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A Parabolic PDE on an evolving curve and surface with finite time singularity

by

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Thesis

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Contents

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Declarations

Chapter 2 is taken from various referenced sources. The material in Section 2.6 is taken from Section 4.1.1 of the author's MSc Thesis (Scott [2011]) and has been edited accordingly. All the other material presented in this thesis is original work of the author, and to the best of his knowledge, new. No material, apart from the edited material of Section 2.6, has been submitted for any other degree at any other University.

Abstract

We consider the heat equation

$$\partial_t^\bullet u + u\nabla_{\mathcal{M}(t)} \cdot v - \Delta_{\mathcal{M}(t)} u = 0$$
$$u(x,0) = u_0 \quad x \in \mathcal{M}(0)$$

on an evolving curve which forms a “kink” in finite time. We describe the behaviour of the solution at the singularity and look to continue the solution past the singularity. We perturb the heat equation and study the effects of a deterministic perturbation and a stochastic perturbation on the solution, before the singularity. We then consider the heat equation on an evolving surface that forms a “cone” singularity in finite time and study the behaviour of the solution at the singularity. We then look to continue the solution past the singularity, in some probabilistic sense. Finally, we consider the heat equation on an evolving curve, where the evolution of the curve is coupled to the solution of the equation on the curve. We prove existence and uniqueness of the solution for small times, before any singularity can occur.

Chapter 1

Introduction

Partial Differential Equations (PDEs) on evolving or stationary surfaces are ubiquitous in the mathematical modelling of real world phenomena. Problems in biology, engineering and image analysis all benefit from the rich mathematical literature on such objects. For example, Neilson et al. [2011] studies chemotaxis in cells and uses a non-linear PDE system to describe the evolution of the quantities on an evolving curve. Moreover, the evolution of the curve is strongly coupled to dynamics of the quantities. Surface dissolution (which has applications in the mining of gold, for example), is modelled by an evolving surface whose evolution is coupled to a non-linear surface PDE (Eilks and Elliott [2008]). In image analysis, motion by mean curvature has been used to describe the graph of a surface which represents an image, to recover the image (Faugeras and Keriven [2002]).

When the surface evolution is a priori known and smooth, meaning that no geometrical singularities nor topological changes occur, the analysis of existence, uniqueness and properties of the solution to the PDE on the surface is standard. (See, for example, Dziuk and Elliott [2007]). Indeed, one may use the method of finite elements to analyse the problem, as in Dziuk and Elliott [2007], or one may map the equation onto a time independent, diffeomorphically equivalent surface. In the case of Stochastic PDEs on an evolving surface, the latter is covered in Scott [2011].

What is not covered in the mathematical literature, to the best of the author's knowledge, is the analysis of a PDE on a surface which undergoes some sort of geometric singularity and the effects this singularity has on the solution of the PDE.

To this end, we consider three problems in this thesis. We give a full description of the results obtained in Section 1.2 and the reader is referred to Chapter 2 for the definition of the differential operators. The numbering of the equations below agrees with the numbering used in the relevant part of the thesis. In the following, it should be noted that $\alpha > 0$.

Problem I: Here we study the heat equation[1]

$$\partial_t^\bullet U + U\nabla_{\mathcal{C}_t} \cdot v - \Delta_{\mathcal{C}_t} U = 0 \tag{2.6.3}$$

with initial condition $U(x,0) := U_0(x)$ on an evolving curve $\mathcal{C}_t = \mathcal{C}_t^\alpha$ given by

$$\mathcal{C}_t^\alpha := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\ y^2 - x^2 = (1-t)^{2\alpha}\}, \quad 0 \leq t \leq 1. \tag{3.1.1}$$

Here, a "kink" forms in finite time ($t = 1$). We analyse the solution at the singularity and look to continue the solution, in some sense, onto

$$\mathcal{C}_t^{\alpha,\text{cont}} := \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 0,\ x^2 - y^2 = (t-1)^{2\alpha}\} \quad 1 \leq t \leq T. \tag{3.1.2}$$

Problem II: We again study the heat equation (2.6.3), however this time we look at the equation on an evolving surface

$$\mathcal{S}_t^\alpha := \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = (1-t)^{2\alpha}\}, \quad 0 \leq t \leq 1 \tag{7.0.1}$$

which forms a "cone" singularity at finite time. We analyse the solution at the singularity and look to continue the solution, in some sense, onto

$$\mathcal{S}_t^{\alpha,\text{cont}} := \{(x,y,z) \in \mathbb{R}^3 \,|\, z^2 - x^2 - y^2 = (t-1)^{2\alpha}\} \quad 1 \leq t \leq T. \tag{9.0.1}$$

Problem III: Finally, we return to studying (2.6.3) on an evolving curve

$$\mathcal{C}_t := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\ y^2 - x^2 = f(t)\}, \quad 0 \leq t \leq T, \tag{10.1.1}$$

for some T small enough, where the evolution of the curve is coupled to the solution of (2.6.3); that is, f is some functional of the solution of (2.6.3). Here we analyse the existence and uniqueness of the

[1] One should note that the heat equation is taken in the evolving surface sense (Dziuk and Elliott [2007]). In a differential geometrical sense, the heat operator is taken as $\partial_t - \Delta_{\mathcal{M}(t)}$, where $\mathcal{M}(t)$ is the evolving manifold and $\Delta_{\mathcal{M}(t)}$ is the Laplace-Beltrami operator (Lee [1997]).

solution for small times away from any possible singularity.

In Problem III, the resulting PDE is a non-linear parabolic PDE. Since we only seek short time existence, the equation is not singular and so we use standard analytical tools to tackle the question of existence and uniquess to the problem.

However, in Problems I and II, the resulting PDE is a linear singular parabolic PDE. The behaviour of the solution at the singularity interestingly depends on the parameter α introduced in the definition of the curve $\mathcal{C}_t^\alpha$ and the surface $\mathcal{S}_t^\alpha$. The parameter models how fast one enters (and so leaves) the singularity and the analytical techniques used depend heavily on α. One should note that the choice of PDE on the curve and surface is natural, for it is derived from a conservation law. Indeed, (2.6.3) models surface concentration of some physical quantity which is conserved and is therefore of physical importance.

Thus, the problems in this thesis sit in the area of linear singular parabolic PDEs, but one should consider the problems as a subclass of surface PDEs in which the underlying surface singularity gives rise to singular coefficients in the PDE, which affect the solution of the PDE. In the following, we review the mathematical literature on linear singular parabolic PDEs. In Section 1.2 we outline the results obtained in each of the different problems.

1.1 Literature Review

Although, as far as the author can find, the mathematical literature is devoid of the analysis of surface PDEs in which the underlying surface singularity (through topology change for example) gives rise to singular coefficients, when one parameterises the surface and transforms the problem into a PDE on a fixed state space, the mathematical literature is rich with such problems. However, as we shall see, none of the problems considered below fit perfectly into the problems that we consider in this thesis. We will not review the literature on the non–linear equations, but mention there has been and continues to be extensive research in this area. For example, an important set of papers on the physically relevant mean curvature flow includes Evans and Spruck [1991, 1992a,b, 1995].

When one transforms (2.6.3) onto a fixed phase–space $\mathbb{R} \times [0, 1)$ using arc–length parameterisation, one derives the following equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} \left(G(l(1-t)^{-\alpha}) u \right) \\ u(l, 0) &= U_0(X(l, 0)). \end{aligned} \tag{3.2.3}$$

The full derivation is given in Chapter 3 together with all the properties of the coefficients. Here, l is the arclength parameter and X is the arclength parameterisation. The function G is given in Chapter 3, equation (3.2.4), but all the reader needs to know at this point is that it is bounded. Using the chain rule, one sees that some of the coefficients of the PDE become singular as $t \to 1^-$, even if $\alpha > 1$. The literature available on analysing the equation at a singular finite time is, in the author's opinion, scarce. Most of the literature considers equations that are either immediately singular or are immediately degenerate. Indeed, the distinction between "singular" and "degenerate" is blurred and some do not specify initial conditions where the singularity occurs.

In the following, we will consider various papers on degenerate and/or singular PDE. The following is not an exhaustive literature review and the reader is encouraged to refer to the references given in the cited papers. The following is a snapshot of the current mathematical standing on the theory of linear singular and/or degenerate parabolic second order PDE in order to state where the results of this thesis sits in the mathematical literature.

We begin our review with Baiocchi [1967]. Here the following equation is studied

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bu) + Au(t) = f(t)$$

in an appropriate Hilbert space with initial condition $Bu(0) = u_0$. Existence and uniqueness of the solution is established, but this equation is degenerate at $t = 0$, not singular. This is because B^{-1} might not exist and at $t = 0$, one only has knowledge of $Bu(0)$, not $u(0)$. A similar equation posed in a Banach space X is studied in Favini [1974]

$$Bu'(t) + Au(t) = f(t)$$

with initial condition u_0 and we require that $\lim_{t\to 0^+} \|u(t) - u_0\|_Y = 0$ where Y is Banach space different from X. Again, existence and uniqueness for the equation are given. On the other hand, Sobolevski [1971] studies

$$a(t)v'(t) + Av(t) = f(t)$$

in a suitable Banach space where a is scalar valued, satisfies $a(0) = 0$ and

$$\int_0^1 a(s)^{-1}\,\mathrm{d}s = +\infty.$$

Here A is "nice" in the sense that it generates the analytic semigroup $\exp(-tA)$.

An example of such an A would be $-\Delta$, the Laplace operator. No initial condition is given and on dividing the studied equation by $a(\cdot)$ one has a singular equation. Having no initial condition raises the question of uniqueness. However, this is addressed and it is shown that under some appropriate growth assumption on a at 0, that there is a unique solution for given f in suitable Hölder spaces.

A more explicit example of an a is given in Lewis and Parenti [1982], with $a(t) = t$, but A is still time independent and the problem is set in a Hilbert space. In this paper, flat solutions are sought; which asks that all the time derivatives of the solution at $t = 0$ vanish. The reader is referred to this paper and references within. A generalisation of Lewis and Parenti [1982] is given in Dore and Venni [1985] where the same equation is considered, but now in a Banach space setting. Here existence, uniqueness as well as regularity properties are given.

The reader will notice that it all the cases above, the operator A is time–independent. However, it turns out that Problem I, II and III all have time–dependent operators. For time–dependent operators, arguably the most influential early paper on an abstract approach to singular parabolic equations comes from Lions and Raviart [1967], where the now classical approach to linear parabolic PDE is adapted to take care of time singularities in the equation at the initial time. The problem is set in an abstract Hilbert space triple $V \subset H \subset V^*$ and a bilinear form $a(t; u, v) := (A(t)u, v)$ is given. Here $(\cdot, \cdot)$ is the Hilbert space H inner product and if $|\cdot|$ denotes the H norm and $\|\cdot\|$ denotes the V norm, it is required that there exists $\lambda(t) \in L^1(0, T; \mathbb{R})$ such that

$$\operatorname{Re}[a(t; v, v)] + \lambda(t)|v|^2 \geq \alpha \|v\|^2$$

where $\alpha > 0$ and this holds for every $v \in V$. One also asks that there exists $\theta \in (0, 1)$ such that $u(t) \mapsto A(t)u(t), A^*(t)u(t)$ is continuous as a map from $L^2(0, T; V) \cap L^\infty(0, T; H) \to L^2(0, T; V^*) + L^{\frac{2}{1+\theta}}(0, T; V^{-1+\theta})$ where $L^2(0, T; V^*) + L^{\frac{2}{1+\theta}}(0, T; V^{-1+\theta})$ is given from the topology of the dual of

$$L^2(0, T; V) + L^{\frac{2}{1-\theta}}(0, T; V^{1-\theta}).$$

Under the above assumptions, Lions and Raviart [1967] establishes the existence and uniqueness of $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ such that

$$u'(t) + A(t)u = f(t)$$

subject to the initial condition $u(0) = u_0$ where u_0 is a given function in H and

$f \in L^2(0,T;V^*)$ is also given. However, this paper does not cover the case where a singularity forms in finite time.

Other papers that consider degenerate time–dependent operators include Friedman and Schuss [1971] where they consider

$$c_1(t)\frac{\mathrm{d}u}{\mathrm{d}t} + c_2(t)A(t)u = f(t) \tag{1.1.1}$$

in a Hilbert space X where $c_1(t) \geq 0$, $c_2(t) \geq 0$ are scalar functions and $c_1(t)+c_2(t) > 0$. It is assumed that $A(t)$ is an unbounded linear operator satisfying the standard assumptions which ensure that the Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(t)u = f(t)$$
$$u(0) = u_0$$

has a unique solution. Again, no initial conditions are explicitly given on the equation involving the c_i, but u_0 does feature in their definition of a weak solution. The paper proves existence, uniqueness and differentiability theorems for solutions to (1.1.1), under the additional restriction that $c_2(t) \equiv 1$ and that $c_1(t), f(t)$ and $A(t)$ have k continuous derivatives. Further, a coercivity result similar to Lions and Raviart [1967] is required along with the assumption that there exist μ_k such that

$$\|[A(t) - \lambda I]^{-1}\|_{\mathcal{L}(X;X)} \leq \frac{\text{const}}{1+|\lambda|}$$

for every $\operatorname{Re}\lambda \leq \mu_k$. With this, it is proven that for every $\varepsilon > 0$ the weak solution belongs to $H^k([\varepsilon,T];X)$. In Schuss [1972], they remove these restrictions and prove that the solution u is infinitely differentiable when $c_1(t)$, $f(t)$ and $A(t)$ are C^∞. The case of $c_1(0) = 0$ and $c_1'(0) > 0$ is considered, giving continuity of the solution u at $t = 0$ along with a formula for the solution. In Baiocchi and Baouendi [1977], the special case of $c_1(t) = t$ is studied in a Hilbert space setting, with applications to equations such as

$$\frac{\partial u}{\partial t} + \lambda(x,t)t^{-1}u - \beta(t)\Delta_x u = f$$

on $\Omega \times [0,T]$ where λ is a bounded function, β is strictly positive on $(0,T]$ but may vanish or become infinite at $t = 0$ and Ω is a smooth compact Riemannian manifold without boundary. They prove, amongst other results, that there exists a unique solution $u \in C^\infty(\Omega \times [0,T])$ to

$$\begin{aligned} \frac{\partial u}{\partial t} + \lambda(x,t)t^{-1}u - \Delta_x u = g \\ u(x,0) = u_0(x) \end{aligned}$$

where $g \in C^\infty(\Omega \times [0,T])$ and $u_0 \in C^\infty$ are given, if and only if $\lambda(x,0)u_0(x) \equiv 0$. This demonstrates that u_0 can be arbitrarily chosen if $\lambda(x,0) \equiv 0$, while u_0 must vanish where $\lambda(x,0)$ does not. Considering (1.1.1), a specific case of $c_1(t) = t$ and $c_2(t) = 1$ is considered in Dore and Guidetti [1986] and Guidetti [1987] in the setting of Banach spaces, however Guidetti [1987] does not require that $A(t)$ are densely defined. Both papers establish the existence of a unique solution in a suitable Banach space, with no initial conditions specified in Dore and Venni [1985]. In Guidetti [1987] they show the existence and uniqueness in a space that forces all $k-1 \in \mathbb{N}$ derivatives of the solution to vanish at 0. Some additional assumptions on the spectrum of the operator are given. The reader is directed to Guidetti [1987], page 493 for the exact conditions. We thus see that the singularity in both cases in fact governs what initial data one can take. We will see in Chapter 5 of this thesis that the "natural" choice for the initial data, is attained in the classical and mild sense, again depending on the singularity.

In Favini [1985], the above problem of Friedman and Schuss [1971] and Schuss [1972] is considered, but now in a Banach space setting. Here they consider

$$\begin{aligned} (A_1(t)u(t))' + A_0(t)u(t) = f(t) \\ \lim_{t\to 0^+} A_1(t)u(t) = 0. \end{aligned}$$

which generalises Friedman and Schuss [1971] and Schuss [1972] not only because Favini [1985] considers a Banach space setting, but A_1 is a suitable closed linear operator between Banach spaces; not just a multiplication operator as in Friedman and Schuss [1971] and Schuss [1972]. Existence and uniqueness of the solution is given, along with (amongst other results) an application to initial-boundary value problems.

Moving slightly away from the abstract equations, Favini et al. [2005] and Favini et al. [2008] consider an L^p approach to singular linear parabolic equations in bounded domains. Here they consider a bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary of class C^2 and the following equation

$$
\begin{aligned}
D_t[m(x)u(x,t)] + A(x,D_x)u(x,t) = f(x,t) \quad (x,t) \in \Omega \times [0,T] \\
u(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,T] \\
m(x)u(x,t) \to m(x)u_0(x) \quad \text{a.e } x \in \Omega, \text{ as } t \to 0^+.
\end{aligned}
$$

Here

$$
A(x,D_x) = -\sum_{i,j=1}^{n} D_{x_j}(a_{ij}(x)D_{x_i}) + \sum_{j=1}^{n} a_j(x)D_{x_j} + a_0(x).
$$

In Favini et al. [2005], $a_j \equiv 0$ and so Favini et al. [2008] is a generalisation of Favini et al. [2005] to include lower order terms. It is assumed that for each $x \in \bar{\Omega}$ we have $(a_{ij}(x))$ is a positive definite matrix. The existence and uniqueness of the solution are given, however, A is not time dependent and Ω is bounded and so the paper does not sit within our regime.

For a problem considering time depending coefficients, we consider Kutev et al. [2010]. Here they consider the solvability of

$$
Lu = u_t - \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + b^i(x,t)u_{x_i} = 0
$$

in $Q := \Omega \times (-t_1, t_2)$ with $\Omega \subset \mathbb{R}^n$ bounded and $t_i > 0$. They ask for $u(x,t) = \varphi(x,t)$ on the parabolic boundary of Q. They prove the existence and uniqueness of a classical solution, where it is assumed that the a^{ij} satisfy

$$
a(t)|\xi|^2 \leq a^{ij}(x,t)\xi_i\xi_j \leq \Lambda a(t)|\xi|^2
$$

for all (x,t) in a larger cylinder Q', for every $\xi \in \mathbb{R}^n$ and $a(t) \geq 0$ with $a \in C([-t_1,t_2];\mathbb{R})$. Here $\Lambda > 0$ is a positive constant. Their model equation is

$$
u_t - t^k u_{xx} + a_1 t^m u_x = 0
$$

in $(-L,L) \times (-t_1,t_2)$ with the assumption that $k \leq 2m+1$. Indeed, for $k > 2m+1$, the equation is not locally solvable for some boundary data, even in the distributional sense. This equation does not quite fit our problems as it does not have a creation term; the coefficient of u is 0. Also, a bounded domain is also considered, whereas the problems in this thesis are considered in an unbounded domain.

The abstract settings all consider singularities that arise from time. However, in Chapter 9 of this thesis we consider the following equation

$$\begin{aligned}\frac{\partial J}{\partial t} &= \frac{\partial^2 J}{\partial l^2} - \frac{\partial}{\partial l}\left(\left(t^{-\alpha}\frac{\coth g}{\sqrt{\cosh 2g}} + \alpha t^{\alpha-1}G\right)J\right)\\ J(l,0) &= \frac{\sqrt{2}}{2}|l|u_0(l) =: J_*(l).\end{aligned} \tag{9.1.4}$$

In Remark 9.1.3 we show that the singularities that we consider are not only in time, but also in the spacial variable. Such examples of a spacial singularity include the Bessel operator

$$A := \frac{\partial^2}{\partial z^2} + \frac{2\nu+1}{z}\frac{\partial}{\partial z}$$

with $\nu > -\frac{1}{2}$. Operators such as these are considered in a series of papers Gorodetskii and Martynyuk [2010a,b] and Gorodetskii and Tupkalo [2011]. Since we do not use classical PDE techniques in Chapter 9 we refer the reader to the relevant references in the mentioned papers and others where Bessel functions are considered.

Interestingly, none of the above papers allow singularities to form in a finite time $s \in (0,T)$ or at $t = T$. Formulation of a singularity at time $t = T$ is precisely the case of Chapter 3 and Chapter 8 of this thesis. However, in Bernardi [1981] such problems are considered. Here, the setting is the Schrödinger equation

$$u'(t) + iA(t)u(t) = f(t)$$

where $A(t)$ is a linear operator between suitable Hilbert spaces. They consider a complex Hilbert space triple $V \subset H \subset V^*$, ask that $V \subset H$ is dense and $A(t) \in W^{1,1}_{\text{loc}}((0,T];\mathcal{L}(V;V^*))$ where $\mathcal{L}(V;V^*)$ is the vector space of all continuous linear mappings from V to V^*. With some coercivity assumptions on $A(t)$, boundedness of the bilinear form $(A(t)u,v)$ with the bound depending on t and boundedness of the bilinear form $(A'(t)u,v)$ with the bound depending on t they show that $\lim_{t\to 0^+} u(t) = 0$ strongly in V^*, but also that this limit holds strongly in H and V. It is discussed how to modify the conditions to consider problems with singularities in an interior point of $(0,T)$, or even at T. However, all our equations in this thesis are real and we go further than the abstract setting in the sense that we are able to describe exactly how the solution behaves near the singularity, when a singularity occurs.

With regards to the behaviour of the solution to (3.2.3) at the singularity as $t \to 1^-$, one may rescale the equation using $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$ if $\alpha \le \frac{1}{2}$ and taking $y = l(1-t)^{-\frac{1}{2}}$ if $\alpha > \frac{1}{2}$. This corresponds to considering stabilisation of solutions of the Cauchy problem, as we are interested in the behaviour of the solution as $\tau \to +\infty$. Such problem is considered in Eidelman et al. [2009] by means of convergence of solutions to the Cauchy problem

$$\frac{\partial u}{\partial t} = \sum_{i.j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N$$

with $N \geq 2$. It is assumed that there exists $\Lambda_1, \Lambda_2 > 0$ such that

$$\Lambda_1(1+|x|)^\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \Lambda_2(1+|x|)^\alpha|\xi|^2$$

with $-N < \alpha < 2$. With the assumption that the initial data is bounded, they prove that $u(x,t) \to A$ as $t \to \infty$ uniformly in $x \in G$ if and only if

$$\lim_{\rho\to\infty} \frac{1}{|B_\rho(0)|}\int_{B_\rho(0)} u_0(y)\,\mathrm{d}y = A.$$

Here G is any compact subset of $\mathbb{R}^N$. The following condition on a_{ij} is needed: for any $i, j = 1, \cdots, N$

$$\rho^{-N-2}\int_0^{\rho^{2-\alpha}}\int_{|x|<\rho} |x|^{-\alpha}\left(a_{ij}(x,t) - \delta^i_j|x|^\alpha\right)^2 \mathrm{d}x\,\mathrm{d}t \to 0$$

as $\rho \to \infty$. Here δ^i_j is the Kronecker delta. If we consider this equation as a rescaling of another equation that forms a singularity in finite time, we see that the behaviour of the initial condition affects the behaviour of the solution at the singular time. However, the work of Eidelman et al. [2009] does not quite fit with our problems, as there are no creation terms; again the coefficient of u is zero, and their problem is set in $\mathbb{R}^N$ with $N \geq 2$; our problems are all one dimensional. However, Eidelman et al. [2009] clearly demonstrates the dependence of the initial data on the long term solution of the equation.

As a concluding note, we mention that papers do exist that consider first order equations, such as Alinhac [1974] and the reader is directed to the references therein. Furthermore, in the weak setting to these singular problems, analogous to the non-singular case, one can consider variational inequalities and perform an analysis using weighted spaces (see Luterotti [1993] and the references therein).

The results of this thesis can be seen as a generalisation of the above, in the sense that we demonstrate how the solution behaves at the singularity and in a neighbourhood of the singularity. Further, we show that for a "natural" choice of the initial condition for the continued PDE, that in the case of the curve, we have classical, weak or mild attainment of the data (according to the parameter α introduced in the definition of $\mathcal{C}_t^\alpha$ and $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ in (3.1.1) and (3.1.2) respectively). We further give a probabilistic interpretation of the continued problem for the case of the surface in Chapter 9 which is not considered in the PDE literature. Further, the techniques employed in this thesis range from probabilistic methods, that yield deep connections between PDE theory and stochastic analysis and stochastic processes, through to the classical use of the Banach fixed point argument and classical techniques in PDE theory. Such an approach to studying these problems in this context has not, as far as the author is aware, been used before. They yield easy to analyse representation formulas in the pointwise sense for the solution to PDEs and together with the analysis of suitable stochastic processes, the methods yield qualitative results about classical solutions to PDEs.

1.2 Outline and results of the thesis

This thesis is organised in the following way. In Chapter 2 we introduce the background material. Here we start with definitions and results from calculus and progress onto the basic theory of classical partial differential equations. Since we will use probabilistic techniques in this thesis, we introduce stochastic differential equations along with some basic existence and uniqueness results and some properties of Itō diffusions. This allows us to introduce stochastic representation of solutions to partial differential equations, where the key results and ideas are given in order to analyse the problems in this thesis. We introduce the Fokker–Planck equation which will enable us to show existence of solutions to a certain class of PDEs. We conclude this chapter by giving a self–contained introduction to partial differential equations on evolving surfaces; deriving the equation of central study

$$\begin{aligned} \partial_t^\bullet u + u\nabla_{\mathcal{M}(t)} \cdot v - \Delta_{\mathcal{M}(t)} u = 0 \\ u(x,0) = u_0 \quad x \in \mathcal{M}(0). \end{aligned} \tag{2.6.3}$$

In Chapter 3 we define the hyperbola

$$\mathcal{C}_t^\alpha := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\ y^2 - x^2 = (1-t)^{2\alpha}\}, \quad 0 \leq t \leq 1 \tag{3.1.1}$$

on which equation (2.6.3) is posed. We formulate the problem in an arc–length parameter l that sees us considering the problem as a PDE on $\mathbb{R} \times [0, 1)$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u\right) \\ u(l,0) &= U_0(X(l,0)). \end{aligned} \tag{3.2.3}$$

We do this in a way so that $l = 0$ and $t = 1$ is where the singularity occurs. We also formulate the problem using so–called flow lines, which will enable easier analysis of the problem in certain regimes of α. We conclude the chapter by further transforming (3.2.3) with the use of so–called self–similar coordinates. Here we take $y := l(1-t)^{-\alpha}$ and $\tau := -\log(1-t)$. We analyse the scaling properties of the resulting PDE and conclude that there are three regimes of interest: $\alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha > \frac{1}{2}$. We refer to the regimes as *subcritical*, *critical* and *supercritical*, respectively. The methods of analysis vary depending on the value of α.

In Chapter 4 we seek to analyse the effects of the geometric singularity of (3.1.1) on the solution to (2.6.3). We achieve this by means of analysing the solution to (3.2.3) as $t \to 1^-$ for $l = 0$ and l close to zero. We first consider $\alpha < \frac{1}{2}$ in Section 4.1 and prove in Theorem 4.1.1 that the solution, u, to (3.2.3) is not only uniformly bounded, but in fact $u(0,t) \longrightarrow 0$ as $t \to 1^-$. We achieve this by using a probabilistic technique which yields a stochastic representation formula for the solution, when (3.2.3) is transformed using the self–similar coordinates. In Theorem 4.1.3 we describe how the solution u behaves in a neighbourhood of $l = 0$ and $t = 1$ by using the flow line formulation and guessing, using a heuristic asymptotic analysis method, as to how the solution should behave near the singularity. We then use a probabilistic technique to give the result.

In Section 4.2 we consider $\alpha = \frac{1}{2}$ and note that some of the techniques of Section 4.1 fail for $\alpha = \frac{1}{2}$. We thus employ a functional analytic technique to gain pointwise bounds of the solution in self–similar coordinates (Theorem 4.2.3). From this, we deduce in Theorem 4.2.4 that the solution u to (3.2.3) vanishes at the singularity at an algebraically fast rate and (from Remark 4.2.5) that the solution is uniformly bounded. As regards the behaviour of u in a neighbourhood of the singularity, we prove a partial result in Theorem 4.2.6, improving this via Theorem 4.2.10 by utilising sub and super–solutions and the Feynman–Kac formula.

We finally consider $\alpha > \frac{1}{2}$ in Section 4.3. The analysis of this section is mainly classical and we consider the problem as a perturbation of the heat equation. Using a generalised Gronwall inequality (Lemma 4.3.1) we prove in Theorem 4.3.3 that the solution u to (3.2.3) is uniformly bounded in l and t. We achieve this by using the variation of constants formula and treating (3.2.3) as a perturbation of the heat equation. We note that this technique fails if $\alpha \leq \frac{1}{2}$. Interestingly, unlike the case of $\alpha \leq \frac{1}{2}$, we see that the solution need *not* vanish at the singularity. This is proven in Theorem 4.3.6 by means of Lemma 4.3.5 where a probabilistic technique is used. Indeed, we remark that for initial data $u_0 \equiv 1$ the solution u does not vanish at the singularity (Remark 4.3.7).

In Chapter 5 we consider continuing the solution onto

$$\mathcal{C}_t^{\alpha,\mathrm{cont}} = \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 0,\, x^2 - y^2 = (t-1)^{2\alpha}\}, \quad 1 \leq t \leq T, \tag{5.0.1}$$

by means of

$$\begin{aligned} \partial_t^\bullet V + V\nabla_\Gamma \cdot v - \Delta_\Gamma V = 0 \quad x \in \Gamma := \mathcal{C}_t^{\alpha,\mathrm{cont}} \\ V(x,1) = U(Bx,1) \quad x \in \mathcal{C}_1^{\alpha,\mathrm{cont}}. \end{aligned} \tag{5.0.2}$$

Here, $B : \mathcal{C}_1^{\alpha,\mathrm{cont}} \to \mathcal{C}_1^\alpha$ is a linear map and we derive the equation that the solution must satisfy on $\mathbb{R} \times (1, T]$ and rescale time $t \mapsto t-1$ giving the following equation posed on $\mathbb{R} \times (0, T]$:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial l^2} - \alpha t^{\alpha-1} \frac{\partial}{\partial l}\left(G(lt^{-\alpha})v\right) \quad (l,t) \in \mathbb{R} \times (0,T) \\ v(l,0) &= v_0(l) \quad l \in \mathbb{R}. \end{aligned} \tag{5.0.3}$$

We see that the equation is now initially singular, as some of the coefficients are singular at $t = 0$. We must now show that a solution exists and attains the initial data in some sense. The initial data is taken as the solution u to (3.2.3) for $t = 1$ chosen up to suitable subsequence if needed. We first consider $\alpha < \frac{1}{2}$ in Section 5.1 where we consider (5.0.3) transformed into flow–variables. We make an ansatz for the solution and observe that the resulting equation is a Fokker–Planck equation. Thus, using probabilistic techniques and transforming the result back to the arc–length coordinates, we show that there exists a classical solution to (5.0.3) (Theorem 5.1.2). Finally, in Theorem 5.1.5 we show that the initial data is attained pointwise using a stochastic representation formula by means of the Feynman–Kac formula.

For the case of $\alpha = \frac{1}{2}$, covered in Section 5.2, we observe that (5.0.3) is also[2] of Fokker–Planck type. We thus show that there exists a unique strong adapted solution to the corresponding stochastic differential equation in Theorem 5.2.1 which allows us to conclude that there is a classical solution to (5.0.3) for $t > 0$ (Remark 5.2.2). Bounds on the trajectory of the solution to the corresponding SDE are proven in Proposition 5.2.3 which yield the weak attainment of the initial data

[2]In fact this is true for any $\alpha > 0$.

(Remark 5.2.5).

To conclude Chapter 5 we finally consider $\alpha > \frac{1}{2}$ in Section 5.3. Here, the techniques of the previous sections fail and indeed the techniques we employ in Section 5.3 fail for the cases of $\alpha \leq \frac{1}{2}$. We treat (5.0.3) as a perturbation of the heat equation and prove in Theorem 5.3.1 that there exists a unique solution to (5.0.3) in the mild sense, which attains the initial data in the sense that

$$\lim_{t \to 0^+} |v(l,t) - v_0(l)| = 0,$$

for every $l \in \mathbb{R}$. We achieve this by selecting the correct Banach space X and defining a contractive linear map $\mathcal{F} : X \to X$. The main tool which is used is the Banach fixed point theorem.

In Chapter 6, we return our attention to (3.2.3) for $\alpha < \frac{1}{2}$, with deterministic and stochastic forcing and ask whether the same results, suitably adapted, hold as in Chapter 4. When perturbing, we use a two–parameter family semigroup to solve the equation. Thus, we start this chapter by considering (3.2.3) started at a time s with rough initial conditions, $f(\cdot, s)$, where $s \in (0,1)$ and run the equation for $s \leq t < 1$. We assume that $f(\cdot, s) \in L^\infty(\mathbb{R})$ for every $s \in (0,1)$ and prove boundedness of the solution u in Theorem 6.1.1. We discuss the challenges of taking $f(\cdot, s) \in L^q(\mathbb{R})$ where $q \neq \infty$ in Remark 6.1.3.

In Section 6.2 we consider perturbing (3.2.3) by studying

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u\right) + f \\ u(l,0) &= u_0(l), \end{aligned} \tag{6.2.1}$$

where we take $f \in C(0,1; L^\infty(\mathbb{R}))$. In Theorem 6.2.3 we prove that the solution has all the properties of the non–perturbed case. In Remark 6.2.5 we state that the same results would also hold if $f \in L^1(0,1; L^\infty(\mathbb{R}))$ and $(1-s)^{-\alpha}\|f(\cdot,s)\|_{L^\infty} \in L^1(0,1;\mathbb{R})$.

We conclude Chapter 6 by stochastically perturbing (3.2.3) in Section 6.3 and so considering a stochastic partial differential equation. For reasons outlined in Section 6.3.1 we only consider a white in time–constant in space perturbation, a coloured in space–white in time perturbation, and space–time white noise with finite rank covariance operator perturbation. The results are given in Theorem 6.3.3 which yield the analogous results to the unperturbed case. However, since we are stochastically perturbing the problem, the results are given in terms of the expectation of the solution squared; pointwise estimates are not available.

We now consider the problem of analysing the solution to (2.6.3) on a surface that forms a singularity in finite time, with Problem II. In Chapter 7 we consider (2.6.3) on the surface[3] $\mathcal{S}_t^\alpha$ as given by

$$\mathcal{S}_t^\alpha := \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = (1-t)^{2\alpha}\} \tag{7.0.1}$$

and formulate the central equation of study for problem II

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{\sqrt{|h|}} \nabla \cdot \left(\sqrt{|h|} h^{-1} \nabla u \right) + \alpha (1-t)^{\alpha-1} \frac{\partial}{\partial l} \left(G(l(1-t)^{-\alpha}) u \right) \\ &\quad + \alpha(1-t)^{-1} \operatorname{sech} 2g(l(1-t)^{-\alpha}) u \\ u(l,0,t) &= u(l,2\pi,t) \\ u(l,\theta,0) &= U_0(X(l,\theta,0)) =: u_0(l,\theta).\end{aligned} \tag{7.1.5}$$

However, we observe that the integral of the solution u to (7.1.5) is not conserved in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{K}} u(l,\theta,t) \,\mathrm{d}l \mathrm{d}\theta \neq 0,$$

where $\mathbb{K} := \mathbb{R} \times [0, 2\pi)$. On the other hand, by Proposition 7.1.6, we see that the integral of $J := u\sqrt{|h|}$ *is* conserved. Here $\sqrt{|h|}$ is the change of area measure from $\mathcal{S}_t^\alpha$ onto $\mathbb{R}^2$ and we call J the density. We thus consider the equation J solves

$$\begin{aligned}\frac{\partial J}{\partial t} &= \nabla \cdot \left(\sqrt{|h|} h^{-1} \nabla \left(\frac{J}{\sqrt{|h|}} \right) \right) + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (GJ) \\ J(l,0,t) &= J(l,2\pi,t) \\ J(l,\theta,0) &= u_0(l,\theta) \cosh g(l)\end{aligned} \tag{7.1.6}$$

and take the simplifying assumption that the initial condition is independent of θ. Our central equations of study are thus

$$\begin{aligned}\frac{\partial J}{\partial t} &= \frac{\partial^2 J}{\partial l^2} - (1-t)^{-\alpha} \frac{\partial}{\partial l} \left(\frac{\tanh g}{\sqrt{\cosh 2g}} J \right) + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (GJ) \\ J(l,0) &= u_0(l) \cosh g(l)\end{aligned} \tag{7.1.7}$$

for the density J and

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + (1-t)^{-\alpha} \frac{\tanh g}{\sqrt{\cosh 2g}} \frac{\partial u}{\partial l} + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (Gu) + \alpha(1-t)^{-1} \operatorname{sech} 2g \, u \\ u(l,0) &= u_0(l)\end{aligned} \tag{7.1.8}$$

[3] which is a hyperboloid of one sheet

for u. We conclude this chapter by considering, in Section 7.2, the scaling properties of (7.1.7) and (7.1.8) for a self–similar coordinate change of $y := l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$. This scaling is well defined if $\alpha \leq \frac{1}{2}$, but becomes unusable if $\alpha > \frac{1}{2}$. In this case, we take $y := l(1-t)^{-\alpha}$ but $\tau := (1-t)^{1-2\alpha}$. We thus have the same subcritical, critical and supercritical cases of α as with the curve.

Our analysis of problem II is given in Chapter 8 and we first consider $\alpha < \frac{1}{2}$ in Section 8.1. In Theorem 8.1.1 we prove the uniform boundedness and vanishing of the solution, J, to (7.1.7). We transform the equation using the self–similar coordinates and use a probabilistic technique that is similar to the approach of the curve. How J behaves near the singularity is given in Theorem 8.1.4 where we again guess an ansatz for the solution near to the singularity and use probabilistic techniques to give the result. The method is very similar to the case of the curve. With regards to the solution u to (7.1.8), we show that u is uniformly bounded as a corollary to the above results for J in Corollary 8.1.5. We finally show, in Theorem 8.1.6 via a probabilistic technique, that the solution u to (7.1.8) need not vanish. We remark that this is at odds with the case of the curve for $\alpha < \frac{1}{2}$.

For the case of $\alpha = \frac{1}{2}$, we show that the solution u to (7.1.8) is bounded in a time–dependent decreasing neighbourhood of $l = 0$, we also show that u is bounded above by a certain function of $|l|$ for $|l| > y_0\sqrt{1-t}$ and we conjecture that this function is bounded. Furthermore, we prove a power law for J, analogous to that of Section 4.2 and show that J vanishes at the singularity. The results are given in Theorem 8.2.1 where we use a perturbation argument for the equation of u to prove the result along with functional analytical and comparison principle techniques. The boundedness result of u is not as strong as in the case of $\alpha < \frac{1}{2}$ and we are unable to prove the uniform boundedness of J.

The case of $\alpha > \frac{1}{2}$ becomes even more challenging, where we can only show the boundedness of the solution J to (7.1.7) in a time–dependent decreasing neighbourhood of $l = 0$ and that J vanishes at the singularity (Theorem 8.3.2). We can show that the solution u to (7.1.8) has a worst–case blow up estimate. That is, we show in Lemma 8.3.1 if the solution u is singular at $t = 1$, then the solution is bounded above by some negative power of $(1-t)$. The challenges that prevent more from being proven are given at the start of Section 8.3 and we conclude this chapter with an heuristic argument for the boundedness of J.

In Chapter 9 we look into continuing the solution obtained in Chapter 8, onto the hyperboloid of two sheets defined by

$$\mathcal{S}_t^{\alpha,\mathrm{cont}} := \{(x,y,z) \in \mathbb{R}^3 \mid z^2 - x^2 - y^2 = (t-1)^{2\alpha}\}. \tag{9.0.1}$$

We derive the continued equation for the solution u to (7.1.8) as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + t^{-\alpha}\frac{\coth g(lt^{-\alpha})}{\sqrt{\cosh 2g(lt^{-\alpha})}}\frac{\partial u}{\partial l} - \alpha t^{\alpha-1}G(lt^{-\alpha})\frac{\partial u}{\partial l} \\ &\quad + \alpha t^{-1}(\operatorname{sech} 2g(lt^{-\alpha}) + \operatorname{sech}^2 2g(lt^{-\alpha}))u \\ u(l,0) &= U(X(l,\theta,1),1) =: u_0(l)\end{aligned} \tag{9.1.3}$$

and we also derive the continued equation for the solution J to (7.1.7) as

$$\begin{aligned}\frac{\partial J}{\partial t} &= \frac{\partial^2 J}{\partial l^2} - \frac{\partial}{\partial l}\left(\left(t^{-\alpha}\frac{\coth g}{\sqrt{\cosh 2g}} + \alpha t^{\alpha-1}G\right)J\right) \\ J(l,0) &= \frac{\sqrt{2}}{2}|l|u_0(l) =: J_*(l).\end{aligned} \tag{9.1.4}$$

The singularity of the drift in equation (9.1.4) is analysed in Remark 9.1.3. Since (9.1.4) is of Fokker–Planck type, it is sensible to consider the problem as continuing a stochastic process, analogous to what was given in Chapter 5. We thus seek a unique strong solution to

$$\begin{aligned}\mathrm{d}X_t &= \left(t^{-\alpha}\frac{\coth g(X_t t^{-\alpha})}{\sqrt{\cosh 2g(X_t t^{-\alpha})}} + \alpha t^{\alpha-1}G(X_t t^{-\alpha})\right)\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t \\ X_0 &= Z\end{aligned} \tag{9.2.1}$$

where $B_\bullet$ is a standard Brownian motion and Z has density J_*. Such a unique, strong solution is given, up to a suitable stopping time, by Theorem 9.2.1. The proof uses estimates on certain stochastic processes and modifying the coefficients of (9.2.1) accordingly. Different regimes of α are considered in the proof. The stopping time is needed due to the singular drift in (9.2.1). But we show in Theorem 9.2.3 that, with probability 1, one can extend the solution for all future times, regardless of the value of $\alpha \in (0,\infty)$. The proof uses a Lyapunov function approach and is similar to showing that a two–dimensional Bessel process never touches the origin. We conclude the chapter by giving in Theorem 9.2.4 the existence of a unique classical solution to equation (9.1.4) and show that the initial data is attained in the weak sense.

Our final problem, Problem III, is outlined and analysed in Chapter 10. Here, we consider (2.6.3) on a curve, but assume that the curve evolution is coupled to the solution of the equation on the curve. The problem here is motivated by mathematical biology and we consider the curve $\mathcal{C}_t$ as given by

$$\mathcal{C}_t := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\ y^2 - x^2 = f(t)\}, \quad 0 \leq t \leq T, \tag{10.1.1}$$

for some T small enough. Here f is some functional of u, which we specify. The equation (2.6.3) is transformed using arc–length parameterisation into

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} - \frac{\dot{f}(t)}{2\sqrt{f(t)}} \frac{\partial}{\partial l}\left(G(lf(t)^{-\frac{1}{2}})u\right) \\ u(l,0) &= U_0(X(l,0)). \end{aligned} \tag{10.1.6}$$

This equation is now non–linear due to the dependence of f on u. In Theorem 10.2.1 we show short time existence and uniqueness of a solution in a suitable Banach space and we do not consider any possible singularity. We treat the problem as a perturbation of the heat equation to gain the result. In Remark 10.2.2 we give a heuristic argument for the behaviour of the solution at any possible singularity and remark that considering a singularity in this setting is extremely challenging. We remark that a more general f with suitable regularity conditions may also be considered, but we do not do this in this thesis.

The thesis is concluded by Chapter 11 where open problems raised in the thesis are discussed along with the problems encountered. We also look at some natural questions that are raised from this thesis to guide future work. The appendices contain important results that are referred to throughout this thesis.

Chapter 2

Background Material

This chapter is designed to set up the notation, definitions and results to be used in the thesis, and to provide the reader with a background of these areas as needed. References are given where appropriate.

2.1 Basic Definitions

2.1.1 Calculus on $\mathbb{R}$

Definition 2.1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. If $k > 0$ then $g \in C^k(\mathbb{R})$ if and only if g is k times differentiable with continuous derivative. If $k = 0$ then $g \in C^0(\mathbb{R})$ if and only if g is continuous. In this case, we drop the superscript 0 and write $g \in C(\mathbb{R})$.*

Definition 2.1.1 has a natural generalisation to "time" dependent functions.

Definition 2.1.2. *Let $a, b \in \mathbb{R}$ with $a < b$, $k, m \in \mathbb{N} \cup \{0\}$ and $f : \mathbb{R} \times (a, b) \to \mathbb{R}$. We say that $f \in C^{k,m}(\mathbb{R} \times (a, b); \mathbb{R})$ if and only if the partial derivatives exist and $f(x, \cdot) \in C^k(\mathbb{R}; \mathbb{R})$ and $f(\cdot, t) \in C^m((a, b); \mathbb{R})$.*

Definition 2.1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then the* support *of f, denoted* supp(f) *is defined as*

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}}.$$

Definition 2.1.4. *We define $C_c^\infty(\mathbb{R})$ to be the collection of all $f \in C^\infty(\mathbb{R})$ with compact support.*

2.1.2 Calculus on a Banach Space

Definition 2.1.5. *Let $p \in [1,\infty]$, $q \in [1,\infty]$ and $T \in (0,\infty)$. Then we define for $1 \leq p < \infty$*

$$L^p([0,T);L^q(\mathbb{R})) := \left\{ f : [0,T) \to L^q(\mathbb{R}) \, \text{measurable} \,|\, \left(\int_0^T \|f(s)\|_{L^q}^p \, \mathrm{d}s \right)^{\frac{1}{p}} < \infty \right\}$$

and for when $p = \infty$

$$L^\infty([0,T);L^q(\mathbb{R})) := \left\{ f : [0,T) \to L^q(\mathbb{R}) \, \text{measurable} \,|\, \operatorname{ess} \sup_{s\in[0,T)} \|f(s)\|_{L^q} < \infty \right\}.$$

We define

$$C([0,T);L^q(\mathbb{R})) := \left\{ f : [0,T) \to L^q(\mathbb{R}) \; \text{continuous} \; | \; \sup_{s\in[0,T)} \|f(s)\|_{L^q(\mathbb{R})} < \infty \right\}$$

2.2 Classical Partial Differential Equations

Consider the following Cauchy problem for $(x,t) \in \mathbb{R} \times (0,T)$ where $T \in (0,\infty)$,

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \mathcal{A}(x,t)u(x,t) \\ u(x,0) &= u_0(x) \end{aligned} \tag{2.2.1}$$

where

$$\mathcal{A}(x,t) = \frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t)$$

with b and c are smooth in both arguments and $c \geq 0$ (the latter condition can be neglected). We assume that the initial data, u_0, has $u_0 \in C_c^\infty(\mathbb{R})$. We will use the following notions of solution.

Definition 2.2.1. *We say that $u \in C^{2,1}(\mathbb{R} \times [0,T);\mathbb{R})$ is a* classical *solution of* (2.2.1) *if and only if u satisfies* (2.2.1) *pointwise with*

$$u(x,0) = u_0(x)$$

for every $x \in \mathbb{R}$ and is unique.

Remark 2.2.2. *In the case that one can only prove that $u \in C^{2,1}(\mathbb{R} \times (0,T);\mathbb{R})$, we will require that $\lim_{t\to 0} u(x,t) = u_0(x)$ for every $x \in \mathbb{R}$.*

Definition 2.2.3. *We say that* $u \in C([0,T); L^\infty(\mathbb{R}))$ *is a* mild *solution of* (2.2.1) *if and only if* u *is the unique solution to*

$$u = \mathcal{F}u$$

in $C([0,T); L^\infty(\mathbb{R}))$ *where for* $v \in C([0,T); L^\infty(\mathbb{R}))$

$$\mathcal{F}v(t) = S(t)u_0 + \int_0^t S(t-s)B(s)v(s)\,\mathrm{d}s$$

and

$$B(s)v(s)(x) := b(x,s)\frac{\partial v(s)(x)}{\partial x} - c(x,s)v(s)(x).$$

Here $S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ *is the heat semigroup defined by*

$$(S(t)v)(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} v(y)\,\mathrm{d}y$$

and it is understood that one performs an integration by parts in the definition of $\mathcal{F}$.

We have the following existence and uniqueness theorem for (2.2.1).

Theorem 2.2.4. *Consider* (2.2.1) *and suppose that* $b(\cdot,t)$ *and* $c(\cdot,t)$ *are bounded for every* $t \in [0,T)$ *and globally Hölder continuous, uniformly in* t. *Suppose also that the first spatial derivative of* b *is bounded, continuous and is globally Hölder continuous, uniform in* t. *Then there exists a unique classical solution* u *to* (2.2.1) *which is also the unique mild solution.*

Proof. Since the coefficients of $\mathcal{A}$ are smooth, bounded and globally Hölder continuous uniformly in t along with the second–order term being uniformly elliptic, Theorem 12, page 25 of Friedman [1964] applies so that (2.2.1) is satisfied. Uniqueness follows from Theorem 16, page 29 of Friedman [1964] and the argument that the classical solution is a mild solution is standard. (See, for example, Sell and You [2002]). $\square$

2.3 Stochastic Differential Equations

Of use will be the classical theory for Stochastic Differential Equations. Detailed below are some standard results. We will refer to the adequate text or provide an alternative proof if and when the standard theory is inadequate for our problem.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, rich enough to support Brownian motion. Let the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ be right–continuous and increasing in the sense that $s < t$ implies $\mathcal{F}_s \subset \mathcal{F}_t$. Let W_t denote the Gaussian process such that

1. $W_0 = 0 \; \mathbb{P} - a.s$,
2. $t \mapsto W_t$ is $\mathbb{P} - a.s$ continuous,
3. $W_t - W_s \sim N(0, t-s)$ for every $s < t$,
4. W has independent increments, so that if $a_1 < a_2 < a_3 < a_4$ then $W_{a_2} - W_{a_1}$ and $W_{a_4} - W_{a_3}$ are independent random variables.

By standard results of probability theory, such a process exists (Øksendal [2003], Section 2.2). Consider the following Stochastic Differential Equation on $\mathbb{R}$

$$\begin{aligned} \mathrm{d}X_t &= a(X_t, t)\,\mathrm{d}t + b(X_t, t)\,\mathrm{d}W_t \\ X_0 &= x \end{aligned} \tag{2.3.1}$$

where $x \in \mathbb{R}$ and $t \in [0, T]$ for some $T \in (0, \infty)$. We note that (2.3.1) is interpreted in the following integral equation sense that $\mathbb{P} - a.s$

$$X_t = x + \int_0^t a(X_s, s)\,\mathrm{d}s + \int_0^t b(X_s, s)\,\mathrm{d}W_s,$$

where

$$\int_0^t b(X_s, s)\,\mathrm{d}W_s$$

is the stochastic integral, interpreted in the Itō sense (Øksendal [2003]).

Definition 2.3.1. *We say that the stochastic process $(X_t)_{t\in[0,T]}$ is a* strong *solution to* (2.3.1) *if and only if $\mathbb{P} - a.s$ we have*

$$X_t = x + \int_0^t a(X_s, s)\,\mathrm{d}s + \int_0^t b(X_s, s)\,\mathrm{d}W_s.$$

The most standard existence and uniqueness result for strong solutions to (2.3.1) is given below.

Theorem 2.3.2. *Let $T > 0$ and $a, b : \mathbb{R} \times [0, T] \to \mathbb{R}$ be measurable functions satisfying there exists $C_i > 0$ ($i = 1, 2$) such that*

$$|a(x, t)| + |b(x, t)| \leq C_1(1 + |x|)$$

for every $(x,t) \in \mathbb{R} \times [0,T]$ *and such that*

$$|a(x,t) - a(y,t)| + |b(x,t) - b(y,t)| \leq C_2|x-y|$$

for every $x, y \in \mathbb{R}$, $t \in [0,T]$. *Then, the Stochastic Differential Equation* (2.3.1) *has a unique* $t-$*continuous solution* X_t, *with the property that* X_t *is adapted to the filtration generated by* W_s; $s \leq t$ *and*

$$\mathbb{E}\left[\int_0^T |X_t| \,\mathrm{d}t\right] < \infty.$$

Proof. The proof can be found in Øksendal [2003], pp. 69 – 71. There, a more general statement is given concerning the initial data and the dimension of the state space. Since we will only consider one spatial dimension, the above suffices. □

We have only discussed the Itō stochastic integral above. Of importance will be the Stratonovich interpretation. Indeed, we have the following. See (Øksendal [2003] p.84), for example.

Definition 2.3.3. *Given the Itō Stochastic Differential Equation*

$$\mathrm{d}X_t = b(X_t,t)\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t,$$

this is interpreted in the Stratonovich sense as

$$\mathrm{d}X_t = \left(b(X_t,t) - \frac{1}{2}\sigma(X_t,t)\frac{\partial\sigma}{\partial y}(X_t,t)\right)\,\mathrm{d}t + \sigma(X_t,t) \circ \mathrm{d}W_t.$$

The converse also holds and we say that

$$-\frac{1}{2}\sigma(X_t,t)\frac{\partial\sigma}{\partial y}(X_t,t)\,\mathrm{d}t$$

is the Itō–Stratonovich correction term.

The following formula, referred to as Itō's formula, is crucial in many calculations.

Lemma 2.3.4. *Suppose* (X_t) *is the strong solution to the following Itō Stochastic Differential Equation*

$$\mathrm{d}X_t = b(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t.$$

Let $f \in C^{2,1}(\mathbb{R} \times [0,T];\mathbb{R})$ *and define* $Y_t := f(X_t,t)$*. Then*

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t}(X_t,t)\mathrm{d}t + \frac{\partial f}{\partial y}(X_t,t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial f^2}{\partial y^2}(X_t,t)\,\sigma^2(X_t,t)\,\mathrm{d}t.$$

Proof. A proof of the one–dimensional case and the n dimensional case when b and σ are independent of time is given in Øksendal [2003], p.44 and p.48 respectively. For the case of b and σ dependent on time as well as state space, we refer the reader to Friedman [1975] p.81. □

Lemma 2.3.5. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* (B_t) *be standard Brownian motion. Let* $(\mathcal{F}_t)$ *be the canonical filtration generated by* (B_t)*. Suppose*

$$f : [0,\infty) \times \Omega \to \mathbb{R}$$

satisfies the following conditions.

i) $(t,\omega) \mapsto f(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}$*-measurable, where* $\mathcal{B}$ *is the Borel* σ*-algebra on* $[0,\infty)$*;*

ii) $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted;*

iii) $\mathbb{E}\left[\int_S^T f(t,\omega)^2\,\mathrm{d}t\right] < \infty$*, where* $S,T \in [0,\infty)$ *with* $S < T$*.*

Then

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega)\,\mathrm{d}B_t\right)^2\right] = \mathbb{E}\left[\int_S^T f(t,\omega)^2\,\mathrm{d}t\right]. \tag{2.3.2}$$

Equation (2.3.2) *is referred to as the "Itō Isometry". If* f *is independent of* ω*, then*

$$\mathbb{E}\left[\left(\int_S^T f(t)\,\mathrm{d}B_t\right)^2\right] = \int_S^T f(t)^2\,\mathrm{d}t.$$

Proof. A proof is given in Øksendal [2003], p. 29. □

Remark 2.3.6. *Of importance to note is that* (2.3.1) *also makes sense, via the scaling properties of Brownian motion, for negative times. By this, we mean that we may start the Stochastic Differential Equation at time* $t = -\tau$ *for some* $\tau > 0$ *and run the equation forward for* $t \in (-\tau, 0]$*. Indeed, this notion will be useful when we considering the stochastic representation of solutions to Partial Differential Equations.*

2.4 Stochastic Representation of Solutions to Partial Differential Equations

Of particular interest in the analysis of Partial Differential Equations is pointwise behaviour of the solution. In order to analyse this, we need a representation formula. Classically, one solves the PDE to get the fundamental solution, or perhaps uses the parametrix method as given in Friedman [1964]. However, usually this is hard to do and a much cleaner approach is to use the Feynman–Kac formula. We describe this as follows. Consider

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \mathcal{A}(x,t)u(x,t) \\ u(x,0) &= u_0(x) \end{aligned} \tag{2.4.1}$$

where

$$\mathcal{A}(x,t) = \frac{D(x,t)}{2}\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t)$$

and suppose the hypotheses on b and c as in Theorem 2.2.4 hold, along with $D(x,t) > 0$ for every $(x,t) \in \mathbb{R} \times [0,T]$. Suppose that the initial data has $u_0 \in C_c^\infty(\mathbb{R})$. Consider the Stochastic Differential Equation

$$\begin{aligned} \mathrm{d}Y_s &= b(Y_s,-s)\,\mathrm{d}s + \sqrt{D(Y_s,-s)}\,\mathrm{d}W_s \\ Y_{-t} &= x \end{aligned} \tag{2.4.2}$$

where $x \in \mathbb{R}$ and $-t \le s \le 0$. We have the following Theorem.

Theorem 2.4.1 (The Feynman–Kac representation Theorem)**.** *Suppose there is a unique classical solution, u, to* (2.4.1) *and suppose that one has a unique strong solution, $(Y_s)_{s\ge -t}$ to* (2.4.2)*. Then the solution u is given by*

$$u(x,t) = \mathbb{E}^{(-t,x)}\left[\exp\left(-\int_{-t}^{0} c(Y_s,-s)\,\mathrm{d}s\right) u_0(Y_0)\right] \tag{2.4.3}$$

Here $\mathbb{E}^{(-t,x)}[\,\cdot\,]$ denotes the conditional expectation, conditioned on $Y_{-t} = x$.

Proof. The proof can be found in Friedman [1975], Theorem 5.3, p.148. It is standard to have a terminal condition for the Partial Differential Equation, however a simple change of time can modify this for our purpose. □

Remark 2.4.2. *We also call* (2.4.3) *the "Feynman–Kac Formula".*

When b , c and D become time–independent, one has the following.

Theorem 2.4.3. *Suppose b, c and D in* (2.4.1) *are time independent. Assume that all the other hypotheses on b , c and D hold with $u_0 \in C_c^\infty(\mathbb{R})$ and that there exists a unique classical solution u to* (2.4.1). *Assume one can find a unique strong solution, $(Y_s)_{0\le s\le T}$, to*

$$\begin{aligned} \mathrm{d}Y_s &= b(Y_s)\,\mathrm{d}s + \sqrt{D(Y_s)}\,\mathrm{d}W_s \\ Y_0 &= x \end{aligned} \tag{2.4.4}$$

where $x \in \mathbb{R}$ and $0 \le t \le T$. Then the solution u is given by

$$u(x,t) = \mathbb{E}^{(x)}\left[\exp\left(-\int_0^t c(Y_s)\,\mathrm{d}s\right)u_0(Y_t)\right].$$

Here $\mathbb{E}^{(x)}[\,\cdot\,]$ denotes the conditional expectation, conditioned on $Y_0 = x$.

Proof. The proof can be found, for example, in Øksendal [2003], p. 143. □

2.5 The Fokker–Planck Equation

The Fokker–Planck equation, or Forward Kolmogorov equation, describes the time evolution of the probability density function of a stochastic process satisfying an Itō SDE. More precisely we have the following.

Theorem 2.5.1. *Let $(X_t)_{0\le t\le T}$ be the unique strong solution of the Ito SDE*

$$\begin{aligned} \mathrm{d}X_t &= b(X_t,t)\,\mathrm{d}t + \sqrt{2}\,\sigma(X_t,t)\,\mathrm{d}W_t \\ X_0 &= Z. \end{aligned} \tag{2.5.1}$$

If (X_t) has a smooth density, ρ, with respect to the Lebesgue measure, dy, *then if*

$$a(y,t) := \sigma(y,t)^2, \quad \frac{\partial a}{\partial y}(y,t), \quad b(y,t), \quad \frac{\partial b}{\partial y}(y,t)$$

are bounded, continuous and globally Hölder continuous uniformly in t with σ^2 uniformly elliptic, then ρ classically solves the Fokker–Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial y^2}\left(a(y,t)\rho(y,t)\right) - \frac{\partial}{\partial y}\left(b(y,t)\rho(y,t)\right) \tag{2.5.2}$$

Proof. The proof is given in Friedman [1975], Theorem 5.4, page 149. □

Note that the theorem asserts nothing about attainment of the initial data.

Remark 2.5.2. *The question of whether a density actually exists and is smooth is very delicate. A sufficient condition is given by the probabilistic version of Hormander's Theorem, presented in Malliavin [1978]. Indeed, if b is smooth and bounded in* (2.5.1) *with σ uniformly elliptic then the strong solution to* (2.5.1) *has a smooth density.*

However, if σ is not uniformly elliptic we have the following.

Theorem 2.5.3 (Probabilistic version of Hörmander's Theorem)**.** *Suppose there is a unique strong solution to the following Stratonovich SDE (Øksendal [2003])*

$$\mathrm{d}X_t = V_0(X_t)\,\mathrm{d}t + V_1(X_t) \circ \mathrm{d}W_t. \tag{2.5.3}$$

Define

$$\mathcal{V}_0 := \{V_1\}$$

and iteratively

$$\mathcal{V}_{k+1} := \mathcal{V}_k \cup \{[U, V_j] \,|\, U \in \mathcal{V}_k \,\&\, j \geq 0\},$$

where the commutator $[A, B]$ is defined as

$$[A, B](x) := DB(x)A(x) - DA(x)B(x)$$

and DB is the derivative of B with respect to x. Define

$$\mathcal{V}_k(x) := \mathrm{Span}\{V(x) \,|\, V \in \mathcal{V}_k\}$$

and suppose that

$$\mathbb{R} = \bigcup_{k\geq 1} \mathcal{V}_k(x)$$

for every $x \in \mathbb{R}$. Then the solution to (2.5.3) *has a smooth density with respect to Lebesgue measure.*

Proof. For a clear explanation of the proof, which considers general dimensions, see Hairer [2011]. The original proof was given in the landmark paper Malliavin [1978]. $\square$

Remark 2.5.4. *One should recall the classical deterministic Hormander's Theorem (Hörmander [1967]) which states that for a hypoelliptic operator on $\mathbb{R}^n$*

$$P := X_0 + \sum_{i=1}^{p} X_i^2$$

where $p < n$ and the vector fields $X_i, [X_i, X_j], [X_k, [X_i, X_j]], \ldots$ span $\mathbb{R}^n$, then every distributional solution to $\partial_t u = Pu$ has $u \in C^\infty(\mathbb{R}^n)$.

Remark 2.5.5. *Theorem 2.5.3 holds true for V_i dependent on t as well as x and if V_1 is a constant, depending only on t, then the Itō and Stratonovich interpretations of the Stochastic Integral agree. The conclusion of the above theorem still holds in these cases.*

If $V_0 = 0$ identically, then $\mathcal{V}_k = \mathcal{V}_0$ for every $k \geq 1$ and so we only require $V_1(x,t)$ to be finite at each (x,t).

Further, if we restrict to where only Hörmander's condition holds, i.e

$$A := \{x \in \mathbb{R} \,|\, \mathbb{R} = \cup_{k\geq 1} \mathcal{V}_k(x)\}$$

satisfies $A \neq \emptyset$ and $A \neq \mathbb{R}$, then Hörmander's Theorem is said to hold in A and so for $x \in A$ a smooth density exists.

Importantly, this theorem says nothing about attainment of any initial data.

Indeed, for such σ that are not uniformly bounded we need the following modification of Theorem 2.5.1.

Theorem 2.5.6. *Suppose b and σ satisfy the conditions in Theorem 2.3.2 so that a unique strong solution exists to the SDE in* (2.5.1)*. Suppose also that a smooth density, ρ, exists (for example, by satisfying Theorem 2.5.3). Then ρ satisfies* (2.5.2).

Proof. For $f \in C_c^\infty(\mathbb{R})$ and (X_t) the unique strong solution to (2.5.1), define $Y_t := f(X_t)$. Then, by Itō's formula (omitting the arguments of σ and b for typographic clarity)

$$f(X_t) = f(y) + \int_0^t \big(f'(X_s)b + \sigma^2 f''(X_s)\big)\,\mathrm{d}s + \int_0^t \sqrt{2}f'(X_s)\sigma\,\mathrm{d}W_s,$$

where $y \in \mathbb{R}$ is a realisation of Z. Taking expectations we have

$$\mathbb{E}[f(X_t)] = f(y) + \mathbb{E}\left[\int_0^t \big(f'(X_s)b + \sigma^2 f''(X_s)\big)\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \sqrt{2}f'(X_s)\sigma\,\mathrm{d}W_s\right].$$

However, $f \in C_c^\infty(\mathbb{R})$ and so it follows, as σ grows at most linearly, that there exists $M > 0$ such that

$$|f'(y)\sigma(y,t)| \leq M$$

for every $y \in \mathbb{R}$ and every $t \in [0,T]$. Thus, since any uniformly bounded functional,

h, of an Itō process (Z_t) has

$$\mathbb{E}\left[\int_0^t h(Z_s)\,\mathrm{d}W_s\right] = 0$$

(Karatzas and Shreve [1991]) it follows that we have the following, which is referred to as "Dynkin's Formula" (Øksendal [2003]):

$$\mathbb{E}[f(X_t)] = f(y) + \mathbb{E}\left[\int_0^t \big(f'(X_s)b(X_s,s) + \sigma^2(X_s,s)f''(X_s)\big)\,\mathrm{d}s\right]. \tag{2.5.4}$$

Using that

$$\mathbb{E}f(X_t) = \int_{\mathbb{R}} f(x)\rho(x,t)\,\mathrm{d}x,$$

Fubini's Theorem, integration by parts and differentiating under the integral, we conclude that for every $f \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} f(x)\frac{\partial\rho}{\partial t}(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\left(-\frac{\partial}{\partial x}\left(b(x,t)\rho(x,t)\right) + \frac{\partial^2}{\partial x^2}\big(\sigma^2(x,t)\rho(x,t)\big)\right)\mathrm{d}x$$

and so (2.5.2) holds.

□

We call a PDE of the type (2.5.2) a *Fokker–Planck* equation. Theorem 2.5.6 will give us existence of a classical solution to (2.5.2). However, one will have to show "by hand" that the equation attains its initial data.

2.6 Surface Partial Differential Equations

Our central object of study is the heat equation on an evolving curve and surface. In the following, we introduce the concept of partial differential equations on moving surfaces and derive the equation, which arises from a conservation law, that will be the subject of study throughout this thesis.

Let $\mathcal{M}(t)$ be a hypersurface for each time $t \in [0,T]$ where $T \in (0,\infty)$ is fixed. We need some notion of what it means to have such an object. Unless otherwise stated, the definitions and proofs are found in Deckelnick et al. [2005].

Definition 2.6.1. *Let $k \in \mathbb{N}$. A subset $\Gamma \subset \mathbb{R}^{n+1}$ is called a C^k-hypersurface if for each point $x_0 \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ containing x_0 and a function $\phi \in C^k(U)$ such that*

$$U \cap \Gamma = \{x \in U \mid \phi(x) = 0\} \text{ and } \nabla\phi(x) \neq 0 \text{ for every } x \in U \cap \Gamma.$$

This allows us to define what it means for a function on Γ to be differentiable.

Definition 2.6.2. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a C^1-hypersurface, $x \in \Gamma$. A function $f : \Gamma \to \mathbb{R}$ is called differentiable at x if $f \circ X$ is differentiable at $X^{-1}(x)$ for each parameterisation $X : \Theta \to \mathbb{R}^{n+1}$ of Γ with $x \in X(\Theta)$.*

The following lemma shows us how to interpret the above definition in terms of functions defined on the ambient space.

Lemma 2.6.3. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a C^1-hypersurface with $x \in \Gamma$. A function $f : \Gamma \to \mathbb{R}$ is differentiable at x if and only if there exists an open neighbourhood U in $\mathbb{R}^{n+1}$ and a function $\tilde{f} : U \to \mathbb{R}$ which is differentiable at x and satisfies $\tilde{f}|_{\Gamma \cap U} = f$.*

With the notion of differentiable functions on Γ we now define the tangential gradient, which is the form of the differential operator we will be considering.

Definition 2.6.4. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a C^1-hypersurface, $x \in \Gamma$ and $f : \Gamma \to \mathbb{R}$ differentiable at x. We define the tangential gradient of f at x by*

$$\nabla_\Gamma f(x) := \nabla \tilde{f}(x) - \left(\nabla \tilde{f}(x) \cdot \nu(x)\right) \nu(x).$$

Here $\tilde{f}$ is as in Lemma 2.6.3, ∇ denotes the usual gradient in $\mathbb{R}^{n+1}$ and $\nu(x)$ is a unit normal at x.

This leads to the definition of the Laplace–Beltrami operator on $\Gamma(t)$,

$$\Delta_{\Gamma(t)} f := \nabla_{\Gamma(t)} \cdot \nabla_{\Gamma(t)} f.$$

Before stating the conservation law and deriving the PDE, we need to define a time derivative that takes into account the evolution of the surface, generalise integration by parts and give the so-called transport Theorem.

Definition 2.6.5. *Suppose $\Gamma(t)$ is evolving with normal velocity v_ν. Define the material velocity field $v := v_\nu + v_\tau$ where v_τ is the tangential velocity field. The material derivative of a scalar function $f = f(x,t)$ defined on $\mathcal{G}_T := \cup_{t \in [0,T]} \Gamma(t) \times \{t\}$ is given as*

$$\partial_t^\bullet f := \frac{\partial f}{\partial t} + v \cdot \nabla f.$$

We now give a generalisation of integration by parts for a hypersurface Γ, the proof of which is found in Gilbarg and Trudinger [2001].

Theorem 2.6.6. *Let Γ be a compact C^2-hypersurface with boundary. Suppose that $f \in W^{1,1}(\Gamma;\mathbb{R}^{n+1})$. Then*

$$\int_\Gamma \nabla_\Gamma \cdot f \,\mathrm{d}\mathcal{H}^n = \int_\Gamma f \cdot H\nu \,\mathrm{d}\mathcal{H}^n + \int_{\partial\Gamma} f \cdot \nu_{\partial\Gamma} \,\mathrm{d}\mathcal{H}^{n-1},$$

where $H = \nabla_\Gamma \cdot \nu$ is the mean curvature and $\nu_{\partial\Gamma}$ is the co-normal.

This leads us nicely onto the following lemma which is referred to as the transport Theorem, whose proof is given in Dziuk and Elliott [2007].

Lemma 2.6.7. *Let $\mathcal{C}(t)$ be an evolving surface portion of $\Gamma(t)$ with normal velocity v_ν. Let v_τ be a tangential velocity field on $\mathcal{C}(t)$. Let the boundary $\partial\mathcal{C}(t)$ evolve with the velocity $v = v_\nu + v_\tau$. Assume that f is a function such that all the following quantities exist. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}(t)} f = \int_{\mathcal{C}(t)} \partial_t^\bullet f + f\nabla_\Gamma \cdot v.$$

We now have all the necessary theory to formulate an advection-diffusion equation from the following conservation law.

Let u be the density of a scalar quantity on $\Gamma(t)$ and suppose there is a surface flux q. Consider an arbitrary portion $\mathcal{C}(t)$ of $\Gamma(t)$, which is the image of a portion $\mathcal{C}(0)$ of $\Gamma(0)$, evolving with the prescribed velocity v_ν. The law is that, for every $\mathcal{C}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}(t)} u = -\int_{\partial\mathcal{C}(t)} q \cdot \nu_{\partial\Gamma}, \tag{2.6.1}$$

along with the surface integral of u being conserved. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}(t)} u(z,t)\,\mathrm{d}\sigma(z) = 0. \tag{2.6.2}$$

Observing that components of q normal to $\mathcal{C}(t)$ do not contribute to the flux, we may assume that q is a tangent vector. With this assumption, Theorem 2.6.6 and Lemma 2.6.7 together with defining the flux $q := uv_\tau - \nabla_{\Gamma(t)}u$, one has the PDE

$$\partial_t^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)}u = 0.$$

We now take $\Gamma(t) = \mathcal{M}(t)$ and arrive at the following model PDE on $\mathcal{M}(t)$

$$\begin{aligned} \partial_t^\bullet u + u\nabla_{\mathcal{M}(t)} \cdot v - \Delta_{\mathcal{M}(t)}u &= 0 \\ u(x,0) &= u_0 \quad x \in \mathcal{M}(0). \end{aligned} \tag{2.6.3}$$

Chapter 3

Statement and formulation of Problem I

We are now in a position to consider the first problem. We set the problem and formulate it in an appropriate way to perform the analysis as outlined in the subsequent chapter.

3.1 The Problem

Fix $\alpha \in (0, \infty)$ and consider the two hyperbolae

$$\mathcal{C}_t^\alpha := \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0,\ y^2 - x^2 = (1-t)^{2\alpha}\}, \quad 0 \leq t \leq 1, \tag{3.1.1}$$

and

$$\mathcal{C}_t^{\alpha,\mathrm{cont}} := \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0,\ x^2 - y^2 = (t-1)^{2\alpha}\}, \quad 1 \leq t \leq T, \tag{3.1.2}$$

where $T \in (1, \infty)$. Figure 3.1 highlights $\mathcal{C}_t^\alpha$ (along with the lower half of the hyperbola) for $\alpha = 0.5$ and various $t \in [0, 1]$ near to the origin whilst Figure 3.2 highlights $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ (again, along with the "left" half of the hyperbola) for $\alpha = 0.5$ and various $t \in [1, 2]$, near to the origin.

For $\mathcal{C}_t^\alpha$ and $0 \leq t \leq 1$, denote by $\phi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}$, the level set function of $\mathcal{C}_t^\alpha$ defined by $\phi(x, y, t) := y^2 - x^2 - (1-t)^{2\alpha}$. It can be seen that $\{\phi = 0\} = \mathcal{C}_t^\alpha$ and that the curve undergoes a "kink" at time $t = 1$ at $x = 0$ and so the regularity of the curve decreases. Figure 3.1 shows this for $\alpha = 0.5$ by means of the brown central cross in the graph, whereas Figure 3.2 shows this for $\alpha = 0.5$ by means of a blue central cross in the graph. In particular, one cannot define the tangent space at

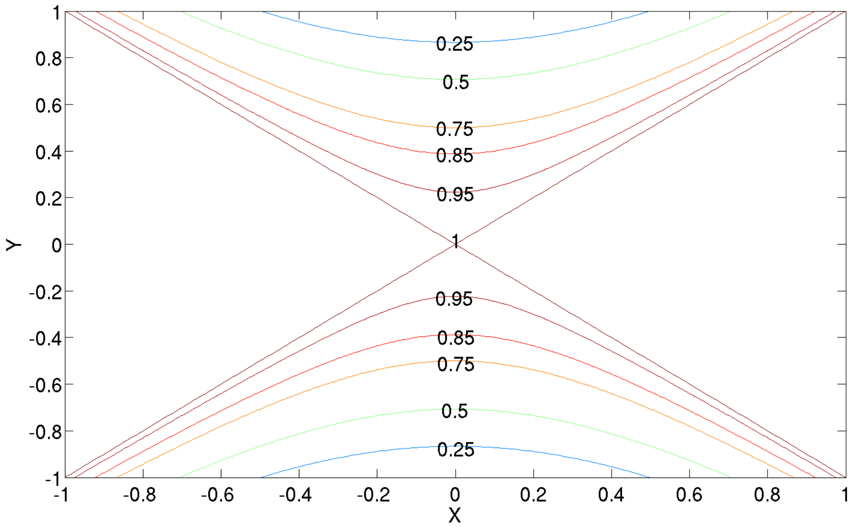

Figure 3.1: Plot of $\mathcal{C}_t^\alpha$ for $\alpha = 0.5$ and various $t \in [0, 1]$.

$(0, 0)$ nor the mean curvature at this point. We are interested in the effects of this geometric singularity to the solution to the following PDE on $\mathcal{C}_t^\alpha$, which is derived from a conservation law (Section 2.6).

$$\begin{aligned} \partial_t^\bullet U + U\nabla_\Gamma \cdot v - \Delta_\Gamma U = 0 \quad x \in \Gamma := \mathcal{C}_t^\alpha \\ U(x, 0) = U_0(x) \quad x \in \mathcal{C}_0^\alpha \end{aligned} \tag{3.1.3}$$

where

$$v := \frac{\phi_t}{|\nabla\phi|}\nu$$

is the prescribed normal velocity of the curve, with the outward pointing unit normal given by

$$\nu := -\frac{\nabla\phi}{|\nabla\phi|}.$$

The time-derivative like term, $\partial_t^\bullet$, is called the material derivative. An introduction to surface Partial Differential Equations is given in Section 2.6.

We are interested in whether the solution can be continued onto $\mathcal{C}_t^{\alpha,\text{cont}}$ for $t \geq 1$ (see Chapter 5) and whether any stochastic perturbation of (3.1.3) for $0 \leq t < 1$ leads to different qualitative results for the solution at the singularity (see Chapter 6).

We will assume that the initial data U_0 is smooth and bounded. Further, we assume that U_0 is also prescribed on the other half of the hyperbola along with the same PDE, making sure that the continuation of the solution is a sensible notion.

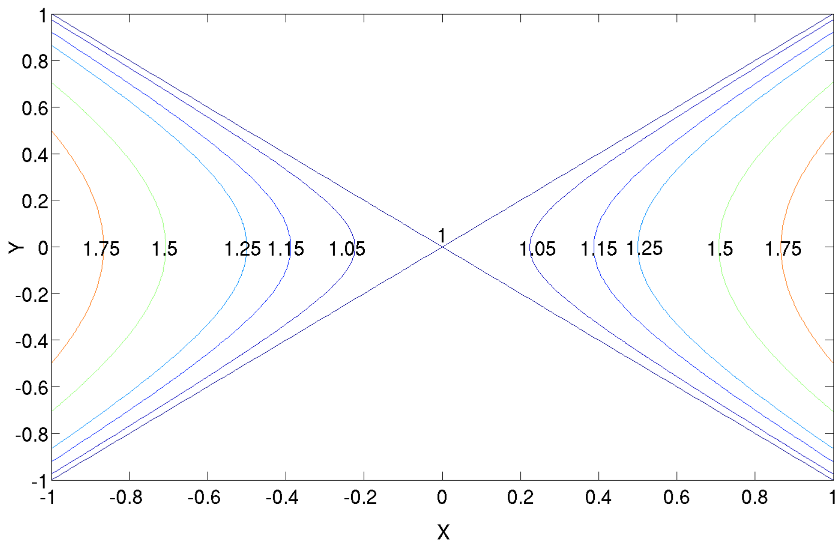

Figure 3.2: Plot of $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ for $\alpha = 0.5$ and various $t \in [1, 2]$.

3.2 Formulation of the Problem in arc-length parameter.

In the following, we transform (3.1.3) into an equation on $\mathbb{R} \times [0, 1)$ using arc-length parameterisation. This is motivated by the fact that the zero level sets of ϕ describe the curve, but in order to arrive at a nice expression for the Laplace-Beltrami term $\Delta_{\mathcal{C}_t^\alpha}$, we will use arc-length parameterisation. We pay the price by introducing a non-physical drift term into the equation, as we will see below.

Let $Y : \mathbb{R} \times [0, 1] \to \mathbb{R}^2$ be defined by $Y(p, t) = (1 - t)^\alpha(\sinh p, \cosh p)$. Denote by Y_p the partial derivative of Y with respect to p. Then by standard hyperbolic identities, we see that Y is a smooth parameterisation of $\mathcal{C}_t^\alpha$. Define $l := \int_0^p |Y_p(u, t)|\, \mathrm{d}u$. Then

$$\frac{l}{(1-t)^\alpha} = \int_0^p \sqrt{\cosh 2u}\, \mathrm{d}u$$

for $t \in [0, 1)$. We see that $l = 0$ corresponds to where the singularity occurs.

Definition 3.2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be the* inverse *of the map $p \mapsto \int_0^p \sqrt{\cosh 2u}\, \mathrm{d}u$.*

Remark 3.2.2. *By the inverse function theorem it follows that, for every $s \in \mathbb{R}$,*

$$\begin{aligned} g'(s) &= \frac{1}{\sqrt{\cosh 2g(s)}}, \\ g(0) &= 0. \end{aligned} \tag{3.2.1}$$

Also, one should note that $g(-s) = -g(s)$ for every $s \in \mathbb{R}$. A plot is given in Figure 3.3.

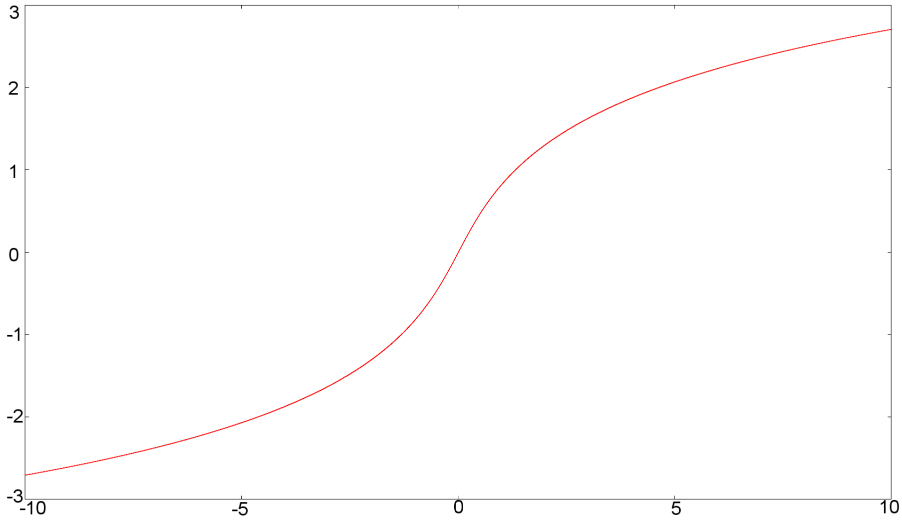

Figure 3.3: Plot of $g(s)$ (vertical axis) for $s \in [-10, 10]$ (horizontal axis).

Define the arc-length parameterisation of $\mathcal{C}_t^\alpha$, denoted by $X : \mathbb{R} \times [0,1] \to \mathbb{R}^2$, by

$$X(l,t) = (1-t)^\alpha \left(\sinh g(l(1-t)^{-\alpha}), \cosh g(l(1-t)^{-\alpha})\right). \tag{3.2.2}$$

We now transform (3.1.3) into a PDE on $\mathbb{R} \times [0,1)$ ready for subsequent analysis. In order to transform (3.1.3) we write $u(l,t) = U(X(l,t),t)$ and compute to find (using the definition of the material derivative in the second equality)

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \nabla U \cdot X_t = \nabla U \cdot (X_t - v) + \partial_t^\bullet U$$

and

$$\frac{\partial u}{\partial l} = \nabla U \cdot X_l.$$

Noting that v is in the normal direction only, and noting the sign[1] of the normal vector, we have that (Dziuk and Elliott [2007], Appendix A)

$$\nabla_\Gamma \cdot v = VH$$

[1] Usually one has $\nabla_\Gamma \cdot v = -VH$, however this is for a curve oriented such that $\nabla\phi/|\nabla\phi|$ is the unit normal. This is not the case for our curve.

where

$$V := \frac{\phi_t}{|\nabla\phi|} \quad \text{and} \quad H := \frac{-1}{|\nabla\phi|}\sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}.$$

Finally,

$$\Delta_\Gamma U(X(l,t),t) = \frac{\partial^2 u}{\partial l^2}(l,t)$$

as $|X_l| = 1$. (See Dziuk and Elliott [2007]). We observe that $X_t - v = \beta\hat{\tau}$ where $\hat{\tau}$ is the unit tangent vector, $\hat{\tau} = \pm X_l$. This is because

$$X_t \cdot \nu - v \cdot \nu = \frac{1}{|\nabla\phi|}\left(X_t \cdot (-\nabla\phi) - \phi_t\right) = \frac{1}{|\nabla\phi|}\left(2\alpha(1-t)^{2\alpha-1} - 2\alpha(1-t)^{2\alpha-1}\right) = 0.$$

Thus, $\beta = X_t \cdot \hat{\tau}$ and so for any orientation of $\hat{\tau}$ we have

$$\nabla U \cdot (X_t - v) = X_t \cdot X_l \frac{\partial u}{\partial l}$$

which is the artificial drift due to the difference in the arc-length parameterisation and the prescribed normal velocity of the curve. One computes and sees that

$$X_t \cdot X_l = \alpha(1-t)^{\alpha-1}\left(\frac{l}{(1-t)^\alpha} - \frac{\sinh 2g(l(1-t)^{-\alpha})}{\sqrt{\cosh 2g(l(1-t)^{-\alpha})}}\right)$$

and

$$VH = \frac{\alpha}{1-t}\operatorname{sech}^2 2g(l(1-t)^{-\alpha}).$$

One notes that there is no $\alpha-$dependence in the coefficient of sech^2 on the power of $1-t$ above and that

$$\frac{\partial}{\partial l}\left(X_t \cdot X_l\right) = -\alpha(1-t)^{-1}\operatorname{sech}^2 2g(l(1-t)^{-\alpha}).$$

Thus, equation (3.1.3) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u\right) \\ u(l,0) &= U_0(X(l,0)) \end{aligned} \tag{3.2.3}$$

where for $s \in \mathbb{R}$

$$G(s) := s - \frac{\sinh 2g(s)}{\sqrt{\cosh 2g(s)}}. \tag{3.2.4}$$

Using Proposition A.0.2 one sees that $G \in L^\infty(\mathbb{R})$ and by Proposition A.0.1 we have $\operatorname{sech}^2 2g(z) \leq \mathcal{O}(z^{-4})$ as $|z| \to \infty$.

Remark 3.2.3. *One notes that changing $\alpha > 0$ still yields a singular curve, but for $\alpha \geq 1$ the resulting PDE seems* non-*singular as $(1-t)^{\alpha-1} \to 0$ as $t \to 1^-$ for $\alpha \geq 1$. However, the spatial derivative of $X_t \cdot X_l$ hides the fact that the purely geometric term, VH, has coefficient independent of α. Changing α changes the speed at which one enters the singularity, as can be seen in the definition of V. However, these changes are quite important, as we shall see, the different values of α means different methods of analysis of the PDE.*

Remark 3.2.4. *We will suppose that $u_0(l) := U(X(l,0))$ has $u_0 \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(l)\,\mathrm{d}l = 1$.*

Standard parabolic theory (Theorem 2.2.4) implies that there exists a smooth solution $u \in C^{2,1}(\mathbb{R} \times [0,1))$ that solves (3.2.3).

3.3 Formulation of the Problem along flow lines

Although the problem in arc-length parameterisation yields a nice constant coefficient second-order term, we will see that it will be be useful in the analysis of the problem for certain α, to have a formulation of the PDE along the flow lines.

Precisely, from basic hyperbolic geometry, if $y^2 - x^2 = \text{const}$ then the curve $xy = \text{const}$ is normal to this curve. Thus we define

$$\begin{aligned} z(l,t) :&= X_1(l,t)X_2(l,t) \\ &= \frac{1}{2}(1-t)^{2\alpha} \sinh 2g(l(1-t)^{-\alpha}), \end{aligned}$$

where X_i are the i^{th} components of the arc-length parameterisation given by (3.2.2). We write $u(l,t) = w(z(l,t),t)$ so that if u solves (3.2.3), it follows after a set of standard calculations that w solves

$$\begin{aligned} \frac{\partial w}{\partial t} &= \sqrt{(1-t)^{4\alpha} + 4z^2}\frac{\partial^2 w}{\partial z^2} - \alpha\frac{(1-t)^{4\alpha-1}}{(1-t)^{4\alpha}+4z^2}w + \frac{2z}{\sqrt{(1-t)^{4\alpha}+4z^2}}\frac{\partial w}{\partial z} \\ w(z,0) &= w_0(z) \end{aligned} \tag{3.3.1}$$

where

$$w_0(\cdot) := u_0(g^{-1}(1/2\,\mathrm{arcsinh}(2\cdot))) \in L^\infty(\mathbb{R}).$$

3.4 Scaling Properties of the PDE

We will vary the value of α and see how the solution behaves around the singularity.

Before we look at specific α, we will get some intuition as to what should occur to the solution to (3.2.3). Naturally, we choose the self–similar coordinate change of

$$y = \frac{l}{(1-t)^{\alpha}}, \qquad \tau = -\log(1-t)$$

and write $v(y,\tau) = u(l,t)$, where u satisfies (3.2.3). Using the chain rule, one concludes that v satisfies

$$\frac{\partial v}{\partial \tau} = e^{-(1-2\alpha)\tau}\frac{\partial^2 v}{\partial y^2} - \alpha y \frac{\partial v}{\partial y} - \alpha \operatorname{sech}^2 2g(y)v + \alpha G(y)\frac{\partial v}{\partial y}.$$

Thus if $\alpha < \frac{1}{2}$, the diffusion term exponentially vanishes as $\tau \to \infty$ and only the drift has an effect. However, if $\alpha > \frac{1}{2}$, the diffusion term exponentially explodes as $\tau \to \infty$ and so this was the wrong scaling. In this case, we let

$$y = \frac{l}{\sqrt{1-t}}, \qquad \tau = -\log(1-t)$$

and write $v(y,\tau) = u(l,t)$ to see that v satisfies (omitting the arguments of G and $\operatorname{sech} 2g$ for typographic clarity)

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} - \frac{y}{2}\frac{\partial v}{\partial y} - \alpha \operatorname{sech}^2 2gv + \alpha e^{-(\alpha-\frac{1}{2})\tau} G\frac{\partial v}{\partial y}.$$

and so the $G\frac{\partial v}{\partial y}$ term exponentially vanishes as $\tau \to \infty$ and only the diffusion takes effect.

We note that the $\alpha = \frac{1}{2}$ case yields

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} - \frac{y}{2}\frac{\partial v}{\partial y} - \frac{1}{2}\operatorname{sech}^2 2g(y)v + \frac{1}{2}G(y)\frac{\partial v}{\partial y}$$

which is arguably the critical case of α.

From these scaling arguments we expect that if $\alpha < \frac{1}{2}$, the solution u will vanish at the singularity, whereas for $\alpha > \frac{1}{2}$ we expect that this is not the case. These expectations are explained via the scaling above, along with the "shearing" effect of the drift in the case of $\alpha < \frac{1}{2}$, which is not present in the $\alpha > \frac{1}{2}$ case. However, in the case of $\alpha = \frac{1}{2}$ it is not a priori obvious what will happen to the solution at the singularity.

We will refer to the $\alpha < \frac{1}{2}$ case as the *Sub–Critical Regime*, the $\alpha = \frac{1}{2}$ case as the *Critical Regime* and the $\alpha > \frac{1}{2}$ case as the *Super–Critical Regime*. The words *regime* and *case* will be interchanged as needed and are understood to mean the same.

To conclude this section, we have the following proposition concerning the properties of the functions present in the self–similar scaled PDE above.

Proposition 3.4.1. *i). Let $F(y) := \alpha \operatorname{sech}^2 2g(y)$. Then F is bounded, uniformly Lipschitz, strictly increasing on $\mathbb{R}_-$, strictly decreasing on $\mathbb{R}_+$ with a global maximum at $y = 0$. Further, for large $|y|$, $F(y) \leq \mathcal{O}(|y|^{-4})$;*

ii). Let $G(y) = y - \frac{\sinh 2g(y)}{\sqrt{\cosh 2g(y)}}$ as in equation (3.2.4). Then G is bounded, $G(0) = 0$ and G is uniformly Lipschitz;

iii). Let $H(y) = y - G(y)$, where G is given in ii) above. Then H grows at most linearly at $\pm\infty$, $H(0) = 0$ and $H(y) \to \pm\infty$ as $y \to \pm\infty$. Further, H is globally Lipschitz.

Proof. i). Since $\operatorname{sech}(x) \leq 1$ for every $x \in \mathbb{R}$, the boundedness of F follows. We calculate $F'(y) = -4\alpha \operatorname{sech}^{\frac{5}{2}} 2g(y) \tanh 2g(y)$ which is uniformly bounded. The uniform Lipschitz property follows. Also, $F'(y) < 0$ for $y > 0$ and $F'(y) > 0$ for $y < 0$, hence the monotone properties on $\mathbb{R}_-$ and $\mathbb{R}_+$ follow. Hence, by continuity, F is maximal at $y = 0$. The asymptotic follows from Proposition A.0.1.

ii). That G is bounded follows from Proposition A.0.1. Since $g(0) = 0$ and $\sinh(0) = 0$ the fact that $G(0) = 0$ follows. Moreover, $G'(y) = -\operatorname{sech}^2 2g(y)$, which is uniformly bounded. Hence the uniform Lipschitz property follows.

iii). Finally, Proposition A.0.1 implies that $H(y) = \operatorname{sgn}(y)(|y| + \kappa) + R(y)$ where $\kappa > 0$ and $R(y) \leq \mathcal{O}(y^{-2})$ for large $|y|$, which shows that H grows at most linearly at $\pm\infty$. $H(0) = 0$ follows from $G(0) = 0$. That $H(y) \to \pm\infty$ as $y \to \pm\infty$ follows as $\operatorname{sgn}(x)|x| = x$ for every $x \in \mathbb{R}$. Finally, $H'(y) = 1 + \operatorname{sech}^2 2g(y)$ and so $|H'(y)| \leq 2$ for every $y \in \mathbb{R}$.

□

Chapter 4

Analysis of Problem I: Before the Singularity

We now analyse the solution $u \in C^{2,1}(\mathbb{R} \times [0,1))$ to (3.2.3) as $t \to 1^-$ for $l = 0$ and l close to 0, according to the three different regimes outlined in Section 3.4. We will see that the three distinct regimes of α demand different tools from analysis and probability.

4.1 Sub–Critical Regime

4.1.1 Vanishing and boundedness of the solution

Consider (3.2.3) with $\alpha < \frac{1}{2}$. We perform the following change of variables. Let

$$y = \frac{l}{(1-t)^\alpha}, \qquad \tau = -\log(1-t),$$

and write $v(y,\tau) = u(l,t)$. Then, as in Section 3.4, we see that v satisfies

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= e^{-\beta\tau}\frac{\partial^2 v}{\partial y^2} + \alpha\left(G(y) - y\right)\frac{\partial v}{\partial y} - \alpha \operatorname{sech}^2 2g(y) v \\ v(y,0) &= u_0(y) \end{aligned} \tag{4.1.1}$$

where $\beta = 1 - 2\alpha > 0$. Thus, we have a vanishing diffusion term and so an efficient way to analyse the qualitative behaviour of this PDE at the singularity is via the Feynman-Kac formula (Theorem 2.4.1).

Consider the following SDE

$$\begin{aligned} \mathrm{d}X_s &= -\alpha H(X_s)\mathrm{d}s + \sqrt{2}e^{\frac{\beta}{2}s}\mathrm{d}W_s \\ X_{-\tau} &= y, \end{aligned} \tag{4.1.2}$$

where $H(y) = y - G(y)$. *If* there is a strong solution to (4.1.2), then the Feynman-Kac formula (Theorem 2.4.1) implies that

$$v(y,\tau) := \mathbb{E}^{(-\tau,y)}\left[\exp\left(-\alpha\int_{-\tau}^{0}\operatorname{sech}^2 2g(X_s)\,\mathrm{d}s\right)u_0(X_0)\right] \tag{4.1.3}$$

solves (4.1.1), given that there exists a unique solution u to (3.2.3) on $\mathbb{R}\times[0,1)$ and so such a unique solution v to (4.1.1) on $\mathbb{R}\times[0,\infty)$.

In the following, let $F(y) := \alpha\operatorname{sech}^2 2g(y)$ and $\mathbb{E}^{(-\tau,y)}[\,\cdot\,]$ denotes the conditional expectation, conditioned on $X_{-\tau} = y$. Any analysis of (4.1.1) will take place using (4.1.3).

We have the following theorem, which is the main results of this thesis.

Theorem 4.1.1. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and let u be the unique smooth solution to* (3.2.3) *with $\alpha < \frac{1}{2}$. Then the solution u is bounded; there exists $C > 0$ such that*

$$|u(l,t)| \le C$$

for every $(l,t) \in \mathbb{R}\times[0,1)$. Furthermore, the solution vanishes at the singularity. That is,

$$u(0,t) \to 0 \quad \text{as } t \to 1^-.$$

Proof. The proof is straightforward once we work with the (y,τ) coordinates. We will first show that the solution v vanishes at $y = 0$ as $\tau\to\infty$. Boundedness of the solution will be seen to be true from the following proof.

We first note that by Proposition 3.4.1, H is globally Lipschitz and there exists $k > 0$ such that

$$|H(y)| \le k(1+|y|)$$

for every $y \in \mathbb{R}$. Also, since $-\tau \le s \le 0$, we have that $e^{\beta s/2} < 1$ and thus appealing to the standard theory of existence and uniqueness of a strong solution to SDEs (Section 2.3), we have that such a unique strong solution exists to (4.1.2) (Theorem 2.3.2).

Consider (4.1.3). Observe that as $F(y) = \alpha \operatorname{sech}^2 2g(y) > 0$, we have that

$$|v(y,\tau)| = \left| \mathbb{E}^{(-\tau,y)} \left[\exp\left(-\int_{-\tau}^{0} F(X_s)\,\mathrm{d}s \right) u_0(X_0) \right] \right| \leq \|u_0\|_{L^\infty(\mathbb{R})}$$

and so the solution is bounded.

Remark 4.1.2. *One should note that the above argument is still true if $\alpha = \frac{1}{2}$.*

We proceed to show that $\lim_{t\to 1^-} u(0,t) = 0$. Indeed, just how u vanishes as a function of l and t is covered in Theorem 4.1.3.

Working in the (y,τ) coordinates, we see from Proposition 3.4.1 that F is even, F is strictly increasing on $\mathbb{R}_-$ and strictly decreasing on $\mathbb{R}_+$ with a global maximum at $y = 0$. With this in mind, define for some $\gamma > 0$ to be chosen later,

$$\Omega_1 := \{\omega \in \Omega \,:\, |X_s| < \gamma,\ \forall -\tau \leq s \leq 0\}$$

and

$$\Omega_2 := \Omega \setminus \Omega_1 = \{\omega \in \Omega \,:\, \exists s_0 \in [-\tau, 0] \text{ s.t} \,|X_{s_0}| \geq \gamma\}.$$

Splitting the expectation over Ω_1 and Ω_2 respectively, we see that if $\omega \in \Omega_1$ then $F(X_s) > F(\gamma)$ for every $-\tau \leq s \leq 0$. Thus,

$$\left| \mathbb{E}^{(-\tau,0)} \left[\exp\left(-\int_{-\tau}^{0} F(X_s)\,\mathrm{d}s \right) u_0(X_0)\chi_{\Omega_1} \right] \right| \leq e^{-\tau F(\gamma)} \|u_0\|_{L^\infty}$$

and

$$\left| \mathbb{E}^{(-\tau,0)} \left[\exp\left(-\int_{-\tau}^{0} F(X_s)\,\mathrm{d}s \right) u_0(X_0)\chi_{\Omega_2} \right] \right| \leq \|u_0\|_{L^\infty}\, \mathbb{P}\left[\Omega_2\right].$$

We have that as $\gamma \to \infty$

$$\mathbb{P}\left[\Omega_2\right] \longrightarrow 0,$$

uniformly in τ. To see this, we note that $\mathbb{P}-$a.s the strong solution to (4.1.2) with $y = 0$ is given by

$$X_s := \alpha \int_{-\tau}^{s} e^{-\alpha s} e^{\alpha r} G(X_r)\,\mathrm{d}r + B_{2e^{-2\alpha s}(e^s - e^{-\tau})}.$$

Here we have used the time change Theorem for martingales (4.6 Theorem, Karatzas and Shreve [1991], p. 174) for the Itō integral, the definition of H (Proposition 3.4.1) and $B_\bullet$ is a standard Brownian motion. Noting that for suitably large $\gamma > 0$,

$$\gamma \leq |X_{s_0}| \implies \gamma \leq \|G\|_{L^\infty} + |B_{\Gamma(s_0,\tau)}|,$$

where

$$\Gamma(s,\tau) := 2e^{-2\alpha s}(e^s - e^{-\tau}).$$

We have

$$\mathbb{P}[\Omega_2] \leq \mathbb{P}\left[|B_{\Gamma(s_0,\tau)}| \geq \gamma - \|G\|_{L^\infty}\right]$$

where we have taken $\gamma > \|G\|_{L^\infty}$. Observing that for a Gaussian random variable Z with zero mean and $b > 0$ we have

$$\mathbb{P}[|Z| > b] = \mathbb{P}[\{Z > b\} \cup \{Z < -b\}] \leq 2\mathbb{P}[Z > b]$$

one concludes that

$$\mathbb{P}[\Omega_2] \leq 2\mathbb{P}[B_{\Gamma(s_0,\tau)} \geq \gamma - \|G\|_{L^\infty}] = \sqrt{\frac{2}{\pi}} \int_{\frac{\gamma - \|G\|}{\sqrt{\Gamma(s_0,\tau)}}}^{+\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

The norm in the lower limit of integration is the $L^\infty(\mathbb{R})$ norm. We now recall that if $b > a > 0$ then

$$\int_b^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x < \int_a^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

With this in mind, since $\Gamma(s_0,\tau) \leq 2e^{(1-2\alpha)s_0}$, $\alpha < \frac{1}{2}$ and $s_0 \leq 0$ it follows that $\Gamma(s_0,\tau) \leq 2$ for every $\tau > 0$ implying that

$$\frac{\gamma - \|G\|_{L^\infty}}{\sqrt{\Gamma(s_0,\tau)}} > \frac{\gamma - \|G\|_{L^\infty}}{\sqrt{2}}$$

and so

$$\mathbb{P}[\Omega_2] \leq \sqrt{\frac{2}{\pi}} \int_{\frac{\gamma - \|G\|}{\sqrt{2}}}^{+\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

Since $\tau \geq 0$ was arbitrary, we now see the claim that as $\gamma \to \infty$

$$\mathbb{P}[\Omega_2] \longrightarrow 0,$$

uniformly in τ. From this, given $\varepsilon > 0$ chose $\gamma_0 > 0$ such that $\gamma > \gamma_0$ implies $\gamma - \|G\|_{L^\infty} > 0$ and

$$\mathbb{P}[\Omega_2] < \frac{\varepsilon}{2\|u_0\|_{L^\infty}}.$$

Thus, if $\gamma > \gamma_0$ then by the above expression for $v(0,\tau)$ we have

$$|v(0,\tau)| \leq e^{-\tau F(\gamma)}\|u_0\|_{L^\infty} + \frac{\varepsilon}{2}.$$

Now take $\tau_0 > 0$ such that $\tau > \tau_0$ implies

$$e^{-\tau F(\gamma)} < \frac{\varepsilon}{2\|u_0\|_{L^\infty}}.$$

Thus, we have that for every $\tau > \tau_0$

$$|v(0,\tau)| < \varepsilon$$

so there exists $\delta > 0$ such that $0 < 1 - t < \delta$ implies

$$|u(0,t)| < \varepsilon.$$

□

4.1.2 Power law for the behaviour of the solution near the singularity

Our aim in this section is to formulate a power law in the arc-length parameter for the solution to (3.2.3) with $\alpha < \frac{1}{2}$. In order to do this, we will use the flow-line formulation of the problem to gather some insight into the problem. However, in the end, we will have a power law in the arc-length parameter.

To gain some insight into the problem, consider (3.3.1) and set $\tau_* = (1-t)$ and relabel z as z_*. Define

$$Lv := \sqrt{\tau_*^{4\alpha} + 4z_*^2}\frac{\partial^2 v}{\partial z_*^2} - \alpha\frac{\tau_*^{4\alpha-1}}{\tau_*^{4\alpha} + 4z_*^2}v + \frac{2z_*}{\sqrt{\tau_*^{4\alpha} + 4z_*^2}}\frac{\partial v}{\partial z_*} + \frac{\partial v}{\partial \tau_*}$$

and fix some $z \in \mathbb{R} \setminus \{0\}$ and $\tau \in (0,1]$ and $\varepsilon > 0$. Let $\tau_* = \varepsilon\tau$ and $z_* = \varepsilon^{2\alpha}z$. Write $\bar{v}(z,\tau) = v(z_*,\tau_*)$ and supposing that $Lv = 0$ we have $L\bar{v} = 0$ and thus a direct computation reveals that

$$0 = \varepsilon^{1-2\alpha}\sqrt{\tau^{4\alpha} + 4z^2}\frac{\partial^2 \bar{v}}{\partial z^2} - \alpha\frac{\tau^{4\alpha-1}}{\tau^{4\alpha} + 4z^2}\bar{v} + 2\varepsilon^{1-2\alpha}\frac{z}{\sqrt{\tau^{4\alpha} + 4z^2}}\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{v}}{\partial \tau}.$$

Since $\alpha < \frac{1}{2}$, one may send $\varepsilon \to 0$, which suggests (but does not prove) that the behaviour of the solution v near the singularity is governed by the equation

$$\frac{\partial \bar{v}}{\partial \tau} = \alpha\frac{\tau^{4\alpha-1}}{\tau^{4\alpha} + 4z^2}\bar{v}$$

whose solution, up to a suitable function $A : \mathbb{R} \to \mathbb{R}$, is given by

$$\bar{v}(z, \tau) = A(z)(\tau^{4\alpha} + 4z^2)^{\frac{1}{4}}.$$

Noting that in arc-length coordinates

$$(\tau^{4\alpha} + 4z^2)^{\frac{1}{4}} = (1-t)^{\alpha} \sqrt{\cosh 2g(l(1-t)^{-\alpha})}$$

we make the ansatz that

$$u(l, t) = (1-t)^{\alpha} \sqrt{\cosh 2g(l(1-t)^{-\alpha})}\, \varphi_*(l, t)$$

for some function φ_*, solves (3.2.3). If we show that φ_* is uniformly bounded then one has a power law in arc-length parameterisation by Proposition A.0.1. Indeed, we have the following.

Theorem 4.1.3. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose that u is the unique solution to* (3.2.3) *with initial data u_0. Then there exists $C > 0$ such that, for every $(l, t) \in \mathbb{R} \times [0, 1)$,*

$$|u(l, t)| \leq C \|u_0\|_{L^\infty} (|l| + \kappa(1-t)^{\alpha}),$$

where $\kappa > 0$.

Remark 4.1.4. *Theorem 4.1.1 implies that u is uniformly bounded in space and time and so Theorem 4.1.3 gives no information for large l. However, for small l, it yields information as to how u vanishes at $(l, t) = (0, 1)$.*

Proof of Theorem 4.1.3. We begin the proof by assuming that u solves (3.2.3) and is given by

$$u(l, t) = (1-t)^{\alpha} \sqrt{\cosh 2g(l(1-t)^{-\alpha})}\, \varphi_*(l, t)$$

for some φ_* to be determined. For ease of calculations, take $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$. Write $v(y, \tau) = u(l, t)$ and $\varphi(y, \tau) = \varphi_*(l, t)$ so that

$$v(y, \tau) = e^{-\alpha\tau} (\cosh 2g(y))^{\frac{1}{2}} \varphi(y, \tau).$$

Thus, as u solves (3.2.3) we have that v satisfies (4.1.1). Thus, the equation for φ

reads

$$\frac{\partial\varphi}{\partial\tau} = e^{-\beta\tau}\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial\varphi}{\partial y}\left(\alpha(G(y)-y) + 2e^{-\beta\tau}\frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}}\right) + \\ + \left(\alpha + 2e^{-\beta\tau}\operatorname{sech}^3 2g(y) + \alpha(G(y)-y)\frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}} - \alpha\operatorname{sech}^2 2g(y)\right)\varphi$$

where $\beta = 1-2\alpha > 0$. However,

$$G(y) - y = -\frac{\sinh 2g(y)}{\sqrt{\cosh 2g(y)}} = -H(y)$$

and so the coefficient of φ is equal to $2e^{-\beta\tau}\operatorname{sech}^3 2g(y)$, by standard hyperbolic identities. The PDE for φ that we must analyse is

$$\begin{aligned}\frac{\partial\varphi}{\partial\tau} &= e^{-\beta\tau}\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial\varphi}{\partial y}\left(-\alpha H(y) + 2e^{-\beta\tau}\frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}}\right) + 2e^{-\beta\tau}\operatorname{sech}^3 2g(y)\varphi \\ \varphi(y,0) &= \frac{v(y,0)}{\sqrt{\cosh 2g(y)}} = \frac{u_0(y)}{\sqrt{\cosh 2g(y)}} =: \varphi_0(y).\end{aligned} \tag{4.1.4}$$

One should note that $\varphi_0 \in L^\infty(\mathbb{R})$ and that a unique classical solution to (4.1.4) exists[1]. In order to show that φ is uniformly bounded, we will use the Feynman-Kac formula. Thus, consider the following SDE

$$\begin{aligned}\mathrm{d}Y_s &= \left(-\alpha H(Y_s) + 2e^{\beta s}\frac{\tanh 2g(Y_s)}{\sqrt{\cosh 2g(Y_s)}}\right)\mathrm{d}s + \sqrt{2}e^{\frac{\beta}{2}s}\,\mathrm{d}W_s \\ Y_{-\tau} &= y\end{aligned} \tag{4.1.5}$$

for $-\tau \le s \le 0$. By Proposition 3.4.1 we know that H is globally Lipschitz. Define

$$b(y) := \frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}};$$

then

$$b'(y) = \frac{3\operatorname{sech}^2 2g(y) - 1}{\cosh 2g(y)} \in L^\infty(\mathbb{R})$$

and so the drift and diffusion of (4.1.5) are globally Lipschitz. Further, by Propo-

[1] One may wish to use the formulation along flow lines and write the resulting PDE as a Fokker-Planck equation and use the probabilistic version of Hörmander's Theorem (Theorem 2.5.3), as we do in Section 5.1 for the continued PDE. The details are, modulo sign, identical to those in Section 5.1.

sition 3.4.1 one has

$$\left| -\alpha H(y) + 2e^{\beta s} \frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}} \right| \leq C(|y| + 1)$$

for every $(y, s) \in \mathbb{R} \times [-\tau, 0]$. Hence the drift and diffusion of (4.1.5) satisfy the standard growth estimate and so there exists a unique strong solution $(Y_s)_{s \geq -\tau}$ on $[-\tau, 0]$ for every $\tau \geq 0$ to (4.1.5) (Theorem 2.3.2). We now apply the Feynman-Kac formula, which says that the unique solution to (4.1.4) is given by

$$\varphi(y, \tau) = \mathbb{E}^{(-\tau, y)} \left[\exp\left(2 \int_{-\tau}^{0} e^{\beta s} \operatorname{sech}^3 2g(Y_s) \, \mathrm{d}s \right) \varphi_0(Y_0) \right].$$

Hence

$$|\varphi(y, \tau)| \leq \exp\left(2 \int_{-\tau}^{0} e^{\beta s} \, \mathrm{d}s \right) \|\varphi_0\|_{L^\infty(\mathbb{R})} \leq e^{\frac{2}{\beta}} \|u_0\|_{L^\infty(\mathbb{R})}.$$

Thus, by transforming back into the (l, t) coordinates, we have that, for every $(l, t) \in \mathbb{R} \times [0, 1)$,

$$\begin{aligned} |u(l, t)| = |v(y, \tau)| &= e^{-\alpha\tau} \sqrt{\cosh 2g(y)} |\varphi(y, \tau)| \\ &\leq C \|u_0\|_{L^\infty(\mathbb{R})} (1 - t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha})}. \end{aligned}$$

To conclude the proof, we note from Proposition A.0.1 that there exists $\kappa > 0$ such that for every $s \in \mathbb{R}$,

$$\cosh 2g(s) = (|s| + \kappa)^2 + R(s)$$

where $R(s) \leq \mathcal{O}(|s|^{-2})$ as $|s| \to \infty$. So,

$$\sqrt{\cosh 2g(s)} = (|s| + \kappa) \sqrt{1 + \frac{R(s)}{(|s| + \kappa)^2}} \leq C(|s| + \kappa)$$

for every $s \in \mathbb{R}$.

$\square$

Remark 4.1.5. *Note that by the estimates above,* $\lim_{\tau \to \infty} \varphi(0, \tau)$ *exists up to a suitable subsequence. This will be useful in the continuation of the solution.*

4.2 Critical Regime

Consider (3.2.3) with $\alpha = \frac{1}{2}$. We perform the following change of variables into self–similar coordinates. Let

$$y = \frac{l}{\sqrt{1-t}}, \qquad \tau = -\log(1-t).$$

Write $v(y,\tau) = u(l,t)$. Then, as in Section 3.4 we see that v satisfies

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial y^2} - \frac{v}{2}\operatorname{sech}^2 2g(y) - \frac{1}{2}H(y)\frac{\partial v}{\partial y} \\ v(y,0) &= u_0(y) \end{aligned} \tag{4.2.1}$$

where $H(y) = y - G(y)$. We will refer to (4.2.1) as the "time-homogeneous problem". To aid the typography later, we define

$$-\bar{A}v := \frac{\partial^2 v}{\partial y^2} - \frac{v}{2}\operatorname{sech}^2 2g(y) - \frac{1}{2}H(y)\frac{\partial v}{\partial y} \tag{4.2.2}$$

and refer to $-\bar{A}$ as the "time-homogeneous operator".

We note that this operator is *not* self–adjoint on $L^2(\mathbb{R})$. A standard trick for such operators in mathematical physics is to perform a so–called ground state transformation. This transforms (4.2.1) into a Schrödinger equation which is self–adjoint on $L^2(\mathbb{R})$. The calculations can be found in Appendix B. This allows easier analysis of the properties of the operator $-\bar{A}$ such as the type of spectrum it has and what asymptotic properties the eigenfunctions possess. Indeed, we have the following important theorem which will be of use later.

Theorem 4.2.1. *The spectrum of $-\bar{A}$ is purely discrete. Further if $\lambda \in \sigma(-\bar{A})$ with corresponding eigenfunction v_λ then the following asymptotic holds as $|y| \to \infty$:*

$$v_\lambda(y) = C_0(|y| + \kappa)^{-2\lambda}(1 + o(1)).$$

Proof. The discreteness of the spectrum follows from Theorem C.0.4. For the asymptotic estimate, we prove the case of $y > 0$. The remaining case is easily obtained from this. Since

$$-\bar{A}v_\lambda = \lambda v_\lambda$$

we have that

$$\bar{A}v_\lambda = -\lambda v_\lambda.$$

Writing $v_\lambda(y) = \varphi(y)w_\lambda(y)$ with φ as in (B.0.1) we have that

$$\tilde{A}w_\lambda = -\lambda w_\lambda$$

where $\tilde{A}$ is given in (B.0.3) and so by Theorem C.0.4, multiplying by φ, given in Appendix B, we have, as $y \to \infty$,

$$v_\lambda(y) = \bar{C}_0(y+\kappa)^{-2\lambda}\varphi(y)\exp\left(-\frac{1}{8}(y+\kappa)^2\right)(1+o(1)).$$

The proof will be complete once we show that

$$C_2(y) := \varphi(y)\exp\left(-\frac{1}{8}(y+\kappa)^2\right)$$

is bounded. Indeed, by Proposition A.0.1 and the definition of φ and H we have

$$\begin{aligned}
\varphi(y)\exp\left(-\frac{1}{8}(y+\kappa)^2\right) &= \exp\left(\frac{1}{4}\int_0^y H(s)\,\mathrm{d}s - \frac{1}{8}(y+\kappa)^2\right) \\
&= \exp\left(\frac{1}{8}(\cosh 2g(y) - 1) - \frac{1}{8}(y+\kappa)^2\right) \\
&= \exp\left(\frac{1}{8}(y+\kappa)^2 - \frac{1}{8} + \bar{R}_2(y) - \frac{1}{8}(y+\kappa)^2\right) \\
&= \exp\left(\bar{R}_2(y) - \frac{1}{8}\right)
\end{aligned}$$

where, for large enough y

$$|\bar{R}_2(y)| \leq C|y|^{-2}.$$

Thus, it follows that

$$\lim_{y\to\infty}\varphi(y)\exp\left(-\frac{1}{8}(y+\kappa)^2\right)$$

exists and so $C_2(\cdot)$ is bounded. $\square$

Remark 4.2.2. *Suppose that $\lambda \in \sigma(\bar{A})$. By considering the Lebesgue space $L^2(\mathbb{R}, \mathrm{d}\mu(y); \mathbb{R})$ with weight*

$$\mathrm{d}\mu(y) = \exp\left(-\frac{1}{2}\int_0^y H(s)\,\mathrm{d}s\right)\mathrm{d}y,$$

it follows via the definition of the inner product and an integration by parts that $\lambda > 0$. Thus, it follows that the maximal eigenvalue of $-\bar{A}$, defined as μ, has $\mu < 0$. The calculation is a simple application of the integration by parts formula and is so *omitted.*

4.2.1 Vanishing of the solution and a power law

We are now ready to prove that the solution u to (3.2.3) vanishes at the singularity and obeys a power law in a neighbourhood of the singularity. As in the sub–critical case, we will work in the (y,τ) self–similar coordinates. Indeed, the long–term dynamics of (4.2.1) at $y=0$ will tell us how the solution u to (3.2.3) behaves. For $y \neq 0$, the long–term dynamics of (4.2.1) will yield a power law in the arc–length parameter for the solution u in a neighbourhood of the singularity, depending on time.

The $\alpha = \frac{1}{2}$ case causes some problems with the analysis. The methods of Section 4.1 fail as $\alpha = \frac{1}{2}$ is where quantities there are no longer integrable. It is also true that the methods we will employ in Section 4.3 also fail. Thus, for this section, we employ a functional analytical technique that shows that the solution multiplied by some time–dependent factor, converges to the projection of the first eigenfunction of an operator, with respect to a certain norm. We then use a Sobolev embedding to yield pointwise estimates on the solution.

We set $\mathcal{H} = -\tilde{A}$ where $-\tilde{A}$ is the ground state transformation of $-\bar{A}$ as given in equation (B.0.3) and denote the domain of $\mathcal{H}$ as $D(\mathcal{H})$. We denote the norm with respect to $D(\mathcal{H})$ as $\|\cdot\|_{\mathcal{H}}$ and note that $D(\mathcal{H}) \subset H^1(\mathbb{R})$.

We have the following estimate on the solution v to (4.2.1).

Theorem 4.2.3. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose that v is the unique smooth solution to* (4.2.1)*. Then, there exists $C_1, C_2 > 0$ and $\mu_1, \mu_2 > 0$ such that, for every $y \in \mathbb{R}$ and $\tau > 0$,*

$$\left|v(y,\tau) - C_1 e^{-\mu_1 \tau} h_1(y)\right| \le C_2 \|u_0\|_{\mathcal{H}} e^{-\mu_2 \tau} \varphi(y), \tag{4.2.3}$$

where $\mu_1 > 0$ is the minimal eigenvalue of $\bar{A}$, $\mu_2 > \mu_1$ is the second eigenvalue, h_1 is the eigenfunction of $\bar{A}$ corresponding to the minimal eigenvalue μ_1 and

$$\varphi(y) = \exp\left(\frac{1}{4}\int_0^y H(s)\,\mathrm{d}s\right).$$

Proof. Recalling the ground state transformation of Appendix B, we have that if v solves (4.2.1) then by writing $v(y,\tau) = \varphi(y) w(y,\tau)$ with φ as above, it follows that w solves

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= -\tilde{A} w \\ w(y,0) &= \varphi^{-1}(y) u_0(y). \end{aligned} \tag{4.2.4}$$

We note that $\tilde{A}$ is self-adjoint on $L^2(\mathbb{R})$ and

$$\tilde{A} = -\frac{\partial^2}{\partial y^2} + \mathcal{V}$$

with $\mathcal{V}(y) \geq -\frac{1}{4}$ for every $y \in \mathbb{R}$ and $\mathcal{V}(y) \to +\infty$ as $|y| \to \infty$. Thus, by Reed and Simon [1978] Theorem XIII.67, p.249, $\tilde{A}$ has a discrete spectrum and a complete orthonormal basis in $D(\tilde{A})$. Let $\{\mu_j\}_{j\in\mathbb{N}}$ be such eigenvalues of $\tilde{A}$, ordered such that $0 < \mu_1 < \mu_2 < \cdots$ with corresponding eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ in $D(\tilde{A})$. (Indeed, by Remark 4.2.2 one can check that $\mu_j > 0$ for every $j \in \mathbb{N}$.) We now express the solution w in terms of the eigenbasis and using the spectral mapping Theorem it follows that

$$w(\tau) = \sum_{j\in\mathbb{N}} \alpha_j e^{-\mu_j \tau} e_j$$

solves (4.2.4) where $\{\alpha_j\}_{j\in\mathbb{N}}$ are such that $w(0) = \sum_{j\in\mathbb{N}} \alpha_j e_j$. Thus, bounding $-\mu_j < -\mu_2$ for every $j \geq 3$ we have

$$\|w - \alpha_1 e_1 e^{-\mu_1\tau}\|_{\mathcal{H}} \leq C\|u_0\|_{\mathcal{H}} e^{-\mu_2\tau}$$

since $\varphi^{-1} \in L^\infty(\mathbb{R})$. Observing that $D(\mathcal{H}) \subset H^1(\mathbb{R})$, we use Sobolev embedding (Grafakos [2009], for example, which states that $H^1(\mathbb{R})$ embeds into $L^\infty(\mathbb{R})$) to conclude that, for the a.e continuous version of $w - \alpha_1 e_1 e^{-\mu_1\tau}$,

$$|w(y,\tau) - \alpha_1 e_1(y)\, e^{-\mu_1\tau}| \leq C\|u_0\|_{\mathcal{H}} e^{-\mu_2\tau}$$

for every $y \in \mathbb{R}$ and every $\tau > 0$. Reversing the ground-state transformation via $w(y,\tau) = \varphi^{-1}(y)v(y,\tau)$ and $e_1(y) = \varphi^{-1}(y)h_1(y)$ we conclude the result on multiplying through by φ.

$\square$

As a consequence, we immediately have the vanishing of the solution u to (3.2.3) by putting $y = 0$ into the above theorem, noting that $u_0 \in C_c^\infty(\mathbb{R})$ implies $u_0 \in D(\mathcal{H})$:

Theorem 4.2.4. *Let $u : \mathbb{R} \times [0,1) \to \mathbb{R}$ be the smooth solution to* (3.2.3) *with initial data $u_0 \in C_c^\infty(\mathbb{R})$. Then*

$$u(0,t) \longrightarrow 0 \quad as \quad t \longrightarrow 1^-$$

at an algebraically fast rate.

Remark 4.2.5. *Appealing to Theorem 4.1.1 with $\alpha = \frac{1}{2}$, we see that the theorem still holds true for the boundedness of the solution via the Feynman-Kac formula. Indeed,*

$$|u(l,t)| \le \|u_0\|_{L^\infty}$$

for every $(l,t) \in \mathbb{R} \times [0,1)$.

The natural question that now arises is whether a power–law which is analogous to that of Theorem 4.1.3 holds.

Due to the exponential weight, φ, in the estimate (4.2.3) we need to make sure that $|y|$ is bounded. The weight is needed to make sure that the resulting ground-state transformation is self-adjoint on $L^2(\mathbb{R})$. Indeed, any other weight fails to yield this property. If φ was polynomial with zero constant term, we may have been able to exploit the exponential decay in τ. As it is, we have the following which yields information about the solution in a time–dependent neighbourhood of $l = 0$.

Theorem 4.2.6. *Let $u : \mathbb{R} \times [0,1) \to \mathbb{R}$ be the smooth solution to* (3.2.3) *with initial data $u_0 \in C_c^\infty(\mathbb{R})$. Then, there exists $C > 0$ such that*

$$|u(l,t)| \le C(1 + \|u_0\|_{\mathcal{H}})(1-t)^{\mu_1} \tag{4.2.5}$$

for every $|l| \le \sqrt{1-t}$, where $\mu_1 > 0$ is given in Theorem 4.2.3.

Proof. By Theorem 4.2.3 above (working in the (y,τ) coordinates) and the triangle inequality, we have, for every $y \in \mathbb{R}$ and $\tau > 0$,

$$|v(y,\tau)| \le C_1 h_1(y) e^{-\mu_1 \tau} + C_2 \|u_0\|_{\mathcal{H}} e^{-\mu_2 \tau} \varphi(y).$$

Now take $|y| \le 1$, recalling that $\mu_2 > \mu_1$ and using the continuity of $y \mapsto h_1(y)$, we conclude that

$$|v(y,\tau)| \le C(1 + \|u_0\|_{\mathcal{H}}) e^{-\mu_1 \tau}.$$

However, we now change back into the (l,t) coordinates to conclude the result. $\square$

Remark 4.2.7. *This may be considered as a partial result, as the arc-length parameter that is considered decreases in size as $t \to 1^-$.*

Remark 4.2.8. *If the initial data u_0 is such that $\varphi^{-1} u_0 \in \operatorname{linspan}\{e_1, \cdots, e_N\}$ for some $N \in \mathbb{N}$, then by the above*

$$w - \alpha_1 e^{-\mu_1 \tau} = \sum_{j=2}^{N} \alpha_j e^{-\mu_j \tau} e_j$$

and so using the above with Theorem 4.2.1 it follows that, for every $|y| > y_0$ *and* $\tau \geq 0$,

$$|v(y,\tau) - \alpha_1 e^{-\mu_1 \tau} h_1(y)| \leq \sum_{j=2}^{N} |C_j|(|y| + \kappa)^{2\mu_j} e^{-\mu_j \tau},$$

where y_0 *depends on* N. *A power law follows on changing back to the* (l,t) *coordinates. Unfortunately, we have no control over the* C_j.

Indeed, asking that $\varphi^{-1} u_0 \in \operatorname{linspan}\{e_1, \cdots, e_N\}$ for some $N \in \mathbb{N}$ is quite a strong condition. We can do better by means of sub and super solutions. We first need an auxiliary lemma.

Lemma 4.2.9. *Consider the self–adjoint operator*

$$H := -\frac{\partial}{\partial y} + V(y)$$

where $V : \mathbb{R} \to \mathbb{R}$. *Suppose there exists* $c \in \mathbb{R}$ *such that* $V(y) \geq c$ *for every* $y \in \mathbb{R}$. *Suppose further that* $V \in L^1_{\text{loc}}(\mathbb{R})$. *Let* λ_1 *be the minimum eigenvalue with eigenfunction* $e_1 \in L^2(\mathbb{R})$. *Then,* e_1 *is continuous, satisfies* $e_1(y) > 0$ *for every* $y \in \mathbb{R}$ *and* e_1 *is locally bounded away from 0.*

Proof. That e_1 is continuous follows from Proposition 3.3 of Carmona [1979]. The argument that shows e_1 is locally bounded away from 0 is given in Remark 4.4 of Carmona [1979]. That λ_1 exists is a consequence of Theorem XIII.67, p.249 of Reed and Simon [1978]. □

We have the following main result.

Theorem 4.2.10. *Suppose* $u : \mathbb{R} \times [0,1) \to \mathbb{R}$ *is the smooth solution to* (3.2.3) *with initial data* $u_0 \in C_c^\infty(\mathbb{R})$. *Then, there exists* $C, y_0 > 0$ *such that*

$$|u(l,t)| \leq C \|u_0\|_{L^\infty} |l|^{2\mu_1}$$

for every $|l| > y_0\sqrt{1-t}$, *where* μ_1 *is the minimal eigenvalue of* $\bar{A}$ *and* $0 < \mu_1 \leq \frac{1}{2}$.

Proof. Let e_1 be the eigenfunction of $\tilde{A}$ with minimal eigenvalue $\mu_1 > 0$ as in the proof of Theorem 4.2.3. Considering the ground–state transformation, $\tilde{A}$, one has that the potential $\mathcal{V}(y)$ has $\mathcal{V}(y) \geq -\frac{1}{4}$ for every $y \in \mathbb{R}$, $\mathcal{V} \in L^1_{\text{loc}}(\mathbb{R})$ and $e_1 \in L^2(\mathbb{R})$. It thus follows from Lemma 4.2.9 that $e_1(y) > 0$ for every $y \in \mathbb{R}$ and e_1 is locally bounded away from 0. Thus, as $h_1(y) = \varphi(y)e_1(y)$ and φ is the exponential function, it follows that $h_1(y) > 0$ for every $y \in \mathbb{R}$ and h_1 is locally bounded away from 0.

Further, $\tilde{v}(y,\tau) := h_1(y)e^{-\mu_1\tau}$ solves

$$\begin{aligned}\frac{\partial\tilde{v}}{\partial\tau} &= -\bar{A}\tilde{v} \qquad (y,\tau)\in\mathbb{R}\times(0,\infty)\\ \tilde{v}(y,0) &= h_1(y) \qquad y\in\mathbb{R}.\end{aligned}$$

Take u to be the unique classical solution to (3.2.3) with initial condition u_0, converting to (y,τ) coordinates, we have that $v(y,\tau) = u(l,t)$ solves

$$\begin{aligned}\frac{\partial v}{\partial\tau} &= -\bar{A}v \qquad (y,\tau)\in\mathbb{R}\times(0,\infty)\\ v(y,0) &= u_0(y) \qquad y\in\mathbb{R}.\end{aligned}$$

Now consider the following: for any unique classical solution of

$$\begin{aligned}\frac{\partial\tilde{v}}{\partial\tau} &= -\bar{A}\tilde{v} \qquad (y,\tau)\in\mathbb{R}\times(0,\infty)\\ \tilde{v}(y,0) &= z_0(y) \qquad y\in\mathbb{R}.\end{aligned} \tag{4.2.6}$$

with $|z_0(y)| \leq A\exp\left(a|y|^2\right)$ (for some constants $A, a > 0$), the Feynman–Kac formula (Theorem 2.4.3) yields that the solution is given by the stochastic representation

$$\tilde{v}(y,\tau) = \mathbb{E}^y\left[\exp\left(-\frac{1}{2}\int_0^\tau \operatorname{sech}^2 2g(X_s)\,\mathrm{d}s\right) z_0(X_\tau)\right]. \tag{4.2.7}$$

Here, $(X_s)_{0\leq s\leq\tau}$ is the unique strong solution to

$$\begin{aligned}\mathrm{d}X_s &= -\frac{1}{2}H(X_s)\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}B_s\\ X_0 &= y\end{aligned}$$

for $0\leq s\leq\tau$. Since $H'(y) = 1 + \operatorname{sech}^2 2g(y)$, it follows from the standard theory of existence and uniqueness of strong solutions to SDEs (Theorem 2.3.2) that such a process $(X_s)_{0\leq s\leq\tau}$ exists. We will now create sub– and super–solutions to bound our solution v and use the Feynman–Kac formula above. To this end, let $w_1(y,\tau) = h_1(y)e^{-\mu_1\tau} - v(y,\tau)$. Then w_1 solves (4.2.6) with $z_0(y) = h_1(y) - u_0(y)$. Thus, if $z_0(y)\geq 0$ for every $y\in\mathbb{R}$ it follows from (4.2.7) that $w_1(y,\tau)\geq 0$ for every $(y,\tau)\in\mathbb{R}\times[0,\infty)$ and so

$$v(y,\tau) \leq h_1(y)e^{-\mu_1\tau}$$

holds for every $(y,\tau)\in\mathbb{R}\times[0,\infty)$. We repeat this argument with $w_2(y,\tau) := v(y,\tau)+h_1(y)e^{-\mu_1\tau}$, which solves (4.2.6) with initial condition $z_0(y) = u_0(y)+h_1(y)$.

Thus, if $z_0(y) \geq 0$ for every $y \in \mathbb{R}$ it follows from (4.2.7) that

$$v(y,\tau) \geq -h_1(y)e^{-\mu_1\tau}$$

holds for every $(y,\tau) \in \mathbb{R}\times[0,\infty)$. So, as $h_1(y) > 0$ for every $y \in \mathbb{R}$, if $|u_0(y)| \leq h_1(y)$ holds for every $y \in \mathbb{R}$, then the above implies

$$|v(y,\tau)| \leq h_1(y)e^{-\mu_1\tau}$$

for every $(y,\tau) \in \mathbb{R}\times[0,\infty)$. Finally, Theorem 4.2.1 implies the existence of $C, y_0 > 0$ such that $|y| > y_0$ implies $h_1(y) \leq C(|y|+\kappa)^{2\mu_1}$. Thus, $|y| > y_0$ implies

$$|v(y,\tau)| \leq C(|y|+\kappa)^{2\mu_1}e^{-\mu_1\tau}$$

for every $\tau \geq 0$. Suppose now that $u_0 \in C_c^\infty(\mathbb{R})$ is arbitrary. Since $\mathrm{supp}(u_0)$ is compact and h_1 is continuous, there exists $a \in (0,\infty)$ such that $h_1(y) \geq a$ for every $y \in \mathrm{supp}(u_0)$. Let

$$K = \frac{2}{a}\|u_0\|_{L^\infty}$$

then

$$\frac{|u_0(y)|}{K} = \frac{a}{2}\frac{|u_0(y)|}{\|u_0\|_{L^\infty}} \leq \frac{a}{2} < a \leq h_1(y)$$

for every $y \in \mathrm{supp}(u_0)$ and

$$\frac{|u_0(y)|}{K} = 0 < h_1(y)$$

for every $y \in \mathbb{R}\setminus\mathrm{supp}(u_0)$. Thus, $v_0(y) := \frac{u_0(y)}{K} \in C_c^\infty(\mathbb{R})$ has $|v_0(y)| \leq h_1(y)$ for every $y \in \mathbb{R}$. We now solve (4.2.6) with $z_0 = v_0$. From the Feynman–Kac formula (4.2.7) and the argument above, we have that, for every $|y| > y_0$ and $\tau \geq 0$,

$$\left|\mathbb{E}^y\left[\exp\left(-\frac{1}{2}\int_0^\tau \mathrm{sech}^2\, 2g(X_s)\,\mathrm{d}s\right)\frac{u_0(X_\tau)}{K}\right]\right| \leq C(|y|+\kappa)^{2\mu_1}e^{-\mu_1\tau}$$

and so recalling our definition of $K > 0$ and absorbing the $2/a$ term into the constant C leads to

$$\left|\mathbb{E}^y\left[\exp\left(-\frac{1}{2}\int_0^\tau \mathrm{sech}^2\, 2g(X_s)\,\mathrm{d}s\right)u_0(X_\tau)\right]\right| \leq C\|u_0\|_{L^\infty}(|y|+\kappa)^{2\mu_1}e^{-\mu_1\tau}.$$

However,

$$v(y,\tau) := \mathbb{E}^y\left[\exp\left(-\frac{1}{2}\int_0^\tau \mathrm{sech}^2\, 2g(X_s)\,\mathrm{d}s\right)u_0(X_\tau)\right]$$

is precisely the solution to (4.2.6) with $z_0 = u_0$. Thus

$$|v(y,\tau)| \leq C\|u_0\|_{L^\infty}(|y|+\kappa)^{2\mu_1}e^{-\mu_1\tau}$$

for every $|y| > y_0$ and every $\tau \geq 0$. We now switch coordinates back to the (l,t) coordinates and take

$$\sqrt{1-t} \leq \frac{|l|}{y_0}$$

to see the result. The estimate on the maximum size of μ_1 follows from Theorem D.0.5. □

Remark 4.2.11. *Theorem 4.2.10 and Theorem 4.2.6 together describe the behaviour of the solution u to* (3.2.3) *for $|l|$ small and t close to 1^-.*

4.3 Super–Critical Regime

4.3.1 Boundedness of the solution

Consider (3.2.3) with $\alpha > \frac{1}{2}$. Let $S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ denote the heat semigroup (Engel and Nagel [2006]). Then for $v \in L^\infty(\mathbb{R})$

$$(S(t)v)(l) := \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(l-y)^2}{4t}}v(y)\,\mathrm{d}y$$

and

$$\|S(t)v\|_{L^\infty} \leq \|v\|_{L^\infty}.$$

By Duhamel's principle

$$u(l,t) = (S(t)u_0)(l) + \alpha\int_0^t\int_{\mathbb{R}}(1-s)^{\alpha-1}K(l,y;t-s)\frac{\partial}{\partial y}\big(G(l(1-s)^{-\alpha})u(y,s)\big)\,\mathrm{d}y\,\mathrm{d}s \tag{4.3.1}$$

solves (3.2.3) in the mild sense of Definition 2.2.3. Here

$$K(l,y;t-s) = \frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(l-y)^2}{4(t-s)}}$$

is the classical heat kernel.

In the following, we will drop the arguments of G and u to make the presentation clearer. We will show that the mild solution u is bounded for every $t \in [0,1]$. Before we do this, we need the following important lemma concerning a Gronwall inequality.

Lemma 4.3.1. *Let $u : [0,1] \to \mathbb{R}_+$ be such that there exists $A, B > 0$ with*

$$u(t) \leq A + B \int_0^t (t-s)^{-\frac{1}{2}}(1-s)^{\alpha-1}u(s)\,\mathrm{d}s$$

where $\alpha > \frac{1}{2}$. Then there exists $C > 0$ such that

$$u(t) \leq C$$

for every $t \in [0,1]$.

Proof. Let $J = \int_0^t (t-s)^{-\frac{1}{2}}(1-s)^{\alpha-1}u(s)\,\mathrm{d}s$. Suppose first that $\alpha \geq 1$. Then by Hölder's inequality with $p, q \in (1,\infty)$ and Hölder conjugate:

$$J \leq \left(\int_0^t (t-s)^{-\frac{p}{2}}\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

However, if $1 < p < 2$ then

$$\int_0^t (t-s)^{-\frac{p}{2}}\,\mathrm{d}s = \frac{1}{1-\frac{p}{2}}t^{1-\frac{p}{2}}$$

and $1 - \frac{p}{2} > 0$. So with $1 < p < 2$ and $q \in (1,\infty)$ Hölder conjugate to p and since $t < 1$

$$J \leq C(p)t^{\frac{1}{p}-\frac{1}{2}} \left(\int_0^t u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \leq C(p)\left(\int_0^t u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

Hence,

$$u^q(t) \leq 2^{q-1}\left(A^q + B^q C(p)^q \int_0^t u^q(s)\,\mathrm{d}s\right) = C_1 + C_2\int_0^t u^q(s)\,\mathrm{d}s.$$

Thus Gronwall's inequality (Gronwall [1919]) implies

$$u^q(t) \leq C_1(1 + C_2 t e^{C_2 t}).$$

Taking $\frac{1}{q}$th powers and bounding t above by 1 concludes the result.

Now suppose $\frac{1}{2} < \alpha < 1$. Then by Hölder's inequality with $p, q \in (1,\infty)$ and Hölder conjugate and noting that for any $\gamma > 0$ and $0 \leq s < t$ it holds that $(1-s)^{-\gamma} \leq (t-s)^{-\gamma}$, we have

$$J \leq \left(\int_0^t (t-s)^{p(\alpha-3/2)}\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}}$$

Provided that $p < (\frac{3}{2} - \alpha)^{-1}$ we have that

$$\int_0^t (t-s)^{p(\alpha-3/2)}\,\mathrm{d}s = C(\alpha,p)\,t^{1+p(\alpha-3/2)}.$$

We note that for $\alpha \in (\frac{1}{2}, 1)$ we have $(\frac{3}{2} - \alpha)^{-1} > 1$ and so if $1 < p < (\frac{3}{2} - \alpha)^{-1}$ and $q \in (1, \infty)$ Hölder conjugate to p then bounding t above by 1

$$J \leq C(\alpha,p)\left(\int_0^t u^q(s)\,\mathrm{d}s\right)^{\frac{1}{q}}.$$

The argument is now identical to the $\alpha \geq 1$ case.

□

Remark 4.3.2. *The above proof does not work if $\alpha \leq \frac{1}{2}$.*

We now have the following, which is the main results of this thesis.

Theorem 4.3.3. *Let u be the unique solution to (3.2.3) with $\alpha > \frac{1}{2}$. Suppose that $u_0 \in C_c^\infty(\mathbb{R})$. Then*

$$\sup_{t\in[0,1]} \|u(\cdot,t)\|_{L^\infty} < \infty,$$

$\lim_{t\to1^-} u(l,t)$ *exists for every $l \in \mathbb{R}$ and*

$$v_0(l) := \lim_{t\to1^-} u(l,t)$$

defines a continuous function on $\mathbb{R}$.

Proof. By (4.3.1), an integration by parts and noting that $G \in L^\infty(\mathbb{R})$, we have that

$$|u(l,t)| \leq \|u_0\|_{L^\infty} + C\int_0^t \int_{\mathbb{R}} |\nabla_y K(l,y;t-s)|\,\|u(\cdot,s)\|_{L^\infty}(1-s)^{\alpha-1}\,\mathrm{d}y\,\mathrm{d}s.$$

However, by Davies [1989], Theorem 6, case 1, we have

$$|\nabla_y K(l,y;t-s)| \leq C(t-s)^{-1}\left(1 + \frac{(l-y)^2}{t-s}\right)e^{-\frac{(l-y)^2}{4(t-s)}}.$$

Hence

$$\begin{aligned} I &:= \int_0^t \int_{\mathbb{R}} |\nabla_y K(l,y;t-s)|\,\|u(\cdot,s)\|_{L^\infty}(1-s)^{\alpha-1}\,\mathrm{d}y\,\mathrm{d}s \\ &\leq C\int_0^t \|u(\cdot,s)\|_{L^\infty}(1-s)^{\alpha-1}(t-s)^{-1}\int_{\mathbb{R}}\left(1+\frac{(l-y)^2}{t-s}\right)e^{-\frac{(l-y)^2}{4(t-s)}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

However, if
$$z = \frac{(l-y)}{\sqrt{t-s}}$$
then
$$\int_{\mathbb{R}} \left(1 + \frac{(l-y)^2}{t-s}\right) e^{-\frac{(l-y)^2}{4(t-s)}} \,\mathrm{d}y = \int_{\mathbb{R}} (1+z^2) e^{-\frac{z^2}{4}} \,\mathrm{d}z (t-s)^{\frac{1}{2}} = C(t-s)^{\frac{1}{2}}.$$
Thus,
$$I \le C \int_0^t (t-s)^{-\frac{1}{2}} (1-s)^{\alpha-1} \|u(\cdot,s)\|_{L^\infty} \,\mathrm{d}s.$$
Putting all this together we have that
$$\|u(\cdot,t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + C \int_0^t (t-s)^{-\frac{1}{2}} (1-s)^{\alpha-1} \|u(\cdot,s)\|_{L^\infty} \,\mathrm{d}s$$
and so by Lemma 4.3.1 we conclude that there exists $K > 0$ such that
$$\|u(\cdot,t)\|_{L^\infty} \le K$$
for every $t \in [0,1]$, which establishes the uniform bound on u. For the second part of the theorem, define $T(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by
$$T(t)(g)(l) := \int_{\mathbb{R}} \nabla_y K(l-y,t) g(y) \,\mathrm{d}y.$$
Then,
$$u(l,t) = S(t)u_0(l) - \alpha \int_0^t T(t-s)(Gu)(1-s)^{\alpha-1} \,\mathrm{d}s(l).$$
Since $S(\cdot)$ and $T(\cdot)$ are defined for every time t, it remains to show that
$$\lim_{t \to 1^-} \int_0^t T(t-s)(Gu)(1-s)^{\alpha-1} \,\mathrm{d}s$$
exists. To this end, by the above result for u, the bound on the derivative of the heat kernel and the techniques above, we have
$$\left|\chi_{[0,t]}(s) T(t-s)(Gu)(1-s)^{\alpha-1}\right| \le C \chi_{[0,t]}(s)(t-s)^{-\frac{1}{2}}(1-s)^{\alpha-1}.$$
However, the right–hand side of this inequality belongs to $L^1([0,1], \mathrm{d}s; \mathbb{R})$ for every

$t \in [0,1]$ and so the dominated convergence theorem yields that

$$\lim_{t \to 1^-} \int_0^t T(t-s)(Gu)(1-s)^{\alpha-1}\,\mathrm{d}s = \int_0^1 T(1-s)(Gu)(1-s)^{\alpha-1}\,\mathrm{d}s.$$

Thus, v_0 exists and

$$v_0(l) = S(1)(u_0)(l) - \alpha \int_0^1 T(1-s)(Gu)(1-s)^{\alpha-1}\,\mathrm{d}s(l).$$

The continuity of $l \mapsto S(1)(u_0)(l)$ is clear from the smoothing property of the heat kernel. Recalling that $T(t)(g)(l) = (\nabla_y K(\cdot,t) * g)(l)$, where $*$ is the spatial convolution operator, the bounds on the derivative of the heat kernel, together with the continuity of $l \mapsto \nabla_y K(l-y, 1-s)$ and the dominated convergence theorem applied twice establishes the continuity of

$$l \mapsto \int_0^1 T(1-s)(Gu)(1-s)^{\alpha-1}\,\mathrm{d}s\,(l).$$

□

Remark 4.3.4. *It will be important in the continuation of the solution in Section 5.3 that the limiting function above, v_0, has $v_0 \in C(\mathbb{R})$.*

4.3.2 Behaviour of the solution at the singularity

To see how the solution u of (3.2.3) behaves at the singularity, we change variables.

Consider (3.2.3) and let

$$y = \frac{l}{\sqrt{1-t}}, \qquad \tau = -\log(1-t).$$

Writing $v(y,\tau) = u(l,t)$, we see that v satisfies

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial y^2} - \frac{y}{2}\frac{\partial v}{\partial y} - \alpha\,\mathrm{sech}^2\,2g(ye^{\beta\tau})v + \alpha e^{-\beta\tau} G(ye^{\beta\tau})\frac{\partial v}{\partial y} \\ v(y,0) &= u_0(y). \end{aligned} \tag{4.3.2}$$

Here $\beta = \alpha - \frac{1}{2} > 0$. Since there exists a unique solution u to (3.2.3) on $\mathbb{R} \times [0,1)$ it follows that a unique solution to (4.3.2) exists on $\mathbb{R} \times [0,\infty)$. By the Feynman-Kac formula one has that

$$v(y,\tau) = \mathbb{E}^{(-\tau,y)}\left[\exp\left(-\int_{-\tau}^{0} F(X_s e^{-\beta s})\,\mathrm{d}s\right) u_0(X_0)\right], \tag{4.3.3}$$

where $(X_s)_{s\geq -\tau}$ is the unique strong solution to

$$\begin{aligned} \mathrm{d}X_s &= \left(-\frac{X_s}{2} + \alpha e^{\beta s} G(X_s e^{-\beta s})\right) \mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s \\ X_{-\tau} &= y \end{aligned} \tag{4.3.4}$$

for $-\tau \leq s \leq 0$. Here $F(y) = \alpha\,\mathrm{sech}^2\,2g(y) \leq \mathcal{O}(|y|^{-4})$ for large $|y|$. The following lemma will be of use.

Lemma 4.3.5. *There exists a unique strong solution to* (4.3.4) *for every* $y \in \mathbb{R}$. *Furthermore, for* $y = 0$, $\mathbb{P}-a.s$

$$0 < \int_{-\infty}^{0} F(X_s e^{-\beta s})\,\mathrm{d}s \, < \infty$$

and hence, depending of course on u_0, $v(0,\tau)$ *need not vanish as* $\tau \to \infty$.

Proof. Let $b(x,s) := -x/2 + \alpha e^{\beta s} G(x e^{-\beta s})$, where $(x,s) \in \mathbb{R} \times [-\tau, 0]$. Then

$$\frac{\partial b}{\partial x} = -\frac{1}{2} - \alpha\,\mathrm{sech}^2\,2g(x e^{-\beta s}),$$

which is bounded uniformly in x and s. Further, $|b(x,s)| \leq C(1+|x|)$ for every $(x,s) \in \mathbb{R} \times [-\tau, 0]$. Thus, the standard existence and uniqueness theorem for SDEs (Theorem 2.3.2) implies the existence and uniqueness of such a strong solution to (4.3.4). Positivity of the integral follows as $F(y) > 0$ for every $y \in \mathbb{R}$. To see the finiteness of the integral, it suffices to show that

$$\mathbb{E}\left[\int_{-\infty}^{0} F(X_s e^{-\beta s})\,\mathrm{d}s\right] < \infty.$$

To achieve this we will show that

$$\int_{-\infty}^{0} \mathbb{E}\left[F(X_s e^{-\beta s})\right] \mathrm{d}s < \infty$$

and use Tonelli's Theorem.

To this end, we consider splitting

$$\mathbb{E}[F(X_s e^{-\beta s})] = \mathbb{E}[F(X_s e^{-\beta s})\chi_{\Omega_A}] + \mathbb{E}[F(X_s e^{-\beta s})\chi_{\Omega_B}]$$

where

$$\Omega_A = \{\omega \in \Omega \,:\, |X_s e^{-\beta s}| \leq e^{-\gamma s}\}$$

for some $\gamma > 0$ to be chosen later, and

$$\Omega_B = \Omega \setminus \Omega_A.$$

By the definition and asymptotics of F it follows that for $s < 0$ and $|s|$ large enough

$$\mathbb{E}[F(X_s e^{-\beta s})] \leq C\left(\mathbb{P}[|X_s| \leq e^{(\beta-\gamma)s}] + e^{4\gamma s}\right).$$

Since G is bounded and $s \leq 0$, consider the solution Y_s to

$$\begin{aligned} \mathrm{d}Y_s &= -\frac{Y_s}{2}\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s \\ Y_{-\tau} &= 0. \end{aligned}$$

Then Y_s is the classical Ornstien-Uhlenbeck (OU) process. Let $Z_s = X_s - Y_s$ and we see that Z_s solves

$$\mathrm{d}Z_s = -\frac{Z_s}{2}\,\mathrm{d}s + \alpha e^{\beta s} G(X_s e^{-\beta s})\,\mathrm{d}s$$

with initial data $Z_{-\tau} = 0$. The solution to this ODE with random coefficients is given by

$$Z_s = \alpha \int_{-\tau}^{s} e^{-\frac{1}{2}(s-r)} e^{\beta r} G(X_s e^{-\beta r})\,\mathrm{d}r.$$

It is easily seen that the following estimate holds, using the boundedness of G

$$|Z_s| \leq C(e^{\beta s} + e^{-\beta\tau})$$

where $C > 0$ is deterministic. We will use this to estimate the probability above. Since $X_s = Y_s + Z_s$ it follows that

$$\mathbb{P}[|X_s| \leq e^{(\beta-\gamma)s}] \leq \mathbb{P}[|Y_s| \leq e^{(\beta-\gamma)s} + C(e^{\beta s} + e^{-\beta\tau})].$$

Recall Y_s is a classical OU process, which is Gaussian. Thus we have for any $b > 0$

$$\mathbb{P}[|Y_s| \leq b] = \frac{1}{\sqrt{4\pi(1 - e^{-\tau}e^{-s})}} \int_{-b}^{b} \exp\left(-\frac{x^2}{4(1 - e^{-\tau}e^{-s})}\right)\,\mathrm{d}x.$$

Hence observing that

$$\int_{-b}^{b} \exp\left(-\frac{x^2}{4(1-e^{-\tau}e^{-s})}\right) \mathrm{d}x \le 2b$$

we have

$$P[|X_s| \le e^{(\beta-\gamma)s}] \le C\left(\frac{e^{(\beta-\gamma)s}+e^{\beta s}+e^{-\beta\tau}}{\sqrt{1-e^{-\tau}e^{-s}}}\right).$$

Choosing $\gamma = \frac{\beta}{2} > 0$ we have

$$\int_{-\tau}^{0} \mathbb{E}[F(X_s e^{-\beta s})]\,\mathrm{d}s \le C\left(\int_{-\tau}^{0} \frac{e^{\frac{\beta}{2}s}+e^{\beta s}+e^{-\beta\tau}}{\sqrt{1-e^{-\tau}e^{-s}}}\,\mathrm{d}s + 1 - e^{-2\beta\tau}\right).$$

Via a standard argument but lengthy calculation, one concludes that for any $\zeta > 0$,

$$\int_{-\tau}^{0} \frac{e^{\zeta s}}{\sqrt{1-e^{-\tau}e^{-s}}}\,\mathrm{d}s \le C\zeta^{-1}\left(\left(1-e^{-\frac{\tau}{2}}\right)^{-\frac{1}{2}}+1\right)$$

and

$$\int_{-\tau}^{0} \frac{e^{-\beta\tau}}{\sqrt{1-e^{-\tau}e^{-s}}}\,\mathrm{d}s \le C(\tau+1)e^{-\beta\tau}(1-e^{-\frac{\tau}{2}})^{-\frac{1}{2}}.$$

Thus, one concludes that

$$\int_{-\infty}^{0} \mathbb{E}[F(X_s e^{-\beta s})]\,\mathrm{d}s < \infty,$$

and so by Tonelli's Theorem,

$$\mathbb{E}\left[\int_{-\infty}^{0} F(X_s e^{-\beta s})\,\mathrm{d}s\right] < \infty$$

thus $\mathbb{P}-a.s$

$$\int_{-\infty}^{0} F(X_s e^{-\beta s})\,\mathrm{d}s < \infty.$$

□

We now have the following theorem that describes the behaviour of the solution to (3.2.3) at the singularity, for a specific initial data.

Theorem 4.3.6. *There exists $u_0 \in L^\infty(\mathbb{R})$ such that the unique solution to* (3.2.3)*, u, does not vanish at the singularity.*

Proof. By the change of variables as above and the Feynman-Kac formula, it suffices to show that $v(0,\tau)$ does not tend to 0 as $\tau \to \infty$, for some specified initial data

u_0. One takes $u_0(l) = 1$ for every $l \in \mathbb{R}$. Then, from Lemma 4.3.5 we see that $v(0, \tau) \geq 1$ for every $\tau \in [0, \infty)$.

$\square$

Remark 4.3.7. *By the finiteness of the integral in Lemma 4.3.5, we see that for initial data $u_0(l) = 1$, the solution to* (3.2.3) *does* not *vanish at $l = 0$ as $t \to 1^-$. By Theorem 2.3.2, such a solution with that initial data exists, is smooth and by Theorem 4.3.3 is uniformly bounded.*

Remark 4.3.8. *For the super–critical regime the behaviour of the initial condition at the singularity dictates what happens to the solution at the singularity. We have seen in Section 4.1 and Section 4.2 that this is* not *the case for the other regimes.*

The possibility of the solution not vanishing at the singularity presents a slight technicality. Recall Chapter 3 and suppose the initial data was not *symmetric about $y > 0$ and $y < 0$ for the curve $\mathcal{C}_t^\alpha$. Then, by the above we have that at $t = 1$, a multivalued function is defined at the singularity. Thus, any notion of continuing the solution is ill-defined. Indeed, this was one of the reasons why we always assume that the initial data is symmetric about $y > 0$ and $y < 0$. Hence Section 5.3* will *make sense as no multivalued functions are taken as initial data since the value at the singularity for $y > 0$ and $y < 0$ agree, by symmetry.*

Chapter 5

Analysis of Problem I: After the Singularity

Our goal is to show that we can continue the solution in all regimes past the singularity. Recall

$$\mathcal{C}_t^{\alpha} = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0,\, y^2 - x^2 = (1-t)^{2\alpha}\}, \quad 0 \leq t \leq 1.$$

Now, the subsequent motion of the curve for $t \geq 1$ is given by

$$\mathcal{C}_t^{\alpha,\mathrm{cont}} = \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 0,\, x^2 - y^2 = (t-1)^{2\alpha}\}, \quad 1 \leq t \leq T \tag{5.0.1}$$

for some $T \in (1,\infty)$. Figure 3.2 illustrates $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ for $\alpha = 0.5$ and various values of $t \in [1,2]$. Using the level set $\phi(x,y,t) = x^2 - y^2 - (t-1)^{2\alpha}$ we take

$$\nu = \frac{\nabla\phi}{|\nabla\phi|}$$

as the *inward* pointing unit normal. We let

$$Y : \mathbb{R} \times [1,T] \to \mathbb{R}^2$$

be a parameterisation of $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ defined by

$$Y(p,t) = (t-1)^{\alpha}\,(\cosh p, \sinh p)\,.$$

Let Y_p denote the partial derivative with respect to p and define

$$l := \int_0^p |Y_p(u,t)|\,\mathrm{d}u.$$

So for $t > 1$

$$\frac{l}{(t-1)^{\alpha}} = \int_0^p \sqrt{\cosh 2u}\,\mathrm{d}u.$$

As before, denote by g the *inverse* of the map

$$p \mapsto \int_0^p \sqrt{\cosh 2u}\,\mathrm{d}u$$

as in Definition 3.2.1. The arc-length parameterisation of $\mathcal{C}_t^{\alpha,\mathrm{cont}}$ is given by

$$X(l,t) = (t-1)^{\alpha}\left(\cosh g(l(t-1)^{-\alpha}), \sinh g(l(t-1)^{-\alpha})\right).$$

We wish to continue the solution to (3.1.3), and thus we wish to study

$$\begin{aligned} \partial_t^{\bullet} V + V\nabla_{\Gamma}\cdot v - \Delta_{\Gamma} V = 0 \quad x \in \Gamma := \mathcal{C}_t^{\alpha,\mathrm{cont}} \\ V(x,1) = U(Bx,1) \quad x \in \mathcal{C}_1^{\alpha,\mathrm{cont}}. \end{aligned} \tag{5.0.2}$$

where

$$v := \frac{\phi_t}{|\nabla\phi|}\nu$$

is the prescribed normal velocity of the curve with ν given above and $U(Bx,1)$ is the solution to (3.1.3) at time $t=1$ in the sense of Chapter 4. Here, $B : \mathcal{C}_1^{\alpha,\mathrm{cont}} \to \mathcal{C}_1^{\alpha}$ is a linear map. Since v is in the normal direction only, it follows as before that

$$\nabla_{\Gamma}\cdot v = VH$$

with

$$V = \frac{\phi_t}{|\nabla\phi|}$$

and

$$-H = \frac{1}{|\nabla\phi|}\sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j},$$

since ν is the *inward* pointing normal. A direct computation reveals that

$$VH = -\alpha(t-1)^{-1}\,\mathrm{sech}^2\, 2g(l(t-1)^{-\alpha}).$$

As in the motion of the curve for $t < 1$ we have the following non-physical correction drift coefficient to yield a nice constant-coefficient second order term,

$$X_t \cdot X_l = \alpha(t-1)^{\alpha-1}\left(-\frac{l}{(t-1)^{\alpha}} + \frac{\sinh 2g(l(t-1)^{-\alpha})}{\sqrt{\cosh 2g(l(t-1)^{-\alpha})}}\right).$$

One should compare this with the expression for the motion of the curve for $t < 1$.

We now write $v(l,t) = V(X(l,t),t)$ and see that v satisfies

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial l^2} - \alpha t^{\alpha-1}\frac{\partial}{\partial l}\left(G(lt^{-\alpha})v\right) \quad (l,t) \in \mathbb{R}\times(0,T) \\
v(l,0) &= v_0(l) \quad l \in \mathbb{R}.
\end{aligned}
\tag{5.0.3}
$$

where G is given in (3.2.4), $v_0(l) = u(l,1)$ with u the limit (up to a suitable subsequence) as in Chapter 4 and we have made the substitution $t \mapsto t-1$.

A major problem is that (5.0.3) is initially singular, whereas the problem before was that (3.2.3) was singular in the limit as $t \to 1^-$. We now look at the analysis of (5.0.3) in each of the different regimes. As the reader will see, the methods of analysis vary depending on the regime.

5.1 Sub–Critical Regime

We adopt a probabilistic approach in showing existence of a continuation of the solution. To this end, consider (5.0.3) and note that by Theorem 4.1.1 we have that the initial data, v_0 is bounded.

In the following, we will use the flow line formulation to show that there exists a unique solution to (5.0.3), but the solution may not attain the initial data of (5.0.3). Indeed, we prove the attainment of the initial data by using a suitable ansatz for the solution v of (5.0.3).

Let $z(l,t) = \frac{1}{2}t^{2\alpha}\sinh 2g(lt^{-\alpha})$ and write $w(z,t) = v(l,t)$. Then, by standard calculations it follows that w solves

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \sqrt{t^{4\alpha}+4z^2}\frac{\partial^2 w}{\partial z^2} + \alpha\frac{t^{4\alpha-1}}{t^{4\alpha}+4z^2}w + \frac{2z}{\sqrt{t^{4\alpha}+4z^2}}\frac{\partial w}{\partial z} \\
w(z,0) &= w_0(z)
\end{aligned}
\tag{5.1.1}
$$

where $w_0(z) = v_0(2\sqrt{|z|}) \in L^\infty(\mathbb{R})$.

We note that this equation is the same as equation (3.3.1), but with $(1-t)$ replaced by t and the sign of the coefficient of w reversed.

By the same asymptotic analysis process as in Section 4.1.2, the important equation near the singularity is

$$
\frac{\partial \tilde{w}}{\partial t} = \alpha\frac{t^{4\alpha-1}}{t^{4\alpha}+4z^2}\tilde{w}
$$

whose solution, up to some function $A : \mathbb{R} \to \mathbb{R}$, is given by

$$\tilde{w}(z,t) = A(z)(t^{4\alpha} + 4z^2)^{1/4}.$$

This provides us with a guess as to what the solution looks like near the singularity. We now write $w(z,t) = (t^{4\alpha} + 4z^2)^{1/4}\varphi(z,t)$ and so φ solves

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \sqrt{t^{4\alpha} + 4z^2}\frac{\partial^2 \varphi}{\partial z^2} + \frac{2t^{4\alpha}}{(t^{4\alpha} + 4z^2)^{3/2}}\varphi + \frac{6z}{\sqrt{t^{4\alpha} + 4z^2}}\frac{\partial \varphi}{\partial z} \\ \varphi(z,0) &= \frac{w(z,0)}{|z|^{1/2}\sqrt{2}} =: \varphi_0(z). \end{aligned} \tag{5.1.2}$$

By Theorem 4.1.3 we have $\varphi_0 \in L^\infty(\mathbb{R})$ and note that by Remark 4.1.5 we have that

$$\lim_{z \to 0} \varphi_0(z)$$

exists and is finite.

Remark 5.1.1 (On the method of analysis for the existence of a solution to (5.1.2))**.** *We note that* (5.1.2) *is* not *uniformly elliptic. Indeed, it is the author's opinion that the method employed in the proof of Theorem 5.1.2 is the best way to treat this equation. This is because the equation is of Fokker–Planck type as given in equation* (2.5.2) *and so the probabilistic approach is the most natural. One should note that the Fokker–Planck approach is only applicable when the equation is of Fokker–Planck type. When the equation is not of this type, the methods in this thesis are not applicable and standard PDE theory should be used. For a discussion of some classical methods of existence in the weak sense, the reader is referred to the relevant references of Section 1.1.*

We have the following existence theorem.

Theorem 5.1.2. *There exists a solution* $\varphi \in C^{2,1}(\mathbb{R} \times (0,T];\mathbb{R})$ *to* (5.1.2) *for every* $0 < T < \infty$ *and so, by changing coordinates from* (z,t) *to* (l,t)*, there exists a solution* $v \in C^{2,1}(\mathbb{R} \times (0,T];\mathbb{R})$ *to* (5.0.3) *for every* $0 < T < \infty$.

Remark 5.1.3. *The following proof does not guarantee that the initial condition is attained. Later, we will show that the initial data* is *attained and so ensuring the uniqueness of such a solution.*

Proof of Theorem 5.1.2. Observe that one may rewrite (5.1.2) as

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= \frac{\partial^2}{\partial z^2}\Big((t^{4\alpha}+4z^2)^{1/2}\varphi\Big) - \frac{\partial}{\partial z}\Big(2z(t^{4\alpha}+4z^2)^{-1/2}\varphi\Big)\\ \varphi(z,0) &= \varphi_0(z)\end{aligned}$$

which is a Fokker-Planck equation. Let $D(z,t) = (t^{4\alpha}+4z^2)^{1/2}$ and $\mu(z,t) = 2z(t^{4\alpha}+4z^2)^{-1/2}$ and consider the SDE in Itō form

$$\begin{aligned}\mathrm{d}X_t &= \mu(X_t,t)\,\mathrm{d}t + \sqrt{2D(X_t,t)}\,\mathrm{d}W_t\\ X_0 &= \Phi\end{aligned}$$

where $\Phi \sim \varphi_0$ which means that Φ is distributed according to φ_0. Indeed, we assume that Φ has a density with respect to the Lebesgue measure, and that density is given by φ_0.
Calculating the drift and diffusion in the above SDE we see that

$$\begin{aligned}\mathrm{d}X_t &= 2X_t(t^{4\alpha}+4X_t^2)^{-\frac{1}{2}}\,\mathrm{d}t + \sqrt{2}\,(t^{4\alpha}+4X_t^2)^{1/4}\,\mathrm{d}W_t\\ X_0 &= \Phi.\end{aligned}\tag{5.1.3}$$

We note that $\mathbb{P}(\Phi = 0) = 0$ and so since Φ is independent of the Brownian motion $W_\bullet$, we may consider $\Phi = y$ for some $y \in \mathbb{R}\setminus\{0\}$. Without loss of generally, assume $y > 0$. The argument for $y < 0$ is analogous. Fix $\delta > 0$ such that $0 < \delta < y$ and define

$$b(t,z) := \begin{cases} 2\delta(t^{4\alpha}+4\delta^2)^{-\frac{1}{2}} & \text{if } -\infty < z < \delta\\ 2z(t^{4\alpha}+4z^2)^{-\frac{1}{2}} & \text{if } \delta \le z < \infty\end{cases}$$

and

$$\sigma(t,z) := \begin{cases} \sqrt{2}\,(t^{4\alpha}+4\delta^2)^{1/4} & \text{if } -\infty < z < \delta\\ \sqrt{2}\,(t^{4\alpha}+4z^2)^{1/4} & \text{if } \delta \le z < \infty.\end{cases}$$

Then, for every $t \ge 0$, $z \mapsto \sigma(t,z)$ is globally Lipschitz, with at most linear growth at infinity. The same holds for $z \mapsto b(t,z)$ and so the standard theory of existence and uniqueness of a strong solution (Theorem 2.3.2) implies

$$X_t = y + \int_0^t b(s,X_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s)\,\mathrm{d}W_s$$

for every $t \ge 0$. If $\tau := \inf\{t \ge 0 \,|\, X_t \le \delta\}$ then $t < \tau$ implies $X_t > \delta$ and so

$$b(t,X_t) = 2X_t(t^{4\alpha}+4X_t^2)^{-\frac{1}{2}}, \qquad \sigma(t,X_t) = \sqrt{2}\,(t^{4\alpha}+4X_t^2)^{1/4}.$$

We thus deduce short time existence to (5.1.3). One stops the evolution at a time $t \in (0, \tau/2)$ and restarts the evolution so giving existence and uniqueness to a strong solution of (5.1.3) for $t \in [0, T]$. This procedure works since for $t > 0$, the coefficients μ and D satisfy the standard conditions for existence and uniqueness of a strong solution. It remains to show that $\mathbb{P}(\tau = 0) = 0$. For $t > 0$,

$$\mathbb{P}(\tau < t) \leq \mathbb{P}(\inf_{0 \leq s \leq t} X_s \leq \delta)$$

and so by the continuity of

$$t \mapsto \inf_{0 \leq s \leq t} X_s$$

one concludes that

$$\lim_{t \to 0} \inf_{0 \leq s \leq t} X_s = y > \delta$$

and so $\mathbb{P}(\tau < t) \to 0$ as $t \to 0$.

Finally, we note that (5.1.3) becomes the following when interpreted in the Stratonovich sense:

$$\begin{aligned} \mathrm{d}X_t &= \left(\mu(X_t, t) - \frac{1}{2} \frac{\partial}{\partial z} D(z, t)|_{z = X_t} \right) \mathrm{d}t - \sqrt{2D(X_t, t)} \circ \mathrm{d}W_t \\ X_0 &= \Phi \end{aligned} \tag{5.1.4}$$

where μ and D are given above. In this case, one may check that

$$\mu(z, t) - \frac{1}{2} \frac{\partial D}{\partial z}(z, t) = 0$$

for every $(z, t) \in \mathbb{R} \times (0, T]$. Thus, appealing to Theorem 2.5.3 and Remark 2.5.5 one takes $V_0(z, t) = 0$ and $V_1(z, t) = \sqrt{2}(t^{4\alpha} + 4z^2)^{\frac{1}{4}}$ to see that $\mathcal{V}_k = \mathcal{V}_0$ for every $k \geq 1$. This implies that $\mathcal{V}_k(z, t) = \mathrm{Span}\{V_1(z, t)\}$. Thus, for fixed $(z, t) \in \mathbb{R} \times (0, T]$, given $y \in \mathbb{R}$, there exists $\alpha \in \mathbb{R}$ such that $y = \alpha V_1(z, t)$. Thus, by Theorem 2.5.3 and Remark 2.5.5 there exists $\varphi \in C^{2,1}(\mathbb{R} \times (0, T]; \mathbb{R})$ such that (5.1.2) is satisfied. Hence a solution to (5.0.3) exists.

□

One should note that the theorem says nothing about the attainment of initial data; this issue is addressed in the next theorem. In order to show that the initial data is attained, one would like to use the Feynman-Kac formula, applied to the process which solves the SDE

$$\begin{aligned}
\mathrm{d}Y_s &= \frac{6Y_s}{((-s)^{4\alpha} + 4Y_s^2)^{1/2}}\,\mathrm{d}s + \sqrt{2}\,((-s)^{4\alpha} + 4Y_s^2)^{1/4}\,\mathrm{d}W_s \\
Y_{-t} &= z.
\end{aligned}$$

However, standard techniques of using local Lipschitz continuity of the coefficients only yield a solution to the above SDE for $-t \le s < 0$. Indeed, there is no a priori way to define the solution at $s = 0$. As the Feynman-Kac formula relies on at least knowing the law of X_0, one should look for a different approach.

This problem is remedied if we use the original (l, t) coordinates. Recall that $(t^{4\alpha} + 4z^2)^{1/4} = t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2}$. So, for (5.0.3) we make the ansatz

$$v(l,t) = t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2}\varphi(l,t).$$

The approach is now to show that considering the equation φ must satisfy, there exists a solution and that φ attains its initial data. If this is true, the following lemma shows that $v(l,t) \to v_0(l)$ as $t \to 0$ for every $l \in \mathbb{R}$.

Lemma 5.1.4. *For every $l \in \mathbb{R}$,*

$$\lim_{t\to 0} t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2} = |l|.$$

Proof. If $l = 0$ then $\cosh 2g(lt^{-\alpha}) = 1$ and so the result is obvious. Suppose that $l \neq 0$. Then Proposition A.0.1 implies

$$(\cosh 2g(lt^{-\alpha}))^{1/2} = \left[\left(\frac{|l|}{t^\alpha} + \kappa\right)^2 + R(lt^{-\alpha})\right]^{1/2}$$

where $R(y) \le \mathcal{O}(|y|^{-4})$ for large $|y|$. Thus, rearranging this expression, one has

$$t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2} = \left[(|l| + \kappa t^\alpha)^2 + t^{2\alpha}R(lt^{-\alpha})\right]^{1/2}.$$

One sends $t \to 0$ to conclude the result. $\square$

From our ansatz of $v(l,t) = t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2}\varphi(l,t)$, a simple calculation shows that φ solves

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial l^2} + \left[2t^{-\alpha}\frac{\tanh 2g(lt^{-\alpha})}{\sqrt{\cosh 2g(lt^{-\alpha})}} - \alpha t^{\alpha-1}G(lt^{-\alpha})\right]\frac{\partial\varphi}{\partial l} + 2t^{-2\alpha}\operatorname{sech}^3 2g(lt^{-\alpha})\varphi$$
$$\varphi(l,0) = \frac{v_0(l)}{|l|} =: \varphi_0(l). \tag{5.1.5}$$

Indeed, Remark 4.1.5 implies that $\lim_{l\to 0}\varphi_0(l)$ exists[1], is finite and Theorem 4.1.3 implies that $\varphi_0 \in L^\infty(\mathbb{R})$. We now have one of the main results of this thesis.

Theorem 5.1.5. *There exists a unique solution to* (5.0.3) *and for every* $l \in \mathbb{R}$,

$$\lim_{t\to 0} v(l,t) = v_0(l).$$

Proof. Consider the ansatz $v(l,t) = t^\alpha(\cosh 2g(lt^{-\alpha}))^{1/2}\varphi(l,t)$. Such a φ exists by Theorem 5.1.2 and so such a v exists. We will show that φ attains its initial data in (5.1.5). We will appeal to the Feynman-Kac formula in the following. To this end, we want to show that there exists a unique strong solution to

$$\mathrm{d}X_s = \left(2(-s)^{-\alpha}\frac{\tanh 2g(X_s\,(-s)^{-\alpha})}{\sqrt{\cosh 2g(X_s\,(-s)^{-\alpha})}} - \alpha(-s)^{\alpha-1}G(X_s\,(-s)^{-\alpha})\right)\mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s$$
$$X_{-t} = l \tag{5.1.6}$$

for $-t \le s \le 0$. To this end, define

$$b(y,s) := \left(2(-s)^{-\alpha}\frac{\tanh 2g(y\,(-s)^{-\alpha})}{\sqrt{\cosh 2g(y\,(-s)^{-\alpha})}} - \alpha(-s)^{\alpha-1}G(y\,(-s)^{-\alpha})\right)$$

for $(y,s) \in \mathbb{R}\times[-t,0)$. Then

$$\frac{\partial b}{\partial y}(y,s) = \alpha(-s)^{-1}\operatorname{sech}^2 2g(y(-s)^{-\alpha}) +$$
$$+ 2(-s)^{-2\alpha}(2\operatorname{sech}^3 2g(y(-s)^{-\alpha}) - \tanh^2 2g(y(-s)^{-\alpha})\operatorname{sech} 2g(y(-s)^{-\alpha})).$$

Thus, it follows that $y \mapsto b(y,s)$ is globally Lipschitz for every $-t \le s < 0$. Further, by the asymptotics of G (Proposition A.0.2) it follows that there exists $C = C(n) > 0$ where $n \in \mathbb{N}$ such that

$$|b(y,s)| \le C(1+|y|)$$

for every $y \in \mathbb{R}$ and every $-\frac{t}{2^n} \le s \le -\frac{t}{2^{n+1}}$ for every $n \in \mathbb{N}$. Hence, by the

[1]We must, of course, take the suitable subsequence in defining $v_0(l)$.

standard existence and uniqueness Theorem for strong solutions (Friedman [1975]) to (5.1.6), one concludes there exists a unique strong solution to (5.1.6), and the solution is given by

$$X_s = l + \int_{-t}^{s} b(X_r, r)\,\mathrm{d}r + \sqrt{2}(B_s - B_{-t})$$

for every $-t \leq s < 0$. We show that the solution can be extended to $-t \leq s \leq 0$. Define

$$Y := l + \int_{-t}^{0} b(X_r, r)\,\mathrm{d}r - \sqrt{2}\,B_{-t}.$$

Then, by the definition of b and the fact that tanh and G are bounded we have

$$|Y| \leq |l| + C\int_{-t}^{0} (-r)^{-\alpha} + (-r)^{\alpha-1}\,\mathrm{d}r + \sqrt{2}\,|B_{-t}|.$$

Since $\alpha < \frac{1}{2}$, it follows that

$$\int_{-t}^{0} (-r)^{-\alpha}\,\mathrm{d}r = \frac{t^{1-\alpha}}{1-\alpha}$$

and

$$\int_{-t}^{0} (-r)^{\alpha-1}\,\mathrm{d}r = \frac{t^{\alpha}}{\alpha}.$$

Thus, $|Y| < \infty$, $\mathbb{P}-$a.s. Also, $\mathbb{P}-$a.s

$$|X_s - Y| \leq C_\alpha((-s)^{1-\alpha} + (-s)^{\alpha}) + \sqrt{2}|B_s| \longrightarrow 0$$

as $s \to 0^-$. Thus

$$X_s = l + \int_{-t}^{s} b(X_r, r)\,\mathrm{d}r + \sqrt{2}(B_s - B_{-t})$$

exists for every $-t \leq s \leq 0$ and also satisfies (5.1.6) for every $-t \leq s \leq 0$.

The Feynman-Kac formula yields that

$$\varphi(l,t) = \mathbb{E}^{(-t,l)}\left[\exp\left(2\int_{-t}^{0} (-s)^{-2\alpha}\operatorname{sech}^3 2g(X_s(-s)^{-\alpha})\,\mathrm{d}s\right)\varphi_0(X_0)\right]$$

solves (5.1.5). Note that as $\alpha < \frac{1}{2}$ we have

$$0 \leq 2\int_{-t}^{0} (-s)^{-2\alpha}\operatorname{sech}^3 2g(X_s(-s)^{-\alpha})\,\mathrm{d}s \leq 2\int_{-t}^{0} (-s)^{-2\alpha}\,\mathrm{d}s = \frac{2t^{1-2\alpha}}{1-2\alpha} \longrightarrow 0$$

as $t \to 0$. So by continuity of $y \mapsto \exp(y)$ we have that

$$\exp\left(2\int_{-t}^{0}(-s)^{-2\alpha}\operatorname{sech}^3 2g(X_s(-s)^{-\alpha})\,\mathrm{d}s\right) \longrightarrow 1$$

as $t \to 0$. We thus conclude by the Dominated Convergence Theorem, that for every $l \in \mathbb{R}$,

$$\varphi(l,t) \longrightarrow \varphi_0(l)$$

as $t \to 0$. Recall

$$v(l,t) = t^{\alpha}(\cosh 2g(lt^{-\alpha}))^{1/2}\,\varphi(l,t)$$

so that by Lemma 5.1.4 above, the product of limits and the definition of φ_0, we conclude that for every $l \in \mathbb{R}$

$$v(l,t) \longrightarrow |l|\varphi_0(l) = v_0(l).$$

Thus, such a solution v to (5.0.3) exists and attains its initial data. Uniqueness follows for if there were two solutions then the difference would have initial data 0. The Feynman-Kac formula implies that the difference of the solutions is zero everywhere and so the two solutions are equal.

□

5.2 Critical Regime

5.2.1 Short–Time Existence

We seek short–time existence to (5.0.3) with $\alpha = \frac{1}{2}$. Since, for $t > 0$ the coefficients of (5.0.3) are bounded and smooth, one can naturally extend the solution to some arbitrary time $T \in (0, \infty)$.

In the following, recall that $G \in L^{\infty}(\mathbb{R})$. The main idea for short time existence to (5.0.3) is to look at the equation as a Fokker-Planck equation and establish short-time existence for the associated Stochastic Differential Equation (SDE). It is standard, via the probabilistic version of Hörmander's Theorem (Theorem 2.5.3) to establish a smooth density and Dynkin's formula (2.5.4), that if a smooth density function exists then it is the solution to the Fokker-Planck equation, or Forward Kolmogorov equation.

To this end, consider (5.0.3), rewritten here as

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial l^2} - \frac{\partial}{\partial l}\left(\frac{1}{2\sqrt{t}}G(l\,t^{-\frac{1}{2}})v\right) \\ v(l,0) &= v_0(l).\end{aligned} \tag{5.2.1}$$

As mentioned above, the Fokker-Planck PDE, or forward Kolmogorov equation, describes the evolution of the probability density function of the Markov process (X_t) satisfying the SDE

$$\begin{aligned}\mathrm{d}X_t &= \frac{1}{2\sqrt{t}}G(X_t\,t^{-\frac{1}{2}})\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t \\ X_0 &= Z\end{aligned} \tag{5.2.2}$$

where Z has probability density v_0. Here (X_t) satisfies the SDE in the strong sense and B_t is standard Brownian motion. Since the coefficient of the noise is constant, the Itō and Stratonovich definition of the stochastic integral coincide.

The idea is now to establish short-time existence to (5.2.2), which yields a Markov process with density that will satisfy (5.2.1) and so showing short-time existence for (5.2.1). Uniqueness is then standard as the L^1 norm is preserved.

Theorem 5.2.1. *There exists a unique strong adapted solution to* (5.2.2) *for all times.*

Proof. Since Z is independent of B_t we may consider (5.2.2) with $Z = y$ with $y \in \mathbb{R}$ fixed. We note that $\mathbb{P}(Z = 0) = 0$. So, suppose first that $y > 0$. The argument for $y < 0$ is analogous. Fix $\delta > 0$ such that $0 < \delta < y$. Define

$$H(z,t) := \begin{cases} G(\delta\,t^{-\frac{1}{2}}), & \text{if } -\infty < z < \delta \\ G(z\,t^{-\frac{1}{2}}), & \text{if } \delta \le z < \infty. \end{cases}$$

Then, by Proposition A.0.2 we have that there exists $C_\delta > 0$ such that, for every $x, y \in \mathbb{R}$,

$$|H(x,t) - H(y,t)| \le tC_\delta |x-y|.$$

Standard Picard iteration for the solution to SDEs (Øksendal [2003]) yields a unique adapted process (X_t) such that $\mathbb{P}$-a.s and for every $t > 0$

$$X_t = y + \int_0^t \frac{1}{2\sqrt{s}}\,H(X_s,s)\,\mathrm{d}s + \sqrt{2}\,B_t.$$

Let $\tau := \inf\{t > 0 \,|\, X_t \le \delta\}$; then for every $t < \tau$ we have $X_t > \delta$ and hence

$H(X_t, t) = G(X_t\, t^{-\frac{1}{2}})$. We thus arrive at

$$X_t = y + \int_0^t \frac{1}{2\sqrt{s}}\, G(X_s\, s^{-\frac{1}{2}})\, \mathrm{d}s + \sqrt{2}\, B_t,$$

which establishes short-time existence and uniqueness to (5.2.2). Long-time existence and uniqueness follows as the drift term is non-singular in time and space after $t > 0$. It remains to show that

$$\mathbb{P}(\tau = 0) = 0.$$

For $t > 0$ it follows that

$$\mathbb{P}(\tau < t) \leq \mathbb{P}(\inf_{0 \leq s \leq t} X_s \leq \delta).$$

However,

$$\begin{aligned}
\inf_{0\leq s\leq t} X_s \leq \delta &\implies \inf_{0\leq s\leq t} \left(\int_0^s \frac{1}{2\sqrt{r}}\, H(X_r, r)\, \mathrm{d}r + \sqrt{2}\, B_s \right) \leq \delta - y \\
&\implies -\inf_{0\leq s\leq t} B_s \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t},
\end{aligned}$$

where $c = \frac{1}{2}\|G\|_{L^\infty(\mathbb{R})} \in (0, \infty)$. Thus

$$\mathbb{P}\left(\inf_{0\leq s\leq t} X_s \leq \delta \right) \leq \mathbb{P}\left(-\inf_{0\leq s\leq t} B_s \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t} \right) = \mathbb{P}\left(\sup_{0\leq s\leq t} B_s \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t} \right)$$

as

$$-\inf_{0\leq s\leq t} B_s \stackrel{d}{=} \sup_{0\leq s\leq t} B_s.$$

The reflection principle of Brownian motion (Karatzas and Shreve [1991], p.79) gives that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} B_s \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t} \right) = 2\mathbb{P}\left(B_t \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t} \right)$$

and since B_t is a Gaussian random variable with mean zero and variance $\sqrt{t}$ we have

$$2\mathbb{P}\left(B_t \geq \frac{y-\delta}{\sqrt{2}} - c\sqrt{t} \right) = \sqrt{\frac{2}{\pi}} \int_{\frac{y-\delta}{\sqrt{2t}} - c}^{\infty} e^{-\frac{x^2}{2}}\, \mathrm{d}x.$$

Putting these all together we have that

$$\mathbb{P}(\tau < t) \leq \sqrt{\frac{2}{\pi}} \int_{\frac{y-\delta}{\sqrt{2t}}-c}^{\infty} e^{-\frac{x^2}{2}} \,\mathrm{d}x$$

and since $y - \delta > 0$ it follows that

$$\sqrt{\frac{2}{\pi}} \int_{\frac{y-\delta}{\sqrt{2t}}-c}^{\infty} e^{-\frac{x^2}{2}} \,\mathrm{d}x \longrightarrow 0$$

as $t \to 0^+$ and so $\mathbb{P}(\tau = 0) = 0$. $\square$

Remark 5.2.2. *We note that all the derivatives of G are bounded. It now follows from Theorem 2.5.3 and Remark 2.5.5 that there exists a density $v \in C^{2,1}(\mathbb{R} \times (0,T];\mathbb{R})$ such that* (5.2.1) *is satisfied for $t > 0$. The aim of the following section is to describe the behaviour of the solution for small times.*

5.2.2 Bounds on the trajectories of the SDE

Although the probabilistic version of Hörmander's Theorem (Theorem 2.5.3) gives existence to such a density u, it cannot tell us whether $v(l,t) \to v_0(l)$ as $t \to 0$.

Here, we prove a bound on the trajectories of the SDE in (5.2.2), which essentially tells us that nothing pathological can happen at the level of the trajectories. This does not imply the stronger result of convergence of the density, however we will show that we have weak convergence of the density to the initial condition. The stronger result was obtained in Section 5.1 and will be seen to be also true in Section 5.3. We conjecture that the result is also true in critical case. However, since $\alpha = \frac{1}{2}$, our analytic and probabilistic tools of Section 5.1 and Section 5.3 fail to produce any results.

Proposition 5.2.3. *Let $f \in C_c^\infty(\mathbb{R})$ and let (X_t) be the strong solution to* (5.2.2) *with initial state Z that is assumed to be distributed according to v_0. Then*

$$\lim_{t\to 0} \mathbb{E}|f(X_t) - f(Z)| = 0.$$

Proof. Observe that from (5.2.2) and the fact that $|B_t|$ is distributed as $\sqrt{t}\,|W|$ where W is a $\mathcal{N}(0,1)$ random variable, we have

$$\mathbb{E}|X_t - Z| \leq Ct^{\frac{1}{2}}.$$

So for fixed $f \in C_c^\infty(\mathbb{R})$ one has by the uniform Lipschitz continuity of f that

$$\mathbb{E}|f(X_t) - f(Z)| \le C_f \mathbb{E}|X_t - Z| \le Ct^{\frac{1}{2}}.$$

The proof is completed by sending $t \to 0$. $\square$

Remark 5.2.4. *Heuristically, one should expect there to be no problem with the density v as t approaches 0, by the following argument. We may think of a particle on $\mathbb{R}$, whose position is modelled by Y_t with density v,* before *the singularity. Theorem 4.2.4 implies that $\mathbb{P}(Y_1 = 0) = 0$ and so with zero probability the particle is at the singularity. So when we restart the evolution, the particle shouldn't "feel" the singularity.*

Remark 5.2.5. *From Remark 5.2.2 it follows that a density $u \in C^{2,1}(\mathbb{R} \times (0,T]; \mathbb{R})$ exists and solves* (5.2.1)*. But, $u(\cdot, t)$ is precisely the density of (X_t) and thus by above, since v_0 is the density of Z,*

$$\mathbb{E}f(X_t) = \int_{\mathbb{R}} f(x)u(x,t)\,\mathrm{d}x \to \int_{\mathbb{R}} f(x)v_0(x)\,\mathrm{d}x = \mathbb{E}f(Z)$$

as $t \to 0$, for every $f \in C_c^\infty$. This precisely shows the weak convergence of the solution to (5.2.1) *to the initial data.*

5.3 Super–Critical Regime

We note that the methods employed in this section fail in the case of $\alpha \le \frac{1}{2}$. Consider (5.0.3) with $\alpha > \frac{1}{2}$. By Theorem 4.3.3 it follows that $v_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. The fact that $v_0 \in C(\mathbb{R})$ will be extremely important in the proof of Theorem 5.3.1 below. Indeed, the problem is now that (5.0.3) is initially singular and so we wish to show that there is a unique solution which attains the initial data. This is provided by the following theorem.

Theorem 5.3.1. *There exists a unique mild solution to* (5.0.3) *that attains the initial data v_0, in the pointwise sense.*

Proof. As in Section 4.3, let $S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ denote the heat semigroup. We wish to solve the following integral equation in $C((0,T]; L^\infty(\mathbb{R}))$:

$$v(l,t) = (S(t)v_0)(l) - \alpha \int_0^t \int_{\mathbb{R}} K(l,y;t-s)s^{\alpha-1}\frac{\partial}{\partial y}\left(G(ys^{-\alpha})v(y,s)\right)\,\mathrm{d}y\,\mathrm{d}s.$$

To do this, we solve in $X := C((0,T_*]; L^\infty(\mathbb{R}))$ for $T_* > 0$ small enough with norm

$$\|\varphi\|_X := \sup_{t\in(0,T_*]} \|\varphi(\cdot,t)\|_{L^\infty},$$

using a fixed point argument, observing that $(X, \|\cdot\|_X)$ is a Banach space. Observing that the coefficients of (5.0.3) are smooth and bounded for $t > T_*$ one extends the solution for $t \in (0,T)$ in the usual fashion.

Define $\mathcal{F} : X \to X$ by

$$\mathcal{F}w(t) = S(t)v_0 - \alpha \int_0^t \int_{\mathbb{R}} K(l,y;t-s)s^{\alpha-1} \frac{\partial}{\partial y}(Gw)\,\mathrm{d}y\,\mathrm{d}s,$$

where K is the heat kernel is given in Section 4.3. Since $v_0 \in L^\infty(\mathbb{R})$ we have, using the estimate of $|\nabla_y K|$ by Davies [1989] as in Section 4.3

$$\begin{aligned}|\mathcal{F}w(t)| &\le \|v_0\|_{L^\infty} + C\|w\|_X \int_0^t \int_{\mathbb{R}} (t-s)^{-1} s^{\alpha-1} \left(1 + \frac{(l-y)^2}{t-s}\right) e^{-\frac{(l-y)^2}{4(t-s)}}\,\mathrm{d}y\,\mathrm{d}s \\ &\le \|v_0\|_{L^\infty} + C\|w\|_X \int_0^t (t-s)^{-\frac{1}{2}} s^{\alpha-1}\,\mathrm{d}s.\end{aligned}$$

Now, using $z = \frac{s}{t}$

$$\int_0^t (t-s)^{-\frac{1}{2}} s^{\alpha-1}\,\mathrm{d}s = t^{\alpha-\frac{1}{2}} \int_0^1 (1-z)^{-\frac{1}{2}} z^{\alpha-1}\,\mathrm{d}z.$$

If $\alpha \in (\frac{1}{2}, 1)$ then

$$\int_0^1 (1-z)^{-\frac{1}{2}} z^{\alpha-1}\,\mathrm{d}z = B(1/2,\alpha)$$

where $B(\gamma_1,\gamma_2)$ is the beta function. Indeed, $B(\frac{1}{2},\alpha) < \infty$ for every $\alpha > \frac{1}{2}$. For $\alpha > 1$ this integral above is clearly bounded. Thus

$$\|\mathcal{F}w\|_X \le \|v_0\|_{L^\infty} + CT_*^{\alpha-\frac{1}{2}}\|w\|_X,$$

for every $\alpha > \frac{1}{2}$. The continuity of $t \mapsto \mathcal{F}w(t)$ follows from the fact that the semigroup $T(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ defined by

$$(T(t)w(\cdot,t))(l) = \int_{\mathbb{R}} \nabla_y K(l,y;t) w(y,t)\,\mathrm{d}y = \int_{\mathbb{R}} \frac{-1}{\sqrt{4\pi t}} e^{-\frac{(l-y)^2}{4t}} \frac{(l-y)}{2t} w(y,t)\,\mathrm{d}y$$

for $w(\cdot,t) \in L^\infty(\mathbb{R})$ is continuous[2] at t_0 for every $t_0 \in (0,T_*]$. Indeed, one may also

[2]Indeed, one can show that $\|T(t)\|_{\mathrm{op}} \le \frac{C}{\sqrt{t}}$

conclude the continuity of $t \mapsto \mathcal{F}w(t)$ via the Dominated Convergence Theorem. From this we conclude that $\mathcal{F} : X \to X$.

We now show that $\mathcal{F}$ is a strict contraction. Fix $u, v \in X$. Then

$$\begin{aligned} |\mathcal{F}u - \mathcal{F}v| &= \alpha \left| \int_0^t \int_{\mathbb{R}} K(l, y; t-s) s^{\alpha-1} \frac{\partial}{\partial y} \left(G(u-v) \right) \mathrm{d}y \, \mathrm{d}s \right| \\ &\leq C \|u - v\|_X \int_0^t (t-s)^{-\frac{1}{2}} s^{\alpha-1} \, \mathrm{d}s \leq C t^{\alpha - \frac{1}{2}} \|u - v\|_X, \end{aligned}$$

by exactly the same argument as above. Thus

$$\|\mathcal{F}u - \mathcal{F}v\|_X \leq C T_*^{\alpha - \frac{1}{2}} \|u - v\|_X.$$

Choosing $T_* > 0$ such that $C T_*^{\alpha - \frac{1}{2}} < 1$ yields that $\mathcal{F}$ is a strict contraction and so by the Contraction Mapping Theorem, there exists a unique $v \in X$ such that

$$v = \mathcal{F}v.$$

We now show that the initial data is attained. By exactly the same argument as above

$$|v(l, t) - (S(t)v_0)(l)| \leq C t^{\alpha - \frac{1}{2}}.$$

From Theorem 4.3.3 we have that $v_0 \in C(\mathbb{R})$. Hence,

$$|S(t)v_0(l) - v_0(l)| \to 0$$

as $t \to 0^+$ and so we conclude the result by the triangle inequality.

□

Chapter 6

Pertubations of Problem I

The analysis presented thus far is for the unperturbed equation (3.2.3). As this was derived from a conservation law, of interest is what happens to the solution at the singularity when one adds extra energy. We are not interested in continuing the solution past the singularity; only whether noise can disturb the fact that the solution vanishes at the singularity in the sub–critical regime. For the critical regime, we recall that we need $u_0 \in D(\mathcal{H})$. This is not satisfied if, for example, $u_0 \in L^\infty(\mathbb{R})$ with no assumptions on whether u_0 is differentiable or not. Indeed, it can be argued that the $\alpha < \frac{1}{2}$ case is more complete than $\alpha = \frac{1}{2}$ and, moreover, the $\alpha > \frac{1}{2}$ case has that the solution need not vanish at the singularity. Therefore, we do not consider the critical regime of $\alpha = \frac{1}{2}$.

In order to carry out the analysis, we model the influx of energy by placing a space-time function on the right hand side of (3.2.3). Since one wants to use a two-parameter semigroup approach for the solution, by appealing to the variation of constants formula, it is clear that we need to look at rougher initial data started at a time $s \in [0, 1)$ and so that we run (3.2.3) for $t \in [s, 1)$. We aim to arrive at analogous results to that of Theorem 4.1.1 and Theorem 4.1.3 for the sub–critical regime with rougher initial conditions and when the equation is suitably perturbed.

The perturbation will be based on what we can prove about the solution with rougher initial conditions. For this reason, we include a section below where one does not necessarily assume that the initial data is smooth.

6.1 Rougher Initial Conditions

Until now, we have only considered initial data $u_0 \in C_c^\infty(\mathbb{R})$ for (3.2.3). In order to perturb (3.2.3), one must consider (3.2.3) with initial condition started at $t = s$

where $s \in [0,1)$ and analyse the results. We will only consider the sub–critical regime.

6.1.1 Sub–Critical Regime

Consider

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u\right), \quad (l,t) \in \mathbb{R}\times[s,1), \\ &u(l,s) = f(l,s), \end{aligned} \tag{6.1.1}$$

where $s \in [0,1)$ and $\alpha < \frac{1}{2}$. Note that we are only running the equation for $t \in [s,1)$ where $s \in [0,1)$ is fixed. For f being suitably integrable or bounded, Theorem 2.2.4 implies the existence of a unique solution $u \in C^{2,1}(\mathbb{R}\times[s,1))$. With this in mind, the results are presented in the following theorem.

Theorem 6.1.1. *For each fixed $s \in [0,1)$, consider the unique solution, u, to* (6.1.1) *and assume that $f(\cdot,s) \in L^\infty(\mathbb{R})$. Then we have that*

i) $|u(l,t)| \le \|f(\cdot,s)\|_{L^\infty}$ *for every* $(l,t) \in \mathbb{R}\times[s,1)$*;*

ii) $u(0,t) \to 0$ *as* $t \to 1^-$*;*

iii) $|u(l,t)| \le (1-s)^{-\alpha}\|f(\cdot,s)\|_{L^\infty}(1-t)^{\alpha}\sqrt{\cosh 2g(l(1-t)^{-\alpha})}$ *for every* $(l,t) \in \mathbb{R}\times[s,1)$.

Remark 6.1.2. *One may make use of the asymptotics of* $\cosh 2g(x)$ *to derive a power law in the arc-length parameter from iii) in the above theorem. This is analogous to the proof of Theorem 4.1.3 and so we omit the details.*

Proof of Theorem 6.1.1. The reader should read this proof in close comparison with the proofs of Theorem 4.1.1 and Theorem 4.1.3.

i) Following the proof and set up of Theorem 4.1.1, changing coordinates via $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$ we have

$$f(l,s) = f(ye^{-\alpha\tau_s}, 1-e^{-\tau_s}),$$

where $\tau_s = -\log(1-s)$. Hence, using the Feynman-Kac representation formula for the solution in (y,τ) coordinates, just as in the proof of Theorem 4.1.1 one concludes that

$$|u(l,t)| = |v(y,\tau)| = \left| \mathbb{E}^{(-\tau,y)} \exp\left(- \int_{-\tau}^{-\tau_s} F(X_r)\,\mathrm{d}r \right) f(X_{-\tau_s} e^{-\alpha\tau_s}, 1 - e^{-\alpha\tau_s}) \right|$$
$$\leq \|f(\cdot,s)\|_{L^\infty}.$$

ii) This follows from the proof of Theorem 4.1.1 with $\|f(\cdot,s)\|_{L^\infty}$ in place of $\|u_0\|_{L^\infty}$.

iii) Working in (y,τ) coordinates we follow the proof of Theorem 4.1.3 and make the ansatz

$$v(y,\tau) = e^{-\alpha\tau}\sqrt{\cosh 2g(y)}\,\varphi(y,\tau)$$

where we take initial condition for φ as

$$\varphi(y,\tau_s) = e^{\alpha\tau_s}\frac{f(ye^{-\alpha\tau_s}, 1 - e^{-\tau_s})}{\sqrt{\cosh 2g(ye^{-\alpha\tau_s})}}.$$

It follows that φ solves (4.1.4) with initial condition $\varphi(y,\tau_s)$ and so via the Feynman-Kac representation formula,

$$|\varphi(y,\tau)| \leq e^{\alpha\tau_s}\|f(\cdot, 1 - e^{-\tau_s})\|_{L^\infty}.$$

Switching back to (l,t) coordinates, we see that

$$|u(l,t)| \leq (1-s)^{-\alpha}\|f(\cdot,s)\|_{L^\infty}\,(1-t)^{\alpha}\sqrt{\cosh 2g(l(1-t)^{-\alpha})}$$

for every $(l,t) \in \mathbb{R} \times [s,1)$.

□

Remark 6.1.3 (On integrable initial data)**.** *The reader will notice that we have not considered the case of $f(\cdot,s) \in L^q(\mathbb{R})$ for every $s \in [0,1)$ where $q \in [1,\infty)$. This is due to the fact that the method we employ to prove Theorem 6.1.1 in the $q = \infty$ case cannot be easily modified for the $q \in [1,\infty)$ case.*

We thus restrict ourselves to the $q = \infty$ case for the rest of this chapter and leave the case of $q \in [1,\infty)$ to further research. This is not detrimental, for the motivation of this section for perturbing the equation really has stochastic perturbation in mind where will be perturb with continuous functions (that turn out to be uniformly bounded).

6.2 Deterministic Perturbation

We will consider the following equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u\right) + f \\ u(l,0) &= u_0(l),\end{aligned} \tag{6.2.1}$$

where $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ is measurable and $\alpha < \frac{1}{2}$.

Definition 6.2.1. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose f is regular enough so that* (6.1.1) *has a classical solution for every $s \in [0,1)$. Then we define the unique mild to* (6.2.1) *as*

$$u(l,t) = \mathcal{U}(t,0)u_0(l) + \int_0^t \mathcal{U}(t,s)f(\cdot,s)(l)\,\mathrm{d}s \tag{6.2.2}$$

where for $0 \le s \le t < 1$ we define $\mathcal{U}$ as $v(l,t) := \mathcal{U}(t,s)g(\cdot,s)(l)$ to be the unique solution to (6.1.1) *for $s \le t < 1$ with initial data $v(l,s) = g(l,s)$.*

Remark 6.2.2. *One can easily check using the estimates in Theorem 6.1.1 that for f and u_0 with suitable regularity, equation* (6.2.2) *does in fact define the unique* classical *solution to* (6.2.1).

6.2.1 Sub–Critical Regime

Consider (6.2.1) with $\alpha < \frac{1}{2}$. We aim to prove the analogue of Theorem 4.1.1 and Theorem 4.1.3 for this equation. Indeed, we have the following.

Theorem 6.2.3. *Let $u_0 \in C_0^\infty(\mathbb{R})$ and suppose that $f \in C([0,1]; L^\infty(\mathbb{R}))$. Then the unique classical solution, u, given in* (6.2.2) *satisfies*

i) There exists $C > 0$ such that $|u(l,t)| \le C$ for every $(l,t) \in \mathbb{R} \times [0,1)$;

ii) It holds that $u(0,t) \to 0$ as $t \to 1^-$;

iii) There exists $C = C(u_0, f) > 0$ such that

$$|u(l,t)| \le C(1-t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha})}$$

for every $(l,t) \in \mathbb{R} \times [0,1)$.

Remark 6.2.4. *Again, as in Theorem 6.1.1, one may use the asymptotics of* $\cosh 2g(x)$ *in iii) above to deduce a power law in the arc-length parameter; analogous to Theorem 4.1.3.*

Proof of Theorem 6.2.3. Using (6.2.2) we have

$$u(l,t) = \mathcal{U}(t,0)u_0(l) + \int_0^t \mathcal{U}(t,s)f(\cdot,s)(l)\,\mathrm{d}s.$$

i) By Theorem 6.1.1,

$$|u(l,t)| \leq \|u_0\|_{L^\infty} + \sup_{s\in[0,1]} \|f(\cdot,s)\|_{L^\infty}.$$

ii) Since $f \in C(0,1;L^\infty(\mathbb{R}))$ then

$$|u(0,t)| \leq |\mathcal{U}(t,0)u_0(0)| + \int_0^1 |\mathcal{U}(t,s)f(\cdot,s)(0)|\chi_{[0,t]}(s)\,\mathrm{d}s.$$

Theorem 6.1.1 implies that

$$|\mathcal{U}(t,s)f(\cdot,s)(0)|\chi_{[0,t]}(s) \leq \|f(\cdot,s)\|_{L^\infty} \leq \sup_{s\in[0,1]} \|f(\cdot,s)\|_{L^\infty} \in L^1(0,1;\mathbb{R}),$$

$$\lim_{t\to 1^-} |\mathcal{U}(t,s)f(\cdot,s)(0)|\chi_{[0,t]}(s) = 0$$

for every $s \in [0,1)$ and

$$\lim_{t\to 1^-} |\mathcal{U}(t,0)u_0(0)| = 0.$$

Applying the Dominated Convergence Theorem yields the result.

iii) By the proof of Theorem 6.1.1, one concludes that

$$|u(l,t)| \leq C\psi(l,t)\left(\|u_0\|_{L^\infty} + \sup_{s\in[0,1]} \|f(\cdot,s)\|_{L^\infty} \int_0^t (1-s)^{-\alpha}\,\mathrm{d}s\right),$$

where

$$\psi(l,t) := (1-t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha}}.$$

However, since $\alpha < \frac{1}{2}$ it follows that $1-\alpha > \frac{1}{2}$ and so

$$\left|\int_0^t (1-s)^{-\alpha}\,\mathrm{d}s\right| < \frac{1}{1-\alpha} < \infty.$$

Thus

$$|u(l,t)| \leq C(1-t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha}} \left(\|u_0\|_{L^\infty} + \sup_{s\in[0,1]} \|f(\cdot,s)\|_{L^\infty}\right).$$

□

Remark 6.2.5. *The reader should note that instead of taking $f \in C([0,1]; L^\infty(\mathbb{R}))$, one could of course take $f \in L^1(0,1; L^\infty(\mathbb{R}))$ to conclude the results of i) and ii) above. If we add the additional assumption that $(1-s)^{-\alpha}\|f(\cdot,s)\|_{L^\infty} \in L^1(0,1;\mathbb{R})$ one may conclude iii). The argument is omitted.*

6.3 Stochastic Perturbation

Until now, we have only considered deterministic perturbations. In the following, we will perturb using a stochastic term. In order to get qualitative results, we will follow the approach of Da Prato and Zabczyk [1992]. In this approach, we need to set the stochastic PDE in a Hilbert space setting. We also want to make sure we have bounded solutions in space. The following lemma is of use.

Lemma 6.3.1. *Let $f \in H^r(\mathbb{R})$ with $r > \frac{1}{2}$. Then there exists $\tilde{f} \in C(\mathbb{R})$ with $\tilde{f} \in L^\infty(\mathbb{R})$, uniformly continuous with $f = \tilde{f}$ a.e.*

Proof. For a proof, the reader is directed to, for example, Grafakos [2009] Theorem 6.2.4. □

Considering $\mathcal{U}(t,s)$ in Definition 6.2.1 as a two-parameter strongly continuous semigroup, we have seen that $\mathcal{U}(t,s) : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ for $0 < s \leq t < 1$. It is natural to ask whether $\mathcal{U}(t,s) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ for every $0 \leq s \leq t \leq 1$. In the following, we will take a heuristic exploration into what noise we can actually take, given we have not been able to show that $\mathcal{U}(t,s) : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ for every $0 \leq s \leq t \leq 1$.

6.3.1 A Heuristic exploration into the noise

Let $W(t)$ be a $H^1(\mathbb{R})$ valued $Q-$Wiener process with $Q = I$. Let $\{\beta_j(t)\}_{j\in\mathbb{N}}$ be a sequence of real-valued independent Brownian motions such that

$$\langle W(t), u\rangle = \sum_{j\in\mathbb{N}} \langle e_j, u\rangle \beta_j(t)$$

for every $u \in H^1(\mathbb{R})$ and for every $t \geq 0$. Here $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $H^1(\mathbb{R})$. Hence, for $u = e_j$ we conclude that $\beta_j = \langle W(t), e_j\rangle$.

Let

$$\mathcal{A}(t) := \frac{\partial^2}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^\alpha)\cdot\right)$$

and consider the Stochastic Partial Differential Equation (SPDE), written as an infinite dimensional SDE (Prévôt and Röckner [2007])

$$\begin{aligned} \mathrm{d}v &= \mathcal{A}(t)v\,\mathrm{d}t + \mathrm{d}W(t) \\ v(0) &= u_0. \end{aligned}$$

Then

$$v(t) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)\,\mathrm{d}W(s)$$

is the mild solution. However, a major problem is that we do not know whether

$$\int_0^t \mathcal{U}(t,s)\,\mathrm{d}W(s) \in H^1(\mathbb{R}).$$

Indeed, if $\beta_j(t) = \langle W(t), e_j\rangle$ then

$$\int_0^t \mathcal{U}(t,s)\,\mathrm{d}W(s) = \int_0^t \sum_{j\in\mathbb{N}} \mathcal{U}(t,s)e_j\,\mathrm{d}\beta_j(s) = \sum_{j\in\mathbb{N}} \int_0^t \mathcal{U}(t,s)e_j\,\mathrm{d}\beta_j(s). \tag{6.3.1}$$

We have formally interchanged the sum and the integral. We note that $\int_0^t \bullet\,\mathrm{d}\beta_j(s)$ is a real Stochastic integral in the Itō sense and $e_j \in H^1(\mathbb{R})$. Let $\tilde{e}_j$ denote the a.e continuous version of e_j. Then, by Lemma 6.3.1 we may consider

$$\mathcal{U}(t,s)e_j = \mathcal{U}(t,s)\tilde{e}_j$$

a.e with $\mathcal{U}(t,s)\tilde{e}_j \in C^{2,1}(\mathbb{R}\times[s,1))$ for every $s \in [0,1)$ and since $\tilde{e}_j \in L^\infty(\mathbb{R})$ it follows that $\mathcal{U}(t,s)\tilde{e}_j \in L^\infty(\mathbb{R}\times[s,1))$ for every $t \in [s,1)$ for every $s \in [0,1)$. However, it does not follow that $\mathcal{U}(t,s)\tilde{e}_j \in H^1(\mathbb{R})$ for every $0 \le s \le t \le 1$; integrability is not guaranteed.

With regard to (6.3.1), it doesn't follow that the above sum is finite. Usually, there is some link between e_j and $\mathcal{A}(t)$ such as the e_j being eigenfunctions of $\mathcal{A}(t)$, with eigenvalues λ_j. This would yield that $\mathcal{U}(t,s)e_j = e^{-\lambda_j}e_j$, and so the sum would converge. In this case, e_j and λ_j would be time dependent, but we would hope for some spectral mapping theorem to relate $\mathcal{U}(t,s)e_j$ and $\mathcal{A}(t)e_j$, via $\mathcal{U}(t,s)e_j = e^{-\lambda_j}e_j$. Since $\mathcal{A}(t)$ is not self-adjoint, there is no hope for such a theorem.

We remark that the main problem is really not knowing how $\mathcal{U}(t,s)$ behaves when acting on an orthonormal basis of $H^1(\mathbb{R})$. Of course, if we were considering a bounded domain $\Gamma \subset \mathbb{R}$, then we would immediately have $\mathcal{U}(t,s) : H^1(\Gamma) \to H^1(\Gamma)$, since integrability is no longer an issue. However, convergence of the sum in (6.3.1) would still be an issue. Thus, we need to weight the expression for the Wiener process W with something that we know is square summable, or take a finite number of terms in the sum for W. This corresponds to taking coloured in space noise or a finite rank covariance operator, Q, respectively. This will guarantee convergence of the sum in (6.3.1).

To see how weighting the expression for W helps, suppose $Q : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is a positive definite bounded linear map. Let $\{e_j\}_{j\in\mathbb{N}}$ be an eigen-basis for Q in $H^1(\mathbb{R})$; that is, we assume

$$Qe_j = \sqrt{\lambda_j} e_j$$

where $\lambda_j > 0$ and assume there exists $C > 0$ such that $|\tilde{e}_j(l)| \leq C$ for every $j \in \mathbb{N}$ and $l \in \mathbb{R}$, where $\tilde{e}_j$ is the uniformly continuous and bounded representative of e_j as given by Lemma 6.3.1. Then

$$\int_0^t \mathcal{U}(t,s)\,\mathrm{d}W(s) = \sum_{j\in\mathbb{N}} \sqrt{\lambda_j} \int_0^t \mathcal{U}(t,s)e_j\,\mathrm{d}\beta_j(s).$$

This series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since, for $n, p \in \mathbb{N}$ arbitrary,

$$\mathbb{E}\left|\sum_{j=n+1}^{n+p} \sqrt{\lambda_j} \int_0^t \mathcal{U}(t,s)e_j\,\mathrm{d}\beta_j(s)\right|^2 \leq \sum_{j=n+1}^{n+p} \lambda_j \int_0^t |\mathcal{U}(t,s)e_j|^2\,\mathrm{d}s$$

by the Itō isometry. However, a.e we have $e_j = \tilde{e}_j$ and so by Theorem 6.1.1 we have that a.e

$$\mathbb{E}\left|\sum_{j=n+1}^{n+p} \sqrt{\lambda_j} \int_0^t \mathcal{U}(t,s)e_j\,\mathrm{d}\beta_j(s)\right|^2 \leq C^2 \sum_{j=n+1}^{n+p} \lambda_j \leq C^2 \sum_{j=1}^{\infty} \lambda_j < \infty$$

as $\lambda_j \in \ell^1(\mathbb{N}; \mathbb{R})$.

6.3.2 Sub–Critical Regime

We will consider three possibilities for the stochastic perturbation:

A) White in time perturbation, constant in space: Here we consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\big(G(l(1-t)^{-\alpha})u\big) + \frac{\mathrm{d}\beta}{\mathrm{d}t}(t) \\ u(l,0) &= u_0(l) \end{aligned} \tag{6.3.2}$$

where $u_0 \in C_c^\infty(\mathbb{R})$, $\beta(t)$ is real-valued Brownian motion and $\frac{\mathrm{d}\beta}{\mathrm{d}t}(t)$ is the distributional derivative of Brownian motion. The unique mild solution is given by

$$u(l,t) = \mathcal{U}(t,0)u_0(l) + \int_0^t \mathcal{U}(t,s)(1)(l)\,\mathrm{d}\beta(s). \tag{6.3.3}$$

Here the integral is the real stochastic integral in the sense of Itō.

Remark 6.3.2 (On the expression (6.3.3))**.** *The reason why we write $\mathcal{U}(t,s)(1)(l)$ in the real valued Itō integral, is because in the definition of the stochastic integral, we take the one-dimensional Wiener process $W(t) = \beta(t)$ where $\beta(\cdot)$ is one-dimensional Brownian motion. Thus, $\mathcal{U}(t,s)$ acts on the function 1, the resulting function, $\mathcal{U}(t,s)(1)$ is continuous and so is to be evaluated at l.*

B) Coloured in space - white in time: Here we consider the following SDE in $H^1(\mathbb{R})$:

$$\begin{aligned} \mathrm{d}u(t) &= \mathcal{A}(t)u(t)\,\mathrm{d}t + \mathrm{d}W_1(s) \\ u(0) &= u_0 \end{aligned} \tag{6.3.4}$$

where $u_0 \in C_c^\infty(\mathbb{R})$ and $\mathcal{A}(t)$ is defined for $u(t) \in \mathcal{D}(\mathcal{A}(t)) := C_b^{2,1}(\mathbb{R})$ with $t \in [0,1)$ by

$$\mathcal{A}(t)u(t)(l) = \frac{\partial^2 u}{\partial l^2}(l,t) + \alpha(1-t)^{\alpha-1}\frac{\partial}{\partial l}\left(G(l(1-t)^{-\alpha})u(l,t)\right).$$

Here, $W_1(t)$ is given by the Karhunen-Loeve expansion

$$W_1(t) = \sum_{j=1}^{\infty}\sqrt{\lambda_j}e_j\beta_j(t)$$

where $\{\beta_j(t)\}_{j\in\mathbb{N}}$ are independent real-valued Brownian motions, $\lambda_j \in \ell^1(\mathbb{N};\mathbb{R})$ and $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $H^1(\mathbb{R})$ such that there exists $C > 0$ such that $|\tilde{e}_j(l)| \le C$ for every $j \in \mathbb{N}$, $l \in \mathbb{R}$. Here $\{\tilde{e}_j\}$ are the a.e uniformly continuous, bounded version of the $\{e_j\}$ as given in Lemma 6.3.1. For this reason, we will consider W_1 but with e_j replaced by $\tilde{e}_j$. Indeed, for the remainder, we assume that the noise is defined via $\tilde{e}_j$ and not e_j. This will make sure that the mild solution is uniquely given by

$$u(t)(l) = \mathcal{U}(t,0)u_0(l) + \int_0^t \mathcal{U}(t,s)\,\mathrm{d}W_1(s)(l)$$

which is interpreted as

$$u(t)(l) = \mathcal{U}(t,0)u_0(l) + \sum_{j\in\mathbb{N}} \sqrt{\lambda_j} \int_0^t \mathcal{U}(t,s)\tilde{e}_j(l)\,\mathrm{d}\beta_j(s) \tag{6.3.5}$$

where the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. However, the resulting solution may *not* be in $H^1(\mathbb{R})$.

C) Space-Time white noise with finite rank covariance operator. Here we consider

$$\begin{aligned} \mathrm{d}u(t) &= \mathcal{A}(t)u(t)\,\mathrm{d}t + \mathrm{d}W_2(s) \\ u(0) &= u_0 \end{aligned} \tag{6.3.6}$$

where $u_0 \in C_c^\infty(\mathbb{R})$ and $\mathcal{A}(t)$ is defined in case B) above. For the noise,

$$W_2(t) = \sum_{j=1}^{N} e_j\,\beta_j(t)$$

where $N \in \mathbb{N}$ with $N \geq 2$. Here, $\{e_j\}$ and $\{\beta_j(t)\}$ are given in case B) above. We will, as in case B) consider the noise defined with $\tilde{e}_j$ instead of e_j. The mild solution is uniquely given by

$$u(t)(l) = \mathcal{U}(t,0)u_0(l) + \int_0^t \mathcal{U}(t,s)\,\mathrm{d}W_2(s)$$

which is interpreted as

$$u(t)(l) = \mathcal{U}(t,0)u_0(l) + \sum_{j=1}^{N} \int_0^t \mathcal{U}(t,s)\tilde{e}_j(l)\,\mathrm{d}\beta_j(s). \tag{6.3.7}$$

Since $N < \infty$, the series automatically makes sense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We now have the following theorem which details the results.

Theorem 6.3.3. *Let $u_0 \in C_0^\infty(\mathbb{R})$ and consider the unique mild solution u to case A, B or C. Then in each of the cases,*

I) There exists $C = C(u_0) > 0$ such that

$$\mathbb{E}|u(l,t)|^2 \leq C$$

for every $(l,t) \in \mathbb{R} \times [0,1)$;

II) It holds that

$$\mathbb{E}|u(0,t)|^2 \to 0$$

as $t \to 1^-$;

III) There exists $C = C(u_0) > 0$ uniform in j such that

$$\mathbb{E}|u(l,t)|^2 \leq C(1-t)^{2\alpha} \cosh 2g(l(1-t)^{-\alpha})$$

for every $(l,t) \in \mathbb{R} \times [0,1)$.

Proof. The main ingredients are the results of Theorem 6.1.1 and the Itō isometry (Lemma 2.3.5).

I) A) Consider (6.3.3). Then

$$\mathbb{E}|u(l,t)|^2 \leq 2\|u_0\|_{L^\infty}^2 + 2\int_0^t |\mathcal{U}(t,s)(1)(l)|^2 \,\mathrm{d}s \leq 2(\|u_0\|_{L^\infty}^2 + 1).$$

B) Consider (6.3.5). Then

$$\mathbb{E}|u(l,t)|^2 \leq 2\|u_0\|_{L^\infty}^2 + 2\sum_{j\in\mathbb{N}} \lambda_j \int_0^t |\mathcal{U}(t,s)\tilde{e}_j(l)|^2 \,\mathrm{d}s \leq C\left(\|u_0\|_{L^\infty}^2 + 1\right),$$

where C contains the ℓ^1 norm of $\lambda = \{\lambda_j\}_{j\in\mathbb{N}}$.

C) Consider (6.3.7). Then

$$\begin{aligned}\mathbb{E}|u(l,t)|^2 &\leq 2\|u_0\|_{L^\infty}^2 + 2\sum_{j=1}^{N} \int_0^t |\mathcal{U}(t,s)\tilde{e}_j(l)|^2 \,\mathrm{d}s \leq 2\|u_0\|_{L^\infty}^2 + 2C^2\sum_{j=1}^{N} 1 \\ &= 2\|u_0\|_{L^\infty}^2 + NC^2(N+1).\end{aligned}$$

II) A) Consider (6.3.3). Then

$$\mathbb{E}|u(0,t)|^2 \leq 2|\mathcal{U}(t,0)u_0(0)|^2 + 2\int_0^t |\mathcal{U}(t,s)(1)(0)|^2 \,\mathrm{d}s.$$

However, $\mathcal{U}(t,0)u_0(0) \to 0$ as $t \to 1^-$ and $\mathcal{U}(t,s)(1)(0) \to 0$ as $t \to 1^-$ for every $s \in [0,1)$. Further,

$$|\mathcal{U}(t,s)(1)(0)|\chi_{[0,t]}(s) \leq 1$$

for every $s \in [0,1)$. Thus the dominated convergence theorem implies

$$\mathbb{E}|u(0,t)|^2 \to 0$$

as $t \to 1^-$.

B) Consider (6.3.5). Then

$$\mathbb{E}|u(0,t)|^2 \leq 2|\mathcal{U}(t,0)u_0(0)|^2 + 2\sum_{j\in\mathbb{N}} \lambda_j \int_0^t |\mathcal{U}(t,s)\tilde{e}_j(0)|^2 \, \mathrm{d}s.$$

Again, $\mathcal{U}(t,0)u_0(0) \to 0$ as $t \to 1^-$ and $\mathcal{U}(t,s)\tilde{e}_j(0) \to 0$ as $t \to 1^-$. Also,

$$|\lambda_j| \int_0^t |\mathcal{U}(t,s)\tilde{e}_j(0)|^2 \, \mathrm{d}s \leq C^2|\lambda_j| \in \ell^1(\mathbb{N};\mathbb{R})$$

for each fixed $t \in [0,1)$ with

$$|\mathcal{U}(t,s)\tilde{e}_j(0)|^2 \chi_{[0,t]}(s) \leq C^2 \in L^1(0,1;\mathrm{d}s;\mathbb{R})$$

for each fixed $t \in [0,1)$. So, applying the dominated convergence theorem twice reveals that

$$\lim_{t\to 1^-} \sum_{j\in\mathbb{N}} \lambda_j \int_0^t |\mathcal{U}(t,s)\tilde{e}_j(0)|^2 \, \mathrm{d}s = 0.$$

The result now follows.

C) Consider (6.3.7). The proof is completely analogous to that of II) B) without the need to use the dominated convergence Theorem twice, as we are considering a finite sum.

III) A) Consider (6.3.3). Then, applying Theorem 6.1.1,

$$\begin{aligned}\mathbb{E}|u(l,t)|^2 &\leq 2|\mathcal{U}(t,0)u_0(0)|^2 + 2\int_0^t |\mathcal{U}(t,s)(1)(l)|^2 \, \mathrm{d}s \\ &\leq C(1-t)^{2\alpha}\cosh 2g(l(1-t)^{-\alpha})\left(\|u_0\|^2_{L^\infty} + \int_0^t (1-s)^{-2\alpha}\, \mathrm{d}s\right).\end{aligned}$$

However, as $\alpha < \frac{1}{2}$ the following integral is finite uniformly in t with

$$\int_0^t (1-s)^{-2\alpha}\, \mathrm{d}s = \frac{1}{1-2\alpha}(1-(1-t)^{1-2\alpha}) \leq \frac{1}{1-2\alpha}.$$

Hence

$$\mathbb{E}|u(l,t)|^2 \leq C\left(\|u_0\|_{L^\infty}^2 + 1\right)(1-t)^{2\alpha}\cosh 2g(l(1-t)^{-\alpha})$$

for every $(l,t) \in \mathbb{R} \times [0,1)$.

B) Consider (6.3.5). Analogously we have

$$\mathbb{E}|u(l,t)|^2 \leq C\left(\|u_0\|_{L^\infty}^2 + \sup_{j\in\mathbb{N}}\|\tilde{e}_j\|_{L^\infty}^2\right)(1-t)^{2\alpha}\cosh 2g(l(1-t)^{-\alpha})$$

for every $(l,t) \in \mathbb{R} \times [0,1)$, where we have used that $\lambda_j \in \ell^1(\mathbb{N};\mathbb{R})$ and that we have a bound uniform in j on the $\tilde{e}_j$.

C) Finally, consider (6.3.7). It follows from exactly the same proof as in II) B) above that for every $(l,t) \in \mathbb{R} \times [0,1)$

$$\mathbb{E}|u(l,t)|^2 \leq C(N)\left(\|u_0\|_{L^\infty}^2 + \sup_{1\leq j\leq N}\|\tilde{e}_j\|_{L^\infty}\right)(1-t)^{2\alpha}\cosh 2g(l(1-t)^{-\alpha}).$$

□

Remark 6.3.4. *It may be argued that the noise chosen is not natural. Since we are ultimately considering a time-dependent curve, it can be argued that the basis functions should depend on time. If we have this and the assumption that*

$$\sup_{s\in[0,1]}\sup_{j\in\mathbb{N}}\|\tilde{e}_j(s)\|_{L^\infty} < \infty$$

where $\tilde{e}_j(t)$, $t \in [0,1]$, are some uniformly continuous, bounded version of some orthonormal basis $\{e_j(t)\}_{j\in\mathbb{N}}$ with $t \in [0,1]$ of $H^1(\mathbb{R})$, then it is clear that all the proofs of Theorem 6.3.3 carry through. Of course, one can construct these $e_j(t)$ from some orthonormal basis of $H^1(\mathcal{C}_t)$.

Chapter 7

Statement and formulation of Problem II

We will now consider the same problem as Chapter 3, but we will replace the curve $\mathcal{C}_t^\alpha$ with a surface. To this end, fix $\alpha \in (0, \infty)$ and consider the hyperboloid of one sheet,

$$\mathcal{S}_t^\alpha := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = (1-t)^{2\alpha}\}. \tag{7.0.1}$$

Figure 7.1 shows $\mathcal{S}_t^\alpha$ with $\alpha = 0.5$ and $t = 0.75$. At $t = 1$ we have that $\mathcal{S}_1^\alpha$ is the cone, which extends to infinity in either z-direction. Thus, at $x = y = 0$ there is no way to define the tangent space. Figure 7.2 illustrates this for $\alpha = 0.5$ and $t = 1$. For a parameterisation-free description of $\mathcal{S}_t^\alpha$, we define $\phi : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}$ by $\phi(x, y, z, t) = x^2 + y^2 - z^2 - (1-t)^{2\alpha}$ so that $\mathcal{S}_t^\alpha = \{(x, y, z) \in \mathbb{R}^3 \mid \phi(x, y, z, t) = 0\}$.

We are interested in the effects of the geometric singularity forming as $t \to 1^-$ on the solution to the following PDE on $\mathcal{S}_t^\alpha$, which is derived from a conservation law (Section 2.6).

$$\begin{aligned} \partial_t^\bullet U + U\nabla_\Gamma \cdot v - \Delta_\Gamma U = 0 \quad x \in \Gamma := \mathcal{S}_t^\alpha \;\; t \in [0, 1). \\ U(x, 0) = U_0(x) \quad x \in \mathcal{S}_0^\alpha. \end{aligned} \tag{7.0.2}$$

Here,

$$v := \frac{\phi_t}{|\nabla \phi|}\nu$$

is the *prescribed* normal velocity of the surface, with *inward* pointing unit normal ν defined

$$\nu := -\frac{\nabla \phi}{|\nabla \phi|}.$$

The time–derivative like term, $\partial_t^\bullet$, is called the material derivative. The reader is

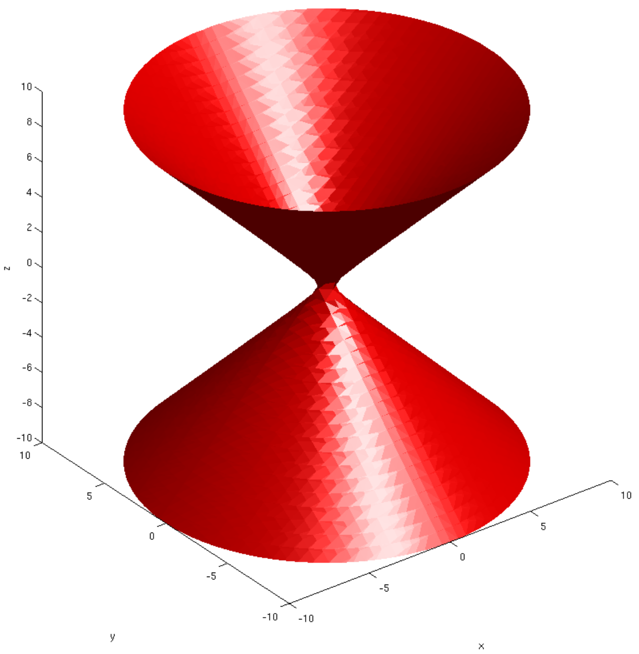

Figure 7.1: Plot of $\mathcal{S}_t^\alpha$ for $\alpha = 0.5$ and $t = 0.75$.

directed to Section 2.6.

We are interested in the qualitative behaviour of the solution as $t \to 1^-$. As with the earlier chapters, we expect that the qualitative results depend on the value of α.

7.1 Parameterisation of the Problem

Since $\mathcal{S}_t^\alpha$ is a surface of revolution, it makes sense to parameterise the hyperbola in the (x, z) plane using arc–length parameterisation. The following were given earlier but are repeated here for convenience.

Definition 7.1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be the inverse of the map $p \mapsto \int_0^p \sqrt{\cosh 2u}\, du$.*

Remark 7.1.2. *By the inverse function Theorem it follows that for every $s \in \mathbb{R}$*

$$\begin{aligned} g'(s) &= \frac{1}{\sqrt{\cosh 2g(s)}} \\ g(0) &= 0. \end{aligned} \tag{7.1.1}$$

Also, one should note that $g(-s) = -g(s)$ for every $s \in \mathbb{R}$.

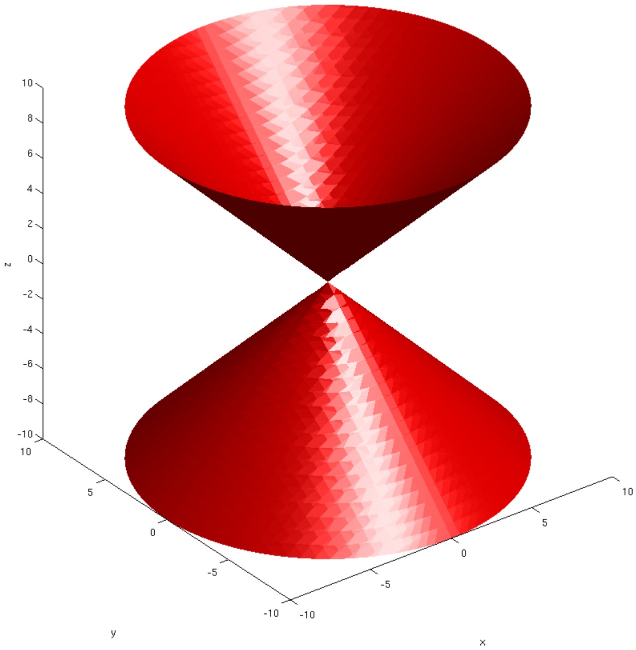

Figure 7.2: Plot of $\mathcal{S}_t^\alpha$ for $\alpha = 0.5$ and $t = 1$.

We parameterise $\mathcal{S}_t^\alpha$ with $X : \mathbb{R} \times [0, 2\pi) \times [0, 1) \to \mathbb{R}^3$, defined as

$$X(l, \theta, t) := (1 - t)^\alpha \left(\cosh g(z) \cos\theta,\ \cosh g(z) \sin\theta,\ \sinh g(z)\right) \tag{7.1.2}$$

where

$$z = l(1 - t)^{-\alpha}.$$

Proposition 7.1.3. *The following results will be of use. The arguments of g, $l(1-t)^{-\alpha}$, are omitted for typographical clarity.*

$$X_l := \frac{\partial X}{\partial l} = g' \left(\sinh g \cos\theta,\ \sinh g \sin\theta,\ \cosh g\right)$$
$$X_\theta := \frac{\partial X}{\partial \theta} = (1 - t)^\alpha \left(-\cosh g \sin\theta,\ \cosh g \cos\theta,\ 0\right)$$
$$X_t := \frac{\partial X}{\partial t} = \begin{pmatrix} \cos\theta(\alpha l(1-t)^{-1} g' \sinh g - \alpha(1-t)^{\alpha-1} \cosh g) \\ \sin\theta(\alpha l(1-t)^{-1} g' \sinh g - \alpha(1-t)^{\alpha-1} \cosh g) \\ \alpha l(1-t)^{-1} g' \cosh g - \alpha(1-t)^{\alpha-1} \sinh g \end{pmatrix}^{\mathrm{T}}$$

Proof. This is a standard calculation and exercise in using the chain rule. $\square$

Definition 7.1.4. *Given a parametrisation $X : \mathbb{R} \times [0, 2\pi) \times [0, 1) \to \mathbb{R}^3$, define the* *metric tensor* $h : \mathbb{R} \times [0, 2\pi) \times [0, 1) \longrightarrow \mathbb{R}^{2\times 2}_{\text{sym}}$ *by*

$$h := \begin{pmatrix} |X_l|^2 & X_l \cdot X_\theta \\ X_\theta \cdot X_l & |X_\theta|^2 \end{pmatrix}.$$

Proposition 7.1.5. *The metric tensor h for $\mathcal{S}^\alpha_t$ with parameterisation X as given in* (7.1.2) *is given by*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & (1-t)^{2\alpha} \cosh^2 g \end{pmatrix}$$

with determinant $|h| := \det(h) = (1-t)^{2\alpha} \cosh^2 g$*. The inverse is given by*

$$h^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (1-t)^{-2\alpha} \operatorname{sech}^2 g \end{pmatrix}.$$

Proof. Follows from the definition of h and Proposition 7.1.3. □

We are now in a position to set up the problem.
Let $u(l, \theta, t) = U(X(l, \theta, t), t)$. Then

$$\frac{\partial u}{\partial t} = \nabla U \cdot (X_t - v) + \partial^\bullet_t U.$$

If $\tau^{(1)} := \pm X_l$ and $\tau^{(2)} := \pm \frac{X_\theta}{|X_\theta|}$ denote the orthogonal unit tangent vectors and if

$$X_t - v = \beta_1 \tau^{(1)} + \beta_2 \tau^{(2)} + \beta_3 \nu,$$

then a simple calculation using Proposition 7.1.3 reveals that $\beta_2 = \beta_3 = 0$ and that $\beta_1 = X_t \cdot \tau^{(1)}$ so that for any orientation of $\tau^{(1)}$ we have

$$X_t - v = (X_t \cdot X_l) X_l.$$

The reader should note that this term is the analogue to the problem considered on the curve of Chapter 3. Noting that

$$\frac{\partial u}{\partial l} = \nabla U \cdot X_l$$

we conclude that

$$\frac{\partial u}{\partial t} = (X_t \cdot X_l) \frac{\partial u}{\partial l} + \partial^\bullet_t U.$$

A simple calculation using Proposition 7.1.3 yields

$$X_t \cdot X_l = \alpha(1-t)^{\alpha-1}\left(\frac{l}{(1-t)^\alpha} - \frac{\sinh 2g(l(1-t)^{-\alpha})}{\sqrt{\cosh 2g(l(1-t)^{-\alpha})}}\right)$$

and thus it is natural to define

$$G(y) := y - \frac{\sinh 2g(y)}{\sqrt{\cosh 2g(y)}}. \tag{7.1.3}$$

The reader should again notice that this is analogous to the curve case. Turning our attention to the Laplace-Beltrami term, $\Delta_\Gamma U$, we have

$$\Delta_\Gamma U = \frac{1}{\sqrt{|h|}}\nabla \cdot \left(\sqrt{|h|}h^{-1}\nabla u\right) \tag{7.1.4}$$

where the gradient is given in local coordinates as

$$\nabla = \left(\frac{\partial}{\partial l}, \frac{\partial}{\partial \theta}\right).$$

Finally, the surface divergence of the velocity, $\nabla_\Gamma \cdot v$, is calculated as

$$\nabla_\Gamma \cdot v = VH$$

where

$$V = \frac{\phi_t}{|\nabla\phi|}$$

is the normal velocity and H is the mean curvature defined, in terms of the level set ϕ as

$$H := \frac{-1}{|\nabla\phi|}\sum_{i,j=1}^{3}\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}$$

with the convention that $\phi_{x_1} \equiv \phi_x, \phi_{x_2} \equiv \phi_y$ and $\phi_{x_3} \equiv \phi_z$. The prefactor of -1 present in the definition of H arises due to the orientation of ν. The reader should note that $\nabla_\Gamma \cdot v = VH$ since the prescribed velocity of the surface is in the normal direction. See Dziuk and Elliott [2007], Appendix A for an alternative (yet equivalent) definition.

One calculates to see that

$$VH = \alpha(1-t)^{-1}\left(\operatorname{sech}^2 2g - \operatorname{sech} 2g\right)$$

where the argument of g is omitted. Noting that $G'(y) = -\operatorname{sech}^2 2g(y)$ we thus study the following PDE on $\mathbb{R} \times [0, 2\pi)$

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{\sqrt{|h|}} \nabla \cdot \left(\sqrt{|h|} h^{-1} \nabla u \right) + \alpha (1-t)^{\alpha-1} \frac{\partial}{\partial l} \left(G(l(1-t)^{-\alpha}) u \right) \\
&\quad + \alpha (1-t)^{-1} \operatorname{sech} 2g(l(1-t)^{-\alpha}) u \qquad (7.1.5) \\
u(l, 0, t) &= u(l, 2\pi, t) \\
u(l, \theta, 0) &= U_0(X(l, \theta, 0)) =: u_0(l, \theta).
\end{aligned}
$$

From (7.1.5) we see that the integral of u is not conserved. In fact, the correct quantity that is conserved is the integral of $u(l, \theta, t)\sqrt{|h|} = u(l, \theta, t)(1-t)^\alpha \cosh g(l(1-t)^{-\alpha})$.

Proposition 7.1.6. *Let $J(l, \theta, t) := u(l, \theta, t)\sqrt{|h|}$ and $\mathbb{K} := \mathbb{R} \times [0, 2\pi)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{K}} J(l, \theta, t) \, \mathrm{d}l \, \mathrm{d}\theta = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}_t^\alpha} U(z, t) \, \mathrm{d}\sigma(z) = 0.$$

Proof. This follows from the definition of the change of area measure and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}_t^\alpha} U(z, t) \, \mathrm{d}\sigma(z) = 0$$

from equation (2.6.2). □

Remark 7.1.7. *Although the proof of Proposition 7.1.6 follows from the conversation law given in equation 2.6.2, assuming enough regularity so that the following is valid, one may directly calculate to see that by the definition of J,*

$$\int_{\mathbb{K}} J(l, \theta, t) \, \mathrm{d}l \, \mathrm{d}\theta = \int_{\mathcal{S}_t^\alpha} U(z, t) \, \mathrm{d}\sigma(z)$$

since the surface measure $\sigma(z) = \sqrt{|h|}$ in local coordinates. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{K}} J(l, \theta, t) \, \mathrm{d}l \, \mathrm{d}\theta = \int_{\mathbb{K}} \frac{\partial J}{\partial t} \, \mathrm{d}l \, \mathrm{d}\theta = \int_{\mathbb{K}} \left(\frac{\partial u}{\partial t} \sqrt{|h|} + u \frac{\partial}{\partial t} \sqrt{|h|} \right) \mathrm{d}l \, \mathrm{d}\theta.$$

Substituting for $\frac{\partial u}{\partial t}$ in (7.1.5), *using the Divergence Theorem and integrating by parts yields*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{K}} J(l, \theta, t) \, \mathrm{d}l \, \mathrm{d}\theta &= \alpha (1-t)^{\alpha-1} \int_{\mathbb{K}} (\tanh 2g \sinh g + \operatorname{sech} 2g \cosh g - \cosh g) \, u \, \mathrm{d}l \, \mathrm{d}\theta \\
&= 0
\end{aligned}
$$

since

$$\cosh x \equiv \tanh 2x \sinh x + \operatorname{sech} 2x \cosh x,$$

for every $x \in \mathbb{R}$.

In light of Proposition 7.1.6, if $J(l,\theta,t) = u(l,\theta,t)\sqrt{|h|}$ then J solves

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \nabla \cdot \left(\sqrt{|h|} h^{-1} \nabla \left(\frac{J}{\sqrt{|h|}} \right) \right) + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (GJ) \\
J(l,0,t) &= J(l,2\pi,t) \\
J(l,\theta,0) &= u_0(l,\theta) \cosh g(l).
\end{aligned} \tag{7.1.6}$$

Since $\mathcal{S}_t^\alpha$ is a surface of revolution, it is natural to assume the following.

Assumption 7.1.8. *We assume that the initial condition* u_0 *is independent of* θ. *This assumption will be in place throughout the rest of this thesis.*

With Assumption 7.1.8, equation (7.1.6) becomes a PDE in one spatial dimension:

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \frac{\partial^2 J}{\partial l^2} - (1-t)^{-\alpha} \frac{\partial}{\partial l} \left(\frac{\tanh g}{\sqrt{\cosh 2g}} J \right) + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (GJ) \\
J(l,0) &= u_0(l) \cosh g(l),
\end{aligned} \tag{7.1.7}$$

which we will refer to as the density equation.

One should compare this equation with the equation derived in Chapter 3 for the curve. We see that we have an extra creation and drift term given by

$$-(1-t)^{-\alpha} \frac{\partial}{\partial l} \left(\frac{\tanh g}{\sqrt{\cosh 2g}} J \right),$$

which comes from the Laplace-Beltrami operator. We thus have the following equation for u which we will refer to as the heat equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + (1-t)^{-\alpha} \frac{\tanh g}{\sqrt{\cosh 2g}} \frac{\partial u}{\partial l} + \alpha(1-t)^{\alpha-1} \frac{\partial}{\partial l} (Gu) + \alpha(1-t)^{-1} \operatorname{sech} 2g \, u \\
u(l,0) &= u_0(l).
\end{aligned} \tag{7.1.8}$$

Remark 7.1.9 (On the growth of the initial data)**.** *We note that Proposition 7.1.6 implies that*

$$\int_{\mathbb{R}} J(l,t) \, \mathrm{d}l = \int_{\mathbb{R}} J(l,0) \, \mathrm{d}l = \int_{\mathbb{R}} u_0(l) \cosh g(l) \, \mathrm{d}l$$

for every $t \in [0,1]$*. We need*

$$\int_{\mathbb{R}} u_0(l) \cosh g(l) \, \mathrm{d}l < \infty$$

which is guaranteed if, for example, $\operatorname{supp}(u_0)$ *is compact. Indeed, any function* $u_0 : \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{|l|\to\infty} u_0(l) \cosh g(l) = 0$ *sufficiently fast will suffice.*

Remark 7.1.10. *Throughout we will assume that* $u_0 \in C_c^\infty(\mathbb{R})$ *and so, as* $\cosh g(\cdot) \in C^\infty(\mathbb{R})$ *it follows that* $J(l,0) \in C_c^\infty(\mathbb{R})$*. Indeed, Theorem 2.2.4 implies there exists* $J, u \in C^{2,1}(\mathbb{R} \times [0,1))$ *such that* (7.1.7) *and* (7.1.8) *are respectively satisfied.*

7.2 Scaling properties of the PDEs

7.2.1 Scaling of the heat equation

As with the curve, the following scaling is important. Suppose u solves (7.1.8) and let $y = l(1-t)^{-\alpha}$, $\tau = -\log(1-t)$ and write $u(l,t) = v(y,\tau)$. Then v solves

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= e^{-\beta\tau} \frac{\partial^2 v}{\partial y^2} + e^{-\beta\tau} \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}} \frac{\partial v}{\partial y} + \alpha(G(y) - y)\frac{\partial v}{\partial y} \\ &\quad + \alpha(\operatorname{sech} 2g(y) - \operatorname{sech}^2 2g(y))v \\ v(y,0) &= u_0(y), \end{aligned} \tag{7.2.1}$$

where $\beta = 1 - 2\alpha$. Thus, if $\alpha < \frac{1}{2}$ we have that $\beta > 0$ and so exponential decay of the diffusion term arising from the Laplace–Beltrami term occurs. We are almost in the case of the curve but crucially, the creation term is positive. This was not the case when considering the curve, and so it is a priori unknown what will happen at the singularity as $\tau \to \infty$.

If $\alpha = \frac{1}{2}$ then $\beta = 0$ and so the problem is time–homogeneous and an interplay of the Laplace–Beltrami term and the drift term will occur. Again, a priori it is unclear what will happen at the singularity.

Finally, if $\alpha > \frac{1}{2}$, exponential explosion of the problem occurs and so this was the incorrect scaling. In this case, we let $y = l(1-t)^{-\alpha}$, $\tau = (1-t)^{1-2\alpha}$ and $\gamma = 2\alpha - 1 > 0$. If u solves (7.1.8), then writing $u(l,t) = v(y,\tau)$ we have that v

solves the following on $\mathbb{R} \times [1, \infty)$

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \gamma^{-1}\frac{\partial^2 v}{\partial y^2} + \gamma^{-1}\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\frac{\partial v}{\partial y} \\
&\quad + \alpha\gamma^{-1}\tau^{-1}(G(y) - y)\frac{\partial v}{\partial y} + \alpha\gamma^{-1}\tau^{-1}(\operatorname{sech} 2g(y) - \operatorname{sech}^2 2g(y))v \qquad (7.2.2) \\
v(y, 1) &= u_0(y).
\end{aligned}$$

We see from this that the drift and creation term vanish as $\tau \to \infty$, leaving the diffusion and drift term from the Laplace–Beltrami term. This scaling was not employed in the curve case and so a priori it is unclear how v behaves at the singularity.

7.2.2 Scaling of the density equation

Consider (7.1.7) and let $y = l(1-t)^{-\alpha}$ while $\tau = -\log(1-t)$. Write $q(y, \tau) = J(l, t)$. Then, if J satisfies (7.1.7), then q satisfies

$$\begin{aligned}
\frac{\partial q}{\partial \tau} &= e^{-(1-2\alpha)\tau}\frac{\partial^2 q}{\partial y^2} - e^{-(1-2\alpha)\tau}\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\frac{\partial q}{\partial y} \\
&\quad - e^{-(1-2\alpha)\tau}\frac{\partial}{\partial y}\left(\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\right) q + \alpha(G(y) - y)\frac{\partial q}{\partial y} - \alpha \operatorname{sech}^2 2g(y)\, q \\
q(y, 0) &= J_0(y).
\end{aligned} \qquad (7.2.3)$$

One observes that if $\alpha < \frac{1}{2}$, then exponential decay of the diffusion term along with a creation term occurs leading to an equation which we expect will have the same qualitative properties as the curve. If $\alpha = \frac{1}{2}$ we have a time-homogeneous problem, however it is a priori unclear what the behaviour of the solution will be at the singularity.

However, if $\alpha > \frac{1}{2}$, then the exponential terms blow up as $\tau \to \infty$ and so this was the incorrect scaling. In this case, we take $y = l(1-t)^{-\alpha}$ and $\tau = (1-t)^{1-2\alpha}$. Let $q(y, \tau) = J(l, t)$ so if J solves (7.1.7), it follows that q solves

$$\begin{aligned}
\frac{\partial q}{\partial \tau} &= \gamma^{-1}\frac{\partial^2 q}{\partial y^2} - \gamma^{-1}\frac{\partial}{\partial y}\left(\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}} q\right) + \alpha\gamma^{-1}\tau^{-1}\frac{\partial}{\partial y}(G(y)q) \\
&\quad - \alpha\gamma^{-1}y\tau^{-1}\frac{\partial q}{\partial y} \qquad (7.2.4) \\
q(y, 1) &= J_0(y)
\end{aligned}$$

where $\gamma = 2\alpha - 1 > 0$ and $(y, \tau) \in \mathbb{R} \times [1, \infty)$. We clearly see that the creation term and one of the drift terms vanish as $\tau \to \infty$, leaving a diffusive term and a drift term. Again this scaling was not employed in the curve case and so a priori it is unclear how q behaves at the singularity.

Chapter 8

Analysis of Problem II: Before the Singularity

As in Chapter 4, we now analyse the problem depending on α. We still refer to $\alpha < \frac{1}{2}$ as the *sub-critical regime*, $\alpha = \frac{1}{2}$ as the *critical regime* and finally $\alpha > \frac{1}{2}$ as the *super-critical regime*. We will interchange the word *case* and *regime* with the understanding that they mean the same.

8.1 Sub–Critical Regime

In the following we set $\alpha < \frac{1}{2}$. We study the qualitative properties of the solution u to (7.1.8) via the equation satisfied by J in (7.1.7). To that end, we have the following theorem

Theorem 8.1.1. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose J is the unique solution to* (7.1.7) *with initial data $J_0(l) := u_0(l)\cosh g(l)$. Then, J is uniform bounded and*

$$J(0,t) \longrightarrow 0$$

as $t \to 1^-$.

Remark 8.1.2. *How J behaves as a function of l and t close to the singularity is discussed in Theorem 8.1.4.*

Proof of Theorem 8.1.1. We will work in the (y,τ) coordinates. To this end, let $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$. Write $q(y,\tau) = J(l,t)$ so that q solves (7.2.3). Such a solution is guaranteed by Theorem 2.2.4. By the Feynman-Kac representation formula (Theorem 2.4.1) the solution is given by

$$q(y,\tau) = \mathbb{E}^{(-\tau,y)}\left[\exp\left(-\int_{-\tau}^{0} F_1(X_s)\,\mathrm{d}s\right)\exp\left(-\int_{-\tau}^{0} e^{\beta s}F_2(X_s)\,\mathrm{d}s\right)J_0(X_0)\right] \tag{8.1.1}$$

where

$$F_1(y) := \alpha\,\mathrm{sech}^2\, 2g(y) \geq 0, \qquad \beta := 1-2\alpha > 0,$$

$$F_2(y) := \mathrm{sech}\, 2g(y)\left(\mathrm{sech}^2\, g(y) - \tanh g(y)\tanh 2g(y)\right).$$

Remark 8.1.3. *The reader should note that the proof below is analogous to Theorem 4.1.1, except for the introduction of a further creation term, $-e^{\beta s}F_2(y)$, in the Feynman–Kac formula above and an additional bounded drift term $-e^{\beta s}\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}$ into the SDE in* (8.1.2)*.*

Returning to equation (8.1.1), we note that $(X_s)_{s\geq -\tau}$ is the unique strong solution to the SDE

$$\begin{aligned}\mathrm{d}X_s &= \left(-e^{\beta s}\frac{\tanh g(X_s)}{\sqrt{\cosh 2g(X_s)}} + \alpha(G(X_s)-X_s)\right)\mathrm{d}s + \sqrt{2}e^{\frac{\beta}{2}s}\,\mathrm{d}W_s \\ X_{-\tau} &= y.\end{aligned} \tag{8.1.2}$$

Here $-\tau \leq s \leq 0$. If $b(y,s) := -e^{\beta s}\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}} + \alpha(G(y)-y)$ then

$$\frac{\partial b}{\partial y}(y,s) = -e^{\beta s}\,\mathrm{sech}\, 2g(y)\left(\mathrm{sech}^2\, g(y) - \tanh g(y)\tanh 2g(y)\right) - \alpha\,\mathrm{sech}^2\, 2g(y) - \alpha$$

which is uniformly bounded in $(y,s) \in \mathbb{R}\times(-\infty,0]$ and so b is globally Lipschitz with at most linear growth at infinity. Further since $s \leq 0$ and $\beta > 0$ it follows that $|\sqrt{2}e^{\frac{\beta}{2}s}| \leq \sqrt{2}$ for every $s \in (-\infty,0]$. Thus, the standard existence and uniqueness result for the unique strong solutions to SDEs (Theorem 2.3.2) implies that such a unique strong solution $(X_s)_{s\geq-\tau}$ exists to (8.1.2). Hence, considering the representation formula, one has

$$|q(y,\tau)| \leq \exp\left(C\int_{-\tau}^{0} e^{\beta s}\,\mathrm{d}s\right)\|J_0\|_{L^\infty} \leq e^{\frac{c}{\beta}}\|J_0\|_{L^\infty}$$

for every $(y,\tau)\in\mathbb{R}\times[0,\infty)$, which establishes, after reverting back to the (l,t) coordinates, the uniformly boundedness of J.

To see that $J(0,t)$ vanishes as $t\to 1^-$, we again work in the (y,τ) coordinates.

Noting that $l = 0$ implies $y = 0$, we set $y = 0$ and note that

$$X_s = \int_{-\tau}^{s} -e^{-\alpha(s-r)} \frac{e^{\beta r} \tanh g(X_r)}{\sqrt{\cosh 2g(X_r)}} + \alpha e^{-\alpha(s-r)} G(X_r) \, \mathrm{d}r + \sqrt{2} \int_{-\tau}^{s} e^{-\alpha(s-r)} e^{\frac{\beta}{2} r} \, \mathrm{d}W_r$$

is the unique strong solution to (8.1.2). By definition of F_2 we have

$$\begin{aligned} |q(0,\tau)| &\leq \exp\left(C \int_{-\tau}^{0} e^{\beta s} \, \mathrm{d}s \right) \mathbb{E}^{(-\tau,0)} \left[\exp\left(- \int_{-\tau}^{0} F_1(X_s) \, \mathrm{d}s \right) |J_0(X_0)| \right] \\ &\leq C \, \mathbb{E}^{(-\tau,0)} \left[\exp\left(- \int_{-\tau}^{0} F_1(X_s) \, \mathrm{d}s \right) |J_0(X_0)| \right]. \end{aligned}$$

We now follow the same method as in the proof of Theorem 4.1.1. Let $\gamma > 0$ to be determined later and define

$$\Omega_1 := \{\omega \in \Omega \,|\, |X_s| < \gamma \quad \forall -\tau \leq s \leq 0\}$$

$$\Omega_2 := \Omega \setminus \Omega_1 = \{\omega \in \Omega \,|\, \exists \, s_0 \in [-\tau, 0] \, : \, |X_s| \geq \gamma\}.$$

We note that if $\omega \in \Omega_1$, then $F_1(X_s) > F_1(\gamma)$ and so splitting the above expectation over the Ω_i we have

$$\mathbb{E}^{(-\tau,0)} \left[\exp\left(- \int_{-\tau}^{0} F_1(X_s) \, \mathrm{d}s \right) |J_0(X_0)| \chi_{\Omega_1} \right] \leq e^{-F_1(\gamma)\tau} \|J_0\|_{L^\infty},$$

$$\mathbb{E}^{(-\tau,0)} \left[\exp\left(- \int_{-\tau}^{0} F_1(X_s) \, \mathrm{d}s \right) |J_0(X_0)| \chi_{\Omega_2} \right] \leq \|J_0\|_{L^\infty} \, \mathbb{P}[\Omega_2].$$

We now show that $\mathbb{P}[\Omega_2] \to 0$ as $\gamma \to \infty$, uniformly in τ.
If there exists $s_0 \in [-\tau, 0]$ such that $|X_{s_0}| \geq \gamma$ then

$$\gamma \leq \|G\|_{L^\infty} + \alpha^{-1} + |Z_{s_0}|$$

where

$$Z_{s_0} := \sqrt{2} \int_{-\tau}^{s_0} e^{-\alpha s_0} e^{\alpha r} e^{\frac{\beta}{2} r} \, \mathrm{d}W_r$$

which is a Gaussian random variable with mean zero and variance

$$\mathbb{E}|Z_{s_0}|^2 = 2e^{-2\alpha s_0} \int_{-\tau}^{s_0} e^{(2\alpha+\beta)r} \, \mathrm{d}r = 2e^{-2\alpha s_0} \left(e^{s_0} - e^{-\tau} \right) \leq 2e^{(1-2\alpha)s_0} \leq 2.$$

The last inequality follows as $\alpha < \frac{1}{2}$ and $s_0 \leq 0$. Thus

$$\mathbb{P}[\Omega_2] \leq \mathbb{P}[|Z_{s_0}| \geq \gamma - \|G\|_{L^\infty} - \alpha^{-1}] \leq 2\mathbb{P}[Z_{s_0} \geq \gamma - \|G\|_{L^\infty} - \alpha^{-1}]$$

where the last inequality holds since Z_{s_0} is a Gaussian random variable with zero mean. Because of this, we have that for some $C > 0$ independent of γ and τ

$$2\mathbb{P}[Z_{s_0} \geq \gamma - \|G\|_{L^\infty} - \alpha^{-1}] = C \int_{\frac{\gamma - \alpha^{-1} - \|G\|}{\sqrt{2e^{-2\alpha s_0}(e^{s_0} - e^{-\tau})}}}^{\infty} e^{-\frac{x^2}{2}} \, \mathrm{d}x.$$

We recall that if $b > a > 0$ then

$$\int_b^\infty e^{-\frac{x^2}{2}} \, \mathrm{d}x < \int_a^\infty e^{-\frac{x^2}{2}} \, \mathrm{d}x.$$

In light of this, take $\gamma > \|G\|_{L^\infty} + \alpha^{-1}$. Observe that for every $s_0 \in [-\tau, 0]$, and for every $\tau > 0$

$$\sqrt{2e^{-2\alpha s_0}(e^{s_0} - e^{-\tau})} < \sqrt{2}$$

by the above and so

$$\frac{\gamma - \alpha^{-1} - \|G\|_{L^\infty}}{\sqrt{2e^{-2\alpha s_0}(e^{s_0} - e^{-\tau})}} > \frac{\gamma - \alpha^{-1} - \|G\|_{L^\infty}}{\sqrt{2}}$$

which implies

$$\mathbb{P}[\Omega_2] \leq C \int_{\frac{\gamma - \alpha^{-1} - \|G\|_{L^\infty}}{\sqrt{2}}}^{\infty} e^{-\frac{x^2}{2}} \, \mathrm{d}x$$

for every $\tau \geq 0$. Given $\varepsilon > 0$, choose $\gamma_0 > 0$ such that $\gamma > \gamma_0$ implies that $\gamma - \|G\|_{L^\infty} - \alpha^{-1} > 0$ and

$$\mathbb{P}[\Omega_2] < \frac{\varepsilon}{2C\|J_0\|_{L^\infty}}.$$

Now choose $\tau_0 > 0$ such that $\tau > \tau_0$ implies

$$e^{-\tau F_1(\gamma)} < \frac{\varepsilon}{2C\|J_0\|_{L^\infty}}.$$

So, $\tau > \tau_0$ implies

$$|q(0, \tau)| < \varepsilon.$$

Thus, changing back into the (l, t) coordinates, given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < 1 - t < \delta$ implies

$$|J(0, t)| < \varepsilon.$$

$\square$

The question remains as to how J behaves as a function of l and t for (l, t) close to $(0, 1)$.

Theorem 8.1.4. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose J is the unique smooth solution to (7.1.7) with initial data $J_0(l) := u_0(l)\cosh g(l)$. Then there exists $C > 0$ such that*

$$|J(l,t)| \le C\|u_0\|_{L^\infty}(1-t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha})}$$

for every $(l,t) \in \mathbb{R} \times [0,1)$.

Proof. Inspired by the analogous theorem (Theorem 4.1.3) for the curve, we make the ansatz

$$J(l,t) = (1-t)^\alpha \sqrt{\cosh 2g(l(1-t)^{-\alpha})}\varphi_*(l,t).$$

Then, after a long but standard calculation, one can show that φ_* solves

$$\begin{aligned}
\frac{\partial \varphi_*}{\partial t} &= \frac{\partial^2 \varphi_*}{\partial l^2} + \left((1-t)^{-\alpha}\left(2\frac{\tanh 2g}{\sqrt{\cosh 2g}} - \frac{\tanh g}{\sqrt{\cosh 2g}}\right) + \alpha(1-t)^{\alpha-1}G\right)\frac{\partial \varphi_*}{\partial l} \\
&\quad + \left(2(1-t)^{-2\alpha}\operatorname{sech}^3 2g - (1-t)^{-2\alpha}\operatorname{sech}^2 g \operatorname{sech} 2g\right)\varphi_* \\
\varphi_*(l,0) &= \frac{J_0(l)}{\cosh^{\frac{1}{2}} 2g(l)} = \frac{u_0(l)\cosh g(l)}{\cosh^{\frac{1}{2}} 2g(l)}.
\end{aligned} \tag{8.1.3}$$

We have omitted the arguments of g and G for typographical clarity. We note that $\varphi_*(\cdot,0) \in L^\infty(\mathbb{R})$, with $\|\varphi_*(\cdot,0)\|_{L^\infty} \le \|u_0\|_{L^\infty}$ and such a solution $\varphi_* \in C^{2,1}(\mathbb{R}\times[0,1))$ exists by Theorem 2.2.4. The reader will observe that the proof is complete once we show that φ_* is uniformly bounded. To this end, we switch to working in the (y,τ) coordinates. Let $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$ and write $\varphi(y,\tau) = \varphi_*(l,t)$. Then φ solves

$$\begin{aligned}
\frac{\partial \varphi}{\partial \tau} &= e^{-\beta\tau}\frac{\partial^2 \varphi}{\partial y^2} + e^{-\beta\tau}\left(2\frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}} - \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\right)\frac{\partial \varphi}{\partial y} + \alpha(G(y)-y)\frac{\partial \varphi}{\partial y} \\
&\quad + 2e^{-\beta\tau}\operatorname{sech}^3 2g(y)\varphi - e^{-\beta\tau}\operatorname{sech}^2 g(y)\operatorname{sech} 2g(y)\varphi \\
\varphi(y,0) &= \varphi_*(y,0).
\end{aligned}$$

Here, $\beta = 1 - 2\alpha > 0$. Since such a φ exists, one may use the Feynman-Kac representation formula which says that the solution φ is given by

$$\varphi(y,\tau) = \mathbb{E}^{(-\tau,y)}\left[\exp\left(\int_{-\tau}^0 e^{\beta s}B(X_s)\,\mathrm{d}s\right)\varphi_0(X_0)\right]$$

where we define

$$B(y) := \operatorname{sech} 2g(y)\left(2\operatorname{sech}^2 2g(y) - \operatorname{sech}^2 g(y)\right)$$

and $(X_s)_{s\geq -\tau}$ is the unique strong solution to the SDE

$$\begin{aligned} \mathrm{d}X_s &= \left(e^{\beta s}\mu(X_s) + \alpha(G(X_s) - X_s)\right)\,\mathrm{d}s + \sqrt{2}e^{\frac{\beta}{2}s}\,\mathrm{d}W_s \\ X_{-\tau} &= y, \end{aligned}$$

with

$$\mu(y) := 2\frac{\tanh 2g(y)}{\sqrt{\cosh 2g(y)}} - \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}.$$

Noting that $\mu \in L^\infty(\mathbb{R})$ with uniformly bounded derivative μ', one may use the standard theory of existence and uniqueness of strong solutions to SDEs (Theorem 2.3.2) to establish such a strong solution. This is because the drift is globally Lipschitz and grows at most linear at infinity. (The argument is analogous to before and so omitted). Since $B \in L^\infty(\mathbb{R})$ we have

$$\begin{aligned} |\varphi(y,\tau)| &\leq \exp\left(C\int_{-\tau}^{0} e^{\beta s}\,\mathrm{d}s\right)\|\varphi_0\|_{L^\infty} \\ &= \exp\left(\frac{C}{\beta}\left(1 - e^{-\beta\tau}\right)\right)\|\varphi_0\|_{L^\infty}. \end{aligned}$$

However, $\alpha < \frac{1}{2}$ and $\beta = 1 - 2\alpha > 0$ so it follows that

$$|\varphi(y,\tau)| \leq C\|\varphi_0\|_{L^\infty} \leq C\|u_0\|_{L^\infty}$$

and so, switching back into the (l,t) coordinates

$$|J(l,t)| \leq C\|u_0\|_{L^\infty}(1-t)^\alpha\sqrt{\cosh 2g(l(1-t)^{-\alpha})}.$$

□

The previous theorem yields the following information about the solution u to (7.1.8).

Corollary 8.1.5. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose u is the unique smooth solution to* (7.1.8) *with initial data u_0. Then there exists $C > 0$ such that*

$$|u(l,t)| \leq C\|u_0\|_{L^\infty}$$

for every $(l,t) \in \mathbb{R}\times[0,1]$.

Proof. Recalling the definition of J and the result of Theorem 8.1.4 above, one has

$$|J(l,t)| = |u(l,t)|(1-t)^\alpha\cosh g(l(1-t)^{-\alpha}) \leq C\|u_0\|_{L^\infty}(1-t)^\alpha\sqrt{\cosh 2g(l(1-t)^{-\alpha})}.$$

Hence,

$$|u(l,t)| \le C\|u_0\|_{L^\infty} \frac{\sqrt{\cosh 2g(l(1-t)^{-\alpha})}}{\cosh g(l(1-t)^{-\alpha})}.$$

Now, for every $x \in \mathbb{R}$,

$$\frac{\sqrt{\cosh 2x}}{\cosh x} = \sqrt{\frac{2\cosh^2 x - 1}{\cosh^2 x}} \le \sqrt{2}.$$

$\square$

Although the density $J(0,t) \to 0$ as $t \to 1^-$, the following shows that there exists admissible initial data u_0, for which the solution u to (7.1.8) does not vanish at the singularity.

Theorem 8.1.6. *There exists $u_0 \in L^\infty(\mathbb{R})$ which is radially symmetric with*

$$\int_{\mathbb{R}} u_0(l) \cosh g(l)\,\mathrm{d}l < \infty$$

such that the unique smooth solution u, with such initial data u_0, does not vanish at the singularity as $t \to 1^-$. That is, there exists $C > 0$ and $\delta > 0$ such that $0 < 1-t < \delta$ implies

$$u(0,t) \ge C.$$

Remark 8.1.7. *The reader should note that this theorem is in fact valid for $\alpha \le \frac{1}{2}$.*

Proof of Theorem 8.1.6. We will work in the (y,τ) coordinates. To this end, let $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$. Write $v(y,\tau) = u(l,t)$. Then, v solves

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= e^{-\beta\tau}\frac{\partial^2 v}{\partial y^2} + e^{-\beta\tau}\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\frac{\partial v}{\partial y} + \alpha(G(y)-y)\frac{\partial v}{\partial y} \\ &\quad + \alpha(\operatorname{sech} 2g(y) - \operatorname{sech}^2 2g(y))v \\ v(y,0) &= u_0(y),\end{aligned} \tag{8.1.4}$$

where $\beta = 1 - 2\alpha > 0$. We note that standard parabolic theory (Theorem 2.2.4) implies that such a $v \in C^{2,1}(\mathbb{R}\times[0,\infty);\mathbb{R})$ exists and so by the Feynman-Kac representation formula, the solution is given by

$$v(y,\tau) = \mathbb{E}^{(-\tau,y)}\left[\exp\left(\int_{-\tau}^{0} \alpha(\operatorname{sech} 2g(X_s) - \operatorname{sech}^2 2g(X_s))\,\mathrm{d}s\right) u_0(X_0)\right]$$

where $(X_s)_{s\geq -\tau}$ is the unique strong solution to the SDE

$$\begin{aligned} \mathrm{d}X_s &= \left(e^{\beta s} \frac{\tanh g(X_s)}{\sqrt{\cosh 2g(X_s)}} + \alpha(G(X_s) - X_s) \right) \mathrm{d}s + \sqrt{2} e^{\frac{\beta}{2} s} \, \mathrm{d}W_s \\ X_{-\tau} &= y. \end{aligned} \tag{8.1.5}$$

The arguments above can be modified for this equation to show that such a unique strong solution exists (Theorem 2.3.2). We make the crucial observation that for every $x \in \mathbb{R}$,

$$\operatorname{sech} 2g(x) - \operatorname{sech}^2 2g(x) = \operatorname{sech} 2g(x)(1 - \operatorname{sech} 2g(x)) \geq 0.$$

For the initial data, u_0, we take

$$u_0(l) = \begin{cases} 1 & \text{if } |l| \leq K \\ 0 & \text{if } |l| > K \end{cases}$$

for some $K \in (0, \infty)$ to be determined later. We have

$$\int_{\mathbb{R}} u_0(l) \cosh g(l) \, \mathrm{d}l = \int_{-K}^{K} \cosh g(l) \, \mathrm{d}l < \infty$$

and so such u_0 is admissible. We note that Theorem 2.2.4 implies that there exists a $u \in C^{2,1}(\mathbb{R} \times [0, 1))$ such that (7.1.8) is satisfied. Working in (y, τ) coordinates and splitting the above expectation over

$$\Omega_1 := \{\omega \in \Omega \,|\, |X_0| \leq K\}$$

and

$$\Omega_2 := \Omega \setminus \Omega_1$$

yields

$$v(0, \tau) \geq \mathbb{E}^{(-\tau,0)} u_0(X_0) \chi_{\Omega_1} = \mathbb{P}[\Omega_1] = 1 - \mathbb{P}[|X_0| > K].$$

However,

$$X_0 = \int_{-\tau}^{0} e^{\alpha r} e^{\beta r} \frac{\tanh g(X_r)}{\sqrt{\cosh 2g(X_r)}} + \alpha e^{\alpha r} G(X_r) \, \mathrm{d}r + \sqrt{2} \int_{-\tau}^{0} e^{\alpha r} e^{\frac{\beta}{2} r} \, \mathrm{d}W_r$$

and so

$$K < |X_0| \implies K < \frac{1}{\alpha} + \|G\|_{L^\infty} + |Z_0|$$

by definition of X_0, where

$$Z_0 := \sqrt{2}\int_{-\tau}^{0} e^{\alpha r} e^{\frac{\beta}{2}r}\,\mathrm{d}W_r$$

is a Gaussian random variable with mean zero and variance

$$\mathbb{E}|Z_0|^2 = 2(1-e^{-\tau}) \leq 2.$$

Thus

$$\begin{aligned}\mathbb{P}[|X_0|>K] \leq 2\mathbb{P}[Z_0 > K-\alpha^{-1}-\|G\|_{L^\infty}] &= \sqrt{\frac{2}{\pi}}\int_{\frac{K-\frac{1}{\alpha}-\|G\|}{\sqrt{2(1-e^{-\tau})}}}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x \\ &\leq \sqrt{\frac{2}{\pi}}\int_{\frac{K-\frac{1}{\alpha}-\|G\|}{\sqrt{2}}}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x.\end{aligned}$$

Now choose $K_0 > 0$ such that $K > K_0$ implies $K - \alpha^{-1} - \|G\|_{L^\infty} > 0$ and

$$\int_{\frac{K-\frac{1}{\alpha}-\|G\|}{\sqrt{2}}}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x < \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Hence, for $K > K_0$

$$\mathbb{P}[|X_0|>K] < \frac{1}{2}$$

and so

$$\mathbb{P}[\Omega_1] = \mathbb{P}[|X_0|\leq K] > 1-\frac{1}{2} = \frac{1}{2}.$$

Hence, working back in the (l,t) coordinates and noting that $y=0$ implies $l=0$ we have that for every $t\in[0,1]$,

$$u(0,t) = v(0,\tau) \geq \mathbb{P}[\Omega_1] = \mathbb{P}[|X_0|\leq K] > 1-\frac{1}{2} = \frac{1}{2}.$$

□

Remark 8.1.8. *The reader will note that $u_0 \notin C^\infty(\mathbb{R})$. However, since it is bounded with compact support, the proof of Theorem 8.1.4 is exactly the same and so Corollary 8.1.5 still holds true. We thus conclude in general that the solution u is bounded, but need not vanish at the singularity. This is at odds with $\alpha < \frac{1}{2}$ for the curve (Theorem 4.1.1) where the solution* always *vanishes at the singularity, regardless of the admissible initial data.*

8.2 Critical Regime

We now show that the solution, u, to (7.1.8) with compactly supported initial data in the case of $\alpha = \frac{1}{2}$ is bounded in a time–dependent neighbourhood of $l = 0$. The reader will note that Theorem 8.1.6, where we establish that there exists bounded radially symmetric initial data such that the solution does not vanish at the singularity, still holds true in the $\alpha = \frac{1}{2}$ case. To show boundedness of the solution u, we use a perturbation argument, improve on the estimates used in Section 4.2 and use the functional analytic and comparison methods of Section 4.2.

In the following, consider (7.2.1) and (7.2.3) and set

$$\begin{aligned} -\bar{A}_0 := \frac{\partial^2}{\partial y^2} &+ \left(\frac{1}{2}(G(y) - y) - \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}} \right) \frac{\partial}{\partial y} \\ &+ \operatorname{sech} 2g(y) \left(\tanh g(y) \tanh 2g(y) - \operatorname{sech}^2 g(y) - \frac{1}{2} \operatorname{sech} 2g(y) \right). \end{aligned}$$

Theorem 8.2.1. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose u is the unique smooth solution to (7.1.8) with $\alpha = \frac{1}{2}$ and initial data u_0. Then, there exists $C, y_0 > 0$ such that, for every $t \in [0,1)$,*

$$|u(l,t)| \le C(1 + \|u_0\|_{L^\infty})$$

for every $|l| \le \sqrt{1-t}$ and

$$|u(l,t)| \le C\|u_0\|_{L^\infty} |l|^{2\bar{\mu}_1 - 1},$$

for every $|l| > y_0\sqrt{1-t}$. Further, if J is the solution to (7.1.7) with initial data $J_0(l) = u_0(l)\cosh g(l)$ then

$$|J(l,t)| \le C\|J_0\|_{L^\infty} |l|^{2\bar{\mu}_1},$$

for every $|l| > y_0\sqrt{1-t}$ and

$$|J(l,t)| \le C(1 + \|J_0\|_{\mathcal{H}_0})(1-t)^{\bar{\mu}_1},$$

for every $|l| \le \sqrt{1-t}$. In particular,

$$J(0,t) \longrightarrow 0$$

as $t \to 1^-$. Here, $\mathcal{H}_0 = -\tilde{A}_0$ where $-\tilde{A}_0$ is the ground–state transformation of $-\bar{A}_0$ above. The norm $\|\cdot\|_{\mathcal{H}_0}$ is the $D(\mathcal{H}_0)$ norm and $\bar{\mu}_1 > 0$ is the minimal eigenvalue *of* $\bar{A}_0$.

Proof. Let $y = l(1-t)^{-\frac{1}{2}}$ and $\tau = -\log(1-t)$. Write $u(l,t) = v(y,\tau)$, so that if u solves (7.1.8) then v solves (8.1.4) with $\alpha = \frac{1}{2}$ and so $\beta = 0$. Consider

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \frac{\partial^2 w}{\partial y^2} + \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}\frac{\partial w}{\partial y} + \frac{1}{2}\left(G(y) - y\right)\frac{\partial w}{\partial y} - \frac{1}{2}\operatorname{sech}^2 2g(y) w \\ w(y,0) &= w_0(y) \end{aligned} \tag{8.2.1}$$

where $w_0 \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We see that (8.1.4) is a perturbation of (8.2.1) and note that such a $w \in C^{2,1}(\mathbb{R} \times [0,\infty); \mathbb{R})$ exists by standard parabolic theory (Theorem 2.2.4). We may use the Feynman–Kac representation formula to see that the solution w is given by

$$w(y,\tau) = \mathbb{E}^y\left[\exp\left(-\frac{1}{2}\int_0^\tau \operatorname{sech}^2 2g(X_s)\,\mathrm{d}s\right) w_0(X_\tau)\right]$$

where $(X_s)_{0\le s\le\tau}$ is the unique strong solution to

$$\begin{aligned} \mathrm{d}X_s &= \left(\frac{\tanh g(X_s)}{\sqrt{\cosh 2g(X_s)}} + \frac{1}{2}\left(G(X_s) - X_s\right)\right)\mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s \\ X_0 &= y. \end{aligned}$$

One may check that the drift and diffusion terms are globally Lipschitz and so standard theory of existence and uniqueness of such a solution may be used (Theorem 2.3.2). Indeed, implicitly the solution is given by

$$X_s = e^{-\frac{1}{2}s}y + \int_0^s e^{-\frac{1}{2}(s-r)}\left(\frac{\tanh g(X_r)}{\sqrt{\cosh 2g(X_r)}} + \frac{1}{2}G(X_r)\right)\mathrm{d}r + \sqrt{2}\int_0^s e^{-\frac{1}{2}(s-r)}\,\mathrm{d}W_r$$

for $0 \le s \le \tau$ for any $\tau > 0$.

We now show that there exists some $f \in L^1(0,\infty;\mathbb{R})$ with

$$|w(y,\tau)| \le f(\tau)\|w_0\|_{L^\infty},$$

for every $|y| < 1$. This estimate is needed, since if $S_0(\tau)v_0(y) =: w(y,\tau)$ solves (8.2.1) with initial data $v_0(y) = u_0(y)$ then the solution v to (8.1.4) with $\alpha = \frac{1}{2}$ is given by

$$v(y,\tau) = S_0(\tau)v_0(y) + \int_0^\tau S_0(\tau - s)\frac{1}{2}\operatorname{sech} 2g(\cdot)\,\mathrm{d}s \tag{8.2.2}$$

and so it is clear that we need some integrability of the operator norm of $S_0(\cdot)$ as a map from $L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$.

Let $\Omega_1 = \{\omega \in \Omega : |X_s| \le \tau^p \ \forall s \in [0,\tau]\}$ for some $p \in (0,\infty)$ to be chosen later and $\Omega_2 := \Omega \setminus \Omega_1 = \{\omega \in \Omega : \exists s_0 \in [0,\tau] : |X_{s_0}| > \tau^p\}$. Then, splitting the expectation over Ω_1 and Ω_2 we have

$$|w(y,\tau)| \le \left(e^{-\tau F(\tau^p)} + \mathbb{P}[\Omega_2]\right) \|w_0\|_{L^\infty}$$

for every $y \in \mathbb{R}$ and every $\tau \in [0,\infty)$. Here, $F(y) := \frac{1}{2}\operatorname{sech}^2 2g(y)$. Observe that if $\omega \in \Omega_2$ then $\tau^p < |X_{s_0}|$. However, from our implicit solution we have

$$|X_{s_0}| \le e^{-\frac{1}{2}s_0}|y| + C\int_0^{s_0} e^{-\frac{1}{2}(s_0-r)}\,\mathrm{d}r + |Z_{s_0}| \le |y| + C + |Z_{s_0}|,$$

where Z_{s_0} is a Gaussian random variable with mean zero and variance

$$\mathbb{E}|Z_{s_0}|^2 = 2\int_0^{s_0} e^{-(s_0-r)}\,\mathrm{d}r = 2(1-e^{-s_0}) \le 2.$$

Thus, $\omega \in \Omega_2$ implies, by the above, $\tau^p - |y| - C < |Z_{s_0}|$ and so as Z_{s_0} is a mean zero Gaussian random variable

$$\mathbb{P}[\Omega_2] \le \mathbb{P}[|Z_{s_0}| > \tau^p - |y| - C] \le 2\mathbb{P}[Z_{s_0} > \tau^p - |y| - C].$$

Thus as $\operatorname{Var}(Z_{s_0}) \le 2$, if $|y| < 1$ it follows for $\tau > 0$ large enough, that

$$\mathbb{P}[\Omega_2] \le C\int_{\frac{\tau^p-1-C}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right)\,\mathrm{d}x.$$

We thus conclude that for $\tau > 0$ large enough

$$|w(y,\tau)| \le C\left(e^{-\tau F(\tau^p)} + \int_{\frac{\tau^p-C}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right)\,\mathrm{d}x\right)\|w_0\|_{L^\infty}$$

for every $|y| < 1$, where we have incorporated the bound on y in the integrand limits into one constant. We now choose $p \in (0,\infty)$. From Proposition A.0.1 we conclude that

$$-\tau F(\tau^p) = -\frac{\tau^{1-4p}}{2\left(\left(1-\frac{c_1}{\tau^p}\right)^2 + \frac{\tilde{R}(\tau)}{\tau^{2p}}\right)^2}$$

where $\tilde{R}(\tau) \le \mathcal{O}(\tau^{-2})$ for large enough τ. Thus choosing $p = \frac{1}{8}$, there exists $\tau_0 > 0$

such that $\tau > \tau_0$ implies

$$\frac{1}{\left(\left(1-\frac{c_1}{\tau^p}\right)^2+\frac{\tilde{R}(\tau)}{\tau^{2p}}\right)^2} > \frac{1}{2}$$

for our choice of p. We thus conclude

$$e^{-\tau F(\tau^{\frac{1}{8}})} \le e^{-\frac{\sqrt{\tau}}{4}}$$

for every $\tau > \tau_0$. We also note that there exists $C, x_0 > 0$ such that $x > x_0$ implies

$$\int_x^\infty \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z \le C \exp\left(-\frac{x^2}{2}\right).$$

Putting all this together, we conclude that there exists $\tau_0 > 0$ such that $\tau > \tau_0$ implies

$$|w(y,\tau)| \le C\left(e^{-\frac{\sqrt{\tau}}{4}} + e^{-\frac{(\sqrt[8]{\tau}-C)^2}{4}}\right)\|w_0\|_{L^\infty},$$

for every $|y| < 1$. Indeed, a naive estimate using the Feynman–Kac representation formula implies that, for every $y \in \mathbb{R}$ and every $\tau > 0$,

$$|w(y,\tau)| \le \|w_0\|_{L^\infty}.$$

Thus if we take

$$f(\tau) := \begin{cases} 1 & \text{if } \tau \le \tau_0 \\ C\left(e^{-\frac{\sqrt{\tau}}{4}} + e^{-\frac{(\sqrt[8]{\tau}-C)^2}{4}}\right) & \text{if } \tau > \tau_0 \end{cases}$$

we see that $f \in L^1(0,\infty;\mathbb{R})$ and satisfies what we need.

We now return to our solution v to (8.1.4) given by (8.2.2). Recall

$$v(y,\tau) = S_0(\tau)u_0(y) + \frac{1}{2}\int_0^\tau S_0(\tau-s)\operatorname{sech} 2g(\cdot)\,\mathrm{d}s.$$

Now suppose that $|y| < 1$ and that $\tau > 0$ is arbitrary. Then, by the above

$$\begin{aligned} |v(y,\tau)| &\le \|u_0\|_{L^\infty} + \frac{1}{2}\|\operatorname{sech} 2g(\cdot)\|_{L^\infty}\int_0^\tau f(\tau-s)\,\mathrm{d}s \\ &\le \|u_0\|_{L^\infty} + \frac{1}{2}\|f\|_{L^1} \\ &\le C(1+\|u_0\|_{L^\infty}). \end{aligned}$$

Changing back into the (l,t) coordinates,

$$|u(l,t)| \le C(1 + \|u_0\|_{L^\infty})$$

for every $|l| < \sqrt{1-t}$, for every $t \in [0,1)$.

To see that J vanishes at the singularity, recall that

$$J(l,t) = u(l,t)\sqrt{1-t}\cosh g(l(1-t)^{-\frac{1}{2}}).$$

Setting $l = 0$ and using the above estimate on $u(l,t)$ we conclude that

$$|J(0,t)| \le C(1 + \|u_0\|_{L^\infty})\sqrt{1-t}.$$

The result follows after sending $t \to 1^-$.

We now proceed to prove the bounds on J. This will enable us to prove the bound on u. Using the (y,τ) coordinates, we analyse the solution q to (7.2.3). By writing $q(y,\tau) = \varphi_0(y)w(y,\tau)$ with

$$\varphi_0(y) := \sqrt{\cosh g(y)}\exp\left(\frac{1}{8}(\cosh 2g(y) - 1)\right)$$

we have that w solves

$$\frac{\partial w}{\partial \tau} = -\tilde{A}_0 w$$

where

$$-\tilde{A}_0 := \frac{\partial^2}{\partial y^2} - \left(\frac{1}{16}H_0^2 - \frac{1}{4}H_0' - F\right),$$

$$H_0(y) := y - G(y) + 2\frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}}$$

and

$$F(y) := \operatorname{sech} 2g(y)\left(\tanh g(y)\tanh 2g(y) - \operatorname{sech}^2 g(y) - \frac{1}{2}\operatorname{sech} 2g(y)\right).$$

We say that $-\tilde{A}_0$ is the groundstate transformation of $-\bar{A}_0$ and one can check that there exists $V_0 > 0$ such that

$$\mathcal{V}_0(y) := \frac{1}{16}H_0^2(y) - \frac{1}{4}H_0'(y) - F(y) \ge V_0$$

for every $y \in \mathbb{R}$ and $\mathcal{V}_0(y) \to +\infty$ as $|y| \to \infty$. Thus, as before (Section 4.2) this implies that the spectrum of $\tilde{A}_0$ is discrete with minimal eigenvalue $\bar{\mu}_1 > 0$, thus the spectrum of $\bar{A}_0$ is also discrete with the same minimal eigenvalue $\bar{\mu}_1$. Further, following the analysis of Section 4.2 one concludes that if $\lambda \in \sigma(\bar{A}_0)$ with corresponding eigenfunction v_λ, then there exists $y_0 > 0$ such that $|y| > y_0$ implies

$$0 < v_\lambda(y) \leq C(|y| + \kappa)^{2\lambda}$$

and v_λ is locally bounded away from 0. (The proof is identical to Section 4.2 and so is omitted). Thus, using the functional analytical techniques from Section 4.2 we conclude that for every $y \in \mathbb{R}$ and every $\tau > 0$

$$|q(y,\tau)| \leq C g_1^0(y) e^{-\bar{\mu}_1 \tau} \varphi_0(y) + C\varphi_0(y) \|J_0\|_{\mathcal{H}_0} e^{-\bar{\mu}_1 \tau}$$

where $g_1^0(y)$ is the eigenfunction of $\bar{A}_0$ with corresponding eigenvalue $\bar{\mu}_1$. Using the triangle inequality with $(|y|+\kappa)^{2\bar{\mu}_1}$ for $|y| \leq 1$ and changing back to (l,t) coordinates we conclude

$$|J(l,t)| \leq C\left((|l| + \kappa\sqrt{1-t})^{2\bar{\mu}_1} + (1-t)^{\bar{\mu}_1}(1 + \|J_0\|_{\mathcal{H}_0})\right) \leq C(1 + \|J_0\|_{\mathcal{H}_0})(1-t)^{\bar{\mu}_1}$$

for every $|l| \leq \sqrt{1-t}$. To see the bound on J for $|l| > y_0\sqrt{1-t}$, one uses the (y,τ) coordinates and the comparison principle as given in the proof of Theorem 4.2.10 (noting that the Feyman–Kac formula works, with suitable modification to the stochastic process used) to conclude that

$$|q(y,\tau)| \leq g_1^0(y) e^{-\bar{\mu}_1 \tau}$$

for every $y \in \mathbb{R}$ and every $\tau > 0$, provided

$$|q(y,0)| \leq g_1^0(y)$$

for every $y \in \mathbb{R}$. However, since $q(y,0)$ has compact support, we thus conclude as we did before that

$$|q(y,\tau)| \leq C\|J_0\|_{L^\infty} g_1^0(y) e^{-\bar{\mu}_1 \tau}$$

for every $(y,\tau) \in \mathbb{R} \times (0,\infty)$. However, for $|y| > y_0$, we have that

$$g_1^0(y) \leq C(|y| + \kappa)^{2\bar{\mu}_1}$$

and so working back in the (l,t) coordinates, for every $|l| > y_0\sqrt{1-t}$

$$|J(l,t)| \leq C\|J_0\|_{L^\infty}(|l| + \kappa\sqrt{1-t})^{2\bar{\mu}_1} \leq C\|J_0\|_{L^\infty}|l|^{2\bar{\mu}_1}.$$

Finally, the bound on u for $|l| > y_0\sqrt{1-t}$ now follows from the above bound on q since

$$q(y,\tau) = v(y,\tau)e^{-\frac{1}{2}\tau}\cosh g(y)$$

and so

$$|v(y,\tau)| \leq C\|u_0\|_{L^\infty} e^{(\frac{1}{2}-\bar{\mu}_1)\tau} \frac{g_1^0(y)}{\cosh g(y)} \tag{8.2.3}$$

for every $(y,\tau) \in \mathbb{R}\times(0,\infty)$. However, for every $|y| > y_0$ we have

$$\frac{g_1^0(y)}{\cosh g(y)} \leq C(|y|+\kappa)^{2\bar{\mu}_1-1}$$

and so, switching back to (l,t) coordinates

$$|u(l,t)| \leq C\|u_0\|_{L^\infty}(|l|+\kappa\sqrt{1-t})^{2\bar{\mu}_1-1} \leq \|u_0\|_{L^\infty}|l|^{2\bar{\mu}_1-1}$$

for every $|l| > y_0\sqrt{1-t}$, since $2\bar{\mu}_1 - 1 \leq 0$ (see Remark 8.2.3). The proof is now complete.

□

Remark 8.2.2. *Indeed, the proof of Theorem 8.2.1 is also true if we take* $|y| \leq 1$.

Remark 8.2.3 (On the sign of $2\bar{\mu}_1 - 1$.)**.** *Recall that Theorem 8.1.6 still holds and so* $2\bar{\mu}_1 - 1 \leq 0$*, since otherwise we would be able to show that* $u(0,t) \to 0$ *as* $t \to 1^-$ *from equation* (8.2.3) *by setting* $y = 0$*, which is a contradiction. Indeed, recalling that* $\mathcal{V}_0(y) \geq V_0$ *for every* $y \in \mathbb{R}$*, numerically one can compute that* $V_0 \approx 0.107$ *and so using a crude estimate in the definition of* $\bar{\mu}_1$ *being an eigenvalue for* $\tilde{A}_0$ *one has that* $0.107 < \bar{\mu}_1 \leq \frac{1}{2}$.

In light of this, we have the following conjecture.

Conjecture 8.2.4. *We conjecture that* $\bar{\mu}_1 = \frac{1}{2}$.

8.3 Super–Critical Regime

If $\alpha > \frac{1}{2}$, then the scaling of $y = l(1-t)^{-\alpha}$ and $\tau = -\log(1-t)$ fails; exponential blow up of the coefficients occur. This is primarily due to the additional drift term that arises due to the Laplace–Beltrami term.

To remedy this, we take $y = l(1-t)^{-\alpha}$ and $\tau = (1-t)^{1-2\alpha}$. Let $\gamma = 2\alpha - 1 > 0$ and write $u(l,t) = v(y,\tau)$. Then, if u solves (7.1.8) it follows that v solves the following on $\mathbb{R} \times [1, \infty)$:

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \gamma^{-1} \frac{\partial^2 v}{\partial y^2} + \gamma^{-1} \frac{\tanh g(y)}{\sqrt{\cosh 2g(y)}} \frac{\partial v}{\partial y} \\
&\quad + \alpha\gamma^{-1}\tau^{-1}(G(y) - y)\frac{\partial v}{\partial y} + \alpha\gamma^{-1}\tau^{-1}(\operatorname{sech} 2g(y) - \operatorname{sech}^2 2g(y))v \\
v(y, 1) &= u_0(y).
\end{aligned} \tag{8.3.1}$$

We wish to show that the solution v to (8.3.1) is uniformly bounded. Since (8.1.4) was a perturbed OU-process, it was fairly straightforward to use probabilistic methods to achieve this. However, now we have a positive creation term with a function of time that is not integrable as a coefficient. Indeed, using the approach of the heat semigroup fails, so does any approach, as far as the author is aware, where one incorporates the homogeneous drift term into the equation. Probabilistic methods also fail due to the nature of the creation term, even when one transforms the equation by a ground-state-like transformation; the form of the resulting potential is hard to analyse as well as problems with the underlying stochastic process that has variance blow-up. Thus, although a bounded lower bound is possible (see Remark 8.3.3), it seems that a bounded upper-bound has evaded us at every turn.

All is not lost; we turn our attention to the *conserved* quantity

$$J(l, t) = u(l, t)(1 - t)^{\alpha} \cosh g(l(1 - t)^{-\alpha}).$$

In (y, τ) coordinates, we write $q(y, \tau) = J(l, t)$ so that

$$q(y, \tau) = v(y, \tau)\tau^{-\alpha\gamma^{-1}} \cosh g(y),$$

where v solves (8.3.1). The following lemma is of use and gives an upper bound on any blow–up of v, should it occur.

Lemma 8.3.1. *Let v be the solution to* (8.3.1)*, which exists by standard parabolic theory (Theorem 2.2.4). If $u_0 \in C_c^{\infty}(\mathbb{R})$ then*

$$|v(y, \tau)| \le \|u_0\|_{L^\infty} \tau^{\rho}$$

for every $\tau \ge 1$, $y \in \mathbb{R}$, where $\rho := \frac{\alpha\gamma^{-1}}{4}$.

Proof. Since such solution v exists, by the Feynman–Kac representation formula,

$$v(y, \tau) = \mathbb{E}^{(-\tau, y)} \left[\exp\left(\alpha\gamma^{-1} \int_{-\tau}^{-1} (-s)^{-1} B(X_s) \, \mathrm{d}s \right) u_0(X_{-1}) \right]$$

where

$$B(y) := \operatorname{sech} 2g(y) - \operatorname{sech}^2 2g(y).$$

Here, (X_s) is the unique strong solution to

$$\begin{aligned} \mathrm{d}X_s &= \gamma^{-1}\left(\frac{\tanh g(X_s)}{\sqrt{\cosh 2g(X_s)}} + \alpha(-s)^{-1}(G(X_s) - X_s)\right)\mathrm{d}s + \sqrt{2\gamma^{-1}}\,\mathrm{d}B_s \\ X_{-\tau} &= y, \end{aligned}$$

for $-\tau \le s \le -1$. Such process exists by an adaption of the proof that a strong solution to (8.1.2) exists, noting that $(-s)^{-1} \le 1$ for every $-\tau \le s \le -1$. To conclude the proof, we note that

$$0 < B(y) \le \frac{1}{4}$$

for every $y \in \mathbb{R}$ and so with ρ as in the statement of the lemma,

$$|v(y,\tau)| \le \|u_0\|_{L^\infty} \exp\left(\rho \int_{-\tau}^{-1} (-s)^{-1}\,\mathrm{d}s\right) = \|u_0\|_{L^\infty}\tau^{\rho}.$$

□

This lemma allows easy proof of the following theorem, which may be seen as a partial result.

Theorem 8.3.2. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose J is the unique solution to* (7.1.7)*. Then, there exists $C > 0$ such that*

$$|J(l,t)| \le C\|u_0\|_{L^\infty}$$

for every $|l| \le (1-t)^\alpha$ and

$$|J(0,t)| \longrightarrow 0$$

as $t \to 1^-$.

Proof. The proof is straightforward in the (y,τ) coordinates. For $|y| \le 1$ we have

$$|q(y,\tau)| \le \cosh g(1)|v(y,\tau)|\tau^{-\alpha\gamma^{-1}} = \cosh g(1)|v(y,\tau)|\tau^{-4\rho}.$$

We now use Lemma 8.3.1 to conclude that

$$|q(y,\tau)| \le \cosh g(1)\|u_0\|_{L^\infty}\tau^{-3\rho}.$$

However, letting $y = 0$, which corresponds to $l = 0$ and sending $\tau \to \infty$, noting that $\rho > 0$, we have

$$|J(0,t)| \longrightarrow 0$$

as $t \to 1^-$.

$\square$

Remark 8.3.3. *One should now be able to see, using the Feynman–Kac representation formula in the proof of Lemma 8.3.1, that for $u_0 \equiv 1$ we have (after changing back to the (l,t) coordinates) $u(l,t) \geq 1$ for every $(l,t) \in \mathbb{R} \times [0,1)$. Indeed, such a solution to the equation with that initial condition is standard (Theorem 2.2.4) and the existence of a strong solution to the associated SDE is also standard (Theorem 2.3.2). Although the initial data is not admissible in the sense of Chapter 7, $u_0(l) = 1$ is a perfectly good initial condition from a PDE point of view. We thus conclude that for $\alpha > \frac{1}{2}$, the solution need not vanish at the singularity.*

However, the density J, does vanish at the singularity, just as it does in the $\alpha \leq \frac{1}{2}$ cases.

A heuristic argument for the boundedness of the solution.

Here we give an heuristic argument as to why we believe that the solution J to (7.1.7) is uniformly bounded in l and t. We outline the challenges of turning this into a proof.

In the following, let $z = g(l(1-t)^{-\alpha})$ and $\tau = (1-t)^{1-2\alpha}$. Write $w(z,\tau) = J(l,t)$, then a short but standard calculation reveals that w solves

$$\begin{aligned}
\frac{\partial w}{\partial \tau} &= \gamma^{-1} \operatorname{sech} 2z \frac{\partial^2 w}{\partial z^2} - \gamma^{-1} \left(\tanh 2z \operatorname{sech} 2z + \tanh z \operatorname{sech} 2z + \alpha\tau^{-1} \tanh 2z\right) \frac{\partial w}{\partial z} \\
&\quad + \gamma^{-1} \left(\tanh z \tanh 2z \operatorname{sech} 2z - \operatorname{sech}^2 z \operatorname{sech} 2z - \alpha \tau^{-1} \operatorname{sech}^2 2z\right) w \\
w(z,1) &= J_0(z).
\end{aligned} \tag{8.3.2}$$

Here $\gamma = 2\alpha - 1 > 0$. Using a probabilistic approach, consider the following SDE, for which it is easy to verify that there exists a unique strong solution for $-\tau \leq s \leq -1$,

$$\begin{aligned}
\mathrm{d}X_s &= -\gamma^{-1} \left(\tanh 2X_s \operatorname{sech} 2X_s + \tanh X_s \operatorname{sech} 2X_s + \alpha(-s)^{-1} \tanh 2X_s\right) \mathrm{d}s \\
&\quad + \sqrt{2\gamma^{-1} \operatorname{sech} 2X_s}\, \mathrm{d}B_s \\
X_{-\tau} &= z.
\end{aligned} \tag{8.3.3}$$

By standard parabolic theory (Theorem 2.2.4) there exists a unique classical solution to (8.3.2) and so by the Feynman-Kac representation formula we have that the solution has the stochastic representation

$$w(z,\tau) = \mathbb{E}^{(-\tau,z)}\left[\exp\left(\gamma^{-1}\int_{-\tau}^{-1} b(X_s,s)\,\mathrm{d}s\right) J_0(X_{-1})\right].$$

where

$$b(z,s) = \tanh z \tanh 2z \operatorname{sech} 2z - \operatorname{sech}^2 z \operatorname{sech} 2z - \alpha(-s)^{-1}\operatorname{sech}^2 2z.$$

The idea is to argue that $\mathbb{P}-$a.s, for every $\tau > 1$,

$$\int_{-\tau}^{-1} b(X_s,s)\,\mathrm{d}s < 0$$

for every initial value z. The following heuristic suggests that this is true, but we are far from a proof. The argument is thus; consider the SDE (8.3.3). Since the drift is always negative for positive values of X_s and positive for negative values of X_s, non–zero initial conditions lead to the solution approaching and passing through zero, for bounded τ. Within a positive and relatively small distance from 0, the noise kicks in and makes the solution fluctuate about zero. The sign of the drift always makes sure we are approaching zero. The noise is extremely small for large values of the process and so we conclude that the process always eventually fluctuates around zero. Figure 8.1 shows one such sample path for $z = 5$ starting at $-\tau = -500$.

Consider b in the above representation formula. It has a global minimum at 0 and two local maxima, either side of zero and the absolute value of the minima is large compared with the maxima. Figure 8.2 illustrates this for $(z,s) \in [-3,3] \times \{-350\}$ with $\alpha = 0.75$. Thus, as the solution to the SDE fluctuates about zero, b picks up mainly negative contributions. Figure 8.3 graphs s versus $b(X_s,s)$ for the sample path given in Figure 8.1. One can clearly see that the process picks up mainly negative contributions. A trapezium rule estimate for the integral yields an approximate value of -12.00. Indeed, if the solution was bounded away from 0, the drift would eventually pull it back towards zero. Any positive contribution is cancelled out by the negative value of the creation term b near 0. We thus conclude, heuristically, that $\mathbb{P}-$a.s, for every $\tau > 1$,

$$\int_{-\tau}^{-1} b(X_s,s)\,\mathrm{d}s < 0.$$

However, one should note that it is unclear whether the above is true for very large

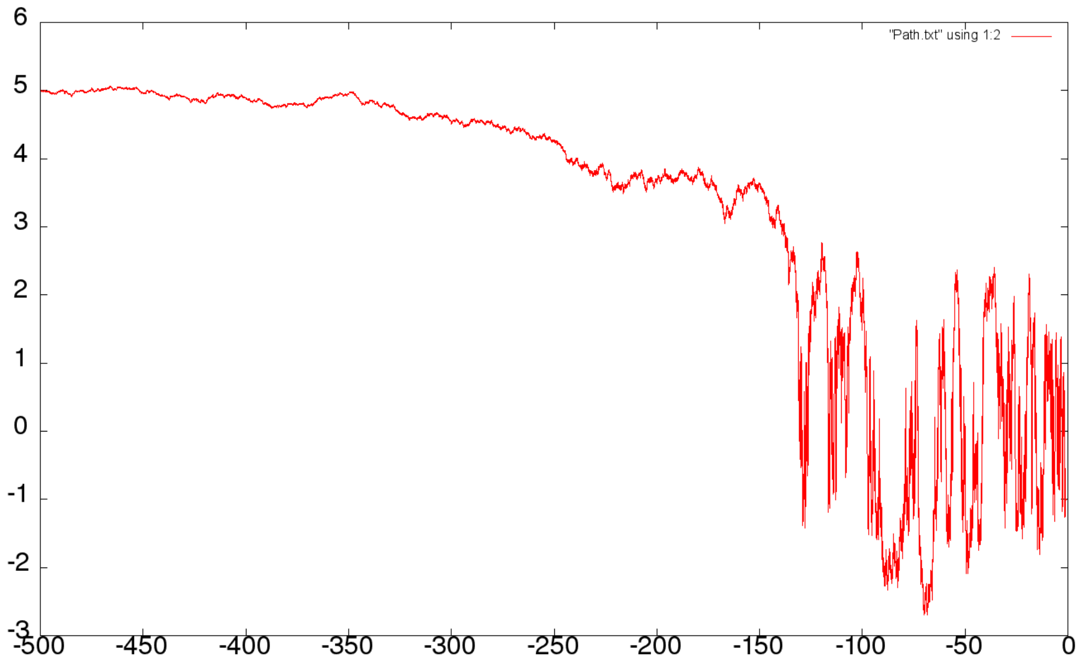

Figure 8.1: A plot of a sample path of the solution to (8.3.3) with $z = 5$, $\tau = 500$ and $\alpha = 0.75$. The horizontal axis is s which corresponds to the time variable of the SDE whereas the vertical axis is the value of the solution.

z and very large τ. Due to the nature of the drift and diffusion, the process may have the tendency to stay where it is. For example, Figure 8.4 shows a sample path for the initial condition $z = 100$ starting at $-\tau = -50000$, with $\alpha = 0.75$. Here we see that the initial condition is large enough so that the noise is effectively zero. Thus, the evolution of the path is now deterministic and we can see that it will take a long time for the solution to be in a small neighbourhood of 0. However, for such z, the integrand b is extremely close to zero. Thus, it is not clear that the solution will then approach 0 and fluctuate as described above, as we increase z and τ.

With regards to non–probabilistic methods, using a semigroup approach based on the equation

$$\frac{\partial v}{\partial \tau} = \gamma^{-1} \operatorname{sech} 2z \frac{\partial^2 v}{\partial z^2}$$

does not work as the time is unbounded; we have a logarithmic divergence at infinity for (8.3.2) in τ when we integrate in time. To conclude this chapter, we leave the following as a conjecture.

Conjecture 8.3.4. *Let $u_0 \in C_c^\infty(\mathbb{R})$ and suppose J is the unique smooth solution to (7.1.7) for $\alpha > \frac{1}{2}$ with initial condition $J_0(l) = u_0(l) \cosh g(l)$. Then there exists*

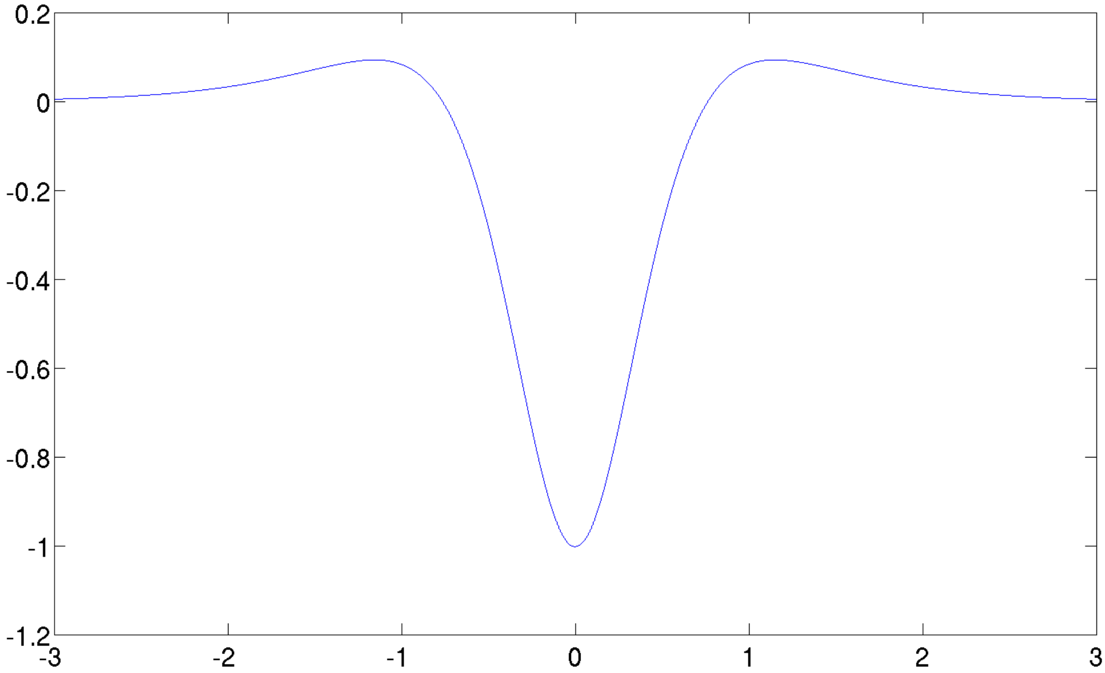

Figure 8.2: A plot of $b(z,s)$ for $(z,s) \in [-3,3] \times \{-350\}$ with $\alpha = 0.75$.

$C > 0$ *such that*

$$|J(l,t)| \leq C \qquad \textit{for every } (l,t) \in \mathbb{R} \times [0,1].$$

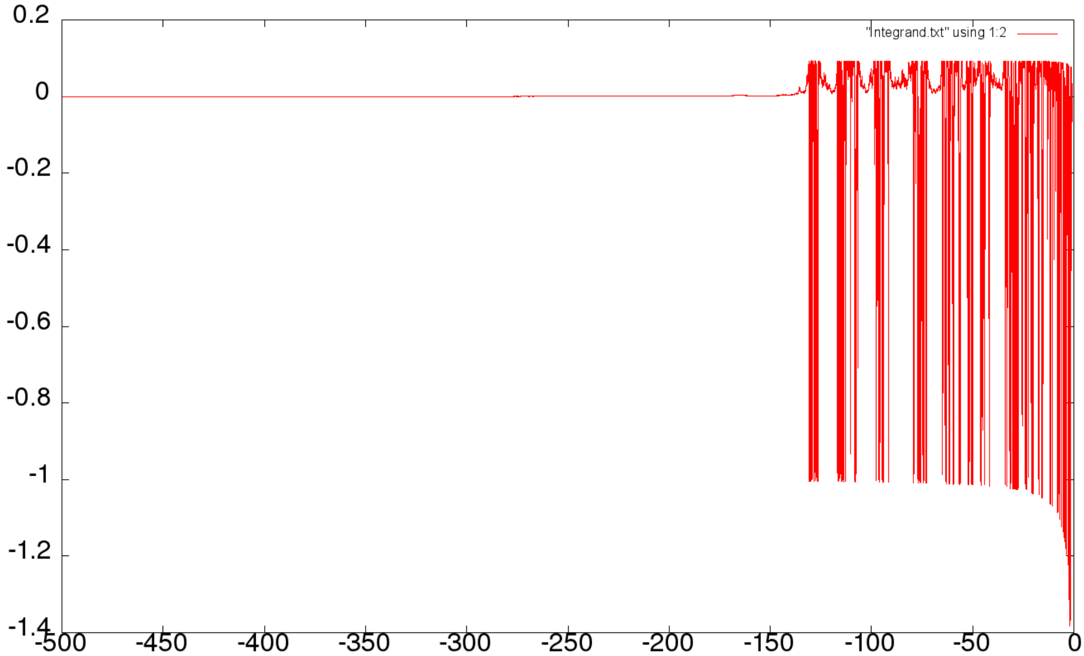

Figure 8.3: A plot of $b(X_s, s)$ for the sample path given in Figure 8.1.

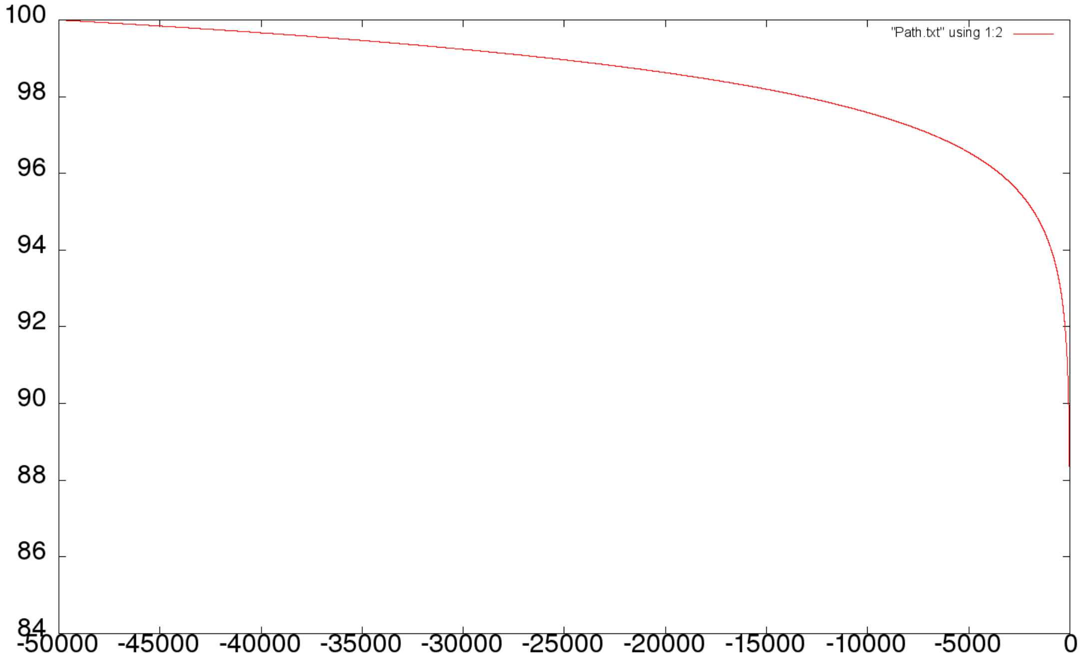

Figure 8.4: A plot of a sample path of the solution to (8.3.3) with $z = 100$, $\tau = 50000$ and $\alpha = 0.75$.

Chapter 9

Analysis of Problem II: After the Singularity

Our goal is to show that we can continue the solution U of (7.0.2), in some sense, past the singularity onto

$$\mathcal{S}_t^{\alpha,\mathrm{cont}} := \{(x, y, z) \in \mathbb{R}^3 \,|\, z^2 - x^2 - y^2 = (t-1)^{2\alpha}\}. \tag{9.0.1}$$

Figure 9.1 illustrates $\mathcal{S}_t^{\alpha,\mathrm{cont}}$ for $\alpha = 0.5$ and $t = 1.25$. We can see that $\mathcal{S}_t^{\alpha,\mathrm{cont}}$ is a hyperboloid of two–sheets. In the following, we shall only consider $z \geq 0$; that is, the upper part of $\mathcal{S}_t^{\alpha,\mathrm{cont}}$.

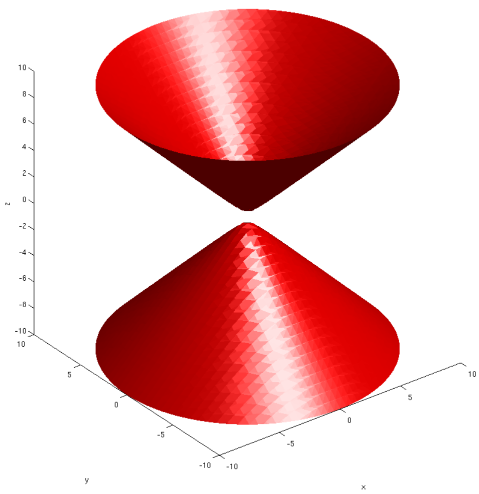

Figure 9.1: Plot of $\mathcal{S}_t^{\alpha,\mathrm{cont}}$ for $\alpha = 0.5$ and $t = 1.25$.

9.1 Formulation of the Continuation of the Solution

Using the level set function $\phi(x, y, z, t) := x^2 + y^2 - z^2 + (t-1)^{2\alpha}$ we take the inward pointing normal as $\nu := -\frac{\nabla\phi}{|\nabla\phi|}$. We parameterise our surface by

$$X(l, \theta, t) := (t-1)^{\alpha} (\sinh g(z) \cos\theta, \sinh g(z) \sin\theta, \cosh g(z))$$

where $g : \mathbb{R} \to \mathbb{R}$ is given in Definition 7.1.1 and

$$z := l(t-1)^{-\alpha}.$$

We now take the metric tensor as

$$h := \begin{pmatrix} 1 & 0 \\ 0 & (t-1)^{2\alpha} \sinh^2 g \end{pmatrix}. \tag{9.1.1}$$

so that

$$|h| := \det(h) = (t-1)^{2\alpha} \sinh^2 g(l(t-1)^{-\alpha}).$$

We are considering the continuation of equation (7.0.2) on $\mathcal{S}_t^{\alpha,\text{cont}}$; that is

$$\begin{aligned} \partial_t^{\bullet} V + V\nabla_\Gamma \cdot v - \Delta_\Gamma V = 0 \quad x \in \Gamma := \mathcal{S}_t^{\alpha,\text{cont}}, \ \ t \in [1, T], \\ V(x, 0) = U(Bx, 1), \quad x \in \mathcal{S}_1^{\alpha,\text{cont}}, \end{aligned} \tag{9.1.2}$$

where U is the solution to (7.0.2) at time $t = 1$ (taken up to some suitable subsequence if necessary) and $B : \mathcal{S}_1^{\alpha,\text{cont}} \to \mathcal{S}_1^{\alpha}$ is some bounded linear map. Here,

$$v := \frac{\phi_t}{|\nabla\phi|}\nu$$

is the *prescribed* normal velocity.

From the results of Chapter 8, equation (9.1.2) is really only well defined if $\alpha < \frac{1}{2}$, in the sense that we can prove the existence of the initial data. However, we will see that a probabilistic approach means that we can make sense of the continuation for every α, if we consider a stochastic interpretation of the problem; namely, the existence of a diffusion process on the subsequent surface.

Recalling that U was radially symmetric, a standard set of calculations, analogous to those before shows that $u(l,t) = u(l,\theta,t) := V(X(l,\theta,t),t)$ solves

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} + t^{-\alpha}\frac{\coth g(lt^{-\alpha})}{\sqrt{\cosh 2g(lt^{-\alpha})}}\frac{\partial u}{\partial l} - \alpha t^{\alpha-1} G(lt^{-\alpha})\frac{\partial u}{\partial l} \\ &\quad + \alpha t^{-1}(\operatorname{sech} 2g(lt^{-\alpha}) + \operatorname{sech}^2 2g(lt^{-\alpha}))u \\ u(l,0) &= U(X(l,\theta,1),1) =: u_0(l).\end{aligned} \tag{9.1.3}$$

We have taken $t \mapsto t-1$ in the above for better presentation and remark that Assumption 7.1.8 still holds so that u_0 is radially symmetric. We also note that G is given in (3.2.4).

Remark 9.1.1. *The major problem now is that the PDE in* (9.1.3) *is initially singular, whereas before,* (7.1.8) *was* eventually *singular. Another problem is that* coth *is always singular; this is a consequence of the parameterisation taken. It is also the problem found when one uses polar spherical coordinates to parameterise the sphere. Remark 9.1.3 below analyses the singularity involving* coth.

We see that the integral of u is not conserved. However, as before, we take $J(l,t) := u(l,t)\sqrt{|h|}$. A standard calculation shows that J solves

$$\begin{aligned}\frac{\partial J}{\partial t} &= \frac{\partial^2 J}{\partial l^2} - \frac{\partial}{\partial l}\left(\left(t^{-\alpha}\frac{\coth g}{\sqrt{\cosh 2g}} + \alpha t^{\alpha-1}G\right)J\right) \\ J(l,0) &= \frac{\sqrt{2}}{2}|l|u_0(l) =: J_*(l).\end{aligned} \tag{9.1.4}$$

Remark 9.1.2. *Note that it follows from Proposition A.0.1 that*

$$\lim_{t\to 0}\sqrt{|h|} = \frac{\sqrt{2}}{2}|l|.$$

We thus see that J is conserved. It is because of this that we consider (9.1.4) as the Fokker–Planck equation of some diffusion process. Thus, if we show that the diffusion process exists, a candidate solution to (9.1.4) is the probability density function (p.d.f) of such a process.

Remark 9.1.3 (On the singularity of the drift in equation (9.1.3))**.** *We briefly describe the nature of the singularity posed by* $\coth g(lt^{-\alpha})$*. Indeed, it is easy to see from the definition of* coth *that*

$$\lim_{s\to 0}\sinh g(s)\coth g(s) = 1.$$

Via the Mean Value Theorem

$$h(s) := \sinh g(s) = h'(s_0)s$$

for some $s_0 \in (-s, s) \setminus \{0\}$. *One calculates to see that*

$$h'(s) = \frac{\cosh g(s)}{\sqrt{\cosh 2g(s)}}$$

and a straightforward calculation shows that, for every $s \in \mathbb{R}$,

$$\frac{\sqrt{2}}{2} < h'(s) \leq 1.$$

This bound together with the definition of the limit shows that there exists $\delta > 0$ *and* $m_1, m_2 > 0$ *such that*

$$m_1 < s \coth g(s) < m_2$$

for every $s \in (-\delta, \delta) \setminus \{0\}$. *With respect to the drift of equation* (9.1.3) *(and so of equation* (9.1.4)*) we now have three cases. In the following, suppose* $l > 0$. *Analogous cases and results there–of remain true for* $l < 0$.

i) If $l \ll t^\alpha$ *then* $lt^{-\alpha} \ll 1$ *and so*

$$\frac{m_1}{l} < t^{-\alpha} \coth g(lt^{-\alpha}) < \frac{m_2}{l}$$

holds. If t *is small, then* $t^{-\alpha} \coth g(lt^{-\alpha})$ *behaves like* $\frac{\text{const.}}{l}$. *This is also the case if* t *is large but* l *is much smaller than* t^α.

ii) If $lt^{-\alpha} = 1$ *then* $t^{-\alpha} \coth g(lt^{-\alpha}) = \frac{\coth g(1)}{l}$. *This is problematic when* $l = t^\alpha$ *and* t *is small. But if* $l = t^\alpha$ *and* t *is large, then the above shows that* $t^{-\alpha} \coth g(lt^{-\alpha})$ *will be small.*

iii) Finally, if $l \gg t^\alpha$ *then* $lt^{-\alpha} \gg 1$. *Recall that* $\coth z \to \pm 1$ *as* $z \to \pm\infty$. *So if* t *is small then* $\coth g(lt^{-\alpha})$ *is of order 1, however* $t^{-\alpha}$ *is large and so* $t^{-\alpha} \coth g(lt^{-\alpha})$ *is of order* $\frac{\text{const}}{t^\alpha}$. *However, if* t *is large, then* $t^{-\alpha} \coth g(lt^{-\alpha})$ *is small.*

The above cases, although given heuristically, highlight the different regimes the $t^{-\alpha} \coth g(lt^{-\alpha})$ *term exhibits. Thus* $t^{-\alpha} \coth g(lt^{-\alpha})$ *cannot always be approximated with a* $\frac{\text{const}}{l}$ *term, order 1 term, small order term or* $t^{-\alpha}$ *order term. We deal with this problem in the following section.*

9.2 Existence of a continued Stochastic Process

The idea here is to view (9.1.4) as a Fokker–Planck equation, or Kolmogorov Forward equation. Consider the following SDE for $0 \leq t \leq T$ for some $T \in (0, \infty)$ fixed:

$$\begin{aligned} \mathrm{d}X_t &= \left(t^{-\alpha} \frac{\coth g(X_t t^{-\alpha})}{\sqrt{\cosh 2g(X_t t^{-\alpha})}} + \alpha t^{\alpha-1} G(X_t t^{-\alpha}) \right) \mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t \\ X_0 &= Z. \end{aligned} \tag{9.2.1}$$

Here, $B_\bullet$ is a standard Brownian motion and Z has density J_*. Here, J_* is taken as the limit of the solution before the singularity as $t \to 1^-$, up to a suitable subsequence as needed. In Chapter 8 we showed that such J_* was uniformly bounded if $\alpha < \frac{1}{2}$, and in all cases of α, we showed that $J_*(0) = 0$. For $\alpha \geq \frac{1}{2}$ we will assume that such a density exists since for $\alpha < \frac{1}{2}$ we already have existence.

As mentioned above, a candidate solution to (9.1.4) is the p.d.f of the process which satisfies (9.2.1) in the strong sense. This is assuming that such a strong solution exists and has a smooth density. Whether a diffusion process has a smooth density is covered by the probabilistic version of Hörmander's Theorem (Malliavin [1978]) (Theorem 2.5.3). Indeed, it follows that where Hörmander's Theorem holds[1], a smooth density exists and so by Dynkin's Formula (2.5.4) satisfies (9.2.1). Thus, we are concerned with the existence of a strong solution to (9.2.1).

We first show that there exists a solution for small times. The major issue here is that the coefficients of (9.2.1) are initially singular in time, but also the problem with the coth singularity occurs for every $t > 0$. Finally we show that we can extend the solution to the whole of $[0, T]$ for any given $T \in (0, \infty)$ (Theorem 9.2.3).

Theorem 9.2.1. *Consider* (9.2.1) *with* $\alpha \in (0, \infty)$ *fixed. Then, there exists a unique strong solution to* (9.2.1) *up to a stopping time* τ *with* $\mathbb{P}(\tau = 0) = 0$.

Proof. Fix $s > 0$ and consider

$$\begin{aligned} \mathrm{d}X_t^s &= \left(t^{-\alpha} \frac{\coth g(X_t^s t^{-\alpha})}{\sqrt{\cosh 2g(X_t^s t^{-\alpha})}} + \alpha t^{\alpha-1} G(X_t^s t^{-\alpha}) \right) \mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t \\ X_s^s &= Z \end{aligned} \tag{9.2.2}$$

where Z is distributed according to J_* and $s \leq t \leq T$. Since Z is independent of $B_\bullet$, we can assume that $Z = y$ for some $y \in \mathbb{R}$. Since $\mathbb{P}(Z = 0) = 0$, we may take $y \in \mathbb{R} \setminus \{0\}$. Assume that $y > 0$. An analogous argument works for $y < 0$

[1] See Remark 2.5.5

and such argument is left to the reader, however, some of the details are given in Remark 9.2.2.
Fix $\delta > 0$ such that $0 < \delta < y$. Define, for $y \in \mathbb{R}$

$$b_1(y) := \frac{\coth g(y)}{\sqrt{\cosh 2g(y)}}, \qquad b_2(y) := G(y).$$

Then

$$b_1'(y) = \big(1 - \coth^2 g(y) - \coth g(y) \tanh 2g(y)\big) \operatorname{sech} 2g(y)$$

and so by the methods of the proof of Proposition A.0.2, one concludes that if $y \in \mathbb{R} \setminus B_\delta(0)$ then there exists $C_\delta > 0$ such that

$$|b_1'(y)| \leq C_\delta |y|^{-2}.$$

Thus, adapting the proof of Proposition A.0.2 for b_1 with the above estimate on b_1', we conclude that if $z_i \in \mathbb{R} \setminus B_\delta(0)$ $(i = 1, 2)$ with $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2)$ then there exists $C_\delta^{(1)} > 0$ such that, for every $t \in [0, T]$,

$$|b_1(z_1 t^{-\alpha}) - b_1(z_2 t^{-\alpha})| \leq C_\delta^{(1)} |z_1 - z_2|. \tag{9.2.3}$$

Using Proposition A.0.2 directly, we have that if $z_i \in \mathbb{R} \setminus B_\delta(0)$ $(i = 1, 2)$ with $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2)$ then there exists $C_\delta^{(2)} > 0$ such that, for every $t \in [0, T]$,

$$|b_2(z_1 t^{-\alpha}) - b_2(z_2 t^{-\alpha})| \leq C_\delta^{(2)} t^{2\alpha} |z_1 - z_2|. \tag{9.2.4}$$

To this end, we define

$$\tilde{b}_1(z, t) := \begin{cases} b_1(\delta t^{-\alpha}) & \text{if } -\infty < z < \delta \\ b_1(z t^{-\alpha}) & \text{if } \delta \leq z < \infty, \end{cases}$$

and

$$\tilde{b}_2(z, t) := \begin{cases} b_2(\delta t^{-\alpha}) & \text{if } -\infty < z < \delta \\ b_2(z t^{-\alpha}) & \text{if } \delta \leq z < \infty. \end{cases}$$

We now consider the following modification of (9.2.2)

$$\begin{aligned} \mathrm{d}X_t^s &= \Big(t^{-\alpha}\, \tilde{b}_1(X_t^s, t) + \alpha t^{\alpha - 1}\, \tilde{b}_2(X_t^s, t)\Big)\, \mathrm{d}t + \sqrt{2}\, \mathrm{d}B_t \\ X_s^s &= y. \end{aligned} \tag{9.2.5}$$

It follows from (9.2.3) and (9.2.4) that there exists $C_\delta^{(i)} > 0$ where $i = 1, 2$ such that for every $x, y \in \mathbb{R}$ and every $t \in [0, T]$

$$t^{-\alpha} \left| \tilde{b}_1(x,t) - \tilde{b}_1(y,t) \right| \leq s^{-\alpha} C_\delta^{(1)} |x - y|$$

and

$$\alpha t^{\alpha - 1} \left| \tilde{b}_2(x,t) - \tilde{b}_2(y,t) \right| \leq t^{3\alpha - 1} C_\delta^{(2)} |x - y|.$$

We bound $t^{3\alpha-1}$ from above in the following way:

$$t^{3\alpha-1} \leq \begin{cases} T^{3\alpha-1} & \text{if } \alpha > \frac{1}{3} \\ 1 & \text{if } \alpha = \frac{1}{3} \\ s^{3\alpha-1} & \text{if } \alpha < \frac{1}{3}. \end{cases}$$

Hence, standard Picard iteration for the solution to SDEs (Øksendal [2003]) yields a unique adapted process $(X_t^s)_{t \in [s,T]}$ which is the unique strong solution to (9.2.5), such that $\mathbb{P}-$a.s and, for every $t \geq s$,

$$X_t^s = y + \int_s^t r^{-\alpha} \tilde{b}_1(X_r^s, r) + \alpha r^{\alpha-1} \tilde{b}_2(X_r^s, r) \, \mathrm{d}r + \sqrt{2}(B_t - B_s).$$

We note that the proof of the existence of such a X_t^s has been independent of the value of α. The idea is to now send $s \to 0$ in the above expression for X_t^s.

Define

$$X_t = y + \int_0^t r^{-\alpha} \tilde{b}_1(X_r, r) + \alpha r^{\alpha-1} \tilde{b}_2(X_r, r) \, \mathrm{d}r + \sqrt{2} B_t.$$

We will split up the cases of α to show that $|X_t| < \infty$ $\mathbb{P}-$a.s. First, let $\alpha \leq \frac{1}{2}$. Noting that $\tilde{b}_i$ are bounded functions in space and time, and $\alpha \leq \frac{1}{2}$ we have that

$$|X_t| \leq |y| + (1 - \alpha)^{-1} \sup_{0 \leq r \leq T} \|\tilde{b}_1(\cdot, r)\|_{L^\infty} t^{1-\alpha} + \sup_{0 \leq r \leq T} \|\tilde{b}_2(\cdot, r)\|_{L^\infty} t^\alpha + \sqrt{2} |B_t|$$

so that $|X_t| < \infty$ $\mathbb{P}-$a.s. Further, $\mathbb{P}$-a.s as $s \to 0$,

$$|X_t - X_t^s| \leq C(\delta, \alpha)(s^{1-\alpha} + s^\alpha) + \sqrt{2}|B_s| \longrightarrow 0.$$

Thus it follows that X_t is the unique strong solution to

$$\begin{aligned} \mathrm{d}X_t &= \Big(t^{-\alpha}\,\tilde{b}_1(X_t,t) + \alpha t^{\alpha-1}\,\tilde{b}_2(X_t,t)\Big)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t \\ X_0 &= y \end{aligned} \tag{9.2.6}$$

for $t \in [0,T]$.

We now turn our attention to $\alpha > \frac{1}{2}$. From the proof of Proposition A.0.2, we have that there exists $C_\delta > 0$ such that, for every $z \in \mathbb{R}$ and every $r > 0$,

$$|\tilde{b}_1(z,r)| \leq C_\delta r^\alpha.$$

This follows from the asymptotics of $\operatorname{sech} 2g(\cdot)$ and the proof is an adaption of that of Proposition A.0.2. Thus, defining X_t as above, and using this bound for $\tilde{b}_1$ we have

$$|X_t| \leq |y| + C_\delta t + C(\delta,\alpha)t^\alpha + \sqrt{2}|B_t|$$

and so $|X_t| < \infty$ $\mathbb{P}-$a.s. Further, $\mathbb{P}-$a.s as $s \to 0$

$$|X_t - X_t^s| \leq C_\delta s + C(\delta,\alpha)s^\alpha + \sqrt{2}|B_s| \longrightarrow 0.$$

Thus, for any $\alpha > 0$ we have that (X_t) is the unique strong solution (9.2.6) for $t \in [0,T]$, for any $T > 0$.

Let $\tau = \tau^\delta := \inf\{t \in (0,T] \,|\, X_t \leq \delta\}$. If $t < \tau$ then $X_t > \delta$ and so

$$\tilde{b}_i(X_r,r) = b_i(X_r r^{-\alpha})$$

for every $0 \leq r \leq t$ and $i = 1,2$. Thus, for $t < \tau$, X_t is the unique strong solution to (9.2.1), for any $\alpha > 0$.

We will finally show that $\mathbb{P}(\tau = 0) = 0$. For $t > 0$ and $\alpha \in (0,\infty)$,

$$\tau < t \implies \inf_{0 \leq s \leq t} X_s \leq \delta.$$

We observe that as $\tilde{b}_1(z,t) \geq 0$ for every $(z,t) \in \mathbb{R} \times [0,T]$ and $\delta > 0$ and that $\tilde{b}_2(z,t) \geq -\kappa^2$ for every $(z,t) \in \mathbb{R} \times [0,T]$, where

$$\kappa^2 := \sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(1/4)} > 0.$$

We thus have that for every $s \in [0, T]$

$$X_s \geq y - \kappa^2 s^\alpha + \sqrt{2} B_s.$$

Hence

$$\begin{aligned}
\mathbb{P}(\tau < t) &\leq \mathbb{P}\left(\inf_{0 \leq s \leq t} \sqrt{2}\, B_s - \kappa^2 s^\alpha \leq \delta - y\right) \\
&\leq \mathbb{P}\left(\inf_{0 \leq s \leq t} B_s \leq \frac{\delta - y}{\sqrt{2}} + \frac{\kappa^2}{\sqrt{2}} t^\alpha\right) \\
&= \mathbb{P}\left(\sup_{0 \leq s \leq t} B_s \geq \frac{y - \delta}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}} t^\alpha\right).
\end{aligned}$$

The last equality follows as

$$- \inf_{0 \leq s \leq t} B_s \stackrel{d}{=} \sup_{0 \leq s \leq t} B_s.$$

By the reflection principle (Karatzas and Shreve [1991], p.79) it follows that

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} B_s \geq b\right) = 2\mathbb{P}(B_t \geq b) = \sqrt{\frac{\pi}{2}} \int_{bt^{-\frac{1}{2}}}^{\infty} e^{-\frac{x^2}{2}}\, \mathrm{d}x,$$

for any $b \in \mathbb{R}$. Thus,

$$\mathbb{P}(\tau < t) \leq \sqrt{\frac{\pi}{2}} \int_{\frac{y-\delta}{\sqrt{2t}} - \frac{\kappa^2}{\sqrt{2}} t^{\alpha - \frac{1}{2}}}^{\infty} e^{-\frac{x^2}{2}}\, \mathrm{d}x.$$

However,

$$\frac{y - \delta}{\sqrt{2t}} - \frac{\kappa^2}{\sqrt{2}} t^{\alpha - \frac{1}{2}} = t^{-\frac{1}{2}} \left(\frac{y - \delta}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}} t^\alpha\right)$$

and $y - \delta > 0$. Thus, the lower limit of the above integration goes to $+\infty$ as $t \to 0^+$ and so it follows that $\mathbb{P}(\tau < t) \to 0$ as $t \to 0^+$.

$\square$

Remark 9.2.2 (Details for the case of $y < 0$)**.** *Fix $\delta > 0$ such that $y < -\delta < 0$ and take $\tilde{b}_i$ as the following:*

$$\tilde{b}_1(z, t) := \begin{cases} b_1(zt^{-\alpha}) & \textit{if } -\infty < z < -\delta \\ b_1(-\delta t^{-\alpha}) & \textit{if } -\delta \leq z < \infty, \end{cases}$$

and

$$\tilde{b}_2(z,t) := \begin{cases} b_2(zt^{-\alpha}) & \text{if } -\infty < z < -\delta \\ b_2(-\delta t^{-\alpha}) & \text{if } -\delta \leq z < \infty. \end{cases}$$

The reader will note that the proof of existence and uniqueness of (X_t) *does not depend on the sign of* y*, only the definition of the stopping time* τ *and the proof that* $\mathbb{P}(\tau = 0) = 0$*. Thus, let* $\tau = \tau^\delta := \inf\{t \in (0,T] \,|\, X_t > -\delta\}$*. If* $\tau^\delta < t$ *then* $\sup_{0\leq s\leq t} X_s \geq -\delta$*. Since*

$$X_s = y + \int_0^s r^{-\alpha}\tilde{b}_1(X_r, r) + \alpha r^{\alpha-1}\tilde{b}_2(X_r, r)\,\mathrm{d}r + \sqrt{2}B_s$$

and $\tilde{b}_1(z,t) \leq 0$ *and* $\tilde{b}_2(z,t) \leq \kappa^2$ *for every* $(z,t) \in \mathbb{R} \times [0,T]$ *for every* $\delta > 0$ *it follows that*

$$X_s \leq y + \kappa^2 t^\alpha + \sqrt{2} \sup_{0\leq s\leq t} B_s$$

and so

$$\mathbb{P}(\tau^\delta < t) \leq \mathbb{P}(\sup_{0\leq s\leq t} B_s \geq \frac{-y-\delta}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}} t^\alpha).$$

The proof now continues as in the case of $y > 0$*, noting that* $-y - \delta > 0$*.*

Theorem 9.2.3. *With probability 1, the solution* (X_t) *to* (9.2.1) *exists for every* $t \in [0,T]$*, where* $T \in (0,\infty)$ *is arbitrary.*

Proof. Again, we will consider the case of $y > 0$. The argument for $y < 0$ is analogous. Recall for the initial condition $y > 0$, $\tau = \tau^\delta = \inf\{t \in (0,T] \,|\, X_t \leq \delta\}$. We need to show that $\mathbb{P}(\tau^\delta < T) \to 0$ as $\delta \to 0$, so that our solution X_t built in Theorem 9.2.1 exists, with probability 1, on $[0,T]$. To this end, we use a Lyapunov function method. Let $V : [0,\infty) \to [1,\infty)$ be defined by

$$V(z) := \begin{cases} W(z) & \text{if } 0 < z < c \\ \text{smoothly decreasing} & \text{if } c \leq z < c+1 \\ 1 & \text{if } z \geq c+1, \end{cases} \tag{9.2.7}$$

where

$$W(z) := \operatorname{arctanh}\left(\frac{\cosh g(z)}{\sqrt{\cosh 2g(z)}}\right) - \sqrt{2}\log\left(\sqrt{2}\cosh g(z) + \sqrt{\cosh 2g(z)}\right).$$

We can easily extend this definition for the case of initial data $y < 0$, by defining, for $z < 0$, $V(z) := V(-z)$. We make sure that V is defined in such a way that

$V \in C^\infty(\mathbb{R})$. The value of c is taken so that

$$W(z) = \operatorname{arctanh}\left(\frac{\cosh g(z)}{\sqrt{\cosh 2g(z)}}\right) - \sqrt{2}\log\left(\sqrt{2}\cosh g(z) + \sqrt{\cosh 2g(z)}\right) \geq 2$$

for every $0 < z < c$. We observe that for $0 < z < c$ we have

$$V'(z) = -\frac{1}{\sinh g(z)}, \qquad V''(z) = \frac{\coth g(z)}{\sinh g(z)\sqrt{\cosh 2g(z)}}.$$

Suppose $0 \leq t < \tau^\delta$ and define $Y_t := V(X_t t^{-\alpha})$. By Itō's formula (Lemma 2.3.4), one has

$$\mathrm{d}Y_t = -\alpha X_t t^{-\alpha-1} V'(X_t t^{-\alpha})\,\mathrm{d}t + t^{-\alpha}V'(X_t t^{-\alpha})\,\mathrm{d}X_t + t^{-2\alpha}V''(X_t t^{-\alpha})\,\mathrm{d}t.$$

Recalling that $G(z) = z - \frac{\sinh 2g(z)}{\sqrt{\cosh 2g(z)}}$ we have the following SDE for Y_t:

$$\begin{aligned}
\mathrm{d}Y_t &= t^{-2\alpha}\frac{\coth g(X_t t^{-\alpha})}{\sqrt{\cosh 2g(X_t t^{-\alpha})}}V'(X_t t^{-\alpha})\,\mathrm{d}t + t^{-2\alpha}V''(X_t t^{-\alpha})\,\mathrm{d}t \\
&\quad - \alpha t^{-1}\frac{\sinh 2g(X_t t^{-\alpha})}{\sqrt{\cosh 2g(X_t t^{-\alpha})}}V'(X_t t^{-\alpha})\,\mathrm{d}t + \sqrt{2}t^{-\alpha}V'(X_t t^{-\alpha})\,\mathrm{d}B_t \\
Y_0 &= 1.
\end{aligned}$$

We note that $X_0 = y > 0$ and so $\lim_{t\to 0} X_t t^{-\alpha} = +\infty$ and so the initial condition for Y_t is $Y_0 = 1$.

For $t < \tau^\delta$ with $X_t t^{-\alpha} > c+1$, we have $V' = V'' = 0$ and so $\mathbb{E}[Y_t] \leq T$ for such t. The first t such that $t < \tau^\delta$ and $c \leq X_t t^{-\alpha} < c+1$ (call it t_1) has $t_1 > 0$. Thus, for such $t_1 < t < \tau^\delta$ with $c \leq X_t t^{-\alpha} < c+1$ has V' and V'' bounded and since $t > t_1$ one has control over the $t^{-2\alpha}$ and t^{-1} terms in the definition of $\mathrm{d}Y_t$. Thus, for $t_1 < t < \tau^\delta$ we have

$$\mathbb{E}[Y_t] \leq C_T^{(1)}.$$

Finally, the first time t with $t < \tau^\delta$ and $X_t t^{-\alpha} < c$ (call it t_2) has $t_2 > 0$. For $t_2 < t < \tau^\delta$, by definition of V we have

$$\mathrm{d}Y_t = \alpha t^{-1}\frac{\sinh 2g(X_t t^{-\alpha})}{\sinh g(X_t t^{-\alpha})\sqrt{\cosh 2g(X_t t^{-\alpha})}}\,\mathrm{d}t - \sqrt{2}\frac{t^{-\alpha}}{\sinh g(X_t t^{-\alpha})}\,\mathrm{d}B_t.$$

Thus, as $t_2 > 0$ and

$$\frac{\sinh 2x}{\sinh x\sqrt{\cosh 2x}} \leq 2$$

for every $x \in \mathbb{R}$, we have that

$$\mathbb{E}[Y_t] \leq C_T^{(2)}.$$

Thus, considering $[0, \tau^\delta)$ as a whole, there exists $C_T > 0$ such that

$$\mathbb{E}[Y_t] \leq C_T.$$

We have used the fact (Karatzas and Shreve [1991]) that if f is some uniformly bounded function of an Itō process X_t and $0 \leq t_* < t^* < \infty$ then

$$\mathbb{E}\left[\int_{t_*}^{t^*} f(X_s, s)\, \mathrm{d}B_s\right] = 0.$$

We are now ready to show that $\mathbb{P}(\tau^\delta < T) \to 0$ as $\delta \to 0$. In the following, we will denote τ^δ by τ. We have

$$\mathbb{P}(\tau < T) = \mathbb{P}(X_\tau \leq \delta) = \mathbb{P}(\tau^{-\alpha} X_\tau \leq \delta\tau^{-\alpha}).$$

From the proof of Theorem 9.2.1 we have, for some $\beta > 0$ to be chosen,

$$\mathbb{P}(\tau < \delta^\beta) \leq \sqrt{\frac{\pi}{2}} \int_{F_{\alpha,\beta}(\delta)}^{\infty} e^{-\frac{x^2}{2}}\, \mathrm{d}x$$

where

$$F_{\alpha,\beta}(\delta) := \delta^{-\frac{\beta}{2}} \left(\frac{y - \delta}{\sqrt{2}} - \frac{\kappa^2}{\sqrt{2}}\delta^{\alpha\beta}\right).$$

(Indeed, one replaces t with δ^β in the proof there. For the case of $y < 0$, we replace y here with $-y$. See Remark 9.2.2). We thus see that

$$\mathbb{P}(\tau < \delta^\beta) \to 0$$

as $\delta \to 0$. Observe that

$$\left\{\tau < \delta^\beta\right\} = \left\{\tau^{-\alpha} > \delta^{-\alpha\beta}\right\}.$$

Observe further that

$$\Omega = \left\{\tau^{-\alpha} > \delta^{-\alpha\beta}\right\} \cup \left\{\tau^{-\alpha} \leq \delta^{-\alpha\beta}\right\}.$$

Thus we have

$$\begin{aligned}\left\{\tau^{-\alpha}X_\tau \leq \delta\tau^{-\alpha}\right\} &= \left\{\tau^{-\alpha}X_\tau \leq \delta\tau^{-\alpha}\right\} \cap \left(\left\{\tau^{-\alpha} \leq \delta^{-\alpha\beta}\right\} \cup \left\{\tau^{-\alpha} > \delta^{-\alpha\beta}\right\}\right)\\ &\subset \left\{\tau^{-\alpha}X_\tau \leq \delta^{1-\alpha\beta}\right\} \cup \left\{\tau^{-\alpha} > \delta^{-\alpha\beta}\right\}.\end{aligned}$$

Thus, as V is decreasing, we have by the observation above, Markov's inequality and the results above that

$$\begin{aligned}\mathbb{P}(\tau < T) &\leq \mathbb{P}\left(V(\tau^{-\alpha}X_\tau) \geq V(\delta^{1-\alpha\beta})\right) + \mathbb{P}(\tau^{-\alpha} > \delta^{-\alpha\beta})\\ &\leq \frac{\mathbb{E}[Y_\tau]}{V(\delta^{1-\alpha\beta})} + \mathbb{P}(\tau < \delta^{\beta})\\ &\leq \frac{C_T}{V(\delta^{1-\alpha\beta})} + \mathbb{P}(\tau < \delta^{\beta}).\end{aligned}$$

We now pick $\beta > 0$ such that $1 - \alpha\beta > 0$ to see that

$$\mathbb{P}(\tau < T) \longrightarrow 0$$

as $\delta \to 0$, since

$$\lim_{z\to 0^+} W(z) = \lim_{z\to 0^+} \operatorname{arctanh}\left(\frac{\cosh g(z)}{\sqrt{\cosh 2g(z)}}\right) = +\infty,$$

as $g(0) = 0$, $\cosh(0) = 1$ and $\lim_{x\to 1^-} \operatorname{arctanh}(x) = +\infty$.

□

We regards to attainment of the initial data for the solution to (9.1.4) we have the following.

Theorem 9.2.4. *There exists a unique classical solution, J, to* (9.1.4) *which attains its initial data in the weak sense. That is,*

$$\int_{\mathbb{R}} J(y,t)f(y)\,\mathrm{d}y \longrightarrow \int_{\mathbb{R}} J_*(y)f(y)\,\mathrm{d}y$$

as $t \to 0^+$ for every $f \in C_c^\infty(\mathbb{R})$.

Proof. A simple calculation reveals that, for $f \in C_c^\infty(\mathbb{R})$,

$$|\mathbb{E}f(X_t) - \mathbb{E}f(Z)| \leq C(t + t^\alpha + \mathbb{E}|B_t|) \leq C(t + t^\alpha + t^{\frac{1}{2}}) \longrightarrow 0$$

as $t \to 0$. So, where the probabilistic version of Hörmander's Theorem (Malliavin

[1978]) holds (that is, outside $|l| = 0$), a density exists and so if $J(\cdot, t)$ is the density of X_t and J_* is the density of Z, it follows that

$$\int_{\mathbb{R}} J(y,t) f(y)\,\mathrm{d}y \longrightarrow \int_{\mathbb{R}} J_*(y) f(y)\,\mathrm{d}y$$

as $t \to 0$, for every $f \in C_c^\infty(\mathbb{R})$. We note that J satisfies (9.1.4) where Hörmander's Theorem holds. This shows the weak convergence of the density to the initial density.

$\square$

Chapter 10

Formulation and Analysis of Problem III

Until now, the evolution of the curve or surface has been completely determined by some time dependent function, which is independent of any equation that actually lives on the curve or surface.

Of interest in mathematical biology is describing cell motion in the presence of another cell emitting some chemotactic signal. Indeed, the following video shows a white blood cell chasing a bacterium:

`http://www.youtube.com/watch?v=JnlULOjUhSQ`

Figure 10.1 is a snapshot of the video which shows that the white blood cell "A" changes its shape according to the presence of the bacterium "B". The stationary objects "C" are the red blood cells.

Figure 10.1: Snapshot of video. Here "A" is the white blood cell, "B" is the bacterium and "C" is one of the stationary red blood cells.

There have been several mathematical models to describe the evolution of a cell in the presence of a chemotactic signal. For example, in Neilson et al. [2011], the authors use a surface partial differential equation with some noise and couple the evolution of the curve to the solution of this equation. This presents many mathematical challenges with respect to showing existence and uniqueness of solutions; even deciding in which space a solution may exist is not clear!

The aim of this chapter is to present a toy model of this sort of problem. We take the hyperbola $\mathcal{C}_t := \{(x,y) \in \mathbb{R}^2 \mid y \geq 0,\ y^2 - x^2 = f(t)\}$ where f depends on the solution to the heat equation on $\mathcal{C}_t$ as before.

The resulting equation is non-linear and by the choice of f, the non-linearity turns out to be time-dependent only. We set out to show short time existence and uniqueness of a mild solution. We do not attempt to analyse the equation at any singularity that may or may not exist, but defer such an analysis to further research.

10.1 The Problem

Consider the curve

$$\mathcal{C}_t := \{(x,y) \in \mathbb{R}^2 \mid y \geq 0,\ y^2 - x^2 = f(t)\}, \quad 0 \leq t \leq T, \tag{10.1.1}$$

for some $T > 0$ to be chosen, where

$$\begin{aligned} \dot{f}(t) &= -u(0,t)^2 \\ f(0) &= \varphi \end{aligned} \tag{10.1.2}$$

with $\varphi > 0$ and $u \in C(0,T; L^\infty(\mathbb{R}))$, for some $T \in (0,\infty)$.

For $\mathcal{C}_t$ and $0 \leq t \leq T$, denote by $\phi : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$, the level set function of $\mathcal{C}_t$ defined by $\phi(x,y,t) := y^2 - x^2 - f(t)$. We are interested in existence and uniqueness of a solution to the following PDE on $\mathcal{C}_t$, which is derived from a conservation law (Section 2.6).

$$\begin{aligned} \partial_t^\bullet U + U\nabla_\Gamma \cdot v - \Delta_\Gamma U = 0 \quad & x \in \Gamma := \mathcal{C}_t \\ U(x,0) = U_0(x) \quad & x \in \mathcal{C}_0 \end{aligned} \tag{10.1.3}$$

where

$$v := \frac{\phi_t}{|\nabla \phi|}\nu$$

is the prescribed normal velocity of the curve, with the outward pointing unit normal given by

$$\nu := -\frac{\nabla \phi}{|\nabla \phi|}.$$

We will assume that the initial data U_0 is smooth, bounded and suitably integrable.

10.1.1 Formulation of the Problem in arc-length parameter.

In the following, we transform (10.1.3) into an equation on $\mathbb{R}\times[0,T]$ using arc-length parameterisation.

Let $Y : \mathbb{R}\times[0,T] \to \mathbb{R}^2$ be defined by $Y(p,t) = \sqrt{f(t)}\,(\sinh p, \cosh p)$. Then by standard hyperbolic identities, we see that Y is a smooth parameterisation of $\mathcal{C}_t$. We also assume that t is sufficiently small to ensure that $f(t) > 0$. Let Y_p denote the partial derivative of Y with respect to p and define $l := \int_0^p |Y_p(u,t)|\,\mathrm{d}u$. Then

$$\frac{l}{\sqrt{f(t)}} = \int_0^p \sqrt{\cosh 2u}\,\mathrm{d}u$$

for $t \in [0,T]$. Define the arc-length parameterisation of $\mathcal{C}_t$, denoted by $X : \mathbb{R}\times[0,T] \to \mathbb{R}^2$, by

$$X(l,t) = \sqrt{f(t)}\left(\sinh g\left(lf(t)^{-\frac{1}{2}}\right), \cosh g\left(lf(t)^{-\frac{1}{2}}\right)\right). \tag{10.1.4}$$

Here, $g : \mathbb{R}\to\mathbb{R}$ is the *inverse* of the map $p \mapsto \int_0^p \sqrt{\cosh 2u}\,\mathrm{d}u$, as introduced earlier.

Remark 10.1.1. *The same properties of g are true as in Chapter 3 and Chapter 7. See, for example, Remark 3.2.2.*

We now transform (10.1.3) into a PDE on $\mathbb{R}\times[0,T]$ ready for subsequent analysis. In order to transform (10.1.3) we write $u(l,t) = U(X(l,t),t)$ and compute to find

$$\frac{\partial u}{\partial t} = \nabla U \cdot (X_t - v) + \partial_t^{\bullet} U$$

and

$$\frac{\partial u}{\partial l} = \nabla U \cdot X_l.$$

Noting that v is in the normal direction only, and noting the sign of the normal vector, we have that (Dziuk and Elliott [2007], Appendix A)

$$\nabla_\Gamma \cdot v = VH$$

where

$$V := \frac{\phi_t}{|\nabla\phi|} \quad \text{and} \quad H := \frac{-1}{|\nabla\phi|}\sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_ix_j}.$$

Finally,

$$\Delta_\Gamma U(X(l,t),t) = \frac{\partial^2 u}{\partial l^2}(l,t)$$

as $|X_l| = 1$. (See Dziuk and Elliott [2007]). We observe that $X_t - v = \beta\hat{\tau}$ where $\hat{\tau}$ is the unit tangent vector, $\hat{\tau} = \pm X_l$. This is because

$$X_t \cdot \nu - v \cdot \nu = \frac{1}{|\nabla\phi|}\left(X_t \cdot (-\nabla\phi) - \phi_t\right) = \frac{1}{2|\nabla\phi|\sqrt{f}}\left(-\dot{f} + \dot{f}\right) = 0.$$

Thus, $\beta = X_t \cdot \hat{\tau}$ and so for any orientation of $\hat{\tau}$ we have

$$\nabla U \cdot (X_t - v) = X_t \cdot X_l \frac{\partial u}{\partial l}.$$

One computes and sees that

$$X_t \cdot X_l = \frac{\dot{f}(t)}{2\sqrt{f(t)}}\left(\frac{\sinh 2g(lf(t)^{-\frac{1}{2}})}{\sqrt{\cosh 2g(lf(t)^{-\frac{1}{2}})}} - \frac{l}{\sqrt{f(t)}}\right)$$

and

$$VH = -\frac{\dot{f}(t)}{2f(t)}\operatorname{sech}^2 2g(lf(t)^{-\frac{1}{2}}).$$

One notes that

$$\frac{\partial}{\partial l}\left(G(lf(t)^{-\frac{1}{2}})u\right) = G(lf(t)^{-\frac{1}{2}})\frac{\partial u}{\partial l} - \frac{1}{\sqrt{f(t)}}\operatorname{sech}^2 2g(lf(t)^{-\frac{1}{2}})u$$

where for $s \in \mathbb{R}$

$$G(s) = s - \frac{\sinh 2g(s)}{\sqrt{\cosh 2g(s)}}. \tag{10.1.5}$$

Thus, equation (10.1.3) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial l^2} - \frac{\dot{f}(t)}{2\sqrt{f(t)}}\frac{\partial}{\partial l}\left(G(lf(t)^{-\frac{1}{2}})u\right) \\ u(l,0) &= U_0(X(l,0)). \end{aligned} \tag{10.1.6}$$

Using Proposition A.0.2 one sees that $G \in L^\infty(\mathbb{R})$ and by Proposition A.0.1 that $\operatorname{sech}^2 2g(z) \leq \mathcal{O}(z^{-4})$ as $|z| \to \infty$.

Remark 10.1.2. *We make the ansatz that the solution to* (10.1.6) *is in the Banach space* $C(0,T; L^\infty(\mathbb{R}))$ *and that* f *depends on the solution. Thus,* (10.1.6) *is a non–linear parabolic PDE. We proceed to show that there is a unique mild solution to* (10.1.6) *in this Banach space.*

10.2 Analysis of the Problem

We have the following main result.

Theorem 10.2.1. *Let $u_0 \in C_c^\infty(\mathbb{R})$. Then, for $T > 0$ small enough, there exists a unique mild solution to* (10.1.6) *for $t \in [0, T]$ with*

$$\sup_{t\in[0,T]} \|u(\cdot, t)\|_{L^\infty} < \infty.$$

Proof. Let $T \in (0, \infty)$ to be chosen later. Define the normed vector space

$$X := C([0, T]; L^\infty(\mathbb{R}))$$

with norm

$$\|v\|_X := \sup_{t\in[0,T]} \|v(\cdot, t)\|_{L^\infty(\mathbb{R})}.$$

Then, $(X, \|\cdot\|_X)$ is a Banach space. Define $\mathcal{F} : X \longrightarrow X$ by

$$\mathcal{F}v(l, t) := S(t)u_0(l) - \frac{1}{2}\int_0^t S(t-s)\frac{\dot{f}(s)}{\sqrt{f(s)}}\frac{\partial}{\partial l}\left(G(\cdot\, f(s)^{-\frac{1}{2}})v\right) \mathrm{d}s$$

where

$$S(t) : L^\infty(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$$

is the standard heat semigroup (Engel and Nagel [2006]). For $R \in (0, \infty)$, define the closed ball in X as

$$B_R := \{x \in X \mid \|x\|_X \leq R\}.$$

Note by definition of f that $|\dot{f}(s)| \leq \|u\|_X^2$ and if $u \in B_R$ then, denoting the dependence of u on f by f_u we have

$$\frac{1}{\sqrt{\varphi + tR}} \leq \frac{1}{\sqrt{f_u(t)}} \leq \frac{1}{\sqrt{\varphi - tR}}$$

which is valid, provided $t < \varphi R^{-1}$. We will now show that $\mathcal{F} : B_R \longrightarrow B_R$ for some choice of R and that $\mathcal{F}$ is a contraction there.

First, an integration by parts and using Davies [1989], Theorem 6, Case 1 we have for $v \in B_R$

$$
\begin{aligned}
|\mathcal{F}v| &\le \|u_0\|_{L^\infty} + CR\int_0^t (t-s)^{-\frac{1}{2}}(\varphi - sR)^{-\frac{1}{2}}\,\mathrm{d}s \\
&= \|u_0\|_{L^\infty} + CR^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}(\varphi R^{-1} - s)^{-\frac{1}{2}}\,\mathrm{d}s.
\end{aligned}
$$

If $t < \frac{1}{2}\varphi R^{-1}$ then $(\varphi R^{-1} - s)^{-\frac{1}{2}} < \frac{\sqrt{2}}{2}\varphi^{-\frac{1}{2}}R^{\frac{1}{2}}$ and so

$$
\int_0^t (t-s)^{-\frac{1}{2}}(\varphi R^{-1} - s)^{-\frac{1}{2}}\,\mathrm{d}s \le C\varphi^{-\frac{1}{2}}R^{\frac{1}{2}}T^{\frac{1}{2}}.
$$

Thus,

$$
\|\mathcal{F}v\|_X \le \|u_0\|_{L^\infty} + CR\varphi^{-\frac{1}{2}}T^{\frac{1}{2}}.
$$

We recall that $t < \frac{1}{2}\varphi R^{-1}$ and so $T^{\frac{1}{2}} < C\varphi^{\frac{1}{2}}R^{-\frac{1}{2}}$ and so

$$
\|\mathcal{F}v\|_X \le \|u_0\|_{L^\infty} + CR^{\frac{1}{2}}.
$$

We thus want to chose $R > 0$ such that $\|u_0\|_{L^\infty} + CR^{\frac{1}{2}} \le R$. Letting $p = R^{\frac{1}{2}}$, this translates into wanting to find $p > 0$ such that

$$
-p^2 + Cp + \|u_0\|_{L^\infty} \le 0. \tag{10.2.1}
$$

However, there exists $p_0 > 0$ such that $p > p_0$ implies (10.2.1) holds true. Hence, for this choice of R, we have that $\mathcal{F} : B_R \to B_R$.

To show that $\mathcal{F}$ is a contraction, let $u, v \in B_R$ be arbitrary. Then

$$
\mathcal{F}u - \mathcal{F}v = \frac{1}{2}\int_0^t S(t-s)\left(\left(\frac{\dot{f}_v}{\sqrt{f_v}} - \frac{\dot{f}_u}{\sqrt{f_u}}\right)\frac{\partial}{\partial l}(Gv) + \frac{\dot{f}_u}{\sqrt{f_u}}\frac{\partial}{\partial l}\left(G(v-u)\right)\right)\mathrm{d}s.
$$

An integration by parts and arguing as before by using Davies [1989], Theorem 6, Case 1 we have

$$
|\mathcal{F}u - \mathcal{F}v| \le C(R)\left(\int_0^t (t-s)^{-\frac{1}{2}}\left|\frac{\dot{f}_u}{\sqrt{f_u}} - \frac{\dot{f}_v}{\sqrt{f_v}}\right|\mathrm{d}s + \left|\frac{\dot{f}_u}{\sqrt{f_u}}\right|\|u-v\|_X t^{\frac{1}{2}}\right).
$$

By the estimate on $\frac{1}{\sqrt{f_w}}$ for $t \in [0, \frac{1}{2}\varphi R^{-1}]$ it follows that there exists $K_1, K_2 > 0$ such that

$$
K_1\frac{\dot{f}_w}{\sqrt{\varphi}} \le \frac{\dot{f}_w}{\sqrt{f_w}} \le K_2\frac{\dot{f}_w}{\sqrt{\varphi}}
$$

for every $w \in X$, recalling that $T < \frac{1}{2}\varphi R^{-1}$. It follows that there exists $C > 0$ such that

$$\left|\frac{\dot{f}_u}{\sqrt{f_u}} - \frac{\dot{f}_v}{\sqrt{f_v}}\right| \leq \frac{C}{\sqrt{\varphi}}\left|\dot{f}_u - \dot{f}_v\right|$$

for every $u, v \in X$. Also, for $u, v \in B_R$,

$$|\dot{f}_u - \dot{f}_v| = |u(0,t)^2 - v(0,t)^2| = |u(0,t) - v(0,t)||u(0,t) + v(0,t)| \leq 2R\|u - v\|_X.$$

Thus, we conclude that for $u, v \in B_R$ arbitrary,

$$|\mathcal{F}u - \mathcal{F}v| \leq C(R,\varphi)\left(T^{\frac{1}{2}} + \int_0^t (t-s)^{-\frac{1}{2}}\,\mathrm{d}s\right)\|u - v\|_X \leq C(R,\varphi)T^{\frac{1}{2}}\|u - v\|_X.$$

Hence

$$\|\mathcal{F}u - \mathcal{F}v\|_X < \frac{1}{2}\|u - v\|_X$$

for every $u, v \in B_R$, provided T is chosen small enough. Thus, by the contraction mapping Theorem, there exists a unique $u \in X$ such that

$$u = \mathcal{F}u.$$

This is precisely the definition of a unique mild solution.

□

Remark 10.2.2 (On the possible behaviour of the solution at a singularity)**.** *Recall Chapter 4 where one takes $f(t) = (1-t)^{2\alpha}$ and so $\dot{f}(t) = -2\alpha(1-t)^{2\alpha-1}$. We recall that if $\alpha \leq \frac{1}{2}$ then $u(0,t) \to 0$ as $t \to 1^-$, whereas if $\alpha > \frac{1}{2}$ we have that u need not tend to 0. These cases correspond to $\dot{f}(t) \to -\infty$ if $\alpha < \frac{1}{2}$ or $\dot{f}(t) \to -1$ if $\alpha = \frac{1}{2}$ as $t \to 1^-$ and $\dot{f}(t) \to 0$ if $\alpha > \frac{1}{2}$ as $t \to 1^-$ respectively. In the case of Problem III, if $u(0,t)^2 \to \infty$ then $\dot{f}(t) \to -\infty$ and so we would expect that $u(0,t) \to 0$ (as in Chapter 4). However, if $u(0,t) \to 0$ then $\dot{f}(t) \to 0$ and so we would expect u not necessarily to converge to 0 (again, as in Chapter 4). In the case that $u(0,t)^2 \to C$ where $C > 0$, then we have $\dot{f}(t) \to -C$ with $-C \neq 0$ and so following Chapter 4 we would expect that $u(0,t) \to 0$. Thus it is perhaps possible by the above arguments that a singularity does not occur in finite time. However, the behaviour of $u(0,t)$ as t approaches some singular time is not clear a priori.*

Chapter 11

Further Research

In the following and final chapter we detail the open questions raised in this thesis. We also look at some problems that are still to be addressed in the area of surface PDEs on evolving surfaces; namely where the evolution of the surface is coupled to the solution to the surface PDE.

11.1 Open problems raised in the thesis

11.1.1 Problem I: Before the singularity

The result of Theorem 4.2.6 is considered partial as the domain in which the result holds decreases with time (Remark 4.2.7). However, together with the results of Theorem 4.2.10 we infer information on the solution in a time–independent neighbourhood of $(0,1)$. The downside is that the proof of Theorem 4.2.10 relied strongly on the compact support of the initial data u_0, otherwise, one needs some growth assumption globally on the initial data in terms of the first eigenfunction of $\bar{A}$. Indeed, this leads us to the following.

Further Research Problem 1. *Prove that Theorem 4.2.6 and Theorem 4.2.10 both hold for initial data that need not have compact support, with estimates involving norms that are weaker than the $D(\mathcal{H})$ norm as given in Theorem 4.2.6.*

A weaker norm will allow extension to the perturbation of the problem.

11.1.2 Problem I: After the singularity

For the $\alpha = \frac{1}{2}$ case, we were only able to show that the initial data to the continued PDE (5.0.3) was attained in the weak sense (Remark 5.2.5). This is at odds with the $\alpha < \frac{1}{2}$ and $\alpha > \frac{1}{2}$ cases, where we showed that the initial data was attained in the classical and mild senses respectively (Theorem 5.1.5 and Theorem 5.3.1 respectively). Naturally, we have

Further Research Problem 2. *Prove an analogous version of Theorem 5.1.5 or Theorem 5.3.1 as appropriate for the $\alpha = \frac{1}{2}$ case.*

As discussed in Chapter 5 the $\alpha = \frac{1}{2}$ case is when the arguments in Section 5.1 and Section 5.3 break down. Thus, the result of Remark 5.2.5 sits naturally between the two cases. However, it is expected that one should be able to prove the attainment of the initial data in a classical sense, however the techniques outlined in this thesis fail to produce the result.

11.1.3 Perturbation of Problem I

We have the following.

Further Research Problem 3. *Attain analogous results to those of Sections 6.1 and 6.2 for initial data $f(\cdot, s) \in L^q(\mathbb{R})$ and perturbation $f(\cdot, s) \in L^q(\mathbb{R})$ where $q \in [1, \infty)$.*

With regards to perturbation, we only considered the case where $\alpha < \frac{1}{2}$ due to the restriction of having the initial data in the $\alpha = \frac{1}{2}$ case belong to a suitable subset of differentiable functions. We would thus want the following.

Further Research Problem 4. *Attain all the results of Further Research Problem 3, for $\alpha = \frac{1}{2}$.*

Linking on from this, we note that in Section 6.3 that we did not take space–time white noise as the stochastic perturbation. This leads onto the following natural problem:

Further Research Problem 5. *Take space–time white noise as the stochastic perturbation in Section 6.3.*

Tackling this problem will be hard; we need to know more information about the two–parameter semigroup as introduced in Definition 6.2.1, such as whether it maps $H^1(\mathbb{R})$ into itself. This was discussed in Section 6.3.1.

11.1.4 Problem II: Before the singularity

In Problem II, we considered the natural conserved quantity J, in Proposition 7.1.6. In the case of $\alpha < \frac{1}{2}$ we were able to prove that J was uniformly bounded in space and time. However, we were unable to establish this result for $\alpha \geq \frac{1}{2}$. Further, the upper bound on u in Theorem 8.2.1 for $|l| > y_0\sqrt{1-t}$ requires that one knows the value of $\bar{\mu}_1$. We naturally have

Further Research Problem 6. *Prove that for every $\alpha \geq \frac{1}{2}$, there exists $C > 0$ such that*

$$|J(l,t)| \leq C$$

for every $(l,t) \in \mathbb{R} \times [0,1]$. Prove further and/or estimate the value of $\bar{\mu}_1$ more accurately.

The limitation of proving this result for the $\alpha = \frac{1}{2}$ case was requiring that, when one is working in the (y,τ) coordinates, we needed that y was bounded (see the proof of Theorem 8.2.1. In particular, the boundedness of y is needed in the probabilistic argument presented there). Also, the functional analytic techniques did not naturally yield L^∞ bounds and the crude estimates on $\bar{\mu}_1$ are not very useful. It is conjectured that $\bar{\mu}_1 = \frac{1}{2}$ in Conjecture 8.2.4. For the boundedness of J when $\alpha > \frac{1}{2}$, discussion is given in Section 8.3 and so omitted here.

We note that the uniform boundedness of J is needed to rule out singularities that behave in the following way: One could have that

$$\lim_{t\to 1^-} \int_{\mathbb{R}} |J(l,t)| \,\mathrm{d}l \qquad \text{exists}$$

and is finite, but

$$\lim_{t\to 1^-} \sup_{l\in\mathbb{R}} |J(l,t)| = +\infty.$$

Figure 11.1 shows an example of such a J in the limit. The singularities at ± 0.02 are approached at an order of $|l|^{-1/2}$ and the tails (not shown) fall like $|l|^{-2}$. Thus, the function is integrable, but is not uniformly bounded. One should immediately note that the bounds in Theorem 8.2.1 rule this figure out for $\alpha = \frac{1}{2}$, but do not rule out such a problem happening "at infinity". However, it is still open as to what happens if $\alpha > \frac{1}{2}$.

Indeed, a solution to Further Research Problem 6 would rule out such problems "at infinity" for $\alpha = \frac{1}{2}$ and in general for the case of $\alpha > \frac{1}{2}$. For $l \neq 0$, $\mathcal{S}_t^\alpha$ is smooth for all $t \in [0,1]$, and we expect that J is indeed uniformly bounded for every $\alpha \geq \frac{1}{2}$.

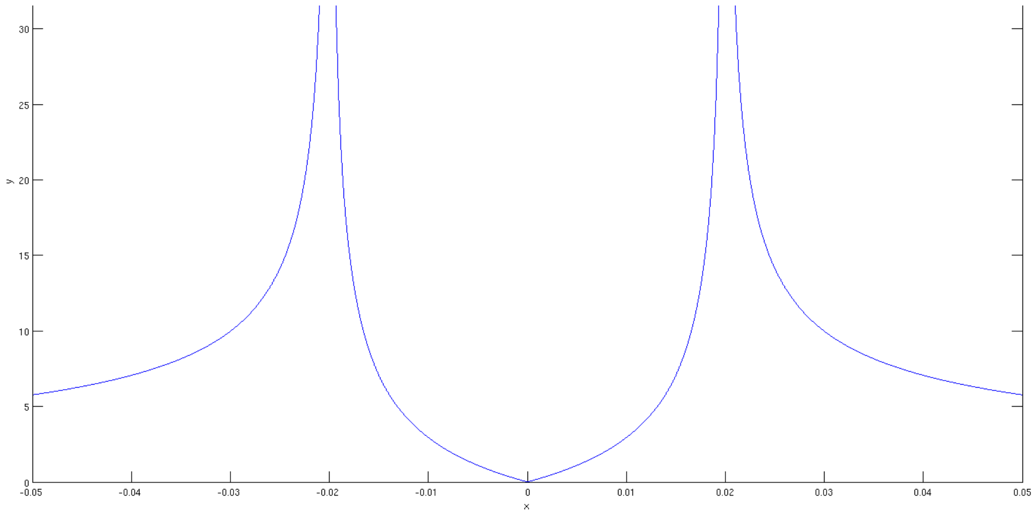

Figure 11.1: Plot of an example of such a limiting J.

11.1.5 Problem II: After the singularity

A major problem with this section was the coth singularity in the parameterisation of the surface.

Further Research Problem 7. *When considering Problem II after the singularity, reformulate and analyse the problem using an alternative method, such as a graph approach.*

This problem was considered, but the coefficients of the PDE were analytically intractable.

We managed to show that the initial data was attained in a weak sense for every $\alpha \in (0, \infty)$. In conjunction with the above Further Research Problem, we pose the following.

Further Research Problem 8. *Show the attainment of the initial data to* (9.1.4) *is achieved in a classical or mild sense (as appropriate) for a given* $\alpha \in (0, \infty)$.

As noted, this should be tackled using an alternative formulation, as the method of "flow variables" as in the curve case fails due to coth being unbounded. The singularity of coth also destroys any argument when one considers (9.1.4) as a perturbation of the heat equation.

11.1.6 Problem III

In problem III, we only considered small times before any singularity occurred. Naturally, we have

Further Research Problem 9. *Analyse and gain understanding about whether a singularity occurs in finite time or not. If one does, how many occur and can we continue the solution past the singularity in some sense?*

It is important to note that equation (10.1.6) is a non–linear PDE and so we cannot use the Feynman–Kac formula together with the Backwards Kolmogorov equation as we did in the linear case, to analyse the behaviour, as it stands. We would have to use a perturbation of the heat equation argument as we did in Theorem 10.2.1, but the argument is limited by the a priori unknown behaviour of $\frac{1}{\sqrt{f(t)}}$.

If any singularity occurs, to continue the solution past the singularity, we cannot use the Fokker–Planck equation approach due to the non-linearity of (10.1.6). Again, we would have to use a perturbation of the heat equation argument and the a priori unknown behaviour of $\frac{1}{\sqrt{f(t)}}$ near the singular time limits the argument.

11.2 Future research problems

The reader will notice that in the case of a surface, we only considered a hyperboloid of one sheet evolving into a hyperboloid of two sheets. Of course, one could consider a hyperboloid of two sheets evolving into a hyperboloid of one sheet. A major problem here is the one realised in Chapter 9; the parameterisation in the arclength parameter is singular. Thus the following problem links into Further Research Problem 7.

Further Research Problem 10. *Find and prove analogous results to those of Chapters 8 and 9 for a hyperboloid of two sheets evolving into a hyperboloid of one sheet.*

With regards to the type of singularities one can find for a smooth curve evolving in the plane, there are only two; a "kink" and a "cusp". An example of each is given in Figure 11.2.
For the curve, we have only been considering a curve that forms a "kink" in finite time and not a cusp. Naturally, this leads to the following.

Further Research Problem 11. *Investigate the effects of a cusp singularity forming in finite time on the heat equation* (2.6.3) *for a curve and a surface of revolution.*

The major problem is finding a suitable curve that forms a cusp with a parameterisation that is analytically tractable.

Finally, linking into Chapter 10:

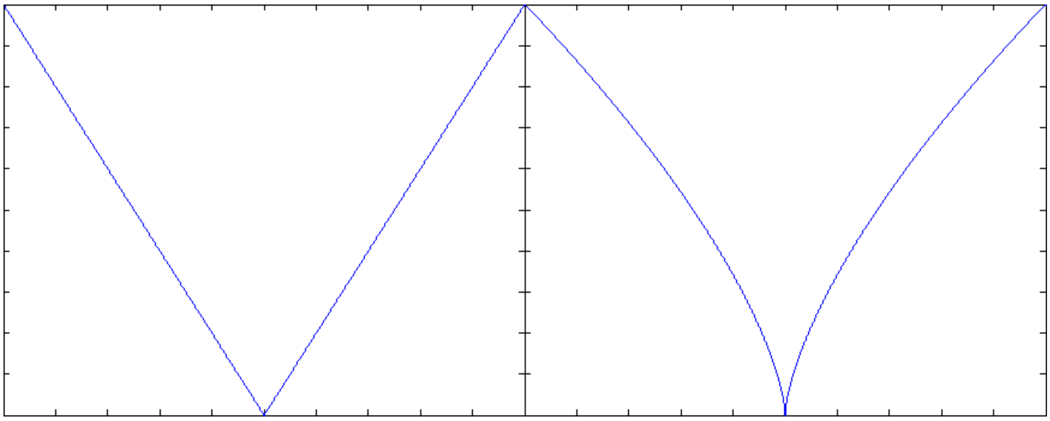

Figure 11.2: An example of a "kink" (left) and a "cusp" (right).

Further Research Problem 12. *Investigate the mathematical theory needed for the problem of the evolution of a surface, where the evolution is coupled to the surface PDE on the surface.*

In this thesis, we only considered the case of a curve, for small times away from any singularity. The mathematical theory would focus on problems of well-posedness and long term properties of the solution to the surface PDE and surface evolution. An example of such a problem would be forced mean curvature flow where the normal velocity, V, of the surface evolves according to

$$V = -H + U$$

where H is the mean curvature and U is the solution to some reaction–diffusion equation such as the heat equation (2.6.3). Such models arise in nature, for example in modelling the movement of a white blood cell in the presence of a bacterium emitting a chemotactic signal (Neilson et al. [2011]).

One would have to establish in what sense we have a solution to these equations, what space the solution is in, whether the solution exists, whether such a solution is unique and the long term properties of solutions to both equations.

We end this thesis with some concluding remarks.

11.3 Concluding remarks

In the author's opinion, this thesis adds to the mathematical knowledge and demonstrates interesting and technically hard problems in the area of linear singular parabolic PDEs that arise from problems in surface PDEs, where the underlying surface undergoes some sort of geometric singularity. The open problems identified at the end of this thesis form a substantial area of future research for the author, but also for the mathematical community as a whole. Indeed, this thesis may be regarded as a starting point in the investigation to singular linear parabolic PDEs arising from underlying geometric singularities.

The areas of mathematics used in this thesis further consolidate the fact that mathematics benefits from different research areas and the interplay of analysis and probability theory here demonstrates this.

Appendix A

Asymptotic Analysis of Functions of the Arc–Length Parameterisation

In the following we prove some asymptotic analysis results for functions of $g : \mathbb{R} \to \mathbb{R}$ as defined in Definition 3.2.1, which prove to be of utter importance to the analysis of the various problems in this thesis.

Proposition A.0.1. *There exists $\tilde{R}_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ measurable such that for every $s \in \mathbb{R}$*

(i) $\sinh 2g(s) = \operatorname{sgn}(s)(|s| + \kappa)^2 + \tilde{R}_1(s)$;

(ii) $\cosh 2g(s) = (|s| + \kappa)^2 + \tilde{R}_2(s)$

where for $i = 1, 2$ and large $|s|$

$$\tilde{R}_i(s) \leq \mathcal{O}(s^{-2}).$$

The constant κ is given by

$$\kappa := -c_1 := -\left(\int_0^\infty \left(\sqrt{\cosh 2u} - \frac{e^u}{\sqrt{2}}\right) \mathrm{d}u - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \approx 0.5991.$$

Proof. In the following, set $c_1 = -\kappa$. We will first consider $s > 0$ noting that $g(-s) = -g(s)$. We recall that g is the inverse of the map $x \mapsto \int_0^x \sqrt{\cosh 2u}\,\mathrm{d}u$. So, set

$$s = s(x) = \int_0^x \sqrt{\cosh 2u}\,\mathrm{d}u.$$

Then, by Taylor's Theorem

$$\sqrt{\cosh 2u} = \frac{e^u}{\sqrt{2}}\left(1 + \frac{e^{-4u}}{2} - \frac{e^{-8u}}{8}(1+\xi_L(u))^{-\frac{3}{2}}\right) \quad \text{for every} \quad u > 0,$$

where $\xi_L(u) \in (0, e^{-4u})$. Thus we have for every $x > 0$

$$\left|\int_0^x \sqrt{\cosh 2u}\,\mathrm{d}u - c_1 - \frac{e^x}{\sqrt{2}} + \frac{1}{6\sqrt{2}}e^{-3x}\right| \leq c_2 e^{-7x},$$

where

$$c_1 = \int_0^\infty \left(\sqrt{\cosh 2u} - \frac{e^u}{\sqrt{2}}\right)\mathrm{d}u - \frac{1}{\sqrt{2}} < 0,$$

and $c_2 > 0$ whose value we do not care about. Thus, we conclude that

$$s = c_1 + \frac{e^x}{\sqrt{2}} + r_0(x)$$

where

$$|r_0(x)| \leq c_3 e^{-3x}$$

gives a first order approximation to s. Write $y = e^x$ and we see that

$$y = \sqrt{2}(s - c_1) + r(y)$$

with

$$|r(y)| \leq c_3 y^{-3}.$$

However,

$$|r(y)| \leq c_3(\sqrt{2}(s - c_1) + r(y))^{-3}$$

so writing

$$y = \sqrt{2}(s - c_1) + r_1(s)$$

we conclude that $r(y) = r_1(s)$ and so

$$|r_1(s)| \leq c_3(\sqrt{2}(s - c_1) + r_1(s))^{-3}$$

which is equivalent to the following holding for every $s > 0$

$$\left(\sqrt{2}|r_1(s)|^{\frac{1}{3}}(s - c_1) + |r_1(s)|^{\frac{1}{3}} r_1(s)\right)^3 \leq c_3.$$

In order for this to hold, we need that

$$|r_1(s)| \leq K_1 |s|^{-3}$$

for some $K_1 > 0$, noting that $r_1(s)$ is already bounded by a constant. We thus conclude that to first order

$$y = \sqrt{2}(s - c_1) + r_1(s)$$

with r_1 bounded as above. We now give a second order approximation

$$\frac{y}{\sqrt{2}} - \frac{1}{6\sqrt{2}} y^{-3} = (s - c_1) + R_2(y)$$

with

$$|R_2(y)| \leq c_2 y^{-7}.$$

Using the above expression for y we conclude that

$$y = \sqrt{2}(s - c_1) + \frac{1}{6}(\sqrt{2}(s - c_1) + r_1(s))^{-3} + R_2(y)$$

and so writing

$$y = \sqrt{2}(s - c_1) + \frac{1}{6}(\sqrt{2}(s - c_1) + r_1(s))^{-3} + r_2(s)$$

as before, we conclude that

$$|r_2(s)| \leq K_2 |s|^{-7}.$$

To deal with the second term in the expression for y we observe that

$$\frac{1}{6}(\sqrt{2}(s - c_1) + r_1(s))^{-3} = \frac{1}{6}(\sqrt{2}(s - c_1))^{-3} + r_3(s)$$

where

$$\begin{aligned} r_3(s) &= \frac{1}{6}(\sqrt{2}(s - c_1) + r_1(s))^{-3} - \frac{1}{6}(\sqrt{2}(s - c_1))^{-3} \\ &= -\frac{1}{6}\left[\frac{3(\sqrt{2}(s - c_1))^2 r_1(s) + 3(\sqrt{2}(s - c_1)) r_1^2(s) - r_1^3(s)}{(\sqrt{2}(s - c_1) + r_1(s))(\sqrt{2}(s - c_1))^3}\right]. \end{aligned}$$

Hence, by the definition of r_1 we have

$$|r_3(s)| \leq K_3 s^{-6}$$

for s sufficiently large. We thus conclude that

$$y = \sqrt{2}(s - c_1) + \frac{1}{6}(\sqrt{2}(s - c_1))^{-3} + r_3(s).$$

Taking the logarithm and recalling that $x = g(s)$ we conclude that for every $s > 0$

$$g(s) = \log\left[\sqrt{2}(s - c_1) + \frac{1}{6}(\sqrt{2}(s - c_1))^{-3} + r_3(s)\right].$$

We are now in a position to prove assertions (i) and (ii). Recall that

$$\sinh 2g(s) = \frac{e^{2g(s)} - e^{-2g(s)}}{2}$$

and

$$\cosh 2g(s) = \frac{e^{2g(s)} + e^{-2g(s)}}{2}.$$

With this in mind, and using the above expression for $g(\cdot)$ we have that

$$\begin{aligned}\sinh 2g(s) = {} & (s - c_1)^2 + \frac{1}{6}(\sqrt{2}(s - c_1))^{-2} + r_4(s) + \\ & - \frac{1}{2}\left((s - c_1)^2 + \frac{1}{6}(\sqrt{2}(s - c_1))^{-2} + r_4(s)\right)^{-1}\end{aligned}$$

where, for large enough $s > 0$,

$$|r_4(s)| \leq K_4 |s|^{-5}.$$

Taking into consideration the fourth term of this expression, we conclude that

$$\sinh 2g(s) = (s - c_1)^2 + r_5(s)$$

with, for large enough $s > 0$,

$$|r_5(s)| \leq K_5 s^{-2}.$$

This yields (i) for $s > 0$. A similar approach yields (ii) for $s > 0$. We note that by symmetry and the fact that $g(-s) = -g(s)$ that (i) and (ii) follow for every $s \in \mathbb{R}$. $\square$

Proposition A.0.2. *Let*

$$\tilde{H}(z) := \frac{\sinh 2g(z)}{\sqrt{\cosh 2g(z)}} - z.$$

Then the following hold

i) $\tilde{H} \in L^{\infty}(\mathbb{R})$ *and is globally Lipschitz;*

ii) $\tilde{H}'(z) = \operatorname{sech}^2 2g(z)$ *for every* $z \in \mathbb{R}$*;*

iii) Fix $\delta > 0$*. Let* $x, y \in \mathbb{R} \setminus B_\delta(0)$ *with* $\operatorname{sgn}(x) = \operatorname{sgn}(y)$ *and suppose* $s > 0$*. Then there exists* $C_\delta > 0$ *depending only on* δ *such that*

$$\left|\tilde{H}(x/\sqrt{s}) - \tilde{H}(y/\sqrt{s})\right| \leq sC_\delta \left|x - y\right|.$$

Proof. i) From Proposition A.0.1 one concludes that for every $z \in \mathbb{R}$

$$\tilde{H}(z) + z = \frac{\operatorname{sgn}(z)(|z| - c_1)^2 + R_1(z)}{\sqrt{(|z| - c_1)^2 + R_2(z)}}$$

with $R_i \leq \mathcal{O}(z^{-2})$ as $|z| \to \infty$. We now simplify and Taylor expand the denominator to see that

$$\tilde{H}(z) + z = \operatorname{sgn}(z)(|z| - c_1) + R(z)$$

with $R \leq \mathcal{O}(|z|^{-3})$ as $|z| \to \infty$ and so

$$\tilde{H}(z) = \operatorname{sgn}(z)|z| - z - \operatorname{sgn}(z)c_1 + R(z).$$

However, $\operatorname{sgn}(z)|z| - z = 0$ for every $z \in \mathbb{R}$ and so

$$\tilde{H}(z) = -\operatorname{sgn}(z)c_1 + R(z).$$

Using the original definition of $\tilde{H}$ we have that $\tilde{H}(0) = 0$. Noting that R is smooth we have that $\tilde{H} \in L^{\infty}(\mathbb{R})$.

The global Lipschitz continuity property follows from (ii) below.

ii) Recalling from (3.2.1) that

$$g'(z) = \frac{1}{\sqrt{\cosh 2g(z)}}$$

we directly compute $\tilde{H}'$ to find that

$$\tilde{H}'(z) = \frac{2\cosh 2g(z) - \tanh 2g(z)\sinh 2g(z)}{\cosh 2g(z)} - 1 = 1 - \tanh^2 2g(z).$$

It is now standard that $1 - \tanh^2 2g(z) = \operatorname{sech}^2 2g(z)$.

The global Lipschitz continuity of $\tilde{H}$ follows from the mean value theorem and the fact that $\tilde{H}'(z) \leq 1$ for every $z \in \mathbb{R}$.

iii) For this proof, we are more careful in bounding the derivative of $\tilde{H}$. Let $\delta > 0$ and suppose $x, y \in \mathbb{R} \setminus B_\delta(0)$ with $\operatorname{sgn}(x) = \operatorname{sgn}(y)$. Then $|x| > \delta$ and $|y| > \delta$. Without loss of generality, assume $x < y$. The mean value theorem implies that there exists $c_{xy} \in (x, y)$ such that, for every $s > 0$,

$$|\tilde{H}(x/\sqrt{s}) - \tilde{H}(y/\sqrt{s})| = \frac{1}{s}|\tilde{H}'(c_{xy}/\sqrt{s})||x - y|.$$

Since $\operatorname{sgn}(x) = \operatorname{sgn}(y)$ and $|x| > \delta$, $|y| > \delta$ it follows that $c_{xy} \neq 0$. Using Proposition A.0.1 we have that, for every $z \in \mathbb{R}$,

$$|\tilde{H}'(z)| = \frac{1}{\left[(|z| - c_1)^2 + \hat{R}(z)\right]^2}$$

where, for large z,

$$\hat{R}(z) \leq \mathcal{O}(z^{-2}).$$

So as $-c_1 > 0$, for $|z| > 0$,

$$|\tilde{H}'(z)| \leq \frac{1}{|z|^4}\left(\frac{1}{1 + \bar{R}(z)}\right)^2$$

with

$$\bar{R}(z) = \frac{\hat{R}(z)}{(|z| - c_1)^2}$$

and thus for large z we have $\bar{R}(z) \leq \mathcal{O}(z^{-4})$. However, since

$$\left(\frac{1}{1 + \bar{R}(z)}\right)^2 \longrightarrow 1$$

as $|z| \to \infty$ and, by Proposition A.0.1,

$$\left(\frac{1}{1 + \bar{R}(z)}\right)^2 = \frac{(|z| - c_1)^4}{\cosh^2 2g(z)} \leq (|z| - c_1)^4$$

which is bounded for bounded z, it follows that there exists $M > 0$ such that, for every $z \in \mathbb{R}$,

$$\left(\frac{1}{1 + \bar{R}(z)}\right)^2 \leq M.$$

Hence, for every $|z| > 0$ we have

$$|\tilde{H}'(z)| \leq \frac{M}{|z|^4}.$$

Thus, taking $z = c_{xy}/\sqrt{s}$ and observing that $|c_{xy}| > \delta$, one has

$$|\tilde{H}'(c_{xy}/\sqrt{s})| \leq \frac{M}{|c_{xy}|^4}s^2 \leq s^2 C_\delta$$

and so the claim follows.

$\square$

Appendix B

Transformation of the Self–Similar Critical Regime for Problem I

Consider (4.2.1) and let $v(y,\tau) = \varphi(y)w(y,\tau)$. Then, a direct computation reveals that

$$\begin{aligned} \frac{\partial^2 v}{\partial y^2} &= \varphi \frac{\partial^2 w}{\partial y^2} + 2\varphi' \frac{\partial w}{\partial y} + \varphi'' w, \\ -\frac{1}{2} H(y) \frac{\partial v}{\partial y} &= -\frac{1}{2} H(y) \varphi \frac{\partial w}{\partial y} - \frac{1}{2} H(y) \varphi' w, \\ -\frac{1}{2} \operatorname{sech}^2 2g(y) v &= -\frac{1}{2} \operatorname{sech}^2 2g(y) \varphi w. \end{aligned}$$

Thus, if v solves (4.2.1) then w solves

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial y} \left[2\frac{\varphi'}{\varphi} - \frac{1}{2} H(y) \right] + w \left[\frac{\varphi''}{\varphi} - \frac{1}{2} H(y) \frac{\varphi'}{\varphi} - \frac{1}{2} \operatorname{sech}^2 2g(y) \right].$$

We choose φ such that the coefficient of the drift term above is zero. That is, we choose φ such that, for any $y \in \mathbb{R}$,

$$\frac{\varphi'}{\varphi} - \frac{1}{4} H(y) = 0.$$

It is easy to see that

$$\varphi(y) := \exp\left(\frac{1}{4} \int_0^y H(s) \, \mathrm{d}s \right) \tag{B.0.1}$$

yields this. Calculating the coefficient of w we see that we have transformed (4.2.1) into the following PDE

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial y^2} + w\left[\frac{1}{4}H'(y) - \frac{1}{16}H^2(y) - \frac{1}{2}\operatorname{sech}^2 2g(y)\right]. \tag{B.0.2}$$

Since one has

$$H'(s) = 1 + \operatorname{sech}^2 2g(s) \qquad H^2(s) = \tanh 2g(s) \sinh 2g(s),$$

we define

$$-\tilde{A} := \frac{\partial^2}{\partial y^2} - \left[\frac{1}{16}\tanh 2g(y) \sinh 2g(y) - \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2 2g(y)\right] \tag{B.0.3}$$

and call $-\tilde{A}$ the "Schrödinger operator" so that

$$\frac{\partial w}{\partial \tau} = -\tilde{A}w$$

is referred to as the "Schrödinger equation". We define the potential associated to the equation as

$$\mathcal{V}(y) := \frac{1}{16}\tanh 2g(y) \sinh 2g(y) - \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2 2g(y). \tag{B.0.4}$$

Appendix C

Analysis of The Eigenfunctions of the Schrödinger Operator for Problem I

Recall $\mathcal{V}$ from (B.0.4). In order to analyse the asymptotics of the eigenfunctions of $-\bar{A}$ we must analyse the asymptotics of the eigenfunctions of $\tilde{A}$ defined above. The following Lemma is of use.

Lemma C.0.3. *For every $y \in \mathbb{R}$ we have that*

$$\mathcal{V}(y) = \frac{1}{16}(|y| + \kappa)^2 - \frac{1}{4} + R(y)$$

where, for, large enough y,

$$|R(y)| \leq Cy^{-2}.$$

Proof. Set

$$R(y) := \mathcal{V}(y) - \left[\frac{1}{16}(|y| + \kappa)^2 - \frac{1}{4}\right].$$

Using the result of Proposition A.0.1 with $c_1 = -\kappa$ and the remainder terms as given in the proof there, we have that for every $s > 0$

$$\begin{aligned}
\mathcal{V}(s) &= \frac{1}{16}\frac{\left((s-c_1)^2 + 2r_5(s) + r_5^2(s)(s-c_1)^{-2}\right)}{1 + r_6(s)(s-c_1)^{-2}} - \frac{1}{4} + \frac{1}{4}((s-c_1)^2 + r_6(s))^{-2} \\
&= \frac{1}{16}(s-c_1)^2 - \frac{1}{4} + r_7(s)
\end{aligned}$$

where, for large enough $s > 0$,

$$|r_7(s)| \leq K_7 s^{-2}.$$

By symmetry, the above holds for $s < 0$ and so evaluating at $s = y$ we see that

$$|R(y)| \leq K_7 y^{-2}$$

for large enough y. $\square$

In light of Lemma C.0.3 we have that

$$\tilde{A} = -\frac{\partial^2 w}{\partial y^2}(y) + \left(\frac{1}{16}(|y| - c_1)^2 - \frac{1}{4} + R(y)\right),$$

where $R(y) \leq \mathcal{O}(|y|^{-2})$.

We will now follow Berezin and Shubin [1991] pp. 69-85 to prove the following.

Theorem C.0.4. *The spectrum of $\tilde{A}$ is discrete and for every $\lambda \in \sigma(\tilde{A})$ with corresponding eigenfunction w_λ there exists $\bar{C}_0 = \bar{C}_0(y_0, \lambda) > 0$ such that the following asymptotic holds*

$$w_\lambda(y) = \bar{C}_0(|y| + \kappa)^{2\lambda} \exp\left(-\frac{1}{8}(|y| + \kappa)^2\right)(1 + o(1)) \quad \textit{as} \quad |y| \to +\infty.$$

Proof. The discreteness of the spectrum follows from Theorem XIII.67 of Reed and Simon [1978], since the potential is bounded below and tends to $+\infty$ as $|y| \to \infty$. We present the proof for the $y > 0$ case. The $y < 0$ case is analogous. Set $c_1 = -\kappa$. We have

$$\tilde{A} w_\lambda = \lambda w_\lambda$$

and so

$$-\frac{\partial^2 w_\lambda}{\partial y^2} + w_\lambda\left(\frac{1}{16}(|y| - c_1)^2 + R(y)\right) = \left(\lambda + \frac{1}{4}\right) w_\lambda.$$

One can easily check that $\lambda > 0$ and so $k^2 := \lambda + \frac{1}{4}$ has $k \in \mathbb{R}$. Define

$$V(y) := \frac{1}{16}(|y| - c_1)^2 + R(y)$$

and choose $y > 0$ large enough so that

$$\Phi(y) := \sqrt{V(y) - k^2}$$

has $\Phi(y) \in \mathbb{R}$. Call this lower bound $\tilde{y}_0$ and assume $\tilde{y}_0$ is large enough so that $y > \tilde{y}_0$ implies

$$|R(y)| \leq C|y|^{-2}.$$

Since $V(y) - k^2 \to \infty$ as $y \to \infty$, there exists some $y_1 > 0$ such that $y > y_1$ implies $V(y) - k^2 > 1$ and so

$$\int_{y_0}^{\infty} \sqrt{V(y) - k^2}\, \mathrm{d}y > \infty.$$

Thus

$$\int_{y_0}^{\infty} \Phi(y)\, \mathrm{d}y = +\infty,$$

where $y_0 = \max\{\tilde{y}_0, y_1\}$. In order to use Theorem 4.6 of Berezin and Shubin [1991], p.84, we need to show that

$$\int_{y_0}^{\infty} \left| \frac{\Phi''}{2\Phi^2} - \frac{3}{4}\frac{(\Phi')^2}{\Phi^3} \right| \mathrm{d}y < \infty.$$

One computes and sees that

$$\left| \frac{\Phi''}{2\Phi^2} - \frac{3}{4}\frac{(\Phi')^2}{\Phi^3} \right| \leq \frac{11}{16}\frac{|V'|^2}{|V - k^2|^{\frac{5}{2}}} + \frac{1}{4}\frac{|V''|}{|V - k^2|^{\frac{3}{2}}}. \tag{C.0.1}$$

However, computing the derivatives for V when V is given as the hyperbolic functions we see that

$$V'(y) = -\tanh 2g(y) \operatorname{sech}^{\frac{5}{2}} 2g(y) + \frac{1}{8}\frac{\sinh 2g(y)}{\sqrt{\cosh 2g(y)}}(1 + \operatorname{sech}^2 2g(y))$$

and so by Proposition A.0.1 we have that $V'(y) = \mathcal{O}(|y|^{-3})$ as $y \to \infty$ and $V'(0) = 0$. In a similar fashion, we compute V'' and see

$$\begin{aligned} V''(y) = &-2\operatorname{sech}^5 2g(y) + 5\tanh^2 2g(y) \operatorname{sech}^3 2g(y) - \frac{1}{2}\sinh^2 2g(y) \operatorname{sech}^4 2g(y) \\ &+ \frac{1}{8}\left(1 + \operatorname{sech}^2 2g(y)\right)^2. \end{aligned}$$

Again, by Proposition A.0.1 we conclude that $|V''(y)|/(V - k^2)^{3/2} = \mathcal{O}(y^{-3})$ as $y \to \infty$ and $V''(0) < \infty$. Thus we see that the right hand side of (C.0.1) is in $L^1(\mathbb{R})$ and appealing to Theorem 4.6 of Berezin and Shubin [1991] we conclude that

$$w_\lambda(y) = (V(y))^{-\frac{1}{4}} \exp\left(-\int_{y_0}^{y} \sqrt{V(s) - k^2}\, \mathrm{d}s \right)(1 + o(1)) \quad \text{as} \quad y \to +\infty. \tag{C.0.2}$$

The idea is now to Taylor expand the various expressions involving V for $y > y_0$ to give the result.

Firstly,

$$\sqrt{V(y)-k^2} = \sqrt{V(y)}\sqrt{1-\frac{k^2}{V(y)}} = \sqrt{V(y)}\left(1-\frac{k^2}{2V(y)}-\frac{k^4}{8V(y)^2}L(y)\right)$$

where

$$L(y) := (1-\xi_L(V(y)))^{-\frac{3}{2}}$$

and

$$\xi_L(V(y)) \in \left(0, \frac{k^2}{2V(y)}\right).$$

Thus simplifying this and Taylor expanding $(V(y))^{\pm\frac{1}{2}}$ we have, for every $y > y_0$,

$$\sqrt{V(y)-k^2} = \frac{1}{4}(y-c_1)+R_1(y)-\frac{k^2}{2}\left(\frac{4}{y-c_1}+R_2(y)\right)-\frac{k^4}{8V(y)^{\frac{3}{2}}}(1-\xi_L(V(y)))^{-\frac{3}{2}}$$

with, for, large enough y,

$$|R_1(y)| \leq C|y|^{-3}$$

and

$$|R_2(y)| \leq C|y|^{-5}.$$

Using the same technique we Taylor expand $(V(y))^{-\frac{1}{4}}$ to see, for $y > y_0$,

$$(V(y))^{-\frac{1}{4}} = \frac{2}{\sqrt{y-c_1}} + R_3(y)$$

with, for, large enough y,

$$|R_3(y)| \leq C|y|^{-\frac{9}{2}}.$$

Substituting this result into (C.0.2) we conclude that, as $y \to \infty$,

$$\begin{aligned} w_\lambda(y) = {} & \frac{2}{\sqrt{y-c_1}} \exp\left(-\int_{y_0}^{y}\left(\frac{1}{4}(s-c_1)-2k^2(s-c_1)^{-1}\right)\,\mathrm{d}s\right) C_1(y)(1+o(1)) \\ & + R_3(y)\exp\left(-\int_{y_0}^{y}\left(\frac{1}{4}(s-c_1)-2k^2(s-c_1)^{-1}\right)\,\mathrm{d}s\right) C_1(y)(1+o(1)), \end{aligned}$$

where

$$C_1(y) = \exp\left(-\int_{y_0}^{y}\left(R_1(s)-\frac{k^2}{2}R_2(s)-\frac{1}{8}(V(s))^{-\frac{3}{2}}(1-\xi_L(V(s)))^{-\frac{3}{2}}\right)\,\mathrm{d}s\right).$$

By definition of the R_i we have that

$$\lim_{y\to\infty} C_1(y) = \ell_0$$

exists with $\ell_0 \in (0,\infty)$.

Now

$$\begin{aligned}
&\exp\left(-\int_{y_0}^{y}\left(\frac{1}{4}(s-c_1) - 2k^2(s-c_1)^{-1}\right)\,\mathrm{d}s\right)\\
&= \tilde{C}_0(y_0,\lambda)\exp\left(-\frac{1}{8}(y-c_1)^2 + \log(y-c_1)^{2k^2}\right)\\
&= \tilde{C}_0(y_0,\lambda)(y-c_1)^{2k^2}\exp\left(-\frac{1}{8}(y-c_1)^2\right).
\end{aligned}$$

Substituting this in we have as $y\to\infty$

$$\begin{aligned}
w_\lambda(y) = 2C_1(y)\tilde{C}_0(y-c_1)^{2k^2-\frac{1}{2}}\exp\left(-\frac{1}{8}(y-c_1)^2\right)(1+o(1))+\\
+\tilde{C}_0R_3(y)C_1(y)(y-c_1)^{2k^2}\exp\left(-\frac{1}{8}(y-c_1)^2\right)(1+o(1)).
\end{aligned}$$

The result follows by recalling that $k^2 = \lambda + \frac{1}{4}$ and so $2k^2 = 2\lambda + \frac{1}{2}$. We now bound C_1 above and drop the last term as the estimation is equivalent and no better than with the term present. $\square$

Appendix D

Upper Bound on the Exponent in the power law in the Critical Case for Problem I

Consider Theorem 4.2.6. We wish to ascertain an upper bound for the exponent, $2\mu_1$, in Theorem 4.2.6.

Theorem D.0.5. *Consider Theorem 4.2.6. Then*

$$2\mu_1 \leq 1.$$

Proof. We have that μ_1 is the minimal eigenvalue of the time-homogeneous operator, $\bar{A}$, in (4.2.2) and so by the ground-state transformation of Section B, μ_1 is also the minimal eigenvalue of $\tilde{A}$, (B.0.3), written below for convenience as

$$\tilde{A} = -\frac{\partial^2}{\partial y^2} + \mathcal{V}(y)$$

where

$$\mathcal{V}(y) = \frac{1}{16}\tanh 2g(y) \sinh 2g(y) - \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2 2g(y).$$

Observe that $\tilde{A}$ is self-adjoint on $L^2(\mathbb{R})$ and since

$$\mathcal{V}(y) = \frac{\sinh^2 2g(y)}{16\cosh 2g(y)} - \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2 2g(y) \geq -\frac{1}{4}$$

for every $y \in \mathbb{R}$, we have that $\tilde{A}$ is semi-bounded. I.e, for every $\varphi \in D(\tilde{A})$,

$$\langle \tilde{A}\varphi, \varphi\rangle = \langle \varphi', \varphi'\rangle + \langle \mathcal{V}\varphi, \varphi\rangle \geq \int_{\mathbb{R}} \mathcal{V}(y)|\varphi(y)|^2 \,\mathrm{d}y \geq -\frac{1}{4}\int_{\mathbb{R}} |\varphi(y)|^2 \,\mathrm{d}y = -\frac{1}{4}\|\varphi\|_{L^2}^2.$$

We may appeal to Theorem XIII.3 of Reed and Simon [1978] p.82 to conclude that

$$\mu_1 \leq \langle \tilde{A}\psi, \psi\rangle$$

where $\psi \in \mathcal{D}(\tilde{A})$ with $\|\psi\|_{L^2}^2 = 1$. In order to make this upper bound sharp, we should take ψ related to the eigenfunction of ψ. Inspired by the asymptotic results of Theorem C.0.4, take

$$\psi(y) := \frac{1}{Z}\exp\left(-\gamma\int_0^y H(s)\,\mathrm{d}s\right),$$

where $\gamma > 0$ is to chosen, H is defined as

$$H(s) = \frac{\sinh 2g(s)}{\sqrt{\cosh 2g(s)}}$$

and Z is the normalisation constant given by

$$Z := \left(\int_{\mathbb{R}} \exp\left(-2\gamma\int_0^y H(s)\,\mathrm{d}s\right)\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

Notice that by definition of H and Proposition A.0.1 we see that $\psi \in L^2(\mathbb{R})$. Elementary calculations reveal that

$$\tilde{A}\psi = \left(\gamma H'(y) - \gamma^2 H^2(y) + \mathcal{V}(y)\right)\psi$$

with

$$H'(y) = 2 - \tanh^2 2g(y).$$

Simplifying, we see that

$$\tilde{A}\psi = \left(2\gamma - (\gamma + \frac{1}{4})\tanh^2 2g(y) + \left(\frac{1}{16} - \gamma^2\right)\frac{\sinh^2 2g(y)}{\cosh 2g(y)}\right)\psi.$$

One is tempted to take $\gamma = -\frac{1}{4}$; the resulting ψ fails to be in $L^2(\mathbb{R})$. Thus, take $\gamma = \frac{1}{4}$ so that

$$\tilde{A}\psi = \frac{1}{2}\operatorname{sech}^2 2g(y)\psi$$

and so
$$\langle \tilde{A}\psi, \psi \rangle = \frac{1}{2Z^2} \int_{\mathbb{R}} \operatorname{sech}^2 2g(y) \exp\left(-\frac{1}{2} \int_0^y H(s)\,\mathrm{d}s \right)\,\mathrm{d}y.$$

However, $\operatorname{sech}^2 2g(y) \leq 1$ for every $y \in \mathbb{R}$ and
$$Z^2 = \int_{\mathbb{R}} \exp\left(-\frac{1}{2} \int_0^y H(s)\,\mathrm{d}s \right)\,\mathrm{d}y.$$

Hence
$$\langle \tilde{A}\psi, \psi \rangle \leq \frac{1}{2} \frac{\int_{\mathbb{R}} \exp\left(-\frac{1}{2} \int_0^y H(s)\,\mathrm{d}s \right)\,\mathrm{d}y}{\int_{\mathbb{R}} \exp\left(-\frac{1}{2} \int_0^y H(s)\,\mathrm{d}s \right)\,\mathrm{d}y} = \frac{1}{2}.$$

Thus
$$\mu_1 \leq \langle \tilde{A}\psi, \psi \rangle \leq \frac{1}{2}.$$

□

Bibliography

S. Alinhac. Problèmes de Cauchy pour des opérateurs singuliers. *Bull. Soc. Math. France*, 102:289–315, 1974.

C. Baiocchi. Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert. *Ann. Mat. Pura Appl. (4)*, 76:233–304, 1967.

C. Baiocchi and M. S. Baouendi. Singular evolution equations. *J. Functional Analysis*, 25(2):103–120, 1977.

F.A. Berezin and M.A. Shubin. *The Schrödinger Equation*, volume 66. Springer, 1991.

M. L. Bernardi. Singular or degenerate equations of Schroedinger type. *Boll. Un. Mat. Ital. B (5)*, 18(3):895–920, 1981.

R. Carmona. Regularity properties of Schrödinger and Dirichlet semigroups. *J. Funct. Anal.*, 33(3):259–296, 1979.

G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 45. Cambridge University Press, 1992.

E.B. Davies. Pointwise bounds on the space and time derivatives of heat kernels. *J. Operator Theory*, 21(2):367–378, 1989.

K. Deckelnick, G. Dziuk, and C.M. Elliott. Computation of geometric partial differential equations and mean curvature flow. *Acta Numerica*, 14:139–232, 2005.

G. Dore and D. Guidetti. On a singular nonautonomous equation in Banach spaces. In *Differential equations in Banach spaces (Bologna, 1985)*, volume 1223 of *Lecture Notes in Math.*, pages 92–101. Springer, Berlin, 1986.

G. Dore and A. Venni. On a singular evolution equation in Banach spaces. *J. Funct. Anal.*, 64(2):227–250, 1985.

G. Dziuk and C.M. Elliott. Finite elements on evolving surfaces. *IMA journal of numerical analysis*, 27(2):262, 2007.

S. D. Eidelman, S. Kamin, and A. F. Tedeev. On stabilization of solutions of the Cauchy problem for linear degenerate parabolic equations. *Adv. Differential Equations*, 14(7-8):621–641, 2009.

C. Eilks and C. M. Elliott. Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method. *Journal of Computational Physics*, 227(23):9727–9741, 2008.

K.J. Engel and R. Nagel. *A short course on operator semigroups*. Springer Verlag, 2006.

L.C. Evans and J. Spruck. Motion of level sets by mean curvature i. *J. Diff. Geom*, 33(3):635–681, 1991.

L.C. Evans and J. Spruck. Motion of level sets by mean curvature ii. *Trans. American Math. Soc*, 330(1), 1992a.

L.C. Evans and J. Spruck. Motion of level sets by mean curvature iii. *Journal of Geometric Analysis*, 2(2):121–150, 1992b.

L.C. Evans and J. Spruck. Motion of level sets by mean curvature iv. *Journal of Geometric Analysis*, 5(1):77–114, 1995.

O. Faugeras and R. Keriven. *Variational principles, surface evolution, PDE's, level set methods and the stereo problem*. IEEE, 2002.

A. Favini. Sulle equazioni differenziali astratte degeneri. *Rend. Sem. Mat. Univ. Padova*, 52:243–263 (1975), 1974.

A. Favini. Degenerate and singular evolution equations in Banach space. *Math. Ann.*, 273(1):17–44, 1985.

A. Favini, A. Lorenzi, H. Tanabe, and A. Yagi. An L^p-approach to singular linear parabolic equations in bounded domains. *Osaka J. Math.*, 42(2):385–406, 2005.

A. Favini, A. Lorenzi, H. Tanabe, and A. Yagi. An L^p-approach to singular linear parabolic equations with lower order terms. *Discrete Contin. Dyn. Syst.*, 22(4): 989–1008, 2008.

A. Friedman. *Partial differential equations of parabolic type. 1964*. Englewood Cliffs, NJ, 1964.

A. Friedman. *Stochastic differential equations and applications*, volume 1. Academic press New York, 1975.

A. Friedman and Z. Schuss. Degenerate evolution equations in Hilbert space. *Trans. Amer. Math. Soc.*, 161:401–427, 1971.

D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 2001.

V. V. Gorodetskii and O. V. Martynyuk. The Cauchy problem for evolution equations with a Bessel operator of infinite order. I. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (6):3–15, 2010a.

V. V. Gorodetskii and O. V. Martynyuk. The Cauchy problem for evolution equations with a Bessel operator of infinite order. II. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (7):31–42, 2010b.

V. V. Gorodetskii and I. S. Tupkalo. The Cauchy problem for evolution equations with infinite-order Bessel operators. *Differ. Uravn.*, 47(3):335–348, 2011.

L. Grafakos. *Modern Fourier Analysis*, volume 250. Springer New York, 2009.

T.H. Gronwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *The Annals of Mathematics*, 20(4):292–296, 1919.

D. Guidetti. Linear singular parabolic equations in banach spaces. *Mathematische Zeitschrift*, 195(4):487–504, 1987.

M. Hairer. On Malliavin's proof of Hörmander's theorem. *Bulletin des sciences mathematiques*, 135(6):650–666, 2011.

L. Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119(1):147–171, 1967.

I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Verlag, 1991.

N. Kutev, A. Oliaro, and P. Popivanov. On the solvability of the characteristic Dirichlet problem for linear degenerate parabolic equations. *Proc. Amer. Math. Soc.*, 138(1):153–163, 2010.

J.M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Springer, Berlin, 1997.

J. E. Lewis and C. Parenti. Abstract singular parabolic equations. *Comm. Partial Differential Equations*, 7(3):279–324, 1982.

J.-L. Lions and P. A. Raviart. Remarques sur la résolution, exacte et approchée, d'équations d'évolution paraboliques à coefficients non bornés. *Calcolo*, 4:221–234, 1967.

F. Luterotti. On some degenerate evolution variational inequalities. *Ann. Mat. Pura Appl. (4)*, 165:239–260, 1993.

P. Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 195–263, New York, 1978. Wiley.

M. P. Neilson, J. A. Mackenzie, S. D. Webb, and R. H. Insall. Modeling cell movement and chemotaxis using pseudopod-based feedback. *SIAM J. Sci. Comput.*, 33(3):1035–1057, 2011.

B.K. Øksendal. *Stochastic differential equations: an introduction with applications.* Springer Verlag, 2003.

C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations.* Number 1905. Springer, 2007.

M. Reed and B. Simon. *IV: Analysis of Operators, Volume 4 (Methods of Modern Mathematical Physics).* Academic Press, 1978.

Z. Schuss. Regularity theorems for solutions of a degenerate evolution equation. *Arch. Rational. Mech. Anal.*, 46:200–211, 1972.

M.R. Scott. Stochastic partial differential equations on an evolving surface. Master's thesis, The University of Warwick, 2011.

G. R Sell and Y. You. *Dynamics of evolutionary equations*, volume 143. Springer, 2002.

P. E. Sobolevski. Degenerate parabolic operators. *Dokl. Akad. Nauk SSSR*, 196: 302–304, 1971.