Intrinsic Variability in Group and Individual Decision-Making[1]

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Abstract

The paper examines the random preference model, which can explain inherent variability of preferences in managerial and individual decision-making, and provides axiomatizations for the utility components of two such models differentiated by the structure of core preferences: expected utility and betweenness-like preferences. We then examine the possibility of violations of weak stochastic transitivity for these models and for a model with core dual-EU preferences. Such violations correspond to the existence of Condorcet cycles and, therefore, the analysis has implications for managerial decision-making and for majority rule voting. The paper also investigates implications of its findings for two popular experimental settings.

1 Introduction

Transitivity of preferences is at the core of most models of decision-making. It is also frequently cast in normative light on the grounds that decision-makers with transitive preferences, whether they are individuals or organizations, are immune to money pumps (Danan 2010). Yet it is common to encounter a pattern of choices that may suggest intransitive behavior (Tversky 1969). For example, three pairwise choices may exhibit a pattern where alternative x is chosen over alternative y once, alternative y is chosen over alternative z once, yet alternative z is chosen over alternative x once. Groups of individuals, such as management teams and boards of directors, that use majority voting to choose between pairs of alternatives, may also be prone to making intransitive choices (Condorcet, 1785).

At the same time, both individual and group choices exhibit substantial degree of variability. On the individual level, there exists an abundant experimental evidence of decision-makers who choose a given lottery A over a certain sum x while, not much later, choose to accept a *smaller* sum y over the same lottery A. Moreover, in retrospect, decision-makers are often not bothered by these seemingly contradicting choices. The observation that decision-makers frequently act differently on similar occasions of choices, even when faced with conditions that are deliberately crafted to be identical, has focused some of the economics and psychology literature on finding explanations for this 'within-subject' variability of choices (Hey, 1995, Otter et al., 2008). Such variability naturally arises when the decision-making entity is comprised of several individuals, as in the case of decision-making by management teams and social choice problems (see the illustrative example in Section 2). This suggests that in order to accommodate the observed variability that is exhibited by decision-makers over time, contexts, and occasions, the assumption that choices are deterministic must be relaxed.

Several competing theories have been proposed to explain non-deterministic behavior. In this paper we focus on the *random preference model* (Luce and Suppes 1965; see also Loomes and Sugden 1995) because it frequently outperforms the other models and because it is considered to be one of the most promising among them (see, e.g., Regenwetter, Dana and

Davis-Stober 2011, Cavagnaro and Davis-Stober 2013).[1] In the random preference model, a decision-maker has a set of deterministic transitive preference orders, called *core preferences*, and on each choice occasion the selection of an alternative is based on a preference order that may seem to be drawn according to some probability distribution over the set of core preferences. For management teams, the deterministic preference relations represent the various team members. For individual decision-makers, they may correspond to different states of mind or reflect various parameters that are hidden from an outside observer. When a decision-maker weighs up various attributes of a choice problem, it may seem as if her decision-making pendulum swings in one direction on one occasion and in the other direction on another occasion.[2] Thus, the random preference approach allows for certain variation in a decision-maker's evaluation of different alternatives. These variations may entail seemingly intransitive behavior even though all of the core preferences are transitive.

The existing literature provides very little axiomatic footing for the random preference model. This limits the practical exports of the model by hindering hypothesis generation and further development of theory. We fill this gap in the literature by providing axiomatizations for the utility components of two classes of models with multiple core preferences. The two classes are differentiated by the structure of core preferences. The first class of models entails core preferences that have an expected utility form. These models are the most common ones in this literature. The second class entails a more general set of core preferences satisfying the betweenness property (Chew 1989 and Dekel 1986). The need for this extension arises from consistently observed violations of the independence axiom, which is the corner stone of all expected utility models. In addition to these two classes of preference structures, we analyze random preference models where core preferences have dual expected utility form (see Quiggin 1982 and Yaari 1987), which is particularly useful for analyzing portfolio choice problems.

The framework developed here is directly applicable to the model of Csaszar and Eggers (2013) which compares three decision-making mechanisms frequently used by management

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[1] Regenwetter et al. (2011) utilize the term mixture model instead of the random preference model.

[2] There are several alternative interpretations of random choice models including limited cognitive ability and limited attention.

teams. They study the performance of majority voting, delegation, and averaging of opinions in a dynamic model with differential flows of information across team members. All of the team members in their model have the same preferences but different knowledge base. In our framework, the reverse holds but the tools developed here can be easily implemented in their framework. Furthermore, the delegation procedure in Csaszar and Eggers (2013) is in exact correspondence to the model developed in the present paper.

To empirically isolate systematic violations of transitivity and, at the same time, to account for the intrinsic variability in choice behavior, the literature has put forth several probabilistic analogues of transitivity. In this paper we study weak stochastic transitivity (WST) (Vail 1953, Davidson and Marschak 1959), which has a prominent role among these analogues. Assume there exists a probability distribution ψ over the set of core preferences and let $\psi(X \succ Y)$ denote the probability that the preference chosen ranks X over Y (see Section 4). Then WST requires that, if $\psi(X \succ Y) \geq 0.5$ and $\psi(Y \succ Z) \geq 0.5$, then $\psi(X \succ Z) \geq 0.5$. Thus, this condition characterizes preferences that are transitive in probabilistic sense.

While it is widely acknowledged that the random preference model can violate WST (see, e.g., Fishburn 1999, Regenwetter et al. 2011), very little is known about the domain restrictions for choice alternatives and core preferences that lead to satisfaction of WST. We contribute to this literature by relating potential violations of WST to the "commonality" of core preferences and "dimensionality" of the choice problem. We show that WST is always satisfied when the set of possible outcomes does not exceed three and all core preferences rank the basic outcomes similarly or share the same risk attitude. When the outcomes are real numbers, this occurs when all core preferences are monotonic with respect to first-order stochastic dominance or are either all risk averse or all risk loving. We also demonstrate that when the set of possible outcomes is greater than three, violations of WST are possible even if one assumes both types of commonality of core preferences.

Although this dimensionality restriction may seem overly restrictive, it is pertinent for a number of popular experiments. We demonstrate it by examining an experiment that consists of a sequence of 2-alternative forced choices where for each element of the sequence a decision-maker chooses between a fixed binary lottery and some certain amount of money

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(Cohen, Jaffray and Said 1987, Holt and Laury 2002). Although on surface this environment may seem not to meet our dimensionality restriction, it is effectively two-dimensional and, hence, WST is satisfied by the random preference model because one of our commonality conditions is satisfied.

In contrast, experiments used to elicit the preference reversal phenomenon (Lichtenstein and Slovic 1971, Lindman 1971, Grether and Plott 1979) violate the dimensionality restriction. We find that in this environment the decision-maker modeled in this paper can violate WST which is consistent with the preference reversals frequently observed in this type of experiments. We should note, however, that violations of WST are rare in some other experimental settings (see, e.g., the discussion in Rieskamp et al. 2006) which was one of the reasons we set out to explore forces that may lead to satisfaction of WST.

There is another practical benefit from identifying conditions under which WST is expected to hold for the random preference model. Violations of WST are problematic for elicitation of net benefits to inform various organizational policies because they may lead to systematic cyclic choices. If cycles are likely, then policy prescriptions may be sensitive to the specifics of an elicitation procedure including the sequence of choices made during the procedure and whether different policies are compared directly or indirectly, e.g. through their elicited certainty equivalents.

Our interest in WST also stems from its central role in collective choice. When a group of individuals uses Condorcet's procedure, i.e., a sequential choice between pairs of alternatives via majority voting, WST is equivalent to the absence of Condorcet cycles.[3] If a cycle exists then, for example, the member of a management team that sets the agenda for sequentially discarding alternatives via majority voting will be able to induce any alternative in the cycle as the overall winner of Condorcet's procedure. Our dimensionality and commonality conditions illuminate when such agenda setting can be avoided. These conditions are different from the existing conditions, such as single-peakedness, value restriction, and net value restriction, that ensure transitivity of collective choice using Condorcet's procedure (see, e.g., Gehrlein, 1981, 1997, 2002; Gehrlein and Fishburn, 1980; Gehrlein and Lepelley, 1997;

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[3]A Condorcet cycle materializes when a majority of the voters choose alternative A over B, B over C, but C over A.

Mueller, 2003; Riker, 1982; Sen, 1969, 1970, 1999; Tangian, 2000).

We structure the paper as follows. We begin by presenting an illustrative example which is used to motivate our modeling approach and to demonstrate our findings in a later section of the paper. Then, we introduce the framework and present the two representation theorems. We proceed to explore the implications of commonality of core preferences for violations of WST. We then derive implications of our analysis for two experimental settings. After providing an analysis of selected cases of core non-EU preferences, we conclude with some final remarks.

2   Illustrative Example

Consider the following choice problem faced by a team of top managers of a company producing electronic tablets and mobile phones. The choice problem pertains to an allocation of a fixed advertising budget between the firm's two product lines. Suppose, for concreteness sake, the team makes this decision on a monthly basis, the monthly advertising budget is US$30m, and every month the team chooses between two of the following three options.[4] Under option A, $20m is spent on advertising the tablets while the rest is spent on phone ads. Option B is characterized by an equal expenditure on ads for the two product lines. Finally, under option C, $13m is spent on tablet ads and $17m on phone ads.

Naturally, each of these three options involves a considerable level of uncertainty. It is hard to envision a scenario where the management team can perfectly forecast whether an ad will work and, more generally, what the precise effect of different advertisement expenditures will be. Thus, the selection process is akin to a choice from a set of lotteries.

To demonstrate, suppose that for each product type there are two possible changes in that product's revenue net of all costs except for advertising. The feasible changes in the revenue for the tablets are $14m and $22m. For the phones, the feasible changes in the revenue are $12m and $20m. Thus, there are four possible contingencies, $y_1 =$($14m,$12m), $y_2 =$($14m,$20m), $y_3 =$($22m,$12m) and $y_4 =$($22m,$20m). Which of these four contingen-

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[4]We assume away learning that might take place between different occasions of the choice problem and other forms of history dependence.

cies materializes is uncertain. Each of the three options $i = A, B, C$ results in some distinct probability distribution $p^i = (p_1^i, p_2^i, p_3^i, p_4^i)$ over these contingencies, when the corresponding advertising costs are subtracted. Thus, choosing option $i = A, B, C$ is equivalent to choosing one of the following gambles over the changes in the profits for the two product types:

Option A: $\big((\text{-\$6m},\text{\$2m}), p_1^A; (\text{-\$6m},\text{\$10m}), p_2^A; (\text{\$2m},\text{\$2m}), p_3^A; (\text{\$2m},\text{\$10m}), p_4^A\big)$

Option B: $\big((\text{-\$1m},\text{-\$3m}), p_1^B; (\text{-\$1m},\text{\$5m}), p_2^B; (\text{\$7m},\text{-\$3m}), p_3^B; (\text{\$7m},\text{\$5m}), p_4^B\big)$ (1)

Option C: $\big((\text{\$1m},\text{-\$5m}), p_1^C; (\text{\$1m},\text{\$3m}), p_2^C; (\text{\$9m},\text{-\$5m}), p_3^C; (\text{\$9m},\text{\$3m}), p_4^C\big)$

where, for example, the first element under option B corresponds to $y_1 - (\text{\$15m},\text{\$15m})$.

The team may have different rankings of the three alternatives on different occasions of the choice problem. A plethora of characteristics unobservable to an outside observer may contribute to the attractiveness of different options to the team. In addition to uncertainties surrounding the team's deliberation process on each occasion, the unobservable characteristics may include market conditions, behavior of competitors, and new innovations that occur over the period of repeated decision-making. For simplicity of interpretation, imagine a scenario where this information pertains to "exogenous" factors, such as long-run market share, image, and reputation, rather than probabilities of different outcomes and associated net profits. This information may change the ranking of the three options even without affecting the way they are seen by an outside observer (that is, the representations that appear in (1)). Suppose that, on certain occasions of binary choice, the team may possess information that favors spending most of the budget on advertising the tablets. In this case, which is called core preference ABC, the team strictly prefers option A to B to C. On other occasions, the team may have information suggesting that equally sharing the budget is the best option and that spending on phone ads is a strictly better investment than spending on tablet ads. In this case, which is called core preference BCA, the team strictly prefers option B to C to A. Note that both preference rankings ABC and BCA are transitive. Suppose that on each choice occasion the team's core preference is drawn according to distribution $\psi(\cdot)$ whose support is given by the core preferences ABC and BCA.

Consider now the following sequence of pairwise choices by the team. When choosing between options A and B, core preference ABC is realized and, as a result, the team picks option A. On a different occasion, when the team's choice set is comprised of options B

and C, the realization of the core preference is again ABC and, consequently, the team chooses option B. Finally, when the team chooses between options A and C, the realization of the core preference is BCA and, hence, the team's choice is C. Formally, this sequence of decisions exhibits non-transitivity or, in other words, forms a cycle: option A is chosen over B, B is chosen over C, but C is chosen over A. This is in spite of the fact that the team chooses according to a transitive preference ranking on each choice occasion.

When the team uses the Condorcet procedure to choose among the three alternatives, WST is equivalent to the requirement of transitive collective preference of the team. After we introduce our model and present the results, we return to this section's example and the Condorcet procedure, in particular, to elucidate the implications of our formal findings.

3 Representation Theorems

We consider a finite set of n distinct outcomes $X = \{x_1, x_2, ..., x_n\}$, and the set $\mathcal{L} = \Delta(X)$ of all lotteries over it (with the induced topology of $\mathbb{R}^n$). For a lottery $p \in \mathcal{L}$ we use the notation $p_i = p(x_i)$. We consider decision-makers who, when confronted with a choice between two lotteries, must make up their mind and choose one of the two lotteries. This assumption reflects many real life situations and it is in agreement with most experimental designs. A decision-maker (DM) is represented by a binary relation $\succcurlyeq$ over $\mathcal{L}$ with the interpretation that a lottery p is related to a lottery q if there are situations in which p is chosen over q. As an example consider a DM who, when asked to make a choice between two lotteries p and q, draws one utility from the set $\{u^1, u^2, u^3\}$ (defined over X) according to some probability distribution and may choose p if the expected utility of p is not smaller than that of q for the drawn utility. That is, $p \succcurlyeq q$ if there exists $j = 1, 2, 3$ such that $\sum_{i=1}^n u^j(x_i) p_i \geq \sum_{i=1}^n u^j(x_i) q_i$.

The strict asymmetric part $\succ$ and the symmetric part $\sim$ are defined as usual: $p \succ q$ if $p \succcurlyeq q$ and $\neg(q \succcurlyeq p)$; $p \sim q$ if both $p \succcurlyeq q$ and $q \succcurlyeq p$. For the DM discussed above, $p \succ q$ if, for all u^j, $\sum_{i=1}^n u^j(x_i) p_i > \sum_{i=1}^n u^j(x_i) q_i$ and $p \sim q$ if there exist two utilities u^j, u^k (not necessarily identical) such that both $\sum_{i=1}^n u^j(x_i) p_i \geq \sum_{i=1}^n u^j(x_i) q_i$ and $\sum_{i=1}^n u^k(x_i) p_i \leq \sum_{i=1}^n u^k(x_i) q_i$ hold.

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3.1 Probabilistic DMs with core Expected Utility preferences

We start with assumptions on $\succcurlyeq$ that characterize DMs of this type.

(**A.1**) (Completeness) For all $p, q \in \mathcal{L}$ either $p \succcurlyeq q$ or $q \succcurlyeq p$.

Completeness follows from our basic requirement that a choice must always be executed. It implies the equivalence of the two relations $\neg \succcurlyeq$ and $\prec$. Note that transitivity of $\succcurlyeq$ is not assumed.

(**A.2**) (Continuity) For all $q \in \mathcal{L}$ the sets $\{p \in \mathcal{L} \,|p \succcurlyeq q\}$ and $\{p \in \mathcal{L} \,|q \succcurlyeq p\}$ are closed.

(**A.3**) (Independence) For all $p, q, r \in \mathcal{L}$ and $\alpha \in [0, 1]$,

$$p \succcurlyeq q \iff \alpha p + (1 - \alpha)\, r \succcurlyeq \alpha q + (1 - \alpha)\, r \ .$$

This is the familiar independence axiom of the Expected Utility (EU) model.

(**A.4**) (Mixture domination) For all $p, q, r \in \mathcal{L}$ and $\alpha \in [0, 1]$,

$$p \prec q \ \text{ and } \ r \prec q \ \implies \ \alpha p + (1 - \alpha)\, r \prec q \ .$$

This axiom is closely related to the Independence axiom. It requires that if the lotteries p and r are strictly worse than q, then so is the compound lottery that either yields p with probability α or r with probability $1 - \alpha$. It can be shown that given the other axioms, Mixture domination is equivalent to the transitivity of $\prec$. Our preference for (A.4) stems from its role in proving our second, and more general, representation theorem. Lehrer and Teper (2011) use another equivalent assumption to derive a similar representation result in a different framework. The role of the set of possible utilities in our paper is played by the set of probabilities (or beliefs) in their paper. Heller (2012) provides a representation theorem that is closer to ours. In his framework, behavior is characterized by a choice correspondence. In contrast, the present paper operates with rankings of two alternatives.

In the following we identify functions $u \in \mathbb{R}^X$ with $(u\,(x_1)\,, u\,(x_2)\,, ..., u\,(x_n))$ and, with slight abuse of notation, use u to denote this vector. The inner product in $\mathbb{R}^n$ is denoted by '$\cdot$'. We now state our first representation theorem.

Theorem 1 *A binary relation $\succcurlyeq$ satisfies (A.1)-(A.4) if and only if there exists a closed convex cone of utility functions $\mathcal{U} \subset \mathbb{R}^n$ such that*

$$p \succcurlyeq q \iff \exists u \in \mathcal{U} : \ u \cdot p \geq u \cdot q . \tag{2}$$

Proof: See Appendix

Note that by taking negations of (2), the strict relation $\succ$ satisfies

$$p \succ q \iff \forall u \in \mathcal{U} : \ u \cdot p > u \cdot q. \tag{3}$$

To understand the structure of the set $\mathcal{U}$ note that if u satisfies the right hand side inequality of (2), then so does every function v that is derived from u through multiplication by a positive scalar (that is, $v = au$, for some $a > 0$). This explains why $\mathcal{U}$ is a cone. Similarly, convexity of $\mathcal{U}$ is a consequence of the weak inequality on the right hand side of (2). Finally, it is easy to verify that any positive affine transformation of u ($v = au + t$, $a > 0$ and t arbitrary) would also satisfy the right hand side of (2). This illustrates why, following Dubra, Maccheroni and Ok (2004), two cones $\mathcal{U}$ and $\mathcal{U}'$ satisfy (2) if and only if the closure of the set $\{u + t\mathbf{e}|u \in \mathcal{U}, \ t \in \mathbb{R}\}$ is equal to that of the set $\{u' + t\mathbf{e}|u' \in \mathcal{U}', \ t \in \mathbb{R}\}$ (where $\mathbf{e} = (1, ..., 1)$). This property generalizes the uniqueness property (up to positive affine transformation) of the classical expected utility theorem.

Few special cases of the structure of the set $\mathcal{U}$ are worth mentioning. At one extreme is the case $\mathcal{U} = \mathbb{R}^n$, so that the cone $\mathcal{U}$ consists of all real functions defined on X. Under this scenario, $\succcurlyeq$ is trivial in the sense that $p \sim q$ for all $p, q \in \mathcal{L}$ (for every p and q it is possible to find u satisfying $u \cdot p \geqslant u \cdot q$ and u' satisfying the converse inequality) and, moreover, $\succcurlyeq$ is transitive. Another extreme case materializes when the cone $\mathcal{U}$ is a ray. This is the only situation in which $\succcurlyeq$ is transitive while $\succ$ is non-trivial and is, in fact, the standard, transitive, EU preference (in which case there is no real randomness over $\mathcal{U}$). Two other interesting cases emerge when the set of alternatives satisfies $X \subset \mathbb{R}$ (i.e., all x_i's are sums of money). If $\mathcal{U}$ consists of all strictly increasing functions then, by (3), $\succ$ is equal to the *strong* first-order stochastic dominance partial relation $>_1$ defined by

$$p >_1 q \ \text{ if } \ \sum_{i=1}^{j} p_i < \sum_{i=1}^{j} q_i \ \text{ for all } \ j=1, ..., n-1,$$

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where we assume, without loss of generality, that $x_1 < x_2 < \cdots < x_n$. Similarly, if $\mathcal{U}$ consists of all concave functions then $\succ$ is equal to a strong version of the second-order stochastic dominance partial relation.

We will call a DM who acts as if she draws a utility function from a given set of utilities a Probabilistic DM (denoted PDM). To reflect the fact that PDMs who are characterized by Theorem 1 satisfy the Independence axiom, we refer to them as PDMs with core EU preferences. The set $\mathcal{U}$ is called the PDM's core utilities. In Sections 4 and 5 we supplement the preference structure with an additional component, a probability distribution ψ over $\mathcal{U}$ such that the PDM draws a utility function from $\mathcal{U}$ according to ψ and subsequently makes her decision based on the drawn utility. Our modeling of the probability distribution function ψ may seem rather ad hoc in the sense that it is not generated by some behavioral axioms similar to those presented above. However, our results in Sections 4 and 5 hold for all probability distribution functions ψ.

The existing literature does offer some representation results along these lines but in different, and often more complex, frameworks. This literature was originated by Kreps (1979) who, in the context of preferences over menus, derived a subjective (non-unique) mental state space that is analogous to our set of core utilities $\mathcal{U}$. Dekel, Lipman and Rustichini (2001; see also Dekel et al., 2007) axiomatized the existence and uniqueness of Kreps' subjective state space but their model did not pin down a unique probability distribution function (the analogue of ψ) over this space. Gul and Pesendorfer (2006) took a different approach and gave necessary and sufficient conditions for a random choice rule to maximize a random utility function. Subsequently, Ahn and Sarver (2013) synthesized the menu choice model of Dekel, Lipman and Rustichini (2001) and the random choice model of Gul and Pesendorfer (2006) to obtain a representation of a two-stage decision process in which, in the first stage, decision-makers choose among menus according to the former model and, in the second stage, they make a stochastic choice from the chosen menu according to the latter representation. The probability distribution function ψ is assumed to be equal to the empirical distribution observed by the experimenter in the second stage. Finally, Karni and Safra (2014) derived a unique distribution function ψ by adding a layer of hypothetical

lotteries over the mental state space.

Note that each $u \in \mathcal{U}$ characterizes an EU functional V^u defined by $V^u(p) = u \cdot p$ and that for such PDMs $p \succcurlyeq q$ if and only if there exists $u \in \mathcal{U}$ such that $V^u(p) \geq V^u(q)$. Similarly, $p \succ q$ if and only if, for all $u \in \mathcal{U}$, $V^u(p) > V^u(q)$. Clearly, PDMs satisfying (A.1)-(A.4) become the usual EU decision-makers when the transitivity of $\succcurlyeq$ is also required.

Our framework in this section is also related to recent models of incomplete preferences (Dubra, Maccheroni and Ok 2004, Ok, Ortoleva and Riella 2012, Galaabaatar and Karni 2013) where decision-makers are represented by sets of EU preferences but choose an alternative if and only if all of these preferences agree that the alternative is preferred to all of the other feasible alternatives. Although there are similarities at a formal level, the behavioral content of our model is very different.

3.2   Probabilistic DMs with core Betweenness-like preferences

Similarly to the deterministic EU model, the model of a PDM with core EU preferences cannot be reconciled with a number of violations of the independence axiom observed in the lab. For example, let $X = \{0, 3000, 4000\}$ and consider a PDM with core EU preferences who is first asked to choose between $p = (0, 1, 0)$ (3000 for certain) and $q = (0.2, 0, 0.8)$ (a lottery with a 0.8 chance of winning 4000) and then, independent of her first choice, between $\bar{p} = (0.75, 0.25, 0)$ (0.25 chance of winning 3000) and $\bar{q} = (0.8, 0, 0.2)$ (0.2 chance of winning 4000). Let ψ be the PDM distribution over $\mathcal{U}$ and let $\alpha = \psi(u \in \mathcal{U} : u \cdot p > u \cdot q)$ be the probability of drawing a utility u that ranks lottery p strictly higher than lottery q. Since $u \cdot p > u \cdot q$ if and only if $u \cdot \bar{p} > u \cdot \bar{q}$, the probability $\psi(u \in \mathcal{U} : u \cdot \bar{p} > u \cdot \bar{q})$ is, by construction, also equal to α. Hence, for this PDM and irrespective of the probability distribution ψ, the probability that p is chosen from the first pair and $\bar{q}$ is chosen from the second pair must be equal to the probability that $\bar{p}$ is chosen from the second pair and q is chosen from the first pair, since both are equal to $\alpha(1-\alpha)$. However, this contradicts persistent experimental evidence showing that the frequency of the choices p and $\bar{q}$ is statistically significantly greater than the frequency of the choices $\bar{p}$ and q, which is immediately recognized as the famous common-ratio effect.

To address such violations of the EU model, we turn to a more general representation

theorem in which the utility set $\mathcal{U}$ still exists but depends on the lotteries at which choice is made. To accomplish this we replace the Independence axiom (A.3) with a weaker betweenness assumption:

(**B.3**) (Betweenness) For all $p, q \in \mathcal{L}$, $r = p, q$ and $\alpha \in [0, 1]$,

$$p \succcurlyeq q \iff \alpha p + (1 - \alpha) r \succcurlyeq \alpha q + (1 - \alpha) r.$$

To see that this assumption is essentially identical to the Betweenness axiom used in generalized EU models (see Chew 1989 and Dekel 1986), note that, assuming transitivity, (B.3) holds if and only if for all $p, q \in \mathcal{L}$, $\alpha \in [0, 1]$ and $\beta \in (0, 1)$

$$p \succcurlyeq q \implies p \succcurlyeq \alpha p + (1 - \alpha) r \succcurlyeq q,$$

$$p \succ q \implies p \succ \beta p + (1 - \beta) r \succ q.$$

The statement in (B.3) was chosen because it emphasizes its relation to (A.3).

Theorem 2 *A binary relation $\succcurlyeq$ satisfies (A.1), (A.2), (B.3) and (A.4) if and only if for each $q \in \mathcal{L}$ there exists a closed convex cone of utility functions $\mathcal{U}^q$ such that for all p and q*

$$p \succcurlyeq q \iff \exists u \in \mathcal{U}^q : u \cdot p \geq u \cdot q \iff \exists v \in \mathcal{U}^p : v \cdot p \geq v \cdot q \qquad (4)$$

Proof: See Appendix

The main difference between this theorem and Theorem 1 is that here the cones $\mathcal{U}^q$ can vary with the lottery q (see the example below). In addition, and similarly to Theorem 1, the cones $\mathcal{U}^q$ are unique up to the affine operator described above. Finally, the strict relation $\succ$ satisfies

$$p \succ q \iff \forall u \in \mathcal{U}^q \ u \cdot p > u \cdot q \iff \forall v \in \mathcal{U}^p : v \cdot p > v \cdot q. \qquad (5)$$

Example Let $X = \{x_1, x_2, x_3\} = \{0, 1, 2\}$ and let p_i denote the probability of outcome x_i. Also, let $w(x_1) = w(x_3) = 0.5$ and $w(x_2) = 1$. Consider the binary relation satisfying betweenness and defined by

$$p \succcurlyeq q \iff V_j(p) \geq V_j(q) \quad \text{for some} \quad j = 1, 2,$$

where

$$V_1(p) = \frac{\sum\limits_{i=1}^{3} p_i w(x_i) x_i}{\sum\limits_{i=1}^{3} p_i w(x_i)}$$

is a weighted utility function and

$$V_2(p) = \sum_{i=1}^{3} p_i x_i$$

is a function that ranks lotteries according to the expected value of the outcome. Both functions rank outcome 2 at the top and outcome 0 at the bottom. As can be seen in Figure 1, drawn in the (p_1, p_3) plane (where $p_2 = 1 - p_1 - p_3$ is omitted), the cones $\mathcal{U}^q$ vary with the lottery q. At $q' = (0.25, 0.25)$, the cone $\mathcal{U}^{q'}$ satisfies

$$\mathcal{U}^{q'} = \left\{ \lambda \left(-\frac{1}{8}, 0, \frac{1}{8} \right) \mid \lambda > 0 \right\}.$$

The cone $\mathcal{U}'$ in Figure 1, with a vertex at q' and spanned by the utility vector $u' = \left(-\frac{1}{8}, \frac{1}{8} \right)$, is the projection of $\mathcal{U}^{q'}$ into the (p_1, p_3) plane. At $q'' = (0.1, 0.7)$, the cone $\mathcal{U}^{q''}$ satisfies

$$\mathcal{U}^{q''} = \left\{ \lambda_1 \left(-\frac{1}{8}, 0, \frac{1}{8} \right) + \lambda_2 \left(\left(-\frac{1}{15}, 0, \frac{1}{5} \right) - \frac{2}{45} \mathbf{e} \right) \mid \lambda_1, \lambda_2 > 0 \right\},$$

while its projection $\mathcal{U}''$ emanates from q'' and it is spanned by $u' = \left(-\frac{1}{8}, \frac{1}{8} \right)$ and $u'' = \left(-\frac{1}{15}, \frac{1}{5} \right)$. Finally, at $q''' = (0.7, 0.1)$, the cone $\mathcal{U}^{q'''}$ satisfies

$$\mathcal{U}^{q'''} = \left\{ \lambda_1 \left(-\frac{1}{8}, 0, \frac{1}{8} \right) + \lambda_2 \left(\left(-\frac{1}{5}, 0, \frac{1}{15} \right) + \frac{2}{45} \mathbf{e} \right) \mid \lambda_1, \lambda_2 > 0 \right\}.$$

Its projection $\mathcal{U}'''$ is spanned by the utility vectors $u' = \left(-\frac{1}{8}, \frac{1}{8} \right)$ and $u''' = \left(-\frac{1}{5}, \frac{1}{15} \right)$ and is depicted as the cone with a vertex at q'''.

Place Figure 1 here

We say that PDMs characterized by Theorem 2 have core Betweenness-like preferences. This reflects the role the Betweenness axiom plays in deriving the representation. It is immediate to verify that the PDMs of this section become the usual Betweenness decision-makers when the transitivity requirement on $\succcurlyeq$ is added to (A.1), (A,2), (B.3) and (A.4). Note however that, unlike the EU case, a PDM characterized by Theorem 2 may not necessarily be represented by a set of usual transitive Betweenness preferences. In fact, such representation cannot be achieved if transitivity of the strict relation is not assumed (see Safra, 2014).

4 WST with core EU preferences

4.1 Preliminaries

In this section we consider PDMs who are characterized by Theorem 1. That is, a typical PDM has core EU preferences given by a set of utilities $\mathcal{U}$. To examine WST, we supplement the preference structure of the preceding section with an additional component, a probability distribution ψ over $\mathcal{U}$. The PDM draws a utility function from $\mathcal{U}$ according to ψ before she makes her decision.[5]

We denote the probability that lottery $p \in \mathcal{L}$ is preferred to lottery $q \in \mathcal{L}$ by $\psi(p \succ q) = \psi(u \in \mathcal{U} : u \cdot p > u \cdot q)$. That is, the binary choice probability $\psi(p \succ q)$ is the measure of the set of utilities for which the expected utility of p is strictly greater than the expected utility of q. In incomplete expected utility models, lottery p is chosen over lottery q if and only if the expected utility of p is strictly greater than the expected utility of q for all EU functions in $\mathcal{U}$, i.e. $\psi(p \succ q) = 1$. In contrast, in the models of probabilistic choice p may be chosen over q even if $\psi(p \succ q)$ is strictly less than one. In the latter models, the nature of the function $\psi(p \succ q)$ is of key interest. WST (see Davidson and Marschak 1959 and Tversky 1969) has received specific attention of the existing literature on random choice. Formally:

Definition 3 *A PDM with a distribution function ψ satisfies weak stochastic transitivity (WST) with respect to the set of lotteries $\mathcal{P} \subset \mathcal{L}$ if $\forall p, q, r \in \mathcal{P}$*

$$\psi(p \succ q) > 0.5 \text{ and } \psi(q \succ r) > 0.5 \implies \psi(p \succ r) \geq 0.5.$$

Thus, WST requires that a PDM whose probability of choosing p over q is greater than 0.5 and probability of choosing q over r is greater than 0.5 will have a probability of choosing p over r that weakly exceeds 0.5.[6] WST has been one of the most prominent approaches as a probabilistic analogue for a deterministic choice model.[7]

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[5]In what follows, it is assumed, without any loss of generality, that the support of ψ is equal to $\mathcal{U}$.

[6]A typical definition of WST involves only weak inequalities. Our strict version is used to avoid non-generic boundary cases.

[7]Many studies in this strand suffer from inappropriate statistical analysis and erroneous conclusions that

In our framework, WST can be violated unless a combination of restrictions is imposed on the probability distribution ψ and on the set of feasible lotteries $\mathcal{P}$. That is, absent such restrictions there exists a family of EU preferences, a probability distribution over that family, and a collection of three distinct lotteries that will lead to a violation of WST.

To illustrate, consider a Condorcet-like situation with the set of utilities

$$u^1 = (3, 2, 1), \; u^2 = (1, 3, 2), \; u^3 = (2, 1, 3)$$

and the set $\mathcal{P}$ of degenerate lotteries $\{\mathbf{e}^i\}_{i=1}^3 \subset \mathcal{L}$, where $\mathbf{e}^i$ is the ith unit vector of $\mathbb{R}^3$ (hence a vertex of $\mathcal{L}$). It is easy to verify that a uniform probability distribution over $\{u^i\}_{i=1}^3$ violates WST with respect to $\mathcal{P}$. Under the uniform distribution, the probability that the PDM will choose lottery $\mathbf{e}^1$ over lottery $\mathbf{e}^2$ is equal to $\frac{2}{3}$ $\left(\psi\left(\mathbf{e}^1 \succ \mathbf{e}^2\right) = \frac{2}{3}\right)$, the probability that the PDM will choose lottery $\mathbf{e}^2$ over lottery $\mathbf{e}^3$ is equal to $\frac{2}{3}$ $\left(\psi\left(\mathbf{e}^2 \succ \mathbf{e}^3\right) = \frac{2}{3}\right)$, while the probability that the PDM will choose lottery $\mathbf{e}^3$ over lottery $\mathbf{e}^1$ is also equal to $\frac{2}{3}$ $\left(\psi\left(\mathbf{e}^3 \succ \mathbf{e}^1\right) = \frac{2}{3}\right)$.[8]

It should be noted, however, that not all violations of WST need to involve the vertices of $\mathcal{L}$. Moreover, a PDM with a given probability distribution over preferences can violate WST without exhibiting such a violation with respect to the vertices of $\mathcal{L}$. For example, let the four possible outcomes be given by $(x_1, x_2, x_3, x_4) = (10, 19, 25, 40)$ and consider the utility set $\mathcal{U} = \{u^1, u^2, u^3\}$ where $u^1 = (0, 160, 210, 240)$, $u^2 = (0, 114, 186, 240)$ and $u^3 = (0, 114, 168, 240)$. As all utilities agree on the order of the outcomes x_i, all PDMs with these utilities will satisfy WST with respect to the vertices $\mathcal{P} = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4\}$, irrespective of the probability distribution ψ. However, as the following calculations demonstrate, a PDM with a uniform probability distribution over $\{u^1, u^2, u^3\}$ violates WST with respect to

violations of WST imply intransitive preferences. See Regenwetter et al. (2011) for an illuminating discussion of these points.

[8]Note that the uniform distribution is not the only one that violates WST. There is a continuum of other distributions over $\{u^1, u^2, u^3\}$ that don't satisfy WST. On the other hand, if, for example, $\psi\left(\cdot\right)$ places almost all of the weight on one of the core preferences of the PDM, WST will be satisfied.

the set of lotteries $\mathcal{P}^* = \{p^1 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), p^2 = \left(\frac{2}{3}, 0, \frac{1}{3}, 0\right), p^3 = \left(\frac{3}{4}, 0, 0, \frac{1}{4}\right)\}$:

$$
\begin{aligned}
u^1 \cdot p^1 &= 80 > u^1 \cdot p^2 = 70 > u^1 \cdot p^3 = 60, \\
u^2 \cdot p^2 &= 62 > u^2 \cdot p^3 = 60 > u^2 \cdot p^1 = 57, \\
u^3 \cdot p^3 &= 60 > u^3 \cdot p^1 = 57 > u^3 \cdot p^2 = 56.
\end{aligned}
$$

Thus, two out of the three core preferences rank p^1 over p^2, two rank p^2 over p^3, and two rank p^3 over p^1. Note that the example demonstrates that single peakedness of preferences over the set of deterministic outcomes X does not preclude the possibility of encountering violations of WST in the much richer lottery space $\mathcal{L}$.

Both of the above examples feature a finite number of core preferences for the PDM. Appendix contains an example with a continuum of core preferences that violates WST.

The literature on transitivity sometimes uses a different, geometric, approach to analyze WST.[9] To illustrate consider a (fixed) choice set consisting of three distinct lotteries $p, q, r \in \mathcal{L}$ and assume, for simplicity, that only strict core preferences (over this set) are allowed. Consider a PDM with a probability distribution ψ over a set of core preferences $\mathcal{U}$. Out of the 8 feasible complete core preferences over the set $\{p, q, r\}$, 6 correspond to the transitive (linear) orders[10] and 2 represent the Condorcet cycles ([p is preferred to q, q is preferred to r, but r is preferred to p] and [q is preferred to p, p is preferred to r, but r is preferred to q]). For any $\ell, \ell' \in \{p, q, r\}$ let $P_{\ell\ell'}$ denote the binary choice probability that ℓ is preferred to ℓ'. Following Iverson and Falmagne (1985) and Regenwetter et al. (2014), we consider the unit cube of the 3-dimensional space where the axes are spanned by the probabilities P_{pq} (which spans the first axis), P_{pr} (the second axis) and P_{qr} (the third axis); see Figure 2 panel (a). Each vertex of this cube corresponds to one of the 8 feasible core preferences. For example, the origin $(0, 0, 0)$ corresponds to the transitive linear order for which r is preferred to both q and p, and q is preferred to p. The two Condorcet cycles correspond to the vertices $(1, 0, 1)$ and $(0, 1, 0)$. Next, denote $\bar{\psi}_{pq} = \psi(p \succ q)$, $\bar{\psi}_{qr} = \psi(q \succ r)$, $\bar{\psi}_{pr} = \psi(p \succ r)$, and

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[9]We are grateful to one of the reviewers for referring us to this analysis and for inducing us to clarify the relationship between this approach and the propositions of Section 4.2.

[10]An example of a transitve linear order is a preference ranking such that p is preferred to both q and r, and q is preferred to r.

$\bar{\psi} = \left(\bar{\psi}_{pq}, \bar{\psi}_{pr}, \bar{\psi}_{qr} \right)$. It can be verified that a PDM with a probability distribution ψ satisfies WST if and only if the vector $\bar{\psi}$ belongs to the unshaded area in Figure 2 panel (a), that is, to the complement of the union of the two shaded half-cubes.

Place Figure 2 here

To see why WST is violated in the first example, consider the set of lotteries $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ and note that the first core utility $u^1 = (3, 2, 1)$ of the PDM corresponds to the vertex $(1, 1, 1)$ in the space of vectors $(P_{\mathbf{e}^1\mathbf{e}^2}, P_{\mathbf{e}^1\mathbf{e}^3}, P_{\mathbf{e}^2\mathbf{e}^3})$, the second core utility $u^2 = (1, 3, 2)$ corresponds to the vertex $(0, 0, 1)$ and the third utility $u^3 = (2, 1, 3)$ corresponds to $(1, 0, 0)$ (see Figure 2 panel (b)). Next observe that the uniform probability distribution ψ' over these core utilities corresponds to the point $\bar{\psi}' = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)$, the mid point of the triangle formed by the vertices $(1, 1, 1)$, $(0, 0, 1)$, and $(1, 0, 0)$ in Figure 2 panel (b), and WST is violated because this point belongs to the upper shaded half-cube.

Since in our framework the lottery set $\mathcal{L}$ is not finite, the case of three fixed lotteries does not provide a sufficiently good visual representation of our analysis. To visualize an approximation to the infinite case, consider a large number n of distinct lotteries. The dimension of the choice probabilities space is $\binom{n}{2}$, the number of vertices is $2^{\binom{n}{2}}$, while the number of the transitive linear orders is just $n!$. Therefore, there are $2^{\binom{n}{2}} - n!$ many shaded half-unit hypercubes and hence the shaded volume converges to 1 as n tends to infinity. Thus, WST becomes exceedingly restrictive as n becomes arbitrarily large.[11] We elaborate on this in the next section.

A number of primitives can lead to the satisfaction of WST. One approach entails supplementing a transitive deterministic preference relation with an error structure. The models in this strand include the tremble model (Harless and Camerer, 1994) and the Fechner model (Fechner, 1860, Luce and Suppes, 1965, Hey and Orme, 1994). Another approach is to consider a probability measure over a set of core preferences and define an aggregate preference using the Condorcet procedure; alternative p is preferred to q, denoted by $\overset{*}{\succ}$, if and only if

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[11]We thank an anonymous reviewer for emphasizing this point.

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the measure of the set of core preferences for which p is preferred to q exceeds 0.5. Transitivity of the preference order $\overset{*}{\succ}$ is equivalent to WST except for boundary cases. A restriction on core preferences, such as single-peakedness or net value restriction, will ensure that WST holds. Our approach falls into this category of models. According to our knowledge, the domain restrictions studied in the present paper have not been identified before. There are primitives leading to WST in addition to the formulations above. We omit their discussion due to space considerations.

As we have argued above, WST cannot be guaranteed without imposing restrictions on the probability distribution ψ and/or the set of feasible lotteries $\mathcal{P}$. Our main focus in the following subsection is in characterizing such conditions.

4.2   Commonality of preferences

Here we maintain the assumption that PDMs have core EU preferences but we restrict them to satisfy a certain property. First, we examine the case where all preferences in $\mathcal{U}$ agree with a certain linear order of the basic outcomes. Then, we suppose that all have similar risk attitudes (in the sense that will be defined below). Finally, we consider the implications of imposing both of these domain restrictions (agreement on common directions and similar risk attitudes).

4.2.1   Comonotonic preferences

Denote the EU preference relation for the utility function $u \in \mathcal{U}$ by $\succsim^u$, where the asymmetric part is denoted by $\succ^u$. Suppose that all preference relations in $\mathcal{U}$ agree on the order of the basic outcomes. That is, there exists a permutation $\{i_j\}_{j=1}^n$ of $\{1, ..., n\}$ such that $x_{i_1} \succ^u x_{i_2} \succ^u \cdots \succ^u x_{i_n}$ for all $u \in \mathcal{U}$. PDMs who satisfy this condition are called *comonotonic*. A special case of comonotonic PDMs occurs when all x_i's are monetary outcomes and money is desired. The main result here is that WST is satisfied as long as the PDM is comonotonic and the set of available lotteries $\mathcal{P}$ is a subset of the set of all lotteries over three outcomes.

Proposition 4 *Suppose there are no more than three outcomes. For all probability distributions ψ, comonotonic PDMs with core EU preferences and distribution function ψ satisfy WST with respect to all sets $\mathcal{P} \subseteq \mathcal{L}$.*

Proof: See Appendix

It can be easily verified that WST is satisfied when $\mathcal{P}$ is a two-dimensional polygon and the PDM is comonotonic with respect to its vertices. Note also that the condition that there are no more than three outcomes is necessary. See the second example of Section 4.1 for a discrete $\mathcal{U}$ and Appendix for a continuous $\mathcal{U}$.

Next we relate Proposition 4 to the geometric analysis presented in the previous section. Fix n distinct lotteries in the feasible set $\mathcal{L} = \Delta(x_1, x_2, x_3)$ and assume, without loss of generality, that $x_1 \succ^u x_2 \succ^u x_3$ for all $u \in \mathcal{U}$. By construction, the set of strict and transitive linear core preferences over the n lotteries that belong to $\mathcal{U}$ corresponds to a proper subset of the $n!$ vertices which represent all possible transitive linear orders. Allowing for all possible probability distributions ψ over the set of core preferences yields a convex polygon that is equal to the convex hull of these vertices. In order for Proposition 4 to hold, this convex polygon should have an empty intersection with the shaded subset of the hypercube that violates WST. However, as was explained in Section 4.1, for very large n the volume of the shaded subset becomes arbitrarily close to 1 while the volume of the subset that satisfies WST becomes arbitrarily close to 0. Since the latter set is not convex, it is not immediately clear how the convex polygon can be completely nested in it.

To grasp the intuition behind Proposition 4, consider first the case $n = 3$ and assume, for simplicity, that the three lotteries are given by $p = \mathbf{e}^2$, $q = \alpha_1 \mathbf{e}^1 + (1 - \alpha_1) \mathbf{e}^3$, $r = \alpha_2 \mathbf{e}^1 + (1 - \alpha_2) \mathbf{e}^3$, where $1 > \alpha_1 > \alpha_2 > 0$.[12] By comonotonicity, all core preferences rank q over r and hence, for every possible probability distribution ψ, we have $\psi_{qr} = 1$. Hence, in the space spanned by P_{pq}, P_{pr} and P_{qr}, the analysis is restricted to a 2-dimensional face of the cube. Indeed, all three feasible transitive linear orders lie in this face: for the first transitive linear order, which corresponds to the vertex $(1, 1, 1)$ in Figure 3 panel (a), p is

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[12] Note that every EU preference is uniquely determined by the lottery of the form $\alpha_1 \mathbf{e}^1 + (1 - \alpha_1) \mathbf{e}^3$ that is indifferent to $\mathbf{e}^2$.

preferred to both q and r; for the second, which corresponds to the vertex $(0, 0, 1)$, both q and r are preferred to p; and for the third, which corresponds to the vertex $(0, 1, 1)$, p is ranked between q and r. The polygon representing all possible probability distributions that satisfy the requirements of Proposition 4 is given by the triangle with vertices $(0, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ in Figure 3 panel (a) and, as can be seen in the figure, it is nested in the subset of the cube that satisfies WST.

<div align="center">Place Figure 3 here</div>

Next let $n = 4$ and, as above, consider four lotteries given by $p = \mathbf{e}^2$, $q = \alpha_1 \mathbf{e}^1 + (1 - \alpha_1) \mathbf{e}^3$, $r = \alpha_2 \mathbf{e}^1 + (1 - \alpha_2) \mathbf{e}^3$ and $s = \alpha_3 \mathbf{e}^1 + (1 - \alpha_3) \mathbf{e}^3$, where $1 > \alpha_1 > \alpha_2 > \alpha_3 > 0$. Again by comonotonicity, every possible probability distribution ψ satisfies $\psi_{qr} = 1$, $\psi_{rs} = 1$ and $\psi_{qs} = 1$. Therefore, although in the space of binary choice probabilities the hypercube is of dimension $\binom{4}{2} = 6$, the relevant analysis is restricted to the 3-dimensional cube that is spanned by the choice probabilities P_{pq} (first axis in Figure 3 panel (b)), P_{pr} (second axis) and P_{ps} (third axis). There are four feasible transitive linear orders: for the first order, p is preferred to q, r and s (this preference corresponds to the vertex $(1, 1, 1)$ in Figure 3 panel (b)); for the second, both q, r and s are preferred to p (the vertex $(0, 0, 0)$); for the third, q is preferred to p while p is preferred to r and s (the vertex $(0, 1, 1)$); and, for the forth, q and r are preferred to p while p is preferred to s (the vertex $(0, 0, 1)$). As can be seen in the figure, the 3-dimensional polygon created by these vertices is nested in the subset that satisfies WST.

For an arbitrary n and lotteries of the form $p = \mathbf{e}^2$ and $q_i = \alpha_i \mathbf{e}^1 + (1 - \alpha_i) \mathbf{e}^3$, $i = 1, ..., n-1$ and $1 > \alpha_1 > \cdots > \alpha_{n-1} > 0$, the relevant sub-hypercube is of dimension $n-1$ and, out of its 2^{n-1} vertices, only n correspond to the feasible linear orders. Intuitively speaking, since these vertices are not 'spread out' over all faces of the relevant sub-hypercube (again, see Figure 3 panel (b)), the relevant n-dimensional polygon 'manages to avoid' intersecting with the shaded non-WST subset. For a more precise argument, note that a non-empty intersection of the polygon with the interior of the non-WST subset implies the existence of a probability distribution ψ and three distinct lotteries $p', q', r' \in \{p, \{q_i\}_1^{n-1}\}$ satisfying

<div align="center">20</div>

$\psi\left(p' \succ q'\right) > 0.5$, $\psi\left(q' \succ r'\right) > 0.5$ and $\psi\left(r' \succ p'\right) > 0.5$. Therefore, there must exist three utilities $u, v, w \in \mathcal{U}$ satisfying $u \cdot p' > u \cdot q' > u \cdot r'$, $v \cdot q' > v \cdot r' > v \cdot p'$ and $w \cdot r' > w \cdot p' > w \cdot q'$. But this cannot hold: at least two of the lotteries p', q', r' are of the form q_i and q_j, $i < j$ and, by comonotonicity, all utilities must rank q_i higher than q_j.

4.2.2 Common risk attitude

In this subsection all x_i's are taken to be monetary outcomes and we use the common notion of risk aversion: a preference relation exhibits risk aversion (or, risk seeking) if the expected value of every non-degenerate lottery is strictly preferred (less preferred) to that lottery. This is equivalent to the strict concavity (convexity) of the corresponding utility function as well as to the PDM preference being strictly decreasing (increasing) with respect to mean-preserving spreads. PDMs for whom all core EU preferences display risk aversion (risk seeking) are called *risk averse* PDMs (*risk seeking* PDMs).

We use the structure imposed by common risk attitude to prove that, when the set $\mathcal{P}$ is a subset of the set of all lotteries over three outcomes, risk averse and risk seeking PDMs always satisfy WST.

Proposition 5 *Suppose there are no more than three outcomes. For all probability distributions ψ, risk averse (risk seeking) PDMs with core EU preferences and distribution function ψ satisfy WST with respect to all sets $\mathcal{P} \subseteq \mathcal{L}$.*

Proof: See Appendix

Finally, since separately requiring either comonotonicity or risk aversion (seeking) of core preferences ensures WST for any two-dimensional set $\mathcal{P}$, a natural question to ask is whether requiring both warrants WST for sets of lotteries that are of dimension higher than 2. However, as the second example of Section 4.1 demonstrates, this is not the case (note that a PDM with the utilities of this example is comonotonic and risk averse). Thus, even if the dimension of $\mathcal{P}$ is as low as 3, a combination of comonotonicity and common risk attitude can not ensure that, for any probability distribution ψ, WST with distribution function ψ will hold.

4.3 Experimental violations of WST

The empirical literature on possible violations of WST in experimental settings is rather vast.[13] However, a considerable share of this research is either purely descriptive with respect to violations of transitivity (e.g., Brandstätter, Gigerenzer and Hertwig 2006) or based on flawed statistical techniques. The shortcomings of many statistical analyses of WST stem from the fact that the asymptotic distribution of the log-likelihood ratio statistic is a weighted sum of Chi-Squared distributions rather than a Chi-Squared distribution, as postulated in these studies. Iverson and Falmagne (1985) derived an order-constrained inference method to test for WST and demonstrated that the data reported in Tversky (1969) provides very little evidence for systematic WST violations. Regenwetter, Dana and Davis-Stober (2010, 2011) built on Davis-Stober (2009) to characterize a complete order-constrained test of WST and didn't find systematic violations either. In contrast, Myung, Karabatsos and Iverson (2005) developed a Bayesian model selection framework whose results in large part agree with Tversky's (1969) findings.[14]

The random preference model allows for violations of WST. Sceptics of the empirical violations of WST, guided by concerns for a model's predictive power, might favor a framework that always satisfies WST.[15] If WST is indeed satisfied, then such a model could potentially be more effective in terms of making theoretical and empirical predictions. As we demonstrate in this section, such concerns are not fully warranted because (by Propositions 4 and

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[13] A related literature tests WST in collective choice problems (e.g., Felsenthal, Maoz, and Rapoport, 1990, 1993; Regenwetter and Grofman, 1998; Regenwetter et al., 2006, 2009, Regenwetter, 2009). These studies typically use election data to test for presence of Condorcet cycles.

[14] For more on the difference between Myung, Karabatsos and Iverson (2005) and Regenwetter, Dana and Davis-Stober (2010, 2011), see the discussion in Regenwetter, Dana and Davis-Stober (2011).

[15] A number of models preclude violations of WST. For example, most simple fixed utility models (Becker, Degroot, and Marschak, 1963 and Luce and Suppes, 1965) always satisfy WST. However, these models may suffer from violations of first-order stochastic dominance. Recently, Blavatskyy (2011) introduced a model derived from a set of axioms, which include WST and monotonicity with respect to first-order stochastic dominance. However, Loomes, Rodríguez-Puerta, Pinto-Prades (2014) argue that Blavatskyy's (2011) model cannot explain the common consequence effect and certain common ratio effects. In contrast, the random preference model with core betweenness-like preferences, axiomatized in Section 3.2 of the present paper, allows for all such behavioral phenomena.

5), when there are fewer than three outcomes, WST is satisfied by any comonotonic or risk averse PDM with EU preferences. It turns out that for many popular experimental settings, the choice problems are effectively two-dimensional.

In the following subsection, we present such an example: an experiment to elicit certainty equivalents of binary lotteries. We then relate violations of WST to the preference reversal phenomenon. In this section we maintain the assumptions that the outcomes are monetary, the PDM's core preferences are EU, and all their utility functions are increasing in income.

4.3.1 Experiments with binary choice lists

Consider a choice problem where a PDM chooses between a sure income of y and a non-degenerate lottery, denoted by $B = (x, q; z, 1-q)$ paying x with probability $q \in (0,1)$ and z with $(1-q)$, where $x, y, z \in [\underline{y}, \bar{y}] \subseteq \mathbb{R}$. Assume, as in many actual experiments, that y varies in $[\underline{y}, \bar{y}]$ while B is fixed when the PDM chooses between y's and B. It is also assumed that the PDM has core EU preferences that are continuous on the interval $[\underline{y}, \bar{y}]$. That is, we consider a popular experimental design with binary choice lists (Cohen et al. 1987, Holt and Laury 2002). Typically the objective of the experiment is to find the certainty equivalent of lottery B.

For this experimental setup, the overall space is the Cartesian product of the interval $[\underline{y}, \bar{y}]$ and the probability interval $[0,1]$, which is a 2-dimensional set. Since an element $(y, p) \in [\underline{y}, \bar{y}] \times [0,1]$ represents the compound lottery $(B, p; y, 1-p)$, the set of pairs (y, p) can be identified with the probability simplex over the alternatives $\underline{y}, \bar{y}$, and B. Denote this set by $\Delta(\underline{y}, B, \bar{y})$ and note that, as in the case of 3-outcome lotteries, the set is 2-dimensional. Also note that, for any increasing vNM utility $u \in \mathcal{U}$, first-order stochastic dominance implies $u(\underline{y}) < u(B) < u(\bar{y})$ (where $u(B)$ stands for the expected utility of B). Hence, assuming that more money is better, all core EU preferences over $\Delta(\underline{y}, B, \bar{y})$ are comonotonic.

Finally, to be able to use Proposition 4 we need to demonstrate that, for every $u \in \mathcal{U}$, indifference curves of the derived expected utility preference are parallel straight lines. This, however, is implied by the Independence axiom (as it enables us to replace the lottery B by its certainty equivalent $u^{-1}(u(B))$; the formal development of this claim is omitted due to space

considerations). Hence, Proposition 4 implies that WST over $\Delta\left(\underline{y}, B, \bar{y}\right)$ must be satisfied and, therefore, violations of WST may be rarely observed in this popular experimental setting.

More precisely, even when WST is satisfied, a sample of pairwise choices by a PDM characterized in the proposition may contain cycles and may even violate the following condition:

$$\forall p, q, r \in \mathcal{L}, \ \phi\left(p \succ q\right) > 0.5 \text{ and } \phi\left(q \succ r\right) > 0.5 \implies \phi\left(p \succ r\right) \geq 0.5,$$

where $\phi\left(p \succ q\right)$ denotes the observed proportion of choices of p over q for some sample of pairwise choices (and similarly for the other pairs). Thus, satisfaction of WST, expressed in terms of the theoretical probability $\psi\left(\cdot\right)$, is not sufficient for an analogous condition where $\psi\left(\cdot\right)$ is replaced by the proportion function $\phi\left(\cdot\right)$. However, if WST is satisfied and the observed data form an independent and identically distributed random sample, then the WST condition for the proportion function $\phi\left(\cdot\right)$ will be satisfied asymptotically.

4.3.2 $-bet versus P-bet type experiments

In this section we examine the implications of our analysis for an experiment that exhibits the "preference reversal phenomenon" (Lichtenstein and Slovic 1971, Lindman 1971, Grether and Plott 1979). Two lotteries are presented to experimental subjects. A '$-bet' offers a relatively high payoff, denoted by x_1, with a relatively small probability, denoted by p_1. In a typical experiment, p_1 is well below 0.5. A 'P-bet' offers a relatively small payoff, denoted by x_2, with a relatively high probability, denoted by p_2. The probability of winning for the P-bet is higher than the probability of winning for the $-bet: $p_2 > p_1$. In the experiment, certainty equivalents (CEs) of the $-bet and P-bet are elicited from the subjects. We denote these certainty equivalents by $CE_\$$ and CE_P, respectively. Most experimental subjects choose the P-bet over the S-bet while revealing a strictly higher certainty equivalent for the $-bet than for the P-bet. Thus, a typical ordering of the certain outcomes is as follows; $x_1 > x_2 > CE_\$ > CE_P > 0$.

For this setting, the overall space can be identified with the space $\Delta\left(0, x_1, x_2\right) \times \left[0, x_1\right]$, where $\Delta\left(0, x_1, x_2\right)$ denotes the probability simplex over alternatives 0, x_1, and x_2, and $\left[0, x_1\right]$

is the range of possible certainty equivalents. But this implies that lotteries are drawn from a space that is at least three dimensional. It then follows from the results in the preceding sections and the example in the Appendix that our model does not preclude violations of WST in this case.

Our findings are in concert with the relatively high frequency of cycles reported for a variety of $-bet versus P-bet type experiments (see, e.g., Loomes Starmer and Sugden 1991). Note, however, that most studies of the preference reversal phenomenon do not involve repeated choices or have too few repetitions. This makes it often impossible to assess whether individual subjects satisfy WST.

5   WST with core non-EU preferences

In this section we provide further results for lottery sets that are two dimensional ($n = 3$) but where PDMs have certain types of non-EU core preferences. Some of the results we obtain here are for a framework with three outcomes where $\mathcal{L}$ is the two dimensional unit simplex. However, there are other cases of interest that fall under the category of two dimensional lotteries. For example, consider a world that has two possible states of nature s_1 and s_2 with the corresponding fixed probabilities p_1 and p_2. Here the probabilities are fixed but the outcomes z_i can vary, and lotteries are given by the pairs (z_1, z_2). Since there is a one-to-one correspondence between this space and the set of lotteries $\mathcal{L}$ in our basic setup, our results apply to this case.

5.1   Three monetary outcomes and betweenness preferences

By Propositions 4 and 5, a PDM satisfies WST if all core EU preferences are either comonotonic or share the same risk attitude. We now extend these results by relaxing the assumption that core preferences are of the EU type. We say that a PDM has core Betweenness preferences if there exists a set of transitive Betweenness functionals $\{V^\tau\}_{\tau \in T}$ such that each satisfies $V^\tau(p) \geq V^\tau(q) \Leftrightarrow V^\tau(p) \geq V^\tau(\alpha p + (1 - \alpha) q) \geq V^\tau(q)$ for all p, q and $\alpha \in [0, 1]$ (see Chew 1989 and Dekel 1986) and the PDM may choose p over q if and only if there exists $\tau \in T$

such that $V^\tau(p) \geq V^\tau(q)$. Note that these PDMs fall under the category of preferences characterized by Theorem 2.

As with transitive EU preferences, indifference sets of the Betweenness functionals V^τ are given by straight lines in $\mathcal{L}$ (not necessarily parallel to each other). Hence, for every $p \in \mathcal{L}$ there exists a vector $u_p^\tau \in \mathbb{R}^n$, perpendicular to the indifference line through p, that satisfies

$$\forall q \in \mathcal{L} \qquad V^\tau(p) \geq V^\tau(q) \Longleftrightarrow u_p^\tau \cdot p \geq u_p^\tau \cdot q.$$

Following Machina 1982, u_p^τ can be called the local utility of V^τ at p. Note that the following two properties hold for Betweenness functionals: (1) V^τ is increasing with respect to the relation of first-order stochastic dominance if, and only if, all u_p^τ's are increasing and (2) V^τ displays risk aversion if, and only if, all u_p^τ's are concave. We now state our result:

Proposition 6 *Suppose there are no more than three outcomes and that $X \subset \mathbb{R}$. Then, for all probability distributions ψ,*

(1) PDMs with core Betweenness preferences, that increase with respect to the relation of first-order stochastic dominance, and a distribution function ψ satisfy WST with respect to all sets $\mathcal{P} \subseteq \mathcal{L}$ and

(2) Risk averse (risk seeking) PDMs with core Betweenness preferences and a distribution function ψ satisfy WST with respect to all sets $\mathcal{P} \subseteq \mathcal{L}$.

Proof: See Appendix

5.2 Two states of nature and increasing dual EU preferences

Consider a world with two states of Nature s_1 and s_2. The probabilities of the two states are fixed and given by p_1 and p_2, respectively. The pair (p_1, p_2) belongs to the one dimensional unit-simplex but, as probabilities are fixed and only outcomes vary, the relevant elements are identified with pairs of the form (z_1, z_2) where z_i denotes the outcome received in state s_i. We assume that $p_1 = p_2 = \frac{1}{2}$ and that the outcomes belong to the interval $[0, M]$. The relevant space is given by $Y = \{(z_1, z_2) \in \mathbb{R}^2 : 0 \leq z_2 \leq z_1 \leq M\}$. The preference relations

we consider belong to the dual EU set (see Yaari 1987 and Quiggin 1982). On the set Y, these preferences are represented by a function of the form

$$V(z_1, z_2) = f\left(\frac{1}{2}\right) z_1 + \left(1 - f\left(\frac{1}{2}\right)\right) z_2,$$

where $f : [0, 1] \to [0, 1]$ is increasing, $f(0) = 0$, and $f(1) = 1$. Note that f is the dual analogue of the EU utility function. Similarly to the EU case, a dual EU preference relation $\succcurlyeq$ can be identified with the vector $f = (f_1, f_2) = \left(f\left(\frac{1}{2}\right), 1 - f\left(\frac{1}{2}\right)\right)$ and hence the following relationship holds

$$\forall z, z' \in Y, \qquad z \succcurlyeq z' \iff f \cdot z \geq f \cdot z'.$$

By construction, all dual EU preferences are increasing with respect to the relation of first-order stochastic dominance. Risk aversion is characterized by the convexity of the function f, which in our case is equivalent to $f\left(\frac{1}{2}\right) < \frac{1}{2}$ (see Yaari 1987 and Chew Karni and Safra 1987). The next result deals with PDMs with core dual EU preferences.[16] Its proof is omitted since it is similar to that of Proposition 4.

Proposition 7 *Suppose that the set of alternatives is given by Y. For all probability distributions ψ, PDMs with core dual EU preferences and distribution function ψ satisfy WST with respect to all sets $\mathcal{P} \subseteq Y$.*

6 Illustrative Example Revisited

We now return to the illustrative example of Section 2 to demonstrate the implications of our findings. First, consider a scenario where, for all possible core preferences, attractiveness of options A, B, and C depends only on the probability distribution over the *total* profits from the two product lines. One could expect such a restriction on possible rankings if, for example, all monetary and non-monetary benefits of all team members were tied solely to the

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[16] Note that these PDMs are comonotonic. We do not provide a representation result for these PDMs as it is an immediate corollary of Theorem 1 (by swapping the roles of outcomes and probabilities; see Maccheroni 2004).

total profitability of the company and, as a result, the team's preference ranking respected monotonicity with respect to the total profits on each occasion a decision was made. This, of course, does not imply that the options A, B, and C will be ranked similarly on all choice occasions. Feasible core preferences may have different attitudes to risk and, consequently, rank the three options differently.

The probability distributions over the total profits for the three options are given by:

$$\text{Option } A: \quad \left(\text{-\$4m}, p_1^A; \$4m, p_2^A + p_3^A; \$12m, p_4^A\right),$$
$$\text{Option } B: \quad \left(\text{-\$4m}, p_1^B; \$4m, p_2^B + p_3^B; \$12m, p_4^B\right),$$
$$\text{Option } C: \quad \left(\text{-\$4m}, p_1^C; \$4m, p_2^C + p_3^C; \$12m, p_4^C\right).$$

Thus, effectively there are three outcomes, -\$4m, \$4m, \$12m, under this scenario and the three options correspond to different probability distributions over these three outcomes. In addition, all of the core preferences are comonotonic. It then follows immediately from Proposition 4 that WST will be satisfied.[17] The corresponding implication for the example's interpretation in terms of the Condorcet procedure is that the procedure will be void of Condorcet cycles (in the limit) as long as all of the voting members of the team care only about the total profits.

Suppose now that appeal of different options does not stem solely from the likelihoods of the total profits. Rather, some core preferences put more weight on the tablet profits while others favor the division that produces the phones. Under this scenario and absent any additional information, one cannot reduce the set of relevant outcomes to three as in the previous scenario and all twelve outcome pairs should be considered. But then our arguments and the second example in subsection 4.1 imply that even if all core preferences were comonotonic and had similar risk attitudes, a violation of WST would be possible. Correspondingly, the Condorcet procedure may exhibit cycles and there may very well be room for agenda setting.

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[17] If the team members disagree on the probabilities of the respective outcomes then the chances of violating WST increase. However, as long as these disagreements are not severe, our commonality and dimensionality restrictions still apply.

7   Conclusions

A large body of empirical literature (see, e.g., Hey 1995, Otter et al. 2008) reports that the same experimental subject may choose differently in exactly the same choice situation on different occasions even when the interval between different decisions is very short. Probabilistic theories of preferential choice account for what seems like an inherent variability of preferences and can also explain observed variability in managerial decision-making. We examine a subclass of such theories, random preference models, which fall into the category of random utility models (see, e.g., Becker DeGroot and Marschak 1963 and Luce and Suppes 1965). In the random preference model considered in the present paper, a decision-maker (either an individual or a group) is characterized by multiple rational preference structures and behaves as if her choice is made according to the preference ranking randomly drawn from the set of core preferences. We axiomatize the utility components of two classes of models with multiple preference rankings. For the first, core preferences have an expected utility form. The second is the more general preference structure where core preferences satisfy the betweenness condition.

The paper also examines the possibility of violations of WST for the cases where core preferences have EU form, betweenness-like form, and dual EU form. Finally, we present the implications of our results for two popular experimental settings. As violations of WST are related to existence of Condorcet cycles, the analysis has implications to managerial decision-making and to the social choice literature dealing with majority rule voting.

8 Appendix

Proof of Theorem 1: The 'if' part is immediate and its proof is omitted. To simplify the exposition, we normalize the utilities such that they all belong to $H = \{y \in \mathbb{R}^n | \sum y_i = 0\}$. If $\prec$ is empty then (2) trivially holds for $\mathcal{U} = H$. Hence, assume that $\prec$ is non-empty and note that, by (A.3), the relation $\prec$ satisfies the Independence assumption: that is, for all $p, q, r \in \mathcal{L}$ and $\alpha \in (0, 1]$,

$$p \prec q \iff \alpha p + (1 - \alpha) r \prec \alpha q + (1 - \alpha) r. \tag{6}$$

Now fix a lottery q in the interior of $\mathcal{L}$ and consider the set

$$W(q) = \{\lambda (p - q) | \lambda > 0, \ p \in \mathcal{L}, \ p \prec q\} \subseteq H.$$

Note that for all $p \in \mathcal{L}$, $p \prec q \Leftrightarrow p - q \in W(q)$. One direction in this equivalence follows from the definition of $W(q)$ by taking $\lambda = 1$. For the converse, consider $p - q \in W(q)$. By construction there exists p' and $\lambda > 0$ such that $p' \prec q$ and $p - q = \lambda (p' - q)$. If $\lambda \in (0, 1]$ then $p = \lambda p' + (1 - \lambda) q$ and the relation $p \prec q$ follows from (6) by taking $r = q$. If $\lambda > 1$ then $p' = \frac{1}{\lambda} p + \left(1 - \frac{1}{\lambda}\right) q$ and $p \prec q$ follows from (6) by taking $r = q$ and $\alpha = \frac{1}{\lambda}$.

Clearly (A.4) yields the convexity of the strictly positive cone $W(q)$. To show that $W(q)$ is independent of q for interior points of $\mathcal{L}$, consider $q, q', p \in \mathcal{L}$ such that q, q' are interior points and $p \prec q$. By construction, there exists $q'' \in \mathcal{L}$ and $\alpha \in (0, 1)$ such that $q' = \alpha q + (1 - \alpha) q''$. Denote $p' = \alpha p + (1 - \alpha) q''$ and note that, by (6),

$$p \prec q \Rightarrow p' = \alpha p + (1 - \alpha) q'' \prec \alpha q + (1 - \alpha) q'' = q'.$$

But as $p - q = \frac{1}{\alpha} (p' - q')$, $W(q) \subseteq W(q')$ for all such q, q', which implies $W(q) = W(q')$. Note that for a boundary point q (of $\mathcal{L}$) we would still have $W(q) \subseteq W(q')$.

Being a strictly positive convex cone in H, $W(q)$ is equal to the intersection of a family of open half spaces $\{r \in H | u \cdot r < 0\}_{u \in \mathcal{U}}$, where $\mathcal{U} \subseteq H$ is a uniquely defined strictly positive closed convex cone (see Rockafellar 1970). That is,

$$p - q \in W(q) \iff \forall u \in \mathcal{U} : \ u \cdot p < u \cdot q.$$

Hence, for all $p \in \mathcal{L}$,

$$p \prec q \quad \Longleftrightarrow \quad \forall u \in \mathcal{U} : u \cdot p < u \cdot q$$

and, by taking negations and using (A.1),

$$p \succcurlyeq q \quad \Longleftrightarrow \quad \exists u \in \mathcal{U} : u \cdot p \geq u \cdot q.$$

∎

Proof of Theorem 2: The 'if' part is immediate and its proof is omitted. The proof of the converse is similar to that of Theorem 1. If $\prec$ is empty then (4) trivially holds for $\mathcal{U}^q = H$, for all $q \in \mathcal{L}$. Hence assume that $\prec$ is non-empty and note that (B.3) is equivalent to the following: for all $p, q \in \mathcal{L}$, $r = p, q$ and $\alpha \in (0, 1]$,

$$p \prec q \quad \Longleftrightarrow \quad \alpha p + (1 - \alpha) r \prec \alpha q + (1 - \alpha) r. \tag{7}$$

As in the former proof, fix a lottery q in the interior of $\mathcal{L}$, consider the set

$$W(q) = \{\lambda (p - q) \,|\, \lambda > 0, \ p \in \mathcal{L}, \ p \prec q\} \subseteq H$$

and note that, again, for all $p \in \mathcal{L}$, $p \prec q \Leftrightarrow p - q \in W(q)$. Utilizing (A.4), $W(q)$ is a strictly positive convex cone and hence is equal to the intersection of a family of open half spaces $\{r \in H \,|\, u \cdot r < 0\}_{u \in \mathcal{U}^q}$, where $\mathcal{U}^q \subseteq H$ is a uniquely defined positive closed convex cone. Hence, for $p \in \mathcal{L}$,

$$p \prec q \quad \Longleftrightarrow \quad \forall u \in \mathcal{U}^q : u \cdot p < u \cdot q$$

or equivalently (using (A.1)),

$$p \succcurlyeq q \quad \Longleftrightarrow \quad \exists u \in \mathcal{U}^q : u \cdot p \geq u \cdot q.$$

To prove the right-hand side equivalence of (4) let $p \in \mathcal{L}$ be an interior point satisfying $p \succcurlyeq q$ and let p' satisfy $p = \alpha p' + (1 - \alpha) q$ for $\alpha \in (0, 1)$. By (B.3), $p' \succcurlyeq p \Leftrightarrow p \succcurlyeq q$ and, by the preceding argument,

$$p' \succcurlyeq p \quad \Longleftrightarrow \quad \exists v \in \mathcal{U}^p : v \cdot p' \geq v \cdot p.$$

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Hence, since $v \cdot p' \geq v \cdot p \Leftrightarrow v \cdot p \geq v \cdot q$,

$$p \succcurlyeq q \iff \exists v \in \mathcal{U}^p : v \cdot p \geq v \cdot q.$$

To conclude, note that continuity implies the equivalence for non-interior points. ∎

Continuous example of necessity of condition $n \leq 3$ in Proposition 4 for WST:

Suppose that $n = 4$ and utility vectors are drawn from a uniform probability distribution ψ and the set $\mathcal{U}$ is given by a triangle with the following vertices:

$$\begin{aligned} u^A &= (0, 0.370, 0.465, 1), \\ u^B &= (0, 0.417, 0.805, 1), \\ u^C &= (0, 0.713, 0.98, 1). \end{aligned}$$

Note that the utility of the worst outcome is set to 0 while the utility of the best outcome is set to 1. Given that these preferences respect monotonicity with regard to first-order stochastic dominance, these restrictions are without any loss of generality. Figure 4 depicts the projection of the support of the probability distribution into the space of intermediate utility levels (the second and third components of the utility vectors). Since the triangle in Figure 4 lies entirely above the 45^0 line, all of the utility vectors in $\mathcal{U}$ satisfy monotonicity with respect to first-order stochastic dominance.

Place Figure 4 here

Consider the following set of lotteries:

$$\begin{aligned} p^1 &= (p_1^1, p_2^1, p_3^1, p_4^1) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0 \right), \\ p^2 &= (p_1^2, p_2^2, p_3^2, p_4^2) = \left(\frac{2}{3}, 0, \frac{1}{3}, 0 \right), \\ p^3 &= (p_1^3, p_2^3, p_3^3, p_4^3) = \left(\frac{3}{4}, 0, 0, \frac{1}{4} \right). \end{aligned}$$

The probability $\psi(p^i \succ p^j)$ that lottery p^i is preferred to lottery p^j has a simple graphical representation under monotonicity and four possible outcomes. For our example, the vertical straight line at $u_2 = 0.5$ in Figure 4 represents the set of vectors (u_2, u_3) for which

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the expected utility of lottery p^1 is equal to the expected utility of lottery p^3. Moreover, $(0, u_2, u_3, 1) \cdot p^1 \leq (0, u_2, u_3, 1) \cdot p^3$ if and only if $u_2 \leq 0.5$. Thus, if the area of the triangle to the left of the vertical line at 0.5 is greater than the area to the right then $\psi(p^3 \succ p^1) \geq 0.5$. Similarly, $\psi(p^1 \succ p^2) \geq 0.5$ if and only if the area below the line $u_3 = 1.5u_2$ is larger than the area above it while $\psi(p^2 \succ p^3) \geq 0.5$ if and only if the area above the horizontal line $u_3 = 0.75$ is larger than the area below it. Calculating these areas we obtain

$$
\begin{aligned}
\psi\left(p^1 \succ p^2\right) &= 0.556, \\
\psi\left(p^2 \succ p^3\right) &= 0.536, \\
\psi\left(p^3 \succ p^1\right) &= 0.52.
\end{aligned}
$$

Thus, the probability distribution ψ violates WST for lotteries p^1, p^2, and p^3. In this example, all preference structures in the support of the uniform probability distribution are comonotonic. However, since there are four possible outcomes we were able to find a probability distribution that led to a violation of WST with respect to $\mathcal{P}$. Similar examples can be constructed for sets $\mathcal{P}$ of higher dimensions.

Proof of Proposition 4: Consider a comonotonic PDM. Without any loss of generality assume that for all $u \in \mathcal{U}$, $u(x_1) > u(x_2) > u(x_3)$ and that $\mathcal{U} \subset H = \{y \in \mathbb{R}^3 | \sum y_i = 0\}$. Since all u satisfy $u \cdot \mathbf{e}^1 > u \cdot \mathbf{e}^3$, $\mathcal{U}$ is a subset of the half plane $\{y \in H | (\mathbf{e}^1 - \mathbf{e}^3) \cdot y \geq 0\}$.

Assume, by way of negation, that WST is violated. Then there exist three lotteries $p, q, r \in \mathcal{L}$ such that $\psi(p \succ q) > 0.5$, $\psi(q \succ r) > 0.5$ and $\psi(r \succ p) > 0.5$. The first two inequalities imply $\psi(p \succ q \succ r) > 0$ and hence the existence of a utility $u \in \mathcal{U}$ satisfying $u \cdot p > u \cdot q > u \cdot r$. Similarly, the last two inequalities imply the existence of $v \in \mathcal{U}$ satisfying $v \cdot q > v \cdot r > v \cdot p$ and the first and last inequalities imply the existence of $w \in \mathcal{U}$ satisfying $w \cdot r > w \cdot p > w \cdot q$.

This, however, cannot hold: since all utilities belong to a half plane, without any loss of generality, there exist $\alpha, \beta > 0$ such that $w = \alpha u + \beta v$. This implies

$$
w \cdot q = \alpha(u \cdot q) + \beta(v \cdot q) > \alpha(u \cdot r) + \beta(v \cdot r) = w \cdot r.
$$

A contradiction. ∎

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Proof of Proposition 5: Consider a risk averse PDM. Without loss of generality assume that $\mathcal{U} \subset H = \{y \in \mathbb{R}^3 | \sum y_i = 0\}$ and that $x_1 > x_2 > x_3$. Let $t \in (0, 1)$ satisfy $tx_1 + (1 - t) x_3 = x_2$ and consider the lotteries $\mathbf{e}^2$ and $q = (t, 0, 1 - t)$ (the lottery that yields x_1 with probability t and x_3 with probability $(1 - t)$) in $\mathcal{L}$. By risk aversion, every EU preference with utility in $\mathcal{U}$ prefers to move from q to $\mathbf{e}^2$, hence $\mathcal{U}$ is a subset of the half plane $\{y \in H | (\mathbf{e}^2 - q) \cdot y \geq 0\}$. Then, follow the proof of Proposition 4.

The case of a risk seeking PDM is similar. ∎

Proof of Proposition 6: Assume, by way negation, that WST is violated. Hence there exists a triplet of Betweenness functionals $\{V^{\tau_i}\}_{i=1}^3$ and lotteries $\{p, q, r\}$ that satisfy the following rankings

$$V^{\tau_1} (p) > V^{\tau_1} (q) > V^{\tau_1} (r),$$
$$V^{\tau_2} (q) > V^{\tau_2} (r) > V^{\tau_2} (p),$$
$$V^{\tau_3} (r) > V^{\tau_3} (p) > V^{\tau_3} (q).$$

By betweenness, the corresponding local utilities $u_q^{\tau_1}$, $u_r^{\tau_2}$ and $u_p^{\tau_3}$ satisfy

$$u_q^{\tau_1} \cdot p > u_q^{\tau_1} \cdot q > u_q^{\tau_1} \cdot r, \tag{8}$$
$$u_r^{\tau_2} \cdot q > u_r^{\tau_2} \cdot r > u_r^{\tau_2} \cdot p,$$
$$u_p^{\tau_3} \cdot r > u_p^{\tau_3} \cdot p > u_p^{\tau_3} \cdot q.$$

Hence, a PDM with core EU preferences defined by $u_q^{\tau_1}$, $u_r^{\tau_2}$ and $u_p^{\tau_3}$ and a uniform probability distribution ψ violates WST with respect to the lotteries $\{p, q, r\}$.

To see that part (1) holds, note that if all V^{τ_i}s are increasing with respect to the relation of first-order stochastic dominance then $u_q^{\tau_1}$, $u_r^{\tau_2}$ and $u_p^{\tau_3}$ are increasing functions. Together with (8), this contradicts Proposition 4. Part (2) holds because if all $\succeq^{\tau_i}$'s are risk averse (risk seeking) then $u_q^{\tau_1}$, $u_r^{\tau_2}$ and $u_p^{\tau_3}$ are concave (convex) functions. Combine this with (8) to obtain a contradiction to Proposition 5. ∎

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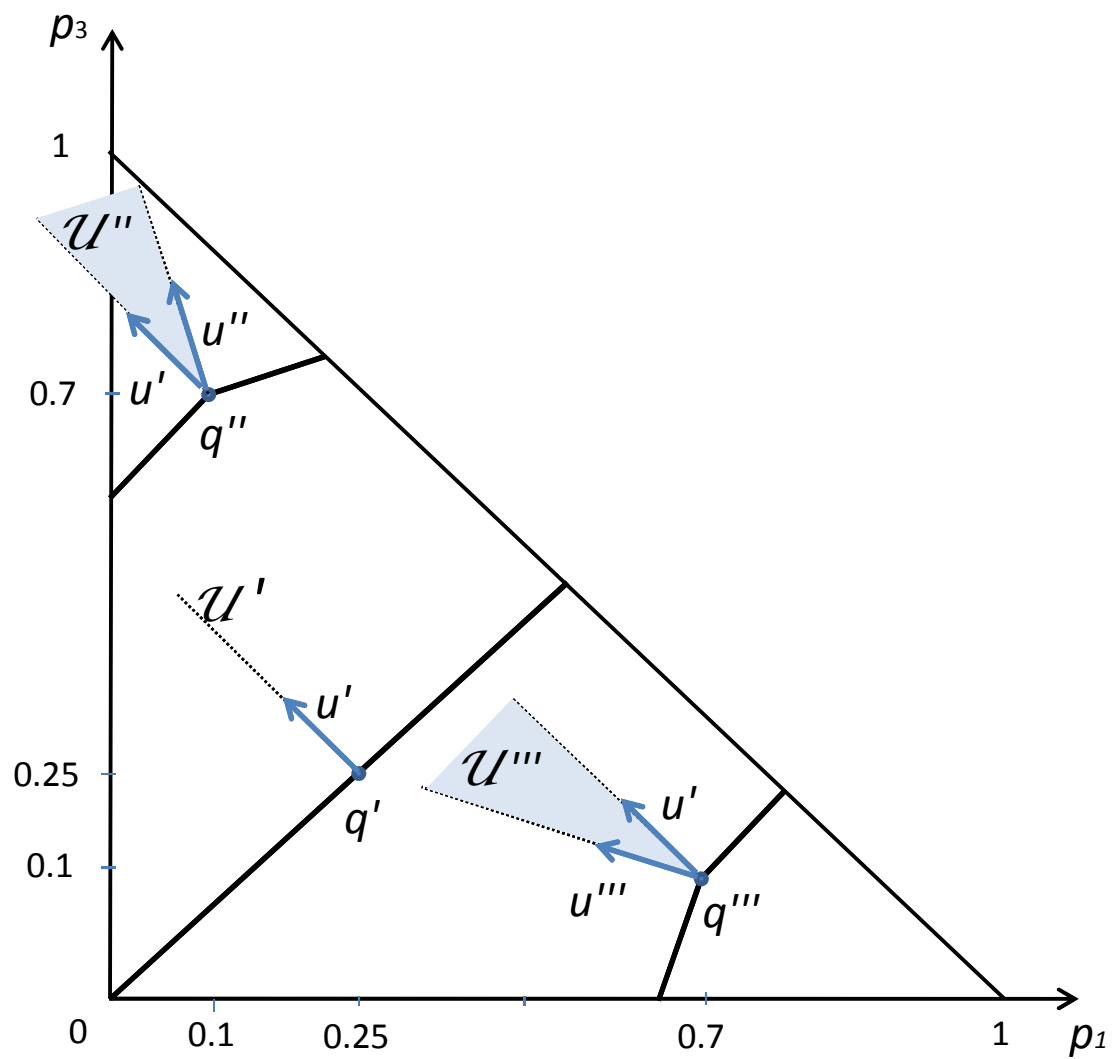
Figure 1 The utility cones $\mathcal{U}'$, $\mathcal{U}''$ and $\mathcal{U}'''$, of the points q', q'' and q''', respectively, are depicted .
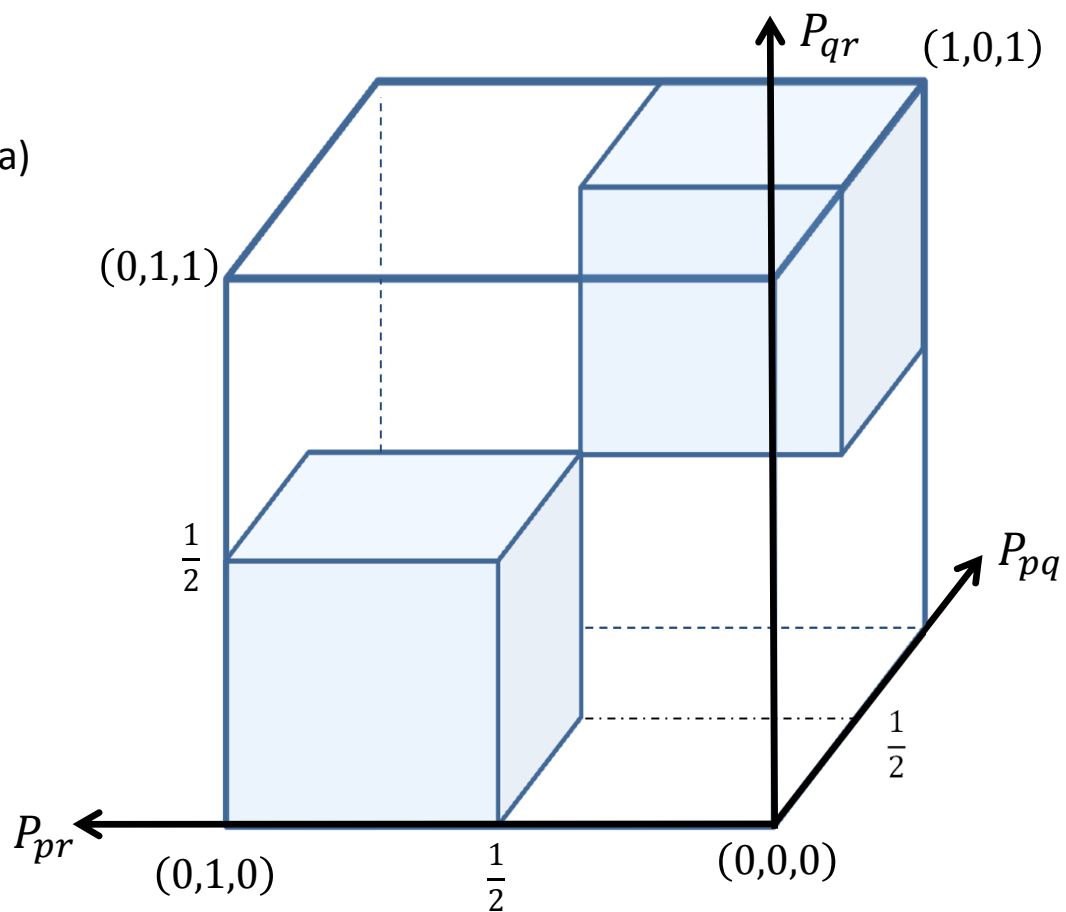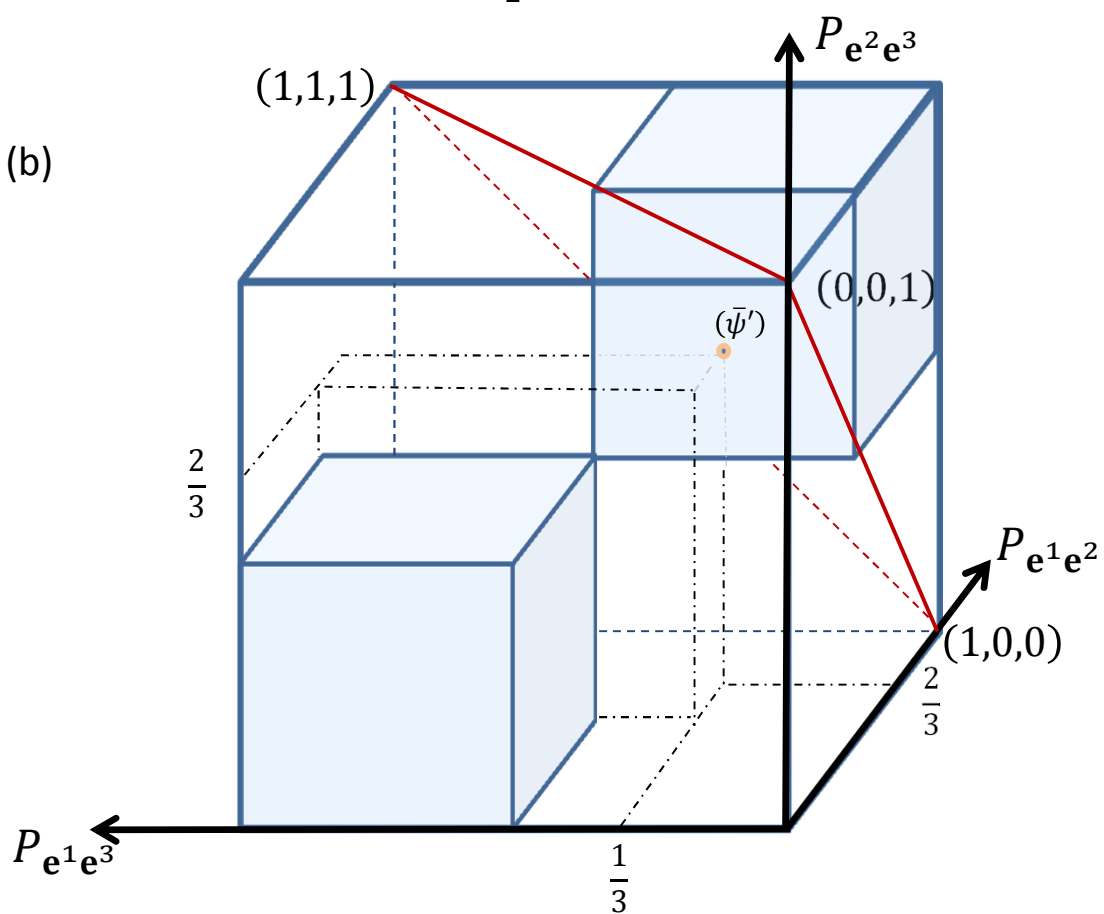
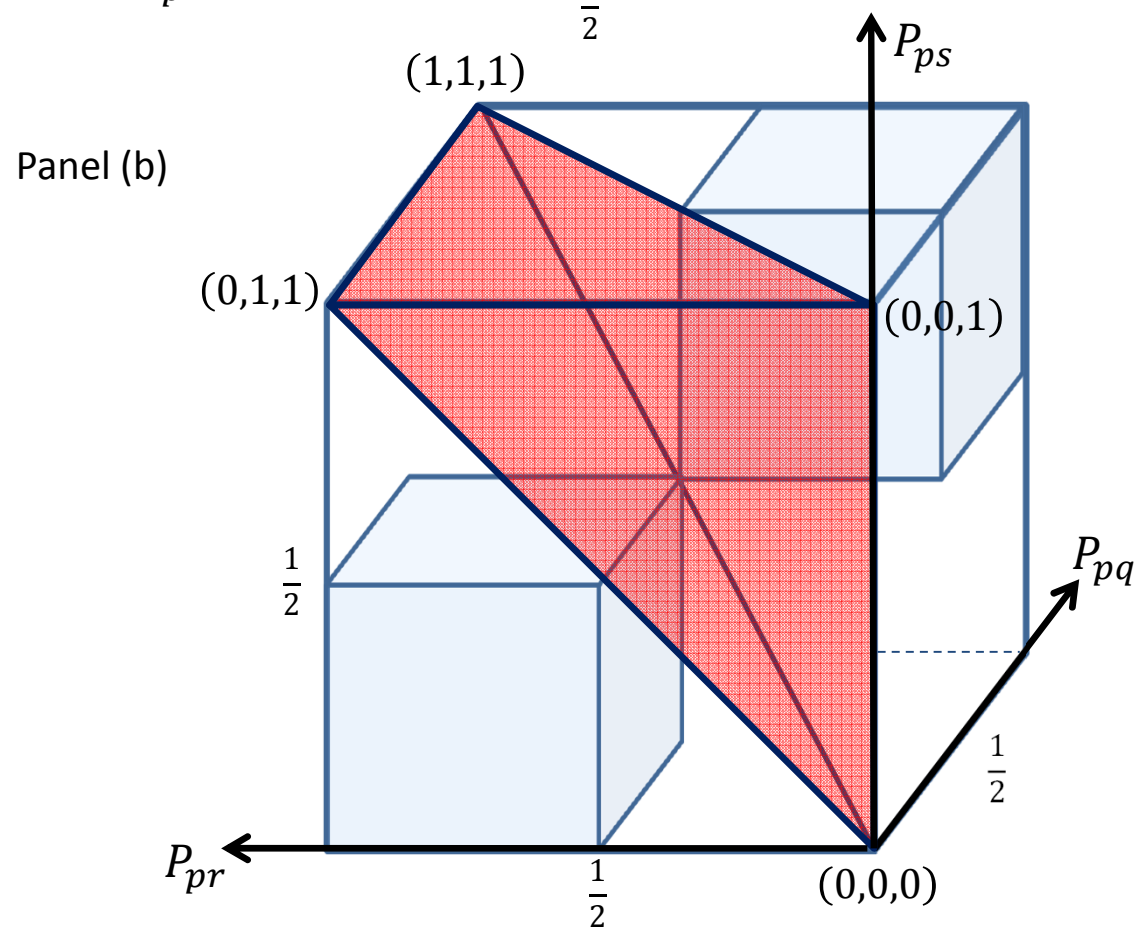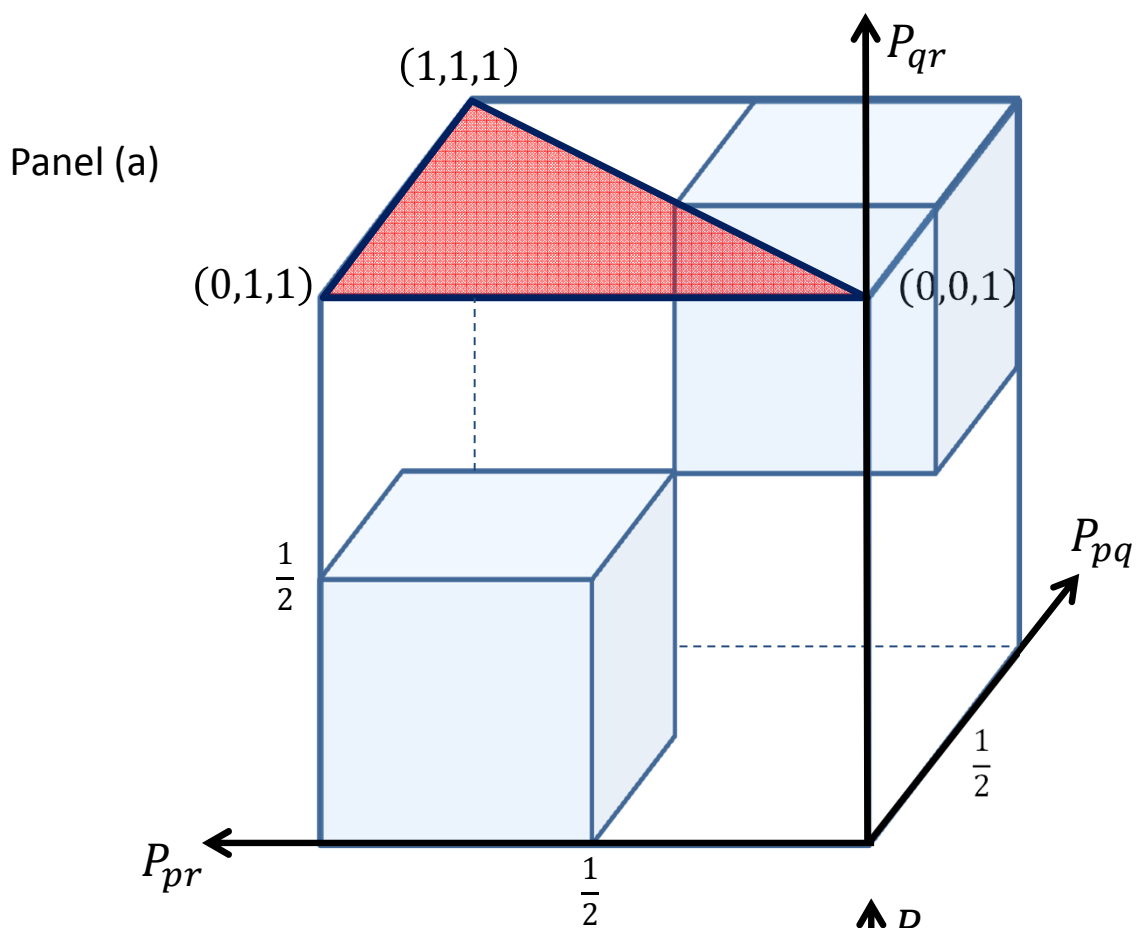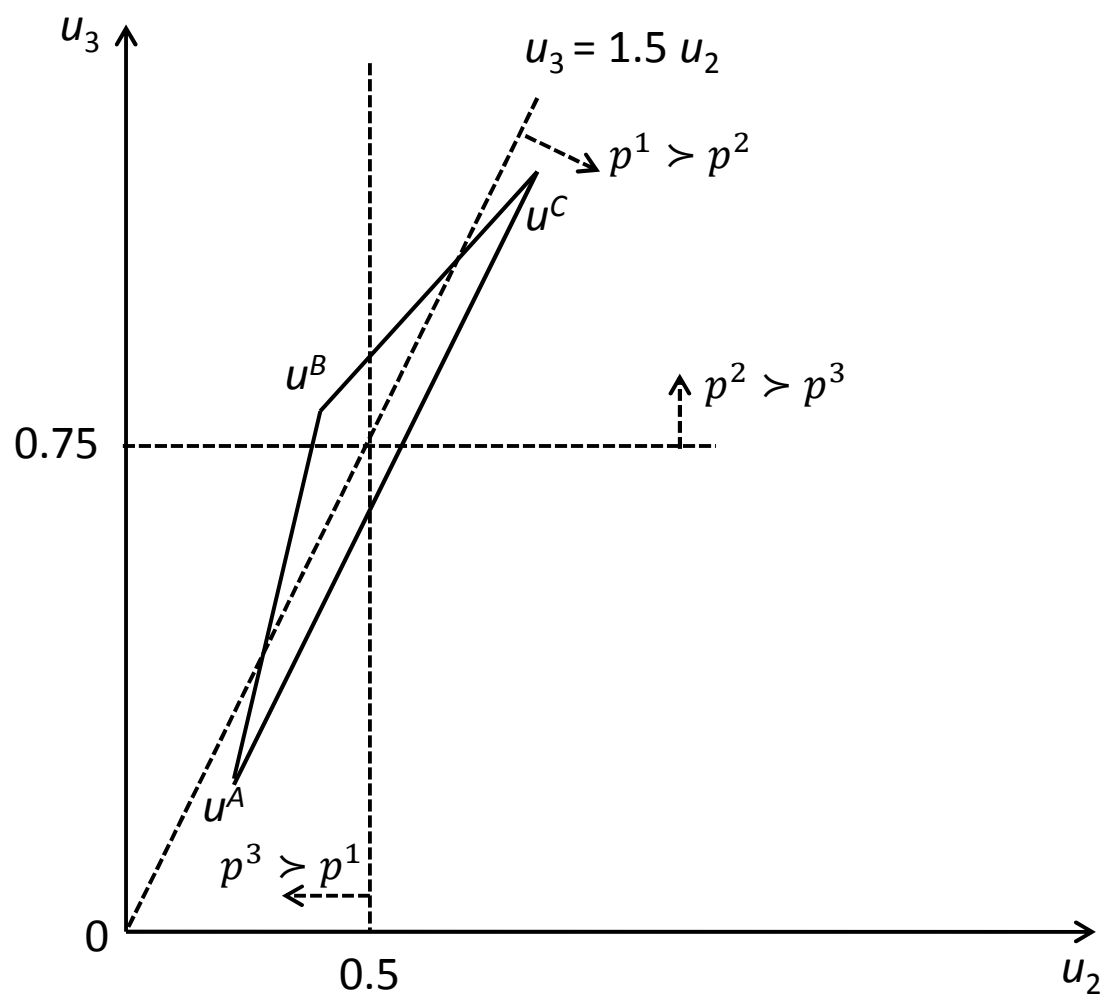**Figure 2.** Geometric Representation of WST

Figure 3. Binary choice probabilities that satisfy WST

Figure 4. Example of a Violation of WST.