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Who should cast the casting vote? Using sequential voting to amalgamate information

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Abstract In this study, we are concerned with how agents can best amalgamate their private information about a binary state of Nature. The agents are heterogeneous in their "ability", the quality of their private information. The agents cannot directly communicate their private information but instead can only vote between the two states (say "Innocent" or "Guilty" on a criminal jury). We first describe possible methods of sequential majority voting, and then we analyze a particular one: the first $n-1$ jurors vote simultaneously and, in the case of a tie, the remaining juror has the casting vote. We prove that when $n = 3$ (a common situation for a tribunal of three judges), the probability of a correct verdict is maximized when the agent of median ability has the casting vote.

Keywords Jury · Sequential voting · Casting vote · Group decision

1 Introduction

Beginning with the celebrated Jury Theorem of Condorcet (1785), the reliability of majority verdicts in secret (or simultaneous) ballots between two alternative states of Nature has been extensively analyzed. The states of Nature might be "Innocent" and "Guilty" in a trial context, or "In" and "Out" for a tennis refereeing team. Condorcet (1785) showed that the reliability of the verdict, that is the probability that the verdict is correct, approaches one as the number of voters (called jurors) goes to infinity. With the notable exceptions of Dekel and Piccione (2000) and Ottaviani and Sørensen (2001), the case of *sequential voting* has not received similar attention. We consider

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the notion of sequential voting quite generally, as covering voting schemes where numbered jurors vote in order, with each voter aware of the votes of a given subset of the earlier voters. For example, in so-called roll-call voting, the given subset is simply all the earlier voters; while in secret or simultaneous ballots, the given subset of each voter is always the empty set. In the casting-vote scheme, extensively studied here, all but one juror announce their votes simultaneously and then the "casting voter" has the deciding vote in a tie. Here, the given subsets for the early voters are empty, whereas that of the casting voter consists of all earlier voters. The casting-vote scheme in action can be observed, for example, in the academic review process for a journal or conference, where often an Associate Editor has the casting vote if recommendations of the two independent referees are in opposite directions.

When the *abilities* of the jurors (to discern the true state of Nature) are heterogeneous, the order of voting in terms of juror ability may well affect the reliability of the verdict. Here we assume that the abilities of the jurors are common knowledge, though the problem of determining these abilities has been considered by Baharad et al. (2012).

As discussed by Ottaviani and Sørensen (2001), different approaches to this problem have been made over time. Taking the term *seniority* to denote higher ability, they contrast the anti-seniority rule (increasing ability order) of the ancient Sanhedrin court to the seniority rule (decreasing ability order) for debating order in the US Supreme court. Alpern and Chen (2017), using the same private information model as here, show numerically that for roll-call voting of three jurors, reliability is maximized for neither of these rules: *it is always best for the juror of median ability to vote first*.

To determine the optimal voting order, we need a strong model of private information and of juror ability: the former is determined by independent signals of real numbers in an interval which are correlated with the state of Nature; the latter is quantified by a number between 0 (no ability) and 1 (maximum ability) that determines the probability densities of these signals. Binary signals, which are common in the literature, are not sufficient to obtain our results.

In this paper, we introduce general sequential voting schemes with our main results centering on the casting-vote scheme of three voters. We show that for honest voting (for the alternative that appears more likely at the time of voting) *reliability is maximized when the median-ability juror has the casting vote*. This contradicts common practice and accepted wisdom that gives the casting vote to the senior judge in a panel of three or gives the tennis umpiring post (with overrule or casting vote) to more senior people than to the linesmen. We also give some results for larger juries.

2 Examples and literature

Let us present some examples of casting-vote schemes with known abilities of the jurors and discuss the small literature on the subject. Throughout the paper, we use the terms "voter" and "juror" interchangeably.

As mentioned in the introduction, an example of interest to academics is the process of refereeing conference paper submissions or journal submissions. In the latter, it is common to have two referees and an editor who can break ties. Usually the editor will know the referees by reputation and can bias her casting vote towards the one with more expertise in the area. For conference paper submissions, the expertise is made more explicit. For example the conference refereeing software *EasyChair* explicitly asks each referee to indicate one of the five possible levels of relevant expertise.

There are numerous examples of three-member casting vote juries. The selection committee for the Master of the Rolls (a senior post in the United Kingdom) is mandated as follows, with the President of the Supreme Court given the casting vote: "The selection panel comprises the President of the Supreme Court or his nominee as Chair, the Lord Chief Justice or his nominee, the Chairman of the JAC or their nominee and a lay member of the JAC. The Chairman of the panel has a casting vote in the event of a tie".

In boxing, three-man juries are common. A famous example, reported by the BBC on February 20, 2000, was the following:

> Marco Antonio Barrera, WBO super-bantamweight champion, and WBC champ Erik Morales squared up in Las Vegas for what will go down as one of the greatest fights in boxing history. Unfortunately, a fantastic contest was spoilt when Morales was handed a controversial split decision by the judges. One judge each voted for Morales and Barrera but the casting vote, of Dalby Shirley, was 115-113 in favour of the Tijuana man.

The literature on what we call jury voting goes back to the so-called Condorcet Jury Theorem (Condorcet 1785). We have not found any analytical work on the casting-vote scheme, but sequential (or roll-call) voting has received some attention. The principal papers in this area are Dekel and Piccione (2000) and Ottaviani and Sørensen (2001). The latter is similar to our model of honest voting because the jurors care about how their reputation will be affected by the correctness (established subsequently) of their opinions stated in debate. However, they do not know their own abilities. See also Alpern and Chen (2017), which contains numerical work on optimal voting order in roll-call voting. For large juries, roll-call voting is clearly far from casting voting, as the first $n-1$ jurors vote sequentially rather than simultaneously. Perhaps for a jury of three the similarities are stronger, as it is only the second juror who has different information in the two schemes. A general investigation of what we call jury voting is given in Ali et al. (2008). Our work would fit into the information amalgamation portion of the survey of Dewan and Shepsle (2011). In discussing the work of Dekel and Piccione (2000), they observe that "because voters condition on the same event, namely that of being pivotal, it makes no difference whether they cast their votes sooner or later." Our contrary results, where voting order matters, is due to the heterogenous abilities of our voters and the continuous nature of their private signals, and so it matters to the later, or casting, voter which early voters went for A and which went for B. If those voting for A were overall of significantly higher abilities, then the casting voter might vote against his weak signal for B (assuming a tie vote). Dewan and Shepsle (2011) take account of this fact in a footnote where they say that "the individual with the casting vote conditions her vote on the set of observed actions."

Dekel and Piccione (2000) also take account of voting order (in sequential voting) and conclude that (p. 48):

> …if voters are endowed *ex ante* with differential information (some voters can be better informed than other) knowing which voters voted in favor and which against can affect the choice of a later voter. It can be shown that, in a common-value and two signal environment (as in Sec. IIIC above), if the player's signals are completely ordered (in the sense of Blackwell), then it is optimal to have the better informed vote earlier. This provides an interesting contrast to the findings of Ottaviani and Sørensen (1998) [subsequently published as Ottaviani and Sørensen (2001)]. They obtain the opposite optimal order in an environment in which information providers care not about the outcome but about appearing to be well informed. It is not difficult, however, to construct examples in which having the best-informed voter vote first is not optimal. Hence it seems unlikely that general insights into this question can be obtained.

Our problem begins with a fixed set of jurors, and our questions are about their voting order. However, if the potential jurors have biases (towards the alternatives such as Innocent or Guilty), then there is a prior question of jury selection that has been analyzed by Brams and Davis (1978) and more generally by DeGroot and Kadane (1980), in terms of so-called "challenges". A related selection problem with vetoes is analyzed in Alpern et al. (2010). In our model, jurors are unbiased and differ only in their abilities to discern the true alternative. However, if we were to give potential jurors differing a priori subjective probabilities (biases) towards the alternatives, then the jury selection problem could be analyzed in terms of both biases and abilities, perhaps with challenges made by Defense, Prosecution and Judge.

We note that Karger et al. (2014) consider an optimization problem that is somehow dual to our problem. Given a target reliability of answers to a number of queries (tasks), they assign (with a cost) to these tasks workers of various abilities from a pool, where a task can have multiple workers, and then combine the answers provided by the assigned workers in an appropriate manner (e.g., majority voting), so that the target reliability of the answers is achieved with minimum cost.

We note that our jurors receive their private information (signals) independently but vote dependently, in the sense that the casting voter knows the previous voting. The significance of dependence in such matters is considered by Nitzan and Paroush (1984). For theoretical work on voting, see Grofman et al. (1983).

3 Sequential voting schemes

This section puts the casting-vote scheme into a more general framework of sequential voting. It can be read before or after the main body of the paper, as it is mainly motivational in approach and has no specific results to be used in the sequel.

3.1 General formalism and scheme counting

In the basic description of voting, we cannot have voter i knowing the vote of voter j and simultaneously have voter j knowing the vote of voter i. More generally, we cannot have a cycle of voters, each of whom knows the vote of his predecessor. A directed graph with no cycles is called acyclic.

Definition 1 A Sequential Voting Scheme is a directed acyclic graph (DAG) on a finite number of nodes. The nodes correspond to voters and an arc from node i to node j indicates that voter j knows the vote of voter i when he votes.

An elementary property of any finite DAG is that all its nodes can be numbered so that any arc is from a lower numbered node to a higher numbered node (see, e.g., Kahn 1962). Therefore, a sequential voting scheme can be alternatively described as follows: Associated with each voter i of $\{1, \ldots, n\}$ is set $K(i) \subseteq \{1, \ldots, i-1\}$, such that when voter i comes to vote he has knowledge of the votes of all voters j in $K(i)$. The possibility of such a time-consistent ordering of the voters enables us to define honest voting in a recursive manner later in the paper.

There are several ways of listing and counting finite DAGs. If the nodes are numbered 1 to n, then the number $d(n)$ of DAGs has been calculated by Robinson (1973) with the recurrence formula $d(n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}2^{k(n-k)}\, d(n-k)$ where $d(0) = 1$. Hence the first five numbers are $d(1) = 1$, $d(2) = 3$, $d(3) = 25$, $d(4) = 543$ and $d(5) = 29281$. However, if we take the aforementioned time-consistent numbering of nodes, these counts can be significantly reduced. For example, in the case of $n = 3$, given that independent specifications of $K(1)$, $K(2)$ and $K(3)$ determine a sequential voting scheme and the fact that there are 2^{i-1} possible subsets of $\{1, \ldots, i-1\}$ for any $i \geq 1$, we have in total $1 \times 2 \times 4 = 8$ possible sequential voting schemes, which we depict in Fig. 1.

As can be seen in Fig. 1, the three DAGs (b)–(d) are isomorphic. So in total there are six non-isomorphic voting schemes. Scheme (a) is secret ballot (or simultaneous voting), (h) is roll-call voting, while (e) is casting-vote scheme, which we analyze in detail later. The remaining three non-isomorphic schemes, which we do not explicitly

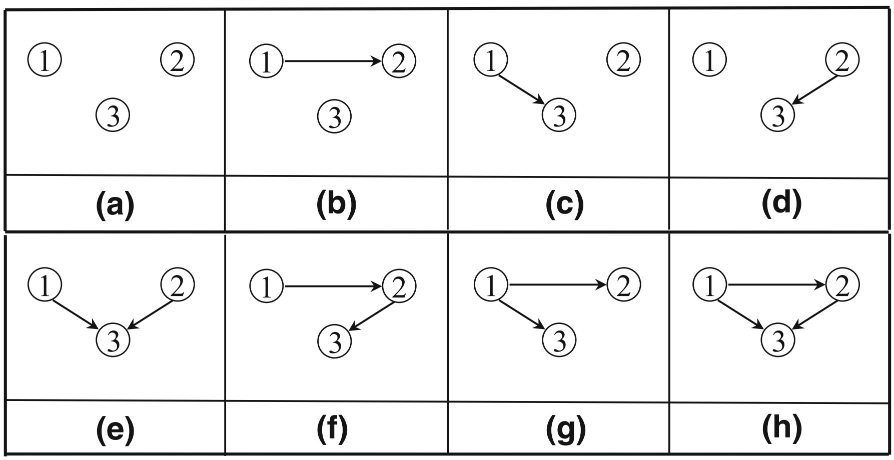

Fig. 1 Sequential voting schemes with three voters

name, look distinctly odd, as they do not appear in actual voting systems. However, we nonetheless keep them since the applicability of our analysis in the paper extends to cases where nodes do not have to represent usual jurors or voters, but might be sensors of locations in a neural network. Their “private information” might be the reading they take of their environment or their chemical surrounding.

3.2 Scheme complexity

For a given sequential voting scheme, let $k(i)$ denote the number of voting histories that can be witnessed by voter i and that do not already determine a majority verdict. For each of these histories, he must have a potentially distinct strategy. For example, in secret voting $k(i) = 1$ for every voter i, as he always observes the null voting history and needs only one strategy (which may of course depend on his private information). Consider roll-call voting for $n = 3$. Here as for secret ballot, $k(1) = 1$, but as voter 2 can see the vote of voter 1 (either A or B), he has two possible voting histories, so $k(2) = 2$. Finally, voter 3 can observe four histories AA, AB, BA and BB, but of these only two (AB and BA) leave him with a verdict-relevant vote, so also $k(3) = 2$. Therefore, in analyzing roll-call voting of three voters, if each strategy is a threshold (in terms of his private information, which is a number in [0, 1]), one must consider a total of $1+2+2 = 5$ thresholds. We call this *complexity* 5, by defining more generally the complexity $\mathcal{C}(n)$ of a given sequential voting scheme of n voters as

$$\mathcal{C}(n) = \sum_{i=1}^{n} k(i).$$

Thus for $n = 3$ the secret ballot has complexity 3, roll-call voting has complexity 5 and the casting-vote scheme has complexity 4. This explains in part why we are in this paper able to give an algebraic solution to the voting order problem for the casting-vote scheme while the similar analysis for roll-call voting can be carried out only with numerical analysis as in Alpern and Chen (2017).

4 The model

Our model is one of majority voting between two alternative states of Nature, A or B. There are an odd number n of jurors, or voters. First $n-1$ of them vote simultaneously. Then the casting juror votes, with knowledge of the earlier voting. It does not matter if the casting voter only votes in the case of a tie or always votes. To specify the model, we have to define what we mean by the ability of each juror, and how his ability (and the state of Nature) determines the distribution of signals that he receives as private information. We then define threshold strategies that determine a juror’s vote, based on his signal and any prior voting he is aware of (if he is the casting voter). Finally, we define what we mean by honest voting.

4.1 Signals and abilities

We assume two states of Nature A and B, considered as negation of A, with *a priori* probability of A given by $\Pr(A) = \theta_0$. To simplify the analysis we will assume the equiprobable case $\theta_0 = 1/2$, although our results are robust for θ_0 values around $1/2$. Individuals have private information about the state of Nature modeled as a signal s in the *signal interval* $[-1, +1]$. Positive signals are indications of A; negative signals B. The signal $s = 0$ is neutral. Higher positive signals indicate A more strongly; similarly for negative signals and B. Thus a better signal is one with a higher absolute value.

Individual jurors have an ability a in the *ability interval* [0, 1], where individuals of higher ability are generally (but not always) able to make better guesses about the state of Nature. When Nature is in state A (resp. B), jurors receive independent signals $s \in [-1, 1]$ with probability density given by $f_a(s)$ (resp. $g_a(s)$) if they have ability a. We make the simplest nontrivial assumption on $f_a(s)$ and $g_a(s)$, namely that they are linear in s. The slope of the density functions $f_a(s)$ and $g_a(s)$ for a juror of ability a is proportional to a. Given that f_a and $g_a(s)$ are density functions on $[-1, +1]$, they take the following form:

$$f_a(s) = (1 + as)/2, \ -1 \le s \le +1, \text{ when Nature is } A;$$
$$g_a(s) = (1 - as)/2, \ -1 \le s \le +1, \text{ when Nature is } B.$$

It is easily checked that $f_a(\cdot)$ and $g_a(\cdot)$ defined above are indeed density functions for any $a \in [0, 1]$. The density functions for ability $a = 1/2$ are shown in Fig. 2. The probability of a correct signal, that is positive when Nature is A, is the area under the $f_{0.5}$ line (and above the s axis) to the right of $s = 0$. When $a = 0$, such an area is $1/2$, showing that a juror with ability $a = 0$ is just guessing (by flipping a fair coin to determine the state of Nature).

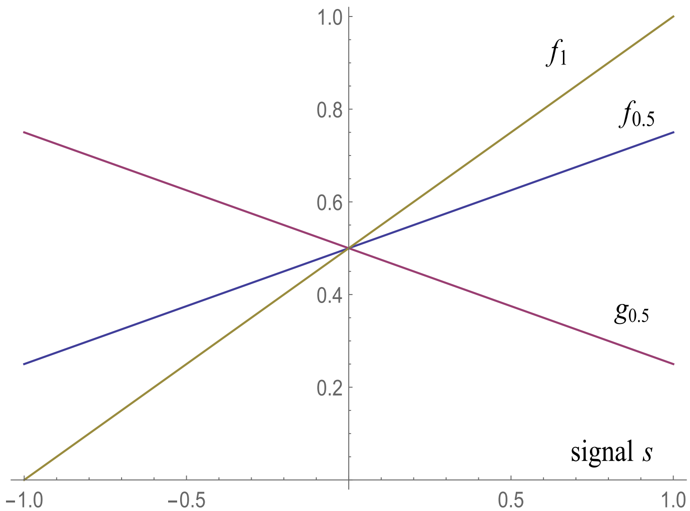

Fig. 2 Plots of signal densities $f_a(a)$ for A and $g_a(s)$ for B

The corresponding cumulative distributions of the signal s when Nature is A or B are given by

$$\begin{cases} F_a(s) = (s+1)(as-a+2)/4, \ -1 \le s \le +1, \ \text{when Nature is } A; \\ G_a(s) = (s+1)(a-as+2)/4, \ -1 \le s \le +1, \ \text{when Nature is } B. \end{cases} \tag{1}$$

Given prior probability θ_0 of A and only his signal s, a juror of ability a has a posterior probability θ' of A, as given by Bayes' Law:

$$\begin{aligned}\theta' = \Pr(A \backslash s) &= \frac{\theta_0 f_a(s)}{\theta_0 f_a(s) + (1-\theta_0) g_a(s)} = \frac{\theta_0 + as\theta_0}{2as\theta_0 - as + 1} \\ &= \frac{as+1}{2} \ (\text{if } \theta_0 = 1/2).\end{aligned} \tag{2}$$

Note that for a juror of ability 0, we have $\theta' = \theta_0$ for any received signal s, reinforcing our notion that ability 0 is no ability at all. A juror of ability 0 can do no more than guess. If we wish to view our juror of ability a as a Condorcet juror, we would say that his probability of a correct signal (positive when Nature is A or negative when Nature is B) is given by

$$\int_0^1 f_a(s)\, ds = (2+a)/4. \tag{3}$$

In particular a juror of ability 0 has only a 50% probability of a correct sign signal, while a boffin of maximum ability 1 gets it right 75% of the time.

4.2 Threshold strategies and honest strategies

A strategy for a juror is a threshold τ, depending on previous voting, if any, such that the juror votes A with signal $s \ge \tau$ and B with signal $s < \tau$ (see Fig. 3 for an illustration). A strategy profile is a list of strategies for each juror. So in our model of three jurors, with a casting vote, the two first voters have single thresholds x and y. (We can number the jurors by their voting order, although in our model the order of the first $n-1$ is arbitrary.) The casting voter has two thresholds z_{AB} (if the prior voting was AB) and z_{BA} (if the prior voting was BA). We can ignore the case of prior voting AA or BB, because in that case the last vote does not matter. So a strategy profile is a four-tuple (x, y, z_{AB}, z_{BA}). We recall from Sect. 3.2 that this agrees with the calculation of complexity $\mathcal{C}(3) = 4$ for the casting-vote scheme.

Now let us formally introduce the notion of voting behavior by first defining the most common or simplest one.

Definition 2 A strategy profile is said **honest** (or **naive**) if the thresholds are such that every juror votes for the alternative that he believes is more likely, given the *a priori* probability of A, his private signal, and any prior voting.

In our casting vote model, with neutral (equiprobable) alternatives A and B, honest voting requires that the two first jurors have their thresholds $x = y = 0$ since for

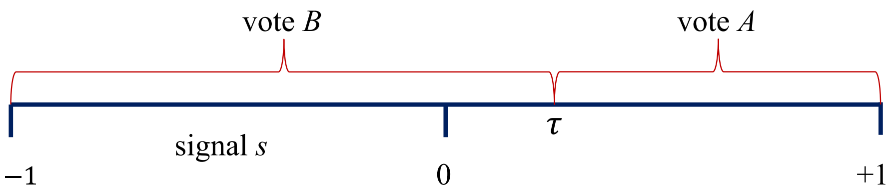

Fig. 3 Voting based on signal and threshold

$\theta_0 = 1/2$ we have by (2) that $\Pr(A/s) = 1/2 + as/2 > 1/2$ if and only if $s > 0$. This should be clear in any case from the symmetry of our model with respect to A and B. In other words, with honest voting early jurors (all but casting voter) each vote A with a positive signal and B with a negative signal. The situation for the casting voter is a bit different. We may relabel the alternatives so that the higher-ability early juror voted A and the lower-ability early juror voted B. If the casting voter gets a positive signal (for A) then obviously he votes A and that is the majority verdict. If however he agrees with the early voter of lower ability (he gets a negative signal), then his honest vote depends on the strength of his signal versus the ability discrepancy of the early voters. Given their abilities, he will have a negative threshold $\tau < 0$. If his own signal is less than τ (even more negative than τ), he follows his signal and votes B. Otherwise he will base his vote on the fact that while the early voters split their votes, those of higher ability voted for A and he will also vote A. This is the crux of the matter—the only case in which the casting voter will vote differently than if he had been voting simultaneously with the others.

4.3 Honest thresholds of the casting voter

In this section, we analyze the problem faced by the casting voter, who knows the voting, abilities and thresholds of the first two voters. What is his honest threshold? We assume the first two voters have ability a with threshold x and ability b with threshold y. We now determine the optimal threshold of the third juror of ability $c > 0$ (the case of $c = 0$ will be considered shortly after), under the assumption that the *a priori* probability of A (before the casting voter receives his signal s) is θ, given the previous voting and thresholds. If the casting voter has signal s, then his posteriori probability of A is given by θ' as given by equation (2) with a replaced by c and θ_0 by θ. Hence

$$\theta' = \frac{\theta + cs\theta}{2cs\theta - cs + 1}.$$

The honest threshold z is the value of s for which $\theta' = 1/2$, or

$$\frac{1}{2} = \frac{\theta + cs\theta}{2cs\theta - cs + 1}.$$

Solving for s and making this value the honest threshold z gives

$$z = \frac{1 - 2\theta}{c}. \tag{4}$$

Of course, if $(1 - 2\theta)/c > 1$ this means always vote B (same as threshold $z = 1$), and if $(1 - 2\theta)/c < -1$ this means always vote A (same as threshold $z = -1$). Such phenomenon is known as *herding* behavior, where agents ignore their own private information and follow prior agents. We can take the limit as $c \to 0+$ to make the same arguments if $c = 0$.

Now let us consider how to determine the above value of θ given the prior voting sequence AB (i.e., the voter of ability a votes A and the voter of ability b votes B). This value is given by

$$\theta = \theta(AB) = \frac{\theta_0(1 - F_a(x))F_b(y))}{\theta_0(1 - F_a(x))F_b(x) + (1 - \theta_0)(1 - G_a(x))G_b(y)} \tag{5}$$

$$= \frac{(2 + a + ax)(2 + b(-1 + y))}{2(4 + ab(1 + x)(-1 + y))}, \tag{6}$$

where the second equality is due to (1) for $\theta_0 = 1/2$. For the case of honest voting, the thresholds x and y of the early voters are 0, so (6) reduces to

$$\theta = \theta(AB) = \frac{(2 + a)(2 - b)}{8 - 2ab}. \tag{7}$$

According to (4) we then have

$$z_{AB} = \frac{1 - 2\theta(AB)}{c} = w(a, b, c) \equiv \frac{2(b - a)}{c(4 - ab)}, \tag{8}$$

and similarly

$$z_{BA} = -z_{AB} = -w(a, b, c) = \frac{2(a - b)}{c(4 - ab)}. \tag{9}$$

To illustrate the importance of these calculations, consider the threshold of the casting voter when the early voters have similar abilities and the ability of the casting voter is large. For example suppose the early voters have abilities 0.5 and 0.6 and the ability of the casting voter is 0.8. Then if the voter of ability 0.5 votes A and the voter of ability 0.6 votes B, the threshold for the casting voter of ability 0.8 is given by equation (8) as $2(0.1)/(0.8(4 - 0.3)) \approx 0.068$. Thus, the signal of the casting voter has to be just a bit above neutral 0 for him to vote A. However, if the early voters have widely different abilities, say 0.1 and 0.9, while the casting voter has ability 0.2, the threshold of the casting voter will be $2(0.8)/(0.2(4-(0.1)(0.9))) = 2.046$. Since this is greater than 1, it means the casting voter will always copy the vote of the stronger early voter, regardless of his own signal, a case of herding as alluded to earlier.

4.4 Reliability

We define the *reliability* of a voting scheme as the probability that the majority verdict is correct under this voting scheme. With equiprobable alternatives, a simple symmetry argument shows this is the same as the probability of majority verdict A when Nature is in state A. It is easy to calculate the reliability under honest voting where the early voters have abilities $a_1, a_2, \ldots, a_{n-1}$ and the casting voter has ability a_n. We ask the simple question: Given a set of n abilities, which one of these should have the casting vote if we wish to maximize the reliability of honest voting? For a jury of size three, we will show that honest-voting reliability is maximized when the juror of median ability has the casting vote. An alternative approach, also within the purview of our model, ascribes costs to each type of voting error (verdict A when Nature is B and *vice versa*) and minimizes the expected cost.

We now evaluate reliability $Q(a, b, c)$, the probability of a correct verdict when $\theta_0 = 1/2$, where the jurors have abilities a, b and c (in voting order) and the earlier voter have honest thresholds $x = y = 0$. As the theoretical voting order of the early voters (who vote simultaneously) does not matter, we clearly have

$$Q(a, b, c) = Q(b, a, c), \text{ for all } 0 \leq a, b, c \leq 1. \tag{10}$$

Let q_A (resp. q_B) denote the probability of majority verdict A (resp. B) when Nature is A (resp. B). Then for an arbitrary *a priori* probability θ_0 of A we have that the reliability $Q(a, b, c)$ is given by

$$Q(a, b, c) = \theta_0\, q_A(a, b, c) + (1 - \theta_0)\, q_B(a, b, c).$$

Hence with neutral alternatives $\theta_0 = 1/2$, we have

$$Q(a, b, c) = \frac{1}{2}(q_A(a, b, c) + q_B(a, b, c)),$$

and symmetry gives the simpler formula

$$q(a, b, c) = q_A(a, b, c) = q_B(a, b, c). \tag{11}$$

From now on we assume the case of neutral alternatives. As long as $|z_{AB}| < 1$, the formula for $q_A(a, b, c)$ is given by summing up the probabilities of voting patterns AA, ABA and BAA when Nature is A. Thus

$$\begin{aligned} q_A(a, b, c) = {} & (1 - F(a, 0))(1 - F(b, 0)) + (1 - F(a, 0))F(b, 0)(1 - F(c, z_{AB}))) \\ & + F(a, 0)(1 - F(b, 0))(1 - F(c, z_{BA}))), \end{aligned} \tag{12}$$

with a similar formula for q_B. Then according to (8), (9), and (11) provided $|z_{AB}| < 1$, we have $Q(a, b, c) = q(a, b, c)$, where

$$q(a, b, c) \equiv q_A(a, b, c) = \frac{1}{32}\left(4(4 + a + b) + \frac{4(a - b)^2}{(4 - ab)c} + (4 - ab)c\right). \tag{13}$$

In the case $|z_{AB}| \geq 1$, the casting voter follows the vote of the early voter of maximum ability, that is, the one of ability $\max\{a, b\}$ in our notation. We have calculated the probability that a juror gets the correct sign signal in (3). So in the above calculation of $q(a, b, c)$, by replacing z_{AB} with -1 and $+1$ respectively if $z_{AB} \leq -1$ and $z_{AB} \geq +1$, we get the more general reliability formula

$$Q(a, b, c) = \begin{cases} q(a, b, c), & \text{if } |w(a, b, c)| < 1; \\ (\max\{a, b\} + 2)/4, & \text{otherwise.} \end{cases} \tag{14}$$

This indicates the fact that in the case of $|z_{AB}| \geq 1$ the private signal of the early juror of higher ability determines the verdict, as his vote will be copied by the casting voter. Thus Q applies even when the casting voter has extreme thresholds whereby he can vote without taking his private signal into account.

For example, suppose we partition the ability interval [0, 1] into three subintervals of length $1/3$, and take a jury with one juror in the middle of each of these subintervals. That is, we have a uniformly distributed jury of abilities $1/6$, $1/2$ and $5/6$. If the high-ability juror has the casting vote ($c = 5/6$ in the above notation), then $z_{AB} \approx 0.204$, which lies within the signal interval $[-1, 1]$ and so reliability is given by the formula $q(1/6, 1/2, 5/6) \approx 0.690$. Similarly, if the juror of middle-ability $1/2$ has the casting vote, then $z_{AB} \approx 0.691$. As this is also in the signal interval, the reliability is given by $q(1/6, 5/6, 1/2) \approx 0.714$. Finally, if the weakest juror has the casting vote, then $|z_{AB}| = 48/43 > 1$. This means that the casting voter follows the vote of the juror of ability $5/6 = \max\{1/2, 5/6\}$, who is correct with probability $(2 + 5/6)/4 \approx 0.708$ by formula (14) or (3). So in this case we have calculated that giving the casting vote to the median-ability juror is best, to the lowest-ability voter is second best, and to the highest-ability voter is worst. This result about the median voter is generalized in Theorem 1.

Note that $q(a, b, c) = (\max\{a, b\} + 2)/4$ when $|z_{AB}| = 1$ according to (8) and (13). On the other hand, it is easy to verify that $|w(x, y, z)| \leq w(\ell, 1, \ell) = 1$ over $\{(x, y, z) : \ell \leq x, y, z \leq 1\}$ where $\ell = 3 - \sqrt{7} \approx 0.354$. We obtain from (14) that

$$Q(a, b, c) = q(a, b, c) \ \text{ if } \ell \leq a, b, c \leq 1. \tag{15}$$

5 Mechanism design

We now consider the main question of the paper, the problem faced by a designer who is given a fixed set of jurors with known abilities and must decide to whom to give the casting vote. (Perhaps he is organizing a sporting event and has three volunteers for refereeing, who come with eyesight certificates. Or maybe he is writing the constitution of the International Court of Justice and has to say which judge has the casting vote.) We suppose here that there are three jurors and their abilities a, b, c are labeled so that $a \leq b \leq c$. (Note that we have changed our labelling conventions from the last section, where c was always the ability of the casting voter.) In the last section we gave an example with uniformly distributed abilities $1/6$, $1/2$ and $5/6$, where we showed that it was best for the juror of ability $1/2$ to have the casting vote, with the juror of ability $1/6$ second best. Here we will discuss the problem more generally.

To aid the intuition, we carry out a thought experiment where the casting-vote scheme is conducted in another equivalent way. We let all the jurors vote simultaneously. If the verdict is close (2 to 1, or $(n+1)/2$ to $(n-1)/2$) we are allowed to pick one of the jurors and let him decide whether to change his vote after viewing the other votes (and with knowledge of everyone's ability). If he voted with the minority, changing his vote will not affect the verdict, so we assume he voted with the majority, say A. This means the others voted equally for A and B. He will only change his vote if he now thinks B is more likely than A, despite his private positive signal for A. This will occur if one of the following two things occurs:

1. His positive signal s for A is weak (close 0).
2. The overall abilities of the B voters are significantly higher than that of the A voters.

We realize that the second condition is vague. Condition 1 is most likely to be satisfied when the juror has the smallest ability. This is because in our model small abilities are more likely to produce weak signals. Condition 2 is most likely, for a jury of three, when the abilities of the other two jurors are as far apart as possible. That is, when they have the two extreme abilities. This occurs when the selected juror has the middle ability. So this intuitive and qualitative analysis leads us to believe that in general the casting voter should have a low or middle signal. Our later analysis for larger juries indeed bears this out. Here we show that for a jury of three the forces (condition 2) favoring the middle- or median-ability juror outweigh the forces (condition 1) favoring the low ability juror. In particular we have our following main result.

Theorem 1 *Under casting-vote scheme suppose that A and B are equiprobable and we have three honest jurors of abilities* a, b, c *with* $0 \le a \le b \le c \le 1$. *Then the reliability* Q *is maximized when the juror of median ability b has the casting vote.*

Proof The idea of our proof is simple: starting with the case where the juror of ability b has the casting vote, we show that reliability cannot increase when he is replaced in that role by the juror of either higher ability c or lower ability a. Let $S = \{(a, b, c) : 0 \le a \le b \le c \le 1\}$. Denote

$$\Delta_1(a, b, c) = Q(a, c, b) - Q(a, b, c);$$
$$\Delta_2(a, b, c) = Q(a, c, b) - Q(b, c, a).$$

Then due to (10), our aforementioned simple idea is implemented by showing that $\Delta_1(a, b, c)$ and $\Delta_2(a, b, c)$ are both non-negative for any $(a, b, c) \in S$. First of all, the following are straightforward according to definition (8) for any given $(a, b, c) \in S$:

$$w(b, c, a) \ge 1 \Leftrightarrow P_1(a, b, c) \equiv abc - 4a - 2b + 2c \ge 0; \tag{16}$$
$$w(a, c, b) \ge 1 \Leftrightarrow P_2(a, b, c) \equiv abc - 2a - 4b + 2c \ge 0; \tag{17}$$
$$w(a, b, c) = 2(b - a)/(c(4 - ab)) \le (2/3)(b/c) < 1. \tag{18}$$

Depending on the magnitude of $w(a, c, b) \geq 0$ for any given $(a, b, c) \in S$, we consider two possible cases separately.

Case 1 $w(a, c, b) \geq 1$.
According to (14), we have

$$Q(a, c, b) = (c + 2)/4. \tag{19}$$

Since $P_1(a, b, c) - P_2(a, b, c) \geq 2(b - a) \geq 0$, it follows from (16) and (17) that

$$w(a, c, b) \geq 1 \Rightarrow w(b, c, a) \geq 1,$$

which together with (14) implies that $Q(b, c, a) = (c + 2)/4$, and hence $\Delta_2(a, b, c) = Q(a, c, b) - Q(b, c, a) = 0$.

Next we prove the more difficult result $\Delta_1(a, b, c) \geq 0$. According to (18) and (14) we have $Q(a, b, c) = q(a, b, c)$. It follows from (19) that

$$\Delta_1(a, b, c) = \frac{c + 2}{4} - q(a, b, c) = \frac{d_3(a, b, c)}{32c(4 - ab)},$$

where

$$\begin{aligned} d_3(a, b, c) \equiv& -4a^2 + 8ab - 4b^2 - 16ac - 16bc \\ &+ 4a^2bc + 4ab^2c + 16c^2 - a^2b^2c^2. \end{aligned}$$

So it remains only to establish that $d_3(a, b, c) \geq 0$ given the additional condition $2b \leq c$, which is implied by (17). In Sect. A.2 of the Appendix we show that $d_3(a, b, c)$ has a minimum of 0 over the set $S \cap \{(a, b, c) : 2b \leq c\}$, which is attained uniquely at the point $a = b = c = 0$.

Case 2 $w(a, c, b) < 1$.
According to (14), we have $Q(a, c, b) = q(a, c, b)$, which together with (18) implies that

$$\Delta_1(a, b, c) = q(a, c, b) - q(a, b, c) = \frac{(c - b)d_1(a, b, c)}{8bc(4 - ab)(4 - ac)},$$

where

$$\begin{aligned} d_1(a, b, c) \equiv& 4a^2 - 8ab + 4b^2 - 8ac + 4bc \\ &+ 2a^2bc - ab^2c + 4c^2 - abc^2. \end{aligned}$$

By taking partial derivatives, we can easily see that $d_1(a, b, c)$ is monotonically decreasing in b and c for all $(a, b, c) \in S$, which implies that $d_1(a, b, c) \geq d_1(a, a, a) = 0$ and hence $\Delta_1(a, b, c) \geq 0$. Let us now show $\Delta_2(a, b, c) \geq 0$. If $w(b, c, a) \geq 1$, then $Q(b, c, a) = (c + 2)/4$ and we have

$$\Delta_2(a, b, c) = q(a, c, b) - \frac{c+2}{4} = \frac{(P_2(a, b, c))^2}{32b(4 - ac)} \geq 0.$$

If $w(b, c, a) < 1$ (note that we always have $w(b, c, a) \geq 0$), with (14) we have $Q(b, c, a) = q(b, c, a)$ and hence

$$\Delta_2(a, b, c) = q(a, c, b) - q(b, c, a) = \frac{(b - a)d_2(a, b, c)}{8ab(4 - ac)(4 - bc)},$$

where

$$d_2(a, b, c) \equiv -4a^2 - 4ab - 4b^2 + 8ac + 8bc \\ + a^2bc + ab^2c - 4c^2 - 2abc^2,$$

In Sect. A.1 of the Appendix we show that the minimum of $d_2(a, b, c)$ over the intersection of S and the set $\{(a, b, c) : w(b, c, a) \leq 1\}$ is 0. Hence $\Delta_2(a, b, c) \geq 0$. □

Remarks As we pointed out at the beginning of Sect. 4.1, the result in Theorem 1 is robust for θ_0 (the *a priori* probability of A) values around $1/2$. However, if θ_0 deviates significantly from $1/2$, the result in Theorem 1 is no longer true. As the following two examples show, unique optimality is achieved when the casting voter is either of the lowest (first example) or highest (second example) ability: If $\theta_0 = 7/10$ and $(a, b, c) = (2/5, 3/5, 4/5)$, then

$$Q(\theta_0, b, c, a) > 0.76 > Q(\theta_0, c, a, b) > 0.75 > Q(\theta_0, a, b, c).$$

On the other hand, if $\theta_0 = 4/5$ and $(a, b, c) = (1/5, 2/5, 3/5)$, then

$$Q(\theta_0, b, c, a) = Q(\theta_0, c, a, b) = 4/5 < 33/40 = Q(\theta_0, a, b, c).$$

5.1 Minimizing expected cost

Let us consider an alternative voting goal. Instead of maximizing the reliability, suppose we wish to minimize the expected cost of making both types of error: (I) verdict B when Nature is A (e.g., acquittal of a guilty defendant); and (II) verdict A when Nature is B (e.g., conviction of an innocent defendant). Similar to equation (12), we can calculate the probability of making either type of error, $\Pr[B/A]$ or $\Pr[A/B]$. Recall that in Sect. 4.4 we defined q_A and q_B as the probability of correct verdict when Nature is in state A and B, respectively. Therefore,

$$\Pr[B/A] = 1 - q_A, \quad \text{and} \quad \Pr[A/B] = 1 - q_B.$$

Let $k_1, k_2 \geq 0$ denote the cost of type-I and type-II error, respectively. Then the total expected cost of making both types of error under honest voting with voting order (a, b, c) (casting vote to c) is given by

$$K(a,b,c) = k_1 \Pr[B/A] + k_2 \Pr[A/B]$$
$$= k_1(1 - q_A(a,b,c)) + k_2(1 - q_B(a,b,c)) = (k_1 + k_2)(1 - q(a,b,c)),$$

where the last equality is due to (11), with which we have shown the following.

Proposition 1 *For a three-member jury under honest voting, the voting order with maximum reliability also minimizes the expected cost of incorrect verdict, making both types of error.*

5.2 Choosing referees for two articles

We have already mentioned the "jury" consisting of an editor and two honest referees in the context of a casting-vote scheme for evaluating an article submitted to an academic journal. Now assume that the editor is a fixed agent of ability e who has to assign four referees of abilities $a \le b \le c \le d \le 1$ to two articles (two to each). For simplicity and practicality, we assume $a \ge \ell = 3 - \sqrt{7} \approx 0.354$ (chosen referees are supposed to be experts in the relevant area of expertise). There are three ways to do this, with $\{a, b\}$, $\{a, c\}$ or $\{a, d\}$ as the abilities of one refereeing team. The editor's aim is to maximize the expected number of articles that receive a correct evaluation. According to (15), the three values of this expectation are given by

$$E_1 = q(a,b,e) + q(c,d,e),$$
$$E_2 = q(a,c,e) + q(b,d,e),$$
$$E_3 = q(a,d,e) + q(b,c,e),$$

where function $q(\cdot,\cdot,\cdot)$ is given by (13). It turns out that the partition of referee abilities into middles $\{b, c\}$ and extremes $\{a, d\}$ produces the highest expectation E_3.

Proposition 2 *For any referee abilities $\ell \le a < b < c < d \le 1$ and any editor ability $\ell \le e \le 1$ we have*

$$E_3 > \max\{E_1, E_2\}.$$

That is, to maximize expected number of correctly evaluated articles, the extreme-ability referees should be assigned to one article and the middle-ability referees to the other.

Proof Let $T = \{(a,b,c,d,e) : \ell \le a < b < c < d \le 1, \ell \le e \le 1\}$. We first show that $E_3 - E_2 > 0$ on T. Note that

$$E_3 - E_2 = q(a,d,e) + q(b,c,e) - (q(a,c,e) + q(b,d,e))$$
$$= (q(b,c,e) - q(a,c,e)) - (q(b,d,e) - q(a,d,e)).$$

Hence it suffices to show that the incremental reliability of improving one of the early voters (e.g., $q(b,x,e) - q(a,x,e)$) is a strictly decreasing function of the ability x of the other early voter. More precisely, we have

$$\begin{aligned}E_3 - E_2 &= \int_{x=a}^{b} \frac{\partial q(x,c,e)}{\partial x} dx - \int_{x=a}^{b} \frac{\partial q(x,d,e)}{\partial x} dx \\ &= \int_{x=a}^{b} \left(\frac{\partial q(x,c,e)}{\partial x} - \frac{\partial q(x,d,e)}{\partial x} \right) dx \\ &= - \int_{x=a}^{b} \int_{y=c}^{d} \frac{\partial^2 q(x,y,e)}{\partial x\, \partial y}\, dx\, dy,\end{aligned}$$

which is positive if

$$\frac{\partial^2 q(x,y,z)}{\partial x\, \partial y} < 0 \tag{20}$$

for any $(x, y, z) \in T$. In fact, according to (13) we have

$$\frac{\partial^2 q(x,y,z)}{\partial x\, \partial y} = \frac{N(x,y,z)}{32(xy-4)^3 z}, \tag{21}$$

where

$$\begin{aligned}N(x,y,z) = 128 &- x^3y^3z^2 + 4x^3y + 12x^2y^2z^2 - 48x^2 + 4xy^3 \\ &- 48xyz^2 + 32xy - 48y^2 + 64z^2.\end{aligned}$$

Since the denominator in (21) is negative and the numerator

$$\begin{aligned}N(x,y,z) &= 128 + z^2(-x^3y^2 + 12x^2y^2 - 48xy + 64) \\ &\quad + 4xy(8 + x^2 + y^2) - 48(x^2 + y^2) \\ &\geq 128 + (64 - 48 - 1)z^2 - 2 \times 48 \geq 128 - 2 \times 48 = 72,\end{aligned}$$

we have shown that (20) holds and hence $E_3 > E_2$ on T. Similarly,

$$\begin{aligned}E_3 - E_1 &= (q(a,d,e) + q(b,c,e)) - (q(a,b,e) + q(c,d,e)) \\ &= (q(d,a,e) - q(b,a,e)) - (q(d,c,e) - q(b,c,e)) \\ &= - \int_{x=b}^{d} \int_{y=a}^{c} \frac{\partial^2 q(x,y,e)}{\partial x\, \partial y}\, dx\, dy,\end{aligned}$$

which is positive according to (20). Hence $E_3 - E_1 > 0$ on T. □

We can easily extend the above result to the problem faced by an editor of ability e who has m articles to evaluate and a pool of $n = 2m$ referees of abilities $\ell \leq a_1 < \cdots < a_n \leq 1$. (If there are more than $2m$ potential referees, clearly it is best to ignore those not in the top $2m$ abilities.) Any refereeing pairing is determined by a permutation π of $N = \{1, 2, \ldots, n\}$ of order 2 (i.e., it satisfies the idempotent equation $\pi^2 = I$, where I is the identical permutation), so that the referee of ability a_i (or we simply say referee i) is paired with the one of ability $a_{\pi(i)}$. For a pairing π, the expected number of correctly evaluated articles is given by

$$\mathbb{E}(\pi) = \frac{1}{2}\sum_{i=1}^{n} q(a_i, a_{\pi(i)}, e), \tag{22}$$

where the factor $1/2$ is needed because each article appears twice in the sum.

Corollary 1 *When an editor of ability e ($\ell \le e \le 1$) has $n = 2m$ referees of abilities $\ell \le a_1 < \cdots < a_n \le 1$ and m articles to be reviewed, the unique referee pairing π that maximizes the expected number $\mathbb{E}(\pi)$ of correctly evaluated articles is given by $\hat{\pi}(i) = n-i+1$, with the associated referee pairings $\{1, n\}, \{2, n-1\}, \ldots, \{m, m+1\}$.*

Proof Since this is a finite problem, it has an optimal referee pairing π. Suppose $\pi \ne \hat{\pi}$. Then there is a minimum j ($1 \le j \le m$) such that $\pi(j) \ne \hat{\pi}(j)$. Hence $\pi(i) = \hat{\pi}(i) = n-i+1$ for $i = 1, \ldots, j-1$, which implies that all indices larger than $\hat{\pi}(j)$ are paired with those smaller than j, which in turn implies that $\pi(j) < \hat{\pi}(j)$. The same reasoning concludes that $\hat{\pi}(j) = \pi(k)$, for some k with $j < k < \hat{\pi}(j)$. Thus we have four indices $j, \pi(j), \hat{\pi}(j) = \pi(k)$ and k, with $j < \pi(j) < \hat{\pi}(j)$ and $j < k < \hat{\pi}(j)$. Of these four indices, the extreme indices (corresponding to the extreme abilities) are j and $\hat{\pi}(j)$. Since these two are not paired by π, it follows from Proposition 2 that the sum of the two terms in (22) (or four terms, counting duplication) containing the four relevant indices can be improved by re-pairing these four indices as $\{j, \hat{\pi}(j)\}$ and $\{\pi(j), k\}$. Thus the given pairing π cannot maximize $\mathbb{E}(\cdot)$, a contradiction. So the unique optimal pairing must be $\hat{\pi}$. □

One might think that our optimization objective of maximizing (22) is incompatible with the editor's moral duty. Perhaps he should adopt a referee pairing $\bar{\pi}$ that maximizes the minimum reliability $\bar{r}$ given to the evaluation of any article. We claim that our solution $\hat{\pi}$ has this maximin property. Suppose that in pairing $\hat{\pi}$ the article with least reliability of evaluation is refereed by referees k and $\hat{\pi}(k)$. So the minimum reliability of our pairing $\hat{\pi}$ is $q(a_k, a_{\hat{\pi}(k)}, e) = \hat{r}$. Suppose that in $\bar{\pi}$ referee k is paired with referee $j = \bar{\pi}(k)$. If $j \le \hat{\pi}(k)$ then $q(a_k, a_j, e) \le q(a_k, a_{\hat{\pi}(k)}, e) = \hat{r}$ by monotonicity of q, indicating that the minimum reliability $\bar{r}$ corresponding to pairing $\bar{\pi}$ cannot be higher than $\hat{r}$, and we are done. So we can assume $j > \hat{\pi}(k)$. It follows that in $\bar{\pi}$ one of the referees $i = 1, \ldots, k-1$, say j, must be paired with a referee $l = \bar{\pi}(j)$ with $l \le \hat{\pi}(k)$, since these $k-1$ referees cannot all be paired with the $k-2$ referees $\{\hat{\pi}(k) + 1 = n - k + 2, \ldots, n\}\backslash\{j\}$. Hence we now have $\bar{r} \le q(a_j, a_l, e) \le q(a_k, a_{\hat{\pi}(k)}, e) = \hat{r}$ because $j < k$ and $l \le \hat{\pi}(k)$. It follows that $\bar{r} \le \hat{r}$.

For a different perspective on the reliability of the refereeing process relating to information that should be available to referees, see Ben-Yashar and Nitzan (2001).

6 Larger juries with uniformly distributed abilities

Our main result of Theorem 1 has been established algebraically only for juries of size three. Such analysis seems out of reach for larger juries with arbitrary sets of

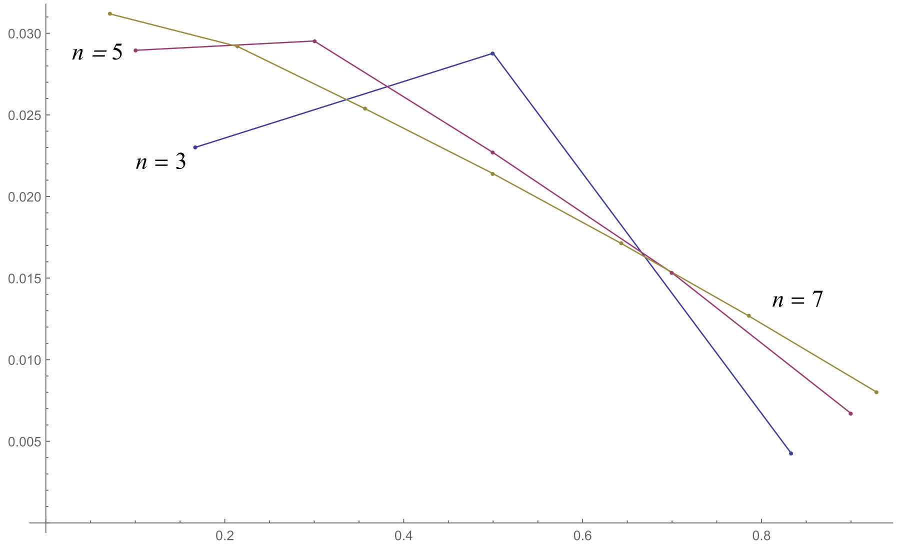

Fig. 4 Incremental reliability as a function of casting voter ability

abilities due to the complexity $\mathcal{C}(n)$ discussed in Sect. 3.2. However, if we take a jury of size n with uniformly distributed abilities, we can determine numerically which juror should be given the casting vote to maximize reliability under honest voting. We divide the ability interval [0, 1] into n subintervals of length $1/n$ and give one juror i the ability of the midpoint of the ith interval, so that $a_i = (2i - 1)/(2n)$ for the ith juror in the jury of size n. As an example, when $n = 5$, the abilities of the five jurors are 0.1, 0.3, 0.5, 0.7 and 0.9. For each jury of size n, let $\bar{Q}_n$ denote the reliability of simultaneous voting and $Q_n[i]$ denote the reliability under the casting-vote scheme with the casting vote given to the ith juror, the one of ability a_i. We then define the non-negative quantities

$$\delta(n, a_i) = Q_n[i] - \bar{Q}_n$$

as the incremental reliability of casting voting. It turns out that calculating $\delta(n, a_i)$ is easier than calculating $Q_n[i]$ directly. For fixed n, the reliability of giving the casting vote to juror i is maximized when $\delta(n, a_i)$ is maximized over a_i. Figure 4 plots for $n = 3, 5, 7$ the incremental reliability $\delta(n, a_i)$ when the casting vote on the jury of size n is given to the juror of ability a_i, $i = 1, \ldots, n$. For each n, the plotted points are connected by straight lines to make the plot easier to read.

The curve for $n = 3$ has three plot points at abilities $1/6$, $1/2$ and $5/6$. As known from Theorem 1, the highest value will be for the median ability $1/2$, as shown clearly in the curve for $n = 3$. For the jury of size $n = 5$, the abilities of the jurors are 0.1, 0.3, 0.5, 0.7, 0.9, and the incremental reliability (and hence the absolute reliability) is maximized when the second lowest ability juror (ability 0.3) has the casting vote. For $n = 7$, reliability is maximized when the juror of lowest ability is given the casting vote. The pattern for $n = 7$ is continued for larger juries, as shown in

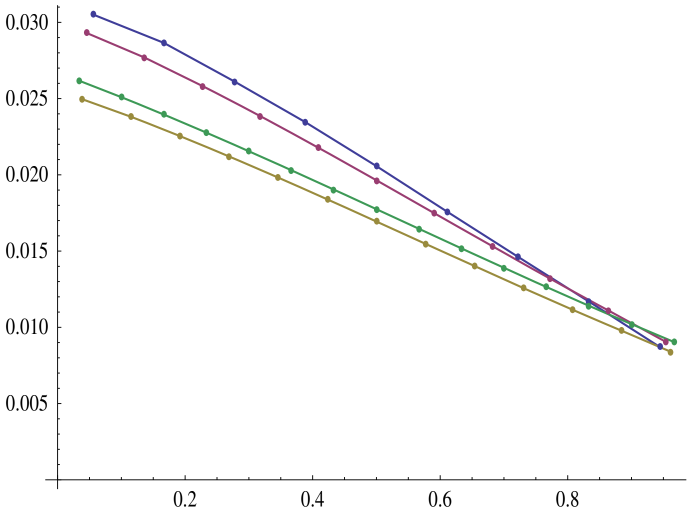

Fig. 5 Plots of $\delta(n,\cdot)$, $n = 9, 11, 13, 15$

Fig. 5 for juries of size $n = 9, 11, 13, 15$, where incremental (or absolute) reliability is decreasing in the ability of the casting voter. To distinguish between the curves for different values of n, note that at their left points, the curves are $n = 9, 11, 15, 13$, counting from the top. Also observe that these figures are not useful for comparing reliability of different size juries, as they have different base points $\bar{Q}_n$. The idea that larger juries have higher reliability goes back to Condorcet, but that is not our point of discussion here.

The mathematical analysis required to calculate the incremental reliabilities is presented in Sect. A.3 of the Appendix.

7 Conclusions

In this paper, we have formally introduced sequential voting schemes as directed acyclic graphs, where the voters are represented by nodes and a voter's knowledge of another voter's (prior) vote is indicated by an arc from the latter to the former. Since we are interested in majority verdict of a jury of an odd size, the simplest nontrivial case is that of a jury of three. For such juries there are essentially three sequential voting schemes: secret ballot (equivalent to simultaneous voting), roll-call voting and what we call the casting-vote scheme, where the first two jurors announce their votes simultaneously and then the third (casting voter) determines the majority verdict in the case of a tie. We define a notion of complexity of a scheme, in which roll-call voting has complexity five and the casting-vote scheme has complexity four. This relates to the difficulty of algebraically determining the optimal (most reliable) voting order, for a fixed jury of heterogeneous abilities. For this reason, we have succeeded here in giving a complete algebraic solution to the voting order problem for the casting-vote scheme. We have shown algebraically that for any fixed abilities of a three-member honest jury, giving the casting vote to the juror of median ability maximizes the reliability of the majority verdict (i.e., the probability that the verdict is correct). For larger juries of uniformly distributed abilities, simulations have suggested that it is best to give the casting vote to the least able juror and it is worst to give the casting vote to the most able juror, results that are in stark contrast to the conventional wisdom (and practice) of giving the casting vote to the most able juror.

We believe our methods can be extended to solve the voting order problem for other sequential voting schemes, particularly roll-call voting. An important role in our technique is to model private signals and individual abilities not by binary variable but by real valued parameters in an interval.

In summary, as the main contribution of this paper, we have established that, for a three-member jury, it is best to give the median-ability juror the casting vote. However, this result does not hold for larger juries or for asymmetric binary alternatives. This paper considers only majority verdicts, as with the original Condorcet model. However, it raises an important question as to the comparative reliability of other decision rules such an requiring unanimous assent for a particular alternative or giving the last voter the deciding vote even when both early voters are in agreement. The question of optimal voting order for these other rules is also of interest.

Appendix

A.1: Proof of $d_2(a, b, c) \geq 0$ in Theorem 1

For minimization of $d_2(a, b, c)$ subject to $(a, b, c) \in S$ and $P_2(a, b, c) \leq 0$, the Kuhn-Tucker conditions for potential minimizers are as follows:

$$
\begin{aligned}
&2abc - 8a + b^2c - 2bc^2 + \lambda_1(bc - 4) - 4b + 8c - \lambda_2 + \lambda_3 = 0,\\
&a^2c + 2abc - 2ac^2 + \lambda_1(ac - 2) - 4a - 8b + 8c - \lambda_3 + \lambda_4 = 0,\\
&a^2b + ab^2 - 4abc + \lambda_1(ab + 2) + 8a + 8b - 8c - \lambda_4 + \lambda_5 = 0,\\
&\lambda_1(abc - 4a - 2b + 2c) = 0,\\
&a\lambda_2 = 0,\ \ \lambda_3(b - a) = 0,\ \ \lambda_4(c - b) = 0,\ \ (1 - c)\lambda_5 = 0.\\
&\lambda_1, \dots, \lambda_5 \geq 0;\ \ (a, b, c) \in S \text{ and } P_2(a, b, c) \leq 0.
\end{aligned}
$$

All solutions $(a, b, c, \lambda_1, \dots, \lambda_5)$ of the above system projected to the (a, b, c) space form the following set $S_{\min}$:

$$
\begin{aligned}
S_{\min} = \{(0, b, b) : 0 \leq b \leq 1\} \cup \{(a_\lambda, a_\lambda, a_\lambda) : 0 < \lambda \leq 3\}\\
\cup\{(a, a, a/(a^2 + 2)) : 0 < a \leq 3 - \sqrt{7}\}
\end{aligned}
$$

where a_λ is the middle root of equation $a(4-a^2)=\lambda$. Since $d_2(a,b,c)=0$ for any $(a,b,c)\in S_{\min}$, all elements of $S_{\min}$ are minimizers of $d_2(a,b,c)$.

A.2: Proof of $d_3(a, b, c) \geq 0$ in Theorem 1

For minimization of $d_3(a,b,c)$ subject to $(a,b,c)\in S$ and $c-2b\geq 0$, the Kuhn-Tucker conditions for potential minimizers are as follows:

$$\begin{aligned}
&-2ab^2c^2+8abc-8a+4b^2c+8b-16c-\lambda_2+\lambda_3=0,\\
&-2a^2bc^2+4a^2c+8abc+8a-8b-16c+2\lambda_1-\lambda_3=0,\\
&-2a^2b^2c+4a^2b+4ab^2-16a-16b+32c-\lambda_1+\lambda_4=0,\\
&\lambda_1(c-2b)=0,\ a\lambda_2=0,\ \lambda_3(b-a)=0,\ (1-c)\lambda_4=0.\\
&\lambda_1,\ldots,\lambda_4\geq 0;\ (a,b,c)\in S \text{ and } c-2b\geq 0.
\end{aligned}$$

The above system has a unique solution of $(0,0,0,0,0,0,0)$ and $d_3(0,0,0)=0$. Hence we have $d_3(a,b,c)\geq 0$.

A.3: Analysis of large juries for Sect. 6

Here, we give the analysis of the casting-vote scheme for an arbitrary odd number of jurors which is used to derive Figs. 4 and 5 in Sect. 6. Suppose we have $n=2m+1$ jurors with abilities, in nondecreasing order, given as the n-vector $\mathbf{a}=(a_1,a_2,\ldots,a_n)$. Let $N=\{1,\ldots,n\}$ and $N^i=N\backslash\{i\}$. If juror i has the casting vote, then the jurors N^i who vote first are ordered as the $(n-1)$-vector $\mathbf{a}^i=(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$. Let $\mathcal{S}^i$ denote the set of all m-subsets of N^i. A set $S\in\mathcal{S}^i$ can be interpreted as the set of jurors who vote for alternative A in the first round when there is a tie vote and juror i has the casting vote. The conditional probability of A in this case is denoted θ_S. If $\theta_S=1/2$ then $Q[i]=\bar{Q}$, where $\bar{Q}$ is the reliability of simultaneous voting (with abilities $\mathbf{a}$) and $Q[i]$ is the reliability of casting voting where juror i has the casting vote. Of course, we have $Q[i]\geq\bar{Q}$ for all $i\in N$. If $\theta_S>1/2$ then for negative signals close to 0, the casting voter i will still vote for A. The condition $\theta_S>1/2$ says roughly that those m jurors who voted for A have collectively stronger abilities than those m jurors who voted for B. For any $j\in S$, let $r_j=1-F(a_j,0)$ be the individual reliability of juror j, the probability that he gets a positive signal and hence votes A when A is the state of Nature (or the probability that juror j gets a negative signal given B). The probability that he gets a negative signal given A (or a positive signal given B) is $F(a_j,0)=1-r_j$. Therefore, we have

$$\theta_S=\frac{\prod\limits_{j\in S} r_j \prod\limits_{k\in N^i\backslash S}(1-r_k)}{\prod\limits_{j\in S} r_j \prod\limits_{k\in N^i\backslash S}(1-r_k)+\prod\limits_{j\in S}(1-r_j)\prod\limits_{k\in N^i\backslash S} r_k}. \tag{23}$$

Consequently, the honest threshold for casting voter i, given S, is

$$\tau_i(S) = \left.\frac{1-2\theta_S}{a_i}\right|_{[-1,1]}, \tag{24}$$

where $z|_{[-1,1]}$ denotes the projection of z onto $[-1, 1]$. We can also calculate the probability that those voting A in the first round constitute a particular set $S \in \mathcal{S}^i$, given Nature is A:

$$\Pr(S/A) = \prod_{j\in S} r_j \prod_{k\in N^i\setminus S} (1-r_k), \text{ for } S \in \mathcal{S}^i. \tag{25}$$

To evaluate $Q[i] - \bar{Q}$, we see that the verdict with casting voter i will be different from that of simultaneous voting only if both of the following two conditions hold: (i) a tie vote (i.e., those voting A, $\{j : s_j > 0\}$, form a set $S \in \mathcal{S}^i$), and (ii) small signal for casting voter (i.e.,juror i gets a signal s_i between his threshold $\tau_i(S)$ and 0). Taking equiprobable alternatives $\theta_0 = 1/2$ with juror i as the casting voter, let $S = \{j \in N^i : s_j > 0\}$, those who vote A in the first round. Then the verdict is A if either $|S| > m$ or $|S| = m$ and $s_i \geq \tau_i(S)$. Therefore, we have the following formula for the reliability of voting with casting voter i:

$$Q[i] = \sum_{S\subset N^i,\,|S|\geq m+1} \Pr(S/A) + \sum_{S\subset N^i,\,|S|=m} \Pr(S/A)(1 - F(a_i, \tau_i(S))).$$

Similarly, for simultaneous voting we can separate out the voting of juror i to obtain the asymmetric formula for the reliability $\bar{Q}$:

$$\bar{Q} = \sum_{S\subset N^i,\,|S|\geq m+1} \Pr(S/A) + \sum_{S\subset N^i,\,|S|=m} \Pr(S/A)(1 - F(a_i, 0)).$$

The difference between the above two formulae is that the latter involves only individual reliabilities, whereas the former takes into account linear density functions on the full signal distribution of the casting voter. In particular we have (noting that the conditions $S \subset N^i$ and $|S| = m$ are the same as $S \in \mathcal{S}^i$):

$$\begin{aligned} Q[i] - \bar{Q} &= \sum_{S\in\mathcal{S}^i} \Pr(S/A)((1 - F(a_i, \tau_i)) - (1 - F(a_i, 0))) \\ &= \sum_{S\in\mathcal{S}^i} \Pr(S/A)(F(a_i, 0) - F(a_i, \tau_i(S))). \end{aligned} \tag{26}$$

In the case of $n = 3$ with $i = 3$ having the casting vote and $a_1 = a$, $a_2 = b$, $a_3 = c$, there are two sets in $\mathcal{S}^3$, namely $\{1\}$ (which is voting pattern AB) and $\{2\}$ (which is BA). There is one set $S \subseteq \{1, 2\}$ with $|S| = m + 1 = 2$, namely $\{1, 2\}$. So evaluating the general formula for $Q[i] = Q[3]$ with the terms S in order $\{1, 2\}$, $\{1\}$, $\{2\}$ gives (with $\tau_3(\{1\}) = z_{AB}$)

$$\begin{aligned} Q(a,b,c) = & (1-F(a,0))(1-F(b,0)) \\ & + [(1-F(a,0))(F(b,0))](1-F(c,z_{AB})) \\ & + ((F(a,0))(1-F(b,0)))(1-F(c,z_{BA})) \end{aligned}$$

Note that the advantage of formula (26) is that we have fewer terms to evaluate to see which is the best juror i to have the casting vote.

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