/2-\alpha)}\Big(\frac{s_1}{s_2}\Big)^m.$$

Therefore we get an upper bound for the second term (3.29):

$$\Big(\frac{s_1}{2}\Big)^m\Big(\int_1^{1+m\delta}+\int_{-1-m\delta}^{-1}\Big)\big|(\ell_{\xi*}^m-\mathcal{T})f(x)\big|\mathrm{d}x\le m2^{m(1/2-\alpha)}\Big(\frac{s_1^2}{s_2}\Big)^m. \quad (3.34)$$

The third term (3.30) is a little more complicated. We split the sum into two terms: long branches and all other intervals. Let $\mathfrak{a}^{(m)}$ be a partition of $\mathbb{R}$ by the points of discontinuity (cf. (3.19)) and let $\mathfrak{a}_n^{(m)} = (a_n^{(m)}, a_{n+1}^{(m)})$ be its intervals. Let $\mathfrak{a}_{n_l} = (-1, a_{n_l+1}^{(m)})$ and $\mathfrak{a}_{n_r} = (a_{n_r-1}^{(m)}, 1)$ be the most left and the most right intervals of the partition inside the interval $[-1,1]$. Let E_m be the m-escaping set as defined by (3.21) above. By definition of ℓ_{ξ^*} and $\mathcal{T}$,

$$2^{-m} \sum_{i=N_l^2}^{N_r^2} \frac{1}{|\Omega_i^2|} \int_{\Omega_i^2} |\ell_{\xi*}^m f(x) - \mathcal{T} f(x)| \mathrm{d}x =$$

$$= \sum_{i=N_l^2}^{N_r^2} \frac{2^{-m}}{|\Omega_i^2|} \int_{\Omega_i^2} \Big| \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) f(y) - \sum_{\hat{y} \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(\hat{y}) f(\hat{y}) \Big| \mathrm{d}x.$$

Let us introduce two functions

$$h(j,x) \colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}; \qquad h(j,x) = \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) \chi_{\mathfrak{a}_j^{(m)}}(y) f(y);$$

and

$$\widehat{h}(j,x) \colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}; \qquad \widehat{h}(j,x) = \sum_{\hat{y} \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(\hat{y}) \chi_{\mathfrak{a}_j^{(m)}}(\hat{y}) f(\hat{y}).$$

Then we see that

$$\sum_{j \in \mathbb{Z}} h(j,x) = \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) f(y);$$

and

$$\sum_{j \in \mathbb{Z}} \widehat{h}(j,x) = \sum_{\hat{y} \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(\hat{y}) f(\hat{y});$$

both sums are well-defined, because they have finite number of non-zero terms, since by Corollary 2 of Lemma 3.2.12 the total number of preimages of a point is not more than

$m^3 2^{m+1} s_1^m s_2^{-m}$. Therefore we may write

$$2^{-m}\sum_{i=N_l^2}^{N_r^2}\frac{1}{|\Omega_i^2|}\int_{\Omega_i^2}|\ell_{\xi*}^m f(x)-\mathcal{T}f(x)|\mathrm{d}x=\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Big|\sum_{j\in\mathbb{Z}}\big(h(j,x)-\widehat{h}(j,x)\big)\Big|\mathrm{d}x=$$

$$=\Big(\sum_{j<n_l^{(m)}}+\sum_{j=n_l^{(m)}}^{n_r^{(m)}}+\sum_{j>n_r^{(m)}}\Big)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Big|h(j,x)-\widehat{h}(j,x)\Big|\mathrm{d}x=$$

$$=\Big(\sum_{j<n_l^{(m)}}+\sum_{\mathfrak{a}_j^{(m)}\subset E_m}+\sum_{\mathfrak{a}_j^{(m)}\not\subset E_m}+\sum_{j>n_r^{(m)}}\Big)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Big|h(j,x)-\widehat{h}(j,x)\Big|\mathrm{d}x. \quad (3.35)$$

First we estimate the finite sums:

$$\Big(\sum_{j<n_l^{(m)}}+\sum_{j>n_r^{(m)}}\Big)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Big|h(j,x)-\widehat{h}(j,x)\Big|\mathrm{d}x\le$$

$$\le\Big(\sum_{k=N_l^1-mN_\delta}^{N_l^1}+\sum_{k=N_r^1}^{N_r^1+mN_\delta}\Big)\sum_{\mathfrak{a}_j^{(m)}\subset\Omega_k^1}\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\big|h(j,x)\big|+\big|\widehat{h}(j,x)\big|\mathrm{d}x\le$$

$$\le 2mN_\delta\cdot\sup|\tau_{ij}|\cdot\sup|f|\le 2mN_\delta\Big(\frac{s_1}{s_2}\Big)^m\|g\|_\infty\le 2m\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m; \quad (3.36)$$

for all $s_1\ge 2$.

Observe that for any domain of a main branch $\mathfrak{a}_j^{(m)}\not\subset E_m$ and $y,\hat{y}\in\mathfrak{a}_j^{(m)}$, such that $\ell_\xi^m(y)=\widehat{\ell_\xi^m}(\hat{y})$ we have that $\mathrm{sgn}(\ell_\xi^m)'(y)=\mathrm{sgn}(\widehat{\ell_\xi^m})'(\hat{y})=1$ and $(\ell_\xi^m)'(y)>s_2^m$. As before, let $\mathfrak{a}_j^{(m)}=(a_j^{(m)},a_{j+1}^{(m)})$. Then

$$|\hat{y}-y|\le\frac{1}{\inf|(\ell_\xi^m)'|}\max\big(\widehat{\ell_\xi^m}(a_j^{(m)})-\ell_\xi^m(a_j^{(m)}),\ell_\xi^m(a_{j+1}^{(m)})-\widehat{\ell_\xi^m}(a_{j+1}^{(m)})\big)\le\frac{1}{s_2^{2m}}.$$

Hence for any $f\in W_\delta(\mathcal{L}_1(\mathbb{R}))$ we see that

$$|f(\hat{y})-f(y)|\le\frac{1}{s_2^{2m}}\sup(W_\delta g)'\le\frac{\|g\|_\infty}{s_2^{2m}\delta}\le\frac{2^{m/2}}{\delta s_2^{2m}}.$$

Summing up, since the total number of main branches is not more than 2^m, we get for the first term of (3.35):

$$\sum_{\mathfrak{a}_j^{(m)}\not\subset E_m}\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\big|h(j,x)-\widehat{h}(j,x)\big|\mathrm{d}x\le\frac{2^{-m}}{\inf|\Omega_i^2|}\int_{-1}^{1}\frac{2^{3m/2}}{\delta s_2^{2m}}\mathrm{d}x\le 2\Big(\frac{s_1 2^{1/2+\alpha}}{s_2^2}\Big)^m. \quad (3.37)$$

To estimate the last term, we introduce two sets of indices

$$D \stackrel{\text{def}}{=} \{(s,t) \in \square \mid \Omega_s^2 \subset [-1,1] \cap \ell_\xi^m(\Omega_t^1 \cap E_m)\};$$

$$\widehat{D} \stackrel{\text{def}}{=} \{(s,t) \in \square \mid \Omega_s^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_t^1 \cap E_m)\}.$$

By Lemma 3.2.13 and its Corollary 1, we see $\#D \le m^2 \delta s_1^{2m}$ and $\#\widehat{D} \le m^2 \delta s_1^{2m}$. Observe that

$$\bigcup_{i,j}\{(\mathfrak{a}_j^{(m)} \times \Omega_i^2) \mid \mathfrak{a}_j^{(m)} \subset E_m,\, \Omega_i^2 \subset \widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)} \cap [-1,1])\} \subset$$

$$\{(\Omega_t^1 \times \Omega_s^2) \mid (s,t) \in \square,\, \Omega_s^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_t^1 \cap E_m)\}.$$

along with

$$\bigcup_{i,j}\{(\mathfrak{a}_j^{(m)} \times \Omega_i^2) \mid \mathfrak{a}_j^{(m)} \subset E_m,\, \Omega_i^2 \subset \ell_\xi^m(\mathfrak{a}_j^{(m)} \cap [-1,1])\} \subset$$

$$\{(\Omega_t^1 \times \Omega_s^2) \mid (s,t) \in \square,\, \Omega_s^2 \subset [-1,1] \cap \ell_\xi^m(\Omega_t^1 \cap E_m)\}.$$

. Hence we calculate an upper bound for the second term of (3.35):

$$\begin{aligned}
&2^{-m} \sum_{\mathfrak{a}_j^{(m)} \subset E_m} \sum_{i=N_l^2}^{N_r^2} \frac{1}{|\Omega_i^2|} \int_{\Omega_i^2} \big|h(j,x) - \widehat{h}(j,x)\big| \mathrm{d}x \le \\
&\le 2^{-m} \sup|f| \sum_{\mathfrak{a}_j^{(m)} \subset E_m} \sum_{i=N_l^2}^{N_r^2} \frac{1}{|\Omega_i^2|} \int_{\Omega_i^2} \Big(\sum_{y \in \ell_\xi^{-m}(x)} \chi_{\mathfrak{a}_j^{(m)}}(y) + \sum_{\hat{y} \in \widehat{\ell}_\xi^{-m}(x)} \chi_{\mathfrak{a}_j^{(m)}}(\hat{y}) \Big) \mathrm{d}x \le \\
&\le 2^{-m} \sup|g| \sum_{(i,j) \in D \cup \widehat{D}} \frac{1}{|\Omega_i^2|} \int_{\Omega_i^2} |\tau_{ij}| \mathrm{d}x \le 2^{-m} \sup|g| \cdot \sup|\tau_{ij}| \cdot \#(D \cup \widehat{D}) \le \\
&\le \frac{1}{2^m} \cdot \|g\|_\infty \cdot \left(\frac{s_1}{s_2}\right)^m \cdot m^2 \delta s_1^{2m} \le m^2 \cdot \left(\frac{s_1^3}{2^{1/2+\alpha} s_2}\right)^m . \qquad (3.38)
\end{aligned}$$

Now we collect the four estimates (3.33), (3.34), (3.36), (3.37), and (3.38) together and get for any function g with $\|g\|_1 = 1$:

$$\|\ell_{\xi*}^m W_\delta g - \widehat{\ell}_{\xi*}^m W_\delta g\|_{\mathcal{L}_1,\Omega^2} \leq$$
$$\leq 16\frac{mN_\delta}{s_2^m\delta} + m\Big(\frac{2^{(1/2-\alpha)}s_1^2}{s_2}\Big)^m + 2\Big(\frac{s_1 2^{1/2+\alpha}}{s_2^2}\Big)^m + 2m^2\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m \leq$$
$$\leq 3m^2\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m. \quad (3.39)$$

for m large enough and $s_2 < 2 < s_1$ chosen such that $s_1 s_2^2 \geq 2^{1/2+2\alpha}$.

Now we turn our attention to the supremum norm. We may write

$$\sup_x\big|\ell_{\xi*}^m f(x) - \widehat{\ell}_{\xi*}^m f(x)\big| = \sup_x\Big|\sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big| \leq$$
$$\leq \sup_x\Big|\sum_{i\in\mathbb{Z}}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| \leq$$
$$\leq \sup_x\Big|\Big(\sum_{-\infty}^{N_l^1-mN_\delta} + \sum_{N_l^1-mN_\delta}^{+\infty}\Big)\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| +$$
$$+ \sup_x\Big|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big|. \quad (3.40)$$

Observe that

$$\sup_x\Big|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| \leq$$
$$\leq 2\sup_x \sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta} \#\{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1\}\sup_{\Omega_i}|f(y)| \leq 2\sup_x \sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}|\tau_{ij}|\sup_{\Omega_i}|f(y)| \leq$$
$$\leq \sup|\tau_{ij}|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\sup_{\Omega_i^1}|f(y)|. \quad (3.41)$$

Our goal is to estimate the last sum from above via weighted $\mathcal{L}_1$-norm. Recall that $f = W_\delta g$.

By definition of the weighted $\mathcal{L}_1$ norm, we see

$$\|W_\delta g\|_1 \geq \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \frac{2^{-m}}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| =$$

$$= 2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big(\frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| - \sup_{\Omega_i^1} |W_\delta g| \Big) + \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} 2^{-m} \sup_{\Omega_i^1} |W_\delta g|;$$

in particular,

$$2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \sup_{\Omega_i^1} |W_\delta g| \leq \|W_\delta g\|_1 + 2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big(\sup_{\Omega_i^1} |W_\delta g| - \frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| \Big). \tag{3.42}$$

We know that for any bounded, continuous, absolutely integrable, and piecewise differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ and any finite interval I

$$\Big| \sup_I f - \frac{1}{|I|} \int_I f \Big| \leq \int_I |f'|.$$

Therefore

$$\sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big| \sup_{\Omega_i^1} |W_\delta g| - \frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| \Big| \leq \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \int_{\Omega_i^1} \Big| \frac{\mathrm{d}}{\mathrm{d}x} |W_\delta g(x)| \Big| \mathrm{d}x <$$

$$< \int_{-2}^{2} \Big| \frac{\mathrm{d}}{\mathrm{d}x} |W_\delta g(x)| \Big| \leq \int_{\mathbb{R}} \Big| \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} w_\delta(x-t) |g(t)| \mathrm{d}t \Big| \mathrm{d}x \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| \frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x} \Big| \cdot |g(t)| \mathrm{d}t \mathrm{d}x =$$

$$= \int_{\mathbb{R}} |g(t)| \int_{\mathbb{R}} \Big| \frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x} \Big| \mathrm{d}x \mathrm{d}t \leq \frac{1}{\sqrt{2\pi}\delta} \sum_{j \in \mathbb{Z}} \int_{\Omega_j^1} |g(t)| \mathrm{d}t \leq \frac{2^m}{\delta} \sup |\Omega_j^1| \cdot \|g\|_{\mathcal{L}_1, \Omega^1}. \tag{3.43}$$

Hence, substituting (3.43) to (3.42), and using Lemma 3.3.3

$$\sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \sup_{\Omega_i^1} |W_\delta g| \leq 2^m \|W_\delta g\|_1 + \frac{2^m \sup |\Omega_i^1|}{\delta} \|g\|_1 \leq \Big(\frac{2^m N_\delta}{s_2^m \delta} + \frac{2^m}{s_2^m \delta} \Big) \|g\|_1 \leq$$

$$\leq \frac{2^{m+1} N_\delta}{s_2^m \delta} \|g\|_1. \tag{3.44}$$

Finally, taking into account $\|g\|_{\Omega_1} = 1$, we substitute (3.44) to (3.41) and get for the second term of (3.40)

$$2^{-m/2} \sup_x \Big| \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big(\sum_{y \in \widehat{\ell}_\xi^{-m}(x) \cap \Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y) f(y) - \sum_{y \in \ell_\xi^{-m}(x) \cap \Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) f(y) \Big) \Big| \leq$$

$$\leq \sup |\tau_{ij}| \frac{2^{m/2+1} N_\delta}{s_2^m \delta} \leq \frac{2m^2 N_\delta}{\delta} \Big(\frac{2^{1/2} s_1}{s_2^2} \Big)^m. \tag{3.45}$$

Let us define $A(x) \stackrel{\text{def}}{=} (x - s_2^{-m}, x + s_2^{-m})$. We have the following upper bound for the first sum in (3.40):

$$\sup_x \Big| \Big(\sum_{-\infty}^{N_l^1 - mN_\delta} + \sum_{N_l^1 - mN_\delta}^{+\infty} \Big) \Big(\sum_{y \in \widehat{\ell}_\xi^{-m}(x) \cap \Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y) f(y) - \sum_{y \in \ell_\xi^{-m}(x) \cap \Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) f(y) \Big) \Big| \leq$$

$$\leq \sup_{x \in \mathbb{R}} \sup_{y_1, y_2 \in A(x)} |f(y_1) - f(y_2)| \leq \sup_{|y_1 - y_2| \leq 2 s_2^{-m}} |f(y_1) - f(y_2)| \leq \frac{\sup |f|}{2\delta s_2^m} \leq \frac{2^{m/2}}{2\delta s_2^m}. \quad (3.46)$$

Summing up (3.45) and (3.46), we get in (3.40)

$$2^{-m/2} \sup_x \big| \ell_{\xi*}^m f(x) - \widehat{\ell}_{\xi*}^m f(x) \big| \leq \frac{2m^2 N_\delta}{\delta} \Big(\frac{2^{1/2} s_1}{s_2^2} \Big)^m + \frac{1}{2\delta s_2^m} \leq 3m \Big(\frac{s_1^3}{2^{1/2+\alpha} s_2} \Big)^m, \quad (3.47)$$

(by straightforward calculation).

■

3.3 Invariant cone in Φ.

In this section we construct an invariant cone in the space of essentially bounded and absolutely integrable functions Φ for the operator $W_{\frac{\delta}{2m}} \ell_{\xi*}^m W_{\frac{\delta}{2m}}$, which is independent of the choice of $\|\xi\| \leq \delta$. We exploit the properties of the Weierstrass transform that we prove below.

3.3.1 Discretization and the Weierstrass transform toolbox

Here we prove a few estimates showing that the image of the Weierstrass transform with Gaussian kernel of a large variance compared to the size of elements of a partition may be very well approximated by a step function on the partition.

Definition 13. Given a partition Ω of the class $\mathcal{G}$ we define *a linear discretization opera-*

tor D_Ω:

$$D_\Omega\colon \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R}) \to \Phi_\Omega \cap \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R});$$

$$D_\Omega\colon f \mapsto \sum_{j\in\mathbb{Z}} d_j \chi_{\Omega_j}, \quad d_j = \frac{1}{2}\big(\max_{\Omega_j} f(x) + \min_{\Omega_j} f(x)\big). \tag{3.48}$$

Definition 14. *The Weierstrass transform* W_δ is a convolution with the Gaussian kernel with variance δ^2

$$W_\delta\colon f \mapsto w_\delta * f, \text{ where } w_\delta(x) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{x^2}{2\delta^2}}. \tag{3.49}$$

Lemma 3.3.1. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a differentiable function. Then*

$$\|f - D_\Omega f\|_{\Omega,\mathcal{L}_1} \le 2^{-m-1} \int_{\mathbb{R}} \Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big| \mathrm{d}x. \tag{3.50}$$

Proof. Indeed, by straightforward calculation,

$$\begin{aligned}
\|f - D_\Omega f\|_{\Omega,\mathcal{L}_1} &= \sum_{k\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_k|} \int_{\Omega_k} |f(x) - D_\Omega f(x)| \mathrm{d}x \le \\
&\le \sum_{k\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_k|} \int_{\Omega_k} |\max_{\Omega_k} f(x) - \min_{\Omega_k} f(x)| \mathrm{d}x = \\
&\le 2^{-m} \sum_{k\in\mathbb{Z}} |\max_{\Omega_k} f(x) - \min_{\Omega_k} f(x)| \le 2^{-m} \sum_{k\in\mathbb{Z}} \int_{\Omega_k} \Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big| \mathrm{d}x = \\
&= 2^{-m} \int_{\mathbb{R}} \Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big| \mathrm{d}x.
\end{aligned}$$

■

Lemma 3.3.2. *Let* Ω^1 *and* Ω^2 *be two partitions of the class* $\mathcal{G}(m, \delta, s_1, s_2)$*. Let* D_{Ω^1} *be a discretization operator and let* W_δ *be the Weierstrass transform defined above. Then for any bounded integrable function* f

$$\|D_{\Omega^1} W_\delta f - W_\delta f\|_1 \le \frac{\max(\sup|\Omega_j^1|, \sup|\Omega_j^2|)}{\delta} \|f\|_2 \le \frac{1}{s_2^m \delta} \|f\|_2.$$

Remark 5. The dispersion δ in the Gaussian kernel is the same δ as in the definition of a partition of the class $\mathcal{G}$.

Proof. We begin with estimation of the $\mathcal{L}_\infty$-norm. Let $D_{\Omega^1}W_\delta f = \sum_{j\in\mathbb{Z}} d_j\chi_{\Omega^1_j}$. Then

$$
\begin{aligned}
\|D_{\Omega^1}W_\delta f - W_\delta f\|_\infty &= \sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t - \sum_{j\in\mathbb{Z}} d_j\chi_{\Omega^1_j}(x)\Big| = \\
&= \frac{1}{2}\sup_{k\in\mathbb{Z}}\Big|\max_{\Omega^1_k}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t - \min_{\Omega^1_k}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big| \le \\
&\le \frac{1}{2}\sup_{k\in\mathbb{Z}}\Big(|\Omega^1_k|\cdot\max_{\Omega^1_k}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big|\Big) \le \\
&\le \sup|\Omega^1_k|\sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}x}w_\delta(x-t)f(t)\mathrm{d}t\Big| \le \\
&\le \sup|\Omega^1_k|\sup|f(x)|\sup_{x\in\mathbb{R}}\frac{1}{\sqrt{2\pi}\delta}\Big(e^{-\frac{(x-t)^2}{2\delta^2}}\Big|_{t=-\infty}^{x} - e^{-\frac{(x-t)^2}{2\delta^2}}\Big|_{x}^{t=+\infty}\Big) \le \\
&\le \sup\frac{|\Omega^1_k|}{\delta}\|f\|_\infty.
\end{aligned}
$$

Now we proceed to the weighted $\mathcal{L}_1$-norm. Using Lemma 3.3.1 we get

$$
\begin{aligned}
\|D_{\Omega^1}W_\delta f - W_\delta f\|_1 &\le 2^{-m-1}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}}{\mathrm{d}x}W_\delta f(x)\Big|\mathrm{d}x = \\
&= 2^{-m-1}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big|\mathrm{d}x \le 2^{-m}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x}\Big|\cdot|f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m-1}\int_{\mathbb{R}}|f(t)|\int_{\mathbb{R}}\Big|\frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x}\Big|\mathrm{d}x\mathrm{d}t = \frac{2^{-m}}{\sqrt{2\pi}\delta}\sum_{j\in\mathbb{Z}}\int_{\Omega_j}|f(t)|\mathrm{d}t \le \frac{\sup|\Omega^2_j|}{\delta}\|f\|_{\Omega^2}.
\end{aligned}
$$

∎

Lemma 3.3.3. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(\delta, s_1, s_2)$. Then an upper bound of the norm of the Weierstrass transform is given by*

$$\|W_\delta f\|_2 \le 2m\cdot\sup|\Omega^1_j|\cdot\frac{N_\delta}{\delta}\|f\|_1 \le \frac{mN_\delta}{s_2^m\delta}\|f\|_1. \tag{3.51}$$

Proof. We estimate the norm of the operator W_δ on step functions first. Let $\phi\in\Phi_{\Omega^1}$ be a step function on Ω^1. Assume that $\phi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j}$ and $\|\phi\|_{\Omega^1} = 1$, that is

$$\max\Big(2^{-m}\sum_{j\in\mathbb{Z}}|c_j|, 2^{-m/2}\sup|c_j|\Big) = 1;$$

which implies

$$\sum_{j\in\mathbb{Z}}|c_j| \le 2^m, \quad \sup|c_j| \le 2^{m/2}.$$

Then

$$W_\delta \phi(x) = \sum_{j\in\mathbb{Z}} \int_{\Omega_j^2} c_j w_\delta(x-t)\mathrm{d}t.$$

So we calculate

$$\begin{aligned}\|W_\delta\phi\|_{\Omega^2,\mathcal{L}_1} &= \sum_{k\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_k^2|}\int_{\Omega_k^2}\Big|\sum_{j\in\mathbb{Z}} c_j \int_{\Omega_j^1} w_\delta(x-t)\mathrm{d}t\Big|\mathrm{d}x \le \\ &\le \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \frac{|c_j|}{2^m|\Omega_k^2|}\int_{\Omega_k^2}\int_{\Omega_j^1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x = \\ &= \sum_{j\in\mathbb{Z}} \frac{|c_j|}{2^m}\int_{\Omega_j^1}\sum_{k\in\mathbb{Z}}\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x\mathrm{d}t = \\ &= \sum_{j\in\mathbb{Z}} \frac{|c_j|}{2^m}\int_{\Omega_j^1}\Big(\sum_{|\Omega_k^2-\Omega_j^1|>m\delta} + \sum_{|\Omega_k^2-\Omega_j^1|<m\delta}\Big)\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We know that

$$\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x < \frac{1}{\delta}.$$

We also observe that for any $t\in\Omega_j^1$

$$\sum_{|\Omega_k^2-\Omega_j^1|>m\delta}\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x \le \frac{1}{\inf|\Omega_k^2|}\Big(\int_{-\infty}^{-t-m\delta} w_\delta(x-t) + \int_{t+m\delta}^{+\infty} w_\delta(x-t)\mathrm{d}x\Big)\mathrm{d}x \le \frac{e^{-m}}{\inf|\Omega_k^2|}.$$

Therefore, taking into account that $2^{-m}\sum_{j\in\mathbb{Z}}|c_j|\le 1$,

$$\|W_\delta\phi\|_{\Omega^2,\mathcal{L}_1} \le \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\frac{|c_j|}{2^m}\Big(\frac{e^{-m}}{\inf|\Omega_k^2|} + \frac{mN_\delta}{\delta}\Big)|\Omega_j^1| \le \sup|\Omega_j^1|\Big(\frac{e^{-m}}{\inf|\Omega_k^2|} + \frac{mN_\delta}{\delta}\Big).$$

Now we consider arbitrary function $f\in\mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R})$ with $\|f\|_1 = 1$. Then

$$\begin{aligned}\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &= \sum_{k\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_k^2|}\int_{\Omega_k^2}\Big|\sum_{j\in\mathbb{Z}}\int_{\Omega_j^1} w_\delta(x-t)f(t)\mathrm{d}t\Big|\mathrm{d}x \le \\ &\le 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega_j^1}|f(t)|\sum_{k\in\mathbb{Z}}\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x\mathrm{d}t = \\ &= 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega_j^1}|f(t)|\Big(\sum_{|\Omega_k^2-\Omega_j^1|>m\delta} + \sum_{|\Omega_k^2-\Omega_j^1|<m\delta}\Big)\frac{1}{|\Omega_k^2|}\int_{\Omega_k^2} w_\delta(x-t)\mathrm{d}x\mathrm{d}t \le \\ &\le 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega_j^1}|f(t)|\Big(\frac{e^{-m}}{\inf|\Omega_k^2|} + \frac{3mN_\delta}{\delta}\Big)\mathrm{d}t \le \\ &\le \sup|\Omega_j^1|\Big(\frac{e^{-m}}{\inf|\Omega_k^2|} + \frac{3mN_\delta}{\delta}\Big).\end{aligned}$$

In the last inequality we take into account that

$$\|f\|_{\Omega^1,\mathcal{L}_1} = 2^{-m}\sum_{j\in\mathbb{Z}}\frac{1}{|\Omega_j^1|}\int_{\Omega_j^1}|f(t)|\mathrm{d}t \le 1.$$

Now we recall that $\inf|\Omega_j^1| \ge s_1^{-m}/m$ and therefore, for $s_1 < e$

$$\frac{e^{-m}}{\inf|\Omega_k^2|} = \left(\frac{s_1}{e}\right)^m \ll 1,$$

while

$$\frac{N_\delta}{\delta} = 2^{m(1-\alpha\log_{s_1}2+\alpha)} > 2^m.$$

Therefore we conclude

$$\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} \le 2\sup|\Omega_j^1|\cdot\frac{3mN_\delta}{\delta}.$$

The upper bound of the supremum norm is easy.

$$\|W_\delta f\|_\infty = \sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big| \le \sup_{x\in\mathbb{R}}|f(x)|.$$

■

Lemma 3.3.4. *Let Ω be a partition of the class $\mathcal{G}(s_1,s_2,\delta,m)$ where the parameters s_1 and $\delta = 2^{-m\alpha}$ satisfy the inequality $\log_2 s_1 < 2\alpha$ then*

$$\|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_\Omega \le 2^{-m/2}. \tag{3.52}$$

Proof. Obviously, $\sup|W_\delta\chi_{[-1,1]}(x) - \chi_{[-1,1]}(x)| \le 1$. Now we have to find an upper bound for $\|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_{\Omega,\mathcal{L}_1}$.

$$\|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_{\Omega,\mathcal{L}_1} = \sum_{j\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big|\int_{\mathbb{R}} w_\delta(x-t)\chi_{[-1,1]}(t)\mathrm{d}t - \chi_{[-1,1]}(x)\Big|\mathrm{d}x =$$

We split the sum into two parts: over the intervals inside $[-1,1]$ and the rest

$$= \sum_{j=N_l}^{N_r}\frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x + \Big(\sum_{j>N_r}+\sum_{j<N_l}\Big)\frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x. \tag{3.53}$$

We begin with the first term of (3.53), that is the sum of the intervals of partition inside the interval $[-1,1]$.

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x =$$
$$= \Big(\sum_{j=N_l}^{N_l+mN_\delta} + \sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} + \sum_{j=N_r-mN_\delta}^{N_r} \Big) \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x. \quad (3.54)$$

We estimate each term separately. The first term of (3.54) has only mN_δ elements:

$$\sum_{j=N_l}^{N_l+mN_\delta} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le \sum_{j=N_l}^{N_l+mN_\delta} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(t+1)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le m2^{-m}N_\delta\Big(1 - \int_{0}^{2} w_\delta(t)\mathrm{d}t\Big).$$

We have the following upper bound for the second term of (3.54), since for $|t| < 1 - m\delta$ the integral $\int_{-1}^{1} w_\delta(x-t)\mathrm{d}x$ is close to 1:

$$\sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le \sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(1-m\delta-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le 2^{-m}(N_r - N_l - 2mN_\delta)\Big(1 - \int_{-m\delta}^{2-m\delta} w_\delta(t)\mathrm{d}t\Big).$$

The third term of (3.54) has only mN_δ elements, so we write

$$\sum_{j=N_r-mN_\delta}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le \sum_{j=N_r-mN_\delta}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(1-t)\mathrm{d}t\Big)\mathrm{d}x \le mN_\delta 2^{-m}\Big(1 - \int_{0}^{2} w_\delta(t)\mathrm{d}t\Big).$$

Putting all three inequalities together, we get the following upper bound for the first term of (3.53):

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \leq$$
$$\leq \frac{2N_\delta}{2^m} \cdot \Big(\frac{1}{2} + \delta\Big) + \Big(1 - \frac{N_\delta}{2^{m-1}}\Big)\Big(1 - \int_{-m\delta}^{2-m\delta} w_\delta(t)\mathrm{d}t\Big) \leq$$
$$\leq \frac{2N_\delta}{2^m} \cdot \Big(\frac{1}{2} + \delta\Big) + \Big(1 - \frac{N_\delta}{2^m}\Big)\Big(\int_{-\infty}^{-m\delta} w_\delta(t)\mathrm{d}t + \int_{2-m\delta}^{+\infty} w_\delta(t)\mathrm{d}t\Big) \leq$$
$$\leq \frac{2N_\delta}{2^m} \cdot \Big(\frac{1}{2} + \delta\Big) + \Big(1 - \frac{N_\delta}{2^m}\Big)\Big(\frac{e^{-m} + e^{-1/\delta}}{\sqrt{\pi}}\Big) \leq \frac{2N_\delta}{2^m}.$$

Recall that $N_\delta \leq 2^{m(1-\alpha\log_{s_1} 2)}$ by definition of the partition of the class $\mathcal{G}$. Therefore we complete the estimation of the first term of (3.53) :

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \leq 2^{-m\alpha\log_{s_1} 2} \leq 2^{-m/2}. \tag{3.55}$$

Now we proceed to the upper bound for the second term of (3.53).

$$\Big(\sum_{j>N_r} + \sum_{j<N_l}\Big)\frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \leq$$
$$\leq \frac{2^{-m}}{\inf|\Omega_j|}\Big(\int_{-1-m\delta}^{-1} + \int_{1}^{1+m\delta}\Big)\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x + \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big)\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \leq$$
$$\leq \frac{m\delta}{2^m \inf|\Omega_j|}\int_0^2 w_\delta(t)\mathrm{d}t + \int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(x+t)\mathrm{d}x\mathrm{d}t + \int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(x-t)\mathrm{d}x\mathrm{d}t \leq$$
$$\leq \frac{2m\delta}{2^m \inf|\Omega_j|}\Big(\frac{1}{2} - \delta\Big) + \int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(t-1)\mathrm{d}t\mathrm{d}x + \int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(t-1)\mathrm{d}t\mathrm{d}x \leq$$
$$\leq \frac{2m\delta}{2^m \inf|\Omega_j|} + 2e^{-m}.$$

We observe that

$$\frac{2m\delta}{2^m \inf|\Omega_j|} + 2e^{-m} = \frac{2ms_1^m}{2^{m(1+\alpha)}} + e^{-m} \leq 2^{-m/2-1},$$

under condition that $s_1 < 2^{1/2+\alpha}$. Therefore, we get the following upper bound for the second term of (3.53)

$$\Big(\sum_{j>N_r} + \sum_{j<N_l}\Big)\frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \leq 2^{-m/2-1}. \tag{3.56}$$

Summing up (3.55) with (3.56), we get (3.52). ■

Proposition 3.3.4. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(\delta)$. Let $\varepsilon_1 = 2^{m(\gamma-1/2)}$. Let $\phi \in \mathrm{Cone}\left(\varepsilon_1, \Omega^1\right)$ be a step function. (See p. 37 for a general definition of cones). Then*

$$\|D_{\Omega^2} W_\delta \phi\|_2 > \frac{1}{4}\|\phi\|_1. \tag{3.57}$$

Proof. By Lemma 3.3.3 above, for any $\phi \in \mathrm{Cone}\left(\varepsilon_1, \Omega^1\right)$,

$$\|W_\delta \phi\|_2 \leq \frac{mN_\delta}{s_2^m \delta}\|\phi\|_1.$$

By Lemma 3.3.2,

$$\|D_{\Omega^2} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_\infty \leq 2. \tag{3.58}$$

We can find an upper bound for the weighted $\mathcal{L}_1$-norm using Lemma 3.3.1,

$$\begin{aligned} &\|D_{\Omega^2} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_{\Omega^2,\mathcal{L}_1} \leq 2^{-m-1}\int_{\mathbb{R}}\left|\frac{\mathrm{d}}{\mathrm{d}x} W_\delta \chi_{[-1,1]}(x)\right|\mathrm{d}x = \\ &= 2^{-m-1}\int_{\mathbb{R}}\left|\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x} w_\delta(x-t)\mathrm{d}t\right|\mathrm{d}x = 2^{-m-1}\int_{\mathbb{R}}|w_\delta(x+1) - w_\delta(x-1)|\mathrm{d}x \leq 2^{-m}. \end{aligned} \tag{3.59}$$

Therefore

$$\|D_{\Omega^2} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_2 \leq 2^{1-m/2}. \tag{3.60}$$

Using Lemma 3.3.4,

$$\|W_\delta \chi_{[-1,1]}\|_{\Omega^1} \geq \|\chi_{[-1,1]}\|_2 - \|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_2 \geq 1 - 2^{-m/2}.$$

Consider a step function $\eta = d\chi_{[-1,1]} + \psi \in \mathrm{Cone}\left(\varepsilon_1, \Omega^1\right)$, with $\|\psi\|_1 \leq d$. By Lemma 3.3.2

$$\|W_\delta \psi - D_{\Omega^2} W_\delta \psi\|_2 \leq \frac{1}{s_2^m \delta}\|\psi\|_1 \leq d\frac{2^{m(\gamma_1-1/2)}}{s_2^m \delta}; \tag{3.61}$$

and by Lemma 3.3.3

$$\|W_\delta \psi\|_2 \leq \frac{mN_\delta}{s_2^m \delta}\|\psi\|_1 \leq d\frac{N_\delta 2^{m(\gamma_1-1/2)}}{s_2^m \delta}; \tag{3.62}$$

summing up the last two (3.61) and (3.62) together

$$\|D_{\Omega^2} W_\delta \psi\|_2 \leq d 2^{m(\gamma_1-1/2)}\frac{N_\delta + 1}{s_2^m \delta}.$$

We have the following upper bound for the error of approximation for a function from the cone $\mathrm{Cone}\left(\varepsilon_1, \Omega^2\right)$, using the inequality (3.59), (3.60), and (3.61),

$$\|W_\delta\phi - D_{\Omega^2}W_\delta\phi\|_2 \leq d\|W_\delta\chi_{[-1,1]} - D_{\Omega^2}W_\delta\chi_{[-1,1]}\|_2 + \|W_\delta\psi - D_{\Omega^2}W_\delta\psi\|_2 \leq$$
$$\leq d\Big(2^{1-m/2} + \frac{2^{m(\gamma_1-1/2)}}{s_2^m\delta}\Big). \quad (3.63)$$

We may also write using and Lemma 3.3.4 and (3.62)

$$\|W_\delta\phi\|_2 = \|dW_\delta\chi_{[-1,1]} + W_\delta\psi\|_2 \geq d\|W_\delta\chi_{[-1,1]}\|_2 - \|W_\delta\psi\|_2 \geq$$
$$\geq d\big(\|\chi_{[-1,1]}\|_2 - \|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_2\big) - \|W_\delta\psi\|_2 \geq$$
$$\geq d\Big(\frac{1}{2} - 2^{-m/2} - \frac{N_\delta 2^{m(\gamma_1-1/2)}}{s_2^m\delta}\Big). \quad (3.64)$$

Hence we deduce from (3.63) and (3.64)

$$\|D_{\Omega^2}W_\delta\phi\|_2 \geq \|W_\delta\phi\|_2 - \|W_\delta\phi - D_{\Omega^2}W_\delta\phi\|_2 \geq$$
$$\geq d\Big(\frac{1}{2} - 2^{-m/2} - 2^{1-m/2} - 2^{m(\gamma_1-1/2)}\frac{(N_\delta+1)}{s_2^m\delta}\Big).$$

We can simplify and write, dividing by d,

$$\|D_{\Omega^2}W_\delta\phi\|_2 > \frac{1}{4}\|\phi\|_1.$$

■

3.3.2 Constructing an invariant cone

We shall construct an invariant cone around the cones for the discretized operator $\mathcal{T}$. First we extend the cones from Φ_{Ω^i} to Φ and obtain a pair of cones for $W_\delta\mathcal{T}$; which depend on the choice of the first partition and hence on the sequence ξ. Then we get rid of this dependence using estimates from the previous Subsection.

Proposition 3.3.5. *Let Ω^1, Ω^2, Ω^3 be partitions of the class $\mathcal{G}(\delta)$. Let $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a generalised toy dynamo. There exists a number $\frac{15}{16} < \alpha < 1$ such that for $\delta = 2^{-m\alpha}$ we may choose $\gamma_2 \stackrel{\text{def}}{=} \gamma_1 + \alpha(1 - \log_{s_1} 2) < 1/2$, and then for any $\eta \in \text{Cone}\left(1, \Omega^1\right)$ we have*

$$D_{\Omega^3} W_\delta \mathcal{T}\colon \text{Cone}\left(1, \Omega^1\right) \to \text{Cone}\left(\frac{2^{m(\gamma_2+1/2)}}{s_2^m}, \Omega^3\right) \tag{3.65}$$

$$\|D_{\Omega^3} W_\delta \mathcal{T} \eta\|_3 \geq 2^{m-3} \|\eta\|_1 \tag{3.66}$$

(See p. 37 for definition of the cone).

Proof. We define an operator $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ as before in (3.16). According to Theorem 3 p. 45, we know that $\mathcal{T}\colon \text{Cone}\left(1, \Omega^1\right) \to \text{Cone}\left(2^{m(\gamma_1-1/2)}, \Omega^2\right)$. Consider a step function $\eta = d\chi_{[-1,1]} + \psi \in \text{Cone}\left(1, \Omega^1\right)$. Then $\mathcal{T}\eta = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1$, where the norm is bounded $\|\psi_1\|_2 = \|\mathcal{E}\psi + (\mathcal{T} - \mathcal{E})\eta\|_2 \leq d2^{m(1/2+\gamma_1)}$. We may write

$$D_{\Omega^3} W_\delta \mathcal{T} \eta = D_{\Omega^3} W_\delta (d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1).$$

Using Lemmas 3.3.2 and 3.3.3

$$\|D_{\Omega^3} W_\delta \psi_1\|_3 \leq \|W_\delta \psi_1\|_3 + \|D_{\Omega^3} W_\delta \psi_1 - W_\delta \psi_1\|_3 \leq \frac{mN_\delta + 1}{s_2^m \delta} \|\psi_1\|_2 \leq$$
$$\leq d2^{m(1/2+\gamma_1)} \frac{mN_\delta + 1}{s_2^m \delta}. \tag{3.67}$$

So we conclude using Lemma 3.3.4 that

$$\|D_{\Omega^3} W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq$$
$$\leq \|D_{\Omega^3} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_3 + \|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq 3 \cdot 2^{-m/2} \tag{3.68}$$

Then we may write

$$D_{\Omega^3} W_\delta (d(N_r^1 - N_l^1)\chi_{[-1,1]}) = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2,$$

where $\psi_2 \in \Phi_{\Omega^3}$ and (3.68)

$$\|\psi_2\|_3 \leq d(N_r^1 - N_l^1)\|D_{\Omega^3} W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq d(N_r^1 - N_l^1) 2^{1-m/2}.$$

Hence

$$D_{\Omega^3} W_\delta \mathcal{T}\eta = D_{\Omega^3} W_\delta (d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1) = D_{\Omega^3} W_\delta \psi_1 + d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2,$$

where using (3.67)

$$\frac{\|D_{\Omega^3} W_\delta \psi_1 + \psi_2\|_3}{d(N_r^1 - N_l^1)} \leq \frac{\|\psi_2\|_3 + \|D_{\Omega^3} W_\delta \psi_1\|_3}{d(N_r^1 - N_l^1)} \leq 2^{1-m/2} + \frac{2^{(\gamma_1+1/2)m}}{N_r^1 - N_l^1} \cdot \frac{mN_\delta + 1}{s_2^m \delta} \leq$$
$$\leq 2^{1-m/2} + \frac{2^{(\gamma_1-1/2)m+3} N_\delta}{s_2^m \delta}.$$

Substituting $\delta = 2^{-\alpha m}$ and $N_\delta = 2^{m(1-\alpha \log_{s_1} 2)}$, we set $\gamma_2 \colon = \gamma_1 + \alpha(1 - \log_{s_1} 2)$ and get

$$D_3 W_\delta \mathcal{T}\eta = \tilde{d}\chi_{[-1,1]} + \psi_3, \text{ where } \|\psi_3\| \leq \tilde{d} \cdot \frac{2^{m(\gamma_2+1/2)}}{s_2^m}.$$

■

Definition 15. We extend the operator $\mathcal{E}$ defined between two spaces of step functions by (3.16) to bounded integrable functions. Given a partition Ω^1 of the class $\mathcal{G}$ we consider a map $g_0 \colon \mathbb{R} \to \mathbb{R}$ by

$$g_0(x) = \begin{cases} 1 + \frac{2x-2b}{a-b}, & \text{if } a < x < b \text{ for some interval } (a,b) = \Omega_j^1 \subset [-1,1] \\ x, & \text{otherwise.} \end{cases} \tag{3.69}$$

and introduce *a linear operator* $\mathcal{E} \colon \mathcal{L}_1(\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R})$ defined by:

$$(\mathcal{E}f)(x) = \sum_{y \in g_0^{-1}(x)} f(y). \tag{3.70}$$

Lemma 3.3.5. *For any bounded integrable function f*

$$\int_{\mathbb{R}} |\ell_{\xi*}^m f(x)| \mathrm{d}x = \int_{\mathbb{R}} |\mathcal{E}f(x)| \mathrm{d}x + \frac{2^{m(3/2+\gamma_1)}}{s_2^m}\|f\|_1.$$

Where $0 < \gamma_1 \le 1/8$ *is chosen such that*

$$m\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}.$$

Remark 6. The statement of Lemma 3.3.5 and the argument below hold true for the map $\widehat{\ell_\xi^m}$ as well.

Proof. Let $\mathfrak{a}^{(m)} \colon = \{-\infty = a_0^{(m)} < a_1^{(m)} < \ldots < a_{N+1}^{(m)} = +\infty\}$ be a set of points of discontinuity of the map $\ell_{\xi*}^m$, and let $\mathfrak{a}_j^{(m)} = (a_j^{(m)}, a_{j+1}^{(m)})$ be intervals of the partition. We can

Let us introduce a set of indices of long branches

$$I_l^{(m)} \stackrel{\text{def}}{=} \{1 \le j \le N \mid \mathfrak{a}_j^{(m)} \text{ is a domain of a long branch of the map } \ell_\xi^m\}.$$

split the integral into two

$$\int_{\mathbb{R}} |\ell_{\xi*}^m f(x)| \mathrm{d}x = \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) |\ell_{\xi*}^m f(x)| \mathrm{d}x + \int_{-1-m\delta}^{1+m\delta} |\ell_{\xi*}^m f(x)| \mathrm{d}x.$$

To estimate the first term we recall that $\ell_{\xi*}^m(x) = (-1)^m x + \sum_{j=1}^m \xi(j)$ for $x < a_0^{(m)}$ and $x > a_N^{(m)}$. Since $\|\xi\|_\infty < \delta$, we see that $\big|\sum_{j=1}^m \xi(j)\big| < m\delta$ and write

$$\begin{aligned}
\Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) |\ell_{\xi*}^m f(x)| \mathrm{d}x &= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}(\ell_\xi^m)'(y) f(y) \Big| \mathrm{d}x = \\
&= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| (-1)^m f\big((-1)^m (x - \sum_{j=1}^m \xi(j))\big) \Big| \mathrm{d}x = \\
&= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| f\big((-1)^m (x - \sum_{j=1}^m \xi(j))\big) \Big| \mathrm{d}x \le \\
&\le \Big(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\Big) |f(x)| \mathrm{d}x = \Big(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\Big) |\mathcal{E} f(x)| \mathrm{d}x.
\end{aligned}$$

Consider the second term.

$$\int_{-1-m\delta}^{1+m\delta}\big|\ell^m_{\xi*}f(x)\big|\mathrm{d}x=\int_{-1-m\delta}^{1+m\delta}\Big|\sum_{y\in\ell_\xi^{-m}(x)}\operatorname{sgn}(\ell^m_\xi)'(y)f(y)\Big|\mathrm{d}x=$$

$$=\int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j=1}^{N}\operatorname{sgn}(\ell^m_\xi)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x=$$

$$=\int_{-1-m\delta}^{1+m\delta}\Big|\Big(\sum_{j\in I_l^{(m)}}+\sum_{j\notin I_l^{(m)}}\Big)\operatorname{sgn}(\ell^m_\xi)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x\le$$

$$\le\int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j\in I_l^{(m)}}f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x+$$

$$+\int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j\notin I_l^{(m)}}\operatorname{sgn}(\ell^m_\xi)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x\le$$

$$\le\sum_{j\in I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\mathrm{d}(\ell^m_\xi(y))+\sum_{j\notin I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\mathrm{d}(\ell^m_\xi(y))\operatorname{sgn}(\ell^m_\xi)'(y)\le$$

$$\le\sum_{j\in I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\frac{|\ell^m_\xi(a_j^{(m)})|}{|\mathfrak{a}_j^{(m)}|}\mathrm{d}y+\sum_{j\notin I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\frac{|\ell^m_\xi(a_j^{(m)})|}{|\mathfrak{a}_j^{(m)}|}\mathrm{d}y\le$$

$$\le\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\frac{|g_0(\Omega_j^1)|}{|\Omega_j^1|}\mathrm{d}y+\sup|f(x)|\sum_{j\in I_l^{(m)}}|\ell^m_\xi(\mathfrak{a}_j^{(m)})|\le$$

$$\le\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\frac{|g_0(\Omega_j^1)|}{|\Omega_j^1|}\mathrm{d}y+\sup|f(x)|\cdot\sup|\tau_{ij}|\cdot\sup|\Omega_j^2|\cdot\#(\mathrm{D_{in}}).$$

Observe that

$$\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\mathrm{d}(g_0(y))=\sum_{j=N_l^1}^{N_r^1}\int_{-1}^{1}|f(g_0^{-1}(x)\cap\Omega_j^1)|\mathrm{d}x=\int_{-1}^{1}|\mathcal{E}f(x)|\mathrm{d}x.$$

So we may proceed

$$\int_{-1-m\delta}^{1+m\delta}\big|\ell^m_{\xi*}f(x)\big|\mathrm{d}x\le2\int_{-1}^{1}|\mathcal{E}f(y)|\mathrm{d}y+\sup|f(x)|\cdot\sup|\tau_{ij}|\cdot\sup|\Omega_j^2|\cdot\#(\mathrm{D_{in}})\le$$

$$\le2\int_{\mathbb{R}}|\mathcal{E}f(x)|\mathrm{d}x+2^{m/2}\|f\|_1\cdot m^2\Big(\frac{s_1}{s_2}\Big)^m\cdot3m^2\delta s_1^{2m}.$$

Recall that $m\delta\cdot\frac{s_1^{3m}}{2^m s_2^m}<2^{m\gamma_1}$ so we may conclude

$$\int_{-1-m\delta}^{1+m\delta}\big|\ell^m_{\xi*}f(x)\big|\mathrm{d}x\le2\int_{\mathbb{R}}|\mathcal{E}f(x)|\mathrm{d}x+\frac{2^{m(3/2+\gamma_1)}}{s_2^m}\|f\|_1.$$

■

Lemma 3.3.6. *Let Ω^1, Ω^2, and Ω^3 be partitions of the class $\mathcal{G}$. Let $\mathcal{T}$ be a linear operator on the main space such that $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^1}$ is generalised toy dynamo. Assume that*

$$\int_{\mathbb{R}} |\mathcal{T} f(x)| \mathrm{d}x \leq \int_{\mathbb{R}} |\mathcal{E} f(x)| \mathrm{d}x + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1;$$

where $0 < \gamma_1 \leq 1/8$ is chosen such that $m\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}$. Then for any essentially bounded and absolutely integrable function f

$$\|W_\delta \mathcal{T} f\|_3 \leq 5m \frac{N_\delta}{\delta} \|f\|_1. \tag{3.71}$$

Proof. We shall show that there exists a polynomial $\widetilde{Q}$ such that

$$\|W_\delta \mathcal{E} f\|_3 \leq \frac{N_\delta}{\delta} \widetilde{Q}(m) \|f\|_1,$$

and the Lemma will follow. By direct calculation, substituting $N_\delta = 2^{m(1-\alpha \log_{s_1} 2)}$ and $\delta = 2^{-\alpha m}$ we see that

$$\frac{2^{m(3/2+\gamma_1)}}{s_2^m} \leq \frac{N_\delta}{\delta},$$

under condition that $2^{1/2+\gamma_1+\alpha(\log_{s_1} 2-1)} \leq s_2$, i.e. for $s_2 < 2$ sufficiently large, or, in other words, for $\varkappa = \log \frac{s_1}{s_2}$ small enough.

By definition of the norm we calculate,

$$2^m \|f\|_1 \geq \sum_{j\in\mathbb{Z}} \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y = \sum_{j=N_l^1}^{N_r^1} \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y +$$
$$+ \Big(\sum_{j=N_l^1-mN_\delta}^{N_l^1} + \sum_{j=N_r^1}^{N_r^1+mN_\delta} \Big) \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y + \Big(\sum_{j<N_l^1-mN_\delta} + \sum_{j>N_r^1+mN_\delta} \Big) \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y. \tag{3.72}$$

We estimate each of three terms separately. For the first term we have the following lower bound, using $|\Omega^1_j|\cdot \mathrm{d}g_0(y)=2$ for any $y\in\Omega^1_j\subset[-1,1]$.

$$\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}\frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y=\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\frac{\mathrm{d}g_0(y)}{2}\mathrm{d}y=\frac{1}{2}\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\mathrm{d}g_0(y)=$$

$$=\frac{1}{2}\sum_{j=N_l^1}^{N_r^1}\int_{-1}^{1}|f(g_0^{-1}(x)\cap\Omega_j^1)|\mathrm{d}x\geq\frac{1}{2}\int_{-1}^{1}|(\mathcal{E}f)(x)\mathrm{d}x.$$

Thus for any function f

$$\int_{-1}^{1}|\mathcal{E}f(x)|\mathrm{d}x\leq 2^{m+1}\|f\|_1. \tag{3.73}$$

Consider the second term of (3.72) now:

$$\Big(\sum_{j=N_l^1-mN_\delta}^{N_l^1}+\sum_{j=N_r^1}^{N_r^1+mN_\delta}\Big)\int_{\Omega_j^1}\frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y\geq\frac{1}{\sup|\Omega_j^1|}\Big(\int_{-1-m\delta}^{-1}+\int_{1}^{1+m\delta}\Big)|f(y)|\mathrm{d}y\geq$$

$$\geq\frac{1}{\sup|\Omega_j^1|}\Big(\int_{-1-m\delta}^{-1}+\int_{1}^{1+m\delta}\Big)|(\mathcal{E}f)(y)|\mathrm{d}y$$

Thus

$$\Big(\int_{-1-m\delta}^{-1}+\int_{1}^{1+m\delta}\Big)|(\mathcal{E}f)(y)|\mathrm{d}y\leq 2^m\cdot\sup|\Omega_j^1|\cdot\|f\|_1. \tag{3.74}$$

We have for the remaining term of (3.72)

$$\Big(\sum_{j<N_l^1-mN_\delta}+\sum_{j>N_r^1+mN_\delta}\Big)\int_{\Omega_j^1}\frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y=2^m\Big(\int_{1+m\delta}^{+\infty}+\int_{-\infty}^{-1-m\delta}\Big)|\mathcal{E}f(y)|\mathrm{d}y. \tag{3.75}$$

Summing up the three inequalities (3.73), (3.74) and (3.75) together, we get

$$\int_{\mathbb{R}}|\mathcal{E}f(y)|\mathrm{d}y\leq 2^{m+2}\|f\|_1. \tag{3.76}$$

Taking the last inequality (3.76) into account, we estimate the norm

$$
\begin{aligned}
\|W_\delta \mathcal{E} f\|_3 &= 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \Big| \int_{\mathbb{R}} w_\delta(x-t)(\mathcal{E}f)(t)\mathrm{d}t \Big| \mathrm{d}x \le \\
&\le 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \int_{\mathbb{R}} w_\delta(x-t)|\mathcal{E}f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \sum_{k\in\mathbb{Z}} \int_{\Omega_k^1} w_\delta(x-t)|\mathcal{E}f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m} \sum_{k\in\mathbb{Z}} \int_{\Omega_k^1} |\mathcal{E}f(t)| \Big(\sum_{|\Omega_j^3-\Omega_k^1|>m\delta} + \sum_{|\Omega_j^3-\Omega_k^1|<m\delta} \Big) \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} w_\delta(x-t)\mathrm{d}x\mathrm{d}t \le \\
&\le 2^{-m} \Big(\frac{e^{-m}}{\inf |\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \int_{\mathbb{R}} |\mathcal{E}f(t)|\mathrm{d}t \le \\
&\le \frac{4mN_\delta}{\delta} \|f\|_1.
\end{aligned}
$$

Taking into account

$$
\int_{\mathbb{R}} |\mathcal{T}f(t)|\mathrm{d}t \le \int_{\mathbb{R}} |\mathcal{E}f(t)|\mathrm{d}t + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1,
$$

we calculate in a similar way

$$
\begin{aligned}
\|W_\delta \mathcal{T} f\|_3 &\le 2^{-m} \Big(\frac{e^{-m}}{\inf |\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \int_{\mathbb{R}} |\mathcal{T}f(t)|\mathrm{d}t \le \\
&\le 2^{-m} \Big(\frac{e^{-m}}{\inf |\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \Big(\int_{\mathbb{R}} |\mathcal{E}f(t)|\mathrm{d}t + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1 \Big) \le \\
&\le \frac{N_\delta}{\delta} \cdot \Big(4m + \frac{2^{m(1/2+\gamma_1)}}{s_2^m} \Big) \|f\|_1 \\
&< 5m \frac{N_\delta}{\delta} \|f\|,
\end{aligned}
$$

for $0 < \gamma_1 < 1/8$ and m large enough. ■

Recall general definition of cones associated to a partition Ω (p. 37):

$$
\operatorname{Cone}(r,\Omega) = \Big\{ \eta = d\chi_{[-1,1]} + \varphi \mid \varphi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\varphi\|_\Omega \le dr \Big\}. \qquad (3.6);
$$

$$
\widehat{\operatorname{Cone}}(r,\varepsilon,\Omega) \stackrel{\text{def}}{=} \Big\{ f = \eta + g,\ \eta \in \operatorname{Cone}(r,\Omega),\ \|g\|_\Omega \le \varepsilon \|\eta\|_\Omega \Big\} \qquad (3.7).
$$

Theorem 5. *Let W_δ be the Weierstrass transform defined by (3.49). Let Ω^1, Ω^2, and Ω^3 be three partitions of the class $\mathcal{G}$. Let a linear operator $\mathcal{T}\colon \mathcal{L}_1(\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R})$ be such that $\mathcal{T}(\Phi_{\Omega^1}) \subset \Phi_{\Omega^2}$ is a generalised toy dynamo. Then for any m sufficiently large and $\varkappa = \log \frac{s_1}{s_2}$ sufficiently small there exists $\frac{3}{4} < \alpha < 1$, $r_2(m) \ll 1$, $\varepsilon_2(m) \ll \varepsilon_1(m) \ll 1$ such that $W_\delta \mathcal{T}(\widehat{\text{Cone}}\,(1, \varepsilon_1, \Omega^1)) \subset \widehat{\text{Cone}}\,(r_2, \varepsilon_2, \Omega^3)$ with $\delta = 2^{-m\alpha}$. Moreover, the norm of any function $f \in \widehat{\text{Cone}}\,(1, \varepsilon_1, \Omega^1)$ grows exponentially fast $\|W_\delta \mathcal{T} f\|_3 \ge 2^{m-5}\|f\|_1$.*

Proof. By Theorem 3 on p. 45 we know that $\mathcal{T}(\text{Cone}\,(1, \Omega^1)) \subset \text{Cone}\,(2^{m(\gamma_1 - 1/2)}, \Omega^2)$. Consider a function $\eta = d\chi_{[-1,1]} + \psi \in \text{Cone}\,(1, \Omega^1)$, such that $\int_{-1}^{1} \mathcal{E}\psi = 0$. By Proposition 3.2.2, for any step function $\varphi \in \Phi_{\Omega^1}$ we have $\|(\mathcal{T} - \mathcal{E})\varphi\|_2 \le 2^{m(1/2+\gamma_1)}\|\varphi\|_1$. Using Lemma 3.2.3, we calculate

$$\|\mathcal{T}\eta\|_2 \ge d\|\mathcal{T}\chi_{[-1,1]}\|_2 - \|\mathcal{T}\psi\|_2 \ge d\|\mathcal{E}\chi_{[-1,1]} + (\mathcal{T} - \mathcal{E})\chi_{[-1,1]}\|_2 - \|(\mathcal{T} - \mathcal{E})\psi + \mathcal{E}\psi\|_2 \ge$$
$$\ge d(N_r^1 - N_l^1) - 2d(2^{m(1/2+\gamma_1)} + 1) > \frac{d}{2}(N_r^1 - N_l^1) \ge d2^{m-3} \quad (3.77)$$

Consider a function $f = \eta + g \in \widehat{\text{Cone}}\,(1, \varepsilon_1, \Omega^1)$, where $\eta \in \text{Cone}\,(1, \Omega^1)$ as above is a piecewise constant part; and $\|g\|_1 < d\varepsilon_1$. We may write $W_\delta \mathcal{T} f = W_\delta \mathcal{T}\eta + W_\delta \mathcal{T} g$.

We shall show that for $\delta = 2^{-m\alpha}$ large enough compared to the size of particles of the partition, $W_\delta \mathcal{T} f$ may be approximated by a step function from Φ_{Ω^3}. We write each term as a sum of a step function with remainder, and estimate the Ω^3 norm of every term. Let

$$W_\delta \mathcal{T}\eta = \phi_1 + g_1, \text{ where } \phi_1 = D_{\Omega^3} W_\delta \mathcal{T}\eta, \text{ and } g_1 = W_\delta \mathcal{T}\eta - D_{\Omega^3} W_\delta \mathcal{T}\eta; \quad (3.78)$$

$$W_\delta \mathcal{T} g = \phi_2 + g_2, \text{ where } \phi_2 = D_{\Omega^3} W_\delta \mathcal{T} g, \text{ and } g_2 = W_\delta \mathcal{T} g - D_{\Omega^3} W_\delta \mathcal{T} g. \quad (3.79)$$

Using Lemma 3.3.2 and Proposition 3.2.2 we estimate the Ω^3 norm of the first remainder term $\|g_1\|$.

$$\|g_1\|_3 = \|W_\delta \mathcal{T}\eta - D_{\Omega^3} W_\delta \mathcal{T}\eta\|_3 \le \frac{\|\mathcal{T}\eta\|_2}{s_2^m \delta} \le \frac{2d(N_r^1 - N_l^1)}{s_2^m \delta} \le \frac{d2^m}{s_2^m \delta}, \quad (3.80)$$

since

$$\|\mathcal{T}\eta\|_2 = \|(\mathcal{T}-\mathcal{E})\eta\|_2 + \|\mathcal{E}\eta\|_2 \leq d2^{m(1/2+\gamma_1)} + d(N_r^1 - N_l^1) \leq 2d(N_r^1 - N_l^1).$$

We also know that $\|\mathcal{T}g\|_2 \leq s_1^m \|g\|_1$, therefore we have the following upper bound for the second remainder term $\|g_2\|_3$:

$$\|g_2\|_3 = \|W_\delta \mathcal{T}g - D_{\Omega^3} W_\delta \mathcal{T}g\|_3 = \frac{\|\mathcal{T}g\|_2}{s_2^m \delta} \leq \frac{d s_1^m \varepsilon_1}{s_2^m \delta}. \tag{3.81}$$

Since $\mathcal{T}\eta \in \operatorname{Cone}\left(2^{(\gamma_1-1/2)}, \Omega^2\right)$ we may apply Proposition 3.3.4 to estimate $\|\phi_1\|_3$, using (3.77)

$$\|\phi_1\|_3 = \|D_{\Omega^3} W_\delta \mathcal{T}\eta\|_3 \geq \frac{1}{4}\|\mathcal{T}\eta\|_2 \geq d2^{m-5}.$$

Finally, for $\|\phi_2\|_3$ we get, using Lemma 3.3.6

$$\|\phi_2\|_3 = \|D_{\Omega^3} W_\delta \mathcal{T}g\|_3 \leq \|W_\delta \mathcal{T}g\|_3 + \|W_\delta \mathcal{T}g - D_{\Omega^3} W_\delta \mathcal{T}g\|_3 \leq$$
$$\leq 5m\frac{N_\delta}{\delta}\|g\|_1 + \|g_2\| \leq d\frac{\varepsilon_1}{\delta}\left(5mN_\delta + \frac{s_1^m}{s_2^m}\right). \tag{3.82}$$

We would like to find a number $0 < r_2(m) \ll 1$ such that for some d_0

$$\phi_1 + \phi_2 = d_0 \chi_{[-1,1]} + \psi \text{ with } \|\psi\|_3 \leq d_0 r_2; \tag{3.83}$$

and two numbers $0 < \varepsilon_2(m) \ll \varepsilon_1(m) < 1$ such that the following inequality holds true

$$\|g_1 + g_2\|_3 \leq d_0 \varepsilon_2. \tag{3.84}$$

We apply Proposition 3.3.5 p. 79 to the function $\eta \in \operatorname{Cone}\left(1, \Omega^1\right)$, and get

$$\phi_1 = D_{\Omega^3} W_\delta \mathcal{T}\eta = \tilde{d}\chi_{[-1,1]} + \psi_1 \text{ where } \|\psi_1\|_3 \leq \tilde{d}\frac{2^{m(\gamma_2+1/2)}}{s_2^m} \text{ and } 2^{m-5}d < \tilde{d} < 2^m d. \tag{3.85}$$

with $\gamma_2 := \gamma_1 + \alpha(1 - \log_{s_1} 2)$. Using the inequalities (3.82) and (3.85) above we write

$$\|\psi\|_3 = \|\phi_2 + \psi_1\|_3 \leq d\frac{\varepsilon_1}{\delta}\left(N_\delta + \frac{s_1^m}{s_2^m}\right) + d2^{m(\gamma_2+3/2)}\frac{1}{s_2^m}. \tag{3.86}$$

Therefore the condition (3.83) on r_2 holds true if

$$\frac{\varepsilon_1}{\delta}\left(N_\delta + \frac{s_1^m}{s_2^m}\right) < r_2 2^{m-3}; \tag{3.87}$$

$$\frac{2^{m(\gamma_2+3/2)}}{s_2^m} < r_2 2^{m-3}. \tag{3.88}$$

We can find a lower bound on d_0 from (3.83), using upper bound for $\|\psi\|_3$ from (3.86)

$$\|d_0\chi_{[-1,1]}\|_3 = \|\phi_1 + \phi_2 - \psi\|_3 = \|\tilde{d}\chi_{[-1,1]} + \psi_1 + \phi_2 - \psi\|_3 \geq$$

$$\geq \|\tilde{d}\chi_{[-1,1]}\|_3 - \|\psi_1 + \phi_2\|_3 - \|\psi\|_3 \geq d2^{m-4} - 2\|\psi\|_3 \geq$$

$$\geq d2^{m-4} - dr_2 2^{m-1} \geq d2^{m-2}, \tag{3.89}$$

for all $r_2 < 1/2$.

We can find an upper bound for $\|g_1 + g_2\|$ summing up (3.80) with (3.81). Then the second inequality (3.84) on ε_2 will follow from

$$\frac{2^m}{\delta s_2^m} + \frac{\varepsilon_1 s_1^m}{\delta s_2^m} \leq 2^{m-2}\varepsilon_2. \tag{3.90}$$

We claim that the three inequalities (3.87), (3.88), (3.90), and conditions of Theorem 3 on p. 45 hold true with $\alpha = \frac{15}{16}$, $\gamma_1 = \frac{1}{8}$, $r_2 = \delta^{\frac{1}{64}}$, and $\varepsilon_1 = r_2^2$, $\varepsilon_2 = r_2^4$, if $\varkappa = \log\frac{s_1}{s_2} \leq \frac{1}{25}$ is small enough. In particular, we get

$$W_\delta\mathcal{T}(\text{Cone}\left(1, r_2^2, \Omega^1\right)) \subset \text{Cone}\left(r_2, r_2^4, \Omega^3\right),$$

for $r_2 = \delta^{\frac{1}{64}}$. The condition on the norm $\|W_\delta\mathcal{T}f\|_3 \geq 2^{m-5}\|f\|_1$ follows from (3.80), (3.81), (3.86) and (3.89). ∎

Corollary 1. *Under the hypotheses and in the notations of Theorem 5 on p. 86, we have for* $r_2 = \delta^{\frac{1}{64}}$*:*

$$W_{\frac{\delta}{m}}\mathcal{T}\colon \text{Cone}\left(1, r_2^2, \Omega^1\right) \to \text{Cone}\left(r_2, r_2^4, \Omega^3\right); \tag{3.91}$$

$$\forall f \in \text{Cone}\left(1, r_2^2, \Omega^1\right) : \|W_{\frac{\delta}{m}}\mathcal{T}f\|_3 \geq 2^{m-5}\|f\|_1. \tag{3.92}$$

Proof. The theorem follows from Propositions 3.3.4 and 3.3.5 and Lemma 3.3.6. If we replace δ in the Gaussian kernel by $\frac{\delta}{m}$, we shall multiply the upper bounds in the inequalities by polynomials. Since the estimates are based on comparison powers of 2, the results still hold true. ■

Theorem 2. *(Invariant cone)* There exist a measure preserving piecewise-smmoth transformation $f\colon \mathbb{R}\to\mathbb{R}$, a cone C in the space Φ of essentially bounded absolutely integrable vector fields on $\mathbb{R}$, and a norm $\|\cdot\|$ in Φ such that for an $m\gg 1$ large enough and any sequence $\|\xi\|_\infty\le\delta$ with $\delta=2^{-m\alpha}$ for $\frac{15}{16}<\alpha<1$ we have (3.1)

$$W_{\frac{\delta}{2m}} f_{\xi*}^m W_{\frac{\delta}{2m}}\colon \overline{C}\to C; \qquad \forall v\in\overline{C}\colon \|W_{\frac{\delta}{2m}} f_{\xi*}^m W_{\frac{\delta}{2m}} f\|\ge\frac{1}{4}\|f_*^m\|\cdot\|v\|.$$

Proof. We choose the transformation to be $f=\ell$, and pick up a partition Ω of the class $\mathcal{G}$. Our goal is to show that there exist four numbers $r_2(m)\ll r_1(m)$ and $\varepsilon_2(m)\ll\varepsilon_1(m)\ll 1$ such that

$$W_\delta\ell_{\xi*}^m W_\delta\colon \widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega)\to\widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega)\subset\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega)\,. \tag{3.93}$$

$$\forall f\in\mathrm{Cone}\,(r_1,\varepsilon_1,\Omega)\,:\,\|W_\delta\ell_{\xi*}^m W_\delta f\|_\Omega\ge 2^{m-2}\|f\|_\Omega. \tag{3.94}$$

Let Ω^1 be the canonical partition of the perturbation ℓ_ξ^m. First of all, we shall find a number r_1 such that for any $\eta\in\mathrm{Cone}\,(r_1,\Omega)$ we have $D_{\Omega^1}W_\delta\eta\in\mathrm{Cone}\,(1,\Omega^1)$.

Since $\eta\in\mathrm{Cone}\,(r_1,\Omega)$, we may write $\eta=d\chi_{[-1,1]}+\psi$, where $\psi=\sum\limits_{j\in\mathbb{Z}}c_j\chi_{\Omega_j}$, $\sum\limits_{j=N_l}^{N_r}c_j=0$; and $\|\psi\|_{\Omega^1}\le dr_1$. Then

$$D_{\Omega^1}W_\delta\eta=dD_{\Omega^1}W_\delta\chi_{[-1,1]}+D_{\Omega^1}W_\delta\psi.$$

Using Lemmas 3.3.2 and 3.3.3 we get

$$\|D_{\Omega^1}W_\delta\psi\|_1\le\|W_\delta\psi\|_1+\|D_{\Omega^1}W_\delta\psi-W_\delta\psi\|_1\le dr_1\frac{mN_\delta+1}{s_2^m\delta}\le dr_1\frac{2mN_\delta}{\delta s_2^m}$$

and for the supremum norm we have $\|D_{\Omega^1}W_\delta\psi\|_\infty \le \|\psi\|_\infty$. Summing up,

$$\|D_{\Omega^1}W_\delta\psi\|_1 \le dr_1\frac{2mN_\delta}{\delta s_2^m}. \tag{3.95}$$

Using Lemma 3.3.4, we calculate

$$\|D_{\Omega^1}W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_1 \le$$
$$\le \|D_{\Omega^1}W_\delta\chi_{[-1,1]} - W_\delta\chi_{[-1,1]}\|_1 + \|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_1 \le 2^{1-m/2}; \tag{3.96}$$

which implies $dD_{\Omega^1}W_\delta\chi_{[-1,1]} = d\chi_{[-1,1]} + \psi_1$, where $\psi_1 \in \Phi_{\Omega^1}$, $\|\psi_1\|_1 \le d2^{1-m/2}$. Hence $D_{\Omega^1}W_\delta\eta = d\chi_{[-1,1]} + D_{\Omega^1}W_\delta\psi + \psi_1$, where

$$\|D_{\Omega^1}W_\delta\psi + \psi_1\|_1 \le dr_1\frac{2mN_\delta}{\delta s_2^m} + d2^{1-m/2}.$$

By Lemma 3.2.2, p. 41, in order to guarantee $D_{\Omega^1}W_\delta\eta \in \mathrm{Cone}\left(1,\Omega^1\right)$, it is sufficient to choose the parameter $r_1 \ll 1$ such that

$$\frac{2mN_\delta}{\delta s_2^m} < \frac{1}{r_1};$$

Let us set

$$r_1 \stackrel{\text{def}}{=} \frac{\delta s_2^m}{4mN_\delta}. \tag{3.97}$$

We can also notice using Lemma 3.3.2, that

$$\|(D_{\Omega^1}W_\delta - W_\delta)\eta\|_1 \le \frac{1}{s_2^m\delta}dr_1 = \frac{d}{4mN_\delta}. \tag{3.98}$$

Taking into account that $D_{\Omega^1}W_\delta\eta \in \mathrm{Cone}\left(1,\Omega^1\right)$ and (3.98) we conclude

$$D_{\Omega^1}W_\delta\eta + (D_{\Omega^1}W_\delta - W_\delta)\eta \in \widehat{\mathrm{Cone}}\left(1, \frac{1}{4mN_\delta}, \Omega^1\right). \tag{3.99}$$

Let $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a generalised toy dynamo, approximating the operator $\ell_{\xi*}^m$, constructed as described in Theorem 4 on p. 62. By straightforward calculation we see that the cone $\widehat{\mathrm{Cone}}\left(1, \frac{1}{4mN_\delta}, \Omega^1\right)$ satisfies the assumptions of Theorem 5 on p. 86 for any $\frac{15}{16} < \alpha < 1$:

$$\frac{1}{4mN_\delta} \le 2^{m(\alpha\log_{s_1}2-1)} < 2^{m(\alpha-1)} < 2^{-\frac{m\alpha}{32}} = \delta^{\frac{1}{32}}.$$

Therefore, by Theorem 5,

$$W_\delta \mathcal{T}(D_{\Omega^1} W_\delta \eta + (D_{\Omega^1} W_\delta - W_\delta)\eta) \in \widehat{\mathrm{Cone}}\left(\delta^{\frac{1}{64}}, \delta^{\frac{1}{16}}, \Omega\right).$$

We may write for any partition Ω^3 of the class $\mathcal{G}$ and for any $f \in \widehat{\mathrm{Cone}}\left(r_1, \varepsilon_1, \Omega^1\right)$

$$W_\delta \ell_{\xi*}^m W_\delta f = W_\delta \ell_{\xi*}^m W_\delta(\eta + g) = W_\delta \mathcal{T} D_{\Omega^1} W_\delta \eta + W_\delta \mathcal{T}(W_\delta - D_{\Omega^1} W_\delta)\eta +$$
$$+ D_{\Omega^3} W_\delta \ell_{\xi*}^m W_\delta g + W_\delta(\ell_{\xi*}^m - \mathcal{T}) W_\delta \eta + (\mathrm{Id} - D_{\Omega^3}) W_\delta \ell_{\xi*}^m W_\delta g. \quad (3.100)$$

We are interested in the coefficient in front of the term $\chi_{[-1,1]}$, which corresponds to the "cone axis". Let $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a linear operator defined by (3.16), p. 42. Then

$$W_\delta \mathcal{T} D_{\Omega^1} W_\delta \eta = W_\delta \mathcal{T}(d\chi_{[-1,1]} + \psi_1) = W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}(d\chi_{[-1,1]} + \psi_1) =$$
$$= W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}\psi_1 + d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}) +$$
$$+ d(N_r^1 - N_l^1)\chi_{[-1,1]} = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2; \quad (3.101)$$

where

$$\psi_2 = W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}\psi_1 + d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}),$$

and its norm may be bounded using Lemmas 3.2.2 p. 41, 3.2.3 p. 42, 3.3.3 p. 72, 3.3.4 p. 74, and Proposition 3.2.2 p. 42:

$$\|\psi_2\|_3 \le \|W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1)\|_3 + \|W_\delta \mathcal{E}\psi_1\|_3 + \|d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]})\|_3 \le$$
$$\le d 2^{m(1/2+\gamma_1)} \frac{m N_\delta}{\delta s_2^m} + 2\frac{m N_\delta}{\delta s_2^m} + d 2^{m-1} 2^{1-m/2} \le d\delta^{\frac{1}{16}} 2^{m-3}; \quad (3.102)$$

for a suitable choice of $s_2 < 2 < s_1$ and $\gamma_1 = \frac{1}{8}$.

By Theorem 4 p. 62 we get, using Lemma 3.3.3

$$\|W_\delta(\ell_{\xi*}^m - \mathcal{T}) W_\delta \eta\|_3 \le \frac{m N_\delta}{s_2^m \delta} \cdot \|(\ell_{\xi*}^m - \mathcal{T}) W_\delta \eta\|_2 \le$$
$$\le \frac{m^2 N_\delta}{s_2^m \delta} \cdot \left(\frac{s_1^3}{2^{1/2+\alpha} s_2}\right)^m \|\eta\|_1 \le d m^2 N_\delta \left(\frac{s_1^3}{2^{1/2} s_2^2}\right)^m. \quad (3.103)$$

Using Lemmas 3.3.2 and 3.3.3 we obtain, taking into account that $\|g\|_\Omega \le d\varepsilon_1$,

$$\|(\mathrm{Id}-D_{\Omega^3})W_\delta \ell_{\xi*}^m W_\delta g\|_3 \le \frac{\|\ell_{\xi*}^m W_\delta g\|_3}{s_2^m\delta} \le \frac{d\varepsilon_1}{s_2^m\delta}\cdot m^2\Big(\frac{2s_1}{s_2}\Big)^m\cdot\frac{mN_\delta}{s_2^m\delta} \le 2d\varepsilon_1\frac{m^3N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m. \tag{3.104}$$

Combining (3.103) and (3.104), we have the following upper bound for the sum of the last two terms in (3.100)

$$\|W_\delta(\ell_{\xi*}^m-\mathcal{T})W_\delta\eta\| + \|(\mathrm{Id}-D_{\Omega^3})W_\delta\ell_{\xi*}^m W_\delta g\| \le$$
$$\le dm^2N_\delta\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m + d\varepsilon_1\frac{m^3N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m. \tag{3.105}$$

Applying Lemma 3.3.3 and Theorem 4 p. 62 again, we get

$$\|W_\delta(\ell_{\xi*}^m-\mathcal{T})W_\delta g\|_3 \le \frac{mN_\delta}{s_2^m\delta}\|(\ell_{\xi*}^m-\mathcal{T})W_\delta g\|_3 \le \frac{m^2N_\delta}{s_2^m\delta}\cdot\frac{s_1^{3m}}{2^{m(1/2+\alpha)}s_2^m}\|g\|_3 \le$$
$$\le d\varepsilon_1 N_\delta m^2\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m.$$

By Lemma 3.3.6, taking into account Lemma 3.3.3,

$$\|W_\delta\mathcal{T}W_\delta g\| \le 5m\frac{N_\delta}{\delta}\|W_\delta g\| \le 5d\varepsilon_1 m^2\frac{N_\delta^2}{s_2^m\delta^2}.$$

Hence summing up the last three inequalities we obtain:

$$\|D_{\Omega^3}W_\delta\ell_{\xi*}^mW_\delta g\|_3 \le \|(\mathrm{Id}-D_{\Omega^3})W_\delta\ell_{\xi*}^mW_\delta g\|_3 + \|W_\delta(\ell_{\xi*}^m-\mathcal{T})W_\delta g\|_3 + \|W_\delta\mathcal{T}W_\delta g\|_3 \le$$
$$\le d\varepsilon_1\frac{m^3N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m + d\varepsilon_1N_\delta m^2\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m + 5d\varepsilon_1m^2\frac{N_\delta^2}{s_2^m\delta^2} \le$$
$$\le d\varepsilon_1m^3\frac{N_\delta}{\delta^2}\cdot\frac{s_1^m}{s_2^{2m}}\Big(\frac{2^m}{s_2^m}+\delta^2\frac{s_1^{2m}}{2^{m/2}}+N_\delta\frac{s_2^m}{s_1^m}\Big).$$

We see that for $\varkappa = \log\frac{s_1}{s_2}$ sufficiently small and α is as chosen above,

$$\delta^2\Big(\frac{s_1^2}{2^{1/2}}\Big)^m \ll 1 \quad\text{and}\quad N_\delta\Big(\frac{s_2}{s_1}\Big)^m \gg 1.$$

Therefore, we may write

$$\|D_{\Omega^3}W_\delta\ell_{\xi*}^mW_\delta g\|_3 \le d\varepsilon_1m^3\frac{N_\delta^2}{\delta^2s_2^m}. \tag{3.106}$$

Therefore we deduce from (3.101), (3.106), and (3.105) that in order to get the inclusion $W_\delta \ell_{\xi*}^m W_\delta f \in \text{Cone}\,(r_2, \varepsilon_2, \Omega)$ we need to make sure that for some $1 \gg \varepsilon_1 > \varepsilon_2$ the following inequalities holds true:

$$2dr_2(N_r^1 - N_l^1) \gg d\varepsilon_1 m^3 \frac{N_\delta^2}{\delta^2 s_2^m}; \tag{3.107}$$

$$2d\varepsilon_2(N_r^1 - N_l^1) \gg dm^2 N_\delta \Big(\frac{s_1^3}{2^{1/2} s_2^2}\Big)^m + d\varepsilon_1 \frac{m^3 N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m. \tag{3.108}$$

We know that $N_r^1 - N_l^1 \geq 2^{m-1}$, therefore we may choose $\varepsilon_1 = \delta^{\frac{1}{32}}$ and get in the first inequality

$$r_2 \geq \delta^{\frac{1}{32}} \frac{m^2 N_\delta^2}{4\delta^2 2^m s_2^m} = \delta^{\frac{1}{32}} \frac{m^2 2^{m2(1-\alpha \log_{s_1} 2)}}{4 s_2^m \cdot 2^{-2\alpha m} \cdot 2^m} = \delta^{\frac{1}{32}} \frac{m^2}{4} \cdot \frac{2^{m(1+2\alpha(1-\log_{s_1} 2))}}{s_2^m}.$$

It holds true, if we set $r_2 = \delta^{\frac{1}{64}}$, as in Theorem 5 on p. 86. Comparing it with the value of $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, we see that $r_2 < r_1$ provided $\log_2 s_2 + \alpha \log_{s_1} 2 > \alpha\frac{63}{64} + 1$.

It remains to check for the second inequality that

$$\varepsilon_2 \geq m^2 N_\delta \Big(\frac{s_1^3}{2^{3/2} s_2^2}\Big)^m + \delta^{\frac{1}{32}} \frac{m^2 N_\delta}{4\delta^2}\Big(\frac{s_1}{s_2^3}\Big)^m. \tag{3.109}$$

We see immediately that we may choose s_1 and s_1 such that $\frac{1}{25} > \log \frac{s_1}{s_2} > \frac{1}{2r_2}$ and then

$$m^2 N_\delta \Big(\frac{s_1^3}{2^{3/2} s_2^2}\Big)^m = m^2 \Big(\frac{s_1^3}{s_2^2 \cdot 2^{1/2+\alpha \log_{s_1} 2}}\Big)^m \leq \delta^{\frac{1}{32}} \frac{m^2 N_\delta}{4\delta^2}\Big(\frac{s_1}{s_2^3}\Big)^m \leq m^2 \delta^{\frac{1}{32}} \Big(\frac{s_1 2^{1+\alpha}}{s_2^3}\Big)^m \ll \delta^{\frac{1}{24}}.$$

Hence we conclude that for $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, $r_2 = \delta^{\frac{1}{64}}$, $\varepsilon_1 = \delta^{\frac{1}{32}}$ and $\varepsilon_2 = \delta^{\frac{1}{24}}$ we have

$$W_\delta \ell_{\xi*}^m W_\delta \colon \widehat{\text{Cone}}\left(r_1, \varepsilon_1, \Omega^1\right) \to \widehat{\text{Cone}}\,(r_2, \varepsilon_2, \Omega) \subset \widehat{\text{Cone}}\,(r_1, \varepsilon_1, \Omega)\,. \tag{3.93}$$

The second inequality on the norm

$$\|W_\delta \ell_{\xi*}^m W_\delta \mid_{\widehat{\text{Cone}}(r_1, \varepsilon_1, \Omega)} \|_\Omega \geq 2^{m-2}$$

follows from (3.107), (3.108) and (3.101) immediately. ■

Corollary 2. *Under the hypotheses and in the notations of Theorem 2 p. 33, let us choose four constants* $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, $r_2 = \delta^{\frac{1}{64}}$, $\varepsilon_1 = \delta^{\frac{1}{32}}$ *and* $\varepsilon_2 = \delta^{\frac{1}{24}}$. *Then we have*

$$W_{\frac{\delta}{m}} \ell_{\xi *}^m W_{\frac{\delta}{m}} : \widehat{\text{Cone}}\,(r_1, \varepsilon_1, \Omega) \to \widehat{\text{Cone}}\,(r_2, \varepsilon_2, \Omega) \subset \widehat{\text{Cone}}\,(r_1, \varepsilon_1, \Omega)$$

$$\forall f \in \text{Cone}\,(r_1, \varepsilon_1, \Omega) : \|W_{\frac{\delta}{m}} \ell_{\xi *}^m W_{\frac{\delta}{m}} f\|_\Omega \geq 2^{m-2} \|f\|_\Omega.$$

The constructive proof of the existence of an invariant cone is complete. Fast Dynamo Theorem now follows as described in the Section 2.2.

Theorem 6 (Fast dynamo on $\mathbb{R}$). *There exist a measure-preserving piecewise-C^2 transformation* $\ell \colon \mathbb{R} \to \mathbb{R}$ *and an essentially bounded, absolutely integrable vector field* v *such that*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \big\| (\exp(\delta \Delta) \ell_*)^n v \big\|_{\mathcal{L}_1} > 0,$$

The map ℓ *may be realised as an induced action of the Poincaré map of the provisional fluid flow on the unstable manifold.*

4 Fast dynamo on the real plane

This Chapter is dedicated to the construction of a piecewise diffeomorphism $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the Invariant Cone Hypothesis 1, p. 28. In perspective, the map T is the Poincaré map of the provisional fluid flow. The main result it the following

Theorem 7. *Let $\|\xi\|_\infty \le \delta$ be a sequence of real vectors. There exists a partition Ω of $\mathbb{R}^2$ and four numbers $r_1(m) \ll r_2(m)$ and $\varepsilon_1(m) \ll \varepsilon_2(m)$ such that*

$$W_{\frac{\delta}{2m}} P^2_{\xi*} W_{\frac{\delta}{2m}} \colon \mathrm{Cone}\,(r_1, \varepsilon_1, \Omega) \to \mathrm{Cone}\,(r_2, \varepsilon_2, \Omega) \subsetneq \mathrm{Cone}\,(r_1, \varepsilon_1, \Omega)\,.$$

$$\left\| W_{\frac{\delta}{2m}} P^2_{\xi*} W_{\frac{\delta}{2m}} \,|_{\mathrm{Cone}(r_1,\varepsilon_1,\Omega)} \right\| \ge 2^{m-5}$$

(See p. 100 for definition of a cone in the space of vector fields).

4.1 Notation

The following notations will be used throughout.

We denote the unit square in the plane $\mathbb{R}^2$ by $\square \overset{\text{def}}{=} [-1,1]^2$.

The Jacobian of a function F we denote by $\mathrm{d}F$, and by $|\mathrm{d}F|$ we denote its determinant. For a function of two variables, by ∂_x we denote the derivative in the first variable and by ∂_y we denote its derivative in the second variable. Similarly, for any point $z \in \mathbb{R}^2$ we denote by z_x and z_y its first and second coordinates.

The indicator function of a set X we denote by χ_X. In particular, $\chi_\square$ is the indicator function of the square $[-1,1]^2$. Given a subset $X \subset \mathbb{R}^2$ and a partition $\Omega = \{\Omega_{ij}\}_{(i,j)\in\mathbb{Z}^2}$ of the plane $\mathbb{R}^2$ we abuse notations and write $(i,j) \in X$ for $\Omega_{ij} \subset X$. We denote by π_x and π_y the natural orthogonal projections

$$\pi_x\colon \mathbb{R}^2 \to \mathbb{R} \qquad \pi_x(z_x, z_y) = z_x, \tag{4.1}$$

$$\pi_y\colon \mathbb{R}^2 \to \mathbb{R} \qquad \pi_y(z_x, z_y) = z_y. \tag{4.2}$$

The length of a vector v we denote by $|v|$ and the n-dimensional Lebesgue measure of a subset $A \in \mathbb{R}^n$ we denote by $|A|$. For any sequence of vectors $\xi \in \ell_\infty(\mathbb{R}^2)$ we denote by $\xi_x \in \ell_\infty(\mathbb{R})$ and $\xi_y \in \ell_\infty(\mathbb{R})$ two sequences of x- and y-coordinates of elements of ξ, respectively. We denote by Σ_δ the subset of sequences with $\|\xi\|_\infty \le \delta$.

The two dimensional Gaussian kernel w_δ is specified by

$$w_\delta(x,y) \stackrel{\text{def}}{=} \frac{1}{2\pi\delta^2} e^{-\frac{x^2+y^2}{2\delta^2}}. \tag{4.3}$$

The Weierstrass transform is a convolution operator with the Gaussian kernel. For any absolutely integrable function f it is given by

$$W_\delta f(z) \stackrel{\text{def}}{=} w_\delta * f(z) = \int_{\mathbb{R}^2} w_\delta(z-t) f(t) \mathrm{d}t. \tag{4.4}$$

For a vector field $v = (v_x, v_y)$ with absolutely Lebesgue-integrable components v_x and v_y the Weierstrass transform is defined by $W_\delta v = (w_\delta * v_x, w_\delta * v_y)$.

The space of essentially bounded vector field in $\mathbb{R}^2$ with absolutely integrable coordinates we denote by $\mathfrak{X}$.

The supremum norm of a matrix A is supremum of absolute values of its elements, we denote it by $\|A\|_\infty \stackrel{\text{def}}{=} \sup_{ij} |A_{ij}|$. The matrices we are dealing with will be bi-infinite.

The following letters are reserved for real constants: M, M_1, μ_1, μ_2, α, $\gamma_{1,2,3,4} > 0$. Suitable intervals of values will be specified later.

4.2 The dynamical system

Here we introduce the dynamical system we will be studying. It consists of the phase space $\mathfrak{X}$; the norm, which is the maximum of weighted $\mathcal{L}_1$ and $\mathcal{L}_\infty$ norms; and the transformation of the phase space, which is an action, induced by a piecewise diffeomorphism of $\mathbb{R}^2$. To define the piecewise diffeomorphism we use a tower construction.

4.2.1 Action on vector fields

A tower of M floors. Let $M > 1$ be a large natural number; and let $0 < \mu_1 < 0.1$, $0 < \mu_2 \ll 1$ be two small real numbers.

Let F_0 be the Baker's map on the unit square

$$F_0(z_x, z_y) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1); 2z_y + 1\right), & \text{if } z_y < 0; \\ \left(\frac{1}{2}(z_x + 1); 2z_y - 1\right), & \text{if } z_y > 0. \end{cases}$$

Consider $M-1$ maps $F_1, \ldots, F_{M-1} \colon \mathbb{R}^2 \setminus \square \to \mathbb{R}^2 \setminus \square$ with the following properties

1. each F_k is a smooth map;
2. each F_k is area-preserving: $|\mathrm{d}F_k| = 1$;
3. the Euclidean norm of the differential is uniformly bounded $\|\mathrm{d}F_k\| \leq 1 + \mu_1$;
4. the Hessian is small $\|\mathrm{d}^2 F_k\| \leq \mu_2$.
5. all F_k are polynomials, most are linear, some are not; the product of degrees of all of them is bounded by a small number d, which is independent of M. In particular, $d^{\frac{2}{M}} \leq 2^{\frac{1}{500}}$. This condition holds true, for example if $F_k \equiv F_j$, for all $1 \leq k \leq j \leq M-1$. We use this a strict assumption only to claim that for any point $z \in \mathbb{R}^2$ $\#\{\pi_x^{-1}(F_1 \circ \ldots \circ F_M(z))\} \leq d$ and $\#\{\pi_y^{-1}(F_1 \circ \ldots \circ F_M(z))\} \leq d$. This bound is required in Proposition 4.3.2 only.

We build a tower $X \subset \mathbb{R}^3$ defined by

$$X \stackrel{\text{def}}{=} \Big(\mathbb{R}^2 \times \{0\}\Big) \bigcup \Big((\mathbb{R}^2 \setminus \Box) \times \{1, 2, \dots, M-1\}\Big)$$

with coordinates (z, n), where $z = (z_x, z_y) \in \mathbb{R}^2$ and $n \in \{0, 1, \dots, M-1\}$. We will abuse notations and identify $\Box \times \{0\} \subset X$ with $\Box$.

The choice of piecewise diffeomorphism. We are ready to introduce a map $F \colon X \to X$ defined by

$$F(z,n) \stackrel{\text{def}}{=} \begin{cases} (F_0(z), 0), & \text{if } n = 0 \text{ and } z \in \Box; \\ (F_{n+1}(z), (n+1) \bmod (M-1)), & \text{otherwise.} \end{cases} \tag{4.5}$$

Consider an extension $\widehat{F} \colon X \times \mathbb{R}^2 \to X$

$$\widehat{F}((z,n), w) \stackrel{\text{def}}{=} \begin{cases} (F_0(z) + w, 0), & \text{if } n = 0 \text{ and } z \in \Box; \\ (F_{M-1}(z) + w, 0), & \text{if } n = M-1; \\ (F_{n+1}(z), (n+1)), & \text{otherwise.} \end{cases} \tag{4.6}$$

Given a sequence $\xi \in \Sigma \subset \ell_\infty(\mathbb{R}^2)$, we define a small random perturbation F_ξ of the map F, as described in Subsection 2.1.1. Then the zero floor $\mathbb{R}^2 \times \{0\}$ is invariant with respect to F_ξ^M and we may consider the M'th iteration as a map $F_\xi^M \colon \mathbb{R}^2 \to \mathbb{R}^2$. We denote by $F_{\underline{0}} \colon X \to X$ the map corresponding to the zero sequence $\xi \equiv 0$.

Remark 7. The inverse map $F_{\xi^k}^{-1}$ is given by

$$F_{\xi^k}^{-1}(z,n) = \begin{cases} (F_0^{-1}(z - \xi^k), 0), & \text{if } z \in \Box + \xi^k \text{ and } n = 0; \\ (F_{M-1}^{-1}(z - \xi^k), M-1), & \text{if } z \notin \Box + \xi^k \text{ and } n = 0; \, i \\ (F_n^{-1}(z), n-1), & \text{otherwise.} \end{cases} \tag{4.7}$$

Also observe that the inverse Baker's map is given by

$$F_0^{-1}(z-\xi^k) = \begin{cases} \left(2z_x+1-2\xi_x^k; \frac{1}{2}(z_y-1)-\frac{1}{2}\xi_y^k\right), & \text{if } z_x < \xi_x^k, \text{ and } z \in (\square + \xi^k); \\ \left(2z_x-1-2\xi_x^k; \frac{1}{2}(z_y+1)-\frac{1}{2}\xi_y^k\right), & \text{if } z_x > \xi_x^k, \text{ and } z \in (\square + \xi^k). \end{cases} \tag{4.8}$$

Let $m_0 \gg 1$ be a large natural number. We set $m = 4Mm_0$ and choose a small real number $\delta = 2^{-m\alpha}$ with $\frac{15}{16} < \alpha \leq 1$. The subset of sequences in $\ell_\infty(\mathbb{R}^2)$ with $\|\xi\|_\infty \leq \delta$ we denote by Σ_δ. Given a sequence $\xi \in \Sigma_\delta$ we may define a map

$$P_\xi \colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad\qquad P_\xi(z) \stackrel{\text{def}}{=} F_\xi^m(z,0). \tag{4.9}$$

The map P_ξ defines induced action on the space $\mathfrak{X}$ according to

$$(P_{\xi*}v)(z) \stackrel{\text{def}}{=} \mathrm{d}P_\xi(P_\xi^{-1}z)v(P_\xi^{-1}z). \tag{4.10}$$

The number of iterations m remains fixed throught the manuscript. We assume it to be sufficiently large so that all inequalities hold true.

4.2.2 The choice of the norm in $\mathfrak{X}$

In this Subsection we introduce a norm in the space of vector fields in $\mathbb{R}^2$. We also give a general definition of a cone in $\mathfrak{X}$.

Given a partition Ω of $\mathbb{R}^2$, we define an associated weighted $(\Omega, \mathcal{L}_1)$-norm of a vector field v on the plane by

$$\|v\|_{\Omega,\mathcal{L}_1} \stackrel{\text{def}}{=} \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v|.$$

Observe that $\|v\|_{\Omega,\mathcal{L}_1}$ is finite if the ordinary $\mathcal{L}_1$-norm is finite and the size of elements of partition is bounded away from zero:

$$\|v\|_{\Omega,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v| \leq \frac{2^{-m}}{\inf|\pi_y(\Omega_{ij})|} \int_{\mathbb{R}^2} |v| = \frac{2^{-m}}{\inf|\pi_y(\Omega_{ij})|} \|v\|_{\mathcal{L}_1}.$$

The supremum norm of a vector field v we denote by $\|v\|_\infty \stackrel{\text{def}}{=} \sup|v|$. We denote by $\mathfrak{X}$ the space of vector fields on the real plane with finite $\mathcal{L}_1$ and supremum norms.

Definition 16 (Norm)**.** We introduce a new norm in $\mathfrak{X}$, associated to the partition Ω, combining the two:

$$\|v\|_\Omega \stackrel{\text{def}}{=} \max\Big(\|v\|_{\Omega,\mathcal{L}_1}, 2^{-m/4}\sup|v|\Big). \tag{4.11}$$

This definition agrees with the general definition in Subsection 2.1.2 with $\alpha = 1/4$.

The subspace of piecewise constant vector fields associated to the partition Ω we denote $\mathfrak{X}_\Omega$. We reserve Greek letters for piecewise constant vector fields. We shall call the basis

$$\Big\{\chi^s_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega_{ij}}; \chi^u_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega_{ij}}\Big\}_{i,j\in\mathbb{Z}}.$$

the canonical basis of the subspace $\mathfrak{X}_\Omega$.

Whenever we are dealing with several partitions Ω^1, Ω^2, and Ω^3, say, we omit Ω in the norm index and write $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, respectively.

We have for the norm of a piecewise constant vector field $\nu = \sum_{ij} \nu^{ij}_s \chi^s_{\Omega_{ij}} + \nu^{ij}_u \chi^u_{\Omega_{ij}}$:

$$\|\nu\|_\Omega \geq \max\Big(2^{-m}\sum_{ij}|\nu^{ij}|, \frac{2^{-m/4}}{\sup|\pi_x(\Omega_{ij})|}\sup|\nu^{ij}|\Big),$$

in particular,

$$\|\nu\|_\Omega = 1 \qquad \Longrightarrow \qquad \sum|\nu_{ij}| < 2^m \quad \text{and} \quad \sup|\nu_{ij}| \leq 2^{-\frac{3}{4}m}. \tag{4.12}$$

Invariant cones. By analogy with one-dimensional part, cones of a special form in the spaces $\mathfrak{X}$ and $\mathfrak{X}_\Omega$ play an important role. We reserve notation for a cone of radius r with main axis $\chi_\Box$ in the subspace of piecewise constant vector fields associated to the partitions Ω^1 and Ω^2:

$$\text{Cone}\left(r, \Omega^1\right) \stackrel{\text{def}}{=} \Big\{\eta = d\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\Box + \varphi \mid \varphi \in \mathfrak{X}_{\Omega^1},\ \|\varphi\|_1 \leq dr,\ \sum_\Box \varphi^{ij}_u = 0\Big\}. \tag{4.13}$$

We extend the cone $\mathrm{Cone}\left(r,\Omega^1\right)$ to include general functions from the main space:

$$\widehat{\mathrm{Cone}}\left(r,\varepsilon,\Omega^1\right) \stackrel{\mathrm{def}}{=} \Big\{ f=\eta+v \mid \eta\in\mathrm{Cone}\left(r,\Omega^1\right),\ \|v\|_1\le\varepsilon\|\eta\|_1 \Big\}. \tag{4.14}$$

This definition agrees with the general definition in Subsection 2.1.3.

4.2.3 The canonical partition

In this subsection we introduce the notion of canonical partition of $\mathbb{R}^2$ associated to a sequence of perturbations $\xi\in\ell_\infty(\mathbb{R}^2)$ as a direct product of a pair of canonical partitions of $\mathbb{R}$ and list the main properties.

Definition 17. *The k'th escaping set* for $k\in\mathbb{Z}$ is defined by

$$E_k \stackrel{\mathrm{def}}{=} \Big\{ z\in\Box\subset X \mid \prod_{j=0}^{k}\chi_{\Box}(F_\xi^j(z))=0\Big\}. \tag{4.15}$$

Obviously, $E_k\subset E_{k+1}$, if $k>0$; and $E_{k+1}\subset E_k$ if $k<0$.

Lemma 4.2.1. *Let $\xi\in\Sigma_\delta\subset\ell_\infty(\mathbb{R}^2)$ be a sequence of small vectors in the plane. Define a sequence $\varsigma(\xi)$ of the length m by $\varsigma^1=-2\xi^{2m}$, $\varsigma^2=-2\xi^{-2m-1}$, ..., $\varsigma^m=-2\xi^{m+1}$. Let p_{ς_x} and p_{ξ_y} be two random perturbations of the doubling map p defined by (3.3) with $s_1=s_2=2$. Then the following diagrams are commutative.*

$$\begin{array}{ccc} \Box\setminus E_{-m} & \xrightarrow{P^{-1}_{\sigma^m\xi}} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi_x} \\ \mathbb{R} & \xrightarrow{p^m_{\varsigma_x}} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \Box\setminus E_{m} & \xrightarrow{P_{\xi}} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle \pi_y} \\ \mathbb{R} & \xrightarrow{p^m_{\xi_y}} & \mathbb{R} \end{array}$$

Proof. Straightforward from definition. The Baker's map preserves the horizontal and vertical foliations, so the second diagram is trivial. For the first diagram, recall that by definition (Subsection 4.2.1) $P_\xi^{-1}=(F_\xi^m)^{-1}=F_{\xi^1}^{-1}F_{\xi^2}^{-1}\dots F_{\xi^m}^{-1}$. Using (4.8) and (4.7), we conclude that the corresponding sequence ς for the doubling map associated to P_ξ^{-1} is as defined in supposition of the Lemma. ∎

We associate a chain $\Upsilon^1, \Upsilon^2, \dots$ of partitions of $\mathbb{R}^2$ to a sequence $\xi \in \Sigma_\delta$.

The first element Υ^1 is defined as follows. Let $\Upsilon^s = \left\{\Upsilon_i^s = \left[\frac{i}{2^m}; \frac{i+1}{2^m}\right]\right\}$, $i \in \mathbb{Z}$, be a partition of $\mathbb{R}$ into equal intervals and let $\Upsilon^u = \{\Upsilon_j^u\}_{j\in\mathbb{Z}}$ be the canonical partition of the map $p^m_{\xi_y}$. Then

$$\Upsilon^1 = \{\Upsilon_{ij}\},\ \Upsilon_{ij} = \Upsilon_i^s \times \Upsilon_j^u.$$

To define partition Υ^k, consider a sequence

$$\varsigma^1 = -2\xi^{2km}, \varsigma^2 = -2\xi^{2km-1}, \dots, \varsigma^m = -2\xi^{(2k-1)m}.$$

Let Υ^s be the canonical partition for the perturbation $p^m_{\varsigma_x}$ of the doubling map, and let Υ^u be the canonical partition of the perturbation $p^m_{\sigma^{2mk}\xi_y}$ of the doubling map. Then Υ^k is given by

$$\Upsilon^k = \{\Upsilon_{ij}\},\ \Upsilon_{ij} = \Upsilon_i^s \times \Upsilon_j^u.$$

Definition 18. We say that a partition Υ of the plane $\mathbb{R}^2$ is *a partition of the class* $\mathcal{G}(m,\delta)$, if there exists a sequence $\xi \in \Sigma_\delta$ such that $\Upsilon = \Upsilon^k$ for some partition Υ^k from the chain of partitions associated to ξ.

Definition 19. *A rectangle* $\left(z_x - \frac{l_x}{2}, z_x + \frac{l_x}{2}\right) \times \left(z_y - \frac{l_y}{2}, z_y + \frac{l_y}{2}\right)$ with centre at z and sides l_x and l_y we denote by $Rec_z(l_x, l_y)$. Whenever location of the centre of the rectangle is of no importance, we omit z and write $Rec(l_x, l_y)$.

Lemma 4.2.2. *Any partition* Υ *of the class* $\mathcal{G}(m,\delta)$ *has the following properties*

1. *The unit square* $\square$ *contains at most* 4^m *and at least* 4^{m-1} *elements of the partition.*

2. *For any element* Υ_{ij} *of the partition* Υ *we have two rectangles*

$$Rec\left(\frac{2^{-m}}{m}, \frac{2^{-m}}{m}\right) \subseteq \Upsilon_{ij} \subseteq Rec\left(2^{1-m}, 2^{1-m}\right).$$

3. *Any square with a side δ may be covered by at most $N_\delta = 4^{m(1-\alpha)+1}$ elements of the partition.*

Proof. Follows from the properties of the canonical partition for perturbation ξ of the doubling map. ∎

4.3 Matrix, approximating the operator $P^2_{\eta*}$.

In this Section we assume that a sequence of vectors $\eta \in \ell_\infty(\mathbb{R}^2)$ is fixed and we study the associated operator $P^2_{\eta*}$ on vector fields on $\mathbb{R}^2$, defined by (4.10), where the map P_η is given by (4.9). Our goal is to show that for any sequence η there exist a pair of subspaces $\mathfrak{X}_{\Omega^1}, \mathfrak{X}_{\Omega^2} \subset \mathfrak{X}$ and a linear operator $\mathcal{A}(\eta)\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega_2}$ with a simple matrix, approximating $P^2_{\eta*} |_{\mathfrak{X}_{\Omega^1}}$ well enough. Given the operator $\mathcal{A}(\eta)$, we construct a pair of cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}(\overline{C_1}) \subset C_2$; $C_2 \ll C_1$ and $\|\mathcal{A}\| |_{C_1} \geq 2^{m-1}$. We begin with the choice of the operator $\mathcal{A}$.

Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta = \{\xi \in \ell_\infty(\mathbb{R}^2) \mid \|\xi\|_\infty \leq \delta\}$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. We introduce two subspaces $\mathfrak{X}_{\Omega^1}$ and $\mathfrak{X}_{\Omega^2}$ of piecewise-constant vector fields in $\mathfrak{X}$, associated to the partitions Ω^1 and Ω^2, respectively. The subspace $\mathfrak{X}_{\Omega^1}$ has the (canonical) basis

$$\chi^s_{\Omega^1_{ij}} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \chi_{\Omega^1_{ij}}, \qquad \chi^u_{\Omega^1_{ij}} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|} \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \chi_{\Omega^1_{ij}};$$

and the (canonical) basis of the subspace $\mathfrak{X}_{\Omega^2}$ is

$$\chi^s_{\Omega^2_{ij}} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^2_{ij})|} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \chi_{\Omega^2_{ij}}, \qquad \chi^u_{\Omega^2_{ij}} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^2_{ij})|} \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \chi_{\Omega^2_{ij}};$$

both bases have $\mathbb{Z}^2$ elements.

Let $\xi \overset{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (see definition of the chain Υ in Subsection 4.2.3, p. 101). We would like to approximate the operator $P^2_{\xi*}\colon \mathfrak{X} \to \mathfrak{X}$ by a linear operator $\mathcal{A}\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ chosen so that the averages along the elements of partition Ω^2 are equal for any field $\nu \in \mathfrak{X}_{\Omega^1}$:

$$\int_{\Omega^2_{kl}} A\nu = \int_{\Omega^2_{kl}} P^2_{\xi*}\nu. \tag{4.16}$$

We write down the action of the operator $\mathcal{A}$ on $\mathfrak{X}_{\Omega^1}$ in matrix form

$$\mathcal{A}(\nu_s^{ij}\chi^s_{\Omega^1_{ij}} + \nu_u^{ij}\chi^u_{\Omega^1_{ij}}) = \sum_{kl} \nu_s^{ij}(SS^{kl}_{ij}\chi^s_{\Omega^2_{kl}} + SU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}(US^{kl}_{ij}\chi^s_{\Omega^2_{kl}} + UU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}), \tag{4.17}$$

where the four matrices SS, SU, US, and UU are specified as follows, so that (4.16) holds true (see Lemma 4.3.14 on p. 128 for details).

$$SS^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|} \int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_x (P^2_\xi)_x(z)\mathrm{d}z; \tag{4.18}$$

$$SU^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|} \int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_x (P^2_\xi)_y(z)\mathrm{d}z; \tag{4.19}$$

$$US^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|} \int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_y (P^2_\xi)_x(z)\mathrm{d}z; \tag{4.20}$$

$$UU^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|} \int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_y (P^2_\xi)_y(z)\mathrm{d}z. \tag{4.21}$$

We observe that

$$SS\colon \langle\chi^s_{\Omega^1_{ij}}\rangle \to \langle\chi^s_{\Omega^2_{ij}}\rangle;\ SU\colon \langle\chi^s_{\Omega^1_{ij}}\rangle \to \langle\chi^u_{\Omega^2_{ij}}\rangle;\ US\colon \langle\chi^u_{\Omega^1_{ij}}\rangle \to \langle\chi^s_{\Omega^2_{ij}}\rangle;\ UU\colon \langle\chi^u_{\Omega^1_{ij}}\rangle \to \langle\chi^u_{\Omega^2_{ij}}\rangle.$$

The matrix UU is the most important as it is responsible for the largest eigenvalue of the operator $\mathcal{A}$. We will study it in a great detail in the next Subsection.

Lemma 4.3.1. *The map $P^2_{\underline{0}}$, corresponding to the zero sequence $\xi \equiv 0$, gives the following matrix elements for any quartet $(i,j,k,l) \in \square \times \square$:* $UU^{kl}_{ij} \equiv 1$; $SS^{kl}_{ij} \equiv 2^{-4m}$; $SU^{kl}_{ij} \equiv 0$; $US^{kl}_{ij} \equiv 0$.

Proof. Each partition of the chain, associated to the zero sequence, is a partition of the unit square $\square$ into 2^{2m+2} equal squares with side length 2^{-m}. Therefore we have that $\Omega^1_{ij} = \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right] \times \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right]$ and $\Omega^2_{kl} = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right] \times \left[\frac{l}{2^m}, \frac{l+1}{2^m}\right]$.

The preimage of an element $\Omega^2_{kl} \subset \square$ of the partition Ω^2 under P_0^{-2} is equal to 2^m disjoint rectangles $Rec(2, 2^{-3m})$ in $\square$. Thus $|P_{\underline{0}}^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}| = 2^{-4m}$. The derivative of $P^2_{\underline{0}}$ on $\square$ is given by the matrix

$$\mathrm{d}P^2_{\underline{0}}(z) \equiv \begin{pmatrix} 2^{-2m} & 0 \\ 0 & 2^{2m} \end{pmatrix} \qquad \text{for all } z \in \square.$$

■

Definition 20. *The matrices*, corresponding to the map $P^2_{\underline{0}}$, we denote by $\overset{\circ}{S}S$, $\overset{\circ}{S}U$, $\overset{\circ}{U}S$, and $\overset{\circ}{U}U$, respectively.

Remark 8. Immediately by definition we see that for any quartet (i, j, k, l) such that $(i, j) \in \square$ and $(k, l) \in \mathbb{R}^2 \setminus \square_{1+m\delta}$ or $(i, j) \in \mathbb{R}^2 \setminus \square_{1+m\delta}$ and $(k, l) \in \square$ we have

$$\overset{\circ}{U}U^{kl}_{ij} = \overset{\circ}{S}U^{kl}_{ij} = \overset{\circ}{U}S^{kl}_{ij} = \overset{\circ}{S}S^{kl}_{ij} = 0 \tag{4.22}$$

In addition, given $\|\mathrm{d}F_k\| \le \mu_1$, from definition of F_k p. 97, we have

$$\max\big(\|\overset{\circ}{U}U\|_\infty, \|\overset{\circ}{S}U\|_\infty, \|\overset{\circ}{U}S\|_\infty, \|\overset{\circ}{S}S\|_\infty\big) \le (1 + \mu_1)^{2m}. \tag{4.23}$$

Remark 9. The condition on the Euclidean norm $\|\mathrm{d}F_k\| \le \mu_1$ implies that there exists a constant M_1 such that for any two partitions Ω^1 and Ω^2 of the class $\mathcal{G}(m, \delta)$,

$$\sup_{(i,j)} \#\big\{(k, l) \in \mathbb{R}^2 \setminus \square_{1+m\delta} \mid P_{\xi}^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij} \neq \varnothing\big\} < M_1 \cdot (\mu_1 + 1)^{2m}. \tag{4.24}$$

Therefore for any pair $(k, l) \subset \mathbb{R}^2 \setminus \square_{1+m\delta}$ there exist not more than $M_1 \cdot (1 + \mu_1)^{2m}$ pairs $(i, j) \subset \mathbb{R}^2 \setminus \square_{1+m\delta}$ such that

$$SS^{kl}_{ij} \cdot SU^{kl}_{ij} \cdot US^{kl}_{ij} \cdot UU^{kl}_{ij} \neq 0.$$

Remark 10. Recall the notations introduced in the beginning of Section 4.3. There exists a constant M_2, independent of m, such that for $R := M_2 m\delta(1 + \mu_1)^{2m} + 1$ and for any quartet (i, j, k, l) where $(i, j) \in \square$ and $(k, l) \in \mathbb{R}^2 \setminus \square_R$ or $(i, j) \in \mathbb{R}^2 \setminus \square_R$ and $(k, l) \in \square$

$$SS^{kl}_{ij} \equiv 0, \quad SU^{kl}_{ij} \equiv 0, \quad US^{kl}_{ij} \equiv 0, \quad UU^{kl}_{ij} \equiv 0.$$

Definition 21. The domains of continuity of the map P_ξ^2 we call (P,ξ)*-domains.*

We split (P,ξ)-domains in $P_\xi^{-2}(\Omega_{kl}^2)\cap\Omega_{ij}^1\subset\square$ in "good" and "bad" parts:

$$(\Delta^G)_{ij}^{kl}\stackrel{\text{def}}{=}\{\Delta\subset P_\xi^{-2}(\Omega_{kl}^2)\cap\Omega_{ij}^1 \mid \Delta \text{ is a } (P,\xi)\text{-domain, and } \forall n\le 2m\colon F_\xi^n(\Delta)\subset\square\}; \quad (4.25)$$

$$(\Delta^B)_{ij}^{kl}\stackrel{\text{def}}{=}\{\Delta\subset P_\xi^{-2}(\Omega_{kl}^2)\cap\Omega_{ij}^1 \mid \Delta \text{ is a } (P,\xi)\text{-domain, and } \exists n\le 2m\colon F_\xi^n(\Delta)\not\subset\square\}. \quad (4.26)$$

Then we may write for $(i,j,k,l)\in\square\times\square$,

$$UU_{ij}^{kl}=(UU^G)_{ij}^{kl}+(UU^B)_{ij}^{kl}, \tag{4.27}$$

where $UU^G, UU^B\in\mathrm{Mat}(2^m,2^m)$ are given by

$$(UU^G)_{ij}^{kl}\stackrel{\text{def}}{=}\frac{1}{|\pi_x(\Omega_{ij}^1)|}\cdot\frac{1}{|\pi_y(\Omega_{kl}^2)|}\cdot\sum_{\Delta\in\Delta^G}\int_\Delta\partial_y(P_\xi^2)_y(z)\mathrm{d}z; \tag{4.28}$$

$$(UU^B)_{ij}^{kl}\stackrel{\text{def}}{=}\frac{1}{|\pi_x(\Omega_{ij}^1)|}\cdot\frac{1}{|\pi_y(\Omega_{kl}^2)|}\cdot\sum_{\Delta\in\Delta^B}\int_\Delta\partial_y(P_\xi^2)_y(z)\mathrm{d}z. \tag{4.29}$$

We define three more pairs of matrices $SU^B+SU^G=SU$, $US^B+US^G=US$, $SS^B+SS^G=SS$ in a similar way.

4.3.1 Properties of the matrix UU

The submatrix $UU\colon\langle\chi^u_{\Omega^1_{ij}}\rangle\to\langle\chi^u_{\Omega^2_{ij}}\rangle$ corresponds to a mapping between two subspaces of vector fields parallel to the expanding direction of the Baker's map and associated to two different partitions. It is also responsible for the norm of the operator $\mathcal{A}$. Our goal is to establish the following two facts about the matrix UU.

Proposition 4.3.1. *The following inequalities hold true for the elements of the matrix UU^G in the canonical bases.*

1. $\|UU^G\|_\infty=\sup|UU_{ij}^{kl}|\le 4$;

2. $\#\{(UU^G)_{ij}^{kl}\ne 1\}\le 2^{4\frac{1}{2}m}\delta.$

Proposition 4.3.2. *There exist a constant $\gamma_1 < 0.01$ such that for M and m sufficiently large and for μ_1 sufficiently small*

$$\max(\|SS\|_\infty, \|US\|_\infty, \|SU\|_\infty, \|UU\|_\infty) \leq 2^{\gamma_1 m}.$$

(Recall Condition 3 on F_k: $\|\mathrm{d}F_k\| \leq 1 + \mu_1$ in the Euclidean operator norm).

By definition, the matrix UU^G is related to subsets of the survivor set $\Box \setminus E_{2m}$. To study the set $\Box \setminus E_{2m}$, we introduce a simplified system, since the map outside of the unit square is of no importance.

Consider a circle S^1 and a cylinder $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{R} \times S^1 \stackrel{\text{def}}{=} \{(x, y),\ x \in \mathbb{R},\ y \in [-1; 1)\}$. Define a map $h\colon \mathcal{C} \to \mathcal{C}$ by

$$h(z) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1), 2z_y + 1\right), & \text{if } -1 \leq z_x \leq 0,\ -1 \leq z_y \leq 1; \\ \left(\frac{1}{2}(z_x + 1), 2z_y - 1\right), & \text{if } 0 \leq z_x \leq 1,\ -1 \leq z_y \leq 1; \\ z, & \text{if } |z_x| > 1. \end{cases} \tag{4.30}$$

Let $\hat{h}\colon \mathcal{C} \times \mathbb{R}^2 \to \mathcal{C}$ be an extension given by

$$\hat{h}(z, w) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1) + w_x, (2z_y + w_y) \bmod 2 - 1\right), & \text{if } -1 \leq z_x \leq 0,\ -1 \leq z_y \leq 1; \\ \left(\frac{1}{2}(z_x + 1) + w_x, (2z_y + w_y) \bmod 2 - 1\right), & \text{if } 0 \leq z_x \leq 1,\ -1 \leq z_y \leq 1; \\ (z_x + w_x, (z_y + w_y) \bmod 2 - 1), & \text{if } |z_x| > 1. \end{cases} \tag{4.31}$$

Using the extension $\hat{h}$, we define a small perturbation h_ξ, as described in Subsection 2.1.1.

We denote the central part of the cylinder by $\odot \stackrel{\text{def}}{=} \{z \in \mathcal{C} \colon |z_x| \leq 1\}$. By rectangle in $\odot$ we understand a subset $Rec(l_x, l_y) = I_x \times I_y$, where $I_x \subset [-1; 1)$ and $I_y \subset S^1 \setminus \{1\}$ are two intervals with $|I_x| = l_x$ and $|I_y| = l_y$.

Lemma 4.3.2. *Given a sequence $\xi \in \Sigma_\delta$, with $\delta = 2^{-m\alpha}$, for any $1 \le k \le m\alpha - 3$ there exist k rectangles $r_1^{k,\xi}, \ldots, r_k^{k,\xi} \subset \odot$ such that*

$$\{z \in \odot \mid \exists 1 \le j \le k\colon h_\xi^j(z) \notin \odot\} \subset \bigcup_{j=1}^{k} r_j^{k,\xi}.$$

Moreover, $r_j^{k,\xi} \subset Rec(2^j\delta, 2^{1-j})$ for all $1 \le j \le k$ and for any $a \in \mathbb{R}^2$ with $|a| \le \delta$ the map h_a^{-1} is continuous on the union of the rectangles $\bigcup_{j=1}^{k} r_j^{k,\xi}$.

Proof. By induction in k. Indeed, the conditions $z \in \odot$ and $h_\xi(z) \notin \odot$ are equivalent to $|\pi_x(h_\xi(z))| > 1$ and $z \in \square$. The latter means

$$z \in r_1^\xi \stackrel{\text{def}}{=} \begin{cases} (-1;-1+2\xi_x^1) \times (-1;0) \subset (-1;-1+2\delta) \times (-1;0), & \text{if } \xi_x < 0, \\ (1-2\xi_x^1;1) \times (0;1) \subset (1-2\delta;1) \times (0;1), & \text{if } \xi_x > 0. \end{cases} \tag{4.32}$$

Thus the statement holds true for $k = 1$. Let us add to the induction assumption the following inclusion which is trivial for $k = 1$:

$$\bigcup_{j=1}^{k} r_j^{k,\xi} \subset (1-2^k\delta;1) \times (-1;1) \cup (-1;-1+2^k\delta) \times (-1;1). \tag{4.33}$$

We may write

$$\begin{aligned} &\{z \in \odot \mid \exists j \le k+1\colon h_\xi^j(z) \notin \odot\} \subset \\ &\qquad \subset \{z \in \odot \mid h_\xi(z) \notin \odot\} \cup \{z \in \odot \mid \exists 1 < j \le k+1\colon h_\xi^j(z) \notin \odot\} \subset \\ &\qquad \subset r_1^\xi \cup \{w = h_{\xi^1}(z) \subset \odot \mid \exists 1 \le j \le k\colon h_{\sigma(\xi)}^j(w) \notin \odot\} \subset r_1^\xi \cup h_{\xi^1}^{-1}\Big(\bigcup_{j=1}^{k} r_j^{k,\sigma(\xi)}\Big). \end{aligned}$$

Therefore we may set $r_1^{k+1,\xi} \stackrel{\text{def}}{=} r_1^\xi$ and $r_{j+1}^{k+1,\xi} \stackrel{\text{def}}{=} h_{\xi^1}^{-1}(r_j^{k,\sigma(\xi)}) \cap \square$ for $j = 1, \ldots, k$. Since h_0^{-1} is continuous on every $(r_j^{k,\sigma(\xi)} - \xi^1) \cap \square$, the sets $r_{j+1}^{k+1,\xi}$ are rectangles. Using supposition (4.33) we conclude

$$\begin{aligned} h_{\xi^1}^{-1}\Big(\bigcup_{j=1}^{k} r_j^{k,\xi}\Big) &\subset h_{\xi^1}^{-1}\Big(\big((1-2^k\delta,1) \cup (-1,-1+2^k\delta)\big) \times (-1,1)\Big) \subset \\ &\subset \big((-1,-1+2^{k+1}\delta) \cup (1-2^k\delta,1)\big) \times (-1,1), \end{aligned} \tag{4.34}$$

and therefore h_a^{-1} is continuous on $\bigcup_{j=1}^{k+1} r_j^{k+1,\xi} - a$ for any $|a| \le \delta$.

Finally, one can check by straightforward calculation that for all $1 \le j \le k$ we have

$$h_{\xi^1}^{-1}(r_j^{k,\sigma(\xi)}) \subset Rec(2^{j+1}\delta, 2^{1-j}).$$

■

Corollary 1. *For any sequence $\xi \in \Sigma_\delta$ there are $\frac{3m}{4}$ rectangles $r_1^\xi, r_2^\xi, \ldots, r_{3m/4}^\xi \subset \odot$ such that*

$$\left\{z \in \odot \mid \exists 1 \le j \le \frac{3m}{4} \colon h_\xi^j(z) \notin \odot\right\} \subset \bigcup_{j=1}^{3m/4} r_j^\xi;$$

and the union $\bigcup_{j=1}^{3m/4} r_j^\xi \subset \square$ may be covered by at most $m^3 2^{2m}\delta$ rectangles $Rec(2^{-5m/4}, 2^{-3m/4})$.

Proof. By Lemma 4.3.2, there exists $\frac{3m}{4}$ rectangles $r_1^\xi, \ldots, r_{3m/4}^\xi \subset \odot$ such that

$$\left\{z \in \odot \mid \exists 1 \le j \le k \colon h_\xi^j(z) \notin \odot\right\} \subset \bigcup_{j=1}^{3m/4} r_j^\xi;$$

moreover, $r_j^\xi \subseteq Rec(2^j\delta, 2^{1-j})$. Therefore, each r_j^ξ may be covered by at most

$$m^2\big((2^{5m/4} \cdot 2^j\delta) \cdot (2^{3m/4} \cdot 2^{1-j}) + 2^{m/4} \cdot 2^{2-j} + 2^{3m/4+j+1}\delta\big) \le 2^{2m}m^3\delta$$

rectangles $Rec(\frac{2^{-5m/4}}{m}, \frac{2^{-3m/4}}{m})$. Since there are $\frac{3m}{4}$ rectangles $r_1^\xi, \ldots, r_{3m/4}^\xi$ their union may be covered by not more than $2^{2m}m^4\delta$ rectangles $Rec(2^{-5m/4}, 2^{-3m/4})$. ■

We may identify a rectangle on the cylinder $I_x \times I_y \subset \odot$ with a rectangle on the plane $I_x \times I_y \subset \square \subset \mathbb{R}^2$, since we agreed that $I_y \subset S^1 \setminus \{1\}$.

Lemma 4.3.3. *Under the hypothesis and in the notations of Lemma 4.3.2 the set $\bigcup_{j=1}^{m/4} r_j^{m/4,\xi}$ may be covered by at most $2^{2m}m^3\delta$ elements of a partition of the class $\mathcal{G}(m,\delta)$.*

Proof. By definition, all elements of a partition of the class $\mathcal{G}(m,\delta)$ are rectangles. By the second part of Lemma 4.2.2, $Rec(\frac{2^{-m}}{m}, \frac{2^{-m}}{m}) \subseteq \Omega_{ij} \subset Rec(2^{1-m}, 2^{1-m})$. Therefore any rectangle $Rec(2^k\delta, 2^{1-k})$ may be covered by at most

$$m^2(2^{k+m}\delta \cdot 2^{1+m-k} + 2^{m+2} \cdot 2^{-k} + 2^{m+k+1}\delta) \leq m^2 2^{2m+2}\delta$$

elements of the partition. Then all $\frac{m}{4}$ rectangles may be covered by at most $2^{2m}m^3\delta$ elements.

■

We lift the map $h\colon \mathcal{C} \to \mathcal{C}$ to the plane $\mathbb{R}^2$ and obtain

$$H(z) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1), 2z_y + 1\right), & \text{if } z \in \square,\ -1 \leq z_y \leq 0; \\ \left(\frac{1}{2}(z_x + 1), 2z_y - 1\right), & \text{if } z \in \square,\ 0 \leq z_y \leq 1; \\ z, & \text{if } z \notin \square. \end{cases} \tag{4.35}$$

Let $\widehat{H}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be an extension given by

$$\widehat{H}(z,w) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1), 2z_y + 1\right) + w, & \text{if } z \in \square \text{ and } -1 \leq z_y \leq 0; \\ \left(\frac{1}{2}(z_x + 1), 2z_y - 1\right) + w, & \text{if } z \in \square \text{ and } 0 \leq z_y \leq 1; \\ z + w, & \text{if } z \notin \square. \end{cases} \tag{4.36}$$

Given a sequence $\xi \subset \Sigma_\delta \subset \ell_\infty(\mathbb{R}^2)$ and extension $\widehat{H}$, we define a small perturbation H_ξ, as described in Subsection 2.1.1.

Remark 11. Observe that $z \in E_k$ if and only if $\prod_{j=1}^{k} \chi_\square(H_\xi^j(z)) = 0$; where E_k is the k'th escaping set defined by (4.15), p. 101.

Remark 12. Let p be the doubling map defined by (3.3) with $s_1 = s_2 = 2$. Let ξ and ς be two sequences defined as in Lemma 4.2.1. Then for any $k > 0$ the following two diagrams are commutative.

$$\begin{array}{ccc} \square \setminus E_k & \xrightarrow{H_\xi^k} & \mathbb{R}^2 \\ \pi_y \big\downarrow & & \big\downarrow \pi_y \\ \mathbb{R} & \xrightarrow{p_{\xi y}} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \square \setminus E_{-k} & \xrightarrow{H_\xi^{-k}} & \mathbb{R}^2 \\ \pi_x \big\downarrow & & \big\downarrow \pi_x \\ \mathbb{R} & \xrightarrow{p_{\varsigma x}} & \mathbb{R} \end{array}$$

Recall the settings, introduced in the beginning of the Section 4.3, p. 103. Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ, and let $\xi = \sigma^{2mk}\eta$ be a shifted sequence.

Lemma 4.3.4. *The number of elements of the partition Ω^1 inside the square $\square$ possibly escaping in the first $\frac{m}{4}$ iterations is bounded by $2^{\frac{9m}{4}+1}\delta$:*

$$\#\left\{\Omega^1_{ij} \subset \square \mid \exists\, 1 \le k \le \frac{m}{4} \colon H^k_\xi(\Omega^1_{ij}) \not\subset \square\right\} \le 2^{\frac{9m}{4}+1}\delta.$$

Proof. By Lemma 4.3.3

$$\#\left\{\Omega^1_{ij} \subset \odot \mid \exists\, 1 \le k \le \frac{m}{4} \colon h^k_\xi(\Omega^1_{ij}) \not\subset \odot\right\} \le 2^{2m} \cdot m^3\delta,$$

which is equivalent to

$$\#\left\{\Omega^1_{ij} \subset \square \mid \exists\, 1 \le k \le \frac{m}{4} \colon \pi_x(H^k_\xi(\Omega^1_{ij})) \not\subset [-1;1]\right\} \le 2^{2m} \cdot m^3\delta.$$

Recall the doublin map p defined by (3.3) with $s_1 = s_2 = 2$. Let $p^k_{\xi_y}$ be a small perturbation as in Lemma 3.2.7. Then the map $p_{\xi_y}\ p^k_{\xi_y}$ has exactly 2^k long branches for all $k \le m\alpha$. Therefore we get an upper bound

$$\#\left\{\Omega^1_{ij} \subset \square \mid \forall\, 1 \le k \le \frac{m}{4} \colon \pi_x(H^k_\xi(\Omega^1_{ij})) \subset [-1;1] \text{ and } \right.$$
$$\left. \exists\, 1 \le k \le \frac{m}{4} \colon \pi_y(H^k_\xi(\Omega^1_{ij})) \not\subset [-1;1]\right\} \le$$
$$\le 2^m \cdot \#\left\{\Omega^1_j \subset [-1;1] \mid \exists\, 1 \le k \le \frac{m}{4} \colon p^k_{\xi_y}(\Omega^1_j) \not\subset [-1;1]\right\} \le 2^{5m/4+1},$$

By supposition on α, we know that $2^{2m}m^3\delta \ll 2^{5m/4}$. (In other words, assume that for some $\Omega^1_j \subset [-1;1]$ we have $p^k_{\xi_y}(\Omega^1_j) \subset [-1;1]$ for all $k < k_0$ and $p^{k_0}_{\xi_y}(\Omega^1_j) \not\subset [-1;1]$. Then Ω^1_j is a subset of the domain of a long branch of $p^k_{\xi_y}$ for all $k < k_0$; and the subset of the domain of a main branch that may escape at the iteration k is an interval, i.e. a connected set, of the measure at most $2^{-k}\delta$, which contains at most $2^{m-k}\delta$ intervals of the canonical partition of the perturbation of the doubling map $p^m_{\xi_y}$.) ■

Remark 13. In Lemma 4.3.4 above, an alternative upper bound would be $2^{\frac{5m}{4}} \cdot C_\delta$, where C_δ is the maximum number of intervals of the canonical partition for the doubling map in the interval of the length δ. In our case all intervals have the length $|\pi_y(\Omega_{ij})| \leq 2^{1-m}$, therefore $2^m\delta > C_\delta > 2^{m-1}\delta$.

Lemma 4.3.5. *There exists at least* $2^{2m} - 2^{9m/4+2}\delta$ *elements* $\Omega^1_{ij} \subset \Box$ *of the partition* Ω^1 *such that* $H^k_\xi(\Omega^1_{ij}) \subset \Box$ *for all* $1 \leq k \leq m$ *and*

$$Rec\Big(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|\Big) = H^{m/4}_\xi(\Omega^1_{ij}) \subset \Box.$$

Proof. By Lemma 4.3.4 we know that there are at most $2^{9m/4+2}\delta$ elements of the partition Ω^1 such that $H^{m/4}_\xi(\Omega^1_{ij}) \not\subset \Box$. We shall show now that there are at most $2^{5m/4}$ elements of Ω^1 such that $H^k_\xi(\Omega^1_{ij}) \subset \Box$ for all $1 \leq k \leq \frac{m}{4}$, and yet

$$H^{m/4}_\xi(\Omega^1_{ij}) \not\supseteq Rec\Big(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|\Big).$$

If $H^{m/4}_\xi(\Omega^1_{ij})$ is connected, then $H^{m/4}_\xi(\Omega^1_{ij}) = Rec\Big(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|\Big)$. Thus without loss of generality we may assume that $H^{m/4}_\xi(\Omega^1_{ij})$ is not a connected set. The latter implies $H^k_\xi(\Omega^1_{ij}) \cap \{z_y = 0\} \neq \varnothing$ for some $1 \leq k \leq m/4$. Recall the doublin map p defined by (3.3) with $s_1 = s_2 = 2$. Let $p^k_{\xi_y}$ be a small perturbation as in Lemma 3.2.7. Since by supposition $H^k_\xi(\Omega^1_{ij}) \subset \Box$ for all $1 \leq k \leq \frac{m}{4}$, we conclude that $\Omega^1_j := \pi_y(\Omega^1_{ij})$ belongs to a main branch of the map $p^{m/4}_{\xi_y}$. We know that the map $p^k_{\xi_y}$ has at most 2^k main branches, and if $\{0\} \in p^{k_1}_{\xi_y}(\Omega^1_j)$, then $\{0\} \notin p^{k_2}_{\xi_y}(\Omega^1_j)$ for all $k_1 < k_2 \leq \frac{m}{4}$. So there are at most $2^{m/4+1}$ elements Ω^1_j such that $\{0\} \in p^k_{\xi_y}(\Omega^1_j)$ for some $1 \leq k \leq \frac{m}{4}$. Thus there are at most $2^{5m/4}$ elements Ω^1_{ij} such that $H^k_\xi(\Omega^1_{ij}) \cap \{y = 0\} \neq \varnothing$ for some $1 \leq k \leq \frac{m}{4}$ and $H^k_\xi(\Omega^1_{ij}) \subset \Box$ for all $1 \leq k \leq \frac{m}{4}$. ∎

Corollary 1. *There exists at least* $2^{2m} - 2^{9m/4}\delta$ *elements* $\Omega^1_{ij} \subset \Box$ *of the partition* Ω^1 *such* *that* $F_\xi^k(\Omega_{ij})) \subset \Box$ *for all* $1 \le k \le \frac{m}{4}$ *and*

$$Rec\Big(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|\Big) = F_\xi^{m/4}(\Omega^1_{ij}) \subset \Box.$$

We need the following fact about small perturbations of the doubling map p.

Lemma 4.3.6. *For any* $\frac{m}{2} \le k \le m\alpha - 2$ *the perturbation of the doubling map* p_ξ^m *with* $\|\xi\|_\infty$ *has at most* 2^{k+2} *main branches such that their domains* $\mathfrak{a}_j^{(m)}$ *satisfy* $|p^m_{\xi_y}(\mathfrak{a}_j^{(m)})| < 2 - 2^{m-k}\delta$.

Proof. Let $\mathfrak{a}_j^{(m)} = (a_j^{(m)}; a_{j+1}^{(m)})$ be the domain of a main branch of the map p_ξ^m such that $|p_\xi^m(\mathfrak{a}_j^{(m)})| < 2 - 2^{m-k}\delta$.

We shall show that the interval $\overline{\mathfrak{a}_j^{(m)}}$ is not contained in a domain of a main branch of the map p_ξ^{k+2}.

Assume for a contradiction that for some $\frac{m}{2} \le k \le m\alpha - 2$ there exists a main branch $\mathfrak{a}_i^{(k+2)} \supset \overline{\mathfrak{a}_j^{(m)}}$ of the map p_ξ^{k+2}. By assumption, $a_j^{(m)}$ and $a_{j+1}^{(m)}$ are points of discontinuity of the map p_ξ^m. Since p_ξ^{k+2} is continuous on $\mathfrak{a}_j^{(k+2)}$, we deduce that there exist $k_1, k_2 \ge k+2$ such that $p_\xi^{k_1}(a_j^{(m)}) = 0$ and $p_\xi^{k_2}(a_{j+1}^{(m)}) = 0$. Since $p_\xi^m(\mathfrak{a}_j^{(m)})$ is an interval, we see that either $|p_\xi^m(a_j^{(m)}) + 1| > 2^{m-k-1}\delta$ or $|p_\xi^m(a_{j+1}^{(m)}) - 1| > 2^{m-k-1}\delta$. Without loss of generality, assume the first. Then

$$p_\xi^m(a_j^{(m)}) = p_\xi^{m-k_1}(0) = p_\xi^{m-k_1-1}(-1 + \xi(k_1+1)),$$

and, therefore, $|p_\xi^m(a_{j+1}^{(m)}) + 1| \le 2^{m-k_1+1}\delta$. Thus $k_1 < k+2$. We deduce that the map p_ξ^{k+2} is not continuous on $\overline{\mathfrak{a}_j^{(m)}}$. We know by Lemma 3.2.7, that for any $1 \le k \le m\alpha$ the map p_ξ^k has exactly 2^k main branches and the Lemma follows.

■

Lemma 4.3.7. *There exist at least* $2^{2m} - 2^{\frac{3}{2}m}$ *elements of the partition* Ω^1 *in the unit square* $\Box$ *such that for some* $\breve{\Omega}^1_{ij} \subset \Omega^1_{ij}$ *we have* $H_\xi^n(\breve{\Omega}^1_{ij}) \subset \Box$ *for all* $1 \le n \le m$ *and*

$$H_\xi^m(\breve{\Omega}^1_{ij}) = Rec\big(2^{-m}|\pi_x(\Omega^1_{ij})|, 2 - 2^{\frac{m}{2}}\delta\big).$$

Proof. Let $\eta = \sigma^{m/4}(\xi)$ and let r_j^η, $1 \le j \le \frac{3m}{4}$ be rectangles covering the escaping set $E_{\frac{3m}{4}}$ of the map $F_\eta^{3m/4}$, defined according to Corollary 1 of Lemma 4.3.2. According to Lemma 4.3.5 there exist at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta$ elements of the partition Ω^1 such that $H_\xi^k(\Omega_{ij}^1) \subset \Box$ for all $1 \le k \le \frac{m}{4}$, and there exists a rectangle $Rec(2^{-m/4}|\pi_x(\Omega_{ij}^1)|, 2^{m/4}|\pi_y(\Omega_{ij}^1)|) = H_\xi^{m/4}(\Omega_{ij}^1) \subset \Box$. It follows from Corollary 1 of Lemma 4.3.2, that among these elements of the partition Ω^1 one can find at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta - 2^{2m}m^4\delta$ elements that satisfy $H_\xi^{m/4}(\Omega_{ij}^1) \cap (\bigcup_{j=1}^{3m/4} r_j^\eta) = \varnothing$.

The condition $H_\xi^{m/4}(\Omega_{ij}^1) \cap (\bigcup_{j=1}^{3m/4} r_j^\eta) = \varnothing$ implies $\pi_x(H_\xi^k(\Omega_{ij}^1)) \subset [-1;1]$ for all $1 \le k \le m$, and it follows that $|\pi_x(H_\xi^m(\Omega_{ij}^1))| = 2^{-m}|\pi_x(\Omega_{ij}^1)|$. Therefore, $H_\xi^k(\Omega_{ij}^1) \not\subset \Box$ for some $\frac{m}{4} < k \le m$ if and only if $\pi_y(H_\xi^k(\Omega_{ij}^1)) \not\subset [-1;1]$. By construction, $\Omega_j^1 = \pi_y(\Omega_{ij}^1)$ is an element of the canonical partition of the map $p_{\xi_y}^m$. By Lemma 4.3.6 with $k = \frac{m}{2}$, there map $p_{\xi_y}^m$ has at most $2^{\frac{m}{2}+2}$ main branches such that $|p_{\xi_y}^m(\mathfrak{a}_j^{(m)})| \le 2 - 2^{\frac{m}{2}}\delta$. For every Ω_{ij}^1, such that $\pi_x(H_\xi^k(\Omega_{ij}^1)) \subset [-1;1]$ and $\pi_y(\Omega_{ij}^1)$ contains the domain $\mathfrak{a}_j^{(m)}$ of a main branch of the map $p_{\xi_y}^m$ with $|p_{\xi_y}^m(\mathfrak{a}_j^{(m)})| \ge 2 - 2^{\frac{m}{2}}\delta$, there exists a rectangle $\breve{\Omega}_{ij}^1 \stackrel{\text{def}}{=} \pi_x(\Omega_{ij}^1) \times \mathfrak{a}_j^{(m)} \subset \Omega_{ij}^1$ with the property $H_\xi^k(\breve{\Omega}_{ij}^1) \subset \Box$ for all $1 \le k \le m$, and, moreover

$$H_\xi^m(\Omega_{ij}^1) \supset H_\xi^m(\breve{\Omega}_{ij}^1) \supset Rec\Big(2^{-m}|\pi_x(\Omega_{ij}^1)|, 2 - 2^{m/2}\delta\Big).$$

Therefore, there are at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta - 2^{2m}m^4\delta - 2^{\frac{3}{2}m+2} \ge 2^{2m} - 2^{\frac{3}{2}m+3}$ elements of the partition Ω^1 such that for some $\breve{\Omega}_{ij}^1 \subset \Omega_{ij}^1$ which satisfies $H_\xi^k(\breve{\Omega}_{ij}^1) \subset \Box$ for all $1 \le k \le m$ we have

$$H_\xi^m(\breve{\Omega}_{ij}^1) = Rec\big(2^{-m}|\pi_x(\Omega_{ij}^1)|, 2 - 2^{\frac{m}{2}}\delta\big).$$

In other words, the map H_ξ^m has at least $2^{2m} - 2^{\frac{3}{2}m+3}$ main branches. ■

Corollary 1. *There exist at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements of the partition Ω^2 such that for some $\breve{\Omega}_{ij}^2 \subset \Omega_{ij}^2$ we have $H_{\sigma^m\xi}^{-k}(\breve{\Omega}_{ij}^2) \subset \Box$ for all $1 \le k \le m$ and*

$$H_\xi^{-m}(\breve{\Omega}_{ij}^2) = Rec(2 - 2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega_{ij}^2)|).$$

Definition 22. The rectangles $\breve{\Omega}^1_{ij}$ and $\breve{\Omega}^2_{ij}$, constructed in Lemma 4.3.7 and Corollary 1 of Lemma 4.3.7 we call *domains of the long branches* of the maps P_ξ and $P^{-1}_{\sigma^m \xi}$, respectively. Their images we call *long branches*.

Lemma 4.3.8. *For any element Ω^1_{ij} of the partition Ω^1, the set $\Omega^1_{ij} \setminus E_m$ is a union of (disjoint) rectangles. The number of rectangles is equal to the number of main branches of the perturbation $p^m_{\xi_y}$ of the doubling map p.*

Proof. We split the argument into several steps.

Claim 1. The projection $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a union of domains of main branches of the small perturbation $p^m_{\xi_y}$ of the doubling map. First we shall show that the image of the projection $p^n_{\xi_y}(\pi_y(\Omega^1_{ij} \setminus E_m)) \subset [-1;1]$ for all $1 \leq n \leq m$. Indeed, assume for a contradiction that for some $1 < n < m$ we have $p^n_{\xi y}(\pi_y(\Omega^1_{ij} \setminus E_m)) \notin [-1;1]$, and n is the smallest number with this property. Since the horizontal lines $\{y = \text{const}\} \cap \square \setminus E_{m-1}$ are invariant under H^n_ξ, we may conclude that $H^n_\xi(\Omega^1_{ij} \setminus E_m) \not\subset \square$, which is a contradiction. Therefore $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a subset of the domain of a main branch. Let an interval $(a,b) \supset \pi_y(\Omega^1_{ij} \setminus E_m)$ be the domain of the main branch. We shall show that $\Omega^1_i \times (a,b) \subset \Omega^1_{ij} \setminus E_m$. Assume that there exists $z \in \Omega^1_i \times (a,b)$ such that $H^n_\xi(z) \notin \square$ for some $1 \leq n \leq m$. Since $\pi_y(H^n_\xi(z)) = p^n_{\xi_y}(z_y) \in [-1;1]$, we conclude $\pi_x(H^n_\xi(z)) \notin (-1;1)$. Observe that, the lines $\{x = \text{const}\} \cap \square \setminus E_{m-1}$ are invariant with respect to H^n_ξ, we get $H^n_\xi(z_x, \pi_y(\Omega^1_{ij} \setminus E_m)) \notin \square$, which is a contradiction. Therefore $(a,b) \subset \pi_y(\Omega^1_{ij} \setminus E_m)$ and hence $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a union of domains of main main branches.

Claim 2. The set $\{y = \text{const}\} \cap (\Omega^1_{ij} \setminus E_m)$ is connected. Indeed, assume that there are three points $z, u, w \in \{y = \text{const}\} \cap (\Omega^1_{ij} \setminus E_m)$ such that $z_x < u_x < w_x$, with $z, w \in \Omega^1_{ij} \setminus E_m$ and $u \notin \Omega^1_{ij} \setminus E_m$. Then there exists $1 \leq n \leq m$ such that $H^n_\xi(u) \notin \square$, and we may assume that n is the smallest number with such property. Then by invariance of $\{y = \text{const}\} \cap \Omega^1_{ij} \setminus E_{n-1}$ with respect to H^n_ξ, we conclude that either $H^n_\xi(z) \notin \square$ or $H^n_\xi(w) \notin \square$, which is a contradiction.

Claim 3. For any two points $z, w \in \Omega^1_{ij} \setminus E_m$ such that z_y and w_y belong to the same domain of a main branch of $p^m_{\xi_y}$, we have $(z_x, w_y), (w_x, z_y) \in \Omega^1_{ij} \setminus E_m$. Indeed, assume for a contradiction that $(z_x, w_y) \notin \Omega^1_{ij} \setminus E_m$. Then choose the smallest n such that $H^n_\xi(z_x, w_y) \notin \Box$. It follows that either $\pi_y(H^n_\xi(z_x, w_y)) \notin [-1; 1]$ or $\pi_x(H^n_\xi(z_x, w_y)) \notin [-1; 1]$, or both. Without loss of generality suppose that projection of the image $\pi_y(H^n_\xi(z_x, w_y) \notin [-1; 1]$. Then due to invariance of $\{x = \text{const}\} \cap \Box \setminus E_{m-1}$ we have $\pi_x(H^n_\xi(z)) \notin [-1; 1]$, which is a contradiction.

Summing up, we conclude that the set $\Omega^1_{ij} \setminus E_m$ is a union of rectangles and the number of rectangles is equal to the number of main branches of the map $p^m_{\xi_y}$ in Ω^1_j.

■

Corollary 1. *In the notation of Lemma 4.2.1, the set $\Omega^2_{ij} \setminus E_{-m}$ is a union of (disjoint) rectangles for any element Ω^2_{ij} of the partition Ω^2. The number of rectangles is equal to the number of main branches of the perturbation $p^m_{\varsigma_x}$ of the doubling map p.*

Lemma 4.3.9. *There exist at most $2^{4m}\delta$ quartets (i, j, k, l) such that $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ has more than one (P, ξ)-domain Δ that satisfies $H^n_\xi(\Delta) \subset \Box$ for all $1 \le n \le 2m$. For any quartet (i, j, k, l) the set $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ has at most four (P, ξ)-domains with this property.*

Proof. Let Δ be a (P, ξ)-domain in $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ such that $H^n_\xi(\Delta) \subset \Box$ for all $1 \le n \le 2m$. Then

$$\#\{\Delta \subset H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij} \mid H^n_\xi(\Delta) \subset \Box \text{ for all } 1 \le n \le 2m\} =$$

$$= \#\{\Delta \subset H^{-m}_{\sigma^m\xi}(\Omega^2_{kl}) \cap H^m_\xi(\Omega^1_{ij}) \mid H^n_\xi(\Delta) \subset \Box \text{ for all } -m \le n \le m\} =$$

$$= \#\{\Delta \subset \big(\Omega^2_{kl} \setminus E_{-m}\big) \cap \big(\Omega^1_{ij} \setminus E_m\big)\}.$$

By Lemma 4.3.8 and Corollary 1 of Lemma 4.3.8, both sets $\Omega^2_{kl} \setminus E_{-m}$ and $\Omega^1_{ij} \setminus E_m$ are unions of rectangles, and the number of rectangles equal to the number of main branches of the corresponding doubling maps on the associated intervals. By Lemma 3.2.5 there are at most $2^m\delta$ intervals Ω_i or Ω_l that contain two main branches. Thus there are at most $2^{4m}\delta$ quartets (i,j,k,l) such that Ω_i or Ω_l or both contain two main branches of the maps $p^m_{\varsigma_x}$ and $p^m_{\xi_y}$, respectively; and the Lemma follows. ∎

Using Lemmas 4.3.7 and 4.3.9 and Corollary 1 of Lemma 4.3.7, we get

Corollary 1. *Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ, and let $\xi = \sigma^{2m(k-1)}\eta$ be a shifted sequence. Then*

1. *There exist at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements Ω^1_{ij} such that for some $\breve{\Omega}^1_{ij} \subset \Omega^1_{ij}$ we have*
$$P_\xi(\breve{\Omega}_{ij}) = Rec\Big(2^{-m}|\pi_x(\Omega_{ij})|, 2 - 2^{\frac{m}{2}}\delta\Big) \qquad \text{and} \qquad \mathrm{d}P_\xi|_{\breve{\Omega}_{ij}} = \begin{pmatrix} 2^{-m} & 0 \\ 0 & 2^m \end{pmatrix}.$$
2. *There exist at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements Ω^2_{ij} such that for some $\breve{\Omega}^2_{kl} \subset \Omega^2_{kl}$ we have*
$$P^{-1}_\xi(\breve{\Omega}_{ij}) = Rec\Big(2 - 2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega_{ij})|\Big) \qquad \text{and} \qquad \mathrm{d}P^{-1}_\xi|_{\breve{\Omega}_{ij}} = \begin{pmatrix} 2^{m} & 0 \\ 0 & 2^{-m} \end{pmatrix}.$$
3. *There exists at most $2^{4m}\delta$ quartets (i,j,k,l) such that the set $P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ contains more than one (P,ξ)-domain Δ that satisfies $d_y(P^2_\xi)_y|_\Delta = 2^{2m}$.*

Proof. Observe that for any $1 \le k \le 2m$ and for any $z \in \square \setminus E_k$ we have $F^k_\xi(z) = H^k_\xi(z)$. ∎

Lemma 4.3.10. *The area of a good (P,ξ)-domain Δ is very small. More precisely, we have an upper bound $|\Delta| \le 2^{2-4m}$.*

Proof. Recall the definition of good connected components (4.25) and observe

$$\begin{aligned}(\Delta^G)^{kl}_{ij} =& \{\Delta \subset P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij} \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \forall 1 \le n \le 2m : F^n_\xi(\Delta) \subset \square\} = \\ =& \{\Delta \subset P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}) \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \forall 1 \le n \le 2m : H^n_\xi(\Delta) \subset \square\}.\end{aligned}$$

We shall show that for any $\Delta \in \Delta^G$ the area $|\Delta| \le 2^{-2m} \cdot |\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|$. Indeed, consider the image $\Delta' = P_\xi(\Delta)$. Since P_ξ is area-preserving, $|\Delta'| = |\Delta|$. Since $P_\xi(\Delta') \subset \Omega^2_{kl}$, the length $|\pi_y(\Delta')| \le 2^{-m} \cdot |\pi_y(\Omega^2_{kl})|$; and $P^{-1}_{\sigma^m\xi}(\Delta') \subset \Omega^1_{ij}$ implies $|\pi_x(\Delta')| \le 2^{-m} \cdot |\pi_x(\Omega^1_{ij})|$. Thus

$$|\Delta| = |\Delta'| \le 2^{-2m}|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})| \le 2^{2-4m}. \tag{4.37}$$

■

Corollary 1. *The matrix SS^G is small. More precisely,*

$$\sum_{\square}\sum_{\square}|(SS^G)^{kl}_{ij}| \le 2^{4-2m}.$$

Proof. By straightforward calculation, using Lemma 4.3.10,

$$\sum_{\square}\sum_{\square}|(SS^G)^{kl}_{ij}| = \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G}\int_{\Delta}\partial_x(P^2_\xi)_x(z)\mathrm{d}z \le$$
$$\le \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G}2^{-4m}|\Delta| \le$$
$$\le \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot 4(2^{-2m}|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|)\cdot 2^{-4m} \le 2^{4-2m}.$$

■

Now we are ready to prove

Proposition 4.3.1. The matrix UU^G has the following properties

1. $\|UU^G\|_\infty \le 4$;
2. $\#\{(UU^G)^{kl}_{ij} \ne 1\} \le 2^{4\frac{1}{2}m}\delta$.

Proof. By Lemma 4.3.9, for any $(i,j,k,l) \in \square\times\square$ we have $\#(\Delta^G)^{kl}_{ij} \le 4$, and by Lemma 4.3.10 we know $|\Delta| \le 2^{-2m}\cdot|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|$. We calculate

$$|(UU^G)^{kl}_{ij}| \le \sum_{\Delta\in(\Delta^G)^{kl}_{ij}}|\Delta|\cdot|\partial_y(P^2_\xi)_y|\cdot|\pi_x(\Omega^1_{ij})|^{-1}\cdot|\pi_y(\Omega^2_{kl})|^{-1} \le$$
$$\le 4\cdot(2^{-2m}|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|)\cdot 2^{2m}\cdot|\pi_x(\Omega^1_{ij})|^{-1}\cdot|\pi_y(\Omega^2_{kl})|^{-1} = 4.$$

To prove the second part, we recall that by Lemma 4.3.7 there are at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements of the first partition Ω^1_{ij} such that for some $\breve{\Omega}_{ij} \subset \Omega^1_{ij}$ the image is a rectangle $P_\xi(\breve{\Omega}_{ij}) = Rec(2^{-m}|\pi_x(\Omega_{ij})|), 2 - 2^{\frac{m}{2}}\delta)$ and $H^n_\xi(\breve{\Omega}_{ij}) \subset \square$ for all $1 \le n \le m$. Similarly by Corollary 1 of Lemma 4.3.7, there are at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements Ω^2_{kl} such that for some small rectangle $\breve{\Omega}_{kl} \subset \Omega^2_{kl}$ the preimage $P^{-1}_\xi(\breve{\Omega}_{kl}) = Rec(2 - 2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega_{ij})|)$ and $H^{-n}_\xi(\breve{\Omega}_{kl}) \subset \square$ for all $1 \le n \le m$. Then there are at least $(2^{2m} - 2^{\frac{3}{2}m+3} - 2^{\frac{5}{2}m}\delta)^2$ pairs Ω^1_{ij}, Ω^2_{kl} such that $P_\xi(\breve{\Omega}_{ij}) \cap P^{-1}_\xi(\breve{\Omega}_{kl}) \neq \varnothing$ which correspond to $(UU^G)^{kl}_{ij} \neq 0$. If $(\Delta^G)^{kl}_{ij}$ has only one element, then it is $\Delta = P_\xi(\breve{\Omega}_{ij}) \cap P^{-1}_{\sigma^m\xi}(\breve{\Omega}_{kl})$ and $|\Delta| = 2^{-2m} \cdot |\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|$. Therefore

$$(UU^G)^{kl}_{ij} = \frac{1}{|\pi_x(\Omega^1_{ij})|} \cdot \frac{1}{|\pi_y(\Omega^2_{kl})|} \int_\Delta 2^{2m} = 1.$$

Summing up, there are at least $2^{4m} - 2^{\frac{9}{2}m+1}\delta$ elements $(UU^G)^{kl}_{ij} = 1$. By Lemma 4.3.9 the set $(\Delta^G)^{kl}_{ij}$ has more than one connected component for not more that $2^{4m}\delta$ quartets (i, j, k, l). Therefore at most $2^{4m}\delta$ elements satisfy $1 < (UU^G)^{kl}_{ij} \le 4$. The other elements are zeros. ∎

Now we proceed to the supremum norm of the matrix UU. Our goal is to prove the following

Proposition 4.3.2. There exist a constant $\gamma_1 < 0.01$ such that for M and m sufficiently large and for μ sufficiently small

$$\max(\|SS\|_\infty, \|SU\|_\infty, \|US\|_\infty, \|UU\|_\infty) \le 2^{\gamma_1 m}.$$

We define two functions on the unit square

$$t_{\text{in}} \colon \square \to \mathbb{N} \qquad t_{\text{in}}(z) = \sum_{j=0}^{2m} \chi_\square(F^j_\xi(z)); \tag{4.38}$$

$$t_{\text{ex}} \colon \square \to \mathbb{N} \cap \left[1; \frac{2m}{M}\right] \qquad t_{\text{ex}}(z) = \#\{1 \le n \le 2m \colon F^{n-1}_\xi(z) \in \square \text{ and } F^n_\xi(z) \notin \square\}. \tag{4.39}$$

Given a sequence $\imath \in \{0,1\}^{\mathbb{N}}$ we define a subset of the unit square $\square$

$$\Delta_\imath \stackrel{\text{def}}{=} \big\{ z \in \square \colon \chi_\square(F_\xi^n(z)) = \imath_n \text{ for all } n \in \{0,1,\dots,2m\} \big\}.$$

Note that some of $\Delta_\imath$ may be empty and they are not necessary connected.

Lemma 4.3.11. *There are at most* $\frac{2m}{M} 2e^{\frac{2m+M}{2+M}}$ *non-empty disjoint subsets* $\Delta_\imath \subset \square$*.*

Proof. We know the total number of sequences that correspond to the points with $t_{\text{ex}} \equiv s$:

$$\#\{\imath \in \{0,1\}^{\mathbb{N}} \mid t_{\text{ex}}(\imath) = s\} = \binom{2m-(s-1)M}{s}.$$

Observe that the number of disjoint subsets $\Delta_\imath \subset \Delta$ is equal to the number of different sequences, which we can estimate in the following way. It is well known that $\binom{2n}{n} > \binom{k}{s}$ for all $1 \le k \le 2n$ and $1 \le s \le k$. The equality $2m - (s-1)M = 2s$ has the solution $s_0 = \frac{2m+M}{2+M}$ so we conclude $\binom{2m-(s-1)M}{s} \le \binom{2s_0}{s_0}$ for all $s > s_0 = \frac{2m+M}{2+M}$. Using the Stirling formula, we calculate

$$\binom{2s_0}{s_0} \le \text{const} \cdot \frac{(2s_0)^{2s_0}}{s_0^{2s_0}} = \text{const} \cdot 2^{2s_0} = \text{const} \cdot 2^{\frac{4m+2M}{2+M}}$$

We also may write for all $s < s_0$

$$\binom{2m-(s-1)M}{s} = \frac{(2m-(s-1)M)!}{s!(2m-(s-1)M-s)!} \le (2m-(s-1)M)^s \Big(\frac{e}{s}\Big)^s.$$

By straightforward calculation

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{(2m-(s-1)M)e}{s}\Big)^s = \\ = \Big(\frac{(2m-(s-1)M)e}{s}\Big)^s \cdot \Big(\ln\frac{2m-(s-1)M}{s} - \frac{s}{2m-(s-1)M}\Big) > 0$$

for all $s \in (1; s_0)$, because

$$\ln\frac{2m-(s-1)M}{s} > \ln\frac{2m-(s_0-1)M}{s_0} = \ln 2 > \\ > \frac{1}{2} = \frac{s_0}{2m-(s_0-1)M} > \frac{s}{2m-(s-1)M}.$$

We conclude that for $s < s_0$

$$\binom{2m-(s-1)M}{s} \le (2s_0)^{s_0}\frac{e^{s_0}}{s_0^{s_0}} = (2e)^{\frac{2m+M}{2+M}}.$$

Summing up,

$$\sum_{j=1}^{\frac{m}{M}} \binom{2m-(s-1)M}{s} \le \frac{m}{M}(2e)^{\frac{2m+M}{2+M}}.$$

■

Given a sequence $\jmath \in \{-1,0,1\}^{\mathbb{N}}$ we define a subset of the unit square

$$\Delta_\jmath \stackrel{\text{def}}{=} \left\{ z \in \square \colon \chi_\square(F_\xi^n(z)) \cdot \operatorname{sgn} \pi_y(F_\xi^n(z)) = \jmath_n \text{ for all } n \in \{0,1,\ldots,2m\} \right\}.$$

Note that some of $\Delta_\jmath$ may be empty, and they are not necessary connected.

Definition 23. We introduce to projections of the tower to the zero floor:

$$\pi_x \colon X \to X \qquad \pi_x(z,n) = ((z_x,0),0);$$

$$\pi_y \colon X \to X \qquad \pi_y(z,n) = ((0,z_y),0).$$

Lemma 4.3.12. *Given a quartet (i,j,k,l) and a subset $B_\imath \stackrel{\text{def}}{=} \Delta_\imath \cap \Omega^1_{ij} \cap P_\xi^{-2}(\Omega^2_{kl})$, there are at most $6^{\frac{2m}{M}}$ disjoint subsets $\Delta_\jmath$ such that $\Delta_\jmath \cap B_\imath \neq \varnothing$.*

Proof. Consider a first half of the sequence $\imath$ of the length m, the subsequence $\imath_1, \imath_2, \ldots, \imath_m$. It may contain not more than $\frac{m}{M}$ "blocks" of 1's. We shall show by induction in number of blocks that

1. There are not more than $6^{\frac{m}{M}}$ different sequences $\jmath_1, \ldots, \jmath_m$ such that $\Delta_\jmath \cap B_\imath \neq \varnothing$.
2. The projection of the image $\pi_y(P_\xi(B_\imath))$ may be covered by not more than $6^{\frac{m}{M}}$ intervals of the total length not more than 2.

In order to use induction, we need to study the original map $F\colon X \to X$ of the tower X defined on p. 98; we also recall that by definition $P_\xi = F_\xi^m \colon \mathbb{R}^2 \to \mathbb{R}^2$.

Given a sequence $\imath$, there are two possibilities.

Case 1. All blocks of 1's in $\imath$ are not longer than $m\alpha - 1$.

Case 2. There are blocks of 1's in $\imath$ of the length $m\alpha$ or longer.

Case 1. Assume that all blocks of 1's in the sequence $\imath$ are not longer than $m\alpha - 1$.

The base of induction. Assume that there is only one block of 1's. Then there exist two numbers $1 \leq t_1 \leq s_1 \leq m$, $s_1 - t_1 \leq m\alpha$:

$$\imath_k = \begin{cases} 1, & \text{if } t_1 \leq k \leq s_1; \\ 0, & \text{otherwise.} \end{cases}$$

We deduce that $\pi_y(P_\xi^{t_1-1}(\Delta_\imath))$ belongs to a union of domains of main branches of the perturbation $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ of the doubling map p. We know by Lemma 3.2.7 that the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ has exactly $2^{s_1-t_1}$ main branches, all of them are long and their domains have the length at least $2^{t_1-s_1} > 2^{-m\alpha}$. In addition, since

$$\operatorname{diam}(B_\imath) = \operatorname{diam}(\Delta_\imath \cap \Omega^1_{ij} \cap P_\xi^{-2}(\Omega^2_{kl})) \leq \operatorname{diam}(\Omega^1_{ij}) < 2^{2-m}$$

we conclude that there exists an interval $I \subset [-1;1]$ such that $\pi_y(F_\xi^{t_1}(B_\imath)) \subset I$ and[1] the length $|I| < 2^{2-m} \cdot (1+\mu_1)^{t_1-1} < 2^{-m\alpha} < 2^{t_1-s_1}$. Thus the interval I may intersect not more than 2 domains of main branches of the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ and therefore there are not more than 4 sequences $\jmath_k$, $1 \leq k \leq m$ corresponding to the sequence $\imath_k$, $1 \leq k \leq m$. In addition, we observe that the image $\pi_y(F_\xi^{s_1}(B_\imath))$ may be covered by 4 intervals of the total length not more than $2^{-m} \cdot 2^{s_1-t_1} \cdot (1+\mu_1)^m$.

Now assume that there are n blocks of 1's. Namely, there exist

$$1 \leq t_1 \leq s_1 < t_2 \leq s_2 < \ldots < t_n \leq s_n \leq m \tag{4.40}$$

[1] We may safely assume that $2^\alpha > 1 + \mu_1$.

such that $t_{i+1} - s_i \geq M$ and $s_i - t_i \leq m\alpha - 1$, where

$$\imath_k = \begin{cases} 1, & \text{if } t_i \leq k \leq s_i, \text{ for } i = 1, \dots, n; \\ 0, & \text{otherwise }. \end{cases} \tag{4.41}$$

Since $s_n - t_n < m\alpha$, by Lemma 3.2.7 the doubling map $p^{s_n - t_n}_{\sigma^{t_n - 1}\xi}$ has exactly $2^{s_n - t_n}$ main branches, all of which are long, and their domains have length at least $2^{t_n - s_n}$. By induction assumption, the set $\pi_y(F^{t_n - 1}_{\xi}(B_\imath))$ may be covered by 4^{n-1} intervals of the total length

$$2^{-m} \cdot \prod_{k=1}^{n-1} 2^{s_k - t_k} \cdot (1 + \mu_1)^m \leq 2^{-m} \cdot 2^{m - (s_n - t_n) - M(n-1)} = 2^{t_n - s_n} \cdot 2^{-M(n-1)}.$$

Therefore it may intersect not more than $\min(2 \cdot 4^{n-1}, 2^{s_n - t_n})$ domains of the main branches of the map $p^{s_n - t_n}_{\sigma^{t_n - 1}\xi}$. Consequently, there are at most 4^n different sequences $\jmath$ of the length m and the projection of the image $\pi_y(F^{s_n}_{\xi}(B_\imath))$ may be covered by 4^n intervals of the total length $2^{-m} \cdot \prod_{k=1}^{n} 2^{s_k - t_k} \cdot (1 + \mu_1)^m \leq 2^{-M(n-1)}(1 + \mu_1)^m$.

Case 2. There exists a subsequence of 1's of the length $m\alpha$ or longer. Then there is only one subsequence with this property (since $\alpha > \frac{15}{16}$). There are two possibilities.

(2A) In the notations introduced in (4.40) and (4.41) above, $s_1 - t_1 > m\alpha$.

(2B) In the notations introduced in (4.40) and (4.41) above, $s_n - t_n > m\alpha$ for some $n > 1$.

In the case 2A, the map $p^{s_1 - t_1}_{\sigma^{t_1 - 1}\xi}$ has at least $2^{s_1 - t_1 - 2}$ long branches, and their domains have length at least $2^{t_1 - s_1}$. At the same time the projection of the image $\pi_y(F^{t_1 - 1}_{\xi}(B_\imath))$ is contained in an interval I of the length $|I| < 2^{-m} \cdot (1 + \mu_1)^{t_1} < 2^{t_1 - s_1}$. By Lemma 3.2.7, the distance between any two domains of the main branches of the map $p^{s_1 - t_1}_{\sigma^{s_1 - 1}\xi}$ which are not long, is at least $2^{m(\alpha - 1)} > 2^{t_1 - s_1}$. Therefore the interval I may intersect not more than three domains of main branches (two long and one more) of the map $p^{s_1 - t_1}_{\sigma^{t_1 - 1}\xi}$. Thus we conclude that there are not more than 6 different sequences $\jmath_{t_1}, \dots \jmath_{s_1}$, corresponding to the sequence $\imath_{t_1}, \dots, \imath_{s_1}$. The induction step then follows as above, giving $6^{\frac{m}{M}}$ sequences.

In the case 2B, the map $p^{s_n - t_n}_{\sigma^{t_n - 1}\xi}$ has at least $2^{s_n - t_n - 2}$ long branches, and their domains have length at least $2^{t_n - s_n}$. Then by induction from the Case 1, we know that there are $4^{n-1} < 4^{m(1-\alpha)-M}$ sequences corresponding to the sequence $\imath_1, \dots, \imath_{t_n - 1}$ and the image of the set $\pi_y(P^{t_n-1}_{\xi}(B_\imath))$ may be covered by 4^{n-1} intervals of the total length not more than $2^{t_n - s_n - M}$. We see that the total number of long branches of the map $p^{s_n - t_n}_{\sigma^{t_n - 1}\xi}$ is greater than the number of intervals covering the image

$$2^{m\alpha - 2} > 4^{m(1-\alpha)-M},$$

and the total length of intervals is shorter than a domain of any long branch. Therefore, each of the intervals may intersect not more than three domains of main branches, and we get at most $6 \cdot 4^{k-1}$ different sequences. In addition, we notice that the image $\pi_y(F^{s_n}_{\xi}(B_\imath))$ may be covered by $6 \cdot 4^{k-1}$ intervals.

To complete the proof of the Lemma, we need to calculate number of different sequences $\jmath_{m+1}, \dots, \jmath_{2m}$ such that $\Delta_\jmath \cap B_\imath \neq \varnothing$. We would like to apply the argument above to the inverse map $F^{-m}_{\sigma^m \xi} = P^{-1}_{\sigma^m \xi}$. Let us consider the image $P_\xi(B_\jmath) \subset \Omega^2_{kl}$. Define a sequence $\jmath'$, associated to the iterations of the inverse map $P_{\sigma^m}\xi^{-1}$.

$$\jmath' \colon z \to \{-1, 0, 1\}^{\mathbb{N}} \qquad \jmath'_k(z) = \begin{cases} 1, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \in \square + \xi^{2m+1-k}_x, \; z_x > \xi^{2m-k}_x; \\ -1, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \in \square + \xi^{2m+1-k}_x, \; z_x < \xi^{2m-k}_x; \\ 0, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \notin \square + \xi^{2m+1-k}_x. \end{cases} \tag{4.42}$$

We see that

$$\jmath'_k(P^2_\xi z) = \jmath_{2m-k+1}(z) \text{ for all } 0 \leq k \leq m.$$

We may associate the sequence $\jmath'$ to main branches of the doubling map p_{ς_x}, defined as in Lemma 4.2.1 p. 101, in the following way.

$$\left\{\jmath'_k \equiv 1,\ \text{for } 0 \le t_1 \le k \le t_2 \le m,\ t_1 < t_2\right\} \Longleftrightarrow$$
$$\left\{\pi_x(F^{-t_1}_{\sigma^{2m-t_1}}(z)) \text{ in a domain of a main branch of } p^{t_2-t_1-1}_{\sigma^{t_1}\varsigma_x}\right\}.$$

Indeed, if, say, $\jmath'_{t_1} = 1$, then by definition, $F^{-t_1+1}_{\sigma^{2m-t_1}\xi}(z) \in \square + \xi_x^{2m+1-t_1}$ and $z_x > \xi_x^{2m-t_1}$. Consequently, $F^{-l}_{\sigma^{2m-l-1}\xi}(z) \in \square$ for all $t_1 \le l \le t_2$, and therefore $\pi_x(P^{-t_1}_{\sigma^{2m-t_1-1}\xi}(z))$ is in a domain of a long branch of $p^{t_2-t_1}_{\sigma^{2m-l-1}\varsigma_x}$.

In the case $t_1 = t_2 = 1$, i.e. a block of the length 1, we get two sequences corresponding to a given $\jmath_{t_1} = 1$ and $\jmath = -1$, similarly to the previous case.

It follows that to any sequence $\imath$ of the length $2m$ correspond $6^{2m/M}$ sequences $\jmath$.

■

Corollary 1. *Among all sequences $\jmath$, there are at most $\frac{2m}{M} \cdot 6^{\frac{2m}{M}} (2e)^{\frac{2m+M}{2+M}}$ pairwise disjoint segments $\Delta_\jmath$ such that $P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij} \cap \Delta_\jmath \neq \varnothing$.*

Now we are ready to prove

Proposition 4.3.2. There exist a constant $\gamma_1 < 0.01$ such that for M and m sufficiently large

$$\max(\|UU\|_\infty, \|SU\|_\infty, \|US\|_\infty, \|SS\|_\infty) \le 2^{\gamma_1 m}.$$

Proof. Recall the definition of the matrices, for instance

$$UU^{kl}_{ij} = \frac{1}{|\pi_x(\Omega^1_{ij})|} \cdot \frac{1}{|\pi_y(\Omega^2_{kl})|} \int_{P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}} \partial_y (P^2_\xi)_y(z) \mathrm{d}z$$

and the other three are defined using another three partial derivatives, according to (4.18)–(4.20). Consider a vertical line segment $\Delta_c = \{z_x = c\} \cap P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}$. Recall that according to condition 5 the composition of maps outside of the unit square $F_{i_1} \circ \ldots \circ F_{i_M}$, where $i_1, \ldots, i_M \in \{1, \ldots, M\}$ is a polynomial of degree at most d. Since P_ξ^2 is smooth on each $\Delta_\jmath \cap \Delta_c$ and $P_\xi^2(\Delta_\jmath \cap \Delta_c) \subset \Omega_{kl}^2$. We can estimate the length of the image using condition 5, p. 97:

$$|P_\xi^2(\Delta_\jmath \cap \Delta_c)| \leq \mathrm{diam}(\Omega_{kl}^2) \cdot d^{\frac{2m}{M}} \leq 2^{\frac{m}{500}}; \tag{4.43}$$

since the preimage with respect to any of the orthogonal projections π_x and π_y has at most $d^{\frac{2m}{M}}$ connected components.

$$\max\Big(\int_{\Delta_\jmath} |\partial_y (P_\xi^2)_y(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_x (P_\xi^2)_y(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_y (P_\xi^2)_x(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_x (P_\xi^2)_x(z)| \mathrm{d}z\Big) \leq$$
$$\leq \big|P_\xi^2(\Delta_\jmath \cap \Delta_c)\big| \leq d^{\frac{2m}{M}} \mathrm{diam}(\Omega_{kl}^2).$$

Therefore

$$\int_{P_\xi^{-2}(\Omega_{kl}^2) \cap \Omega_{ij}^1} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z =$$
$$= \int_{\pi_y(\Omega_{ij}^1)} \int_{\Delta_c} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \mathrm{d}c =$$
$$= \int_{\pi_y(\Omega_{ij}^1)} \sum_{\Delta_\jmath \subset \Delta_c} \int_{\Delta_\jmath} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \mathrm{d}c \leq$$
$$\leq \frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot \mathrm{diam}(\Omega_{kl}^2) \cdot d^{\frac{2m}{M}} \cdot |\pi_y(\Omega_{ij}^1)|.$$

Finally,

$$\int_{P_\xi^{-2}(\Omega_{ij}^1) \cap \Omega_{kl}^2} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \leq$$
$$\leq |\pi_x(\Omega_{ij}^1)| \cdot |(\pi_y(\Omega_{kl}^2)| \cdot \frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot d^{\frac{2m}{M}}.$$

We can choose μ_1 and μ_2 sufficiently small so that for m and M large enough and for some $\gamma_1 \leq 0.01$

$$\frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot d^{\frac{2m}{M}} \leq 2^{\gamma_1 m}.$$

■

Lemma 4.3.13. *The sum of elements of the matrix* $|(UU^B)^{kl}_{ij}|$ *with* $(i,j,k,l) \in \Box \times \Box$ *is at most* $2^{2m} \cdot 8m\delta$.

Proof. Indeed, recall that for any $\Delta \subset \Delta^B_\xi$ there exists $1 \le n \le 2m$ such that $F^n_\xi(\Delta) \not\subset \Box$ and thus

$$\bigcup_{ij}\bigcup_{kl}\bigcup_{(\Delta^B)^{kl}_{ij}} \Delta = \bigcup_{ij}\bigcup_{kl}\Big\{\Delta \text{ is a } (P,\xi)\text{-domain}\big| \, F^n_\xi(\Delta) \not\subset \Box \text{ for some } 1 \le n \le 2m\Big\} =$$

$$= \{z \in \Box \mid \exists 1 \le n \le 2m \colon F^n_\xi(z) \notin \Box\} =: B.$$

We get $|B| \le 8m\delta$ by induction in number of iterations and conclude

$$\sum_{ij}\sum_{kl}|(UU^B)^{kl}_{ij}| \le \int_B |\partial_y(P^2_\xi)_y(z)|\mathrm{d}z \le 2^{2m} \cdot 8m\delta.$$

■

Remark 14. It follows from the condition 3 on the map F (see p. 97) that partial derivatives are essentially bounded $\|\partial_y(P^2_\xi)_x\|_\infty \le (1+\mu)^{2m}$, $\|\partial_x(P^2_\xi)_y\|_\infty \le (1+\mu)^{2m}$, and, finally, $\|\partial_x(P^2_\xi)_x\|_\infty \le (1+\mu)^{2m}$. Thus by the same argument as in Lemma 4.3.13 we get

$$\sum_{\Box}\sum_{\Box}|(US^B)^{kl}_{ij}| \le (1+\mu)^{2m} m\delta; \tag{4.44}$$

$$\sum_{\Box}\sum_{\Box}|(SU^B)^{kl}_{ij}| \le (1+\mu)^{2m} m\delta; \tag{4.45}$$

$$\sum_{\Box}\sum_{\Box}|(SS^B)^{kl}_{ij}| \le (1+\mu)^{2m} m\delta. \tag{4.46}$$

4.3.2 The operators $W_\delta\mathcal{A}$ and $W_\delta P^2_{\xi*}$ are close on $\mathfrak{X}$

We keep the notation introduced in the first paragraph of this Section.

Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. Let $\xi \overset{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (cf. Definition of the chain Υ in subsection 4.2.3, p. 101). Let $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$ be a linear operator, approximating the operator $P^2_{\xi *}$, defined according to (4.17).

In this section we establish the following

Proposition 4.3.3. *The operators $W_\delta \mathcal{A}$ and $W_\delta P^2_{\xi *}$ are close. Namely, for any $\nu \in \mathfrak{X}_{\Omega^1}$,*

$$\|W_\delta(P^2_{\xi *} - \mathcal{A})\nu\|_{\Omega^2,\mathcal{L}_1} \le \frac{4\sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{2m}, \tag{4.47}$$

$$\|W_\delta(P^2_{\xi *} - \mathcal{A})\nu\|_\infty \le \frac{4\sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{(2+\gamma_1)m}; \tag{4.48}$$

where γ_1 is defined by Proposition 4.3.2.

We start with

Lemma 4.3.14. *For any element Ω^2_{kl} of the partition Ω^2, and for any $\nu \in \mathfrak{X}_{\Omega^1}$,*

$$\int_{\Omega^2_{kl}} P^2_{\xi *}\nu = \int_{\Omega^2_{kl}} \mathcal{A}\nu.$$

Proof. Let $\nu = \sum_{ij} \nu^{ij}_s \chi^s_{\Omega^1_{ij}} + \sum_{ij} \nu^{ij}_u \chi^u_{\Omega^1_{ij}}$. Then

$$\begin{aligned}
P^2_{\xi *}\nu(z) &= \mathrm{d}P^2_\xi(P^{-2}_\xi z) \cdot \nu(P^{-2}_\xi z) = \\
&= \sum_{ij} \nu^{ij}_s \mathrm{d}P^2_\xi(P^{-2}_\xi z) \chi^s_{\Omega^1_{ij}}(P^{-2}_\xi z) + \sum_{ij} \nu^{ij}_u \mathrm{d}P^2_\xi(P^{-2}_\xi z) \chi^u_{\Omega^1_{ij}}(P^{-2}_\xi z) = \\
&= \sum_{ij} \nu^{ij}_s \big(\partial_x (P^2_\xi)_x(P^{-2}_\xi z) + \partial_x (P^2_\xi)_y(P^{-2}_\xi z)\big) \cdot \frac{1}{|\pi_x(\Omega^1_{ij})|} \chi_{\Omega^1_{ij}}(P^{-2}_\xi z) + \\
&\quad + \sum_{ij} \nu^{ij}_u \big(\partial_y (P^2_\xi)_x(P^{-2}_\xi z) + \partial_y (P^2_\xi)_y(P^{-2}_\xi z)\big) \cdot \frac{1}{|\pi_x(\Omega^1_{ij})|} \chi_{\Omega^1_{ij}}(P^{-2}_\xi z).
\end{aligned}$$

We may integrate

$$\begin{aligned}
&\frac{1}{|\Omega^2_{kl}|} \int_{\Omega^2_{kl}} \partial_x (P^2_\xi)_x(P^{-2}_\xi z) \cdot \frac{1}{|\pi_x(\Omega^1_{ij})|} \cdot \chi_{\Omega^1_{ij}}(P^{-2}_\xi z) \mathrm{d}z = \\
&\quad = \frac{1}{|\pi_x(\Omega^2_{kl})|} \cdot \frac{1}{|\pi_y(\Omega^2_{kl})|} \cdot \frac{1}{|\pi_x(\Omega^1_{ij})|} \int_{P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}} \partial_x (P^2_\xi)_x(z) \mathrm{d}z = \frac{1}{|\pi_x(\Omega^2_{kl})|} SS^{kl}_{ij}.
\end{aligned}$$

Similarly,

$$\frac{1}{|\pi_x(\Omega^2_{kl})|}US^{kl}_{ij} = \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_y(P^2_\xi)_x(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z;$$
$$\frac{1}{|\pi_x(\Omega^2_{kl})|}SU^{kl}_{ij} = \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_x(P^2_\xi)_y(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z;$$
$$\frac{1}{|\pi_x(\Omega^2_{kl})|}UU^{kl}_{ij} = \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_y(P^2_\xi)_y(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z.$$

So we may write

$$\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} P^2_{\xi *}\nu(z)\mathrm{d}z = \sum_{ij}\nu^{ij}_s\frac{1}{|\pi_x(\Omega^2_{kl})|}(SS^{kl}_{ij}+US^{kl}_{ij}) + \sum_{ij}\nu^{ij}_u\frac{1}{|\pi_x(\Omega^2_{kl})|}(SU^{kl}_{ij}+UU^{kl}_{ij}).$$

Observe that for any Ω^2_{kl}, by definition of the operator $\mathcal{A}$ (4.17) on p. 104,

$$\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \mathcal{A}\nu = \frac{1}{|\pi_x(\Omega^2_{kl})|}\Big(\sum_{ij}\nu^{ij}_s(SS^{kl}_{ij}+US^{kl}_{ij}) + \sum_{ij}\nu^{ij}_u(SU^{kl}_{ij}+UU^{kl}_{ij})\Big).$$

■

Lemma 4.3.15. *For any partition Ω of the plane $\mathbb{R}^2$ into rectangles we have*

$$\int_{\mathbb{R}^2}\big|\max_{t\in\Omega_{ij}} w_\delta(z-t) - \min_{t\in\Omega_{ij}} w_\delta(z-t)\big|\mathrm{d}z \le \frac{4\sup\mathrm{diam}(\Omega_{ij})}{\pi\delta}.$$

Proof. Given a compact convex subset $A\subset\mathbb{R}^2$, let $\gamma(A)$ be the longest line segment connecting the points where the function $w_\delta(t)$ achieves its maximum and minimum in A. By straightforward calculation

$$\max_{t\in\Omega_{ij}} w_\delta(z-t) - \min_{t\in\Omega_{ij}} w_\delta(z-t) = \max_{t\in\Omega_{ij}-z} w_\delta(t) - \min_{t\in\Omega_{ij}-z} w_\delta(t) \le \int_{\gamma(\Omega_{ij}-z)} |\nabla w_\delta(t)|\mathrm{d}t.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^2}\big|\max_{\Omega_{ij}} w_\delta(z-t)-\min_{t\in\Omega_{ij}} w_\delta(z-t)\big|\mathrm{d}z &\le \int_{\mathbb{R}^2}\int_{\gamma(\Omega_{ij}-z)}\big|\nabla w_\delta(t)\big|\mathrm{d}t\mathrm{d}z = \\ &= \int_{\mathbb{R}^2}\int_{\gamma(\Omega_{ij})}\big|\nabla w_\delta(t-z)\big|\mathrm{d}t\mathrm{d}z = \int_{\gamma(\Omega_{ij})}\int_{\mathbb{R}^2}\big|\nabla w_\delta(t-z)\big|\mathrm{d}z\mathrm{d}t = \\ &= \int_{\gamma(\Omega_{ij})}\int_{\mathbb{R}^2}\big|\nabla w_\delta(z)\big|\mathrm{d}z\mathrm{d}t \le \operatorname{diam}(\Omega_{ij})\int_{\mathbb{R}^2}\big|\nabla w_\delta(z)\big|\mathrm{d}z = \\ &= \operatorname{diam}(\Omega_{ij})\int_{\mathbb{R}^2}\frac{1}{\pi^2\delta^4}\sqrt{z_x^2+z_y^2}\cdot e^{\frac{-z_x^2-z_y^2}{2\delta^2}}\mathrm{d}z \le \\ &\le \operatorname{diam}(\Omega_{ij})\int_{\mathbb{R}^2}\frac{1}{\pi^2\delta^4}(|z_x|+|z_y|)\cdot e^{\frac{-z_x^2-z_y^2}{2\delta^2}}\mathrm{d}z \le \\ &\le \operatorname{diam}(\Omega_{ij})\Big(\int_{\mathbb{R}}\frac{|z_x|}{\pi^2\delta^3}e^{-\frac{z_x^2}{2\delta^2}}\mathrm{d}z_x+\int_{\mathbb{R}}\frac{|z_y|}{\pi^2\delta^3}e^{-\frac{z_y^2}{2\delta^2}}\mathrm{d}z_y\Big) \le \\ &\le \frac{4\operatorname{diam}(\Omega_{ij})}{\pi\delta}.\end{aligned}$$

■

Lemma 4.3.16. *Let $f\colon \mathbb{R}^2\to\mathbb{R}^2$ be a bounded integrable function. Assume that for any element Ω^1_{ij} of a partition Ω^1 of the class (m,δ) we have $\int_{\Omega_{ij}} f\equiv 0$. Then for any partition Ω^2 of the class (m,δ)*

$$\|W_\delta f\|_{\Omega^2,\mathcal{L}_1}\le 8\frac{\sup\operatorname{diam}(\Omega^1_{ij})}{\delta}\|f\|_{\Omega^1,\mathcal{L}_1}; \tag{4.3.16.1}$$

$$\|W_\delta f\|_\infty\le 8\frac{\sup\operatorname{diam}(\Omega^1_{ij})}{\delta}\|f\|_\infty. \tag{4.3.16.2}$$

Proof. By straightforward calculation

$$\|W_\delta f\|_{\mathcal{L}_1}=\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t\Big|\mathrm{d}z=\int_{\mathbb{R}^2}\Big|\sum_{ij}\int_{z-\Omega_{ij}} w_\delta(t)f(z-t)\mathrm{d}t\Big|\mathrm{d}z.$$

We recall $\int_{z-\Omega_{ij}} f(z-t)\mathrm{d}t=\int_{\Omega_{ij}} f(t)\mathrm{d}t=0$ and so $\int_{z-\Omega_{ij}} f(z-t)\int_{z-\Omega_{ij}} w_\delta(s)\mathrm{d}s\mathrm{d}t=0$.

Hence we conclude

$$\begin{aligned}
\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\Big|\sum_{ij}\int_{z-\Omega^1_{ij}}\Big(w_\delta(t)-\frac{1}{|\Omega^1_{ij}|}\int_{z-\Omega^1_{ij}}w_\delta(s)\mathrm{d}s\Big)f(z-t)\mathrm{d}t\Big|\mathrm{d}z\leq\\
&\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\int_{z-\Omega^1_{ij}}\Big|w_\delta(t)-\frac{1}{|\Omega^1_{ij}|}\int_{z-\Omega^1_{ij}}w_\delta(s)\mathrm{d}s\Big|\cdot\big|f(z-t)\big|\mathrm{d}t\mathrm{d}z\leq\\
&\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\int_{z-\Omega^1_{ij}}\big|\max_{s\in z-\Omega^1_{ij}}w_\delta(s)-\min_{s\in z-\Omega^1_{ij}}w_\delta(s)\big|\cdot|f(z-t)|\mathrm{d}t\mathrm{d}z\leq\\
&\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\big|\max_{s\in z-\Omega^1_{ij}}w_\delta(s)-\min_{s\in z-\Omega^1_{ij}}w_\delta(s)\big|\cdot\int_{z-\Omega^1_{ij}}|f(z-t)|\mathrm{d}t\mathrm{d}z=\\
&= \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\big|\max_{s\in z-\Omega^1_{ij}}w_\delta(s)-\min_{s\in z-\Omega^1_{ij}}w_\delta(s)\big|\cdot\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t\mathrm{d}z=\\
&\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\sum_{ij}\int_{\mathbb{R}^2}\big|\max_{s\in z-\Omega^1_{ij}}w_\delta(s)-\min_{s\in z-\Omega^1_{ij}}w_\delta(s)\big|\cdot\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t\mathrm{d}z\leq\\
&\leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\sum_{ij}\frac{4\mathrm{diam}(\Omega^1_{ij})}{\pi\delta}\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t\leq\\
&\leq \frac{\sup|\pi_y(\Omega^1_{ij})|}{\inf|\pi_y(\Omega^2_{kl})|}\cdot\frac{4\sup\mathrm{diam}(\Omega^1_{ij})}{\pi\delta}\|f\|_{\Omega^1,\mathcal{L}_1},
\end{aligned}$$

by Lemma 4.3.15.

Similarly for the supremum norm

$$\begin{aligned}
\sup|W_\delta f| &= \sup_z\Big|\int_{\mathbb{R}^2}w_\delta(z-t)f(t)\mathrm{d}t\Big|\leq\sup_z\Big|\sum_{ij}\int_{\Omega^1_{ij}}w_\delta(z-t)f(t)\mathrm{d}t\Big|=\\
&=\sup_z\Big|\sum_{ij}\Big(\int_{\Omega^1_{ij}}w_\delta(z-t)-\frac{1}{|\Omega^1_{ij}|}\int_{\Omega^1_{ij}}w_\delta(z-s)\mathrm{d}s\Big)f(t)\mathrm{d}t\Big|\leq\\
&\leq\sup_z\sum_{ij}\int_{\Omega^1_{ij}}\big|\max_{t\in\Omega^1_{ij}}w_\delta(z-t)-\min_{t\in\Omega^1_{ij}}w_\delta(z-t)\big|\cdot|f(t)|\mathrm{d}t\leq\\
&\leq\sup|f|\sup_z\sum_{ij}|\Omega^1_{ij}|\cdot\big|\max_{t\in\Omega^1_{ij}}w_\delta(z-t)-\min_{t\in\Omega^1_{ij}}w_\delta(z-t)\big|\leq\\
&\leq\sup|f|\sup_z\sum_{ij}|\Omega^1_{ij}|\cdot\sup_{t\in\Omega^1_{ij}}|\nabla_z w_\delta(z-t)|\cdot\mathrm{diam}(\Omega^1_{ij})\leq\\
&\leq\sup|f|\sup\mathrm{diam}(\Omega^1_{ij})\cdot\int_{\mathbb{R}^2}|\nabla_z w_\delta(z)|\mathrm{d}z\leq\frac{\sup\mathrm{diam}(\Omega^1_{ij})}{\delta}\cdot\sup|f|.
\end{aligned}$$

■

Lemma 4.3.17. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(m,\delta)$. Then for any sequence $\xi \in \ell_\infty(\mathbb{R}^2)$ we have*

$$\|(P^2_{\xi*}\nu)\|_2 \leq 2^{2m+2}\|\nu\|_1,$$

Proof. Upper bound for the supremum norm is obvious. Indeed, we have for the first coordinate

$$\begin{aligned}\|(P^2_{\xi*}\nu)_y\|_{\mathcal{L}_1} &= \int_{\mathbb{R}^2}\left|(P^2_{\xi*}\nu)_y(z)\right|\mathrm{d}z = \\ &= \int_{\mathbb{R}^2}\left|\partial_x(P^2_\xi)_y(P^{-2}_\xi z)\nu_s(P^{-2}_\xi z) + \partial_y(P^2_\xi)_y(P^{-2}_\xi z)\nu_u(P^{-2}_\xi z)\right|\mathrm{d}z = \\ = \int_{\mathbb{R}^2}&\left|\partial_x(P^2_\xi)_y(z)\nu_s(z) + \partial_y(P^2_\xi)_y(z)\nu_u(z)\right|\mathrm{d}z \leq 2^{2m+1}\int_{\mathbb{R}^2}|\nu_s(z)| + |\nu_u(z)|\mathrm{d}z.\end{aligned} \tag{4.49}$$

For the second coordinate we have got

$$\begin{aligned}\|(P^2_{\xi*}\nu)_x\|_{\mathcal{L}_1} &= \int_{\mathbb{R}^2}\left|(P^2_{\xi*}\nu)_x(z)\right|\mathrm{d}z = \\ &= \int_{\mathbb{R}^2}\left|\partial_x(P^2_\xi)_x(P^{-2}_\xi z)\nu_s(P^{-2}_\xi z) + \partial_y(P^2_\xi)_x(P^{-2}_\xi z)\nu_u(P^{-2}_\xi z)\right|\mathrm{d}z = \\ = \int_{\mathbb{R}^2}&\left|\partial_x(P^2_\xi)_x(z)\nu_s(z) + \mathrm{d}_y(P^2_\xi)_x(z)\nu_u(z)\right|\mathrm{d}z \leq 2^{2m+1}\int_{\mathbb{R}^2}|\nu_s(z)| + |\nu_u(z)|\mathrm{d}z.\end{aligned} \tag{4.50}$$

Therefore

$$\|P^2_{\xi*}\nu\|_2 = \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega_{ij}}\left|P^2_{\xi*}\nu(z)\right|\mathrm{d}z \leq \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{ij})|}\|P^2_{\xi*}\nu\|_{\mathcal{L}_1} \leq m2^{2m+1}\|\nu\|_1.$$

■

Lemma 4.3.18. *In the notations introduced in the beginning of this subsection 4.3.2, p. 127, the following inequalities on the norm of operators hold true for M and m large enough.*

$$\|UU\nu_u\|_{\Omega^2,\mathcal{L}_1} \leq 4\cdot 2^{2m}\|\nu\|_1, \tag{4.3.18.1}$$

$$\max(\|SU\nu_u\|_{\Omega^2,\mathcal{L}_1}, \|US\nu_s\|_{\Omega^2,\mathcal{L}_1}, \|SS\nu_s\|_{\Omega^2,\mathcal{L}_1}) \leq 2^{\gamma_2 m}\|\nu\|_1; \tag{4.3.18.2}$$

where the constant γ_2 satisfies

$$1 < \gamma_2 = \frac{9}{4} + \gamma_1 + 2\log_2(1+\mu_1) - \alpha < \frac{3}{2}. \tag{4.51}$$

Proof. Let $\nu_u = \sum_{ij} \nu_u^{ij} \chi^u_{\Omega^1_{ij}} \in \Phi_{\Omega^1}$ be the y-component of a field with the unit norm

$$\|\nu_u\| = \max\Big(\sum_{ij} |\nu_u^{ij}| \cdot |\pi_y(\Omega_{ij})|, 2^{\frac{3}{4}m} \sup |\nu_u^{ij}|\Big) = 1,$$

therefore we will be assuming that $\sum_{ij} |\nu_u^{ij}| \leq 2^{m-1}$ and $\sup |\nu_u^{ij}| \leq 2^{-\frac{3}{4}m}$. We write down the formal action of the operator UU on ν_u

$$UU\nu_u = \sum_{kl}\sum_{ij} UU^{kl}_{ij} \nu_u^{ij} \chi^u_{\Omega^2_{kl}} = \sum_{\square}\sum_{\square}(UU^{kl}_{ij} - 1)\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{\square}\sum_{\square} \nu_u^{ij}\chi^u_{\Omega^2_{kl}} +$$
$$+ \Big(\sum_{\mathbb{R}^2\backslash\square}\sum_{\square} + \sum_{\square}\sum_{\mathbb{R}^2\backslash\square}\Big) UU^{kl}_{ij}\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{\mathbb{R}^2\backslash\square}\sum_{\mathbb{R}^2\backslash\square} UU^{kl}_{ij}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}. \tag{4.52}$$

We estimate the norm of each of the four terms separately. Recall that by the choice of the basis $\chi^u_{\Omega^2_{kl}} = \frac{1}{|\pi_x(\Omega^2_{kl})|}\chi_{\Omega^2_{kl}}\binom{0}{1}$ and therefore

$$\|\chi^u_{\Omega^2_{kl}}\|_{\Omega^2,\mathcal{L}_1} = \frac{2^{-m}}{|\pi_y(\Omega^2_{kl})|}\int_{\Omega^2_{kl}} \chi^u_{\Omega^2_{kl}} = 2^{-m}.$$

$$\Big\|\sum_{\square}\sum_{\square}(UU^{kl}_{ij} - 1)\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = \sum_{\square}\Big|\sum_{\square}(UU^{kl}_{ij} - 1)\nu_u^{ij}\Big| \cdot 2^{-m} \leq$$
$$\leq 2^{-m}\sum_{\square}\sum_{\square}|UU^{kl}_{ij} - 1| \cdot |\nu_u^{ij}| \leq 2^{-m}\sup|\nu_u^{ij}|\sum_{\square}\sum_{\square}|UU^{kl}_{ij} - 1| \leq$$
$$\leq 2^{1-\frac{7}{4}m} \cdot \sum_{\square}\sum_{\square}|(UU^G)^{kl}_{ij} + (UU^B)^{kl}_{ij} - 1| \leq$$
$$\leq 2^{1-\frac{7}{4}m} \cdot \sum_{\square}\sum_{\square}|(UU^G)^{kl}_{ij} - 1| + |(UU^B)^{kl}_{ij}| \leq$$
$$\leq 2^{-\frac{7}{4}m}(2^{2m}\delta + 2^{\frac{9}{2}m}\delta) \leq 2^{2\frac{3}{4}m}\delta, \tag{4.53}$$

using Lemma 4.3.13 and the second part of Proposition 4.3.1.

The second part of (4.52) has the following upper bound, since $\sum_{ij}|\nu_u^{ij}| \leq 2^m$,

$$\Big\|\sum_{\square}\sum_{\square}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = 2^{-m}\sum_{\square}\Big|\sum_{\square}\nu_u^{ij}\Big| \leq 2^{2m}\cdot 2^m \cdot 2^{1-m} \leq 2^{2m+1}.$$

The last sum has only finite number of non-zero terms and can be estimated via the supremum norm. Recall Remark 10: for $R = M_2(1+\mu_1)^{2m} \cdot m\delta + 1$, any quartet (i,j,k,l) such that $(i,j) \in \square$ and $(k,l) \in \mathbb{R}^2 \setminus \square_R$ or $(i,j) \in \mathbb{R}^2 \setminus \square_R$ and $(k,l) \in \square$

$$SS_{ij}^{kl} \equiv 0, \quad SU_{ij}^{kl} \equiv 0, \quad US_{ij}^{kl} \equiv 0, \quad UU_{ij}^{kl} \equiv 0.$$

Therefore

$$\Big\| \Big(\sum_{\mathbb{R}^2 \setminus \square} \sum_{\square} + \sum_{\square} \sum_{\mathbb{R}^2 \setminus \square} \Big) UU_{ij}^{kl} \nu_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} =$$

$$= \Big\| \Big(\sum_{\square_R \setminus \square} \sum_{\square} + \sum_{\square} \sum_{\square_R \setminus \square} \Big) UU_{ij}^{kl} \nu_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} \leq$$

$$\leq \Big(\sum_{\square_R \setminus \square} \sum_{\square} + \sum_{\square} \sum_{\square_R \setminus \square} \Big) \sup |UU_{ij}^{kl}| \cdot \sup |\nu_u^{ij}| 2^{-m} \leq$$

$$\leq 4(R^2-1) m^4 2^{4m} \cdot 2^{\gamma_1 m} \cdot 2^{-\frac{3}{4}m} \cdot 2^{1-m} \leq$$

$$\leq M_2 m^5 \delta 2^{(\gamma_1 + \frac{9}{4})m} (1+\mu_1)^{2m}. \quad (4.54)$$

We have for the last term, using the bound (4.24) (p. 105)

$$\Big\| \sum_{\mathbb{R}^2 \setminus \square} \sum_{\mathbb{R}^2 \setminus \square} UU_{ij}^{kl} \nu_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} \leq \sum_{\mathbb{R}^2 \setminus \square} |\nu_u^{ij}| \cdot 2^{-m} \cdot \sup |UU_{ij}^{kl}| \cdot M_1 (1+\mu_1)^{2m} \leq$$

$$\leq 2^{m-1} \cdot 2^{-m} \cdot 2^{\gamma_1 m} \cdot M_1 (1+\mu_1)^{2m} = M_1 \cdot 2^{\gamma_1 m} \cdot (1+\mu_1)^{2m}. \quad (4.55)$$

Summing up the last four together, we get an upper bound $\|UU\nu_u\|_{\Omega^2, \mathcal{L}_1} \leq 2^{2+2m}$.

Now we proceed to the last inequality (4.3.18.2). We would like to show that there exists a constant γ_2 satisfying (4.51) such that for M and m large enough:

$$\max\big(\|SU\nu_u\|_{\Omega^2, \mathcal{L}_1}, \|US\nu_s\|_{\Omega^2, \mathcal{L}_1}, \|SS\nu_s\|_{\Omega^2, \mathcal{L}_1} \big) \leq 2^{\gamma_2 m} \|\nu\|_{\Omega^1, \mathcal{L}_1}.$$

We shall show that it holds true for the matrix SU, the argument for the matrix US is similar.

As before, we may assume for the first component of the vector field $\nu_s \in \Phi_{\Omega^1}$ that[1]

$$\max\Big(\sum_{ij} \nu_s^{ij} \cdot |\pi_y(\Omega_{ij})|, 2^{\frac{3}{4}m} \sup |\nu_s^{ij}|\Big) = 1,$$

and, consequently, $\sum\limits_{ij} |\nu_s^{ij}| \leq 2^{m-1}$ and $\sup |\nu_s^{ij}| \leq 2^{-\frac{3}{4}m}$. We recall the definition of "good" and "bad" connected components (4.25) and (4.26) :

$$(\Delta^G)_{ij}^{kl} \stackrel{\text{def}}{=} \{\Delta \subset P_\xi^{-2}(\Omega_{kl}^2) \cap \Omega_{ij}^1 \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \forall 1 \leq n \leq 2m : F_\xi^n(\Delta) \subset \square\};$$

$$(\Delta^B)_{ij}^{kl} \stackrel{\text{def}}{=} \{\Delta \subset P_\xi^{-2}(\Omega_{kl}^2) \cap \Omega_{ij}^1 \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \exists 1 \leq n \leq 2m : F_\xi^n(\Delta) \not\subset \square\}.$$

We may write, similarly to (4.27)

$$(SU)_{ij}^{kl} = (SU^G)_{ij}^{kl} + (SU^B)_{ij}^{kl},$$

where

$$(SU^G)_{ij}^{kl} := \frac{1}{|\pi_x(\Omega_{ij}^1)|} \cdot \frac{1}{|\pi_y(\Omega_{kl}^2)|} \cdot \sum_{\Delta \in \Delta^G} \int_\Delta \partial_y (P_\xi^2)_y(z) \mathrm{d}z;$$

$$(SU^B)_{ij}^{kl} := \frac{1}{|\pi_x(\Omega_{ij}^1)|} \cdot \frac{1}{|\pi_y(\Omega_{kl}^2)|} \cdot \sum_{\Delta \in \Delta^B} \int_\Delta \partial_y (P_\xi^2)_y(z) \mathrm{d}z.$$

Obviously, $(SU^G)_{ij}^{kl} \equiv 0$. We also recall $B = \{z \in \square \mid \exists 1 \leq n \leq 2m \colon F_\xi^n(z) \not\in \square\}$ and observe that

$$\sum_{\square} \sum_{\square} |(SU^B)_{ij}^{kl}| \leq \int_B |\partial_x (P_\xi^2)_y(z)| \mathrm{d}z = 2^{2m} \cdot 8m\delta.$$

We may write the action of SU on ν_s

$$SU\nu_s = \sum_{kl} \sum_{ij} SU_{ij}^{kl} \nu_s^{ij} \chi_{\Omega_{kl}^2}^u = \sum_{\square} \sum_{\square} (SU^B)_{ij}^{kl} \nu_s^{ij} \chi_{\Omega_{kl}^2}^u +$$

$$+ \sum_{\mathbb{R}^2 \setminus \square} \sum_{\square} SU_{ij}^{kl} \nu_s^{ij} \chi_{\Omega_{kl}^2}^u + \sum_{\square} \sum_{\mathbb{R}^2 \setminus \square} SU_{ij}^{kl} \nu_s^{ij} \chi_{\Omega_{kl}^2}^u + \sum_{\mathbb{R}^2 \setminus \square} \sum_{\mathbb{R}^2 \setminus \square} SU_{ij}^{kl} \nu_s^{ij} \chi_{\Omega_{kl}^2}^u.$$

[1] We denote the space of essentially bounded, absolutely integrable, piece-wise constant functions, associated to the partition Ω^1 of $\mathbb{R}$ by Φ_{Ω^1}.

We have the following upper bound for the first term, corresponding to the central part of the matrix

$$\Big\|\sum_{\square}\sum_{\square}(SU^B)_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}=\sum_{\square}\sum_{\square}\big|(SU^B)_{ij}^{kl}\big|\cdot|\nu_s^{ij}|\cdot 2^{-m}\leq$$

$$\leq \sup|\nu_s^{ij}|\cdot 2^{-m}\cdot\sum_{\square}\sum_{\square}\big|(SU^B)_{ij}^{kl}\big|\leq 2^{2m}m\delta\cdot 2^{-\frac{3}{4}m}\cdot 2^{-m}\leq 2^{\frac{m}{4}}m\delta.$$

Repeating the estimates (4.54) and (4.55) above, since $\|SU\|_\infty\leq\|UU\|_\infty\leq 2^{\gamma_1 m}$ and using the upper bounds $\|\nu_s\|_\infty\leq 2^{-\frac{3}{4}m}$ we obtain

$$\Big\|\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\mathbb{R}^2\setminus\square}+\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\Big)SU_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}\leq$$

$$\leq\sup\big|SU_{ij}^{kl}\big|\cdot\sup|\nu_s^{ij}|\cdot(1+\mu_1)^{2m}(M_1+M_2m^5\delta\cdot 2^{\frac{5}{2}m})\leq$$

$$\leq 2^{\gamma_1 m}\cdot 2^{\frac{5}{2}m-\frac{3}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta\leq 2^{\gamma_1 m}\cdot 2^{\frac{7}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta.$$

Summing up altogether, we get

$$\|SU\nu_s\|_{\Omega^2,\mathcal{L}_1}\leq 2^{\gamma_1 m}\cdot 2^{\frac{9}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta+2^{\frac{m}{4}}m\delta\leq 2^{\gamma_2 m}.$$

Similarly, $\|US\nu_y\|\leq 2^{\gamma_2 m}$. It only remains to show that for $\gamma_2=\gamma_1+\frac{9}{4}+2\log_2(1+\mu_1)-\alpha$ and for M and m sufficiently large

$$\|SS\nu_s\|_{\Omega^2,\mathcal{L}_1}\leq 2^{\gamma_2 m}. \tag{4.56}$$

Recall Corollary 1 of Lemma 4.3.10:

$$\sum_{\square}\sum_{\square}\big|(SS^G)_{ij}^{kl}\big|\leq 2^{4-2m}.$$

We can get an upper bound for the central part

$$\Big\|\sum_{\square}\sum_{\square}(SS^G)_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}=\sum_{\square}\sum_{\square}\big|(SS^G)_{ij}^{kl}\big|\cdot|\nu_s^{ij}|\cdot 2^{-m}\leq$$

$$\leq\sup|\nu_s^{ij}|\cdot 2^{-m}\cdot\sum_{\square}\sum_{\square}\big|(SS^G)_{ij}^{kl}\big|\leq 2^{4-2m}\cdot 2^{-\frac{3}{4}m}\cdot 2^{-m}<4\cdot 2^{-3m/2}.$$

Repeating the estimates (4.54) and (4.55) for the matrix SS and taking into account an upper bound $\|SS\|_\infty \le 2^{\gamma_1 m}$ from Proposition 4.3.2, we get

$$\Big\|\Big(\sum_{\mathbb{R}^2\setminus\Box}\sum_{\Box}+\sum_{\Box}\sum_{\mathbb{R}^2\setminus\Box}+\sum_{\mathbb{R}^2\setminus\Box}\sum_{\mathbb{R}^2\setminus\Box}\Big)SS_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} \le$$

$$\le \sup\big|SU_{ij}^{kl}\big|\cdot(1+\mu_1)^{2m}(M_1+M_2m^5 2^{\frac{5}{2}m}\delta) \le 2^{(\gamma_1+\frac{5}{2})m}\cdot(1+\mu_1)^{2m}\cdot M_2m^5\delta.$$

Thus

$$\big\|SS\nu_s\big\|_{\Omega^2,\mathcal{L}_1} = \Big\|\sum_{kl}\sum_{ij}SS_{ij}^{kl}\nu_s^{ij}\chi^s_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} \le$$

$$\le 2^{(\gamma_1+\frac{5}{2})m}\cdot(1+\mu_1)^{2m}\cdot M_2m^5\delta + 2^{3-3m/2} \le 2^{\gamma_2 m}.$$

■

Corollary 1. *Under the hypothesis and in the notations of Lemma 4.3.18, the norm of the operator* $\|\mathcal{A}\|_{\Omega^2} \le 2^{2m+2}$. *Namely,* $\|\mathcal{A}\nu\|_2 \le 2^{2m+2}\|\nu\|_1$.

Proof. Recall the definition (4.17) of the operator $\mathcal{A}\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$

$$\mathcal{A}\nu = \sum_{ij}\mathcal{A}\big(\nu_s^{ij}\chi^s_{\Omega^1_{ij}} + \nu_u^{ij}\chi^u_{\Omega^1_{ij}}\big) =$$

$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}\big(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}\big) + \nu_u^{ij}\big(US_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + UU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}\big)\Big),$$

The upper bound for $\mathcal{L}_1$-norm follows from the parts 4.3.18.1 and 4.3.18.2 of Lemma 4.3.18.

Now we proceed to the supremum norm.

$$\sup_z |\mathcal{A}\nu(z)| =$$
$$= \sup_z \Big| \sum_{ij} \sum_{kl} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) + SU_{ij}^{kl} \chi^u_{\Omega^2_{kl}}(z) \big) + \nu_u^{ij} \big(US_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) + UU_{ij}^{kl} \chi^u_{\Omega^2_{kl}}(z) \big) \Big) \Big| \le$$
$$\le \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \sup_{kl} \Big| \sum_{ij} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} + SU_{ij}^{kl} \big) + \nu_u^{ij} \big(US_{ij}^{kl} + UU_{ij}^{kl} \big) \Big) \Big| \le$$
$$\le \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \cdot (\|SS\|_\infty + \|SU\|_\infty + \|US\|_\infty + \|UU\|_\infty) \cdot \Big(\sum_{ij} (|\nu_s^{ij}| + |\nu_u^{ij}|) \Big) \le$$
$$\le 2^m \cdot 4 \cdot 2^{\gamma_1 m} \cdot 2^m \le 2^{2+(2+\gamma_1)m}.$$

The Corollary follows from the definition of the norm on p. 100. ■

The result we were seeking follows immediately

Proposition 4.3.3. The operators $W_\delta \mathcal{A}$ and $W_\delta P^2_{\xi*}$ are close. Namely,

$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_{\Omega^2, \mathcal{L}_1} \le \frac{4 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot \frac{\sup |\pi_y(\Omega^1_{ij})|}{\inf |\pi_y(\Omega^2_{kl})|} \cdot 2^{2m} \|\nu\|_1; \tag{4.57}$$

$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_\infty \le \frac{4 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{(2+\gamma_1)m} \|\nu\|_1. \tag{4.58}$$

Proof. Follows from Lemma 4.3.14, Lemma 4.3.16, the first and second parts of Lemma 4.3.18, and Corollary 1 of Lemma 4.3.18. ■

Corollary 2.

$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_2 \le \frac{8 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{2m} \|\nu\|_1$$

4.3.3 A pair of cones for the operator $\mathcal{A}$

In this Subsection we construct two cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}(\overline{C_1}) \subset C_2$, $C_2 \ll C_1$, and $\|\mathcal{A}|_{C_1}\| \ge 2^{m-1}$. This is the main result of Section 4.3, which is presented in Preliminary Dynamo Theorem 8 below.

Lemma 4.3.19. *The operator* UU *is a small perturbation of the operator* $\overset{\circ}{UU}$. *Namely*

$$\|(UU - \overset{\circ}{UU})\nu\|_2 \le 2^{(\gamma_1 + 2\frac{3}{4})m}\delta\|\nu\|_1.$$

Proof. We begin with $(\Omega^2, \mathcal{L}_1)$-norm. Consider a vector field $\nu \in \mathfrak{X}_{\Omega_1}$ with $\|\nu\|_1 = 1$. We may assume that $\sum_{ij} |\nu_u^{ij}| \le 2^m$ and $\sup |\nu_u^{ij}| \le 2^{-\frac{3}{4}m}$. Then

$$\|(UU - \overset{\circ}{UU})\nu\|_{\Omega^2,\mathcal{L}_1} = \Big\|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} =$$

$$= \sum_{kl}\Big|\sum_{ij}(UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl})\nu_u^{ij}\Big| \cdot 2^{-m} \le \sum_{kl}\sum_{ij}\big|UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} \le$$

$$\le \sum_{\Box}\sum_{\Box}\big|UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} +$$

$$+ \Big(\sum_{\mathbb{R}^2\setminus\Box}\sum_{\Box} + \sum_{\Box}\sum_{\mathbb{R}^2\setminus\Box} + \sum_{\mathbb{R}^2\setminus\Box}\sum_{\mathbb{R}^2\setminus\Box}\Big)\big|UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m}$$

We have for the first term

$$\sum_{\Box}\sum_{\Box}\big|UU_{ij}^{kl} - \overset{\circ}{UU}{}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} = \sum_{\Box}\sum_{\Box}\big|UU_{ij}^{kl} - 1\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} \le$$

$$\le \|UU\|_\infty \cdot \#\{(i,j,k,l) \in \Box \times \Box \mid UU_{ij}^{kl} \ne 1\} \cdot \sup|\nu_u^{ij}| \cdot 2^{-m} \le$$

$$\le 2^{\gamma_1 m} \cdot 2^{4\frac{1}{2}m}\delta \cdot 2^{-\frac{3}{4}m} \cdot 2^{-m} \le 2^{(2\frac{3}{4} + \gamma_1 - \alpha)m}.$$

Recall Remark 10: for $R = M_2(1+\mu_1)^{2m} \cdot m\delta + 1$, any quartet (i,j,k,l) such that $(i,j) \in \Box$ and $(k,l) \in \mathbb{R}^2 \setminus \Box_R$ or $(i,j) \in \mathbb{R}^2 \setminus \Box_R$ and $(k,l) \in \Box$

$$SS_{ij}^{kl} \equiv 0, \quad SU_{ij}^{kl} \equiv 0, \quad US_{ij}^{kl} \equiv 0, \quad UU_{ij}^{kl} \equiv 0.$$

Since $\overset{\circ}{UU}{}_{ij}^{kl} \equiv 0$ for all $(i,j,k,l) \in \Box \times (\mathbb{R}^2 \setminus \Box) \cup (\mathbb{R}^2 \setminus \Box) \times \Box$ we may write for the second term

$$\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\mathbb{R}^2\setminus\square}\Big)\big|UU_{ij}^{kl}-\mathring{U}U_{ij}^{kl}\big|\cdot\big|\nu_u^{ij}\big|\cdot|\pi_y(\Omega_{kl}^2)|=$$

$$=\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\mathbb{R}^2\setminus\square}\Big)\big|UU_{ij}^{kl}\big|\cdot\big|\nu_u^{ij}\big|\cdot|\pi_y(\Omega_{kl}^2)|=$$

$$=\Big(\sum_{\square_R\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\square_R\setminus\square}\Big)\big|UU_{ij}^{kl}\big|\cdot\big|\nu_u^{ij}\big|\cdot|\pi_y(\Omega_{kl}^2)|\leq$$

$$\leq 2^{1-m}\#\big\{(i,j,k,l)\in\square\times(\square_R\setminus\square)\cup(\square_R\setminus\square)\times\square\big\}\cdot\|UU\|_\infty\cdot\sup|\nu_u^{ij}|\leq$$

$$\leq 2^{1-m}\cdot 2^{4m}(1+\mu_1)^{2m}\cdot M_2m^5\delta\cdot 2^{\gamma_1 m}\cdot 2^{-\frac{3}{4}m}\leq m^2 2^{(\frac{9}{4}+\gamma_1-\alpha)m},$$

where $\gamma_2=\frac{5}{2}+\gamma_1-\alpha+2\log(1+\mu_1)$. Finally, for the last term we calculate

$$\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\big|UU_{ij}^{kl}-\mathring{U}U_{ij}^{kl}\big|\cdot\big|\nu_u^{ij}\big|\cdot 2^{-m}\leq 2^{-m}\cdot\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\Big(\big|UU_{ij}^{kl}\big|+\big|\mathring{U}U_{ij}^{kl}\big|\Big)\cdot\big|\nu_u^{ij}\big|\leq$$

$$\leq 2^{-m}\cdot 2M_1(1+\mu_1)^{2m}\|UU\|_\infty\sum_{\mathbb{R}^2\setminus\square}|\nu_u^{ij}|\leq(1+\mu_1)^{2m}\cdot 2^{\gamma_1 m}.$$

Summing up,

$$\|(UU-\mathring{U}U)\nu\|_{\Omega^2,\mathcal{L}_1}\leq m^2 2^{(2\frac{3}{4}+\gamma_1)}\delta\|\nu\|_1.$$

The upper bound for the supremum norm is easy:

$$\|(UU-\mathring{U}U)\nu\|_\infty=\sup_z\Big\|\sum_{kl}\sum_{ij}(UU_{ij}^{kl}-\mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega_{kl}^2}(z)\Big\|\leq$$

$$\leq\frac{1}{\inf|\pi_x(\Omega_{kl}^2)|}\cdot\sup_{kl}\sum_{ij}\big|UU_{ij}^{kl}-\mathring{U}U_{ij}^{kl}\big|\cdot\big|\nu_u^{ij}\big|\leq$$

$$\leq\frac{2\|UU\|_\infty}{\inf|\pi_x(\Omega_{kl}^2)|}\cdot\sum_{ij}|\nu_u^{ij}|\leq 2^{(\gamma_1+2)m+1}.$$

Then

$$\max(\|(UU-\mathring{U}U)\nu\|_{\Omega^2,\mathcal{L}_1},2^{-\frac{3}{4}m}\|(UU-\mathring{U}U)\nu\|_\infty)\leq 2^{(2\frac{3}{4}+\gamma_1)}\delta\|\nu\|_1.$$

■

Let Υ be a chain of partitions associated to the sequence $\eta\in\Sigma_\delta$. Let $\Omega^1=\Upsilon^k$, $\Omega^2=\Upsilon^{k+1}$, and $\Omega^3=\Upsilon^{k+2}$ be three consecutive partitions from the chain Υ. Consider the sequence

$\xi \stackrel{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (See definition of the chain Υ in subsection 4.2.3, p. 101). Let $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$ be a linear operator, approximating the operator $P^2_{\xi *}$, defined according to (4.17). Consider $\text{Cone}\left(1, \Omega^1\right) \subset \mathfrak{X}_{\Omega^1}$ and $\text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^2\right) \subset \mathfrak{X}_{\Omega^2}$; defined according to the general definition from p. 100.

$$\text{Cone}\left(1, \Omega^1\right) \stackrel{\text{def}}{=} \left\{\nu = \left(\begin{smallmatrix}0\\ d\end{smallmatrix}\right)\chi_{\square} + \psi,\ \psi \in \Omega^1,\ \|\psi\|_1 \le d,\ \sum_{\square} \psi_u^{ij} = 0\right\}; \tag{4.59}$$

$$\text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^2\right) \stackrel{\text{def}}{=} \left\{\nu = \left(\begin{smallmatrix}0\\ d\end{smallmatrix}\right)\chi_{\square} + \psi,\ \psi \in \Omega^2,\ \|\psi\|_2 \le d2^{(\gamma_1+\frac{3}{4}-\alpha)m},\ \sum_{\square} \psi_u^{ij} = 0\right\}. \tag{4.60}$$

Theorem 8 (Preliminary Dynamo Theorem)**.** *In the notations introduced above for arbitrary partition Ω^3 of the class $\mathcal{G}(m, \delta)$,*

$$\mathcal{A}\colon \text{Cone}\left(1, \Omega^1\right) \to \text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^2\right)$$

Proof. Consider a piecewise constant vector field $\nu \in \text{Cone}\left(1, \Omega^1\right)$. By definition of the $\text{Cone}\left(1, \Omega^1\right)$, we may write $\nu = \left(\begin{smallmatrix}0\\ d\end{smallmatrix}\right)\chi_{\square} + \psi$, where $\|\psi\| \le d$ and $\sum_{\square} \psi_u^{ij} = 0$. We deduce $\|\nu_s\| = \|\psi_s\| \le d$ and $\|\psi_u\| \le d$. Moreover, since

$$\int_{\square} \mathring{U}U\psi_u = \int_{\square} \sum_{\square}\sum_{ij} \mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}} = \sum_{\square}\int_{\square}\sum_{kl} \psi_u^{ij}\chi^u_{\Omega^2_{kl}} =$$
$$= \sum_{\square}\psi_u^{ij}\int_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega_{kl})|}\chi_{\Omega^2_{kl}} = \sum_{\square}\psi_u^{ij}\sum_{\square}|\pi_y(\Omega^2_{kl})| = 2^{m+1}\sum_{\square}\psi_u^{ij}. \tag{4.61}$$

We conclude that the condition $\int_{\square} \mathring{U}U\psi_u = 0$ is equivalent to

$$\sum_{\square} \psi_u^{ij} = 0 \tag{4.62}$$

By definition of $\mathcal{A}$ we write

$$\mathcal{A}\nu = \sum_{ij} \mathcal{A}(\nu_s^{ij}\chi^s_{\Omega^1_{ij}} + \nu_u^{ij}\chi^u_{\Omega^1_{ij}}) =$$

$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}(US_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + UU_{ij}^{kl}\chi^u_{\Omega^2_{kl}})\Big) =$$

$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}US_{ij}^{kl}\chi^s_{\Omega^2_{kl}}\Big) +$$

$$+ \sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}. \quad (4.63)$$

By Lemma 4.3.19 we know

$$\Big\|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_2 \le 2^{2\frac{3}{4}+\gamma_1-\alpha}d. \quad (4.64)$$

Using the third equality of Lemma 4.3.18, we get (recall $\gamma_2 = \gamma_1 + 2\frac{1}{4} + 2\log_2(1+\mu_1) - \alpha$)

$$\Big\|\sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}US_{ij}^{kl}\chi^s_{\Omega^2_{kl}}\Big)\Big\|_{\Omega^2,\mathcal{L}_1} \le 3\cdot 2^{\gamma_2 m}d.$$

The supremum norm estimate is similar to the supremum norm of $\mathcal{A}$

$$\sup_z\Big|\sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}}(z) + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}(z)) + \nu_u^{ij}US_{ij}^{kl}\chi^s_{\Omega^2_{kl}}(z)\Big)\Big| \le$$

$$\le \frac{1}{\inf|\pi_x(\Omega^2_{kl})|}\sup_{kl}\Big|\sum_{ij}\Big(\nu_s^{ij}(SS_{ij}^{kl} + SU_{ij}^{kl}) + \nu_u^{ij}US_{ij}^{kl}\Big)\Big| \le$$

$$\le \frac{1}{\inf|\pi_x(\Omega^2_{kl})|}\cdot(\|SS\|_\infty + \|SU\|_\infty + \|US\|_\infty)\cdot\Big(\sum_{ij}(|\nu_s^{ij}| + |\nu_u^{ij}|)\Big) \le$$

$$\le 2^m\cdot 4\cdot 2^{\gamma_1 m}\cdot 2^m d \le 2^{2+(2+\gamma_1)m}d.$$

Thus

$$\Big\|\sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}US_{ij}^{kl}\chi^s_{\Omega^2_{kl}}\Big)\Big\|_2 \le$$

$$\le \max(2^{2+(\frac{3}{2}+\gamma_1)m}, 3\cdot 2^{\gamma_2 m})d = 3\cdot 2^{\gamma_2 m}d. \quad (4.65)$$

We expand $\nu_u = d\chi_\Box + \psi_u$ and observe, using Lemma 4.3.1 and equality (4.22)

$$\mathring{U}U\chi_\Box = \sum_\Box \sum_\Box \mathring{U}U_{ij}^{kl}\chi_{\Omega^2_{kl}} = 2^{2m}\chi_\Box. \tag{4.66}$$

By definition of the $(\Omega^2, \mathcal{L}_1)$-norm,

$$\left\|\chi^u_{\Omega^2_{kl}}\right\|_{\Omega^2,\mathcal{L}_1} = \frac{2^{-m}}{|\pi_y(\Omega_{kl})|}\int_{\Omega^2_{kl}} \frac{\chi_{\Omega^2_{kl}}(z)}{|\pi_x(\Omega^2_{kl})|}\mathrm{d}z = 2^{-m}.$$

Using (4.62), we calculate the norm

$$\begin{aligned}
\Big\|\sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} &\le \Big\|\sum_\Box\sum_\Box \mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} + \qquad (4.67)\\
&+ \Big\|\Big(\sum_{\mathbb{R}^2\setminus\Box}\sum_\Box + \sum_\Box\sum_{\mathbb{R}^2\setminus\Box} + \sum_{\mathbb{R}^2\setminus\Box}\sum_{\mathbb{R}^2\setminus\Box}\Big)\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} \le\\
&\le 2^{-m}\sum_\Box\Big|\sum_\Box\psi_u^{ij}\Big| + 2^{-m}\sum_{\mathbb{R}^2\setminus\Box}\sum_{\mathbb{R}^2\setminus\Box}|\mathring{U}U_{ij}^{kl}|\cdot|\psi_u^{ij}|\cdot|\pi_y(\Omega^2_{kl})| \le\\
&\le 2^{-m}(1+\mu_1)^{2m}\sup_{\mathbb{R}^4\setminus\Box\times\Box}|\mathring{U}U_{ij}^{kl}|\cdot\sup|\psi_u^{ij}| \le 2^{-m}(1+\mu_1)^{2m}\cdot 2^{\gamma_1 m}\cdot d2^{-m/2} \le\\
&\le d2^{-3m/2}(1+\mu_1)^{2m}\cdot 2^{\gamma_1 m}. \qquad (4.68)
\end{aligned}$$

We shall estimate the supremum norm as well

$$\begin{aligned}
\sup_z\Big|\sum_{kl}\sup_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}(z)\Big| &\le \frac{1}{|\pi_x(\Omega^2_{kl})|}\sup_{kl}\Big|\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\Big| \le\\
&\le \frac{1}{|\pi_x(\Omega^2_{kl})|}\cdot\sup|\mathring{U}U_{ij}^{kl}|\cdot\sum_{ij}|\psi_u^{ij}| \le d(1+\mu)^{2m}\cdot 2^{2m}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Big\|\sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2} \le\\
&\qquad \le d\cdot\max\big(2^{-3m/2}(1+\mu_1)^{2m}\cdot 2^{\gamma_1 m}, (1+\mu)^{2m}\cdot 2^{3m/2}\big) = d(1+\mu)^{2m}\cdot 2^{3m/2}. \qquad (4.69)
\end{aligned}$$

Now we substitute (4.64), (4.65), and (4.66) to (4.63) and obtain $\mathcal{A}\nu = d2^{2m}\chi_\square + \psi^1$, where

$$\psi^1 = \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}) + \nu_u^{ij}US_{ij}^{kl}\chi^s_{\Omega^2_{kl}}\Big) + \\ + \sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}. \quad (4.70)$$

with the norm (recall $\gamma_2 = \gamma_1 + \frac{9}{4} + 2\log_2(1+\mu_1) - \alpha$).

$$\|\psi^1\|_{\Omega^2} \le d2^{\gamma_2 m} + d2^{(2\frac{3}{4}+\gamma_1-\alpha)m} + d(1+\mu)^{2m}2^{\frac{3}{2}m} \le d2^{1+(2\frac{3}{4}+\gamma_1-\alpha)m} \le \\ \le d\|\mathring{U}U\chi_\square\|_{\Omega^2}\cdot 2^{(\frac{3}{4}+\gamma_1-\alpha)m}.$$

We would like to write ψ^1_y as a sum $\psi^1_y = b\chi_\square + \phi$ with $\int_\square \mathring{U}U\phi = 0$. We may choose

$$b = \frac{\int_\square \mathring{U}U\psi^1_y}{\int_\square \mathring{U}U\chi_\square}. \quad (4.71)$$

Using (4.66) we get $\int_\square \mathring{U}U\chi_\square = 2^{2m+2}$. Using (4.70) we get

$$\psi^1_y = \sum_{kl}\sum_{ij}\nu_s^{ij}SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}} + \sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi^u_{\Omega^2_{kl}}. \quad (4.72)$$

Apply (4.61) to ψ^1_y

$$\int_\square \psi^1_y = 2^{m+1}\sum_{kl}\sum_{ij}(SU_{ij}^{kl}\nu_u^{ij} + (UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij} + \mathring{U}U_{ij}^{kl}\psi_u^{ij}).$$

We may obtain an upper bound

$$\Big|\int_\square \psi^1_y\Big| \le 2^{m+1}\Big(\Big|\sum_{kl}\sum_{ij}SU_{ij}^{kl}\nu_s^{ij}\Big| + \Big|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\Big| + \Big|\sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\Big|\Big).$$

From Lemma 4.3.19 it follows that

$$\Big|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\Big| \le 2^m\cdot 2^{(2\frac{3}{4}+\gamma_1-\alpha)m}d.$$

Using (4.68) we deduce

$$\Big|\sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\Big| \le d(1+\mu)^{2m}\cdot 2^{-\frac{m}{2}}\cdot 2^{\gamma_1 m}(1+\mu)^{2m}.$$

From the third part (4.3.18.2) of Lemma 4.3.18 we get

$$\Big|\sum_{ij}\sum_{kl} SU_{ij}^{kl}\nu_s^{ij}\Big| \le 2^{(\gamma_2+1)m}d.$$

Summing up the last three together, we get

$$\Big|\int_{\square}\psi_y^1\Big| \le 3d\cdot 2^{(3\frac{3}{4}+\gamma_1-\alpha)m}. \tag{4.73}$$

We conclude that the ratio (4.71) is bounded by $b \le 2^{(1\frac{3}{4}+\gamma_1-\alpha)m} \ll 2^{2m}$.

Therefore $\mathcal{A}\nu = d(2^{2m}+b)\binom{0}{1}\chi_\square + \binom{\psi_x^1}{\phi} \in \text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^2\right)$ and $\|\mathcal{A}\nu\| \ge d2^{2m-1}$.

■

4.4 An invariant cone for the operator $W_{\frac{\delta}{2m}}P_{t*}^2W_{\frac{\delta}{2m}}$

The main goal of this Section is to get rid of the dependence of the sequence in the Preliminary Dynamo Theorem. We exploit properties of the Weierstrass transform, and construct an invariant cone for the operator $W_{\frac{\delta}{2m}}P_{t*}^2W_{\frac{\delta}{2m}}$, which is independent of the choice of $\|t\| \le \delta = 2^{-m\alpha}$.

4.4.1 Discretization and the Weierstrass transform toolbox

In this Subsection we establish the fact that the image of the Weierstrass transform may be very well approximated by piecewise-constant vector fields associated to some canonical partition.

Two-dimensional discretization operator on vector fields on the real plane, associated to a partition Ω, we define by

$$D_\Omega\colon \mathcal{L}(\mathbb{R}^2)\cap\mathcal{L}_\infty(\mathbb{R}^2)\to\mathfrak{X} \qquad D_\Omega v \overset{\text{def}}{=} \sum_{ij}\left(d_s^{ij}\chi_{\Omega_{ij}}^s + d_u^{ij}\chi_{\Omega_{ij}}^u\right), \tag{4.74}$$

where

$$d_s^{ij} \overset{\text{def}}{=} \frac{1}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}} v_s \quad\text{and}\quad d_u^{ij} \overset{\text{def}}{=} \frac{1}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}} v_u. \tag{4.75}$$

In this section we assume that Ω^1, Ω^2, and Ω^3, are three arbitrary partitions of the class $\mathcal{G}(m,\delta)$, defined on p. 102. In particular, all three partitions satisfy Lemma 4.2.2.

Lemma 4.4.1. *Let $\nu \in \mathfrak{X}$ be a bounded vector field with absolutely integrable components in $\mathbb{R}^2$. Then there exists a constant $\gamma_3 > 0$, that depends on δ and on the size of partition elements, such that*

$$\|W_{\frac{\delta}{m}}\nu - D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \leq 2^{-\gamma_3 m}\|\nu\|_1.$$

One may choose $\gamma_3 = 1 - \frac{\log_2\delta}{m} + \frac{2\log_2 m}{m} = 1 - \alpha + \frac{2\log_2 m}{m} < 1 - \alpha + \gamma_1$.

Proof. We shall show that the inequality holds true for any bounded and integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ first. We may write by definition

$$W_\delta f(z) = \int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t,$$

and for the discretization operator we have that

$$D_{\Omega^2}W_\delta f(z) = \sum_{ij}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\int_{\mathbb{R}^2} w_\delta(s-t)f(t)\mathrm{d}t\mathrm{d}s\cdot\chi^u_{\Omega^2_{ij}}(z) =$$

$$= \int_{\mathbb{R}^2} f(t)\sum_{ij}\frac{1}{|\Omega^2_{ij}|}\int_{\Omega^2_{ij}} w_\delta(s-t)\mathrm{d}s\cdot\chi_{\Omega^2_{ij}}(z)\mathrm{d}t.$$

Therefore, $(\Omega^2, \mathcal{L}_1)$ norm may be bounded as following:

$$\|W_\delta f - D_{\Omega^2} W_\delta f\|_{\Omega^2,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \big|W_\delta f(z) - W_\delta D_{\Omega^2} f(z)\big| \mathrm{d}z =$$
$$= \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| \int_{\mathbb{R}^2} f(t) \Big(w_\delta(z-t) - \sum_{kl} \frac{1}{|\Omega^2_{kl}|} \int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s \cdot \chi_{\Omega^2_{kl}}(z) \Big) \mathrm{d}t \Big| \mathrm{d}z \le$$
$$\le \int_{\mathbb{R}^2} |f(t)| \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| w_\delta(z-t) - \sum_{kl} \frac{1}{|\Omega^2_{kl}|} \int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s \cdot \chi_{\Omega^2_{kl}}(z) \Big| \mathrm{d}z\mathrm{d}t \le$$
$$\le \int_{\mathbb{R}^2} |f(t)|\mathrm{d}t \cdot \sup_t \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| w_\delta(z-t) - \sum_{kl} \frac{1}{|\Omega^2_{kl}|} \int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s \cdot \chi_{\Omega^2_{kl}}(z) \Big| \mathrm{d}z \le$$
$$\le \int_{\mathbb{R}^2} |f(t)|\mathrm{d}t \cdot \sup_t \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| w_\delta(z-t) - \frac{1}{|\Omega^2_{ij}|} \int_{\Omega^2_{ij}} w_\delta(s-t)\mathrm{d}s \Big| \mathrm{d}z \le$$
$$\le \int_{\mathbb{R}^2} |f(t)|\mathrm{d}t \cdot \sup_t \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| \max_z w_\delta(z-t) - \min_z w_\delta(z-t) \Big| \mathrm{d}z. \tag{4.76}$$

We have to find an upper bound for the last term:

$$\sup_t \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| \max_{s\in\Omega^2_{ij}} w_\delta(s-t) - \min_{s\in\Omega^2_{ij}} w_\delta(s-t) \Big| \mathrm{d}z \le$$
$$\le 2^{-m} \sup_t \sum_{ij} |\pi_x(\Omega^2_{ij})| \cdot \big| \max_{z\in\Omega^2_{ij}} w_\delta(z-t) - \min_{z\in\Omega^2_{ij}} w_\delta(z-t) \big| \le$$
$$\le 2^{-m} \sup_t \sum_{ij} |\pi_x(\Omega^2_{ij})| \cdot |\mathrm{diam}(\Omega^2_{ij})| \cdot \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| =$$
$$= 2^{-m} \sup_t \sum_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| \cdot |\Omega^2_{ij}| \le$$
$$\le 2^{-m} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \sup_t \sum_{ij} \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| \cdot |\Omega^2_{ij}| \le$$
$$\le 2^{-m} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \int_{\mathbb{R}^2} |\nabla_z w_\delta(z)| \mathrm{d}z \le \frac{2^{-m}}{\delta} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|}. \tag{4.77}$$

Therefore substituting (4.77) to (4.76) we conclude

$$\|W_\delta f - D_{\Omega^2} W_\delta f\|_{\Omega^2,\mathcal{L}_1} \le \frac{\sup|\pi_y(\Omega^1_{ij})|}{\delta} \cdot \sup_{kl} \frac{|\mathrm{diam}(\Omega^2_{kl})|}{|\pi_y(\Omega^2_{kl})|} \|f\|_{\Omega^1,\mathcal{L}_1}. \tag{4.78}$$

Similarly, for the supremum norm

$$\begin{aligned}
&\|D_{\Omega^2}W_\delta\nu_s - W_\delta\nu_s\|_\infty = \\
&\qquad = \sup_s \Big| \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \sum_{ij} \frac{1}{|\Omega^2_{ij}|}\int_{\mathbb{R}^2} w_\delta(z-t)\mathrm{d}t\mathrm{d}z\chi_{\Omega^2_{ij}}(s)\Big| = \\
&\qquad = \sup_{ij}\sup_{s\in\Omega^2_{ij}} \Big| \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \frac{1}{|\Omega^2_{ij}|}\int_{\Omega^2_{ij}}\int_{\mathbb{R}^2} w_\delta(z-t)\nu_s(t)\mathrm{d}t\mathrm{d}z\Big| = \\
&\qquad = \sup_{ij}\Big| \max_{s\in\Omega^2_{ij}} \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \min_{s\in\Omega^2_{ij}} \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t\Big| \le \\
&\qquad \le \sup_{\Omega^2_{ij}} \int_{\gamma(\Omega^2_{ij})} \Big|\nabla \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t\Big|\mathrm{d}s, \qquad (4.79)
\end{aligned}$$

where $\gamma(\Omega^2_{ij})$ is a line segment connecting the points of maxima and minima of the integrand in Ω^2_{ij}. We proceed therefore

$$\begin{aligned}
\|D_{\Omega^2}W_\delta\nu_s - W_\delta\nu_s\|_\infty &\le \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup_s \Big|\nabla_s \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t\Big| \le \\
&\le \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup|\nu| \cdot \sup_s \int_{\mathbb{R}^2} \Big|\nabla_s w_\delta(s-t)\Big|\mathrm{d}t \le \\
&\le \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup|\nu| \cdot \int_{\mathbb{R}^2} \frac{1}{\pi^2\delta^4}\sqrt{t_x^2+t_y^2}\cdot e^{-\frac{t_x^2-t_y^2}{2\delta^2}}\mathrm{d}t \le \\
&\le \frac{\sup \operatorname{diam}(\Omega^2_{ij})}{\pi\delta}\|\nu\|_\infty. \qquad (4.80)
\end{aligned}$$

We put (4.78) and (4.80) together, and conclude that we may find a constant $\gamma_3 > 0$ such that

$$\max\Big(\frac{m\sup|\pi_y(\Omega^1_{ij})|}{\delta}\cdot\sup_{kl}\frac{|\operatorname{diam}(\Omega^2_{kl})|}{|\pi_y(\Omega^2_{kl})|}, \frac{m\sup\operatorname{diam}(\Omega^2_{ij})}{\delta}\Big) = 2^{-\gamma_3 m}.$$

■

Remark 15. It follows from the properties of partitions of the class $\mathcal{G}(m,\delta)$, Lemma 4.2.2, that $\gamma_3 < 1-\alpha$ and it may be chosen arbitrary close to $1-\alpha$.

Lemma 4.4.2. *Let Ω be a partition of $\mathbb{R}^2$ the class $\mathcal{G}(m,\delta)$. Then*

$$\|W_\delta\chi_\square - D_\Omega W_\delta\chi_\square\|_\Omega \le 2^{-m/4}; \tag{4.4.2.1}$$

$$\|W_\delta\chi_\square - \chi_\square\|_\Omega \le 2^{-m/4}. \tag{4.4.2.2}$$

Proof. We start with the first inequality. The upper bound for the supremum norm is trivial. Indeed, observe that for any non-negative integrable function f and any element Ω_{ij}

$$\sup_{\Omega_{ij}} f \ge \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}} f > 0,$$

and, consequently,

$$\sup_{\Omega_{ij}}\Big| f - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}} f\Big| \le \sup_{\Omega_{ij}}|f|.$$

Therefore

$$\sup_z\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \sum_{ij}\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(z-t)\mathrm{d}t\mathrm{d}s\chi_{\Omega_{ij}}(z)\Big| =$$
$$= \sup_{ij}\sup_{z\in\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big| \le \sup_{ij}\sup_{z\in\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t\Big| \le 1.$$

Now we consider $(\Omega,\mathcal{L}_1)$-norm. Let k be such that $e^k > 2^m$ and $k < m$. Introduce three sets of indices:

$$r_1 \colon = \{(i,j)\in\mathbb{Z}^2 \mid \Omega_{ij}\subset\square_{1-k\delta}\};$$

$$r_2 \colon = \{(i,j)\in\mathbb{Z}^2 \mid \Omega_{ij}\subset\square_{1+k\delta},\, \Omega_{ij}\not\subset\square_{1-k\delta}\};$$

$$r_3 \colon = \{(i,j)\in\mathbb{Z}^2 \mid \Omega_{ij}\not\subset\square_{1+k\delta}\}.$$

We split the sum of integrals in three parts:

$$\sum_{kl}\frac{2^{-m}}{|\pi_y(\Omega_{kl}|}\int_{\Omega_{kl}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \sum_{ij}\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\chi_{\Omega_{ij}}(z)\Big|\mathrm{d}z =$$
$$\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega_{ij}|}\int_{\Omega_{ij}}\int_{\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big|\mathrm{d}z$$
$$= \Big(\sum_{r_1}+\sum_{r_2}+\sum_{r_3}\Big)\frac{2^{-m}}{|\pi_y(\Omega_{ij}|}\int_{\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big|\mathrm{d}z. \tag{4.81}$$

We estimate the three sums separately.

Observe that for any $(i,j) \in r_1$ and any $z \in \Omega_{ij} \subset \Box_{1-k\delta}$

$$1 > \int_{\Box} w_\delta(z-t)\mathrm{d}t = \int_{\Box - z} w_\delta(t)\mathrm{d}t \geq \int_{-k\delta}^{k\delta}\int_{-k\delta}^{k\delta} w_\delta(t)\mathrm{d}t_x\mathrm{d}t_y = \int_{-k}^{k}\int_{-k}^{k} w_1(t)\mathrm{d}t \geq 1 - 4e^{-k}.$$

Therefore

$$\sum_{r_1} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}}\Big|\int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big|\mathrm{d}z \leq$$
$$\leq \sum_{r_1} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}}\Big|1 - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}(1-4e^{-k})\Big|\mathrm{d}z \leq \frac{2^{-m}}{\inf|\pi_y(\Omega_{ij})|}\sum_{r_1} 4e^{-k}|\Omega_{ij}| \leq$$
$$\leq \frac{(1-k\delta)^2}{2^m e^k \inf|\pi_y(\Omega_{ij})|} \leq \sup \mathrm{diam}|\Omega_{ij}|. \quad (4.82)$$

Observe that for any $(i,j) \in r_2$ and any $z \in \Omega_{ij}$

$$\sup_{z\in\Omega_{ij}}\Big|\int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big| \leq$$
$$\leq \sup_{z\in\Omega_{ij}}\Big|\nabla_z \int_{\Box} w_\delta(z-t)\mathrm{d}t\Big| \cdot \mathrm{diam}(\Omega_{ij}) \leq \sup_{z\in\Omega_{ij}}\int_{\Box}\Big|\nabla_z w_\delta(z-t)\Big|\mathrm{d}t \cdot \mathrm{diam}(\Omega_{ij}) =$$
$$= \sup_{z\in\Omega_{ij}}\int_{\Box}\frac{1}{\pi^2\delta^4}\sqrt{(z_x-t_x)^2+(z_y-t_y)^2}\cdot e^{-\frac{(z_x-t_x)^2-(z_y-t_y)^2}{2\delta^2}}\mathrm{d}t\cdot\mathrm{diam}(\Omega_{ij}) \leq$$
$$\leq \sup_{z\in\Omega_{ij}}\int_{\Box}\frac{1}{\pi^2\delta^4}\big(|z_x-t_x|+|z_y-t_y|\big)\cdot e^{-\frac{(z_x-t_x)^2-(z_y-t_y)^2}{2\delta^2}}\mathrm{d}t\cdot\mathrm{diam}(\Omega_{ij}) \leq \frac{4\mathrm{diam}(\Omega_{ij})}{\pi^2\delta}.$$

Therefore

$$\sum_{r_2}\int_{\Omega_{ij}}\Big|\int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s\cdot\chi_{\Omega_{ij}}(z)\Big|\mathrm{d}z \leq$$
$$\leq \sum_{r_2}|\Omega_{ij}|\frac{4\mathrm{diam}(\Omega_{ij})}{\delta} \leq \big((1+k\delta)^2-(1-k\delta)^2\big)\frac{4\sup\mathrm{diam}(\Omega_{ij})}{\delta} \leq 16k\sup\mathrm{diam}(\Omega_{ij}).$$
$$(4.83)$$

Finally, for the third term we cut r_3 into squared annuli

$$\mathrm{ai}_n: = \{(i,j) \in r_1, |\ \Omega_{ij} \subset \Box_{1+(k+n)\delta},\ \Omega_{ij} \not\subset \Box_{1+(k+n-1)\delta}\}.$$

Obviously, $\bigcup_{n=0}^{\infty} \mathrm{ai}_n = r_1$, and $\sum_{\mathrm{ai}_n} |\Omega_{ij}| \leq 2\delta + \delta^2(2k+2n-1)$. Therefore,

$$\sum_{r_1} \int_{\Omega_{ij}} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \mathrm{d}z =$$
$$= \sum_{n=0}^{\infty} \sum_{\mathrm{ai}_n} \int_{\Omega_{ij}} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \mathrm{d}z =$$
$$= \sum_{n=0}^{\infty} \sum_{\mathrm{ai}_n} |\Omega_{ij}| \cdot \sup_{z\in\Omega_{ij}} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\Box} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \leq$$
$$\leq \sum_{n=0}^{\infty} \sum_{\mathrm{ai}_n} |\Omega_{ij}| \cdot \mathrm{diam}(\Omega_{ij}) \cdot \sup_{z\in\Omega_{ij}} \int_{\Box} \Big| \nabla_z w_\delta(z-t) \Big| \mathrm{d}t \leq$$
$$\leq \sum_{n=0}^{\infty} \sum_{\mathrm{ai}_n} |\Omega_{ij}| \cdot \mathrm{diam}(\Omega_{ij}) \cdot \frac{4}{\pi^2\delta} \cdot e^{-\frac{(k+n)^2}{2}} \leq 4 \sup \mathrm{diam}(\Omega_{ij}). \quad (4.84)$$

Substituting up (4.82), (4.83), and (4.84) to (4.81):

$$\|W_{\frac{\delta}{m}} \chi_\Box - D_\Omega W_{\frac{\delta}{m}} \chi_\Box\|_\Omega < 32m \sup \mathrm{diam}(\Omega_{ij}).$$

We conclude, using the second part of Lemma 4.2.2: $\Omega_{ij} \subset Rec(2^{1-m}, 2^{1-m})$

$$\|W_\delta \chi_\Box - D_\Omega W_\delta \chi_\Box\|_\Omega \leq \max(32m \sup \mathrm{diam}(\Omega_{ij}), 2^{-m/4}) = 2^{-m/4}.$$

Now we consider the second inequality (4.4.2.2). Obviously, $\|W_\delta\chi_\Box - \chi_\Box\|_\infty \leq 1$. We proceed to the weighted $(\Omega, \mathcal{L}_1)$-norm. We shall show that

$$\|W_\delta\chi_\Box - \chi_\Box\|_{\Omega,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |W_\delta\chi_\Box - \chi_\Box| \leq \frac{12\delta}{2^m \inf |\pi_y(\Omega_{ij})|}. \quad (4.85)$$

By straightforward calculation

$$\sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |W_\delta\chi_\Box - \chi_\Box| = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t - \chi_\Box(z) \Big| \mathrm{d}z \leq$$
$$\leq \frac{2^{-m}}{\inf |\pi_y(\Omega_{ij})|} \cdot \Big(\int_{\mathbb{R}^2\setminus\Box} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t \Big| \mathrm{d}z + \int_{\Box} \Big| \int_{\Box} w_\delta(z-t)\mathrm{d}t - 1 \Big| \mathrm{d}z \Big). \quad (4.86)$$

Recall the error function

$$\mathrm{erf}(z) := \int_0^z \frac{2}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x;$$

and its antiderivative

$$\int \operatorname{erf}(z)\mathrm{d}z = z\operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}.$$

We estimate each of two terms of (4.86) separately.

$$\begin{aligned}
&\int_{\square} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x_1-t_1)^2}{2\delta^2}} \mathrm{d}x_1\mathrm{d}t_1 = \\
&\qquad = \int_{-1}^{1}\int_{-1-t_1}^{1-t_1} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{x_1^2}{2\delta^2}} \mathrm{d}x_1\mathrm{d}t_1 = \int_{-1}^{1}\int_{\frac{-1-t_1}{\sqrt{2}\delta}}^{\frac{1-t_1}{\sqrt{2}\delta}} \frac{1}{\sqrt{\pi}} e^{-x_1^2}\mathrm{d}x_1\mathrm{d}t_1 = \\
&\qquad = \frac{1}{2}\int_{-1}^{1}\Big(\int_{0}^{\frac{1-t_1}{\sqrt{2}\delta}} \frac{2}{\sqrt{\pi}} e^{-x_1^2}\mathrm{d}x_1 + \int_{0}^{\frac{1+t_1}{\sqrt{2}\delta}} \frac{2}{\sqrt{\pi}} e^{-x_1^2}\mathrm{d}x_1\Big)\mathrm{d}t_1 = \\
&\qquad = \frac{1}{2}\int_{-1}^{1} \operatorname{erf}\Big(\frac{1-t_1}{\sqrt{2}\delta}\Big) + \operatorname{erf}\Big(\frac{1+t_1}{\sqrt{2}\delta}\Big)\mathrm{d}t_1 = \\
&\qquad = \frac{\delta}{\sqrt{2}}\Big(\int_{0}^{\sqrt{2}/\delta} \operatorname{erf}(z)\mathrm{d}z - \int_{-\sqrt{2}/\delta}^{0} \operatorname{erf}(z)\mathrm{d}z\Big) = \\
&\qquad = \frac{\delta}{\sqrt{2}}\Big(\big(z\operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}\big)\Big|_{0}^{\sqrt{2}/\delta} - \Big(z\operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}\Big)\Big|_{-\sqrt{2}/\delta}^{0}\Big) = \\
&\qquad = 2\operatorname{erf}\Big(\frac{\sqrt{2}}{\delta}\Big) + \sqrt{\frac{2}{\pi}}\delta(e^{-2/\delta^2} - 1) \geq (2-\delta)(1 - e^{-2/\delta^2}).
\end{aligned}$$

Therefore for the first term of (4.86) we have

$$\begin{aligned}
&\int_{\square}\Big|\int_{\square} w_\delta(z-t)\mathrm{d}t - 1\Big|\mathrm{d}z = \int_{\square}\Big(1 - \int_{\square} w_\delta(z-t)\mathrm{d}t\Big)\mathrm{d}z = \\
&\qquad = 4 - \Big(\int_{\square} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x_1-t_1)^2}{2\delta^2}} \mathrm{d}x_1\mathrm{d}t_1\Big)^2 \leq 4 - (2-\delta)^2(1-e^{-2/\delta^2})^2 \leq 4\delta. \qquad (4.87)
\end{aligned}$$

We claim

$$\int_{\mathbb{R}^2\setminus\square}\int_{\square} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \leq 8\delta. \qquad (4.88)$$

Indeed, using approximation $\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}\mathrm{d}x \geq 1 - e^{-x}$ for large x,

$$\begin{aligned}
&\int_{-1}^{1}\int_{1}^{+\infty} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x_1-t_1)^2}{2\delta^2}} \mathrm{d}x_1\mathrm{d}t_1 = \frac{1}{2}\int_{-1}^{1}\int_{\frac{1-t_1}{\sqrt{2}\delta}}^{+\infty} \sqrt{2\pi} e^{-x_1^2}\mathrm{d}x_1\mathrm{d}t_1 = \\
&= 1 - \frac{1}{2}\int_{-1}^{1} \operatorname{erf}\Big(\frac{1-t_1}{\sqrt{2}\delta}\Big)\mathrm{d}t_1 = 1 + \frac{1}{2}\int_{-\sqrt{2}/\delta}^{0} \operatorname{erf}(z)\mathrm{d}z = 1 + \frac{1}{2}\Big(z\operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}\Big)\Big|_{-\sqrt{2}/\delta}^{0} = \\
&\qquad = 1 + \frac{1}{2}\Big(\frac{1}{\sqrt{\pi}} - \frac{\sqrt{2}}{\delta}\operatorname{erf}\Big(\frac{\sqrt{2}}{\delta}\Big) - \frac{e^{-2/\delta^2}}{\sqrt{\pi}}\Big) \leq 1 - (1 - e^{-2/\delta^2}) + \frac{\delta}{\sqrt{2\pi}} - \frac{e^{-2/\delta^2}}{\sqrt{2\pi}} \leq \delta.
\end{aligned}$$

Therefore,

$$\int_1^{+\infty}\int_1^{+\infty}\int_{\square} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \le 4\delta^2;$$

and, similarly,

$$\int_1^{+\infty}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \le 2\delta.$$

The claim (4.88) follows and hence the inequality (4.85). ■

Lemma 4.4.3. *Let Ω^1 and Ω^2 be two arbitrary partitions of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$. Then*

An upper bound for the norm of the Weierstrass transform is given by

$$\|W_\delta\nu\|_2 \le \sup|\pi_y(\Omega^2_{kl})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\cdot\|\nu\|_1.$$

Proof. Consider a function $f\in\mathcal{L}_1(\mathbb{R}^2)\cap\mathcal{L}_\infty(\mathbb{R}^2)$ with $\|f\|_{\Omega^1}=1$. Then

$$\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^1_{ij})|}\int_{\Omega^1_{ij}}|f|\le 1;\qquad \sup|f|\le 2^{\frac{m}{4}}.$$

By straightforward calculation

$$\begin{aligned}\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &= \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\Big|\sum_{kl}\int_{\Omega^1_{kl}} w_\delta(z-t)f(t)\mathrm{d}t\Big|\mathrm{d}z\le\\ &\le 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}}|f(t)|\sum_{ij}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\mathrm{d}t\le\\ &\le 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}}|f(t)|\Big(\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|>m\delta}+\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|<m\delta}\Big)\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\mathrm{d}t.\end{aligned}\tag{4.89}$$

We have to estimate two sums separately. We know that $\|w_\delta\|_\infty\le\frac{1}{\delta^2}$; thus

$$\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\le\frac{|\pi_x(\Omega^2_{ij})|}{\delta^2}.$$

Therefore, since for a fixed Ω^1_{kl}, the total number of elements of another partition Ω^2_{ij} satisfying $|\Omega^2_{ij}-\Omega^1_{kl}|<m\delta$ is bounded by m^2N_δ:

$$\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|<m\delta}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\le\sup|\pi_x(\Omega^2_{ij})|\cdot m^2\cdot\frac{N_\delta}{\delta^2}.\tag{4.90}$$

We also observe that for any $t \in \Omega^1_{kl}$

$$\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|>m\delta} \frac{1}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z \leq \frac{1}{\inf|\pi_y(\Omega^2_{ij})|} \int_{\mathbb{R}^2\setminus\Box_{1+m\delta}} w_\delta(z-t)\mathrm{d}z \leq \frac{4e^{-m}}{\inf|\pi_y(\Omega^2_{ij})|}. \tag{4.91}$$

Substituting (4.90) and (4.91) to (4.89) we get

$$\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} = 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}} |f(t)|\Big(\frac{4e^{-m}}{\inf|\pi_y(\Omega^2_{ij})|} + \sup|\pi_x(\Omega^2_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\Big)\mathrm{d}t \leq$$

$$\leq \sup|\pi_x(\Omega^2_{ij})|\cdot\sup|\pi_y(\Omega^1_{kl})|\cdot m^2\frac{N_\delta}{\delta^2}\|f\|_{\Omega^1,\mathcal{L}_1}.$$

The upper bound of the supremum norm is easy

$$\|W_\delta f\|_\infty = \sup_{z\in\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t\Big| \leq \sup_{z\in\mathbb{R}^2}|f(z)|.$$

The upper bound for the vector fields follows immediately.

■

4.4.2 Constructing an invariant cone

In this Subsection we use approximations we obtained earlier and two cones constructed for the operator $\mathcal{A}$ (Section 4.3, Theorem 8) to get an invariant cone in the space $\mathfrak{X}$ for the operator $W_{\frac{\delta}{2m}}P^2_{\xi*}W_{\frac{\delta}{2m}}$. The main result is Theorem 7. We shall prove two Lemmas first.

Lemma 4.4.4. *There exists $\gamma_4 > 0$ such that for any $\nu \in \mathrm{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^1\right)$ and for arbitrary partition Ω^2 of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$:*

$$\|D_{\Omega^2}W_\delta\nu\|_2 \geq (1-2^{-\gamma_4 m})\|\nu\|_1.$$

(See p. 100 for a general definition of a cone in $\mathfrak{X}$.)

Proof. Let $\nu \in \mathfrak{X}_{\Omega^2}$ be a bounded and integrable vector field. Then similarly to one-dimensional case, by Lemma 4.4.3

$$\|W_{\frac{\delta}{m}}\nu\|_{\Omega^2,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big| \int_{\mathbb{R}^2} w_\delta(z-t)\nu(t)\mathrm{d}t \Big| \mathrm{d}z \leq$$
$$\leq \sup|\pi_y(\Omega^2_{kl})| \cdot \sup|\pi_x(\Omega^2_{ij})| \cdot m^4 \frac{N_\delta}{\delta^2} \cdot \|\nu\|_{\Omega^1,\mathcal{L}_1}.$$

By Lemma 4.4.2 we know that

$$\|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box - D_{\Omega^2}W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 \leq 2^{-\frac{m}{4}}.$$

Now we find a lower bound for the norm of $\|D_{\Omega^2}W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_{\Omega^2}$. Observe that the integral over the unit square $\int_\Box w_\delta(z)\mathrm{d}z \geq 1 - e^{-1/\delta^2}$.

$$\|D_{\Omega^2}W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 \geq \|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 - \|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box - D_{\Omega^2}W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 \geq 1 - 2^{-\frac{m}{4}} - e^{-1/\delta^2}.$$

Consider $\psi \in \mathfrak{X}_{\Omega^1}$, with $\|\psi\|_1 \leq d2^{(\frac{3}{4}+\gamma_1-\alpha)m}$, $\int_\Box \mathring{U}U\psi_u = 0$. Then by Lemma 4.4.1

$$\|W_{\frac{\delta}{m}}\psi - D_{\Omega^2}W_{\frac{\delta}{m}}\psi\|_2 \leq d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}, \tag{4.92}$$

where $\gamma_3 = 1 - \alpha + \frac{2\log_2 m}{m}$; and thus by Lemma 4.4.3

$$\|D_{\Omega^2}W_{\frac{\delta}{m}}\psi\|_2 \leq \|W_{\frac{\delta}{m}}\psi\|_2 + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m} \leq$$
$$\leq d \cdot \sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot \frac{m^2 N_\delta}{\delta^2} + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}. \tag{4.93}$$

We use Lemma 4.4.2 and (4.92), to estimate the approximation error for the field $W_\delta\nu$:

$$\|W_{\frac{\delta}{m}}\nu - D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \leq d\|W_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box - D_{\Omega^2}W_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 + \|W_{\frac{\delta}{m}}\psi - D_{\Omega^2}W_{\frac{\delta}{m}}\psi\|_2 \leq$$
$$\leq d2^{-m/4} + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}.$$

Observe that by Lemma 4.4.3, since $\|\psi\| \leq 2^{\frac{3}{4}+\gamma_1-\alpha}$,

$$\|W_{\frac{\delta}{m}}\nu\|_2 = \|dW_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box + W_{\frac{\delta}{m}}\psi\|_2 \geq \|dW_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\Box\|_2 - \|W_{\frac{\delta}{m}}\psi\|_2 \geq$$
$$\geq d(1 - e^{-m^2/\delta^2}) - \sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot \frac{N_\delta}{\delta^2} m^2 \|\psi\|_1 \geq$$
$$\geq d(1 - e^{-m^2/\delta^2}) - d\sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot 2^{(\frac{3}{4}+\gamma_1)m} \frac{m^2 N_\delta}{\delta}.$$

Summing up altogether

$$\|D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \geq \|W_{\frac{\delta}{m}}\nu\|_2 - \|W_{\frac{\delta}{m}}\nu - D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \geq d\big(1-2^{-m/4}-e^{-m^2/\delta^2}\big)-$$
$$- d\Big(\sup|\pi_y(\Omega^2_{ij})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot 2^{(\frac{3}{4}+\gamma_1)m}\cdot\frac{m^2N_\delta}{\delta}+2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}\Big).$$

We know that $\|\nu\| \leq d(1+2^{(\gamma_1+\frac{3}{4}-\alpha)m})$. Hence

$$\|D_{\Omega^2}W_\delta\nu\| \geq (1-2^{-\gamma_4 m})\|\nu\|,$$

where $\gamma_4 > 0$ has been chosen such that

$$\sup|\pi_y(\Omega^2_{ij})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot 2^{(\gamma_1+\frac{3}{4})m}\cdot\frac{m^2N_\delta}{\delta}+2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}\leq 2^{-\gamma_4 m}.$$

■

Remark 16. It follows from Lemma 4.2.2 and Remark 15 that we can choose the constant γ_4 to be $0 < \gamma_4 < \frac{1}{4} - \gamma_1 < \frac{1}{4}$.

Proposition 4.4.4. *Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. Let $\xi \overset{\text{def}}{=} \sigma^{2m(k-1)}\eta$ Consider a linear operator $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$, approximating the operator $P^2_{\xi*}$, defined according to (4.17). Let Ω^3 be another partition of the class $\mathcal{G}(m,\delta)$.*

$$D_{\Omega^3}W_{\frac{\delta}{m}}\mathcal{A}\colon \overline{\text{Cone}\,(1,\Omega^1)} \to \text{Cone}\,\big(2^{-\gamma_4 m},\Omega^3\big)\,.$$

(See p. 100 for definition of a cone and the chain Υ.)

Proof. According to Theorem 8 p. 141, $\mathcal{A}\colon \text{Cone}\,(1,\Omega^1) \to \text{Cone}\,\Big(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\Big)$. We may write then

$$\mathcal{A}\nu = 2^{2m}\big(\begin{smallmatrix}0\\ d\end{smallmatrix}\big)\chi_\square + \psi,\ \ \psi\in\mathfrak{X}_{\Omega_2},\ \ \|\psi\|_2 \leq d2^{(2\frac{3}{4}+\gamma_1-\alpha)m},\ \ \sum_\square \psi^{ij}_u = 0.$$

By straightforward calculation

$$D_{\Omega^3} W_{\frac{\delta}{m}} \mathcal{A}\nu = 2^{2m} D_{\Omega^3} W_{\frac{\delta}{m}} \left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right)\chi_\square + D_{\Omega^3} W_{\frac{\delta}{m}} \psi.$$

Using Lemma 4.4.1

$$\|D_{\Omega^3} W_{\frac{\delta}{m}} \psi - W_{\frac{\delta}{m}} \psi\|_3 \le 2^{-\gamma_3 m} \|\psi\|_2 \le d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}.$$

Thus introducing γ_4 defined by Lemma 4.4.4 and using Lemma 4.4.3,

$$\begin{aligned} &\|D_{\Omega^3} W_{\frac{\delta}{m}} \psi\|_3 \le \|W_{\frac{\delta}{m}} \psi\|_3 + \|D_{\Omega^3} W_{\frac{\delta}{m}} \psi - W_{\frac{\delta}{m}} \psi\|_3 \le \\ &\le d2^{(\gamma_1+2\frac{3}{4})m} \sup |\pi_y(\Omega^3_{ij})| \cdot \sup |\pi_x(\Omega^2_{ij})| \cdot m^2 \frac{N_\delta}{\delta} + d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m} \le d2^{(2-\gamma_4)m}. \end{aligned} \tag{4.94}$$

By Lemma 4.4.2 we deduce

$$\|D_{\Omega^3} W_{\frac{\delta}{m}} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\chi_\square - \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\chi_\square\|_3 \le 2^{-m/4}$$

Thus we may conclude

$$d2^{2m} D_{\Omega^3} W_{\frac{\delta}{m}} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\chi_\square = d2^{2m} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\chi_\square + \varphi \in \mathfrak{X}_{\Omega^3},$$

where $\|\varphi\|_3 \le d2^{3m/2}$. Together with (4.94) we get the result. ∎

Theorem 7. Let Ω be a partition of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$; and let $\|\xi\|_\infty \le \delta$ be a sequence of real vectors. There exists $r_1(m) \ll r_2(m)$ and $\varepsilon_1(m) \ll \varepsilon_2(m)$ such that

$$W_{\frac{\delta}{2m}} P^2_{\xi *} W_{\frac{\delta}{2m}} : \operatorname{Cone}(r_1, \varepsilon_1, \Omega) \to \operatorname{Cone}(r_2, \varepsilon_2, \Omega) \subsetneq \operatorname{Cone}(r_1, \varepsilon_1, \Omega).$$

$$\left\| W_{\frac{\delta}{2m}} P^2_{\xi *} W_{\frac{\delta}{2m}} |_{\operatorname{Cone}(r_1,\varepsilon_1,\Omega)} \right\| \ge 2^{m-5}$$

(See p. 100 for definition of a cone in the space of vector fields).

Proof. Let Ω^1 be a canonical partition for the map P^2_ξ. First of all we shall find a number r_1 such that for any $\eta \in \operatorname{Cone}(r_1,\Omega)$ we have $D_{\Omega_1} W_{\frac{\delta}{2m}} \eta \in \operatorname{Cone}(1,\Omega^1)$. We may write $\eta = \left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right)\chi_\square + \psi$, with $\sum_\square \psi^{ij}_y = 0$ and $\|\psi\|_\Omega \le dr_1$. Then

$$D_{\Omega^1} W_{\frac{\delta}{2m}} \eta = \left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right) D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\square + D_{\Omega^1} W_{\frac{\delta}{2m}} \psi;$$

and using Lemmas 4.4.1 and 4.4.3, we calculate

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \psi\|_1 \leq \|W_\delta \psi\|_1 + \|D_{\Omega^1} W_{\frac{\delta}{2m}} \psi - W_{\frac{\delta}{2m}} \psi\|_1 \leq$$
$$\leq \left(2^{-\gamma_3 m} + 2^{2-2m} m^4 \frac{N_\delta}{\delta^2}\right) \|\psi\|_\Omega \leq 5 d r_1 m^4 2^{-2m} \frac{N_\delta}{\delta^2}; \quad (4.95)$$

Using Lemma 4.4.2, we calculate

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\square - \chi_\square\|_1 \leq 2^{1-m/4}, \tag{4.96}$$

which implies $D_{\Omega^1} W_\delta \binom{0}{d} \chi_\square = \binom{0}{d} \chi_\square + \psi_1$, where $\psi_1 \in \mathfrak{X}_{\Omega^1}$ and $\|\psi_1\|_1 \leq 2^{1-m/4}$. Hence $D_{\Omega^1} W_{\frac{\delta}{2m}} \eta = \binom{0}{d} \chi_\square + D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\square + \psi_1$, where

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\square + \psi_1\|_1 \leq d r_1 \Big(m^4 2^{-2m} \frac{N_\delta}{\delta^2} + 2^{1-m/4}\Big).$$

In order to guarantee $D_{\Omega_1} W_{\frac{\delta}{2m}} \eta \in \operatorname{Cone}\left(1, \Omega^1\right)$ it is sufficient to choose r_1 such that

$$m^4 2^{-2m} \frac{N_\delta}{\delta^2} \leq \frac{1}{r_1}.$$

We set

$$r_1 \stackrel{\text{def}}{=} \frac{2^{2m} \delta^2}{4 m^4 N_\delta}. \tag{4.97}$$

We can also notice using Lemma 4.4.1 that

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \eta - W_{\frac{\delta}{2m}} \eta\|_1 \leq d r_1 2^{-\gamma_3 m}.$$

Taking into account $D_{\Omega^1} W_{\frac{\delta}{2m}} \eta \in \operatorname{Cone}\left(1, \Omega^1\right)$ we deduce $W_{\frac{\delta}{2m}} \eta \in \widehat{\operatorname{Cone}}\left(1, r_1 2^{-\gamma_3 m}, \Omega^1\right)$. We also observe that by Lemma 4.4.3 for any $v = \eta + g \in \widehat{\operatorname{Cone}}\left(r_1, \varepsilon_1, \Omega\right)$ we have

$$\left\|W_{\frac{\delta}{2m}} g\right\| \leq 4 \varepsilon_1 m^2 \frac{N_\delta}{2^{2m} \delta^2} = \frac{16 \varepsilon_1}{m^2 r_1} =: \widetilde{\varepsilon_1}.$$

We will be assuming that $\widetilde{\varepsilon_1} \geq r_1 2^{-\gamma_3 m}$. Then without loss of generality

$$W_{\frac{\delta}{2m}} : \widehat{\operatorname{Cone}}\left(r_1, \varepsilon_1, \Omega\right) \to \widehat{\operatorname{Cone}}\left(1, \widetilde{\varepsilon_1}, \Omega\right). \tag{4.98}$$

Let $\mathcal{A}\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ be a linear operator approximating $P^2_{\xi *}$ and defined by (4.17), p. 104. It follows from Theorem 8 p. 141, that $\mathcal{A}\colon \mathrm{Cone}\left(1, \Omega^1\right) \to \mathrm{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^1\right) \subset \mathfrak{X}_{\Omega^2}$; moreover, the norm is growing exponentially with number of iterations $\|\mathcal{A}\mid_{\mathrm{Cone}(1,\Omega^1)}\| \geq 2^{2m-1}$. In particular, we see that for any vector field $\nu \in \mathrm{Cone}\left(1, \Omega^1\right)$,

$$\|\mathcal{A}\nu\|_2 = \|\mathcal{A}((\begin{smallmatrix}0\\d\end{smallmatrix})\chi_\square + \psi)\|_2 \geq d\|\mathcal{A}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 - \|\mathcal{A}\psi\|_2 \geq d2^{2m}\left(1 - 2^{(\gamma_1+\frac{3}{4}-\alpha)m}\right)$$

Consider a vector field $v = \nu + g \in \widehat{\mathrm{Cone}}\left(1, \widetilde{\varepsilon}_1, \Omega^1\right)$, where $\nu \in \mathrm{Cone}\left(1, \Omega^1\right) \subset \mathfrak{X}_{\Omega^1}$ is a piecewise constant part with the norm $\|\nu\|_1 \leq d$ and $\|g\|_1 < \widetilde{\varepsilon}_1 d$. Then by linearity $P^2_{\xi *}v = P^2_{\xi *}\nu + P^2_{\xi *}g$. By inequality (4.3.17) of Lemma 4.3.17,

$$\|P^2_{\xi *}g\|_\Omega \leq m2^{2m+2}\|g\|_1 \leq md\widetilde{\varepsilon}_1 2^{2m+2}. \tag{4.99}$$

By Proposition 4.3.3 for $\nu \in \mathrm{Cone}\left(1, \Omega^1\right) \subset \mathfrak{X}_{\Omega^1}$

$$\|W_{\frac{\delta}{2m}}(P^2_{\xi *} - \mathcal{A})\nu\|_\Omega \leq 8\frac{\sup \mathrm{diam}(\Omega_{ij})}{\delta}2^{2m}\|\nu\|_1 \leq d2^{m+4}\delta. \tag{4.100}$$

We have decomposition

$$W_{\frac{\delta}{2m}}P^2_{\xi *}v = W_{\frac{\delta}{2m}}P^2_{\xi *}\nu + W_{\frac{\delta}{2m}}P^2_{\xi *}g = W_{\frac{\delta}{2m}}(P^2_{\xi *} - \mathcal{A})\nu + W_{\frac{\delta}{2m}}\mathcal{A}\nu + W_{\frac{\delta}{2m}}P^2_{\xi *}g. \tag{4.101}$$

We write $W_{\frac{\delta}{2m}}\mathcal{A}\nu$ and $W_{\frac{\delta}{2m}}P^2_{\xi *}u$ as a sum of piecewise-constant part and a remainder

$$W_{\frac{\delta}{2m}}\mathcal{A}\nu = \nu_1 + g_1, \text{ where } \nu_1 = D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu \in \mathfrak{X}_\Omega, \text{ and } g_1 = W_{\frac{\delta}{2m}}\mathcal{A}\nu - D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu; \tag{4.102}$$

$$W_{\frac{\delta}{2m}}P^2_{\xi *}g = \nu_2 + g_2, \text{ where } \nu_2 = D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi *}g \in \mathfrak{X}_\Omega, \text{ and } g_2 = W_{\frac{\delta}{2m}}P^2_{\xi *}g - D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi *}g. \tag{4.103}$$

We estimate all four terms separately.

Using Lemmas 4.4.1 and 4.3.18, since $\|\nu\|_1 \leq d$, we get

$$\|g_1\|_\Omega = \|W_{\frac{\delta}{2m}}\mathcal{A}\nu - D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu\|_\Omega \leq 2^{-\gamma_3 m}\|\mathcal{A}\nu\|_\Omega \leq d2^{(2-\gamma_3)m}. \tag{4.104}$$

By Lemmas 4.4.1 and 4.3.18, using $\|g\|_1 \le d\widetilde{\varepsilon}_1$, and (4.99)

$$\|g_2\|_\Omega = \|W_{\frac{\delta}{2m}} P^2_{\xi^*} g - D_\Omega W_{\frac{\delta}{2m}} P^2_{\xi^*} g\|_\Omega \le 2^{-\gamma_3 m} \|P^2_{\xi^*} g\|_\Omega \le m d\widetilde{\varepsilon}_1 2^{(2-\gamma_3)m+2}. \tag{4.105}$$

Finally, using (4.99) and (4.105),

$$\|\nu_2\|_\Omega = \|D_\Omega W_{\frac{\delta}{2m}} P^2_{\xi^*} g\|_\Omega \le \|P^2_{\xi^*} g\|_\Omega + \|W_{\frac{\delta}{2m}} P^2_{\xi^*} g - D_\Omega W_{\frac{\delta}{2m}} P^2_{\xi^*} g\|_\Omega \le$$
$$\le m d\widetilde{\varepsilon}_1 2^{2m+2}\left(1 + 2^{-\gamma_3 m}\right). \tag{4.106}$$

We now need a lower bound for the norm of ν_1 defined by (4.102). By Theorem 8 p. 141 we have $\mathcal{A}\nu \in \mathrm{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^1\right)$, and Lemma 4.4.4 is applicable:

$$\|\nu_1\|_\Omega = \|D_\Omega W_{\frac{\delta}{2m}} \mathcal{A}\nu\|_\Omega \ge (1 - 2^{-\gamma_4 m}) \cdot \|\mathcal{A}\nu\|_2 \ge d 2^{2m}\left(1 - 2^{(\frac{3}{4}+\gamma_1-\alpha)m}\right)(1 - 2^{-\gamma_4 m}). \tag{4.107}$$

We need to check that

$$\nu_1 + \nu_2 = D_\Omega W_{\frac{\delta}{2m}} \mathcal{A}\nu + D_\Omega W_{\frac{\delta}{2m}} P^2_{\xi^*} g \in \mathrm{Cone}\,(r_2, \Omega)\,; \tag{4.108}$$

and to verify the inequality

$$\|g_1\|_\Omega + \|g_2\|_\Omega + \|W_{\frac{\delta}{2m}} (P^2_{\xi*} - \mathcal{A})\nu\|_\Omega \le \|\nu_1 + \nu_2\|_\Omega \cdot \varepsilon_2. \tag{4.109}$$

Consider a vector field $\nu = \binom{0}{d}\chi_\square + \psi \in \mathrm{Cone}\left(1, \Omega^1\right)$ with $\|\psi\|_1 \le d$ and $\sum\limits_{\square} \psi^{ij}_u = 0$. Using Theorem 8 p. 141 we write $\mathcal{A}\nu = d2^{2m}\binom{0}{1}\chi_\square + \varphi$, where $\varphi \in \mathfrak{X}_{\Omega^2}$, and $\|\varphi\|_2 \le 2^{(2\frac{3}{4}+\gamma_1-\alpha)m}$.

For the first inclusion (4.108), we expand $D_\Omega W_{\frac{\delta}{2m}} \mathcal{A}\nu$ as following.

$$D_\Omega W_{\frac{\delta}{2m}} \mathcal{A}\nu = D_\Omega W_{\frac{\delta}{2m}} (d2^{2m}\tbinom{0}{1}\chi_\square + \varphi) = d2^{2m} D_\Omega W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square + D_\Omega W_{\frac{\delta}{2m}} \varphi =$$
$$= d2^{2m}\tbinom{0}{1}\chi_\square + d2^{2m}\Big(D_\Omega W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square - W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square + W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square - \tbinom{0}{1}\chi_\square\Big) +$$
$$+ (D_\Omega W_{\frac{\delta}{2m}} \varphi - W_{\frac{\delta}{2m}} \varphi) + W_{\frac{\delta}{2m}} \varphi.$$

We see that by Lemma 4.4.2

$$d2^{2m}\|D_\Omega W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square - W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square\|_\Omega \le d2^{\frac{7}{4}m}; \tag{4.110}$$

$$d2^{2m}\|W_{\frac{\delta}{2m}} \tbinom{0}{1}\chi_\square - \tbinom{0}{1}\chi_\square\|_\Omega \le d2^{\frac{7}{4}m}. \tag{4.111}$$

By Lemma 4.4.1 again, since $\|\varphi\|_2 \leq d2^{(2\frac{3}{4}+\gamma_1-\alpha)m}$

$$\|D_\Omega W_{\frac{\delta}{2m}}\varphi - W_{\frac{\delta}{2m}}\varphi\|_\Omega \leq d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}. \tag{4.112}$$

Therefore we may write

$$D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu = d2^{2m}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square + \varphi \in \mathfrak{X}_\Omega, \tag{4.113}$$

where

$$\phi = d2^{2m}(D_\Omega W_{\frac{\delta}{2m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square) + D_\Omega W_{\frac{\delta}{2m}}\varphi \in \mathfrak{X}_\Omega;$$

with the norm that can be bounded using (4.110), (4.111) and (4.112)

$$\begin{aligned}\|\phi\|_\Omega &\leq d2^{2m}\|D_\Omega W_{\frac{\delta}{2m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square\|_\Omega + \|D_\Omega W_{\frac{\delta}{2m}}\varphi - W_{\frac{\delta}{2m}}\varphi\|_\Omega + \|W_{\frac{\delta}{2m}}\varphi\|_\Omega \leq \\ &\leq d\Big(2^{\frac{7}{4}m+1} + 2^{(\gamma_1+2\frac{3}{4}-\alpha)m}\cdot\Big(2^{-\gamma_3 m} + \sup|\pi_y(\Omega^2_{kl})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\Big)\Big) \leq \\ &\leq 4d\cdot 2^{(2-\gamma_4)m}.\end{aligned} \tag{4.114}$$

Thus using (4.113) and (4.102), (4.103), we write

$$D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu + D_\Omega W_{\frac{\delta}{2m}}P^{2m}_{\xi*}g = \nu_1+\nu_2 = d2^{2m}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square + \phi + \nu_2. \tag{4.115}$$

Then the condition (4.108): $\nu_1+\nu_2 \in \widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega)$ is equivalent to $\|\phi+\nu_2\|_\Omega \leq dr_2 2^{2m}$. We see using (4.114) and (4.106) that

$$\begin{aligned}\|\phi+\nu_2\|_\Omega \leq \|\phi\|_\Omega + \|\nu_2\|_\Omega &\leq 4d\cdot 2^{(2-\gamma_4)m} + 4dm\widetilde{\varepsilon_1}2^{2m}\big(1+2^{-\gamma_3 m}\big) = \\ &= 4d2^{2m}\big(2^{-\gamma_4 m} + m\widetilde{\varepsilon_1}\big(1+2^{-\gamma_3 m}\big)\big)\end{aligned} \tag{4.116}$$

Now recall the second inequality (4.109)

$$\|g_1\|_\Omega + \|g_2\|_\Omega + \|W_{\frac{\delta}{2m}}(P^2_{\xi*}-\mathcal{A})\nu\|_\Omega \leq \varepsilon_2\|\nu_1+\nu_2\|_\Omega. \tag{4.117}$$

We know already from (4.100), (4.104) and (4.105),

$$\|g_1\|_\Omega + \|g_2\|_\Omega + \|W_{\frac{\delta}{2m}}(P^2_{\xi*}-\mathcal{A})\nu\|_\Omega \leq d2^{2m}\Big(2^{-\gamma_3 m} + \widetilde{\varepsilon_1}2^{-\gamma_3 m} + 2^{(\alpha-1)m+1}\Big) \leq 3d\widetilde{\varepsilon_1}2^{(2-\gamma_3)m}.$$

Using (4.107) and (4.106), we deduce, taking into account Remark 15 and Remark 16 $\gamma_3 < 1-\alpha$ and $\gamma_4 < \frac{1}{4} - \gamma_1$, and $\alpha = \frac{15}{16}$:

$$\|\nu_1 + \nu_2\| \geq d2^{2m}\left(1 - 2^{(\frac{3}{4}+\gamma_1-\alpha)m}\right)\left(1 - 2^{-\gamma_4 m}\right) - d2^{2m}\widetilde{\varepsilon_1}\left(1 + 2^{-\gamma_3 m}\right) \geq$$
$$\geq d2^{2m}\left(1 - 2^{(\frac{3}{4}+\gamma_1-\alpha)m} - 2^{-\gamma_4 m} - \widetilde{\varepsilon_1}2^{-\gamma_3 m}\right) \geq d2^{2m}\left(1 - \widetilde{\varepsilon_1}2^{-\frac{m}{24}}\right) \quad (4.118)$$

Therefore (4.108) and (4.109) would follow from

$$3\widetilde{\varepsilon_1}2^{-\gamma_3 m} \leq \varepsilon_2\left(1 - \widetilde{\varepsilon_1}2^{-\frac{m}{24}}\right) \quad (4.119)$$

$$2^{-\gamma_4 m} + \widetilde{\varepsilon_1} + \widetilde{\varepsilon_1}2^{-\gamma_3 m} < r_2. \quad (4.120)$$

Recall now that $\widetilde{\varepsilon_1} = 4\varepsilon_1 m^2 \frac{N_\delta}{2^{2m}\delta^2}$. We may choose the following parameters for the cones $r_2 = 2^{-m\frac{1-\alpha}{4}} = 2^{-\frac{m\alpha}{64}}$, $\varepsilon_1 = 2^{-m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{32}}$, and $\varepsilon_2 = 2^{-2m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{16}}$. It is clear that $r_2 \ll r_1 = \frac{2^{2m}\delta^2}{4m^4 N_\delta}$ and the second condition on the norm follows immediately from (4.115), (4.116), and (4.117).

■

The proof of the existence of an invariant cone is complete. The fast dynamo theorem in dimension two follows as shown in Section 2.2. It is the main result of the present work.

Theorem 9. *There exists a volume preserving piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\|(\exp(\varepsilon\Delta)F_*)^n B_0\|_{\mathcal{L}_1} > 0.$$

The map F may be realised as a Poincaré map of an incompressible fluid flow filling a compact domain in $\mathbb{R}^3$ (an immersed 3-dimensional manifold with a boundary).

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