The sharp interface limit of the Stochastic Allen-Cahn equation

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy in Mathematics

Mathematics Institute

June 2014

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ACKNOWLEDGEMENTS
My PhD was supported by a scholarship from the EPSRC, for which I am very grateful.

First and foremost I would like to thank my supervisor Martin Hairer for being a great supervisor. His deep mathematical intuition, never-ending enthusiasm for the subject, combined with an ability to explain matters to those of less ingenious minds have been very inspiring. There are many academics, both at Warwick and elsewhere, from whom my research has benefited; this includes, but is not limited to Dirk Blömker, Maria Westdickenberg, Hendrik Weber, Roger Tribe and Tadahisa Funaki. Many other researchers of Warwick have, if not directly, at least indirectly helped by creating a great research environment with many interesting seminar talks, conferences and colloquiums. The many other PhD students whose presence made life in the quiet English Midlands much more enjoyable shall not be forgotten either. Finally, my friends and family were a great support over the last three and a half years, despite being located all over the world.

DECLARATIONS

I declare that, to the best of my knowledge, all material contained in this thesis is my own original work, unless otherwise stated, cited or commonly known.

I have not published any material in this thesis before or submitted it for a degree at another university.

ABSTRACT

We study the Allen-Cahn equation

$$u_t = \epsilon^2 u_{xx} + f(u) + \epsilon^\gamma \dot{W}$$

with an additive noise term $\epsilon^\gamma \dot{W}$ for small $\epsilon > 0$, and in particular the limit $\epsilon \to 0$. This is a reaction-diffusion equation, where $f(\cdot)$ is the negative derivative of a symmetric double-well potential.

We study this equation in the interval $(0, 1)$ with symmetric boundary conditions, and relatively general initial conditions, for $\dot{W}$ we take space-time white noise.

Brassesco et al., Funaki and other authors showed (with different boundary conditions) that if we can project the solution of the equation to an energy-optimal deterministic solution with just one zero, then in the sharp interface limit $\epsilon \to 0$ of the solution appropriately rescaled in time is a standard Brownian motion.

In this work, we extend these results to a much more general case: We start with fairly general initial conditions, show that after some time we are able to project the solution onto energy-optimal deterministic solutions with finitely many zeroes, after which we derive a semimartingale representation for the interfaces; this representation holds until two interfaces get close to each other and annihilate. In the sharp interface limit $\epsilon \to 0$, the appropriately time-rescaled interface position of the solution converges weakly to annihilating independent standard Brownian motions. We also derive an analogous result for smooth noise with trace-class covariance operator, in this case the phenomenon happens on a different timescale than for space-time white noise.

1. INTRODUCTION AND OVERVIEW OF RESULTS

The Allen-Cahn equation was first studied in [AC79] as a microscopic diffusional theory for the motion of a curved antiphase boundary. It is one of the most simple ways to model phase separation, and is commonly used as a simplification of other models of phase separation, for example of the Cahn-Hilliard equation, which essentially behaves in the same way, but with energy conservation. In this work we study the effect of small noise on this equation, and derive some new results on the dynamics in one space-dimension, in particular of a scaling limit known as the sharp interface limit. Heuristically speaking, the classical way of deriving the Allen-Cahn equation is to derive the L^2-gradient flow of the Ginzburg-Landau-Wilson free energy functional.

In the deterministic equation for fixed $\epsilon > 0$, this is a parameter of the equation that loosely speaking determines the width of the interfaces; the solution will after an initial "relaxation time" have a modulus of approximately 1 in most of the range; in one space dimension this leaves a discrete set of interfaces, where each interface moves towards its nearest neighbour at an exponentially slow speed until they are sufficiently close to annihilate within order $|\log \epsilon|$ time. In $N \geq 2$ space dimensions the interfaces form a manifold of dimension $N-1$ and in the sharp interface limit perform mean curvature flow. A reference in one dimension is [Che04], and in higher dimensions [Che92].

In this chapter we formulate the equation, define the coordinate system used to find a better description of it and give an overview of the results presented in this work.

1.1 Deterministic equation

For a physical system which attains values in $[-1, 1]$, but mostly close to ± 1 a natural choice is the following potential energy:

$$\int F\left(u\left(x\right)\right) dx$$

where $F(\cdot)$ is a symmetric double-well potential with minima at ± 1, a classical example would be $F(u) = \frac{1}{4}\left(u^2 - 1\right)^2$. This description of the physical phenomena is quite local and needs another term to smoothen out the otherwise quite rough transitions from ± 1 to ∓ 1 with a kinetic energy:

$$\int \frac{1}{2}\epsilon^2 \left|\nabla u\right|^2 dx,$$

the energy functional thus becomes

$$E(u) = \int \frac{1}{2}\epsilon^2 |\nabla u|^2 dx + \int F(u(x))\, dx.$$

In order to now arrive at an equation for u, the assumption is made that the system is quickly drawn towards its energy minimisers, i.e.

$$\frac{d}{dt}u = -\frac{\partial}{\partial u}E(u) = -\frac{\partial}{\partial u}[\int \frac{1}{2}\epsilon^2 |\nabla u|^2 dx + \int F(u(x))dx]$$
$$= -\frac{\partial}{\partial u}\left[-\int \frac{1}{2}\epsilon^2 u\Delta u dx + \int F(u(x))\, dx\right] = \epsilon^2\Delta u - F'(u).$$

This yields the Allen-Cahn equation:

$$\partial_t u(x,t) = \epsilon^2 \Delta u(x,t) - F'(u(x,t)).$$

Alternatively, one may note that

$$\frac{\partial}{\partial t}E(u) = -\int \epsilon^2 \Delta u \cdot u_t dx + \int F'(u(x))\, u_t dx = -\int \left(\epsilon^2\Delta u - F'(u(x))\right) u_t dx.$$

Mathematically speaking, this is the L^2 gradient flow of the energy $E(u)$, since for all $t > 0$ we have the weak characterisation

$$\langle \partial_t u, u\rangle_{L^2} = -\langle E'(u), u\rangle_{L^2}.$$

While one could indeed consider the case $|u| > 1$, this is not meaningful to physicists who tend to be interested in systems with values in $[-1,+1]$, and indeed any bounded initial configuration u_0 will have the property $|u| \leq 1 + \mathcal{O}(\epsilon)$ after a time of order $|\log\epsilon|$. We will for this reason not restrict our solution to $[-1,+1]$ when introducing the noise.

As a remark, if one was to choose the H^{-1}-inner product rather than L^2 one would obtain the Cahn-Hilliard equation as the gradient flow of the energy, which was introduced at an earlier point in [CH58]:

$$\partial_t u = \Delta(\Delta u - F'(u)).$$

So far this discussion was valid in arbitrarily many space-dimensions. From now on we talk about the case of one space-dimension.

We now give an overview of the behaviour of the Allen-Cahn equation for small ϵ. The obvious first observation is that $u(x) = +1$ and $u(x) = -1$ minimise the energy. If we have a solution that needs to pass from -1 to $+1$ at some point and is fixed to these values at its boundaries we observe

$$E(u) = \int_0^1 \left(\frac{1}{2}\epsilon^2 (\partial_x u(x))^2 + F(u(x))\right) dx$$
$$= \int_0^1 \left(\frac{1}{2}\left(\epsilon\partial_x u(x) + \sqrt{2F(u(x))}\right)^2 - \epsilon\sqrt{2F(u(x))}\right)\partial_x u dx$$

$$\geq \int_{-1}^{1} \sqrt{2F(u)}du = S_\infty.$$

For $F(u) = \frac{1}{4}\left(u^2-1\right)^2$ we have $S_\infty = \frac{2\sqrt{2}}{3}$. One way to view S_∞ is as the minimum energy required for a transition from ± 1 to ∓ 1.

This is obtained exactly when

$$\epsilon \partial_x u(x) + \sqrt{2F(u(x))} = 0$$

or equivalently

$$\epsilon^2 \partial_{xx} u(x) - F'(u(x)) = 0,$$

i.e. it is a time-invariant solution of the Allen-Cahn equation.

For $F(u) = \frac{1}{4}\left(u^2-1\right)^2$ these solutions take the explicit form $u^h(x) = \tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)$ if the equation is posed on the real line:

$E\left(\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)$
$= \int_{-\infty}^{\infty} \left[\frac{1}{2}\epsilon^2 \left(\partial_x \tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^2 + \frac{1}{4}\left(\left(\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^2 - 1\right)^2\right] dx$
$= \int_{-\infty}^{\infty} \left[\frac{1}{2}\epsilon^2 \cdot \frac{1}{2\epsilon^2}\left(1 - \left(\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^2\right)^2 + \frac{1}{4}\left(\left(\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^2 - 1\right)^2\right] dx$
$= \int_{-\infty}^{\infty} \left(\frac{3}{4}\left(\left(\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^2 - 1\right)^2 dx = \int_{-\infty}^{\infty} \frac{3}{4}\left(sech\left(\frac{x-h}{\sqrt{2}\epsilon}\right)\right)^4\right) dx = S_\infty.$

As $E(u)$ is non-negative, this clearly shows that $\tanh\left(\frac{x-h}{\sqrt{2}\epsilon}\right)$ is an energy minimiser.

A key notion introduced in section 1.2 will be to "glue" these invariant solutions (or energy minimisers) together to obtain quasi-invariant solutions indexed by the position of their zeroes, onto which we can project the actual solution, given it is close enough. Except for a size $k\epsilon$ neighbourhood of h, u^h is in a neighbourhood of ± 1 that is exponentially small in ϵ, so most energy is concentrated around such an interface.

Using this idea of quasi-invariant solutions, we can at a hand-waving level explain the behaviour of the equation as initially being quickly drawn to a neighbourhood of a quasi-invariant solution (with N zeroes) exponentially small in ϵ, the energy will then be almost constant at about NS_∞ for a time of order $e^{\frac{-d}{\epsilon}}$, where d is the minimum distance of two neighbouring interfaces. Then we quickly see the annihilation of two interfaces to obtain the new energy of approximately $(N-2)S_\infty$, and see the same dynamics again until we finally reach either one or no interfaces. A very detailed account of these dynamics can be found in [Che04]. For the motion of well-separated zeroes the ideas and notations in [CP89] have also been very influential in this present work. In [OR07] similar deterministic results are obtained using energy techniques.

What happens in the Cahn-Hilliard equation in one space-dimension (with Neumann boundary conditions) is related, but not completely identical and more complicated, as there is "mass conservation", i.e. the integral of the solution of the interval stays constant: While we do also see exponentially slow motion for well separated interfaces, the system of ordinary differential equation for the interface positions is (up to small errors) essentially the same, except that the crucial terms gets divided by the distance to the next interfaces - this is barely noticeable for well-separated interfaces, but leads to a notably different behaviour at short distances. For a detailed reference on the Cahn-Hilliard equation, see, e.g., [BX94] and [BX95].

1.2 Stochastic equation

We formulate the Stochastic Allen-Cahn equation for $x \in (0,1)$, $t > 0$ as

$$u_t(x,t) = \epsilon^2 u_{xx}(x,t) + f(u(x,t)) + \epsilon^\gamma \dot{W} \tag{1.1}$$

with periodic boundary conditions

$$u(x,t) = u(x+1,t) \text{ for } t \geq 0.$$

$\dot{W}$ is space-time white noise defined as the derivative of a cylindrical Wiener process W. We have $\gamma > 2$.

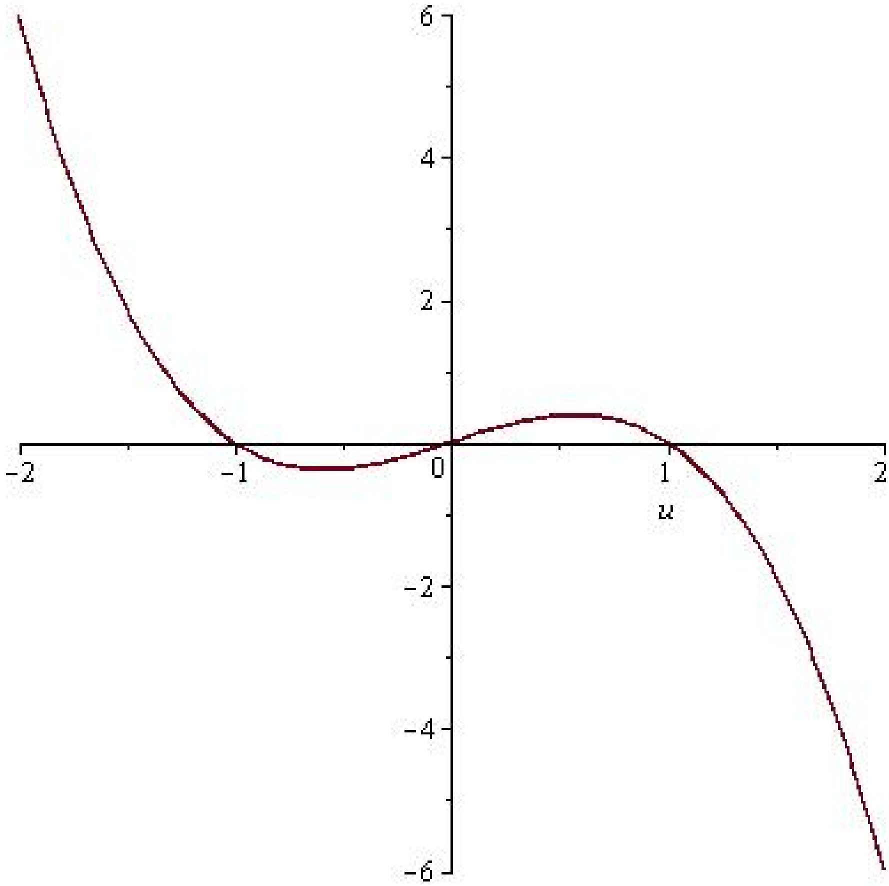
$f \in C^2(\mathbb{R})$ is a function such that
(a) $f(\pm 1) = f(0) = 0$ are the only zeros
(b) $f(x) = -f(-x)$
(c) $\exists c_1, c_2, p > 0$ s.t. $|f(x)| \leq c_1(1 + |x|^p)$ and $f'(x) \leq c_2 \ \forall x \in \mathbb{R}$
(d) $f'(1) = f'(-1) < 0$
(e) $f'(x) \leq f'(0) \ \forall x \in \mathbb{R}$

Condition (e) is not needed in principle, but in place to make the sketch of the argument of phase separation in chapter 2 less involved. Condition (b), while not necessary for fixed finite ϵ, is needed to take the sharp interface limit (otherwise the deterministic part of the drift does not converge to 0); this is indeed the typical definition of the reaction term in the (stochastic) Allen-Cahn equation found in the literature and identical to the negative derivative of a symmetric double-well potential.

Later on we will rescale time to $t' = S_\infty^{-1} \epsilon^{2\gamma+1} t$ (for S_∞ as defined before) to take the limit.

The associated potential is $-F(x)$, s.t. $\frac{dF(x)}{dx} = -f(x)$ and $F(-1) = F(1) = 0$.

A typical example is $f(u) = u - u^3$:



Remark:

The reason that we require $\gamma > 2$ is that in order to take the sharp interface limit in chapter 3, we need the drift of the stochastic differential equation converging to 0. Given that we only have control over the L^2-norm and not the L^∞-norm of v (the quasi-orthogonal distance to the slow manifold - see next section), $\|v\|$ may not be greater or equal than $\epsilon^{3/2}$. A qualitative way to think of this is that in most of chapter 3, $\|v\|$ is roughly the same size as the L^2-norm of the linearisation of (1.1) starting from a profile constant at 0 on timescales polynomial in ϵ^{-1}. This however is exactly the case when $\gamma > 2$.

The results should also hold in the more general case $\gamma > 3/2$ since the drift converges to 0 if $\|v\|_\infty$ is not greater or equal than $\epsilon^{3/2}$, proving the results in such generality however is beyond the scope of this work, as it would require control over the L^∞-norm of v throughout the motion studied in chapter 3.

1.3 The approximate slow manifold

We now introduce the natural coordinate system for studying this problem. We follow the approach of [CP89] quite closely, which at its core is about joining invariant solutions together to form quasi-invariant solutions, which (together with the approximately orthogonal distance) we will use as the coordinate system; given a sufficiently small distance to the manifold of these quasi-invariant solutions we can indeed map onto the coordinate system smoothly. This is very elegant in our case, since despite our equation being perturbed by infinite-dimensional noise, we have a smooth coordinate system.

We consider $a > 0$ such that $f'(u) < 0 \, \forall u$ s.t. $|u \pm 1| < a$. Then, given an $\epsilon > 0$, for l sufficiently large a unique solution $\phi = \phi(x, l, \pm 1)$ exists for the following stationary Dirichlet problem

$$\epsilon^2 \phi_{xx} - f(\phi) = 0 \tag{1.2}$$

$$\phi = 0 \text{ at } x = \pm l/2,$$

which satisfies:

$$(a)\ \phi(x, l, +1) > 0 \text{ for } |x| < l/2 \text{ and } |\phi(0) - 1| < a$$

$$(b)\ \phi(x, l, -1) < 0 \text{ for } |x| < l/2 \text{ and } |\phi(0) + 1| < a$$

A proof can be found e.g. in [CP89]. Schematically, these solutions have the following shape on the relevant interval:

However, ϕ is a periodic function on the real line with period $2l$, we have $\phi(x) = -\phi(x + l)$.

Consider the slowly evolving solutions to the PDE (no noise) case with N well separated layers and define the set of admissible positions h of interfaces as Ω_l.

Denote by S a circle of circumference 1 and with h_1 the first interface of an up-front (change from -1 to $+1$) in the anti-clockwise direction from 0. Then

$$h \in \Omega_l := S^N \text{ and } l > h_j - h_{j-1} \tag{1.3}$$

for $1 \leq j \leq N+1$.

While the vector h includes the coordinates of $h_1, ..., h_N$ we additionally define the recursion $h_{1+N} = h_1 + 1 \; \forall N \in \mathbb{Z}$ so that the profile is indeed periodic.

Any statements made from now on involving h are ommitting $\mod 1$ in the coordinates; this is to make the statements easier to read.

These interfaces evolve in time and are expected to have a width of order ϵ. Therefore the distance between interfaces should be bounded below by $\rho\epsilon$ for some $\rho > \rho^{**} > \rho^*$ where $\rho^* = \sup_l \left\{\rho : \phi(\rho\epsilon - 1/2, l, +1) = \frac{3}{4}\right\}$ (cf. [Che04] Theorem 7.1 for the analogous on the real line). The constant ρ^{**} must be large enough for Claim 3.7 to hold true.

Denote the midpoints between the interfaces by $m_j := \frac{h_{j-1}+h_j}{2}$ for $j = 1, ..., N+1$.

Furthermore, we define the function u^h on each interval $I_j := [m_j, m_{j+1}]$ as

$$u^h(x) = \left[1 - \chi\left(\frac{x - h_j}{\epsilon}\right)\right] \cdot \phi\left(x - m_j, h_j - h_{j-1}, (-1)^j\right)$$

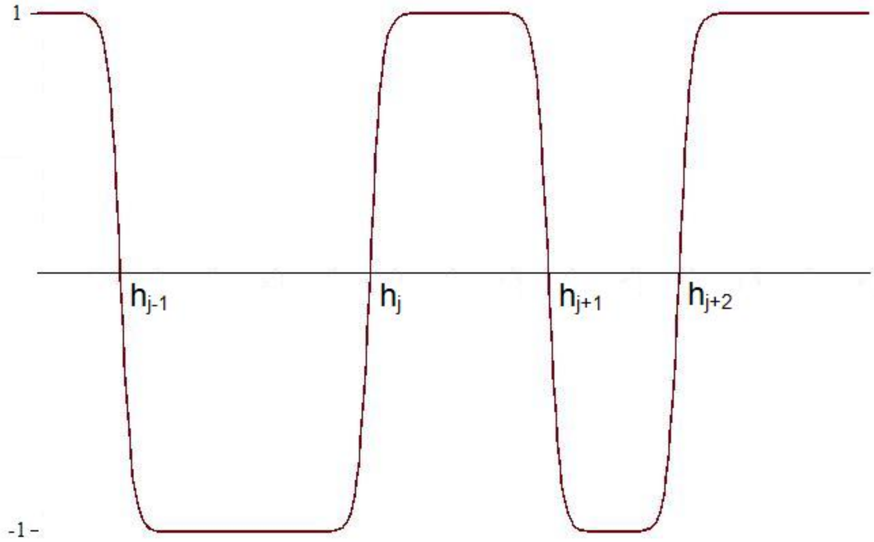
$$+\chi\left(\frac{x - h_j}{\epsilon}\right) \cdot \phi\left(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1}\right)$$

on $I_j := [m_j, m_{j+1}]$,

where $\chi : \mathbb{R} \to [0, 1]$ is a C^∞ cut-off function s.t. $\chi = 1$ on $[1, \infty)$ and $\chi = 0$ on $(-\infty, -1]$; we denote the first up-front in clockwise direction from "twelve o'clock" as h_1.

Due to the appropriate use of indicator functions in the expression u^h is smooth, which would not be the case if one was to position invariant solutions of different interval lengths next to each other.

This function $u^h(x)$ schematically has the following shape:



Definition 1.1. *The approximate slow manifold of the solution to the stochastic Allen-Cahn equation is defined as*

$$\mathcal{M}_\epsilon := \left\{ u^h : h \in \Omega_{\rho^{**}\epsilon} \right\}.$$

We use the following coordinate system around $\mathcal{M} : u \to (h, v)$ *, where we write* u *as the sum*

$$u(\cdot) := u^h(\cdot) + v(\cdot) \tag{1.4}$$

with $\langle v, \tau_j^h \rangle = 0$ *for* $j = 1, ..., N$*, where we define* τ_j^h *as*

$$\tau_j^h := -\gamma^j(x)\, u_x^h(x),$$

with

$$\gamma^j(x) = \chi\left((x - m_j - \epsilon)/\epsilon\right)\left[1 - \chi\left((x - m_{j+1} + \epsilon)/\epsilon\right)\right].$$

We furthermore use the notation

$$u_j^h(x) := \frac{\partial u^h}{\partial h_j}, u_{jk}^h = \frac{\partial^2 u^h}{\partial h_j \partial h_k}, u_{jkl}^h = \frac{\partial^3 u^h}{\partial h_j \partial h_k \partial h_l}.$$

The use of τ_j^h in the definition in the same spirit as in [CP89], and in particular, had we used u_j^h (which is approximately the same), we had to prove the existence of a mapping ourselves rather than use the theorem of [CP89], and this proof would also have been more involved. This is not an orthogonal projection onto the manifold, but a "quasi-orthogonal" one, i.e. up to small error terms the two projections are the same.

If the L^2-norm between u and u^h is bounded above by $c\epsilon$ for some small $\epsilon > 0$, there exists a smooth mapping from the set of such u to the approximate slow manifold.

We call this mapping

$$\mathcal{H} : \Gamma'_\epsilon \to \Omega_{\rho\epsilon}.$$

Γ'_ϵ is defined to be

$$\Big\{ u : \text{we can denote } u = u^h + v \text{ with } h \in \Omega_{\rho^{**}\epsilon}, \\ \|v\| \le C_{map}\epsilon^{1/2}, \langle v, \tau_j^h \rangle = 0\ \forall j \Big\}$$

The existence of this mapping is proved in Claim 7.7.

Since $u^h(x)$ is pieced together of time-invariant solutions of the PDE problem, one can view this as a decomposition into a part $u^h(x)$, which is quasi-invariant under the deterministic flow, and a small approximately orthogonal component v.

Remark: The name "slow manifold" is used in analogy to the PDE case, in which the solution converges to an exponentially small neighbourhood of exactly this slow manifold, after which an exponentially slow motion occurs until two interfaces are sufficiently close to each other to annihilate.

Definition 1.2. *W is a cylindric Wiener process in the underlying Hilbert space $H = L^2(0,1)$ (cf. [DPZ92]),the covariance operator is the identity operator. For an orthonormal basis of H denoted $\{e_j(\cdot)\}_{j=1}^{\infty}$ and a sequence of independent standard Brownian motions $\{\beta_j(t)\}_{j=1}^{\infty}$ we can therefore denote*

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k.$$

For simplicity of presentation we also denote $dV = \epsilon^{\gamma} dW$.

Notation. From now on, $\|\cdot\|$ refers to the L^2 inner product in space.

Now we define the slow channel and extended slow channel; a solution will need to be in the extended slow channel so that we can denote its interfaces as a semimartingale and in the slow channel so that we can take the sharp interface limit.

Definition 1.3. *Denote*

$$\Gamma_{c,\epsilon} := \left\{ u^h + v \,:\, h \in \Omega_c, \|v\| \le \epsilon^{3/2+\delta} \right\}.$$

for some fixed $\delta > 0$ s.t. $\gamma - 2 > \delta > 0$.
We define $\xi_\epsilon = \frac{(2\gamma+\beta)\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ for some small $1 > \beta > 0$.
The slow channel is defined as $\Gamma_{\xi_\epsilon,\epsilon}$.

Definition 1.4. *The extended slow channel is defined as*

$$\Gamma'_\epsilon := \{\ u : \text{we can denote } u = u^h + v \text{ with } h \in \Omega_{\rho^{**}\epsilon}, \|v\| \le C_{map}\epsilon^{1/2}, \\ \left\langle v, \tau_j^h \right\rangle = 0\ \forall j\ \}.$$

1.4 Literature review

The first works of the stochastic equation can be found in the Physics community, i.e. [KO82], the results are not completely rigorous however. The first rigorous results can be found in [Fun95] and [BB98], Brassesco subsequently published further works with varying coauthors. These works concern the sharp interface limit of a solution with one interface. Subsequent works, i.e. [Web10], [OWW13] have considered the case of several interfaces (the latter on exponentially large intervals), however only the invariant measure. The invariant measure of the Allen-Cahn equation is in fact a Gibbs measure with respect to a Brownian motion of the appropriate boundary conditions, which is a nice further perspective from which to view the stochastic Allen-Cahn equation

One key idea is presented in [ABK12] to obtain the motion of several interfaces in the Cahn-Hilliard equation with noise of high spatial regularity, upon which chapter 3 in this work builds. An interesting recent preprint [Bar12] studies the transition from one constant profile ± 1 to ∓ 1 on exponential timescales, which can be viewed as a first step towards rigorously understanding the nucleation of interface pairs on exponential timescales.

Besides the previously mentioned phase separation (see also [Bra91]), there are other physical interpretations/applications to this equation, i.e. the behaviour of an elastic string in a viscous stochastic environment submitted to a potential, see [Fun83] - note that in this interpretation our space variable becomes a parameter of the string. There are also interpretations in quantum field theory, see for example [FJL82], [COP86]. Another interpretation would be that for certain initial conditions a suitable random function on the one-dimensional Glauber spin flip process converges to the solution of the stochastic Allen-Cahn equation, for a reference see [GLP98]. For existence and uniqueness there are a number of possible references, for example [Zab89], [DPZ92], [DPZ96] or [GP93], we will also present a simplified sketch of argument in 7.2.

1.5 Choice and motivation of noise

The most classical choice of noise is, taking applications in mind, typically white noise, for what reason this was chosen for the main result of this work. It is also intuitively easier to understand, as it can be viewed as the limit of a many-particle system with independent interactions.

However, if one was to consider problems in statistical mechanics with external fields, then the noise may have strong correlations to the extent that the covariance operator is trace class. For that reason chapter 6 considers the case of smooth noise. Interestingly, the time-scale on which we may take the sharp interface limit is different to the one for space-time white noise.

While there are examples in the literature where equations of this type were studied with more exotic forms of noise (e.g. Levy noise, fractional Brownian motion, or even fractional Brownian motion with jumps), not much is known of how this relates to applications. Thus, nothing in this direction was pursued further in this work.

The reason of studying the noise-perturbed equation in the first place is motivated by applications, some references are given in the previous section. In particular, in the classical example of phase separation, computations of dendrites showed that models with thermal noise give much better approximation to the real phenomenon than determinisic computations, cf. [NDG05].

1.6 Overview of results

If our solution initially satisfies the condition $u_0(x) \in X_\epsilon^{h,0}$ (cf. chapter 2 for the definition - essentially we are in an ϵ-neighbourhood of a differentiable function independent of ϵ), the solution $u(x,t)$ to the Stochastic Allen-Cahn Equation is in separated phases (i.e. except for $\mathcal{O}(\epsilon)$-neighbourhoods of the interfaces the modulus is bounded below by $1/2$) after an $\mathcal{O}(|\log\epsilon|)$ time, and after further $\mathcal{O}(|\log\epsilon|)$ time enters $\Gamma_{\xi_\epsilon,\epsilon}$, in which it follows the following semimartingale representation (the error bounds being in the L^∞ norm) until two interfaces are

ξ_ϵ near each other:

$$dh_k = o\left(\epsilon^{2\gamma+1}\right)dt + \epsilon^\gamma \left\langle \frac{\epsilon}{S_\infty} u_k^h + o\left(\epsilon^{1/2}\right), dW \right\rangle.$$

The bound on the drift requires the fact that $\|v\| \leq o\left(\epsilon^{3/2}\right)$. This is the reason why the results are only shown for $\gamma > 2$.

We have $S_\infty = \int_{-1}^{1} \sqrt{-2F(u)}du$, which for $f(u) = u - u^3$ is $S_\infty = \frac{2\sqrt{2}}{3}$. The order terms are in the L^∞-norm and in the sense that with a probability converging to 1 as $\epsilon \to 0$ they hold true.

These dynamics are with a high probability stable until two interfaces are ξ_ϵ near each other, which w.h.p. takes a time of order $\mathcal{O}\left(\epsilon^{-2\gamma-1}\right)$; then w.h.p. they annihilate each other in $\mathcal{O}\left(\epsilon^{-2\gamma+1-\kappa}\right)$ time (for some small $\kappa > 0$), after which we are back in the regime of the semimartingale.

After a timechange to the fast timescale $t' = S_\infty^{-1}\epsilon^{2\gamma+1}t$ we do obtain annihilating Brownian motions in the sharp interface limit $\epsilon \to 0$. The intuition of this result is that the only thing not converging to 0 is $\int \left\langle \frac{\sqrt{\epsilon}}{\sqrt{S_\infty}} u_k^h, dW \right\rangle$, which in the limit can be thought of as the square root of a Dirac Delta function centred at h_k integrated up against space-time white noise.

For a more detailed exposition of the results, the reader shall refer to chapter 2 for the process of entering $\Gamma_{\xi_\epsilon,\epsilon}$, chapter 3 for the motion in the slow channel, chapter 4 for the annihilation of interfaces and chapter 5 for the sharp interface limit.

In chapter 6 the analogous results for noise that is smooth in space may be found.

2. ATTRACTIVITY OF THE MANIFOLD

As outlined in the previous chapter, the solution of the stochastic Allen-Cahn equation needs to be sufficiently close to the slow manifold in order to be projected orthogonally onto it, thus allowing the problem to be solved in the new coordinate system.

In this chapter, we obtain the necessary results to show that if at $t = 0$, $u = u_0(\cdot)$ is inside an ϵ-neighbourhood of a periodic function in $C^1(0,1)$ independent of ϵ with finitely many $x \in (0,1)$ s.t. $u_0(x) = 0$, u will be in the slow channel $\Gamma_{\xi_\epsilon,\epsilon}$ after $\mathcal{O}(|\log \epsilon|)$ time with a probability converging to 1 as $\epsilon \to 0$. The idea behind the sketch of arguments is a simple perturbation argument: On timescales of order 1 we have very good control over the L^∞-norms of the stochastic Allen-Cahn equation, linearised at its stable points. Subtracting this linearisation from the actual stochastic Allen-Cahn equation yields the deterministic Allen-Cahn equation perturbed by the linearisation of the stochastic Allen-Cahn equation. We can then iteratively apply the deterministic results (which we know on the real line from [Che04], while the necessary results for us are only slightly different), to reach the point where for a fixed ϵ we have achieved the required distance in L^∞-norm and then apply the deterministic result on the short last interval to obtain a time for ϵ arbitrarily small, due to the exponential rate of convergence, this time will be of logarithmic order in ϵ.

Denote

$$\begin{aligned} C^h_{0,C^1} = \{\ u \mid u \in C^1\left(S^1\right), u'(0) = u'(1) \text{ and } u \text{ having finitely many zeroes,} \\ \text{located at } h; \text{ we have } |u(x)| + |u'(x)| > 0\ \forall x\ \}, \end{aligned}$$

$$C_{0,\epsilon} = \left\{ u | u \text{ is a continuous function s.t. } \frac{\|u\|_\infty}{\epsilon} \to 0 \text{ and } u(0) = u(1) \right\}.$$

We thus construct our set of initial conditions

$$X^{h,0}_\epsilon = \left\{ u_{0,C^1} + u_{0,\epsilon} | u_{0,C^1} \in C^h_{0,C^1} \text{ and } u_{0,\epsilon} \in C_{0,\epsilon} \right\}$$

and our set of phase-separated solutions

$$X^{h,\rho}_\epsilon = \left\{ \phi \in C\left(\left(S^1\right) \to [-2,2]\right) \mid |\phi| \geq 1/2 \text{ on } (0,1) \setminus \cup_i (h_i - \rho\epsilon, h_i + \rho\epsilon) \right\}$$

for $\rho > 0$.

We now state our main result, the sketch of argument being split up into Claims 2.2 and 2.3.

Denote $\delta_h = \min\{|h_j - h_{j+1}| \text{ for } j = 0, ...N\}$.

Claim 2.1. *fix $h_0 \in [0,1]^N$ and let $u(\cdot,t)$ for $t > 0$ solve (1.1).*

Then there exists a suitable $C \geq \frac{1}{|f'(0)|}$ depending only on $f(\cdot)$ and h_0 and a function C_p depending only on $f(\cdot)$, s.t. for $t_1 = C|\log\epsilon|$ we have

$$\sup_{u_0 \in X_\epsilon^{h_0,0}} \mathbb{P}\left[u(\cdot,t_1) \notin \Gamma_{\xi_\epsilon,\epsilon}\right] \leq C_p \epsilon^p,$$

$h' = \mathcal{H}(u(\cdot,t_1))$ associated to $u(\cdot,t_1) \in \Gamma_{\xi_\epsilon,\epsilon}$ fulfils

$$\mathbb{P}\left[\max_{0\leq i\leq N}\left|h_0^i - h'^i\right| > k\epsilon\sqrt{|\log\epsilon|}\right] \leq C_p\epsilon^p$$

for some $k > 0$.

Sketch of Argument. This is a consequence of Claims 2.3 and 2.4 since our initial profile satisfies exactly the conditions for applying Claim 2.3, which in turn leads to a solution in $X_\epsilon^{h,\rho}$ for some h, ρ, which has w.h.p. a modulus bounded by 2,

However it is easy to see that Claim 2.3 in fact also holds for all larger constants $1 > c > 0$ instead of $\frac{1}{2}$, thus upon applying Claim 2.3 we have a profile $u\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right)$ s.t. $\left|u\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right)\right| \to_{\epsilon\to 0} 1$ outside a finite number of neighbourhoods of size $\rho\epsilon$, in which the profile is bounded by 2 with high probability. Hence $\left\|u\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right)\right\| \to_{\epsilon\to 0} 0$, and in particular there exists $u^{h'}$ s.t. $\left\|u\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right) - u^{h'}\right\| \to_{\epsilon\to 0} 0$.

Thus we may apply Claim 2.4 and obtain the result.

We consider as a linearisation of the stochastic Allen-Cahn equation at its stable points the stochastic heat equation

$$\frac{\partial z}{\partial t} = \epsilon^2 \frac{\partial^2 z}{\partial x^2} + f'(1)\, z + \epsilon^\gamma \dot{W} \tag{2.1}$$

where $x \in (0,1)$ with periodic boundary conditions and the same noise as (1.1).

Denote $\bar{u} = u - z$ so that

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u} + z) - f'(1)\, z \tag{2.2}$$

for $x \in (0,1)$.

Let at $t = 0$ the initial profile of z be $z_0(\cdot) = 0$.

We note that with a probability converging to 1 as $\epsilon \to 0$ we have that $\|u(\cdot,t)\|_\infty$ is of order $\mathcal{O}(1)$ on polynomial timescales (cf. eg. [Fen06]), and $\|z(\cdot,t)\|_\infty$ is of smaller order (see Claim 2.2) in ϵ, for what reason $\|\bar{u}(\cdot,t)\|_\infty$ is of order $\mathcal{O}(1)$ on polynomial timescales.

Claim 2.2. *Let $z(\cdot,0)$ be constantly equal to 0 and $z(\cdot,t)$ solve (2.1) for $t>0$. For all $\nu,q,r>0$ there exist $C_{r,\nu}, D_{q,r,\nu}>0$ independent of ϵ s.t.*

$$\mathbb{P}\left[\sup_{t\in\left[0,r\epsilon^{-(2\gamma+1)}\right]}\|z(\cdot,t)\|_\infty > C_{r,\nu}\epsilon^{\gamma-1/2-\nu}\right] \leq D_{q,r,\nu}\epsilon^q \;\forall q>0. \tag{2.3}$$

Remark: This obviously also holds in L^2 norm.

Sketch of Argument. The solution is the following stochastic convolution:

$$z(x,t) = \epsilon^\gamma \int_0^t e^{(t-s)\left(\epsilon^2\partial_{xx}+f'(1)\right)} dW(x,s) \tag{2.4}$$

This is clearly a Gaussian process, since our initial profile is constantly equal to 0. Firstly we show $\sup_{t\leq r\epsilon^{-2\gamma-1}} E\left[\|z\|^2\right] \leq C\epsilon^{2\gamma-1}$ by taking the Fourier transform of (2.1):

$$\frac{\partial \hat{z}}{\partial t} = \left(f'(1) - k^2\epsilon^2\right)\hat{z} + \epsilon^\gamma \dot{\hat{W}}(k,t)$$

However, similarly to the classical Ornstein-Uhlenbeck process, this equation has the explicit solution

$$\hat{z}(k,t) = \epsilon^\gamma \int_0^t e^{\left(k^2\epsilon^2 - f'(1)\right)(s-t)} d\hat{W}(k,t),$$

which is normally distributed with mean $\frac{f'(1)}{k^2\epsilon^2}(1-\exp(-k^2\epsilon^2 t))$ and variance $\frac{\epsilon^{2\gamma-2}}{2k^2\epsilon^2}\left(1-\exp(-2k^2\epsilon^2 t)\right)$.

Therefore we have by the properties of Gaussian processes that

$$\sup_{t\leq r\epsilon^{-2\gamma-1}} \mathbb{E}\left[\|z(x,t)\|^2\right] \leq \sup_{t\leq r\epsilon^{-2\gamma-1}} \sum_{k\neq 0} \mathbb{E}\left[(\hat{z}(h,t))^2\right] \leq C'\epsilon^{2\gamma-1}$$

for some $C'>0$ independent of ϵ.

We now show that $\mathbb{E}\left[\sup_{t\leq r\epsilon^{-2\gamma-1}} \|z\|_\infty\right] \leq C\epsilon^{\gamma-1/2}$:

Firstly, we prove the result on time 1. To do so, we note that by [Adl90] Corollary 4.15 we have

$$\mathbb{E}\left[\sup_{0\leq t\leq 1}\|z\|_\infty\right] \leq K\int_0^\infty \{\log N_\epsilon(r)\}^{1/2} dr,$$

where $N(r)$ is the minimal number of balls of radius r needed to cover $\{(x,t) : x\in(0,1), 0\leq t\leq 1\}$ with the following metric:

$$d((x,t),(y,s)) = \sqrt{\mathbb{E}\left[(z(x,t)-z(y,s))^2\right]}$$

It can be shown (cf. eg. [Wal81] Prop. 4.2) that

$$\mathbb{E}\left[(z(x,t)-z(y,t))^2\right] \le c_1\epsilon^{2\gamma-1}\,|x-y| \;\; \forall t\ge 0;\; x,y\in(0,1)$$

and

$$\mathbb{E}\left[(z(x,s)-z(x,t))^2\right] \le c_2\epsilon^{2\gamma-1}\,|t-s|^{1/2} \;\; \forall x,y\in(0,1), |s|,|t|\le T>0.$$

Using these bounds we can now easily see that there exists $k\ge 0$ s.t.

$$N_\epsilon(r) \le \max\left\{1, k\epsilon^{3\gamma-3/2}r^{-3}\right\}.$$

Thus (combining this with the previous expression) we know that there exists $C>0$ s.t.

$$\mathbb{E}\left[\sup_{0\le t\le 1}\|z\|_\infty\right] \le C\epsilon^{\gamma-1/2}.$$

We note that if $z(\cdot,0)$ is Gaussian with

$$\|z(\cdot,0)\|\epsilon^{-\gamma+1/2+\nu} \to_{\epsilon\to 0} 0 \;\forall \nu>0,$$

the result also holds, since the solution may be expressed as

$$z(x,t) = e^{t\left(\epsilon^2\partial_{xx}+f'(1)\right)}z_0 + \epsilon^\gamma\int_0^t e^{(t-s)\left(\epsilon^2\partial_{xx}+f'(1)\right)}dW(x,s).$$

This is because $e^{t\left(\epsilon^2\partial_{xx}+f'(1)\right)}$ is a strongly continuous semigroup, so the expression is the sum of two terms which we can bound, which means that we have the bound on all time intervals of the form $[i,i+1]$ where $i>0$. The bound on $\|z(\cdot,i)\|_\infty$ follows inductively from the bound on $\sup_{0\le t\le 1}\|z(\cdot,t)\|_\infty$ obtained above. To see this we note that since we have the bound at time 1, we in particular also have it at time $1/2$. This then means that over the interval $[0,2]$ we obtain the bound $\exp(-f(1)/2)C\epsilon^{\gamma-1/2}+C\epsilon^{\gamma-1/2}$; the rest follows iteratively.

Due to Gaussianity we then conclude that

$$\mathbb{E}\left[\sup_{[i,i+1]}\|z\|_\infty^p\right] \le C_p\epsilon^{(\gamma-1/2)p}.$$

Now we note since for finite $p>0$ we have $\|\cdot\|_{l^\infty}\le\|\cdot\|_{l^p}$ the following:

$$\mathbb{E}\left[\sup_{t\le r\epsilon^{-2\gamma-1}}\|z\|_\infty\right] \le \mathbb{E}\left[\left\{\sum_{i=0}^{\lfloor r\epsilon^{-2\gamma-1}\rfloor}\sup_{i\le t\le i+1}\|z\|_\infty^p\right\}^{1/p}\right]$$

$$\leq \left\{ \mathbb{E}\left[\sum_{i=0}^{\lfloor r\epsilon^{-2\gamma-1} \rfloor} \sup_{i\leq t\leq i+1} \|z\|_\infty^p \right] \right\}^{1/p} = \left\{ \sum_{i=0}^{\lfloor r\epsilon^{-2\gamma-1} \rfloor} \mathbb{E}\left[\sup_{i\leq t\leq i+1} \|z\|_\infty^p \right] \right\}^{1/p}$$

$$\leq \left\{ \sum_{i=0}^{\lfloor r\epsilon^{-2\gamma-1} \rfloor} C_p \epsilon^{(\gamma-1/2)p} \right\}^{1/p} = \left(r\epsilon^{-2\gamma-1} \right)^{1/p} C_p' \epsilon^{\gamma-1/2}$$

$$= C_p' r^{1/p} \epsilon^{\gamma-1/2-\frac{2\gamma-1}{p}} \to_{p\to\infty} C_p' \epsilon^{\gamma-1/2}.$$

Using Borell's inequality (cf. eg. [Adl90]) we conclude that

$$\mathbb{P}\left[\sup_{t\in\left[0,r\epsilon^{-(2\gamma+1)}\right]} \|z(\cdot,t)\| > C_{r,\nu}\epsilon^{\gamma-1/2-\nu} \right] \leq D_{q,r,\nu}\epsilon^{q\nu} \ \forall \nu, q > 0. \qquad (2.5)$$

We now have all the ingredients for a perturbative sketch of argument of the following claim, where we use a deterministic result that may be found in the appendix.

Claim 2.3. *Phase separation*

Fix $h_0 \in [0,1]^N$ and let $u(\cdot,t)$ for $t > 0$ solve (1.1).

Then there exist $h \in [0,1]^N$ and $\rho > 0$, both depending only on $u_0(\cdot)$ and $f(\cdot)$ and a function C_q only depending on $u_0(\cdot)$ and $f(\cdot)$, s.t.

$$\sup_{u_0 \in X_\epsilon^{h_0,0}} \mathbb{P}\left[u\left(\frac{|\log \epsilon|}{|f'(0)|} \right) \notin X_\epsilon^{h,\rho} \right] \leq C_q \epsilon^q \ \forall q > 0.$$

We have

$$\mathbb{P}\left[|h - h_0| > k'\epsilon\sqrt{|\log \epsilon|} \right] \leq C_q \epsilon^q \ \forall q > 0$$

for some $k' > 0$.

Sketch of Argument. As in (2.2) we have for $\bar{u} = u - z$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u} + z) - f'(1)\, z \\ &= \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u}) + C(x,t)\, z \end{aligned}$$

with $C(x,t)$ being bounded by C' with a probability converging to 1 as $\epsilon \to 0$.

Denoting by $w(\cdot,t)$ the solution to the deterministic Allen-Cahn equation with the same initial condition $u_0(\cdot)$, we get

$$\frac{\partial(\bar{u} - w)}{\partial t} = \epsilon^2 \frac{\partial^2 (\bar{u} - w)}{\partial x^2} + f(\bar{u}) - f(w) + C(x,t)\, z$$

so that using subdifferentials leads to an expression for the time-derivative of the L^∞-norm:

$$\begin{aligned}\frac{\partial \|\bar{u}-w\|_\infty}{\partial t} &\leq \sup_{l\in\partial\|\bar{u}-w\|} \left\langle l, \epsilon^2 \frac{\partial^2 (\bar{u}-w)}{\partial x^2} + f(\bar{u}) - f(w) + C(x,t)\, z \right\rangle \\ &\leq \sup_{l\in\partial\|\bar{u}-w\|} \left\langle l, f(w + (\bar{u}-w)) - f(w) + C' z \right\rangle \\ &\leq |f'(0)|\, \|\bar{u}-w\|_\infty + C' \|z\|_\infty .\end{aligned}$$

The convex analysis technique employed here may be found, for example, in [MIT03].

This expression allows the use of Gronwall's inequality so that

$$\|(\bar{u}-w)(t)\|_\infty \leq C \sup_{0\leq s\leq t} \{\|z(s)\|_\infty\} \exp\left(|f'(0)|\, t\right),$$

thus, at $t = \frac{|\log \epsilon|}{|f'(0)|}$ we get the estimate

$$\|(\bar{u}-w)(t)\|_\infty \leq C \sup_{0\leq s\leq t} \|z(s)\|_\infty \, \epsilon^{-1}.$$

Then, using Claim 7.1 and Claim 2.2 (which tells us that $\sup_{0\leq s\leq t} \|z\|_\infty < \mathcal{O}\left(\epsilon^{\gamma-1/2}\right)$ w.h.p.), the result follows for C^1 initial conditions with finitely many zeroes. This is because $\gamma > 2$, and thus with a high probability we have $\left\|\bar{u}-w\right\|_\infty \leq C\epsilon^{1/2}$.

To now finally prove the result in an ϵ-Neighbourhood of a C^1-function, we consider the stochastic Allen-Cahn equation with two different initial conditions: $u_{0,C^1} \in C^1(0,1)$ and $v_0 = u_{0,C^1} + u_{0,\epsilon}$, where $u_{0,\epsilon} \in C(0,1)$ with $|u_{0,\epsilon}| \leq o(\epsilon)$:

As before, we denote their SPDEs at time t by

$$\partial_t u = \epsilon^2 \partial_{xx} u + f(u) + \epsilon^\gamma \dot{W}$$

and

$$\partial_t v = \epsilon^2 \partial_{xx} v + f(v) + \epsilon^\gamma \dot{W}.$$

We then immediately have

$$\partial_t (u-v) = \epsilon^2 \partial_{xx} (u-v) + f(v + (u-v)) - f(v).$$

Due to the smoothing of the Laplacian in space, $\|u-v\|_\infty$ is differentiable in time.

By the definition of the subdifferential and the fact that $u-v$ is (for positive times) twice differentiable in space we know that at the x-value of its supremum its second derivative is negative, thus implying the following:

$$\partial_t \|u-v\|_\infty = \sup_{l\in\partial\|u-v\|_\infty} \left\langle l, \epsilon^2 \partial_{xx}(u-v) + f(v + (u-v)) - f(v) \right\rangle$$

$$\leq \sup_{l \in \partial \|u-v\|_\infty} \langle l, f(v + (u - v)) - f(v) \rangle \leq |f'(0)| \, \|u - v\|_\infty$$

Using Gronwall's inequality this gives

$$\|u - v\|_\infty \leq \|u_{0,\epsilon}\|_\infty \, e^{|f'(0)|t}.$$

Therefore we obtain for $t = \frac{|\log \epsilon|}{|f'(0)|}$ the following bound:

$$\left\| (u - v) \left(\frac{|\log \epsilon|}{|f'(0)|} \right) \right\|_\infty \leq \|u_{0,\epsilon}\|_\infty \, \epsilon^{-1}$$

And hence (by the definition of $C_{0,\epsilon}$) we have

$$\left\| (u - v) \left(\frac{|\log \epsilon|}{|f'(0)|} \right) \right\|_\infty \to_{\epsilon \to 0} 0,$$

which proves the result for fixed u_0. Taking the supremum over $X_\epsilon^{h_0,0}$ completes the sketch of argument.

Claim 2.4. *Generation of metastable patterns*

Let $u_0(\cdot) \in X_\epsilon^{h_0,\rho}$ *with* ρ *as in Claim 2.3 so that* $\left\| u_0(\cdot) - u^{h'}(\cdot) \right\| \to_{\epsilon \to 0} 0$ *for some* $u^{h'}(\cdot)$ *being the orthogonal projection of* $u_0(\cdot)$ *in the* L^2 *norm.*

Let $t_2 = C |\log \epsilon|$ *for a suitable* C *(depending only on* $f(\cdot)$ *and* ρ*),* $u(\cdot, t)$ *solves (1.1) for* $t > 0$*.*

Then there exists a function C_q *depending only on* $f(\cdot)$ *and* ρ *such that*

$$\sup_{u_0 \in X_\epsilon^{h_0,\rho}} \mathbb{P}\left[u(\cdot, t_2) \notin \Gamma_{\xi_\epsilon,\epsilon} \right] \leq C_q \epsilon^q \ \forall q > 0,$$

$h' = \mathcal{H}(u(t_2))$ *associated to* $u(t_2) \in \Gamma_{\xi_\epsilon,\epsilon}$ *fulfils*

$$\mathbb{P}\left[\max_{i \in N} \left| h_0^i - h'^i \right| > k\epsilon \, |\log \epsilon| \right] \leq C_q \epsilon^q \ \forall q > 0$$

for some $k > 0$ *depending.*

Sketch of Argument. Our random PDE used for proving the result is the deterministic Allen-Cahn equation with an $\mathcal{O}\left(\sup_{t \in [0,\epsilon^{-r}]} \|z\|_\infty \right) = \mathcal{O}\left(\epsilon^{\gamma - 1/2 - \nu} \right)$ perturbation w.h.p. As in (2.2) we have for $\bar{u} = u - z$

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u} + z) - f'(1) \, z$$

$$= \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u}) + C(x, t) \, z$$

for a function $C(x, t)$ bounded by the constant C' with a probability converging to 1 as $\epsilon \to 0$.

Since $\bar{u} - w$ (where w is the deterministic Allen-Cahn equation) has the property

$$\partial_t (\bar{u} - w) = \epsilon^2 \partial_{xx} (\bar{u} - w) + f(w + (\bar{u} - w)) - f(w) + C(x,t) z =$$
$$= \epsilon^2 \partial_{xx} (\bar{u} - w) + C'(x,t) (\bar{u} - w) + C(x,t) z$$

where $C'(x,t)$ is bounded by a constant, we note that $\|\bar{u} - w\|_\infty$ converges to 0 as $\epsilon \to 0$. Due to the smoothing property of the Laplacian in space, $\|\bar{u} - w\|_\infty$ is differentiable as a function of time.

Clearly, by the definition of subdifferentials and the fact that a twice differentiable function has a negative second derivative at its maximum we have

$$\begin{aligned}
&\frac{\partial \|\bar{u} - w\|_\infty}{\partial t} \\
&\leq \sup_{l \in \partial \|\bar{u} - w\|_\infty} \left\langle l, \epsilon^2 \partial_{xx} (\bar{u} - w) + f(w + (\bar{u} - w)) - f(w) + C(x,t) z \right\rangle \\
&\leq |f'(0)| \|\bar{u} - w\|_\infty + C' \|z\|_\infty ,
\end{aligned}$$

so that Gronwall's inequality implies

$$\|(\bar{u} - w)(t)\|_\infty \leq C \sup_{0 \leq s \leq t} \|z(s)\|_\infty e^{|f'(0)|t.}$$

Since w.h.p. $\sup_{0 \leq s \leq t} \|z\| \leq \epsilon^{\gamma - 1/2 - \mu}$ $\forall \mu > 0$, this means that for a time up to $\frac{|\log \epsilon| \hat{\delta}}{|f'(0)|}$ for $0 < \hat{\delta} < \gamma - 2$ the error between the orthogonal distance to the slow manifold of the deterministic and stochastic equation is smaller than the size of the orthogonal distance to the slow manifold in the slow channel, and thus small enough to "track" the behaviour of the stochastic equation using the deterministic one. We may use Claim 7.4 due to the assumption made on the initial profile.

Claim 7.4 implies directly that there is a deterministic time τ after which we have $\left\| w(\tau, \cdot) - w^{\mathcal{H}(u(\tau,\cdot))}(\cdot) \right\| \leq \frac{1}{2} \left\| w_0(\cdot) - w^{\mathcal{H}(u_0(\cdot))}(\cdot) \right\|$.

Thus (due to Claim 2.2) with a probability converging to 1 as $\epsilon \to 0$ after this time τ, we have the same property for the random PDE and can iteratively apply the result.

Denote the original distance in L^2-norm by d. Until we have $\left\| u(T, \cdot) - u^{\mathcal{H}(u(T,\cdot))}(T, \cdot) \right\| \leq \epsilon^{3/2+\delta}$, the n-th step yields a distance $\frac{d}{2^n}$ in the deterministic equation; we note that this term converges to 0 as $n \to \infty$, and thus there is a finite time T, after which $\mathbb{P}\left[\left\| \bar{u}(T, \cdot) - \bar{u}^{\mathcal{H}(\bar{u}(T,\cdot))}(T, \cdot) \right\| < \frac{d}{2^n} < \epsilon^{3/2+\delta} \right] \leq C_q \epsilon^q$ for a fixed ϵ. This will be after $C |\log \epsilon|$ iterations, for some $C > 0$.

In the last step, the same exponential rate of Claim 7.4 lets us go from a distance of order 1 to a distance of polynomial order in ϵ in a time of order $|\log \epsilon|$. To avoid blowup, we iterate the result on time-intervals smaller than $\frac{|\log \epsilon| \hat{\delta}}{|f'(0)|}$ for some $0 < \hat{\delta} < \gamma - 2$. As explained earlier, up to this time the difference between stochastic and deterministic equation is small enough, at the end of it we restart the "determinstic approximation" with the the solution of the stochastic Allen-Cahn equation at that point. We continue this until the solution has entered the slow channel.

Noting that by Claim 2.2, the difference between u and $\bar{u}$ is with high probability of order $\epsilon^{\gamma-1/2}$ on timescales of logarithmic order in ϵ with high probability completes the sketch of argument that $\mathbb{P}\left[u\left(t_2\right) \notin \Gamma_{\xi_\epsilon,\epsilon}\right] \leq C_q \epsilon^q \ \forall q > 0$. Taking the supremum over $X_\epsilon^{h_0,\rho}$ yields the first inequality.

$\mathbb{P}\left[\max_{i \in N}\left|h_i - h_i'\right| > k\epsilon\left|\log\epsilon\right|\right] \leq C_q \epsilon^q \ \forall q > 0$ similarly follows from the deterministic equation by applying the same iteration scheme.

Finally, by the existence of a mapping (see [Che04] Theorem 5.1 and [Che04] Theorem 7.1) the result follows.

3. INTERFACE MOTION IN THE SLOW CHANNEL

In this chapter, we obtain, based on the ideas presented in [ABK12] - but presented in a different context and extended, an explicit description of the multi-kink behaviour of the Stochastic Allen-Cahn equation while the solution is near the slow manifold. In the sharp interface limit on the fast timescale this will be the only phenomenon whose duration does not converge to 0.

We recall the critical distance after which annihilation occurs (with a probability converging to 1 as $\epsilon \to 0$) as $\xi_\epsilon = \frac{|\log \epsilon|(2\gamma+\beta)\epsilon}{\sqrt{-f'(1)}}$; as in last chapter z is the solution of the linearisation of (1.1) with an initial profile constantly equal to 0.

We now state the main result, which is a consequence of the somewhat lengthy cleim in Section 3.1:

Claim 3.1. *Sharp interface limit in the slow channel*

Let our initial profile $u_0(\cdot)$ at $t = 0$ be inside $\Gamma_{\xi_\epsilon,\epsilon}$. Denote its interface configuration at $t = 0$ by $h_0 = \mathcal{H}(u_0(\cdot))$. For $t > 0$ $u(\cdot, t)$ solves (1.1). We assume that $\delta_{h_0} \geq \xi_\epsilon$ with all other distances between neighbouring interfaces bounded below by $c > 0$.

Denote the interfaces on the fast timescale as $\hat{h}(t) = h\left(S_\infty \epsilon^{-1-2\gamma} t\right) = \mathcal{H}\left(u\left(S_\infty \epsilon^{-1-2\gamma} t, \cdot\right)\right)$ and let $C > 0$.

Then $\hat{h}(t)$ stopped at

$$\hat{\tau}^* = C \wedge \inf \{ \ t > 0 : \delta_{\hat{h}(t)} = \xi_\epsilon \ \text{or} \ \|v(t)\| > \epsilon^{3/2+\delta}$$

$$\text{or} \ \|v(t)\|_\infty \geq 1 \ \text{or} \ \|z(t)\| > \epsilon^{3/2+\delta} \ \},$$

converge in law to standard Brownian motions on S starting at h_0 and stopped at $C \wedge \mu$, where μ is the first hitting time of two neighbouring interfaces.

Sketch of Argument. All initial conditions fulfil the second part of Claim 3.2, of which this result is the special case of the limit $\epsilon \to 0$.

3.1 The Semimartingale representation

In essence, the following Claim gives us the semimartingale notation of our interfaces inside the extended slow channel, and in particular the convergence to Brownian motion if we are inside the slow channel. We recall that in order to uniquely map $u \to u^h + v$ we require $\|v\| \leq C_{map}\epsilon^{1/2}$.

Claim 3.2. *Interface motion inside the slow channel*

Let $T_\epsilon = CS_\infty \epsilon^{-2\gamma-1}$, for a large $C > 0$ independent of ϵ. Let furthermore our initial profile $u_0(\cdot)$ at $t = 0$ be inside Γ'_ϵ. Denote its interfaces at $t = 0$ by $h_0 = \mathcal{H}(u_0(\cdot))$. For $t > 0$ $u(\cdot,t)$ solves (1.1). We assume that $\delta_{h_0} \geq \rho^{**}\epsilon$ with all other distances of neighbouring interfaces bounded below by $c > 0$.

Then up to the first exit time

$$\tau^{**} = T_\epsilon \wedge \inf \{ t > 0 : \delta_{h(t)} = \epsilon\rho^{**} \\ \text{ or } \|v\| > C_{map}\epsilon^{1/2} \text{ or } \|v|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{3/2+\delta} \}$$

the interface position of u, defined as $h = \mathcal{H}(u(\cdot, t\wedge\tau^{**}))$, is a semimartingale on $[0,\tau^{**}]$ given by

$$dh_k = b_k(h,v)\,dt + \epsilon^\gamma \langle \sigma_k(h,v), dW\rangle$$

where

$$\sigma_r(h,v) = \sum_i A_{ri}^{-1}(h,v)\,\tau_i^h$$

and

$$b_r(h,v) = \sum_i A_{ri}^{-1}(v,h)\left\langle \mathcal{L}\left(u^h+v\right), \tau_i^h\right\rangle \\ +\epsilon^{2\gamma}\sum_{i,l,k} A_{ri}^{-1}(v,h)\left[\frac{1}{2}\left\langle v,\tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h,\tau_i^h\right\rangle - \left\langle u_k^h,\tau_{il}^h\right\rangle\right]\langle\sigma_k(h,v),\sigma_l(h,v)\rangle \\ +\epsilon^{2\gamma}\sum_i A_{ri}^{-1}(h,v)\sum_j\left\langle \tau_{ij}^h,\sigma_j(h,v)\right\rangle.$$

We have $\mathcal{L}(u) := \epsilon^2 u_{xx} + f(u)$ and $A_{ri}^{-1}(h,v)$ being the inverse matrix of $A_{ri}(h,v) = \left\langle u_r^h,\tau_i^h\right\rangle - \left\langle v,\tau_{ri}^h\right\rangle$.

We have the following probability:

$$\mathbb{P}\left[\|v(\tau^{**})\| > C_{map}\epsilon^{1/2} \text{ or } \|v(\tau^{**})\|_\infty \geq 1\right] \leq C_q\epsilon^q \;\forall q > 0,$$

where the function C_q depends only on $f(\cdot)$.

Now define

$$\tau^* = T_\epsilon \wedge \inf\left\{t > 0 : \delta_{h(t)} = \xi_\epsilon \text{ or } \|v\| > \epsilon^{3/2+\delta} \text{ or } \|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{3/2+\delta}\right\}.$$

If at $t = 0$ our $u_0(\cdot)$ is in $\Gamma_{\xi_\epsilon,\epsilon}$ we can write down the behaviour of its interfaces $h(t\wedge\tau^*)$ up to τ^* by the same semimartingale notation as above.

Before the stopping time the following determinstic bound holds:

$$|b_r(h,v)| \leq k\epsilon\exp\left\{\frac{-\sqrt{f'(1)}\delta_h}{\epsilon}\right\} + o\left(\epsilon^{2\gamma+1}\right)$$

for some $k > 0$ independent of ϵ.

After the time-change $\hat{h}_k(t) = h_k\left(S_\infty \epsilon^{-1-2\gamma} t\right)$, the stopping time becomes $\hat{\tau}^* = S_\infty^{-1}\epsilon^{1+2\gamma}\tau^*$ and the equation becomes

$$d\hat{h}_k = \frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}} dt' + \left\langle \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}} \sigma_k\left(\hat{h}, v\right), dW \right\rangle$$

where the deterministic bound

$$\sup_{t\in[0,\tau^*]} \left\| \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}} \sigma_k\left(\hat{h}, v\right) - \frac{\sqrt{\epsilon}}{\sqrt{S_\infty}} u_k^h \right\|_\infty \leq F\epsilon^{1/2}$$

for some $F > 0$ independent of ϵ holds up to τ^*, as we are inside the slow channel.

In the sharp interface limit as $\epsilon \to 0$, the law of $\left(\hat{h}_1, ...\hat{h}_N\right)$ stopped at $\hat{\tau}^*$ converges weakly to that of independent Brownian motions $(M_1, ..., M_N)$ stopped at the minimum of their first hitting time and C.

Sketch of Argument. The derivation of the semimartingale expression, based on the assumption that the interfaces are a semimartingale, is in the section after this sketch of argument. The sketch of argument of the assumption that the interfaces have a semimartingale notation is given after the sketch of argument of Claim 3.7.

Starting in Γ'_ϵ the asymptotically small probability of leaving the slow channel other than by reaching the critical distance (or time $S_\infty C\epsilon^{-2\gamma-1}$) follows from Claim 4.4. This completes the sketch of argument of the first part of the Claim where we are in Γ'_ϵ.

Whenever we are from now on in this sketch of argument applying Claim 3.7, we use the special case of $\tau^* = \psi_{\xi_\epsilon}$ instead of the general ψ_d.

From now on, we consider the regime starting in $\Gamma_{\xi_\epsilon,\epsilon}$, which allows us to use the Claims from section 3.2. The asymptotic expansion of the "diffusion coefficient" $\sigma(h, v)$ follows from Claim 3.8 combined with Claim 3.7; for the drift $b(h, v)$ the bound follows from Claim 3.9 combined with Claim 3.7. The stability of this system is a consequence of Claim 3.7, i.e. the probability of $\|v\| > \epsilon^{3/2+\delta}$ or $\|v\|_\infty \geq 1$ or $\|z\| > \epsilon^{3/2+\delta}$ is asymptotically small.

We now show that after a time-rescaling onto $t' = S_\infty^{-1}\epsilon^{2\gamma+1}t$ the h_k weakly converge in law to a Brownian motion starting at h_0 stopped at $\hat{\tau}^* = S_\infty^{-1}\epsilon^{2\gamma+1}\tau^*$. The intuition of it is that $\left(\sqrt{\epsilon}u_r^h\right)^2$ is close to a Dirac delta in the limit $\epsilon \to 0$, so that since our noise is white in space, integrating it against the square root of a Dirac delta function leads to a Brownian motion.

Denote by $\hat{h}_k(t) = h_k\left(S_\infty\epsilon^{-2\gamma-1}t\right)$ and $\hat{\tau}^* = S_\infty^{-1}\epsilon^{2\gamma+1}\tau^*$ our interfaces and the stopping time on the fast timescale.

We firstly show that $\hat{h}_k$ stopped at τ^* is tight on $C([0, \hat{\tau}^*], S)$:

Let $t, s > 0$ and $p > 0$ be even. Then

$$\mathbb{E}\left[\left|\hat{h}_k(t \wedge \hat{\tau}^*) - \hat{h}_k(s \wedge \hat{\tau}^*)\right|^p\right]$$

$$= \mathbb{E}\left[\left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}} dr + \int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \left\langle \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}} \sigma_k\left(\hat{h}, v\right), dW\left(x, r\right)\right\rangle\right)^p\right]$$

$$\leq C_p \mathbb{E}\left[\left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}} dr)^p + (\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \left\langle \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}} \sigma_k\left(\hat{h}, v\right), dW\left(x, r\right)\right\rangle\right)^p\right]$$

$$\leq C_p \mathbb{E}\left[\left(\left|t\wedge\hat{\tau}^* - s\wedge\hat{\tau}^*\right| \sup_{r\in[s\wedge\hat{\tau}^*, t\wedge\hat{\tau}^*]} \left|\frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}}\right|\right)^p\right.$$

$$\left. + \int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \left(\frac{S_\infty}{\epsilon}\left\langle \sigma_k\left(\hat{h}, v\right), \sigma_k\left(\hat{h}, v\right)\right\rangle\right)^{p/2} dt\right]$$

$$\leq C_p \mathbb{E}\left[\left(\left|t\wedge\hat{\tau}^* - s\wedge\hat{\tau}^*\right| \sup_{r\in[s\wedge\hat{\tau}^*, t\wedge\hat{\tau}^*]} \left|\frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}}\right|\right)^p\right.$$

$$\left. + \left(C_p' \left|t\wedge\hat{\tau}^* - s\wedge\hat{\tau}^*\right|\right)^{p/2}\right]$$

(the bound on the stochastic integral follows using the Burkholder-Davis-Gundy inequality)

$$\leq D_p \left|t\wedge C - s\wedge C\right|^{p/2}.$$

The last inequality follows because $\tau^* \leq C$.

We used Claim 3.9 to bound $\left|\frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}}\right|$ (since we are inside $\Gamma_{\xi_\epsilon,\epsilon}$) and the fact that $\hat{\tau}^* \leq C$ (by definition of τ^*) on the fast timescale to obtain the bound.

We can therefore apply the Kolmogorov continuity theorem to obtain tightness and hence the existence of a convergent subsequence. Subsequently, we will show that all convergent subsequences have the same limit, and thus, the limit $\epsilon \to 0$ is unique.

The following stochastic process is clearly a martingale:

$$M_k^\epsilon(t) = \hat{h}_k\left(t\wedge\hat{\tau}^*\right) - \int_0^{t\wedge\hat{\tau}^*} \frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}} ds$$

However, as we are on the fast timescale, this means for times $0 \leq t' \leq \hat{\tau}^*$ that

$$\left|\hat{h}_k\left(t\wedge\hat{\tau}^*\right) - M_k^\epsilon(t)\right| = \left|\int_0^t \frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}} ds\right| \leq \hat{\tau}^* \sup_{s\in[0,\tau^*]} \left|\frac{S_\infty b_k\left(\hat{h}, v\right)}{\epsilon^{2\gamma+1}}\right|$$

Thus using Claim 3.9 and Claim 3.7 we get for some $C' > 0$ that

$$\mathbb{P}\left[\sup_{t'\in[0,\tau^*]} \left|\hat{h}_k(t') - M_k^\epsilon(t')\right| > C'\left\{\epsilon^\beta + \epsilon^\delta\right\}\right] \leq C_q \epsilon^q \;\forall q > 0$$

and hence $\hat{h}_k$ has a subsequence that weakly converges to the same limit as $M_k^\epsilon(t)$ in the sharp interface limit, which is a martingale we shall call M_k.

The fact that it is a martingale follows (c.f. e.g. [EK09] Theorem 8.10 in the chapter "Generators and Markov Processes") because for $s > 0$ we have

$$\mathbb{E}\left[\frac{S_\infty}{\epsilon}\left\langle \sigma_k\left(\hat{h}\left((t+s\wedge\hat{\tau}^*)\right),v\right),\sigma_k\left(\hat{h}\left((t+s\wedge\hat{\tau}^*)\right),v\right)\right\rangle\right.$$

$$\left.-\frac{S_\infty}{\epsilon}\left\langle \sigma_k\left(\hat{h}\left(t\wedge\hat{\tau}^*\right),v\right),\sigma_k\left(\hat{h}\left(t\wedge\hat{\tau}^*\right),v\right)\right\rangle\right]$$

$$=\mathbb{E}\left[\frac{S_\infty}{\epsilon}\int_0^1\left[\frac{\epsilon}{S_\infty}\tau^{h_k(t+s\wedge\hat{\tau}^*)}+\pi\left(t+s\wedge\hat{\tau}^*,\epsilon\right)\right]^2dx\right.$$

$$\left.-\frac{S_\infty}{\epsilon}\int_0^1\left(\frac{\epsilon}{S_\infty}\tau^{h_k(t\wedge\hat{\tau}^*)}+\pi\left(t\wedge\hat{\tau}^*,\epsilon\right)\right)^2dx\right]$$

$$=\mathbb{E}\left[\frac{S_\infty}{\epsilon}\frac{\epsilon^2}{S_\infty^2}\frac{S_\infty}{\epsilon}+o\left(\pi\left(t+s\wedge\hat{\tau}^*,\epsilon\right)\right)-\frac{S_\infty}{\epsilon}\frac{\epsilon^2}{S_\infty^2}\frac{S_\infty}{\epsilon}+o\left(\pi\left(t\wedge\hat{\tau}^*,\epsilon\right)\right)\right]$$

$$\leq o\left(\epsilon\right)\rightarrow_{\epsilon\rightarrow 0}0$$

(we used Claim 3.8; π has the property $\sup_t\left|\pi\left(t,\epsilon\right)\right|\leq o\left(\epsilon\right)$,

which means that the inner product of the diffusion with itself converges to a constant as $\epsilon\rightarrow 0$, which is required to apply the theorem from [EK09].

We now want to prove convergence to a stopped Brownian motion by showing that the extension of the limiting process beyond its stopping time by a Brownian motion is a Brownian motion; for this we use the Levy characterisation.

Denote by τ the weak limit of $\hat{\tau}^*$ as $\epsilon\rightarrow 0$. We will show in the end of the sketch of argument that this limit exists. Now we consider the following stochastic processes:

$$Z_k\left(t\right)=\begin{cases}M_k\left(t\right) & \text{for } t\leq\tau\\ M_k\left(\tau\right)+B_k\left(t-\tau\right) & \text{for } t>\tau\end{cases}$$

$$H_k\left(t\right)=\begin{cases}\hat{h}_k\left(t\right) & \text{for } t\leq\tau\\ \hat{h}_k\left(\tau\right)+B_k\left(t-\tau\right) & \text{for } t>\tau\end{cases}$$

where $B_k(t)$ are independent standard Brownian motions.

We observe that for $j\neq k$ the quadratic covariation for times $t<\hat{\tau}^*$ is

$$\left(d\hat{h}_k,d\hat{h}_j\right)$$

$$=\left(\frac{S_\infty b_k\left(h,v\right)}{\epsilon^{2\gamma+1}}dt'+\left\langle\frac{\sqrt{S_\infty}}{\sqrt{\epsilon}}\sigma_k\left(h,v\right),dW\right\rangle,\frac{S_\infty b_j\left(h,v\right)}{\epsilon^{2\gamma+1}}dt'+\left\langle\frac{\sqrt{S_\infty}}{\sqrt{\epsilon}}\sigma_j\left(h,v\right),dW\right\rangle\right)$$

$= \left(\left\langle \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}}\sigma_k(h,v), dW\right\rangle, \left\langle \frac{\sqrt{S_\infty}}{\sqrt{\epsilon}}\sigma_j(h,v), dW\right\rangle\right)$

$= \epsilon^{-1}S_\infty \langle \sigma_k(h,v), \sigma_j(h,v)\rangle\, dt'$

Noting that using the time-rescaled version of Claim 3.8 and Claim 3.7 we have

$$\mathbb{P}\left[\sup_{t\in[0,\tau^*]}\left\|S_\infty^{1/2}\epsilon^{-1/2}\sigma_k(h,v) - \frac{\sqrt{\epsilon}}{\sqrt{S_\infty}}\tau_k^h\right\|_\infty \geq F\sqrt{\epsilon}\right] \leq C_q\epsilon^q \ \forall q > 0$$

and that inside the slow channel by [CP89] Proposition 2.3 we have for $k \neq j$ that

$$\frac{\epsilon}{S_\infty}\left\langle \tau_k^h, \tau_j^h\right\rangle < \frac{\epsilon}{S_\infty}C\epsilon^{2\gamma-3/2},$$

it follows that $\forall T > 0$

$$\mathbb{P}\left[\left|\int_0^T (dH_k, dH_j)\right| \geq F'\epsilon^{1/2}\right] \leq C_q\epsilon^q$$

for some $F' > 0$. If on the other hand we have $k = j$ and use Claim 3.3 and Claim 3.7 we get $\forall T > 0$

$$\mathbb{P}\left[\left|\int_0^{T\wedge\tau^*} (dH_k, dH_k) - T\wedge\tau^*\right| \geq \epsilon^{1/2}\right]$$

$$= \mathbb{P}\left[\left|\int_0^{T\wedge\tau^*} S_\infty\epsilon^{-1}\langle\sigma_k(h,v), \sigma_k(h,v)\rangle\, dt'\right.\right.$$
$$\left. -T\wedge\tau^* - ((T - T\wedge\tau^*) - (T - T\wedge\tau^*)) \geq \epsilon^{1/2} \leq C_q\epsilon^q.\right.$$

We note using Levy's characterisation of N-dimensional Brownian motion that $(Z_1(t), ..., Z_N(t))$ is an N-dimensional Brownian motion.

This is implies that $\left(\hat{h}_1(t), ..., \hat{h}_N(t)\right)$ stopped at $\hat{\tau}^*$ weakly converges to an N-dimensional Brownian motion stopped at τ, which is their first hitting time, unless C is smaller.

We now prove that it is not a later hitting time, since if one has a minimum distance $g(\epsilon) > \xi_\epsilon$ (greater than the stopping distance but still with the property $\lim_{\epsilon\to 0} g(\epsilon) \to 0$) before the stopping time, we have the following:

Denote $\psi_j(t) = g(\epsilon) + \int_0^t \epsilon^{-1/2}\left\langle \sigma_{j+1}\left(\hat{h}, v\right) - \sigma_j(\hat{h}, v), dW\right\rangle$; this is approximately the difference between the position of the $j+1$-th and the j-th interface, started at $g(\epsilon)$ and without the drift.

We note that this is nothing else than a time-changed Brownian motion starting at $g(\epsilon)$, i.e. if $B(\cdot)$ is a standard Brownian motion, we can denote ψ_j as $g(\epsilon) + B\left(\epsilon^{-1}\int_0^t\left[\left\langle\sigma_{j+1}\left(\hat{h}, v\right), \sigma_{j+1}\left(\hat{h}, v\right)\right\rangle\right.\right.$
$\left.\left. -2\left\langle\sigma_{j+1}\left(\hat{h}, v\right), \sigma_j\left(\hat{h}, v\right)\right\rangle + \left\langle\sigma_j\left(\hat{h}, v\right), \sigma_j\left(\hat{h}, v\right)\right\rangle\right] ds\right)$

For shorthand notation, we denote
$Y_\epsilon = \epsilon^{-1}\int_0^t \left[\left\langle \sigma_{j+1}\left(\hat{h},v\right),\sigma_{j+1}\left(\hat{h},v\right)\right\rangle\right.$
$\left.-2\left\langle \sigma_{j+1}\left(\hat{h},v\right),\sigma_j\left(\hat{h},v\right)\right\rangle + \left\langle \sigma_j\left(\hat{h},v\right),\sigma_j\left(\hat{h},v\right)\right\rangle ds.\right.$

We will show that with a probability converging to 1 as $\epsilon \to 0$, the minimum of this quantity after time $(g(\epsilon))^{2-\kappa}$ for some small $\kappa > 0$ is ξ_ϵ, while the probability of this quantity becoming $\mathcal{O}(1)$ large converges to 0 as $\epsilon \to 0$.

Since Brownian fluctuations scale like square roots of their time increments, it is sufficient to show that with high probability $\psi_j(t)$ will hit ξ_ϵ after time $(g(\epsilon))^{2-\kappa}$ (plus a term accounting for the difference between this expression and the actual difference between the interfaaces) for some small $\kappa > 0$, with its maximum converging to 0 as $\epsilon \to 0$.

By [RY99] V§1 together with the fact that in the slow channel we have $0 < Y_\epsilon \le K_2\epsilon$ (using Claim 3.8) and applying Doob's martingale inequality we have for some small $\kappa > 0$:

$$\mathbb{P}\left[\inf_{t\in\left[0,(g(\epsilon))^{2-\kappa}+\left[C\sup_{s\in[0,C]}\left|\frac{S_\infty b_k\left(\hat{h},v\right)}{\epsilon^{2\gamma+1}}\right|\right]^{2-\kappa}\wedge\hat{\tau}^*\right]}\psi_j(t) \le \xi_\epsilon - C\sup_{s\in[0,C]}\left|\frac{S_\infty b_k\left(\hat{h},v\right)}{\epsilon^{2\gamma+1}}\right|\right] \geq 1 - C_q\epsilon^q \ \forall q > 0.$$

However we note that by Claim 3.7 if $\hat{h}_{j+1} - \hat{h}_j$ has the same initial value as $\psi_j(t)$ where

$$\mathbb{P}\left[\sup_{t\in\left[0,(g(\epsilon))^{2-\kappa}+\left[C\sup_{s\in[0,C]}\left|\frac{S_\infty b_k\left(\hat{h},v\right)}{\epsilon^{2\gamma+1}}\right|\right]^{2-\kappa}\wedge\hat{\tau}^*\right]}\left|\psi_j(t) - \left(\hat{h}_{j+1}-\hat{h}_j\right)\right| \le C\sup_{s\in[0,C]}\left|\frac{S_\infty b_k\left(\hat{h},v\right)}{\epsilon^{2\gamma+1}}\right|\right] \geq 1 - C_q\epsilon^q \ \forall q > 0,$$

and hence

$$\mathbb{P}\left[\inf_{t\in\left[0,(g(\epsilon))^{2-\kappa}+\left[C\sup_{s\in[0,C]}\left|\frac{S_\infty b_k\left(\hat{h},v\right)}{\epsilon^{2\gamma+1}}\right|\right]^{2-\kappa}\wedge\hat{\tau}\right]}\hat{h}_{j+1}-\hat{h}_j \le \xi_\epsilon\right] \geq 1 - C_q\epsilon^q \ \forall q > 0,$$

which trivially implies

$$\mathbb{P}\left[\inf_{t\in[0,\hat{\tau}^*]}\hat{h}_{j+1}-\hat{h}_j \le \xi_\epsilon\right] \geq 1 - C_q\epsilon^q \ \forall q > 0.$$

Therefore, as $\epsilon \to 0$, any starting configuration with the minimum distance converging to 0 as $\epsilon \to 0$ ends up at the critical stopping distance ξ_ϵ with high probability within a time that converges to 0 on the fast timescale in the sharp interface limit. Thus it is necessarily the first hitting time (unless C is smaller than the first hitting time).

We now show the convergence of $\hat{\tau}^*$ to a limit τ in the sharp interface limit: Firstly we recall the definition

$$\hat{\tau}^* = C \wedge \inf \Big\{ t > 0 : \min_{i \neq j} \left| \hat{h}_i(t) - \hat{h}_j(t) \right| = \xi_\epsilon$$

$$\text{or } \|v\| > \epsilon^{3/2+\delta} \text{ or } \|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{3/2+\delta} \Big\}$$

and note that since the probability of $\|v\| > \epsilon^{3/2+\delta}$ or $\|v\|_\infty \geq 1$ or $\|z\| > \epsilon^{3/2+\delta}$ is asymptotically small in ϵ, and converges to 0 as $\epsilon \to 0$, the limit of $\hat{\tau}^*$ - if it exists - is the same as the limit of $C \wedge \inf \left\{ t > 0 : \min_{i \neq j} \left| \hat{h}_i(t) - \hat{h}_j(t) \right| = \xi_\epsilon \right\}$.

Secondly define

$$\Sigma_0 \left(f_1(\cdot), ..., f_N(\cdot) \right) = \inf \left\{ t : \min_{i \neq j} \left| f_i(t) - f_j(t) \mod 1 \right| = 0 \right\}.$$

We also define $\Xi = \inf\{t : \delta_h \leq \frac{4\gamma\epsilon|\log \epsilon|}{\sqrt{-f'(1)}}\}$.

Then for all $\epsilon > 0$ we trivially have $(2\gamma + \beta)\epsilon |\log \epsilon| < 4\gamma\epsilon |\log \epsilon|$ and hence we have by Claim 3.7

$$\mathbb{P}\left[\hat{\tau}^* \geq \Sigma_0 \left(\hat{h}_1(t \wedge \hat{\tau}^* \wedge \Xi), ..., \hat{h}_N(t \wedge \hat{\tau}^* \wedge \Xi) \right) \wedge C \right] \geq 1 - C_q \epsilon^q \ \forall q > 0,$$

since this event could only happen if $\|v(\hat{\tau}^*)\| > \epsilon^{3/2+\delta}$ or $\|v(\hat{\tau}^*)\|_\infty \geq 1$.

We note that $\Sigma_0 \left(\hat{h}_1(t \wedge \Xi \wedge \hat{\tau}^*), ..., \hat{h}_N(t \wedge \Xi \wedge \tau^*) \right) \to_{\epsilon \to 0} \Sigma_0(M_1, ..., M_N)$ weakly, since $\Sigma_0(M_1, ..., M_N)$ is only discontinuous on a set of Lebesgue measure 0 so we may apply [Bil99] Theorem 2.7.

Hence in the limit

$$\mathbb{P}\left[\tau \geq C \wedge \Sigma_0(M_1, ..., M_N) \right] = 1.$$

However we also have $0 < (2\gamma + \beta)\epsilon |\log \epsilon|$ for $\epsilon > 0$ and thus by the same argument

$$\mathbb{P}\left[C \wedge \Sigma_0 \left(\hat{h}_1, ..., \hat{h}_N \right) \geq \hat{\tau}^* \right] \geq 1 - C_q \epsilon^q \ \forall q > 0,$$

which implies in the limit that

$$\mathbb{P}\left[C \wedge \Sigma_0(M_1, ..., M_N) \geq \tau \right] = 1.$$

We have now shown that

$$\mathbb{P}\left[C \wedge \Sigma_0(M_1, ..., M_N) = \tau \right] = 1,$$

i.e. that $\hat{\tau}^*$ weakly converges to $C \wedge \Sigma_0 (M_1, ..., M_N)$, which is the first hitting time, unless C is smaller.

We now finally note that τ is positive and finite: This is an easy observation, since the definition gives the upper bound $\hat{\tau}^* \leq C$ on the fast timescale, while due to the finite moments of M_k it almost surely takes at least finite time for two interfaces to attain the distance 0. Therefore τ is positive and finite with probability 1.

Notation

For the ease of presentation, we denote the quadratic covariation of x and y as (dx, dy).

Derivation of the semimartingale expression

We assume that the interfaces h_k have a semimartingale notation. This will be properly proven after the sketch of argument of Claim 3.7, to first be able to guess what this semimartingale notation is we assume this and prove it later on.

$$dh_k = b_k (h, v)\, dt + \langle \sigma_k (h, v), dV \rangle$$

By the Itô formula, we have

$$du = \sum_{j=1}^{N} u_j^h dh_j + \frac{1}{2} \sum_{1 \leq k,l \leq N} u_{kl}^h (dh_k, dh_l) + dv \tag{3.1}$$

with $u_{kl}^h = \frac{\partial^2 u^h}{\partial h_k \partial h_l}$.

We now take the inner product in space of (1.1) with τ_i^h to get for any $i = 1, ..., N$

$$\left\langle \tau_i^h, du \right\rangle = \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle,$$

where we defined $\mathcal{L}(u) := \epsilon^2 u_{xx} + f(u)$.

Taking the inner product with (3.1), we obtain

$$\left\langle \tau_i^h, du \right\rangle = \sum_{j=1}^{N} \left\langle u_j^h, \tau_i^h \right\rangle dh_j + \frac{1}{2} \sum_{1 \leq k,l \leq N} \left\langle u_{kl}^h, \tau_i^h \right\rangle (dh_k, dh_l) + \left\langle \tau_i^h, dv \right\rangle. \tag{3.2}$$

In the remainder of this chapter, any summation is on $1, 2, ..., N$ for any index.

To eliminate dv, we apply the Itô-formula to the orthogonality condition $\left\langle v, \tau_i^h \right\rangle = 0$, and obtain

$$\begin{aligned}\left\langle \tau_i^h, dv \right\rangle &= -\left\langle v, d\tau_i^h \right\rangle - \left\langle dv, d\tau_i^h \right\rangle \\ &= -\sum_j \left\langle v, \tau_{ij}^h \right\rangle dh_j - \frac{1}{2} \sum_{j,k} \left\langle v, \tau_{ijk}^h \right\rangle (dh_j, dh_k) - \sum_j \left\langle \tau_{ij}^h, dv \right\rangle dh_j.\end{aligned}$$

Now we use $dv = du - du^h$ and $(dt, dt) = 0$ as well as $(dV, dt) = 0$, i.e.

$$
\begin{aligned}
&-\sum_j \left(\left\langle \tau_{ij}^h, dv\right\rangle, dh_j\right) \\
&= -\sum_j \left(\left\langle \tau_{ij}^h, du\right\rangle, dh_j\right) + \sum_j \left(\left\langle \tau_{ij}^h, du^h\right\rangle, dh_j\right) \\
&= -\sum_j \left\langle \tau_{ij}^h, \mathcal{L}(u)\right\rangle (dt, dh_j) - \sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) + \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h\right\rangle (dh_k, dh_j) \\
&= -\sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) + \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h\right\rangle (dh_k, dh_j), \qquad (3.3)
\end{aligned}
$$

where the inner product of the stochastic Allen-Cahn equation with u_{ij}^h was taken and the following used:

$$
(dh_j, dt) = b_j(h, v)(dt, dt) + (\langle \sigma_j(h, v), dV\rangle, dt) = 0.
$$

Hence we have

$$
\begin{aligned}
\left\langle \tau_i^h, dv\right\rangle = &-\sum_j \left\langle v, \tau_{ij}^h\right\rangle dh_j - \frac{1}{2}\sum_{j,k} \left\langle v, \tau_{ijk}^h\right\rangle (dh_j, dh_k) \\
&-\sum_j \left(\left\langle dV, \tau_{ij}^h\right\rangle, dh_j\right) + \sum_{j,k} \left\langle u_k^h, \tau_{ij}^h\right\rangle (dh_j, dh_k). \qquad (3.4)
\end{aligned}
$$

Combining this with (3.2) and (3.3) yields

$$
\begin{aligned}
&\sum_j \left[\left\langle u_j^h, \tau_i^h\right\rangle - \left\langle v, \tau_{ij}^h\right\rangle\right] dh_j = \left\langle \mathcal{L}(u), \tau_i^h\right\rangle dt \\
&+\sum_{l,k} \left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right] (dh_l, dh_k) \\
&+\sum_j \left(\left\langle dV, \tau_{ij}^h\right\rangle, dh_j\right) + \left\langle \tau_i^h, dV\right\rangle. \qquad (3.5)
\end{aligned}
$$

Claim 3.3. *For all $k, l \le N$ we have*

$$
(\langle \sigma_k(h, v), dV\rangle, \langle \sigma_l(h, v), dV\rangle) = \epsilon^{2\gamma} \langle \sigma_k(h, v), \sigma_l(h, v)\rangle dt.
$$

Sketch of Argument. Since $(d\beta_j, d\beta_i) = \delta_{ij} dt$ and $W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$ for an orthonormal basis $\{e_k(\cdot)\}_{k=1}^{\infty}$ of $L^2(0,1)$ and independent Brownian motions $\{\beta_k\}_{k=1}^{\infty}$, Parseval's identity yields the following:

$$
\begin{aligned}
&(\langle \sigma_k(h, v), dV\rangle, \langle \sigma_l(h, v), dV\rangle) \\
&= \epsilon^{2\gamma} \sum_{i,j} \langle \sigma_k(h, v), e_i\rangle \langle \sigma_l(h, v), e_j\rangle (d\beta_j, d\beta_i)
\end{aligned}
$$

$$= \epsilon^{2\gamma} \sum_j \langle \alpha_k(h,v), e_j \rangle \langle \sigma_l(h,v), e_j \rangle dt$$

$$= \epsilon^{2\gamma} \sum_j \langle \sigma_k(h,v), e_j \rangle \langle \sigma_l(h,v), e_j \rangle dt = \epsilon^{2\gamma} \langle \sigma_k(h,v), \sigma_l(h,v) \rangle dt.$$

In an analogous way to this claim one can easily obtain (using $(dt, dV) = 0$)

$$\left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) = \left(\left\langle \tau_{ij}^h, dV \right\rangle, \langle \sigma_j(h,v), dV \rangle \right) = \epsilon^{2\gamma} \left\langle \tau_{ij}^h, \sigma_j(h,v) \right\rangle dt.$$

For short-hand notation, we define the matrix $A(h,v) = (A_{ij}(h,v)) \in \mathbb{R}^{N\times N}$ by

$$A_{ij}(h,v) = \left\langle u_j^h, \tau_i^h \right\rangle - \left\langle v, \tau_{ij}^h \right\rangle, \tag{3.6}$$

For an invertibility condition of this matrix, see Claim 3.6; the stability shown in Claim 4.4 implies that the condition of Claim 3.6 does indeed hold true in Γ'_ϵ with a probability converging to 1 as $\epsilon \to 0$ until two interfaces are at distance ξ_ϵ. The inverse matrix of $A(h,v)$ is then denoted by $A^{-1}(h,v) = \left(A_{ij}^{-1}(h,v)\right) \in \mathbb{R}^{N\times N}$.

We now arrive for all $i \in \{1, ..., N\}$ at

$$\sum_j A_{ij}(h,v)\, dh_j$$

$$= \left\langle \mathcal{L}\left(u^h + v\right), \tau_i^h \right\rangle dt + \sum_{l,k} \left[\frac{1}{2} \left\langle v, \tau_{ilk}^h \right\rangle - \frac{1}{2} \left\langle u_{kl}^h, \tau_i^h \right\rangle - \left\langle u_k^h, \tau_{il}^h \right\rangle \right] \epsilon^{2\gamma} \langle \sigma_k(h,v), \sigma_l(h,v) \rangle dt$$

$$+ \sum_j \epsilon^{2\gamma} \left\langle u_{ij}^h, \sigma_j(h,v) \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle.$$

To obtain the equation for dh we use that $dh = A(h,v)^{-1} A(h,v)\, dh$.

Therefore the final equation for h (given that u is inside the slow channel) is given for any $r = 1, ...N$ by

$$\begin{aligned} dh_r = & \sum_i A_{ri}^{-1}(h,v) \left\langle \mathcal{L}\left(u^h + v\right), \tau_i^h \right\rangle dt \\ & + \epsilon^{2\gamma} \sum_{i,l,k} A_{ri}^{-1}(h,v) \left[\frac{1}{2} \left\langle v, \tau_{ilk}^h \right\rangle - \frac{1}{2} \left\langle u_{kl}^h, \tau_i^h \right\rangle - \left\langle u_k^h, \tau_{il}^h \right\rangle \right] \langle \sigma_k(h,v), \sigma_l(h,v) \rangle dt \\ & + \epsilon^{2\gamma} \sum_i A_{ri}^{-1}(h,v) \sum_j \left\langle \tau_{ij}^h, \sigma_j(h,v) \right\rangle dt + \sum_i A_{ri}^{-1}(h,v) \left\langle \tau_i^h, dV \right\rangle. \end{aligned} \tag{3.7}$$

Relating this to the original ansatz means

$$\sigma_r(h,v) = \sum_i A_{ri}^{-1}(h,v)\, \tau_i^h. \tag{3.8}$$

Therefore one can write down $b_r(h,v)$ in terms of $\sigma_j(h,v)$:

$$b_r(h,v) = \sum_i A_{ri}^{-1}(h,v) \left\langle \mathcal{L}\left(u^h + v\right), \tau_i^h \right\rangle$$

$$+\sum_{i,l,k} A_{ri}^{-1}(h,v)\left[\frac{1}{2}\left\langle v,\tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h,\tau_i^h\right\rangle - \left\langle u_k^h,\tau_{il}^h\right\rangle\right]\epsilon^{2\gamma}\left\langle\sigma_k(h,v),\sigma_l(h,v)\right\rangle$$
$$+\sum_i A_{ri}^{-1}(h,v)\,\epsilon^{2\gamma}\sum_j\left\langle\tau_{ij}^h,\sigma_j(h,v)\right\rangle \tag{3.9}$$

3.2 Stability of the manifold

This section gives many key ingredients to the sketch of argument of Claim 3.2.

Recall $\xi_\epsilon = \frac{(2\gamma+\beta)\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ and that the slow channel $\Gamma_{\xi_\epsilon,\epsilon}$ is defined

$$\Gamma_{\xi_\epsilon,\epsilon} := \left\{u^h + v \,:\, h\in\Omega_{\xi_\epsilon}, \|v\| \le \epsilon^{3/2+\delta}\right\}.$$

for some fixed $\delta > 0$ s.t. $\gamma - 1/2 < 3/2+\delta$

We will now give sufficient conditions for the derivation in the previous step to actually hold true under the assumption of h being a semimartingale, namely invertibility of the matrix $A(h,v)$; further to this an asymptotic expansion of the drift and diffusion "coefficients" will be obtained inside the slow channel. Finally it will be shown that the probability of u leaving the slow channel $\Gamma_{\xi_\epsilon,\epsilon}$ before two interfaces are at distance ξ_ϵ is asymptotically small in ϵ; afterwards we will also show that h is indeed a semimartingale and thus the derivation in the previous part was valid. The more general stability for $\delta_h > \rho^{**}\epsilon$ is shown in Claim 4.4.

Now we recall the definition of τ^* as the first exit time (below the threshold T_ϵ) of u from $\Gamma_{\xi_\epsilon,\epsilon}$, which is the stopping time

$$\tau^* = CS_\infty\epsilon^{-2\gamma-1}\wedge\inf\Big\{t>0 : \delta_h = \xi_\epsilon \text{ or } \|v\| > \epsilon^{3/2+\delta}$$
$$\text{or } \|v\|_\infty \ge 1 \text{ or } \|z\| > \epsilon^{3/2+\delta}\Big\}.$$

Definition 3.4. *We say that a term is $\mathcal{O}(e_\epsilon)$ if it is asymptotically smaller than any polynomial uniformly for times $t \le \tau^*$.*

Definition 3.5. $l_j = h_j - h_{j-1},\ l := \min\{l_1,...,l_N\},\ r := \epsilon/l,\ \beta := 1 - \phi(0,l,+)$ *and* $\alpha(r) := F(\phi(0,l,+)),\ \phi^j(x) := \phi\left(x-m_j,l_j,(-1)^j\right),\ r_j := \epsilon/l_j$

$\beta^j(r) := \beta(r_j)$ *and* $\beta(r) := \max_j \beta^j(r)$.

$\alpha^j(r) := \alpha(r_j)$ *and* $\alpha(r) := \max_j \alpha^j(r)$.

Note that $\alpha,\beta,\|\mathcal{L}(u^h)\|_\infty$ are $\mathcal{O}(1)$ if $\delta_h > \rho^{**}\epsilon$ and $\mathcal{O}(\epsilon^{2\gamma})$ for $\delta_h > \xi_\epsilon$ (cf. [CP89] Theorem 3.5).

Claim 3.6. *Suppose $h\in\Omega_{\rho^{**}\epsilon}$ for ρ^{**} large enough and $\|v\| \le \zeta\epsilon^{1/2}$ for some $C_{map} > \zeta > 0$ small enough; then we have for $\delta_h > \rho^{**}\epsilon$ the expansion*

$$A_{ij}(h,v) = \begin{cases} \frac{1}{\epsilon}S_\infty + \mathcal{O}\left(\zeta\epsilon^{-1} + \epsilon^{-1}\exp\left\{\frac{-\sqrt{f'(1)}\delta_h}{2\epsilon}\right\}\right) & \text{if } i=j \\ o\left(\zeta\epsilon^{-1}\right) & \text{if } i\neq j \end{cases}$$

and the matrix is invertible, with

$$A_{ij}^{-1}(h,v) = \begin{cases} \frac{\epsilon}{S_\infty + \mathcal{O}\left(\zeta + C\exp\left\{\frac{-\sqrt{f'(1)}\delta_h}{2\epsilon}\right\}\right)} & \text{if } i = j. \\ o(\zeta\epsilon) \ \text{if } i \neq j. \end{cases}$$

We have $S_\infty = \int_{-1}^{1}\sqrt{-2F(u)}du$*, which in our typical case* $f(u) = u - u^3$ *is*

$$S_\infty = \frac{2\sqrt{2}}{3}.$$

Sketch of Argument. We recall $A_{ij} = \langle u_j^h, \tau_i^h\rangle - \langle v, \tau_{ij}^h\rangle$.

We have from [CP89] Theorem 3.5, Proposition 3.4 (note that in this publication the constant ρ denotes the inverse of our constant ρ in $\rho\epsilon$ so that $\rho \to 0^+$ has the equivalent effect as $\epsilon \to 0$) that

$$\langle u_j^h, \tau_j^h\rangle = S_\infty\epsilon^{-1} + \mathcal{O}\left(\epsilon^{-1}e^{-\sqrt{f'(1)}\delta_h/2\epsilon}\right),$$

and by [CP89] Theorem 3.5 for $|j-k| = 1$ that

$$\left|\langle u_j^h, \tau_k^h\rangle\right| \leq C\epsilon^{-1}e^{-\sqrt{f'(1)}\delta_h/2\epsilon},$$

and by [CP89] Theorem 3.5 that for $|j-k| > 1$ we have

$$\left|\langle u_j^h, \tau_k^h\rangle\right| = 0.$$

Using that (cf. [CP89] Proposition 2.3) $\left\|\tau_{jj}^h\right\| \leq C\epsilon^{-3/2}$ for some $C > 0$, and if $j \neq k$, $\left\|\tau_{jk}^h\right\| \leq o(1)\epsilon^{-3/2}$ (by [$CP89$] Proposition 2.3), we obtain for $j = k$ that

$$\left|\langle v, \tau_{ij}^h\rangle\right| \leq \mathcal{O}\left(\zeta\epsilon^{-1}\right)$$

and for $j \neq k$ that

$$\left|\langle v, \tau_{ij}^h\rangle\right| \leq o\left(\zeta\epsilon^{-1}\right).$$

We hence have the expressions

$$A_{ii} = S_\infty\epsilon^{-1} + \mathcal{O}\left(\epsilon^{-1}e^{-\sqrt{f'(1)}\delta_h/2\epsilon}\right) + \mathcal{O}\left(\zeta\epsilon^{-1}\right),$$

$$A_{ij} = \mathcal{O}\left(\epsilon^{-1}e^{-\sqrt{f'(1)}\delta_h/2\epsilon}\right) + o\left(\zeta\epsilon^{-1}\right) \ \text{if } |i-j| = 1,$$

$$A_{ij} = o\left(\zeta\epsilon^{-1}\right) \ \text{if } |i-j| > 1.$$

This gives invertibility of the matrix and the stated formula for its inverse.

Let $\psi_d = CS_\infty\epsilon^{-2\gamma-1} \wedge \inf\{t > 0 : \delta_{h(t)} = d \text{ or } \|v\| > \epsilon^{3/2+\delta}$ $\|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{3/2+\delta}\}$ where $d \geq \frac{2\epsilon|\log\epsilon|}{\sqrt{f'(1)}}$.

Claim 3.7. *Suppose $u_0 \in \Gamma_{\xi_\epsilon,\epsilon}$. Let $u(x,t) = u^h(x,t) + v(x,t)$ solve (1.1) for $t > 0$; τ^* is as previously defined.*

Then there exist constants C'_p, D_p depending only on $f(\cdot)$ such that the following inequalities hold:

$$\mathbb{E}\left[\|v(\psi_d)\|^p\right] \leq \mathbb{E}[\|v(0)\|^p] + C'_p \epsilon^{-2\gamma-1}\epsilon^{(3/2+\delta)p} \leq D_p \epsilon^{-2\gamma-1}\epsilon^{(3/2+\delta)p} \; \forall p \geq 2$$

for some $C'_p, D_p > 0$ independent of ϵ and dependent on p, as well as

$$\mathbb{P}(\|v(\psi_d)\| > \epsilon^{3/2+\delta} \text{ or } \|z(\psi_d)\| > \epsilon^{3/2+\delta}) \leq D_r \epsilon^{-2\gamma-1}\epsilon^{2\delta r} \; \forall r \geq 2,$$

$$\mathbb{P}(\|v(\psi_d)\| > \epsilon^{3/2+\delta} \text{ or } \|z(\psi_d)\| > \epsilon^{3/2+\delta} \text{ or } \|v(\psi_d)\|_\infty \geq 1) \leq D_r \epsilon^{-2\gamma-1}\epsilon^{2\delta r} \; \forall r \geq 2,$$

for some small $\delta > 0$.

Remark: The second inequality implies that the probability of exiting the slow channel within the $\mathcal{O}(\epsilon^{-2\gamma-1})$ time needed for two interfaces to have ξ_ϵ distance is asymptotically small.

Before proving this, we collect some claims that will ease the sketch of argument:

Claim 3.8. *Let $u = u^h + v \in \Gamma_{\xi_\epsilon,\epsilon}$ and $r = 1, ..., N$, then there exists $C > 0$ independent of ϵ s.t. it holds uniformly in ϵ that*

$$\left\|\sigma_r(h,v) - \frac{\epsilon}{S_\infty}\tau_r^h\right\|_\infty \leq o(\epsilon)$$

and

$$\|\sigma_r(h,v)\| = C\sqrt{\epsilon} + o(\epsilon^{1/2}).$$

If instead we only have $u = u^h + v \in \Gamma'_\epsilon$ then we have

$$\left\|\sigma_r(h,v) - \frac{\epsilon}{S_\infty}\tau_r^h\right\|_\infty \leq \mathcal{O}(1)$$

and

$$\|\sigma_r(h,v)\| \leq \mathcal{O}(\epsilon^{1/2}).$$

Sketch of Argument. In the first case we have $\|v\| \leq \epsilon^{3/2+\delta}$ and $\delta_h \geq \xi_\epsilon$. Upon recalling that $\sigma_r(h,v) = \sum_i A^{-1}_{ri}(h,v)\tau_i^h$, we can apply Claim 3.6 to get

$$\sigma_r(h,v) = \left(\frac{\epsilon}{S_\infty + \mathcal{O}\left(\epsilon^{5/2+\delta} + \epsilon^\gamma\right)}\right)\tau_r^h + \sum_{i\neq r} o\left(\epsilon^{2+\delta}\right)\tau_i^h.$$

Thus, upon noting that $\left|\tau_i^h\right| \leq \mathcal{O}(\epsilon^{-1})$ we have

$$\left\|\sigma_r(h,v) - \epsilon S_\infty^{-1}\tau_r^h\right\|_\infty$$

$$= \left\| \left(\frac{\epsilon}{S_\infty + \mathcal{O}\left(\epsilon^\gamma + \epsilon^{5/2+\delta}\right)} - \epsilon S_\infty^{-1} \right) \tau_r^h + \sum_{i \neq r} o\left(\epsilon^{2+\delta}\right) \tau_i^h \right\|_\infty$$

$$\leq o(\epsilon).$$

The L^2 norm follows similarly, upon noting that $\|\tau_i^h\| \leq \mathcal{O}(\epsilon^{-1/2})$:

$$\|\sigma_r(h,v)\| - \left\| \frac{\epsilon}{S_\infty} \tau_r^h \right\| = \left\| \frac{\epsilon}{S_\infty + C\epsilon^\gamma + \epsilon^2} \tau_r^h + \sum_{i \neq r} o(\epsilon) \tau_i^h \right\| - \left\| \frac{\epsilon}{S_\infty} \tau_r^h \right\| \leq o(\epsilon^{1/2}),$$

clearly, by [CP89] Proposition 2.3, $\left\| \frac{\epsilon}{S_\infty} \tau_r^h \right\| = C\epsilon^{1/2} + o(\epsilon^{1/2})$. This gives the bound on the inner product.

If instead we only have $\|v\| \leq C_{map}\epsilon^{1/2}$ then Claim 3.6 implies

$$\sigma_r(h,v) = \left(\epsilon S_\infty^{-1} + \mathcal{O}(\epsilon)\right) \tau_r^h + \sum_{i \neq r} o(\epsilon) \tau_i^h.$$

Once again, recalling that $\|\tau_i^h\|_\infty \leq \mathcal{O}(\epsilon^{-1})$ and $\|\tau_i^h\| \leq \mathcal{O}(\epsilon^{-1/2})$ gives

$$\left\|\sigma_r(h,v) - \epsilon S_\infty^{-1} \tau_r^h\right\|_\infty = \left\| \mathcal{O}(\epsilon) \tau_r^h + \sum_{i \neq r} o(\epsilon) \tau_i^h \right\|_\infty \leq \mathcal{O}\left(\epsilon^{1/2}\right)$$

and

$$\|\sigma_r(h,v)\| = \left\| \left(\epsilon S_\infty^{-1} + \mathcal{O}(\epsilon)\right) \tau_r^h + \sum_{i \neq r} o(\epsilon) \tau_i^h \right\| \leq \mathcal{O}\left(\epsilon^{1/2}\right).$$

We now define (just like [CP89]) w^j for each interval $[m_j, m_{j+1}]$ as

$$w^j(x) = w\left(x - m_j, h_j - h_{j-1}, (-1)^j\right),$$

where for $x \in [-l, l]$ we have w defined as

$$2\phi_l(x, l \pm 1) = -(sgn(x))\phi_x(x, l, \pm 1) + 2w(x, l, \pm 1).$$

We will from now on denote derivatives w.r.t. h_j as $\frac{\partial}{\partial h_j} f = f_j$.

Claim 3.9. *Let $u^h + v \in \Gamma_{\frac{2\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}, \epsilon}$, then we have the following pointwise estimate:*

$$|b_r(h,v)| \leq k\epsilon e^{-\sqrt{f'(1)}\delta_h/\epsilon} + \mathcal{O}\left(\epsilon^{2\gamma+1+\beta}\right)$$

for any $r = 1, ..., N$, some $C > 0$ and a small $\beta > 0$. As a special case, if we have $u^h + v \in \Gamma_{\xi_\epsilon, \epsilon}$ we also have the following uniform estimate:

$$\sup_{h \in \Omega_{\xi_\epsilon}} |b_r(h,v)| \leq C\epsilon^{2\gamma+1+\beta} \tag{3.10}$$

Sketch of Argument. We recall

$$b_r(h,v) = \sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}(u^h+v), \tau_i^h \right\rangle$$

$$+\sum_{i,l,k} A_{ri}^{-1}(h,v)\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right]\epsilon^{2\gamma}\left\langle \sigma_k(h,v), \sigma_l(h,v)\right\rangle$$

$$+\sum_i A_{ri}^{-1}(h,v)\sum_j \epsilon^{2\gamma}\left\langle \tau_{ij}^h, \sigma_j(h,v)\right\rangle.$$

It is easy to check that unless we have $r = i = l = k$ all contributing terms in the second two sums are $\mathcal{O}(\epsilon^{2\gamma+1+\beta})$.

We quote [CP89] Lemma 8.1 and recall the definition of u^h for the following: $u_j^h = -u_x^h + (1-\chi^j)w^j - \chi^j w^{j+1}$ for $x \in I_j$.

We know from [CP89] Lemma 7.9, Proposition 3.4, that $w^j \leq \mathcal{O}(\epsilon)$ and from [CP89] Lemma 7.10, Proposition 3.4 that $w_x^j \leq \mathcal{O}\left(\frac{|\log \epsilon|}{\epsilon}\right)$ for $\delta_h \geq \frac{2|\log \epsilon|}{\sqrt{-f'(0)}}$.

We also have
$u_{jx}^h = -u_{xx}^h + (1-\chi^j)w_x^j - \chi_x^j w^j - \chi_x^j w^{j+1} - \chi^j w_x^{j+1}$
for $x \in I_j$.

Evidently,

$$\begin{aligned} u_{ii}^h &= -u_{xi}^h - \chi_i^i w^i + \left(1-\chi^i\right) w_i^i - \chi_i^i w^{i+1} - \chi^i w_i^{i+1} \\ &= -u_{xx}^h + \left(1-\chi^i\right) w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x^{i+1} \\ &\quad -\chi_i^i w^i + \left(1-\chi^i\right) w_i^i - \chi_i^i w^{i+1} - \chi^i w_i^{i+1}. \end{aligned}$$

We recall $\tau_j^h = \gamma^j u_x^h$ (which is clearly equal to 0 outside of I_j) where

$$\gamma^j = \chi\left((x-m_j)/\epsilon - 1\right)\left[1 - \chi((x-m_{j+1})/\epsilon - 1)\right],$$

and quote from [CP89] p. 564 that

$$\tau_{ii}^h = -\gamma_i^i u_x^h - \gamma^i u_{xi}^h = -\gamma_i^i u_x^h - \gamma^i\left(-u_{xx}^h + \left(1-\chi^i\right) w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x\right)$$

$$= u_{xx}^h - \gamma_i^i u_x^h + \left(\gamma^i - 1\right) u_{xx}^h - \gamma^i\left(\left(1-\chi^i\right) w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x\right),$$

which is nothing else than $\tau_{ii}^h = \gamma^i u_{xx}^i + \mathcal{O}\left(\epsilon^{-1}\right)$.

$u^h = \left(1-\chi^j\right)\phi^j + \chi^j \phi^{j+1}$

For $r = i = k = l$ we obtain the following expressions:

$$\left\langle \tau_{jj}^h, u_j^h \right\rangle = \int_{m_j}^{m_{j+1}} \tau_{jj}^h u_j^h dx$$

$$= \int_{m_j}^{m_j+2\epsilon} \tau_{jj}^h u_j^h dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{jj}^h u_j^h dx + \int_{m_{j+1}-2\epsilon}^{m_{j+1}} \tau_{jj}^h u_j^h dx,$$

$$\left\langle u_{jj}^h, \tau_j^h \right\rangle = \int_{m_j}^{m_{j+1}} u_{jj}^h \tau_j^h dx$$

$$= \int_{m_j}^{m_j+2\epsilon} u_{jj}^h \tau_j^h dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{jj}^h u_j^h dx + \int_{m_{j+1}-2\epsilon}^{m_{j+1}} u_{jj}^h \tau_j^h dx.$$

We calculate further that

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{jj}^h u_j^h dx$$

$$= \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} \left[-u_{xj}^h - \chi_j^j w^j + \left(1-\chi^j\right) w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1} \right] u_j^h dx$$

$$= \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} -u_{xj}^h u_j^h dx$$

$$+ \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} \left[-\chi_j^j w^j + \left(1-\chi^j\right) w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1} \right] u_j^h dx$$

$$= \left[-\frac{1}{2} \left(u_i^h\right)^2 \right]_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}$$

$$+ \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} \left[-\chi_j^j w^j + \left(1-\chi^j\right) w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1} \right]$$

$$\cdot \left[\left(1-\chi^j\right) \phi_j^j + \chi^j \phi_j^{j+1} + \chi_x^j \left(\phi^j - \phi^{j+1}\right) \right] dx,$$

but

$$\left[-\frac{1}{2} \left(u_j^h\right)^2 \right]_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} = -\frac{1}{2} \left(\left(u_j^h \left(m_{j+1} - 2\epsilon\right)\right)^2 - \left(u_j^h \left(m_j + 2\epsilon\right)\right)^2 \right)$$

$$= \frac{1}{2} \left(\left(\phi_x^{j+1} \left(m_{j+1} - 2\epsilon\right) + w^{j+1} \left(m_{j+1} - 2\epsilon\right)\right) \right)^2$$

$$-\frac{1}{2} \left(2 \left(-\phi_x^j \left(m_j + 2\epsilon\right) + w^j \left(\left(m_j + 2\epsilon\right)\right) \right) + \phi_x^{j+1} \left(m_j + 2\epsilon\right) - w^{j+1} (m_j + 2\epsilon \right)^2$$

$$\leq \left(\mathcal{O}\left(\epsilon\right)\right)^2 \leq \mathcal{O}\left(\epsilon^2\right),$$

(the end works in a similar manner to [CP89] Lemma 7.9) and

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} \left[-\chi_j^j w^j + \left(1-\chi^j\right) w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1} \right]$$

$$\cdot\left[\left(1-\chi^j\right)\phi_j^j+\chi^j\phi_j^{j+1}+\chi_x^j\left(\phi^j-\phi^{j+1}\right)\right]dx$$

$$=\int_{m_j+2\epsilon}^{h_j-\epsilon\vee m_{j+1}-2\epsilon} w_j^j\phi_j^j dx+$$

$$+\int_{h_j-\epsilon\vee m_{j+1}-2\epsilon}^{h_j+\epsilon\wedge m_{j+1}-2\epsilon}\left[-\chi_j^j w^j+\left(1-\chi^j\right)w_j^j-\chi_j^j w^{j+1}-\chi^j w^{j+1}\right]$$

$$\cdot\left[\left(1-\chi^j\right)\phi_j^j+\chi^j\phi_j^{j+1}+\chi_x^j\left(\phi^j-\phi^{j+1}\right)\right]dx$$

$$+\int_{h_j+\epsilon\wedge m_{j+1}-2\epsilon}^{m_{j+1}-2\epsilon} -w^{j+1}\phi_j^{j+1}dx=\mathcal{O}\left(\epsilon^{-1+\beta}\right)$$

(cf. [CP89] sections 7 and 8 for more details on how to do the arithmetic).
Now,

$$\int_{m_i}^{m_i+2\epsilon}\tau_{ii}^h u_i^h dx=\int_{m_i}^{m_i+2\epsilon}\left(-\gamma_i^i\phi_x^i+\gamma^i\phi_{xx}^i-\gamma^i w_x^i\right)\left(-\phi_x^i+w^i\right)dx=$$

$$-\int_{m_i}^{m_i+2\epsilon}\gamma^i\phi_x^i\phi_{xx}^i dx$$

$$+\int_{m_i}^{m_i+2\epsilon}\left[\left(-\gamma_i^i\phi_x^i+\gamma^i\phi_{xx}^i-\gamma^i w_x^i\right)w^i+\left(-\gamma_i^i\phi_x^i-\gamma^i w_x^i\right)\left(-\phi_x^i+w^i\right)\right]dx$$

$$=\mathcal{O}(\epsilon^{-1+\beta}).$$

But in a similar manner we may bound

$$\int_{m_i}^{m_i+2\epsilon}u_{ii}^h\tau_i^h dx,\int_{m_{i+1}-2\epsilon}^{m_{i+1}}\tau_{ii}^h u_i^h dx,\int_{m_{i+1}-2\epsilon}^{m_{i+1}}u_{ii}^h\tau_i^h dx,$$

so that we may conclude that

$$\left|\left\langle\tau_{ii}^h,u_i^h\right\rangle\right|,\left|\left\langle u_{ii}^h,\tau_i^h\right\rangle\right|\leq o\left(\epsilon^{-1}\right).$$

We used [CP89] Lemma 7.9 to bound w^j, w^{j+1} and [CP89] Lemma 7.10 to bound w_j^j, w_j^{j+1}.

We now calculate

$$\tau_{iii}^h=-\gamma_{ii}^i u_x^h-\gamma_{ix}^i u_i^h-\gamma_i^i u_{xi}^h-\gamma^i u_{xii}^h.$$

But

$$u_{xii}^h=\frac{\partial}{\partial x}u_{ii}^h$$

$$= \frac{\partial}{\partial x}\left[-u^h_{xx} + \left(1-\chi^i\right) w^i_x - \chi^i_x w^i - \chi^i_x w^{i+1} - \chi^i w^{i+1}_x\right.$$
$$\left. -\chi^i_i w^i + \left(1-\chi^i\right) w^i_i - \chi^i_i w^{i+1} - \chi^i w^{i+1}_i\right]$$
$$= -u^h_{xxx} + \mathcal{O}\left(\epsilon^{-2}\right).$$

However we also have

$$\left\|u^h_{xxx}\right\|$$
$$= \left\|-\chi^j_{xxx}\phi^j - 3\chi^j_{xx}\phi^j_x - 3\chi^j_x\phi^j_{xx} - \chi^j\phi^j_{xxx}\right.$$
$$\left. +\chi^j_{xxx}\phi^{j+1} + 3\chi^j_{xx}\phi^{j+1}_x + 3\chi^j_x\phi^{j+1}_{xx} + \chi^j\phi^{j+1}_{xxx}\right\|$$
$$\leq \mathcal{O}(\epsilon^{-5/2})$$

since

$$\phi^j_{xxx} = -\frac{1}{\epsilon^2} f'\left(\phi^j\right)\phi^j_x.$$

Combining the fact that in the slow channel we have $\|v\| \leq \epsilon^{3/2+\delta}$ and that (cf. [CP89] Proposition 2.3) we have

$$\left\|\tau^h_{iii}\right\| \leq \left\|\gamma^i_{ii} u^h_x\right\| + \left\|\gamma^i_{ix} u^h_i\right\| + \left\|\gamma^i_i u^h_{xi}\right\| + \left\|\gamma^i u^h_{xii}\right\|$$
$$\leq \mathcal{O}\left(\epsilon^{-2}\right)\left\|u^h_x\right\| + \mathcal{O}\left(\epsilon^{-2}\right)\left\|u^h_i\right\| + \mathcal{O}\left(\epsilon^{-1}\right)\left\|u^h_{xi}\right\| + \mathcal{O}\left(1\right)\left\|-u^h_{xxx} + \mathcal{O}(\epsilon^{-2})\right\|$$
$$\leq \mathcal{O}\left(\epsilon^{-2}\right)\mathcal{O}\left(\epsilon^{-1/2}\right) + \mathcal{O}\left(\epsilon^{-2}\right)\mathcal{O}\left(\epsilon^{-1/2}\right) + \mathcal{O}\left(\epsilon^{-1}\right)\mathcal{O}\left(\epsilon^{-3/2}\right) + \mathcal{O}\left(1\right)\left\|u^h_{xxx}\right\| + \mathcal{O}\left(\epsilon^{-2}\right)$$
$$\leq \mathcal{O}\left(\epsilon^{-5/2}\right) + \mathcal{O}\left(\epsilon^{-5/2}\right) \leq \mathcal{O}\left(\epsilon^{-5/2}\right),$$

we get

$$\left|\left\langle v, \tau^h_{iii}\right\rangle\right| \leq \mathcal{O}\left(\epsilon^{-1+\beta}\right).$$

Observe that (combining [CP89] Theorem 6.1, Lemma 3.3, Proposition 3.4)

$$\left\|\sum_i A^{-1}_{ri}\left(h,v\right)\left\langle \mathcal{L}\left(u^h + v\right), u_i\right\rangle\right\|_\infty \leq \mathcal{O}\left(\epsilon e^{-\sqrt{f'(1)}\delta_h/\epsilon}\right).$$

Since $\|\sigma_k\| = \mathcal{O}(\epsilon^{1/2})$ and $A^{-1}_{ij} = \mathcal{O}(\epsilon)$ we thus obtain

$$\left\|b_r(h,v)\right\|_\infty \leq \mathcal{O}\left(\epsilon^{2\gamma+1+\beta}\right) + \mathcal{O}(\epsilon)\left(\mathcal{O}\left(\epsilon^{-1+\beta}\right) + \mathcal{O}\left(\epsilon^{4\gamma-2}\right)\right)\epsilon^{2\gamma}\epsilon + \mathcal{O}(\epsilon)\epsilon^{2\gamma}\mathcal{O}\left(\epsilon^{4\gamma-2}\right)$$
$$\leq C\epsilon^{2\gamma+1+\beta}$$

uniformly in ϵ and pointwise

$$\left|b_r(h,v)\right| \leq \mathcal{O}\left(\epsilon e^{-\sqrt{f'(1)}\delta_h/\epsilon}\right) + \mathcal{O}(\epsilon)\left(\mathcal{O}\left(\epsilon^{-1+\beta}\right) + \mathcal{O}\left(\epsilon^{4\gamma-2}\right)\right)\epsilon^{2\gamma}\epsilon + \mathcal{O}\left(\epsilon\right)\epsilon^{2\gamma}\mathcal{O}\left(\epsilon^{4\gamma-2}\right)$$
$$\leq \mathcal{O}\left(\epsilon e^{-\sqrt{f'(1)}\delta_h/\epsilon}\right) + O\left(\epsilon^{2\gamma+1+\beta}\right).$$

Sketch of Argument of Claim 3.7. Substituting $u(x,t) = u^{h(t)}(x) + v(x,t)$ into the stochastic Allen-Cahn equation upon applying the Itô formula to the LHS yields

$$\sum_i u_i^h dh_i + \frac{1}{2}\sum_{i,j} u_{ij}^h (dh_i, dh_j) + dv = \left[\mathcal{L}u^h + Lv + f_2 v^2\right] dt + dV$$

where $L = \epsilon^2 \partial_{xx} + f'(u^h)$ and $f_2 = \int_0^1 (1-\tau) f''\left(u^h + \tau v\right) d\tau$.

By Claim 3.3, this is the same as

$$\sum_i u_i^h dh_i + \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \langle \sigma_i(h,v), \sigma_j(h,v)\rangle dt + dv = \left[\mathcal{L}u^h + Lv + f_2 v^2\right] dt + \epsilon^\gamma dW.$$

Recall the definition of the SPDE for z on $(0,1)$ with periodic boundary conditions:

$$dz = \left(\epsilon^2 \partial_{xx} + f'(1)\right) z dt + \epsilon^\gamma dW$$

Claim 2.2 implies that with a probability converging to 1 as $\epsilon \to 0$, $\|z\| \le C\epsilon^{\gamma - 1/2 - \nu}$ $\forall \nu > 0$ for times $r\epsilon^{-2\gamma-1}$ where $r > 0$ is arbitrary, for some $C > 0$ independent of ϵ. Note that we can choose ν small enough so that $\gamma - 1/2 - \nu > 3/2$.

Therefore by writing $\bar{v} = v - z$ we obtain

$$\sum_i u_i^h dh_i + \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^2 \langle \sigma_i(h,v), \sigma_j(h,v)\rangle dt$$

$$+ d\bar{v} = \left[\mathcal{L}u^h + L\bar{v} + \left(-f'(1) - f'\left(u^h\right)\right) z + f_2 (v)^2\right] dt,$$

where $f_2 = \int_0^1 (1-\tau) f''\left(u^h + \tau v\right) d\tau$.

Rearranging this gives the following random PDE perturbed by finite-dimensional noise:

$$d\bar{v} = \Bigg[\mathcal{L}u^h + L\bar{v} - \left(f'(1) + f'\left(u^h\right)\right) z + f_2(v)^2 - \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \langle \sigma_i(h,v), \sigma_j(h,v)\rangle$$

$$- \sum_i u_i^h b_i(h,v)\Bigg] dt - \epsilon^{2\gamma} \sum_i u_i^h \langle \sigma_i(h,v), dW\rangle$$

Since the Itô formula yields $d\|\bar{v}\|^2 = d\langle \bar{v}, \bar{v}\rangle = \langle d\bar{v}, \bar{v}\rangle + \langle \bar{v}, d\bar{v}\rangle + \langle (d\bar{v}, d\bar{v})\rangle = 2\langle \bar{v}, d\bar{v}\rangle + \langle 1, (d\bar{v}, d\bar{v})\rangle$ we get

$$d\|\bar{v}\|^2 = 2\left\langle \bar{v}, \mathcal{L}u^h + L(v - z) - \left(f'(1) + f'\left(u^h\right)\right) z + f_2 v^2\right.$$

$$-\frac{1}{2}\sum_{i,j} u^h_{ij}\epsilon^{2\gamma}\langle\sigma_i(h,v),\sigma_j(h,v)\rangle - \sum_i u^h_i b_i(h,v)\Big\rangle\, dt$$

$$-2\left\langle \bar{v}, \epsilon^{2\gamma}\sum_i u^h_i\langle\sigma_i(h,v), dW\rangle\right\rangle + \epsilon^{4\gamma}\sum_{i,j}\left\langle u^h_i, u^h_j\right\rangle\langle\sigma_i(h,v),\sigma_j(h,v)\rangle\, dt.$$

Since we are in the slow channel we know that $\|v\| \leq \epsilon^{3/2+\delta}$, $\|z\| \leq \epsilon^{3/2+\delta}$ and Claim 3.8 tells us that $\|\sigma_r(h,v)\| = C\sqrt{\epsilon} + o\left(\sqrt{\epsilon}\right)$, while Claim 3.9 gives us $|b_r(h,v)| \leq C\epsilon^{2\gamma+1}$, so this becomes

$$d\|\bar{v}\|^2 \leq \epsilon^{3/2+\delta}\left(B\left\|\mathcal{L}u^h\right\| + C\epsilon^{3/2+\delta} + D\epsilon^{3+2\delta}\right)dt - 2\sum_i\left\langle\bar{v},\epsilon u^h_i\right\rangle\langle\sigma_i(h,v), dW\rangle.$$

In this expansion, $\int_0^1 (1-\tau)f''\left(u^h + \tau v\right)d\tau < C$ for some $C > 0$ is the case, as long as $\sup_{t\in[0,\psi_d]}\|v\|_\infty \leq 1$, which we have due to the definition of our stopping time. Taking expectations (noting that the expectation of a stochastic integral is 0) and integrating up to our stopping time gives

$$\mathbb{E}\left[\|\bar{v}(\psi_d)\|^2\right] \leq \mathbb{E}\left[\|v(0)\|^2\right] + CS_\infty\epsilon^{-2\gamma-1}\epsilon^{3+2\delta}.$$

We used [CP89] Theorem 3.5 to bound $\left\|\mathcal{L}u^h\right\| \leq C\epsilon^{2\gamma+1/2}$.

By applying the Itô formula we similarly have for $p \geq 3$ that

$$\begin{aligned} d\|\bar{v}\|^p &= \frac{p}{2}\|\bar{v}\|^{p-2}d\|\bar{v}\|^2 \\ &\quad + \frac{p(p-1)}{8}\|\bar{v}\|^{p-4}\sum_{i,j}\left\langle\bar{v},\epsilon^2u^h_i\right\rangle\left\langle\bar{v},\epsilon^2u^h_j\right\rangle\langle\sigma_i(h,v),\sigma_i(h,v)\rangle\, dt, \end{aligned}$$

and thus

$$\begin{aligned} &\mathbb{E}\left[\|\bar{v}\|^p(\psi_d)\right] - \mathbb{E}\left[\|\bar{v}\|^p(0)\right] = \\ &\mathbb{E}\Bigg[\int_0^{\tau^*}\frac{p}{2}\|\bar{v}\|^{p-2}2\left\langle\bar{v},\mathcal{L}u^h + L\bar{v} - \left(f'(1) + f'\left(u^h\right)\right)z + f_2v^2\right. \\ &\quad\left. -\frac{1}{2}\sum_{i,j}u^h_{ij}\epsilon^2\langle\sigma_i(h,v),\sigma_j(h,v)\rangle - \sum_i u^h_i b_i(h,v)\right\rangle dt\Bigg] \\ &\quad + \mathbb{E}\Bigg[\int_0^{\tau^*}\Bigg(\frac{p}{2}\|\bar{v}\|^{p-2}\epsilon^4\sum_{i,j}\left\langle u^h_i,u^h_j\right\rangle\langle\sigma_i(h,v),\sigma_j(h,v)\rangle\, dt \\ &\quad + \frac{p(p-1)}{8}\|\bar{v}\|^{p-4}\sum_{i,j}\left\langle\bar{v},\epsilon^2u^h_i\right\rangle\left\langle\bar{v},\epsilon^2u^h_j\right\rangle\langle\sigma_i(h,v),\sigma_i(h,v)\rangle\Bigg)dt\Bigg] \end{aligned}$$

and therefore combining all estimates from claims 2.2, 3.8 and 3.9 as well as [CP89] Theorem 3.5 as before and noting that $\|\bar{v}\| = \|v - z\| \leq \|v\| + \|z\| \leq 2\epsilon^{3/2+\delta}$ we conclude

$$\mathbb{E}\left[\|\bar{v}\left(\psi_d\right)\|^p\right] \leq \mathbb{E}[\|v(0)\|^p] + C_p\epsilon^{-2\gamma-1}\epsilon^{(3/2+\delta)p}$$

for all $p \geq 2$.

Markov's inequality then yields

$$\mathbb{P}\left(\|v(\psi_d)\| \geq \epsilon^{3/2+\delta}\right) \leq C_q\epsilon^{-2\gamma-1}\epsilon^{2\delta q}\ \forall q \geq 2.$$

We note that since the probability of $\|v(\psi_d)\|_\infty \geq 1$ is exponentially small (cf. eg. [Fen06] for the fact that the probability of $\|u(\psi_d)\|_\infty \geq 2$ being exponentially small, while $\|u^h\|_\infty \leq 1$ by definition), we have in particular

$$\mathbb{P}\left(\|v(\psi_d)\| \geq \epsilon^{3/2+\delta} \text{ or } \|v(\psi_d)\|_\infty \geq 1 \text{ or } \|z(\psi_d)\| \geq \epsilon^{3/2+\delta}\right) \leq C_q\epsilon^q\ \forall q > 0.$$

Sketch of argument that the interface position is a semimartingale. In the beginning of this chapter we made the assumption that h is a semimartingale. We now prove that this assumption is indeed true. Firstly, we observe that the coupled system for h and v has a solution:

Up to our stopping time τ^*, our drift has a bound of the form

$$|b_k(h, v)| \leq C\epsilon\ \forall k$$

for some $C > 0$. Showing this works in the same way as the sketch of argument of Claim 3.9, except that here we only have $\|v\| \leq C_{map}\epsilon^{1/2}$ and $\delta_h \geq \rho^{**}\epsilon$.

Further, we have

$$\|\sigma_k(h, v)\|_\infty \leq C$$

for some $C > 0$. This is a consequence of Claim 3.8 and the fact that $\tau_k^h \leq C'\epsilon^{-1}$ for some $C' > 0$.

We recall that the equation for v is

$$\begin{aligned} dv &= \\ &= \mathcal{L}\left(u^h + v\right)dt - \sum_i u_i^h\left[b_i\left(h, v\right)dt + \epsilon^\gamma\left\langle\sigma_i(h, v), dW\right\rangle\right] \\ &\quad -\frac{1}{2}\sum_{i,j} u_{ij}^h\epsilon^{2\gamma}\left\langle\sigma_i(h, v), \sigma_j(h, v)\right\rangle dt + \epsilon^\gamma dW. \end{aligned}$$

Up to our stopping time, the term $\mathcal{L}(u^h + v)$, is, as we know from [CP89] Theorem 3.5, bounded by a constant in the L^∞-norm.

Combining what we just saw for h with the fact that $u_i^h \leq C\epsilon^{-1}$ for some $C > 0$, we have that

$$\left\| \sum_i u_i^h b_i(h,v) \right\|_\infty \leq C.$$

We also easily see that

$$\left\| \sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \left\langle \sigma_i(h,v), \sigma_j(h,v) \right\rangle \right\|_\infty \leq C\epsilon^{2\gamma},$$

for some $C > 0$ and that the quadratic variation of $\sum_i u_i^h \epsilon^\gamma \langle \sigma_i(h,v), dW \rangle$ has the same bound in L^∞-norm.

Now clearly,

$$|b_i(h_1,v) - b_i(h_2,v)| \leq C'|h_1 - h_2|$$

and

$$\|\sigma_i(h_1,v) - \sigma_i(h_2,v)\|_\infty \leq C''|h_1 - h_2|.$$

Similarly, for $d_i(h,v,x) = \mathcal{L}(u^h+v) - \sum_i u_i^h b_i(h,v) - \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \left\langle \sigma_i(h,v), \sigma_j(h,v) \right\rangle$ we have

$$|d_i(h,v,x) - d_i(h,v,y)| \leq C|x-y|$$

while the quadratic variation of $\sum_i u_i^h \epsilon^\gamma \langle \sigma_i(h,v), dW \rangle$ is in the slow channel nothing else than

$$\sum_i \epsilon^{2\gamma} \left\langle u_i^h, u_i^h \right\rangle \left\langle \mathcal{Q}\sigma_i(h,v), \sigma_i(h,v) \right\rangle dt + o\left(\epsilon^{2\gamma+1}\right) = \sum_i \epsilon^{2\gamma+1} S_\infty q(0) + o\left(\epsilon^{2\gamma+1}\right),$$

so that we may conclude that we have necessary bounds on
$\sum_i u_i^h \epsilon^\gamma \langle \sigma_i(h,v), dW \rangle$ and ϵ^γ.

We have these L^∞-bounds up to τ^* with all terms being almost surely continuous, in fact Lipschitz continuous (as we are on the interval).
Now we note that we may consider the coupled system of $(v, h_1, ..., h_N)$ as an SPDE in $N+1$ space-dimensions, where in the second to $N+1st$ space dimensions the noise and solution are constant in space. Applying [DPZ92] Theorem 7.4 then gives existence and uniqueness (up to equivalence) of our coupled system.

Secondly, we establish that we do indeed have $\left\langle v, \tau_i^h \right\rangle = 0 \; \forall i$, so that $u^h + v$ solves the equation:

Recall that for h we have that

$$dh_r = \sum_i A_{ri}^{-1}(h,v) \left\langle \mathcal{L}(u^h+v), \tau_i^h \right\rangle dt$$

$$+\epsilon^2 \sum_{i,l,k} A_{ri}^{-1}(h,v) \left[\frac{1}{2}\left\langle v, \tau_{ilk}^h \right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h \right\rangle - \left\langle u_k^h, \tau_{il}^h \right\rangle \right] \left\langle \sigma_k(h,v), \sigma_l(h,v) \right\rangle dt$$

$$+\epsilon^2 \sum_i A_{ri}^{-1}(h,v) \sum_j \left\langle \tau_{ij}^h, \sigma_j(h,v) \right\rangle dt + \sum_i A_{ri}^{-1}(h,v) \left\langle \tau_i^h, dV \right\rangle .$$

Applying $A(h,v)$ on both sides gives

$$\sum_i A_{ij}(h,v) dh_j = \langle \mathcal{L}(u^h + v), \tau_i^h \rangle dt +$$

$$\epsilon^2 \sum_{l,k} \left[\frac{1}{2} \left\langle v, \tau_{ilk}^h \right\rangle - \frac{1}{2} \left\langle u_{kl}^h, \tau_i^h \right\rangle - \left\langle u_k^h, \tau_{il}^h \right\rangle \right] \langle \sigma_k(h,v), \sigma_l(h,v) \rangle \, dt$$

$$+\epsilon^2 \sum_j \left\langle u_{ij}^h, \sigma_j(h,v) \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle .$$

Applying Claim 3.3 and the definition of A we can rewrite this as

$$\sum_j \left[\left\langle \tau_i^h, u_j^h \right\rangle - \left\langle v, \tau_{ij}^h \right\rangle \right] dh_j = \left\langle \mathcal{L}\left(u^h + v\right), \tau_i^h \right\rangle dt$$

$$+\epsilon^2 \sum_{l,k} \left[\frac{1}{2} \left\langle v, \tau_{ilk}^h \right\rangle - \frac{1}{2} \left\langle u_{kl}^h, \tau_i^h \right\rangle - \left\langle u_k^h, \tau_{il}^h \right\rangle \right] (dh_l, dh_k)$$

$$+\sum_j \left\langle dV, \tau_{ij}^h \right\rangle dh_j + \left\langle \tau_i^h, dV \right\rangle$$

Now, taking the inner product of τ_i^h and the left hand side of our SPDE

$$du = \sum_j u_j^h dh_j + \frac{1}{2} \sum_{k,l} u_{kl}^h \left(dh_k, dh_l\right) + dv$$

gives

$$\left\langle \tau_i^h, du \right\rangle = \sum_j \left\langle \tau_i^h, u_j^h \right\rangle dh_j + \frac{1}{2} \sum_{k,l} \left\langle u_{kl}^h, \tau_i^h \right\rangle (dh_k, dh_l) + \left\langle \tau_i^h, dv \right\rangle .$$

On the other hand applying the Itô formula gives

$$d\langle \tau_i^h, v \rangle = \langle \tau_i^h, dv \rangle + \langle v, d\tau_i^h \rangle + \langle dv, d\tau_i^h \rangle$$

$$= \langle \tau_i^h, dv \rangle + \sum_j \langle v, \tau_{ij}^h \rangle dh_j + \frac{1}{2} \sum_{j,k} \left\langle v, \tau_{ijk}^h \right\rangle (dh_j, dh_k) + \sum_j \left\langle \tau_{ij}^h, dv \right\rangle dh_j$$

But since $dv = du - du^h$ we have

$$\sum_j \langle \tau h_{ij}, dv \rangle \, dh_j = \sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) - \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h \right\rangle (dh_k, dh_j)$$

Furthermore taking the inner product of τ_i^h and the right hand side of the SPDE gives

$$\left\langle \tau_i^h, du \right\rangle = \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle$$

Combining the last equations yields

$$d\left\langle \tau_i^h, v \right\rangle = \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle - \sum_j \left\langle \tau_i^h, u_j^h \right\rangle dh_j$$

$$-\frac{1}{2}\sum_{k,l} \left\langle u_{kl}^h, \tau_i^h \right\rangle (dh_k, dh_l) + \sum_j \left\langle v, \tau_{ij}^h \right\rangle dh_j + \frac{1}{2}\sum_{j,k} \left\langle v, \tau_{ijk}^h \right\rangle (dh_j, dh_k)$$

$$+\sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) - \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h \right\rangle (dh_k, dh_j)$$

$$= \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle - \sum_j A_{ij}(h,v) dh_j + \sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right)$$

$$+\sum_{j,k} \left[\frac{1}{2} \left\langle v, \tau_{ijk}^h \right\rangle - \frac{1}{2} \left\langle u_{kj}^h, \tau_i^h \right\rangle - \left\langle \tau_{ij}^h, u_k^h \right\rangle \right] (dh_k, dh_j)$$

$$= \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle - \left[\left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle - \sum_j A_{ij}(h,v) dh_j \right.$$

$$\left. + \sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) + \sum_{j,k} \left[\frac{1}{2} \left\langle v, \tau_{ijk}^h \right\rangle - \frac{1}{2} \left\langle u_{kj}^h, \tau_i^h \right\rangle - \left\langle \tau_{ij}^h, u_k^h \right\rangle \right] (dh_k, dh_j) \right]$$

$$+ \sum_j \left(\langle \tau_{ij}^h, dV \rangle, dh_j \right) + \sum_{j,k} \left[\frac{1}{2} \left\langle v, \tau_{ijk}^h \right\rangle - \frac{1}{2} \left\langle u_{kj}^h, \tau_i^h \right\rangle - \left\langle \tau_{ij}^h, u_k^h \right\rangle \right] (dh_k, dh_j) = 0$$

This completes the sketch of argument that h is indeed a semimartingale.

4. ANNIHILATION

In this chapter we show that once the critical distance $\xi_\epsilon = \frac{(2\gamma+\beta)\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ has been reached between a pair of interfaces of the solution of the stochastic Allen-Cahn equation, they will annihilate w.h.p. within a time converging to 0 on the fast timescale, after which the interfaces of the solution are in a small neighbourhood of the initial configuration without the annihilated pair.

Recall that $\Gamma_{\xi_\epsilon,\epsilon}$ is defined to be $\left\{u^h + v \,:\, h \in \Omega_{\xi_\epsilon}, \|v\| \leq \epsilon^{3/2+\delta}\right\}$ for some fixed $\delta > 0$ s.t. $\gamma - 1/2 < 3/2 + \delta$. We state our main result, followed by the Claims needed to prove it, to ultimately prove this result.

Claim 4.1. *Annihilation*

Let $u_0 \in \Gamma_{\xi_\epsilon,\epsilon}$ where exactly two neighbouring interfaces have distance $\delta_{h_0} = \xi_\epsilon$ to each other, and all other interfaces have a distance bounded below by $c\epsilon^{1/2-\kappa}$ for some $\frac{1}{2} \geq \kappa > 0$ and some $c > 0$. For $t > 0$, $u(\cdot,t)$ solves (1.1).

With a probability converging to 1 as $\epsilon \to 0$, one has that within $\epsilon^{-2\gamma+1-\iota}$ (for some small $\kappa > \iota > 0$) time these two interfaces will be annihilated leading to the new interface configuration $h' = \mathcal{H}\left(u\left(\frac{1}{\epsilon^{2\gamma-1+\iota}}\right)\right)$; with the same probability, the solution reenters $\Gamma_{\xi_\epsilon,\epsilon}$ within this time.

With a probability converging to 1 as $\epsilon \to 0$ this new configuration h' is inside a neighbourhood of size $c'\epsilon$ (for some $c' > 0$) of the initial configuration h, but without the annihilated interfaces.

Before proving this, we collect some ingredients of the sketch of argument.

Firstly, two claims that ensure that the drift is negative below a nearby threshold and blows up to $-\infty$ as $\epsilon \to 0$:

Claim 4.2. *Suppose $\frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}} \leq \delta_h < \frac{2\gamma\epsilon\log\epsilon}{\sqrt{-f'(1)}}$ and the other interface pairs have a distance bounded below by $\epsilon^{1/2-\kappa}$ for $\frac{1}{2} \geq \kappa > 0$ and $u = u^h + v \in \Gamma_{\frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}},\epsilon}$.*

Then if one denotes the two interfaces between which the minimum distance δ_h is attained by h_j and h_{j+1}, we have the following:

$$\sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}\left(u^h+v\right), \tau_i^h\right\rangle\Big|_{r=j+1} - \sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}(u^h+v), \tau_i^h\right\rangle\Big|_{r=j}$$
$$= \frac{\epsilon}{S_\infty}\left[-2a\exp\left\{-\sqrt{-f'(1)}\delta_h/\epsilon\right\}\right] + o\left(\epsilon^{2\gamma+1}\right). \tag{4.1}$$

We have

$$a = -\left[\exp\left\{\int_0^1 \left(\frac{\sqrt{-f'(1)}}{(2F(t))^{1/2}} - \frac{1}{(1-t)}\right) dt\right\}\right]^2 f'(1),$$

and the uniform bound

$$\sup_{u^h+v\in\Gamma'_\epsilon \ s.t. \ \frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}} \le \delta_h \le \frac{2\gamma\epsilon\log\epsilon}{\sqrt{-f'(1)}}} \frac{b_{j+1}(h,v) - b_j(h,v)}{S_\infty \epsilon^{2\gamma+1}} \to_{\epsilon\to 0} k < 0.$$

Sketch of Argument. The formulae for $\sum_i A_{ri}^{-1}(h,v)\langle \mathcal{L}(u^h+v), \tau_i^h\rangle$ follow from combing [CP89] Theorem 6.1, Lemma 3.3, Proposition 3.4. As $f'(1) < 0$, we have $a > 0$, and therefore $\frac{\epsilon}{S_\infty}\left[-2a\exp\left\{-\sqrt{-f'(1)}\delta_h/\epsilon\right\}\right] < 0$.

We recall that

$$b_r(h,v) = \sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}(u^h+v), \tau_i^h\right\rangle$$
$$+ \sum_{i,l,k} A_{ri}^{-1}(h,v)\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right]\epsilon^{2\gamma}\left\langle \sigma_k(h,v), \sigma_l(h,v)\right\rangle$$
$$+ \sum_i A_{ri}^{-1}(h,v)\sum_j \epsilon^{2\gamma}\left\langle \tau_{ij}^h, \sigma_j(h,v)\right\rangle = T_1 + T_2 + T_3$$

Since $u \in \Gamma_{\frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}},\epsilon}$, we have that $\|v\| \le \epsilon^{3/2+\delta}$, for what reason we may use the same bounds for T_2 and T_3 as obtained in the sketch of argument of Claim 3.9:

$|T_2| + |T_3| \le \mathcal{O}(\epsilon)\left(o\left(\epsilon^{-1}\right) + \mathcal{O}\left(\epsilon^{4\gamma-2}\right)\right)\epsilon^{2\gamma}\epsilon + \mathcal{O}(\epsilon)\epsilon^{2\gamma}\mathcal{O}\left(\epsilon^{4\gamma-2}\right) \le o\left(\epsilon^{2\gamma+1}\right)$

The further limits follow from the asymptotic expansion (4.1), which we therefore also have for $b_{j+1}(h,v) - b_j(h,v)$.

Claim 4.3. *Suppose* $\rho^{**}\epsilon \le \delta_h < \frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ *and* $u^h + v \in \Gamma'_\epsilon$.

Then, if one denotes the two interfaces between which the minimum distance is attained by h_j *and* h_{j+1}*, we have the same asymptotic expansion as in (4.1).*

We have

$$\sup_{u^h+v\in\Gamma'_\epsilon \ s.t. \ \rho^{**}\epsilon \le \delta_h < \frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}} \frac{b_{j+1}(h,v) - b_j(h,v)}{S_\infty \epsilon^{2\gamma+1}} \to_{\epsilon\to 0} -\infty.$$

Sketch of Argument. The formulae for T_1 follow from combing [$CP89$] Theorem 6.1, Lemma 3.3, Proposition 3.4.

We recall the expression for the drift $b_j(h,v)$.

By our assumption, we have $\|v\| \le C_{map}\epsilon^{1/2}$.

Thus the T_2 term of the drift thus can be estimated in the following way similarly to Claim 3.9:

$$|T_2| \leq \mathcal{O}(\epsilon)\left(\mathcal{O}\left(\epsilon^{-2}\right)+\mathcal{O}\left(\epsilon^{-2}\right)+\mathcal{O}\left(\epsilon^{-2}\right)\right)\epsilon^{2\gamma}\mathcal{O}(\epsilon) \leq \mathcal{O}\left(\epsilon^{2\gamma}\right).$$

The term T_3 has exactly the same bounds as in Claim 3.9 and we can note that $|T_3| \leq C\epsilon^{3/2}$.

The limits on the fast timescale follow from the asymptotic expansion (4.1), which we therefore also have for $b_{j+1}(h,v) - b_j(h,v)$.

Now we show that the probability of having minimum distance between the interfaces larger than $\rho^{**}\epsilon$ when exiting the extended slow channel Γ'_ϵ is very small:

Claim 4.4. *Suppose $u_0 = u^h + v \in \Gamma'_\epsilon$ where $\|v\| \leq C\epsilon^{1/2}$ for some $C < C_{map}$ and $T_\epsilon < K\epsilon^{-2\gamma-1}$ $K > 0$ independent of ϵ, and suppose that $\mathbb{E}\left[\|v(0)\|^p\right] \leq C_p\epsilon^{p/2}$ $\forall p \geq 2$ for some $C_p > 0$. Let τ^{**} be as in Chapter 3.*

Then we have

$$\mathbb{P}\left[\delta_{h(\tau^{**})} > \rho^{**}\epsilon\right] \leq C_q\epsilon^q \;\forall q > 0.$$

Sketch of Argument. Consider the linearised Allen-Cahn SPDE starting at 0, i.e.

$$\partial_t z = \epsilon^2\partial_{xx}z + f'(1)z + \epsilon^\gamma \dot{W}$$

Then for $\bar{u} = u - z$ we have

$$\partial_t\bar{u} = \epsilon^2\partial_{xx}\bar{u} + f(\bar{u}+z) - f'(1)z = \epsilon^2\partial_{xx}\bar{u} + f(\bar{u}) + C(x,t)z$$

for some bounded function $C(x,t)$.

Except for the last term, which we can view as a perturbation on $\mathcal{O}(1)$ timescales, this is exactly the same as the deterministic Allen-Cahn equation. We will use the fact, that with high probability, the last term is small on $\mathcal{O}(1)$ times, to show that with high probability we have the same behaviour as in the deterministic case.

With w we shall now denote the deterministic Allen-Cahn equation starting from the initial profile u_0.

If we indeed have $\delta_{h(\tau^{**})} > \rho^{**}\epsilon$, then at $t = 0 \vee (\tau^{**} - 1)$ we have

$\delta_{h(0\vee(\tau^{**}-1))} > \rho^{**}\epsilon$ and $\|v(0 \vee (\tau^{**} - 1))\| < C_{map}\epsilon^{1/2}$. However, we know from [CP89] Theorem 5.3 (5.6), that as long as we can map onto the slow manifold, we have for some $\nu_0, C_0 > 0$ the expression

$$\|w(t\vee\tau^{**}) - u_\infty^{\mathcal{H}(w(t\vee\tau^{**}))} \leq \epsilon^{1/2}\sqrt{C'\sum_{j=1}^{N}\left\langle\mathcal{L}(u^h),\tau_j^h\right\rangle^2 + \exp\left(-\frac{1}{2}C''t\right)C'''}.$$

But since $\left\|w(t \vee \tau^{**}) - u^{\mathcal{H}(w(t\vee\tau^{**}))}\right\| \le \left\|w(t \vee \tau^{**}) - u^{\mathcal{H}(w(t\vee\tau^{**}))}\right\|_\infty$, the same inequality follows in L^2-norm.

Claim 2.2 tells us that for all $\nu > 0$ we have

$$\mathbb{P}\left[\sup_{t\in[0,C']} \|z\|_\infty \le \epsilon^{\gamma-1/2-\nu}\right] \le 1 - C_q\epsilon^q \ \forall q > 0.$$

Thus on $\mathcal{O}(1)$ times we have (since L^2 bounds are smaller than L^∞ bounds and $u = \bar{u} + z$)

$$\mathbb{P}\left[\|w - \bar{u}\| \le \epsilon^{\gamma-1/2-\nu}\right] \le 1 - C_q\epsilon^q \ \forall q > 0.$$

Since $\|v(\tau^{**})\| \le \left\|\bar{u}(\tau^{**}) - u^{\mathcal{H}(\bar{u}(\tau^{**}))}\right\| + \|z(\tau^{**})\|$, this implies that at τ^{**} we have

$$\mathbb{P}\left[\|v(\tau^{**})\| \ge C_{map}\epsilon^{1/2}\right] \le C_q\epsilon^q \ \forall q > 0,$$

since (as noted above) for minimal distances larger than $\rho^{**}\epsilon$, $\|v\|$ is bounded above by a term smaller than $C_{map}\epsilon^{1/2}$.

This however means that we did with probability $1 - C_q\epsilon^q \ \forall q > 0$ not reach τ^{**} for a different reason than the minimal distance becoming $\rho^{**}\epsilon$.

This however is a contradiction, and thus we have proven that with probability $1 - C_q\epsilon^q \ \forall q > 0$ we reach τ^{**} because of the minimum distance reaching $\rho^{**}\epsilon$.

Let $K > 0$ be large. Define the stopping time $\nu = K\epsilon^{-2\gamma-1} \wedge \inf\{t > 0 : \delta_{h_0} = \frac{2\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ or $\|v\| \ge \epsilon^{3/2+\delta}$ or $\|v\|_\infty \ge 1$ or $\|z\| \ge \epsilon^{3/2}\}$ for some small $\delta > 0$.

Claim 4.5. *Suppose $u_0 \in \Gamma_{\frac{2\epsilon|\log\epsilon|}{\sqrt{-f'(1)}},\epsilon}$ and $T_\epsilon < K\epsilon^{-2\gamma-1}$, $K > 0$ independent of ϵ, and suppose that $\mathbb{E}\left[\|v(0)\|^p\right] \le C_p\epsilon^{(3/2+\delta)p} \ \forall p \ge 2$ for some $\delta > 0, C_p > 0$. Let $u(x,t) = u^h(x,t) + v(x,t)$ solve (1.1) for $t > 0$; ν is as previously defined.*

Then the following inequalities hold:

$$\mathbb{E}\left[\|v(\nu)\|^p\right] \le \mathbb{E}\left[\|v(0)\|^p\right] + C_p'T_\epsilon\epsilon^{(3/2+\delta)p} \le D_pT_\epsilon\epsilon^{(3/2+\delta)p} \ \forall p \ge 2$$

for some $C_p', D_p > 0$ independent of ϵ and dependent on p, as well as

$$\mathbb{P}\left(\|v(\tau^*)\| \ge \epsilon^{3/2+\delta} \text{ or } \|z(\nu)\| > \epsilon^{3/2+\delta}\right) \le D_rT_\epsilon\epsilon^{2\delta r} \ \forall r \ge 2,$$

$$\mathbb{P}\left(\|v(\nu)\| \ge \epsilon^{3/2+\delta} \text{ or } \|v(\nu)\|_\infty \ge 1 \text{ or } \|z(\nu)\| > \epsilon^{3/2+\delta}\right) = \mathcal{O}(e_\epsilon).$$

Sketch of Argument. This is a special case of Claim 3.7, where we take $\psi_{\frac{2\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}}$ instead of ψ_d.

Now, crucially, we show that the time it takes from having a minimum distance ξ_ϵ between our interfaces to having $\rho^{**}\epsilon$ converges to 0 on the fast timescale:

Claim 4.6. *Let for $t = 0$ $u_0 = u_0^h + v \in \Gamma_{\xi_\epsilon,\epsilon}$ with $\delta_{h_0} = \xi_\epsilon$ and all other distances between neighbouring interfaces bounded below by $\epsilon^{1/2-\kappa}$ for some $\frac{1}{2} \geq \kappa > 0$. Then, with a probability converging to 1 as $\epsilon \to 0$, we have at τ^{**} that $u(\tau^{**}) = u^h + v \in \Gamma'_\epsilon$ with $\delta_h = \rho^{**}\epsilon$. Furthermore, for all $k > 0, \mu > 0$ we have*

$$\mathbb{P}\left[\epsilon^{2\gamma-1+\mu}\tau^{**} < k\right] \geq 1 - C_{q,k,\mu}\epsilon^q \;\forall q > 0.$$

Sketch of Argument. Claims 4.2, 4.3 and 4.4 give us an asymptotic expansion of the semimartingale expression for $\delta_h \leq \frac{2\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ and a probability asymptotically small in ϵ of it leaving Γ'_ϵ before the critical distance is reached.

Recall from Claim 3.2 that our semimartingale expression is

$$dh_k = b_k(h,v)dt + \epsilon^\gamma\langle\sigma_k(h,v), dW\rangle$$

Clearly,

$$H_k(t) = h_k(0) + \int_0^t \epsilon^\gamma\langle\sigma_k(h,v), dW(x,s)\rangle = h_k(t) - \int_0^t b_k(h,v)dt$$

is a martingale. Then by the Dambis-Dubins-Schwartz theorem, we can denote $H_k(t)$ as a time-changed Brownian motion

$$H_k(t) - H_k(0) = B\left(\epsilon^{2\gamma}\int_0^t \langle\sigma_k(h,v), \sigma_k(h,v)\rangle ds\right),$$

where $B(\cdot)$ is a standard Brownian motion. But from Claims 4.2, 4.3, 4.4 we know that our drift points strictly into the direction of the neighbouring interface if we are below the distance $\frac{2\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$; this means that $b_k(h,v) - b_{k-1}(h,v)$ is strictly negative in this case; Claim 3.9 and Claim 3.7 tell us that the drift is with high probability small for larger distances, so we get

$$\mathbb{P}\left[|(h_k(t) - h_{k-1}(t)) - (H_k(t) - H_{k-1}(t))| \geq C\epsilon^{2\gamma+1}\right] \leq C_q\epsilon^q.$$

Clearly, $\hat{H}_k(t) = H_k(t) - H_{k-1}(t)$ can also be denoted as a time-changed Brownian motion $B(\epsilon^{2\gamma}\int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds)$ starting at ξ_ϵ.

We note from the second part of Claim 3.8 that $\forall k$

$$\mathbb{P}\left[\left|\frac{\int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds}{\epsilon}\right| < C\right] \geq 1 - C_q\epsilon^q \;\forall q > 0$$

for some $C > 0$ while

$$\mathbb{P}\left[\left|\frac{\int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds}{\epsilon^p}\right| > M\right] \geq 1 - C_q\epsilon^q \;\forall q > 0 \;\forall p > 1 \;\forall M > 0.$$

We observe that τ^{***}, the first time $\hat{H}_k(t)$ hits the level $\rho^{**}\epsilon$ is strictly greater than τ^{**}, unless τ^{**} occurs because of leaving the slow channel other than by achieving the critical distance. However, this would happen with a probability asymptotically small in ϵ (cf. Claim 3.2).

Let $\mu > 0$. Using Doob's martingale inequality applied to $\hat{H}_k$ (c.f. e.g. [RY99]), we note that for all $k > 0$ we have

$$\mathbb{P}\left[\epsilon^{2\gamma-1+\mu}\tau^{***} < k\right] \geq 1 - C_{q,k}\epsilon^q \;\forall q > 0.$$

However, this immediately implies

$$\mathbb{P}\left[\epsilon^{2\gamma-1+\mu}\tau^{**} < C\right] \geq 1 - C_q\epsilon^q \;\forall q > 0,$$

which completes the sketch of argument.

Sketch of argument of Claim 4.1. Initially, u is still in $\Gamma_{\xi_\epsilon,\epsilon}$ and we can write down the semimartingale expression of its interfaces given in Claim 3.2 by Claim 4.4.

Using Claim 4.6 we note that with a probability converging to 1 as $\epsilon \to 0$, the two interfaces of initial distance ξ_ϵ are moving towards each other until they reach the distance $\rho^{**}\epsilon$. The time taken for this converges to 0 on the fast timescale.

We now have a profile of u where one interface pair has the distance $\rho^{**}\epsilon$, while all other interface pairs are bounded below by $c\epsilon^{1/2-\kappa}$ for some $\frac{1}{2} > \kappa > 0, c > 0$; indeed the probability of two interface pairs having distance $\rho^{**}\epsilon$ converges to 0 as $\epsilon \to 0$.

In a similar manner to chapter 2, we now consider the difference between the stochastic Allen-Cahn equation and the linear stochastic heat equation:

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u}) + \left(f'(\bar{u}) - f'(1)\right) z + z^2 \int_0^1 (1-\tau) f''\left(\bar{u} + \tau z\right) d\tau$$

Like in chapter 2, we denote by w the solution to the deterministic Allen-Cahn PDE and now note

$$\begin{aligned}
\frac{\partial \|\bar{u} - w\|_\infty}{\partial t} &= \sup_{l \in \partial\|\bar{u}-w\|} \langle l, \epsilon^2 \frac{\partial^2(\bar{u} - w)}{\partial x^2} + f(\bar{u}) - f(w) + C(x,t)z\rangle \\
&\leq \sup_{l \in \partial\|\bar{u}-w\|} \langle l, f(w + (\bar{u} - w)) - f(w) + Cz\rangle \\
&\leq |f'(0)|\, \|\bar{u} - w\|_\infty + C\|z\|_\infty.
\end{aligned}$$

Gronwall's inequality implies

$$\|\bar{u} - w\|_\infty \leq C\|z\|_\infty e^{|f'(0)|t},$$

at the time $\frac{\delta|\log \epsilon|}{|f'(0)|}$ for some $\delta < \gamma - 2$ this becomes

$$\|\bar{u} - w\|_\infty \leq C\|z\|\epsilon^{-\delta},$$

so on times of order 1 or up to $\frac{\hat{\delta}|\log\epsilon|}{|f'(0)|}$ for some $\hat{\delta} < \gamma - 2$ the difference between the two is a term within the size of the orthogonal distance in the slow channel - the two are roughly behaving in the same way. This is because with high probability (converging to 1 as $\epsilon \to 0$) $\|z\| \leq \epsilon^{\gamma-1/2-\mu}$ for $\mu > 0$ arbitrarily small, so that with high probability we have $\|\bar{u} - w\|_\infty \leq C\epsilon^{\gamma-1/2-\mu-\hat{\delta}} = C\epsilon^{3/2+\pi}$, where $\pi > 0$ can be arbitrarily small, provided that $\hat{\delta} > 0$ is small enough. Since this term is smaller than the orthogonal distance to the slow manifold in the slow channel, this means that we can "track" the distance to the slow manifold of our actual solution approximately by w with high probability.

Our initial profile is (with probability converging to 1 as $\epsilon \to 0$) in L^2 norm at an $\mathcal{O}(\epsilon^2)$ distance to the slow manifold, except for a $\rho^{**}\epsilon$ neighbourhood, whose L^2 norm is clearly $\mathcal{O}(\epsilon)$ since the solution is bounded by 2 with high probability and therefore there exists $u^{h'}$ s.t. $\left\|u_0 - u^{h'}\right\| \to_{\epsilon\to 0} 0$, thus our profile fulfills the conditions of Claim 7.4; therefore Claim 7.4 immediately implies that there exists an $\mathcal{O}(1)$ time after which the distance of w in L^2 norm to a slow manifold configuration is $\frac{d}{2}$, if we denote the original distance by d. Iterating this eventually leads to a distance smaller than $\epsilon^{3/2}$; for the last interval solving the exponential error bound for time like in the sketch of argument of Claim 2.4 yields that after $C|\log\epsilon|$ time ($C > 0$) we have $\left\|w(x) - u^{h'}(x)\right\| \leq \epsilon^{3/2+\delta}$, where h' denotes the interface configuration of the new slow manifold element with configuration $h' = \mathcal{H}\left(w(C|\log\epsilon|)\right)$ which is in a neighbourhood of size $\epsilon^{1/2-k}$ (for some $k > 0$) of the original configuration. We now use Claim 2.2 and the bound on $\|\bar{u} - w\|_\infty$ to obtain

$$\mathbb{P}\left[\left\|\bar{u}(x) - u^{h'}(x)\right\| < \epsilon^{3/2+\delta}\right] \geq 1 - C_p\epsilon^p$$

for some $C_p > 0$. The logarithmic time in the end is split up into intervals of size $\frac{\delta|\log\epsilon|}{|f'(0)|}$ for $\delta < \gamma - 2$: after each interval we restart the "approximation using the deterministic equation" with the current solution to the stochastic Allen-Cahn equation as initial profile. This way our error terms never get larger than the orthogonal distance to the slow manifold in the slow channel, and the exponential contraction of Claim 7.4 will eventually yield that the solution of the stochastic Allen-Cahn equation is in the slow channel.

Since by the virtue of Claim 2.2, u is w.h.p the same as $\bar{u}$ on each time-interval, the result follows.

For the statement of the interfaces, the fact that the new configuration does not include the annihilated interface pair is trivial. To show that the remaining interfaces are in a neighbourhood of size $c'\epsilon^{1-k}$ for some small $k > 0, c'$ of their initial position, we note, similarly to the sketch of argument of Claim 3.2, that a martingale approximation of the interfaces will with a probability converging to 1 as $\epsilon \to 0$ only move by a distance bounded by ϵ^{2-k} $\forall k > 0$ while the critical interface pair reaches the distance $\rho^{**}\epsilon$. Once the critical interface pair has reached the distance $\rho^{**}\epsilon$, we note that each time we apply Claim 7.4 the new interfaces are in a neighbourhood of size $\rho\epsilon$ around the old interfaces for some $\rho > 0$, while the "stochastic component" only contributes changes of order $\epsilon^{3/2+\delta}$ for some $\delta > 0$. Thus the statement follows.

5. SHARP INTERFACE LIMIT

In this chapter we consider the time-rescaled solution to the stochastic Allen-Cahn equation, to finally obtain the main result of this work, i.e. that in the sharp interface limit the interfaces weakly converge to annihilating Brownian motions.

We now rescale time onto the previously mentioned timescale $t' = S_\infty^{-1}\epsilon^{2\gamma+1}t$ to obtain that the phase separation and generation of metastable patterns take a time of order $o(1)$ with high probability.

From Claim 3.2 we know that the interface motion inside the slow channel converges weakly in law to a Brownian motion stopped at the first hitting time as $\epsilon \to 0$ on the timescale t'.

On the fast timescale, with a probability converging to 1 as $\epsilon \to 0$, annihilations take a time of order $o(1)$.

We now set up the notation:

Firstly, for each $2N$-dimensional initial interface position h_0 we define $B^{h_0,1}(t)$ to be a $2N$-dimensional standard Brownian motion on S^1 starting at h_0; similarly, we define (independently from it) $B^{h'_0,2}(t)$ to be a $2N-2$-dimensional standard Brownian motion on S^1 starting at h'_0 and carry on defining and denoting independent standard Brownian motions in this way until we reach dimension 2.

We define the first hitting time of $B^{h_0,i}$ started at h_0 as

$\sigma^{h_0,i} = \inf_{j\neq k}\left\{t : B_j^{h_0,i}(t) = B_k^{h_0,i}(t)\right\}$

and $h'_0\left(\sigma^{h_0,i}\right) = B^{h_0,i}\left(\sigma^{h_0,i}\right)\backslash\left\{B_K^{h_0,i}\left(\sigma^{h_0,i}\right), B_J^{h_0,i}\left(\sigma^{h_0,i}\right)\right\}$, where $B_K^{h_0,i}\left(\sigma^{h_0,i}\right) = B_J^{h_0,i}\left(\sigma^{h_0,i}\right)$ for some $K = J$.

We define the limiting process of $u\left(S_\infty\epsilon^{-2\gamma-1}t\right)$:

For $0 \leq t \leq \sigma^{h_0,1}$ we have

$$u_s(x,t) = \begin{cases} \dots & \dots \\ -1 & \text{for } B_{2N}^{h_0,1} - 1 \leq x < B_1^{h_0,1}(t) \\ 1 & \text{for } B_1^{h_0,1} \leq x < B_2^{h_0,1}(t) \\ \dots & \dots \end{cases}$$

Similarly, for $\sigma^{h_0,1} \leq t \leq \sigma^{h_0,1} + \sigma^{h'_0(\sigma^{h_0,1}),2}$ we have

$$u_s(x,t) = \begin{cases} \dots & \dots \\ -1 & \text{for } B_{2N}^{h'_0(\sigma^{h_0,1}),2} - 1 \leq x < B_1^{h'_0(\sigma^{h_0,1}),2}(t) \\ 1 & \text{for } B_1^{h'_0(\sigma^{h_0,1}),2} \leq x < B_2^{h'_0(\sigma^{h_0},1),2}(t) \\ \dots & \dots \end{cases}$$

Etc.

We now define a mapping $\mathcal{I}$ which maps the function onto the slow manifold if possible, and otherwise gives the last known such interface configuration.

For $u \in \Gamma_{\xi_\epsilon,\epsilon}$ we define the mapping simply as

$$\mathcal{I}(u) = \mathcal{H}(u),$$

at the initial stage where $u \in X_\epsilon^{h_0,0}$ we define $\mathcal{I}$ as

$$\mathcal{I}(u_0) = h_0.$$

Finally, whenever u leaves $\Gamma_{\xi_\epsilon,\epsilon}$, we define $\mathcal{I}(u)$ to be the last defined value $\mathcal{H}(u)$.

Claim 5.1. *Sharp interface limit*

Given $u_0(x) \in X_\epsilon^{h_0,0}$, for $t > 0$ we have that $u(t)$ solves (1.1).

In the limit $\epsilon \to 0$, its time-rescaled process $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ weakly converges to $u_s(t)$ in law in the L^2 topology for positive times. Between time zero and positive times there is possibly a discontinuity, however not in the position of the interfaces.

In particular, with a probability converging to 1 as $\epsilon \to 0$, within a finite time the solution obtains a constant profile of either $+1$ or -1 with fluctuations converging to 0 as $\epsilon \to 0$.

Sketch of Argument. We prove this by induction.

Firstly, suppose $u_0 \in X_\epsilon^{\emptyset,0}$ has no sign-changing interfaces, i.e. (for $\epsilon \to 0$) we have $u_0 \geq 0\ \forall x \in (0,1)$ or $u_0 \leq 0\ \forall x \in (0,1)$. Then we may apply Claim 2.1 to obtain that there exists a suitable $C \geq \frac{1}{|f'(0)|}$, s.t. for $t_1 = C|\log \epsilon|$ we have

$$\sup_{u_0 \in X_\epsilon^{\emptyset,0}} \mathbb{P}\left[u(t_1) \notin \Gamma_{\xi_\epsilon,\epsilon}\right] \leq C_\epsilon \epsilon^p.$$

However we note that in this case, $\Gamma_{\xi_\epsilon,\epsilon} = \left\{\pm 1 + v : \|v\| \leq \epsilon^{3/2+\delta}\right\}$.

We note that the time-rescaled solution $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ we are considering is thus in the limit $\epsilon \to 0$ instantly converging to ± 1 with probability 1, as the time becomes $C|\log \epsilon|\epsilon^{2\gamma+1}$, which converges to 0 as $\epsilon \to 0$.

We are now in the position to apply Claim 3.1, which tells us that in the limit $\epsilon \to 0$ the minimum of an arbitrarily chosen constant $C > 0$ and the first hitting time of the interfaces (which is ∞, as there are no interfaces) the (non-existing) interfaces perform independent Brownian motions. Hence we will see a constant profile ± 1 until the time 1.

We have shown that in the limit $\epsilon \to 0$, the expression $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ instantly converges to ± 1, and stays constant up to an arbitrarily chosen finite positive stopping time. The sign depends on $\lim_{\epsilon \to 0} u_0$: if $\lim_{\epsilon \to 0} u_0$ is non-negative, then we converge to 1; if $\lim_{\epsilon \to 0} u_0$ is non-positive, we converge to -1.

Thus, we have proven the statement for 0 initial sign-changes.

Now suppose the statement is true for $u_0 \in X_\epsilon^{h_0,0}$ with $|h_0| = 2K$, i.e. $2K$ initial sign-changes.

Consider $u_0 \in X_\epsilon^{h_0',0}$ where $|h_0'| = 2K + 2$.

We may apply Claim 2.1 to obtain that there exists a suitable $C \geq \frac{1}{|f'(0)|}$, s.t. for $t_1 = C|\log \epsilon|$ we have

$$\sup_{u_0 \in X_\epsilon^{h_0',0}} \mathbb{P}\left[u(t_1) \notin \Gamma_{\xi_\epsilon,\epsilon}\right] \leq C_p \epsilon^p.$$

In the limit $\epsilon \to 0$, we have that $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ is instantly inside $\Gamma_{\xi_\epsilon,\epsilon} = \left\{u^h + v : \|v\| \leq \epsilon^{3/2+\delta}\right\}$, as the time becomes $C|\log \epsilon|\epsilon^{2\gamma+1}$ after rescaling. But we know (c.f. e.g. [CP89]) that outside a neighbourhood of size $2\xi_\epsilon$ of h_0'(which is a set of Lebesgue measure zero in the limit $\epsilon \to 0$) , u^h converges to ± 1, the sign depending on which sign $\lim_{\epsilon\to 0} u_0$ has in the interval between the sign-changes. This is exactly the definition of the shape of u_s at this stage.

We are now in the position to apply Claim 3.1 together with Skorokhod's representation theorem, which tells us that in the limit $\epsilon \to 0$, the interfaces $h\left(S_\infty \epsilon^{-2\gamma-1} t\right) = \mathcal{I}\left(u\left(S_\infty \epsilon^{-2\gamma-1} t\right)\right)$ converge with probability 1 to annihilating Brownian motions stopped at the minimum of their first hitting time and an arbitrary constant. Since during this period (by Claim 3.2) we may denote $u(x,t) = u^{h(t)} + v(x,t)$, where $\|v\| \leq \epsilon^{3/2+\delta}$ and $u^{h(t)}$ converging to ± 1 (the sign being the same as the sign of u in this interval) outside a neighbourhood of h of size $2\xi_\epsilon$ (which is in the limit $\epsilon \to 0$ a set of Lebesgue measure 0), it follows that in the the limit $\epsilon \to 0$ the term $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ converges weakly to u_s stopped at $\sigma^{h_0'}$ with probability 1 in the L^2 topology.

For finite ϵ, at the stopping time we obtain (with a probability converging to 1 as $\epsilon \to 0$) that $u \in \Gamma_{\xi_\epsilon,\epsilon}$ where exactly one pair of the interfaces $h\left(S_\infty \epsilon^{-2\gamma-1} t\right) = \mathcal{I}\left(u\left(S_\infty \epsilon^{-2\gamma-1} t\right)\right)$ has distance ξ_ϵ to each other.

We are now in the position to apply Claim 4.1, to note that with a probability converging to 1 as $\epsilon \to 0$ one has that within $o(1)$ (for some small $\iota > 0$) time these two interfaces will be annihilated to obtain a new interface configuration h'; with the same probability, the solution reenters $\Gamma_{\xi_\epsilon,\epsilon}$ within this time.

With a probability converging to 1 as $\epsilon \to 0$ this new configuration h' is inside a neighbourhood of size $\epsilon^{1/2-\kappa}$ (for some $1/2 > \kappa > 0$) of the initial configuration h, but without the annihilated interfaces. Therefore, in the sharp interfaces limit, we instantly see how the two hitting interfaces disappear and all other interfaces stay the same. Thus $u\left(S_\infty \epsilon^{-2\gamma-1} t\right)$ instantly converges to $u_s(t)$ with probability 1 in the L^2 topology. We note that now $|\mathcal{I}(u_s(t))| = 2K$.

Hence, by induction, it follows that the statement it true for all even numbers of sign-changes in the initial condition. This completes the sketch of argument.

6. CORRELATED NOISE

In this work we considered the Allen-Cahn equation perturbed by space-time white noise. However, one could have other forms of noise as well, in particular, noise with a trace class covariance operator.

For some general background of such noise, [Blö05] is a nice reference.

Quite interestingly, to take the sharp interface limit in this case, the time-scale onto which we need to rescale will be different for such smooth noise.

Similarly to the white noise case, the asymptotic analysis for noise with trace-class covariance operator (we will later take slightly more regularity) gives us

$$dh_k = o\left(\epsilon^{2\gamma+2}\right) dt + \epsilon^{\gamma} \left\langle \frac{\epsilon}{S_\infty} u_k^h + o(\epsilon), dW \right\rangle,$$

where the error terms are w.h.p. in the L^∞ norm. Thus, the only term contributing to the limit is

$$\epsilon^{\gamma+1} \left\langle \frac{u^h}{S_\infty}, dW \right\rangle.$$

Now, depending on the type of noise chosen, dW can have a scaling behaviour in space ranging from a square-root (space-time white noise) to a linear mapping (trace-class). We know that in the space-time white noise case the correct timescale is $t' = S_\infty^{-1}\epsilon^{2\gamma+1}t$. For noise smooth in space, in order to see the sharp interface limit, it will be $t' = S_\infty^{-2}S_{\mathcal{Q}}\epsilon^{2\gamma+2}t$ for some and each interface becomes a standard Brownian motion up to the stopping time (correlated to the other interfaces).

The reason we rescale in such a fashion is qualitatively that for space-time white noise, we want to integrate the noise in space against the square root of a Dirac Delta (because of the spatial scaling effect of the noise) to obtain a finite limit, while for noise with strong correlations in space we want to to integrate in space against an actual Dirac Delta. Hence the choice of the powers of ϵ in the time-scales. The constants are chosen so that each interface performs a Brownian motion of diffusion coefficient 1.

More generally, let us denote space-time white noise as $N(x,t)$, and our noise perturbing the equation as $W(x,t) = \mathcal{Q}^{1/2}N(x,t)$. Then

$$\epsilon^{\gamma+1} \left\langle \frac{u^h}{S_\infty}, dW \right\rangle \epsilon^{\gamma+1} \left\langle \frac{u^h}{S_\infty}, dW \right\rangle = \epsilon^{2\gamma+2} \left\langle \frac{\mathcal{Q}u^h}{S_\infty}, \frac{u^h}{S_\infty} \right\rangle dt \approx \frac{S_{\mathcal{Q}}}{S_\infty^2}\epsilon^{2\gamma+1+C}dt,$$

where C depends on $\mathcal{Q}$.

For space-time white noise, we have $C = 1$ (the maximal value) and for trace-class noise $C = 0$ (the minimal value). Depending on how the eigenvalues of $\mathcal{Q}$ decay, all other values in between are possible.

The timescale on which we then see annihilating correlated Brownian motions in the sharp interface limit will clearly be $t' = S_\infty^{-2} S_{\mathcal{Q}} \epsilon^{2\gamma+2} t$.

In other words, the timescale on which our interface motion converges to something finite depends on the choice of noise.

We shall now derive rigorous results for noise with a trace-class covariance operator:

Definition 6.1. *Definition of the noise*

W is a Wiener process in the underlying Hilbert space $H = L^2(0,1)$ (cf. [DPZ92]),the covariance operator is denoted $\mathcal{Q}$. For an orthonormal basis of H denoted $\{e_j(\cdot)\}_{j=1}^{\infty}$ and a sequence of independent standard Brownian motions $\{\beta_j(t)\}_{j=1}^{\infty}$we can denote

$$W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k.$$

where
$\mathcal{Q} e_k = \alpha_k^2 e_k$, $\sum_{k=1}^{\infty} k\alpha_k^2 < \infty$.

Remark: Our results would probably still be true for $trace\mathcal{Q} = \sum_{k=1}^{\infty} \alpha_k^2 < \infty$, however the sketch of argument of convergence would become very technical.

Similarly to white noise, we use a little claim for our various results (although in theory we could do without it, as we can use Itô calculus):

We have

$$\gamma > 1.$$

The following section corresponds to chapter 2:

As before, but with different noise, we denote

$$\partial_t z = \epsilon^2 \partial_{xx} z + f'(1) z + \epsilon^\gamma \dot{W}.$$

Claim 6.2. *Let $z(\cdot,t)$ be constantly equal to 0 at time 0.*

For all $\nu, q, r > 0$ there exist $C, D > 0$ independent of ϵ, ν, r s.t.

$$\mathbb{P}\left[\sup_{t\in\left[0, r\epsilon^{-(2\gamma+2)}\right]} \|z(\cdot,t)\| > D_{q,r,\nu}\epsilon^{\gamma-\nu}\right] \le C_{q,r,\nu}\epsilon^q \ \forall q > 0 \tag{6.1}$$

and

$$\mathbb{P}\left[\sup_{t\in\left[0, r\epsilon^{-(2\gamma+2)}\right]} \|z(\cdot,t)\|_\infty > D_{q,r,\nu}\epsilon^{\gamma-\nu}\right] \le C_{q,r,\nu}\epsilon^q \ \forall q > 0. \tag{6.2}$$

Sketch of Argument. In the same way as Claim 2.2.

We now come to the first proper result, which is very similar, due to the deterministic flow dominating:

Define the new critical distance $\xi'_\epsilon = \frac{(2\gamma+2+\beta)\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$ for some small $2 > \beta > 0$.

Denote

$$\Gamma''_{c,\epsilon} := \left\{u^h + v \,:\, h \in \Omega_c, \|v\| \leq \epsilon^{1+\delta}\right\}$$

for some fixed $\delta > 0$ s.t. $\gamma - 1 > \delta > 0$.

Our slow channel is defined as $\Gamma''_{\xi'_\epsilon,\epsilon}$.

Claim 6.3. *Let $u_0(\cdot) \in X_\epsilon^{h_0,0}$ and $u(\cdot,t)$ for $t > 0$ solve (1.1).*

Then there exists a suitable $C \geq \frac{1}{|f'(0)|}$, s.t. for $t_1 = C|\log\epsilon|$ we have

$$\sup_{u_0 \in X_\epsilon^{h_0,0}} \mathbb{P}\left[u(\cdot,t_1) \notin \Gamma''_{\xi'_\epsilon,\epsilon}\right] \leq C_p\epsilon^p$$

$h' = \mathcal{H}(u(\cdot,t_1))$ associated to $u(\cdot,t_1) \in \Gamma''_{\xi'_\epsilon,\epsilon}$ fulfils

$$\mathbb{P}\left[\max_{1\leq i\leq N} |h_i - h'_i \mod 1| > k\epsilon\sqrt{|\log\epsilon|}\right] \leq C_p\epsilon^p.$$

Sketch of Argument. In the same way as Claim 2.1, except that we use Claim 6.2 rather than Claim 2.2.

This section corresponds to chapter 3:

The "heart" of the work is the main difference in results compared to the white-noise case. We firstly derive the semimartingale expression, based on the the assumption that the interfaces are a semimartingale; later we will prove this.

Derivation of the semimartingale notation

The interfaces h_k are a semimartingale as denoted above. This will be properly proven in the sketch of argument of Claim 6.7, to first be able to guess what this semimartingale notation is we assume this and prove it later on.

$$dh_k = b_k(h,v)dt + \langle\sigma_k(h,v), dV\rangle$$

By the Itô formula, we have

$$du = \sum_{j=1}^{N} u_j^h dh_j + \frac{1}{2}\sum_{1\leq k,l\leq N} u_{kl}^h(dh_k, dh_l) + dv \tag{6.3}$$

with $u_{kl}^h = \frac{\partial^2 u^h}{\partial h_k \partial h_l}$.

We now take the inner product in space of (1.1) with τ_i^h to get for any $i = 1, ..., N$

$$\left\langle \tau_i^h, du \right\rangle = \left\langle \mathcal{L}(u), \tau_i^h \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle,$$

where we defined $\mathcal{L}(u) := \epsilon^2 u_{xx} + f(u)$.

Taking the inner product with (6.3), we obtain

$$\left\langle \tau_i^h, du \right\rangle = \sum_{j=1}^{N} \left\langle u_j^h, \tau_i^h \right\rangle dh_j + \frac{1}{2} \sum_{1 \le k,l \le N} \left\langle u_{kl}^h, \tau_i^h \right\rangle (dh_k, dh_l) + \left\langle \tau_i^h, dv \right\rangle \quad (6.4)$$

In the remainder of this , any summation is on $1, 2, ..., N$ for any index.

To eliminate dv, we apply the Itô-formula to the orthogonality condition $\left\langle v, \tau_i^h \right\rangle = 0$, and obtain

$$\begin{aligned}\left\langle \tau_i^h, dv \right\rangle &= -\left\langle v, d\tau_i^h \right\rangle - \left\langle dv, d\tau_i^h \right\rangle \\ &= -\sum_j \left\langle v, \tau_{ij}^h \right\rangle dh_j - \frac{1}{2} \sum_{j,k} \left\langle v, \tau_{ijk}^h \right\rangle (dh_j, dh_k) - \sum_j \left\langle \tau_{ij}^h, dv \right\rangle dh_j.\end{aligned}$$

Now we use $dv = du - du^h$ and $(dt, dt) = 0$ as well as $(dV, dt) = 0$, i.e.

$$\begin{aligned}&-\sum_j \left(\left\langle \tau_{ij}^h, dv \right\rangle, dh_j \right) \\ &= -\sum_j \left(\left\langle \tau_{ij}^h, du \right\rangle, dh_j \right) + \sum_j \left(\left\langle \tau_{ij}^h, du^h \right\rangle, dh_j \right) \\ &= -\sum_j \left\langle \tau_{ij}^h, \mathcal{L}(u) \right\rangle (dt, dh_j) - \sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) + \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h \right\rangle (dh_k, dh_j) \\ &= -\sum_j \left(\left\langle \tau_{ij}^h, dV \right\rangle, dh_j \right) + \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h \right\rangle (dh_k, dh_j) \qquad (6.5)\end{aligned}$$

where the inner product of the stochastic Allen-Cahn equation with u_{ij}^h was taken and the following used:

$$(dh_j, dt) = b_j\,(h, v)(dt, dt) + (\langle \sigma_j(h, v), dV \rangle, dt) = 0.$$

Hence we have

$$\begin{aligned}\left\langle \tau_i^h, dv \right\rangle = &-\sum_j \left\langle v, \tau_{ij}^h \right\rangle dh_j - \frac{1}{2} \sum_{j,k} \left\langle v, \tau_{ijk}^h \right\rangle (dh_j, dh_k) \\ &- \sum_j \left(\left\langle dV, \tau_{ij}^h \right\rangle, dh_j \right) + \sum_{j,k} \left\langle u_k^h, \tau_{ij}^h \right\rangle (dh_j, dh_k). \qquad (6.6)\end{aligned}$$

Combining this with (24) and (25) yields

$$\sum_j \left[\left\langle u_j^h, \tau_i^h\right\rangle - \left\langle v, \tau_{ij}^h\right\rangle\right] dh_j = \left\langle \mathcal{L}(u), \tau_i^h\right\rangle dt$$

$$+\sum_{l,k}\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right](dh_l, dh_k)$$

$$+\sum_j \left(\left\langle dV, \tau_{ij}^h\right\rangle, dh_j\right) + \left\langle \tau_i^h, dV\right\rangle. \qquad (6.7)$$

Claim 6.4. *For all $k, l \leq N$ we have*

$$(\langle \sigma_k(h,v), dV\rangle, \langle \sigma_l(h,v), dV\rangle) = \epsilon^{2\gamma}\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\rangle dt.$$

Sketch of Argument. Since $(d\beta_j, d\beta_i) = \delta_{ij}dt$ and $W(t) = \sum_{k=1}^{\infty}\alpha_k\beta_k(t)e_k$ for an orthonormal basis $\{e_k(\cdot)\}_{k=1}^{\infty}$ of $L^2(0,1)$ and independent Brownian motions $\{\beta_k\}_{k=1}^{\infty}$, Parseval's identity yields the following:

$$(\langle \sigma_k(h,v), dV\rangle, \langle \sigma_l(h,v), dV\rangle)$$

$$= \epsilon^{2\gamma}\sum_{i,j}\langle \sigma_k(h,v), \alpha_i e_i\rangle\langle \sigma_l(h,v), \alpha_j e_j\rangle(d\beta_j, d\beta_i)$$

$$= \epsilon^{2\gamma}\sum_j \alpha_j^2\langle \sigma_k(h,v), e_j\rangle\langle \sigma_l(h,v), e_j\rangle dt = \epsilon^{2\gamma}\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\rangle dt.$$

In an analogous way to this claim one can easily obtain (using $(dt, dV) = 0$)

$$\left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) = \left(\left\langle \tau_{ij}^h, dV\right\rangle, \langle \sigma_j(h,v), dV\rangle\right) = \epsilon^{2\gamma}\left\langle \mathcal{Q}\tau_{ij}^h, \sigma_j(h,v)\right\rangle dt.$$

For short-hand notation, we define the matrix $A(h,v) = (A_{ij}(h,v)) \in \mathbb{R}^{N\times N}$ by

$$A_{ij}(h,v) = \left\langle u_j^h, \tau_i^h\right\rangle - \left\langle v, \tau_{ij}^h\right\rangle, \qquad (6.8)$$

For an invertibility condition of this matrix, see Claim 3.6; the stability shown in Claim 6.5 implies that the condition of Claim 3.6 does indeed hold true in Γ'_ϵ with a probability converging to 1 as $\epsilon \to 0$ until two interfaces are at distance ξ'_ϵ. The inverse matrix of $A(h,v)$ is then denoted by $A^{-1}(h,v) = \left(A_{ij}^{-1}(h,v)\right) \in \mathbb{R}^{N\times N}$.

We now arrive for all $i \in \{1, ..., N\}$ at

$$\sum_j A_{ij}(h,v)dh_j$$

$$= \langle \mathcal{L}(u^h{+}v), \tau_i^h\rangle dt + \sum_{l,k}\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right]\epsilon^{2\gamma}\left\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\right\rangle dt$$

$$+\sum_j \epsilon^{2\gamma}\left\langle \mathcal{Q}u^h_{ij}, \sigma_j(h,v)\right\rangle dt + \langle \tau^h_i, dV\rangle.$$

To obtain the equation for dh we use that $dh = A(h,v)^{-1}A(h,v)dh$.

Therefore the final equation for h (given that u is inside the slow channel) is given for any $r = 1, ...N$ by

$$\begin{aligned} dh_r = &\sum_i A^{-1}_{ri}(h,v)\langle \mathcal{L}(u^h{+}v), \tau^h_i\rangle dt \\ &+\epsilon^{2\gamma}\sum_{i,l,k} A^{-1}_{ri}(h,v)\left[\frac{1}{2}\left\langle v, \tau^h_{ilk}\right\rangle - \frac{1}{2}\left\langle u^h_{kl}, \tau^h_i\right\rangle - \left\langle u^h_k, \tau^h_{il}\right\rangle\right]\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\rangle dt \\ &+\epsilon^{2\gamma}\sum_i A^{-1}_{ri}(h,v)\sum_j \left\langle \mathcal{Q}\tau^h_{ij}, \sigma_j(h,v)\right\rangle dt + \sum_i A^{-1}_{ri}(h,v)\left\langle \tau^h_i, dV\right\rangle. \end{aligned} \tag{6.9}$$

Relating this to the original ansatz means

$$\sigma_r(h,v) = \sum_i A^{-1}_{ri}(h,v)\tau^h_i. \tag{6.10}$$

Therefore one can write down $b_r(h,v)$ in terms of $\sigma_j(h,v)$:

$$\begin{aligned} b_r(h,v) = &\sum_i A^{-1}_{ri}(h,v)\langle \mathcal{L}(u^h{+}v), \tau^h_i\rangle \\ &+\sum_{i,l,k} A^{-1}_{ri}(h,v)\left[\frac{1}{2}\left\langle v, \tau^h_{ilk}\right\rangle - \frac{1}{2}\left\langle u^h_{kl}, \tau^h_i\right\rangle - \left\langle u^h_k, \tau^h_{il}\right\rangle\right]\epsilon^{2\gamma}\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\rangle \\ &+\sum_i A^{-1}_{ri}(h,v)\epsilon^{2\gamma}\sum_j \left\langle \mathcal{Q}\tau^h_{ij}, \sigma_j(h,v)\right\rangle \end{aligned} \tag{6.11}$$

Claim 6.5. *Sharp interface limit in the slow channel*

Let our initial condition $u_0(\cdot)$ at $t = 0$ be inside $\Gamma''_{\xi'_\epsilon,\epsilon}$. Denote its interface configuration at $t = 0$ by $h_0 = \mathcal{H}(u_0(\cdot))$. For $t > 0$, $u(\cdot,t)$ solves (1.1). We assume that $\delta_{h_0} \geq \xi'_\epsilon$ with all other distances of neighbouring interfaces bounded below by $c > 0$.

Denote $\hat{h}(t) = \mathcal{H}\left(u\left(S_{\mathcal{Q}}S^{-2}_\infty\epsilon^{2+2\gamma}t,\cdot\right)\right)$ and let $C > 0$ be large.

Then as $\epsilon \to 0$, $\hat{h}(t)$ stopped at $\hat{\tau}^* = C\wedge\inf\left\{t > 0 : \delta_{h(t)} = \xi'_\epsilon \text{ or } \|v\| > \epsilon^{1+\delta} \text{ or } \|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{1+\delta}\right\}$, converge in law to correlated Brownian motions on S^1 starting at h_0 and stopped at $C \wedge \mu$, where μ is the first hitting time of two neighbouring interfaces.

Their generator is given as

$$\mathcal{G} = \sum_{i,k,l} S_{\mathcal{Q}kl}S^{-1}_{\mathcal{Q}}\alpha^2_i(-1)^{k+l}e_i(h_k)e_i(h_l)\partial^2_{kl}.$$

We used the notation

$$S_{\mathcal{Q}km}S_{\mathcal{Q}}^{-1} = 4S_{\mathcal{Q}}^{-1}\sum_{r=1}^{\infty}\alpha_r^2 e_r(h_k - h_m)$$

$$S_{\mathcal{Q}} = 4\sum_{r=1}^{\infty}\alpha_r^2 e_r(0)\ .$$

The sketch of argument is a consequence of the more detailed Claim on the semimartingale representation. Note that the sketch of argument of this claim makes extensive use of claims presented afterwards in this section.

Claim 6.6. *Interface motion inside the slow channel*

Let $T_\epsilon = CS_{\mathcal{Q}}^{-1}S_\infty\epsilon^{-2\gamma-2}$, $C > 0$ be large and independent of ϵ. Let furthermore the initial profile $u_0(\cdot)$ at $t = 0$ be inside Γ'_ϵ. Denote its interfaces at $t = 0$ by $h_0 = \mathcal{H}(u_0(\cdot))$. For $t > 0$, $u(\cdot, t)$ solves (1.1). We assume that $\delta_{h_0} \geq \rho^{**}\epsilon$ with all other distances of neighbouring interfaces bounded below by $c > 0$.

Then up to the first exit time

$\tau^{**} = \frac{CS_\infty^2}{S_{\mathcal{Q}}} \wedge \inf\{t > 0 : \delta_{h_0} = \epsilon\rho^{**}$ *or*

$u \notin \Gamma'_\epsilon$ *with* $\langle v, \tau_j^h\rangle = 0\ \forall j$ *or* $\|v\|_\infty \geq 1$ *or* $\|z\| > \epsilon^{1+\delta}\}$

we have that the interfaces of u, $h = \mathcal{H}(u(\cdot, t\wedge\tau^{**}))$ are a semimartingale denoted by

$$dh_k = b_k(h, v)dt + \epsilon^\gamma\langle\sigma_k(h, v), dW\rangle$$

where

$$\sigma_r(h, v) = \sum_i A_{ri}^{-1}(h, v)u_i^h$$

and

$$\begin{aligned} b_r(h, v) = &\sum_i A_{ri}^{-1}(v, h)\langle\mathcal{L}(u^h + v), u_i^h\rangle \\ &+\epsilon^{2\gamma}\sum_{i,l,k}A_{ri}^{-1}(v, h)\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right]\langle\mathcal{Q}\sigma_k(h, v), \sigma_l(h, v)\rangle \\ &+\epsilon^{2\gamma}\sum_i A_{ri}^{-1}(h, v)\sum_j\left\langle\mathcal{Q}\tau_{ij}^h, \sigma_j(h, v)\right\rangle . \end{aligned}$$

We have $\mathcal{L}(u) := \epsilon^2 u_{xx} + f(u)$ and $A_{ri}^{-1}(h, v)$ being the inverse of $A_{ri}(h, v) = \langle u_r^h, \tau_i^h\rangle - \langle v, \tau_{ri}^h\rangle$.

We have the following probability:

$$\mathbb{P}\left[\|v(\tau^{**})\| > C_{map}\epsilon^{1/2}\ \text{or}\ \|v(\tau^{**})\|_\infty \geq 1\right] \leq C_q\epsilon^q\ \forall q > 0.$$

Now define

$$\tau^* = \frac{CS_\infty^2}{S_{\mathcal{Q}}} \wedge \inf\left\{t > 0 : \delta_{h_0} = \xi'_\epsilon\ \text{or}\ \|v\| > \epsilon^{1+\delta}\ \text{or}\ \|v\|_\infty \geq 1\ \text{or}\ \|z\| > \epsilon^{1+\delta}\right\} .$$

If at $t = 0$, the initial profile $u_0(\cdot)$ is in $\Gamma''_{\xi'_\epsilon,\epsilon}$, then we can express the behaviour of its interfaces $\mathcal{H}(u(t \wedge \tau^*, \cdot))$ up to τ^* as before, and before the stopping time the following deterministic result holds:

$$|b_r(h,v)| \leq D\epsilon e^{-\sqrt{-f'(1)}\delta_h/\epsilon} + o\left(\epsilon^{2\gamma+2}\right)$$

for some $D > 0$ independent of ϵ.

Define

$$S_{\mathcal{Q}} = \lim_{\epsilon\to 0}\int_0^1 \left(\int_0^1 \sum_{r=1}^{\infty} \alpha_r^2 e_r(y)\ \tau_k^h(x-y)dy\right)\tau_k^h(x)dx = 4\sum_{r=1}^{\infty}\alpha_r^2 e_r(0).$$

After the time-change $\hat{h}_k(t') = h_k\left(S_\infty^{-2}S_{\mathcal{Q}}\epsilon^{2+2\gamma}t\right)$, the time-rescaled stopping time $\hat{\tau}^* = S_\infty^{-2}S_{\mathcal{Q}}\epsilon^{2+2\gamma}\tau^*$ is finite and the equation becomes

$$d\hat{h}_k = \frac{S_\infty^2 b_k\left(\hat{h},v\right)}{S_{\mathcal{Q}}\epsilon^{2+2\gamma}}dt' + \left\langle \frac{S_\infty}{\epsilon\sqrt{S_{\mathcal{Q}}}}\sigma_k\left(\hat{h},v\right), dW\right\rangle$$

with the following deterministic bound up to $\hat{\tau}^*$:

$$\sup_{t\in[0,\hat{\tau}^*]}\left\|\frac{S_\infty}{\sqrt{S_{\mathcal{Q}}}\epsilon}\sigma_k\left(\hat{h},v\right) - \frac{\epsilon}{\sqrt{S_{\mathcal{Q}}}}u_k^h\right\|_\infty \leq F\epsilon^{1/2}$$

for some $F > 0$ independent of ϵ.

In the sharp interface limit as $\epsilon \to 0$, the law of $\left(\hat{h}_1, ..., \hat{h}_N\right)$ stopped at $\hat{\tau}^*$ converges weakly to that of correlated Brownian motions $(M_1, ..., M_N)$

stopped at the minimum of their first hitting time and C.

M_k and M_m have the correlation

$$S_{\mathcal{Q}km}S_{\mathcal{Q}}^{-1} = 4S_{\mathcal{Q}}^{-1}\sum_{r=1}^{\infty}\alpha_r^2 e_r(h_k - h_m),$$

where

$$S_{\mathcal{Q}} = \sum_{r=1}^{\infty} 4\alpha_r^2 e_r(0).$$

The generator is given as

$$\mathcal{G} = \sum S_{\mathcal{Q}kl}S_{\mathcal{Q}}^{-1}\alpha_i^2(-1)^{k+l}e_i(h_k)e_i(h_l)\partial_{kl}^2.$$

Sketch of Argument. The fact that the interfaces are a semimartingale, which is exactly as stated here, follows from the derivation (based on the assumption that the interfaces are a semimartingale) and the sketch of argument that the interfaces are a semimartingale, given later in this section.

Starting in Γ'_ϵ the asymptotically small probability of leaving the slow channel other than by reaching the critical distance (or time $CS_\infty^{-2}S_{\mathcal{Q}}\epsilon^{-2\gamma-2}$) follows from Claim 6.11 later in this . This completes the sketch of argument of the first part of the Claim.

Whenever we will in the remainder of this sketch of argument apply Claim 6.7, we will apply it by setting $d = \xi'_\epsilon$ in the definition of ψ_d.

From now on, we consider the regime starting in $\Gamma''_{\xi'_\epsilon,\epsilon}$, which allows us to use a number of Claims. The asymptotic expansion of the diffusion coefficient σ follows from Claim 3.8 combined with Claim 6.7; for the drift b it follows from Claim 6.8 combined with Claim 6.7. The stability of this system is a consequence of Claim 6.7, i.e. the probability of $\|v\| \geq \epsilon^{1+\delta}$ or $\|v\|_\infty \geq 1$ or $\|z\| > \epsilon^{1+\delta}$ is asymptotically small.

We now show that after a time-rescaling onto $t' = S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}t$ the h_k weakly converge in law to Brownian motions with correlation structure $S_{\mathcal{Q}kj}S_\mathcal{Q}^{-1}$ stopped at τ^*. The intuition of it is that u_r^h is close to a Dirac delta in the limit $\epsilon \to 0$, so that since our noise is smooth in space, integrating it against a Dirac delta function leads to a Brownian motion.

Denote by $\hat{h}_k(t) = h_k\left(S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}t\right)$ and $\hat{\tau}^* = S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}\tau^*$ our interfaces and the stopping time on the fast timescale.

We firstly show that $\hat{h}_k$ stopped at τ^* is tight on $C\left([0,T], S^1\right)$:

Let $t, s > 0$ and $p > 0$ be even. Then

$$\mathbb{E}\left[\left|\hat{h}_k\left(t\wedge\hat{\tau}^*\right) - \hat{h}_k\left(s\wedge\hat{\tau}^*\right)\right|^p\right]$$

$$= \mathbb{E}\left[\left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \frac{b_k\left(\hat{h},v\right)}{S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}}dr + \int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*}\left\langle \frac{S_\infty}{\sqrt{S_\mathcal{Q}}\epsilon}\sigma_k\left(\hat{h},v\right), dW(x,r)\right\rangle\right)^p\right]$$

$$\leq C_p\mathbb{E}\left[\left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \frac{b_k\left(\hat{h},v\right)}{S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}}dr\right)^p + \left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*}\left\langle \frac{1}{\sqrt{S_\mathcal{Q}}S_\infty^{-1}\epsilon}\sigma_k\left(\hat{h},v\right), dW(x,r)\right\rangle\right)^p\right]$$

$$\leq C_p\mathbb{E}\left[\left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*} \frac{b_k\left(\hat{h},v\right)}{S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}}dr\right)^p + \left(\int_{s\wedge\hat{\tau}^*}^{t\wedge\hat{\tau}^*}\frac{1}{S_\mathcal{Q} S_\infty^{-1}\epsilon^2}\left\langle \sigma_k\left(\hat{h},v\right), \mathcal{Q}\sigma_k\left(\hat{h},v\right)\right\rangle dt\right)^{p/2}\right]$$

$$\leq C_p\mathbb{E}\left[\left(\left|t\wedge\hat{\tau}^* - s\wedge\hat{\tau}^*\right| \sup_{r\in[s\wedge\hat{\tau}^*, t\wedge\hat{\tau}^*]}\left|\frac{b_k\left(\hat{h},v\right)}{S_\mathcal{Q} S_\infty^{-2}\epsilon^{2\gamma+2}}\right|\right)^p + \left(C'_p\left|t\wedge\hat{\tau}^* - s\wedge\hat{\tau}^*\right|\right)^{p/2}\right]$$

(the bound on the stochastic integral follows using the Burkholder-Davis-Gundy inequality)

$$\leq D_p\left|t\wedge C - s\wedge C\right|^{p/2}$$

(the last inequality follows because $\tau^* \leq C$).

We used Claim 6.8 to bound $\left|\frac{b_k\left(\hat{h},v\right)}{S_\infty^{-2}S_\mathcal{Q}\epsilon^{2\gamma+2}}\right|$ (since we are inside $\Gamma''_{\xi'_\epsilon,\epsilon}$) and the fact that $\hat{\tau}^* \leq C$ (by definition of τ^*) on the fast timescale to obtain the bound.

We can therefore apply the Kolmogorov continuity theorem to obtain tightness and hence the existence of a convergent subsequence. Subsequently, we will show that all convergent subsequences have the same limit, and thus, the limit $\epsilon \to 0$ is unique.

The following stochastic process is clearly a martingale:

$$M_k^\epsilon(t') = \hat{h}_k(t') - \int_0^{t'} \frac{b_k\left(\hat{h}, v\right)}{S_\infty^{-2} S_\mathcal{Q} \epsilon^{2\gamma+2}} ds$$

However, as we are on the fast timescale, this means for times $0 \leq t' \leq \hat{\tau}^*$ that

$$\left|\hat{h}_k(t') - M_k^\epsilon(t')\right| = \left|\int_0^{t'} \frac{b_k\left(\hat{h}, v\right)}{S_\infty^{-2} S_\mathcal{Q} \epsilon^{2\gamma+2}} ds\right| \leq \hat{\tau}^* \sup_{s\in[0,\hat{\tau}^*]} \left|\frac{b_k\left(\hat{h}, v\right)}{S_\infty^{-2} S_\mathcal{Q} \epsilon^{2\gamma+2}}\right|$$

Thus using Claim 3.9 and Claim 3.7 we get for some $C' > 0$ that

$$\mathbb{P}\left[\sup_{t'\in[0,\hat{\tau}^*]} \left|\hat{h}_k(t') - M_k^\epsilon(t')\right| > C'\left\{\epsilon^\beta + \epsilon^\delta\right\}\right] \leq C_q \epsilon^q \ \forall q > 0$$

and hence $\hat{h}_k$ has a subsequence that weakly converges to the same limit as $M_k^\epsilon(t)$ in the sharp interface limit, which is a martingale we shall call M_k.

Thus using Claim 6.8 and Claim 6.7 we get for some $C' > 0$ that

$$\mathbb{P}\left[\sup_{t'\in[0,t'']} \left|\hat{h}_k(t') - M_k^\epsilon(t')\right| > C'\epsilon^\delta\right] \leq C_q \epsilon^q \ \forall q > 0$$

for some $\delta > 0$ and hence $\hat{h}_k$ has a subsequence that weakly converges to the same limit as $M_k^\epsilon(t')$ in the sharp interface limit, which is a martingale we shall call M_k.

The fact that it is a martingale follows (c.f. e.g. [EK09] 8.10) because for $s > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2}\left\langle \sigma_k\left(\hat{h}\left((t+s\wedge\tau^*)\right), v\right), \sigma_k\left(\hat{h}\left((t+s\wedge\tau^*)\right), v\right)\right\rangle\right. \\
&\qquad \left. - \frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2}\left\langle \sigma_k\left(\hat{h}\left(t\wedge\tau^*\right), v\right), \sigma_k\left(\hat{h}\left(t\wedge\tau^*\right), v\right)\right\rangle\right] \\
&= \mathbb{E}\left[\frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2}\int_0^1 \left[\frac{\epsilon}{S_\infty}\tau^{h_k(t+s\wedge\tau^*)} + \pi\left(t+s\wedge\tau^*, \epsilon\right)\right]^2 dx\right. \\
&\qquad \left. - \frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2}\int_0^1 \left[\frac{\epsilon}{S_\infty}\tau^{h_k(t\wedge\tau^*)} + o\left(t\wedge\tau^*, \epsilon\right)\right]^2 dx\right]
\end{aligned}$$

$$= \mathbb{E}\left[\frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2} \frac{\epsilon^2}{S_\infty^2} \frac{S_\infty}{\epsilon} + o(1) - \frac{1}{S_\infty^{-2} S_\mathcal{Q} \epsilon^2} \frac{\epsilon^2}{S_\infty^2} \frac{S_\infty}{\epsilon} + o(1)\right] \leq o(1) \to_{\epsilon\to 0} 0$$

(we used Claim 3.8, π has the property $|\pi(t,x)| \leq o(\epsilon)$ in the slow channel).

We now want to prove convergence to a stopped Brownian motion by showing that the extension of the limiting process beyond its stopping time by a Brownian motion is a Brownian motion; for this we use the Levy characterisation for each coordinate (in general our interfaces are correlated!).

Denote by τ the limit as $\epsilon \to 0$ of $\hat{\tau}^*$. We will show in the end of the sketch of argument that this limit exists. Now we consider the following stochastic process:

$$Z_k(t) = \begin{cases} M_k(t) \text{ for } t \leq \tau \\ M_k(\tau) + B_k(t-\tau) \text{ for } t > \tau \end{cases}$$

where $B_k(t)$ are Brownian motions with correlations $S_{\mathcal{Q}jk} S_\mathcal{Q}^{-1}$.

If we have $k = j$ and use Claim 3.6 and Claim 6.7 we get $\forall T > 0$

$$\mathbb{P}\left[\left|\int_0^{T\wedge\hat{\tau}^*} \left(d\hat{h}_k, d\hat{h}_k\right) - T\wedge\hat{\tau}^*\right| \geq \epsilon^{1/2}\right]$$

$$= \mathbb{P}\left[\left|\int_0^{T\wedge\hat{\tau}^*} S_\mathcal{Q}^{-1}\epsilon^{-2}\left\langle \mathcal{Q}\sigma_k(h,v), \sigma_k(h,v)\right\rangle dt' - T\wedge\hat{\tau}^*\right| \geq \epsilon^{1/2}\right] \leq C_q\epsilon^q$$

with a probability converging to 1 as $\epsilon \to 0$.

We used that inside the slow channel we have by Claim 3.8 that

$$\epsilon^{-2}\left\langle \mathcal{Q}\sigma_k(h,v), \sigma_k(h,v)\right\rangle = \epsilon^{-2}\left\langle \mathcal{Q}\left[\epsilon\tau_k^h + o(\epsilon)\right], \epsilon\tau_k^h + o(\epsilon)\right\rangle$$
$$= \left\langle \mathcal{Q}\tau_k^h, \tau_k^h\right\rangle + o(1) = \left\langle q*\tau_k^h, \tau_k^h\right\rangle + o(1),$$

where $q(x) = \sum_{i=1}^\infty \alpha_i^2 e_i^2(x)$ is the kernel of $\mathcal{Q}$. We used the following:

$$\left\langle q*\tau_k^h, \tau_k^h\right\rangle = \int_{m_j}^{m_j+2\epsilon} \left(q*\tau_k^h\right)\tau_k^h dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} \left(q*\tau_k^h\right)\left(-\frac{du^h}{dx}\right) dx$$
$$+ \int_{m_{j+1}-2\epsilon}^{m_{j+1}} \left(q*\tau_k^h\right)\tau_k^h dx$$

$$u_j^h = -u_x^h + \left(1-\chi^j\right)w^j - \chi^j w^{j+1}$$
$$\tau_j^h = \gamma^j u_x^h$$

$$q*\tau_k^h = \int_{m_j}^{m_{j+1}} q(t)\tau_k^h(x-t)dt$$

$$= \int_{m_j}^{m_j+2\epsilon} q(t)\tau_k^h(x-t)dt + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} q(t)\left(-\frac{du^h}{dx}(x-t)\right) dt$$

$$+\int_{m_{j+1}-2\epsilon}^{m_{j+1}} q(t)\tau_k^h(x-t)dt$$

But (by a similar argument to [CP89] Lemma 8.1 combined with [CP89] Lemma 7.9)

$$\left|\int_{m_j}^{m_j+2\epsilon} q(t)\tau_k^h(x-t)dt\right|, \left|\int_{m_{j+1}-2\epsilon}^{m_{j+1}} q(t)\tau_k^h(x-t)dt\right|$$

$$\leq C\epsilon^{-1}\alpha \leq C\epsilon^{-1}\epsilon^{2\gamma+2+\beta} \to_{\epsilon\to 0} 0.$$

Now, we note by the definition of u^h that

$$\frac{du^h}{dx} = \left(1-\chi^j\right)\phi_x^j - \chi_x^j\phi^j + \chi^j\phi_x^{j+1} + \chi_x^j\phi^{j+1}.$$

Due to the definition of χ^j we have

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} q(t)\left(-\frac{du^h}{dx}(x-t)\right)dt$$

$$= \int_{m_j+2\epsilon}^{h_j-\epsilon} q(t)\left(-\phi_x^j(x-t)\right)dt$$

$$+ \int_{h_j-\epsilon}^{h_j+\epsilon} q(t)\left(-\left(1-\chi^j\right)\phi_x^j(x-t) + \chi_x^j\phi^j(x-t) - \chi^j\phi_x^{j+1}(x-t) - \chi_x^j\phi^{j+1}(x-t)\right)dt$$

$$+ \int_{h_j+\epsilon}^{m_{j+1}-2\epsilon} q(t)\left(-\phi_x^{j+1}(x-t)\right)dt.$$

[CP89] Lemma 8.2 tells us that in $[h_j-\epsilon, h_j+\epsilon]$ we have $\left|\phi^j - \phi^{j+1}\right|, \left|\phi_x^j - \phi_x^{j+1}\right| \leq C\epsilon^{-1}\left|\alpha_j - \alpha_{j+1}\right| \leq C'\epsilon^{2\gamma+1+\beta}$ so that

$\int_{h_j-\epsilon}^{h_j+\epsilon} q(t)\left(-\left(1-\chi^j\right)\phi_x^j + \chi_x^j\phi^j(x-t) - \chi^j\phi_x^{j+1}(x-t) - \chi_x^j\phi^{j+1(x-t)}\right)dt = \int_{h_j-\epsilon}^{h_j+\epsilon} q(t)\left(-\phi_x^j(x-t) + \mathcal{O}\left(\epsilon^{2\gamma+1+\beta}\right)\right)dt.$

Since ϕ^j is monotone on $[m_j, h_j], [h_j, m_{j+1}]$ and $|\phi \pm 1| < \delta$ by [CP89] Lemma 7.2, we can conclude that $|\phi_x^j| \leq C$ on these intervals for a suitable $C > 0$. However as $\epsilon \to 0$, this constant becomes arbitrarily small. Thus $\left|\int_{m_j+2\epsilon}^{h_j-\epsilon} q(t)\left(-\phi_x^j(x-t)\right)dt\right|, \left|\int_{h_j+\epsilon}^{m_{j+1}-2\epsilon} q(t)\left(-\phi_x^{j+1}(x-t)\right)dt\right| \to_{\epsilon\to 0} 0.$

To show that $q * \tau_k^h = -2q\left(x-h_k\right)$, we approximate q in the interval $[h_k-\epsilon, h_k+\epsilon]$ as $q(h_k)$:

We note that for each $c > 0$, there exists a small enough $\epsilon > 0$ s.t. $|t-h_k| \geq c$ implies

$$\left\langle q(h_k), -\phi_x^j(x-t)\right\rangle = q(h_k)\int_{h_j-\epsilon}^{h_j+\epsilon} -\phi_x^j(x-t)dx = 0$$

while

$$\left\langle q(h_k), -\phi_x^j(x-h_k)\right\rangle = q(h_k)\int_{h_j-\epsilon}^{h_j+\epsilon} -\phi_x^j(x-t)dx = -2q(h_k).$$

Since as $\epsilon \to 0$, the constant approximation of q converges to q, this shows that

$$q * \tau_k^h \to_{\epsilon\to 0} q\left(x-h_k\right).$$

To complete the sketch of argument, we now only need to show that

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} 2q(x-h_k)\frac{du^h}{dx}dx \to_{\epsilon\to 0} q(0).$$

Similarly to before, we have that for each $c>0$ there exists $\epsilon>0$ small enough s.t. $\frac{du^h}{dx}<c$ outside of $[h_j-\epsilon, h_j+\epsilon]$. Thus it suffices to show that

$$\int_{h_j-\epsilon}^{h_j+\epsilon} 2q(x-h_k)\frac{du^h}{dx}dx \to_{\epsilon\to 0} 4q(0).$$

By employing the same argument as before of constantly approximating q in $[h_j-\epsilon, h_j+\epsilon]$ we conclude that

$$\langle \mathcal{Q}\sigma_k(h,v), \sigma_k(h,v)\rangle_{\epsilon\to 0} 4q(0)$$

in the slow channel.

We note using Levy's characterisation of Brownian motion that $Z_k(t)$ is a Brownian motion.

We note for each pair j,k that $\forall T>0$

$$\mathbb{P}\left[\left|\int_0^T \left(d\hat{h}_j, d\hat{h}_k\right) - S_{\mathcal{Q}jk}S_{\mathcal{Q}}T\right| \geq \epsilon^{1/2}\right] =$$

$$= \mathbb{P}\left[\left|\int_0^T S_{\mathcal{Q}}^{-1}\epsilon^{-2}\langle \mathcal{Q}\sigma_j(h,v), \sigma_k(h,v)\rangle dt' - S_{\mathcal{Q}jk}S_{\mathcal{Q}}T\right| \geq \epsilon^{1/2}\right] \leq C_q\epsilon^q,$$

in addition the generator of this diffusion is Lipschitz and thus weakly converges to a unique solution of a stochastic differential equation.

This is implies that $(\hat{h}_1(t), ..., \hat{h}_N(t))$ stopped at $\hat{\tau}^*$ weakly converges to an N-dimensional Brownian motion with correlations $S_{\mathcal{Q}kj}S_{\mathcal{Q}}^{-1}$ between the k-th and j-th coordinate stopped at τ, which is a hitting time, unless $S_{\mathcal{Q}}S_\infty^{-2}C$ is smaller.

We now prove that it is not a later hitting time, since if one has a minimum distance $g(\epsilon) > \xi'_\epsilon$ (greater than the stopping distance but still with the property $\lim_{\epsilon\to 0} g(\epsilon) \to 0$) before the stopping time, we have the following:

Denote $\psi_j(t) = g(\epsilon) + \int_0^t \epsilon^{-1}\left\langle \sigma_{j+1}\left(\hat{h}, v\right) - \sigma_j\left(\hat{h}, v\right), dW\right\rangle$, this is essentially the difference between the position of the $j+1$-th interface and the j-th interface starting at $g(\epsilon)$, but without the (basically irrelevant) drift.

We now show that this will with a probability converging to 1 as $\epsilon \to 0$ have a minimum value of ξ'_ϵ after time $(g(\epsilon))^{2-\kappa}$ while the probability of becoming $\mathcal{O}(1)$ large converges to 0 as $\epsilon \to 0$:

We note that this is nothing else than a time-changed Brownian motion starting at $g(\epsilon)$, i.e. if $B(\cdot)$ is a standard Brownian motion, we can denote ψ_j as

$$g(\epsilon) + B\left(\epsilon^{-2}\int_0^t \left[\left\langle \mathcal{Q}\sigma_{j+1}\left(\hat{h},v\right), \sigma_{j+1}\left(\hat{h},v\right)\right\rangle\right.\right.$$
$$\left. -2\left\langle \mathcal{Q}\sigma_{j+1}\left(\hat{h},v\right), \sigma_j\left(\hat{h},v\right)\right\rangle + \left\langle \mathcal{Q}\sigma_j\left(\hat{h},v\right), \sigma_j\left(\hat{h},v\right)\right\rangle dt\right.$$

Since Brownian fluctuations scale like square roots of their time increments, it is sufficient to show that with high probability $\psi_j(t)$ will hit ξ'_ϵ after time $(g(\epsilon))^{2-\kappa}$ (plus an aditional term for the difference between this martingale and the actual difference between the interfaces) for some small $\kappa > 0$.

By $[RY99]$ V §1 (together with the fact that in the slow channel we have

$$0 < \int_0^t \left\langle \mathcal{Q}\sigma_{j+1}\left(\hat{h},v\right), \sigma_{j+1}\left(\hat{h},v\right)\right\rangle - 2\left\langle \mathcal{Q}\sigma_{j+1}\left(\hat{h},v\right), \sigma_j\left(\hat{h},v\right)\right\rangle$$
$$+ \left\langle \mathcal{Q}\sigma_j\left(\hat{h},v\right), \sigma_j\left(\hat{h},v\right)\right\rangle dt = \mathcal{O}(\epsilon)$$

using Claim 3.8 and Claim 6.7) we have for some small $\kappa > 0$ that

$$\mathbb{P}\left[\min_{t\in\left[0,(g(\epsilon))^{2-\kappa}+\left[C\sup_{s\in[0,C]}\left|\frac{S_\infty^2 b_k(\hat{h},v)}{S_\mathcal{Q}\epsilon^{2\gamma+2}}\right|\right]^{2-\kappa}\wedge\hat{\tau}^*\right]} \psi_j(t) \le \xi'_\epsilon - C\sup_{s\in[0,C]}\left|\frac{S_\infty^2 b_k\left(\hat{h},v\right)}{S_\mathcal{Q}\epsilon^{2\gamma+2}}\right|\right]$$

$\geq 1 - C_q\epsilon^q \;\forall q > 0$.

However we note that by Claim 6.7 that

$$\mathbb{P}\left[\sup_{t\in\left[0,(g(\epsilon))^{2-\kappa}+\left[C\sup_{s\in[0,C]}\left|\frac{S_\infty^2 b_k(\hat{h},v)}{S_\mathcal{Q}\epsilon^{2\gamma+2}}\right|\right]^{2-\kappa}\wedge\hat{\tau}^*\right]} \left|\psi_j(t) - \left(\hat{h}_{j+1} - \hat{h}_j\right)\right| \le C\left(g(\epsilon)\right)^{2-\kappa}\right]$$

$\geq 1 - C_q\epsilon^q \;\forall q > 0$

and hence

$$\mathbb{P}\left[\min_{t\in\left[0,(g(\epsilon))^{2-\kappa}\wedge\hat{\tau}^*\right]} \hat{h}_{j+1} - \hat{h}_j \le \xi_\epsilon\right] \geq 1 - C_q\epsilon^q \;\forall q > 0,$$

which trivially implies

$$\mathbb{P}\left[\min_{t\in[0,\hat{\tau}^*]} \hat{h}_{j+1} - \hat{h}_j \le \xi_\epsilon\right] \geq 1 - C_q\epsilon^q \;\forall q > 0.$$

Therefore, as $\epsilon \to 0$, any starting configuration with the minimum distance converging to 0 as $\epsilon \to 0$ ends up at the critical stopping distance ξ'_ϵ with high probability within a time that converges to 0 on the fast timescale in the sharp interface limit. Thus it is necessarily the first hitting time (unless C is smaller than the first hitting time).

We now show the convergence of $\hat{\tau}^*$ to a limit τ in the sharp interface limit: Firstly we recall the definition

$$\hat{\tau}^* = C \wedge \inf \left\{ t > 0 : \delta_{\hat{h}(t)} = \xi'_\epsilon \text{ or } \|v\| > \epsilon^{1+\delta} \text{ or } \|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{1+\delta} \right\}$$

and note that since the probability of $\|v\| > \epsilon^{1+\delta}$ or $\|v\|_\infty \geq 1$ or $\|z\| > \epsilon^{1+\delta}$ is asymptotically small in ϵ, and converges to 0 as $\epsilon \to 0$, the limit of $\hat{\tau}^*$ - if it exists - is the same as the limit of $C \wedge \inf \left\{ t > 0 : \delta_{\hat{h}(t)} = \xi'_\epsilon \right\}$.

Secondly define $\Sigma_a(f_1(\cdot), ..., f_N(\cdot)) = \inf \{ t : \min_{i \neq j} |f_i(t) - f_j(t) \mod 1| = a \}$. Then for all $\epsilon > 0$ we have $(2\gamma + 2 + \beta)\epsilon |\log \epsilon| < 4\gamma\epsilon|\log \epsilon|$ and hence we have by Claim 6.7

$$\mathbb{P}\left[\hat{\tau}^* \geq C \wedge \Sigma_0 \left(\hat{h}_1(t \wedge \Sigma_{\frac{4\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}}), ..., \hat{h}_N(t \wedge \Sigma_{\frac{4\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}}) \right) \right] \leq C_q \epsilon^q \ \forall q > 0,$$

since this event could only happen if $\|v(\hat{\tau}^*)\| \geq \epsilon^{1+\delta}$ or $\|v(\hat{\tau}^*)\|_\infty \geq 1$ or $\|z\| > \epsilon^{1+\delta}$.

We note that $\Sigma_0 \left(\hat{h}_1(t \wedge \Sigma_{\frac{4\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}}), ..., \hat{h}_N(t \wedge \Sigma_{\frac{4\gamma\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}}) \right)$ weakly, since $\Sigma_0(M_1, ..., M_N)$ is only discontinuous on a set of Lebesgue measure 0; this follows from [Bil99] Theorem 2.7.

Hence in the limit

$$\mathbb{P}\left[\tau \geq C \wedge \Sigma_0(M_1, ..., M_N)\right] = 1.$$

However we also have $0 < (2\gamma + 2 + \beta)\epsilon|\log \epsilon|$ for $\epsilon > 0$ and thus by the same argument

$$\mathbb{P}\left[C \wedge \Sigma_0 \left(\hat{h}_1, ..., \hat{h}_N \right) \geq \hat{\tau}^* \right] \leq C_q \epsilon^q \ \forall q > 0,$$

which implies in the limit that

$$\mathbb{P}\left[C \wedge \Sigma_0(M_1, ..., M_N) \geq \tau\right] = 1.$$

In both cases we used Lebesgue's dominated convergence theorem to obtain convergence to the limit.

We have now shown that

$$\mathbb{P}[C \wedge \Sigma_0(M_1, ..., M_N) = \tau] = 1,$$

i.e. that

$$\hat{\tau}^* \to_{\epsilon \to 0} C \wedge \Sigma_0(M_1, ..., M_N) \text{ weakly};$$

this limit is the first hitting time, unless C is smaller.

We now finally note that τ is positive and finite: This is an easy observation, since the definition gives the upper bound $\hat{\tau}^* \leq C$ on the fast timescale, while due to the finite moments of M_k it almost surely takes at least finite time for two interfaces to attain the distance 0. Therefore τ is positive and finite with probability 1.

In the beginning of this we made the assumption that h is a semimartingale. We now prove that this assumption is indeed true. Firstly, we observe that the coupled system for h and v has a solution. This works in the exact same way as for space-time white noise: We note that all terms are bounded up to the stopping times and Lipschitz continuous. Considering the couples system as an SPDE with several space-dimensions then gives existence and uniqueness by the use of [DPZ92] Theorem 7.4.

We prove this in the same way as we did for white noise, except that we use the appropriate bounds on drift and diffusion.

Secondly, we establish that we do indeed have $\langle v, \tau_i^h \rangle = 0 \; \forall i$, so that $u = u^h + v$ solves the equation:

Recall that for h we have

$$
\begin{aligned}
dh_r = &\sum_i A_{ri}^{-1}(h,v) \langle \mathcal{L}(u^h + v), \tau_i^h \rangle dt \\
&+ \epsilon^{2\gamma} \sum_{i,l,k} A_{ri}^{-1}(h,v) \left[\frac{1}{2} \langle v, \tau_{ilk}^h \rangle - \frac{1}{2} \langle u_{kl}^h, \tau_i^h \rangle - \langle u_k^h, \tau_{il}^h \rangle \right] \langle \sigma_k(h,v), \sigma_l(h,v) \rangle dt \\
&+ \epsilon^{2\gamma} \sum_i A_{ri}^{-1}(h,v) \sum_j \left\langle \tau_{ij}^h, \sigma_j(h,v) \right\rangle dt + \sum_i A_{ri}^{-1}(h,v) \langle \tau_i^h, dV \rangle.
\end{aligned}
$$

Applying $A(h,v)$ on both sides gives

$$
\begin{aligned}
&\sum_i A_{ij}(h,v) dh_j = \langle \mathcal{L}(u^h + v), \tau_i^h \rangle dt + \\
&\epsilon^{2\gamma} \sum_{l,k} \left[\frac{1}{2} \langle v, \tau_{ilk}^h \rangle - \frac{1}{2} \langle u_{kl}^h, \tau_i^h \rangle - \langle u_k^h, \tau_{il}^h \rangle \right] \langle \mathcal{Q} \sigma_k(h,v), \sigma_l(h,v) \rangle dt \\
&+ \epsilon^{2\gamma} \sum_j \left\langle \mathcal{Q} u_{ij}^h, \sigma_j(h,v) \right\rangle dt + \left\langle \tau_i^h, dV \right\rangle .
\end{aligned}
$$

Applying Claim 6.4 and the definition of A we can rewrite this as

$$
\begin{aligned}
&\sum_j \left[\left\langle \tau_i^h, u_j^h \right\rangle - \left\langle v, \tau_{ij}^h \right\rangle \right] dh_j = \langle \mathcal{L}(u^h + v), \tau_i^h \rangle dt \\
&+ \epsilon^{2\gamma} \sum_{l,k} \left[\frac{1}{2} \langle v, \tau_{ilk}^h \rangle - \frac{1}{2} \langle u_{kl}^h, \tau_i^h \rangle - \langle u_k^h, \tau_{il}^h \rangle \right] (dh_l, dh_k) \\
&+ \sum_j \left\langle dV, \tau_{ij}^h \right\rangle dh_j + \langle \tau_i^h, dV \rangle
\end{aligned}
$$

Now, taking the inner product of τ_i^h and the left hand side of our SPDE

$$du = \sum_j u_j^h dh_j + \frac{1}{2}\sum_{k,l} u_{kl}^h (dh_k, dh_l) + dv$$

gives

$$\langle \tau_i^h, du\rangle = \sum_j \left\langle \tau_i^h, u_j^h \right\rangle dh_j + \frac{1}{2}\sum_{k,l} \langle u_{kl}^h, \tau_i^h\rangle (dh_k, dh_l) + \langle \tau_i^h, dv\rangle.$$

On the other hand applying the Itô formula gives

$$\begin{aligned} d\langle \tau_i^h, v\rangle &= \langle \tau_i^h, dv\rangle + \langle v, d\tau_i^h\rangle + \langle dv, d\tau_i^h\rangle \\ &= \langle \tau_i^h, dv\rangle + \sum_j \left\langle v, \tau_{ij}^h\right\rangle dh_j + \frac{1}{2}\sum_{j,k} \left\langle v, \tau_{ijk}^h\right\rangle (dh_j, dh_k) + \sum_j \left\langle \tau_{ij}^h, dv\right\rangle dh_j \end{aligned}$$

But since $dv = du - du^h$ we have

$$\sum_j \langle \tau h_{ij}, dv\rangle dh_j = \sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) - \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h\right\rangle (dh_k, dh_j)$$

Furthermore taking the inner product of τ_i^h and the right hand side of the SPDE gives

$$\langle \tau_i^h, du\rangle = \langle \mathcal{L}(u), \tau_i^h\rangle dt + \langle \tau_i^h, dV\rangle$$

Combining the last equations yields

$$\begin{aligned} d\langle \tau_i^h, v\rangle &= \langle \mathcal{L}(u), \tau_i^h\rangle dt + \langle \tau_i^h, dV\rangle - \sum_j \langle \tau_i^h, u_j^h\rangle dh_j \\ &\quad - \frac{1}{2}\sum_{k,l} \langle u_{kl}^h, \tau_i^h\rangle (dh_k, dh_l) + \sum_j \left\langle v, \tau_{ij}^h\right\rangle dh_j + \frac{1}{2}\sum_{j,k} \left\langle v, \tau_{ijk}^h\right\rangle (dh_j, dh_k) \\ &\quad + \sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) - \sum_{j,k} \left\langle \tau_{ij}^h, u_k^h\right\rangle (dh_k, dh_j) \\ &\quad = \langle \mathcal{L}(u), \tau_i^h\rangle dt + \langle \tau_i^h, dV\rangle - \sum_j A_{ij}(h,v) dh_j + \sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) \\ &\quad + \sum_{j,k} \left[\frac{1}{2}\left\langle v, \tau_{ijk}^h\right\rangle - \frac{1}{2}\left\langle u_{kj}^h, \tau_i^h\right\rangle - \left\langle \tau_{ij}^h, u_k^h\right\rangle\right] (dh_k, dh_j) \\ &= \langle \mathcal{L}(u), \tau_i^h\rangle dt + \langle \tau_i^h, dV\rangle - \Bigg[\langle \mathcal{L}(u), \tau_i^h\rangle dt + \langle \tau_i^h, dV\rangle - \sum_j A_{ij}(h,v) dh_j \end{aligned}$$

$$+\sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) + \sum_{j,k}\left[\frac{1}{2}\left\langle v, \tau_{ijk}^h\right\rangle - \frac{1}{2}\left\langle u_{kj}^h, \tau_i^h\right\rangle - \left\langle \tau_{ij}^h, u_k^h\right\rangle\right](dh_k, dh_j)\Bigg]$$

$$+\sum_j \left(\left\langle \tau_{ij}^h, dV\right\rangle, dh_j\right) + \sum_{j,k}\left[\frac{1}{2}\left\langle v, \tau_{ijk}^h\right\rangle - \frac{1}{2}\left\langle u_{kj}^h, \tau_i^h\right\rangle - \left\langle \tau_{ij}^h, u_k^h\right\rangle\right](dh_k, dh_j) = 0$$

This completes the sketch of argument that h is indeed a semimartingale notation.

Define

$$\psi_d = \frac{CS_\infty^2}{S_\mathcal{Q}} \wedge \inf\left\{t > 0 : \delta_{h_0} = d \text{ or } \|v\| > \epsilon^{1+\delta} \text{ or } \|v\|_\infty \geq 1 \text{ or } \|z\| > \epsilon^{1+\delta}\right\}$$

where $\psi_d \geq \frac{\epsilon|\log\epsilon|}{\sqrt{-f'(1)}}$.

Claim 6.7. *Suppose $u_0 \in \Gamma''_{\xi'_\epsilon,\epsilon}$. Let $u(x,t) = u^h(x,t) + v(x,t)$ solve (1.1) for $t > 0$.*

Then the following inequalities hold:

$$\mathbb{E}\left[\|v(\psi_d)\|^p\right] \leq \mathbb{E}\left[\|v(0)\|^p\right] + C'_p\epsilon^{-2\gamma-2}\epsilon^{p\gamma} \leq D_p\epsilon^{-2\gamma-2}\epsilon^{(1+\delta)p}\ \forall p \geq 2$$

for some $C'_p, D_p > 0$ independent of ϵ and polynomial in p, as well as

$$\mathbb{P}\left(\|v(\psi_d)\| \geq \epsilon^{1+\delta}\right) \leq D_r\epsilon^{-2\gamma-2}\epsilon^{2\delta r}\ \forall r \geq 2,$$

$$\mathbb{P}\left(\|v(\psi_d)\| \geq \epsilon^{1+\delta} \text{ or } \|v(\psi_d)\|_\infty \geq 1\right) \leq D_r\epsilon^{-2\gamma-2}\epsilon^{2\delta r}\ \forall r \geq 2.$$

Remark: The second inequality implies that the probability of exiting the slow channel within the $\mathcal{O}\left(\epsilon^{-2\gamma-2}\right)$ time needed for two interfaces to have ξ'_ϵ distance is asymptotically small.

Claim 6.8. *Let $u = u^h + v \in \Gamma''_{\xi'_\epsilon,\epsilon}$, then we have uniformly in ϵ that*

$$|b_r(h,v)| \leq C\epsilon^{2\gamma+2+\beta} \tag{6.12}$$

for any $r = 1, ..., N$, some $C > 0$. We also have the following pointwise estimate:

$$|b_r(h,v)| \leq C\epsilon e^{-\delta_h/\epsilon} + \mathcal{O}\left(\epsilon^{2\gamma+2+\beta}\right)$$

Sketch of Argument. We recall

$$b_r(h,v) = \sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}(u^h+v), \tau_i^h\right\rangle$$

$$+\sum_{i,l,k} A_{ri}^{-1}(h,v)\left[\frac{1}{2}\left\langle v, \tau_{ilk}^h\right\rangle - \frac{1}{2}\left\langle u_{kl}^h, \tau_i^h\right\rangle - \left\langle u_k^h, \tau_{il}^h\right\rangle\right]\epsilon^{2\gamma}\langle \mathcal{Q}\sigma_k(h,v), \sigma_l(h,v)\rangle$$

$$+\sum_i A_{ri}^{-1}(h,v)\sum_j \epsilon^{2\gamma}\left\langle \mathcal{Q}\tau_{ij}^h,\sigma_j(h,v)\right\rangle.$$

It is easy to check that unless we have $r=i=l=k$ all contributing terms in the second two sums are $\mathcal{O}\left(\epsilon^{2\gamma+2+\beta}\right)$.

We also have
$u_{jx}^h = -u_{xx}^h + \left(1-\chi^j\right)w_x^j - \chi_x^j w^j - \chi_x^j w^{j+1} - \chi^j w_x^{j+1}$
for $x\in I_j$.
Evidently,

$$u_{ii}^h = -u_{xi}^h - \chi_i^i w^i + (1-\chi^i)w_i^i - \chi_i^i w^{i+1} - \chi^i w_i^{i+1}$$

$$= -u_{xx}^h + \left(1-\chi^i\right)w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x^{i+1} - \chi_i^i w^i + \left(1-\chi^i\right)w_i^i - \chi_i^i w^{i+1} - \chi^i w_i^{i+1}.$$

We recall $\tau_j^h = \gamma^j u_x^h$ (which is clearly equal to 0 outside of I_j) where

$$\gamma^j = \chi\left((x-m_j)/\epsilon - 1\right)\left[1-\chi((x-m_{j+1})/\epsilon - 1)\right],$$

and quote from [CP89] p. 564 that

$$\tau_{ii}^h = -\gamma_i^i u_x^h - \gamma^i u_{xi}^h = -\gamma_i^i u_x^h - \gamma^i\left(-u_{xx}^h + \left(1-\chi^i\right)w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x\right)$$

$$= u_{xx}^h - \gamma_i^i u_x^h + \left(\gamma^i - 1\right)u_{xx}^h - \gamma^i\left(\left(1-\chi^i\right)w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x\right),$$

which is nothing else than $\tau_{ii}^h = \gamma^i u_{xx}^i + \mathcal{O}\left(\epsilon^{-1}\right)$.
$u^h = \left(1-\chi^j\right)\phi^j + \chi^j\phi^{j+1}$
For $r=i=k=l$ we obtain the following expressions:

$$\left\langle \tau_{ii}^h, u_i^h\right\rangle = \int_{m_j}^{m_{j+1}} \tau_{ii}^h u_i^h\,dx = \int_{m_j}^{m_j+2\epsilon}\tau_{ii}^h u_i^h\,dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{ii}^h u_i^h\,dx + \int_{m_{j+1}-2\epsilon}^{m_{j+1}}\tau_{ii}^h u_i^h\,dx,$$

$$\left\langle u_{ii}^h, \tau_i^h\right\rangle = \int_{m_j}^{m_{j+1}} u_{ii}^h \tau_i^h\,dx = \int_{m_j}^{m_j+2\epsilon} u_{ii}^h\tau_i^h\,dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{ii}^h u_i^h\,dx + \int_{m_{j+1}-2\epsilon}^{m_{j+1}} u_{ii}^h\tau_i^h\,dx.$$

We calculate further that

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} u_{ii}^h u_i^h\,dx = \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}\left[-u_{xj}^h - \chi_j^j w^j + \left(1-\chi^j\right)w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1}\right]u_j^h\,dx$$

$$= \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} -u_{xj}^h u_j^h\,dx + \int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}\left[-\chi_j^j w^j + \left(1-\chi^j\right)w_j^j - \chi_j^j w^{j+1} - \chi^j w^{j+1}\right]u_j^h\,dx$$

$$= \left[-\frac{1}{2}\left(u_i^h\right)^2\right]_{m_j+2\epsilon}^{m_{j+1}-2\epsilon} +$$

$$+\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}\left[-\chi_j^j w^j+\left(1-\chi^j\right)w_j^j-\chi_j^j w^{j+1}-\chi^j w^{j+1}\right]$$

$$\cdot\left[\left(1-\chi^j\right)\phi_j^j+\chi^j\phi_j^{j+1}+\chi_x^j\left(\phi^j-\phi^{j+1}\right)\right]dx,$$

but

$$\left[-\frac{1}{2}\left(u_j^h\right)^2\right]_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}=-\frac{1}{2}\left(\left(u_j^h\left(m_{j+1}-2\epsilon\right)\right)^2-\left(u_j^h\left(m_j+2\epsilon\right)\right)^2\right)$$

$$=\frac{1}{2}\left(\left(\phi_x^{j+1}\left(m_{j+1}-2\epsilon\right)+w^{j+1}\left(m_{j+1}-2\epsilon\right)\right)\right)^2$$

$$-\frac{1}{2}\left(2\left(-\phi_x^j(m_j+2\epsilon)+w^j(m_j+2\epsilon)\right)\right)+\phi_x^{j+1}(m_j+2\epsilon)-w^{j+1}(m_j+2\epsilon))^2$$

$$\leq\left(\mathcal{O}\left(\epsilon\right)\right)^2\leq\mathcal{O}\left(\epsilon^2\right)$$

(the end works in a similar manner to [CP89] Lemma 7.9) and

$$\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}\left[-\chi_j^j w^j+\left(1-\chi^j\right)w_j^j-\chi_j^j w^{j+1}-\chi^j w^{j+1}\right]$$

$$\cdot\left[\left(1-\chi^j\right)\phi_j^j+\chi^j\phi_j^{j+1}+\chi_x^j\left(\phi^j-\phi^{j+1}\right)\right]dx$$

$$=\int_{m_j+2\epsilon}^{h_j-\epsilon\vee m_{j+1}-2\epsilon}w_j^j\phi_j^j dx+$$

$$+\int_{h_j-\epsilon\vee m_{j+1}-2\epsilon}^{h_j+\epsilon\wedge m_{j+1}-2\epsilon}\left[-\chi_j^j w^j+\left(1-\chi^j\right)w_j^j-\chi_j^j w^{j+1}-\chi^j w^{j+1}\right]$$

$$[\left(1-\chi^j\right)\phi_j^j+\chi^j\phi_j^{j+1}+\chi_x^j\left(\phi^j-\phi^{j+1}\right)dx$$

$$+\int_{h_j+\epsilon\wedge m_{j+1}-2\epsilon}^{m_{j+1}-2\epsilon}-w^{j+1}\phi_j^{j+1}dx=\mathcal{O}\left(\epsilon^{-1+\beta}\right)$$

(cf. [CP89] sections 7 and 8 for more details on how to do the arithmetic). Now,

$$\int_{m_i}^{m_i+2\epsilon}\tau_{ii}^h u_i^h dx=\int_{m_i}^{m_i+2\epsilon}\left(-\gamma_i^i\phi_x^i+\gamma^i\phi_{xx}^i-\gamma^i w_x^i\right)\left(-\phi_x^i+w^i\right)dx=$$

$$-\int_{m_i}^{m_i+2\epsilon}\gamma^i\phi_x^i\phi_{xx}^i dx$$

$$+\int_{m_i}^{m_i+2\epsilon}\left[\left(-\gamma_i^i\phi_x^i+\gamma^i\phi_{xx}^i-\gamma^i w_x^i\right)w^i+\left(-\gamma_i^i\phi_x^i-\gamma^i w_x^i\right)\left(-\phi_x^i+w^i\right)\right]dx$$

$= \mathcal{O}\left(\epsilon^{-1+\beta}\right).$

But in a similar manner we may bound $\int_{m_i}^{m_i+2\epsilon} u_{ii}^h \tau_i^h dx, \int_{m_{i+1}-2\epsilon}^{m_{i+1}} \tau_{ii}^h u_i^h dx, \int_{m_{i+1}-2\epsilon}^{m_{i+1}} u_{ii}^h \tau_i^h dx$, so that we may conclude that

$$\left|\left\langle \tau_{ii}^h, u_i^h \right\rangle\right|, \left|\left\langle u_{ii}^h, \tau_i^h \right\rangle\right| \leq \mathcal{O}\left(\epsilon^{-1+\beta}\right).$$

We used [CP89] Lemma 7.9 to bound w^j, w^{j+1} and [CP89] Lemma 7.10 to bound w_j^j, w_j^{j+1}.

We now calculate

$$\tau_{iii}^h = -\gamma_{ii}^i u_x^h - \gamma_{ix}^i u_i^h - \gamma_i^i u_{xi}^h - \gamma^i u_{xii}^h.$$

But

$$u_{xii}^h = \frac{\partial}{\partial x} u_{ii}^h$$
$$= \frac{\partial}{\partial x}\left[-u_{xx}^h + \left(1-\chi^i\right) w_x^i - \chi_x^i w^i - \chi_x^i w^{i+1} - \chi^i w_x^{i+1}\right.$$
$$-\chi_i^i w^i + \left(1-\chi^i\right) w_i^i - \chi_i^i w^{i+1} - \chi^i w_i^{i+1}$$
$$= -u_{xxx}^h + \mathcal{O}\left(\epsilon^{-2}\right).$$

However,

$$\left\|u_{xxx}^h\right\|$$
$$= \left\|-\chi_{xxx}^j \phi^j - 3\chi_{xx}^j \phi_x^j - 3\chi_x^j \phi_{xx}^j - \chi^j \phi_{xxx}^j + \chi_{xxx}^j \phi^{j+1} + 3\chi_{xx}^j \phi_x^{j+1} + 3\chi_x^j \phi_{xx}^{j+1} + \chi^j \phi_{xxx}^{j+1}\right\|$$
$$\leq \mathcal{O}\left(\epsilon^{-5/2}\right)$$

since

$$\phi_{xxx}^j = -\frac{1}{\epsilon^2} f'\left(\phi^j\right) \phi_x^j.$$

Combining the fact that in the slow channel we have $\|v\| \leq \epsilon^{1+\delta}$ and that (cf. [CP89] Proposition 2.3) we have

$$\begin{aligned}
\left\|\tau_{iii}^h\right\| &\leq \left\|\gamma_{ii}^i u_x^h\right\| + \left\|\gamma_{ix}^i u_i^h\right\| + \left\|\gamma_i^i u_{xi}^h\right\| + \left\|\gamma^i u_{xii}^h\right\| \\
&\leq \mathcal{O}\left(\epsilon^{-2}\right)\left\|u_x^h\right\| + \mathcal{O}\left(\epsilon^{-2}\right)\left\|u_i^h\right\| + \mathcal{O}\left(\epsilon^{-1}\right)\left\|u_{xi}^h\right\| + \mathcal{O}(1)\left\|-u_{xxx}^h + \mathcal{O}\left(\epsilon^{-2}\right)\right\| \\
&\leq \mathcal{O}\left(\epsilon^{-2}\right)\mathcal{O}\left(\epsilon^{-1/2}\right) + \mathcal{O}\left(\epsilon^{-2}\right)\mathcal{O}\left(\epsilon^{-1/2}\right) + \mathcal{O}\left(\epsilon^{-1}\right)\mathcal{O}\left(\epsilon^{-3/2}\right) + \mathcal{O}(1)\left\|u_{xxx}^h\right\| + \mathcal{O}\left(\epsilon^{-2}\right) \\
&\leq \mathcal{O}\left(\epsilon^{-5/2}\right) + \mathcal{O}\left(\epsilon^{-5/2}\right) \leq \mathcal{O}\left(\epsilon^{-5/2}\right),
\end{aligned}$$

we get

$$\left|\left\langle v, \tau_{iii}^h \right\rangle\right| \leq \mathcal{O}\left(\epsilon^{-3/2+\delta}\right).$$

Observe that (combing [CP89] Theorem 6.1, Lemma 3.3, Proposition 3.4)

$$\left|\sum_i A_{ri}^{-1}(h,v)\left\langle \mathcal{L}(u^h+v),u_i\right\rangle\right| \leq \mathcal{O}\left(\epsilon e^{-\delta_h/\epsilon}\right).$$

In a similar manner to how we showed in the sketch of argument of Claim 6.5 that $\left\langle \mathcal{Q}\tau_i^h,\tau_i^h\right\rangle \to_{\epsilon\to 0} 4q(0)$ we note that

$$\left\langle \mathcal{Q}\sigma_k(h,v),\sigma_l(h,v)\right\rangle \leq \mathcal{O}\left(\epsilon^2\right).$$

We now calculate our last necessary bound: In the slow channel we have

$$\epsilon^{-1}\left\langle \mathcal{Q}\tau_{ij}^h,\sigma_k(h,v)\right\rangle = \epsilon^{-1}\left\langle \mathcal{Q}\tau_{ij}^h,\frac{\epsilon}{S_\infty}\tau_k^h+o(\epsilon)\right\rangle$$

$$=\frac{1}{S_\infty}\left\langle \tau_{ij}^h,\mathcal{Q}\tau_k^h\right\rangle+o(\epsilon)=\frac{1}{S_\infty}\left\langle \tau_{ij}^h,q*\tau_k^h\right\rangle+o(1),$$

we showed in the sketch of argument of Claim 6.5 that $q*\tau_k^h \to_{\epsilon\to 0} 2q(x-h_k)$, so it suffices to show that $\frac{1}{S_\infty}\left\langle \tau_{ij}^h,2q(x-h_k)\right\rangle \to_{\epsilon\to 0} 0$.

This is (since τ_{ij}^h is defined on I_j) the same as

$$\left|\int_{m_j}^{m_{j+1}}\frac{1}{S_\infty}\tau_{ij}^h(x)2q(x-h_k)dx\right| \leq \left|\int_{m_j}^{m_{j+1}}\frac{1}{S_\infty}\tau_{ii}^h(x)2q(x-h_i)dx\right|.$$

We note that due to the definition of τ_i^h this is the same as

$$\int_{m_j}^{m_j+2\epsilon}\frac{1}{S_\infty}\tau_{ii}^h(x)2q(x-h_i)dx+\int_{m_j+2\epsilon}^{m_{j+1}-2\epsilon}\frac{1}{S_\infty}\left(\frac{-d^2u^h}{dx^2}\right)2q(x-h_i)dx$$
$$+\int_{m_{j+1}-2\epsilon}^{m_{j+1}}\frac{1}{S_\infty}\tau_{ii}^h(x)2q(x-h_i)dx.$$

We recall

$$\begin{aligned}\tau_{ii}^h &= -\gamma_i^i u_x^h-\gamma^i u_{xi}^h = -\gamma_i^i u_x^h-\gamma^i\left(-u_{xx}^h+\left(1-\chi^i\right)w_x^i-\chi_x^i w^i-\chi_x^i w^{i+1}-\chi^i w_x\right)\\ &= u_{xx}^h-\gamma_i^i u_x^h+\left(\gamma^i-1\right)u_{xx}^h-\gamma^i\left(\left(1-\chi^i\right)w_x^i-\chi_x^i w^i-\chi_x^i w^{i+1}-\chi^i w_x\right)\\ &= \gamma^i u_{xx}^h+o\left(\epsilon^{-1}\right),\end{aligned}$$

but similarly to [CP89] Lemma 8.1, we have on the intervals $[m_j,m_j+2\epsilon]$, $[m_{j+1}-2\epsilon,m_{j+1}]$ that

$$\left|u_{xx}^h\right| \leq \mathcal{O}\left(\epsilon^{-1}\max\left\{w^j,w^{j+1}\right\}\right) \leq \mathcal{O}\left(\epsilon^{-1}\alpha\right) \leq o\left(\epsilon^{-1}\right),$$

which means that in those interval we have $|\tau_{ii}^h| \leq o(\epsilon^{-1})$, and in particular that

$$\left|\int_{m_j}^{m_j+2\epsilon}\frac{1}{S_\infty}\tau_{ii}^h(x)2q(x-h_i)dx\right|,\left|\int_{m_{j+1}-2\epsilon}^{m_{j+1}}\frac{1}{S_\infty}\tau_{ii}^h(x)2q(x-h_i)dx\right|$$

$$\leq 2\epsilon\frac{1}{S_\infty}o\left(\epsilon^{-1}\right)\mathcal{O}(1)\rightarrow_{\epsilon\to 0} 0.$$

Now since $u^h=\left(1-\chi^j\right)\phi^j+\chi^j\phi^{j+1}$ we have

$$\frac{d^2u^h}{dx^2}=\chi_{xx}^j\left(\phi^{j+1}-\phi^j\right)+2\chi_x^j\left(\phi_x^{j+1}-\phi_x^j\right)+\chi^j\left(\phi_{xx}^{j+1}-\phi_{xx}^j\right)-\phi_{xx}^j.$$

However using [CP89] Lemma 8.2 for $\left|\phi^{j+1}-\phi^j\right|$ and $\left|\phi_x^{j+1}-\phi_x^j\right|$ and a similar argument for $\phi_{xx}^{j+1}-\phi_{xx}^j$ yields

$$\frac{d^2u^h}{dx^2}=-\phi_{xx}^j+o(1),$$

while ϕ_{xx}^j is a function that integrates up to 0 up to an $o(1)$ error. As with ϕ_x^j, we note that this function is arbitrarily small outside the interval $[h_j-\epsilon,h_j+\epsilon]$, so that the term we are interested in has the same limit as $\epsilon\to 0$ as

$$\int_{h_j-\epsilon}^{h_j+\epsilon}-\frac{1}{S_\infty}\phi_{xx}^j2q(x-h_i)dx,$$

if we approximate $q(x-h_i)$ on $[h_j-\epsilon,h_j+\epsilon]$ as $q(0)$, the expression has the limit 0, however as this approximation converges to q as $\epsilon\to 0$, we have shown that

$$\epsilon^{-1}\left\langle\mathcal{Q}\tau_{ij}^h,\sigma_k(h,v)\right\rangle\rightarrow_{\epsilon\to 0} 0.$$

Recall that $A_{ij}^{-1}=\mathcal{O}(\epsilon)$ so that we finally obtain

$$\begin{aligned}|b_r(h,v)|&\leq\mathcal{O}\left(\epsilon e^{-\delta_h/\epsilon}\right)+\mathcal{O}(\epsilon)\left(\mathcal{O}\left(\epsilon^{-1+\beta}\right)+\mathcal{O}\left(\epsilon^{4\gamma-2}\right)\right)\epsilon^{2\gamma}\epsilon^2+\mathcal{O}(\epsilon)\epsilon^{2\gamma}o(\epsilon)\\&\leq C\epsilon e^{-\delta_h/\epsilon}+o\left(\epsilon^{2\gamma+2+\beta}\right)\end{aligned}$$

uniformly in ϵ and pointwise

$$\begin{aligned}|b_r(h,v)|&\leq\mathcal{O}\left(\epsilon^{2\gamma+2+\delta}\right)+\mathcal{O}(\epsilon)\left(\mathcal{O}\left(\epsilon^{-1+\beta}\right)+\mathcal{O}\left(\epsilon^{4\gamma-2}\right)\right)\epsilon^{2\gamma}\epsilon^2+\mathcal{O}(\epsilon)\epsilon^{2\gamma}o(\epsilon)\\&\leq o\left(\epsilon^{2\gamma+2+\beta}\right).\end{aligned}$$

Sketch of argument of Claim 6.7. Substituting $u(x,t)=u^{h(t)}(x)+v(x,t)$ into the stochastic Allen-Cahn equation upon applying the Itô formula to the LHS yields

$$\sum_i u_i^h dh_i + \frac{1}{2}\sum_{i,j} u_{ij}^h (dh_i, dh_j) + dv = \left[\mathcal{L}u^h + Lv + f_2 v^2\right] dt + dV$$

where $L = \epsilon^2 \partial_{xx} + f'\left(u^h\right)$ and $f_2 = \int_0^1 (1-\tau) f''\left(u^h + \tau v\right) d\tau$.
By Claim 3.3, this is the same as

$$\sum_i u_i^h dh_i + \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \langle \sigma_i(h,v), \sigma_j(h,v)\rangle dt + dv = \left[\mathcal{L}u^h + Lv + f_2 v^2\right] dt + \epsilon^\gamma dW,$$

this may be rearranged to

$$dv = \left[\mathcal{L}u^h + Lv + f_2 v^2\right] dt - \sum_i u_i^h dh_i - \frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \left\langle \sigma_i(h,v), \sigma_j(h,v)\right\rangle dt + \epsilon^\gamma dW,$$

Since the Itô formula yields $d\|v\|^2 = d\langle v, v\rangle = \langle dv, v\rangle + \langle v, dv\rangle + \langle (dv, dv)\rangle = 2\langle v, dv\rangle + \langle 1, (dv, dv)\rangle$ we get

$$\begin{aligned}
&d\|v\|^2 = 2\langle v, \mathcal{L}u^h + Lv + f'(1)v + f_2 v^2 \\
&-\frac{1}{2}\sum_{i,j} u_{ij}^h \epsilon^{2\gamma} \langle \mathcal{Q}\sigma_i(h,v), \sigma_j(h,v)\rangle - \sum_i u_i^h b_i(h,v)\rangle dt \\
&-2\langle v, \epsilon^{2\gamma} \sum_i u_i^h \langle \sigma_i(h,v), dW\rangle + \epsilon^\gamma dW\rangle \\
&+\epsilon^{4\gamma} \sum_{i,j} \left\langle u_i^h, u_j^h\right\rangle \left\langle \mathcal{Q}\sigma_i(h,v), \sigma_j(h,v)\right\rangle dt + tr\mathcal{Q}\epsilon^{2\gamma} dt
\end{aligned}$$

Since we are in the slow channel we know that $\|v\| \le \epsilon^{1+\delta}$ for some $\delta > 0$ and Claim 3.3 tells us that $\|\sigma_r(h,v)\| = C\sqrt{\epsilon} + o(\epsilon)$, while Claim 6.8 gives us $|b_r(h,v)| \le C\epsilon^{2\gamma+2}$; therefore upon taking expectations and integrating up to our stopping time this becomes

$$\mathbb{E}\left[\|v(\tau^{**})\|^2\right] \le \mathbb{E}\left[\|v(0)\|^2\right] + C'\epsilon^{-2\gamma-2}\epsilon^{2\gamma+\delta}$$

for some $C' > 0$.

In this expansion, $\int_0^1 (1-\tau) f''(u^h + \tau v) d\tau < C$ for some $C > 0$ is the case, as long as $\sup_{t\in[0,\tau^*]} \|v\|_\infty \le 1$, which we have due to the definition of our stopping time. By applying the Itô formula we similarly have for $p \ge 3$ that

$$\begin{aligned}
&d\|v\|^p = \frac{p}{2}\|v\|^{p-2} d\|v\|^2 \\
&\quad + \frac{p(p-1)}{8}\|v\|^{p-4}\Big(\sum_{i,j} \langle v, \epsilon^{2\gamma} u_i^h\rangle \left\langle v, \epsilon^{2\gamma} u_j^h\right\rangle \left\langle \mathcal{Q}\sigma_i(h,v), \sigma_i(h,v)\right\rangle + \langle v, v\rangle \epsilon^{4\gamma}
\end{aligned}$$

$$+\epsilon^{4\gamma}\sum_i \langle u_i^h, v\rangle\langle\sigma_i(h,v),v\rangle tr\mathcal{Q})dt,$$

and thus

$$\mathbb{E}[\|v\|^p(\tau^*)] - \mathbb{E}[\|v\|^p(0)] =$$

$$\mathbb{E}\left[\int_0^{\tau^*} \frac{p}{2}\|v\|^{p-2}2\left\langle v, \mathcal{L}u^h + Lv + f'(1)v + f_2v^2\right.\right.$$

$$\left.\left. -\frac{1}{2}\sum_{i,j} u_{ij}^h\epsilon^{2\gamma}\left\langle \mathcal{Q}\sigma_i(h,v),\sigma_j(h,v)\right\rangle - \sum_i u_i^h b_i(h,v)\right\rangle dt\right]$$

$$+\mathbb{E}\left[\int_0^{\tau^*}\left(\frac{p}{2}\|v\|^{p-2}\epsilon^{4\gamma}\sum_{i,j}\left\langle u_i^h, u_j^h\right\rangle\left\langle\mathcal{Q}\sigma_i(h,v),\sigma_j(h,v)\right\rangle dt\right.\right.$$

$$+\frac{p(p-1)}{8}\|v\|^{p-4}(\sum_{i,j}\langle v,\epsilon^{2\gamma}u_i^h\rangle\left\langle v,\epsilon^{2\gamma}u_j^h\right\rangle\left\langle\mathcal{Q}\sigma_i(h,v),\sigma_i(h,v)\right\rangle + \langle v,v\rangle\epsilon^{4\gamma}$$

$$\left.\left. +\epsilon^{4\gamma}\sum_i\langle u_i^h,v\rangle\langle\sigma_i(h,v),v\rangle tr\mathcal{Q})\rangle dt\right)\right]$$

and therefore combining all estimates from Claim 6.2, 3.3 and 3.4 as well as [CP89] Theorem 3.5 we conclude

$$\mathbb{E}\left[\|v(\psi_d)\|^p\right] \leq \mathbb{E}[\|v(0)\|^p] + C_p\epsilon^{-2\gamma-2}\epsilon^{(1+\delta)p}$$

for all $p \geq 2$.

Markov's inequality then yields

$$\mathbb{P}(\|v(\psi_d)\| \geq \epsilon^{1+\delta}) \leq C_pT_\epsilon\epsilon^{2\delta q}\ \forall p \geq 2.$$

We note that since the probability of $\|v(\psi_d)\|_\infty \geq 1$ is exponentially small (cf. eg. [Fen06] for the fact that the probability of $\|u(\psi_d)\|_\infty \geq 2$ being exponentially small, while $\|u^h\|_\infty \leq 1$ by definition), we have in particular

$$\mathbb{P}(\|v(\psi_d)\| \geq \epsilon^{1+\delta} \text{ or } \|v(\psi_d)\|_\infty \geq 1) \leq C_q\epsilon^q\ \forall q > 0$$

and we clearly also have

$$\mathbb{P}(\|v(\psi_d)\| \geq \epsilon^{1+\delta} \text{ or } \|v(\psi_d)\|_\infty \geq 1 \text{ or } \|z(\psi_d)\| > \epsilon^{1+\delta}) \leq C_q\epsilon^q\ \forall q > 0.$$

The following corresponds to chapter 4:

Claim 6.9. *Annihilation*

Let $u_0 \in \Gamma''_{\xi'_\epsilon}$ where exactly two neighbouring interfaces have distance $\delta_{h_0} = \xi'_\epsilon$ to each other, and all other interfaces have a distance bounded below by $\epsilon^{1/2-\kappa}$ for some $\frac{1}{2} \geq \kappa > 0$. For $t > 0$, $u(\cdot, t)$ solves (1.1).

With a probability converging to 1 as $\epsilon \to 0$ one has that within $\epsilon^{-2\gamma-\iota}$ (for some small $\kappa > \iota > 0$) time these two interfaces will be annihilated leading to the new interface configuration $h' = \mathcal{H}\left(u\left(\frac{1}{\epsilon^{2\gamma+\iota}}\right)\right)$; with the same probability, the solution reenters $\Gamma''_{\xi'_\epsilon,\epsilon}$ within this time.

With a probability converging to 1 as $\epsilon \to 0$ this new configuration h' is inside a neighbourhood of size $k\epsilon$ (for some $k > 0$) of the initial configuration h_0, but without the annihilated interfaces.

Before proving this, we collect some ingredients of the sketch of argument:

Claim 6.10. *Suppose $\epsilon\rho^{**} \leq \delta_h < \frac{(2\gamma+2)\epsilon|\log\epsilon|}{\sqrt{f'(1)}}$ and the other interface distances bounded below by $\epsilon^{1/2-\kappa}$ for $\frac{1}{2} \geq \kappa > 0$ and $u = u^h + v \in \Gamma'_\epsilon$.*

Then if one denotes the two interfaces between which the minimum distance δ_h is attained by h_j and h_{j+1}, we have the following:

$$\sum_i A^{-1}_{ri}(h,v)\langle\mathcal{L}(u^h+v),\tau^h_i\rangle\Big|_{r=j+1} - \sum_i A^{-1}_{ri}(h,v)\langle\mathcal{L}(u^h+v),\tau^h_i\rangle\Big|_{r=j}$$
$$= \frac{\epsilon}{S_\infty}\left[-2a\exp\left\{-\sqrt{-f'(1)}\delta_h/\epsilon\right\}\right] + o\left(\epsilon^{2\gamma+2}\right). \tag{6.13}$$

This bound is uniformly in ϵ.

We have

$$a = -\left[\exp\left\{\int_0^1\left(\frac{\sqrt{-f'(1)}}{(2F(t))^{1/2}} - \frac{1}{(1-t)}\right)dt\right\}\right]^2 f'(1).$$

In addition, we have

$$\sup_{u^h+v\in\Gamma'_\epsilon \text{ s.t. } \rho^{**}\epsilon=\delta_h} \frac{b_{j+1}(h,v) - b_j(h,v)}{S_\infty^{-2}S_{\mathcal{Q}}\epsilon^{2\gamma+2}} \to_{\epsilon\to 0} -\infty.$$

Sketch of Argument. The formulae for $\sum_i A^{-1}_{ri}(h,v)\langle\mathcal{L}(u^h+v),\tau^h_i\rangle$ follow from combing [CP89] Theorem 6.1, Lemma 3.3, Proposition 3.4.

We recall that

$$b_r(h,v) = \sum_i A^{-1}_{ri}(h,v)\langle\mathcal{L}(u^h+v),\tau^h_i\rangle$$
$$+ \sum_{i,l,k} A^{-1}_{ri}(h,v)\left[\frac{1}{2}\langle v,\tau^h_{ilk}\rangle - \frac{1}{2}\langle u^h_{kl},\tau^h_i\rangle - \langle u^h_k,\tau^h_{il}\rangle\right]\epsilon^{2\gamma}\langle\mathcal{Q}\sigma_k(h,v),\sigma_l(h,v)\rangle$$

$$+\sum_i A_{ri}^{-1}(h,v)\sum_j \epsilon^{2\gamma}\left\langle \mathcal{Q}\tau_{ij}^h, \sigma_j(h,v)\right\rangle = T_1 + T_2 + T_3$$

We can bound T_2 and T_3 in a similar way to how it was done in Claim 6.8, except for the different bound on $\|v\|$ and differences arising from the shorter minimal distance:
$|T_2| \leq o\left(\epsilon^{2\gamma+2}\right)$
Similarly, we have
$|T_3| \leq o\left(\epsilon^{2\gamma+2}\right)$, which completes the sketch of argument of our bound.
The further limits on the fast timescale follow from the asymptotic expansion (6.13), which we therefore also have for $b_{j+1}(h,v) - b_j(h,v)$.

The following Claim tells us that the probability of exiting the extended slow channel is small, unless the critical distance $\rho^{**}\epsilon$ is achieved:

Claim 6.11. *Suppose $u_0 = u^h + v \in \Gamma'_\epsilon$ with $\|v\| \leq C\epsilon^{1/2}$ for some $C < C_{map}$ and $T_\epsilon < K\epsilon^{-2\gamma-2}$, $K > 0$ independent of ϵ, and suppose that $\mathbb{E}[\|v(0)\|^p] \leq C_p\epsilon^{p/2}\ \forall p \geq 2$ for some $\delta > 0, C_p > 0$. Let τ^{**} be as in Chapter 3.*
Then we have

$$\mathbb{P}\left[\delta_{h(\tau^{**})} > \rho^{**}\epsilon\right] \leq C_q\epsilon^q\ \forall q > 0.$$

Sketch of argument of Claim 6.11. Consider the linearised Allen-Cahn SPDE starting at 0, i.e.

$$\partial_t z = \epsilon^2\partial_{xx}z + f'(1)z + \epsilon^\gamma \dot{W}$$

Then for $\bar{u} = u - z$ we have

$$\partial_t\bar{u} = \epsilon^2\partial_{xx}\bar{u} + f(\bar{u}+z) - f'(1)z = \epsilon^2\partial_{xx}\bar{u} + f(\bar{u}) + C(x,t)z$$

for some bounded function $C(x,t)$.
Except for the last term, which we can view as a perturbation on $\mathcal{O}(1)$ timescales, this is exactly the same as the deterministic Allen-Cahn equation. We will use the fact, that with high probability, the last term is small on $\mathcal{O}(1)$ times, to show that with high probability we have the same behaviour as in the deterministic case.
With w we shall now denote the deterministic Allen-Cahn equation started at u_0.
If we indeed have $\delta_{h(\tau^{**})} > \rho^{**}\epsilon$, then at $t = 0 \vee (\tau^{**}-1)$ we have $\delta_{h(0\vee(\tau^{**}-1))} > \rho^*\epsilon$ and $\|v(0\vee(\tau^{**}-1))\| < C_{map}\epsilon^{1/2}$. However, we know from [CP89] Theorem 5.3 (5.6), that up to $\mathcal{O}(1)$ times we have for some $\nu_0, C_0 > 0$ the expression

$$\left\|w(t) - u^{\mathcal{H}(w(t))}\right\|_\infty \leq \epsilon^{1/2}\sqrt{C_0\sum_{j=1}^N\left\langle\mathcal{L}(u^h), \tau_j^h\right\rangle^2 + \exp\left(-\frac{1}{2}\nu_0 t\right)C'}.$$

But since $\|\cdot\| \leq \|\cdot\|_\infty$ we also have this for the L^2-norm.

Claim 6.2 tells us that for all $\nu > 0$ we have

$$\mathbb{P}\left[\sup_{t\in[0,1]} \|z\|_\infty > \epsilon^{\gamma-\nu}\right] \leq C_q\epsilon^q \ \forall q > 0.$$

Thus on $\mathcal{O}(1)$ times we have

$$\mathbb{P}[\|w - \bar{u}\|_\infty \geq \epsilon^\gamma] \leq C_q\epsilon^q \ \forall q > 0.$$

Since $\|v(\tau^{**})\| \leq \|v(\tau^{**})\|_\infty \leq \left\|\bar{u}(\tau^{**}) - u^{\mathcal{H}(\bar{u}(\tau^{**}))}\right\|_\infty + \|z(\tau^{**})\|_\infty$, this implies that at τ^{**} we have

$$\mathbb{P}\left[\|v(\tau^{**})\| \leq C_{map}\epsilon^{1/2}\right] \leq C_q\epsilon^q \ \forall q > 0.$$

The following Claim tells us that with a probability asymptotically close to 1, it takes a time that converges to 0 on the fast timescale to go from the initial minimum distance $\delta_h = \xi_\epsilon$ to the minimum distance $\delta_h = \rho^{**}\epsilon$.

Claim 6.12. *Let for $t = 0$ $u_0 = u^h + v \in \Gamma''_{\xi'_\epsilon,\epsilon}$ with $\delta_h = \xi_\epsilon$ and all other distances between neighbouring interfaces bounded below by $\epsilon^{1/2-\kappa}$ for some $\frac{1}{2} \geq \kappa > 0$. Then, with a probability converging to 1 as $\epsilon \to 0$, we have at τ^{**} that $u(\tau^{**}) = u^h + v \in \Gamma'_\epsilon$ with $\delta_h = \rho^{**}\epsilon$. Furthermore, for all $k > 0$ we have*

$$\mathbb{P}\left[\epsilon^{2\gamma+\mu}\tau^{**} < k\right] \geq 1 - C_{q,k}\epsilon^q \ \forall q > 0 \ \forall \mu > 0.$$

Sketch of Argument. Claims 6.8 and 6.9 (combined with Claim 6.11) give us an asymptotic expansion of the semimartingale expression and a probability asymptotically small in ϵ of it leaving Γ'_ϵ before the critical distance is reached.

Recall from Claim 6.6 that our semimartingale expression is

$$dh_k = b_k(h,v)dt + \epsilon^\gamma\langle\sigma_k(h,v), dW\rangle.$$

Clearly,

$$H_k(t) = h_k(0) + \int_0^t \epsilon^\gamma\langle\sigma_k(h,v), dW(x,s)\rangle = h_k(t) - \int_0^t b_k(h,v)dt$$

is a martingale. Then by the Dambis-Dubins-Schwartz theorem, we can denote $H_k(t)$ as a time-changed Brownian motion:

$$H_k(t) - H_k(0) = B\left(\epsilon^{2\gamma}\int_0^t \langle\sigma_k(h,v), \sigma_k(h,v)\rangle ds\right),$$

where $B(\cdot)$ is a standard Brownian motion. But from Claims 6.8, 6.9 (combined with Claim 6.11) we know that our drift points strictly into the direction of the neighbouring interface if we are below the critical distance, this means that

$b_k(h,v) - b_{k-1}(h,v)$ is strictly negative; Claim 6.8 and Claim 6.7 tell us that it is with high probability small for larger distances, so we get

$$\mathbb{P}\left[|(h_k(t) - h_{k-1}(t)) - (H_k(t) - H_{k-1}(t))| \geq C\epsilon^{2\gamma+2}\right] \leq C_q\epsilon^q.$$

Clearly, $\hat{H}_k(t) = H_k(t) - H_{k-1}(t)$ can also be denoted as time-changed Brownian motion $B\left(\epsilon^2 \int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds\right)$ starting at ξ_ϵ. Clearly, we have $h_k(t) - h_{k-1}(t) \leq H_k(t) - H_{k-1}(t)$.

We note from the second part of Claim 3.3 that $\forall k$

$$\mathbb{P}\left[|\frac{\int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds}{\epsilon}| < C\right] \geq 1 - C_q\epsilon^q \ \forall q > 0$$

for some $C > 0$ while

$$\mathbb{P}\left[|\frac{\int_0^t \|\sigma_k(h,v) - \sigma_{k-1}(h,v)\|^2 ds}{\epsilon^p}| > M\right] \geq 1 - C_q\epsilon^q \ \forall q > 0 \ \forall p > 1 \ \forall M > 0.$$

We observe that τ^{***}, the first time $H_k(t)$ hits the level $\rho^{**}\epsilon$ is strictly greater than τ^{**}, unless τ^{**} occurs because of leaving the slow channel other than by achieving the critical distance. However, this would happen with a probability asymptotically small in ϵ (cf. Claim 6.6).

Let $\mu > 0$. Using standard results about time-changed Brownian motion, (c.f. e.g. [RY99]), Doob's martingale inequality in particular, we note that for some $C > 0$ we have

$$\mathbb{P}\left[\epsilon^{2\gamma-1+\mu}\tau^{***} < C\right] \geq 1 - C_q\epsilon^q \ \forall q > 0.$$

However, this immediately implies

$$\mathbb{P}\left[\epsilon^{2\gamma-1+\mu}\tau^{**} < C\right] \geq 1 - C_q\epsilon^q \ \forall q > 0,$$

which completes the sketch of argument.

Sketch of argument of Claim 6.9. Initially, u is still in $\Gamma_{\xi_\epsilon,\epsilon}$ and we can write down the semimartingale expression of the interfaces by Claim 6.6.

Using Claims 6.11 and 6.12 we note that with a probability converging to 1 as $\epsilon \to 0$, the two interfaces of initial distance ξ_ϵ are moving towards each other until they reach the distance $\rho^{**}\epsilon$.

In a similar manner to phase separation and convergence to the slow manifold, we now consider the difference between the stochastic Allen-Cahn equation and the linear stochastic heat equation:

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^2 \frac{\partial^2 \bar{u}}{\partial x^2} + f(\bar{u}) + (f'(\bar{u}) - f'(1))\, z + z^2 \int_0^1 (1-\tau) f''(\bar{u} + \tau z) d\tau$$

Like before, we denote by w the solution to the deterministic Allen-Cahn PDE and now note

$$\frac{\partial \|\bar{u}-w\|_\infty}{\partial t} = \sup_{l\in\partial\|\bar{u}-w\|} \left\langle l, \epsilon^2 \frac{\partial^2(\bar{u}-w)}{\partial x^2} + f(\bar{u}) - f(w) + C(x,t)z \right\rangle$$
$$\le \sup_{l\in\partial\|\bar{u}-w\|} \langle l, f(w+(\bar{u}-w)) - f(w) + Cz\rangle$$
$$\le |f'(0)|\|\bar{u}-w\|_\infty + C\|z\|_\infty.$$

Gronwall's inequality implies

$$\|\bar{u}-w\|_\infty \le C\|z\|_\infty e^{|f'(0)|t}.$$

Our initial condition is with high probability bounded by 2 in modulus, at an $\mathcal{O}(\epsilon^2)$ distance to the slow manifold in L^2 norm except for an $\mathcal{O}(\epsilon)$, thus there exists $u^{h'}$ s.t. $\left\|u_0 - u^{h'}\right\| \to_{\epsilon\to 0} 0$; thus Claim 7.4 immediately implies that there exists an $\mathcal{O}(1)$ time after which the distance in L^2 norm of w to the slow manifold is $\frac{d}{2}$, if we denote the original distance by d. Iterating this and for the last interval solving for time like in the sketch of argument of Claim 6.2 yields that after $C|\log\epsilon|$ time ($C>0$) we have $\left\|w(x) - u^{h'}(x)\right\| < \epsilon$, where h' denotes the interface configuration of the new slow manifold element with configuration $h' = \mathcal{H}(w(C|\log\epsilon|))$ which is in a neighbourhood of size $\epsilon^{1/2-\kappa}$ ($1/2 > \kappa > 0$) of the original configuration. To avoid blowup, the final time interval is being split up into intervals of size up to $\frac{|\log\epsilon|\delta}{|f'(0)|}$ for $\delta < \gamma - 1$, the reason is the same as in the sketch of argument of Claim 4.1

We now use Claim 6.2 and the bound on $\|\bar{u}-w\|_\infty$ to obtain

$$\mathbb{P}\left[\left\|u(x) - u^{h'}(x)\right\| < \epsilon\right] \ge 1 - C_p\epsilon^p$$

for some $C_p > 0$.

Since by the virtue of Claim 6.2, u is w.h.p the same as $\bar{u}$, the result follows. The statement about the interfaces follows in the same way.

The following corresponds to chapter 5:

Let R be the set of integer-valued measures on our circle S^1, where each measure μ has a total variation bounded by 2 and the property $\int \mu = 0$.

We now define the correlated annihilating Brownian motion to which our interfaces converge in the limit.

Their generator is given as

$$\mathcal{G} = \sum_{i,k,l} S_{\mathcal{Q}kl} S_{\mathcal{Q}}^{-1} \alpha_i^2 (-1)^{k+l} e_i(h_k) e_i(h_l) \partial_{kl}^2.$$

We used the notation

$$S_{\mathcal{Q}km} S_{\mathcal{Q}}^{-1} = 4 S_{\mathcal{Q}}^{-1} \sum_{r=1}^{\infty} \alpha_r^2 e_r(B_k - B_m)$$

$$S_{\mathcal{Q}} = 4\sum_{r=1}^{\infty} \alpha_r^2 e_r(0) \ .$$

We now set up the notation:

Firstly, for each initial interface position h_0 we define $B^{h_0,1}(t)$ to be a $2N$-dimensional Brownian motion with generator $\mathcal{G}$ on S^1 starting at h_0; similarly, we define (independently from it) $B^{h_0',2}(t)$ to be a $2N-2$-dimensional independent Brownian motion with generator $\mathcal{G}$ on S^1 starting at h_0' and carry on defining and denoting independent Brownian motions with generator $\mathcal{G}$ in this way until we reach dimension 2.

We define the first hitting time of $B^{h_0,i}$ started at h_0 as

$\sigma^{h_0,i} = \inf_{j\neq k}\left\{t : B_j^{h_0,i}(t) = B_k^{h_0,i}(t)\right\}$

and $h_0'\left(\sigma^{h_0,i}\right) = B^{h_0,i}\left(\sigma^{h_0,i}\right)\backslash\left\{B_K^{h_0,i}\left(\sigma^{h_0,i}\right), B_J^{h_0,i}\left(\sigma^{h_0,i}\right)\right\}$, where $B_K^{h_0,i}\left(\sigma^{h_0,i}\right) = B_J^{h_0,i}\left(\sigma^{h_0,i}\right)$ for some $K = J$.

We define the limiting process of $u\left(S_\infty \epsilon^{-2\gamma-1}t\right)$:

For $0 \leq t \leq \sigma^{h_0,1}$ we have

$$u_s(x,t) = \begin{cases} ... & ... \\ -1 & \text{for } B_{2N}^{h_0,1} - 1 \leq x < B_1^{h_0,1}(t) \\ 1 & \text{for } B_1^{h_0,1} \leq x < B_2^{h_0,1}(t) \\ ... & ... \end{cases}$$

Similarly, for $\sigma^{h_0,1} \leq t \leq \sigma^{h_0,1} + \sigma^{h_0'(\sigma^{h_0,1}),2}$ we have

$$u_s(x,t) = \begin{cases} ... & ... \\ -1 & \text{for } B_{2N}^{h_0'(\sigma^{h_0,1}),2} - 1 \leq x < B_1^{h_0'(\sigma^{h_0,1}),2}(t) \\ 1 & \text{for } B_1^{h_0'(\sigma^{h_0,1}),2} \leq x < B_2^{h_0'(\sigma^{h_0},1),2}(t) \\ ... & ... \end{cases}$$

Etc.

We now define a mapping $\mathcal{I}$ which maps the function onto the slow manifold if possible, and otherwise gives the last known such interface configuration.

For $u \in \Gamma_{\xi_\epsilon,\epsilon}$ we define the mapping simply as

$$\mathcal{I}(u) = \mathcal{H}(u),$$

at the initial stage where $u \in X_\epsilon^{h_0,0}$ we define $\mathcal{I}$ as

$$\mathcal{I}(u_0) = h_0.$$

Finally, whenever u leaves $\Gamma_{\xi_\epsilon,\epsilon}$, we define $\mathcal{I}(u)$ to be the last defined value $\mathcal{H}(u)$.

Claim 6.13. *Sharp interface limit*

Given $u_0(x) \in X_\epsilon^{h_0,0}$, for $t > 0$ we have that $u(t)$ solves (1.1).

In the limit $\epsilon \to 0$, its time-rescaled process $u\left(\frac{S_\infty^2 t}{S_{\mathcal{Q}}\epsilon^{2\gamma+2}}\right)$ weakly converges to $u_s(t)$ in law with probability 1 in the L^2 topology for positive times. Between *time* 0 *and positive times there is a possible discontinuity (although not for the interfaces).*

In particular, with a probability converging to 1 *as* $\epsilon \to 0$*, within a finite time the solution obtains a constant profile of either* $+1$ *or* -1 *with fluctuations converging to* 0 *as* $\epsilon \to 0$.

Sketch of Argument. This works like the sketch of argument of Claim 5.1. The differences are that we are on the timescale $\frac{S_\infty^2 t}{S_{\mathcal{Q}}\epsilon^{2\gamma+2}}$ rather than $\frac{S_\infty t}{\epsilon^{2\gamma+1}}$, and the interfaces perform annihilating Brownian motions with generator $\mathcal{G}$ rather than independent standard Brownian motions.

Instead of Claim 2.1, we use Claim 6.6; instead of Claim 3.1 and Claim 3.2 we use Claim 6.9 and Claim 6.10. For the annihilation we use Claim 6.9 rather than Claim 2.4.

6.1 Noise that neither has a trace-class covariance operator nor is white

Firstly, it is clear that the semimartingale representation is the same as for the smooth noise case presented in section 6.2.

As before, the noise is defined to be

$$W(x,t) = \sum_{j=1}^{\infty} \alpha_j \beta_j(t) e_j(x)$$

where $\beta_1, \beta_2, ...$ are independent standard Brownian motions and $\{e_1(\cdot), e_2(\cdot), ...\}$ is an orthonormal basis of $L^2(0,1)$.

Define the "noise strength" as

$$K_{\mathcal{Q}} = \min\left\{1, \sup\left\{j > 0 : \lim_{k\to\infty} k^j \alpha_k < \infty\right\}\right\}.$$

We conjecture that the appropriate timescale on which a sharp interface limit may be taken is

$$t' = S_{\mathcal{Q}} S_\infty^{-2} \epsilon^{2\gamma+1+K_{\mathcal{Q}}} t$$

with

$$S_{\mathcal{Q}} = \lim_{\epsilon\to 0} \epsilon^{1-K_{\mathcal{Q}}} \left\langle \mathcal{Q} u_j^h, u_j^h \right\rangle.$$

We conjecture that in the limit $\epsilon \to 0$ the interfaces converge weakly to independent Brownian motions. This is because the following correlation terms appear to converge to 0 for $j \neq k$:

$$S_{\mathcal{Q}kj} = \lim_{\epsilon\to 0} \epsilon^{1-K_{\mathcal{Q}}} \left\langle \mathcal{Q} u_k^h, u_j^h \right\rangle.$$

7. APPENDIX

7.1 Results (and generalisations) from [Che04]

The following results concern the solution of the deterministic Allen-Cahn equation:

$$\partial_t u = \epsilon^2 \partial_{xx} u + f(u), \tag{7.1}$$

where $u(\cdot, 0) = u_0(\cdot)$.

The equation is posed on $\mathbb{R}$ in the case of Claim 7.1, Claim 7.2, 7.3,7.5, 7.6.

Otherwise the equation is posed in the interval $(0, 1)$ with periodic boundary conditions.

The following is based on ***[Che04] Theorem 2.1:***

Claim 7.1. *Assume that $u_0(x)$ satisfies $u_0 \in C^1(0, 1)$ with $u_0(0) = u_0(1)$ and $u_{0x}(0) = u_{0x}(1)$, is independent of ϵ and bounded, we require finitely many roots. We require $|u_0(x)| + |u_0'(x)| > 0 \; \forall x$.*

Let $h_0 = \left(h_0^1, ..., h_0^n\right)$ be all roots to $u_0(\cdot) = 0$, arranged in increasing order. Then there exists a positive constant ρ, depending only on f, such that for every sufficiently small positive

$\epsilon, u(\cdot, t_1) \in \left\{u(x) : |u(x)| > 1/2 \text{ on } (0, 1) \setminus \cup_i \left(h_i^1 - \rho\epsilon, h_i^1 + \rho\epsilon\right)\right\}$ for $t_1 = \frac{|\log \epsilon|}{|f'(0)|}$ and some $h = h_0 + \mathcal{O}\left(\epsilon\sqrt{|\log \epsilon|}\right)$.

To prove this, we need some claims:

"Decomposition" into reaction and diffusion.

Claim 7.2. *Let $W(t)$ be a solution to*

$$W_t = f(W)$$

$$W(a, 0) = a$$

The following holds:

(i) $W \in C^2(\mathbb{R} \times [0, \infty))$ and $W_a := \frac{\partial}{\partial a} W(a, t) > 0$ for all $a \in \mathbb{R}$ and $t \geq 0$;

(ii) For all $\epsilon \in (0, 1], \pm W\left(a, \frac{|\log \epsilon|}{|f'(0)|}\right) \geq \frac{1}{2}$ if $\pm a \geq a^* \epsilon$ where

$$a^* = \exp\left\{\int_{-1/2}^{1/2} \frac{|f(s) - f'(0)s|}{|f(s)s|} ds\right\};$$

(iii) For every $m \geq 1$, there exists $M_0(m) > 0$ such that

$$\left|\frac{W_{aa}(a,t)}{W_a(a,t)}\right| \le M_0(m)\left(e^{|f'(0)|t}-1\right) \quad \forall a\in[-m,m], t\ge 0.$$

Claim 7.3. *Let $U(x,t)=W(g(x,t),t)$. Then $U(x,t)$ is a subsolution/ supersolution to $u_t-\epsilon^2 u_{xx}-f(u)=0$ if and only if*

$$g_t-\epsilon^2 g_{xx}+\epsilon^2\frac{W_{aa}(g,t)}{W_a(g,t)}g_x^2\le 0 \ / \ \ge 0.$$

Sketch of argument of Claim 7.2. The choice of $f(\cdot)$ implies $f>0$ in $(-\infty,-1)\cup(0,1)$ and $f<0$ in $(-1,0)\cup(1,\infty)$. Hence, for $\pm a>0, W(a,t)\to\pm 1$ monotonically as $t\to\infty$. In addition, $W\in C^2([\mathbb{R}\times[0,\infty))$.

Differentiating $W_t=f(W)$ w.r.t. a and solving the obtained equation, we get $W_a=\exp\left\{\int_0^t f'(W(a,\tau))d\tau\right\}$, as well as property (i).

If $|a|\ge 1/2, \pm W(a,t)\ge 1/2$ for all $t\ge 0$. If $|a|\in(0,1/2)$, we have

$$W(a,t)=a\exp(f'(0)t)\exp(\beta(t)),\ \beta(t):=\int_a^W\frac{f(s)-f'(0)s}{f(s)s}ds.$$

This is because

$$\begin{aligned}
W &= a\exp(f'(0)t)\frac{W}{a}\exp(-f'(0)t)\\
&= a\exp(f'(0)t)\exp\left(\int_a^W\frac{1}{s}ds\right)\exp\left(\int_0^t -f(s)\frac{f'(0)}{f(s)}ds\right)\\
&= a\exp(f'(0)t)\exp\left(\int_a^W\frac{1}{s}ds\right)\exp\left(\int_0^t -\frac{f'(0)}{f(W)}W_r dr\right)\\
&= a\exp(f'(0)t)\exp\left(\int_a^W\frac{1}{s}ds\right)\exp\left(\int_a^W -\frac{f'(0)}{f(s)}ds\right)\\
&= a\exp(f'(0)t)\exp\left(\int_a^W\frac{f(s)-f'(0)s}{f(s)s}ds\right).
\end{aligned}$$

Property (ii) then follows by the definition of a^*.

Now we differentiate $t=-\int_a^W\frac{ds}{f(s)}$ w.r.t. a to get

$$\frac{W_{aa}}{W_a}=\begin{cases}\frac{1}{f(a)}\int_a^W f''(s)ds & \text{if } f(a)\ne 0,\\ \frac{f''(a)}{f'(a)}\left(\exp\{f'(a)t\}-1\right) & \text{if } f(a)=0.\end{cases}$$

One can easily check that $\beta(t)$ is bounded above, uniformly in $a\in[-m,m]$. Property (iii) can now be verified: If $f(a)=0$, then on $[-m,m]$ we have

$\left|\frac{W_{aa}}{W_a}\right| = \left|\frac{f''(a)}{f'(a)}\left(\exp\left\{f'(a)t\right\}-1\right)\right| \leq M(m)\left(\exp\left(\left|f'(0)\right|t\right)-1\right)$ by the properties of $f(\cdot), \exp(\cdot)$ and the properties of continuous functions.

If $f(a) \neq 0$, then

$$\left|\frac{W_{aa}}{W_a}\right| = \left|\frac{1}{f(a)}\int_a^{W(t)} f''(s)ds\right| = \left|\frac{1}{f(a)}\left[f'(r)\right]_a^{W(t)}\right| =$$

$$\left|\frac{1}{f(a)}(f'(W(t)) - f'(a))\right| = \left|\frac{f'(a+[a\exp(f'(0)t)\exp(\beta(t))-a]) - f'(a)}{f(a)}\right|$$

$$\leq M'(m)\left(\exp\left(\left|f'(0)\right|t\right)-1\right)$$

by the properties of $f(\cdot)$ and by considering the Taylor expansion around the points $a = 0, \pm 1$.

Sketch of argument of Claim 7.3. Differentiation and noting that $W_t = f(W)$ leads to

$$U_t - \epsilon^2 U_{xx} - f(U) = W_a\left(g_t - \epsilon^2 g_{xx} + \epsilon^2\frac{W_{aa}}{W_a}g_x^2\right).$$

However we have that $W_a > 0$, and hence the sketch of argument is complete.

A diffusion problem

Let u_0 satisfy the conditions of Claim 7.1. We look at the heat equation

$$\begin{cases} \bar{u}_\tau = \bar{u}_{xx} & \text{for } x \in (0,1), \tau > 0 \\ \bar{u}\ (x,0) = u_0(x) & \text{for } x \in (0,1). \end{cases}$$

with periodic boundary conditions.

Our aim is to consider the zero level set of $\bar{u}$. One can easily check that we have for all $x \in (0,1)$ and $\tau > 0$,

$$\bar{u}(x,\tau) := \int_0^1 K(y,\tau)\, u_0(x-y)dy = \int_0^1 K(\eta,1)\, u_0\left(x - \sqrt{\tau}\eta\right) d\eta,$$

where $K(y,\tau)$ is the fundamental solution of $K_\tau = K_{yy}$.

Denote $m_0 = \|u_0\|_\infty$ and $m_1 = \|u_0'\|_\infty$. Then $\|\bar{u}\|_{C^0([0,1]\times[0,\infty))} \leq m_0$. But due to the continuity of $\bar{u}_x(x,\tau)$ we note that $\|\bar{u}_x\|_{C^0([0,1]\times[0,\infty))} \leq m_1$. We now denote by $h_0^1, ..., h_0^n$ the zeroes of u_0 where it changes its sign. Due to the assumptions made on u_0 in Claim 7.1, we get $(-1)^i u_0'(h_0^i) > 0$. Due to continuity, we obtain the existence of a constant $\eta > 0$ and continuous functions $\bar{x}^i(\cdot)$ such that $\bar{x}^i(0) = h_0^i$ as well as

$$\bar{u}(\bar{x}^i(\tau),\tau) = 0, (-1)^i\bar{u}_x > \eta \text{ in } [\bar{x}^i(\tau)-\eta, \bar{x}^i(\tau)+\eta]\ \forall i = 1,...,n$$

$$(-1)^i\bar{u} > \eta^2 \text{ in } (\bar{x}^i(\tau)+\eta, \bar{x}^{i+1}(\tau)-\eta)\ \forall i = 1,...,n$$

for all $\tau \in [0,\eta]$; we define $\bar{x}^0 = \bar{x}^n - 1$ and $\bar{x}^{n+1} = \bar{x}^1 + 1$. As $\bar{u}_{xx} = \frac{1}{\sqrt{\tau}} \int_0^1 K_\eta(\eta,1) u_0'(x-\sqrt{\tau\eta}) d\eta$, we have $\|\bar{u}_\tau(\cdot,\tau)\|_\infty \leq \frac{m_1}{\sqrt{\pi\tau}}$ for all $\tau > 0$. Hence, by the Implicit Function Theorem, $\left|\frac{d}{d\tau}\bar{x}^i\right| \leq \frac{C_1}{\sqrt{\tau}}$ with $C_1 = \frac{m_1}{\eta\sqrt{\pi}}$. Consequently, $\left|\bar{x}^i(\tau) - x_0^i\right| \leq 2C_1\sqrt{\tau}$ for all $\tau \in [0,\eta]$ and all $i = 1, ..., n$.

Similarly, if $u_0 \in C^2(\mathbb{R})$, then $\frac{d}{d\tau}\bar{x}^i$ is bounded and $\left|\bar{x}^i(\tau) - z_0^i\right| = \mathcal{O}(\tau)$ for all $\tau \in [0, \eta_1]$.

Sketch of argument of Claim 7.1. Define
$g^\pm(x,t) = \begin{matrix}\min\\ \max\end{matrix} \left\{ \pm m_0, \bar{u}\left(x, \epsilon^2 t\right) \pm \epsilon^2 m_1^2 M_0(m_0) \frac{\exp\left(|f'(0)|t\right)-1}{|f'(0)|} \right\} \forall x \in [0,1], t \in [0,\infty)$ with m_0, m_1 as defined above and M_0 as in the statement of Claim 7.2.

Using Claim 7.2 (iii), we obtain that $g^\pm$ is a super/sub-solution to the equality in the statement of Claim 7.2. Thus,

$$W(g^-(x,t),t) \leq u(x,t) \leq W\left(g^+(x,t),t\right) \ \forall x \in (0,1), t \in [0,\infty).$$

We now complete the sketch of argument by using Claim 7.3 (ii): We note that

$$\left\| g^+\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right) - \bar{u}\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right)\right\|_\infty \leq C_2\epsilon;\ C_2 := \frac{m_1^2 M_0(m_0)}{|f'(0)|}.$$

Due to the obtained property for $\bar{u}$, we have for sufficiently small $\epsilon > 0$

$$(-1)^i \bar{u}\left(\cdot, \epsilon^2 \frac{|\log\epsilon|}{|f'(0)|}\right) > (a^* + C_2)\epsilon$$

$$\text{in } \left(\bar{x}^i\left(\epsilon^2\frac{|\log\epsilon|}{|f'(0)|}\right) + \frac{a^*+C_2}{\eta}\epsilon, \bar{x}^{i+1}\left(\epsilon^2\frac{|\log\epsilon|}{|f'(0)|}\right) - \frac{a^*+C_2}{\eta}\epsilon\right).$$

This implies that $(-1)^i g^\pm\left(\cdot, \frac{|\log\epsilon|}{|f'(0)|}\right) > a^*\epsilon$ in the interval $\left(\bar{x}^i\left(\epsilon^2\frac{|\log\epsilon|}{|f'(0)|}\right) + \frac{a^*+C_2}{\eta}, x^{i+1}\left(\epsilon^2\frac{|\log\epsilon|}{|f'(0)|}\right) - \frac{a^*+C_2}{\eta}\epsilon\right)$. However the assumption of the claim, with $h^i = \bar{x}^i\left(\epsilon^2\frac{|\log\epsilon|}{|f'(0)|}\right)$ and $C = 2\frac{a^*+C_2}{\eta}$ follows from the previous inequality for W and Claim 7.2 (ii).

The following is based on ***[Che04] Theorem 3.1.*** Note that there is a typographical error in the statement of the theorem in [Che04], the maximum value of the valid time interval should be divided by ϵ.

Claim 7.4. *Suppose* $\left\|\phi(\cdot,0) - u^{h'}(\cdot)\right\| \to_{\epsilon\to 0} 0$ *for some* $u^{h'}(\cdot)$ *being the orthogonal projection of* $\phi(\cdot,0)$ *in the* L^2 *norm.*

$u(\cdot,t)$ has the initial profile $\phi(\cdot,0)$ and solves (1.1) for positive t.

There exist positive constants ν_0, κ_0, C_0 and ρ_0, all depending only on $f(\cdot)$ and a positive constant $K_0 \geq 1$, depending on $f(\cdot)$ and ρ, such that the following holds:

For $\epsilon \in (0,1]$, let u be the solution to (7.1) with initial condition $u_0(x) \in X_\epsilon^{h_0,\rho}$ for some $\rho > 0$ and h_0 satisfying $\min_i |h_0^{i+1} - h_0^i \mod 1| > 2\rho\epsilon$. Set

$$I := \left\{ i \in \{1, ..., n\} | u_0(h^i - \rho\epsilon) u_0(h^i + \rho\epsilon) < 0 \right\} = \{i_1, ..., i_n\}.$$

Then there exists $(h^{i_1}, ..., h^{i_n})$ depending on $h_0, \rho, f(\cdot)$ such that

$$\max_{i \in I} |h^i - h_0^i| \leq (\rho + \rho_0)\epsilon,$$

$$\left\| u(\cdot, t) - u^h(\cdot) \right\| \leq K_0 \left\| u_0 - u^h(\cdot) \right\| e^{-\nu_0 t} + C_0 e^{-C/\epsilon}$$
$$\textit{for all } t \in \left[0, \kappa_0 \left(\delta_h \epsilon^{-1} - 2\rho\right)/\epsilon\right],$$

for some $C > 0$ depending on $f(\cdot)$ and the number of interfaces.

If $I = \emptyset$ (i.e. no sign-changes in the initial profile), then $|u^h| = 1$.

Sketch of argument of Claim 7.4. Take any $i \in \{1, ..., N\}$.

We set $\phi(y,t) = u(h_0^i + y), a = h_0^{i-1} - h_0^i + \rho$ and $b = h_0^{i+1} - h_0^i - \rho$. Applying Claim 7.6 with $r = \frac{\delta_{h_0}}{2} - \rho$ we can now conclude $\left\| \phi(y,t) - u^h(y) \right\| \leq K_0 \left\| \phi_0 - u^h(y) \right\| e^{-\nu_0 t} + K_{FM} e^{-D/\epsilon - \nu' t}$ for all $x \in \left(h_0^{i-1/2}, h_0^{i+1/2} \right)$ and $t \in \left[0, \left(\kappa_0 \delta_{h_0} \epsilon^{-1} - 2\rho\right) \epsilon^{-1}\right]$. As we are on the interval rather than the real line, we obtain the error term $C_0 e^{-C/\epsilon}$ in our expression. The term u^h we used here is the invariant solution of (7.1) on each interval, and thus up to exponentially small errors the same as u^h in chapter 1. This completes the sketch of argument.

In the following two claims we denote the solution of the deterministic Allen-Cahn equation as $\phi(x,t)$ to underline that we are in the single-interface case.

This is based on ***[Che04] Proposition 3.1***; in [Che04] however the sketch of argument of the statement just refers to [FM77], which in fact only proves the convergence to the standing wave but not to constant profiles.

Claim 7.5. *Suppose $\left\| \phi(\cdot, 0) - u^{h'}(\cdot) \right\| \to_{\epsilon \to 0} 0$ for some $u^{h'}(\cdot)$ being the orthogonal projection of $\phi(\cdot, 0)$ in the L^2 norm.*

There exist positive constants ν_0, ν' and ρ_0, C, D, c, ν', depending only on $f(\cdot)$ as well as $h = h_0 + \alpha(t)$ with $|h| \leq (\rho + \rho_0)\epsilon, |\alpha(t)| \leq \epsilon c e^{-\nu' t}$ and $K \geq 1$, depending on $f(\cdot)$ and ρ, such that if

$\phi_0 \in \left\{ \phi \in C(-\infty, \infty) \to [-2, 2]) | |\phi| \geq \frac{1}{2} \text{ on } \mathbb{R} \setminus (-\rho\epsilon, \rho\epsilon) \right\}$ for some $\rho > 0$, then the solution ϕ of (7.1)

satisfies

$$\left\| \phi(\cdot, t) - u^h(\cdot) \right\| \leq K \left\| \phi(\cdot, 0) - u^h(\cdot) \right\| e^{-\nu_0 t} + C e^{-D/\epsilon} \; \forall t \geq 0;$$

if $h = \emptyset$, then $u^h = 1$.

Sketch of Argument. This result relies on knowing that ϕ converges to the standing wave, which is proven on the real line in [FM77]. To obtain it here, we simply apply the result to our initial condition, which we constantly extrapolate; up to an exponentially small error the same result thus holds.

The paper [FM77] does not show the case of convergence to ± 1, however this may easily be shown by replacing the wave profiles with ± 1 in their sketch of argument.

We denote $y(x,t) = \phi(x,t) - \phi^*(x-h)$ where $h = h_0 + \alpha(t)$ and ϕ^* is the invariant solution.

If there is no sign-change, then $u^h = \pm 1$, if there is a sign-change this is defined so that $\langle y, u_x^h \rangle = 0$ for large times .

Since we have

$$\partial_t \phi = \epsilon^2 \partial_{xx} \phi + f(\phi)$$

we thus obtain the following equation for y:

$$y_t = \phi_t + \alpha' \left(u^h\right)' = \epsilon^2 \phi_{xx} + f(\phi) - \epsilon^2 \left(u^h\right)'' - f\left(u^h\right) + \alpha' \left(u^h\right)'$$

$$= \epsilon^2 y_{xx} + f'\left(u^h\right) y + \alpha' \left(u^h\right)' + \mathcal{O}\left(y^2\right)$$

Now we set $Ly = -\epsilon^2 y_{xx} - f'\left(u^h\right) y$ with appropriate domain in $L^2(-\infty,\infty)$, this operator is self-adjoint and has a continuous strictly positive spectrum to the right of $-f'(1)$ and a discrete spectrum to the left, we also know that 0 lies in the discrete spectrum (e.g. by differentiating the equation for the invariant solution) with eigenfunction u^h; due to the constant sign of its derivative, 0 must be simple and the least eigenvalue; y is clearly included in $L^2(-\infty,\infty)$.

We now integrate the equation up against y to obtain due to the orthogonality $\langle y, u_x^h \rangle = 0$ that

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \le (-Ly, y) + \mathcal{O}\left(\|y\|^3\right).$$

But y is orthogonal to the eigenfunction $(u^h)'$ corresponding to the zero eigenvalue of L, so (to see that the ϵ^2 term disappears we rescale space onto $x' = \epsilon^{-1} x$, approximate the operator as $(-(-\partial_{xx} - f'(u^h))y, y) \le -M\|y\|^2$ and then rescale back) this becomes

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \le -M\|y\|^2 + \mathcal{O}\left(\|y\|^3\right),$$

where $M > 0$ is a constant independent of t. Since we have $y \to 0$ uniformly as $t \to \infty$ and for small enough $\epsilon > 0$ $\|y\|$ will itself be small enough we finally get

$$\|y(t)\| \frac{d\|y(t)\|}{dt} = \frac{1}{2}\frac{d}{dt}\|y(t)\|^2 \le -M'\|y(t)\|^2.$$

for some $M' > 0$.

This means for $\|y(t)\| > 0$ that

$$\frac{d\|y(t)\|}{dt} \leq -M'\|y(t)\|.$$

Integrating this equation yields

$$\|y(t)\| \leq K'\|y_0\|e^{-\nu t}$$

for some $K' \geq 1$, $\nu > 0$.

Reworking the sketch of argument in [FM77] with appropriate scaling shows that $|\alpha(t)| \leq \epsilon c e^{-\nu' t}$ for some $c, \nu' > 0$. The final expression follows because of the exponentially small error between ϕ^* and u^h.

This is inspired by ***[Che04] Lemma 3.1*** (in [Che04] the interval sizes are slightly wrong: in his scaling it should be $\left[\epsilon^{-1}a, \epsilon^{-1}b\right]$ rather than $[a, b]$):

Claim 7.6. *Suppose* $\left\|\phi(\cdot,0) - u^{h'}(\cdot)\right\| \to_{\epsilon\to 0} 0$ *for some* $u^{h'}(\cdot)$ *being the orthogonal projection of* $\phi(\cdot,0)$ *in the* L^2 *norm.*

There exist positive constants $\kappa_0, \nu_0, \rho_0, C', C$ *depending only on* $f(\cdot)$ *and a positive constant* $K_0 \geq 1$ *depending on* $f(\cdot)$ *and* ρ, *and* h *depending on* $h_0, \rho, f(\cdot)$ *with*$|h| \leq (\rho + \rho_0)\epsilon$, *s.t. if*

$\phi_0 \in \left\{\phi \in C(\mathbb{R}) \to [-2,2] | |\phi| \geq \frac{1}{2} \text{ on } [a,b] \backslash (-\rho\epsilon, \rho\epsilon)\right\}$ *with* $\min\{-a, b\} > \rho > 0$, *the solution* ϕ *to the equation of (7.1) satisfies, for any* $r \in [0, \min\{-a, b\}]$,

$$\sqrt{\int_{a+r\epsilon}^{b-r\epsilon} |\phi(y,t) - u^h(y)|^2 \, dy} \leq K_0 \left\|(\phi_0 - u^h(\cdot))\right\| e^{-\nu_0 t} + C' e^{-C/\epsilon} \; \forall t \in \left[0, \frac{2\kappa_0 r}{\epsilon}\right]$$

Sketch of Argument. Let $\tilde{\phi}$ be the solution to the equation of Claim 3.1 with initial profile $\tilde{\phi}(y,0) = \phi_0(y)$ for $y \in (a,b), = sgn(\phi(a))$ for $y \leq a$, and $= sgn(\phi_0(b))$ for $y \geq b$. Then, $\tilde{\phi}(\cdot,0) \in X_\epsilon^{\rho,0}$. Hence, by Claim 7.5, $\tilde{\phi}$ satisfies the inequality of Claim 7.5. We now need to estimate the quantity $v := \phi - \tilde{\phi}$ to obtain the desired result:

Firstly we observe that $v_t = \epsilon^2 v_{yy} + a(y,t)v$ for some $a(y,t) \leq \max_{s\in[-2,2]} \{-f'(s)\} \leq |f'(0)|$ for all $y \in \mathbb{R}$ and $t \geq 0$.

Therefore, by the comparison principle for positive solutions of the heat equation we have

$$|v(y,t)| \leq e^{|f'(0)|t} \int_{-\infty}^{\infty} K(z,t)|v(y-z,0)|dz,$$

where $K(z,t) = (4\pi)^{-1/2} e^{\frac{-z^2}{\epsilon^2 4t}}$ is the usual heat kernel of our equation.

This straightforwardly implies through squaring and integrating over the interval $[a\epsilon + r\epsilon, b\epsilon - r\epsilon]$ that

$$\sqrt{\int_{a+r\epsilon}^{b-r\epsilon} |v(y,t)|^2 dy} \leq \sqrt{\int_{a+r\epsilon}^{b-r\epsilon} e^{2|f'(0)|t} (\int_{-\infty}^{\infty} K(y-z,t)|v(y-z,0)|dz)^2 dy}.$$

Given the definition of ϕ as the difference between the actual solution of our equation and the one with an initial profile set to ± 1 outside $[a,b]$, we easily observe that $v(y,0)=0$ for all $y \in [a,b]$.

Then (using the Cauchy-Schwartz inequality) we have

$$\sqrt{\int_{a+r\epsilon}^{b-r\epsilon} |v(y,t)|^2 dy} \leq e^{|f'(0)|t} \|v(\cdot,0)\| \int_{|z|\geq r} K(z,t)dz$$
$$\leq \|v(\cdot,0)\| e^{|f'(0)|t - r^2/(4\epsilon^2 t)}.$$

But now it follows for all $t \geq 0$ that

$$\sqrt{\int_{a+\epsilon r}^{b-r\epsilon} \left|\phi(y,t) - u^h(y)\right|^2 dy}$$
$$\leq K_{FM} \left\|\phi(\cdot,0) - u^h(\cdot)\right\| e^{-\nu_0 t} + C'' e^{-D/\epsilon} + \|v(\cdot,0)\| e^{|f'(0)|t - r^2/(4\epsilon^2 t)}$$

for some $C'', D > 0$.

We now choose $\kappa_0 = [16(A+\nu_0)]^{-1/2}$. This implies $At - r^2/(4\epsilon^2 t) \leq -\nu_0 t$ for all $t \in (0, \frac{2\kappa_0 r}{\epsilon}]$.

We note that

$\|v(\cdot,0)\| \leq \left\|\phi(\cdot,0) - u^h(\cdot)\right\| + \left\|\mathbb{I}_{(-\infty,a)\cup(b,\infty)} - u^h(\cdot)\right\| \leq \left\|\phi(\cdot,0) - u^h(\cdot)\right\| + C' e^{-C/\epsilon}$ for some $C, C' > 0$, the exponential correction appears because $\phi(\cdot,0)$ can be closer to u^h than ± 1 outside $[a,b]$.

If we now choose K_0 in such a way that $K_0 \geq K_{FM} + 1$, and add the aforementioned exponential correction term the assertion of our Claim follows.

This is inspired by ***[CP89] Theorem 2.4***:

Claim 7.7. *For some $0 < \rho < \rho^{**}$ and $C_{map} > 0$ independent of ϵ there exists a smooth function (depending on ϵ) $\mathcal{H} : \Gamma'_\epsilon \to \Omega_{\rho\epsilon}$ and a constant $C > 0$ such that provided $h = \mathcal{H}(u)$ we have*

$$\left\langle u - u^h, \tau_j^h \right\rangle = 0 \text{ for } j = 1, ..., N$$

and

$$\left\|u - u^h\right\| \leq C \left\{\left\|u - u^k\right\| : k \in \Omega_\rho\right\}.$$

Sketch of Argument. We will achieve this sketch of argument through using the implicit function theorem. For this we prove the local result stated in Claim 7.8:

$\mathcal{H}(u)$ will be determined from a zero of the function

$$\mathcal{F} : \Omega_{\rho^{**}\epsilon} \times \Gamma'_\epsilon \to [0,1]^N \text{ given by } \mathcal{F}(h,u) = \left(\left\langle u^h - u, \tau_j^h \right\rangle\right).$$

$\mathcal{F}$ is clearly smooth in (h,u) and $\mathcal{F}\left(h,u^h\right)=0$ for all $h\in\Omega_{\rho^{**}\epsilon}$. We will apply a contraction mapping principle in the first argument of the function $G:\Omega_{\rho^{**}}\times\Gamma'_\epsilon\times\Omega_\rho\to[0,1]^N$ given by

$$G(h,u,h^*)=h-(D\mathcal{F}_*)^{-1}\mathcal{F}(h,u),$$

where

$$(D\mathcal{F}_*)_{jk}=\frac{\partial\mathcal{F}_j}{\partial h_k}\left(h^*,u^{h^*}\right)=\left\langle u_k^{h^*},\tau_j^{h^*}\right\rangle.$$

From the estimates of [CP89] subsestion 2.5 and 3.2, if ρ^{**} is sufficiently large we may estimate the Lipschitz constant of G by estimating

$$\begin{aligned}\left|\frac{\partial G}{\partial h}\right| &\le \left|(D\mathcal{F}_*)^{-1}\right|\cdot\left|\left(\left\langle u_k^{h^*},\tau_j^{h^*}\right\rangle-\left\langle u_k^h,\tau_j^h\right\rangle+\left\langle u-u^h,\tau_{jk}^h\right\rangle\right)\right|\\ &\le C\left(b'(\rho)+\left\|u-u^h\right\|\epsilon^{-1/2}\right)\end{aligned}$$

where $b'(\rho)=o(1)$ as $\rho\to\infty$.

From the estimate $\left\|u^h-u^{h^*}\right\|\le C\epsilon^{-1/2}|h-h^*|$ which follows from [CP89] subsection 2.5, we have

$$\left|\frac{\partial G}{\partial h}\right|\le C\left(b'(\rho^{**})+\left\|u-u^{h^*}\right\|\epsilon^{-1/2}+\epsilon^{-1}|h-h^*|\right).$$

If $\left\|u-u^{h^*}\right\|\le C_{map}\epsilon^{1/2}$, $|h-h^*|<\epsilon b$ and $\frac{1}{\rho^{**}},C_{map},b$ are sufficiently small (independent of ϵ), we find $\left|\frac{\partial G}{\partial h}\right|<\frac{1}{4}$. If ρ^{**} is sufficiently large, $G(\cdot,u,h^*)$ is defined for all $h\in B_{\epsilon b}(h^*)$ (a ball of radius ϵb around h^*), and

$$\begin{aligned}&|G(h,u,h^*)-h^*|\\ &\le\left|G(h,u,h^*)-G\left(h,u^{h^*},h^*\right)\right|+\left|G\left(h,u^{h^*},h^*\right)-G\left(h^*,u^{h^*},h^*\right)\right|\\ &\le\left|(D\mathcal{F}_*)^{-1}\right|\left|\left\langle u^{h^*}-u,\tau_j^h\right\rangle\right|+\frac{1}{4}|h-h^*|\\ &\le C\left\|u-u^{h^*}\right\|\max_j\left\|\tau_j^h\right\|+\frac{1}{4}\epsilon b\le\epsilon\left(C\cdot C_{map}+\frac{1}{4}b\right)<\frac{1}{2}\epsilon b,\end{aligned}$$

provided $C_{map}<\frac{b}{4C}$. Thus we have a local result by applying the contraction mapping principle.

Global control follows from Claim 7.9.

Claim 7.8. *Assume ρ,ρ^{**},C_{map} and b are as required above; then if $h^*\in\Omega_\rho$ and $u\in\Gamma'_\epsilon$, there exists a unique solution $h=\mathcal{H}(u,h^*)$ to $\mathcal{F}(h,u)=0$ in $B_{\epsilon b}(h^*)$. (In fact, $|h-h^*|<\frac{1}{2}\epsilon b$). This function $\mathcal{H}(u,h^*)$ is smooth in its first argument. (Here the notation $B_\sigma(w)$ means $\{u\in L^\infty|\|u-w\|_\infty<\sigma\}$.)*

Claim 7.9. *Given b as above, if ρ is sufficiently large, there exists C'_{map}, $0 < C'_{map} \leq C_{map}$, such that if h^* and h^{**} lie in $\Omega_{\rho^{**}}$ and $\left\|u^{h^*} - u^{h^{**}}\right\| \leq 2C'_{map}\epsilon^{1/2}$, then $|h^* - h^{**}| < \frac{1}{2}\epsilon b$.*

Sketch of Argument. We may take $\rho > b^{-1}$ so that $h_j - h_{j-1} > \epsilon b$ for $h \in \Omega_\rho$. We claim that if ρ^{**} and ρ are sufficiently large, then for each $k \in \{1, ..., N\}$ there exists a unique $j(k) \in \{1, ..., N\}$ such that $\left|h^*_k - h^{**}_{j(k)}\right| < \frac{1}{2}\epsilon b$.

To show this, we argue as follows. There exists $c > 0$ s.t. $|\Phi(x)| > c$ (Φ is the invariant solution of the PDE on the real line with interface 0) if $|x| > \frac{1}{2}\epsilon b$. Choose $\delta_1 = 1/2$, H_1 large, δ_2 with $0 < 2\delta_2 < c$, and apply [CP89] Proposition 2.2. Set $C'_{map} = \frac{1}{2}\min\left(\delta_2, \frac{1}{2}\right)$. From [CP89] subsection 2.3 it follows that if ρ is sufficiently large, $h \in \Omega_\rho$, and $|x - h_j| \geq \frac{1}{2}\epsilon b$ for all j, then $\left\|u^h(x)\right\| > 2C'_{map}\epsilon^{1/2}$. Now if $\left|h^*_k - h^{**}_j\right| \geq \frac{1}{2}\epsilon b$ for all j, then $\left\|\left(u^{h^*} - u^{h^{**}}\right)(h^*_k)\right\| > 2C'_{map}\epsilon^{1/2}$ contradicting the hypothesis, thus establishing the claim, since $h^{**}_j - h^{**}_{j-1} > \epsilon b$ for all j yields uniqueness. The mapping $k \to j(k)$ is one to one, hence onto, and it is increasing, so $k = j(k)$. Hence $|h^* - h^{**}| < \frac{1}{2}\epsilon b$, as desired.

Now to define $\mathcal{H}(u)$ in Claim 7.7, take $C_{map} = C'_{map}$. Given $u \in \Gamma'_\epsilon$, there exists $h^* \in \Omega_\rho$ with $\left\|u - u^{h^*}\right\| < C'_{map}\epsilon^{1/2}$; we set $\mathcal{H}(u) = \mathcal{H}(u, h^*)$. To show that this is well-defined, suppose also $h^{**} \in \Omega_\rho$ with $\left\|u - u^{h^{**}}\right\| < C'_{map}\epsilon^{1/2}$. To see that $\mathcal{H}(u, h^*) = \mathcal{H}(u, h^{**})$, it suffices to show that $|\mathcal{H}(u, h^{**}) - h^*| < \epsilon b$ by the uniqueness assertion of Claim 7.8. But

$$|\mathcal{H}(u, h^{**}) - h^{**}| + |h^{**} - h^*| < \frac{1}{2}\epsilon b + \frac{1}{2}\epsilon b$$

as required, since $\left\|u^{h^*} - u^{h^{**}}\right\| < 2C'_{map}\epsilon^{1/2}$. The claim is proved.

7.2 Existence and uniqueness of the stochastic Allen-Cahn equation

For the convenience of the reader, we here review existence and uniqueness of the stochastic Allen-Cahn equation. These results are relatively well known and can in better detail, for example, be found in [Zab89], [DPZ92], [DPZ96] or [GP93].

We consider the system

$$u_t = \epsilon^2 u_{xx} + f(u) + \epsilon^\gamma \dot{W} \; x \in (0, 1)$$
$$u(\cdot, 0) = u_0(\cdot), \; u(x) = u(x + 1).$$

f is as defined in section 1.2, W is a cylindric Wiener process as in definition 1.2. We may view this equation as a stochastic differential equation in some infinite-dimensional function space. A problem in interpreting the "meaning" of this equation is that $\dot{W}$ is a not a function, but rather an object that attains

values in some distribution space. To avoid this problem of interpreting the equation, we can pose it in mild form

$$u(\cdot,t) = S(t)u_0 + \epsilon^\gamma \int_0^t S(t-s)dW(s) + \int_0^t S(t-s)f(u(s))ds.$$

$S(t)$ denotes the associated heat semigroup, i.e. $S(t)w_0$ solves $w_t = \epsilon^2 w_{xx}$; $w(\cdot,0) = w_0(\cdot), w(x) = w(x+1)$.

The formula written down can be viewed as a stochastic analogue of the variations of constants formula. Even though our noise takes values in a "rougher" space, the stochastic convolution takes values in the space of continuous functions due to the smoothing properties of $S(t)$. The solution to the nonlinear equation then works using a fixed point argument in this space. One danger of this approach is that our mild form only formally corresponds to the equation, and particular care needs to be taken, if one wants to apply Itô's formula.

Before we show its existence and uniqueness, we will look at the linear case:

$$z_t = \epsilon^2 z_{xx} + \epsilon^\gamma \dot{W}$$

As in chapter 1, we denote by $\{e_j(\cdot)\}_{j=1}^\infty$ an orthonormal basis of $L^2[0,1]$ consisting of the eigenvectors of the Laplacian with periodic boundary conditions so that the associated eigenvalues are $\lambda_k = -(k\epsilon\pi)^2$. Then we have $z = \sum_{j=1}^{j=\infty} x_k(t)e_k(x)$ where

$$dx_k(t) = -\lambda_k x_k dt + \epsilon^\gamma dW^k(t)$$

where we denote $W^k = \langle W(t), e_k\rangle$, which are independent Brownian motions. The convergence of the sum $z = \sum x_k e_k$ in $C([0,T], L^2[0,1])$follows immediately by use of a maximal inequality; application of the Kolmogorov-Chentsov criterion gives convergence in $C([0,T], C_0[0,1])$. We will obtain that $z(x,t)$ are locally α-Hölder-continuous in x for all $\alpha < \frac{1}{2}$ for fixed t and locally β-Hölder-continuous in t for all $\beta < \frac{1}{4}$ for fixed x. Unlike for an Ornstein-Uhlenbeck process in finite dimensions, for each $x \in (0,1)$, $(z(x,t), t \geq 0)$ is not a semimartingale. However, the process is reversible, and the reversible measure is a normal distribution $\mathcal{N}\left(0, \frac{\epsilon^{2\gamma}}{2\lambda_k}\right)$; this in turn implies that the entire process is a normal distribution on L^2 with covariance operator $(-\Delta)^{-1}$, which may be interpreted as a Brownian bridge. We may write down the solution also in the mild solution form

$$z = S(t)z_0 + \epsilon^\gamma \int_0^t S(t-s)dW_s.$$

Denoting the heat semigroup in terms of its eigenvector basis, we can verify that this is the same as the solution above.

Now we shall consider the nonlinear case. Our nonlinearity defines a continuous operator from C_0 onto itself. To be precise, we have the following: If u and v satisfy $\|u\|_\infty, \|v\|_\infty \leq C$, $C > 0$, then one has for all $x \in [0,1]$ that

$$|f(u(x)) - f(v(x))| \leq \sup_{|w|\leq C} |f'(w)|\, 2C,$$

so that the operator $\mathcal{F} : C_0[-1,1] \to C_0[-1,1]$ *defined by* $\mathcal{F}(u)(x) = f(u(x))$ *is locally Lipschitz. Now, we define the operator* $H : C_0[-1,1] \to \mathbb{R}$ *by* $H(u) = -\int_{-1}^{1} F(u(x))dx$*; we may interpret* $\mathcal{F}$ *as the* L^2*-gradient:* H *is Frechet-differentiable, and for each* $u, h \in C_0[-1,1]$ *one has*

$$DH[u](h) := \lim_{\delta \to 0} \frac{H(u+\delta h) - H(u)}{\delta} = \int_{-1}^{1} f(u(x))h(x)dx = -\langle \mathcal{F}(u), h \rangle.$$

Let us now construct a global solution. We shall inspect the mild solution starting at v_0 *rather than* u_0*, i.e.* v *s.t.*

$$v(t) = S(t)v_0 + \epsilon^\gamma \int_0^t S(t-s)dW_s + \int_0^t S(t-s)f(u(s))ds$$

To obtain local existence we choose the path $z(t) = S(t)v_0 + \epsilon^\gamma \int_0^t S(t-s)dW_s \in C([0,T], C_0[-1,1])$*, we have continuity because* $\epsilon^\gamma \int_0^t S(t-s)dW_s$ *is a Gaussian process (cf. e.g. [Adl90] for the continuity of Gaussian processes) and* $S(t)v_0$ *is the solution to the heat equation started at* v_0 *(c.f. [Eva10] for its continuity); we look for solutions of the mild equation in a pathwise manner. Observe that the mapping*

$$\Theta v(t) = z(t) + \int_0^t S(t-s)f(u(s))ds$$

maps $C([0,T], C_0[-1,1])$ *onto itself. Because of the maximum principle one has* $\|S(t)\|_{C_0 \to C_0} \leq 1$*, and hence can conclude*

$$\|\Theta v(t) - z(t)\|_\infty \leq \int_0^t \|f(u(s))\|_\infty ds.$$

Together with the Lipschitz property of f *this means that for a fixed* C *and* $T = T(C)$ *small enough* Θ *maps*

$$\bar{B}_r(z) = \left\{ u(t)([0,T], C_0[-1,1]) : \sup_{t \in [0,T]} \|u(t) - z(t)\| \leq C \right\}$$

into itself. Similarly, for $u, v \in \bar{B}_r(z)$*, the lipschitz property gives*

$$\|\Theta(u(t)) - \Theta(v(t))\|_\infty \leq \int_0^t \|\mathcal{F}(u(s)) - \mathcal{F}(v(s))\|_\infty ds \leq C' t \sup_{s \in [0,t]} \|u(s) - v(s)\|_\infty$$

so that (potentially involving a smaller choice of t*)* Θ *is a contraction. By the Banach Fixed Point Theorem we then have local existence and uniqueness of mild solutions. To obtain global uniqueness, it is sufficient to show that on a finite time interval* $[0,T]$ *our norm* $\|u(t)\|_\infty$ *cannot blow up. Now for a fixed path* $z(t)$ *denote* $\bar{v} = v - z$*. On a formal level, we want to conclude that* $\bar{v}$ *solves the random PDE*

$$\bar{v}_t(x,t) = \epsilon^2 \bar{v}_{xx}(x,t) + f(\bar{v}(x,t) + z) + z(x,t).$$

Since $f(u)$ obtains only negative values for $|u|$ sufficiently large, one could conclude non-blow-up of $\|\bar{v}\|_\infty$ on $[0,T]$ using a comparison principle.

However, since we only know that the mild formulation of the equation holds, we do not know if this formulation of the equation holds. To complete the sketch of argument, we replace the laplacian in the mild formulation by the Yoshida approximation. Then one can proceed to prove non-blow-up of these regularized solutions. The maximum principle will be replaced by the Hille-Yoshida theorem yielding that the Yoshida approximation of the laplacian generates a contraction semigroup on $C_0[-1,1]$. Since the solutions of the approximated equations converge to the solution of the original equation, we may finally conclude non-blow-up for the original equation. In [DPZ92] these calculations can be found in detail. The Markov property then follows from the pathwise uniqueness property.

7.3 Open problems

While this work has given the first general view of the stochastic Allen-Cahn equation with finitely many interfaces, there are still some optimisations that could be done and related open problems:

- The results proven for white noise should also hold for slightly stronger noise, i.e. $\gamma > \frac{3}{2}$ instead of $\gamma > 2$. This should be achievable by replacing the condition $\|v\| \le C_{map}\epsilon^{1/2}$ with $\|v\|_\infty \le c_{map}$ for some $c_{map} > 0$ and formulate and prove the Theorems and lemmas accordingly in terms of this norm.

- The results should also hold for the initial condition simply being a continuous bounded function satisfying periodic boundary conditions. Since a deterministic result of this exists in [Che04], this should be possible through generalising the proof found in [Che04].

- One could prove the results using energy methods, that have already led to very detailed knowledge of the invariant measure; an interesting approach for this could be the one taken in [OR07].

- A very challenging problem is proving that on exponential timescales nucleation of interface pairs occurs. One work in this direction is [Bar12].

- What exactly happens if the white noise is so strong (i.e. $\gamma \le \frac{3}{2}$) that the sharp interface limit does not yield annihilating Brownian motions, but (presumably) something like the marked Brownian web? See [FINR04] for more about the (marked) Brownian web.

- Using [OW14], it should be possible to extend the results of [ABK12] to something similar to chapter 6.

- Related to fractional Brownian motion is the question of the interface behaviour in the Cahn-Hilliard equation perturbed by white noise. In [BBB14] the behaviour of one interface has been derived, and an extension similar to what was done in this work would be very interesting.

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