MAXIMAL TORI IN FINITE GROUPS OF LIE TYPE

by

Philip Charles Gager

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ABSTRACT.

It is believed that a unified approach to a study of the representation theory of the finite groups of Lie type should begin with a study of the regular characters of the maximal tori of these groups. This thesis is directed towards determining the structure of the maximal tori in the finite groups of Lie type.

Chapter 1 is a general introduction to the properties of Chevalley groups, together with the consequences of a result of Springer and Steinberg. This result establishes a correspondence between the conjugacy classes of maximal tori and certain equivalence classes of the associated Weyl group. In certain cases, these classes are the conjugacy classes, and Chapter 2 begins with a review of Carter's unified approach to the conjugacy classes of Weyl groups. Chapter 2 also includes some results on automorphisms of Weyl groups in relation to Carter's approach.

The finite Chevalley groups are the first to be considered. Chapter 3 studies those of type A_ℓ, and Chapter 4 simultaneously considers the Chevalley groups of types B_ℓ, C_ℓ and D_ℓ. Finally, Chapter 5 presents the results for the Chevalley groups of exceptional type.

The finite groups of twisted type are the last to be discussed. Chapter 6 begins with a general description of the classes of the Weyl group in these types and concludes with the results for the Steinberg groups of types ${}^2A_\ell$, ${}^2D_\ell$ and ${}^2E_6$. The Steinberg groups of type ${}^3D_4$ are left until the end of Chapter 7, after a discussion of the Ree and Suzuki groups.

The thesis concludes with a note on the representation theory and a description of the regular characters of the maximal tori.

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INTRODUCTION

In 1955, Chevalley showed how to construct analogues of the complex simple Lie groups over arbitrary fields. These Chevalley groups were found to be simple and were identified by Ree with some families of classical simple groups over finite fields. However, there were families of classical groups which could not be identified in this way, for example the unitary groups. Independently, Steinberg and Tits modified Chevalley's construction to obtain more finite simple groups of Lie type, the "twisted groups" or Steinberg groups. These groups accounted for the remaining classical groups over finite fields and, in fact, added to the list of previously known finite simple groups. However, there were found to be three types of Chevalley group which suggested that they might lead to further families of finite simple groups, although Steinberg's construction did not yield such results. Then, in 1960, Suzuki discovered a new family of simple groups using properties of centralisers of elements. Ree immediately interpreted these as a new type of twisted group, and then proved the existence of two further families. These three families are the Ree and Suzuki groups.

These finite groups of Lie type have evoked considerable interest since their introduction and rapid progress has been made in the general theory. However, a present unsolved problem is a unified approach to the representation theory of these finite groups, and it is believed that a study of certain characters of a particular type of abelian subgroup (called maximal tori) will lead to a solution of the problem. The representation theory problem has been solved in a few particular cases, notably the work of Green on $GL_n(q)$, and also by Srinivasan in the case.

of $Sp_4(q)$, Lehrer in the case of $SL_n(q)$ and Ree and Chang in the case of $G_2(q)$. In such a light, the purpose of this thesis is to determine the structure of these maximal tori in the finite groups of Lie type.

One of the more rewarding viewpoints of the Chevalley groups is as split-forms of linear algebraic groups, and it is from this standpoint that the required results are determined. From a theorem of Lang on the fixed-point group of an endomorphism of a linear algebraic group, Springer and Steinberg have shown in [19] that there is a bijection between the conjugacy classes of maximal tori in the finite group and certain equivalence classes in the corresponding Weyl group. This result, and its consequences, is discussed in Chapter 1, together with a discussion of automorphisms of Chevalley groups. In certain cases, the above equivalence classes are just the conjugacy classes of the Weyl group. Since certain ideas and results from it will be needed throughout the thesis, a brief survey of Carter's paper on the conjugacy classes [6] is included in Chapter 2, together with some results on automorphisms of Weyl groups which are related to op.cit.

A working knowledge of most standard facts on linear algebraic groups will be assumed, and these can be found in [1] and [8]. Carter's recently published "Simple groups of Lie type" also provides an ideal reference to the finite groups considered in this thesis. For the construction of, and the main results of, Chevalley groups, the reference is §§1-3 of [2]. Some of this is treated in greater detail by Steinberg in [20], but the thesis will be based upon the notation of the former. Thus, associated with the triple $\{\mathfrak{g}, \pi, k\}$ consisting of a complex, semi-simple Lie algebra $\underline{g}$, a faithful representation $\pi: \underline{g} \longrightarrow \underline{gl}(E)$ of $\underline{g}$ over a complex vector space E , and the field k , there exists a corresponding Chevalley group $G_{\pi,k}$. Chevalley has shown in [8] that, if K is an algebraically closed field, every connected, semi-simple, linear algebraic group over K is isomorphic to one of the groups $G_{\pi,K}$.

Throughout the thesis, K_o denotes the finite field $GF(q)$ of $q = p^n$ elements, for some prime p, and the algebraic closure of K_o is denoted by K . Then G_{π,K_o} is a finite Chevalley group. Let $G = G_{\pi,K}$, so that G is a subgroup of $GL_n(K)$ for some n , and is also an algebraic set in the affine space which is determined by the n^2 matric coefficients and is subject to the Zariski topology. Let $G(q)$ denote the K_o-rational points of G, ie. $G(q) = G_{\pi,K} \cap GL(E_Z \otimes_Z K_o)$, where E_Z is an admissible Z-form of E with a basis of eigenvectors of the Cartan subalgebra $\underline{h}$ of $\underline{g}$. Then it is known that G_{π,K_o} is the derived group of $G(q)$. In the case that G is simply connected, ie. π is the sum of the representations having the fundamental weights as highest weights, then G_{π,K_o} , is equal to $G(q)$.

The group $G(K)$, of K-rational points of G , can (and will) be identified with G , and a <u>torus</u> in G is defined to be a closed subgroup which is isomorphic to a direct product of r copies of K^* , for some $r > 0$, where K^* is the multiplicative group of K . A <u>maximal torus</u> of G is one contained in no other, so that r is equal to ℓ , the <u>rank</u> of G . In [8] , Chevalley shows that the maximal tori of G are all conjugate in G .

Now let σ be an endomorphism of G onto itself and let G_σ be the group of σ-fixed points. Then Lang has shown that if G_σ is finite, any element of G can be written as $g.\sigma(g)^{-1}$ for some $g \in G$. This is discussed in Chapter 1. The justification for this approach is that the groups $G(q)$ can be realised as groups of type G_σ. For, if σ is the automorphism of G (induced from the Frobenius automorphism of K) which raises every matric entry of an element $g \in G$ to its q^{th} power, then G_σ is finite and is just $G(q)$, the group of K_0-rational points of G. These are the finite Chevalley groups. The finite twisted groups of Lie type are constructed by combining the above automorphism with another type of automorphism and are the groups G_σ again. This is discussed in Chapters 1, 6 and 7.

A <u>torus</u> of G_σ is defined to be the group T_σ of σ-fixed points of a σ-fixed torus T of G, and a <u>maximal torus</u> to be a subgroup obtained in the same way from a maximal torus of G. Then a maximal torus in $G(q)$ is the group of K_0-rational points of a K_0-maximal torus (ie. one defined over K_0) in G. It is to be noted that the maximal tori of G_σ need not be maximal in the set of tori of G_σ, and certainly they need not be isomorphic to direct products of copies of K_0. For example, there always exists a maximal torus of $G(q)$ which comes from a K_0-split maximal torus (diagonalisable over K_0) of G. When $q = 2$ this consists solely of the identity element and is thus contained in other (non-maximal) tori. The aim of this thesis is to determine the structure of the maximal tori of the groups G_σ, when G is a simple Chevalley group $G_{\pi,K}$ and σ is an automorphism of G.

Given $\underline{g}$ and k , there is a collection $\mathcal{G}$ of groups $\{G_{\pi_1,k},\ldots,G_{\pi_n,k}\}$ corresponding to different representations π_i of $\underline{g}$. $\mathcal{G}$ is called the <u>isogeny class</u> of type $\{\underline{g}\ ,\ k\}$ and there exists (under certain conditions depending upon π_i , π_j) an isogeny $\pi_{ij} : G_{\pi_i,k} \longrightarrow G_{\pi_i,k}$, where an isogeny is a surjective k-rational homomorphism with finite kernel. In most cases, the cardinality of this set is either 2 (when π_1 is the simply connected representation and π_2 is the adjoint representation of $\underline{g}$ on itself) or 1 (when π_1 and π_2 coincide). However, in groups of type A_ℓ and D_ℓ , the cardinality of $\mathcal{G}$ is greater than 2 . One might expect the results of Chapter 1 to imply that the results of the thesis are independent of π . In fact, it is shown in Proposition 1.4. that the order of a maximal torus T_σ of $G_{\pi,K}(K_o)$ is independent of the representation π . It is further proved in [1] that the order of $G_{\pi,K}(K_o)$ is independent of π , as in the case of groups of type A_ℓ , where $G_{ad,K}(K_o) \cong PGL_{\ell+1}(q)$ and $G_{sc,K}(K_o) \cong SL_{\ell+1}(q)$ have the same order. This, of course, although strongly suggestive, does not imply that the structure of the maximal torus is independent of π , and the dependence upon π in types A_ℓ and D_ℓ will be seen. It is to be noted however that in the adjoint groups it is the representation theory which is well-behaved, whereas in the simply-connected groups it is the conjugacy classes which are well-behaved.

The finite Chevalley groups of type A_ℓ are discussed in Chapter 3, and those of type B_ℓ , C_ℓ and D_ℓ in Chapter 4. The reason for this simultaneous treatment becomes obvious when one considers the results of

[6]. The finite Chevalley groups of exceptional type are discussed in Chapter 5. At present, only partial results for the groups of type E_n are presented, due to the size of the groups $W(E_n)$. However, the results for the most interesting cases, viz. the semi-Coxeter tori, are presented.

As remarked earlier, when σ is modified from a pure field automorphism to include a graph automorphism, then other finite groups due to Steinberg, and further groups due to Ree and Suzuki, are recovered. The determination of the maximal tori in the former type is discussed in Chapter 6, and the latter in Chapter 7. Because of the connection between types D_4 and F_4, the Steinberg groups of type ${}^3D_4$ are discussed with the Ree groups of type ${}^2F_4$ in Chapter 7.

Finally, the thesis is concluded with a brief note on the representation theory of the groups G_σ. This outlines Springer's recent work towards a unified approach.

To the best of the author's knowledge, those results in this thesis which are not otherwise attributed are original.

CHAPTER 1. Preliminaries

§1.1 Definition and properties of Chevalley groups.

For any group G and any subset $S \subseteq G$, we denote by $Z_G(S)$ the centraliser of S in G, the normaliser of S in G by $N_G(S)$, and for any $x \in G$ we denote $x.S.x^{-1}$ by ${}^xS$. For any $g \in G$, we denote by i_g the inner automorphism, $i_g : x \longmapsto {}^gx$, of G, and by $\mathcal{C}(G)$, the set of conjugacy classes of G. Let N be the monoid of natural numbers, Z the ring of integers, and R (resp. C) the field of real (resp. complex) numbers.

If $\underline{h}$ is the Cartan subalgebra of the complex, semi-simple Lie algebra $\underline{g}$ corresponding to the Cartan decomposition of $\underline{g}$, and $\{h_a, e_b : a \in \Pi, b \in \Sigma\}$ is a Chevalley basis of $\underline{g}$ with root system Σ and fundamental roots Π, and if $\underline{h}^*$ is the dual space of $\underline{h}$, then $m \in \underline{h}^*$ is a <u>weight</u> if and only if $m(h_a) \in Z$ for all $a \in \Pi$. The weights form a Z-form of $\underline{h}^*$ which we call Δ_{sc}, and the fundamental weights, i.e. those $m_a \in \underline{h}^*$ for which $m_a(h_b) = \delta_{ab}$ for $a, b \in \Pi$, form a basis of this. The subgroup Δ_{ad} generated by the roots has finite index in Δ_{sc}, and in fact Δ_{sc}/Δ_{ad} has relation matrix (A_{ij}), the Cartan matrix of Π. If (π, E) is a finite dimensional $\underline{g}$-module and $m \in \underline{h}^*$, we let $E_m = \{v \in E : \pi(h).v = m(h).v \text{ for all } h \in \underline{h}\}$. Those $m \in \underline{h}^*$ for which $E_m \neq 0$ are the weights of π, and the space E is the direct sum of the E_m, where m runs through the set $P(\pi)$ of the weights of π. We let Δ_π denote the subgroup of $\underline{h}^*$ generated by the weights of π . If π is faithful , then $\Delta_{sc} \supset \Delta_\pi \supset \Delta_{ad}$, and we know from the representation theory that , given Δ between Δ_{sc} and Δ_{ad} , there always exists a faithful representation π of $\underline{g}$ such that $\Delta_\pi = \Delta$.

Let (π, E) be any $\underline{g}$-module. An <u>admissible Z-form</u> of E is a Z-form which is stable under $\pi(e_a^j / j!)$ for $a \in \Sigma$ and $j \in N$. If $\pi : \underline{g} \longrightarrow \underline{gl}\,(E)$ represents $\underline{g}$ in the Lie algebra of endomorphisms of a complex vector space E , then we identify E with C^n by means of a basis of an admissible Z-form E_Z , which consists of eigenvectors of $\underline{h}$. Then $x_a^{(\pi)}(t) = \exp\,(t.\pi(e_a))$ defines an automorphism of $E_{(k)} = E_Z \otimes_Z k$ for every $t \in k$. Then , the Chevalley group associated with $\{\underline{g}\ , \pi ,\ k\}$ is $G_{\pi,k} = \langle x_a^{(\pi)}(t) : a \in \Sigma,\ t \in k \rangle$, and this is a subgroup of $GL\,(E_Z \otimes_Z k)$. If there is no ambiguity , we shall write $x_a(t)$ for $x_a^{(\pi)}(t)$. By $\langle S : R \rangle$, we mean the group generated by the set S , subject to the relation set R .

For each $a \in \Sigma$, there is a unique homomorphism $\mu_a : SL_2(k) \longrightarrow G_{\pi,k}$ which maps $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$) onto $x_a^{(\pi)}(t)$ (resp. $x_{-a}^{(\pi)}(t)$) for every $t \in k$. Let $h_a(t)$ be the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, for $t \in k^*$, under μ_a . The group $H = \langle h_a(t) : a \in \Pi,\ t \in k^* \rangle$ is a maximal torus of $G_{\pi,k}$.

If Δ is a Z-form of $\underline{h}^*$, we let Δ^* denote the dual Z-form of $\underline{h}$, i.e.

$$\Delta^* = \{ h \in \underline{h} : a(h) \in Z \text{ for } a \in \Delta \} .$$

If $m \in P(\pi)$ is a weight of π, then, putting

$E_{(k),m} = (E_Z \wedge E_m) \otimes_Z k$, we have

$$h_a(t) \,.\, x = t^{m(h_a)} \,.\, x \text{, for } x \in E_{(k),m} \,;\; t \in k^* \,;\; a \in \Sigma \,.$$

To m there is associated a homomorphism of H into k^* characterised by $m\left(\prod_{a \in \pi} h_a(t_a)\right) = \prod_{a \in \pi} t_a^{m(h_a)}$. (1) .

Let w_a be the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under μ_a, for $a \in \Sigma$. Then w_a normalises H. We let $N = \langle H, w_a : a \in \Sigma \rangle$. Let $U = U^+ = \langle x_a(t) : t \in k, a > 0 \rangle$ and $U^- = \langle x_a(t) : t \in k, a < 0 \rangle$. Then the group $U^{\pm}$ is unipotent. Let $a_1, a_2, \ldots, a_N$ be the positive roots. Then $\nu_{\pm} : (t_1, \ldots, t_N) \longmapsto x_{\pm a_1}(t_1). \ldots .x_{\pm a_N}(t_N)$, is a bijection of k^N onto $U^{\pm}$.

The group $U^{\pm}$ is normalised by H in the following way :-

$$h.x_a(t) \,.\, h^{-1} = x_a\,(a(h).t) \quad \text{for } t \in k \,;\; h \in H \,;\; a \in \Sigma.$$

Let $B = B^+ = H.U$, $B^- = H \,.\, U^-$. Then (B,N) is a B.N-pair in $G_{\pi,k}$ with root system Σ, and B is a Borel subgroup.

Let ρ be another faithful representation of $\underline{g}$. If $\Delta_\pi \supset \Delta_\rho$, then the map $x_a^{(\pi)}(t) \longmapsto x_a^{(\rho)}(t)$ for $t \in k$; $a \in \Sigma$ extends to a surjective homomorphism $\lambda_{\rho,\pi} : G_{\pi,k} \longrightarrow G_{\rho,k}$.

Suppose now that $k = K$ is algebraically closed. Then the kernel of $\lambda_{\rho,\pi}$ is isomorphic to $\Delta_\rho^* / \Delta_\pi^*$ modulo p-torsion. Furthermore, H is a K_0-split maximal torus of $G_{\pi,K}$.

For $m \in P(\pi)$, the homomorphism of H into K^* associated to m by (1) is a rational character of H. Let $X(H) = \mathrm{Mor}\,(H, K^*)$ be the group of rational characters of H. Then the map $P(\pi) \longrightarrow X(H)$ defined by (1) induces an isomorphism of Δ_π onto $X(H)$. We shall identify $X(H)$ with Δ_π via this isomorphism. Furthermore, we shall henceforth use the notation h^m for $m(h)$, where $m \in P(\pi)$ and $h \in H$.

The Weyl group $W = N/_H$ acts on $\underline{h}$, $\underline{h}^*$ and on $P(\pi)$. We have $\langle w(m), w(h) \rangle = \langle m, h \rangle$, where $m \in \underline{h}^*$; $h \in \underline{h}$; $w \in W$ and $\langle , \rangle$ is the canonical pairing (the non-degenerate restriction to $\underline{h}$ of the Killing form on $\underline{g}$.). Also, W acts on H via inner automorphisms and on $X(H)$ by $h^{w(x)} = (h^w)^x$ for $w \in W$ and $x \in X(H)$. From now, $(\ ,\ )$ denotes a fixed W - invariant, positive-definite bilinear form on $X(H)$.

Henceforth, we let $\underline{g}$ be semi-simple, and let $G = G_{\pi,K}$. Then the conjugates of H under G are all the maximal tori of G [8], and we let T be any such maximal torus of G. If $N = N_G(T)$, then $W = N/_T$ is the Weyl group of G relative to T, and n_w denotes a representative for $w \in W$ in N. If $X(T) = \mathrm{Mor}\,(T, K^*)$ then we know that there exists an isomorphism between Δ_π and $X(T)$.

Definition 1.1. If A is a group upon which the endomorphism σ acts, we let $H^1(\sigma, A)$ denote A modulo the equivalence relation :-

(R) $a \underset{R}{\sim} b$ if and only if $a = c.b.\sigma(c)^{-1}$, for some $c \in A$.

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§1.2. Automorphisms of G.

In this section, we consider automorphisms σ of the Groups $G_{\pi,k}$, for any field k . Steinberg shows [20] that automorphisms of $G_{\pi,k}$ are of the following types :-

(1). Field automorphisms . If $\gamma \in \mathrm{Aut}(k)$, then the map $\bar{\gamma} : G_{\pi,k} \longrightarrow G_{\pi,k}$, defined by $\bar{\gamma} : x_a^{(\pi)}(t) \longmapsto x_a^{(\pi)}(t^{\gamma})$ on generators , extends to an automorphism of $G_{\pi,k}$.

(2). Diagonal automorphisms . Let $f : \Pi \longrightarrow k^*$ be a map associating an element $f_a \in k^*$ to each fundamental root $a \in \Pi$. Let f be extended to a homomorphism of Δ_{ad} into k^* . Then $\bar{f} : G_{\pi,k} \longrightarrow G_{\pi,k}$ such that $\bar{f} : x_a^{(\pi)}(t) \longmapsto x_a^{(\pi)}(f_a . t)$ extends to a unique automorphism of $G_{\pi,k}$. If k is algebraically closed , then every diagonal automorphism can be realised as conjugation by a semi-simple element , i.e. an inner automorphism .

(3). Inner automorphisms .

(4). Graph automorphisms . Let Σ be an indecomposable root system and ρ an angle-preserving permutation of the fundamental roots such that $\rho \neq 1$. If all roots are equal in length , then ρ extends to an automorphism of Σ . If not , and $p = (a, a)/(b, b)$ for a long and b short , then ρ must interchange long and short roots and ρ extends to a permutation of all roots which also interchanges long and short roots , and is such that the map $\bar{\rho} : \left\{ \begin{matrix} a \longmapsto \rho a \text{ for } a \text{ long} \\ a \longmapsto p\rho a \text{ for } a \text{ short} \end{matrix} \right\}$ is an isomorphism of root systems . Let k be a field and let $G_{\pi,k}$ be constructed from a simple Lie algebra $\mathfrak{g}$ with root system Σ . If two root lengths occur , then we assume that k is perfect of characteristic p . If $\mathfrak{g}$ is of type D_{2n} , then we must assume that $\rho(\Delta_\pi) = \Delta_\pi$.

Then there exists an automorphism Q of G and signs ε_a (such that $\varepsilon_a = 1$ if $\pm a \in \pi$) such that

$$Q\, x_a(t) = \begin{cases} x_{\rho a}(\varepsilon_a t) & \text{if } a \text{ is long or all roots are of one length} \\ x_{\rho a}(\varepsilon_a t^p) & \text{if } a \text{ is short} \end{cases} .$$

<u>NOTE.</u> Referring to § 4.2, we see that if $\mathfrak{g}$ is of type D_{2n} and $\pi = \pi_2$, then $Q : G_{\pi_2,K} \longrightarrow G_{\pi_3,K}$ is the isomorphism mentioned , since $\rho(\Delta_{\pi_2}) = \Delta_{\pi_3}$. Hence , such groups have only the identity as graph automorphisms . (We always include the possibility of the identity graph automorphism.).

Now we want to consider the groups G_σ of fixed points under $\sigma \in \mathrm{Aut}(G)$, and Steinberg has shown [20] that $\sigma = i.d.f.g$, where $i \in \mathrm{Inn}(G)$, $d \in \mathrm{Diag}(G)$, $f \in \mathrm{Field}(G)$ and $g \in \mathrm{Graph}(G)$.

In the case of $G_{\pi,K}$, $\mathrm{Inn}(G)$ and $\mathrm{Diag}(G)$ are uninteresting since their fixed-point groups are group centralisers of certain elements of G . However , f on its own gives a Chevalley group of the same type but over a different field , (as we saw in the introduction) and such groups are the finite groups of (normal) Chevalley type which we shall discuss in Chapters 3,4 and 5 . Similarly , g on its own gives G_σ as an embedding of a Chevalley group of one type in another Chevalley group of different type , e.g. $Sp_n(K)$ or $SO_n(K)$ in SL_{2n} or SL_{2n+1} . If we take σ to be a certain combination of f and g , then we get new simple groups - the Steinberg groups of Chapter 6 , and the Suzuki and Ree groups of Chapter 7 , i.e. the group G_σ is a finite group of twisted type . An example is the embedding of $SU_n(q)$ in $SL_n(q)$.

We have the following identification with the Classical groups:-

TABLE 1.1.

$g \backslash \pi$	ad	sc	π_1
A_ℓ	$PSL_{\ell+1}$	$SL_{\ell+1}$	
B_ℓ	$PSO_{2\ell+1} = SO_{2\ell+1}$	$Spin_{2\ell+1}$	
C_ℓ	$PSp_{2\ell}$	$Sp_{2\ell}$	
D_ℓ	$PSO_{2\ell}$	$Spin_{2\ell}$	$SO_{2\ell}$
${}^2A_\ell$	$PSU_{\ell+1}$	$SU_{\ell+1}$	
${}^2D_\ell$	$P\Omega_{2\ell}$		

NOTES. (1) π_1 is defined in Proposition 4.3.

(2) Ω is the commutator subgroup of the orthogonal group leaving invariant a quadratic form of index ℓ relative to K and index $(\ell-1)$ relative to K_o.

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§1.3. The Basic Theorem and its consequences.

An essential part of our work rests upon the following important extension of a theorem of Lang [19] :

Theorem 1.1. Let $G = G_{\pi,K}$ be semi-simple and σ an endomorphism of G onto G such that G_σ is finite. Then the map $f : x \longmapsto x.\sigma(x)^{-1}$ of G into G is surjective. $\square$

Under these conditions, we can show that σ fixes a Borel subgroup B and a maximal torus T contained in B. Also, any two such couples are conjugate in G_σ. Now, since σ fixes N, it follows that σ fixes W and we have a natural action of σ on W. If G is semi-simple, then we can show that the Bruhat decomposition exists for G_σ with W_σ, N_σ etc. in place of W, N etc.

Since K is algebraically closed, we know that all maximal tori of $G(K)$ are conjugate. However, in the case of a finite field K_o, the situation is described in Theorem 1.2., where $\mathcal{E}(G_\sigma, T)$ is the set of G_σ-conjugacy classes of the σ-fixed maximal tori of G.

<u>Theorem 1.2.</u> Let G and σ be as in Theorem 1.1. Then :-

(a). G contains a maximal torus fixed by σ;

(b). If T is such a torus and $W = N/T$ is its Weyl group, then there is a 1-1 correspondence $Q : H^1(\sigma, W) \longrightarrow \mathcal{E}(G_\sigma, T)$;

(c). If σ fixes each element of W, i.e. commutes with the action of W on T, then the classes $\mathcal{E}(G_\sigma, T)$ correspond to the conjugacy classes of W. $\square$

For a proof of this and other results above, we refer to [21].

<u>Definition 1.2.</u> "<u>Twisting</u>". Let T be as in Theorem 1.2., $w \in W$ and correspondingly $n_w \in N$. By Theorem 1.1., there exists some $g \in G$ with $n_w = g^{-1}.\sigma(g)$. Hence σ fixes the maximal torus $T' = {}^gT$.

Also, every σ-fixed maximal torus can be obtained in this way, by "<u>twisting</u>" by some $w \in W$. For, suppose that T_1 is fixed by σ.

Then $T_1 = {}^{g_1}T$ for some $g_1 \in G$. Hence, $g_1^{-1}.\sigma(g_1)$ normalises T, and so corresponds to some $w_1 \in W$. This is the correspondence Q in Theorem 1.2. (b) ; see [19] .

If we identify T', above, with T according to the isomorphism i_g, then the original action of σ on T' is equivalent to that of $w_o \sigma$ on T. For if $t \in T$, and correspondingly $t' \in T'$ such that $i_g(t) = t'$, then :-

$$i_g \circ (w \circ \sigma)(t) = i_g \circ (w \circ \sigma) \circ i_{g^{-1}}(t') = i_{g.n_w.\sigma(g)^{-1}}(\sigma(t')) = \sigma \circ i_g(t)$$

Hence the following diagram commutes :-

$$\begin{tikzcd} T \arrow[r, "i_g"] \arrow[d, "w_o\sigma"'] & T' \arrow[d, "\sigma"] \\ T \arrow[r, "i_g"] & T' \end{tikzcd}$$

If we replace w by an element equivalent to it in $H^1(\sigma, W)$, then this amounts to replacing $w_o\sigma$ by something conjugate to it under W. For if $w_1 = w' . w . \sigma(w')^{-1}$ for $w' \in W$, then $w'.(w_o\sigma).w'^{-1} = w_1 \circ \sigma$ on T.

If we let G, σ, T, be as in Theorem 1.2 , then we denote the induced action of σ on $X(T)$, the (discrete) character group of T, by σ^*. If $u \in X(T) = \mathrm{Mor}(T, K^*)$, then we write the image of $t \in T$ under u as t^u, and W acts on $X(T)$ by $(t^w)^u = t^{w(u)}$, where $t^w = {}^{n_w}t$.

Since $\sigma(T) = T$, we can consider $\sigma_T = \sigma|_T$, so that σ^*, mapping u to $u \circ \sigma_T$, is an automorphism of $X(T)$. Thus $t^{\sigma^*(u)} = \sigma(t)^u$ for $t \in T$

and $u \in X(T)$, and this action extends to $X(T)_R = R \otimes_Z X(T)$, the real extension of $X(T)$. From now , $V = X(T)_R$.

Now the isomorphism i_g from T to T' induces , naturally , the dual isomorphism $g^* : X(T') \longrightarrow X(T)$ defined by $t^{g^*(u')} = ({}^g t)^{u'}$ for $u' \in X(T')$ and $t \in T$. Hence , $t^{\sigma^*.w.g^*(u')} = (i_{g.n_w} \circ \sigma(t))^{u'}$ $=(i_{\sigma(g)} \circ \sigma(t))^{u'} = t^{g^*.\sigma(u')}$ for all $t \in T$, $u' \in X(T')$, and so the action of σ^* on $X(T')$ is that of $\sigma^*{}_o w$ on $X(T)$. Thus the following diagram commutes :-

$$\begin{tikzcd} X(T) \arrow[d, "\sigma^*{}_o w"'] & X(T') \arrow[l, "g^*"'] \arrow[d, "\sigma^*"] \\ X(T) & X(T') \arrow[l, "g^*"] \end{tikzcd}$$

Also , i_g extends naturally to an isomorphism from $N = N(T)$ to $N' = N(T')$. Hence $i_g \circ w = w \circ i_g$ on T and $g.n_w.g^{-1} = n'_w.t'$. for $n_w \in N$ and $n'_w \in N'$, and some $t' \in T'$.

Finally, we make the convention that if T is a fixed , σ-fixed maximal torus of G , and T' is obtained from T by twisting by $w \in W$, then we denote T' by $\overline{T}_w$, and also $(T')_\sigma$ by T_w .

With the above notation , we have the following [19] :-

<u>Proposition 1.3</u>. (i) Suppose that G is simple and that T is a fixed K_o-split maximal torus of G . Let σ be a combination of a field automorphism γ and a graph automorphism ρ as in § 1.2. , where we allow ρ to be the identity . Then $\sigma = q\tau$, for some $q > 1$ and some isometry τ of V .

(ii) Further, given any σ-fixed torus T' of G, we can find $w \in W$ such that $T' = \overline{T}_w$, and the action of σ^* on $X(T')$ is given by $\sigma^* = q.w\tau$.

(iii) If ρ is the identity graph automorphism, then G_σ is a finite group of normal Chevalley type, and τ is the identity. Then, the set $\mathcal{C}(G_\sigma, T)$ is in 1-1 correspondence with the set $\mathcal{C}(W)$ of conjugacy classes of W under the action of twisting.

Proof. (i) Since G is simple, then Σ is an indecomposable root system. Now if all the roots of Σ have equal length, then ρ is an automorphism of Σ and an isometry of V. For T we may take the maximal torus generated by the $h_a(t)$, as in § 1.1., and we let $\gamma : c \longmapsto c^q$ be the Frobenius automorphism of K.

Let $t \in T$, so $t = \prod_{a\in\pi} h_a(t_a)$ for some $t_a \in K$. Then σ acts on T by $\sigma(t) = \prod_{a\in\pi} h_a(t_a^q)$. Now let $u \in \Delta_\pi$. Then $t^{\sigma^* u} = \sigma(t)^u = \prod_{a\in\pi} t_a^{qu(h_{\rho a})} = \prod_{a\in\pi} t_a^{q\rho u(h_a)}$. However, $t^{\sigma^* u} = \prod_{a\in\pi} t_a^{\sigma^* u(h_a)}$, and this must be true for all $t \in T$, i.e. for all $t_a \in K$. Hence $\sigma^* u = q.\rho u$, and $\sigma^* = q\rho$ on $X(T)$. Since ρ is an isometry of V in this case, the result follows with $\tau = \rho$.

On the other hand, if roots of different lengths occur, then the map

$$\overline{\rho} : \begin{Bmatrix} a \longmapsto \rho a & \text{for a long} \\ a \longmapsto p\rho a & \text{for a short} \end{Bmatrix}$$ is an isomorphism of root systems for $p = (a, a)/(b, b)$ if a is long and b is short. However, $\overline{\rho}$ is <u>not</u> an isometry of V. Now, in such a case, we shall show in

Chapter 7 that we must restrict K_o to fields of a certain type, viz. such that K_o has characteristic $p = (a, a)/(b, b)$ as above, and also that $|K_o| = p^{2m+1}$, for some $m > 0$. Then we must take γ to be the automorphism $\gamma : c \longmapsto c^{p^m}$ for $c \in K$, since ρ has order 2 in this case. Now $\Delta_\pi = \Delta_{ad}$ for such cases, whatever the value of π; as we shall see in Chapter 7. Thus :-

$$\sigma : \left\{ \begin{array}{lll} x_a(t) \longmapsto x_{\rho a}(t^{p^m}) & \text{if} & a \in \Pi \text{ is long} \\ x_a(t) \longmapsto x_{\rho a}(t^{p^{m+1}}) & \text{if} & a \in \Pi \text{ is short} \end{array} \right\} \text{ defines } \sigma \text{ on}$$

generators of G.

Suppose that $u \in \Delta_\pi$ and that ρ acts on Π in cycles $C_i = (a_i\ b_i)$ of order 2, with a_i long and b_i short. Let $h_C = h_a(t_a).h_b(t_b)$, so that $h_C^{\sigma^* u} = (h_b(t_a^{p^m}) . h_a(t_b^{p^{m+1}}))^u = t_a^{p^m . \langle u, b\rangle} . t_b^{p^{m+1} . \langle u, a\rangle}$.
Also, $h_C^{\sigma^* u} = t_a^{\langle \sigma^* u, a\rangle} . t_b^{\langle \sigma^* u, b\rangle}$, and this must be true for all $t_a, t_b \in K$.

Thus, $\langle \sigma^* u, a\rangle = \langle p^m.u, b\rangle = \langle p^{m+\frac{1}{2}}.\tau u, a\rangle$, where $\tau a = p^{\frac{1}{2}}.\rho a = p^{\frac{1}{2}}.b$. Similarly,
$\langle \sigma^* u, b\rangle = \langle p^{m+1}.u, a\rangle = \langle p^{m+\frac{1}{2}}.\tau u, b\rangle$, where $\tau b = p^{-\frac{1}{2}}.\rho b = p^{-\frac{1}{2}}.a$. Let $q = p^{m+\frac{1}{2}}$, so $\langle \sigma^* u - q.u, a\rangle = 0$ for all $a \in \Pi$. Hence, $\sigma^* u = q.\tau u$ for all $u \in \Delta_\pi$, and so $\sigma^* = q.\tau$ on $X(T)$.

Now, $(\tau a, \tau a) = p(b, b) = (a, a)$, if $a \in \Pi$ is long and $b = \rho a$, and $(\tau b, \tau b) = p^{-1}(a, a) = (b, b)$. Hence,

τ is an isometry of V , and $\sigma^* = q.\tau$ for some $q > 1$ and isometry τ , in all cases .

(ii). We have seen from Definition 2.that given any σ-fixed maximal torus T' of G , we can find $w \in W$ such that $T' = \overline{T}_w$, and then the action of σ on T' is that of $w_o\sigma$ on T . Further , the action of σ^* on $X(T')$ is that of $\sigma^*{}_o w$ on $X(T)$, ie. of $q.\tau w$. Since replacing w by an equivalent element in $H^1(\sigma , W)$ amounts to replacing $\sigma^*{}_o w$ by an element conjugate under W , and since $\sigma(w)$ is equivalent to w , we can take the action of σ^* on $X(T')$ to be that of $w.\sigma^*{}_o w.w^{-1}$, ie. $q.w\tau$ on $X(T)$.

(iii). If ρ is the identity , then $\sigma(t) = t^q$ for all $t \in T$. Hence , $w_o\sigma(t) = w(t^q) = w(t)^q = \sigma_o w(t)$ for all $w \in W$. Hence σ commutes with the action of W on T and by Theorem 1.2(c) , the bijection $Q : \mathcal{C}(W) \longrightarrow \mathcal{C}(G_\sigma , T)$, since $H^1(\sigma , W) = \mathcal{C}(W)$. $\square$

<u>Definition 1.3</u>. Following Proposition 1.3 , we say that the triple (Σ , X , τ) of a root system in V , a lattice $X = \Delta_\pi$ in V and an isometry τ of V , is the <u>type</u> of (G , σ) , and that q is the <u>parameter</u> of (G , σ) .

<u>Proposition 1.4</u>. Assuming the notation and situation of Proposition 1.3. , with $X = X(\overline{T}_w)$ and (Σ , X , τ) the type of (G , σ) , we have :-

(a). T_w is in duality with, and hence isomorphic to $X / (\sigma^* - I)X$. Thus the matrix $(\sigma^* - I)$ is a relation matrix for the group T_w .

(b). If $f_{w\tau}(t)$ is the characteristic polynomial of $w\tau$ on $X(\overline{T}_w)_R$, then the order of T_w is $\left| f_{w\tau}(q) \right|$. [21]

Proof. (a). Consider σ^* (modulo p) relative to X , ie. the action of σ^* on $X_{\{p\}} = X \otimes_Z Z_p$, the localisation of X at p . Then , by Lemma 1.3. , $\sigma^{*m} = q^m$, where m is the order of $w\tau \in \langle W , \tau \rangle = W^*$. Hence, if n is the exponent of W^*, then $\sigma^{*n} \equiv 0$ (modulo p) , and σ^* is nilpotent on $X_{\{p\}}$.

Now, if $(\sigma^* - I)u = 0$ for $u \in X_{\{p\}}$, then $\sigma^* u = u$ and $(\sigma^*)^n u = u$. Hence , $u = 0$, so that $(\sigma^* - I)$ is injective on $X_{\{p\}}$.

Consider the exact sequence $X \xrightarrow{(\sigma^* - I)} X \xrightarrow{\pi} X / (\sigma^* - I)X \longrightarrow 0$. Then the following sequence is also exact :-

$X_{\{p\}} \xrightarrow{(\sigma^* - I)} X_{\{p\}} \xrightarrow{\pi'} (X / (\sigma^* - I)X)_{\{p\}} \longrightarrow 0$. Now , $(\sigma^* - I)$ is injective on $X_{\{p\}}$, so that the following sequence is short exact:-

$$0 \longrightarrow X_{\{p\}} \xrightarrow{(\sigma^* - I)} X_{\{p\}} \xrightarrow{\pi'} (X / (\sigma^* - I)X)_{\{p\}} \longrightarrow 0 .$$

But $X_{\{p\}}$ is finite , so that $(\sigma^* - I)$ is an isomorphism on $X_{\{p\}}$ and so π' is the zero map .

Hence , $(X / (\sigma^* - I)X) \otimes Z_p = 0$. Now $X / (\sigma^* - I)X$ is isomorphic to a direct sum of cyclic groups Z_{q_i} , for integers q_i , and since $Z_m \otimes Z_n$ is isomorphic to $Z_{(m,n)}$, then $(q_i , p) = 1$ for all i . Hence , $X / (\sigma^* - I)X$ has finite order prime to p .

By the elementary divisor theorem , and by choosing compatible bases for X and $(\sigma^* - I)X$, we have $\left| X / (\sigma^* - I)X \right| = \left| \det(\sigma^* - I) \right|$. Under the duality between X and $\overline{T}_W$, the annihilator of $(\sigma^* - I)X$ in

$\overline{T}_w$ is isomorphic to the dual of $X / (\sigma^* - I)X$. Since the latter is a finite abelian group which has order prime to p , it is isomorphic to its own dual .

However , the annihilator of $(\sigma^* - I)X$ in $\overline{T}_w$ is just T_w . For , if $t \in \overline{T}_w$ and t annihilates $(\sigma^* - I)X$, then $t^{(\sigma^* - I)u} = 1$ for all $u \in X$. Thus $(\sigma(t).t^{-1})^u = 1$ for all $u \in X$. Hence $\sigma(t) = t$ and $t \in T_w$. Converseley , if $t \in T_w$, then t annihilates $(\sigma^* - I)X$.

Thus , T_w is isomorphic to $X / (\sigma^* - I)X$, and $\left|T_w\right| = \left|\det(\sigma^* - I)\right|$.

(b). Since $\sigma^* = q.w\tau$ on $X(\overline{T}_w)_R$, then $\left|T_w\right| = \left|\det(q.w\tau - I)\right|$.

Now $(w\tau)$ is an isometry of $X(\overline{T}_w)_R$, so that $\left|\det w\tau\right| = 1$ and $(w\tau)^{-1} = {}^t(w\tau)$. Thus

$$\begin{aligned} \left|T_w\right| &= \left|\det(q.I - (w\tau)^{-1})\right| . \left|\det(w\tau)\right| \\ &= \left|\det {}^t(qI - w\tau)\right| = \left|f_{w\tau}(q)\right| . \quad \square \end{aligned}$$

Hence , the characteristic polynomial of $w\tau$ evaluated at q yields the order of T_w , and the corresponding matrix $(q.(w\tau)_\Lambda - I)$ is a relation matrix for the group T_w , where Λ is some basis of $X(\overline{T}_w) = \Delta_\pi$. All the groups T_w corresponding to some class $h \in H^1(\sigma , W)$ form a conjugacy class of maximal tori of G_σ by Theorem 1.2. , and hence have the same structure . Thus we need only determine the relation matrix $(q.(w\tau)_\Lambda - I)$ for a representative w from each class of $H^1(\sigma , W)$ in order to determine the structure of the maximal tori of G_σ .

Now , T_w is an abelian group with ℓ generators and with relation matrix $(q.(w\tau)_\Lambda - I)$. This relation matrix is equivalent to a diagonal matrix D , in the sense that there exist integral , uni-modular matrices X , Y such that $X.(q.(w\tau)_\Lambda - I).Y = D$. According to the theory of abelian groups , [24] , if $D = \mathrm{diag}(e_1, e_2, \ldots, e_\ell)$, then these e_i are called the <u>elementary divisors</u> of T_w , and T_w is isomorphic to the direct product of cyclic groups $C_{e_1} \times C_{e_2} \times \ldots \times C_{e_k}$, where $\{e_1, \ldots, e_k\}$ are the non-unit elementary divisors of T_w . Hence , we must determine those elementary divisors corresponding to a representative w of each class of $H^1(\sigma, W)$, by diagonalising the matrix $(q.(w\tau)_\Lambda - I)$ over Z .

<u>Notation</u>. By the above , we mean that given an integral matrix A which is a relation matrix for a group G , we reduce A to a diagonal matrix (or diagonalise A) in order to find the elementary divisors of G (we freely make use of also calling these the elementary divisors of A) . This reduction process involves the use of elementary transformations of the matrix A . We use the following transformations :-

(i). ar_i to r_j for $i \neq j$. This amounts to replacing the generator R_j of G by $R_j + a R_i$, where $a \in Z$, and is the operation of replacing j^{th} row of A by adding a times the i^{th} row to the j^{th} row .

(ii). $\varepsilon \times r_i$, where ε is a unit of Z , ie. replacing the generator R_i by its inverse , which multiples the i^{th} row of A by -1 .

(iii). $a\,c_i$ to c_j for $i \neq j$. This is the operation of replacing the j^{th} column of A by adding a times the i^{th} column to the j^{th} column , for $a \in Z$.

(iv). Remove (r_i, c_i) . This is when A is of the form $\begin{bmatrix} \underline{+1} & \underline{0} \\ \underline{0} & A' \end{bmatrix}$, and we can remove the unit elementary divisor $\underline{+}1$, reducing A to A' . Such an elementary divisor corresponds to a generator of T_W being made redundant by A .

Finally we denote a reduction of A by one of these transformations as $A \longrightarrow B$, with the relevant transformation described with the arrow , according to the notation above. We sometimes say that B is <u>equivalent</u> to A .

<u>§1.4. Some results on weight lattices .</u>

In this section , we prove a result which reduces the number of cases to be considered . We let Δ be any Z-module and $\Delta^* = \mathrm{Hom}_Z(\Delta, Z)$ its dual . If $x \in \Delta$ and $x^* \in \Delta^*$, then we denote $x^*(x) \in Z$ by $\langle x , x^* \rangle$. If Δ' is another Z-module , and u is a linear map $u: \Delta \longrightarrow \Delta'$, then we denote the transpose of u by ${}^t u$, so ${}^t u : \Delta'^* \longrightarrow \Delta^*$ such that ${}^t u : y^* \longmapsto y^* {}_o u$ for $y^* \in \Delta'^*$. Hence

$$\langle u(x) , y^* \rangle = \langle x , {}^t u(y^*) \rangle \quad \text{for } x \in \Delta , \ y^* \in \Delta'^* .$$

Let Λ be a basis of Δ and $V = \Delta \otimes_Z R$. The matrix of u with respect to Λ is denoted by u_Λ . Let Σ be a system of roots in V , $W(\Sigma)$ the Weyl group of Σ , and let $(\ , \ )$ be a non-degenerate , symmetric bilinear form on V , which is W-invariant . Then identify V with $V^* = \mathrm{Hom}_R(V,R) = \Delta^* \otimes_Z R$ according to this form , ie. given $y^* \in V^*$, there exists a unique $y \in V$ with $y^*(x) = (x,y)$ for all $x \in V$ and we identify $y^* \longleftrightarrow y$.

If Δ is the weight lattice Δ_π for some faithful representation π of a complex , simple Lie algebra g with root system Σ , and u is an isometry of Δ with respect to (,) , then u $(= u \otimes 1)$ acts on V , and hence on V^* as ${}^t u$ by $(u(x) , y) = (x , {}^t u(y))$.

Lemma 1.5. If Λ^* is the dual basis of Λ , then ${}^t u_{\Lambda^*} = u_\Lambda^{-1}$.

Proof. If $\Lambda = \{\gamma_i\}_{i=1}^{\ell}$, then $\Lambda^* = \{\gamma_j^* : (\gamma_i,\gamma_j^*) = \delta_{ij}\}$ is a basis of Δ^* , the dual basis of Λ . Since u is an isometry , then $(u(\gamma_i) , u(\gamma_j^*)) = (\gamma_i , \gamma_j^*) = \delta_{ij}$.

Further , $(u(\gamma_i) , u(\gamma_j^*)) = (\sum_{k=1}^{\ell} (u_\Lambda)_{ki}.\gamma_k , \sum_{r=1}^{\ell} (u_{\Lambda^*})_{rj}.\gamma_r^*)$

$$= \sum_{k=1}^{\ell} (u_\Lambda)_{ki} . (u_{\Lambda^*})_{kj} = ({}^t u_\Lambda . u_{\Lambda^*})_{ij} .$$

Hence , ${}^t u_\Lambda . u_{\Lambda^*} = I_\ell$, and ${}^t u_{\Lambda^*} = u_\Lambda^{-1}$.

$\square$

Now for every $a \in \Sigma$, there exists an element $a^\vee \in V^*$ such that $a^\vee(a) = 2$. Then $a^\vee = \frac{2a}{(a,a)}$ under the identification . By [4] :

Lemma 1.6. If $\Sigma^\vee = \{a^\vee : a \in \Sigma\}$, then $\Sigma^\vee$ is a root system in V^* (ie. in V) , and $a^{\vee\vee} = a$. The map $Q : W(\Sigma) \longrightarrow W(\Sigma^\vee)$ defined by $Q : w \longmapsto {}^t w^{-1}$ is an isomorphism of groups .

Proof. Both statements follow since ${}^t w_a^{-1} = w_{a^\vee}$, as the axioms for a root system are readily verified .

$\square$

Lemma 1.7. Let $\Delta^\vee_{ad}$ be the Z-module generated by the root system $\Sigma^\vee$. Then $\Delta_{sc} = (\Delta^\vee_{ad})^*$.

Proof. This follows since $\Delta_{sc} = \{x \in V : (x, y^*) \in Z \text{ for all } y^* \in \Delta^{v}_{ad}\}$.

$\square$

If g is a simple Lie algebra with root system Σ, then let g^{v} denote a simple Lie algebra with root system Σ^{v}. By Lemma 1.7, if $\Delta_{ad} \subset \Delta \subset \Delta_{sc}$, then $\Delta^{v}_{sc} \supset \Delta^{*} \supset \Delta^{v}_{ad}$. Hence, if $\Delta = \Delta_{\pi}$ for some faithful representation π of g, then $\Delta^{*} = \Delta_{\pi^*}$ for some faithful representation π^* of g^{v}.

If Σ is not of type B_ℓ or C_ℓ, then $\Sigma^{v} = \Sigma$, so $\Delta_{sc} = \Delta^{*}_{ad}$, and the map Q of Lemma 1.6 becomes the identity when induced to $\hat{Q} : \mathcal{L}(W) \longrightarrow \mathcal{L}(W)$, since Q is the identity.

However, if Σ is of type B_ℓ, then Σ^{v} is of type C_ℓ. Also, W^{v} is isomorphic with W and $\hat{Q} : \mathcal{L}(W) \longrightarrow \mathcal{L}(W^{v})$ is the identity.

Lemma 1.8. Let $q \in Z$ and $w \in W$. Then $\Delta/_{(qw-I)\Delta} \cong \Delta^{*}/_{(qw-I)\Delta^{*}}$.

Proof. It suffices to show that $(q. w_{\Lambda^*} - I)$ and $(q. w_{\Lambda} - I)$ are equivalent in the sense of the previous section. We know, by [6], that w is conjugate to w^{-1} in W. Since $|\det w| = 1$ for $w \in W$ and w is known to be integral [4], it follows that there exists an integral unimodular matrix X_1 with $X_1 . w_{\Lambda}^{-1} . X_1^{-1} = w_{\Lambda}$.

Suppose that $X\,(q.w_{\Lambda} - I)\,.\,Y = D$, where X, Y are integral unimodular, and D is a diagonal matrix. Then,

$$(X\,X_1)\,.\,(q\,w_{\Lambda}^{-1} - I)\,.\,(X_1^{-1}\,Y) = D\,.$$

Hence, $(X\,X_1)\,.\,(q.{}^{t}w_{\Lambda^*} - I)\,.\,(X_1^{-1}\,Y) = D$, by Lemma 1.5, so $\quad {}^t(X_1^{-1}Y)\ .\ (q.w_{\Lambda^*} - I)\ .\ {}^t(X\,X_1) = {}^tD = D$.

Thus , $(q.w_{\Lambda} - I)$ and $(q.w_{\Lambda^*} - I)$ are equivalent to the same diagonal matrix and the result follows . $\square$

Given the group $G_{\pi,K}$ corresponding to the triple $\{\mathfrak{g}, \pi, K\}$, we let $G^{v}_{\pi^*,K}$ be the Chevalley group corresponding to the triple $\{\mathfrak{g}^{v}, \pi^*, K\}$.

<u>Corollary 1.8</u>. If T (resp. T^*) is a maximal K_o-split torus of $G_{\pi,K}$ (resp. $G^{v}_{\pi^*,K}$) , and w is a representative of some element of $\mathcal{L}(W)$, then $T_w \cong T^*_{Q(w)}$, and so the structures of the tori are identical in the groups $G_{\pi,K}(K_o)$ and $G^{v}_{\pi^*,K}(K_o)$.

<u>Proof</u>. This follows from Proposition 1.4 since $T_w \cong \Delta/(q.w - I)\Delta$. $\square$

This means that we do not have to determine the structure of the maximal tori of $G^{v}_{\pi^*,K}$ separately . For example , if $\mathfrak{L}$ is not of type B_ℓ or C_ℓ , then the results for $G_{ad,K}(K_o)$ and $G_{sc,K}(K_o)$ are identical. This is discussed in Chapter 4 .

<u>§1.5. A result on normalisers of maximal tori .</u>

We conclude this chapter with a result concerning the normaliser of a maximal torus T_w in G_σ . In Chapter 8 , we note that in considering the representation theory of the finite groups we are interested in $N_{G_\sigma}(\overline{T}_w) / T_w$ which is always isomorphic to W_σ . Let T be a σ-fixed maximal torus .

Lemma 1.9. Let $\bar{T}_w$ be a maximal torus of G obtained from T by twisting by $w \in W$, and let $h_w \in H^1(\sigma, W)$ be the class containing w. Let W be the Weyl group of G relative to $\bar{T}_w$. Then there is a bijection

$$W / W_\sigma \longleftrightarrow h_w .$$

Proof. We know that σ acts on W as conjugation by $w\tau$, so that

$$W_\sigma = \{ w_1 \in W : (w\tau)^{-1}.w_1.(w\tau) = w_1 \}$$

$$= \{ w_1 \in W : w_1.w.\ \sigma(w_1)^{-1} = w \} .$$

Thus the map $Q : W / W_\sigma \longrightarrow h_w$ such that $w'.W_\sigma \longrightarrow w'.w.\sigma(w')^{-1}$ is a bijection. This follows from the Orbit-Stabiliser Theorem. $\square$

This bijection is in fact a generalisation of the bijection between $W / \mathbb{C}_W(w)$ and C_w, where $\mathbb{C}_W(w)$ is the centraliser of w in W and C_w is the conjugacy class of W containing w. This is the special case of the Lemma when σ is just a field automorphism of K, in which case W_σ is isomorphic with $\mathbb{C}_W(w)$. Although we cannot in general say what the structure of W_σ is, we do know that in the case when w is in the Coxeter class of W, then $\mathbb{C}_W(w)$ is a cyclic group of order h, the Coxeter number of W, see [6]. Furthermore, in groups of small order viz. those discussed in Chapter 7, we are able by virtue of Lemma 1.9 to state the order of the group W_σ corresponding to the maximal torus T_w of G_σ.

CHAPTER 2. Conjugacy classes in the Weyl group.

Let $G = G_{\pi,K}$ be a simple Chevalley group, and let W be the corresponding Weyl group. Suppose now that σ is the field automorphism of G induced by the Frobenius automorphism of K, so that τ is the identity. Then G_σ is a finite group of (normal) Chevalley type, and we have seen in Proposition 1.3.(iii) that, in this case, the classes $H^1(\sigma, W)$ are just the conjugacy classes $\mathcal{C}(W)$ of W. Hence, in order to determine the structure of the maximal tori T_w of G_σ, we need only consider twisting the fixed K_0-split maximal torus T by a representative from each conjugacy class of W, and to diagonalise over Z the matrix $(q.w_\Lambda - I)$. Also the σ-fixed maximal tori of G are just the K_0-maximal tori, ie. those defined over K_0.

Hence, we need a uniform description of the conjugacy classes of the various Weyl groups, and this has been discussed by R.W. Carter in [6], where he describes the conjugacy classes of W by using its structure as a reflection group, hence obtaining a uniform description. We present a brief summary of this work in §2.1..

To determine the elementary divisors of T_w, we need to consider the matrix of w with respect to a suitable basis Λ of Δ_π. In fact, great care should be taken to ensure that Λ is a basis of Δ_π and not a basis of V which generates some Z-module in V other than Δ_π, in which case it follows from Lemma 1.4 that $(q.w_\Lambda - I)$ is not a relation matrix for T_w. In § 2.2, we show how we can find a suitable basis

Φ of Δ_{ad} from the description of §2.1 , for which the matrix w_Φ is readily obtainable . Due to the results of Chapter 1 (principally Lemmas 1.7 and 1.8) , it turns out that we need only consider Δ_π for π = ad .

In §2.3 , we consider some examples of Φ from §2.2 , and in §2.4. we discuss certain automorphisms of W . Finally , §2.5 contains a discussion of why we are forced to consider all the conjugacy classes of W and not just those contained in no Weyl subgroup , as described in §§2.1 and 2.2.

The material in this chapter is aimed principally at Chapter 5 , although we do use the results and ideas in other chapters .

§2.1. Description of the conjugacy classes of W.

Referring to [6] , we see that the basis of the work rests upon the fact that one can express $w \in W$ as a minimal-length product of reflections of W , ie. $w = w_{r_1} . \ldots . w_{r_k}$ with $k \leq \ell$, and then split the corresponding set of roots $\Phi = \{r_1, \ldots, r_k\}$ into two subsets of mutually orthogonal roots , so that $w = (w_{a_1} . \ldots . w_{a_h}).(w_{a_{h+1}} \ldots . w_{a_k})$ is a product of two involutions . Correspondingly , we define a graph Γ with k nodes , where two nodes a and b are joined by a bond of strength $n_{ab} . n_{ba}$, where $n_{ab} = 2(a,b).(a,a)^{-1}$. Then , Γ represents the conjugacy class of w , and the admissible graphs corresponding to all elements $w \in W$ can then be determined , as in [6] . In fact , it is only necessary to determine those admissible graphs Γ for which the corresponding conjugacy class is not contained in any Weyl subgroup W' of W, and these are tabulated in [6]. There is not a 1-1 correspondence between conjugacy classes and admissible graphs, and this breakdown is discussed fully in [6].

If A_w is the matrix, with coefficients $n_{a_i a_j}$ in the position (i,j), which corresponds to w, then A_w is determined by Γ to within alterations obtained by replacing certain roots by their negatives, provided we know which nodes of Γ correspond to long roots and which to short roots. Hence, care must be taken in assigning signs to the bonds, a task made easier by the result that a subgraph of Γ which is a cycle contains an even number of nodes, and an odd number of acute angles between roots adjacent in Γ. Also, the characteristic polynomial of w is determined by Γ. Now, if w is as above, with Φ split as $\{a_1, \ldots, a_h\} \cup \{a_{h+1}, \ldots, a_k\}$ and we order the roots in this way, then they are linearly independent and so span a subspace U of $V = X_R$ of dimension k. Then, it is shown in [6], that :-

<u>Proposition 2.1</u> The matrix A_w is a block matrix :

$A_w = \begin{bmatrix} 2I_h & B \\ C & 2I_{k-h} \end{bmatrix}$, and also the matrix, w_Φ, of w on U with respect to the (ordered) basis Φ is $\begin{bmatrix} BC - I_h & B \\ -C & -I_{k-h} \end{bmatrix}$.

$\square$

---

§2.2. Choosing a basis of Δ_{ad} .

We have already mentioned the importance of choosing a suitable basis Λ of Δ_{ad} , and in order to employ the above result for obtaining the matrix w_{Φ} of the action of w on Δ_{ad} , we must ensure that Φ is a basis of Δ_{ad} , (which it certainly need not be , even when rank $\Phi = \ell$.) . Now, a natural basis of Δ_{ad} is any fundamental system of roots Π , and we must consider the matrix E_{Φ} corresponding to the change of basis of V from Π to Φ . In order that Φ be a basis of Δ_{ad} it is necessary and sufficient that the matrix E_{Φ} be integral and unimodular . In fact , since Φ is contained in Δ_{ad} , it is only necessary to check that E_{Φ} is unimodular . Hence , in order to employ the result of Proposition 2.1 to obtain the matrix w_{Φ} as the action of w on Δ_{ad} , we must check that the corresponding E_{Φ} is unimodular for every Φ-type of W .

Certainly E_{Φ} will not be unimodular in the case when rank $\Phi < \ell$, for then U has dimension strictly less than that of V . Even if rank $\Phi = \ell$, it does not always happen that E_{Φ} is unimodular (e.g. in the class $A_2 + \tilde{A}_2$ in $W(F_4)$.) . However ,we do have :-

Proposition 2.2. If Φ corresponds to a conjugacy class of W which is not contained in any Weyl subgroup W' of W , then the corresponding matrix E_{Φ} is unimodular .

Proof. For every root system Σ , $W(\Sigma)$ has at least one conjugacy class which is not contained in any Weyl subgroup , viz. the Coxeter class, where Φ is just π . (For more details see [4] .) . Hence , in this case , E_Φ is the identity matrix and the result follows trivially .

If $\mathcal{L}$ is of type A_ℓ , B_ℓ , C_ℓ or G_2 , then there are no conjugacy classes satisfying the hypothesis of the Proposition , apart from the Coxeter class . In those cases , we use methods other than those employing the result of Proposition 2.1. . Similarly , we use other methods for $\mathcal{L}$ of type D_ℓ , although we do prove the Proposition for this case . In fact the remainder of the proof follows a "case - by - case" treatment , and we consider the list of Φ-types , corresponding to conjugacy classes of W not contained in any Weyl subgroup and excluding the Coxeter class , which appears in [6] . We call such a Φ-type a <u>semi-Coxeter type</u> . In these cases , except for types $E_7(a_4)$, $E_8(a_6)$, $E_8(a_7)$ and $E_8(a_8)$, it can readily be seen that by removing one root from Φ , and replacing it by another root such that Γ becomes the Dynkin diagram Γ' of the group W , the new system Φ' remains linearly independent and assumes the property that the scalar product of any two of its elements is non-positive . Hence , this new system Φ' becomes a fundamental system , π say . Then we can check that E_Φ is unimodular by checking that , in the expression for the removed root in terms of the elements of π , the coefficient of the new root is ± 1 .

Then ,

$$E_\Phi = \begin{bmatrix} I_{\ell-1} & * \\ 0 & \pm 1 \end{bmatrix} \quad \text{, which is unimodular .}$$

In the cases $E_7(a_4)$, $E_8(a_6)$, $E_8(a_7)$ and $E_8(a_8)$, it is necessary to replace more than one node of Γ to make Γ' the Dynkin diagram of W . Hence , it will be necessary to compute the determinant of E_{Φ} directly , to check that E_{Φ} is unimodular .

We omit the details of this " case - by - case " treatment , but we give two examples of the method in § 2.3. . $\square$

Hence , by Proposition 2.2, we may use the result of Proposition 2.1 to obtain the matrix w_{Φ} of the action of w upon Δ_{ad} , in the case where Φ is a semi-Coxeter type . Then we can find the elementary divisors of the group T_w for a representative w of the conjugacy class corresponding to Φ , by diagonalising the matrix $(q.w_{\Phi} - I)$ over Z . In fact, as we shall see in Chapter 5 , the matrix $(q.w_{\Phi} - I)$ is diagonalisable over the ring Z[q] of polynomials in q in every case except one . This exceptional case occurs in groups of type E_8 , and is dealt with in § 5.5. .

Before proceeding with the details of the proof of Proposition 2.2 , we give an example of the need to take care over the choice of basis for Δ_{ad} in the case of Φ-type $D_{\ell}(a_j)$, where a particularly simple expression for w_{Φ} can be found by considering the action of w with respect to an orthonormal basis , Φ , of V . However , E_{Φ} is not unimodular in this case , and , in fact , if this basis is considered , then one can show that T_w is isomorphic to $C_{(q^i+1)} \times C_{(q^{\ell-i}+1)}$ for i = j+1 , which is inconsistent with the result obtained by considering the matrix w_π , with respect to a fundamental system π , and certainly inconsistent with the isomorphism between $W(A_3)$ and $W(D_3)$.

NOTES. (1). Since we must take care of the lengths of roots, we make the notation that in any graph Γ , we denote long roots by nodes of the form $\bullet$, and short roots by nodes of the form $\circ$. If Γ represents a class of a Weyl group W whose root system $\mathcal{S}$ has roots of two lengths , and the system Φ corresponding to Γ has all its roots of the same length , then we denote Γ by X if the roots are long and by $\tilde{X}$ if the roots are short , where X is the type of Γ .

(2). Although the results for the classical groups could be derived entirely in terms of root systems , as explained above , we find it more convenient to follow [6] and to use the language of permutation groups in Chapters 3 and 4 .

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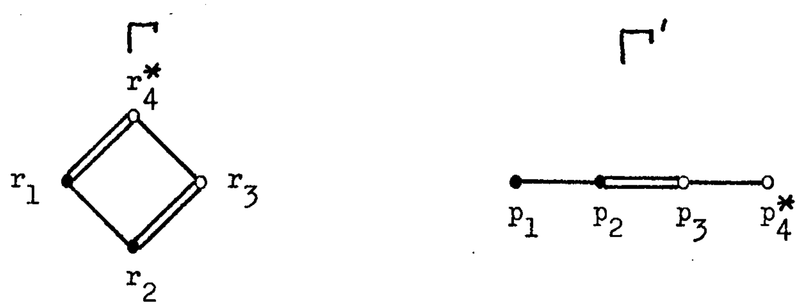
§2.3. Examples of the methods to prove Proposition 2.2.

(i) $F_4(a_1)$.



A system of roots Φ with graph Γ is :-

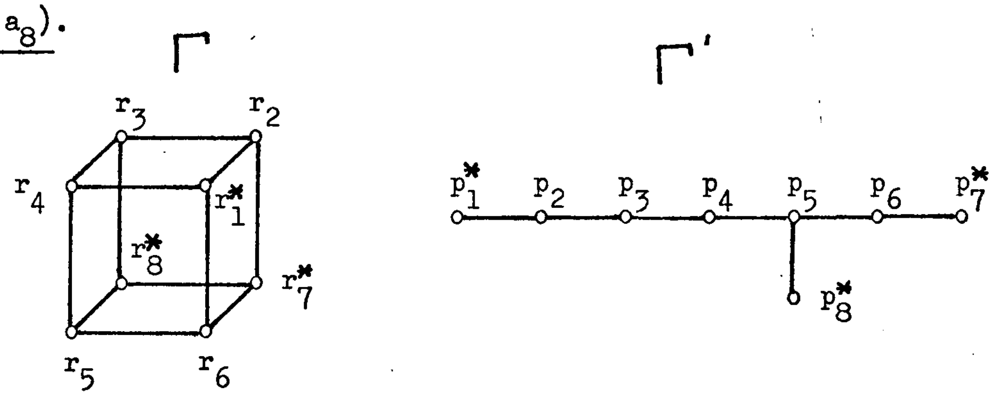
$\Phi = \left\{ r_1 = e_2-e_3 \text{ , } r_2 = e_3-e_4 \text{ , } r_3 = e_4 \text{ , } r_4 = \tfrac{1}{2}(e_1-e_2+e_3+e_4) \right\}$, where $\{ e_i \}$ is an orthonormal basis of X_R .

A system of roots Π with graph Γ' is :-

$$\Pi = \{ p_1 = e_2 - e_3 \ , \ p_2 = e_3 - e_4 \ , \ p_3 = e_4 \ , \ p_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4) \} \ .$$

Then $r_4 = p_2 + 2p_3 + p_4$, and , since the coefficient of p_4 is $+1$, the change of basis from Φ to Π is integral unimodular .

(ii). $\underline{E_8(a_8)}$.



In this case , we must replace the three nodes r_1 , r_7 , r_8 . A system of roots Φ with graph Γ is $\Phi = \{r_i\}_{i=1}^{8}$, and a system of fundamental roots Π with graph Γ' is $\Pi = \{p_i\}_{i=1}^{8}$, where :-

$r_2 = p_2 = e_2 - e_3$, $\qquad p_1 = e_1 - e_2$,

$r_3 = p_3 = e_3 - e_4$,

$r_4 = p_4 = e_4 - e_5$, $\qquad p_7 = -\tfrac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8)$,

$r_5 = p_5 = e_5 - e_6$,

$r_6 = p_6 = e_6 + e_7$, $\qquad p_8 = e_6 - e_7$.

Then we find :-

$$r_1 = \tfrac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 + e_6 + e_7 + e_8) \ ,$$

$$r_7 = \tfrac{1}{2}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8) \ ,$$

$$r_8 = \tfrac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 + e_6 - e_7 - e_8) \ .$$

Thus ,

$$r_1 = -p_3 - 2p_4 - 2p_5 - p_6 - p_7 - p_8 \ ,$$

$$r_7 = p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_5 + p_6 + p_7 + p_8 \ ,$$

$$r_8 = p_1 + 2p_2 + 3p_3 + 3p_4 + 3p_5 + 2p_6 + p_7 + 2p_8 \ .$$

If we order Φ into two mutually orthogonal subsets as $\{r_1, r_3, r_5, r_7; r_2, r_4, r_6, r_8\}$ and π as $\{p_1, p_3, p_5, p_7; p_2, p_4, p_6, p_8\}$, then $w_\pi = E_\Phi^{-1} . w_\Phi . E_\Phi$, where

$$E_\Phi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 & 0 & 0 & 0 & 3 \\ -2 & 0 & 1 & 2 & 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 & 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}, \text{ and } \det E_\Phi = 1 .$$

Similarly, we can check that, for each Φ-type satisfying the hypothesis of Proposition 2.2, the matrix E_Φ is unimodular. Hence we have proved Proposition 2.2, which we use in Chapter 5.

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§2.4. Some automorphisms of W.

In this section, we combine the work of Carter described in the previous sections and a result of Burnside, to obtain a useful result. Let Σ be any irreducible root system in a real vector space V, and let $A(\Sigma)$ be the group of all automorphisms of V which leave Σ invariant. Further, let π be a fundamental system for Σ and $D(\Sigma)$ the subgroup of $A(\Sigma)$ formed by elements leaving π invariant. Then by [4]:

Lemma 2.3 $W(\Sigma)$ is a normal subgroup of $A(\Sigma)$, and $A(\Sigma)$ is the semi-direct product of $W(\Sigma)$ by $D(\Sigma)$. Moreover, $D(\Sigma)$ is the group of automorphisms of the graph of Σ, which is thus isomorphic to $A(\Sigma) / W(\Sigma)$.

Proof. If $a \in \Sigma$ and $t \in A(\Sigma)$, then $t.w_a.t^{-1} = w_{t(a)}$. Since

$W(\precsim)$ is generated by the set $\{ w_a : a \in \precsim \}$, it follows that $W(\precsim) \lhd A(\precsim)$. Now t transforms Π into another fundamental system Π', and, since $W(\precsim)$ operates simply transitively on the set of fundamental systems, t can be written uniquely as $w.d$ for $w \in W(\precsim)$ and $d \in D(\precsim)$. Hence, $A(\precsim) = W(\precsim).D(\precsim)$.

Finally, since the graph of $\precsim$ has nodes in 1-1 correspondence with the elements of Π, it follows easily that $D(\precsim)$ is the group of automorphisms of this graph. $\square$

Now let $\tau \in D(\precsim)$. Then τ acts on $W(\precsim)$ by conjugation in $A(\precsim)$, and let $\mathrm{ord}(\tau) = \delta$. Then τ corresponds, by §1.2, to a permutation π of $\precsim$.

<u>Lemma 2.4</u>. π acts on the set $\mathcal{C}(W)$, as a permutation of order δ, by permuting the set of admissible graphs.

<u>Proof</u>. Since $\pi \in \mathrm{Aut}(W)$, then π acts on the set $\mathcal{C}(W)$ by conjugation. We can see this action by considering how π acts on the graph Γ_C of a class $C \in \mathcal{C}(W)$. Let Γ_C have nodes representing the set $\Phi = \{ a_1, \ldots, a_h ; b_1, \ldots, b_k \}$, and a bond of strength $n_{ab} \cdot n_{ba} = 2 \frac{(a,b)}{(a,a)} \cdot 2 \frac{(b,a)}{(b,b)}$ joining the node a to node b. Then π acts on the nodes of Γ_C by its action on $\precsim$, and we claim that bond strengths are preserved. Then $\Gamma_{\pi(C)}$ has nodes representing the set $\Phi^{\pi} = \{\pi(a_1), \ldots, \pi(a_h); \pi(b_1), \ldots, \pi(b_k)\}$.

If the roots a, b have the same length, then $\pi = \tau$ is an isometry of the space they generate . So $n_{\pi(a)\pi(b)} \cdot n_{\pi(b)\pi(a)} = n_{ab} \cdot n_{ba}$. If a is short , and b is long , then $\tau a = p^{\frac{1}{2}} . \pi a$, and $\tau b = p^{-\frac{1}{2}} . \pi b$, by Proposition 1.3. Hence ,

$$\begin{aligned} n_{\pi(a)\pi(b)} \cdot n_{\pi(b)\pi(a)} &= 2 \frac{(\pi a, \pi b)}{(\pi a, \pi a)} \cdot 2 \frac{(\pi b, \pi a)}{(\pi b, \pi b)} \\ &= 2 \frac{(p^{-\frac{1}{2}} . \tau a \ , \ p^{\frac{1}{2}} . \tau b)}{(p^{-\frac{1}{2}} . \tau a \ , \ p^{-\frac{1}{2}} . \tau a)} \cdot 2 \frac{(p^{\frac{1}{2}} . \tau b \ , \ p^{-\frac{1}{2}} . \tau a)}{(p^{\frac{1}{2}} . \tau b \ , \ p^{\frac{1}{2}} . \tau b)} \\ &= 2 \frac{(a,b)}{(a,a)} \cdot 2 \frac{(b,a)}{(b,b)} \quad \text{since } \tau \text{ is an isometry .} \end{aligned}$$

Hence , bond strengths are preserved , and $\Gamma_{\pi(C)} = \pi(\Gamma_C)$. $\square$

Hence , we can see precisely how π acts on $\mathcal{C}(W)$ by seeing how π acts on the set $\Gamma(W) = \{ \Gamma_C : C \in \mathcal{C}(W) \}$ of admissible graphs of W .

<u>Notation</u>. If G is any group and $C \in \mathcal{C}(G)$ with $g \in C$, we specify C by writing C as C_g if there is no ambiguity . Otherwise , or in case of ambiguity , if $H \leq G$, we write the H-conjugacy class containing $g \in G$ as $\{g\}^H$.

We now prove a result (due to Burnside [5]) , which proves to be very useful in later chapters . Let $W^* = \langle W , \tau \rangle = W . \langle \tau \rangle$.

<u>Proposition 2.5</u>. Suppose that $|\mathcal{C}(W)| = c$, and that the action of π on $\mathcal{C}(W)$ leaves c_1 classes fixed . Then π permutes the remaining $(c - c_1)$ classes in orbits of length δ , and the number of conjugacy classes of W^* which are in any non-trivial coset of W in W^* is just c_1 . Moreover , the total number of conjugacy classes of W is $(\delta c_1 + \frac{c - c_1}{\delta})$.

<u>Proof</u>. If $W = \{w_1, \ldots, w_n\}$, then let $\Omega = \{w_1\tau, \ldots, w_n\tau\}$. Then, the elements of Ω, when transformed by conjugation by any element of W^*, are permuted amongst themselves, and so W^* acts as a permutation group on the coset $W\tau = \Omega$. We suppose that Ω splits into t orbits under this action as $\Omega = \Omega_1 \cup \cdots \cup \Omega_t$.

Suppose $w \in W$ and that $w \in C$ for some $C \in \mathcal{C}(W)$ with $\tau.C.\tau^{-1} = C$. Then $\tau.w.\tau^{-1} = w_j.w.w_j^{-1}$ for some fixed $j \in \{1, \ldots, n\}$, since $\tau.w.\tau^{-1} \in C$. Hence, any element $w_i\tau \in \Omega$ is fixed by $w \iff w.w_i\tau.w^{-1} = w_i\tau$,

$$\iff w_i^{-1}.w.w_i = \tau.w.\tau^{-1},$$

$$\iff w_i^{-1}.w.w_i = w_j.w.w_j^{-1},$$

$$\iff w_i.w_j \in \mathbb{C}_W(w),$$

$$\iff w_i \in \mathbb{C}_W(w).w_j^{-1}.$$

Hence, there are precisely $|\mathbb{C}_W(w)| = |W| / |C|$ such elements $w_i\tau \in \Omega$.

On the other hand, if $w \in W$ and $w \in C$ for some $C \in \mathcal{C}(W)$ with $\tau.C.\tau^{-1} \neq C$, then $w_i\tau \in \Omega$ is fixed by $w \iff w.w_i\tau.w^{-1} = w_i\tau$,

$$\iff w_i^{-1}.w.w_i = \tau.w.\tau^{-1} \notin C,$$

which is not true for any w_i. Hence, no element of Ω is fixed by w. Hence, when Ω is acted upon by all the permutations of W, the total number F of fixed symbols is $c_1.|C|.(|W|/|C|) = c_1.|W|$.

Now W is transitive on each Ω_i, so consider the action of W on $\Omega_i = \{a_1, \ldots, a_k\}$. Let $\mu_i(w) = |\{a_j \in \Omega_i : w.a_j = a_j\}|$ for $w \in W$, and $\lambda_i(a_j) = \left|\{w \in W : w.a_j = a_j\}\right|$. Certainly , we know that

$$\sum_{j=1}^{k} \lambda_i(a_j) = \sum_{w \in W} \mu_i(w) \quad . \qquad (*) \ .$$

Now $\lambda_i(a_j) = |S_j|$, where $S_j = \mathrm{stab}_W(a_j)$, and since W is transitive on Ω_i , we have $k = |\Omega_i| = |W / S_j|$ for all j . So $|W| = k.|S_j|$, and $|S_j| = |S_1|$ for all j . Hence ,

$$\sum_{j=1}^{k} \lambda_i(a_j) = \sum_{j=1}^{k} |S_j| = k.|S_1| = |W| \ .$$

Thus , $\sum_{w \in W} \mu_i(w) = |W|$, by $(*)$. This is true for all Ω_i , so if $\nu(w) = \left|\{a \in \Omega : w.a = a\}\right|$, then

$$F = \sum_{w \in W} \nu(w) = \sum_{w \in W} \left(\sum_{i=1}^{t} \mu_i(w) \right) = \sum_{i=1}^{t} \left(\sum_{w \in W} \mu_i(w) \right) = \sum_{i=1}^{t} |W| = t|W| \ .$$

However , we have already seen that $F = c_1.|W|$, so that $t = c_1$. Suppose $\Omega_i = \{w_1'\tau , \ldots , w_k'\tau\}$. Then $\Omega_i = \{w_1'\tau\}^W$, and

$$\Omega_i^{\tau} = \{\tau.w_1'\tau.\tau^{-1}\}^{W^{\tau}} = \{\tau w_1'\}^W = \{w_1'^{-1}.w_1'\tau.w_1'\}^W = \{w_1'\tau\}^W = \Omega_i \ .$$

Hence , τ fixes each Ω_i for $i \in \{1,\ldots,t\}$, and so each Ω_i is invariant under the action of any element of W^* . Hence , the sets $\Omega_1 , \Omega_2 , \ldots , \Omega_t$ are just the W^*-conjugacy classes in the coset $W\tau$. (This is a union of conjugacy classes since τ is an automorphism of W.). Hence , the set $\Omega = W\tau$ falls into c_1 W^*-conjugacy classes . Similarly with the other cosets $W\tau^2 , \ldots , W\tau^{\delta-1}$.

Finally , since τ permutes the $(c - c_1)$ classes of $\mathcal{C}(W)$, which are not fixed by τ , in orbits of order δ , these fuse in sets of order δ to give $\frac{c - c_1}{\delta}$ W^*-conjugacy classes, whereas the c_1 classes of $\mathcal{E}(W)$ which are fixed by τ, remain as W^*-conjugacy classes. Hence the total number of W^*-conjugacy classes of W^* is

$$c_1 + (\delta - 1)c_1 + \left(\frac{c - c_1}{\delta}\right) = \delta c_1 + \left(\frac{c - c_1}{\delta}\right) . \qquad \square$$

Hence, by combining Lemma 2.4 and Proposition 2.5 with the work described in §2.1, we obtain a useful result about the number of conjugacy classes of W^* by considering the action of π on the set $\Gamma(W)$ of admissible diagrams of W. In Chapter 6, we show how this result gives us the number of equivalence classes in $H^1(\sigma,W)$ when σ is not a pure field automorphism, and we use this in Chapters 6 and 7. Although we need to know representative elements of the classes of $H^1(\sigma,W)$ in order to determine the structure of the maximal tori, this result does tell us when we have all the representatives.

§2.5. Maximal tori corresponding to Weyl subgroups of W.

In §2.1, we mentioned the fact that is only necessary in the work of [6] to determine those admissible graphs of a Weyl group W for which the corresponding conjugacy class is not contained in any Weyl subgroup W' of W. Furthermore, in Proposition 2.2, we saw that the matrix E_{Φ} corresponding to such classes is unimodular, thus enabling us to readily compute the elementary divisors of the maximal tori T_w corresponding to these classes from the graph.

Suppose that $w \in W'$, a proper Weyl subgroup of W of rank ℓ'. Then we can choose W' so that w belongs to no proper Weyl subgroup of W', and then the conjugacy class containing w has as graph an admissible graph Γ' of W'. Then the roots Φ mentioned in §2.1 span a subspace U of V of dimension ℓ', and the characteristic polynomial of w is $f_w(t) = f_{\Gamma'}(t) \cdot (q-1)^{\ell-\ell'}$, [6]. One might therefore expect that T_w would be isomorphic to $T'_w \times \overbrace{C_{q-1} \times \ldots \times C_{q-1}}^{\ell-\ell'}$, where T'_w is the maximal torus of the corresponding Chevalley subgroup G' of G. Although this may be true in certain cases, one only needs to refer to Table 5.2 to see that this is certainly not so in general. We now discuss this situation. Even if the above were to be valid one would have to be careful of the isogeny type of G' with respect to that of G, a problem which involves the extension problem of finite abelian groups.

Suppose that W' is a Weyl subgroup of W of rank ℓ' and that $w \in W'$. Let T be a maximal torus of $G_{\pi,K}$ twisted with respect to w, and let $X = X(T)$. If $X' = \{x \in X : x^w = x \text{ for all } w \in W'\}$, then it easily follows that $X/_{X'}$ has no torsion. Hence, $(X')^{\perp} = T'$ is a subtorus of T, see [8], and $X/_{X'} \cong X(T')$.

Now, although w acts trivially on X', it is not clear that there exists a w-invariant complement of X' in X, and indeed such a complement does not exist in general. However, if it did, then the action of w on X could be represented as a block matrix $\begin{bmatrix} I_{\ell-\ell'} & 0 \\ 0 & w' \end{bmatrix}$, where w' is the matrix of w on the subgroup $X(T')$. Then we should certainly be able to say that T_w is isomorphic to $C_{q-1} \times \ldots \times C_{q-1} \times T'_w$. Hence, we must direct our study, not only to the semi-Coxeter classes of W , but to all the conjugacy classes of W .

A simple counter example is in the adjoint group of type A_2 , where w is in the conjugacy class corresponding to the partition $[1,2]$. Then $w \in W' \lneq W$, where $W' \cong W(A_1)$, and by Proposition 3.3 , $T_w \cong C_{q^2-1}$, which is not isomorphic with $C_{q-1} \times C_{q+1}$ when q is an odd prime power .

CHAPTER 3. Chevalley groups of Type A_ℓ

§3.1. In this chapter, we discuss the groups $G_{\pi,K}$ when $\mathfrak{g}$ is a simple complex Lie algebra of type A_ℓ. These groups are of especial interest because $G_{sc,K} \cong SL_{\ell+1}(K)$ (as groups), and generally they are the first to be investigated with regard to conjectures, and their properties often bear fruit for generalisations to Chevalley groups. However, although these groups generally prove to be easier to handle, in this case the problem has not been solved completely for all cases. We proceed to give an account of the partial results that we have so far been able to obtain.

In this situation, the isogeny class $\mathcal{A}_\ell$ of simple groups of type A_ℓ contains groups other than $G_{ad,K}$ and $G_{sc,K}$. For then the finite group Δ_{sc}/Δ_{ad} is isomorphic to the cyclic group $C_{\ell+1}$ of order $\ell+1$. Hence, given any divisor d of $(\ell+1)$, we can find a lattice Δ such that $\Delta_{sc} \supset \Delta \supset \Delta_{ad}$ and $\Delta_{sc}/\Delta \cong C_d$. Then, by §1.1, there exists a faithful representation π^d of $\mathfrak{g}$ with $\Delta = \Delta_{\pi^d}$, and hence a corresponding group $G_{\pi^d, K}$. If ρ and π are faithful representations of $\mathfrak{g}$, then the kernel of the homomorphism $\lambda_{\rho,\pi} : G_\pi(K) \longrightarrow G_\rho(K)$, mentioned in §1.1, is isomorphic with the group Δ_ρ/Δ_π modulo p-torsion. Hence, there may be some repetitions or collapsing in the groups $G_{\pi^d,K}$, depending upon the value of $(\ell+1,p)$. If there is no collapsing, then there is a radical isogeny between $G_{\pi^d,K}$ and $G_{sc,K}/Z_d$, where Z_d is the unique central subgroup of $G_{sc,K}$ of order d. From now, we denote $\Delta_{\pi d}$ by Δ_d.

Lemma 3.1. The lattice Δ_d has a basis $\Lambda_d = \{r_1, r_2, \ldots, r_\ell\}$, where $r_1 = d(e_1 - \frac{1}{\ell+1}\sum_{j=1}^{\ell+1} e_j)$ and $r_i = e_1 - e_i$ for $i \in \{2, \ldots, \ell\}$.

Proof. Now $\Pi = A.\Omega$, where A is the Cartan matrix of the root system of type A_ℓ, Π is a system of fundamental roots of the root system Σ of type A_ℓ, and Ω is the corresponding system of fundamental weights. Since $\Delta_{sc}/\Delta_{ad} \cong C_{\ell+1}$, we can diagonalise A to $A' = \begin{bmatrix} I_{\ell-1} & \underline{0} \\ \underline{0} & \ell+1 \end{bmatrix}$ over Z by the basis theorem for abelian groups [24] to find a basis Π' of Δ_{ad} and a basis Ω' of Δ_{sc} such that $\Pi' = A'.\Omega'$. Now $A' = R.A.C$, where R is the product of the elementary row operations on A and C is the product of the elementary column operations. Hence, if $\Pi' = R.\Pi$ and $\Omega' = C^{-1}.\Omega$, then $\Pi' = A'.\Omega'$ as required.

If $\{e_i\}_{i=1}^{\ell+1}$ is an orthonormal basis of a real vector space V, then we can embed Δ_{ad} and Δ_{sc} in the hyperplane $\{\sum_{i=1}^{\ell+1} \xi_i . e_i : \sum_{i=1}^{\ell+1} \xi_i = 0\}$, and realise the bases Π and Ω as $\Pi = \{p_i = e_i - e_{i+1} : i \in \{1, \ldots, \ell\}\}$, and $\Omega = \{q_i = \sum_{j=1}^{i} e_j - \frac{i}{\ell+1}\sum_{j=1}^{\ell+1} e_j : i \in \{1, \ldots, \ell\}\}$.

Recalling that A is the matrix $\begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \underline{0} \\ \underline{0} & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$, it follows that

$R = \left[\begin{array}{c:cccc} \underline{0} & & I_{\ell-1} & & \\ \hdashline 1 & 2 & 3 & \cdots & \ell \end{array}\right]$, so that $\Pi' = \{p_1, p_2, \ldots, p_{\ell-1}, (\ell+1).q_1\}$ and

$\Omega' = \{p_1, p_2, \ldots, p_{\ell-1}, q_1\}$.

Hence , for any divisor d of $(\ell+1)$, there corresponds a proper sublattice of Δ_{sc} containing Δ_{ad} denoted Δ_d , and Δ_d is generated by the basis $\Lambda'_d = \{p_1, p_2, \ldots p_{\ell-1}, dq_1\}$.

The lemma now follows since Λ_d and Λ'_d are integrally equivalent in the sense that an element of one can be expressed as an integral combination of the elements of the other , and vice versa .

Explicitly , $r_1 = dq_1$, and $r_i = \sum_{j=1}^{i-1} p_j$ for $i \in \{2, \ldots, \ell\}$. Conversely , $p_i = r_{i+1} - r_i$.

$\square$

Hence , by Chapter 1 , we must find the elementary divisors of the matrix $(q.w_d - I)$ for a representative element w from each class of $H^1(\sigma, W)$, where w_d is the matrix of the action of w on Δ_d with respect to the basis Λ_d.

---

§3.2. The matrix $L_d(\lambda)$.

We have seen in Chapter 2 that in the case of the Chevalley groups , the set $H^1(\sigma, W)$ of equivalence classes corresponds to the set $\mathcal{C}(W)$ of conjugacy classes of W . Now , in groups of type A_ℓ, $W \cong \mathfrak{S}_{\ell+1}$, the symmetric group on $(\ell+1)$ letters , and this is another reason why this group tends to be the first to be investigated .

It is well known that there is a 1-1 correspondence between the conjugacy classes of $\mathfrak{S}_{\ell+1}$ and partitions of $\ell+1$, due to Young . In fact the group W acts upon V by permutations of the basis $\{e_i\}_{i=1}^{\ell+1}$.

Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t]$ be a general partition of $\ell+1$ such that

$\sum_{i=1}^{t} \lambda_i = \ell + 1$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_t$. Then we denote the corresponding conjugacy class of W by C_λ . Following the work of Carter [6] , this has a graph of the form $A_{\lambda_1 - 1} + A_{\lambda_2 - 1} + \ldots + A_{\lambda_t - 1}$.

A representative element of C_λ is the element w such that :-

$$\omega : \begin{cases} e_1 \mapsto e_2 \mapsto \cdots \mapsto e_{\lambda_1} \mapsto e_1 \\ e_{\lambda_1+1} \mapsto e_{\lambda_1+2} \mapsto \cdots \mapsto e_{\lambda_1+\lambda_2} \mapsto e_{\lambda_1+1} \\ e_{\lambda_1+\lambda_2+1} \mapsto e_{\lambda_1+\lambda_2+2} \mapsto \cdots \mapsto e_{\lambda_1+\lambda_2+\lambda_3} \mapsto e_{\lambda_1+\lambda_2+1} \\ \vdots \\ e_{\lambda_1+\cdots+\lambda_{t-1}+1} \mapsto e_{\lambda_1+\cdots+\lambda_{t-1}+2} \mapsto \cdots \to e_{\lambda_1+\cdots+\lambda_t} \mapsto e_{\lambda_1+\cdots+\lambda_{t-1}+1} \end{cases} .$$

Then w acts on Δ_d in the following way with respect to the basis Λ_d :-

$$w_d = \begin{array}{c} \begin{bmatrix} 1 & & & & & & & & & k \\ -d & -1 & \cdots & -1 & -1 & \cdots & \cdots & \cdots & -1 & -2 \\ & 1 & 0 & & & & & & & -1 \\ & & \ddots & \ddots & & & & & & \vdots \\ & & & 1 \;\; 0 & & & & & & \vdots \\ & & & & 0 & & 1 & & & \vdots \\ & & & & 1 & \ddots & & & & \vdots \\ & & & & & \ddots & \ddots & & & \vdots \\ & & & & & & 1 \;\; 0 & & & \vdots \\ & & & & & & & \ddots & & \vdots \\ & & & & & & & & 0 & \vdots \\ & & & & & & & & 1 \;\; \ddots & \vdots \\ & & & & & & & & \ddots \;\; 0 & -1 \\ & & & & & & & & 1 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \} \lambda_1 \\ \\ \\ \\ \} \lambda_2 \\ \\ \\ \\ \\ \} \lambda_t - 1 \\ \\ \\ \end{array}$$

, where $kd = \ell + 1$.

Hence , the matrix $(q.w_d - I)$ can be obtained and this reduces to the matrix $L_d(\lambda)$ upon diagonalisation over Z , where

$$L_d(\lambda) = \begin{bmatrix} q-1 & & & & & & -k \\ d & f_{\lambda_1}(q) & f_{\lambda_2}(q) & f_{\lambda_3}(q) & \cdots & & z \\ & & q^{\lambda_2}-1 & & & & f_{\lambda_2}(q) \\ & & & q^{\lambda_3}-1 & & & \vdots \\ & & & & \ddots & & \\ & & & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & & & f_{\lambda_t}(q) \end{bmatrix},$$

where $z = \sum_{i=1}^{t} \left[q^{\lambda_i-2} + 2q^{\lambda_i-3} + \ldots + (\lambda_i-2)q + (\lambda_i-1)\right]$ and $f_n(q) = q^{n-1} + \ldots + 1$. We note that $(q-1)z = \sum_{i=1}^{t} f_{\lambda_i}(q) - (\ell+1)$ (*).

The matrix $L_d(\lambda)$ appears to resist all attempts at a general diagonalisation which accounts for the incomplete results for these groups. However, in the next few sections, we consider some particular cases, beginning with the groups $G_{sc,K}$.

§3.3. The groups $G_{ad,K}$ and $G_{sc,K}$.

We consider the groups $G_{sc,K}$ and, equivalently because of Corollary 1.8, the groups $G_{ad,K}$. These groups correspond to the situation $d = 1$, and then the matrix $L_d(\lambda)$ quickly reduces to the matrix

$$L_1(\lambda) = \begin{bmatrix} q^{\lambda_1}-1 & & & & f_{\lambda_1}(q) \\ & q^{\lambda_2}-1 & & & f_{\lambda_2}(q) \\ & & \ddots & & \vdots \\ & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & f_{\lambda_t}(q) \end{bmatrix},$$

by the following operations :-

(i) $(q-1)r_2$ to r_1 ; (ii) $\sum_{i=3}^{t+1} r_i$ to r_1 , using the expression (*) ;

(iii) Removing (r_2, c_1) . To diagonalise this matrix $L_1(\lambda)$, we need the following Lemma .

Lemma 3.2. Consider the submatrix $L = \begin{bmatrix} q^{\lambda_1}-1 & 0 & f_{\lambda_1}(q) \\ 0 & q^{\lambda_2}-1 & f_{\lambda_2}(q) \\ 0 & 0 & f_{\lambda_t}(q) \end{bmatrix}$ of $L_1(\lambda)$,

and let $v_1 = (\lambda_1, \lambda_2)$ be the greatest common divisor of λ_1 and λ_2 .

Then L reduces to the matrix $\begin{bmatrix} \dfrac{(q^{\lambda_1}-1)(q^{\lambda_2}-1)}{(q^{v_1}-1)} & 0 & 0 \\ 0 & q^{v_1}-1 & f_{v_1}(q) \\ 0 & 0 & f_{\lambda_t}(q) \end{bmatrix}$

Proof. If

$$\lambda_2 = a_1.\lambda_1 + b_1$$
$$\lambda_1 = a_2.b_1 + b_2$$
$$b_1 = a_3.b_2 + b_3$$
$$\vdots$$
$$b_{n-2} = a_n.b_{n-1} + b_n$$
$$b_{n-1} = a_{n+1}.b_n \quad , \text{ so that } b_n = v_1, \text{ and } b_{n+1} = 0 .$$

Let $c_i = -q^{b_i}\left[(q^{b_i-1})^{a_i-1} + \ldots + (q^{b_i-1}) + 1\right]$. Then by the sequence of operations (1) $c_i.r_1$ to r_2 if i is odd ,

(2) $c_i.r_2$ to r_1 if i is even , beginning with $i = 1$,

we can reduce the matrix L to a matrix of the form

$$L_i = \begin{bmatrix} q^{b_i}-1 & \dfrac{(q^{\lambda_2}-1)(q^{\lambda_1}-q^{b_i})}{(q^{b_i}-1)} & f_{b_i}(q) \\ q^{b_{i+1}}-1 & \dfrac{(q^{\lambda_2}-1)(q^{\lambda_1}-q^{b_{i+1}})}{(q^{b_{i+1}}-1)} & f_{b_{i+1}}(q) \\ 0 & 0 & f_{\lambda_t}(q) \end{bmatrix} .$$

After the $(n+1)^{th}$ operation , we have the matrix L_{n+1} , where $b_{n+1} = 0$. Hence ,

$$L_{n+1} = \begin{bmatrix} q^{v_1}-1 & 0 & f_{v_1}(q) \\ 0 & \frac{(q^{\lambda_1}-1)(q^{\lambda_2}-1)}{(q^{v_1}-1)} & 0 \\ 0 & 0 & f_{\lambda_t}(q) \end{bmatrix}$$

, and the result follows .

$\square$

We are now able to solve the problem for the groups $G_{sc,K}$, after introducing some notation . Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t]$ be a partition of $(\ell+1)$, and let $v = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be the greatest common divisor of the λ_i . Then define $v_1 = (\lambda_1, \lambda_2)$ and $v_i = (v_{i-1}, \lambda_{i+1})$ for $i = 2, \ldots, s$, so that $v_s = (v_{s-1}, \lambda_{s+1}) = v$. For , certainly such an s with $1 \leq s < t$ exists , by definition of v .

Proposition 3.3. Let w be a representative element of the conjugacy class C_λ of $W(A_\ell)$, where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t]$. Then , with the above notation , the corresponding maximal torus T_w of the group $G_{\pi,K}(q)$ has elementary divisors , when π is ad or sc , equal to :

$$e_1 = \frac{(q^{\lambda_1}-1)(q^{\lambda_2}-1)}{(q^{v_1}-1)} , \quad e_2 = \frac{(q^{v_1}-1)(q^{\lambda_3}-1)}{(q^{v_2}-1)}, \ldots, e_s = \frac{(q^{v_{s-1}}-1)(q^{\lambda_{s+1}}-1)}{(q^{v}-1)} ,$$

$$e_{s+1} = q^{v-1}+\ldots+q+1 , \quad e_{s+2} = (q^{\lambda_{s+2}}-1) , \ldots, e_t = (q^{\lambda_t}-1) .$$

Proof. By Lemma 3.2 , it follows that we can reduce the matrix $L_1(\lambda)$ to obtain the first elementary divisor $e_1 = \frac{(q^{\lambda_1}-1)(q^{\lambda_2}-1)}{(q^{v_1}-1)}$ and we are left with the matrix

$$L_1(\lambda)_1 = \begin{bmatrix} q^{v_1}-1 & & & & f_{v_1}(q) \\ & q^{\lambda_3}-1 & & & f_{\lambda_3}(q) \\ & & \ddots & & \vdots \\ & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & f_{\lambda_t}(q) \end{bmatrix} .$$

We can continue this process until we have obtained the first s elementary divisors $e_i = \dfrac{(q^{v_{i-1}}-1)(q^{\lambda_{i+1}}-1)}{(q^{v_i}-1)}$ for $i = 2,\ldots,s$. Then we are left with the matrix

$$L_1(\lambda)_s = \begin{bmatrix} q^{v}-1 & & & & f_{v}(q) \\ & q^{\lambda_{s+3}}-1 & & & f_{\lambda_{s+3}}(q) \\ & & \ddots & & \vdots \\ & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & f_{\lambda_t}(q) \end{bmatrix} \text{, since } v_s = v .$$

Now if we subtract the final column from the first column $(q-1)$ times , and add the intermediate columns to the first column , then we can reduce this matrix to

$$L_1(\lambda)'_s = \begin{bmatrix} 0 & & & & f_{v}(q) \\ & q^{\lambda_{s+3}}-1 & & & f_{\lambda_{s+3}}(q) \\ & & \ddots & & \vdots \\ & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ q^{\lambda_t}-1 & & & & f_{\lambda_t}(q) \end{bmatrix} .$$

Now , since $v = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, it follows from Corollary 4.6 that since $v \mid \lambda_i$ for all i , then $f_v(q) \mid f_{\lambda_i}(q)$ for $i = 1, \ldots, t$ and in particular for $i = s+3, \ldots, t$. Hence , by operations on the first row of $L_1(\lambda)'_s$, we can reduce the matrix to

$$\operatorname{diag}\ (f_v(q)\ ,\ q^{\lambda_{s+3}}-1, \ldots, q^{\lambda_{t-1}}-1, q^{\lambda_t}-1)\ .$$

Hence , the result follows. $\square$

Although the results in this case are not quite as simple as one might expect , in practice it is generally possible to arrange the sequence $\{\lambda_1, \ldots, \lambda_t\}$ so that the sequence $\{v_1, \ldots, v_s\}$ becomes quite short .

Examples (i). If $\lambda = [\ell+1]$, so that $t = 1$, it follows that $v = \ell+1$. Since $s = 0$, it follows that there is just one elementary divisor in this case , viz. $e_1 = q^{\ell}+\ldots+q+1$.

(ii). If $\lambda = [4,6,8]$, then $v = 2$. Now $v_1 = (4,6) = v$, so that $s = 1$. Hence the elementary divisors in this case are

$$e_1 = \frac{(q^4-1)(q^6-1)}{(q^2-1)} = (q^2+1)(q^6-1)\ ,\quad e_2 = (q+1)\ ,\quad e_3 = (q^8-1)\ .$$

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§3.4. The Coxeter tori in the groups $G_{\pi,K}$.

Considering the previous results and Corollary 1.8 , the fact that the tori in all of the groups $G_{\pi,K}$ corresponding to a particular class of $\mathcal{E}(W)$ have the same order (a consequence of Proposition 1.4) might seem to suggest that such tori have the same structure also . Although we have been unable to obtain complete results for the groups A_ℓ , we now consider a special case which demonstrates that the structure of the tori corresponding to some class of $\mathcal{L}(W)$ does depend upon the isogeny type of the group concerned .

We have discussed the Coxeter class of W in Proposition 2.2 and we define a <u>Coxeter torus</u> of $G_{\pi,K}$ to be a maximal torus of $G_{\pi,K}$ which corresponds to the Coxeter class of $\mathcal{L}(W)$ under the bijection Q of Theorem 1.2. We now discuss the structure of the Coxeter tori in the various groups $G_{\pi^d,K}$ for d a divisor of $(\ell+1)$, where G is of type A_ℓ . Now the Coxeter class of W corresponds to the partition $\lambda=[\lambda_1]$ such that $\lambda_1 = (\ell+1)$, since this is the only partition of $(\ell+1)$ which corresponds to elements not lying in any Weyl subgroup of W .

<u>Lemma 3.4</u> The matrix $L_d(\lambda)$ in this case reduces to the matrix

$$\begin{bmatrix} q-1 & (q^d)^{k-1}+\dots+q^d+1 \\ q^{d-1}+\dots+q+1 & 0 \end{bmatrix}$$

<u>Proof</u>. The matrix $L_d(\lambda) = \begin{bmatrix} q-1 & k \\ -d & z \end{bmatrix}$ in this case , where

$$z = (q^{\ell-1}+ 2q^{\ell-2}+\dots+(\ell-1)q+\ell) .$$

Let $y = ((q^d)^{k-2}+ 2(q^d)^{k-3}+\dots+(k-2)q^d+(k-1)) . (q^{d-1}+\dots+q+1)$, so that the operation $y.c_1$ to c_2 reduces $L_d(\lambda)$ to

$$\begin{bmatrix} (q-1) & (q^d)^{k-1}+(q^d)^{k-2}+\dots+q^d + 1 \\ -d & ((q^d)^{k-1}+\dots+q^d+1) . (q^{d-2}+ 2q^{d-3}+\dots+(d-2)q+(d-1)) \end{bmatrix} .$$

By the operation $-(q^{d-2}+ 2q^{d-3}+\dots+(d-2)q+(d-1)).r_1$ to r_2 , and then multiplying r_2 by the unit -1 of Z , this matrix reduces to

$$\begin{bmatrix} q-1 & (q^d)^{k-1}+\dots+q^d+1 \\ q^{d-1}+\dots+q+1 & 0 \end{bmatrix} .$$

□

Lemma 3.5. Suppose that $B = \begin{bmatrix} a_1 & 0 \\ a_3 & a_2 \end{bmatrix}$ is a relation matrix for an abelian group G, and that $e = (a_1, a_2, a_3)$ - the greatest common divisor of the three integers a_1, a_2, a_3. Then B is equivalent (in the sense of relation matrices) to the diagonal matrix diag $(a_1.a_2.e^{-1}, e)$, and, consequently, G is isomorphic to the group $C_e \times C_{a_1.a_2.e^{-1}}$.

Proof. Let $a_i = a_i'.e$ for $i = 1,2,3$ so that $(a_1', a_2', a_3') = 1$, and then $B = eI . B'$, where $B' = \begin{bmatrix} a_1' & 0 \\ a_3' & a_2' \end{bmatrix}$. Since the matrix eI is in the centre of the group $GL_2(Z)$, we may just consider reducing the matrix B'.

We assume that $a_1 > 0$, and let $a_{ij}' = (a_i', a_j')$ so that $(a_{ij}', a_k) = 1$ for i,j,k distinct. Then we proceed by induction on the modulus of the leading term of B'.

Suppose that $a_1' = 1$, so that B' easily reduces to $\begin{bmatrix} 1 & 0 \\ 0 & a_1'.a_2' \end{bmatrix}$. Assume now that $a_1' > 1$. Then $a_{13}' \leq a_1'$.

Suppose first that $a_{13}' = a_1'$. Then $a_1' \mid a_3'$, and B easily reduces to the matrix $\begin{bmatrix} a_1' & 0 \\ 0 & a_2' \end{bmatrix}$. However $(a_1', a_2') = (a_1', a_2', a_3') = 1$, so that there exist $u, v \in Z$ with $u.a_1' + v.a_2' = 1$. Hence

$$\begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} . \begin{bmatrix} a_1' & 0 \\ 0 & a_2' \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} = \begin{bmatrix} a_1' & 0 \\ 1 & a_2' \end{bmatrix}, \text{ which reduces to } \begin{bmatrix} 1 & 0 \\ 0 & a_1'\, a_2' \end{bmatrix}$$

Suppose now that $a_{13}' < a_1'$, so that there exist $u, v \in Z$ with $u.a_3' + v.a_1' = a_{13}'$. Then $\begin{bmatrix} v & u \\ -a_3'/a_{13}' & a_1'/a_{13}' \end{bmatrix} . \begin{bmatrix} a_1' & 0 \\ a_3' & a_2' \end{bmatrix} = \begin{bmatrix} a_{13}' & ua_2' \\ 0 & a_1'.a_2'/a_{13}' \end{bmatrix}$, and we have reduced B' to a matrix $B'' = \begin{bmatrix} b_1 & 0 \\ b_3 & b_2 \end{bmatrix}$ such that $0 < b_1 < a_1'$

and $(b_1,b_2,b_3) = (a'_{13}, a'_1.a'_2/a'_{13}, ua'_2) = (a'_{13}, a'_2) = 1$.

We assume for our induction principle that for matrices $X = \begin{bmatrix} x_1 & 0 \\ x_3 & x_2 \end{bmatrix}$ with $(x_1,x_2,x_3) = 1$ and $0 < x_1 < a'_1$, then X is equivalent to the diagonal matrix $\begin{bmatrix} 1 & 0 \\ 0 & x_1x_2 \end{bmatrix}$. Thus , by induction , B' is equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & b_1b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a'_1.a'_2 \end{bmatrix} = A$. Thus we have found integral unimodular matrices P,Q such that $P.B'.Q = A$. Hence ,

$$P.B.Q = P.(eI).P^{-1}.\ P.B'.Q = eI\ .\ A = \begin{bmatrix} e & 0 \\ 0 & a'_1a'_2e \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & a_1a_2e^{-1} \end{bmatrix} .$$

The final part follows obviously .

$\square$

<u>Lemma 3.6</u>. The Coxeter tori of the groups $G_{\pi^d,K}(q)$ have the structure $C_{e_1} \times C_{e_2}$, with elementary divisors $e_1 = (d, \frac{\ell+1}{d}, q-1)$ and $e_2 = (q^\ell+\dots+1).e_1^{-1}$.

<u>Proof</u>. By Lemmas 3.4 and 3.5 , it follows that the matrix $L_d(\lambda)$ is equivalent to the diagonal matrix $\begin{bmatrix} e & 0 \\ 0 & (q^\ell+\dots+1).e^{-1} \end{bmatrix}$, where $e = (q^{d-1}+\dots+1,\ (q^d)^{k-1}+\dots+q^d+1\ ,\ q-1) = (d,k,q-1)$.

If w is a representative of the Coxeter class of W , then it follows from §3.2 that the matrix $(q.w_d-I)$ reduces to $\begin{bmatrix} e & 0 \\ 0 & (q^\ell+\dots+1).e^{-1} \end{bmatrix}$ Hence the Coxeter tori have elementary divisors $e_1 = (d, \frac{\ell+1}{d}, q-1)$ and $e_2 = (q^\ell+\dots+1).e_1^{-1}$.

$\square$

<u>Example</u>. Obviously , in the groups $G_{sc,K}$ when $d = 1$, it follows that $e_1 = 1$, as we have already seen . The first non-trivial example

occurs in groups of type A_3, where $\Delta_{sc}/\Delta_{ad} \cong C_4$. If we let $K_o = GF(5)$, then there is no collapsing in the groups $G_{\pi^d,K}$. The only proper divisor d of $(\ell+1)$ is 2, so that $G_{\pi^2,K}$ is a Chevalley group of type A_3 distinct from $G_{ad,K}$ and $G_{sc,K}$. In this case, $e_1 = (2,2,4) = 2$, and $e_2 = (5^3+5^2+5+1).2^{-1} = 78$. Hence, the structure of the Coxeter tori of the group $G_{\pi^2,K}(5)$ is $C_2 \times C_{78}$. However, in the groups $G_{sc,K}(5)$ and $G_{ad,K}(5)$, the Coxeter tori have structure $C_{156} \cong C_4 \times C_{39}$, as in Example (i) of §3.3.

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§3.5. A generalisation.

Finally, we discuss the case of those maximal tori of the groups $G_{\pi^d,K}$ which correspond to the conjugacy classes of $W(A_\ell)$ with elements inside a proper Weyl subgroup of type $W(A_{\ell-1}) \cong \mathfrak{S}_\ell$. These classes correspond to partitions $\lambda' = [\lambda_1,\lambda_2,\ldots,\lambda_{t+1}]$ in which some $\lambda_i = 1$, and we may assume that $\lambda_{t+1} = 1$. This amounts to assuming that the action of a representative element w of C_λ on V leaves one co-ordinate axis fixed. Then we let $\lambda = [\lambda_1,\lambda_2,\ldots,\lambda_t]$.

Lemma 3.7. The matrix $L_d(\lambda')$ in this case reduces to the matrix

$$L_d(\lambda')' = \begin{bmatrix} q-1 & & & & & d \\ & q^{\lambda_1}-1 & & & & f_{\lambda_1}(q) \\ & & q^{\lambda_2}-1 & & & f_{\lambda_2}(q) \\ & & & \ddots & & \vdots \\ & & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & & f_{\lambda_t}(q) \end{bmatrix}$$

Proof. From §3.2, the polynomial entry $f_{\lambda_{t+1}}(q)$ of $L_d(\lambda')$ becomes $f_{\lambda_{t+1}}(q) = 1$, when $\lambda_{t+1}=1$. Hence, by elementary operations, we may remove (r_{t+2}, c_{t+2}) and transpose r_1 and c_1 to reduce $L_d(\lambda')$ to the matrix $L_d(\lambda')'$ above. $\square$

Proposition 3.8. Let w be a representative element of the conjugacy class $C_{\lambda'}$ of $W(A_\ell)$, where $\lambda' = [\lambda_1,\ldots,\lambda_{t+1}]$ and $\lambda_{t+1} = 1$. With λ defined as above and the notation of Proposition 3.3, the corresponding maximal torus T_w has elementary divisors

$$e_1 = \frac{(q^{\lambda_1}-1)(q^{\lambda_2}-1)}{(q^{v_1}-1)}, \quad e_2 = \frac{(q^{v_1}-1)(q^{\lambda_3}-1)}{(q^{v_2}-1)}, \quad \ldots \quad e_s = \frac{(q^{v_{s-1}}-1)(q^{\lambda_{s+1}}-1)}{(q^{v}-1)},$$

$$e_{s+1} = e, \quad e_{s+2} = (q^v-1).e^{-1}, \quad e_{s+3} = (q^{\lambda_{s+2}}-1), \ldots, \quad e_{t+1} = (q^{\lambda_t}-1),$$

where $e = (q-1, v, d)$.

Proof. We can proceed as in proposition 3.3 until we reach the stage

$$L_d(\lambda')_s = \begin{bmatrix} q-1 & & & & & d \\ & q^v-1 & & & & f_v(q) \\ & & q^{\lambda_{s+2}}-1 & & & f_{\lambda_{s+2}}(q) \\ & & & \ddots & & \vdots \\ & & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & & & & & f_{\lambda_t}(q) \end{bmatrix}.$$

If, as before, we subtract the final column from the second column $(q-1)$ times, and add the intermediate columns to the second column, then adding the first column to the second column d times , we can reduce the matrix to

$$L_d(\lambda')'_s = \begin{bmatrix} q-1 & & & & & d \\ & 0 & & & & f_v(q) \\ & & q^{\lambda_{s+2}}-1 & & & f_{\lambda_{s+2}}(q) \\ & & & \ddots & & \vdots \\ & & & & q^{\lambda_{t-1}}-1 & f_{\lambda_{t-1}}(q) \\ & q^{\lambda_t}-1 & & & & f_{\lambda_t}(q) \end{bmatrix} .$$

Proceeding as in Proposition 3.3, we can reduce this matrix to

$$\operatorname{diag}\,(q^{\lambda_{s+2}}-1\ ,\ldots,\ q^{\lambda_t}-1\ ,\ L\ )\ ,\ \text{where}$$

$$L = \begin{bmatrix} q-1 & d \\ 0 & f_v(q) \end{bmatrix} .$$

By Lemma 3.5 , L reduces to $\operatorname{diag}\,(e,(q^v-1).e^{-1})$, where $e = (q-1,f_v(q),d) = (q-1,v,d)$, and the result follows . $\square$

We conclude this chapter by considering the isomorphism between groups of type A_3 and groups of type D_3 . In both of these , there is a group $G_{\pi,K}$ distinct from $G_{ad,K}$ and $G_{sc,K}$ since $\Delta_{sc}/\Delta_{ad} \cong C_4$. Using the results of §3.4 , we see that the Coxeter torus of the group of type A_3 has the structure $C_e \times C_{(q^3+q^2+q+1)e^{-1}}$, where $e = (q-1,2,2)$. So $e = 2$ in the case of K having odd characteristic , and

$$T_w \cong C_2 \times C_{\frac{(q^3+q^2+q+1)}{2}} .$$

This class corresponds to the class with signed cycle-type $[2,1]$ in the group $W(D_3)$ under the above isomorphism, and the results of § 4.4 show us that the corresponding torus T_w is isomorphic to $C_{q+1} \times C_{q^2+1}$. This does not contradict the previous paragraph since $(q+1, q^2+1) = 2$ if q is odd. However, this does suggest that the result of §3.4 may not be in the best form, since, in this case, the polynomial $(q^{\ell}+q^{\ell-1}+\ldots+q+1)$ factorises as $(q^d+\ldots+q+1).((q^d)^{k-1}+\ldots+q^d+1)$.

That this does not happen (and therefore that the results of § 3.4 are in the best form) is demonstrated in groups of type A_{35}, where $k = d = 6$, for the field $GF(5)$. Our results show that the Coxeter torus has the structure $C_2 \times C_t$, where $t = \frac{5^{36}-1}{8}$. However, a torus with elementary divisors of the form $(q^d+\ldots+q+1)$ and $((q^d)^{k-1}+\ldots+q^d+1)$ would have a subgroup isomorphic to $C_6 \times C_6$.

CHAPTER 4. Chevalley Groups of type B_ℓ, C_ℓ and D_ℓ.

§4.1. In this chapter, we consider the groups $G_{\pi,K}$, where π is a faithful representation of a complex, simple Lie algebra of type B_ℓ, C_ℓ or D_ℓ. The reason for the simultaneous treatment of these groups becomes apparent when one considers the description of the conjugacy classes of their respective Weyl groups, and we refer to [6]. For, let $W = W(B_\ell) = W(C_\ell)$. The elements of $W(C_\ell)$ operate on an orthonormal basis $\{e_i\}_{i=1}^{\ell}$ of X_R by means of permutations and sign changes. Each element $w \in W$ determines a permutation of the set $\{1,\ldots,\ell\}$ which can be expressed as a product of disjoint cycles, and if $(j_1 j_2 \ldots j_r)$ is one such cycle, then w operates as

$$e_{j_1} \longmapsto \pm e_{j_2} \longmapsto \pm e_{j_3} \longmapsto \quad \cdots \quad \longrightarrow \pm e_{j_r} \longmapsto \pm e_{j_1} .$$

The cycle is said to be <u>positive</u> if $w^r(e_{j_1}) = e_{j_1}$, and <u>negative</u> if $w^r(e_{j_1}) = - e_{j_1}$. The lengths of the cycles together with their signs give a set of positive or negative integers called the <u>signed cycle-type</u> of w. Two elements of W are conjugate if and only if they have the same signed cycle-type. A positive λ_i-cycle $[\lambda_i]$ is a Coxeter element of a Weyl subgroup $W(A_{\lambda_i - 1})$ so is represented by the admissible graph $A_{\lambda_i - 1}$ (with A_0 the empty set), whereas a negative μ_j-cycle $[\bar{\mu}_j]$ is a Coxeter element of a Weyl subgroup $W(C_{\mu_j})$ so is represented by the graph C_{μ_j}. If we define the partition $[\lambda,\bar{\mu}]$ by $\lambda = [\lambda_1,\ldots,\lambda_t]$ and $\bar{\mu} = [\bar{\mu}_1,\ldots,\bar{\mu}_s]$, then we recover Young's classical result that there is a 1-1 correspondence between the set $\mathcal{L}(W(C_\ell))$ and pairs of partitions (λ,μ) such that $|\lambda| + |\mu| = \ell$.

Now let $W_1 = W(D_\ell)$. Then W_1 is a subgroup of W , and an element of W lies in W_1 if and only if it has an even number of negative cycles in its signed cycle-type , ie. s is even . Two elements of W_1 are conjugate if and only if they have the same signed cycle-type , except that if all the cycles are even and positive then there are two conjugacy classes . The admissible graphs representing the classes are as follows . The positive i-cycle $[i]$ has graph A_{i-1} and the pair of negative cycles $[\bar{i}\bar{j}]$, with $i \geqslant j \geqslant 1$, has graph $D_{i+j}(a_{j-1})$, where $D_k(a_o) = D_k$. A general graph is obtained by combining such graphs .

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§4.2. Modules in V .

We know that , given any maximal torus T of $G_{\pi,K}$, $X(T) \cong \Delta_\pi$ and is generated , as a Z-module , by the weights $P(\pi)$ of π . Whichever representation π is , X_R is a real vector space , say V , of dimension ℓ . Let $\{e_1,\dots,e_\ell\}$ be the natural basis of V , with scalar product (x,y) on V for which this basis is orthonormal , and identify V^* with V by this product . Then X is embedded as a lattice in V , and we define certain Z-modules in V as follows .

(i). Let M_1 be the Z-module with basis $\{e_i\}$, ie. $M_1 = \bigoplus_{i=1}^{\ell} Ze_i$.

(ii). Let M_2 be the submodule of M_1 consisting of elements $x = \sum_{i=1}^{\ell} \xi_i . e_i$ such that $\sum_{i=1}^{\ell} \xi_i$ is even . Let M_2' be the submodule of M_2 generated by the set $\{e_i \pm e_j\}$. Then $\sum_{i=1}^{\ell} \xi_i e_i \equiv (\sum_{i=1}^{\ell} \xi_i) e_\ell$ modulo M_2' , and since $2e_\ell \in M_2'$, then $\sum_{i=1}^{\ell} \xi_i e_i \in M_2'$ if $\sum_{i=1}^{\ell} \xi_i$ is even . Hence $M_2' = M_2$. Since $M_1 = \langle M_2, e_1 \rangle$, it follows that $M_1 / M_2 \cong C_2$.

(iii). Let M_3 be the Z-module generated by M_1 and $\rho_\ell = \frac{1}{2} \sum_{i=1}^{\ell} e_i$. Then $x = \sum_{i=1}^{\ell} \xi_i e_i \in M_3$ if and only if

(a). $2\xi_i \in Z$ and (b). $(\xi_i - \xi_j) \in Z$ for all $i, j \in \{1, \dots, \ell\}$.

Also , $M_3 / M_1 \cong C_2$.

(iv). Let M_4 be the Z-module generated by M_2 and ρ_ℓ . Then it is clear that $M_4 / M_2 \cong C_2$. Furthermore , if ℓ is a multiple of 4 , then M_4 is the set of elements $\sum_{i=1}^{\ell} \xi_i e_i$ which satisfy (a), (b) and

(c). $\sum_{i=1}^{\ell} \xi_i \in 2Z$.

Lemma 4.1. If $M^* = \mathrm{Hom}_Z(M, Z)$ is the dual module of a module M, then :-
$M_1^* = M_1$; $M_2^* = M_3$; and $M_4^* = M_4$ if ℓ is a multiple of 4 .

Proof. Trivially , M_1 is self-dual because the basis $\{e_i\}$ is orthonormal under (,) .

Suppose $x = \sum_{i=1}^{\ell} \xi_i e_i \in M_2^*$. Then , since M_2 is generated by $\{e_i \pm e_j\}$, we have $(\xi_i \pm \xi_j) \in Z$ for all $i, j \in \{1, \dots, \ell\}$. Hence , since this is equivalent to (a) and (b) , it follows that $M_2^* = M_3$ and M_2 , M_3 are dual .

Finally , $M_4^* = \{ x = \sum_{i=1}^{\ell} \xi_i e_i \in M_3 : \sum_{i=1}^{\ell} \xi_i \in 2Z \}$. If ℓ is a multiple of 4 , then M_4 is self-dual . $\square$

We consider the modules M_i to be embedded in V , and let $\tau : V \longrightarrow V$ be the involutive automorphism $\tau : e_\ell \longmapsto -e_\ell$.

<u>Lemma 4.2</u>. (i). If ℓ is odd, then there is precisely one Z-module M satisfying $M_2 \subset M \subset M_3$. This is M_1.

(ii). If ℓ is even, there are precisely 3 Z-modules M satisfying $M_2 \subset M \subset M_3$. They are M_1, M_4 and $\tau(M_4)$.

<u>Proof</u>. It is clear that the abelian group M_3 / M_2 has order 4, and there are two cases.

(i). If ℓ is odd, then $n.\rho_\ell \in M_2 \Longleftrightarrow n \equiv 0$ (modulo 4). Hence, $M_3 / M_2 \cong C_4$, and there is precisely one subgroup M. This is generated by M_2 and $2\rho_\ell = \sum_{i=1}^{\ell} e_i$, so since ℓ is odd it follows that $M = M_1$.

(ii). If ℓ is even, then $n.\rho_\ell \in M_2 \Longleftrightarrow n$ is even, and $n.\tau(\rho_\ell) \in M_2 \Longleftrightarrow n$ is even. Since $\rho_\ell + M_2 \neq \tau(\rho_\ell) + M_2$, it follows that $M_3 / M_2 \cong C_2 \times C_2$, and M_3 / M_2 contains 3 distinct, proper subgroups. Since $(\rho_\ell + M_2) + (\tau(\rho_\ell) + M_2) = e_\ell + M_2$, it follows that these are $\langle M_2, \rho_\ell \rangle_Z = M_4$, $\langle M_2, \tau(\rho_\ell) \rangle_Z = \tau(M_4)$, $\langle M_2, e_\ell \rangle_Z = M_1$.

Hence, in this case, since τ fixes M_2 and M_3, then τ acts on the subgroup lattice of M_3 / M_2 by permuting M_4 and $\tau(M_4)$, and fixing M_1. $\square$

<u>Definition</u>. We say that M_1 is situated <u>diagonally</u> between M_2 and M_3.

Consider now the fundamental roots $\Pi = \{p_i\}$ and the fundamental weights $\Omega = \{q_j\}$ of a root system Σ. So that $\Delta_{ad} = Z\Pi$ and $\Delta_{sc} = Z\Omega$.

Proposition 4.3. If $G_{\pi,K}$ is a group of type B_ℓ, C_ℓ or D_ℓ, then Δ_π is one of the modules M_1, M_2, M_3, M_4 or $\tau(M_4)$ as follows.

G \ M	M_1	M_2	M_3	M_4	$\tau(M_4)$
B_ℓ	Δ_{ad}		Δ_{sc}		
C_ℓ	Δ_{sc}	Δ_{ad}			
D_ℓ ℓ even	Δ_{π_1}	Δ_{ad}	Δ_{sc}	Δ_{π_2}	Δ_{π_3}
D_ℓ ℓ odd	Δ_{π_1}	Δ_{ad}	Δ_{sc}		

, where π_1, π_2 and π_3 are 3 faithful representations of a complex, simple Lie algebra of type D_ℓ which we describe below.

Proof. (i). If Σ is of type B_ℓ, then $p_i = e_i - e_{i+1}$ for $i<\ell$ and $p_\ell = e_\ell$. Hence, $\Delta_{ad} = M_1$. Also, $q_j = \sum_{k=1}^{j} e_k$ for $j<\ell$ and $q_\ell = \rho_\ell$. Hence $\Delta_{sc} = M_3$.

(ii). If Σ is of type C_ℓ, then $p_i = e_i - e_{i+1}$ for $i<\ell$ and $p_\ell = 2e_\ell$. Hence, $\Delta_{ad} = M_2$. Also $q_j = \sum_{k=1}^{j} e_k$ for all $j \leq \ell$, so $\Delta_{sc} = M_1$.

(iii). If Σ is of type D_ℓ, then $p_i = e_i - e_{i+1}$ for $i < \ell$ and $p_\ell = e_{\ell-1}+e_\ell$. Hence $\Delta_{ad} = M_2$. Also, $q_j = \sum_{k=1}^{j} e_k$ for $j<\ell-1$, $q_{\ell-1} = \tau(\rho_\ell)$ and $q_\ell = \rho_\ell$. Hence $\Delta_{sc} = M_3$.

When $\underline{g}$ is simple of type B_ℓ or C_ℓ, then $\Delta_{sc}/\Delta_{ad} \cong C_2$ and there are no submodules Δ such that $\Delta_{ad} \subset \Delta \subset \Delta_{sc}$. Hence the isogeny class $\mathcal{G}$ consists solely of the groups $G_{ad,K}$ and $G_{sc,K}$.

However, when $\underline{g}$ is simple of type D_ℓ, then $\mathcal{G}$ contains groups other than $G_{ad,K}$ and $G_{sc,K}$. For then $\Delta_{sc} / \Delta_{ad} \cong M_3 / M_2$, and, by Lemma 4.2, there exist submodules $\Delta_{ad} \subset \Delta \subset \Delta_{sc}$ such that $\Delta_{sc} / \Delta \cong C_2$. Then $\Delta = \Delta_\pi$ for some faithful representation π of $\underline{g}$, and, correspondingly, we have the group $G_{\pi,K}$ which is distinct from $G_{ad,K}$ and $G_{sc,K}$.

Following Lemma 4.2, we have 3 representations π_1, π_2 and π_3 which we define by $\Delta_{\pi_1} = M_1$, $\Delta_{\pi_2} = M_4$ and $\Delta_{\pi_3} = \tau(M_4)$. We note that only the first representation occurs if ℓ is odd. $\square$

By §1.1, there is an isomorphism between $\zeta = \ker(\lambda_{ad,sc})$ and the group $\Delta_{sc} / \Delta_{ad}$ modulo p-torsion, so there may be some repetitions or collapsing in the groups $G_{\pi_i,K}$. Assuming otherwise, if ζ_i is the subgroup of ζ corresponding to π_i, there is a radical isogeny between $G_{\pi_i,K}$ and $G_{sc,K}/\zeta_i$. In fact, we have :-

if ℓ is even, then $\zeta = \langle z, z' : z^2 = z'^2 = [z,z'] = 1 \rangle \cong C_2 \times C_2$,

and if ℓ is odd, then $\zeta = \langle z : z^4 = 1 \rangle \cong C_4$.

In the case when ℓ is even, then the two elements z, z' play a symmetric role, in that there exists an automorphism (inducing the automorphism τ of Lemma 4.2) of $G_{sc,K}$ which interchanges z and z'. Hence, if $\zeta_2 = \langle z \rangle$ and $\zeta_3 = \langle z' \rangle$, then there exists an automorphism between $G_{\pi_2,K}$ and $G_{\pi_3,K}$ which induces $\tau : \Delta_{\pi_2} \longrightarrow \Delta_{\pi_3}$. Hence, we need only consider the submodule $\Delta_{\pi_2} = M_4$.

On the other hand, the element (zz'), which generates $\mathfrak{Z}_1$ and is fixed by the above automorphism, is essentially different from z and z' (except when $\ell = 4$, in which case $G_{sc,K} \cong Spin_8(K)$ has an automorphism which cyclically permutes z, z' and zz' - see Chapter 7.). We say that $\mathfrak{Z}_1$ is situated diagonally in $\mathfrak{Z}$.

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§4.3. Description of the weight lattices

Suppose that Δ_π has Z-basis Λ, and that with respect to this basis, the action of $w \in W$ is given by the matrix w_Λ. Then, in order to determine the elementary divisors of the maximal torus T_w in the groups $G_{\pi,K}(K_0)$, we must take a representative w from each element of $\mathcal{L}(W)$ and compute the elementary divisors of the matrix $(q.w_\Lambda - I)$. We have seen in §4.1. that such a representative is given by a pair of partitions $[\lambda, \bar{\mu}]$ of ℓ. So, for this chapter, we take a general element of $\mathcal{L}(W)$ with representative w and signed cycle-type $[\lambda, \bar{\mu}]$. Furthermore, by Proposition 4.3, in the groups under consideration, Δ_π is one of M_1, M_2, M_3 or M_4, since we can exclude $\tau(M_4)$ because of the automorphism. By Lemmas 1.7 and 1.8, we may exclude one of a pair of dual modules, since the elementary divisors will be identical in such cases. Hence, we need only to consider the action of w on M_1, M_3 and M_4 to determine the elementary divisors of the maximal tori in the groups $G_{\pi,K}(K_0)$ of type B_ℓ, C_ℓ or D_ℓ. In fact :-

Proposition 4.4. Let w have signed cycle-type $[\lambda, \bar{\mu}]$. Then the elementary divisors of $(q.w-I)$ acting on the module M are those of the corresponding tori T_w in the groups $G_{\pi,K}(K_o)$ according to the table :-

G \ M	M_1	M_3	M_4
B_ℓ	ad	sc	
C_ℓ	sc	ad	
D_ℓ ℓ even	π_1	ad, sc	π_2, π_3
D_ℓ ℓ odd	π_1	ad, sc	

, where a given type G and module M fixes the representation π .

Proof. From the above , and Proposition 4.3. □

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§4.4. The module M_1 .

In this section, we consider the action of w upon the module M_1 with basis $\Lambda_1 = \{e_i\}_{i=1}^{\ell}$, and we make the convention that if w is as above , then w acts on Λ_1 as :-

$$w:\begin{cases} e_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+1} \mapsto e_{\lambda_1+\cdots+\lambda_{i-1}+2} \mapsto \cdots \mapsto e_{\lambda_1+\cdots+\lambda_i} \mapsto e_{\lambda_1+\cdots+\lambda_{i-1}+1} \\ e_{|\lambda|+\mu_1+\cdots+\mu_{j-1}+1} \mapsto e_{|\lambda|+\mu_1+\cdots+\mu_{j-1}+2} \mapsto \cdots \mapsto e_{|\lambda|+\mu_1+\cdots+\mu_j} \mapsto -e_{|\lambda|+\mu_1+\cdots+\mu_{j-1}+1} \end{cases}$$

for all $i \in \{1,\ldots,t\}$ and $j \in \{1,\ldots,s\}$, where $\lambda_0 = \mu_0 = 0$.

Proposition 4.5. The elementary divisors of $(q.w_{\Lambda_1}-I)$ are

$$e_1 = q^{\lambda_1}-1 \ , \ e_2 = q^{\lambda_2}-1 \ , \ \ldots \ , \ e_t = q^{\lambda_t}-1 \ , \ e_{t+1} = q^{\mu_1}+1 \ , \ldots \ , \ e_{t+s} = q^{\mu_s}+1 .$$

Proof. Let $M^{\lambda_i} = \langle e_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+1}, e_{\lambda_1+\cdots+\lambda_{i-1}+2}, \ldots, e_{\lambda_1+\cdots+\lambda_i}\rangle$,

and $M^{\mu_j} = \langle e_{|\lambda|+\mu_1+\cdots+\mu_{j-1}+1}, e_{|\lambda|+\mu_1+\cdots+\mu_{j-1}+2}, \ldots, e_{|\lambda|+\mu_1+\cdots+\mu_j}\rangle$.

Then each M^{λ_i} , M^{μ_j} is w-invariant and so is (q.w-I)-invariant. Also , $M_1 = (\bigoplus_{i=1}^{t} M^{\lambda_i}) \oplus (\bigoplus_{j=1}^{s} M^{\mu_j})$. Hence , $(q.w_{\Lambda_1}-I)$ is equal to the diagonal block matrix $\mathrm{diag}((q.w_{\Lambda_1}-I)\big|_{M^{\lambda_1}}, \ldots, (q.w_{\Lambda_1}-I)\big|_{M^{\mu_s}})$.

Now $(q.w_{\Lambda_1}-I)\big|_{M^{\lambda_i}} = \begin{bmatrix} -1 & & & q \\ q & -1 & & \\ & q & \ddots & \\ & & q & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & q^{\lambda_i}-1 \end{bmatrix}$, and

$$(q.w_{\Lambda_1}-I)\big|_{M^{\mu_j}} = \begin{bmatrix} -1 & & & -q \\ q & -1 & & \\ & q & \ddots & \\ & & q & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & q^{\mu_j}+1 \end{bmatrix} .$$

Hence , $(q.w_{\Lambda_1}-I) \longrightarrow \mathrm{diag}(q^{\lambda_1}-1, \ldots q^{\lambda_t}-1, q^{\mu_1}+1, \ldots, q^{\mu_s}+1)$. $\square$

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§4.5. The module M_3 .

In this section, we consider the action of w upon the module M_3 , which is generated by M_1 and ρ_ℓ . Then , a Z-basis Λ of M_3 is $\Lambda = \{e_1, \ldots, e_{\ell-1}, \rho_\ell\}$, since $e_\ell = 2\rho_\ell - \sum_{j=1}^{\ell-1} e_j$. We now consider different cases .

Case (I). In this case we assume that $s \geq 1$, ie. that there is at least one negative cycle , and for the moment we also assume that there exists some $j \in \{1, \ldots, s\}$ for which $\mu_j \neq 1$. Hence , we may assume that $\mu_s > 1$.

We let M^{λ_i} and M^{μ_j} be as in §4.4 , except that for $j = s$, we have $M^{\mu_s} = \langle e_{|\lambda|+\mu_1+\cdots+\mu_{s-1}+1}, \ldots, e_{|\lambda|+|\mu|-1} = e_{\ell-1}, \rho_\ell \rangle$.
Then M^{λ_i} and M^{μ_j} are w-invariant for all $i \in \{1, \ldots, t\}$ and all $j \in \{1, \ldots, s-1\}$. However , w acts on M^{μ_s} in the following way :-

$$\omega : \begin{cases} e_{|\lambda|+\mu_1+\cdots+\mu_{s-1}+1} \longmapsto e_{|\lambda|+\mu_1+\cdots+\mu_{s-1}+2} \\ \quad \vdots \qquad\qquad\qquad \vdots \\ e_{\ell-2} \longmapsto e_{\ell-1} \\ e_{\ell-1} \longmapsto e_\ell = -e_1-e_2-\cdots-e_{\ell-1}+2\rho_\ell \\ \rho_\ell \longmapsto -\sum_{j=0}^{s-1} e_{|\lambda|+\mu_0+\cdots+\mu_j+1} + \rho_\ell \end{cases} .$$

As before , $(qw-I)\Big|_{M^{\lambda_i}} = \begin{bmatrix} -1 & & & q \\ q & \ddots & & \\ & \ddots & \ddots & \\ & & q & -1 \end{bmatrix}$ and $(qw-I)\Big|_{M^{\mu_j}} = \begin{bmatrix} -1 & & & -q \\ q & \ddots & & \\ & \ddots & \ddots & \\ & & q & -1 \end{bmatrix}$

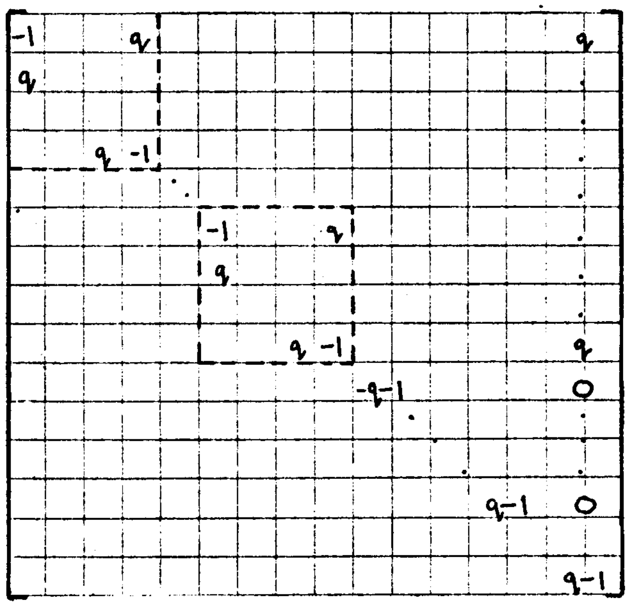
whenever $j \neq s$. Then it follows that the matrix $(qw-I)$ reduces to the matrix

$$M(\lambda,\bar{\mu}) = \begin{bmatrix} q^{\lambda_1}-1 & & & & & & & & f_{\lambda_1}(q) \\ & q^{\lambda_2}-1 & & & & & & & f_{\lambda_2}(q) \\ & & \ddots & & & & & & \vdots \\ & & & q^{\lambda_r}-1 & & & & & f_{\lambda_r}(q) \\ & & & & q^{\mu_1}+1 & & & & f_{\mu_1+\mu_s}(q) \\ & & & & & q^{\mu_2}+1 & & & f_{\mu_2+\mu_s}(q) \\ & & & & & & \ddots & & \vdots \\ & & & & & & & q^{\mu_{s-1}}+1 & f_{\mu_{s-1}+\mu_s}(q) \\ & & & & & & & & q^{\mu_s}+1 \end{bmatrix} ,$$

where $f_n(q) = q^{n-1}+\ldots+q+1$, for any positive integer n , as in Chapter 3 .

We have assumed that $\mu_s > 1$, so now let us assume that $\mu_j = 1$ for all $j \in \{1,\ldots,s\}$. Then $w : e_j \longmapsto e_j$ for all $j \in \{|\lambda|+1,\ldots,\ell\}$, so that $w\Big|_{M^{\mu_j}} = -I$ for $j \neq s$. Now $M^{\mu_s} = \langle \rho_\ell \rangle_Z$, so

$w : \rho_\ell \longmapsto e_1+e_2+\dots+e_{|\lambda|} - \rho_\ell$. Then

$(qw-I) =$  ,

which can be diagonalised to $M(\lambda,\bar{\mu})$. Hence we do not need to consider this case separately .

As in Chapter 3 , for any sequence $\{a_1,\dots,a_n\}$ of integers , we denote the ideal of Z that they generate by $\langle a_1,\dots,a_n\rangle$. We also denote g.c.d. $\{a_1,\dots,a_n\}$ by $(a_1,\dots,a_n)$.

Considering $M(\lambda,\bar{\mu})$, we see that , in order to diagonalise it , we need to consider whether (i) $f_{\lambda_i}(q) \in \langle q^{\lambda_i}-1 , q^{\mu_s}+1\rangle$

and (ii) $f_{\mu_j+\mu_s}(q) \in \langle q^{\mu_j}+1 , q^{\mu_s}+1\rangle$.

So we make the following notation for any non-negative integers a , b . Let $J_{a,b} = \langle q^a+1 , q^b+1\rangle$ and $\bar{J}_{a,b} = \langle q^a-1 , q^b+1\rangle$.

Proposition 4.6. For any non-negative integers a, b such that $d = (a,b)$, we have (i) $J_{a,b} = \langle q^d+1 , 1 + (-1)^{\frac{a+b}{d}-1}\rangle$; and

(ii) $\bar{J}_{a,b} = \left\langle q^d+1 \ , \ 1+(-1)^{\frac{a}{d}-1} \right\rangle$.

Proof. (ii) Assume that (i) is true . Then we have

$$\bar{J}_{a,b} = \langle q^a-1, q^b+1\rangle = \langle q^{a+b}+1, q^b+1\rangle = \left\langle q^d+1, 1+(-1)^{\frac{a+2b}{d}-1}\right\rangle \text{ by (i)}$$

$$= \left\langle q^d+1, 1+(-1)^{\frac{a}{d}-1}\right\rangle .$$

(i). Suppose that (i) is true for $(a,b) = 1$, ie. suppose that

$J_{a,b} = \langle q+1 \ , \ 1+(-1)^{a+b-1}\rangle$. (1) . Now let a,b be any non-negative integers such that $(a,b) = d$. Then $a = a'd$ and $b = b'd$ and $(a',b') = 1$. Hence ,

$J_{a,b} = \langle (q^d)^{a'}+1 \ , \ (q^d)^{b'}+1\rangle = \langle q^d+1 \ , \ 1+(-1)^{a'+b'-1}\rangle$, by (1) .

Thus , $J_{a,b} = \left\langle q^d+1, \ 1+(-1)^{\frac{a+b}{d}-1}\right\rangle$, and it remains for us to show (1) .

Consider the Euclidean algorithm for finding the greatest common divisor of a,b. Then

$$\begin{aligned} a &= s_1.b+r_2 \\ b &= s_2.r_2+r_3 \\ r_2 &= s_3.r_3+r_4 \\ &\vdots \\ r_{n-1} &= s_n.r_n+r_{n+1} \\ r_n &= s_{n+1}.r_{n+1} \quad , \text{ where } r_{n+1} = 1 . \end{aligned}$$

Then $\langle q^a+1 \ , \ q^b+1\rangle = \langle (q^b)^{s_1}.q^{r_2}+1 \ , \ q^b+1\rangle = \langle (-1)^{s_1}q^{r_2}+1 \ , \ q^b+1\rangle$

$$= \langle q^b+1 \ , \ q^{r_2}+(-1)^{s_1}\rangle$$
$$\vdots$$
$$= \langle q+1 \ , \ 1+(-1)^{a+b-1}\rangle , \quad \text{by this process} .$$

Hence the result follows . $\square$

Corollary 4.6. $\langle q^a-1 \ , \ q^b-1\rangle = \langle q^d-1\rangle$.

Proof. Assume first that $(a,b) = 1$. Then , if both a and b are odd , it follows from Proposition 4.6 (i), by replacing q with $-q$, that $\langle q^a-1, q^b-1\rangle = \langle q^d-1\rangle$. On the other hand, if one of a and b is even and the other is odd, we may choose a to be even, then by replacing q with $-q$ in Proposition 4.6 (ii), it follows that $\langle q^a-1, q^b-1\rangle = \langle q^d-1\rangle$. The result follows as in the proof of Proposition 4.6 (i). $\square$

<u>Proposition 4.7</u>. Let a,b be any non-negative integers such that $d = (a,b)$. Then,

$$f_a(q) \equiv \begin{cases} 1 \text{ if } a \text{ is odd and } q \text{ is odd} \\ 0 \text{ otherwise} \end{cases} \text{ modulo } \bar{J}_{a,b}.$$

<u>Proof</u>. We have two cases :-

(i). Suppose that $a/d = 2c$ is even. Then $\bar{J}_{a,b} = \langle q^d+1\rangle$, and

$$f_a(q)=q^{a-1}+\ldots+q+1 = \frac{q^a-1}{q-1} = \frac{(q^{2d})^c-1}{q-1} = (q^d+1).\frac{(q^d-1)}{q-1}.\left[(q^{2d})^{c-1}+\ldots+q^{2d}+1\right]$$

$$\equiv 0 \text{ modulo } \bar{J}_{a,b}.$$

(ii). Suppose that a/d is odd, so that $\bar{J}_{a,b} = \langle q^d+1, 2\rangle$. Then we must consider the parity of q.

If q is odd, then $\bar{J}_{a,b} = \langle 2\rangle$, so $f_a(q) \equiv a$ modulo $\bar{J}_{a,b} = \langle 2\rangle$.

On the other hand, if q is even, then $\bar{J}_{a,b} = \langle 1\rangle = Z$, so that $f_a(q) \equiv 0$ modulo $\bar{J}_{a,b}$. $\square$

<u>Corollary 4.7</u>. With a,b as above, we have

$$f_{a+b}(q) \equiv \begin{cases} 1 \text{ if } (a+b) \text{ is odd and } q \text{ is odd} \\ 0 \text{ otherwise} \end{cases} \text{ modulo } J_{a,b}$$

<u>Proof</u>. This follows since $\bar{J}_{a+b,b} = J_{a,b}$. $\square$

Suppose , for the moment , that we have fixed μ_s . Then we order the $\{\lambda_i\}$ such that $\lambda_1,\ldots,\lambda_g$ are odd and $\lambda_{g+1},\ldots,\lambda_t$ are even . We also order the $\{\mu_j\}$ such that for $j\in\{1,\ldots,h\}$, we have $(\mu_j+\mu_s)$ is odd ; and for $j\in\{h+1,\ldots,s-1\}$, we have $(\mu_j+\mu_s)$ is even .

Returning to $M(\lambda,\bar{\mu})$, we can perform elementary row and column operations using r_i and c_{t+s} to alter the $(i,s+t)$-entry of $M(\lambda,\bar{\mu})$, for $i\in\{1,\ldots,t+s-1\}$. Then we can use Proposition 4.7 and its Corollary to obtain :-

<u>Lemma 4.8</u>. (i). If q is even , then

$$M(\lambda,\bar{\mu}) \longrightarrow \mathrm{diag}(q^{\lambda_1}-1,\ldots,q^{\lambda_t}-1\ ,\ q^{\mu_1}+1,\ldots,q^{\mu_s}+1) .$$

(ii). If q is odd, then

$$M(\lambda,\bar{\mu}) \longrightarrow \mathrm{diag}(q^{\lambda_{g+1}}-1,\ldots,q^{\lambda_t}-1,\ q^{\mu_{h+1}}+1,\ldots,q^{\mu_{s-1}}+1\ ,\ M(\lambda,\bar{\mu})') ,$$ where

$$M(\lambda,\bar{\mu})' = \begin{bmatrix} q^{\lambda_1}-1 & & & & & & & 1\\ & q^{\lambda_2}-1 & & & & & & 1 \\ & & \ddots & & & & & \vdots \\ & & & q^{\lambda_g}-1 & & & & 1\\ & & & & q^{\mu_1}+1 & & & 1 \\ & & & & & \ddots & & \vdots\\ & & & & & & q^{\mu_h}+1 & 1\\ & & & & & & & q^{\mu_s}+1\end{bmatrix}$$

<u>Proof</u>. (i). If q is even , then , by Proposition 4.7 and Cor.4.7 ,

$$f_{\lambda_i}(q) \equiv 0 \ \text{modulo}\ \bar{J}_{\lambda_i,\mu_s} \ ; \text{ and }\ f_{\mu_j+\mu_s}(q) \equiv 0 \ \text{modulo}\ J_{\mu_j,\mu_s} .$$

Hence we can eliminate the $(i,t+s)$-entry of $M(\lambda,\bar{\mu})$, for all

$i \in \{1,\ldots,t+s-1\}$, and the result follows .

(ii). If q is odd , then , in the same way , we can reduce the $(k,t+s)$-entry of $M(\lambda,\bar{\mu})$ to :-

$$\begin{cases} \lambda_k \ (\text{modulo } 2) & \text{if } k \in \{1,\ldots,t\} , \\ \mu_{k-t}+\mu_s \ (\text{modulo } 2) & \text{if } k \in \{t+1,\ldots,t+s-1\}. \end{cases}$$

Hence the result follows . $\square$

Henceforth , we shall assume that q is odd . Then it remains to consider the matrix $M(\lambda,\bar{\mu})'$, and we assume , for the moment , that $g \neq 0$. Then we can reduce $M(\lambda,\bar{\mu})'$ to

$$M(\lambda,\bar{\mu})'' \quad \begin{bmatrix} q^{\lambda_2}-1 & & & & & & & q^{\lambda_1}-1 \\ & q^{\lambda_3}-1 & & & & & & q^{\lambda_1}-1 \\ & & \ddots & & & & & \vdots \\ & & & q^{\lambda_g}-1 & & & & q^{\lambda_1}-1 \\ & & & & q^{\mu_1}+1 & & & q^{\lambda_1}-1 \\ & & & & & \ddots & & \vdots \\ & & & & & & q^{\mu_h}+1 & q^{\lambda_1}-1 \\ & & & & & & & (q^{\mu_s}+1)(q^{\lambda_1}-1) \\ & & & & & & & \end{bmatrix} .$$

<u>Lemma 4.9</u>. Let a,b be any odd integers with $d = (a,b)$, let c be any integer , and let $\bar{\bar{I}}_{a,b,c} = \langle q^a-1 \, , \, (q^b-1)(q^c+1) \rangle$. Then $(q^b-1) \equiv 0$ modulo $\bar{\bar{I}}_{a,b,c}$, as long as q is odd .

<u>Proof</u>. By Corollary 4.6 , $\langle q^a-1 \, , \, q^b-1 \rangle = \langle q^d-1 \rangle$, so that if $a = a'd$, then $\bar{\bar{I}}_{a,b,c} = (q^d-1) \, . \, \langle q^c+1 \, , \, f_{a'}(q^d) \rangle$.

Now , $\langle q^c+1, f_{a'}(q^d)\rangle \supseteq \langle q^c+1, q^a-1\rangle = \langle q^{(a,c)}+1, 2\rangle$ by Proposition 4.6

$= \langle 2\rangle$ since q is odd .

However $f_{a'}(q^d) \equiv a' \equiv 1$ modulo $\langle 2\rangle$. Hence ,

$\langle q^c+1 \ , \ f_{a'}(q^d)\rangle = Z$ and $\bar{\bar{I}}_{a,b,c} = \langle q^d-1\rangle$.

Thus $(q^b-1) \equiv 0$ modulo $\bar{\bar{I}}_{a,b,c}$, as claimed . $\square$

<u>Lemma 4.10</u>. Let a be any even integer , and b,c any odd integers with $(a,b) = d$. Let $\bar{I}_{a,b,c} = \langle q^a+1 \ , \ (q^b-1)(q^c+1)\rangle$. Then , as long as q is odd , $(q^b-1) \equiv 0$ modulo $\bar{I}_{a,b,c}$.

<u>Proof</u>. By Proposition 4.6 , $\bar{I}_{a,b,c} = (q^d+1,2).\left\langle \frac{q^a+1}{(q^d+1,2)} \ , \ q^c+1\right\rangle$

$= 2\left\langle \frac{q^a+1}{2} \ , \ q^c+1\right\rangle$.

Let $e = (a,c)$. Then

$\left\langle \frac{q^a+1}{2}, q^c+1\right\rangle \supseteq \langle q^a+1, q^c+1\rangle = \langle q^e+1, 2\rangle$ by Proposition 4.6

$= \langle 2\rangle$ since q is odd .

However , since q is odd and a is even , then $q^a+1 \equiv 2 \pmod 4$.

Hence , $\frac{q^a+1}{2} \equiv 1$ modulo $\langle 2\rangle$, and $\left\langle \frac{q^a+1}{2} \ , \ q^c+1\right\rangle = Z$.

Hence , $\bar{I}_{a,b,c} = \langle 2\rangle$ and $(q^b-1) \in \bar{I}_{a,b,c}$. $\square$

<u>Lemma 4.11</u>. When q is odd , then $M(\lambda,\bar{\mu})'$ diagonalises to

$$\mathrm{diag}(q^{\lambda_2}-1,\ldots,q^{\lambda_g}-1,q^{\mu_1}+1,\ldots,q^{\mu_h}+1,(q^{\lambda_1}-1).(q^{\mu_s}+1)) \ .$$

<u>Proof</u>. By considering $M(\lambda,\bar{\mu})''$ and Lemma 4.9 with $a = \lambda_i$ for $i \in \{2,\ldots,g\}$, $b = \lambda_1$ and $c = \mu_s$, we can eliminate the first $(g-1)$ entries of the final column .

Suppose now that μ_j is even for all $j \in \{1,\dots,s\}$. Then μ_s would be even so that $h = 0$, and this case can be treated as above (since c is artibrary in 4.9.) . Hence we may assume that some μ_j is odd , and we choose μ_s to be odd .

Then , for $j \in \{1,\dots,h\}$, we have μ_j is even . By considering $M(\lambda,\bar{\mu})''$ and Lemma 4.10 with $a = \mu_j$ for $j \in \{1,\dots,h\}$, $b = \lambda_1$ and $c = \mu_s$, we can eliminate the remaining non-diagonal entries of the final column . Hence the result follows . $\square$

We have assumed that $g \neq 0$, and if we assume otherwise then

$$M(\lambda,\bar{\mu})' = \begin{bmatrix} q^{\mu_1}+1 & & & 1 \\ & \ddots & & \vdots \\ & & q^{\mu_h}+1 & 1 \\ & & & q^{\mu_s}+1 \end{bmatrix}$$

, which diagonalises to

$$M(\lambda,\bar{\mu})'' = \begin{bmatrix} q^{\mu_2}+1 & & & q^{\mu_1}+1 \\ & \ddots & & \vdots \\ & & q^{\mu_h}+1 & q^{\mu_1}+1 \\ & & & (q^{\mu_1}+1)(q^{\mu_s}+1) \end{bmatrix} .$$

<u>Lemma 4.12.</u> Let a,b be even integers with $d = (a,b)$, and let c be an odd integer . Let $I_{a,b,c} = \langle q^a+1 , (q^b+1)(q^c+1) \rangle$. Then $(q^b+1) \equiv 0$ modulo $I_{a,b,c}$, as long as q is odd .

Proof. We have to consider two cases :-

(i). When $\frac{a+b}{d}$ is even , and $I_{a,b,c} = (q^d+1).\left\langle \frac{q^a+1}{q^d+1}, q^c+1 \right\rangle = (q^d+1).I$, say .

(ii). When $\frac{a+b}{d}$ is odd , and $I_{a,b,c} = (q^d+1,2)\left\langle \frac{q^a+1}{(q^d+1,2)}, q^c+1 \right\rangle$

$$= 2.\left\langle \frac{q^a+1}{2}, q^c+1 \right\rangle = 2.J \text{ , say , since}$$

q is odd .

Now $I,J \supseteq \langle q^a+1, q^c+1 \rangle = \langle q^{(a,c)}+1, 2 \rangle = \langle 2 \rangle$, since q is odd .

Now a is even so that $\frac{q^a+1}{2} = 1$ modulo $\langle 2 \rangle$ as in Lemma 4.10 , and hence $\frac{q^a+1}{q^d+1} \equiv 1$ modulo $\langle 2 \rangle$. Thus $I = J = Z$, and so $I_{a,b,c} = \langle q^d+1 \rangle$ in case (i) and $I_{a,b,c} = \langle 2 \rangle$ in case (ii) . Hence $q^b+1 \cong 0$ modulo $I_{a,b,c}$. $\square$

Lemma 4.13. If $g = 0$ and q is odd , then $M(\lambda,\bar{\mu})'$ diagonalises to

$$\text{diag}\,(q^{\mu_2}+1,\ldots,q^{\mu_h}+1,(q^{\mu_1}+1).(q^{\mu_s}+1)) .$$

Proof. If μ_j is even for all $j \in \{1,\ldots,s\}$, then $h = 0$ and $M(\lambda,\bar{\mu})'$ is just the 1 x 1 matrix $(q^{\mu_s}+1)$. Hence , we can choose μ_s to be odd , so that μ_j is even for all $j \in \{1,\ldots,h\}$. By considering $M(\lambda,\bar{\mu})'''$ and Lemma 4.12 with $a = \mu_j$ for $j \in \{2,\ldots,h\}$, $b = \mu_1$ and $c = \mu_s$, we can eliminate the non-diagonal entries of the final column . $\square$

We collect these results in :-

Proposition 4.14. The elementary divisors of (qw-I) are :-

(i). $\left\{q^{\lambda_1}-1, q^{\lambda_2}-1, \ldots, q^{\lambda_t}-1, q^{\mu_1}+1, \ldots, q^{\mu_s}+1\right\}$ if q is even, or if all λ_i are even and all μ_j have the same parity .

(ii). $\left\{q^{\lambda_2}-1, \ldots, q^{\lambda_t}-1, q^{\mu_1}+1, \ldots, q^{\mu_{s-1}}+1, (q^{\lambda_1}-1)(q^{\mu_s}+1)\right\}$ if there exists some λ_i , say λ_1, which is odd , and μ_s is odd (unless all μ_j are even , when μ_s is even.) .

(iii). $\left\{q^{\lambda_1}-1, \ldots, q^{\lambda_t}-1, q^{\mu_2}+1, \ldots, q^{\mu_{s-1}}+1, (q^{\mu_1}+1)(q^{\mu_s}+1)\right\}$ if all λ_i are even , and there exists some μ_j , say μ_s , which is odd , and μ_1 , which is even . $\square$

This completes case (I) , and now we turn to :-

<u>Case (II)</u>. In this case , we assume that $s = 0$, ie. that all cycles are positive . Then we can assume that $\lambda_i > 1$ for some i , so we choose $\lambda_t > 1$. Then we define M^{λ_i} as in §4.4 , for $i \in \{1, \ldots, t-1\}$, but we define

$$M^{\lambda_t} = \left\langle e_{\lambda_1+\ldots+\lambda_{t-1}+1}, \ldots, e_{\ell-1}, \rho_\ell \right\rangle_Z .$$

Then $(qw-I)\Big|_{M^{\lambda_i}} = \begin{bmatrix} -1 & & & q \\ q & \ddots & & \\ & & \ddots & \\ & & q & -1 \end{bmatrix}$ for $i \in \{1, \ldots, t-1\}$, and on M^{λ_t}

w acts as :-

$$w : \left\{ \begin{array}{ccc} e_{\lambda_1+\ldots+\lambda_{t-1}+1} & \longmapsto & e_{\lambda_1+\ldots+\lambda_{t-1}+2} \\ \vdots & & \vdots \\ e_{\ell-2} & \longmapsto & e_{\ell-1} \\ e_{\ell-1} & \longmapsto & -e_1-e_2-\ldots-e_{\ell-1}+2\rho_\ell \\ \rho_\ell & \longmapsto & \rho_\ell \end{array} \right\} .$$

Then $(qw-I)$ diagonalises to

$$M(\lambda) = \begin{bmatrix} q^{\lambda_1}-1 & & & & & f_{\lambda_1}(q) & 0 \\ & q^{\lambda_2}-1 & & & & f_{\lambda_2}(q) & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & q^{\lambda_{t-1}}-1 & & f_{\lambda_{t-1}}(q) & 0 \\ & & & & & f_{\lambda_t}(q) & 0 \\ & & & & & q+1 & q-1 \end{bmatrix} .$$

<u>Lemma 4.15</u>. Suppose that not all cycles are even . Then $M(\lambda)$ diagonalises to $\operatorname{diag}(q^{\lambda_1}-1,\dots,q^{\lambda_t}-1)$.

<u>Proof</u>. If not all cycles are even , then we can choose λ_t to be odd , and then

$$M(\lambda) \longrightarrow \begin{bmatrix} q^{\lambda_1}-1 & & & & (q^{\lambda_1}-1)f_{\nu}(q^2) \\ & q^{\lambda_2}-1 & & & (q^{\lambda_2}-1)f_{\nu}(q^2) \\ & & \ddots & & \vdots \\ & & & q^{\lambda_{t-1}}-1 & (q^{\lambda_{t-1}}-1)f_{\nu}(q^2) \\ & & & & q^{\lambda_t}-1 \end{bmatrix} ,$$

and the result follows . □

Hence the only remaining case is when w has all cycles even and positive , the case which causes the correspondence between pairs of partitions of ℓ and the conjugacy classes of $W(D_\ell)$ to be not bijective. So we assume this situation from now , and we have the following lemma .

Lemma 4.16. (i). Let $\{a_1,\ldots,a_n\}$ be any set of integers, and let $d_{ij} = (a_i,a_j)$. If we write $a_i = d_{ij}a'_{ij}$, then there exists some $k \in \{1,\ldots,n\}$ such that a'_{kj} is odd for all $j \in \{1,\ldots,n\}$.

(ii). If, further, q is odd, and $I'_{a_i,a_j} = \left\langle \frac{q^{a_i}-1}{2}, q^{a_j}-1 \right\rangle$, then $\frac{q^{a_j}-1}{2} \equiv 0$ modulo I'_{a_k,a_j} for all $j \in \{1,\ldots,n\}$, and k as in (i).

Proof. (i). Let $d = (a_1,\ldots,a_n)$. Then a_i/d must be odd for some $i \in \{1,\ldots,n\}$, say k. Since $d \mid d_{kj}$ for all $j \in \{1,\ldots,n\}$, then $a'_{kj} \mid a_k/d$, so a'_{kj} is odd.

(ii). By Corollary 4.6, $I'_{a_k,a_j} = \frac{q^{d_{kj}}-1}{2} \cdot \left\langle f_{a'_{kj}}(q), 2 f_{a'_{jk}}(q) \right\rangle$.

Since $(a'_{ij},a'_{ji}) = 1$ for all $i,j \in \{1,\ldots,n\}$, then

$$\left\langle f_{a'_{ij}}(q), 2 f_{a'_{ji}}(q) \right\rangle = \left\langle f_{a'_{ij}}(q), 2 \right\rangle \quad \text{by Corollary 4.6}.$$

$$= \left\langle a'_{ij}, 2 \right\rangle, \quad \text{since } q \text{ is odd}.$$

Since a'_{kj} is odd, then $I'_{a_k,a_j} = \left\langle \frac{q^{d_{kj}}-1}{2} \right\rangle$. Hence,

$$\frac{q^{a_j}-1}{2} \equiv 0 \text{ modulo } I'_{a_k,a_j}. \qquad \square$$

Lemma 4.17. (i) If q is even, then $M(\lambda)$ diagonalises to $\operatorname{diag}(q^{\lambda_1}-1,\ldots,q^{\lambda_t}-1)$.

(ii). If q is odd, then $M(\lambda)$ diagonalises to $\operatorname{diag}(q^{\lambda_1}-1,\ldots,\frac{q^{\lambda_t}-1}{2},2)$, where λ_t is such that $\lambda_t.(\lambda_i,\lambda_t)^{-1}$ is odd for all $i \in \{1,\ldots,t-1\}$.

Proof. (i). This follows by elementary row and column operations, since $(q+1,q-1) = 1$.

(ii). In this case , $M(\lambda)$ diagonalises to $\operatorname{diag}(2, M(\lambda)')$, where

$$M(\lambda)' = \begin{bmatrix} q^{\lambda_1}-1 & & & & & \frac{q^{\lambda_1}-1}{2} \\ & q^{\lambda_2}-1 & & & & \frac{q^{\lambda_2}-1}{2} \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & q^{\lambda_{t-1}}-1 & \frac{q^{\lambda_{t-1}}-1}{2} \\ & & & & & \frac{q^{\lambda_t}-1}{2} \end{bmatrix}.$$

By Lemma 4.16(i) , we know that there exists $k \in \{1,\ldots,t\}$ such that $\lambda_k.(\lambda_i,\lambda_k)^{-1}$ is odd for all $i \in \{1,\ldots,t\}$. Choose $k = t$, and then Lemma 4.16 (ii) shows that we can eliminate the (j,t)-entry of $M(\lambda)'$ for all $j \in \{1,\ldots,t-1\}$, by elementary row and column operations with c_j and r_t . Hence the result follows . $\square$

This completes the section for M_3 , and we have :-

<u>Proposition 4.18</u>. The elementary divisors of $(q.w-I)$ with respect to M_3 are :-

(i). $\{q^{\lambda_1}-1, q^{\lambda_2}-1, \ldots, q^{\lambda_t}-1, q^{\mu_1}+1, \ldots, q^{\mu_s}+1\}$ if q is even , or if all λ_i are even and $s \geqslant 1$ with all μ_j having the same parity .

(ii). $\{q^{\lambda_2}-1, \ldots, q^{\lambda_t}-1, q^{\mu_1}+1, \ldots, q^{\mu_{s-1}}+1, (q^{\lambda_1}-1).(q^{\mu_s}+1)\}$ if there exists some λ_i , say λ_1 , which is odd , and μ_s is odd (unless all μ_j are even , when μ_s is even) .

(iii). $\{q^{\lambda_1}-1, \ldots, q^{\lambda_t}-1, q^{\mu_2}+1, \ldots, q^{\mu_{s-1}}+1, (q^{\mu_1}+1).(q^{\mu_s}+1)\}$ if all λ_i are even , and there exists some μ_j , say μ_s , which is odd , and μ_1 which is even .

(iv). $\left\{ q^{\lambda_1}-1, q^{\lambda_2}-1, \ldots, \frac{q^{\lambda_t}-1}{2}, 2 \right\}$ if all cycles are even and positive , where λ_t is such that $\lambda_t.(\lambda_i,\lambda_t)^{-1}$ is odd for all $\lambda_i \neq \lambda_t$.

Proof. From Proposition 4.14 and Lemmas 4.15 , 4.17 . $\square$

This concludes the section relating to the modules M_2 and M_3 . From now , we cease to make use of the notation $f_a(q) = q^{a-1}+\ldots+q+1$.

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§4.6. The module M_4 .

In this section , we consider the groups $G_{\pi,K}$, where G is a group of type D_ℓ for ℓ even , and π is one of the pair of "dual" representations π_2 and π_3 of g mentioned in §4.3 . We have seen that we need only consider one of these groups since they are isomorphic . In fact , these groups tend to be rather ill behaved , and , for reasons very similar to those of § 3.4. we have been unable to obtain complete results for these groups . This is due to the fact that , with the rather complicated basis of Δ_π , viz. the module M_4 , the matrix $(q.w_\lambda-I)$ does not appear to be readily diagonalisable over $Z[q]$ to give a general result , although it obviously diagonalises over Z in any particular case .

However, we do have results for the classes corresponding to the semi-Coxeter classes of $W(D_\ell)$. These , in themselves , are rather interesting in that they show how these particular groups behave differently depending upon the value of ℓ (modulo 4) . We saw how M_4 is self-dual if ℓ is a multiple of 4 in Lemma 4.1.

Lemma 4.19. The set $\Lambda = \{r_1, \ldots, r_\ell\}$ is a basis for the Z-module M_4, where $r_1 = \rho_\ell = \frac{1}{2}\sum_{i=1}^{\ell} e_i$, $r_i = e_i - e_{i+1}$ for $i \in \{2, \ldots, \ell-1\}$ and $r_\ell = 2e_\ell$.

Proof. This follows immediately by definition of M_4 as in §4.2, and since M_2 is generated by $\{e_1-e_2, r_2, \ldots r_\ell\}$. We note that $\frac{1}{2}\ell \in Z$ and that $e_1-e_2 = 2r_1 - 2r_2 - 3r_3 - \ldots - (\ell-1)r_{\ell-1} - \frac{1}{2}\ell\, r_\ell$. $\square$

We now consider the case where w is a representative of the particular conjugacy class corresponding to $[\lambda, \bar{\mu}]$, where $t = 0$, $s = 2$ and $\bar{\mu} = [\bar{i}\ \ \overline{\ell-i}]$. This is denoted in [6] by $D_\ell(a_{i-1})$, with $D_\ell(a_0) = D_\ell$ the Coxeter class of $W(D_\ell)$.

Then we can choose w to be the element which acts on $\{e_i\}$ by :-

$$w : \left\{ \begin{array}{l} e_1 \longmapsto e_2 \longmapsto \quad \ldots\ldots\ldots \quad \longrightarrow e_i \longmapsto -e_1 \\ e_{i+1} \longmapsto e_{i+2} \longmapsto \quad \ldots\ldots\ldots \quad \longrightarrow e_\ell \longmapsto -e_{i+1} \end{array} \right\}.$$

Then w acts on Λ by :-

$$\omega : \left\{ \begin{array}{l} r_1 \longmapsto r_1 - e_1 - e_{i+1} = -r_1 + r_2 + 2r_3 + \ldots + (i-1)r_i + (i-1)r_{i+1} + \ldots + (\ell-3)r_{\ell-1} + \left(\frac{\ell-2}{2}\right)r_\ell \\ r_2 \longmapsto e_3 - e_4 = r_3 \\ r_3 \longmapsto e_4 - e_5 = r_4 \\ \quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ r_{i-1} \longmapsto e_1 + e_i = 2r_1 - r_2 - 2r_3 - \ldots - (i-2)r_{i-1} - (i-2)r_i - \ldots - (\ell-3)r_{\ell-1} - \left(\frac{\ell-2}{2}\right)r_\ell \\ r_i \longmapsto -e_1 - e_{i+2} = -2r_1 + r_2 + 2r_3 + \ldots + i\,r_{i+1} + i\,r_{i+2} + \ldots + (\ell-3)r_{\ell-1} + \left(\frac{\ell-2}{2}\right)r_\ell \\ r_{i+1} \longmapsto e_{i+2} - e_{i+3} = r_{i+2} \\ \quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ r_{\ell-1} \longmapsto e_{i+1} + e_\ell = r_{i+1} + r_{i+2} + \ldots + r_{\ell-1} + r_\ell \\ r_\ell \longmapsto -2e_{i+1} = -2r_{i+1} - 2r_{i+2} - \ldots - 2r_{\ell-1} - r_\ell . \end{array} \right.$$

Then the matrix $(q.w\ -I)$ is

$$
\begin{array}{c}
\\ \\ \\ \\ \\ \\ \\ i \\ \\ \\ \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|ccccc}
 & & & & & & i & & & & & \\
\end{array}\\
\left[\begin{array}{ccccccc|ccccc}
-q-1 & & & & & 2q & -2q & & & & & \\
q & -1 & & & & -q & q & & & & & \\
2q & q & -1 & & & -2q & 2q & & & & & \\
\vdots & & q & \ddots & & \vdots & \vdots & & & & & \\
\vdots & & & \ddots & & \vdots & \vdots & & & & & \\
\vdots & & & & -1 & \vdots & \vdots & & & & & \\
\vdots & & & & q & -(i-2)q-1 & (i-2)q & & & & & \\
(i-1)q & & & & & -(i-2)q & (i-1)q-1 & & & & & \\
\hline
(i-1)q & & & & & -(i-1)q & iq & -1 & & & q & -2q \\
iq & & & & & -iq & iq & q & \ddots & & \vdots & \vdots \\
\vdots & & & & & \vdots & \vdots & & \ddots & & \vdots & \vdots \\
\vdots & & & & & \vdots & \vdots & & & -1 & q & -2q \\
(\ell-3)q & & & & & -(\ell-3)q & (\ell-3)q & & & q & q-1 & -2q \\
\left(\frac{\ell-2}{2}\right)q & & & & & -\left(\frac{\ell-2}{2}\right)q & \left(\frac{\ell-2}{2}\right)q & & & & q & -q-1
\end{array}\right]
\end{array}
$$

This readily diagonalises over $Z[q]$ to the matrix

$$X = \begin{bmatrix} q^i+1 & 0 \\ x & q^{\ell-i}+1 \end{bmatrix} ,$$

where $x = \left[q^{\ell-2} + 2q^{\ell-3} + \ldots + (\ell-2)q + (\ell-1)\right] - \frac{1}{2}\ell\left[q^{\ell-i-1} + \ldots + q + 1\right]$.

To diagonalise this, we need to know when $x \in \langle q^i+1, q^{\ell-i}+1\rangle = J_{i,\ell-i}$, in the notation of Proposition 4.6 . Then we have :-

<u>Proposition 4.20</u>. In the groups $G_{\pi,K}(q)$, where π is either π_2 or π_3 , then the torus T_w for w corresponding to the class $D_\ell(a_{i-1})$ has elementary divisors as follows :-

(i). $e_1 = (q^i+1).(q^{\ell-i}+1)$ if $\ell \equiv 2\ (4)$, $i \equiv 0\ (2)$ and q is odd .

(ii). $e_1 = (q^i+1)$, $e_2 = (q^{\ell-i}+1)$ otherwise .

<u>Proof</u>. By Proposition 4.6 , $J_{i,\ell-i} = \langle q^d+1\ ,\ 1+(-1)^{\frac{\ell}{d}-1}\rangle$, where

$d = (\ell,i)$. Then two cases arise , viz. either ℓ/d is odd or ℓ/d is even . Let $J = J_{i,\ell-i}$.

(a). If ℓ/d is odd , then $J = \langle q^d+1, 2\rangle$ and we have to consider the parity of q . If q is even , then $J = \langle 1\rangle = Z$, so $x \in J$ and so X has two elementary divisors $e_1 = (q^i+1)$, $e_2 = (q^{\ell-i}+1)$. However , this case does not really occur since Δ_{sc}/Δ_{ad} modulo 2-torsion is trivial and the groups $G_{\pi,K}$ all collapse to $G_{ad,K}$. This accords with the results of Propositions 4.5 and 4.18 .

On the other hand , if q is odd , then $J = \langle 2\rangle = 2Z$. Then

$$x \equiv \left[1+2+\ldots+(\ell-1)\right] - \tfrac{1}{2}\ell(\ell-i) \quad (\text{modulo } J)$$

$$= \frac{\ell(i-1)}{2} \quad (\text{modulo } J) .$$

So $x \equiv \left\{\begin{array}{ll} 1 \text{ (modulo J)} & \text{if } \ell \equiv 2(4) \text{ and } i \equiv 0(2) \\ 0 \text{ (modulo J)} & \text{otherwise} \end{array}\right\}$.

Hence , the elementary divisors of the maximal torus T_w are :-

$$\left\{\begin{array}{ll} e_1 = (q^i+1).(q^{\ell-i}+1) & \text{if } \ell \equiv 2(4) , \ i \equiv 0(2) \text{ and } q \text{ is odd} \\ e_1 = (q^i+1) , \ e_2 = (q^{\ell-i}+1) & \text{otherwise} \end{array}\right\} .$$

(b). If ℓ/d is even , say $\ell = 2kd$, then i/d is odd , say $i = (2m+1)d$. Hence , $J = \langle q^d+1\rangle$. Let $n = k-m-1$.

Now , $q^{\ell-i-1}+\ldots+q+1 = q^d(q^{2nd-1}+\ldots+1) + (q^{d-1}+\ldots+1)$

$= q^d(q^d+1).(\frac{q^d-1}{q-1}).(q^{2n-2}+q^{2n-4}+\ldots+q^2+1) + (q^{d-1}+\ldots+1)$, as in Proposition 4.7 .

Hence , $\tfrac{1}{2}\ell\left[q^{\ell-i-1}+\ldots+q+1\right] \equiv \tfrac{1}{2}\ell(q^{d-1}+\ldots+1)$ modulo J. (1)

Furthermore , $\left[q^{\ell-2}+2q^{\ell-3}+\ldots+(\ell-2)q+(\ell-1)\right] = y(q^d+1) + kd(q^{d-1}+\ldots+1)$,(2)

where

$$\begin{array}{llll}
y = & q^{\ell-d-2} + & 2q^{\ell-d-3}+3q^{\ell-d-4}+ & \ldots\ldots\ldots+dq^{\ell-2d-1} \\
& +dq^{\ell-2d-2} + & dq^{\ell-2d-3} + & \ldots\ldots\ldots+dq^{\ell-3d-1} \\
& +(d+1)q^{\ell-3d-2} + & (d+2)q^{\ell-3d-3} + & \ldots\ldots\ldots+2dq^{\ell-4d-1} \\
& +2dq^{\ell-4d-2} + & 2dq^{\ell-4d-3} + & \ldots\ldots\ldots+2dq^{\ell-5d-1} \\
& +(2d+1)q^{\ell-5d-2} + & (2d+2)q^{\ell-5d-3} + & \ldots\ldots\ldots+3dq^{\ell-6d-1} \\
& \vdots & \vdots & \vdots \\
& +(k-1)dq^{2d-2} & +(k-1)dq^{2d-3} + & \ldots\ldots\ldots+(k-1)dq^{d-1} \\
& +((k-1)d+1)q^{d-2} & +((k-1)d+2)q^{d-3} + & \ldots\ldots\ldots+(kd-1)
\end{array}$$

By (1) and (2) , and since $kd = \frac{1}{2}\ell$, it follows that

$$x = \left[q^{\ell-2}+2q^{\ell-3}+\ldots+(\ell-2)q+(\ell-1)\right] - \frac{\ell}{2}\left[q^{\ell-i-1}+\ldots+q+1\right]$$

$$\equiv \frac{\ell}{2}(q^{d-1}+\ldots+q+1) - \frac{\ell}{2}(q^{d-1}+\ldots+1) \equiv 0 \quad (\text{modulo } J) .$$

So , in this case , X diagonalises over $Z[q]$ to give the two elementary divisors $e_1 = (q^i+1)$ and $e_2 = (q^{\ell-i}+1)$.

Since the condition $\ell \equiv 2(4)$ and $i \equiv 0(2)$ implies that $\frac{\ell}{d}$ is odd , the result follows as stated . $\square$

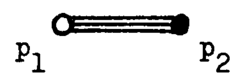
This concludes the chapter on Chevalley groups of type B_ℓ, C_ℓ, and D_ℓ .

CHAPTER 5. Chevalley groups of type G_2, F_4, E_6, E_7 and E_8.

§5.1. In this chapter, we consider the groups $G_{\pi,K}$, where $\mathfrak{g}$ is one of the exceptional complex, simple Lie Algebras, ie. of type G_2, F_4, E_6, E_7 or E_8. In such cases, $G_{\pi,K}$ is either $G_{ad,K}$ or $G_{sc,K}$, and the latter case arises as a distinct group only in the case where $\mathfrak{g}$ is of type E_6 or E_7. Hence, since $\Sigma^{v} = \Sigma$ in these cases, Lemmas 1.7 and 1.8 show that in all these cases we need only consider the action of w on Δ_{ad}, where w is a representative of each conjugacy class of W.

In this chapter, we make use of Propositions 2.1 and 2.2 to find the matrix w_{Φ} from the graph Γ of Φ, in those cases for which we have shown that Φ generates Δ_{ad}, ie. for the Coxeter and semi-Coxeter classes of W. Using further methods, the results are complete for groups of type G_2 or F_4, but not for groups of type E_n, due to the size of the groups $W(E_n)$. In fact, we are only able to give the results for the Coxeter and semi-Coxeter types in E_6 and E_7. We also show how many of the results for groups of type E_8 can be obtained in terms of the results for the groups of type D_8, which are not situated diagonally, viz. the groups treated in §4.6. However, the results there not being complete, we are only able to present slightly more results than in the other two cases. We omit most of the details of diagonalisation.

§5.2. Type G_2 . This case is straightforward since $W(G_2)$ is isomorphic to the dihedral group of order 12 , and so has 6 conjugacy classes . If we let Π be the system of fundamental roots $\{p_1, p_2\}$ with corresponding diagram



,

then $W = \langle w_1, w_2 : w_1{}^2 = w_2{}^2 = (w_1 w_2)^6 = 1 \rangle$, where $w_i = w_{p_i}$. Moreover , representatives for the set $\mathcal{C}(W)$ are $R = \{1 , (w_1 w_2) , (w_1 w_2)^2 , (w_1 w_2)^3 , w_1 , w_2\}$, with respective graphs $\{\emptyset , G_2 , A_2 , A_1 + \tilde{A}_1 , \tilde{A}_1 , A_1\}$, where X denotes a graph consisting of long roots and $\tilde{X}$ a graph of short roots .

Considering the action of each $w \in R$ upon the basis Π of Δ_{ad} , we obtain the following results :-

TABLE 5.1.

Φ-type	Elementary divisors e_i
$\emptyset$	$e_1 = e_2 = (q-1)$.
A_1	$e_1 = (q^2-1)$.
$\tilde{A}_1$	$e_1 = (q^2-1)$.
$A_1+\tilde{A}_1$	$e_1 = e_2 = (q+1)$.
A_2	$e_1 = (q^2+q+1)$.
G_2	$e_1 = (q^2-q+1)$.

NOTE . The class $A_1 + \tilde{A}_1$ consists of the unique non-identity central element $w_0 = -1$, where w_0 is the unique element of W of maximal length .

§5.3. Type F_4. The group $W(F_4)$ is soluble of order 1152, and it has 25 conjugacy classes with admissible diagrams as in [6]. A system of fundamental roots of type F_4 is

$\pi = \{ p_1 = e_2-e_3 , p_2 = e_3-e_4 , p_3 = e_4 , p_4 = \frac{1}{2}(e_1-e_2-e_3-e_4) \}$ embedded in a real 4-dimensional vector space V with natural basis $\{e_i\}_{i=1}^{4}$. Hence, Δ_{ad} is generated, over Z, by $\{e_1, e_2, e_3, \rho_4\}$, so that $\Delta_{ad} \cong M_3$.

Now, the root system Σ of type F_4 has a sub-system Σ' of type B_4, and so $W = W(F_4)$ has a Weyl subgroup W' isomorphic to $W(B_4)$, (in fact $(W:W') = 3$.). Hence, for any conjugacy class C_Γ of W with admissible diagram Γ such that Γ is also admissible for $W(B_4)$, we can find a representative $w \in C_\Gamma$ such that $w \in W'$. Now, a fundamental system π' for Σ' is $\pi' = \{ r_1 = e_1-e_2 , e_2-e_3 , e_3-e_4 , e_4 \}$ so that W' acts on Δ_{ad} by permutations of the $\{e_i\}$ and by sign changes. Hence, the action of such an element w is known from §4.5, and is noted in Table 5.2.

From Table 5.2, we see that there are precisely 7 classes (see §7.5) not dealt with in this way, viz. $\mathcal{C}^* = \{7,13,16,18,20,24,25\}$. Now there are 3 maximal Weyl subgroups of W, viz. $W' \cong W(B_4)$, $W'' \cong W(C_3) \times W(A_1)$ and $W''' \cong W(A_2) \times W(\tilde{A}_2)$, (see [3]). Then Class Nos. 13,20 are entirely contained in W'', Class Nos. 7,16,18 are entirely contained in W''', and Class Nos. 24,25, being the Coxeter and semi-Coxeter classes respectively, are contained in no proper Weyl subgroup.

Now , in the case of Class Nos. 7,13,16,18 and 20 , the corresponding Φ does not generate Δ_{ad} , and so we cannot use the results of Propositions 2.1 and 2.2 . Hence , we must calculate w_π directly in these cases . However , by Proposition 2.2 , we know that Φ does generate Δ_{ad} in the case where Φ corresponds to a Coxeter or semi-Coxeter class . Hence we may calculate w_Φ in the cases of Class Nos. 24,25 from the graph , as in Proposition 2.1 . In the remaining cases , we refer to Proposition 4.18 .

It is well known(§7.4.) , that there exists an involutive automorphism τ of Δ_{ad} in this case such that τ normalises W . In fact , by Lemma 2.4 , τ acts on $\mathcal{L}(W)$ by permuting the Γ in orbits of length 1 or 2 . Then we have :

__Lemma 5.1__. Let w be a representative from a conjugacy class Γ , and w' a representative from the class Γ^τ . Then $T_w \cong T_{w'}$.

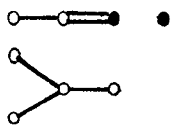
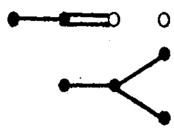
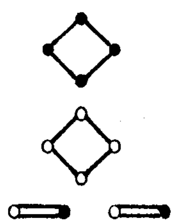
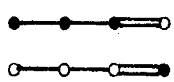
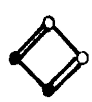
__Proof__. It is sufficient to show that $(q.w_\pi - I)$ and $(q.w'_\pi - I)$ are equivalent , (in the sense of §1.3.) , since then $\Delta_{ad}/(q.w-I)\,\Delta_{ad} \cong \Delta_{ad}/(q.w'-I)\,\Delta_{ad}$. Now we may choose w' to be $w^\tau = \tau.w.\tau^{-1}$, and then $(q.w'_\pi - I) = \tau_\pi.(q\ w_\pi - I).\tau_\pi^{-1}$. Since τ is an isometry , then τ is integral unimodular , and the result follows . $\square$

__Proposition 5.2__. The elementary divisors of the Chevalley groups $G_{\pi,K}(K_o)$ of type F_4 are those of Table 5.2 .

<u>TABLE 5.2.</u>

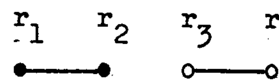
No. of Class C	Φ-type of C	Γ	$\lvert C\rvert$	Signed cycle-type	Elementary divisors e_i	C^{τ}
1	ϕ	ϕ	1	$[1111]$	$e_1=e_2=e_3=e_4=(q-1)$.	1
2	A_1	●	12	$[112]$	$e_1=e_2=(q-1), e_3=(q^2-1)$.	3
3	$\tilde{A}_1$	○	12	$[111\bar{1}]$	$e_1=e_2=(q-1), e_3=(q^2-1)$.	2
4	$2A_1;2\tilde{A}_1$	● ● ○ ○	18	$[22]$ $[11\overline{11}]$	$e_1=(q-1), e_2=(q+1), e_3=(q^2-1)$	4
5	$A_1+\tilde{A}_1$	● ○	72	$[12\bar{1}]$	$e_1=e_2=(q^2-1)$.	5
6	A_2	●—●	32	$[13]$	$e_1=(q-1), e_2=(q^3-1)$.	7
7	$\tilde{A}_2$	○—○	32		$e_1=(q-1), e_2=(q^3-1)$.	6
8	B_2	○═●	36	$[11\bar{2}]$	$e_1=(q-1), e_2=(q-1)(q^2+1)$.	8
9	$3A_1:2\tilde{A}_1+A_1$	● ● ● ○ ○ ●	12	$[2\overline{11}]$	$e_1=e_2=(q+1), e_3=(q^2-1)$.	10
10	$2A_1+\tilde{A}_1;3\tilde{A}_1$	● ● ○ ○ ○ ○	12	$[1\overline{111}]$	$e_1=e_2=(q+1), e_3=(q^2-1)$.	9
11	$A_3;\tilde{A}_1+B_2$	●—●—● ○ ○═●	72	$[4]$ $[\overline{112}]$	$e_1=(q^2-1), e_2=(q^2+1)$.	12
12	$\tilde{A}_3;A_1+B_2$	○—○—○ ● ●═○	72	$[22]$	$e_1=(q^2-1), e_2=(q^2+1)$.	11
13	C_3	○—○═●	96		$e_1=(q-1)(q^3+1)$.	14
14	B_3	●—●═○	96	$[1\bar{3}]$	$e_1=(q-1)(q^3+1)$.	13
15	$A_2+\tilde{A}_1$	●—● ○	96	$[3\bar{1}]$	$e_1=(q^3-1)(q+1)$.	16
16	$\tilde{A}_2+A_1$	○—○ ●	96		$e_1=(q^3-1)(q+1)$.	15
17	$4A_1:4\tilde{A}_1$ $2A_1+2\tilde{A}_1$	● ● ● ● ○ ○ ○ ○ ● ● ○ ○	1	$[\overline{1111}]$	$e_1=e_2=e_3=e_4=(q+1)$.	17
18	$A_2+\tilde{A}_2$	●—● ○—○	16		$e_1=e_2=(q^2+q+1)$.	18
19	$\tilde{A}_3+A_1$ $A_3+\tilde{A}_1$ B_2+2A_1 $B_2+2\tilde{A}_1$	○—○—○ ● ●—●—● ○ ○═● ● ● ●═○ ○ ○	36	$[\overline{11}2]$	$e_1=(q+1), e_2=(q+1)(q^2+1)$.	19

TABLE 5.2.(continued)

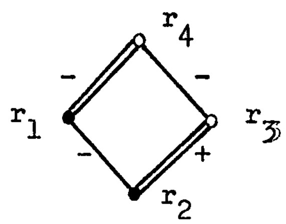
No.of Class C	Φ-type of C	Γ	$\lvert C\rvert$	Signed cycle-type	Elementary divisors e_i	C^{τ}
20	C_3+A_1; $\tilde{D}_4$		32		$e_1=(q+1), e_2=(q^3+1)$.	21
21	$B_3+\tilde{A}_1$; D_4		32	$[\overline{13}]$	$e_1=(q+1), e_2=(q^3+1)$.	20
22	$D_4(a_1)$; $\tilde{D}_4(a_1)$; B_2+B_2		12	$[\overline{22}]$	$e_1=e_2=(q^2+1)$	22
23	B_4 ; C_4		144	$[\overline{4}]$	$e_1=(q^4+1)$.	23
24	F_4		96		$e_1=(q^4-q^2+1)$.	24
25	$F_4(a_1)$		16		$e_1=e_2=(q^2-q+1)$.	25

Proof.

(i). Class Nos. 7,13,16,20. In these cases, Γ^{τ} is class No. 6,14,15 and 21 respectively. Hence, we may use Lemma 5.1 to determine the e_i for such classes.

(ii). Class No. 18. In this case, Γ is fixed by τ, so we cannot use this method. However, Γ is the graph  , and we may choose a system $\Phi = \{r_1 = -\delta,\ r_2 = p_1,\ r_3 = p_3,\ r_4 = p_4\}$, where δ is the highest root of Σ. A representative of Γ is $w = w_{r_4} w_{r_2} . w_{r_1} w_{r_3}$, and so we may calculate w_{π} directly.

(iii). Class Nos. 24,25. In these cases, we use Proposition 2.1, to calculate w_{Φ} directly from the graph, knowing that Φ generates Δ_{ad}, and we give an example here in the case of Φ-type $F_4(a_1)$. Then Γ is



with signs assigned to the bonds as described in Chapter 2.

$$\text{Then } A_W = \begin{array}{cc} \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} & \\ \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{2I} & -1 & -1 \\ & & 2 & -1 \\ \hline -1 & 1 & \multicolumn{2}{c}{2I} \\ -2 & -1 & & \end{array}\right] & \begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \end{array} \end{array}, \text{ so that } BC = 3I_2 .$$

$$\text{Thus } w_{\Phi} = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 2 & -1 \\ 1 & -1 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix} \text{ and } (q.w_{\Phi}-I) = \begin{bmatrix} 2q-1 & 0 & -q & -q \\ 0 & 2q-1 & 2q & -q \\ q & -q & -q-1 & 0 \\ 2q & q & 0 & -q-1 \end{bmatrix}.$$

So we diagonalise $(q.w_{\Phi}-I)$ by the following operations :-

$$\xrightarrow{-r_4 \text{ to } r_1} \begin{bmatrix} -1 & -q & -q & 1 \\ 0 & 2q-1 & 2q & -q \\ q & -q & -q-1 & 0 \\ 2q & q & 0 & -q-1 \end{bmatrix} \xrightarrow[\substack{\text{(ii). } 2q.r_1 \text{ to } r_4 \\ \text{(iii). Remove } (r_1,c_1)}]{\text{(i). } q.r_1 \text{ to } r_3} \begin{bmatrix} 2q-1 & 2q & -q \\ -q^2-q & -q^2-q-1 & q \\ -2q^2+q & -2q^2 & q-1 \end{bmatrix}$$

$$\xrightarrow[\text{(ii). } r_2 \text{ to } r_1]{\text{(i). } q.r_1 \text{ to } r_3} \begin{bmatrix} -q^2+q-1 & -q^2+q-1 & 0 \\ -q^2-q & -q^2-q-1 & q \\ 0 & 0 & -q^2+q-1 \end{bmatrix} \xrightarrow[\text{(ii). Remove } (r_2,c_2)]{\text{(i). } -c_1 \text{ to } c_2} \begin{bmatrix} -q^2+q-1 & 0 \\ 0 & -q^2+q-1 \end{bmatrix}$$

Hence, the two non-unit elementary divisors of $(q.w_{\Phi} - I)$ are both (q^2-q+1).

(iv). <u>Remaining Class Nos.</u> In these cases, we refer to Proposition 4.18, where the elementary divisors for T_w are calculated for an element w with signed cycle-type $[\lambda,\bar{\mu}]$ acting on M_3. These are straightforward to calculate, but we note that in Class Nos. 4 and 11, there are two cycle-types corresponding to each class. In these classes, one of the cycle-types has all its cycles even and positive, a case which necessitates individual treatment in Lemma 4.17. As an example, in Class No. 4, according to Proposition 4.18, if we take

(a). $[\lambda,\bar{\mu}] = [22]$, then $e_1 = 2$, $e_2 = \frac{q^2-1}{2}$, $e_3 = q^2-1$;

(b). $[\lambda,\bar{\mu}] = [11\bar{1}\bar{1}]$, then $e_1' = (q-1)$, $e_2' = (q+1)$ and $e_3 = (q-1)(q+1)$.

Since $(q+1,\frac{q-1}{2}) = 1$, and $C_m \times C_n \cong C_{mn}$ if $(m,n) = 1$, then we may rewrite $e_1 = 2.\frac{q-1}{2} = q-1 = e_1'$, and $e_2 = q+1 = e_2'$, if $q \not\equiv (4)$.
Similarly if $q \not\equiv 3(4)$ by taking $\frac{q+1}{2}$.

In case q is even, then for w of signed cycle-type $[\lambda,\bar{\mu}]$, $(q.w_\pi - I)$ has elementary divisors $\{q^{\lambda_1}-1,\ldots,q^{\lambda_t}-1,q^{\mu_1}+1,\ldots q^{\mu_s}+1\}$.

However , that causes no disagreement with Table 5.2 , since, for example in Class No. 15 , $(q^3-1,q+1) = 1$ so that $C_{q^3-1} \times C_{q+1} \cong C_{(q^3-1)(q+1)}$. Similarly for class Nos. 3,8,13,14,15,16 and 19 . Hence the results of Table 5.2 hold for all values of q . $\square$

This completes the section on F_4 .

---

§5.4. Types E_6 and E_7 . For the groups of type E_6 or E_7 , the results here are incomplete , due to the large order of the Weyl groups , since $|\mathcal{L}(W)|$ is 25 and 60 respectively , and the structure of the maximal tori T_w is determined only for w a representative of the semi-Coxeter or Coxeter classes of the corresponding Weyl group . To determine the elementary divisors of T_w in these cases , we use the results of Propositions 2.1 and 2.2. , and present the results in the following table :-

TABLE 5.3.

Φ-type	Elementary divisors e_i
E_6	$e_1 = (q^2+q+1).(q^4-q^2+1)$.
$E_6(a_1)$	$e_1 = (q^6+q^3+1)$.
$E_6(a_2)$	$e_1 = (q^2-q+1)$, $e_2 = (q^2-q+1).(q^2+q+1)$.
E_7	$e_1 = (q+1).(q^6-q^3+1)$.
$E_7(a_1)$	$e_1 = (q^7+1)$.
$E_7(a_2), E_7(b_2)$	$e_1 = (q^3+1).(q^4-q^2+1)$.
$E_7(a_3)$	$e_1 = (q^3+1).(q^4-q^3+q^2-q+1)$.
$E_7(a_4)$	$e_1 = e_2 = (q^2-q+1)$, $e_3 = (q^3+1)$.

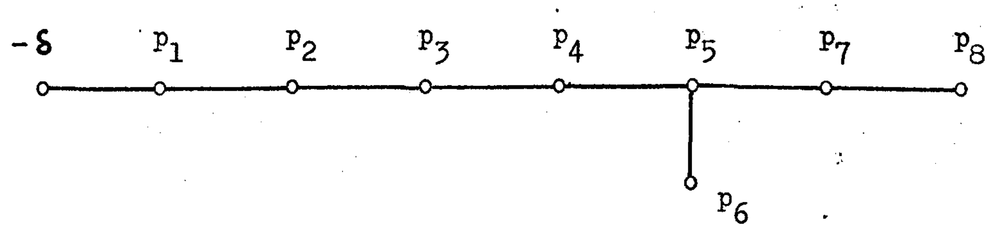
These groups are the only ones considered in this chapter where the isogeny class $\mathcal{G}$ contains more than one distinct element, viz. $G_{ad,K}$ and $G_{sc,K}$. By Corollary 1.8, the results are identical for these two situations.

---

§5.5. Type E_8. In groups of type E_8, the results are more detailed because of the nature of $\Delta_{ad} = \Delta_{sc}$, even though $|\mathcal{Z}(W)| = 112$ in this case. Let Σ be a root system of type E_8 embedded in a real vector space V with natural basis $\{e_i\}_{i=1}^{8}$, and let π be the fundamental system

$$\pi = \{ p_i = e_i - e_{i+1} \text{ for } 1 \leqslant i \leqslant 6 \ ; \ p_7 = e_6 + e_7 \ ; \ p_8 = -\rho_8 = -\tfrac{1}{2}\sum_{i=1}^{8} e_i \} .$$

Then the corresponding Dynkin diagram is



and the highest root is $\delta = e_1 - e_8 = 2p_1+3p_2+4p_3+5p_4+6p_5+3p_6+4p_7+2p_8$.

Hence, by [3], there are five maximal Weyl subgroups of $W = W(E_8)$ viz. $W_1 \cong W(D_8)$, $W_2 \cong W(A_8)$, $W_3 \cong W(A_4) \times W(A_4)$, $W_4 \cong W(E_6) \times W(A_2)$, $W_5 \cong W(E_7) \times W(A_1)$. As in §5.3, for any conjugacy class C_Γ of W with admissible diagram Γ such that Γ is also admissible for W_1, we can find a representative $w \in C_\Gamma$ such that $w \in W_1$.

Although $(W:W_1) = 3^3.5$, W_1 is "large" in the sense that it meets exactly half (ie. 56) of the classes of W in this way . If Σ_1 is the subsystem of Σ corresponding to W_1 , with fundamental system $\Pi_1 = \{p_1, p_2, \dots, p_7, -\delta\}$, then W_1 acts on Δ_{ad} by permutations and sign changes and every element w corresponds to a pair of partitions $[\lambda, \bar{\mu}]$ of 8 , as in §4.1 . It is clear that $\Delta_{ad} \cong M_4$, so that the action of $w \in W_1$ on Δ_{ad} is known from §4.6 .

Although the results are incomplete in that case , §4.6 does give the elementary divisors of the maximal torus T_w when w is a representative of a class of W which corresponds to a semi-Coxeter class of W_1 . Furthermore , for the 9 semi-Coxeter classes of W , we may use the results of Propositions 2.1 and 2.2 to obtain w_{Φ} from the corresponding graph Γ , since Φ generates Δ_{ad} in such cases .

Lemma 5.3. The matrices $(q.w_{\Phi} - I)$ are diagonalisable over $Z[q]$, with the exception of Φ-type $E_8(a_4)$.

Proof. This follows a case-by-case argument. In the exceptional case $E_8(a_4)$, the matrix $(q.w_{\Phi} - I)$ diagonalises over $Z[q]$ to the matrix

$$P = \begin{bmatrix} q^6-q^3+1 & -q-1 \\ 0 & q^2-q+1 \end{bmatrix} .$$

P is not diagonalisable over $Z[q]$, since $f_i(q) = h_{ij}(q).f_j(q) + 3$ for some $h_{ij}(q) \in Z[q]$, if $\deg f_i > \deg f_j$, where the $f_i(q)$ are the polynomial entries of P . Hence , the diagonalisation of P depends upon the value of $(q+1,3)$:-

(i). If $(3,q+1) = 1$, then there exists some $n \in Z$ with $(q+1)+3.n = \pm 1$. Now P reduces to $\begin{bmatrix} q^6-q^3+1 & -q-1 \\ q^7+2q^6-q^4-2q^3+q-2 & 3 \end{bmatrix}$, and this reduces to a matrix with one elementary divisor $e_1 = (q^2-q+1).(q^6-q^3+1)$ on premultiplying by the integral unimodular matrix $\begin{bmatrix} 3 & q+1 \\ 1 & -n \end{bmatrix}$.

(ii). If $3 \mid (q+1)$, then $(q+1) = 3.m$ for some $m \in Z$, and hence , as above , $(q^6-q^3+1) - h(q).(q^2-q+1) = 3$. Thus $m(q^6-q^3+1) - m.h(q).(q^2-q+1) = (q+1)$, and P reduces to a matrix with elementary divisors $e_1 = (q^2-q+1)$ and $e_2 = (q^6-q^3+1)$.

Thus , in the case of Φ-type $E_8(a_4)$, T_w is isomorphic to $C_{(q^2-q+1).(q^6-q^3+1)}$ if $(3,q+1) = 1$, and to $C_{(q^2-q+1)} \times C_{(q^6-q^3+1)}$ if $3 \mid (q+1)$.

However , suppose that $(3,q+1) = 1$. Then , for $d = (q^6-q^3+1, q^2-q+1)$, $d = (3, q^2-q+1)$. Then :-

if $q+1 \equiv 1$ (modulo 3) , then $q \equiv 0 \pmod 3$ and $d = (3,1) = 1$,

if $q+1 \equiv 2$ (modulo 3) , then $q \equiv 1 \pmod 3$ and $d = (3,1) = 1$,

again. Hence , $C_{(q^2-q+1).(q^6-q^3+1)}$ is isomorphic to $C_{(q^2-q+1)} \times C_{(q^6-q^3+1)}$.

Also , if $3 \mid (q+1)$, then $q \equiv 2 \pmod 3$ and $q^2 \equiv 1 \pmod 3$. Hence , $d = (3,0) = 3$. Thus , whatever the value of q , it is clear that T_w is always isomorphic to $C_{(q^2-q+1)} \times C_{(q^6-q^3+1)}$.

$\square$

We collect the results for the groups of type E_8 in the following table :-

TABLE 5.4.

Φ-type	Elementary divisors e_i
D_8	$e_1 = (q+1)$, $e_2 = (q^7+1)$.
$D_8(a_1)$	$e_1 = (q^2+1)$. $e_2 = (q^6+1)$.
$D_8(a_2)$	$e_1 = (q^3+1)$, $e_2 = (q^5+1)$.
$D_8(a_3)$	$e_1 = (q^4+1)$, $e_2 = (q^4+1)$.
E_8	$e_1 = (q^8+q^7-q^5-q^4-q^3+q+1)$.
$E_8(a_1)$	$e_1 = (q^8-q^4+1)$.
$E_8(a_2)$	$e_1 = (q^8-q^6+q^4-q^2+1)$.
$E_8(a_3)$	$e_1 = e_2 = (q^4-q^2+1)$.
$E_8(a_4)$	$e_1 = (q^2-q+1)$, $e_2 = (q^6-q^3+1)$.
$E_8(a_5)$	$e_1 = (q^8-q^7+q^5-q^4+q^3-q+1)$.
$E_8(a_6)$	$e_1 = e_2 = (q^4-q^3+q^2-q+1)$.
$E_8(a_7)$	$e_1 = e_2 = (q^2-q+1)$, $e_3 = (q^4-q^2+1)$.
$E_8(a_8)$	$e_1 = e_2 = e_3 = e_4 = (q^2-q+1)$.

CHAPTER 6. Steinberg groups of type $^2A_\ell$, $^2D_\ell$, $^2E_6$.

It was mentioned in §1.2, that the Steinberg groups are the groups G_σ for σ a combination of a field automorphism and a graph automorphism of G. Precisely how this is achieved, we explain now, and then we spend the rest of this chapter in determining the structure of the maximal tori of such finite groups.

Following Tits [23], we let ${}_{K_o}G$ be any semi-simple algebraic group defined over K_o and we extend the base field to obtain a group $\overline{G}$ defined over K. Then Chevalley has shown that $\overline{G} = G_{\pi,K}$, where π is a faithful representation of a semi-simple Lie algebra $\underline{g}$. We say that ${}_{K_o}G$ is a $\underline{K_o\text{-form}}$ of $G_{\pi,K}$. We are interested in two particular types of K_o-forms, viz.

(i). <u>Normal (Chevalley, split) forms</u>. We say that ${}_{K_o}G$ is <u>normal (split)</u> over K_o if every conjugacy class of parabolic subgroups contains at least one subgroup defined over K_o. Then every simple algebraic group defined over K has one and (up to isomorphism) only one normal K_o-form, viz. G_{π,K_o} - the <u>Chevalley form</u>, and this we have studied in the previous chapters.

(ii). <u>Exterior (semi-split) forms</u>. We say that ${}_{K_o}G$ is <u>exterior (semi-split)</u> if it has Borel subgroups defined over K_o. By Theorem 1.1., it follows that every semi-simple group ${}_{K_o}G$ is exterior. We will follow the normal practice of giving these exterior forms the name

Steinberg groups , since $K_0 = GF(q)$ throughout .

§6.1. Exterior forms . In this Chapter we are interested in the exterior forms and we let $G = G_{\pi,K}$.

Theorem 6.1. (a). The Borel subgroups of G are all conjugate ;

(b). The parabolic subgroups of G are all connected ;

(c). The lattice of parabolic subgroups containing a given Borel subgroup B is isomorphic to the lattice of subsets of a finite set . In other words , if $P^{(i)}$, for $i \in I$, denote the maximal subgroups of G which contain B , then the index set I is finite , and every subgroup containing B is uniquely an intersection of the form $P^{(i_1)} \wedge P^{(i_2)} \wedge \cdots \wedge P^{(i_m)}$ for $(i_1,\ldots,i_m) \in I^m$;

(d). Two parabolic subgroups containing the same Borel subgroup B are never conjugate .

It follows that the classes of conjugate parabolic subgroups are in canonical 1-1 correspondence with the subsets of the finite set I . See [23] . $\square$

Construction of the diagram $\mathcal{D}$ of G .

For all subsets $J \subsetneq I$, we let $d_J = \dim (\bigwedge_{i \notin J} P^{(i)})$, so $d_\emptyset = \dim B$. Then , by 6.1 , it follows that $d_{i,j} - d_\emptyset \geqslant 2$ and $d_i - d_\emptyset = 1$ for $i,j \in I$. So we construct $\mathcal{D}$ by nodes corresponding to each $i \in I$, and those corresponding to $i,j \in I$ are joined by a bond of strength $(d_{ij} - d_\emptyset - 2)$.

Suppose that ${}_{K_0}G$ is any exterior form of a semi-simple group $G_{\pi,K}$. Then the Galois group Γ of K / K_0 acts on G by $\gamma(a_{ij}) = ((\gamma(a_{ij}))_{ij})$ for $\gamma \in \Gamma$ and some embedding of G in $GL_n(K)$. Suppose that B is a Borel subgroup of ${}_{K_0}G$ defined over K_0. Then Γ preserves B, and hence permutes among themselves the $P^{(i)}$. Since these are represented by the nodes of $\mathcal{D}$, we can say that Γ operates on $\mathcal{D}$, and it is clear that the elements of Γ induce automorphisms of $\mathcal{D}$ since $d_{\gamma(i),\gamma(j)} = d_{i,j}$.

If Γ operates trivially on $\mathcal{D}$, then the $P^{(i)}$ are defined over K_0, and so ${}_{K_0}G$ is a normal form, by Theorem 6.1. More generally, if Γ_1 denotes the group of all elements of Γ which induce the identity on $\mathcal{D}$, then the field K_1 of the invariants of Γ_1 is the smallest extension of K_0 for which ${}_{K_1}G$ is a normal form. We call K_1 the <u>splitting field</u> of ${}_{K_0}G$. Then K_1 is a finite Galois extension of K_0, and its Galois group $\Delta = \Gamma / \Gamma_1$ operates faithfully on $\mathcal{D}$.

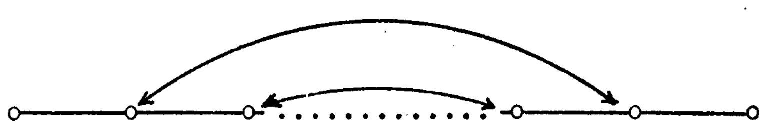
<u>Theorem 6.2</u>. Let $\mathcal{D}$ be the diagram of a simple group $G_{\pi,K}$, K_1 a finite Galois extension of K_0, and suppose we are given a faithful representation $\rho : \Delta \longrightarrow \mathrm{Aut}(\mathcal{D})$, where $\Delta = \mathrm{Gal}\ (K_1 / K_0)$. Then $G_{\pi,K}$ has one exterior K_0-form and (up to isomorphism) only one having K_1 as splitting field, and such that, with respect to this form, Δ operates on $\mathcal{D}$ in the given way. See [23].

$\square$

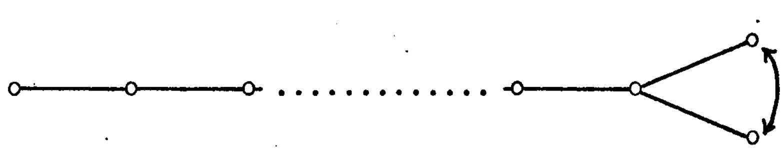
NOTE. Since K_o is a finite field, then Δ must be a cyclic group, and this excludes the case ${}^6D_4$, where $\Delta \cong \mathrm{Aut}(\mathcal{D}) \cong \mathfrak{S}_3$.

In Table 6.1 are represented all the groups of non-trivial automorphisms of diagrams of simple groups. By 6.2, to each of them corresponds a type of exterior form. Once the type $G = A_\ell, B_\ell, \ldots$ of the diagram $\mathcal{D}$ and the order δ of the group Δ is known, then this group Δ and its action on $\mathcal{D}$ are fully determined, so we may denote the type of forms in question by ${}^\delta G$. In the case we consider, it is clear that K_1 is well-determined as $GF(q^\delta)$. Hence, since a given exterior form ${}_{K_o}G$ of type ${}^\delta G$ is fully determined if we are given K_o and K_1, we may denote this exterior form by ${}^\delta_{K_o}G$, and its group of K_o-rational points by ${}^\delta G(K_o)$.

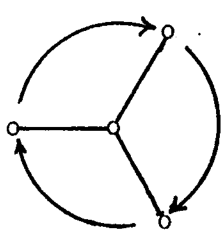
TABLE 6.1.



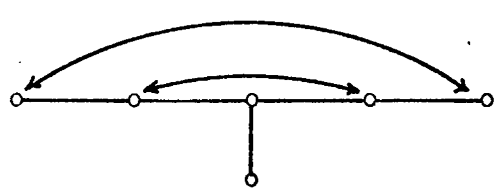
${}^2A_\ell$



${}^2D_\ell$



${}^3D_4$



${}^2E_6$

We are interested in the groups ${}^\delta G(K_o)$, and these groups can be obtained from the groups $G(K_1)$ in the following way, where $G = G_{\pi,K}$.

The group Γ_1 operates trivially on the group $G(K_1)$ of

K_1-rational points of the normal form ${}_{K_1}G$. Consequently , $\Delta = \Gamma/\Gamma_1$ acts on $G(K_1)$, and the group of Δ -invariant points of $G(K_1)$ is ${}^{\delta}G(K_o)$.

---

§6.2. The set $H^1(\sigma,W)$.

In this chapter , we are interested in the Steinberg groups ${}^2A_\ell$, ${}^2D_\ell$ and ${}^2E_6$, where $\delta = 2$. In [7] , it is shown that $G(K_1)$ admits an automorphism $\sigma : x_a(t) \longmapsto x_{\tau a}(t^q)$, where τ is the permutation of the fundamental roots (ie. graph automorphism) of Table 6.1 . Then ${}^2G(K_o) = G(K_1)_\sigma$, the σ -fixed points of $G(K_1)$. If we let $\sigma = q\tau$ be the corresponding automorphism of $G = G_{\pi,K}$ (ie. the combination of field and graph automorphisms mentioned in § 1.2), then $G_{\sigma^2} = G(K_1)$. Since $(G_{\sigma^2})_\sigma = G_\sigma$, then ${}^2G(K_o) = G_\sigma$, a situation we are equipped to deal with , by Chapter 1 .

Let Σ be an irreducible root system in a real vector space V , and let $A(\Sigma)$ and $D(\Sigma)$ be as in §2.4 . Then , as in [4] :-

<u>Lemma 6.3</u>. The group $A(\Sigma) / W(\Sigma)$ (and hence $D(\Sigma)$) operates naturally on Δ_{sc}/Δ_{ad} .

<u>Proof</u>. If $q \in \Delta_{sc}$ and $w \in W(\Sigma)$, then $q-w(q) \in \Delta_{ad}$. For , let $w = w_a$. Then $q-w(q) = \langle q,a^\vee\rangle . a \in Za \subset \Delta_{ad}$, and for $w = w_{a_1} \dots . w_{a_r}$, we repeat this .

Now $A(\Sigma)$ fixes Δ_{sc} and Δ_{ad} , and so operates on Δ_{sc}/Δ_{ad} . By the above , $W(\Sigma)$ operates trivially on Δ_{sc}/Δ_{ad} , so that the result follows . $\square$

This justifies our statements in §1.2 , where we impose the condition on $\tau \in D(\precsim)$ that $\tau(\Delta_\pi) = \Delta_\pi$. For , τ operates on the subgroup lattice of Δ_{sc}/Δ_{ad} by Lemma 6.3 , and hence permutes the Δ_π such that $\Delta_{ad} \subset \Delta_\pi \subset \Delta_{sc}$.

So we now consider the situation of Chapter 1 , with T a K_o-split maximal torus of $G = G_{\pi,K}$, and suppose that $\overline{T}_w$ is a σ-fixed maximal torus of G which is twisted from T by $w \in W$. From the previous section we have seen that we must restrict attention to G of type A_ℓ, D_ℓ or E_6 , and that the representation π is such that $\tau(\Delta_\pi) = \Delta_\pi$. Then the action of σ induced to $X(T) \cong \Delta_\pi$ is $\sigma^* = q\tau$. Furthermore , the action of σ induced to $X(\overline{T}_w)$ is given by $w_o\sigma^* = qw\tau$, by Proposition 1.3 . By Theorem 1.2 , the G_σ-conjugacy classes of σ-fixed maximal tori of $G_{\pi,K}$ are in 1-1 correspondence with the elements of $H^1(\sigma,W)$. So we must investigate the nature of $H^1(\sigma,W)$, when W is of type A_ℓ, D_ℓ or E_6 and σ is as above .

Now σ acts on W according to the way $\tau \in D(\precsim)$ acts on W in the semi-direct product $W(\precsim).D(\precsim) = A(\precsim)$, viz. by conjugation in $A(\precsim)$. So we let $W^* = \langle W, \tau \rangle \leq A(\precsim)$, so that $W^* = W.\langle\tau\rangle$ (semi-direct) .

Now in the case of a normal form , $H^1(\sigma,W) = \mathcal{E}(W)$. However , in the case of an exterior form , when σ acts as $q\tau$ on V , we have :-

Lemma 6.4. There is a bijection $\psi : H^1(\sigma,W) \longrightarrow \mathcal{L}_\tau(W^*)$, where $\mathcal{L}_\tau(W^*)$ is the set of conjugacy classes of W^* which are contained in the coset $W\tau$.

Proof. Now $\mathcal{L}_\tau(W^*)$ is well-defined since $W\tau$ is a union of conjugacy classes of W^*. For $(w_1\tau)^w = w_1{}^w.w.w^{-\tau}.\tau \in W\tau$ and $(w_1\tau)^\tau = w_1^\tau.\tau \in W\tau$.

$$
\begin{aligned}
\text{Now} \quad w_1 \underset{R}{\sim} w_2 &\Longleftrightarrow \exists\, w \in W \text{ with } w_1 = w.w_2.\sigma(w)^{-1} \\
&\Longleftrightarrow \exists\, w \in W \text{ with } w_1 = w.w_2.\tau.w^{-1}.\tau^{-1} \\
&\Longleftrightarrow \exists\, w \in W \text{ with } w_1.\tau = w.w_2\tau.w^{-1} \\
&\Longleftrightarrow w_1\tau \underset{W}{\sim} w_2\tau , \text{ where } \underset{W}{\sim} \text{ denotes W-conjugacy .}
\end{aligned}
$$

Thus , there is a bijection $\psi : H^1(\sigma,w) \longrightarrow \{ \{w.\tau\}^W : w \in W \}$, ie. the set of W-conjugacy classes of $W\tau$.

$$
\begin{aligned}
\text{Now} \quad \{w\tau\}^{W\tau} &= \{ w\tau^{w_1\tau} : w_1 \in W \} = \{ w_1\tau.w\tau.\tau^{-1}w_1^{-1} : w_1 \in W \} \\
&= \{ w_1w^{-1}.w\tau.ww_1^{-1} : w_1 \in W \} = \{w\tau\}^W .
\end{aligned}
$$

Hence $\{w\tau\}^{W^*} = \{w\tau\}^W \cup \{w\tau\}^{W\tau} = \{w\tau\}^W$, and the W^*-conjugacy classes of $W\tau$ are identical with the W-conjugacy classes of $W\tau$.

Thus , $\psi : H^1(\sigma,W) \longrightarrow \mathcal{L}_\tau(W^*)$ is a bijection such that if w is a representative of $h \in H^1(\sigma,W)$, we have $\psi(h) = \{w\tau\}^{W^*}$ - the W^*-conjugacy class containing $w\tau$.

$\square$

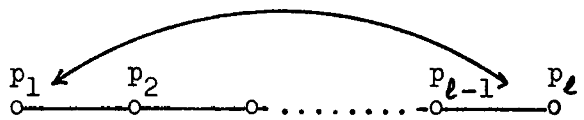
Hence , to find the structure of the maximal tori T_w of ${}^2G(K_o)$, we must determine the elementary divisors of $\Delta_\pi/(q.w\tau - I)\Delta_\pi$, where $G = G_{\pi,K}$ and $w\tau$ runs through the representatives of the classes of $\mathcal{L}_\tau(W^*)$. We now consider the groups W^* for the groups ${}^2A_\ell$, ${}^2D_\ell$ and

${}^2E_6$, in which case $\tau^2 = 1$ and $W\tau$ is the unique non-trivial coset of W in W^* .

---

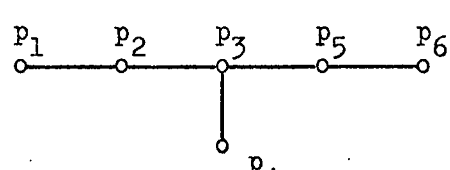
§6.3. The root systems of type A_ℓ, D_ℓ and E_6 .

Type A_ℓ . $(\ell \geqslant 2)$. We must assume that $\ell \geqslant 2$ because τ is the identity in the case A_1 . Let Σ be a root system of type A_ℓ . For $\ell \geqslant 2$, let $\tilde{\tau} \in A(\Sigma)$ be the automorphism $\tilde{\tau} : p_i \longmapsto p_{\ell+1-i}$, where $\Pi = \{p_i\}_{i=1}^{\ell}$ is a fixed fundamental system in Σ . It is clear that the automorphism



of $D(\Sigma)$ induced by $\tilde{\tau}$ is the unique non-trivial automorphism τ of the graph of Σ . The group $A(\Sigma) / W(\Sigma)$ is isomorphic to Z_2 . Since $-1 \in A(\Sigma)$ always , but $-1 \notin W(\Sigma)$ in this case , we see that $A(\Sigma) \cong W(\Sigma) \times \{1,-1\}$, and $w_0 = -\tilde{\tau}$. Furthermore , the unique non-trivial element τ of $A(\Sigma) / W(\Sigma)$ acts on $\Delta_{sc} / \Delta_{ad}$ by the automorphism $x \longmapsto -x$, by Lemma 6.3 .

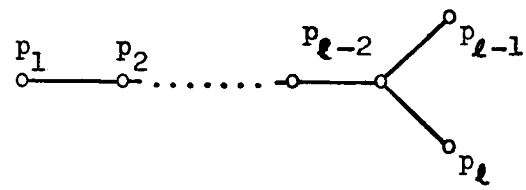
Type E_6 . Let Σ be a root system of type E_6 with fundamental system $\Pi = \{p_i\}_{i=1}^{6}$ such that Σ has graph



.

Let $\tilde{\tau} \in A(\Sigma)$ be the automorphism which maps $p_1, p_2, p_3, p_4, p_5, p_6$ into $p_6, p_5, p_3, p_4, p_2, p_1$ respectively . As for A_ℓ , it is clear that the automorphism of $D(\Sigma)$ induced by $\tilde{\tau}$ is the unique non-trivial automorphism τ of the graph of Σ . Also $-1 \notin W(\Sigma)$, so

$A(\Sigma) \cong W(\Sigma) \times \{1,-1\}$, and $w_o = -\tilde{\tau}$. Further , the unique non-trivial element $\tau \in A(\Sigma) / W(\Sigma)$ acts on Δ_{sc}/Δ_{ad} by the automorphism $x \longmapsto -x$.

Type D_ℓ . Let Σ be a root system of type D_ℓ with fundamental system $\pi = \{p_i\}_{i=1}^{\ell}$ such that Σ has graph 

.

Let $\tilde{\tau} \in A(\Sigma)$ be the automorphism of V which maps $p_{\ell-1} \longleftrightarrow p_\ell$ and fixes p_i for $i \in \{1,\ldots,\ell-2\}$. If $p_i = e_i - e_{i+1}$ for $i \in \{1,\ldots,\ell-1\}$ and $p_\ell = e_{\ell-1} + e_\ell$, where $\{e_i\}_{i=1}^{\ell}$ is an orthonormal basis of V , then $\tilde{\tau}$ is the automorphism such that $\tilde{\tau}$ fixes e_i for $i \in \{1,\ldots,\ell-1\}$ and $\tilde{\tau} : e_\ell \longmapsto -e_\ell$, (see § 4.2.) .

Assume that $\ell \geqslant 5$. Then the automorphism of $D(\Sigma)$ induced by $\tilde{\tau}$ is the unique non-trivial automorphism τ of the graph of Σ . Hence $A(\Sigma) / W(\Sigma)$ is isomorphic to Z_2 .

Now , $W(D_\ell)$ acts on $\{e_i\}_{i=1}^{\ell}$ by permutations and an even number of sign changes . Hence we have :-

<u>ℓ even</u>. Then $-1 \in W(D_\ell)$, so $w_o = -1$ by uniqueness .

<u>ℓ odd</u>. Then $-1 \notin W(D_\ell)$, so $A(D_\ell) \cong W(D_\ell) \times \{1,-1\}$ and $w_o = -\tilde{\tau}$.

Now $A(\Sigma) / W(\Sigma)$ acts on Δ_{sc}/Δ_{ad} as described in § 4.2 , in particular :-

(i). if ℓ is even , then τ fixes Δ_{π_1} and permutes Δ_{π_2} and Δ_{π_3} ,

(ii). if ℓ is odd , then τ fixes Δ_{π_1} .

Lemma 6.5. $W(B_\ell) \cong A(D_\ell) = W(D_\ell) . \langle \tau \rangle = W^*$, for $\ell \geqslant 5$.

Proof. As above , let $w_{p_i} = w_i$. Since $\tau . w_a . \tau^{-1} = w_{\tau(a)}$, then centralises w_i for $i \in \{1,\ldots,\ell-2\}$ and permutes $w_{\ell-1}$ and w_ℓ . Now $W^* = \langle W(D) , \tau \rangle = \langle w_1,\ldots,w_{\ell-1}, \tau : R \rangle$, where R is the set of defining relations :-

$$R = \{ w_i^2 = \tau^2 = 1 \ ; \ (w_i w_{i+1})^3 = 1 \ \text{for} \ i < \ell-1 \ ; \ (w_i \tau)^2 = 1 \ \text{for} \ i < \ell-1 \ ;$$
$$(w_i w_j)^2 = 1 \ \text{for} \ j \neq i+1 \ ; \ (w_{\ell-1}\tau)^4 = 1 \ \} ,$$

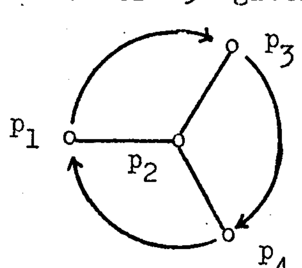
since $(w_{\ell-1}\tau)^2 = w_{\ell-1} . \tau w_{\ell-1} \tau = w_{\ell-1} . w_\ell$.

Hence , W^* is a Coxeter group satisfying the relations of $W(B_\ell)$, so W^* is isomorphic to a factor group of $W(B_\ell)$. Since $|W^*| = |W(B_\ell)| = 2^\ell . \ell!$, then $W^* \cong W(B_\ell)$. $\square$

Corollary 6.5. The set $\mathcal{C}_\tau(W^*)$ of conjugacy classes is in 1-1 correspondence with the pairs of partitions $[\lambda, \bar{\mu}]$ of ℓ , consisting of an odd numbers of negative cycles .

Proof. By Lemma 6.5. , the classes of $\mathcal{C}_\tau(W^*)$ are those in $W(B_\ell) \cong W(C_\ell)$ which are not in $W(D_\ell)$. By § 4.1 , these are the classes of $W(C_\ell)$ which correspond to pairs of partitions $[\lambda, \bar{\mu}]$ of ℓ , consisting of an odd number of negative cycles . $\square$

NOTE. As we see in Table 6.1. , when $\ell = 4$, there is a further automorphism $\rho \in D(\precsim)$ of order 3 given by $\rho : p_1 \mapsto p_3 \mapsto p_4 \mapsto p_1$ and $p_2 \mapsto p_2$,



In this case, we still have the automorphism τ interchanging p_3 and p_4, and $D(D_4) = \langle \rho, \tau : \tau^2 = \rho^3 = (\tau\rho)^2 = 1 \rangle \cong \mathfrak{S}_3$. So that, although $W^* < A(D_4)$ (strictly), we still have
$W^* = W(D_4).\langle\tau\rangle \cong W(B_4)$.

Then ρ acts on Δ_{sc}/Δ_{ad} by permuting the Δ_{π_i} for $i = 1,2,3$ in a cycle of order 3. However, we leave the Steinberg groups ${}^3D_4(q)$ until Chapter 7, for reasons which become obvious there.

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§6.4. The groups ${}^2A_\ell$, ${}^2D_\ell$ (ℓ odd) and ${}^2E_6$.

In all these cases, τ acts on Δ_{sc}/Δ_{ad} by $\tau : x \longmapsto -x$, and hence τ fixes the subgroup lattice of Δ_{sc}/Δ_{ad}. Hence, ${}^2G(K_o)$ is defined for the group $G_{\pi,K} = G$ for all possible faithful representations π of $\mathfrak{g}$ (of type A_ℓ, D_ℓ (ℓ odd) or E_6), since then $\tau(\Delta_\pi) = \Delta_\pi$ for all Δ_π with $\Delta_{ad} \subset \Delta_\pi \subset \Delta_{sc}$.

Lemma 6.6. For $\mathfrak{L}$ of type A_ℓ, D_ℓ (ℓ odd) or E_6, $\tau = -w_o$ and $W^* \cong W \times \{1,-1\}$. Further, $|\mathcal{C}(W^*)| = 2\,|\mathcal{C}(W)|$ and $Z(W) = 1$. The set $\mathcal{C}_\tau(W^*)$ is just the set $\{-C : C \in \mathcal{C}(W)\}$, and the classes $H^1(\sigma,W)$ are in 1-1 correspondence with the W-conjugacy classes of W ie. the set $\mathcal{C}(W)$ under the map $\bar{\psi} = -\psi$, (for ψ as in Lemma 6.4). So that
$\bar{\psi}^{-1} : \mathcal{C}(W) \longrightarrow H^1(\sigma, w)$ by $\bar{\psi}^{-1} : C \longmapsto C.w_o$, for $C \in \mathcal{C}(W)$.

Proof. The first statement follows from the investigation of the root systems above. Secondly, we know from §2.4 that τ acts on the set of graphs $\{\Gamma_C : C \in \mathcal{E}(W)\}$. In the case where $\lesssim$ is of type A_ℓ or E_6 , there is a 1-1 correspondence between admissible graphs and conjugacy classes in W . Hence , τ must fix each class $C \in \mathcal{E}(W)$. However , in groups of type D_ℓ, there is no such correspondence , but for ℓ odd , there is a 1-1 correspondence between conjugacy classes in W and signed cycle-types $[\lambda, \bar{\mu}]$ with s even . Now τ acts on the class C corresponding to the cycle-type $[\lambda, \bar{\mu}]$ by mapping w to w^τ . Let γ be the cycle of $\{e_i\}_{i=1}^{\ell}$ under w which contains e_ℓ . Then τ fixes all other cycles , and acts on γ in the following way :-

if $\gamma = (\ldots\ldots e_i e_\ell e_j \ldots\ldots)$, then $\gamma^\tau = (\ldots\ldots e_i -e_\ell e_j \ldots\ldots)$ and

$$\operatorname{sgn}(\gamma^\tau) = \operatorname{sgn}(\gamma) .$$

Hence , w^τ has the same cycle type as w , and τ fixes the classes $\mathcal{E}(W(D_\ell))$ for ℓ odd . In fact , this follows directly since $\tau = -w_o$, but we need the fact that τ fixes the signed cycle-type of any $w \in W(D_\ell)$ for any ℓ , in §6.5 .

By Proposition 2.5 , $|\mathcal{E}_\tau(W^*)| = |\mathcal{E}(W)|$ and $\mathcal{E}_\tau(W^*) = \{-C : C \in \mathcal{E}(W)\}$. Then , by Lemma 6.4 , ψ is the map $\psi : \mathcal{E}(W) \longrightarrow H^1(\sigma, w)$ such that $\psi : C \longmapsto C\tau = -Cw_o$. So we put $\bar{\psi} = -\psi$. $\square$

<u>Corollary 6.6</u>. Let $\mathcal{E}(W) = \{C_i\}_{i=1}^{r}$, and let $w_i w_o$ be a representative element of C_i , for a subset $\{w_i : i = 1,\ldots r\} \subset W$. Then a representative for the corresponding element $h_i \in H^1(\sigma, W)$ is w_i .

<u>Proof</u>. By Lemmas 6.4 and 6.6 . $\square$

Proposition 6.7. Let $G = G_{\pi,K}$ be a simple group of type A_ℓ, D_ℓ (ℓ odd) or E_6, and let $w_i \in W$ be the representative of the element $h_i \in H^1(\sigma, W)$, as above. Let $E_i = \{ e_{ij} = f_{ij}(q) : j$ runs over some subset $N_i \subset N \}$, so that $E_i \subset Z(q)$, be the elementary divisors for the torus $T_{w_i w_o}$ in the corresponding Chevalley form $G(K_o)$. Then the elementary divisors for the maximal torus ${}^2T_{w_i}$ of the group ${}^2G(K_o)$ in the conjugacy class corresponding to h_i under Q are ${}^2E_i = \{ {}^2e_{ij} = |f_{ij}(-q)| : j \in N_i \}$.

Proof. Given a K_o-split maximal torus T of $G = G_{\pi,K}$, and a maximal torus $\overline{T}_w$, twisted from T by $w \in W$, the action of σ^* on $X(\overline{T}_w) \cong \Delta_\pi$ is given by $q.w\tau = -q.ww_o$, by Proposition 1.3.

Hence, ${}^2T_w = (\overline{T}_w)_\sigma \cong \Delta_\pi / ((-q).ww_o - I)\Delta_\pi$, by Proposition 1.4.

Thus, for $w = w_i$, ${}^2e_{ij} = |f_{ij}(-q)|$. $\square$

Hence we may obtain the sets ${}^2E_i$ for the group ${}^2G(K_o)$ by changing the sign of q in the sets E_i for the corresponding group $G(K_o)$, and we have determined these latter in Chapters 3,4,5.

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§6.5. The groups ${}^2D_\ell$ for ℓ even.

In this case, τ acts on Δ_{sc}/Δ_{ad} by permuting Δ_{π_2} and Δ_{π_3}, so that, by §1.2, we may only consider G to be one of $G_{ad,K}$, $G_{sc,K}$ or $G_{\pi_1,K}$. We have seen, in the proof of Lemma 6.6, that τ acts on $\mathcal{C}(W)$ by mapping the class with signed cycle-type $[\lambda, \bar{\mu}]$ to another class with the same signed cycle-type. When ℓ is odd, we have seen that this just means that τ fixes every element of $\mathcal{L}(W)$, but when ℓ is even , there are two classes corresponding to the signed cycle-type $[\lambda]$, where λ_i is even for all $i \in \{1,\ldots,t\}$. Hence , this allows the possibility of τ permuting such elements of $\mathcal{L}(W)$. In fact ,

<u>Lemma 6.8</u>. If C and C' are two distinct classes in $\mathcal{L}(W)$ with the same signed cycle-type , then $C^{\tau} = C'$.

<u>Proof</u>. By Lemma 6.5 , we know that $W^{*} \cong W(C_{\ell})$, and $Z(W) = \{1,-1\}$. Now , there is a 1-1 correspondence between classes and signed cycle-types in W^{*} , by [6] . Hence , the two classes C and C' cannot be distinct inside W^{*} and so must fuse under action by W^{*} . Thus C and C' are permuted by W^{*}/W , ie. by τ . $\square$

<u>Corollary 6.8</u>. Let $|\mathcal{L}(W)| = r$, and suppose that there are r_2 distinct "even-positive" partitions of ℓ . Then there are r_1 classes fixed by τ acting on $\mathcal{L}(W)$, and $2r_2$ classes which are permuted in cycles of order 2 , where $r = r_1 + 2r_2$. Then the number of classes of W^{*} in the coset $W\tau$ is r_1 , and the total number of conjugacy classes of $W^{*} \cong W(B_{\ell})$ is $(2r_1 + r_2) = 2r - 3r_2$.

<u>Proof</u>. By the proof of Lemma 6.6 , all the classes of $\mathcal{L}(W)$ are fixed except those corresponding to "even-positive" partitions . The result follows from Lemma 6.8 , using Proposition 2.5 . $\square$

Hence, we see that by Corollary 6.5, if $G = G_{\pi,K}$, then the maximal tori of the group ${}^2G(K_o)$ of type D_ℓ (ℓ even) have the same structure as those of the corresponding (via lattices) Chevalley group $B_\ell(K_o)$ which do not correspond to conjugacy classes of $W(D_\ell)$. The correspondence here is

if π is ad or sc, we take $(B_\ell)_{sc,K}(K_o)$,

if π is π_1, we take $(B_\ell)_{ad,K}(K_o)$, (see Proposition 4.4).

We put this on a more formal basis in the following way. First we make two definitions to determine the classes of $H^1(\sigma, W)$.

Definition 6.1. Let $\mathcal{P}$ be the partition of W defined by associating with each $p_i \in \mathcal{P}$ an ordered n-tuple $N_i = [\nu_1, \nu_2, \ldots, \nu_n]$ of positive and negative integers with $\sum_{j=1}^{n} |\nu_j| = \ell$, such that $w \in p_i$ if w acts as a ν_j-cycle (taking care of the sign) on the subset $\{e_{\nu_1+\ldots+_{j-1}+k} : 1 \leq k \leq \nu_j\}$ of the natural basis $\{e_j\}_{j=1}^{\ell}$ of V. (This is a finer partition than that effected by the signed-cycle type).

Definition 6.2. Let $\mathcal{N} = \{N : N$ is an ordered n-tuple of integers$\}$ and let Λ be the set of pairs of partitions $[\lambda, \bar{\mu}]$ of ℓ. We define a function $g : \mathcal{N} \longrightarrow \Lambda$ which maps an ordered n-tuple N onto the (unordered) n-tuple $g(N)$. Let $\bar{N} = [\nu_1, \nu_2, \ldots, -\nu_n]$.

Lemma 6.9. Let $[\lambda, \bar{\mu}]$ be any signed cycle-type corresponding to a class $C \in \mathcal{E}_\tau(W^*)$ and consider all those $p_i \in P$ such that $g(\bar{N}_i) = [\lambda, \bar{\mu}]$.

Let $h = \bigcup p_i$. Then $h \in H^1(\sigma,W)$ and $\psi(h) = C$.

<u>Proof</u>. Let $w \in p_i$ with associated $N_i = [\nu_1,\nu_2,\ldots,\nu_n]$. Then, since τ fixes e_j for $j < \ell$ and $\tau : e_\ell \longmapsto -e_\ell$, it is clear that $w\tau$ has associated with it the n-tuple $\bar{N}_i = [\nu_1,\nu_2,\ldots,\nu_{n-1},-\nu_n]$.

Let $w_i \in p_i$ and $w_j \in p_j$ for p_i , p_j as above . Then

$$w_i \underset{R}{\curlyvee} w_j \Longleftrightarrow w_i\tau ,\ w_j\tau \in C .$$
$$\Longleftrightarrow g(\bar{N}_i) = g(\bar{N}_j) = [\lambda,\bar{\mu}] .$$

Hence $h = \bigcup p_i$. $\square$

This determines the classes of $H^1(\sigma,W)$, and we finally have :-

<u>Proposition 6.10</u>. Let $w \in p$ such that $g(\bar{N}) = [\lambda,\bar{\mu}]$, and suppose that $p \subset h$ for $h \in H^1(\sigma,W)$. Then the maximal tori ${}^2T_w$ of the group ${}^2G_{\pi,K}(K_o)$ of type D_ℓ (ℓ even) corresponding to the class h have elementary divisors given by :-

(i). Proposition 4.5 , if $\pi = \pi_1$,

(ii). Proposition 4.18 , if π = ad or sc .

<u>Proof</u>. This follows from Lemma 6.9 . $\square$

CHAPTER 7. Finite groups of twisted type ${}^2G_2(q^2)$, ${}^2B_2(q^2)$, ${}^2F_4(q^2)$ and ${}^3D_4(q)$.

If one does not take account of the type of nodes in the set of diagrams of simple groups $G_{\pi,K}$, then the list of graph automorphisms in Table 6.1 is completed by the addition of three cases , represented in Table 7.1 . These automorphisms are described in § 1.2 , where we note that such automorphisms only occur when $p = 2$ or 3 , depending upon the type of the root system Σ of $\mathfrak{g}$. Due to the nature of these automorphisms , the diagrams of Table 7.1 do not give rise to new exterior forms of $G_{\pi,K}$. However , R. Ree [14] showed that one can , under certain conditions , associate with them abstract groups which are analogues of groups of rational points ${}^{\delta}G(K_o)$ of exterior forms . These are the Ree groups ${}^2G_2(q^2)$ and ${}^2F_4(q^2)$, and the Suzuki groups ${}^2B_2(q^2)$ which were originally discovered by M. Suzuki [22] in a very different light .

In this Chapter , we complete our discussion by finding the structure of the maximal tori in the remaining finite groups of Lie type , viz. the Ree and Suzuki groups and the Steinberg groups ${}^3D_4(q)$, and we begin by examining the procedure for obtaining the Ree and Suzuki groups in § 7.1. The reason for including the groups ${}^3D_4(q)$ in this Chapter is that the groups ${}^3D_4(q)$ and ${}^2F_4(q^2)$ are very closely related and , had we not already considered the groups ${}^2D_4(q)$ in Chapter 6 , it would be convenient to also discuss those in this Chapter . In fact , if Σ is a root system of type F_4 then both the long roots Σ_{ℓ} and the short roots

Σ_s form root systems of type D_4 .

§7.1. Construction of the Ree and Suzuki groups .

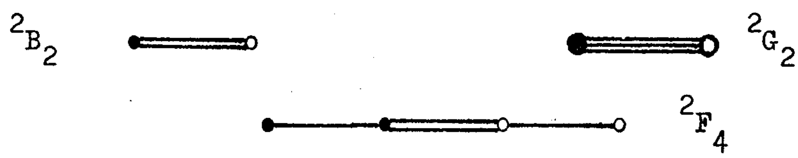
If one attemps to reproduce the construction of § 6.1 by taking a figure ^2G from Table 7.1 , then , as in §1.2 , one can prove that for certain fields K_1 , (of characteristic $p = 2$ if $G = B_2$ or F_4 , and of characteristic 3 if $G = G_2$) , the group $G(K_1)$ has automorphisms σ of order 2 . By analogy with the exterior forms , we shall call a group of points fixed under such an automorphism σ a twisted group of type $^2G(K_1)$. The essential difference between the two types is that here , no field K_o plays the rôle of "base field" .

Another automorphism σ' of $G(K_1)$ of order 2 , which possesses the same properties as σ , will be said to be equivalent to σ if it is the transform of σ by an inner automorphism of $G(K_1)$. The classification of the automorphisms , (and hence the Ree groups) , is given by the following Theorem , due to Ree , [14] .

Theorem 7.1. If $G = B_2$ or F_4 (resp. G_2) , the equivalence classes of automorphisms of order 2 of $G(K_1)$, which act in the way shown in Table 7.1 , are in canonical 1-1 correspondence with the automorphisms θ of K_1 such that $(x^2)^{\theta^2} = x$, (resp. $(x^3)^{\theta^2} = x$) , for all $x \in K_1$. $\square$

Thus to every automorphism θ of K_1 satisfying this relation , there is associated a group of type $^2G(K_1)$, which one could write as $^2G(\theta,K_1)$, although we will not need to make use of such a notation .

TABLE 7.1.



The field $K_1 = GF(q)$ of $q = p^n$ elements has an automorphism θ such that $(x^p)^{\theta^2} = x$ if and only if $n = 2m+1$, for any $m \geqslant 0$. Then θ must be the automorphism $x^\theta = x^{p^m}$, and the uniqueness of this automorphism will permit us , in the case of a finite field K_1 , to speak without ambiguity of <u>the</u> Ree group $^2G(K_1)$, where $K_1 = GF(p^{2m+1})$.

Specifically , by [7] , we have σ defined on generators as :-

$$\sigma : \left\{ \begin{array}{lll} x_a(t) \longmapsto x_{\rho a}(t^{p^m}) & \text{if } a \in \pi & \text{is long} \\ x_a(t) \longmapsto x_{\rho a}(t^{p^{m+1}}) & \text{if } a \in \pi & \text{is short} \end{array} \right\} , \text{ where } \rho$$

is the permutation of π described in the corresponding diagram ^2G of Table 7.1 .

Now let K be the algebraic closure of the field $K_1 = GF(p^{2m+1})$, where $p = 2$ (resp. 3) and let $G = G_{\pi,K}$ be a simple algebraic group of type B_2 or F_4 (resp. G_2) . Then , with σ defined as above , we know that $G_{\sigma^2} = G(K_1)$. Hence , since $(G_{\sigma^2})_\sigma = G_\sigma$, we know that $G_\sigma = {}^2G(K_1)$, the twisted group of type G . Thus , we may use the results of Chapter 1 to find the structure of the maximal tori of the Ree and Suzuki groups .

Let Σ be a root system of type B_2 , G_2 or F_4 embedded in a real vector space V . Now σ induces , by Proposition 1.3 , an isometry τ of V and also , via the permutation ρ of π , an automorphism of W . Hence τ acts on W and we consider the group $W^* = \langle W,\tau \rangle$, which is the semi-direct product of W by τ . Then W^* is a group of automorphisms of V , although $A(\Sigma) = W(\Sigma)$ is a proper , normal subgroup of W^* since τ does not preserve Σ . In fact , τ is an isomorphism of root systems and maps Σ into a distinct root system Σ' of the same type embedded in V .

We recall Lemma 6.4 , which sets up a bijection $\psi : H^1(\sigma,W) \longrightarrow \mathcal{L}_\tau(W^*)$, and this holds in this case . We proceed by considering each group in turn , but first we prove a general result on Dihedral groups . The reason for proving this is that if Σ is of type B_2 or G_2 , then W is a Dihedral group of order divisible by 4 .

<u>Lemma 7.2</u>. Let W be the Dihedral group , $W = \langle x,y : x^2 = y^2 = (xy)^{2n} = 1 \rangle$, of order $4n$, and let τ be the involutive automorphism of W which permutes x and y . Then $|\mathcal{L}_\tau(W^*)| = (n+1)$, and a set of representatives of the elements of $\mathcal{L}_\tau(W^*)$ is $\{ \tau, x\tau, (xy)x\tau, (xy)^2x\tau, \ldots, (xy)^{n-1}.x\tau \}$.

<u>Proof</u>. Now $\mathcal{L}(W) = (n+3)$, with classes represented by

$$C_0 = \{1\}^W ,\ C_1 = \{xy\}^W ,\ C_2 = \{(xy)^2\}^W, \ldots, C_n = \{(xy)^n\}^W ,\ C_{n+1} = \{x\}^W$$

and $C_{n+2} = \{y\}^W$.

As in Lemma 2.4 , τ acts on $\mathcal{L}(W)$, and fixes all the classes except the last two . Hence , by Proposition 2.5 , the number of classes of $\mathcal{L}_\tau(W^*)$ is just $(n+1)$.

As we remarked earlier , this does not supply us with a list of representatives of elements of $\mathcal{L}_\tau(W^*)$, but it does serve as a useful check . To determine representatives , we see that the W-conjugacy classes of $W\tau$ are :-

$$\{\tau\}^W = \{\tau,\ (xy)\tau\ ,\ (xy)^2\tau, \ldots, (xy)^{2n-1}\tau\},$$

$$\{(xy)^s x\tau\}^W = \{(xy)^s x\tau\ ,\ (xy)^{2n-s-1}x\tau\} \text{ for } s \in \{0\ ,\ 1\ , \ldots,\ n-1\}\ .$$

Now , we have seen in the proof of Lemma 6.4 that the W^*-conjugacy classes of W^* are the W-conjugacy classes of W^* . Hence we have the result as claimed . $\square$

NOTE. We could get information by looking at the action of τ on $\{\Gamma_C : C \in \mathcal{L}(W)\}$ as in Lemma 2.4 , but this would not enable us to find the representatives of $H^1(\sigma,W)$, as Lemma 7.2 does .

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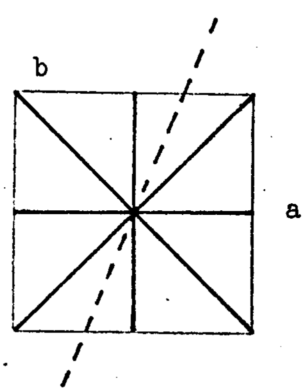
§7.2. The Suzuki groups ${}^2B_2(q^2)$, where $q^2 = 2^{2m+1}$.

In this case , τ is the operation of reflection in the dotted line shown at an angle of $67\frac{1}{2}^o$ with the root a , where $\Pi = \{a,b\}$ is a fundamental system for Σ . Furthermore , in this case , $\mathrm{Hom}(\Delta_{sc}/\Delta_{ad},K^*)$ is trivial , since K has



characteristic 2 . Hence , the centre of $G_{sc,K}$ is trivial , and the isogeny $G_{sc,K} \longrightarrow G_{ad,K}$ becomes a radical isogeny . Thus we need only consider (up to radical isogeny) the groups $G_{ad,K}$, by considering the action of σ^* on Δ_{ad} .

Now , in this case , $W = \langle w_a, w_b : w_a^2 = w_b^2 = (w_a w_b)^4 = 1 \rangle$ is a Dihedral group of order 8 . Hence , by Lemma 7.2 , $|\mathcal{L}_\tau(W^*)| = 3$ and representatives of the 3 elements of $\mathcal{L}_\tau(W^*)$ are $\{\tau, w_a\tau, (w_a w_b)w_a\tau\}$. Thus , by Lemma 6.4 , representatives of the 3 elements of $H^1(\sigma,W)$ are $\{1, w_a, w_a w_b w_a\}$. By Propositions 1.3 and 1.4 , a maximal torus of ${}^2B_2(K_1)$ corresponding to the representative w has relation matrix $(q.(w\tau)_\pi - I)$. Then , by Theorem 1.2 , there are 3 conjugacy classes of maximal tori in ${}^2B_2(K_1)$ with relation matrices corresponding to the 3 representatives above . Hence we must determine the action of $w\tau$ on Δ_{ad} , for $w \in \{1 , w_a, w_a w_b w_a\}$. Then we have :

Proposition 7.3. There are 3 conjugacy classes of maximal tori in the groups ${}^2B_2(q^2)$, and these are described in the following table together with the order of the corresponding group W_σ .

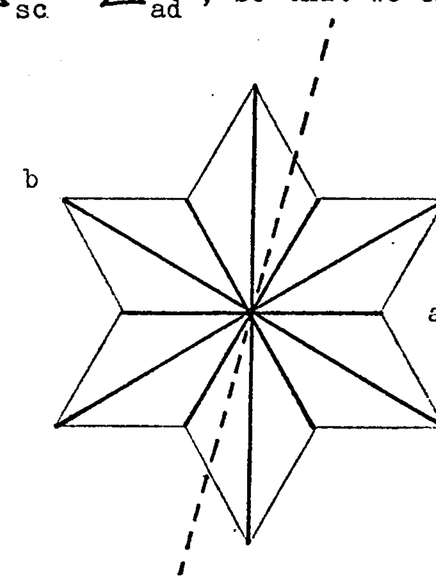
Representative element	Order of class	$\lvert W_\sigma \rvert$	Elementary divisors
1	4	2	$e_1 = (q^2-1)$.
w_a	2	4	$e_1 = (q^2-\sqrt{2}q+1)$.
$w_a w_b w_a$	2	4	$e_1 = (q^2+\sqrt{2}q+1)$.

Proof. We omit the details but note that the matrices $(q.(w\tau)_\pi - I)$ diagonalise over the ring $Z[q/\sqrt{2}]$. $\square$

This completes the case for the groups $^2B_2(K_1)$.

§7.3. The Ree groups $^2G_2(q^2)$, where $q^2 = 3^{2m+1}$.

In this case , τ is the operation of reflection in the dotted line shown at an angle of 75^o with the root a , where $\Pi = \{a,b\}$ is a fundamental system for Σ . Also , $\Delta_{sc} = \Delta_{ad}$, so that we only need consider the action of σ^* on Δ_{ad} .



When Σ is of type G_2 , then W is a Dihedral group $\langle w_a, w_b : w_a^2 = w_b^2 = (w_a w_b)^6 = 1 \rangle$ of order 12 . Hence , by Lemma 7.2 , $|\mathcal{L}_\tau(W^*)| = 4$ and representatives of the 4 elements of $\mathcal{L}_\tau(W^*)$ are $\{\tau, w_a\tau, w_a w_b w_a\tau, (w_a w_b)^2 w_a\tau\}$.

As in §7.2 , we have :-

Proposition 7.4. There are 4 conjugacy classes of maximal tori in the groups $^2G_2(q^2)$, and these are described in the following table together with the order of the corresponding group W_σ .

Representative element	Order of class	$\lvert W_\sigma \rvert$	Elementary divisors
1	6	2	$e_1 = (q^2-1)$.
w_a	2	6	$e_1 = (q^2-\sqrt{3}q+1)$.
$w_a w_b w_a$	2	6	$e_1=e_2=2, e_3=\frac{1}{4}(q^2+1)$.
$(w_a w_b)^2 w_a$	2	6	$e_1 = (q^2+\sqrt{3}q+1)$.

Proof. We omit the details but note that the matrices $(q.(w\tau)_\pi - I)$ diagonalise over the ring $Z[q/\sqrt{3}]$ only in cases (i),(ii) and (iv) . In case (iii) , however , the matrix $(q.(w\tau)_\pi - I)$ reduces to

$$\begin{bmatrix} q/\sqrt{3}+1 & 2 \\ -2 & \sqrt{3}q-3 \end{bmatrix} \xrightarrow[\text{to } c_1]{-\frac{1}{2}(q/\sqrt{3}+1)c_2} \begin{bmatrix} 0 & 2 \\ -\frac{1}{2}(q^2+1) & \sqrt{3}q-3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 2 \\ \frac{1}{2}(q^2+1) & 0 \end{bmatrix} ,$$

over Z since $q/\sqrt{3} = 3^m$. This result is non-trivial since (q^2+1) is divisible by 4 , and by no higher power of 2 , when $q^2 = 3^{2m+1}$. Hence , $C_2 \times C_{\frac{1}{2}(q^2+1)}$ is isomorphic to $C_2 \times C_2 \times C_{\frac{1}{4}(q^2+1)}$. □

NOTE. In these two cases , the groups C_{q^2-1} correspond to the K_1-split tori of $G(K_1)$. For groups of type B_2 and G_2 we may consider the action of τ on $\{ \Gamma_C : C \in \mathcal{E}(W) \}$. Then we have

$\Gamma(B_2) = \{ \emptyset , \circ , \bullet , \circ\ \bullet , \circ\!\Rightarrow\!\bullet \}$

and $\Gamma(G_2) = \{ \emptyset , \circ , \bullet , \circ\ \bullet , \bullet\!-\!\bullet , \circ\!\Rrightarrow\!\bullet \}$.

Hence , by Lemma 2.4 , τ fixes each graph of $\Gamma(W)$ except $\circ$ and $\bullet$, which are permuted , (since $\circ\!-\!\circ$ and $\bullet\!-\!\bullet$ represent the same class in

$W(G_2)$.) . Hence , $\left|\mathcal{E}_{\tau}(W^*)\right| = n + 1$, as in Lemma 7.2 . In these cases , $2n = h$ - the Coxeter number , [4] .

Also , in the list of representatives of Lemma 7.2 , the final representative can be replaced by $(xy)^n . x\tau$, ie. by $(xy)^{h/2} . x\tau$ in these cases (see the proof of Lemma 7.2.) . But $(xy)^{h/2} = z$ is the unique non-identity central element of W , (see [10] .) . Further , $z = -1$ in these cases , so that for $w = (w_a w_b)^{n-1} w_a$, then $(q.(w\tau)_\pi - I)$ is equivalent to the matrix $(q.(zw_a\tau)_\pi - I) = ((-q).(w_a\tau)_\pi - I)$. Since the matrix for the representative w_a of the class (ii) is diagonalisable over $Z[q / \sqrt{p}]$ in each case to give elementary divisors $e_i = f_i(q) \in Z[q / \sqrt{p}]$, then the elementary divisors corresponding to the final class are $\bar{e}_i = f_i(-q)$.

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§7.4. The Ree groups of type ${}^2F_4(q^2)$, where $q^2 = 2^{2m+1}$.

In this case , it is much more difficult to see geometrically how τ acts on Σ , since Σ is embedded in a 4-dimensional real vector space . However , according to the diagram of Table 7.1 , τ acts by exchanging long roots with short roots , and vice-versa . Hence , τ acts on W by mapping a reflection due to a short root onto a reflection corresponding to a long root and vice-versa . Thus , τ acts on $\mathcal{E}(W)$ by mapping the graph Γ_C representing $C \in \mathcal{E}(W)$ to the graph $\Gamma_{\tau(C)}$ representing $\tau(C) \in \mathcal{E}(W)$, where $\Gamma_{\tau(C)}$ is obtained from Γ_C by exchanging each node • by the node o and vice-versa . Hence , we can find $\tau(C)$ for every class $C \in \mathcal{C}(W)$ by looking at the graphs Γ_C listed in Table 5.2. In $W(F_4)$, we do have the situation where, in some cases, more than one graph represents a given class C. Hence, we do need the full list of admissible graphs and their respective classes in order to determine precisely where τ maps each graph Γ. I take this opportunity of thanking Professor R.W. Carter for supplying me with this information. We also list the graphs Γ^{τ} in Table 5. 2.

When $\underline{g}$ is of type F_4, then $\Delta_{sc} = \Delta_{ad}$ so that we only have one type of group in the isogeny class, viz. $G_{ad,K}$. Hence we need only consider the action of w on Δ_{ad}, where w is a representative of each element of $H^1(\sigma,W)$, in order to determine the elementary divisors of the matrix $(q.(w\tau)_{\pi} - I)$.

<u>Proposition 7.5</u>. Let $W = W(F_4)$, then $\left|H^1(\sigma,W)\right| = 11$.

<u>Proof</u>. If we look at Table 5.2, we see that there are precisely 11 conjugacy classes C such that $\tau(C) = C$. Hence, by Proposition 2.5, since τ is of order 2,

$$\left|\mathcal{C}(W^*)\right| = 2 \times 11 + \frac{25-11}{2} = 29 \quad \text{and} \quad \left|\mathcal{C}_{\tau}(W^*)\right| = 11 .$$

Now, by Lemma 6.4, there is a bijection $\psi: H^1(\sigma,W) \longrightarrow \mathcal{C}_{\tau}(W^*)$, so the result follows.

$\square$

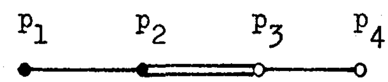
Although this result does not supply us with representative elements for the classes of $H^1(\sigma,W)$, it does at least inform us when we have

TABLE 7.2.

Positive root i	Co-ordinates	$w_1(i)$	$w_2(i)$	$w_3(i)$	$w_4(i)$	Length of root	$\tau(i)$
1	(0,1,-1,0)	-1	7	1	1	ℓ	2
2	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	2	2	8	-2	s	1
3	(0,0,1,-1)	7	-3	5	3	ℓ	4
4	(0,0,0,1)	4	6	-4	8	s	3
5	(0,0,1,1)	11	5	3	9	ℓ	6
6	(0,0,1,0)	10	4	6	12	s	5
7	(0,1,0,-1)	3	1	11	7	ℓ	8
8	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	8	12	2	4	s	7
9	(1,-1,0,0)	15	9	9	5	ℓ	10
10	(0,1,0,0)	6	10	10	16	s	9
11	(0,1,0,1)	5	13	7	15	ℓ	12
12	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	16	8	14	6	s	11
13	(0,1,1,0)	13	11	13	17	ℓ	14
14	$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	18	14	12	14	s	13
15	(1,0,-1,0)	9	17	15	11	ℓ	16
16	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	12	16	18	10	s	15
17	(1,0,0,-1)	17	15	19	13	ℓ	18
18	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	14	20	16	18	s	17
19	(1,0,0,1)	19	21	17	19	ℓ	20
20	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	20	18	22	20	s	19
21	(1,0,1,0)	23	19	21	21	ℓ	22
22	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	22	22	20	24	s	21
23	(1,1,0,0)	21	23	23	23	ℓ	24
24	(1,0,0,0)	24	24	24	22	s	23

completed our search for such representatives . Since ψ is just post-multiplication by τ , we direct our search to finding representatives of the 11 classes of $\mathcal{L}_\tau(W^*)$. Hence , we must investigate the nature of the group W^* .

Let Σ be a root system of type F_4 embedded in a 4-dimensional real vector space V with natural basis $\{e_1, e_2, e_3, e_4\}$. A fundamental system π for Σ is $\pi = \{p_1 = e_2 - e_3 \ , \ p_2 = e_3 - e_4 \ , \ p_3 = e_4 \ , \ p_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$, with corresponding Dynkin diagram



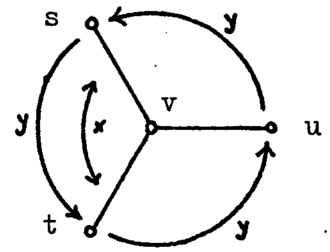
. Then, with respect to the basis $\{e_i\}$, the 24 positive roots of Σ are given in Table 7.2 , along with the action of τ on Σ and the action of the fundamental reflections $w_i = w_{p_i}$ on Σ . The action of τ is obtained since we know from Table 7.1 that τ permutes p_1 with p_4 and p_2 with p_3 .

Here , we label the positive roots as r_i for $i \in \{1,\ldots,24\}$ and for abbreviation in the Table , we denote r_i by i . So $w_1(i)$ is the image of r_i under the fundamental reflection w_1 , and if $w_1(r_i) = -r_j$, we write $w_1(i) = -j$. Also , we have $r_1 = p_1$, $r_2 = p_4$, $r_3 = p_2$, $r_4 = p_3$.

Then , if Σ_s is the set of short roots of Σ and Σ_ℓ the set of long roots of Σ , both Σ_s and Σ_ℓ are root systems of type D_4 and $\tau(\Sigma_s) = \Sigma_\ell$.

There are several ways we can investigate the group $W(F_4)$ (and hence the group W^*) , and we shall begin by considering these approaches .

First we consider W in the context of reflection groups, following [13]. Let V be a 4-dimensional real vector space with orthonormal basis $\{e_i\}_{i=1}^{4}$, and consider the four vectors of the set $\Pi_1 = \{r_3, r_5, r_9, r_1\}$. Then Π_1 is a fundamental system for the root system $\Sigma_\ell = \Sigma_1$, say, which is of type D_4. Let s,t,u,v be the corresponding reflections in $\mathrm{Aut}(V)$, ie. s is the reflection in the plane $e_3 = e_4$. Then the group $W_1 = \langle s,t,u,v : R_1 \rangle$ with defining relations $R_1 = \{ s^2=t^2=u^2=v^2=(sv)^3=(tv)^3=(uv)^3=1; st=ts,\ u=ut, su=us \}$ is a Weyl group of type D_4 with corresponding graph



Consider the group $S = D(\Sigma_1) = \langle x,y : x^2 = y^3 = 1,\ xyx^{-1} = y^{-1} \rangle$ acting on the graph in the following way :-

$R_2 = \{ x.s.x^{-1}=t, xu=ux, xv=vx, y.s.y^{-1}=t, y.t.y^{-1}=u, y.u.y^{-1}=s, y.v.y^{-1}=v \}$.

Then $S \cong \mathfrak{S}_3$, the symmetric group on three letters (viz. r_3, r_5, r_9), and $S \leq A(\Sigma_1)$. So S acts as above.

From R_2, we see that S is an automorphism group of W_1, and we let $W = W_1 . S$ (semi-direct), so that $W = A(\Sigma_1)$ by Lemma 2.3. Then we have :-

Proposition 7.6. $W \cong W(F_4)$.

Proof. From R_1 and R_2 , it follows that W is generated by s,v,x and xy . Hence , $W = \langle s,v,x,xy : R_3 \rangle$ with defining relations

$$R_3 = \{ s^2=v^2=x^2=(xy)^2=(sv)^3=(x.xy)^3=(sx)^4=1, vx=xv, v(xy)=(xy)v, s(xy)=(xy)s \} .$$

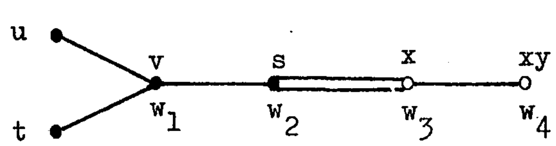
This is a set of defining relations for $W(F_4)$, so $W \lesssim W(F_4)$. Since $|W| = |W(F_4)| = 1152$, the result follows . $\square$

Now let $W_2 = \langle s,t,u,v,x : R_1 \cup R_2 \rangle$. Then we have :-

Proposition 7.7. $W_2 \cong W(B_4)$.

Proof. Since $x.s.x^{-1} = t$, it follows that $W_2 = \langle s,u,v,x : R_1 \cup R_2 \rangle$ and the result follows by a similar argument to the proof of 7.6 . $\square$

Hence we can regard $W(F_4)$ as the full group of automorphisms of V which preserve the root system Σ_1 of type D_4 , ie. the group $A(\Sigma_1)$. Then $W(D_4) \triangleleft W(F_4)$ and $W(F_4) / W(D_4) \cong \mathfrak{S}_3$. Furthermore , $W(B_4) \leqslant W(F_4)$, and this subgroup is not normal since $y.x.y^{-1} = y^{-1}x$. This situation is represented by the following graph :-



where each of the groups W_1, W_2, W corresponds to a certain subgraph containing 4 nodes . This corresponds to the reverse process described by Dynkin [12] in order to find the subsystems of maximal rank of Σ , so that u is the reflection corresponding to the highest root of Σ , and

t that corresponding to the highest root of $\Sigma_2 = \Sigma(B_4)$. Let τ be the isometry of V defined by the graph automorphism of Σ as in Proposition 1.3 and let $W^* = \langle W, \tau \rangle$. Then τ is the involutive automorphism of W which is defined by

$$R_4 = \{ \tau . v . \tau^{-1} = xy \quad \text{and} \quad \tau . s . \tau^{-1} = x \} .$$

<u>Proposition 7.8</u>. (i). $N = W_1^{\tau} \cap W_1$ is a normal subgroup of W of order 32 , and , by definition , N is a normal subgroup of W^* ,

(ii). $W / N \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ and this extension splits ,

(iii). The commutator subgroup W' of W is generated by $\{ N, y, e \}$, where $e = (stuv)^2$. Then , $W = W^1 \cup W^1 v \cup W^1 x \cup W^1 xv$.

<u>Proof</u>. See [13] . $\square$

Since we wish to find the conjugacy classes of W^*, and we know that a normal subgroup is a union of conjugacy classes , we could begin by considering all the normal subgroups of W^* . We know that W^* is a soluble group of order $2304 = 2^8 . 3^2$, and $W^* = \langle s, v, \tau \rangle$ with defining relations given by $R_3 \cup R_4$.

<u>Proposition 7.9</u>. (i). $O_2(W^*) = O_2(W) = N$.

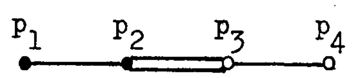
(ii). $O_3(W^*) = 1$.

<u>Proof</u>. Since $A \times B \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ has no normal 2-subgroups , then $O_2(W) = N$. Now W^* / N has no normal 2-subgroups , so that $O_2(W^*) = O_2(W)$.

Part (ii) follows because of the nature of τ . $\square$

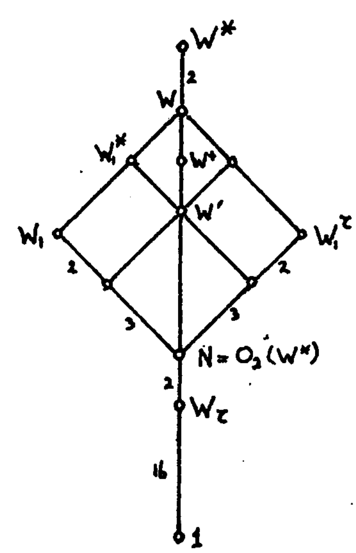
If we denote the group , $\{ w : \tau w \tau^{-1} = w \}$, of τ-fixed points of W by W_τ , then :-

Proposition 7.10. $W_\tau = \langle a,b : a^2=b^2=(ab)^8 = 1 \rangle \cong D_{16}$, the Dihedral group of order 16 , where $a = v.xy$ and $b = (s.x)^2$.

Proof. The Dynkin diagram of Σ is 

, and π splits into two τ-orbits :- $\bar{w}_1 = \{ p_1, \sqrt{2}p_4 \}$ and $\bar{w}_2 = \{ p_2, \sqrt{2}p_3 \}$. Let $\alpha = av(\bar{w}_1) = \frac{1}{2}(p_1+\sqrt{2}p_4)$ and $\beta = av(\bar{w}_2) = \frac{1}{2}(p_2+\sqrt{2}p_3)$. Then Steinberg [20] shows how $w_\alpha = v.xy$, $w_\beta = (s.x)^2$ and
$W_\tau = \langle w_\alpha, w_\beta : w_\alpha^2 = w_\beta^2 = (w_\alpha w_\beta)^8 = 1 \rangle$. $\square$

Hence W^* has the following structure :-



Suppose that M is a maximal normal subgroup of W^*. Then W^*/M is abelian , so $(W^*)' \leq M$.

Lemma 7.11. There are 3 maximal normal subgroups of W^*, and $W^*/(W^*)' \cong C_2 \times C_2$.

Proof. $W^*/(W^*)' = \langle \bar{s}, \bar{v}, \bar{\tau} \rangle$ with defining relations $R_5 = \{ \bar{s}^2 = \bar{v}^2 = \bar{\tau}^2 = \overline{sv}^2 = \bar{1} \}$, where $\bar{x} = (W^*)'x$.
Hence $W^*/(W^*)' \cong C_2 \times C_2$ with maximal normal subgroups

$$\langle \bar{1}, \bar{s} \rangle , \langle \bar{1}, \bar{\tau} \rangle \text{ and } \langle \bar{1}, \overline{s\tau} \rangle .$$ $\square$

Coxeter [9,10] discusses W as the symmetry group of the self-reciprocal 24-cell $\{3,4,3\}$ in Euclidean 4-space . This offers a geometrical illustration of many of the properties of W and W^*. This regular polytope is quite remarkable , being another peculiarity of 4-dimensional Euclidean space in that it has no analogue in any other dimension . In fact , we can see that Σ is a "skeleton" of $\{3,4,3\}$ with the long roots corresponding to the vertices of $\{3,4,3\}$ and the short roots to the mid-points of the 24 cells of $\{3,4,3\}$. Then we can easily see that τ is the dual map of this self-reciprocal figure .

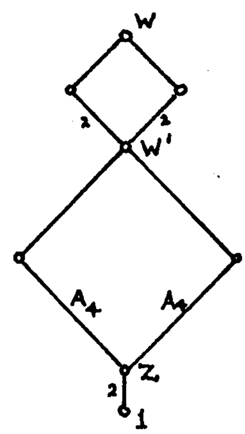
The polytope $\{3,4,3\}$ arises by truncating β_4 , the 4-dimensional hyperoctahedron . So it has 24 octahedra as cells , and consequently 24 vertices , viz. the centres of the edges of β_4 . The subgroup of W that leaves fixed a vertex of $\{3,4,3\}$ is the symmetry group of the vertex figure $\{4,3\}$, ie. the cube , so this group is the hyperoctahedral group $W(B_3)$. The 24 cosets of $W(B_3)$ in $W(F_4)$ correspond to the 24 vertices of $\{3,4,3\}$. This construction , due to Cesàro , exhibits $W(C_4)$ as a subgroup of W of index 3 , and the 24 octahedra fall into 3 sets of 8 , which are the vertex figures of 3 distinct hyperoctahedra lying in the bounding hyperplanes of 3 hypercubes.

We also have the reciprocal construction , due to Gosset , from two equal hypercubes , which is the analogy of the construction for the rhombic dodecahedron . In fact , the 24 vertices of $\{3,4,3\}$ represent the 24 units of Hurwitz's integral quaternions .

Coxeter shows the link between W as the symmetry group of $\{3,4,3\}$ and as a reflection group, and then provides some additional information in [10]. Since the defining relations, R_3, for W involve an even number of generators, each element of W is either even or odd, according as any expression for it has an even number of generators or an odd number. The even elements of W form a subgroup W^+ of index 2, called the <u>even subgroup</u>.

In a dual paper, [11], on groups of the form $\langle a,b : a^{\ell} = b^m = (a^{-1}b^{-1}ab)^p = 1\rangle$, the authors show that the subgroup of W generated by $\{a = st$ and $b = uv\}$ has defining relations $R = \{a^3 = b^3 = (a^{-1}b^{-1}ab)^2 = 1\}$, and is a subgroup of index 4. Since a and b are commutators, it follows that this subgroup is W'. In [10], Coxeter shows that W contains a unique central inversion which reverses every vector in V. This is $z = (stuv)^6$, which generates the central subgroup Z.

Coxeter also shows, [11], that the central quotient group of W' is isomorphic to $A_4 \times A_4$, where A_4 is the alternating group of degree 4. Hence we have the structure shown alongside.



Lemma 7.12. (i). $(W^*)' = W^+$.

(ii). Each coset of W^+ in W^*, viz. $\{W^+, W^+s, W^+\tau, W^+s\tau\}$ is a union of conjugacy classes of W^*.

Proof. (i). Let $w \in (W^*)'$, so w is a product of terms $[g_1, g_2]$ with $g_i \in W^*$. Now, $W^* = W \cup W\tau$, so we have three situations :-

(a). $g_1, g_2 \in W$ and then $[g_1, g_2] \in W^+$ since $W' \leq W^+$.

(b). $g_1 \in W$, $g_2 \in W\tau$ so that $g_1 = w_1$, $g_2 = w_2\tau$ for $w_1, w_2 \in W$. Then $[g_1, g_2] = w_1\tau.w_2.\tau^{-1}w_1^{-1}.w_2^{-1} = w_1.w_2^{\tau}.w_1^{-1}w_2^{-1} \in W^+$ since $\ell(w_2^{\tau}) = \ell(w_2)$, where $\ell(w)$ is the number of generators in a reduced word for w.

(c). $g_1, g_2 \in W\tau$ so that $g_i = w_i\tau$ for $w_i \in W$. Then $[g_1, g_2] = w_1.\tau w_2\tau.\tau w_1^{-1}\tau w_2^{-1} = w_1 w_2^{\tau}.w_1^{-\tau}.w_2^{-1} \in W^+$ as in (b).

Hence $w \in W^+$, and $(W^*)' \leq W^+$. By Lemma 7.11 and by definitions of the groups W^+ and W^*, both $(W^*)'$ and W^+ have index 4 in W^*. Hence $(W^*)' = W^+$.

(ii). Let G be any group and let $G/G' = \{G', G'x_1, \dots, G'x_n\}$. Let $g \in G$. Then $g.G'x_i.g^{-1} = g.G'.g^{-1}.(g\,x_i g^{-1}x_i^{-1}).x_i = G'.[g, x_i].x_i = G'x_i$. Thus each coset $G'x_i$ is a normal subset of G, and is thus a union of conjugacy classes of G. When $G = W^*$, then $(W^*)' = W^+$ and the four cosets W^+, W^+s, $W^+\tau$ and $W^+s\tau$ are each a union of conjugacy classes of W.

$\square$

We have already established in Proposition 7.5 that we are searching for the 11 conjugacy classes of W^* which are contained in $W\tau$, so we have :-

Corollary 7.12. The 11 classes of W^* contained in $W\tau$ are partitioned into the two cosets $W^+\tau$ and $W^+s\tau$ of W^*/W^+.

Proof. By Lemma 7.12. $\square$

Another way of considering W is suggested by considering the vector spaces $V_n(3)$ of dimension n over the field $GF(3)$. Then $V_n(3)$ can be thought of as a "skeleton" of an n-dimensional cube in the sense that each point of $V_n(3)$ represents the mid-point of some j-dimensional face of the cube, for $0 \leq j \leq n$. The orthogonal group $O_n^+(3)$ is the group of transformations of $V_n(3)$ which fix the origin and preserve distances. The group $O_n^+(3)$ is generated by reflections in the hyperplanes which pass through the origin and are orthogonal to the non-isotropic vectors of $V_n(3)$. It is clear that if $n = 2$ or 3, then $O_n^+(3) \cong W(B_n)$ - the symmetry group of the n-cube. However, such an analogy soon breaks down in 4-dimensional space because of the unit vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$. For we can embed $W(B_n)$ in $O_n^+(Z)$ and reduce (modulo 3) to embed in $O_n^+(3)$. This gives a monomorphism $W(B_n) \hookrightarrow O_n^+(Z) \xrightarrow{\psi} O_n^+(3)$ so that $W(B_n) \leq O_n^+(3)$. However, when $n = 2$ or 3, any $g \in O_n^+(3)$ can be "lifted" uniquely to $\psi^{-1}(g) \in O_n^+(Z)$, and this breaks down for $n = 4$ because of the unit vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$ so that $W(B_4) < O_4^+(3)$.

Now $W(B_4) \cong C_2 \wr \mathfrak{S}_4$, and we can think of the $C_2 \times C_2 \times C_2 \times C_2$ as a split maximal torus T inside $GL_4(3)$ with the $\mathfrak{S}_4$ as Weyl group $W(T)$. Then $W(B_4)$ is just the normaliser $N(T)$ inside $GL_4(3)$, ie. the subgroup of monomial matrices .

Lemma 7.13. $(O_4^+(3) : W(B_4)) = 3$

Proof. It is straightforward to check that there are twice as many orthogonal matrices that cannot be "lifted" from $O_4^+(3)$ to $O_4^+(Z)$ as there are monomial matrices . $\square$

We recall that $(W(F_4) : W(B_4)) = 3$ also , and this suggests a possible isomorphism between $W(F_4)$ and $O_4^+(3)$.

Proposition 7.14. $W(F_4) \cong O_4^+(3)$.

Proof. Let $Z^* = \langle Z , \frac{1}{2} \rangle$ and let $Q : Z^* \longrightarrow Z_3$ be reduction modulo 3 . Then $Z^*/ 3Z^* \cong Z_3$. By the nature of W as a reflection group and of $\lesssim$, we know that we can embed W in $O_4^+(Z^*)$, where the basis of V is $\{ e_i \}$ and the roots of $\lesssim$ are as in Table 7.2 .

If $m \in O_4^+(Z^*)$, then either $m \in M$ is a monomial matrix in $O_4^+(3)$, or every entry of m is $\pm\frac{1}{2}$. Now consider the sequence

$$\psi : W \hookrightarrow O_4^+(Z^*) \xrightarrow{\tilde{Q}} O_4^+(3) \text{ , where } \tilde{Q}(a_{ij}) = (Q(a_{ij})) .$$

Let $m \in \ker\psi$ and suppose $m \in O_4^+(Z^*)$ is a monomial . Then , since

$\tilde{Q}(M) = I_4$, we must have $m = 1$. If , on the other hand , m has every entry equal to $\pm\frac{1}{2}$, then correspondingly , $\tilde{Q}(m)$ will have every entry ∓ 1 , so that $\tilde{Q}(m) \neq I_4$. Hence , $\ker\psi = 1$ and ψ is a monomorphism . We have already seen that $|W| = |O_4^+(3)|$, although it is easy to see that $\tilde{Q}$ can be inverted to a "lifting" map . Hence , ψ is an isomorphism .

□

Now we have shown W to be isomorphic to $O_4^+(3)$, we may describe the group W^* in the same context .

Proposition 7.15. $W^* \leq O_4^+(3^2)$.

Proof. Let $k = GF(3)$ and consider $K = k(i)$, where i is the positive root of the polynomial $x^2+1 \in k[x]$. Then $K \cong GF(3^2)$. Identify W with $O_4^+(3)$ as in 7.14 , and consider all matrices with respect to the natural basis $\{e_i\}_{i=1}^4$. Consider the action of τ on V and let the corresponding matrix be T . Then we can show that

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} , \quad \text{where} \quad E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Now let $Y = Z^*(\sqrt{2})$, then obviously $T \in O_4^+(Y)$ and if $Q : Y \longrightarrow K$ is reduction modulo 3 , then $Y / 3Y \cong K$. Then , just as in Proposition 7.14 , we have $W^* \cong \langle O_4^+(3), T \rangle \leq O_4^+(3^2)$. So we may identify W^* with the subgroup $\langle O_4^+(3), T \rangle = O_4^+(3)^*$ of $O_4^+(3^2)$.

□

Now that we have established some of the various rôles of the group W , we may freely use each rôle to aid us in our solution of the problem .

First, we prove a result concerning the characteristic polynomials corresponding to the classes $\mathcal{L}_\tau(W^*)$.

<u>Proposition 7.16</u>. Let $f_x(t) \in Y[t]$ be the characteristic polynomial of a representative element x of some class of $\mathcal{L}_\tau(W^*)$ acting on V . Then $Q\ (f_x(t)) = g_x(t) \in K[t]$ is one of the following :-

(i)	t^4+1	(ii).	t^4+it^3+it+1	(iii).	t^4-it^3-it+1
(iv).	t^4+t^2+1	(v).	$t^4+it^3+t^2+it+1$	(iv).	$t^4-it^3+t^2-it+1$
(vii).	t^4-t^2+1	(viii).	$t^4+it^3-t^2+it+1$	(ix).	$t^4-it^3-t^2-it+1$
(x).	t^4-1	(xi).	t^4+it^3-it-1	(xii).	t^4-it^3+it-1 ,

where Q is as in the proof of Proposition 7.15 .

<u>Proof</u> . Consider W^* embedded in $O_4^+(3^2)$ and let $x \in W^* \setminus W$. Then $x = T.w$ for some $w \in W$. Hence $x = i\ .\ B$, where $B = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} . w \in GL_4(3)$. Now $x \in O_4^+(3^2)$, so ${}^tx\ .\ x = I_4$. Hence , ${}^tB\ .\ B = i^2I_4 = -I_4$.

Now let $g_A(t) \in k[t]$ be the characteristic polynomial of the matrix A on $V_4(3)$. Then $g_B(t) = \det(t.B-I) = t^4.\det B.g_B(-t^{-1})$. Since $\det \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = 1$, then $g_B(t) = \pm\ t^4.g_B(-t^{-1})$. (1)

Also , $g_x(t) = g_{iB}(t) = \det(it.B-I) = g_B(it)$, so that $g_x(t) = g_B(it)$. (2)

Let $g_B(t) \in K[t]$, so that $g_B(t) = t^4+a_3t^3+a_2t^2+a_1t+a_0$, with $a_i \in k$.

In order to satisfy (1) , we must have

$$t^4+a_3t^3+a_2t^2+a_1t+a_0 = \varepsilon\,(a_0t^4-a_1t^3+a_2t^2-a_3t+1)\ , \qquad (3)$$

where $\varepsilon = \pm 1$. This set of equations has two solutions :-

(i). if $\varepsilon = +1$, then $a_0 = 1$ and $a_1 = -a_3$,

(ii). if $\varepsilon = -1$, then $a_0 = -1$, $a_1 = a_3$ and $a_2 = 0$.

Hence , if ${}^{t}B.B = -I_4$, then $g_B(t)$ has one of the two forms :-

(1) $g_B(t) = t^4+at^3+bt^2-at+1$, or

(2) $g_B(t) = t^4+at^3+at-1$, where $a,b \in GF(3)$.

Correspondingly , if $x \in W^* \setminus W$, then $g_x(t)$ has one of the two forms :-

(3) $g_x(t) = t^4-ait^3-bt^2-ait+1$ or

(4) $g_x(t) = t^4-ait^3+ait-1$, where $a,b \in GF(3)$.

This follows from equation (2) . Since $a,b \in GF(3)$, equation (3) results in the nine possible polynomials (i) - (ix) , and equation (4) supplies the further three polynomials (x) - (xii) . $\square$

Hence , we know that if $f_x(t)$ is the characteristic polynomial of the element x acting on V , where x belongs to some class of $\mathcal{L}_\tau(W^*)$, then its reduction modulo 3 , ie. $g_x(t)$, must be one of the polynomials of $K[t]$ listed in Proposition 7.16 . Although there is not a 1-1 correspondence between the classes of $\mathcal{L}_\tau(W^*)$ and these polynomials , this result proves useful in finding the classes .

We now proceed with the detailed search for the eleven classes of $\mathcal{L}_\tau(W^*)$, making use of the results so far obtained . However , we first prove a few useful lemmas .

Lemma 7.17. Let z be the central involution of W, and let $x \in W^*$, then $f_{zx}(t) = f_x(-t)$.

Proof. It is well known [4], that in the group $W(F_4)$, $w_0 = -1$, so that $z = -1$ is the unique central involution of W. Furthermore, z commutes with τ, so that z is the unique central involution of W^* also.

Hence, $f_{zx}(t) = \det(t.zx - I) = \det((-t)x - I) = f_x(-t)$. □

This result is extremely useful in our search for the classes, for we know that two elements from the same conjugacy class of W^* have the same characteristic polynomial. Hence, if we find an element x with characteristic polynomial $f_x(t)$ such that $f_x(t) \neq f_x(-t)$, then we know that the conjugacy class, C_x, of W^* containing the element x is distinct from C_{zx}.

Lemma 7.18. Let $C \in \mathcal{L}_\tau(W^*)$ and let x be a representative of C, so that $C = C_x$. If $\mathbb{C}_W(x)$ denotes the centraliser of x in W, and $s_x = |\mathbb{C}_W(x)|$, then $|C_x| = |W| / s_x$. Furthermore, $|C_{zx}| = |C_x|$.

Proof. In the proof of Lemma 6.4, we saw that $C_x = \{x\}^{W^*} = \{x\}^W$. By letting W act on the coset $W\tau$, it follows from the orbit-stabiliser theorem that there is a bijection between C_x and $W / \mathrm{stab}_W(x) = W / \mathbb{C}_W(x)$. The final statement follows since $\mathbb{C}_W(zx) = \mathbb{C}_W(x)$.

□

With this result we are able to calculate , by working with the subgroup $O_4^+(3)$ in $GL_4(3)$, the size of each class of $\mathcal{L}_\tau(W^*)$ once we have a representative of the class . This merely involves the solution of several simultaneous equations , and , by Lemma 1.9 , this fact enables us to calculate $|W_\sigma|$, which is useful for the representation theory . As we discover each class C_i for $i \in \{1,\ldots,11\}$, we shall list a representative element x , the order $\mathrm{ord}(x)$ of the elements of the class C_i , the order of the class $|C_i|$, the coset of W^+ in W^* to which the class belongs , and its characteristic polynomial $f_x(t)$, together with the structure of the corresponding torus $T_{x\tau}$. This latter we shall calculate by taking the representative x of C_i . Under the map ψ of Lemma 6.4 , the corresponding class $\psi^{-1}(C_i)$ of $H^1(\sigma,W)$ has $x\tau \in W$ as a representative . Then the maximal torus of G_σ corresponding to the class $\psi^{-1}(C_i)$ of $H^1(\sigma,W)$ is an abelian group with elementary divisors $\{e_1,\ldots,e_k\}$ determined by diagonalising the matrix $(q.(x\tau)\tau-I) = (q.x_\pi-I)$ over Z , where x_π is the action of x on the lattice $\Delta_{ad} = M_3$.

The methods will be omitted in most cases , except to illustrate the method in one case , and to note that the matrices $(q.x_\pi -I)$ are diagonalisable over $Z[q/\sqrt{2}]$.

NOTE. The notation of $f_i(t)$ for the characteristic polynomial corresponding to the class C_i is not to be confused with the notation $f_n(q) = q^{n-1}+\ldots+1$ used only in Chapters 3 and 4

Lemma 7.19. The class C_1 has as representative the element τ , and $C_1 \subset W^+\tau$. Also , $\mathrm{ord}(\tau) = 2$, $|C_1| = 72$ and $f_1(t) = (t^2-1)^2$. Further , $e_1 = e_2 = (q^2-1)$.

Proof. Certainly we know that τ must belong to some class of $\mathcal{L}_\tau(W^*)$, and $\tau = 1.\tau \in W^+\tau$. In fact , $C_\tau = [W,\tau].\tau$. In the case of the groups B_2 and G_2 , $[W,\tau] = W^+$, but that is not true in this case .

Although the general method of calculating $|C_i|$ is to use Lemma 7.18 with W identified with $O_4^+(3)$, in this case it is easy to see that $\mathbb{C}_W(\tau) = \{ w \in W : w.\tau.w^{-1} = \tau \} = W_\tau \cong D_{16}$, so the result follows .

$$\text{Also , } (q.\tau_\pi - I) = \begin{bmatrix} -1 & 0 & 0 & q/\sqrt{2} \\ 0 & -1 & q/\sqrt{2} & 0 \\ 0 & \sqrt{2}q & -1 & 0 \\ \sqrt{2}q & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} q^2-1 & 0 \\ 0 & q^2-1 \end{bmatrix}$$

$\square$

We note that $z\tau \in C_1$. For , if we let $c = (w_1w_2w_3w_4)$, an element of the Coxeter class of W , then

$z = c^6 = c^3.(\tau.c^{-3}.\tau^{-1}) = [c^3,\tau] \in [W,\tau]$. Hence $z\tau \in [W,\tau]\tau = C_\tau$.

Since elements of the same class must have the same order , we find this a useful guide in our search also .

Lemma 7.20. The class C_2 has as a representative the element $w_2\tau$, and $C_2 \subset W^+s\tau$. Also , $\mathrm{ord}(w_2\tau) = 8$, $|C_2| = 144$ and $f_2(t) = t^4-\sqrt{2}t^3+\sqrt{2}t - 1$. Further , $e_1 = q^4-\sqrt{2}q^3+\sqrt{2}q - 1$ and $w_3\tau \in C_2$.

Proof. Certainly $w_2\tau \in W^+s\tau$, and also , $(w_2\tau)^2 = w_2\tau w_2\tau = w_2w_3$, an element of order 4 . Hence , $\mathrm{ord}(w_2\tau) = 8$.

To find $|C_2|$, we use Lemma 7.18 . The matrix of w_2 with respect to the basis $\{e_i\}_{i=1}^4$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so the matrix of τw_2 is

$$i \times \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = i \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = iC \text{ , as in}$$

Proposition 7.15 .

Then $\mathbb{C}_W(w_2\tau) = \{A \in O_4^+(3) : AC = CA\}$, and if we let $A = (a_{ij})$, then the condition $AC = CA$ is equivalent to the set of 16 equations .

$a_{12} = a_{21}$; $a_{22} = a_{11}+a_{12}$; $a_{23} = -a_{14}$; $a_{24} = a_{13}$;

$a_{11} = -a_{21}+a_{22}$; $a_{21} = a_{12}$; $a_{13} = -a_{23}-a_{24}$; $a_{14} = a_{23}-a_{24}$;

$a_{41} = a_{32}$; $a_{42} = a_{31}+a_{32}$; $a_{43} = -a_{34}$; $a_{44} = a_{33}$;

$a_{31} = -a_{42}$; $a_{32} = -a_{41}-a_{42}$; $a_{33} = a_{44}$; $a_{34} = -a_{43}$.

Hence , $a_{13} = a_{14} = a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = 0$, and

$$\mathbb{C}_W(w_2\tau) = \left\{ \begin{bmatrix} a & b & 0 & 0 \\ b & a+b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{bmatrix} \in O_4^+(3) : a,b,c,d \in GF(3) \right\} .$$

Now $A = \begin{bmatrix} a & b & 0 & 0 \\ b & a+b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{bmatrix} \in O_4^+(3) \Longleftrightarrow {}^tA.A = I_4$, and this condition

is equivalent to the set of 3 equations :-

$$\{ a^2+b^2 = 1 \ , \ b^2-ab = 0 \ , \ c^2+d^2 = 1 \} \ .$$

Since $b^2-ab = 0$ implies that either $a = b$ or $b = 0$, this set of equations is equivalent to the two sets :-

(1). $\{ a^2+b^2 = 1 \ , \ c^2+d^2 = 1 \ , \ b = a \}$ and

(2). $\{ a^2+b^2 = 1 \ , \ c^2+d^2 = 1 \ , \ b = 0 \}$.

Now set (1) is impossible as this would imply that $a^2 = -1$, and this has no solution in $GF(3)$. Hence the condition ${}^tA.A = I_4$ is equivalent to the set of equations (2) . Hence , the only possibilities for the ordered quadruple (a,b,c,d) are $(\pm1,0,\pm1,0)$ and $(\pm1,0,0,\pm1)$.

Thus , $|\mathbb{C}_W(w_2\tau)| = 8$, and $|C_2| = 144$, by Lemma 7.18 .

Since $\tau.w_2\tau^{-1}.\tau = w_3$, it follows that $w_3\tau \in C_2$. The remaining facts follow by diagonalisation . $\square$

<u>Lemma 7.21</u>. The class C_3 has as a representative the element $zw_2\tau$, and $C_3 \subset W^+s\tau$. Also , $\mathrm{ord}(zw_2\tau) = 8$, $|C_3| = 144$ and $f_3(t) = t^4+\sqrt{2}t^3-\sqrt{2}t - 1$. Further , $e_1 = q^4+\sqrt{2}q^3-\sqrt{2}q - 1$.

<u>Proof</u>. Since $f_2(t) \neq f_2(-t)$, then $zw_2\tau \notin C_2$ by Lemma 7.17. Hence C_3 is distinct from C_2 , and $C_3 \subset W^+s\tau$, since $w_2 \in W^+s$. By Lemma 7.18 , $|C_3| = |C_2| = 144$ and $f_3(t) = f_2(-t) = t^4+\sqrt{2}t^3 -\sqrt{2}t - 1$.

Also , $(zw_2\tau)^n = z^n.(w_2\tau)^n$ so that $\mathrm{ord}(zw_2\tau) = 8$, since $\mathrm{ord}(\tau) = 2$. $\square$

<u>Lemma 7.22</u>. The class C_4 has as a representative the element $w_1\tau$, and $C_4 \subset W^+s\tau$. Also , $\mathrm{ord}(w_1\tau) = 4$, $|C_4| = 288$ and $f_4(t) = t^4 -1$.

Further , $e_1 = q^4-1$ and $w_4\tau \in C_4$.

Proof. Certainly $w_1\tau \in W^+s\tau$, and also $(w_1\tau)^2 = w_1.\tau w_1\tau = w_1w_4$, an element of order 2 . Hence , $\mathrm{ord}(w_1\tau) = 4$. The remaining results follow as above . □

Lemma 7.23. The coset $W^+s\tau$ is the union of the three conjugacy classes C_2 , C_3 and C_4 .

Proof. We have seen in Lemma 7.12 that $W^*/W^+ \cong C_2 \times C_2$. Hence , each coset of W^+ in W^* has order 576 , in particular the coset $W^+s\tau$.

Since $|C_2| + |C_3| + |C_4| = 576$ and $C_2 , C_3 , C_4 \subset W^+s\tau$, it follows that $W^+s\tau$ is the union of these three classes . □

So , by Corollary 7.12 , we know that the remaining 7 classes of $\mathcal{L}_\tau(W^*)$ lie in $W^+\tau$. In order to determine these , we consider the possible orders of their representatives .

Lemma 7.24. The only possibilities for the orders of elements of $W\tau$ are 2 , 4 , 6 , 8 , 12 , 16 , 24 .

Proof. Suppose that $x \in W\tau$, so that $x = w\tau$ for $w \in W$. Then $x^2 = w\tau.w\tau = w.\tau w\tau^{-1} = w.w^\tau$. Hence , $\mathrm{ord}(x) = 2e$, where $e = \mathrm{ord}(w.w^\tau)$. Since $(w.w^\tau) \in W$, the only possibilities for e are the orders of elements of W , viz. 1 , 2 , 3 , 4 , 6 , 8 , 12 , [6] .

□

Corollary 7.24. The representatives of the classes C_x and C_{zx} , for $x \in W\tau$, have the same order .

Proof. Since $\mathrm{ord}(x) = 2e$, for some e , it follows that $(zx)^n = z^n x^n = 1$. Now $\mathrm{ord}(z) = 2$, so that $\mathrm{ord}(zx) = 2e$ also . □

We now have the possibilities for $\mathrm{ord}(x)$, where $x \in W\tau$, and we may be able to reduce the list of Lemma 7.24 even further if we can decide just how the set $E = \{w.w^{\tau} : w \in W\}$ intersects the conjugacy classes of W . For the possibilities of e are precisely those belonging to the set $\{\mathrm{ord}(x) : x \in E\}$. Although we have found the coset $W^+ s\tau$ as a union of conjugacy classes of W^* , we can in fact show that there is no element of order 2 in this coset .

Lemma 7.25. There is no element of order 2 in the coset $W^+ s\tau$.

Proof. Suppose $w \in W^+ s$ such that $\mathrm{ord}(w\tau) = 2$. Then $w.w^{\tau} = 1$, ie. $w^{\tau} = w^{-1}$. Now w is conjugate to w^{-1}, [6] , so that τ must fix the graph Γ_w . Since $w \notin W^+$, then Γ must have either 1 or 3 nodes . There are 10 such graphs in Table 5.2 , none of which is fixed by τ . Hence there is no such element . □

Lemma 7.26. There is no element of order 6 in $W\tau$.

Proof. Suppose that $\mathrm{ord}(w\tau) = 6$ for some $w \in W$. Then the element

$w.w^{\tau} \in E$ has order 3 , and so would have to belong to one of the classes A_2 , $\tilde{A}_2$ or $A_2+\tilde{A}_2$. However , $w.w^{\tau}$ cannot belong to A_2 or $\tilde{A}_2$ since then w would have to be a reflection w_r , and then $w.w^{\tau} = w_r.w_{\tau(r)}$ would belong to $A_1+\tilde{A}_1$.

Now for $w\tau$ to belong to $A_2+\tilde{A}_2$, we must have $w = w_r w_s$ such that (i). r is long , s is short ; (ii). $(r,s) = 0$; (iii). $(\tau(r),r) = (\tau(s),s) = 0$; (iv). $(\tau(r),s) = \pm\frac{1}{2}$; (v). $(\tau(s),r) = \pm 1$. Then $(w\tau)^2 = w_r w_s . w_{\tau(r)} w_{\tau(s)} \in A_2+\tilde{A}_2$.

For r to satisfy (i) and (iii) , we see from Table 7.2 that r must be one of r_1 , r_{13} , r_{17} , r_{19} . Then we have $\tau(r) = r_2, r_{14}, r_{18}, r_{20}$ respectively . Similarly , for s to satisfy (i) and (iii) , s must belong to the set $r_2, r_{14}, r_{18}, r_{20}$, with $\tau(s) = r_1, r_{13}, r_{17}, r_{19}$ respectively . It soon follows that none of these pairings of r and s satisfy (ii), (iv) and (v) also . Hence no such element w exists . $\square$

Lemma 7.27. The class C_5 has as a representative the element $w_1 w_2 \tau$, and $C_5 \subset W^+\tau$. Also , $\mathrm{ord}(w_1 w_2 \tau) = 24$, $|C_5| = 96$ and $f_5(t) = t^4-\sqrt{2}t^3+t^2-\sqrt{2}t + 1$. Further , $e_1 = q^4-\sqrt{2}q^3+q^2-\sqrt{2}q + 1$.

Proof. Now $w_1 w_2 \tau \in W^+\tau$, and $(w_1 w_2 \tau)^2 = w_1 w_2 (w_1 w_2)^{\tau} = w_1 w_2 w_4 w_3$, which belongs to the Coxeter class of W and so has order 12 . Hence , $\mathrm{ord}(w_1 w_2 \tau) = 24$ and C_5 is distinct from C_1 . The remaining results follow as in Lemmas 7.19 and 7.20 . $\square$

Lemma 7.28. The class C_6 has as a representative the element $zw_1w_2\tau$, and $C_6 \subset W^+\tau$. Also, $\mathrm{ord}(zw_1w_2\tau) = 24$, $|C_6| = 96$ and $f_6(t) = t^4+\sqrt{2}t^3+t^2+\sqrt{2}t + 1$. Further, $e_1 = q^4+\sqrt{2}q^3+q^2+\sqrt{2}q + 1$.

Proof. Since $f_5(t) \neq f_5(-t)$, then $zw_1w_2\tau \notin C_5$ by Lemma 7.17. Hence C_6 is distinct from C_5, and $C_6 \subset W^+\tau$ since $z \in W^+$. By Lemma 7.18, $|C_6| = |C_5| = 96$ and $f_6(t) = f_5(-t) = t^4+\sqrt{2}t^3+t^2+\sqrt{2}t + 1$. By Corollary 7.24, it follows that $\mathrm{ord}(zw_1w_2\tau) = 24$. $\square$

We recall the notation of Proposition 7.10. Then :-

Lemma 7.29. The class C_7 has as a representative the element $ab\tau$, and $C_7 \subset W^+\tau$. Also, $\mathrm{ord}(ab\tau) = 8$, $|C_7| = 12$ and $f_7(t) = (t^2-\sqrt{2}t+1)^2$. Further, $e_1 = e_2 = (q^2-\sqrt{2}q+1)$.

Proof. We recall that $a = w_1w_4$ and $b = (w_2w_3)^2$, so that $ab\tau \in W^+\tau$. Also, $(ab\tau)^2 = (ab)^2$ since $\langle a,b\rangle = W_\tau$. Now $\mathrm{ord}(ab) = 8$, so that $\mathrm{ord}(ab\tau) = 8$ also. The final result follows as in Lemma 7.19.

As in Lemma 7.20, we can show that

$$\mathbb{C}_W(ab\tau) = \{(a_{ij}) \in O_4^+(3) : a_{31} = -a_{13} ; a_{32} = -a_{14} ; a_{33} = a_{11} ; a_{34} = a_{12} ; a_{41} = -a_{23} ; a_{42} = -a_{24} ; a_{43} = a_{21} ; a_{44} = a_{22}\}.$$

Hence, $A \in \mathbb{C}_W(ab\tau) \Longleftrightarrow A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ -c & -d & a & b \\ -g & -h & e & f \end{bmatrix} \in O_4^+(3)$

$\Longleftrightarrow (a,b,c,d)$ and (e,f,g,h) are orthonormal vectors of $V_4(3)$

satisfying $ag + bh - ce - df = 0$, (1) .

If the vectors are monomials , then we may choose $a = \pm 1$ and $b = c = d = 0$. Hence , $e = g = 0$ and $f^2+h^2 = 1$. This gives 8 combinations for each choice of one of a,b,c,d to be non-zero . Hence there are 32 possibilities among the monomials .

Otherwise , $a^2=b^2=c^2=d^2=e^2=f^2=g^2=h^2 = 1$, and we may choose one of the 16 possibilities for (a,b,c,d) . Then , the orthogonality condition is

$$ae + bf + cg + dh = 0 , \quad (2) .$$

Adding equations (1) and (2) , we have

$$(a-c)e + (b-d)f + (a+c)g + (b+d)h = 0 \quad (3) .$$

Now , in the vector (a,b,c,d) we have chosen , just one of $\{a-c , a+c\}$ is zero and just one of $\{b-d , b+d\}$ is zero . So equation (3) gives one of $\{e,f,g,h\}$ as a multiple ε of one other element of the set , and $\varepsilon = \pm 1$. Then equation (2) gives one of the remaning two in the set $\{e,f,g,h\}$ as a multiple of the other . Hence ,we have freedom of choice for two of the set $\{e,f,g,h\}$, a total of 4 matrices for each first choice . Hence there are 64 such matrices altogether , and $|\mathbb{C}_W(ab\tau)| = 96$. Then $|C_7| = 12$ by Lemma 7.18 .

$\square$

<u>Lemma 7.30</u>. The class C_8 has as a representative the element $zab\tau$, and $C_8 \subset W^+\tau$. Also , $\mathrm{ord}(zab\tau) = 8$, $|C_8| = 12$ and $f_8(t) = (t^2+\sqrt{2}t+1)^2$. Further , $e_1 = e_2 = (q^2+\sqrt{2}q + 1)$.

Proof. Since $f_7(t) \neq f_7(-t)$, then $zab\tau \notin C_7$ by Lemma 7.17. Hence C_8 is distinct from C_7, and $C_8 \subset W^+\tau$ since $z \in W^+$. By Lemma 7.18, $|C_8| = |C_7| = 12$ and $f_8(t) = f_7(-t) = (t^2+\sqrt{2}t+1)^2$. By Corollary 7.24, it follows that $\mathrm{ord}(zab\tau) = 8$. $\square$

Lemma 7.31. There exist elements $w.w^\tau \in E$ of order 2, and $(ab)^2$ is one such.

Proof. The classes of W containing elements of order 2 are $4A_1$, $2A_1$, A_1, $\tilde{A}_1$, $3\tilde{A}_1$, $A_1+\tilde{A}_1$. If $w.w^\tau = w'$, then $\Gamma_{w'}$ has an equal number of long and short nodes. Hence, the only possibilities are $A_1+\tilde{A}_1$ and $4A_1 = 2A_1+2\tilde{A}_1$. Now, if $w' \in A_1+\tilde{A}_1$, then w is a reflection and so belongs to W^+s. Hence, we must look in the class $2A_1+2\tilde{A}_1$.

Consulting Table 7.2, we see that $z = w_{r_1}.w_{r_2}.w_{r_{19}}.w_{r_{20}}$ is the central element of W in the class $2A_1+2\tilde{A}_1$, and a product of commuting reflections. Hence, $(w_{r_1}w_{r_{19}}\tau)^2 = w_{r_1}w_{r_{19}}.w_{r_2}w_{r_{20}} = z$. Since z has order 2, then $\mathrm{ord}(w_{r_1}w_{r_{19}}\tau) = 4$.

Now $w_3w_4w_1w_2w_3(w_{r_{19}}) = r_3$, so that

$v = w_{r_1}w_{r_{19}} = w_3w_2.(w_1w_4).(w_2w_3)^2.(w_1w_4).w_2w_3$, which is conjugate to $(ab)^2$. $\square$

Lemma 7.32. The class C_9 has as a representative the element $(ab)^2\tau$, and $C_9 \subset W^+\tau$. Also, $\mathrm{ord}((ab)^2\tau) = 4$, $|C_9| = 24$ and $f_9(t) = (t^2+1)^2$. Further, $e_1 = e_2 = (q^2+1)$.

Proof. Since $(ab)^2 \in W_\tau$, it follows that $((ab)^2\tau)^4 = (ab)^8 = 1$, so that $\mathrm{ord}((ab)^2\tau) = 4$. The remaining results follow as in Lemmas 7.19 and 7.20 .

$\square$

Lemma 7.33. There exist elements $w.w^\tau \in E$ of order 6 .

Proof. The classes of W containing elements of order 6 are D_4 , $\tilde{D}_4$, $F_4(a_1)$, B_3 , C_3 , $A_2+\tilde{A}_1$, $\tilde{A}_2+A_1$. If $w.w^\tau = w'$, then $\Gamma_{w'}$ has an equal number of long and short nodes . Hence the only possibility is the class $F_4(a_1)$.

Suppose $w = w_r w_s$, with (i). r long and s short . If $w.w^\tau$ is to belong to the class $F_4(a_1)$, then (ii). $(r,s) = (\tau(r),\tau(s)) = 0$, (iii). $(r,\tau(r)) = \pm(s,\tau(s)) = \pm 1$, (iv). $(r,\tau(s)) = \pm 1$, (v). $(\tau(r),s) = \pm \frac{1}{2}$.

Let $r = r_3$ and $\tau(r) = r_4$. Then the possibilities for the pair $\{s,\tau(s)\}$ to satisfy (i) and (ii) are $\{10,9\}$, $\{2,1\}$, $\{14,13\}$, $\{16,15\}$ $\{22,21\}$, $\{24,23\}$, and the only pair to satisfy all the conditions is the pair $\{16,15\}$. So we let $w = w_{r_3}.w_{r_{16}}$, and then $w.w^\tau$ belongs to the class $F_4(a_1)$ and so has order 6 .

By Table 7.2 , it follows that $w_2w_1w_4(r_{16}) = p_3$ and $r_3 = p_2$, so that $w\tau = w_2w_4w_1w_2w_3w_2w_1w_4\tau$, which is conjugate in W^* to $w_2w_1w_3w_2\tau$, or $c(w_2w_3)\tau$, where $c = w_1w_2w_3w_4$ is a Coxeter element .

$\square$

Lemma 7.34. The class C_{10} has as a representative element $c(w_2w_3)\tau$, and $C_{10} \subset W^+\tau$. Also, $\mathrm{ord}(c(w_2w_3)\tau) = 12$, $|C_{10}| = 192$ and $f_{10}(t) = t^4 - t^2 + 1$. Further, $e_1 = (q^4-q^2+1)$.

Proof. By Lemma 7.33, $c(w_2w_3)\tau$ is conjugate to the element $w\tau$ which has order 12. The remaining results follow as in Lemmas 7.19 and 7.20.

$\square$

We have so far found 10 of the 11 conjugacy classes of $\mathcal{C}_\tau(W^*)$ and to determine the remaining class, we consider the characteristic polynomials of the classes we have found, taking note of Proposition 7.16.

Lemma 7.35. In the notation of Proposition 7.16, the characteristic polynomials $g_x(t) \in K[t]$ for the classes C_j for $j = 1,\ldots,10$ are :-

Class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
Polynomial	(iv).	(xii).	(xi).	(x).	(vi).	(v).	(v).	(vi).	(vii).	(vii).

Proof. Since $g_x(t) = Q(f_x(t))$, we just reduce the coefficients of each polynomial $f_j(t)$ modulo 3.

$\square$

We saw in Proposition 7.16, that the last three polynomials (x), (xi), (xii) of the list correspond to elements of the coset $W^+s\tau$, and since we know (Lemma 7.23) that the remaining class is in the coset $W^+\tau$, the missing characteristic polynomial $g_{11}(t)$ must be one of the list (i),...,(ix).

Lemma 7.36. $g_{11}(t)$ is not one of the polynomials (ii), (iii), (viii) or (ix) .

Proof. These polynomials occur in pairs , so that if the final class C_{11} had a representative element x and characteristic polynomial one of these , then $f(t) \neq f(-t)$ so that zx would be a representative element of a new class C_{12} , which is clearly a contradiction . $\square$

Although there may be repetitions as in (v), (vi) and (vii) , it appears that the most likely candidate is (i). t^4+1 .

Lemma 7.37. There exists an element $w \in W$ such that $f_{wz}(t) = t^4+1$, and w belongs to the Coxeter class of W . Such an element is $c.b = (w_1w_2w_3w_4).(w_2w_3)^2$.

Proof. The companion matrix for $f(t) = t^4+1$ is $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$,

and we want to find a matrix $B \in GL_4(3)$ such that ${}^tB.B = -I_4$. In order to preserve the characteristic polynomial $f(t)$ of C , so that $f_B(t) = f(t)$, we must endeavour to transform C into B as $B = E.C.E^{-1}$, where E is a product of elementary matrices . It is easy to see that

$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ satisfies all the necessary conditions .

Then $B = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} . w$ for some $w \in O_4^+(3)$, so that

$$w = \begin{bmatrix} E^{-1} & 0 \\ 0 & E^{-1} \end{bmatrix} .B \quad = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \quad \in \quad O_4^+(3) \cong W .$$

Furthermore , it is easy to see that $\operatorname{ord}(w) = 12$, so that w belongs to the Coxeter class of W . It can be shown that $w\tau$ is W-conjugate to the element $cb\tau$, where $c = w_1w_2w_3w_4$ belongs to the Coxeter class of W and $b = (w_2w_3)^2$, so that we may take cb to be the required $w \in W$. Since $f_{cb\tau}(t) = t^4+1$, it follows that $\operatorname{ord}(cb\tau) = 8$. $\square$

Lemma 7.38. The class C_{11} has as a representative element $cb\tau$, and $C_{11} \subset W^+\tau$. Also , $\operatorname{ord}(cb\tau) = 8$, $|C_{11}| = 72$ and $f_{11}(t) = t^4+1$. Further , $e_1 = (q^4+1)$.

Proof. Since $cb = (w_1w_2w_3w_4).(w_2w_3)^2$, it follows that $C_{11} \subset W^+\tau$. By Lemma 7.37 it follows that $f_{11}(t) = t^4+1$ and that $\operatorname{ord}(cb\tau) = 8$. The remaining results follow as in Lemmas 7.18 and 7.19 . $\square$

Lemma 7.39. The coset $W^+\tau$ of W^* is the union of the 8 conjugacy classes C_1 , C_5 , C_6 , C_7 , C_8 , C_9 , C_{10} , C_{11} .

Proof. As in Lemma 7.23 , the coset $W^+\tau$ has order 576 , and by the information of Lemmas 7.27 to 7.38 ,

$$\left|C_1 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11}\right| = 72 + 96 + 96 + 12 + 12 + 24 + 192 + 72$$
$$= 576$$

Since all these classes are contained in $W^{+}\tau$, the result follows .

$\square$

We recall from Chapter 6 that the classes of $H^1(\sigma, W)$ are $\{C_i.\tau : C_i \in \mathcal{C}_\tau(W^*)\}$.

Proposition 7.40. The elementary divisors of the maximal tori T_w and the order of the corresponding group W_σ in the groups ${}^2F_4(q^2)$ are as in the following Table 7.3 .

TABLE 7.3.

Class	Representative w	Order of $w\tau$	Order of Class	Elementary Divisors	$\lvert W_\sigma \rvert$
$C_2.\tau$	w_2	8	144	$e_1 = q^4-\sqrt{2}q^3+\sqrt{2}q-1.$	8
$C_3.\tau$	zw_2	8	144	$e_1 = q^4+\sqrt{2}q^3-\sqrt{2}q-1.$	8
$C_4.\tau$	w_1	4	288	$e_1 = q^4-1$.	4
$C_1.\tau$	1	2	72	$e_1 = e_2 = (q^2-1)$.	16
$C_5.\tau$	w_1w_2	24	96	$e_1 = q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1.$	12
$C_6.\tau$	zw_1w_2	24	96	$e_1 = q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1.$	12
$C_7.\tau$	ab	8	12	$e_1 = e_2 = (q^2-\sqrt{2}q+1).$	96
$C_8.\tau$	zab	8	12	$e_1 = e_2 = (q^2+\sqrt{2}q+1)$.	96
$C_9.\tau$	$(ab)^2$	4	24	$e_1 = e_2 = (q^2+1)$.	48
$C_{10}.\tau$	$c(w_2w_3)$	12	192	$e_1 = q^4-q^2+1$.	6
$C_{11}.\tau$	cb	8	72	$e_1 = q^4+1$.	16

Here , $a = (w_1 w_4)$, $b = (w_2 w_3)^2$, $c = w_1 w_2 w_3 w_4$ and $z = (ab)^4$ is the non-trivial central element of W . The representatives are those of the classes $H^1(\sigma, W)$.

Proof. This follows from Lemmas 7.19 - 7.39 , and by Lemma 1.9 . $\square$

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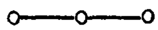
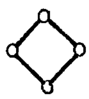
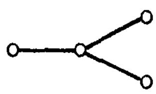
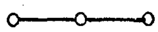
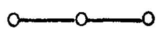
§7.5. The Steinberg groups of type $^3D_4(q)$.

We discuss these groups in this Chapter because of the way in which $W(F_4)$ was constructed from $W(D_4)$. Referring to § 6.1 , we see that K_1 is a finite field of the form $GF(q^3)$, $G_{\pi,K}$ is either of the groups $G_{ad,K}$ or $G_{sc,K}$, and τ is the triality automorphism of the Dynkin diagram of D_4 illustrated in Table 6.1 , which corresponds to the automorphism of $Spin_8(K)$ cyclically permuting z, z' and zz'. This is discussed in §4.2 . Hence , $\tau(\Delta_\pi) = \Delta_\pi$ if and only if π = ad or sc . Furthermore , by Corollary 1.8 , we need only discuss the case when π is the simply connected representation of $\underline{g}$, since the results are identical in the cases π = ad or sc .

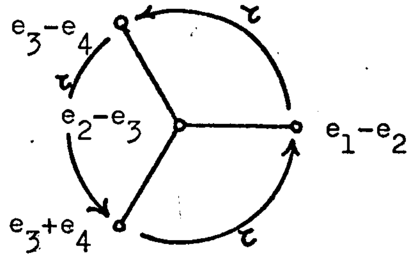
Now Σ_1 is a root system of type D_4 embedded in V , and W_1 is the corresponding Weyl group , as in §7.4 . Furthermore , τ corresponds to the element $y \in D(\Sigma_1)$, and $W_1^* = \langle W_1, \tau \rangle$ is a normal subgroup of W of index 2 , as in the diagram after proposition 7.10 . Since τ is a permutation of Σ_1 , it follows that τ maps any graph Γ_C onto an identical graph , although we have no easy algorithm for deciding how τ acts on $\mathcal{L}(W_1)$ in the case of one graph representing distinct classes . We list the conjugacy classes for W_1 in Table 7.4 [6] , and we notice how there are two graphs which each occur 3 times , suggesting that each triple may be cyclically permuted by τ . That this is so, we prove now .

TABLE 7.4.

Class Number	Φ-type	Γ	Order of Class	Cycle-type	C^{τ}
C_1	ϕ	ϕ	1	$[1\ 1\ 1\ 1]$	C_1
C_2	D_2		6	$[\bar{1}\ 1\ \bar{1}\ \bar{1}]$	C_7
C_3	D_2+D_2		1	$[\bar{1}\ \bar{1}\ \bar{1}\ \bar{1}]$	C_3
C_4	A_1		12	$[2\ 1\ 1]$	C_4
C_5	A_1+D_2		12	$[2\ \bar{1}\ \bar{1}]$	C_5
C_6	D_3		24	$[\bar{2}\ 1\ \bar{1}]$	C_{12}
C_7	$(A_1+A_1)'$		6	$[2\ 2]$	C_8
C_8	$(A_1+A_1)''$		6	$[2\ 2]$	C_2
C_9	$D_4(a_1)$		12	$[\bar{2}\ \bar{2}]$	C_9
C_{10}	A_2		32	$[3\ 1]$	C_{10}
C_{11}	D_4		32	$[\bar{3}\ \bar{1}]$	C_{11}
C_{12}	A_3'		24	$[4]$	C_{13}
C_{13}	A_3''		24	$[4]$	C_6

Lemma 7.40. The automorphism τ of W_1 cyclically permutes the triples (C_2, C_7, C_8) and (C_6, C_{12}, C_{13}) in its action on $\mathcal{C}(W_1)$.

Proof. Now τ acts on Σ_1 in the manner illustrated, so that the matrix of τ with respect to the normal basis



is
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$
So take any element $w \in C_2$, for example

$$w = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \quad \text{Then } w^{\tau} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } w^{\tau^{-1}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then w^{τ} and $w^{\tau^{-1}}$ both have signed cycle type $[22]$. However, they are conjugate only by an element of $W(C_4)$, and hence belong to different classes of W_1, viz. C_7 and C_8.

Similarly for the triple (C_6, C_{12}, C_{13}). □

Now $W_1 = \langle s,t,u,v \rangle$ as in §7.4 and is the semi-direct product of the group $C_2 \times C_2 \times C_2$ by the group $\mathfrak{S}_4$, [4]. Coxeter [9] describes it as the symmetry group of the half-measure polytope $h\gamma_4$. Also, there are three cosets of W_1 in W_1^*, each of which is a union of conjugacy classes of W_1^*, by proposition 2.5, and it is the coset $W\tau$ in which we are interested, by Lemma 6.4.

Proposition 7.41. If $W_1 \cong W(D_4)$, then $|H^1(\sigma, W_1)| = 7$, where σ is the triality automorphism of W_1 .

Proof. In this case , there is only one graph for each class so that if the graph Γ_C corresponding to the class C does not occur elsewhere in the list $\{ \Gamma_C : C \in \mathcal{C}(W) \}$ of Table 7.4 , then τ must fix that class . Hence , the only classes not fixed by τ are those described in Lemma 7.40 , and so there are 7 classes fixed by τ .

Thus , by Proposition 2.5 , since τ is of order 3 ,

$|\mathcal{C}(W_1{}^*)| = 3 \times 7 + \frac{13-7}{3} = 23$ and $|\mathcal{C}_\tau(W_1{}^*)| = 7$.

Now , by Lemma 6.4 , there is a bijection $\psi : H^1(\sigma, W_1) \longrightarrow \mathcal{C}_\tau(W_1{}^*)$, so the result follows .

$\square$

We now prove some more results about $W_1{}^*$.

Proposition 7.42. (i). $(W_1)_\tau = \langle (stu) , v : (stu)^2 = v^2 = (stuv)^6 = 1 \rangle$ is isomorphic to D_{12} , the Dihedral group of order 12 , and contains the central element $z' = (stuv)^3$.

(ii). The commutator subgroups of W_1 and $W_1{}^*$ are identical , and equal to the even subgroup $W_1{}^+$. Also , the quotient group $W_1{}^* / W_1{}^+$ is cyclic of order 6 .

(iii). $|W_1{}^+| = 96$ and z is the central element of $W_1{}^*$.

Proof. The fundamental system Π_1 splits into two τ orbits as in Proposition 7.10 , and (i) follows similarly . In fact $(W_1)_\tau \cong W(G_2)$ and $(\Sigma_1)_\tau$ is a root system of type G_2 . See [20] .

By Coxeter [10] , since all the branches of the graph of Σ_1 have odd numbers attached , it follows that $W_1' = W_1^+$. Now , by the above , $|W_1| = 192$, so $|W_1'| = |W_1^+| = 96$. Then , as in Lemma 7.11 , we can see that $W_1^*/(W_1^*)'$ is isomorphic to C_6 . Hence , $|(W_1^*)'| = 96$. Since $W_1' \leq (W_1^*)'$, it follows that all the groups W_1', W_1^+ and $(W_1^*)'$ are identical

Since $z' = (stuv)^3 \in (W_1)_\tau$, then z' is the unique central element of W_1^*. $\square$

Corollary 7.43. Each coset of W_1^+ in W_1^* is a union of conjugacy classes of W_1^* . In particular , the 7 classes of W_1^* contained in $W_1\tau$ are partitioned into the two cosets $W_1^+\tau$ and $W_1^+s\tau$, each of order 96 .

Proof. This follows from the fact that $W_1^+ = (W_1^*)'$ as above , and by Lemma 7.12 (ii) . $\square$

Proposition 7.44. Let $S = (W_1y \cup W_1y^{-1})$ be the union of the two cosets W_1y and W_1y^{-1} of W_1 in W . Then S is the union of the 7 conjugacy classes of the set $\mathcal{C}^*$ (mentioned in §5.3) which do not meet W_2 .

Proof. Now $|S| = 2|W_1| = 384$, and the union of the conjugacy classes of $\mathcal{C}^*$ has order $(32+96+96+16+32+96+16) = 384$ also .

Furthermore , S is a normal subset of W , since W_1y is a normal subset of W_1^* , by Corollary 7.43 , and since $x.W_1y.x^{-1} = W_1y^{-1}$.

Finally , $S \wedge W_2 = \phi$ by definition of W_2 as $\langle s,t,u,v,x \rangle$. Now , as we have seen in §5.3 , the only normal subsets of W not intersecting W_2 are the subsets of the set $\mathcal{C}^*$. Hence , by considering the size of these , S is just the union of these classes .

$\square$

Hence , the union of the two non-trivial cosets of W_1 in W_1^* is the union of 7 conjugacy classes of W . We see precisely how these classes split into W_1^*-conjugacy classes in the next lemma .

Lemma 7.45. Each of the 7 classes constituting S splits into two W_1^*-conjugacy classes of equal size , one of which lies in W_1y and the other in W_1y^{-1} .

Proof. We recall how W_1y and W_1y^{-1} are dual under conjugation by x . Also , by Corollary 7.43 , W_1y is a union of W_1^*-conjugacy classes , so let C' be one such . Then $x.C'.x^{-1} \subset W_1y^{-1}$ and $|x.C'.x^{-1}| = |C'|$.

Further , $(C' \cup x.C'.x^{-1}) = C$ is a conjugacy class of W , and since it lies in S , it follows that $C \in \mathcal{C}^*$.

$\square$

We retain the notation that if $C_i \in \mathcal{L}^*$, then C_i' is the corresponding W_1^*-conjugacy class contained in $W_1 y$. In fact $i \in \{7,13,16,18,20,24,25\}$ from Table 5.2 .

Lemma 7.46. The coset $W_1\tau$ is the union of the 7 W_1^*-conjugacy classes $\{C_i' : C_i \in \mathcal{L}^*\}$.

Proof. We have seen how $|C_i'| = \frac{1}{2}|C_i|$ and how $|W_1\tau| = \frac{1}{2}|S|$. Since each $C_i' \subset W_1\tau$, it follows that $W_1\tau = \bigcup C_i'$. $\square$

Proposition 7.47. The elementary divisors of the torus T_w of ${}^3D_4(q)$, where $w \in W_1$ is a representative element of the class C_i' , are precisely those of the torus T_w of $F_4(q)$ corresponding to the class $C_i \subset W$.

Proof. As already stated , it follows from Corollary 1.8 that we need only consider the case $G_{\pi,K}$ where π is the adjoint representation or the simply connected representation , since the results are identical in both cases . However , since Δ_{sc} for the group D_4 is identical with Δ_{ad} for the group F_4 when they are embedded in V , (both being the lattice M_3 described in §4.2) , then our obvious choice is $G_{sc,K}$. For then , if w is a representative element of some class $h = \psi^{-1}(C_i')$ of $H^1(\sigma, W_1)$, we need to find the elementary divisors of the matrix $(q.(w\tau)_\Omega - I)$, by Chapter 1 , where $(w\tau)_\Omega$ is the matrix of the action of the element $w\tau \in W_1^*$ with respect to a basis Ω of Δ_{sc} for the group

D_4. However, $w\tau \in C_i$, which is a W-conjugacy class as we have seen in Lemmas 7.45 and 7.46. Hence, $(w\tau)_\Omega$ is the matrix of the action of $w\tau \in W$ with respect to a basis Ω of Δ_{ad} for the group F_4, corresponding to the conjugacy class C_i of W. Hence, the elementary divisors of the matrix $(q.(w\tau)_\Omega - I)$ are precisely those corresponding to the classes $C_i \in \mathcal{C}^*$, which we have already calculated in Table 5.2, and the result follows. $\square$

We conclude this chapter with a table of results for the groups ${}^3D_4(q)$ which lists the elementary divisors of T_w corresponding to the conjugacy classes C'_i of $\mathcal{C}_\tau(W_1^*)$, together with the order of the corresponding group W_σ, using Proposition 1.9.

TABLE 7.5.

Class	Order of class	Elementary divisors	$\lvert W_\sigma \rvert$
C'_{13}	48	$e_1 = (q-1)(q^3+1)$.	4
C'_{16}	48	$e_1 = (q+1)(q^3-1)$.	4
C'_{24}	48	$e_1 = (q^4-q^2+1)$.	4
C'_7	16	$e_1 = (q-1)$, $e_2 = (q^3-1)$.	12
C'_{20}	16	$e_1 = (q+1)$, $e_2 = (q^3+1)$.	12
C'_{18}	8	$e_1 = e_2 = (q^2+q+1)$.	24
C'_{25}	8	$e_1 = e_2 = (q^2-q+1)$.	24

CHAPTER 8. Conclusion.

In concluding this thesis, we consider some work of Springer [17] in which he shows that, under favourable circumstances, a σ-fixed maximal torus T of G and a regular character ϕ of the corresponding finite group T_w determine an irreducible character $\chi_{T,\phi}$ of G_σ. The construction of the character $\chi_{T,\phi}$ uses ideas from the theory of exceptional characters of finite groups. It is believed that a further study of the character theory of the groups G_σ should begin with a closer investigation of the characters $\chi_{T,\phi}$.

Let $G = G_{\pi,K}$, and choose any σ-fixed maximal torus T of G. Then we let T_σ denote its fixed-point group, and let $\hat{T}_\sigma$ denote the character group of T_σ. Let $N = N_G(T)$, so that $W_\sigma = (N / T)_\sigma$ and W_σ acts on T_σ and $\hat{T}_\sigma$. In fact, if σ is the Frobenius endomorphism, it easily follows that $W_\sigma = Z_W(w)$, where T is twisted with respect to w, when σ is a pure field automorphism.

We say that $x \in T_\sigma$ (resp. $\phi \in \hat{T}_\sigma$) is <u>regular</u> if the isotropy group of x (resp. ϕ) in W_σ is reduced to the identity. If $x \in T_\sigma$ is regular, then T is the unique maximal torus containing x, and $Z_G(x)^o = T$.

If $w \in W$, we denote the subgroup of elements of T fixed by w by ${}_wT$. If $w \in W_\sigma$, then ${}_wT$ is σ-fixed. Also, any non-regular element of T_σ is contained in $({}_wT)_\sigma$ for some non-trivial $w \in W_\sigma$.

Let S be a subset of T_σ which satisfies the following condition :-

(I). for all $x \in G_\sigma \setminus N_\sigma$, the intersection $T_\sigma \cap {}^xT_\sigma$ lies in S.

Then S consists of non-regular elements of T_σ.

We say that two characters ϕ, $\phi' \in \hat{T}_\sigma$ are <u>equivalent</u> if they have the same restriction to S. We denote the equivalence class of ϕ by $c(\phi)$, and the set of regular characters of T_σ by $\hat{T}_{\sigma,r}$. If H is a subgroup of G and ϕ is a character of H, then we denote the character induced to G by $i_{H \longrightarrow G}\phi$.

Let $\phi \in \hat{T}_{\sigma,r}$ satisfy the following condition :-

(II). $c = c(\phi)$ meets at least 3 orbits of W_σ in $\hat{T}_\sigma$.

Then it can be shown that there exists a unique sign $\varepsilon_c = \pm 1$ and unique irreducible characters $\chi_{T,\phi}$' (independent of the choice of S) of G_σ such that

$$i_{T \longrightarrow G}(\phi - \phi') = \varepsilon_c \, (\chi_{T,\phi} - \chi_{T,\phi'}) .$$

Also, if $\phi, \phi' \in \hat{T}_{\sigma,r}$ satisfy (II), then $\chi_{T,\phi} = \chi_{T,\phi'}$ if and only if ϕ' lies in the W_σ-orbit of ϕ.

Also, if T and T_1 are two σ-fixed maximal tori which are not conjugate by an element of G_σ, and if $S_1 \subset T_1$ satisfies (I), then it can be shown for regular characters $\phi \in \hat{T}_{\sigma,r}$ and $\phi_1 \in (\hat{T}_1)_{\sigma,r}$ satisfying (II) that $\chi_{T,\phi} \neq \chi_{T_1,\phi_1}$.

In [17], there is also a discussion of asymptotic formulae for the number of regular elements of T_σ and $\hat{T}_\sigma$, and also of the distinct irreducible characters of G_σ constructed in this way .

There is a conjecture of MacDonald [18] concerning the values of the characters $\chi_{T,\phi}$ on the regular semi-simple elements of G_σ , viz.

$$\chi_{T,\phi}(t) = (-1)^{\ell} \sum_{w \in W} \phi({}^{w}t) \quad , \qquad \text{where } t \in T_\sigma \text{ is regular}$$

and the sum is extended over all distinct W-conjugates of t . MacDonald has also conjectured in the case where G_σ is of Chevalley type that the degree of $\chi_{T,\phi}$ should be

$$|T_\sigma|^{-1} \cdot \prod_{i=1}^{\ell} (q^{d_i}-1) \quad , \quad \text{where the } d_i \text{ are the basic}$$

invariants of W , see [20] .

It seems necessary to obtain explicit values of the characters $\chi_{T,\phi}$, and , in a recent paper [15] , Ree and Chang have done this for the groups $G_2(q)$ when $q \neq 2,3$. In fact , $G_2(2)$ is the only group of Chevalley type which does not contain regular elements in the Coxeter tori .

In general , if w belongs to the Coxeter class of W , and h is the Coxeter number of W , then the non-regular elements of T_σ are those of the groups $({}_{w^d}T)_\sigma$, where d is a divisor of h .

If $L(G)$ is the Lie algebra of G , then Chevalley has shown [16] that if G is a semi-simple adjoint group , then $L(G)(K_o)$ contains regular elements , and in fact these elements lie in $L(T)(K_o)$, where T is a Coxeter torus of G . Later , A. Borel and T.A. Springer [Tohoku Math. Journal No.20] showed this to be true for any reductive Group G .

If we consider regular semi-simple elements in the group $G(K_o)$, it is easy to show that, for any maximal torus T of G which is twisted with respect to $w \in W$, regular elements lie in T_σ if and only if $(\sigma^*-I)X$ contains no root of Σ, ie. that $(qw-I)X$ contains no root.(III).

Certainly, to satisfy this condition, w must not lie in any Weyl subgroup W' of rank less than that of W, and we can check that this condition is satisfied in every group $G(K_o)$ except $G_2(2)$. Referring to § 7.3, we know that w must belong to the Coxeter class of W, and a representative of this class is a rotation anticlockwise through 60° about the origin. Then, $(2.w-I).a = 2.(b+2a)-a = 2b+3a$, the highest root of $\Sigma(G_2)$. Hence, the condition (III) is not satisfied, and the Coxeter tori of $G_2(2)$, which are isomorphic to C_3, do not contain regular elements. In fact, in this group, the group $({}_{w^2}T)$, which is a sub-torus corresponding to a long root of Σ, is such that $({}_{w^2}T)_\sigma$ coincides with T_σ. We have already remarked that the groups $({}_{w^d}T)_\sigma$ consist of the non-regular elements of T_σ.

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