Price of Anarchy for Congestion Games with Stochastic Demands

by

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Contents

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Declarations

I hereby declare that this thesis is a presentation of my own research, and has not been submitted to anywhere else for any degree.

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Abstract

The price of anarchy is a game-theoretical concept and it measures system degradation caused by players' selfish behaviours. This thesis extends models of congestion games to take stochastic demands into account and studies the price of anarchy on the basis of generalised models developed in this research. In the presence of stochastic demands, the models developed in this study better reflect the reality of a transportation network. The study would help provide a theoretical foundation and insights into mechanism design of transportation games and traffic control in practice.

This thesis is concerned with both non-atomic and atomic congestion games, which involve an infinite and finite number of travellers respectively. We introduce the notions of user equilibrium and system optimum under stochastic demands and investigate the behaviours of travellers and central coordinators in a stochastic environment. At a user equilibrium, travellers choose routes independently and aim to minimise their own expected travel costs, while at a system optimum, traffic is fully coordinated to minimise the expected total cost over the whole network.

We extend two existing methods of bounding the price of anarchy and compute the quality upper bounds for polynomial cost functions and very general settings of demand distributions. More specifically, we consider positive-valued distributions and normal distributions for non-atomic congestion games, and positive-valued discrete distributions for atomic congestion games. Our results show that the price of anarchy depends on the class of cost functions, demand distributions and, to some extent, network topologies. All the upper bounds are tight in some special cases, including the case of deterministic demands. The two bounding methods are also compared.

Chapter 1

Introduction

1.1 Introduction

Congestion games are a class of games in game theory, first proposed by Rosenthal [1973], which illustrate non-cooperative situations involving players competing for a finite set of resources. In a congestion game, each player chooses a combination of resources, and the payoff associated with each resource depends on how many players involve that resource in their choices. Congestion games have applications in various fields, for example transportation science, telecommunications [Roughgarden and Tardos, 2002] and ecology [Milinski, 1979].

Routing problem in transportation networks [Wardrop, 1952; Beckmann et al., 1956] is a very important application of congestion games. Although each traveller cares only about his/her own travel cost (time) and chooses his/her respective cheapest path selfishly, the cost required to travel along a given path depends not only on his/her own choice but also on the amount of traffic congestion. The more congested the path, the longer it takes to pass through it. This congestion effect generates interdependencies between travellers' decisions: we may expect to see a steady state in which no traveller can improve his/her own cost by switching unilaterally to other paths. In other words, all travellers in a the same origin-destination pair experience the same and cheapest travel cost. Such an assignment is actually a Nash equilibrium of the congestion game. We use the standard terminology of traffic networks and call this a *user equilibrium* (UE).

Inefficiency is a well-known characteristic of a Nash equilibrium. For example, the unique Nash equilibrium in the famous Prisoner's Dilemma is Pareto inefficient. The disadvantage of the UE is obvious in real applications: scarce traffic resources (street and road capacity) may be used in an inefficient way [Koutsoupias and Papadimitriou, 1999; Helbing et al., 2005]. Another classic model, the *system optimum* (SO), describes the most efficient assignment of traffic by assuming that all traffic is perfectly coordinated by a central authority. Although the SO overcomes the shortcoming of system inefficiency, it introduces the problem of unfairness, as some travellers have to take longer detours to contribute to system efficiency. It is hard to implement the SO in the real world without forcing travellers to submit to the coordination of a central authority. It is based on the rather unrealistic behavioural assumption that travellers will sacrifice their own benefits and cooperate with each other to meet the system objective [Moreno-Quintero, 2006]. Hence the SO is only an ideal model, and the UE is more realistic for practical applications.

There is an apparent bottleneck preventing improvements in the efficiency of traffic networks under these two classic models. Ideally, users would be allowed to make their own decisions in the hope that the outcomes would approximate to the system objective. From this arises an interesting problem: how bad is the UE, and how much can system efficiency be improved by implementing traffic coordination?

Quantifying the inefficiency of UE has emerged as a major line of research in algorithmic game theory. Quantitative methods reveal factors influencing the inefficiency of UE and thus provide directions and guidelines to improve the efficiency of traffic networks. This issue is especially crucial when UEs are unacceptably inefficient and where directly imposing an optimal solution is impractical.

Quantifying the inefficiency of UE enables us to deem certain outcomes of a game optimal or approximately optimal. This also makes it possible to identify conditions that may make equilibria in routing games optimal or approximately optimal, meaning that the benefit of imposing a controlling authority is relatively small. In addition, quantifying the inefficiency of UE may contribute to the design of mechanism design of routing games, namely designing a new game or modifying an existing game to minimise the inefficiency of their equilibria.

The price of anarchy (PoA), first introduced by Koutsoupias and Papadimitriou [1999] on a load-balancing game, is the most popular measure of system degradation due to lack of coordination. It is defined as the ratio between the worst objective function value of a user equilibrium and that of a system optimum. Successful attempts to bound the PoA for congestion games have been described in the literature. According to whether or not the amount of traffic controlled by each traveller is negligible, congestion games can be categorised into non-atomic and atomic. The next section provides a review of the literature on PoA for these two streams.

1.2 Literature Review

1.2.1 Non-atomic Congestion Games

A non-atomic congestion game refers to the assumption of a large number of travellers, each controlling a negligible fraction of the overall traffic. Thus, the congestion effect caused by a single traveller can be ignored.

Roughgarden and Tardos [2002] initiated the study of bounding the PoA for non-atomic congestion games. They bounded the PoA when the link cost functions are separable, semi-convex and differentiable [Roughgarden and Tardos, 2002, 2004] and proved that the PoA is dependent only on the class of the cost functions, and independent of the network topology [Roughgarden, 2003]. In particular, the PoA with affine cost functions is tightly bounded by 4/3.

The main developments in PoA research have been extensions to networks with a broader range of cost functions. Chau and Sim [2003] generalised Roughgarden and Tardos' results to cases with symmetric cost functions. Correa et al. [2004, 2008] provided a geometric proof of the upper bound of the PoA with cost functions that are non-convex, non-differentiable, and even discontinuous. Perakis [2007] extended the work to asymmetric cost functions and bounded the PoA by two parameters of asymmetry and nonlinearity. Sheffi [1985] introduced the notion of stochastic user equilibrium (SUE), which describes travellers' selfish routing decisions based on subjectively perceived travel costs by involving stochastic cost functions. The PoA on logit-based SUE was bounded by Guo et al. [2010] on the basis of Sheffi's model.

Another line of development in the study of PoA is to improve the setting of the traffic demand to better reflect reality. Chau and Sim [2003] presented a weaker upper bound on the PoA with elastic demands. Although the study of the PoA with stochastic demands is still quite new, efforts have been made to model UE and SO involving demand uncertainty. It has been assumed that the objective of selfish travellers is to choose a path that minimises the mean travel cost [Sumalee and Xu, 2011] or weighted sum of the mean and the variance of the travel cost [Sumalee and Xu, 2011; Bell and Cassir, 2002], with risk-neutral and risk-averse travellers respectively. A travel time budget has also been considered in the equilibrium condition on the basis of reliability [Lo et al., 2006; Shao et al., 2006]. However, in order to deduce the distributions of the path and link flows, all of these studies have relied on some assumptions, such as that all the path flows follow the same type of distribution as the demand and have the same ratio of variance (or standard deviation) to mean [Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008], and that all the path flows are independent [Clark and Watling, 2005; Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008]. These assumptions are open to question regarding the relationship between path flows and demands, not only because of a lack of supporting empirical data but also because they violate the demand feasibility constraint even in simple networks. In order to produce a more reliable result for the PoA, we need to relax the aforementioned assumptions and establish a new equilibrium condition.

1.2.2 Atomic Congestion Games

Although congestion games were first introduced in an atomic setting, the PoA was studied later for atomic congestion games than that for non-atomic ones. In an atomic congestion game, there is a finite number of travellers, each of whom must consider the congestion effect of his/her own traffic when making a routing decision. Owing to the discreteness of the atomic model, bounding the PoA is more difficult than in a non-atomic model. The PoA for atomic congestion games was first studied in networks with a simple structure, for example parallel link networks [Koutsoupias and Papadimitriou, 1999] and ring networks [Chen et al., 2010]. In a general network, the upper bounds of the PoA have been obtained by Awerbuch et al. [2005] and Christodoulou and Koutsoupias [2005] independently. In particular, the PoA with affine cost functions is bounded by 2.5 precisely for unweighted demand and 2.618 for weighted demand, which are both proved to be tight. The PoA with polynomial cost functions of degree m is bounded by $m^{\Theta(m)}$. These results hold for both pure strategies and mixed strategies.

Aland et al. [2011] have improved the work of Awerbuch et al. [2005] and Christodoulou and Koutsoupias [2005], giving exact bounds of the PoA with polynomial cost functions. For the weighted case, the PoA with polynomial cost functions with degree at most m is bounded by Φ_m^{m+1}, where Φ_m is defined as the unique non-negative real solution to $(x+1)^m = x^{m+1}$ (which is the golden ratio when $m = 1$). For the unweighted case, the worst case PoA is

$$\frac{(r+1)^{2m+1} - r^{m+1}(r+2)^m}{(r+1)^{m+1} - (r+2)^m + (r+1)^m - r^{m+1}},$$

where $r = \lfloor \Phi_m \rfloor$.

Roughgarden [2009] has integrated preceding works on the PoA (including studies by Awerbuch et al. [2005]; Christodoulou and Koutsoupias [2005]; Aland et al. [2011]) and has given a canonical sufficient condition to bound the PoA with pure equilibrium, known as the "smoothness argument". It has been found that the upper bounds of the PoA obtained via a smoothness argument can be extended automatically to mixed equilibrium and correlated equilibrium with no quantitative degradation [Roughgarden, 2009]. The upper bounds in [Aland et al., 2011] also hold for coarse correlated equilibrium.

To the best of our knowledge, uncertainty is still new in the study of PoA for atomic congestion games. In this thesis, we will consider stochastic demand as the source of uncertainty, as traffic demand varies day-to-day in the real world.

1.3 Outline and Main Contributions

This section presents an outline of this study and an overview of its main contributions. This thesis focuses on modelling congestion games theoretically in more complex situations of stochastic demand, which are thus closer to reality, and on proving the upper bounds of the PoA. We shall consider both non-atomic settings (Chapters 3 and 4) and atomic settings (Chapters 5 and 6). For each of these, we shall extend the literature by theoretically investigating how travellers make routing choices, at what point they reach a steady state of equilibrium, how bad their selfish routing decisions are, and what affects network inefficiency in stochastic environments. We explicitly address these research questions and their relationships in a chapter-by-chapter discussion as follows.

Chapter 2: Preliminaries

Chapter 2 provides a review of deterministic work on congestion games. We first introduce a basic traffic model in a general network, and then present fundamental concepts such as the user equilibrium, the system optimum, and the PoA of non-atomic and atomic congestion games. We also review classic results for the upper bounds of the PoA and discuss some simple examples to show the tightness of the upper bounds. The central concepts and bounding methods reviewed in this chapter will be extended in later chapters. In this respect, Chapter 2 is a foundation for all subsequent chapters.

Chapter 3: Non-atomic Model

It is well known that traffic demand varies from day to day. In order to reflect demand fluctuation in the real world, in Chapter 3 we consider networks with stochastic demands. We assume that the traffic demands are publicly known in probabilities, as travellers are able to draw on historical data from their own experiences. The actual demand level is assumed to be known in a deterministic model, enabling travellers easily to find the cheapest (shortest) paths. However this is no longer true in networks with demand uncertainty; thus, we must find new ways to model travellers' routing behaviours and to define the user equilibrium and the system optimum.

This becomes more difficult and complex in the case of stochastic demands because path and link flows and travel costs will all be random. Chapter 3 presents an analytical method to compute path and link flows under given demand distributions and, from a practical perspective, to describe travellers' behaviours by path choice probabilities. We generalise the deterministic UE condition to a stochastic version with risk-neutral travellers. A user equilibrium with stochastic demand (UE-SD) is defined as a steady state in which no traveller can improve his/her own expected travel cost by unilateral derivation. In contrast to the deterministic non-atomic model, the model with stochastic demands may have multiple equilibria. We also present system optimum with stochastic demand (SO-SD), in which the expected total cost in the network is minimised.

Bibliographic Information: This chapter is based on a research article by Wang et al. [2014a].

Chapter 4: PoA for Non-atomic Congestion Games

Chapter 4 defines the PoA on the basis of the UE-SD and SO-SD models in Chapter 3, and bounds the PoA with stochastic demands. Roughgarden and Tardos [2002, 2004] use the convexity of the total cost functions and bound the deterministic PoA, while Correa et al. [2008] provide a geometric proof of the upper bound of the PoA. We extend both the methods to bound the PoA in our stochastic model. For convenience, we refer to the methods from [Roughgarden and Tardos, 2002, 2004] and [Correa et al., 2008] as the *convexity method* and the *geometry method*.

We first prove the convexity and geometry bounds of the PoA with stochastic demands with general cost functions and general demand distributions. As stated in the deterministic literature, with no restriction on link cost functions the PoA is unbounded, and polynomial cost functions are usually considered in bounding the PoA. We also focus on polynomial cost functions and compute exact upper bounds for the PoA with stochastic demands. As traffic demand is positive in the real world and normal distributions are also widely used in the literature to simulate traffic demands, we discuss the PoA when demands follow positive-valued distributions and normal distributions in Sections 4.3.1 and 4.3.2 respectively.

In deterministic work, the PoA depends only on the class of the cost functions [Roughgarden, 2003], while in the presence of stochastic demand, as we shall show in Chapter 4, the PoA depends on the class of cost functions, distributions of traffic demands and, to some extent, network topologies. Moreover, all the upper bounds discussed in Chapter 4 are tight in some special cases, including deterministic demands. Thus Chapter 4 generally extends the work of [Roughgarden and Tardos, 2002, 2004] and [Correa et al., 2008] by investigating stochastic traffic demands.

We also compare the geometry bound and convexity bound. From numerical comparison, we find that the former is generally tighter than the latter. However, in special cases of affine cost functions and single commodity networks, the convexity bound is tighter.

Bibliographic Information: This chapter is based on a research article by Wang et al. [2014a].

Chapter 5: Atomic Model

Chapters 3 and 4 will consider non-atomic congestion games with an infinite number of travellers, in which the amount of traffic controlled by a single traveller can be ignored. In Chapter 5, we focus on a different setting of atomic games, which assumes a finite number of travellers. In an atomic game, the traffic controlled by each traveller is non-negligible and has a significant effect on traffic congestion. It is known from the literature that atomic problems are more complex and difficult due to their discreteness.

Chapter 5 extends the literature on unweighted atomic congestion games by considering stochastic demands. Similarly to Chapter 4 for non-atomic games, we also assume that the traffic demand distributions are publicly known, and that all travellers will make routing decisions on the basis of the whole distribution of traffic demands to minimise their own expected travel costs. As each traveller controls one unit of traffic in the unweighted setting, the atomic congestion game with stochastic demands described in this thesis is actually a game with random players.

Chapter 5 is organised in a similar way to Chapter 3. We first use mixed strategies to model travellers' routing behaviours and show how to derive random path and link flows. Then we establish models of user equilibrium and system optimum with stochastic demands (UE-SD and SO-SD). The existence and uniqueness of the UE-SD will also be discussed.

Bibliographic Information: This chapter is based on a research article by Wang et al. [2014b].

Chapter 6: PoA for Atomic Congestion Games

Chapter 6 defines and analyses the PoA with stochastic demands on the basis of the models of UE-SD and SO-SD described in Chapter 5. It is interesting that in the case of stochastic demands, the discreteness of the atomic games in our study vanishes. Motivated by the work discussed in Chapter 4 on non-atomic work, we extend Roughgarden and Tardos' (2002; 2004) convexity method and Correa et al.'s (2008) geometry method to bound the PoA for atomic congestion games with stochastic demands.

As mentioned in Chapter 5, the traffic demand in atomic congestion games represents the number of travellers, which should be a positive integer in the real world. Thus, we focus on positive-valued discrete demand distributions in bounding the PoA. In Chapter 6, we first present both convexity and geometry upper bounds with general cost functions and general positive-valued demand distributions, and then compute exact upper bounds for affine and polynomial cost functions. For affine cost functions, we present both convexity and geometry upper bounds with general positive-valued demand distributions in general networks. For polynomial cost functions, we compute the two upper bounds with general positive-valued demand distributions in single commodity networks. All the upper bounds in this chapter are asymptotically tight when the demands approach infinity. From numerical comparisons, we find that the convexity upper bound is tighter than the geometry one.

Bibliographic Information: This chapter is based on a research article by Wang et al. [2014b].

Chapter 2

Preliminaries

In this chapter we lay the ground work and introduce the notation that will be used throughout the thesis. Sections 2.2 and 2.3 introduce non-atomic and atomic work in deterministic models respectively. In each section, we present the definition of user equilibrium and system optimum, and illustrate the main results on bounding the PoA. The models we shall discuss in the following chapters are generalisations of the basic models in this chapter.

2.1 Basic Traffic Model

Consider a general network $G = (N, E)$, where N and E denote the set of nodes and links, respectively. To each link $e \in E$, we associate a (link) cost function $c_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, which is assumed to be non-decreasing in its argument, the link flow. Costs have concrete interpretations in terms of money or the delay incurred in a network, A subset of nodes form a set of origin-destination (O-D) pairs, denoted by I. We call an O-D pair $i \in I$ a *commodity*. Every O-D pair, $i \in I$, is associated with a traffic demand, $d_i > 0$. We use vector $\mathbf{d} = (d_i : i \in I)$ to denote the demands in the whole network. Parallel links are allowed and a node can be in multiple O-D pairs. Denote P_i as the set of all possible paths connecting an O-D pair $i \in I$. Each traveller from O-D pair $i \in I$ has to choose a path from P_i. After all s make their routing choices, the output in the network is the resulting traffic flows, denoted as follows:

- f_k^i: traffic flow on path $k \in P_i$, $i \in I$;
- $\mathbf{f}$: vector of path flows, i.e., $\mathbf{f} = (f_k^i : k \in P_i, i \in I)$;
- v_e: traffic flow on link $e \in E$;
- $\mathbf{v}$: vector of link flows, i.e., $\mathbf{v} = (v_e : e \in E)$.

Given demands vector $\mathbf{d}$, flow $\mathbf{f}$ is said to be feasible if

$$\sum_{k \in P_i} f_k^i = d_i, \quad \forall\, i \in I. \tag{2.1}$$

It is clear that the flow on each link is the sum of flows on all the paths that include the link:

$$v_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i f_k^i, \quad e \in E,$$

where $\delta_{k,e}^i$ is the link-path indicator, which is 1 when link e is included in path $k \in P_i$ and 0 otherwise. The path cost is simply the sum of the cost of those links that constitute the path, i.e.,

$$c_k^i(\mathbf{f}) = \sum_{e \in E} \delta_{k,e}^i c_e(v_e), \quad \forall\, k \in P_i,\ \forall\, i \in I.$$

Given the link costs, we can also compute the total (social) cost as follows:

$$C(\mathbf{f}) = \sum_{e \in E} c_e(v_e).$$

We denote any instance of a congestion game by a triple $(G, \mathbf{d}, \mathbf{c})$, where G is the underlying network, $\mathbf{d}$ and $\mathbf{c}$ are the vectors of demands and cost functions, respectively. For notational simplicity, we do not specify the dimension of vector $\mathbf{c}$ with paths or links. We use $\mathbf{c}(\mathbf{f})$ and $\mathbf{c}(\mathbf{v})$ to distinguish vectors of path costs and link costs.

2.2 Deterministic Non-atomic Model

2.2.1 User Equilibrium

Recall that we assume that an infinite number of players participate in the non-atomic congestion games, and traffic controlled by a single player is negligible. Every traveller wants to minimise his/her own travel cost selfishly, and would deviate if he/she finds a cheaper path. With the non-atomic setting, we usually only consider that travellers play pure strategies. As any flow assignment induced by mixed strategies can also be attained by assigning travellers to different paths, according to corresponding proportions. The travel costs depend only on the flow assignment in the network. Thus, from the perspective of traffic flow assignment, also called *action distribution* [Roughgarden and Tardos, 2002], pure strategies and mixed strategies are the same thing in a non-atomic congestion game.

Next we will introduce the definition of user equilibrium (UE) in non-atomic congestion games. As it was first formally defined by Wardrop [1952] in his first principle, equilibrium flows are also called *Wardrop equilibrium.* At a user equilibrium, all the travellers are travelling along the cheapest path.

Definition 2.1. (Non-atomic UE) [Wardrop, 1952] Let $\mathbf{f}$ be a feasible flow for non-atomic instance $(G, \mathbf{d}, \mathbf{c})$. The flow is a UE if, for every commodity $i \in I$ and every pair $k, k' \in P_i$ with $f_k^i > 0$

$$c_k^i(\mathbf{f}) \leq c_{k'}^i(\mathbf{f})$$

In non-atomic congestion games with continuous and non-decreasing cost functions, a user equilibrium defined above is equivalent to a Nash equilibrium [Palma and Nesterov, 1998].

The UE condition can be reformulated as a minimization problem, which helps to illustrate the relationship between the equilibrium flow and optimal flow.

Proposition 2.1. *[Beckmann et al., 1956] Let* $(G, \mathbf{d}, \mathbf{c})$ *be a non-atomic instance. Then* $\mathbf{f}$ *is a UE flow, if and only if it solves the following minimization problem:*

$$\begin{aligned} \min_{\mathbf{f}} \quad & \sum_{e \in E} \int_0^{v_e} c_e(x)dx \\ s.t. \quad & \sum_{k \in P_i} f_k^i = d_i, \qquad \forall\, i \in I \\ & f_k^i \geq 0, \qquad \forall\, k \in P_i,\ \forall\, i \in I. \end{aligned}$$

More generally, Wardrop conditions are stated as the finite-dimensional variational inequality(VI) problem.

Proposition 2.2. *[Nagurney, 1998] Let* $(G, \mathbf{d}, \mathbf{c})$ *be a non-atomic instance. Then feasible* $\mathbf{f}^*$ *is a UE flow, if and only if it solves the following variational inequality (VI) problem:*

$$\mathbf{c}(\mathbf{f}^*)^T(\mathbf{f} - \mathbf{f}^*) \geq 0, \qquad \text{for any feasible } \mathbf{f} \geq 0,$$

where we recall that flow is said to be feasible when (2.1) *is satisfied.*

From the equivalent minimization problem in Proposition 2.1, the existence and uniqueness conditions of the UE can be easily proved.

Proposition 2.3 (Existence and Uniqueness of UE)**.** *[Beckmann et al., 1956] Let* $(G, \mathbf{d}, \mathbf{c})$ *be a non-atomic instance.*
(a)If cost functions $\mathbf{c}$ *are continuous, instance* $(G, \mathbf{d}, \mathbf{c})$ *admits at least one*

equilibrium flow.

(b) If $\mathbf{f}$ *and* $\mathbf{f}'$ *are both UE flows, then* $c_e(v_e) = c_e(v'_e)$ *for every* $e \in E$*, where* v_e *and* v'_e *are corresponding link flows on* $e \in E$ *for* $\mathbf{f}$ *and* $\mathbf{f}'$ *respectively.*

When link cost functions are strictly increasing, the objective function in Proposition 2.1 is convex, together with the compactness of the feasible domain of flows, the link flows of the UE can be guaranteed to be unique. But as one link could be included in multiple paths, the path flows may not be unique. The same result can also be proved via the VI problem in Proposition 2.2.

2.2.2 System Optimum

System optimum is simply defined as the optimal usage of the social resource. We adopt the social objective as minimising the total (average) travel cost in the network.

Definition 2.2 (System optimum)**.** Let $(G, \mathbf{d}, \mathbf{c})$ be an non-atomic instance. Feasible flow $\mathbf{f}$ is a system optimum if it solves the following minimization problem:

$$\begin{aligned} \min_{\mathbf{f}} \quad & \sum_{e \in E} c_e(v_e) v_e \\ \text{s.t.} \quad & \sum_{k \in P_i} f_k^i = d_i, \quad \forall\ i \in I, \\ & f_k^i \geq 0, \qquad \forall\ k \in P_i,\ \forall\ i \in I. \end{aligned}$$

Recall Proposition 2.1, UE condition can be also formulated as a minimization problem, connection between equilibrium and optimal flows can be established as they are optimal solutions for different objective functions. When $c_e(v_e) \cdot v_e$, $e \in E$ is convex and continuously differentiable, a UE flow is actually a SO flow for modified link cost functions (see [Nisan et al., 2007] for details).

2.2.3 Price of Anarchy

As mentioned in Chapter 1, the PoA is the worst-case ratio between the social welfare at a user equilibrium and at a system optimum [Koutsoupias and Papadimitriou, 1999]. In this section we introduce the definition of the PoA based on the model presented in preceding sections.

Definition 2.3. Let $(G, \mathbf{d}, \mathbf{c})$ be a non-atomic instance. The PoA is defined as

$$\mathrm{PoA}(G, \mathbf{d}, \mathbf{c}) = \max \left\{ \frac{C(\bar{\mathbf{f}})}{C(\mathbf{f}^*)} : \bar{\mathbf{f}} \text{ is a UE and } \mathbf{f}^* \text{ is a SO} \right\}.$$

Let $\mathcal{I}$ be a set of non-atomic instances. Then the PoA with respect to $\mathcal{I}$ is defined as

$$\mathrm{PoA}(\mathcal{I}) = \max_{(G,\mathbf{d},\mathbf{c}) \in \mathcal{I}} \mathrm{PoA}(G, \mathbf{d}, \mathbf{c}).$$

Let us look at a classic example introduced by Pigou [1920] to show how to compute the PoA

Example 2.1 (Pigou's example)**.** Consider a two-link network in Figure 2.1. One unit of traffic wants to travel from s to t, i.e., $d = 1$. The cost function on the upper link is a constant 1, and that on the lower link is x. We can expect that all the travellers will choose the lower link, as it is never worse than the upper link even when it is fully congested. Thus the total cost at the UE is 1. If the traffic is controlled by a central authority, the whole demand will be divided into halves and assigned onto each links, which can be found by solving the minimization problem in Definition 2.2. The total cost at the SO is $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$. Hence the PoA can be computed as 4/3.

It is interesting that the simple network shown in Pigou's example just gives the worse case of the PoA. Roughgarden and Tardos [2002, 2004] first prove the result by using the fact that the link cost functions are semi-convex and differentiable.

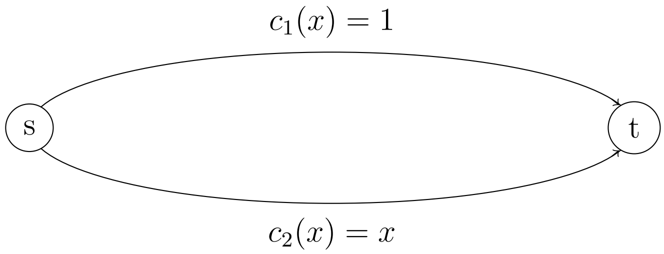


Figure 2.1: Pigou's example

Definition 2.4. If $c(\cdot)$ is a continuous differentiable and semi-convex cost function, then

$$\gamma(c) = \inf_{x \geq 0}(\mu + 1 - \lambda),$$

where $\lambda \in [0, 1]$ solves $\frac{d}{dx}c(\lambda x)\lambda x = c(x)$, $\mu = c(\lambda x)\lambda/c(x)$, and $0/0$ is defined to be 1. The anarchy value $\gamma(\mathcal{C})$ of a set $\mathcal{C}$ of cost functions is

$$\gamma(\mathcal{C}) = \inf_{0 \neq c \in \mathcal{C}} \gamma(c).$$

The value of $1/\gamma(\mathcal{C})$ is the *anarchy value* in [Roughgarden and Tardos, 2004], which is actually the tight upper bound of the PoA. The upper bound can be attained via a simple two-link network as in Example 2.1.

Theorem 2.4. *Let* $(G, \mathbf{d}, \mathbf{c})$ *be a instance with cost functions in* $\mathcal{C}$*. Then*

$$PoA(G, \mathbf{d}, \mathbf{c}) \leq \frac{1}{\gamma(\mathcal{C})}.$$

From the definition of the anarchy value, we can see it only depends on the class of the functions; thus the PoA is independent of the network topology [Roughgarden, 2003]. Note that the anarchy value need not be finite. Without any restriction on the cost functions, the PoA is unbounded. When link cost functions are affine, the upper bound of the PoA is $4/3$, which matches the lower bound in Example 2.1. When link cost functions are (non-zero) polynomials with highest degree at m, the PoA is bounded by $(1 - m(m+1)^{-(m+1)/m})^{-1}$. As the convexity of $c(x)x$ in Definition 2.4 plays an important role in bounding the PoA, we refer Roughgarden and Tardos' (2002; 2004) method as *convexity method* and the upper bound in Theorem 2.5 as *convexity bound*.

Correa et al. [2008] used a geometric method and obtained the same upper bound of the PoA.

Definition 2.5. [Correa et al., 2004] For cost function c, define

$$\beta(c) = \sup_{x \geq 0, y \geq 0} \frac{x(c(y) - c(x))}{y \cdot c(y)}.$$

where $0/0 = 0$ by convention. For a set of cost functions $\mathcal{C}$, define

$$\beta(\mathcal{C}) = \sup_{0 \neq c \in \mathcal{C}} \beta(c).$$

Correa et al. [2008] removed the assumption of semi-convex and differentiable functions. Next we show the geometric meaning of $\beta(c)$ in Figure 2.2. Easy to find, for any non-decreasing function c

$$\beta(c) = \sup_{0 \leq x \leq y} \frac{x(c(y) - c(x))}{y \cdot c(y)},$$

as $c(y) - c(x) < 0$ when $x > y$. The shaded rectangle represents $x(c(y) - c(x))$, which is within the big rectangle representing $yc(y)$. Thus $\beta(c)$ is the maximum ratio of the size of shaded rectangle to that of the big rectangle.

Due to the geometric explanation of $\beta(c)$, we refer this method as *geometry method* and the upper bound in [Correa et al., 2008] as *geometry upper bound*.

Theorem 2.5 (Geometry upper bound)**.** *Let* $(G, \mathbf{d}, \mathbf{c})$ *be a instance with cost*

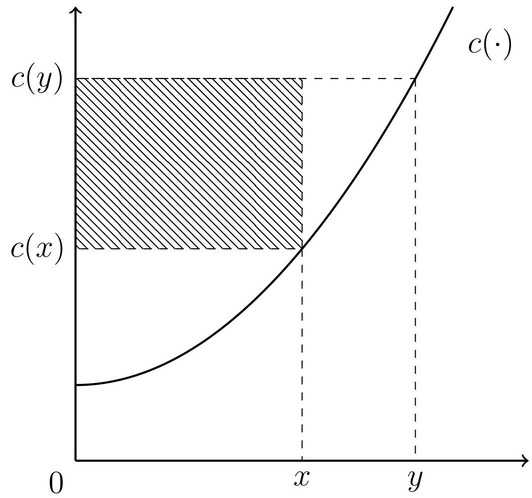


Figure 2.2: Illustration of $\beta(c)$

functions in $\mathcal{C}$. Then

$$PoA(G, \mathbf{d}, \mathbf{c}) \leq (1 - \beta(\mathcal{C}))^{-1}.$$

When cost functions in $\mathcal{C}$ are continuous, semi-convex and differentiable, the geometry bound is equivalent to the convexity bound, namely $1 - \beta(\mathcal{C}) = \gamma(\mathcal{C})$.

2.3 Deterministic Atomic Model

In atomic congestion games, the number of travellers is finite and each player controls a non-negligible amount of traffic. Different from non-atomic games, every player needs to take the congestion effect of his/her own traffic load into consideration when making the routing decision. Let J_i denote the set of players between O-D pair $i \in I$, and w_j be the amount of traffic controlled by player $j \in J_i$. In particular, when each player controls the same amount of traffic, it is called unweighted demand and usually modelled by $w_j = 1$ for all $j \in J_i$, $i \in I$ [Awerbuch et al., 2005; Christodoulou and Koutsoupias, 2005].

The general case is called weighted demand. The atomic works are divided into two streams on the basis that whether a single player is allowed to split his/her traffic onto different paths, i.e., splittable and unsplittable models. We focus on unsplittable games in this thesis, namely every traveller has to select one path to arrive the destination. The demand d_i is simply the number of travellers from O-D pair $i \in I$ under the atomic setting, i.e., $d_i = |J_i|$.

2.3.1 Use Equilibrium

As we have mentioned, the traveller needs to take the congestion effect of his own traffic into account, thus pure strategy and mixed strategy will make a difference in atomic congestion games. We will introduce the definitions of user equilibrium on the basis of both pure strategies and mixed strategies.

Let us first assume all the travellers play pure strategies. Each player chooses one path and assigns all his weight on it in order to minimise his own travel cost. Let $x_j \in P_i$ be a strategy of player $j \in J_i$. Let $\mathbf{x}$ be the tuple of strategy profile of all the players, which describes the outcome of a game. Let $\mathbf{x}_{-j}$ be the tuple of the strategies of all other players except player j. The cost $c_j(\cdot)$ incurred by player j in a game is a function of the strategies chosen by all the players in the network. For example, we use $c_j(y_j, \mathbf{x}_{-j})$ to denote the cost experienced by player j when he adopts strategy y_j and all the other players play according to $\mathbf{x}_{-j}$. The pure Nash equilibrium describes a steady state that, with all the other players' strategies fixed, no player can reduce his own travel cost by a unilateral deviation.

Definition 2.6 (Pure Nash equilibrium)**.** A pure strategy profile $\mathbf{x}$ is said to be a Nash equilibrium if, for any strategy $y_j \in P_i$ for player $j \in J_i$, $i \in I$,

$$c_j(\mathbf{x}) \leq c_j(y_j, \mathbf{x}_{-j}).$$

Let $\mathbf{v}$ be the flow formed by players' strategy profile $\mathbf{x}$. Then the costs in Definition 2.6 can be written as

$$c_j(\mathbf{x}) = \sum_{e \in x_j} c_e(v_e),$$
$$c_j(y_j, \mathbf{x}_{-j}) = \sum_{e \in y_j} c_e(y_j, \mathbf{x}_{-j}) = \sum_{e \in x_j \cap y_j} c_e(v_e) + \sum_{e \in y_j \setminus x_j} c_e(v_e + w_j).$$

For a mixed strategy, player $j \in J_i$ selects a probability distribution $\mathbf{p}_j = (p^i_{k,j} : \; k \in P_i)^T$ over the set of all pure strategies $x_j \in P_i$ between O-D pair $i \in I$. Let $\mathbf{p}$ be the tuple of mixed strategies of all the players and $\mathbf{p}_{-j}$ be the tuple of mixed strategies of all other players except player j. We use $EC_j(\cdot)$ to denote the expected travel cost incurred by traveller j, which is a function of the mixed strategies adopted by all the players. For example, $EC_j(\mathbf{p})$ is the expected travel cost of player j when all the players play according to strategy profile $\mathbf{p}$, and $EC_j(x_j = k, \mathbf{p}_{-j})$ is the expected cost for player j when he chooses path k and all the other players adopt strategies $\mathbf{p}_{-j}$. Then

$$EC_j(\mathbf{p}) = \sum_{k \in P_i} p^i_{k,j} EC_j(x_j = k, \mathbf{p}_{-j}), \quad j \in J_i, \; i \in I. \tag{2.2}$$

The players reach a mixed Nash equilibrium when no player can reduce his own expected travel cost by unilaterally changing his strategy. Due to linearity of mixed strategies, it is equivalent to define with a player's unilateral deviation to a pure strategy [Nash, 1951].

Definition 2.7 (Mixed Nash equilibrium)**.** [Roughgarden and Schoppmann, 2011] The mixed strategy profile $\mathbf{p}$ is a mixed equilibrium if, for any player $j \in J_i, \; i \in I$, and any pure strategy $y_j \in P_i$,

$$EC_j(\mathbf{p}) \leq EC_j(y_j, \mathbf{p}_{-j}).$$

At a mixed equilibrium, no player can decrease his own expected travel cost by switching to any single path. As every pure strategy is a special mixed strategy, equation (2.2) implies that no player can improve his expected travel cost by unilaterally switching to another mixed strategy.

In order to show how to calculate the expected costs in Definition 2.7, let us introduce some notations first. Let random binary variables $\{X^i_{k,j} : k \in P_i,\ j \in J_i,\ i \in I\}$ indicate whether player $j \in J_i$ chooses path k between O-D pair $i \in I$, i.e., $\mathbb{P}[X^i_{k,j} = 1] = p^i_{k,j}$ and $\mathbb{P}[X^i_{k,j} = 0] = 1 - p^i_{k,j}$. Every player has to choose one path to allocate his traffic, i.e.,

$$\sum_{k \in P_i} X^i_{k,j} = 1, \quad \forall\ j \in J_i,\ i \in I. \tag{2.3}$$

Let random variables F^i_k indicate the total flow load on path $k \in P_i$, i.e.,

$$F^i_k = \sum_{j \in J_i} X^i_{k,j} w_j.$$

As players are assumed to behave independently in the game, the flows on paths connecting different O-D pairs are independent. But the path flows from a same O-D pair may be dependent, due to feasible conservation (2.3) for every single player.

Let $X_{e,j}$ be a random binary variable indicating whether player $j \in J_i$ chooses link $e \in E$, i.e., $X_{e,j} = \sum_{k \in P_i} \delta^i_{k,e} X^i_{k,j}$. Define $p^j_e = \sum_{k \in P_i} \delta^i_{k,e} p^i_{k,j}$, then $\mathbb{P}[X_{e,j} = 1] = p^j_e$. The link flow is

$$V_e = \sum_{i \in I} \sum_{j \in J_i} X_{e,j} w_j,\ \forall\ e \in E.$$

We still have the following conservations between link and path flows:

$$V_e = \sum_{i \in I} \sum_{k \in P_i} \delta^i_{k,e} F^i_k, \qquad \forall\, e \in E. \tag{2.4}$$

Let $\mathbf{F} = (F^i_k : k \in P_i,\ i \in I)^T$ be the corresponding flow of strategy profile $\mathbf{p}$. Let $c_e(\cdot)$ be a function of the link flow to denote the travel cost on link $e \in E$. Let $c^i_k(\cdot)$ be a function of the flow to denote the travel cost on path $k \in P_i$, which is the sum of the cost of those links that are included in path k, i.e.,

$$c^i_k(\mathbf{F}) = \sum_{e \in k} c_e(V_e), \quad \forall\, k \in P_i,\ \forall\, i \in I.$$

The expected costs in Definition 2.7 can be calculated as follows:

$$EC_j(y_j = l, \mathbf{p}_{-j}) = \mathbb{E}[c^i_l(\mathbf{F}) \mid X^i_{l,j} = 1], \quad l \in P_i,\ i \in I,$$
$$EC_j(\mathbf{p}) = \sum_{k \in P_i} p^i_{k,j} EC_j(y_j = k, \mathbf{p}_{-j}) = \sum_{k \in P_i} p^i_{k,j} \mathbb{E}[c^i_k(\mathbf{F}) \mid X^i_{k,j} = 1].$$

At a mixed user equilibrium, every path with positive probability must yield the same expected cost for every player. Because if it is not true, then the player's expected cost can be decreased by switching to the pure strategy with lower expected cost. Thus for a mixed Nash equilibrium $\mathbf{p}$ with $p^i_{k,j} > 0$ for path $k \in P_i$, $i \in I$,

$$EC_j(\mathbf{p}) = \mathbb{E}[c^i_k(\mathbf{F}) \mid X^i_{k,j} = 1]. \tag{2.5}$$

This is to say, at a mixed Nash equilibrium, each pure strategy involved in it is a best response itself, and yields the same expected payoff. Then we can rewrite the mixed equilibrium condition as the following corollary.

Corollary 2.6. *[Awerbuch et al., 2005] Strategy profile* $\mathbf{p}$ *is a mixed equilib-*

rium if and only if, for any $k, l \in P_i$, with $p^i_{k,j} > 0$,

$$\mathbb{E}[c^i_k(\mathbf{F}) \mid X^i_{k,j} = 1] \leq \mathbb{E}[c^i_l(\mathbf{F}) \mid X^i_{l,j} = 1], \quad \forall\, j \in J_i,\ i \in I.$$

A pure Nash equilibrium may not exist in weighted atomic congestion games [Harks and Klimm, 2010; Libman and Orda, 2001], while it always exists in unweighted atomic congestion games [Rosenthal, 1973]. Every atomic congestion game admits at least one mixed Nash equilibrium [Nash, 1951].

2.3.2 System Optimum

System optimum is defined as the optimal usage of the traffic resources in the whole network, which is understood as a result of the well coordination by a central authority. Unlike the UE, the traffic is well coordinated by a central authority, who will assign each traveller to a specific path. Thus the routing choice of each traveller can be regarded as a pure strategy.

Definition 2.8. In an atomic instance $(G, \mathbf{d}, \mathbf{c})$, a pure strategy profile $\mathbf{x}$ is a system optimum if, and only if, it minimises the total cost $\sum_{i \in I} \sum_{j \in J_i} c_j(\mathbf{x}) \cdot \omega_j$.

2.3.3 Price of Anarchy

The PoA is the worst-case ratio between the total cost at a UE and at a SO introduced in Sections 2.3.1 and 2.3.2. We can still look at Definition 2.3 for the definition of the PoA for atomic congestion games.

As we have reviewed in Section 1.2.2, the PoA with affine cost functions is bounded by 2.5 for unweighted demand and 2.618 for weighted demand, which are both proved to be tight. For polynomial cost functions with degree no more than m, the PoA is bounded by Φ_m^{m+1} and

$$\frac{(r+1)^{2m+1} - r^{m+1}(r+2)^m}{(r+1)^{m+1} - (r+2)^m + (r+1)^m - r^{m+1}},$$

respectively, for weighed and unweighted demands, where Φ_m is defined as the unique non-negative real solution to $(x+1)^m = x^{m+1}$ (which is the golden ratio when $m = 1$) and $r = \lfloor \Phi_m \rfloor$ [Aland et al., 2011].

For a more detailed analysis on the PoA, we refer to Awerbuch et al. [2005], Christodoulou and Koutsoupias [2005] and Aland et al. [2011].

Chapter 3

Non-atomic Model

This chapter is a generalisation of the deterministic model of non-atomic congestion games in Section 2.2. We consider a more realistic setting of stochastic demand. We present the routing strategies in the stochastic environment and generalise notions of user equilibrium and system optimum.

This chapter is based on a research article by Wang et al. [2014a].

3.1 Introduction

Day-to-day variability of traffic demands is considered as the source of uncertainty. We assume that the demand distributions are given and publicly known, which is based on the fact that a traveller, especially a commuter, has knowledge of the probabilities of possible demand levels from his or her own experiences, although the actual current demand level is unknowable. A similar assumption can be found in the model with deterministic demands, which states that travellers have perfect knowledge of the fixed demand in the network [Wardrop, 1952]. The demands of different O-D pairs are assumed to be independent. We adopt the following notation, where capital and lower-case letters are used to express random variables and, if applicable, the corresponding mean values, respectively.

- $\mathbf{D}$: vector of random traffic demands with component D_i as the random demand between O-D pair $i \in I$;
- $\mathbf{d}$: vector of mean traffic demands with component $d_i > 0$ as the mean demand between O-D pair $i \in I$;
- σ_i^2: variance of D_i, $i \in I$;
- ϵ_i: coefficient of demand variation, i.e., $\epsilon_i = \sigma_i / d_i$, $i \in I$;
- $\overline{\epsilon}$: maximum coefficient of demand variation, i.e., $\overline{\epsilon} = \max_{i \in I}\{\epsilon_i\}$;
- $\underline{\epsilon}$: minimum coefficient of demand variation, i.e., $\underline{\epsilon} = \min_{i \in I}\{\epsilon_i\}$;
- F_k^i: random traffic flow on path $k \in P_i$, $i \in I$;
- f_k^i: mean traffic flow on path $k \in P_i$, $i \in I$;
- $\mathbf{F}$: vector of random path flows, i.e., $\mathbf{F} = (F_k^i : k \in P_i, i \in I)$;
- $\mathbf{f}$: vector of mean path flows, i.e., $\mathbf{f} = (f_k^i : k \in P_i, i \in I)$;
- V_e: random traffic flow on link $e \in E$;
- v_e: mean traffic flow on link $e \in E$;
- $\mathbf{V}$: vector of random link flows, i.e., $\mathbf{V} = (V_e : e \in E)$;
- $\mathbf{v}$: vector of mean link flows, i.e., $\mathbf{v} = (v_e : e \in E)$;
- $\delta_{k,e}^i$: link-path incidence indicator, which is 1 if link e is included in path $k \in P_i$ and 0 otherwise, $e \in E,\ i \in I$;
- δ_e^i: link-commodity incidence indicator, i.e., $\delta_e^i = \max_{k \in P_i} \delta_{k,e}^i$, $e \in E$, $i \in I$;
- n_e: number of O-D pairs that use link $e \in E$ in their paths, i.e., $n_e = \sum_{i \in I} \delta_e^i$;
- n: $n = \max_{e \in E}\{n_e\}$. Hence $n \leq |I|$.

On parameter n defined above, we note that, if $n = 1$, then every link is used by only a single O-D pair, which implies that the whole network can be separated into $|I|$ single-commodity sub-networks. Therefore, as far as our problem is concerned for system stability and optimality (to be defined more precisely in Sections 3.3 and 3.4), our problem is reduced to the problem with a single commodity when $n = 1$.

Similar to deterministic work, we denote any instance of a non-atomic congestion game by a triple $(G, \mathbf{D}, \mathbf{c})$, where G is the underlying network, $\mathbf{D}$ and $\mathbf{c}$ are the vectors of random demands and (link) cost functions, respectively.

3.2 Routing Strategies

Under the deterministic setting, i.e., $D_i = d_i$ for all $i \in I$, the continuum of players of each O-D pair $i \in I$ is represented by the interval $[0, d_i]$ endowed with the Lebesgue measure. The set of mixed strategies of each player from O-D pair $i \in I$ is

$$\Omega_i = \left\{ \mathbf{p}^i = (p_k^i \geq 0 : k \in P_i) : \textstyle\sum_{k \in P_i} p_k^i = 1 \right\},$$

where p_k^i is the probability that path $k \in P_i$ is chosen. According to [Schmeidler, 1973], a *strategy profile* is a (Lebesgue) measurable function q^i from $[0, d_i]$ to Ω_i, i.e, for each player $x \in [0, d_i]$, $q^i(x) \in \Omega_i$ is his/her mixed strategy. A strategy profile q^i induces the vector $\mathbf{f}^i$ of path flows, $\mathbf{f}^i = (f_k^i : k \in P_i)$, which is called an *action distribution* in [Roughgarden and Tardos, 2002], as follows:

$$f_k^i = \int_0^{d_i} q_k^i(x) dx, \quad \forall\, k \in P_i,$$

where $q_k^i(x)$ is the probability that path $k \in P_i$ is chosen by the player x from O-D pair $i \in I$. Clearly, $\sum_{k \in P_i} f_k^i = d_i$ since $q^i(x) \in \Omega_i$ for all $x \in [0, d_i]$, $i \in I$. [Roughgarden and Tardos, 2002] focused on flow assignments, i.e., action distributions, instead of strategy profiles with the argument that every flow assignment can be induced by some strategy profile and the costs depend only on the flow assignment of a strategy profile. Under the stochastic setting, realized path flows depend on not only the chosen strategy profile but also the realized demand. Therefore, it is necessary for us to work with strategy profiles as primary variables instead of flow assignments. Given that the demands are stochastic, it is reasonable to assume that all the players of a same O-D pair play the same strategy at an equilibrium in such an environment with incomplete information [Myerson, 1998; Ashlagi et al., 2006]. Indeed it is unrealistic for a player to know the routing choices of all other players or to distinguish players from a same O-D pair when the demand is uncertain. According to [Myerson, 1998], players can only form perceptions about how other players make routing decisions solely depending on the information of which O-D pairs these players belong to. Mathematically, we assume that for any two different players x and x' in the (random) interval $[0, D_i]$ of the same O-D pair $i \in I$,

$$q_k^i(x) = q_k^i(x') = p_k^i, \quad \forall\, k \in P_i,$$

where $\mathbf{p}^i$ is some mixed strategy in Ω_i. Under this assumption, each strategy profile for players from O-D pair $i \in I$ is now represented by a single mixed strategy $\mathbf{p}^i \in \Omega_i$. Let $\Omega = \prod_{i \in I} \Omega_i$. Then each vector $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$ represents a strategy profile of players from all O-D pairs.

Now let us define random path flows and link flows for our stochastic model. Given a strategy profile represented by $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$, the random path flows can be calculated as follows:

$$F_k^i = \int_0^{D_i} q_k^i(x)dx = \int_0^{D_i} p_k^i dx = p_k^i \cdot D_i, \quad \forall\ k \in P_i,\ i \in I. \tag{3.1}$$

Since $\mathbf{p}^i \in \Omega_i$, we have:

$$\sum_{k \in P_i} F_k^i = D_i, \quad \forall\, i \in I. \tag{3.2}$$

It is clear that the flow on each link is the sum of flows on all the paths that include the link:

$$V_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i F_k^i, \quad \forall\, e \in E.$$

Applying (3.1), we obtain the following formulation for random link flows:

$$V_e = \sum_{i \in I} p_e^i \cdot D_i, \quad \forall\, e \in E, \tag{3.3}$$

where $p_e^i = \sum_{k \in P_i} \delta_{k,e}^i p_k^i$ is the (link) choice probability of link $e \in E$ for the players from $i \in I$.

Given the link cost functions, the random path cost is simply the sum of the costs of those links that constitute the path, i.e.,

$$c_k^i(\mathbf{F}) = \sum_{e \in E} \delta_{k,e}^i c_e(V_e), \quad \forall\, k \in P_i,\ \forall\, i \in I. \tag{3.4}$$

We can also compute the total (social) cost as follows:

$$C(\mathbf{F}) = \sum_{e \in E} c_e(V_e) V_e. \tag{3.5}$$

Remark 3.1. It is commonly assumed in the literature [Clark and Watling, 2005; Sumalee and Xu, 2011; Shao et al., 2006; Zhou and Chen, 2008] that all path flows $\{F_k^i : k \in P_i, i \in I\}$ are independent, which apparently violates the flow constraints (3.2). In our study, dependent path flows from a same O-D pair are considered as they should be according to (3.2) and we only assume that demands $\{D_i : i \in I\}$ are independent. From (3.1) we can see that the path flows from different O-D pairs are independent, i.e., for any $i, i' \in I$, $i \neq i'$ and any $k \in P_i, k' \in P_{i'}$, path flows F_k^i and $F_{k'}^{i'}$ are independent of each other.

3.3 Equilibrium under Stochastic Demands (UE-SD)

As discussed in the previous section, under stochastic traffic demands, we assume that risk-neutral travellers between a same O-D pair will use the same strategy at a steady state. We define our equilibrium condition such that travellers cannot improve their expected travel costs by unilaterally changing their routing choice strategies.

Definition 3.1 (UE-SD condition)**.** Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, strategy profile $\mathbf{p} \in \Omega$ is said to be a user equilibrium (UE-SD) if and only if

$$\mathbb{E}[c_k^i(\mathbf{F})] \leq \mathbb{E}[c_\ell^i(\mathbf{F})], \quad \forall\ k, \ell \in P_i, i \in I \text{ with } p_k^i > 0. \tag{3.6}$$

From the definition we see that, at any UE-SD, all the paths with positive probabilities for the same O-D pair have the equal and minimum expected travel cost. When all travellers play mixed strategies according to the UE-SD condition, the expected travel costs are guaranteed to be at minimum. To solve the equilibrium problem, let us reformulate the UE-SD condition as a variational inequality (VI).

Proposition 3.1. *Given a transportation game* $(G, \mathbf{D}, \mathbf{c})$ *with stochastic demands, let* $\bar{\mathbf{p}} \in \Omega$ *be a strategy profile. Then* $\bar{\mathbf{p}}$ *is a UE-SD if and only if it satisfies the following VI condition: for any strategy profile* $\mathbf{p} \in \Omega$*,*

$$(\mathbf{f} - \bar{\mathbf{f}})^T \mathbb{E}\left[c(\bar{\mathbf{F}})\right] \geq 0, \tag{3.7}$$

where $\bar{\mathbf{F}}$ *is the vector of path flows corresponding to* $\bar{\mathbf{p}}$*, and* $\bar{\mathbf{f}}$ *and* $\mathbf{f}$ *are, respectively, the vector of the mean path flows corresponding to* $\bar{\mathbf{p}}$ *and* $\mathbf{p}$.

Proof. Taking the expectation in (3.1), we have $f_k^i = p_k^i d_i$. Since demand $d_i > 0$ for every $i \in I$, we can write the UE-SD condition (3.6) as follows:

$$\mathbb{E}[c_k^i(\mathbf{F})] \leq \mathbb{E}[c_\ell^i(\mathbf{F})], \quad \forall\, k, \ell \in P_i, i \in I \text{ with } f_k^i > 0. \tag{3.8}$$

Let $\pi_i = \min_{\ell \in P_i} \mathbb{E}[c_\ell^i(\mathbf{F})]$ for any $i \in I$, then (3.8) is equivalent to

$$\begin{cases} f_k^i(\mathbb{E}[c_k^i(\mathbf{F})] - \pi_i) = 0, \\ f_k^i \geq 0, \end{cases} \qquad \forall\, k \in P_i,\ \forall\, i \in I.$$

Let $\bar{\mathbf{p}}$, $\bar{\mathbf{F}}$ and $\bar{\mathbf{f}}$ be the vectors of path choice probabilities and the corresponding path flows, mean path flows at a UE-SD, respectively. Then

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) = 0,$$

where $\bar{\pi}_i = \min_{\ell \in P_i} \mathbb{E}[c_\ell^i(\bar{\mathbf{F}})]$. For any $\mathbf{f} = (f_k^i \geq 0 : k \in P_i, i \in I)$, we also have

$$\sum_{i \in I} \sum_{k \in P_i} f_k^i(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) \geq 0.$$

Thus

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i) \leq \sum_{i \in I} \sum_{k \in P_i} f_k^i(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \bar{\pi}_i). \tag{3.9}$$

From condition (3.2) we have $\sum_{k \in P_i} f_k^i = \sum_{k \in P_i} (\bar{f}_k^i) = d_i$ for every $i \in I$. Hence

$$\sum_{i \in I} \sum_{k \in P_i} (\bar{f}_k^i)\bar{\pi}_i = \sum_{i \in I} \sum_{k \in P_i} f_k^i \bar{\pi}_i,$$

which together with (3.9) implies (3.7):

$$\sum_{i\in I}\sum_{k\in P_i}(\bar{f}_k^i)\mathbb{E}[c_k^i(\bar{\mathbf{F}})] \le \sum_{i\in I}\sum_{k\in P_i} f_k^i \mathbb{E}[c_k^i(\bar{\mathbf{F}})].$$

On the other hand, observe that as the first order optimality condition, the solution of VI problem (3.7) also solves the following LP problem:

$$\begin{aligned} \min \quad & \mathbf{f}^T\mathbb{E}[\mathbf{c}(\bar{\mathbf{F}})] \\ \text{s.t.} \quad & \sum_{k\in P_i} f_k^i = d_i, \quad i\in I, \\ & f_k^i \ge 0, \qquad\quad k\in P_i,\, i\in I, \end{aligned}$$

the duality of which is

$$\begin{aligned} \max \quad & \lambda^T\mathbf{d} \\ \text{s.t.} \quad & \lambda_i \le \mathbb{E}[c_k^i(\bar{\mathbf{F}})], \quad k\in P_i,\, i\in I. \end{aligned}$$

Therefore, we have the following complementary slackness conditions:

$$(\mathbb{E}[c_k^i(\bar{\mathbf{F}})] - \lambda_i)f_k^i = 0,\ \ k\in P_i,\, i\in I,$$

which imply (3.6). □

Remark 3.2. With equations (3.3) and (3.4) we can rewrite the VI condition (3.7) in terms of link flows: $\bar{\mathbf{p}}\in\Omega$ is a UE-SD if and only if it satisfies the condition that, for any vector $\mathbf{p}\in\Omega$ of path choice probabilities,

$$\sum_{e\in E}(v_e-\bar{v}_e)\mathbb{E}[c_e(\bar{V}_e)] \ge 0, \tag{3.10}$$

where $\bar{V}_e$ is the link flow on link $e\in E$ corresponding to $\bar{\mathbf{p}}$, and v_e and $\bar{v}_e$ are, respectively, the mean link flows on link $e\in E$ corresponding to $\mathbf{p}$ and $\bar{\mathbf{p}}$.

An equivalence between the UE-SD condition and a minimization problem can also be established if the link cost functions are affine, which is stated in the following proposition.

Proposition 3.2. *Given a transportation game* $(G, \mathbf{D}, \mathbf{c})$ *with stochastic demands and affine cost functions, let* $\bar{\mathbf{p}} \in \Omega$ *be a vector of path choice probabilities. Then* $\bar{\mathbf{p}}$ *is a UE-SD if and only if it solves the following minimization problem*

$$\min_{\mathbf{p} \in \Omega} Z(\mathbf{p}) \equiv \sum_{e \in E} \int_0^{v_e} c_e(x)dx, \tag{3.11}$$

where, as we recall, $v_e = \sum_{i \in I} \sum_{k \in P_i} \delta^i_{k,e} p^i_k d_i$.

Proof. We prove this proposition by verifying the equivalence between VI problem (3.7) and minimization problem (3.11). Note that, since the link cost function $c_e(x)$ is continuously differentiable and non-decreasing, function $\int_0^{v_e} c_e(x)dx$ is convex (with respect to v_e) for any $e \in E$. Convexity is invariant under affine maps; therefore, the objective function $Z(\mathbf{p})$ in (3.11) is convex (with respect to $\mathbf{p}$). In addition, feasible region Ω is convex and compact. Thus minimization problem (3.11) is a convex optimization problem. It is then necessary and sufficient for $\bar{\mathbf{p}}$ to satisfy the first order optimality condition of (3.11) [Bertsekas, 1999, Proposition 2.1.2]:

$$(\mathbf{p} - \bar{\mathbf{p}})^T \nabla Z(\bar{\mathbf{p}}) \geq 0. \tag{3.12}$$

We have:

$$\frac{\partial Z(\mathbf{p})}{\partial p^i_k} = \sum_{e \in E} c_e(v_e) \frac{\partial v_e}{\partial p^i_k} = \sum_{e \in E} c_e(v_e)(\delta^i_{k,e} d_i) = \ c^i_k(f) d_i.$$

In addition, we have $\bar{f}^i_k = \bar{p}^i_k d_i$ by taking the expectation in (3.1). Thus, condition (3.12) is equivalent to

$$(\mathbf{f} - \bar{\mathbf{f}})^T \mathbf{c}(\bar{\mathbf{f}}) \geq 0,$$

which in turn is equivalent to (3.7) when the link cost functions are affine. □

Proposition 3.2 establishes that the VI condition for a UE-SD is just a restatement of the first order necessary and sufficient condition of a minimization problem, if the cost functions $\mathbf{c}$ are affine. For general link cost functions, we can rewrite condition (3.7) in the following form by substituting $f_k^i = p_k^i d_i$ and $\bar{f}_k^i = \bar{p}_k^i d_i$:

$$(\mathbf{p} - \bar{\mathbf{p}})^T \mathbf{S}(\bar{\mathbf{p}}) \geq 0, \ \mathbf{p} \in \Omega, \tag{3.13}$$

where $\mathbf{S}(\mathbf{p})$ is a vector with the same dimension as $\mathbb{E}[\mathbf{c}(\mathbf{F})]$, obtained by replacing element $\mathbb{E}[c_k^i(\mathbf{F})]$ in vector $\mathbb{E}[\mathbf{c}(\mathbf{F})]$ with $\mathbb{E}[c_k^i(\mathbf{F})]d_i$ for every $k \in P_i, \ i \in I$. When link cost functions are continuous, the game admits at least one UE-SD. This is due to the fact that existence of a solution $\bar{\mathbf{p}} \in \Omega$ to VI problem (3.13) is guaranteed by the continuity of $\mathbf{S}(\mathbf{p})$ and the compactness of Ω.

Let us conclude this section with a discussion on non-uniqueness of user equilibria in transportation games with stochastic demands. In deterministic models, the user equilibrium is unique with respect to link flows under the assumption of separable and strictly increasing link cost functions [Beckmann et al., 1956; Dafermos and Sparrow, 1969]. As one link flow can correspond to many path flows in general networks, the path flow of a deterministic user equilibrium is not unique. In our stochastic model, path and link flows are random and determined by path and link choice probabilities respectively. The following example shows that multiple UE-SDs may exist even under the assumption of separable and strictly increasing link cost functions.

Example 3.1. Consider the network in Figure 3.1. There are two O-D pairs in the network, (s_1, t) and (s_2, t). Each O-D pair is connected by two paths,

paths 1 and 2 from s_1 to t and paths 3 and 4 from s_2 to t, where Path 1 consists of links 1 and 3, Path 2 of links 1 and 4, Path 3 of links 2 and 3, and Path 4 of links 2 and 4. The cost function on each link is also indicated in the figure.

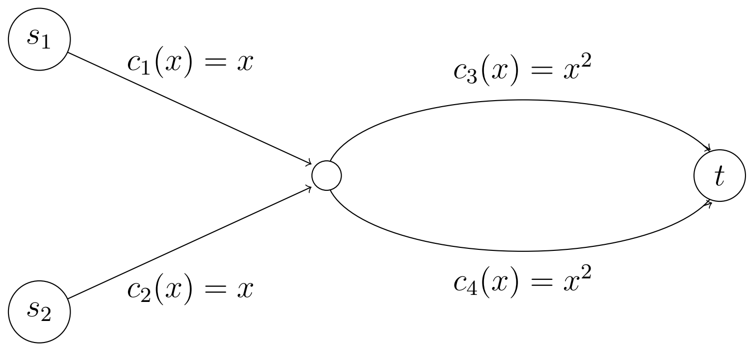


Figure 3.1: Network with multiple UE-SDs

The demand D_1 from s_1 to t follows a distribution with a mean $d_1 = 1$ and variance $\sigma_1^2 = 1$, while the demand D_2 from s_2 to t follows a different distribution with a mean $d_2 = 1$ and variance $\sigma_2^2 = 4$. Given the definition of UE-SD, a feasible strategy profile is clearly a UE-SD when both paths from each O-D pair have the same expected travel cost, i.e., when

$$\begin{cases} \mathbb{E}[c_1(V_1)] + \mathbb{E}[c_3(V_3)] = \mathbb{E}[c_1(V_1)] + \mathbb{E}[c_4(V_4)], \\ \mathbb{E}[c_2(V_2)] + \mathbb{E}[c_3(V_3)] = \mathbb{E}[c_2(V_2)] + \mathbb{E}[c_4(V_4)], \end{cases}$$

which are equivalent to

$$\mathbb{E}[c_3(V_3)] = \mathbb{E}[c_4(V_4)]. \tag{3.14}$$

Based on condition (3.14), we can find many UE-SDs. Here we present two of them for comparison: $\bar{\mathbf{p}}_\mathbf{1} = (1, 0, 0.25, 0.75)$ and $\bar{\mathbf{p}}_\mathbf{2} = (0.5, 0.5, 0.5, 0.5)$. From (3.3) we can calculate means and variances of link flows and expected link costs for each of the two strategy profiles as shown in Table 3.1, from which satisfaction of condition (3.14) at each strategy profile confirms that they are both UE-SDs.

At $\bar{\mathbf{p}}_1$	v_e	$\mathrm{Var}[V_e]$	$\mathbb{E}[c_e(V_e)]$	At $\bar{\mathbf{p}}_2$	v_e	$\mathrm{Var}[V_e]$	$\mathbb{E}[c_e(V_e)]$
Link 1	1	1	1	Link 1	1	1	1
Link 2	1	4	1	Link 2	1	4	1
Link 3	1.25	1.25	2.8125	Link 3	1	1.25	2.25
Link 4	0.75	2.25	2.8125	Link 4	1	1.25	2.25

Table 3.1: Means and variances of link flows, expected link costs at $\bar{\mathbf{p}}_1 = (1, 0, 0.25, 0.75)$ and $\bar{\mathbf{p}}_2 = (0.5, 0.5, 0.5, 0.5)$

It is easy to see that at the two UE-SDs the mean link flows on links 3 and 4 are different and the link choice probabilities are also different. For example, the choice probability of link 3 is 1 for travellers from s_1 to t and 0.25 from s_2 to t in the first UE-SD, while it becomes 0.5 for both O-D pairs in the second UE-SD. Furthermore, in terms of the expected total cost $\mathbb{E}[C(\mathbf{F})]$ (see definition (3.5)), they are also different at the two UE-SDs as shown in the following calculations (assuming that $\mathbb{E}[D_i^3]$ is finite for $i = 1, 2$), where $\bar{\mathbf{F}}_1$ and $\bar{\mathbf{F}}_2$ are path flows resulted from $\bar{\mathbf{p}}_1$ and $\bar{\mathbf{p}}_2$, respectively:

$$\begin{aligned}
\mathbb{E}[C(\bar{\mathbf{F}}_1)] &= \mathbb{E}[V_1^2] + \mathbb{E}[V_2^2] + \mathbb{E}[V_3^3] + \mathbb{E}[V_4^3] \\
&= \mathbb{E}[D_1^2] + \mathbb{E}[D_2^2] + \mathbb{E}[(p_1 D_1 + p_3 D_2)^3] + \mathbb{E}[(p_2 D_1 + p_4 D_2)^3] \\
&= 9.4375 + \mathbb{E}[D_1^3] + 0.4375\ \mathbb{E}[D_2^3], \\
\mathbb{E}[C(\bar{\mathbf{F}}_2)] &= \mathbb{E}[V_1^2] + \mathbb{E}[V_2^2] + \mathbb{E}[V_3^3] + \mathbb{E}[V_4^3] \\
&= 12.25 + 0.25\mathbb{E}[D_1^3] + 0.25\ \mathbb{E}[D_2^3].
\end{aligned}$$

Clearly, $\mathbb{E}[C(\bar{\mathbf{F}}_1)] \neq \mathbb{E}[C(\bar{\mathbf{F}}_2)]$ when $\mathbb{E}[D_1^3] + 0.25\ \mathbb{E}[D_2^3] \neq 3.75$. □

Example 3.1 shows that multiple UE-SDs with different mean link flows, link choice probabilities, and expected total costs can exist. If cost functions are further restricted to being affine, as addressed in Proposition 3.2, the UE-SD condition can be reformulated as a minimization problem, which is actually in the same form as deterministic user equilibrium condition [Beckmann et al., 1956] with respect to mean link flows. Thus under the same condition of separable and strictly increasing cost functions, the mean link flows of UE-SD are unique. However, link choice probabilities and expected total cost are still non-unique in general, which can be shown by modifying Example 3.1 with all settings remaining the same except that all link cost functions are affine, $c_e(x) = x$ for all $e = 1, \ldots, 4$.

By solving the minimization problem (3.11), a strategy profile is a UE-SD if and only if $v_1 = v_2 = v_3 = v_4 = 1$, which can be expressed as follows:

$$p_1 + p_2 = p_3 + p_4 = p_1 + p_3 = p_2 + p_4 = 1.$$

We can find multiple UE-SD strategy profiles from this system of equations, such as $(1, 0, 0, 1)$ and $(0.5, 0.5, 0.5, 0.5)$. The choice probability of link 3 is 1 from s_1 to t and 0 from s_2 to t in the former UE-SD, while both become 0.5 in the latter UE-SD. The expected total cost in the whole network can be calculated as 14 and 11.5 for the two UE-SDs respectively.

3.4 System Optimum under Stochastic Demand (SO-SD)

At a system optimum (SO-SD), traffic is coordinated by a central authority according to mixed strategies. It should be noted in the case of coordination that traffic is assigned according to path choice probabilities rather than by traffic proportions. This is due to the fact that demand is cumulative over the time period concerned, while traffic allocation needs to be made once a traffic flow arrives at a route entrance. The central authority has to implement traffic coordination without full knowledge of the actual demand. The objective of the coordinator is to minimise the expectation of the total travel cost at an SO-SD. This gives rise to our following definition.

Definition 3.2 (SO-SD condition)**.** Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, a vector $\mathbf{p} \in \Omega$ of path choice probabilities is said to be an SO-SD strategy if it solves the following minimization problem:

$$\min_{\mathbf{p}\in\Omega} T(\mathbf{p}) \equiv \mathbb{E}\left[C(\mathbf{F})\right] = \mathbb{E}\left[\sum_{e\in E} c_e(V_e)V_e\right], \tag{3.15}$$

where V_e is computed from $\mathbf{p}$ according to (3.3).

Generally, an SO-SD may not be unique as the optimization problem (3.15) may have more than one optimal solution, all of which, however, must yield the same expected total cost in the whole network.

Chapter 4

Price of Anarchy for Non-atomic Congestion Games

In this chapter we investigate the price of anarchy (PoA) to be defined below based on the model presented in Chapter 3 with the expected total cost $T(\cdot)$ defined in the network by (3.15) as the social (system) objective function. Given a transportation game $(G, \mathbf{D}, \mathbf{c})$ with stochastic demands, the corresponding PoA is defined as the worst-case ratio between expected total costs at a UE-SD and at an SO-SD:

$$\mathrm{PoA}(G, \mathbf{D}, \mathbf{c}) := \max\left\{\frac{T(\mathbf{p})}{T(\mathbf{q})} : \mathbf{p}, \mathbf{q} \in \Omega, \mathbf{p} \text{ is a UE-SD and } \mathbf{q} \text{ is an SO-SD}\right\}.$$

Here and in the remainder of the thesis, it is understood that the corresponding ratio is infinity whenever the denominator is zero.

Let $\mathcal{I}$ be any given set of instances $(G, \mathbf{D}, \mathbf{c})$ of transportation games with stochastic demands, then the PoA with respect to $\mathcal{I}$ is defined as

$$\mathrm{PoA}(\mathcal{I}) := \max_{(G,\mathbf{D},\mathbf{c})\in\mathcal{I}} \mathrm{PoA}(G, \mathbf{D}, \mathbf{c}).$$

Note that even for deterministic demands (i.e., $\mathbf{D}$ is particularly deterministic), the PoA is already unbounded if the link cost functions $\mathbf{c}$ are unrestricted [Roughgarden and Tardos, 2002]. In this chapter, we will establish upper bounds on the PoA for a fixed set $\mathcal{C}_m$ of link cost functions, the class of polynomial cost functions with degree at most m. As reviewed in Section 2.2, the PoA in deterministic models were bounded by two methods, i.e., convexity method and geometry method. In this chapter, we extend both of them to our stochastic model. We first establish the convexity and geometry bounds for general (link) cost functions and general demand distributions in Section 4.1 and then compute both the upper bounds for polynomial cost functions in Section 4.2. In Section 4.3, we compute the upper bounds for two different settings of the demand distributions, i.e., positive-valued distributions and normal distributions. Finally in Section 4.4, we discuss two special cases of affine cost functions and single commodity networks.

This chapter is based on a research article by Wang et al. [2014a].

4.1 General Upper Bounds

Both convexity and geometry methods we mentioned above for deterministic models require general bounds on the total cost function $\sum_{e\in E} c_e(v_e)v_e$. In our stochastic model, the expected total cost function is $\mathbb{E}\left[\sum_{e\in E} c_e(V_e)V_e\right]$, which in general is not solely a function of the mean link flows v_e, $e \in E$. In order to extend the bounding techniques to our stochastic model, we make the following general assumption. Denote by $\mathcal{C}(\mathcal{I})$ the class of all link cost functions $\{c_e(\cdot) : e \in E\}$ used in game instances $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$.

Assumption 4.1. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, there exist non-decreasing functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\overline{t}_e(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$, such that $\underline{s}_e(0) = \overline{s}_e(0) = 0$ and $\underline{t}_e(0) = \overline{t}_e(0) = c_e(0)$; and for any random link flows V_e ($e \in E$)

as defined in (3.3) with $v_e > 0$,

$$0 < \underline{s}_e(v_e) \le \mathbb{E}[c_e(V_e)V_e] \le \overline{s}_e(v_e), \tag{4.1}$$

$$0 < \underline{t}_e(v_e) \le \mathbb{E}[c_e(V_e)] \le \overline{t}_e(v_e). \tag{4.2}$$

Note that when $v_e = 0$, we can derive $V_e = 0$ from (3.3) and the fact that $d_i > 0$, $i \in I$. Hence with $\mathbb{E}[c_e(V_e)] = c_e(0)$ and $\mathbb{E}[c_e(V_e)V_e] = 0$, we can still use $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$ $\underline{t}_e(\cdot)$ and $\overline{t}_e(\cdot)$ to bound $\mathbb{E}[c_e(V_e)V_e]$ and $\mathbb{E}[c_e(V_e)]$ when $v_e = 0$.

The above assumption is satisfied under some mild conditions, which we will discuss in detail later. Based on Definition 2.5, we make the following definitions.

Definition 4.1. Under Assumption 4.1, for each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define

$$\beta(c_e, \mathcal{I}) = \sup_{x \ge 0,\, y > 0} \frac{x\left(\overline{t}_e(y) - \underline{t}_e(x)\right)}{y\, \overline{t}_e(y)},$$

and

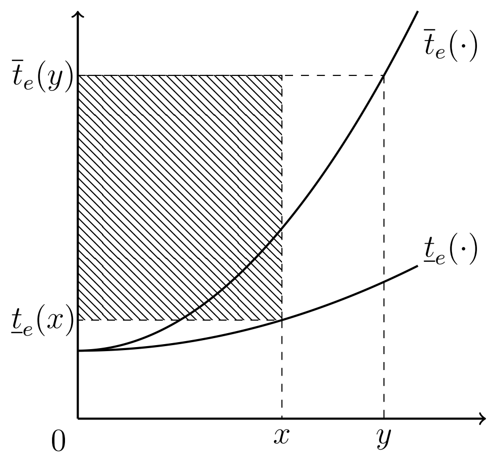
$$\beta(\mathcal{I}) = \sup_{c \in \mathcal{C}(\mathcal{I})} \beta(c, \mathcal{I}).$$

Note that the definition above is a generalisation of Definition 2.5 in deterministic work. When demands are particularly deterministic, we can choose $\underline{t}_e(\cdot) = \overline{t}_e(\cdot) = c_e(\cdot)$ and obtain

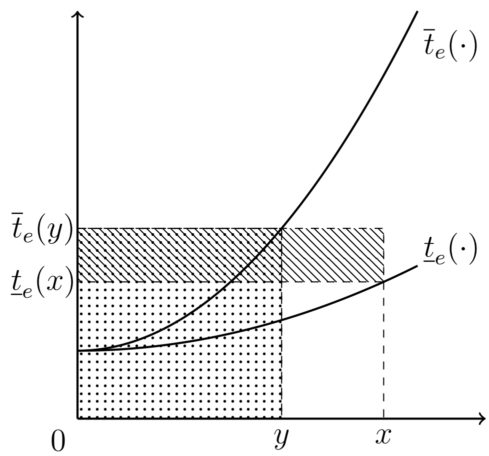
$$\beta(c_e, \mathcal{I}) = \sup_{x \ge 0,\, y > 0} \frac{x\left(c_e(y) - c_e(x)\right)}{y\, c_e(y)} = \sup_{y > x \ge 0} \frac{x\left(c_e(y) - c_e(x)\right)}{y\, c_e(y)},$$

which is proved geometrically to be less than 1 (see Section 2.2.3 for the details). However with stochastic demands, we can no longer guarantee that $\beta(\mathcal{I})$ in Definition 4.1 is always less than 1. The supremum in the definition of $\beta(c_e, \mathcal{I})$ can be attained under the condition $x > y$. We demonstrate this point later in our study of the PoA. We now use Figure 4.1 to show how

$(x\,(\bar{t}_e(y) - \underline{t}_e(x)))/(y\,\bar{t}_e(y))$ can be interpreted geometrically under both conditions, $x \le y$ and $x > y$. As shown in panel (a), when $x \le y$, the shaded rectangle of area $x\,(\bar{t}_e(y) - \underline{t}_e(x))$ is within the big rectangle of area $y\,\bar{t}_e(y)$. However, in panel (b), the shaded rectangle of area $x\,(\bar{t}_e(y) - \underline{t}_e(x))$ is not completely within the dotted rectangle of area $y\,\bar{t}_e(y)$ due to the possibility that $\bar{t}_e(y) > \underline{t}_e(x)$, which implies $\beta(c_e, \mathcal{I})$ could be more than 1. It shows that the geometric meaning of this ratio is not as clear as its counterpart in the deterministic setting. However, we still use the word "geometry" to refer to the bounding technique motivated from [Correa et al., 2008] to indicate the significance of the motivating work.



(a) $x \le y$



(b) $x > y$

Figure 4.1: Geometric interpretation of $\dfrac{x\,(\bar{t}_e(y) - \underline{t}_e(x))}{y\,\bar{t}_e(y)}$ in the definition of $\beta(c_e, \mathcal{I})$

Definition 4.2. Under Assumption 4.1, for each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define for each $e \in E$ functions $\phi_e(\cdot)$ and $\eta_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\phi_e(x) = \frac{x\,\underline{t}_e(x)}{\bar{s}_e(x)}, \quad \eta_e(x) = \frac{x\,\bar{t}_e(x)}{\underline{s}_e(x)}.$$

Let

$$\underline{\alpha}(c_e, \mathcal{I}) = \inf_{x>0} \phi_e(x), \quad \overline{\alpha}(c_e, \mathcal{I}) = \sup_{x>0} \eta_e(x),$$

and

$$\underline{\alpha}(\mathcal{I}) = \inf_{c \in \mathcal{C}(\mathcal{I})} \underline{\alpha}(c, \mathcal{I}), \quad \overline{\alpha}(\mathcal{I}) = \sup_{c \in \mathcal{C}(\mathcal{I})} \overline{\alpha}(c, \mathcal{I}).$$

Now we are ready to show our first bound by the geometry method.

Proposition 4.1 (General geometry bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ *be any game instance. Under Assumption 4.1, if* $\beta(\mathcal{I}) < 1$, *then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq (1 - \beta(\mathcal{I}))^{-1} \cdot \frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ be a UE-SD with $\bar{\mathbf{V}}, \bar{\mathbf{v}}$ as the corresponding link flows and mean link flows. Let $\mathbf{p}^*$ be an SO-SD with $\mathbf{V}^*, \mathbf{v}^*$ as the corresponding link flows and mean link flows. Given that $\mathbf{d} > 0$, we have $\bar{\mathbf{v}}$, $\mathbf{v}^* \geq 0$. From UE-SD condition (3.10), we have

$$\begin{aligned}\sum_{e \in E} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)] &\leq \sum_{e \in E} v_e^* \mathbb{E}[c_e(\bar{V}_e)] \\ &= \sum_{e \in E} v_e^* \mathbb{E}[c_e(V_e^*)] + \sum_{e \in E} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right] \right),\end{aligned}$$

which can be rearranged as

$$(1 - R) \sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right] \leq \sum_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right],$$

where, with $\{e \in E : \bar{v}_e > 0\} \neq \emptyset$,

$$\begin{aligned} R &= \frac{\sum\limits_{e \in E} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right] \right)}{\sum\limits_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \\ &\leq \frac{\sum\limits_{e \in E: \bar{v}_e > 0} v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right] \right)}{\sum\limits_{e \in E: \bar{v}_e > 0} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \\ &\leq \max_{e \in E: \bar{v}_e > 0} \frac{v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right] \right)}{\bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]}. \end{aligned}$$

The first inequality above is due to $v_e^* \left(\mathbb{E}\left[c_e(\bar{V}_e)\right] - \mathbb{E}\left[c_e(V_e^*)\right] \right) \leq 0$ when $\bar{v}_e = 0$ as can be seen from the facts that $\bar{V}_e = 0$ when $\bar{v}_e = 0$ and $\underline{t}_e(\cdot)$ is non-decreasing (Assumption 4.1). Now we have:

$$\begin{aligned} R &\leq \max_{e \in E: \bar{v}_e > 0} \left\{ \frac{v_e^*}{\bar{v}_e} - \frac{v_e^* \mathbb{E}[c_e(V_e^*)]}{\bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]} \right\} \leq \max_{e \in E: \bar{v}_e > 0} \left\{ \frac{v_e^*}{\bar{v}_e} - \frac{v_e^* \underline{t}_e(v_e^*)}{\bar{v}_e \bar{t}_e(\bar{v}_e)} \right\} \\ &= \max_{e \in E: \bar{v}_e > 0} \frac{v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*) \right)}{\bar{v}_e \bar{t}_e(\bar{v}_e)} \leq \beta(\mathcal{I}). \end{aligned}$$

Hence

$$(1 - \beta(\mathcal{I})) \sum_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right] \leq \sum_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right]. \tag{4.3}$$

We have

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} = \frac{\sum\limits_{e \in E} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]}{\sum\limits_{e \in E} \mathbb{E}[c_e(V_e^*)V_e^*]} = R_1 \cdot {R_2}^{-1} \cdot R_3,$$

where

$$R_1 = \frac{\sum\limits_{e \in E} \bar{v}_e \mathbb{E}\left[c_e(\bar{V}_e)\right]}{\sum\limits_{e \in E} v_e^* \mathbb{E}\left[c_e(V_e^*)\right]} \leq (1 - \beta(\mathcal{I}))^{-1},$$

according to inequality (4.3), and

$$
\begin{aligned}
R_2 &= \frac{\sum_{e\in E} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\sum_{e\in E} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} = \frac{\sum_{e\in E:\bar{v}_e>0} \bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\sum_{e\in E:\bar{v}_e>0} \mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} \\
&\geq \min_{e\in E:\bar{v}_e>0} \frac{\bar{v}_e \mathbb{E}[c_e(\bar{V}_e)]}{\mathbb{E}[c_e(\bar{V}_e)\bar{V}_e]} \geq \underline{\alpha}(\mathcal{I}), \qquad (4.4) \\
R_3 &= \frac{\sum_{e\in E} v_e^* \mathbb{E}[c_e(V_e^*)]}{\sum_{e\in E} \mathbb{E}[c_e(V_e^*)V_e^*]} = \frac{\sum_{e\in E:v_e^*>0} v_e^* \mathbb{E}[c_e(V_e^*)]}{\sum_{e\in E:v_e^*>0} \mathbb{E}[c_e(V_e^*)V_e^*]} \\
&\leq \max_{e\in E:v_e^*>0} \frac{v_e^* \mathbb{E}[c_e(V_e^*)]}{\mathbb{E}[c_e(V_e^*)V_e^*]} \leq \overline{\alpha}(\mathcal{I}). \qquad (4.5)
\end{aligned}
$$

The second equations in (4.4) and (4.5) hold because $\bar{V}_e = 0$ and $V_e^* = 0$ when $\bar{v}_e = 0$ and $v_e^* = 0$, respectively. Therefore,

$$
\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq (1-\beta(\mathcal{I}))^{-1} \cdot \frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})},
$$

for any pair $\bar{\mathbf{p}}, \mathbf{p}^* \in \Omega$ of a UE-SD and an SO-SD. □

In considering polynomial link cost functions, Roughgarden and Tardos [2004] used the fact that link cost functions are differentiable and semi-convex in their bounding techniques (or more exactly, the convexity of the function $x\, c_e(x)$). In extending their method to our stochastic model, we make the following assumption, which we will show later is satisfied for polynomial link cost functions.

Assumption 4.2. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, the function $\underline{s}_e(\cdot)$ in Assumption 4.1 is convex and differentiable. In addition, there exists a function $\lambda_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\underline{s}_e'(\lambda_e(x)x) = \bar{t}_e(x)$ for all $x \geq 0$, where $\underline{s}_e'(\cdot)$ is the derivative of $\underline{s}_e(\cdot)$.

Definition 4.3. Under Assumptions 4.1 and 4.2, for each link cost function

$c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, define for $e \in E$ functions $\psi_e(\cdot)$ and $\mu_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\psi_e(x) = \frac{x\,\bar{t}_e(x)}{\bar{s}_e(x)}, \quad \mu_e(x) = \frac{\underline{s}_e(\lambda_e(x)\,x)}{\bar{s}_e(x)}.$$

Using $\phi_e(\cdot)$ defined in Definition 4.2 in addition to $\psi_e(\cdot)$ and $\mu_e(\cdot)$, we define

$$\gamma(c_e, \mathcal{I}) = \inf_{x>0} \left\{\mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x)\right\},$$

Let

$$\gamma(\mathcal{I}) = \inf_{c\in\mathcal{C}(\mathcal{I})} \gamma(c, \mathcal{I}).$$

Now let us present a bound by the convexity method in the following proposition.

Proposition 4.2 (General convexity bound)**.** *Let $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ be any game instance. Under Assumptions 4.1 and 4.2, if $\gamma(\mathcal{I}) > 0$, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \le \frac{1}{\gamma(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ and $\mathbf{p}^* \in \Omega$ be respectively a UE-SD and an SO-SD, with $\bar{\mathbf{v}}, \mathbf{v}^* \ge 0$ as the corresponding mean link flows. Then

$$\begin{aligned} T(\mathbf{p}^*) &= \sum_{e\in E} \mathbb{E}\left[c_e(V_e^*)V_e^*\right] \ge \sum_{e\in E} \underline{s}_e(v_e^*) \\ &\ge \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\underline{s}_e'(\lambda_e(\bar{v}_e)\bar{v}_e)\right) \\ &= \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\bar{t}_e(\bar{v}_e)\right) \\ &= \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + v_e^*\bar{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right), \end{aligned}$$

where the first inequality follows from (4.1), and the second last equation follows from Assumption 4.2.

Applying UE-SD condition (3.10) and inequalities (4.2) in the last line above leads to

$$
\begin{aligned}
T(\mathbf{p}^*) &\geq \sum_{e \in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e \underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e \bar{t}_e(\bar{v}_e) \right) \\
&= \sum_{e \in E : \bar{v}_e > 0} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e \underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e \bar{t}_e(\bar{v}_e) \right) \\
&= \sum_{e \in E : \bar{v}_e > 0} \left(\mu_e(\bar{v}_e) + \phi_e(\bar{v}_e) - \psi_e(\bar{v}_e)\lambda_e(\bar{v}_e) \right) \bar{s}_e(\bar{v}_e) \\
&\geq \gamma(\mathcal{I}) \sum_{e \in E : \bar{v}_e > 0} \bar{s}_e(\bar{v}_e) = \gamma(\mathcal{I}) \sum_{e \in E} \bar{s}_e(\bar{v}_e) \geq \gamma(\mathcal{I})\, T(\bar{\mathbf{p}}).
\end{aligned}
$$

where the first equation follows from $\underline{s}_e(0) = 0$ (Assumption 4.1) and the second equation follows from Definitions 4.2 and 4.3. Given that $\gamma(\mathcal{I}) > 0$, we have

$$
\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq \frac{1}{\gamma(\mathcal{I})},
$$

for any pairs $\bar{\mathbf{p}}, \mathbf{p}^* \in \Omega$ of a UE-SD and an SO-SD. $\square$

Remark 4.1. When demands are deterministic, Propositions 4.1 and 4.2 yield the PoA bounds in Theorems 2.5 and 2.4, respectively, by choosing $\underline{s}_e(v_e) = \bar{s}_e(v_e) = c_e(v_e)v_e$ and $\underline{t}_e(v_e) = \bar{t}_e(v_e) = c_e(v_e)$ for $e \in E$, which implies $\underline{\alpha}(c_e, \mathcal{I}) = \overline{\alpha}(c_e, \mathcal{I}) = 1$. As we have mentioned before, $\beta(c_e, \mathcal{I}) < 1$ always holds for nondecreasing cost functions according to Figure 2.2. Similarly, condition $\gamma(c_e, \mathcal{I}) > 0$ is satisfied since

$$
\mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x) = \mu_e(x) + 1 - \lambda_e(x) > 0,
$$

due to the fact that $\mu_e(x) > 0$ and $\lambda_e(x) \leq 1$ for $x > 0$ for nonzero cost functions (see Section 2.2.3 and Roughgarden and Tardos [2004] for details).

4.2 Polynomial Cost Functions

As mentioned in Section 2.2.3, for deterministic models, both convexity and geometry methods lead to the same PoA upper bound with polynomial link cost functions. For our stochastic model, this is no longer true. After establishing two general PoA upper bounds in this section, we will show respectively in the next two sections that for polynomial link cost functions, the geometry bound on the PoA is better (but not tight) in general, while the convexity bound is better and indeed tight in some special cases.

We consider the set $\mathcal{I}_m$ of game instances for any fixed $m \in \mathbb{Z}_+$ ($m \geq 1$) with (non-zero) polynomial link cost functions in the form of

$$c_e(x) = \sum_{j=0}^{m} b_{ej} x^j, \;\; b_{ej} \geq 0, \;\; j = 0, 1, \ldots, m \text{ and } \sum_{j=0}^{m} b_{ej} > 0; \;\; e \in E.$$

In other words, $\mathcal{C}(\mathcal{I}_m) = \mathcal{C}_m$, the set of (non-zero) polynomial functions with nonnegative coefficients and degree at most m. Let $\widetilde{\mathcal{C}}_m$ be the subset of $\mathcal{C}_m$ consisting of only one term, namely $\widetilde{\mathcal{C}}_m = \cup_{0 \leq j \leq m} \widetilde{\mathcal{C}}_m^j$, where $\widetilde{\mathcal{C}}_m^j = \{bx^j : b > 0\}$ for all $j = 0, 1, \ldots, m$. Let $\widetilde{\mathcal{I}}_m$ be the subset of game instances in $\mathcal{I}_m$ with link cost functions in $\widetilde{\mathcal{C}}_m$. The following lemma shows we can focus on the subset $\widetilde{\mathcal{I}}_m$ when bounding the PoA for instances in $\mathcal{I}_m$.

Lemma 4.3. *For any instance* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$*, we have*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq PoA(\widetilde{\mathcal{I}}_m).$$

Proof. Any instance $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$ can be transformed into an equivalent instance with link cost functions in $\widetilde{\mathcal{C}}_m$ by replacing any link $e \in E$ of cost $c_e(x) = \sum_{j=0}^{m} b_{ej} x^j$ with a directed path consisting of no more than $m + 1$ links of costs $\widetilde{c}_{e,j}(x) = b_{ej} x^j$ ($0 \leq j \leq m$) such that $b_{ej} > 0$. This equivalent instance clearly belongs to $\widetilde{\mathcal{I}}_m$. The result then follows immediately. □

A similar lemma can be found in [Roughgarden, 2005] for calculating the anarchy value of polynomial cost functions in deterministic models.

We now consider monomial link cost functions in $\widetilde{\mathcal{C}}_m$. Given link cost function $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^j$, i.e., $c_e(x) = b_{ej}x^j$ with $b_{ej} > 0$ for a fixed $j \leq m$, we have:

$$\mathbb{E}\left[c_e(V_e)V_e\right] = b_{ej}\mathbb{E}\left[V_e^{j+1}\right].$$

If $j = 0$, then $\mathbb{E}\left[c_e(V_e)V_e\right] = b_{ej}v_e$, which is a function of the mean link flow v_e. For $j \geq 1$, in order to compute $\mathbb{E}\left[c_e(V_e)V_e\right]$, we need the moment $\mathbb{E}\left[V_e^{j+1}\right]$ of V_e to be finite. Given that $V_e = \sum_{i\in I} p_e^i \cdot D_i$ and $\{D_i : i \in I\}$ are independent, we then need the first $j+1$ moments of D_i to be finite. In addition, in order to construct functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, and $\overline{t}_e(\cdot)$ in Assumption 4.1, we make the following assumption.

Assumption 4.3. The first $m+1$ moments of random demands D_i $(i \in I)$ are all finite and positive. In addition, for $j = 2, \ldots, m+1$, there exist $0 < l_j \leq h_j$ such that

$$0 \leq l_j v_e^j \leq \mathbb{E}\left[V_e^j\right] \leq h_j v_e^j, \quad \forall\, e \in E.$$

Positivity of moments is satisfied in general if we consider positive-valued demand distributions, which is reasonable to assume. We also consider normal distributions later since they are widely used in the literature to simulate traffic demands, especially the ones with large (positive) means or relatively small variances, although negative tails are contained [Clark and Watling, 2005; Asakura and Kashiwadani, 1991]. Positivity of higher moments for normal distributions is again satisfied easily under the assumption of positive means. With respect to the parameters l_j and h_j, for consistency we define $l_0 = h_0 = l_1 = h_1 = 1$ since $\mathbb{E}\left[V_e^j\right] = v_e^j$ for $j = 0, 1$. We will show later how to compute l_j and h_j for $j > 1$ for both positive-valued demand distributions and normal distributions (with positive means).

Under Assumption 4.3, we can now show that there exist functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\overline{t}_e(\cdot)$, and $\lambda_e(\cdot)$ with which both Assumptions 4.1 and 4.2 are satisfied for monomial link cost functions.

Definition 4.4. For a fixed j $(0 \le j \le m)$, let $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^j$. Let

$$\begin{cases} \overline{t}_e(x) = h_j b_{ej} x^j, \\ \underline{t}_e(x) = l_j b_{ej} x^j, \end{cases} \quad \begin{cases} \overline{s}_e(x) = h_{j+1} b_{ej} x^{j+1}, \\ \underline{s}_e(x) = l_{j+1} b_{ej} x^{j+1}, \end{cases}$$

where h_j and l_j are taken from Assumption 4.3. In addition, let

$$\lambda_e(x) = \begin{cases} \left(\dfrac{h_j}{(j+1) l_{j+1}} \right)^{1/j}, & j > 0, \\ 1, & j = 0. \end{cases}$$

With the functions defined in Definition 4.4, it is easy to show that Assumption 4.1 is satisfied. As for Assumption 4.2, it is indeed that $\underline{s}_e(\cdot)$ is convex and differentiable. We can also check easily that $\underline{s}_e'(\lambda_e(x)x) = \overline{t}_e(x)$ for all $x \ge 0$. We are now ready to compute all necessary parameters to provide specific upper bounds on the PoA.

Lemma 4.4. *Under Assumption 4.3, we have*

$$\underline{\alpha}(\widetilde{\mathcal{I}}_m) = \min_{0 \le j \le m} \frac{l_j}{h_{j+1}}, \quad \overline{\alpha}(\widetilde{\mathcal{I}}_m) = \max_{0 \le j \le m} \frac{h_j}{l_{j+1}}, \tag{4.6}$$

$$\beta(\widetilde{\mathcal{I}}_m) = \max_{1 \le j \le m} \left\{ \frac{j}{j+1} \cdot \left(\frac{h_j}{(j+1) l_j} \right)^{1/j} \right\}, \tag{4.7}$$

$$\gamma(\widetilde{\mathcal{I}}_m) = \min_{1 \le j \le m} \left\{ \frac{l_j}{h_{j+1}} - \frac{j}{j+1} \cdot \frac{h_j}{h_{j+1}} \cdot \left(\frac{h_j}{(j+1) l_{j+1}} \right)^{1/j} \right\}. \tag{4.8}$$

Proof. We have

$$\phi_e(x) = \frac{x\, \underline{t}_e(x)}{\overline{s}_e(x)} = \frac{l_j}{h_{j+1}}, \quad \eta_e(x) = \frac{x\, \overline{t}_e(x)}{\underline{s}_e(x)} = \frac{h_j}{l_{j+1}}.$$

Hence

$$\underline{\alpha}(c_e, \widetilde{\mathcal{I}}_m) = \frac{l_j}{h_{j+1}}, \quad \overline{\alpha}(c_e, \widetilde{\mathcal{I}}_m) = \frac{h_j}{l_{j+1}}.$$

Since $\mathcal{C}(\widetilde{\mathcal{I}}_m) = \widetilde{\mathcal{C}}_m = \cup_{0 \le j \le m} \widetilde{\mathcal{C}}_m^j$, we have

$$\begin{aligned}
\underline{\alpha}(\widetilde{\mathcal{I}}_m) &= \min_{0 \le j \le m} \inf_{c \in \widetilde{\mathcal{C}}_m^j} \underline{\alpha}(c, \widetilde{\mathcal{I}}_m) = \min_{0 \le j \le m} \frac{l_j}{h_{j+1}}, \\
\overline{\alpha}(\widetilde{\mathcal{I}}_m) &= \max_{0 \le j \le m} \sup_{c \in \widetilde{\mathcal{C}}_m^j} \overline{\alpha}(c, \widetilde{\mathcal{I}}_m) = \max_{0 \le j \le m} \frac{h_j}{l_{j+1}}.
\end{aligned}$$

For parameter $\beta(c_e, \widetilde{\mathcal{I}}_m)$, we have

$$\frac{x\left(\overline{t}_e(y) - \underline{t}_e(x)\right)}{y\,\overline{t}_e(y)} = \frac{x}{y}\left(1 - \frac{l_j}{h_j}\left(\frac{x}{y}\right)^j\right) \equiv f_j(z),$$

where $z = x/y$, which implies that

$$\beta(\widetilde{\mathcal{I}}_m) = \max_{0 \le j \le m} \sup_{c \in \widetilde{\mathcal{C}}_m^j} \beta(c, \widetilde{\mathcal{I}}_m) = \max_{0 \le j \le m} \sup_{z > 0} f_j(z).$$

For $1 \le j \le m$, elementary calculus gives

$$\sup_{z>0} f_j(z) = \frac{j}{j+1} \cdot \left(\frac{h_j}{(j+1)l_j}\right)^{1/j},$$

which together with the fact that $f_0(\cdot) \equiv 0$ implies equation (4.7).

We now consider $\gamma(c_e, \widetilde{\mathcal{I}}_m)$. If $j = 0$, we have: $\lambda_e(x) = \mu_e(x) = 1 = \phi_e(x) = \eta_e(x)$. Thus we have $\gamma(c_e, \widetilde{\mathcal{I}}_m) = 1$ for $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^0$. When $j > 0$ we have

$$\lambda_e(x) = \left(\frac{h_j}{(j+1)l_{j+1}}\right)^{1/j}.$$

Thus

$$\mu_e(x) = \frac{\underline{s}_e(\lambda_e(x)x)}{\overline{s}_e(x)} = \frac{l_{j+1}}{h_{j+1}}\left(\frac{h_j}{(j+1)l_{j+1}}\right)^{1/j+1}.$$

Similar to $\phi_e(\cdot)$ and $\eta_e(\cdot)$, we have $\psi_e(x) = h_j/h_{j+1}$. All three functions, $\lambda_e(\cdot)$,

$\mu_e(\cdot)$, and $\psi_e(\cdot)$ are constants, leading to the following:

$$\begin{aligned}\gamma(c_e, \widetilde{\mathcal{I}}_m) &= \mu_e(x) + \phi_e(x) - \psi_e(x)\lambda_e(x) \\ &= \frac{l_{j+1}}{h_{j+1}}\left(\frac{h_j}{(j+1)l_{j+1}}\right)^{1/j+1} + \frac{l_j}{h_{j+1}} - \frac{h_j}{h_{j+1}}\left(\frac{h_j}{(j+1)l_{j+1}}\right)^{1/j} \\ &= \frac{l_j}{h_{j+1}} - \frac{j}{j+1}\cdot\frac{h_j}{h_{j+1}}\cdot\left(\frac{h_j}{(j+1)l_{j+1}}\right)^{1/j}.\end{aligned}$$

Since specifically for $j = 1$, we have $l_1/h_2 = 1/h_2 \leq 1$, which implies $\gamma(c_e, \widetilde{\mathcal{I}}_m) \leq 1$ for $c_e(\cdot) \in \widetilde{\mathcal{C}}_m^1$, according to the definition of $\gamma(\widetilde{\mathcal{I}}_m)$, we obtain equation (4.8). □

In the following two theorems we present specific geometry and convexity bounds on the PoA by applying Propositions 4.1 and 4.2 to game instances in $\mathcal{I}_m$.

Theorem 4.5 (Geometry upper bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$*. Under Assumption 4.3, if*

$$\frac{h_j}{l_j} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots, m, \tag{4.9}$$

then

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \left(1 - \max_{1\leq j\leq m}\left\{\frac{j}{j+1}\cdot\left(\frac{h_j}{(j+1)l_j}\right)^{1/j}\right\}\right)^{-1} \cdot \frac{\max_{0\leq j\leq m} h_j/l_{j+1}}{\min_{0\leq j\leq m} l_j/h_{j+1}}.$$

Proof. The proof of the theorem is straightforward by applying Proposition 4.1 for $\widetilde{\mathcal{I}}_m$ combined with Lemma 4.4. Note that Assumption 4.1 is satisfied for functions defined in Definition 4.4 under Assumption 4.3. Lemma 4.3 is then used to bound the PoA for game instances in $\mathcal{I}_m$. □

Theorem 4.6 (Convexity upper bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$*. Under Assumption 4.3, if*

$$\frac{h_j}{l_{j+1}}\left(\frac{h_j}{l_j}\right)^j < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots, m, \tag{4.10}$$

then

$$PoA(G,\mathbf{D},\mathbf{c}) \leq \max_{1\leq j\leq m} \left(\frac{l_j}{h_{j+1}} - \frac{j}{j+1} \cdot \frac{h_j}{h_{j+1}} \cdot \left(\frac{h_j}{(j+1)l_{j+1}} \right)^{1/j} \right)^{-1}.$$

Remark 4.2. Theorems 4.5 and 4.6 both generalise the PoA bounds provided by Roughgarden and Tardos [2002, 2004] and Correa et al. [2008] for deterministic models.

When the traffic demands return to being deterministic, we can choose $l_j = h_j = 1$ for all integer $j \leq m+1$, so that Assumption 4.3 is satisfied. Both conditions (4.9) and (4.10) clearly hold since $(j+1)^{j+1}/j^j > 1$ for $1 \leq j \leq m$. Both the geometry and convexity bounds become $\left(1 - m(m+1)^{-(m+1)/m}\right)^{-1}$, which matches the tight upper bound of the PoA in deterministic models.

On the other hand, unlike in deterministic models, conditions (4.9) and (4.10) are necessary for our stochastic model. It is due to the fact that parameters we use to bound the PoA in our stochastic model now depend on not only the cost functions but also demand distributions and to some extent, the network structure. Both conditions are constructed based on functional approximations of $\mathbb{E}\left[c_e(V_e)\right]$ and $\mathbb{E}\left[c_e(V_e)V_e\right]$. In general it is difficult to determine what set $\mathcal{I}$ of instances of transportation games with stochastic demands for which these conditions are always satisfied. However, we will derive these two conditions in the next section with specific demand distributions.

4.3 Specific Demand Distributions

4.3.1 General Positive-valued Distributions

It is natural to consider general positive-valued distributions for demands $\{D_i : i \in I\}$. It was clear that we need to assume the finiteness of the first $m+1$ moments of D_i ($i \in I$) when considering game instances in $\mathcal{I}_m$. These moments are non-negative when demands follow positive-valued distributions. Let

$$\theta_i^{(j)} = \frac{\mathbb{E}[D_i^j]}{d_i^j} > 0, \quad \forall\, i \in I,\ \forall\, j = 0, 1, \ldots, m+1. \tag{4.11}$$

Lemma 4.7. *For any $s, t \in \mathbb{Z}_+$ and any $i \in I$,*

$$\theta_i^{(s+t)} \geq \theta_i^{(s)} \cdot \theta_i^{(t)}. \tag{4.12}$$

Proof. We have

$$\mathbb{E}[D_i^{s+t}] = \mathbb{E}[D_i^s D_i^t] = \mathbb{E}[D_i^s] \cdot \mathbb{E}[D_i^t] + \mathrm{Cov}(D_i^s, D_i^t), \quad i \in I.$$

Since D_i is a positive random variable, $\mathrm{Cov}(D_i^s, D_i^t) \geq 0$ (see, e.g., Schmidt [2014]). Thus

$$\mathbb{E}[D_i^{s+t}] \geq \mathbb{E}[D_i^s] \cdot \mathbb{E}[D_i^t], \quad \forall\, i \in I,$$

which leads to

$$\frac{\mathbb{E}\left[D_i^{s+t}\right]}{\mathbb{E}[D_i]^{s+t}} \geq \frac{\mathbb{E}[D_i^s]}{\mathbb{E}[D_i]^s} \cdot \frac{\mathbb{E}[D_i^t]}{\mathbb{E}[D_i]^t}, \quad \forall\, i \in I.$$

We then have $\theta_i^{(s+t)} \geq \theta_i^{(s)} \cdot \theta_i^{(t)}$ for all $i \in I$. □

We will need Minkowski's inequality, which is stated in the following lemma.

Lemma 4.8 (Minkowski's Inequality)**.** *Let X and Y be random variables. Then for $1 \leq q < \infty$,*

$$\left(\mathbb{E}[|X+Y|^q]\right)^{1/q} \leq \left(\mathbb{E}[|X|^q]\right)^{1/q} + \left(\mathbb{E}[|Y|^q]\right)^{1/q}.$$

Denote $\overline{\theta}^{(j)} = \max_{i \in I}\{\theta_i^{(j)}\}$ for $j = 0, 1, \ldots, m$. The following lemma shows positive-valued distributions do satisfy Assumption 4.3 with $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 2, \ldots, m+1$. Note that for $j = 0, 1$, we also have $l_j = h_j =$

$\overline{\theta}^{(j)} = 1$.

Lemma 4.9. *For any transportation game* $(G, \mathbf{D}, \mathbf{c})$ *in which random demands* $\{D_i : i \in I\}$ *follow positive-valued distributions, the moments of link flows can be bounded as follows:*

$$0 \le v_e^j \le \mathbb{E}[V_e^j] \le \overline{\theta}^{(j)} v_e^j, \quad \forall\ j = 2, \ldots, m+1,\ e \in E.$$

Proof. According to (3.3), V_e is a non-negative random variable since $\{D_i : i \in I\}$ follow positive-valued distributions. Due to the convexity of x^j on $[0, \infty)$ for $j \ge 2$, the middle inequality follows directly from Jensen's inequality. For the last inequality in the lemma, we have

$$\begin{aligned}\left(\mathbb{E}[V_e^j]\right)^{1/j} &= \left(\mathbb{E}\left[\left(\sum_{i\in I} \delta_e^i p_e^i D_i\right)^j\right]\right)^{1/j} \le \sum_{i \in I} \delta_e^i \left(\mathbb{E}\left[\left(p_e^i D_i\right)^j\right]\right)^{1/j} \\ &= \sum_{i\in I} \delta_e^i p_e^i \left(\mathbb{E}[D_i^j]\right)^{1/j} = \sum_{i \in I} \delta_e^i p_e^i d_i \left(\theta_i^{(j)}\right)^{1/j} \le \left(\overline{\theta}^{(j)}\right)^{1/j} v_e,\end{aligned}$$

where the first inequality follows Minkowski's inequality. We then have

$$\mathbb{E}[V_e^j] \le \overline{\theta}^{(j)} v_e^j, \quad \forall\ e \in E,$$

which completes our proof. □

Substituting $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 0, 1, \ldots, m$ in Theorem 4.5, we obtain the following specific geometry bound on the PoA.

Proposition 4.10. *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_m$. *If* $\{D_i : i \in I\}$ *follow positive-valued distributions with finite first* $m+1$ *moments and*

$$\overline{\theta}^{(j)} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\ j = 1, \ldots m, \tag{4.13}$$

then

$$PoA(G,\mathbf{D},\mathbf{c}) \leq \max_{1\leq j\leq m} \left(1 - \frac{j}{j+1} \cdot \left(\frac{\overline{\theta}^{(j)}}{j+1}\right)^{1/j}\right)^{-1} \cdot \overline{\theta}^{(m)} \overline{\theta}^{(m+1)}.$$

Proof. We only need to show that $\underline{\alpha}(\widetilde{\mathcal{I}}_m) = 1/\overline{\theta}^{(m+1)}$ and $\overline{\alpha}(\widetilde{\mathcal{I}}_m) = \overline{\theta}^{(m)}$. By setting $s = t+1$ in (4.12), it is easy to prove that $\overline{\theta}^{(j)}$ is nondecreasing in j. Then

$$\underline{\alpha}(\widetilde{\mathcal{I}}_m) = \min_{0\leq j\leq m} \frac{1}{\overline{\theta}^{(j+1)}} = \frac{1}{\overline{\theta}^{(m+1)}},$$
$$\overline{\alpha}(\widetilde{\mathcal{I}}_m) = \max_{0\leq j\leq m} \overline{\theta}^{(j)} = \overline{\theta}^{(m)}.$$

□

Similarly, we can substitute $h_j = \overline{\theta}^{(j)}$ and $l_j = 1$ for $j = 0, 1, \dots, m$ in Theorem 4.6 to obtain the following specific convexity bound on the PoA.

Proposition 4.11. *Let $(G,\mathbf{D},\mathbf{c}) \in \mathcal{I}_m$. If $\{D_i : i \in I\}$ follow positive-valued distributions with finite first $m+1$ moments and*

$$\overline{\theta}^{(m)} < \frac{m+1}{m^{m/(m+1)}}, \tag{4.14}$$

then

$$PoA(G,\mathbf{D},\mathbf{c}) \leq \max_{1\leq j\leq m} \left(\frac{1}{\overline{\theta}^{(j+1)}} - \frac{j}{j+1} \cdot \frac{\overline{\theta}^{(j)}}{\overline{\theta}^{(j+1)}} \cdot \left(\frac{\overline{\theta}^{(j)}}{j+1}\right)^{1/j}\right)^{-1}.$$

Remark 4.3. Since $(j+1)/j^{j/(j+1)}$ is decreasing in j, condition (4.14) implies condition (4.13) for $m \geq 1$. When $m = 1$ both conditions are always satisfied and we will discuss this special case in Section 4.4. So let us consider the two conditions for $m = 2, 3, 4$, as the highest power of a link cost function is seldom greater than 4 in practice [Clark and Watling, 2005; Sumalee and Xu, 2011]. Table 4.1 shows applicable ranges of moments up to degree 4 for two conditions (4.13) and (4.14). The results indicate that condition (4.14) for the specific convexity bound in Proposition 4.11 is much less applicable for polynomial cost functions with higher degrees.

Degree	$j = 2$	$j = 3$	$j = 4$
Geometry condition (4.13):	$\overline{\theta}^{(2)} < 6.75$	$\overline{\theta}^{(3)} < 9.48$	$\overline{\theta}^{(4)} < 12.21$
Convexity condition (4.14):	$\overline{\theta}^{(2)} < 1.89$	$\overline{\theta}^{(3)} < 1.75$	$\overline{\theta}^{(4)} < 1.65$

Table 4.1: Applicable ranges of moments for the two PoA bounds

Example 4.1. We provide an example with log-normal distributions for the comparison in the above remark. Assume for $i \in I$ that $D_i \sim \ln N(\mu_i, \omega_i)$, i.e., D_i follows a log-normal distribution with mean $d_i = e^{\mu_i + \omega_i^2/2}$ and variance $\sigma_i^2 = (e^{\omega_i^2} - 1)d_i^2$, which means that the coefficient of demand variation $\epsilon_i = \sigma_i/d_i = (e^{\omega_i^2} - 1)^{1/2}$. The moments of D_i are $\mathbb{E}\left[D_i^j\right] = e^{j\mu_i + j^2\omega_i^2/2}$. Thus

$$\theta_i^{(j)} = \frac{\mathbb{E}\left[D_i^j\right]}{d_i^j} = e^{j(j-1)\omega_i^2/2} = (\epsilon_i^2 + 1)^{j(j-1)/2}, \quad \forall\, i \in I,\, j \in \mathbb{Z}^+.$$

We then have $\overline{\theta}^{(j)} = (\bar{\epsilon}^2 + 1)^{j(j-1)/2}$ for all $j \in \mathbb{Z}^+$. Conditions (4.13) and (4.14) can now be expressed in terms of applicable ranges of the maximum coefficient of variation $\bar{\epsilon}$ for different classes of polynomial link cost functions. Table 4.2 shows maximum values of $\bar{\epsilon}$ for $\mathcal{C}_m$ with $m = 2, 3$, and 4.

Class of cost functions	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Geometry condition (4.13):	2.40	1.06	0.72
Convexity condition (4.14):	0.94	0.45	0.29

Table 4.2: Maximum values of coefficient of variation $\bar{\epsilon}$ of log-normal distributions for the two upper bounds

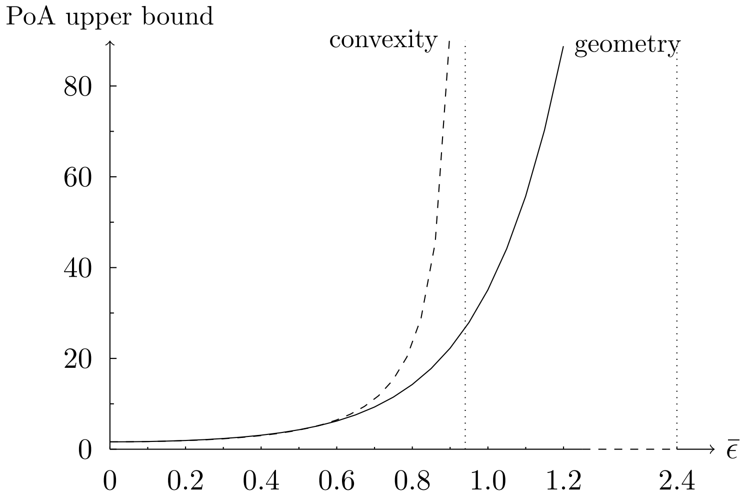


Figure 4.2: The two PoA upper bounds with quadratic cost functions ($m = 2$) and log-normal distributions

In Figure 4.2 we also plot the two specific PoA bounds presented in Propositions 4.10 and 4.11 when demands follow log-normal distributions for quadratic cost functions ($m = 2$).

As can be seen, the geometry bound is better and applicable for a wider range of game instances. The vertical dotted (asymptotical) lines help to show that the convexity and geometry bound approach infinity when $\bar{\epsilon} \to 0.94$ and 2.40 respectively. Note that when the demand variation is very small ($\bar{\epsilon} \leq 0.54$ in this case), the convexity bound can be slightly better than the geometry bound although the overall improvement is insignificant. Similar results can be obtained for $m = 3$ and $m = 4$. $\square$

4.3.2 Normal Distributions

As previously mentioned, normal distributions can be used to approximate traffic demands, especially those with large positive means or relatively small variances. As the second class of specific demand distributions, let us consider $D_i \sim N(d_i, \sigma_i^2)$ for $i \in I$, i.e., D_i follows a normal distribution with mean

d_i and variance σ_i^2 for $i \in I$. We assume $d_i > 0$ for $i \in I$. Note that this assumption guarantees the non-negativity of mean link flows, which is needed to derive general upper bounds on the PoA in Propositions 4.1 and 4.2.

Given that demands $\{D_i : i \in I\}$ are independent, clearly V_e also follows a normal distribution for $e \in E$. The mean v_e and variance σ_e^2 of V_e can be derived from (3.3) as follows, which is applicable for any independent demand distributions:

$$v_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i p_k^i d_i,$$

and

$$\sigma_e^2 = \text{Var}\left[\sum_{i \in I,\, k \in P_i} \delta_{k,e}^i p_k^i D_i\right] = \text{Var}\left[\sum_{i \in I} \delta_e^i p_e^i D_i\right] = \sum_{i \in I} \delta_e^i (p_e^i)^2 \sigma_i^2.$$

Since $V_e \sim N(v_e, \sigma_e^2)$, the jth moment of the link flow V_e can be written as follows:

$$\mathbb{E}[V_e^j] = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\sigma_e)^r (v_e)^{j-r} (r-1)!!, \quad \forall\, e \in E, \tag{4.15}$$

where $j \in \mathbb{N}$ is the power degree, $(r-1)!!$ is the double factorial of $r-1$, i.e., $(r-1)!! = (r-1)(r-3)\cdots 1$ (if r is even) with the understanding that $(-1)!! = 1$, and $\binom{j}{r} = j!/((j-r)!r!)$ is a binomial coefficient. (Note that moment formula (4.15) for the normal distribution can be found in standard texts, e.g., in [Patel and Read, 1996; Ross, 2002, p. 396 (47)].)

In order to bound moments of V_e, we first bound its variance σ_e^2 using the following lemma.

Lemma 4.12. *The mean v_e and variance σ_e^2 of random link flow V_e ($e \in E$) satisfy the following inequalities:*

$$\frac{\underline{\epsilon}^2}{n} v_e^2 \le \sigma_e^2 \le \overline{\epsilon}^2 v_e^2,$$

where n, $\underline{\epsilon}$, and $\overline{\epsilon}$ are defined in Section 3.1.

Proof. By definition $\epsilon_i = \sigma_i/d_i$, we can bound σ_e^2 from above:

$$\begin{aligned}
\sigma_e^2 &= \sum_{i\in I}\delta_e^i(p_e^i)^2\epsilon_i^2 d_i^2 \leq \left(\max_{i\in I}\{\epsilon_i\}\right)^2 \sum_{i\in I}\delta_e^i(p_e^i d_i)^2 \\
&\leq \overline{\epsilon}^2\left(\sum_{i\in I,\, k\in P_i}\delta_{k,e}^i p_k^i d_i\right)^2 = \overline{\epsilon}^2 v_e^2,
\end{aligned}$$

and bound σ_e^2 from below:

$$\begin{aligned}
\sigma_e^2 &\geq \left(\min_{i\in I}\{\epsilon_i\}\right)^2 \sum_{i\in I}\delta_e^i(p_e^i d_i)^2 \geq \frac{\underline{\epsilon}^2}{n_e}\left(\sum_{i\in I}\delta_e^i p_e^i d_i\right)^2 \\
&= \frac{\underline{\epsilon}^2}{n_e}\left(\sum_{i\in I,\, k\in P_i}\delta_{k,e}^i p_k^i d_i\right)^2 \geq \frac{\underline{\epsilon}^2}{n} v_e^2,
\end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality with $n_e = \sum_{i\in I}\delta_e^i$ as defined in Section 3.1. □

We are now ready to bound moments of link flows and show that Assumption 4.3 is satisfied.

Lemma 4.13. *For any transportation game $(G, \mathbf{D}, \mathbf{c})$ in which $\{D_i : i \in I\}$ follow normal distributions with positive mean demands, Assumption 4.3 is satisfied with*

$$l_j = \sum_{r=0,\, r=even}^{j}\binom{j}{r}\left(\frac{\underline{\epsilon}^2}{n}\right)^{r/2}(r-1)!!, \quad h_j = \overline{\theta}^{(j)}, \quad \forall\, j = 2, \ldots, m+1. \tag{4.16}$$

Proof. Since $D_i \sim N(d_i, \sigma_i^2)$ with $d_i > 0$ and finite σ_i for all $i \in I$, it is clear that all moments of D_i are finite and positive according to (4.15). We need to show that

$$l_j v_e^j \leq \mathbb{E}[V_e^j] \leq h_j v_e^j,\ \forall\, j = 2, \ldots, m+1,\ e \in E,$$

where l_j and h_j are defined in (4.16). Applying Lemma 4.12 in (4.15), we obtain

$$\mathbb{E}[V_e^j] \geq \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} \left(\frac{\underline{\epsilon}^2}{n}\right)^{r/2} (v_e)^j (r-1)!!,\ e \in E.$$

On the other hand, observe that

$$\theta_i^{(j)} = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\epsilon_i)^r (r-1)!!,\ \forall\, i \in I,$$

which implies

$$\overline{\theta}^{(j)} = \sum_{r=0,\, r=\text{even}}^{j} \binom{j}{r} (\overline{\epsilon})^r (r-1)!!,$$

which together with Lemma 4.12 implies the upper bound. □

Lemma 4.13 indicates that we can apply Theorems 4.5 and 4.6 for transportation games $(G, \mathbf{D}, \mathbf{c})$ in which $\{D_i : i \in I\}$ follow normal distributions with positive means by using values of l_j and h_j in (4.16). Since l_j depends on $\underline{\epsilon}$ and $\underline{\epsilon} \to 0$ implies $l_j \to 1$, without any restriction on $\underline{\epsilon}$ (i.e., no positive lower bound), we would have the same PoA upper bounds as with any positive-valued distributions (with the same moments as those of normal distributions). This is due to the fact that $h_j = \overline{\theta}^{(j)}$ in both settings.

Additionally l_j depends also on $n = \max_{e \in E} n_e$, where n_e as defined in Section 3.1 is the number of O-D pairs that use link e. Clearly, n is a network-related parameter, which means for normal distributions, the two upper bounds (and conditions of their applicability) are not network independent as in deterministic models in general. However the effect of n is limited, as we can also derive upper bounds independent of n by setting $n \to \infty$. From (4.16), $n \to \infty$ also implies $l_j \to 1$. Thus in such an extreme case, the PoA upper bounds (and conditions of their applicability) for normal distributions would return to the same as those for positive-valued distributions (with the

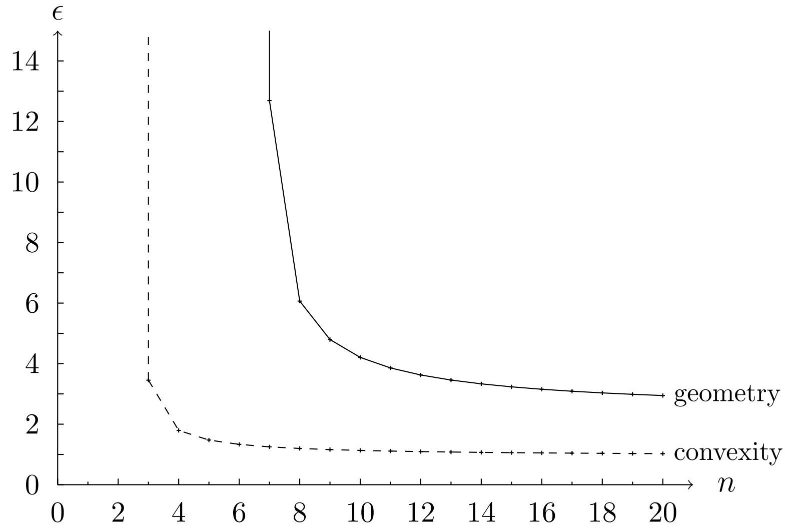


Figure 4.3: Maximum values of coefficient of variation, ϵ, of normal distributions for the two PoA upper bounds with quadratic cost functions ($m = 2$)

same moments as those of normal distributions). Then from Table 4.1 we can derive maximum applicable $\overline{\epsilon}$ of normal distributions for both geometry and convexity upper bound, as shown in Table 4.3. For polynomial cost functions with degree no more than 4, we can bound the PoA using the geometry method for arbitrary network as long as $\overline{\epsilon} < 1.08$. This condition is actually not restrictive, as only normal distributions with relatively small variance are usually used in practice to simulate traffic demands.

Class of cost functions	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Geometry Condition (4.9):	2.40	1.68	1.08
Convexity Condition (4.10):	0.94	0.50	0.32

Table 4.3: Maximum applicable $\overline{\epsilon}$ of normal distributions for the two upper bounds when $n \to \infty$

In order to demonstrate the effect of small n, we use the case of $\underline{\epsilon} = \overline{\epsilon} = \epsilon$ for simplicity. Figure 4.3 shows the maximum applicable values of ϵ when $m = 2$ for our geometry and convexity bounds. We can see that condition (4.9) is always satisfied for $n < 7$ and so is condition (4.10) for $n < 3$. When n is large

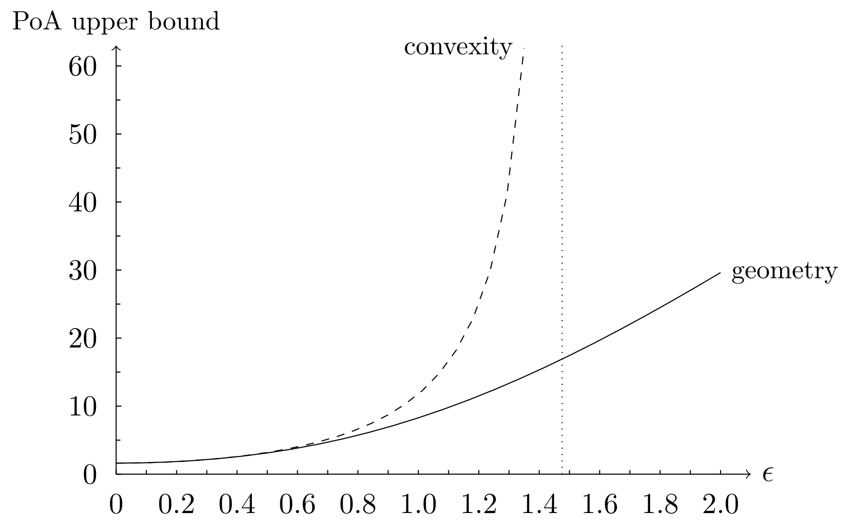


Figure 4.4: The two PoA upper bounds with quadratic cost functions ($m = 2$) and normal distributions and $n = 5$

enough, the applicable ranges of ϵ remain almost constant and as discussed above, the upper bounds of these ranges converge to the corresponding values reported in Table 4.3. Similar results can be found for $m = 3$ and $m = 4$. As noted in Section 3.1, the case of $n = 1$ is equivalent to the case of single commodity, which we will treat as a special simple case later in Section 4.4, as we will do for the case of $m = 1$.

We also compare the two upper bounds for different values of ϵ, which is illustrated in Figure 4.4 for $m = 2$ and $n = 5$ with clearly better a quality of the geometry bound.

4.4 Two Special Cases

We have provided two general upper bounds on the PoA for transportation games with general networks and general polynomial cost functions under two specific classes of demand distributions, namely, general positive-valued distributions and normal distributions. With these settings, the geometry method leads to a better upper bound under less stringent conditions. In this section we will investigate two special cases in which the convexity method will lead to a better (and in fact tight) upper bound.

The first special case is with single-commodity networks ($|I| = 1$), or equivalently $n = 1$ as noted before, while the second special case is of affine cost functions. Interestingly, both conditions (4.9) and (4.10) in these two special cases are satisfied automatically as in deterministic models.

4.4.1 Single Commodity Networks

Consider any transportation game $(G, D, \mathbf{c}) \in \mathcal{I}_m$ such that G has a single O-D pair. Since $|I| = 1$, we will drop the subscript i in writing relevant parameters, such as writing D instead of D_i. In order to satisfy Assumption 4.3, we assume the first $m+1$ moments of D are finite and positive. Then $V_e = p_e \cdot D$. Thus $\mathbb{E}\left[V_e^j\right] = \theta^{(j)} v_e^j$, where $\theta^{(j)} = \mathbb{E}\left[D^j\right]/d^j$. We can then select $l_j = h_j = \theta^{(j)}$ for all $j = 0, 1, \ldots, m+1$ and hence Assumption 4.3 is satisfied.

Condition (4.9) is satisfied since $h_j/l_j = 1$ for all $j = 1, \ldots, m$. The geometry bound in Theorem 4.5 can be calculated as follows:

$$\begin{aligned}\mathrm{PoA}(G, D, \mathbf{c}) &\leq (1 - \max_{1 \leq j \leq m} j(j+1)^{-(j+1)/j})^{-1} \cdot \frac{\max\limits_{0 \leq j \leq m} \theta^{(j)}/\theta^{(j+1)}}{\min\limits_{0 \leq j \leq m} \theta^{(j)}/\theta^{(j+1)}} \\ &= (1 - m(m+1)^{-(m+1)/m})^{-1} \cdot \frac{\max\limits_{0 \leq j \leq m} \theta^{(j)}/\theta^{(j+1)}}{\min\limits_{0 \leq j \leq m} \theta^{(j)}/\theta^{(j+1)}}.\end{aligned}$$

We claim that $\theta^{(j+1)} \geq \theta^{(j)} \geq 1$ for all j in both settings for the demand distributions, general positive-valued distributions and normal distributions. For positive-valued distributions, it follows directly from (4.12). For normal distributions, we can use (4.15) to derive the result. Thus $\max_{0 \leq j \leq m} \theta^{(j)}/\theta^{(j+1)} =$

$\theta^{(0)}/\theta^{(1)} = 1$. The geometry bound can then be simplified further as follows:

$$\text{PoA}(G, D, \mathbf{c}) \leq (1 - m(m+1)^{-(m+1)/m})^{-1} \cdot \max_{0 \leq j \leq m} \frac{\theta^{(j+1)}}{\theta^{(j)}}. \tag{4.17}$$

Condition (4.10) becomes

$$\frac{\theta^{(j)}}{\theta^{(j+1)}} < \frac{(j+1)^{j+1}}{j^j}, \quad \forall\, j = 1, \ldots, m,$$

which is also satisfied given that $\theta^{(j+1)} \geq \theta^{(j)} \geq 1$ for all j. The convexity bound in Theorem 4.6 becomes:

$$\text{PoA}(G, D, \mathbf{c}) \leq \max_{1 \leq j \leq m} \left\{ \left(1 - \frac{j}{j+1} \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1} \frac{\theta^{(j+1)}}{\theta^{(j)}} \right\}. \tag{4.18}$$

We claim that the convexity bound in (4.18) is better than the geometry bound in (4.17). In fact,

$$\begin{aligned}
&\max_{1 \leq j \leq m} \left\{ \left(1 - \frac{j}{j+1} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1} \cdot \frac{\theta^{(j+1)}}{\theta^{(j)}} \right\} \\
&\quad \leq \max_{1 \leq j \leq m} \left(1 - \frac{j}{j+1} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}}\right)^{1/j}\right)^{-1} \cdot \max_{1 \leq j \leq m} \frac{\theta^{(j+1)}}{\theta^{(j)}} \\
&\quad \leq \max_{1 \leq j \leq m} \left(1 - \frac{j}{j+1} \left(\frac{1}{j+1}\right)^{1/j}\right)^{-1} \cdot \max_{0 \leq j \leq m} \frac{\theta^{(j+1)}}{\theta^{(j)}} \\
&\quad = (1 - m(m+1)^{-(1+m)/m})^{-1} \cdot \max_{0 \leq j \leq m} \frac{\theta^{(j+1)}}{\theta^{(j)}},
\end{aligned}$$

where the second inequality is due to $\theta^{(j+1)}/\theta^{(j)} \geq 1$ and $\theta^{(0)} = \theta^{(1)} = 1$.

Figure 4.5 shows these two upper bounds for the log-normal distributions discussed in Section 4.3.1 for quadratic link cost functions ($m = 2$), in which the convexity bound is strictly better than the geometry bound. In what follows we provide an example to show that the convexity bound in (4.18) is

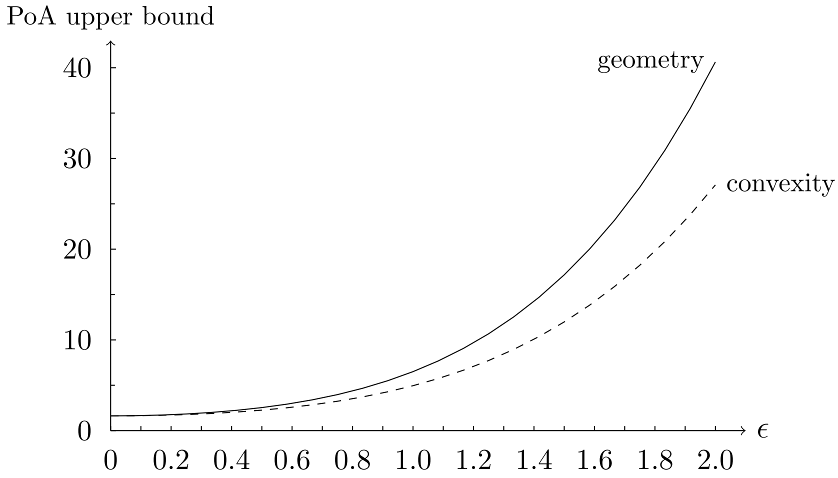


Figure 4.5: The two PoA upper bounds for single commodity networks with log-normal distributions and quadratic cost functions ($m = 2$)

actually tight.

Example 4.2. Consider a two-link network in Figure 4.6. The cost function on the upper link is a constant, $c_1(x) = \mathbb{E}[D^j]$, and that on the lower link is a polynomial function, $c_2(x) = x^j$ for a fixed j, $1 \leq j \leq m$.

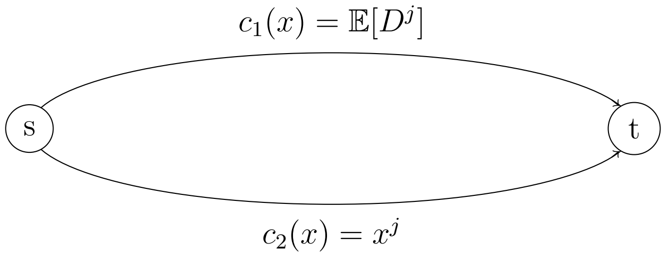


Figure 4.6: Two-link network with polynomial cost functions

As the expected total cost on the lower link is never greater than that on the upper link, strategy $\bar{\mathbf{p}} = (0, 1)$ is a UE-SD. We have

$$T(\bar{\mathbf{p}}) = \mathbb{E}[D^{j+1}] = \theta^{(j+1)} d^{j+1}.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*)$ be an SO-SD strategy, which minimises the expected total cost

$$T(\mathbf{p}) = p_1 \theta^{(j)} d^{j+1} + (p_2)^{j+1}\, \theta^{(j+1)} d^{j+1}.$$

Hence

$$p_1^* = 1 - \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \quad \text{and} \quad p_2^* = \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j},$$

which lead to

$$T(\mathbf{p}^*) = \left(1 - \frac{j}{j+1} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \right) \theta^{(j)} d^{j+1}.$$

Thus, for this instance,

$$\text{PoA} \geq \left(1 - \frac{j}{(j+1)} \cdot \left(\frac{\theta^{(j)}}{(j+1)\theta^{(j+1)}} \right)^{1/j} \right)^{-1} \cdot \frac{\theta^{(j+1)}}{\theta^{(j)}}, \tag{4.19}$$

which shows that the convexity bound in (4.18) is tight. □

4.4.2 Affine Cost Functions

We now consider a transportation game $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}_1$, i.e., all link cost functions belong to $\mathcal{C}_1$, the set of all non-zero affine functions with non-negative coefficients:

$$c_e(x) = a_e x + b_e, \text{ where } a_e, b_e \geq 0 \text{ and } a_e + b_e > 0, \ \forall\, e \in E.$$

Assume that $\{D_i : i \in I\}$ have positive means and finite second moments. From Lemma 4.12, we can choose $h_2 = 1 + \bar{\epsilon}^2$ and $l_2 = 1 + \underline{\epsilon}^2/n$. Hence Assumption 4.3 is satisfied.

Condition (4.9) is satisfied since $h_1 = l_1 = 1$. With the chosen values of h_2 and l_2, the geometry bound in Theorem 4.5 can be simplified:

$$\mathrm{PoA}(G, \mathbf{D}, \mathbf{c}) \le \frac{4}{3}(1 + \overline{\epsilon}^2). \tag{4.20}$$

Condition (4.10) reduces to $(1 + \underline{\epsilon}^2/n)^{-1} < 4$, which is always satisfied. The convexity bound in Theorem 4.6 is simplified as follows:

$$\begin{aligned}
\mathrm{PoA}(G, \mathbf{D}, \mathbf{c}) &\le \left(\frac{1}{1 + \overline{\epsilon}^2} - \frac{1}{2} \cdot \frac{1}{1 + \overline{\epsilon}^2} \cdot \frac{1}{2(1 + \underline{\epsilon}^2/n)} \right)^{-1} \\
&= \frac{4}{3} \left(1 + \overline{\epsilon}^2\right) \left(\frac{1 + \underline{\epsilon}^2/n}{1 + (4/3) \cdot \underline{\epsilon}^2/n} \right).
\end{aligned} \tag{4.21}$$

It is apparent that the convexity bound in (4.21) is better than the geometry bound in (4.20). In addition, the bound in (4.21) indicates that it is network dependent in general.

Figure 4.7 shows the two bounds with different values of n for affine cost functions when $\underline{\epsilon} = \overline{\epsilon} = \epsilon$. We can see that the geometry bound is the limiting convexity bound when n tends to infinity.

We conclude our consideration of the special case of affine cost functions by noting that the convexity bound in (4.21) is actually tight when $n = 1$ as can be easily verified by direct computation with the special case $j = 1$ of Example 4.2.

4.5 Concluding Remarks

In this chapter, we have extended two existing bounding techniques and established two different upper bounds on the PoA for our stochastic model, namely, the convexity bound and geometry bound, respectively. Unlike in the deterministic models, the two upper bounds are applicable in general only under certain conditions. In our opinion, these conditions are technical limitations of the bounding techniques we have used. We believe that in general, if these

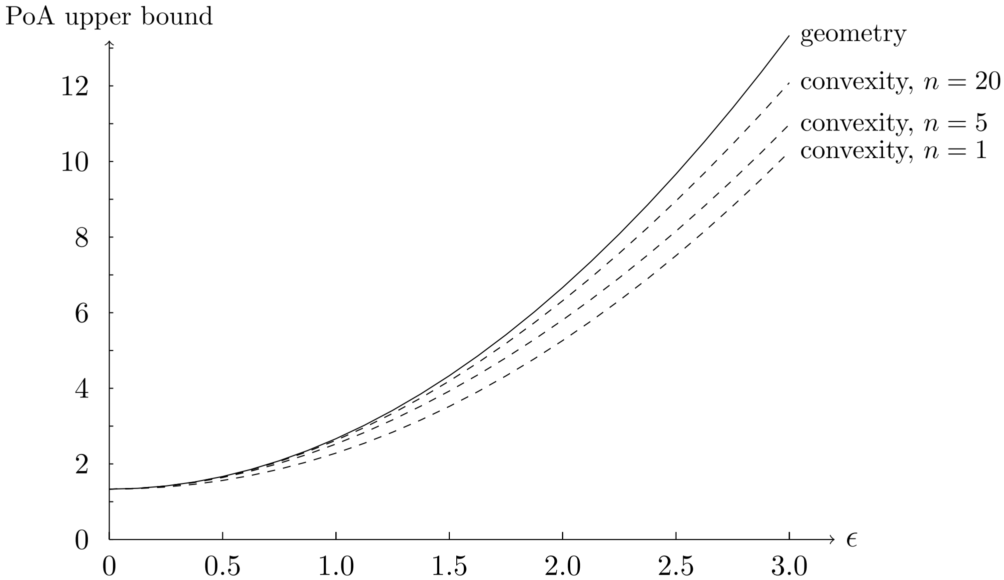


Figure 4.7: The two PoA upper bounds with affine cost functions for different values of n

conditions are not met, the PoA can still be bounded even though we are not able to prove it at present.

We have derived two specific PoA upper bounds for the class of polynomial link cost functions with positive-valued demand distributions as well as normal distributions, which are commonly used to approximate demand distributions. Numerical results show that in general the geometry bound is better and more applicable than the convexity bound. However, for single-commodity networks, the convexity bound is tight (and hence better than the geometry bound). Similarly, when only affine link cost functions are considered, the convexity bound is again better than the geometry one. One possible explanation is that the convexity method relies on convex under-approximation, which is less effective with highly non-linear (convex) cost functions. On the other hand, it seems this method is more effective if we have a good approximation of $\mathbb{E}[c_e(V_e)V_e]$, which is indeed case when the network is of a single commodity (and hence no approximation is needed) or when the demand variation is very small. In general, both upper bounds can be improved if we can have better approximation of $\mathbb{E}\left[c_e(V_e)V_e\right]$ for some specific types of demand distributions. For the class of polynomial link cost functions, better approximation of $\mathbb{E}\left[c_e(V_e)V_e\right]$ means larger l_j and smaller h_j for $j \in \mathbb{Z}^+$ in Assumption 4.3.

All upper bounds obtained for our stochastic model under various specific settings generalise the corresponding upper bounds obtained by Roughgarden and Tardos [2002, 2004] and Correa et al. [2008] for deterministic demands. The stochasticity of demands plays an important role in the formulation of these upper bounds in our model. Unlike in the deterministic models, these upper bounds can go to infinity when the demand variation tends to infinity. It shows that travellers' selfish routing can cause serious system degradation with stochastic demands. In addition, while the upper bounds in the deterministic models are network independent, those in our stochastic model can be network dependent (through the number n of O-D pairs whose paths share a particular link in the network).

Chapter 5

Atomic Model

Chapters 3 and 4 have studied non-atomic congestion games with stochastic demands. This chapter focuses on a different setting of atomic, which considers a finite number of travellers. Similar to Chapter 3, we consider day-to-day demand variation as the source of uncertainty and establish new models of user equilibrium and system optimum to describe travellers' and coordinators' behaviours in the stochastic environment. This chapter is a generalisation of the deterministic model of atomic congestion games in Section 2.3.

This chapter is based on a research article by Wang et al. [2014b].

5.1 Introduction

We follow the notation introduced in Section 3.1 for our study on atomic games. We denote an instance of an atomic congestion game by a triple $(G, \mathbf{D}, \mathbf{c})$, where G is the underlying network, $\mathbf{D}$ and $\mathbf{c}$ are the vectors of random demands and (link) cost functions, respectively. We focus on unweighted demand, thus an atomic game with stochastic demands in this thesis is actually a game with random players. Similar to the non-atomic model in Chapter 3, we also assume that the demand distributions are given and publicly known, and the demands of different O-D pairs are assumed to be independent.

5.2 Routing Strategies

In deterministic atomic work, both pure strategy and mixed strategy are considered to model travellers' routing choices. In order to find the best pure strategy, every single player needs to know all the others' routing choices. But this assumption does not hold in our stochastic model as the number of players is random. We adopt mixed strategies in this study, i.e., each player selects a probability distribution over the set of all paths for his O-D pair.

As mentioned in Section 3.2, players can only form perceptions about how other players make routing decisions solely depending on the information of which O-D pairs these players belong to, since there is no way for a player to know the routing choices of all other players or to distinguish players from a same O-D pair when the demand is uncertain [Myerson, 1998]. Thus we assume that under stochastic demands, all the players of a same O-D pair play the same strategy at an equilibrium [Myerson, 1998; Ashlagi et al., 2006].

Let p_k^i be the probability that path $k \in P_i$ is chosen. The set of mixed strategies of each player from O-D pair $i \in I$ is

$$\Omega_i = \left\{ \mathbf{p}^i = (p_k^i \geq 0 : k \in P_i) : \sum_{k \in P_i} p_k^i = 1 \right\}.$$

Let $\Omega = \prod_{i \in I} \Omega_i$. Then each vector $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$ represents a strategy profile of players from all O-D pairs. Let random binary variables $\{X_{k,j}^i : \ 1 \leq j \leq D_i, \ k \in P_i, \ i \in I\}$ indicate whether player j from O-D pair $i \in I$ chooses path k, i.e., $\mathbb{P}[X_{k,j}^i = 1] = p_k^i$ and $\mathbb{P}[X_{k,j}^i = 0] = 1 - p_k^i$. Every player has to choose one path for his traffic, i.e.,

$$\sum_{k \in P_i} X^i_{k,j} = 1, \quad \forall\ 1 \leq j \leq D_i. \tag{5.1}$$

The total traffic load on path k can be written as

$$F^i_k = \sum_{j=1}^{D_i} X^i_{k,j}, \quad k \in P_i,\ i \in I, \tag{5.2}$$

which is a *compound* random variable [Ross, 2002]. When demand D_i is realized at y, the conditional path flow on $k \in P_i$ follows binomial distribution $B(y, p^i_k)$. Then the unconditional path flow F^i_k in (5.2) can be identified by the *total probability theorem* with a given demand distribution.

Given that demands of different O-D pairs are independent, the flows on paths connecting different O-D pairs are independent. However, the path flows from a same O-D pair are dependent due to flow conservation constraint (5.1).

Let $X^i_{e,j}$ be a random binary variable indicating whether player j ($1 \leq j \leq D_i,\ i \in I$) chooses link $e \in E$, i.e., $X^i_{e,j} = \sum_{k \in P_i} \delta^i_{k,e} X^i_{k,j}$. Define $p^i_e = \sum_{k \in P_i} \delta^i_{k,e} p^i_k$, then $\mathbb{P}[X^i_{e,j} = 1] = p^i_e$ for any $1 \leq j \leq D_i$. The link flow V_e is a result of independent choices of all the players on link e:

$$V_e = \sum_{i \in I} \sum_{j=1}^{D_i} X^i_{e,j},\ \forall\ e \in E. \tag{5.3}$$

Clearly $\sum_{j=1}^{D_i} X^i_{e,j}$ is also a compound random variable, which follows Binomial distribution $B(D_i, p^i_e)$ with D_i itself a random variable. Thus the distribution of $\sum_{j=1}^{D_i} X^i_{e,j}$ can be identified given the distributions of $\mathbf{D}$ and the mixed strategy profile $\mathbf{p}$. The link flow in (5.3) is the sum of independent distributions $\sum_{j=1}^{D_i} X^i_{e,j}$ over all O-D pairs. From (5.2) and (5.3), we have the following conservations between link and path flows:

$$V_e = \sum_{i \in I} \sum_{k \in P_i} \delta^i_{k,e} F^i_k, \qquad \forall\, e \in E.$$

Given the link cost functions, the random path cost is simply the sum of the costs of those links that constitute the path, i.e.,

$$c^i_k(\mathbf{F}) = \sum_{e \in E} \delta^i_{k,e} c_e(V_e), \quad \forall\, k \in P_i,\ \forall\, i \in I. \tag{5.4}$$

5.3 User Equilibrium with Stochastic Demands (UE-SD)

At a mixed Nash equilibrium, every path of any given O-D pair with positive probability must incur the same expected cost for every player from the O-D pair, since otherwise the expected cost of any of the players from the O-D pair can be decreased by taking the lower-cost path with a higher probability. Given strategy profile $\mathbf{p}$ with the corresponding path flows $\mathbf{F}$, the expected cost of taking path $k \in P_i$ for a single player j in O-D pair $i \in I$ can be expressed as the following conditional expectation

$$\mathbb{E}[c^i_k(\mathbf{F}) \mid X^i_{k,j} = 1], \quad \forall\, 1 \le j \le D_i,\ i \in I. \tag{5.5}$$

Since at a mixed Nash equilibrium, each pure strategy involved (i.e., with positive probability) in the mixed strategy is a best response itself and yields the same expected cost, we arrive at the following definition of a user equilibrium with stochastic demands.

Definition 5.1 (UE-SD condition)**.** Strategy profile $\mathbf{p}$ with the corresponding path flows $\mathbf{F}$ is a UE-SD if and only if, for any $k, l \in P_i$, with $p_k^i > 0$,

$$\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1] \leq \mathbb{E}[c_l^i(\mathbf{F}) \mid X_{l,j}^i = 1], \tag{5.6}$$

for arbitrary traveller j ($1 \leq j \leq D_i$, $i \in I$).

Let us calculate the conditional expectation in (5.5). We have

$$\begin{aligned}
\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1] &= \mathbb{E}\left[\sum_{e \in E} \delta_{k,e}^i c_e(V_e) \mid X_{k,j}^i = 1\right] \\
&= \sum_{e \in E} \delta_{k,e}^i \mathbb{E}\left[c_e\left(\sum_{i' \in I}\sum_{j'=1}^{D_{i'}} X_{e,j'}^{i'}\right) \mid X_{k,j}^i = 1\right] \\
&= \sum_{e \in E} \delta_{k,e}^i \mathbb{E}\left[c_e\left(1 + \sum_{j'=1}^{D_i-1} X_{e,j'}^i + \sum_{i' \neq i}\sum_{j'=1}^{D_{i'}} X_{e,j'}^{i'}\right)\right],
\end{aligned}$$

which implies that $\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1]$ is independent of the choice of player j of O-D pair i. Hence we can drop subscript j by denoting $t_k^i(\mathbf{p}) = \mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1]$. Let

$$V_e(\mathbf{D} - \mathbf{e}_i) = \sum_{j'=1}^{D_i-1} X_{e,j'}^i + \sum_{i' \neq i}\sum_{j'=1}^{D_{i'}} X_{e,j'}^{i'}, \tag{5.7}$$

where $\mathbf{e}_i \in \{0,1\}^{|I|}$ is a unit vector with 0 for all its components except 1 for its i-th component. Thus we can write

$$t_k^i(\mathbf{p}) = \sum_{e \in k} \mathbb{E}[c_e(V_e(\mathbf{D} - \mathbf{e}_i) + 1)], \quad \forall\, k \in P_i,\ \forall\, i \in I.$$

Then we can reformulate the UE-SD condition as a VI program as follows.

Proposition 5.1. *A mixed strategy profile* $\mathbf{p}^*$ *is a UE-SD if and only if it satisfies the following VI problem:*

$$(\mathbf{f} - \mathbf{f}^*)^T \mathbf{t}(\mathbf{p}^*) \geq 0,\ \forall \mathbf{p} \in \Omega, \tag{5.8}$$

where $\mathbf{t}(\mathbf{p}^*) = (t_k^i(\mathbf{p}^*) : k \in P_i, i \in I)$*,* $\mathbf{f}$ *and* $\mathbf{f}^*$ *are the mean flows corresponding to strategy profiles* $\mathbf{p}$ *and* $\mathbf{p}^*$*, respectively.*

Proof. From the definition of the UE-SD, $\forall\ k, l \in P_i$, $\forall\ i \in I$, with $(p_k^i)^* > 0$, we have

$$t_k^i(\mathbf{p}^*) \leq t_l^i(\mathbf{p}^*).$$

Let $\pi_i = \min_{l \in P_i} t_l^i(\mathbf{p}^*)$ for $i \in I$. The UE-SD condition is equivalent to

$$(p_l^i)^* \left(t_l^i(\mathbf{p}^*) - \pi_i\right) = 0, \quad \forall\ l \in P_i,\ \forall\ i \in I.$$

Multiplying both sides of the above by $d_i > 0$, we obtain

$$(f_l^i)^* \left(t_l^i(\mathbf{p}^*) - \pi_i\right) = 0, \quad \forall\ l \in P_i,\ \forall\ i \in I.$$

Summing up the above over all the paths, we get

$$\sum_{i \in I} \sum_{l \in P_i} (f_l^i)^* \left(t_l^i(\mathbf{p}^*) - \pi_i\right) = 0. \tag{5.9}$$

On the other hand, for any feasible strategy profile $\mathbf{p}$, as $f_l^i \geq 0$ for any $l \in P_I$, $i \in I$,

$$\sum_{i \in I} \sum_{l \in P_i} f_l^i \left(t_l^i(\mathbf{p}^*) - \pi_i\right) \geq 0,$$

which together with (5.9) leads to

$$\sum_{i \in I} \sum_{l \in P_i} \left((f_l^i)^* - f_l^i\right) \left(t_l^i(\mathbf{p}^*) - \pi_i\right) \leq 0. \tag{5.10}$$

From the feasibility of the mixed strategies we have

$$\sum_{l \in P_i} f_l^i \pi_i = \sum_{l \in P_i} (f_l^i)^* \pi_i = \pi_i d_i,\ \forall\ i \in I.$$

Substituting the above into (5.10), we obtain

$$\sum_{i \in I} \sum_{l \in P_i} \left((f_l^i)^* - f_l^i \right) t_l^i(\mathbf{p}^*) \leq 0,$$

which is (5.8).

Next assume $\mathbf{p}^*$ satisfies (5.8). We show that it also satisfies the UE-SD condition. First with the first order optimality condition we observe that $\mathbf{p}^*$ is an optimal solution to the following linear program (LP):

$$\begin{aligned} \min \quad & \mathbf{f}^T \mathbf{t}(\mathbf{p}^*) \\ \text{s.t.} \quad & \sum_{k \in P_i} f_k^i = d_i, \quad \forall\, i \in I, \\ & f_k^i \geq 0, \qquad\quad \forall\, k \in P_i,\ \forall\, i \in I. \end{aligned}$$

With LP duality we have

$$\begin{aligned} \max \quad & \lambda^T \mathbf{d} \\ \text{s.t.} \quad & \lambda_i \leq t_k^i(\mathbf{p}^*), \quad \forall\, k \in P_i,\ \forall\, i \in I. \end{aligned}$$

Then the complementary slackness conditions lead us to

$$f_k^i(t_k^i(\mathbf{p}^*) - \lambda_i) = 0, \quad \forall\, k \in P_i, \forall\, i \in I,$$

which implies satisfaction of the UE-SD condition. □

Remark 5.1. From the VI formulation in Proposition 5.1, it is interesting to find that the discreteness of atomic games vanishes, as the UE-SD is determined by the path (link) choice probabilities, and the probabilities can be arbitrary value in $[0, 1]$. This is due to the fact that we assume all the travellers in a same O-D pair adopt an identical mixed strategy in the stochastic environment. While in the deterministic model, travellers in a same O-D pair may use different mixed strategies.

Let us check the existence of UE-SDs from the VI formula. For general link cost functions, we can rewrite condition (5.8) in the following form by substituting $f_k^i = p_k^i d_i$ and $\bar{f}_k^i = \bar{p}_k^i d_i$:

$$(\mathbf{p} - \bar{\mathbf{p}})^T \mathbf{S}(\bar{\mathbf{p}}) \geq 0, \ \mathbf{p} \in \Omega, \tag{5.11}$$

where $\mathbf{S}(\mathbf{p})$ is a vector with the same dimension as $\mathbf{t}(\mathbf{p})$, obtained by replacing element $t_k^i(\mathbf{p})$ in $\mathbf{t}(\mathbf{p})$ with $t_k^i(\mathbf{p})d_i$ for every $k \in P_i,\ i \in I$. When link cost functions are continuous, the game admits at least one UE-SD, due to the fact that existence of a solution $\bar{\mathbf{p}} \in \Omega$ to VI problem (5.11) is guaranteed by the continuity of $\mathbf{S}(\mathbf{p})$ and the compactness of Ω.

Under deterministic demands, multiple user equilibria can exist even if each O-D pair consists of a single player. The deterministic model with one single traveller in each O-D pair is a special case of our stochastic model. Note that the assumption that all the travellers from each O-D pair adopt the same strategy is clearly satisfied in this case. Therefore we can use a similar example as in [Awerbuch et al., 2013] to demonstrate the non-uniqueness of UE-SDs (see Example 5.1).

Example 5.1. Consider four O-D pairs in Figure 5.1: A-B, A-C, B-C and C-B, and each of them contains one traveller. Each traveller has two options to travel to his destination: either the one-hop path or the two-hop path. When all travellers choose the one-hop paths (i.e., A-B, A-C, B-C and C-B), they reach a UE-SD. If they all choose the two-hop paths (i.e., A-C-B, A-B-C, B-A-C and C-A-B), they reach another UE-SD.

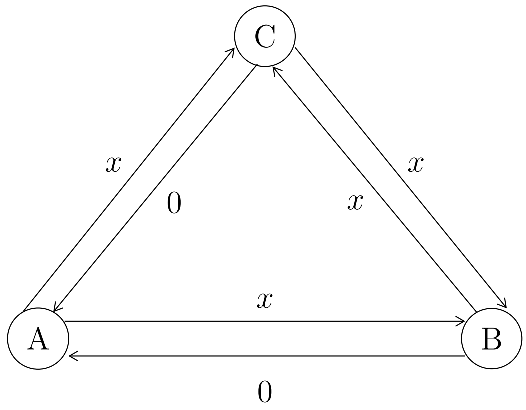


Figure 5.1: Multiple UE-SDs

5.4 System Optimum with Stochastic Demands (SO-SD)

At a system optimum the traffics are centrally coordinated and assigned so that the expected total social cost is at minimum. With deterministic demands, the system optimum can always be reached by an assignment with pure strategies, namely each player is allocated to a certain path. However, under stochastic demands, such an assignment is no longer possible, as the central coordinator can only know the demands in probabilities. We assume the central coordinator treats all the travellers of a same O-D pair equally and assigns them according to an identical mixed strategy.

Definition 5.2 (SO-SD condition)**.** A strategy profile $\mathbf{p}$ is at an SO-SD for transportation game $(G, \mathbf{D}, \mathbf{c})$ if and only if it solves the following minimization problem:

$$\min_{\mathbf{p}\in\Omega} T(\mathbf{p}) \equiv \mathbb{E}[\sum_{e\in E} c_e(V_e)V_e]. \tag{5.12}$$

Chapter 6

Price of Anarchy for Atomic Congestion Games

This chapter investigates the PoA on the basis of the atomic model with stochastic demands presented in Chapter 5. Given an atomic instance $(G, \mathbf{D}, \mathbf{c})$, the corresponding PoA is defined as:

$$\text{PoA}(G, \mathbf{D}, \mathbf{c}) := \max \left\{ \frac{T(\mathbf{p})}{T(\mathbf{q})} : \mathbf{p}, \mathbf{q} \in \Omega, \mathbf{p} \text{ is a UE-SD; } \mathbf{q} \text{ is an SO-SD} \right\},$$

where $T(\cdot)$ is the expected total cost defined in (5.12) as the social (system) objective function. Given any set $\mathcal{I}$ of instances of atomic congestion games, the PoA with respect to $\mathcal{I}$ is defined as

$$\text{PoA}(\mathcal{I}) := \max_{(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}} \text{PoA}(G, \mathbf{D}, \mathbf{c}).$$

In this chapter, we extend both the convexity and geometry methods in non-atomic works and bound the PoA for our atomic model with stochastic demands. We present analytical bounds for general cost functions and general demand distributions in Section 6.1 first, and then compute specific upper bounds for affine cost functions and polynomial cost functions in Sections 6.2

and 6.3, respectively. In particular, Section 6.2 studies affine cost functions and general positive-valued demand distributions in general networks. Section 6.3 focuses on polynomial cost functions and general positive-valued demand distributions in single commodity networks. Comparisons between two bounding methods are also addressed.

This chapter is based on a research article by Wang et al. [2014b].

6.1 General Upper Bounds

As we can see from Proposition 5.1, the UE-SD condition is equivalent to a path-based VI problem. The following two lemmas provide bounds with link-based expectations.

Lemma 6.1. *Given a transportation game* $(G, \mathbf{D}, \mathbf{c})$, *for any* $i \in I$ *with* $\delta_e^i = 1$, *the following bounds hold:*

$$\mathbb{E}[c_e(V_e)] \leq \mathbb{E}[c_e(V_e(\mathbf{D} - \mathbf{e}_i) + 1)] \leq \mathbb{E}[c_e(V_e + 1)], \ \forall \ e \in E.$$

Proof. Since $0 \leq X_{e,j}^i \leq 1$, with (5.3) and (5.7) we have

$$V_e \leq V_e(\mathbf{D} - \mathbf{e}_i) + 1 \leq V_e + 1,$$

which together with that $c_e(\cdot)$ is nondecreasing implies the inequalities in the lemma. □

As F_k^i follows a compound distribution, we can calculate

$$\text{Var}\left[F_k^i\right] = d_i p_k^i \left(1 - p_k^i\right) + \sigma_i^2 \left(p_k^i\right)^2 = \left(\sigma_i^2 - d_i\right) \left(p_k^i\right)^2 + d_i p_k^i.$$

Similarly we have

$$\operatorname{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i\right] = \left(\sigma_i^2 - d_i\right)\left(p_e^i\right)^2 + d_i p_e^i, \quad \forall\, i \in I.$$

Denote $\mathrm{VMR}_i = \sigma_i^2/d_i$ as the variance-to-mean ratio for any $i \in I$. Since demands of different O-D pairs are independent, we have

$$\begin{aligned}\operatorname{Var}\left[V_e\right] &= \sum_{i\in I}\delta_e^i \operatorname{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i\right] = \sum_{i\in I}\delta_e^i\left(\sigma_i^2 - d_i\right)\left(p_e^i\right)^2 + \sum_{i\in I}\delta_e^i d_i p_e^i \\ &= \sum_{i\in I}\delta_e^i(\mathrm{VMR}_i - 1)d_i\left(p_e^i\right)^2 + v_e. \end{aligned} \tag{6.1}$$

Lemma 6.2. *For any transportation game* $(G, \mathbf{D}, \mathbf{c})$*, we have*

$$l \cdot v_e^2 + v_e \le \mathbb{E}[V_e^2] \le h \cdot v_e^2 + v_e, \quad \forall\, e \in E,$$

where

$$l = \begin{cases} \dfrac{1}{n}\min\limits_{i\in I}\dfrac{VMR_i - 1}{d_i} + 1, & \text{if } \min\limits_{i\in I} VMR_i \ge 1, \\ \min\limits_{i\in I}\dfrac{VMR_i - 1}{d_i} + 1, & \text{otherwise}; \end{cases}$$

$$h = \begin{cases} \max\limits_{i\in I}\dfrac{VMR_i - 1}{d_i} + 1, & \text{if } \max\limits_{i\in I} VMR_i \ge 1, \\ \dfrac{1}{n}\max\limits_{i\in I}\dfrac{VMR_i - 1}{d_i} + 1, & \text{otherwise}, \end{cases}$$

and n is defined in Section 3.1.

Proof. From Cauchy-Schwarz inequality and $v_e = \sum_{i\in I} p_e^i d_i$, we have

$$\frac{1}{n}v_e^2 \le \sum_{i\in I}\delta_e^i\left(p_e^i d_i\right)^2 \le v_e^2, \quad \forall\, e \in E,$$

which together with (6.1) implies

$$\begin{aligned}\mathrm{Var}[V_e] &\leq \max_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\}\sum_{i\in I}\left(p_e^i d_i\right)^2 + v_e \\ &\leq \begin{cases}\max_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\} v_e^2 + v_e, & \text{if } \max_{i\in I}\mathrm{VMR}_i \geq 1; \\ \frac{1}{n}\max_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\} v_e^2 + v_e, & \text{otherwise.}\end{cases}\end{aligned}$$

Similarly we have

$$\begin{aligned}\mathrm{Var}[V_e] &\geq \min_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\}\sum_{i\in I}\left(p_e^i d_i\right)^2 + v_e \\ &\geq \begin{cases}\frac{1}{n}\min_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\} v_e^2 + v_e, & \text{if } \min_{i\in I}\mathrm{VMR}_i \geq 1; \\ \min_{i\in I}\left\{\frac{\mathrm{VMR}_i - 1}{d_i}\right\} v_e^2 + v_e, & \text{otherwise.}\end{cases}\end{aligned}$$

Together with $\mathbb{E}[V_e^2] = \mathrm{Var}[V_e] + v_e^2$, the lemma is implied. □

Recall $\mathcal{C}(\mathcal{I})$ denoting the class of link cost functions $\{c_e(\cdot) : e \in E\}$ used in game instances $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$. In establishing our general upper bounds on the PoA, we make a few general assumptions about existence of some bounding functions, which we will identify in computing specific bounds later.

Assumption 6.1. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, there exist non-decreasing functions $\underline{s}_e(\cdot)$, $\overline{s}_e(\cdot)$, such that $\underline{s}_e(0) = \overline{s}_e(0) = 0$, and for any random link flows V_e $(e \in E)$ with $v_e > 0$,

$$0 < \underline{s}_e(v_e) \leq \mathbb{E}[c_e(V_e)V_e] \leq \overline{s}_e(v_e).$$

There also exist non-decreasing functions $\underline{t}_e(\cdot)$, $\overline{t}_e(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$, such that

$\forall\ e \in E,\ \forall\ i \in I$ with $\delta_e^i = 1$,

$$0 \leq \underline{t}_e(v_e) \leq \mathbb{E}[c_e(V_e(\mathbf{D} - \mathbf{e}_i) + 1)] \leq \bar{t}_e(v_e). \qquad (6.2)$$

With Assumption 6.1, we can derive the following inequality from UE-SD condition (5.8).

Lemma 6.3. *Let* $\bar{\mathbf{p}}$ *be a strategy profile at the UE-SD, and* $\bar{\mathbf{v}}$ *be the vector of the corresponding mean link flows. Let* $\mathbf{p} \in \Omega$ *be any strategy profile with* $\mathbf{v}$ *as the vector of the corresponding mean path flows. Then*

$$\sum_{e \in E} \bar{v}_e \underline{t}_e(\bar{v}_e) \leq \sum_{e \in E} v_e \bar{t}_e(\bar{v}_e). \qquad (6.3)$$

Proof. From Assumption 6.1, we have

$$\sum_{e \in E} \delta_{k,e}^i \underline{t}_(v_e) \leq t_k^i(\mathbf{p}) \leq \sum_{e \in E} \delta_{k,e}^i \bar{t}_(v_e),$$

which together with (5.8) leads to

$$\sum_{k \in P_i} f_k^i \left(\sum_{e \in E} \delta_{k,e}^i \bar{t}_(\bar{v}_e) \right) \geq \sum_{k \in P_i} \bar{f}_k^i \left(\sum_{e \in E} \delta_{k,e}^i \underline{t}_(\bar{v}_e) \right).$$

Rearranging the above inequality gives (6.3). □

Assumption 6.2. For each link cost function $c_e(\cdot) \in \mathcal{C}(\mathcal{I})$, the function $\underline{s}_e(\cdot)$ in Assumption 6.1 is convex and differentiable. In addition, there exists a function $\lambda_e(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\underline{s}_e'(\lambda_e(x)x) = \bar{t}_e(x)$ for all $x \geq 0$, where $\underline{s}_e'(\cdot)$ is the derivative of $\underline{s}_e(\cdot)$.

Definition 6.1. Under Assumptions 6.1 and 6.2, let

$$\gamma(c_e, \mathcal{I}) = \inf_{x>0} \left\{ \frac{\underline{s}_e(\lambda(x)x) + x\underline{t}_e(x) - \lambda(x)x\bar{t}_e(x)}{\bar{s}_e(x)} \right\},\ c_e(\cdot) \in \mathcal{C}(\mathcal{I})$$

and

$$\gamma(\mathcal{I}) = \inf_{c \in \mathcal{C}(\mathcal{I})} \gamma(c, \mathcal{I}).$$

Now we are ready to present our first general upper bound on the PoA. We also call it *convexity bound*, as the bounding method is an adaption and extension from the convexity method in Chapter 4 for non-atomic models.

Theorem 6.4 (General convexity bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ *be any transportation game. Under Assumptions 6.1 and 6.2, if* $\gamma(\mathcal{I}) > 0$*, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \frac{1}{\gamma(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ and $\mathbf{p}^* \in \Omega$ be respectively a UE-SD and an SO-SD, with $\bar{\mathbf{v}}$ and $\mathbf{v}^*$ as the corresponding mean link flows. Then

$$\begin{aligned}
T(\mathbf{p}^*) &= \sum_{e \in E} \mathbb{E}\left[c_e(V_e^*)V_e^*\right] \geq \sum_{e \in E} \underline{s}_e(v_e^*) \\
&\geq \sum_{e \in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\underline{s}_e'(\lambda_e(\bar{v}_e)\bar{v}_e)\right) \\
&= \sum_{e \in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + (v_e^* - \lambda_e(\bar{v}_e)\bar{v}_e)\bar{t}_e(\bar{v}_e)\right) \\
&= \sum_{e \in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + v_e^*\bar{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right),
\end{aligned}$$

where the first inequality follows from Assumption 6.1 and the second inequality is due to the convexity of $\underline{s}_e(\cdot)$, while the the second equality follows from Assumption 6.2. On the other hand, applying (6.3) for the last line above leads to

$$\begin{aligned} T(\mathbf{p}^*) &\geq \sum_{e\in E} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e\underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right) \\ &= \sum_{e\in E:\bar{v}_e>0} \left(\underline{s}_e(\lambda_e(\bar{v}_e)\bar{v}_e) + \bar{v}_e\underline{t}_e(\bar{v}_e) - \lambda_e(\bar{v}_e)\bar{v}_e\bar{t}_e(\bar{v}_e)\right) \\ &\geq \gamma(\mathcal{I}) \sum_{e\in E:\bar{v}_e>0} \overline{s}_e(\bar{v}_e) = \gamma(\mathcal{I}) \sum_{e\in E} \overline{s}_e(\bar{v}_e) \geq \gamma(\mathcal{I}) T(\bar{\mathbf{p}}), \end{aligned}$$

where the first equality follows from $\underline{s}_e(0) = 0$ (Assumption 6.1), and the second equality is according to Definition 6.1. Given that $\gamma(\mathcal{I}) > 0$, we have

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq \frac{1}{\gamma(\mathcal{I})}.$$

The above inequality is true for all pairs $(\bar{\mathbf{p}}, \mathbf{p}^*)$, which implies that $1/\gamma(\mathcal{I})$ is an upper bound of the PoA. □

Definition 6.2. Under Assumption 6.1, let

$$\beta(c_e, \mathcal{I}) = \sup_{x\geq 0,\, y>0} \left\{ \frac{x(\bar{t}_e(y) - \underline{t}_e(x))}{y\underline{t}_e(y)} \right\},\ \ c_e(\cdot) \in \mathcal{C}(\mathcal{I})$$

and

$$\beta(\mathcal{I}) = \sup_{c\in\mathcal{C}(\mathcal{I})} \beta(c, \mathcal{I}).$$

Let

$$\phi_e(x) = \frac{x\underline{t}_e(x)}{\overline{s}_e(x)}, \quad \eta_e(x) = \frac{x\underline{t}_e(x)}{\underline{s}_e(x)}.$$

Define

$$\underline{\alpha}(c_e, \mathcal{I}) = \inf_{x>0} \phi_e(x), \quad \overline{\alpha}(c_e, \mathcal{I}) = \sup_{x>0} \eta_e(x);$$

and

$$\underline{\alpha}(\mathcal{I}) = \inf_{c\in\mathcal{C}(\mathcal{I})} \underline{\alpha}(c, \mathcal{I}), \quad \overline{\alpha}(\mathcal{I}) = \sup_{c\in\mathcal{C}(\mathcal{I})} \overline{\alpha}(c, \mathcal{I}).$$

Assumption 6.3. $\underline{t}_e(0) = \bar{t}_e(0)$, for any $e \in E$.

We still call the bound presented below a *geometry bound*, as it is based on a bounding method initially of a geometric argument for non-atomic models, although it has lost its pure geometric base for our atomic models. The bounding method is an adaption and extension from what is used in Chapter 4 for non-atomic models.

Theorem 6.5 (General geometry bound)**.** *Let $(G, \mathbf{D}, \mathbf{c}) \in \mathcal{I}$ be any transportation game. Under Assumptions 6.1 and 6.3, if $\beta(\mathcal{I}) < 1$, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq (1 - \beta(\mathcal{I}))^{-1} \cdot \frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})}.$$

Proof. Let $\bar{\mathbf{p}}$ and $\mathbf{p}^* \in \Omega$ be respectively a UE-SD and an SO-SD, with $\bar{\mathbf{v}}$ and $\mathbf{v}^*$ as the corresponding mean link flows. From Lemma 6.3, we have

$$\begin{aligned}\sum_{e \in E} \bar{v}_e \underline{t}_e(\bar{v}_e) &\leq \sum_{e \in E} v_e^* \bar{t}_e(\bar{v}_e) \\ &= \sum_{e \in E} v_e^* \underline{t}_e(v_e^*) + \sum_{e \in E} v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right),\end{aligned}$$

which can be rearranged as

$$(1 - R) \sum_{e \in E} \bar{v}_e \underline{t}_e(\bar{v}_e) \leq \sum_{e \in E} v_e^* \underline{t}_e(v_e^*),$$

where with $\bar{E} \equiv \{e \in E : \bar{v}_e > 0\} \neq \emptyset$,

$$\begin{aligned}R \equiv \frac{\sum_{e \in E} v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right)}{\sum_{e \in E} \bar{v}_e \underline{t}_e(\bar{v}_e)} &\leq \frac{\sum_{e \in \bar{E}} v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right)}{\sum_{e \in \bar{E}} \bar{v}_e \underline{t}_e(\bar{v}_e)} \\ \leq \max_{e \in \bar{E}} \frac{v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right)}{\bar{v}_e \underline{t}_e(\bar{v}_e)} &\leq \beta(\mathcal{I}).\end{aligned}$$

The first inequality above is due to $v_e^* \left(\bar{t}_e(\bar{v}_e) - \underline{t}_e(v_e^*)\right) \leq 0$ when $\bar{v}_e = 0$ as can be seen from $\underline{t}_e(v_e^*) \geq \underline{t}_e(0) = \bar{t}_e(0)$ (Assumptions 6.1 and 6.3). Hence

$$(1-\beta(\mathcal{I}))\sum_{e\in E}\bar{v}_e\underline{t}_e(\bar{v}_e) \leq \sum_{e\in E} v_e^*\underline{t}_e(v_e^*). \tag{6.4}$$

We have

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \equiv R_1 \cdot R_2^{-1} \cdot R_3,$$

where

$$R_1 \equiv \frac{\sum_{e\in E}\bar{v}_e\underline{t}_e(\bar{v}_e)}{\sum_{e\in E} v_e^*\underline{t}_e(v_e^*)} \leq (1-\beta(\mathcal{I}))^{-1},$$

according to (6.4), and

$$\begin{aligned} R_2 &\equiv \frac{\sum_{e\in E}\bar{v}_e\underline{t}_e(\bar{v}_e)}{T(\bar{\mathbf{p}})} \geq \frac{\sum_{e\in E}\bar{v}_e\underline{t}_e(\bar{v}_e)}{\sum_{e\in E}\bar{s}_e(\bar{v}_e)} \qquad \text{(Assumption 6.1)} \\ &= \frac{\sum_{e\in \bar{E}}\bar{v}_e\underline{t}_e(\bar{v}_e)}{\sum_{e\in \bar{E}}\bar{s}_e(\bar{v}_e)} \geq \min_{e\in\bar{E}}\frac{\bar{v}_e\underline{t}_e(\bar{v}_e)}{\bar{s}_e(\bar{v}_e)} \geq \underline{\alpha}(\mathcal{I}), \end{aligned}$$

and with $E^* \equiv \{e \in E : v_e^* > 0\} \neq \emptyset$,

$$\begin{aligned} R_3 &\equiv \frac{\sum_{e\in E} v_e^*\underline{t}_e(v_e^*)}{T(\mathbf{p}^*)} \leq \frac{\sum_{e\in E} v_e^*\underline{t}_e(v_e^*)}{\sum_{e\in E}\underline{s}_e(v_e^*)} \qquad \text{(Assumption 6.1)} \\ &= \frac{\sum_{e\in E^*} v_e^*\underline{t}_e(v_e^*)}{\sum_{e\in E^*}\underline{s}_e(v_e^*)} = \max_{e\in E^*}\frac{v_e^*\underline{t}_e(v_e^*)}{\underline{s}_e(v_e^*)} \leq \overline{\alpha}(\mathcal{I}). \end{aligned}$$

Therefore

$$\frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} \leq (1-\beta(\mathcal{I}))^{-1}\frac{\overline{\alpha}(\mathcal{I})}{\underline{\alpha}(\mathcal{I})}.$$

for any pair $\bar{\mathbf{p}}, \mathbf{p}^* \in \Omega$ of a UE-SD and an SO-SD. □

Remark 6.1. Assumption 6.3 is a technical assumption to achieve finite geometry bound in Theorem 6.5. In definition 6.2, the value of $\beta(c_e, \mathcal{I})$ is bigger than that in Definition 4.1 for the non-atomic model, due to the approximate link-base inequality (6.3) of the UE-SD condition (5.6).

6.2 Affine Cost Functions

Having established two upper bounds on the PoA for general link cost functions, let us now compute specific upper bounds for affine link cost functions, i.e.,

$$c_e(x) = a_e x + b_e, \quad a_e, b_e \geq 0 \text{ and } a_e + b_e > 0, \ e \in E. \tag{6.5}$$

Let us choose specific functions of $\overline{t}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\overline{s}_e(\cdot)$ and $\underline{s}_e(\cdot)$ satisfying Assumptions 6.1 and 6.2 to compute the convexity bound of the PoA in Theorem 6.4.

Definition 6.3. Given affine cost functions in (6.5), for each $e \in E$, let

$$\begin{cases} \overline{s}_e(x) = h a_e x^2 + a_e x + b_e x, \\ \underline{s}_e(x) = l a_e x^2 + a_e x + b_e x, \end{cases}$$

and

$$\begin{cases} \overline{t}_e(x) = c_e(x+1) = a_e x + a_e + b_e, \\ \underline{t}_e(x) = c_e(x) = a_e x + b_e \end{cases}$$

where l and h are defined in Lemma 6.2. In addition, let $\lambda_e(x) = 1/(2l)$.

Remark 6.2. Since $d_i \geq 1$, $i \in I$, we have $h \geq l \geq 0$. With Lemmas 6.1 and 6.2, it is easy to check Assumptions 6.1 and 6.2 are satisfied under Definition 6.3. We have noted that $\lambda_e(x)$ goes to infinity when $l = 0$. But it will be only attained when $D_i = d_i = 1$ and $n = 1$, namely only one player in a single commodity network, which is not of interest for the PoA study. Moreover the special case of $l = 0$ will be excluded in our later study on both the convexity and geometry bounds due to the restrictive conditions $\gamma(\mathcal{C}) > 0$ and $\beta(\mathcal{C}) < 1$ (see Theorems 6.4 and 6.5).

Proposition 6.6. *Let* $(G, \mathbf{D}, \mathbf{c})$ *be a transportation game with affine cost functions as in* (6.5) *and* l *and* h *as defined in Lemma 6.2. If* $l > 1/4$*, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \max\left\{\frac{4hl}{4l-1},\ 1 + \max_{e\in E}\frac{a_e}{b_e}\right\}.$$

Proof. From Definitions 6.1 and 6.3, we have

$$\gamma(c_e, \mathcal{I}) = \inf_{x>0} L(x) \equiv \frac{(1 - 1/(4l))\, a_e x + b_e}{h a_e x + a_e + b_e}. \tag{6.6}$$

In order to have $\gamma(c_e, \mathcal{I}) > 0$, we assume $1 - 1/(4l)$, i.e., $l > 1/4$. Since the ratio of any two affine functions achieves its extreme values at the boundary of its domain, we obtain

$$\gamma(c_e, \mathcal{I}) = \min\left\{\lim_{x\to\infty} L(x),\ \lim_{x\to 0} L(x)\right\} = \min\left\{\frac{4l-1}{4hl},\ \frac{b_e}{a_e + b_e}\right\},$$

which together with Theorem 6.4 concludes our proof. □

Proposition 6.6 involves new parameters a_e and b_e and fails to bound the PoA for the special case of linear cost functions ($b_e = 0$). We overcome this difficulty in the next proposition by choosing another bounding function $\underline{t}_e(\cdot)$ as follows:

$$\underline{t}_e(x) = a_e g x + a_e + b_e,\ \ e \in E, \tag{6.7}$$

where

$$g = 1 - 1/\underline{d} \ \text{ and } \ \underline{d} = \min_{i\in I} d_i. \tag{6.8}$$

Clearly we have $0 \leq g < 1$ since $\underline{d} \geq 1$. We make sure in the next lemma that Assumption 6.1 is satisfied with the new choice of functions.

Lemma 6.7. *Given transportation game* $(G, \mathbf{D}, \mathbf{c})$*, for any* $e \in E$ *we have*

$$\mathbb{E}[V_e(\mathbf{D} - \mathbf{e}_i) + 1] \geq g \cdot v_e + 1, \quad \forall\ i \in I \text{ with } \delta_e^i = 1.$$

Proof. Since $v_e = \sum_{i \in I} p_e^i d_i$, we have

$$\begin{aligned}
\mathbb{E}[V_e(\mathbf{D} - \mathbf{e}_i) + 1] &= \mathbb{E}\left[\sum_{j=0}^{D_i - 1} X_{e,j}^i\right] + \sum_{i' \in I \setminus \{i\}} \delta_e^i p_e^{i'} d_{i'} + 1 \\
&= p_e^i (d_i - 1) + \sum_{i' \in I \setminus \{i\}} \delta_e^i p_e^{i'} d_{i'} + 1 \\
&\geq v_e - p_e^i + 1 \geq v_e - \sum_{i \in I} \delta_e^i p_e^i + 1 \geq v_e - \frac{v_e}{\underline{d}} + 1,
\end{aligned}$$

from which the lemma follows. □

With $\underline{t}_e(\cdot)$ defined in (6.7) and $\bar{t}_e(\cdot)$, $\bar{s}_e(\cdot)$, $\underline{s}_e(\cdot)$, and $\lambda_e(\cdot)$ in Definition 6.3, Assumptions 6.1–6.3 are all satisfied. We have the following alternative bound on the PoA.

Proposition 6.8. *Let* $(G, \mathbf{D}, \mathbf{c})$ *be a transportation game with affine cost functions as in* (6.5), *l and h as defined in Lemma 6.2, and g in* (6.8). *If* $4gl > 1$, *then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \frac{4hl}{4gl - 1}.$$

Proof. According to the definitions of l, h and g, we can easily verify $h > g - 1/(4l)$. Hence we have

$$\begin{aligned}
\gamma(c_e, \mathcal{I}) &= \inf_{x > 0} \frac{a_e\,(g - 1/(4l))\,x + (a_e + b_e)}{h a_e x + a_e + b_e} \\
&= \lim_{x \to \infty} \frac{a_e\,(g - 1/(4l))\,x + (a_e + b_e)}{h a_e x + a_e + b_e} \\
&= \frac{g - 1/(4l)}{h} = \frac{4gl - 1}{4hl} > 0,
\end{aligned}$$

which together with Theorem 6.4 implies the proposition. □

Combining Propositions 6.6 and 6.8, we obtain a convexity upper bound specific to affine cost functions.

Theorem 6.9 (Convexity bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c})$ *be a transportation game with affine cost functions as in* (6.5)*,* l *and* h *as defined in Lemma 6.2, and* g *in* (6.8)*. Then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \begin{cases} W \equiv \max\left\{\dfrac{4hl}{4l-1},\, 1 + \max_{e \in E} \dfrac{a_e}{b_e}\right\}, & \textit{if } \dfrac{1}{4} < l \leq \dfrac{1}{4g}; \\ \min\left\{W,\, \dfrac{4hl}{4gl-1}\right\}, & \textit{if } l > \dfrac{1}{4g}. \end{cases}$$

Remark 6.3. In Theorem 6.9, the applicability condition, $l > 1/4$, of the convexity bound is equivalent to that, for any $i \in I$, either $d_i > 4/3$ or ($d_i \leq 4/3$ and $\sigma_i^2 > d_i - 3d_i^2/4$). Apparently, they are mild as d_i represents the average number of travellers for O-D pair i.

Our geometry upper bound specific to affine cost functions are provided in the next theorem.

Theorem 6.10 (Geometry bound)**.** *Let* $(G, \mathbf{D}, \mathbf{c})$ *be a transportation game with affine cost functions as in* (6.5)*,* l *and* h *as defined in Lemma 6.2, and* g *in* (6.8)*. If* $\underline{d} > 2$*, then*

$$PoA(G, \mathbf{D}, \mathbf{c}) \leq \frac{4g^2}{4g^2 - 1} \cdot \frac{\max\{1,\, g/l\}}{\min\{1,\, g/h\}}.$$

Proof. We use $\bar{t}_e(\cdot)$, $\bar{s}_e(\cdot)$, $\underline{s}_e(\cdot)$ in Definition 6.3 and $\underline{t}_e(\cdot)$ in (6.7). With Assumptions 6.1 and 6.3 both satisfied, Theorem 6.5 will be applicable. We have

$$\begin{aligned} \beta(c_e, \mathcal{I}) &= \sup_{x \geq 0,\, y > 0} \frac{x((a_e y + a_e + b_e) - (a_e g x + a_e + b_e))}{y(a_e g y + a_e + b_e)} \\ &= \sup_{x \geq 0,\, y > 0} \frac{a_e x\,(y - gx)}{y(a_e g y + a_e + b_e)} = \sup_{y > 0} \frac{a_e y}{4g(a_e g y + a_e + b_e)} = \frac{1}{4g^2}, \end{aligned}$$

where the third equality is obtained by setting $x = y/(2g)$. In order to have

$\beta(\mathcal{I}) < 1$, we assume $1/(4g^2) < 1$, i.e., $\underline{d} > 2$. Also we have

$$\underline{\alpha}(c_e, \mathcal{I}) = \inf_{x>0} \frac{x\,(a_e g x + a_e + b_e)}{h a_e x^2 + a_e x + b_e x} = \min\{g/h,\ 1\},$$
$$\overline{\alpha}(c_e, \mathcal{I}) = \sup_{x>0} \frac{x(a_e g x + a_e + b_e)}{l a_e x^2 + a_e x + b_e x} = \max\{1, g/l\}.$$

Substituting the above into Theorem 6.5 completes our proof. □

Remark 6.4. When $d_i \to \infty$ and $\sigma_i^2 \to 0$, $i \in I$, we have $h = l = g \to 1$. Both the convexity and geometry bounds in Theorems 6.9 and 6.10 become 4/3. Example 6.1 shows that our upper bounds are asymptotically tight. It is interesting that our upper bounds in such an extreme case also match the tight upper bound on the PoA for non-atomic transportation games of deterministic demands [Roughgarden and Tardos, 2004], which can be regarded as a limit of our model when $d_i \to \infty$ and $\sigma_i^2 \to 0$, $i \in I$. On the other hand, when d_i is finite and $\sigma_i^2 = 0$, our upper bounds do not match the upper bound of 5/2 for deterministic atomic models in Awerbuch et al. [2005]. This is mainly because in the deterministic atomic models, different players can choose different strategies at an equilibrium even if they are from a same O-D pair, while in our UE-SD model, all the players from a same O-D pair adopt the same mixed strategy.

Example 6.1. Consider the two-link network in Figure 6.1. Let D be the demand from s to t, with $\mathbb{E}[D] = d$ and $\mathrm{Var}[D] = \sigma^2$.

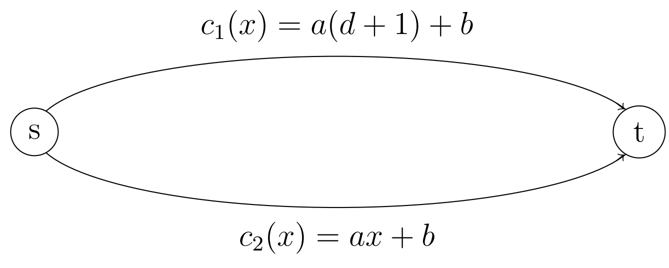


Figure 6.1: Two-link network

From the UE-SD condition, strategy $\mathbf{p}^T = (0,1)^T$ is a UE-SD. The expected total cost is

$$T(\mathbf{p}) = \mathbb{E}[aD^2 + bD] = ad^2 + a\sigma^2 + bd.$$

Let $\mathbf{p}^*$ be the SO-SD strategy. Then $h = 1 + (\sigma^2 - d)/d^2 \to 1$ when $d \to \infty$ and $\sigma^2 = o(d^2)$. We have $\mathbf{p}^* = (1 - 1/(2h), 1/(2h))$ by solving

$$\min_{\mathbf{p}\in\Omega} T(\mathbf{p}) \equiv \mathbb{E}[(a(d+1)+b)V_1] + \mathbb{E}[(aV_2+b)V_2].$$

Then the expected total cost is

$$T(\mathbf{p}^*) = a\left(1 - \frac{1}{4h}\right)d^2 + ad + bd.$$

Thus

$$\text{PoA} = \frac{ad^2 + a\sigma^2 + bd}{a\left(1 - 1/(4h)\right)d^2 + ad + bd},$$

which approaches 4/3 as $d \to \infty$ and $\sigma^2 = o(d^2)$.

We will conclude this section with a numerical comparison between the convexity and geometry bounds. We consider two types of demand distributions, i.e., discrete uniform distributions and zero-truncated Poisson (ZTP) distributions.

Let us start with discrete uniform distributions. For simplicity we consider demands all following identical uniform distribution $U[1,z]$ with $z \geq 7$. The mean and variance are

$$d_i = \frac{1+z}{2}, \text{ and } \sigma_i^2 = \frac{z^2-1}{12}, \qquad \forall\, i \in I.$$

We have

$$l = 1 + \frac{z-7}{3(z+1)n}, \quad h = \frac{4(z-1)}{3(z+1)}, \quad g = \frac{z-1}{z+1}.$$

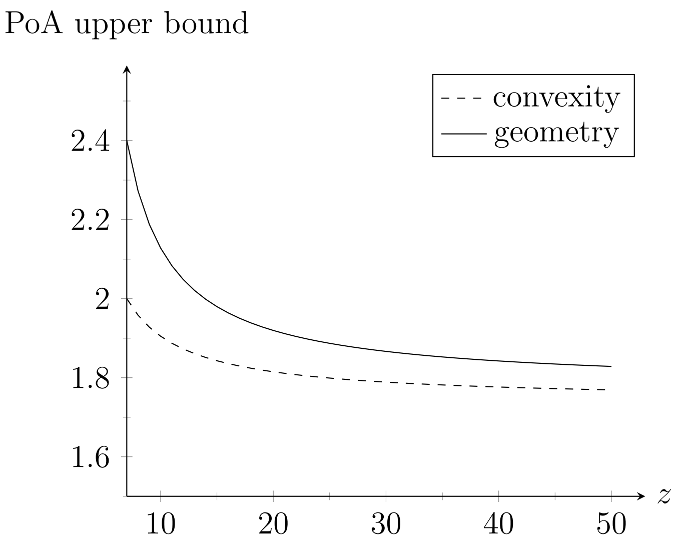


Figure 6.2: The two PoA upper bounds with with affine cost functions, discrete uniform distributions $U[1, z]$ and $n = 5$

Note that condition $l > 1/(4g)$ (see Theorem 6.9) is always satisfied. Thus the convexity bound is no more than

$$\frac{4hl}{4gl - 1}. \tag{6.9}$$

Figure 6.2 compares (6.9) with the geometry bound in Theorem 6.10 when $n = 5$. It shows that the convexity bound in this case even with the value (6.9) is tighter than the geometry bound. Similar results can be found for different values of n.

Next let us consider zero-truncated Poisson (ZTP) distributions. For simplicity, we take demands all following identical distribution ZTP(λ), where λ is the mean of the original Poisson distribution. From [Johnson et al., 2005], the expectation and variance of D_i, $i \in I$, are

$$d_i = \frac{\lambda e^\lambda}{e^\lambda - 1}, \quad \sigma_i^2 = \frac{\lambda e^\lambda}{e^\lambda - 1}\left(1 - \frac{\lambda}{e^\lambda - 1}\right),$$

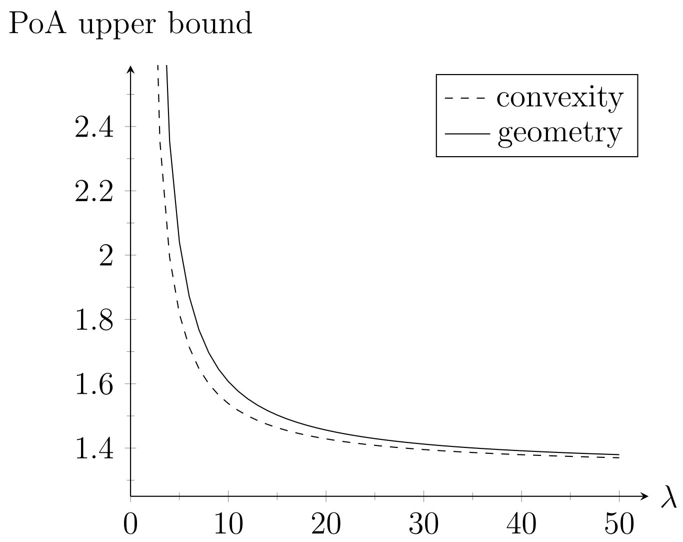


Figure 6.3: The two PoA upper bounds with affine cost functions, ZTP distributions and $n = 5$

which implies

$$\frac{\sigma_i^2 - d_i}{d_i^2} = -e^{-\lambda}.$$

Thus we have

$$h = 1 - \frac{e^{-\lambda}}{n}, \quad l = 1 - e^{-\lambda}, \quad g = \frac{e^{-x} - 1}{x} + 1.$$

Condition $l > 1/(4g)$ in Theorem 6.9 reduces to $\lambda > 1.06$. Similar to Figure 6.2 for uniform distributions, we look at (6.9) for the convexity bound. Figure 6.3 compares (6.9) with the geometry bound in Theorem 6.10 when $n = 5$. It shows that the convexity bound is tighter than the geometry bound. Similar results can be found for different values of n.

In both numerical comparisons of the two bounds above, we have found the convexity bound is tighter. Note that, the convexity bound is applicable for a wide range of demand distributions, since condition $l > 1/4$ is always satisfied when $\underline{d} \geq 4/3$ (Remark 6.3).

6.3 Polynomial Cost Functions

In this section, we consider single commodity networks and compute the upper bounds in Theorems 6.4 and 6.5 for general positive-valued demand distributions and (non-zero) polynomial link cost functions in the form of

$$c_e(x) = \sum_{j=0}^{m} b_{ej}x^j,\ b_{ej} \geq 0,\ j = 0, 1, \ldots, m \text{ and } \sum_{j=0}^{m} b_{ej} > 0;\ e \in E. \quad (6.10)$$

In a single commodity network, all the links are used in only one O-D pair, thus we can drop superscript $i \in I$ in this section for simplicity. The link flow can be written as

$$V_e = \sum_{j=0}^{D} X_{e,j}, \quad \forall\ e \in E,$$

where D is the random demand and $X_{e,j}$ is a random binary variable indicating whether player j $(1 \leq j \leq D)$ chooses link $e \in E$. As introduced in Section 5.2, V_e here is a compound random variable, for which the m-th moment can be expressed as [Grubbstrom and Tang, 2006]:

$$\mathbb{E}[V_e^m] = \sum_{i=0}^{m} \mathbb{E}\left[\binom{D}{i}\right] \left\{\sum_{r=0}^{i} \binom{i}{r} (-1)^{i-r} \mathbb{E}\left[\left(\sum_{j=1}^{r} X_{e,j}\right)^m\right]\right\}. \quad (6.11)$$

We can see that $\mathbb{E}[V_e^m]$ are functions of moments of D and $X_{e,j}$. Actually from [Grubbstrom and Tang, 2006], we can express $\mathbb{E}[V_e^m]$ as functions of v_e for any $m \in \mathbb{Z}^+$. But we will focus on polynomial cost functions with degree no more than 4, as we have done in Section 4.2 for the non-atomic model. As $\mathbb{E}[c_e(V_e)V_e]$ is required for computing the expected total cost, we need to consider the first five moments of V_e when $m \leq 4$. Following from [Grubbstrom and Tang, 2006], we have, for any $e \in E$:

$$\begin{aligned}
\mathbb{E}[V_e^2] &= \mathbb{E}[D]p_e + (\mathbb{E}[D^2] - \mathbb{E}[D])p_e^2, \\
\mathbb{E}[V_e^3] &= \mathbb{E}[D]p_e + 3(\mathbb{E}[D^2] - \mathbb{E}[D])p_e^2 + (\mathbb{E}[D^3] - 3\mathbb{E}[D^2] + 2\mathbb{E}[D])p_e^3, \\
\mathbb{E}[V_e^4] &= \mathbb{E}[D]p_e + 7(\mathbb{E}[D^2] - \mathbb{E}[D])p_e^2 + 6(\mathbb{E}[D^3] - 3\mathbb{E}[D^2] + 2\mathbb{E}[D])p_e^3 \\
&\quad + (\mathbb{E}[D^4] - 6\mathbb{E}[D^3] + 11\mathbb{E}[D^2] - 6\mathbb{E}[D])p_e^4, \\
\mathbb{E}[v_e^5] &= \mathbb{E}[D]p_e + 15(\mathbb{E}[D^2] - \mathbb{E}[D])p_e^2 + 25(\mathbb{E}[D^3] - 3\mathbb{E}[D^2] + 2\mathbb{E}[D])p_e^3, \\
&\quad + 10(\mathbb{E}[D^4] - 6\mathbb{E}[D^3] + 11\mathbb{E}[D^2] - 6\mathbb{E}[D])p_e^4 \\
&\quad + (\mathbb{E}[D^5] - 10\mathbb{E}[D^4] + 35\mathbb{E}[D^3] - 50\mathbb{E}[D^2] + 24\mathbb{E}[D])p_e^5,
\end{aligned}$$

which can be simplified as

$$\begin{aligned}
\mathbb{E}[V_e^2] &= \mu_1 v_e + \mu_2 v_e^2, \\
\mathbb{E}[V_e^3] &= \mu_1 v_e + 3\mu_2 v_e^2 + \mu_3 v_e^3, \\
\mathbb{E}[V_e^4] &= \mu_1 v_e + 7\mu_2 v_e^2 + 6\mu_3 v_e^3 + \mu_4 v_e^4, \\
\mathbb{E}[V_e^5] &= \mu_1 v_e + 15\mu_2 v_e^2 + 25\mu_3 v_e^3 + 10\mu_4 v_e^4 + \mu_5 v_e^5,
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= 1 \\
\mu_2 &= (\mathbb{E}[D^2] - \mathbb{E}[D])/d^2 \geq 0, \\
\mu_3 &= (\mathbb{E}[D^3] - 3\mathbb{E}[D^2] + 2\mathbb{E}[D])/d^3, \\
\mu_4 &= (\mathbb{E}[D^4] - 6\mathbb{E}[D^3] + 11\mathbb{E}[D^2] - 6\mathbb{E}[D])/d^4, \\
\mu_5 &= (\mathbb{E}[D^5] - 10\mathbb{E}[D^4] + 35\mathbb{E}[D^3] - 50\mathbb{E}[D^2] + 24\mathbb{E}[D])/d^5.
\end{aligned} \tag{6.12}$$

Given $D \geq 1$, we can also compute $\mathbb{E}\left[\left(\sum_{j=0}^{D-1} X_{e,j} + 1\right)^m\right]$. Denote

$$\mathbb{E}\left[(V_e(D-1)+1)\right] = \mathbb{E}\left[\left(\sum_{j=0}^{D-1} X_{e,j} + 1\right)^m\right].$$

Then we have

$$\begin{aligned}
\mathbb{E}[V_e(D-1)+1] &= \omega_0 + \omega_1 v_e, \\
\mathbb{E}[(V_e(D-1)+1)^2] &= \omega_0 + 3\omega_1 v_e + \omega_2 v_e^2, \\
\mathbb{E}[(V_e(D-1)+1)^3] &= \omega_0 + 7\omega_1 v_e + 6\omega_2 v_e^2 + \omega_3 v_e^3, \\
\mathbb{E}[(V_e(D-1)+1)^4] &= \omega_0 + 15\omega_1 v_e + 25\omega_2 v_e^2 + 10\omega_3 v_e^3 + \omega_4 v_e^4,
\end{aligned}$$

where

$$\begin{aligned}
\omega_0 &= 1 \\
\omega_1 &= (d-1)/d, \\
\omega_2 &= (\mathbb{E}[D^2] - 3d + 2)/d^2, \\
\omega_3 &= (\mathbb{E}[D^3] - 6\mathbb{E}[D^2] + 11d - 6)/d^3, \\
\omega_4 &= (\mathbb{E}[D^4] - 10\mathbb{E}[D^3] + 35\mathbb{E}[D^2] - 50d + 24)/d^4.
\end{aligned} \tag{6.13}$$

From the above computation, we have, for any $j = 0, 1, \ldots, 4$,

$$\begin{aligned}
\mathbb{E}(V_e(D-1)+1)^j] &= \sum_{r=0}^{j} a_{jr}\omega_r v_e^r, \\
\mathbb{E}[V_e^{j+1}] &= \sum_{r=0}^{j} a_{jr}\mu_{r+1} v_e^{r+1},
\end{aligned}$$

in which a_{jr} is defined in Table 6.3.

Now we are ready to define specific functions of $\bar{t}_e(\cdot)$, $\underline{t}_e(\cdot)$, $\bar{s}_e(\cdot)$ and $\underline{s}_e(\cdot)$ satisfying Assumptions 6.1–6.3. As mentioned in the non-atomic part, we

j	r				
	0	1	2	3	4
0	1				
1	1	1			
2	1	3	1		
3	1	7	6	1	
4	1	15	25	10	1

Table 6.1: Coefficient a_{jr} for $r \leq j \leq 4$

only need to consider monomial cost functions to bound the PoA for general polynomial cost functions (see Lemma 4.3).

Definition 6.4. For any monomial cost function $c_e(x) = b_{ej}x^j$, $e \in E$ with $b_{ej} > 0$, $j = 0, 1, \ldots, m$ ($m \leq 4$) we define:

$$\bar{t}_e(x) = \underline{t}_e(x) = \sum_{r=0}^{j} b_{ej} a_{jr} \omega_r v_e^r,$$

$$\bar{s}_e(x) = \underline{s}_e(x) = \sum_{r=0}^{j} b_{ej} a_{jr} \mu_{r+1} v_e^{r+1}.$$

For notational simplicity, let $t_e(x) = \bar{t}_e(x) = \underline{t}_e(x)$ and $s_e(x) = \bar{s}_e(x) = \underline{s}_e(x)$.

Recall Section 4.2, $\widetilde{\mathcal{I}}_m$ is a set of game instances for any fixed $m \in \mathbb{Z}_+$ ($m \geq 1$) with (non-zero) monomial link cost functions in the form of bx^j with $b > 0$ and $j = 0, 1, \cdots, m$. Following Definition 6.2, we can compute

$$\beta(c_e, \widetilde{\mathcal{I}}_m) = \sup_{x>0, y>0} \frac{x(t_e(y) - t_e(x))}{y t_e(y)},$$

which is actually the same thing as that in the deterministic model (see Definition 2.5), thus

$$\beta(\widetilde{\mathcal{I}}_m) = \frac{m}{(m+1)^{1+1/m}}.$$

We can also compute

$$\begin{aligned}
\underline{\alpha}(c_e, \widetilde{\mathcal{I}}_m) &= \inf_{x>0} \frac{x t_e(x)}{s_e(x)} = \inf_{x>0} \frac{\sum_{r=0}^{j} b_{ej} a_{jr} \omega_r x^{r+1}}{\sum_{r=0}^{j} b_{ej} a_{jr} \mu_{r+1} x^{r+1}} \\
&\geq \min_{0\leq r\leq j} \frac{\omega_r}{\mu_{r+1}}, \\
\underline{\alpha}(\widetilde{\mathcal{I}}_m) &\geq \min_{0\leq r\leq m} \frac{\omega_r}{\mu_{r+1}}.
\end{aligned}$$

Similarly, we have

$$\overline{\alpha}(\widetilde{\mathcal{I}}_m) \leq \max_{0\leq r\leq m} \frac{\omega_r}{\mu_{r+1}}.$$

Substituting the above into Theorem 6.5, we arrive the following geometry bound of the PoA.

Proposition 6.11 (Geometry upper bound)**.** *Let* $(G, D, \mathbf{c})$ *be an atomic instance, in which* G *is a single commodity network,* D *is a positive-valued distribution, and* $c_e(\cdot) \in \mathcal{C}$ *for any* $e \in E$*. If* $\mu_r, \omega_r \geq 0$ *for any* $r = 0, 1, \cdots, m$*, then*

$$PoA(G, D, \mathbf{c}) \leq \frac{(m+1)^{1+1/m}}{(m+1)^{1+1/m} - m} \cdot \frac{\max_{0\leq r\leq m} \{\omega_r/\mu_{r+1}\}}{\min_{0\leq r\leq m} \{\omega_r/\mu_{r+1}\}}.$$

Next we compute the convexity upper bound of the PoA. From Definition 6.4, we can derive

$$s_e'(x) = \sum_{r=0}^{j} (r+1) a_e b_{jr} \mu_{r+1} v_e^r.$$

As addressed in Section 6.1, we need to find $\lambda(x)$ which solves $s_e'(\lambda(x)x) = t_e(x)$. Although such a function exists, it is much more complicated than that in non-atomic work because both $t_e(x)$ and $s_e'(x)$ are non-homogeneous polynomial functions even when $c_e(x)$ is simply a monomial function. This will increase the difficulty of computing the convexity upper bound in Theorem 6.4 significantly. Observe that if we split $t_e(x)$ and $s_e'(x)$ into monomial functions, each term of them only differ in coefficients. Next we introduce a modified method to bound the PoA.

Let $t_{e,jr}(x) = a_e b_{jr} \omega_r x^r$ and $s_{e,jr}(x) = a_e b_{jr} \mu_{r+1} x^{r+1}$. Then $t_e(x) = \sum_{r=0}^{j} t_{e,jr}(x)$ and $s_e(x) = \sum_{r=0}^{j} s_{e,jr}(x)$. Observe that

$$\lambda_r = \left(\frac{\omega_r}{(r+1)\mu_{r+1}} \right)^{1/r}, \qquad r = 0, 1, \ldots, j,$$

solves $s'_{e,jr}(\lambda_r x) = t_{e,jr}(x)$, in which $s'_{e,jr}(x)$ is the derivative of $s_{e,jr}(x)$ on x. Next we use constant λ_r to derive a simplified convexity upper bound.

Proposition 6.12 (Convexity upper bound)**.** *Let* $(G, D, \mathbf{c})$ *be an atomic instance, in which* G *is a single commodity network,* D *is a positive-valued distribution, and* $\mathbf{c}$ *are polynomial functions in the form of* (6.10)*. If* $\mu_r, \omega_r \geq 0$ *for any* $r = 0, 1, \cdots, m$*, and*

$$\min_{0 \leq r \leq m} \left\{ \mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r \right\} \geq 0,$$

Then

$$PoA(G, D, \mathbf{c}) \leq \max_{0 \leq r \leq m} \frac{\mu_{r+1}}{\omega_r} \left(1 - \frac{r}{r+1} \left(\frac{\omega_r}{(r+1)\mu_{r+1}} \right)^{1/r} \right)^{-1}.$$

Proof. Let $\bar{\mathbf{v}}$ and $\mathbf{v}^*$ be mean link flows at a UE-SD and at an SO-SD, respectively. From the convexity of $s_{e,jr}(\cdot)$,

$$\begin{aligned} s_{e,jr}(v_e^*) &\geq s_{e,jr}(\lambda_r \bar{v}_e) + (v_e^* - \lambda_r \bar{v}_e) s'_{e,jr}(\lambda_r \bar{v}_e) \\ &= s_{e,jr}(\lambda_r \bar{v}_e) + (v_e^* - \lambda_r \bar{v}_e) t_{e,jr}(\bar{v}_e). \end{aligned}$$

Summing over $r = 0, 1, \ldots, j$ leads to

$$s_e(v_e^*) \geq \sum_{r=0}^{j} s_{e,jr}(\lambda_r \bar{v}_e) + v_e^* t_e(\bar{v}_e) - \sum_{r=0}^{j} \lambda_r \bar{v}_e t_{e,jr}(\bar{v}_e).$$

Thus, we have

$$\begin{aligned} T(\mathbf{p}^*) &= \sum_{e \in E} s_e(v_e^*) \\ &\geq \sum_{e \in E} \sum_{r=0}^{j} s_{e,jr}(\lambda_r \bar{v}_e) + \sum_{e \in E} v_e^* t_e(\bar{v}_e) - \sum_{e \in E} \sum_{r=0}^{j} \lambda_r \bar{v}_e t_{e,jr}(\bar{v}_e) \\ &\geq \sum_{e \in E} \sum_{r=0}^{j} s_{e,jr}(\lambda_r \bar{v}_e) + \sum_{e \in E} \bar{v}_e t_e(\bar{v}_e) - \sum_{e \in E} \sum_{r=0}^{j} \lambda_r \bar{v}_e t_{e,jr}(\bar{v}_e), \end{aligned}$$

where the last inequality follows from Lemma 6.3.

Then

$$\begin{aligned} T(\mathbf{p}^*) &\geq \sum_{e \in E} \sum_{r=0}^{j} \left(s_{e,jr}(\lambda_r \bar{v}_e) + \bar{v}_e t_{e,jr}(\bar{v}_e) - \lambda_r \bar{v}_e t_{e,jr}(\bar{v}_e) \right) \\ &= \geq \sum_{e \in E} \sum_{r=0}^{j} a_e b_{jr} (\mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r) \bar{v}_e^{r+1}. \end{aligned}$$

If $\mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r \geq 0$, for any $r = 0, 1, \ldots, m$, we have

$$\begin{aligned} \frac{T(\bar{\mathbf{p}})}{T(\mathbf{p}^*)} &\leq \frac{\sum_{e \in E} s_e(\bar{v}_e)}{\sum_{e \in E} \sum_{r=0}^{j} a_e b_{jr} (\mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r) \bar{v}_e^{r+1}} \\ &= \frac{\sum_{e \in E} \sum_{r=0}^{j} a_e b_{jr} \mu_{r+1} \bar{v}_e^{r+1}}{\sum_{e \in E} \sum_{r=0}^{j} a_e b_{jr} (\mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r) \bar{v}_e^{r+1}} \\ &\leq \max_{0 \leq r \leq m} \frac{\mu_{r+1}}{\mu_{r+1} \lambda_r^{r+1} + \omega_r - \omega_r \lambda_r} \\ &= \max_{0 \leq r \leq m} \frac{\mu_{r+1}}{\omega_r} \left(1 - \frac{r}{r+1} \left(\frac{\omega_r}{(r+1)\mu_{r+1}} \right)^{1/r} \right)^{-1}. \end{aligned}$$

□

Remark 6.5. When demand returns to deterministic and approaches infinity, we have $\mu_{r+1} = \omega_r = 1$ for any $0 \leq r \leq m$. Thus both the convexity and geometry upper bound will match the tight upper bound of the PoA in the deterministic non-atomic work:

$$\frac{(m+1)^{1+1/m}}{(m+1)^{1+1/m} - m}.$$

This is to say that the upper bounds in Propositions 6.11 and 6.12 are asymptotically tight.

Remark 6.6. Specially when $m = 1$, the upper bounds in Propositions 6.11 and 6.12 do not match those in Theorems 6.9 and 6.10 for affine cost functions with $n = 1$. This is because the upper bounds are obtained via different definitions of $\bar{t}_e(\cdot)$ and $\underline{t}_e(\cdot)$. Clearly $\bar{t}_e(\cdot)$ in this section is tighter than that in Section 6.2, which thus provides tighter upper bounds.

We will conclude this section with a numerical comparison between the geometry bound in Proposition 6.11 and the convexity bound in Proposition 6.12. Similar to Section 6.2, we consider two types of demand distributions, i.e., discrete uniform distributions and zero-truncated Poisson (ZTP) distributions.

Let D follow discrete uniform distribution $U[1, z]$, where $z \in \mathbb{Z}^+$. We can compute

$$d = \frac{1+z}{2}, \quad \mathbb{E}[D^2] = \frac{1}{6}(z+1)(2z+1), \quad \mathbb{E}[D^3] = \frac{1}{4}z(z+1)^2,$$

from which we can derive

$$\mu_2 = \frac{4(z-1)}{3(z+1)}, \qquad \mu_3 = \frac{2(z-1)(z-2)}{(z+1)^2},$$
$$\omega_1 = \frac{z-1}{z+1}, \qquad \omega_2 = \frac{4(z-1)(z-2)}{3(z+1)^2}.$$

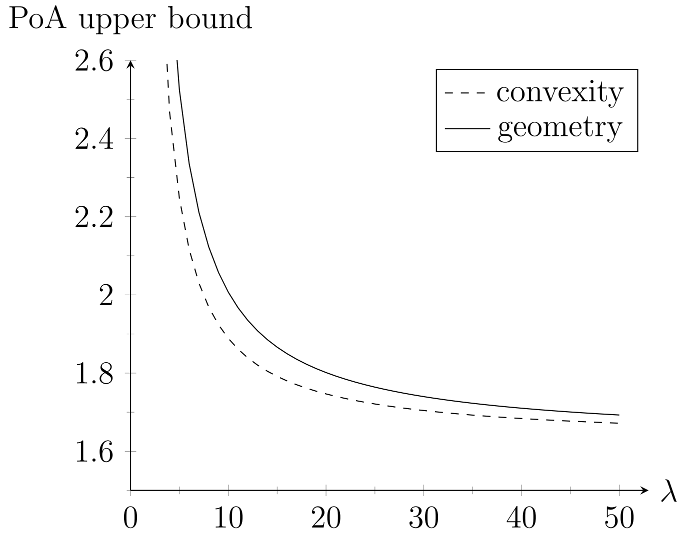


Figure 6.4: The two PoA upper bounds for single commodity networks with quadratic cost functions ($m = 2$) and ZTP distributions

From Propositions 6.11 and 6.12, we can compute both the geometry and convexity bounds when $z \geq 2$. The geometry and convexity bounds are 2.44 and 2.19 for quadratic cost functions (m=2), respectively. Clearly the convexity bound is tighter than the geometry bound.

Next we consider demand D following ZTP distribution. Let λ be the mean of the original Poisson distribution. We have

$$d = \frac{e^\lambda}{e^\lambda - 1}\lambda, \qquad \mathbb{E}[D^2] = \frac{e^\lambda}{e^\lambda - 1}(\lambda^2 + \lambda),$$
$$\mathbb{E}[D^3] = \frac{e^\lambda}{e^\lambda - 1}(\lambda^3 + 3\lambda^2 + \lambda).$$

Thus the values of μ_r and ω_r can be computed from (6.12) and (6.13). Both the convexity and geometry bounds are applicable when $\lambda \geq 1.48$. Figure 6.4 shows the convexity bound is also tighter than the geometry bound when demands following ZTP distributions and cost functions are quadratic.

6.4 Concluding Remarks

Based on our reformulation of the user equilibrium condition as a variational inequality problem, we have extended two bounding techniques for models of non-atomic traffic, to establish two general PoA bounds for our models of atomic traffic. As is expected, our bounds depend not only on the class of cost functions, but also demand distributions. We have also found that the bounds are related, although weakly, to the network topology (via parameter n in the definitions of h and l).

We have computed our two bounds particularly for affine cost functions in general networks and polynomial cost functions in single commodity networks. All the upper bounds in this chapter are asymptotically tight with the increase of the number of travellers. Given the definitions of h, l and g, it is difficult to compare these two upper bounds in general. From our numerical experiments on discrete uniform distributions and zero-truncated Poisson distributions, the convexity bound is tighter than the geometry one. This is consistent with our finding in the two special cases (affine cost functions and single commodity networks) in Non-atomic work (see Section 4.4 for the details).

As we have derived in Chapter 5, in a general multi-commodity network, the link flow in our atomic model with stochastic demands is a sum of independent compound random variables, which is much more complex than that in the non-atomic model in Chapter 3. This increases the difficulty of computing higher moments of link flows, and in consequence makes it very complicated to compute the two upper bounds for polynomial cost functions. It is a challenge to find proper functional approximation satisfying Assumptions 6.1–6.3 for our general setting of positive-valued demand distributions, although we believe such functions exist. Possible work may be done for a narrowed set of distributions, or a specific distribution.

Chapter 7

Conclusion

In a transportation network, people choose the shortest (cheapest) route selfishly. This is beneficial to individual travel but may cause system inefficiency due to lack of coordination. Central coordination is impractical, since travellers have the final say in routing choice and may not follow coordinators' suggestions at the expense of their own interests. The PoA is defined to measure the system inefficiency caused by travellers' selfish behaviours. In a perfect case the PoA is just one, which means people will reach the system optimality automatically, without traffic coordination. Unfortunately, this is not true of most cases in practice. Studying the PoA determines how far the system is from optimality if no central coordinator is imposed. This question has been answered for non-atomic and atomic models with deterministic demands, which deal with infinite and finite numbers of travellers respectively. This thesis contributes to extending the literature to a more general setting of stochastic demand and produces models and results which better reflect reality.

We have presented general models for both non-atomic and atomic congestion games with stochastic demands. We have provided analytical upper bounds by two methods developed from deterministic non-atomic work, and have also computed the specific upper bounds for very general settings of link cost functions and demand distributions. All the upper bounds are proved to be (asymptotically) tight in some special cases.

Our study of PoA with stochastic demands is a general extension of that in the existing deterministic literature, and provides some new insights into network problems. It is well known that deterministic PoA depends only on the class of cost functions. In contrast, our analysis shows that the PoA depends also on demand distributions and, to some extent, network topologies. In particular, the upper bound of the PoA goes up when the degree of cost functions increases, which is consistent with the findings of deterministic work [Roughgarden and Tardos, 2004] . When the demand variation increases, our PoA upper bounds also rise. In the extreme case of infinite demand variation, the PoA will be unbounded, while in deterministic work it has been proved that the PoA is always bounded for polynomial cost functions. Thus, system inefficiency has been somewhat underestimated in deterministic studies. In addition, there is a network-related parameter n in our PoA results; thus, network topology may also affect system gradation.

We conclude this thesis with open questions for future research. Firstly, unlike deterministic work, for each upper bound in our study we have a restrictive condition for applicable demand variations, which is caused by the technical limitations of our bounding methods. When the conditions are not satisfied, we believe that the PoA is still finite, but our method is unable to bound it. Thus, a new method must be sought to bound the PoA for more general demand distributions. Secondly, functional approximations of the expected travel cost (see Assumptions in Sections 4.1 and 6.1) play a significant role in bounding the PoA for both non-atomic and atomic congestion games. Different approximations may lead to different values of the upper bounds of the PoA. The functions in our study were established on the basis of very general settings of demand distributions, i.e. positive-valued (discrete) distributions. We might find better functional approximation satisfying the assumptions in Section 4.1 or 6.1 after applying more specific distributions, which would consequently improve the PoA upper bounds. Finally, all upper bounds obtained in this study can be reached under the assumption of separable cost functions. Extension of this study to non-separable cost functions is a potential future research direction. In addition, it would be interesting to determine whether novel methods might be found to bound the PoA, when the necessary conditions proposed in this study are relaxed.

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