Topics in Risk-sensitive stochastic control.

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

The work presented in the first topic is joint work with Prof. S.D. Jacka and it has been accepted in *Risk and Decision Analysis*, refer [9].

The work presented in the second topic is an individual work of mine and has been accepted in *Journal of Applied Probability*, refer [10].

The work presented in the third topic is an individual work of mine and has been accepted in *Stochastic analysis and applications*, refer [11].

Abstract

This thesis consists of three topics whose over-arching theme is based on risk-sensitive stochastic control. In the first topic (chapter 2), we study a problem on benchmark out-performance. We model this as a zero-sum risk-sensitive stochastic game between an investor who as a player wants to maximize the risk-sensitive criterion while the other player (a stochastic benchmark) tries to minimize this maximum risk-sensitive criterion. We obtain an explicit expression for the strategies for both these two players. In the second topic (chapter 3), we consider a finite horizon risk-sensitive asset management problem. We study it in the context of a zero-sum stochastic game between an investor and the second player called the "market world" which provides a probability measure. Via this game, we connect two (somewhat) disparate areas in stochastics; namely, stochastic stability and risk-sensitive stochastic control in mathematical finance. The connection is through the Föllmer-Schweizer minimal martingale measure. We discuss the impact of this measure on the investor's optimal strategy. In the third topic (chapter 4), we study the sufficient stochastic maximum principle of semi-Markov modulated jump diffusion. We study its application in the context of a quadratic loss minimization problem. We also study the finite-horizon risk-sensitive optimization in relation to the underlying sufficient stochastic maximum principle of a semi-markov modulated diffusion.

Chapter 1

Introduction.

In this chapter, we first introduce the reader to the concept of sufficient stochastic maximum principle. Here we provide this principle for a class of diffusion process. We also introduce concepts of risk-sensitive stochastic control optimization, Föllmer-Schweizer minimal martingale measure, stochastic stability, zero-sum two player stochastic differential game and semi-Markov modulated jump diffusion in that order. These ideas form key ingredients to the chapters to follow. Towards the end of this chapter we briefly mention how each subsequent chapter relates to these concepts.

1 Some key concepts in Stochastic Control.

The basic idea of stochastic control is to consider a family of controlled problems by varying the initial state values and to derive some relations between the values of the associated value function. This is called the dynamic programming principle. This approach yields a certain nonlinear PDE of second order called the the Hamilton-Jacobi-Bellman or (HJB) equation. The first fundamental theorem in stochastic control describes what is known as "necessary" stochastic maximum principle. Roughly speaking, it states that if there exists an optimal control, then it is simply associated to the easier problem of finding the maximum of a certain real function in a particular control space. On the contrary, the sufficient stochastic maximum states that if a certain real function is maximum for a particular control, then that control is optimal. This is also termed the sufficient stochastic maximum principle and constitutes what is called the second fundamental theorem in stochastic control. Thus in short the sufficient stochastic maximum principle validates the optimality of the candidate solution to the HJB equations. We elaborate on the sufficient stochastic maximum principle for a controlled diffusion as follows. We borrow the following material from Pham [31].

Consider a controlled SDE modeled in $\mathbb{R}^n$ as

$$dX_s = b(X_s, u_s)ds + \sigma(X_s, u_s)dW_s, \quad X_0 = x; \tag{1.1}$$

where W is a d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_{\{t\geq 0\}}, \mathbb{P})$. $u = (u_s)$ is a process (progressively measurable w.r.t to $\mathbb{F}$) taking values in $A \subset \mathbb{R}^m$. The measurable functions $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipschitz condition in $A :$ *i.e.* $\exists K \geq 0, \ \forall x, y \in \mathbb{R}^n, \forall\ a \in A,$

$$|b(x, a) - b(y, a)| + |\sigma(x, a) - \sigma(y, a)| \leq K|x - y|. \tag{1.2}$$

for $0 \leq t \leq T \leq \infty$.

Finite horizon problem.

We fix a finite horizon $0 < T < \infty$. We denote by $\mathcal{A}$ the set of control processes u such that

$$E\int_0^T [|b(0, u_t)|^2 + |\sigma(0, u_t)|^2]dt < \infty. \tag{1.3}$$

It is known that the conditions given by (1.2) and (1.3) ensure that for $u \in \mathcal{A}$ and for any initial conditions $(t, x) \in [0, T] \times \mathbb{R}^n$, existence and uniqueness of a strong solution to the SDE (1.1) (with random coefficients) starting from x at $s = t$ is guaranteed.

Functional objective Let $f : [0, T]\times\mathbb{R}^n\times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions where g is lower-bounded or g satisfies a quadratic growth condition given by $|g(x)| \leq C(1 + |x|^2) \quad \forall\ x \in \mathbb{R}^n$ for some constant C independent of x. For $(t, x) \in [0, T] \times \mathbb{R}^n$ we denote by $\mathcal{A}(t, x)$ the non-empty subset of controls u in $\mathcal{A}$ such that

$$E[\int_t^T |f(s, X_s^{t,x}, u_s)|ds] < \infty. \tag{1.4}$$

We can then define the gain function:

$$J(t, x, u) = E[\int_t^T f(s, X_s^{t,x}, u_s)ds + g(X_T^{t,x})], \tag{1.5}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $u \in \mathcal{A}(t,x)$. The objective is to maximize over control processes the gain function J by introducing the associated value function

$$\nu(t,x) = \sup_{u \in \mathcal{A}(t,x)} J(t,x,u). \tag{1.6}$$

Remarks

1. Given an initial condition $(t,x) \in [0,T) \times \mathbb{R}^n$, we say that $\hat{u} \in \mathcal{A}(t,x)$ is an optimal control if $\nu(t,x) = J(t,x,\hat{u})$

2. A control process u of the form $u_s = u(s, X_s^{t,x})$ for some measurable function u from $[0,T] \times \mathbb{R}^n$ into A, is called a Markovian control.

Dynamic programming principle (DPP)

It is an important concept in the theory of stochastic control. It is formulated in the context of controlled diffusion for the finite time horizon as follows:

Theorem 1. Let $(t,x) \in [0,T] \times \mathbb{R}^n$. Then we have

$$\nu(t,x) \quad = \sup_{u \in \mathcal{A}(t,x)} \inf_{\theta \in T_t} E[\int_t^\theta f(s, X_s^{t,x}, u_s)ds + \nu(\theta, X_\theta^{t,x})] \tag{1.7}$$

where T_t is the set of stopping times valued in $[t,T]$. The interpretation of the DPP is that the optimization problem can now be split in two parts: an optimal control on the whole time interval $[t,T]$ may be obtained by first searching for an optimal control from time θ given the state value $X_\theta^{t,x}$, i.e compute $\nu(\theta, X_\theta^{t,x})$ and then maximizing over controls on $[t,\theta]$ the quantity $E[\int_t^\theta f(s, X_s^{t,x}, u_s)ds + \nu(\theta, X_\theta^{t,x})]$.

The HJB equation is the infinitesimal version of the dynamic programming principle: it describes the local behavior of the value function when we send the stopping time θ to t in (1.7). The HJB equation is also called the dynamic programming equation or DPE and is given by,

$$-\frac{\partial \nu}{\partial t}(t,x) - \mathcal{H}(t,x,D_x\nu(t,x), D_x^2\nu(t,x)) = 0 \;\; \forall \;\; (t,x) \in [0,T) \times \mathbb{R}^n, \tag{1.8}$$

where for $(t,x,p,M) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$

$$\begin{aligned} \mathcal{H}(t,x,p,M) &= \sup_{a \in A}[b(x,a) \cdot p + \frac{1}{2} tr(\sigma\sigma'(x,a)M) + f(t,x,a)] \\ \nu(T,x) &= g(x) \quad \forall \; x \in \mathbb{R}^n. \end{aligned} \tag{1.9}$$

The function $\mathcal{H}$ is called the Hamiltonian of the associated control problem. The crucial step in the classical approach to dynamic programming consists of proving that given a smooth solution to the HJB equation, this candidate coincides with the value function. As stated earlier, this result is called the verification lemma and allows us to obtain an optimal Markovian control as a byproduct. Given the diffusion dynamics presented earlier, the following would be the corresponding verification lemma.

Let $C^{1,2}([0,T)\times\mathbb{R}^n)$ be the space of functions of once differentiable in t and twice differentiable in x while $C^0([0,T)\times\mathbb{R}^n)$ is the space of continuous functions.

Theorem 2. Define $\mathcal{L}^a$ to be the generator of the controlled diffusion process (1.1). Let w be a $C^{1,2}([0,T)\times\mathbb{R}^n)\cap C^0([0,T]\times\mathbb{R}^n)$, functions satisfying the growth condition , i.e. there exist a constant C such that

$$|w(t,x)| \le C(1+|x|^2). \ \ \forall \quad (t,x)\in[0,T]\times\mathbb{R}^n.$$

(i) Suppose that

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}[\mathcal{L}^a w(t,x) + f(t,x,a)] \ge 0, \quad (t,x)\in[0,T)\times\mathbb{R}^n, \tag{1.10}$$

$$w(T,x)\ge g(x) \quad x\in\mathbb{R}^n. \tag{1.11}$$

Then $w \ge \nu$ on $[0,T]\times\mathbb{R}^n$.

(ii) Suppose further that $w(T,\cdot) = g(\cdot)$, and there exists a measurable function $\hat{u}(t,x)\in[0,T)\times\mathbb{R}^n$, valued in A such that

$$\begin{aligned} - & \frac{\partial w}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] \\ = & -\frac{\partial w}{\partial t}(t,x) - \mathcal{L}^{\hat{u}(t,x)}w(t,x) - f(t,x,\hat{u}(t,x)) = 0. \end{aligned} \tag{1.12}$$

and the SDE

$$dX_s = b(X_s,\hat{u}(s,X_s))ds + \sigma(X_s,\hat{u}(s,X_s))dW_s$$

admits a unique solution, denoted by $\hat{X}_s^{t,x}$, given an initial condition $X_t = x$, and the process $\{\hat{u}(t,\hat{X}_s^{t,x}; t\le s\le T\}$ lies in $\mathcal{A}(t,x)$. Then

$$w=\nu \quad on \ \ [0,T]\times\mathbb{R}^n,$$

and $\hat{u}$ is an optimal Markovian control.

Note that when the control space A is reduced to a singleton, the verification lemma is a version of the Feynman-Kac formula.

The usual steps to derive the optimal control are the following:
Step 1. Derive the HJB equation.
Step 2. Verfication step: Show that the smooth solution is the value function.
Step 3. As a byproduct, obtain an optimal feedback control.

These steps will be a recurring theme in the chapters to follow.

2 Risk-sensitive control optimization

The risk-sensitive control-optimization or(RSCO) balances an investor's interest in maximizing the expected growth rate of wealth against his aversion to risk due to deviations of the realized rate from the expectation. The subjective notion of investor's risk aversion is parameterized by a single variable, say θ. More formally, we write the finite horizon risk-sensitive optimization criterion as :

$$\max \quad J_{T,h} := -\frac{1}{\theta} \log E[e^{-\theta F(T,h)}], \tag{2.1}$$

where $F(T, h)$ is the time-T- value reward function corresponding to control h. In the optimal investment problem we take $F(T, h) = \log V(T)$ where $V(t)$ is the time t-value of the portfolio corresponding to portfolio asset allocation h.

An asymptotic expansion around $\theta = 0$ for the above criterion with $F := F(t, h)$ yields

$$\begin{aligned} J_{T,h} &= -\frac{1}{\theta} \log [1 - \theta E[F] + \frac{\theta^2}{2} E[F^2] + ...], \\ &= -\frac{1}{\theta} \log [1 - (\theta E[F] - \frac{\theta^2}{2} E[F^2]) + ...], \\ &= \frac{1}{\theta} [(\theta E[F] + \frac{\theta^2}{2} E[F^2]) + \frac{1}{2} (\theta E[F] + \frac{\theta^2}{2} E[F^2])^2 + ...], \\ &= E[F] - \frac{\theta}{2} E[F^2] + \frac{\theta}{2} (E[F])^2, \\ &= E[F(T,h)] - \frac{\theta}{2} Var(F(T,h)) + O(\theta^2). \end{aligned}$$

where $O(\cdot)$ as usual denotes the big-O symbol. From this expression it is clear this criterion compromises between maximizing the portfolio return while penalizing its riskiness. Values of $\theta > 0$ correspond to a risk-averse investor, $\theta < 0$ to a risk-seeking investor and $\theta = 0$ to a risk-neutral investor who maximizes

$$J_{T,h} := E[F(T,h)].$$

The optimal expected utility function depends on θ and is a generalization of the traditional stochastic control approach to utility optimization in the sense that now the degree of risk aversion of the investor is explicitly parameterized through θ rather than importing it into the problem via an exogenous utility function. There has been a substantial amount of research on the infinite-time horizon ergodic problem:

$$\begin{aligned} &\max \quad \bar{J}_\infty \quad \text{where} \\ &\bar{J}_\infty = \liminf_{t\to\infty} -\frac{1}{\theta} t^{-1} \log E[e^{-\theta F(t,h)}]. \end{aligned} \tag{2.2}$$

However, in this thesis we will restrict to studying the finite horizon risk sensitive control-optimization for the case when V is a diffusion process.

3 Föllmer and Schweizer minimal martingale measure

We now discuss in brief the concept of minimal martingale measure. The material is borrowed from the work of Föllmer and Schweizer [17]. Let $X = (X_t)_{0\le t\le T}$ be a semi-martingale with continuous paths on some probability space $(\Omega, \mathcal{F}, P)$ with right-continuous filtration $(\mathcal{F}_t)_{0\le t\le T}$. The Doob-Meyer decomposition states that

$$X = X_0 + M + A. \tag{3.1}$$

where $M = (M_t)_{0\le t\le T}$ is a local martingale and a predictable process $A = (A_t)_{0\le t\le T}$ with paths of bounded variation. In particular M is a square integrable martingale under P. Consider a contingent claim at time T given by a random variable

$$H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P). \tag{3.2}$$

In order to hedge against this claim, we want to use a portfolio strategy which involves a stock X and a riskless bond $B \equiv 1$, which yields a random payment H at the terminal time T. Let ξ_t and η_t denote the amounts of stock and bond, respectively held at time t. The value of the resulting portfolio at time t is given by

$$V_t = \xi_t X_t + \eta_t \quad 0 \le t \le T, \tag{3.3}$$

and the cost accumulated up to time t by

$$C_t = V_t - \int_0^t \xi_s dX_s \quad 0 \le t \le T. \tag{3.4}$$

We only admit strategies (ξ, η) such that the process $V = V_{t0 \leq t \leq T}$ and $C = C_{t(0 \leq t \leq T)}$ are square -integrable and have right continuous paths and satisfy

$$V_T = H, \quad P - a.s. \tag{3.5}$$

We also require the integrability condition

$$E\Big[\int_0^T {\xi^2}_s d < X >_s + (\int_0^T |\xi_s| d|A|_s)^2\Big] < \infty, \tag{3.6}$$

where $< X >$ denotes pathwise-defined quadratic variation of the process X and $|A|$ the total variation of the process A. This ensures that the stochastic integral in (3.4) is well-defined. Such strategies are called admissible. For example suppose our claim H admits the following representation

$$H = H_0 + \int_0^T \xi_s^H dX_s, \quad P - a.s. \tag{3.7}$$

Then for the choice of strategy

$$\xi := \xi^H; \eta := V - \xi \cdot X; V_t := H_0 + \int_0^t \xi_s^H dX_s \ \ 0 \leq t \leq T. \tag{3.8}$$

This strategy is admissible and is self-financing i.e. $C_t = C_T = H_0$.

If we suppose that the market is complete then every contingent claim is attainable. This allows for complete elimination of risk involved in handling an option. This is no longer possible in an incomplete market and a typical claim will carry an intrinsic risk, and the problem consists in finding a dynamic portfolio strategy which reduces the actual risk to that intrinsic component. We discuss this further.

In the absence of arbitrage there exists an equivalent probability measure P^* such that X is P^*-martingale; which implies that X is a P-semimartingale. First let us start with the simple case wherein X is a P-martingale i.e $P^* = P$. In that context Föllmer-Sondermann introduced the following risk-minimization criterion wherein one looks for an admissible strategy that minimizes at each time t the remaining risk

$$R_t(\pi) := E[(\tilde{C_T}(\pi) - \tilde{C_t}(\pi))^2 \mid \mathcal{F}_t], \tag{3.9}$$

for any other admissible strategy π. We say that an admissible strategy π^* is risk minimizing if

$$R_t(\pi^*) \leq R_t(\pi). \tag{3.10}$$

H is attainable provided this remaining risk is brought down to 0. However the risk-minimizing strategy will no longer be self-financing but it will be mean self-financing in the sense that

$$E[C_T(\pi) - C_t(\pi)|\mathcal{F}_t] = 0 \quad 0 \leq t \leq T. \tag{3.11}$$

i.e. C is a martingale. There exists a unique risk-minimizing strategy and it is tied down to the existence of the Kunita-Watanabe decomposition given by

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^h,$$

with $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$, where $L^H = (L_t^H)_{0\leq t\leq T}$ is a square integrable martingale orthogonal to X under P.

Let us now consider the general incomplete case where $P \approx P^*$, but where P itself is no longer a martingale measure. In this situation in general there exist NO risk-minimizing strategy because unlike the martingale case discussed before, the issue of determining hedging strategies gets subtle and a locally risk-minimizing criterion is introduced. Intuitively the reason for the failure is a compatibility issue. We minimize $R_t(\pi)$ over all admissible continuations from t to obtain the t-optimal continuation. But for $s < t$, the s- optimal continuation for (s,T] $\supset$ (t,T] is different from t-optimal continuation when X is not P-martingale. Hence we must minimize the residual risk "locally". This condition may be thought of as an infinitesimal analogue of condition (3.10). We say a strategy is locally risk-minimizing if for any $t < T$, the remaining risk is minimal under all infinitesimal perturbations of the strategy at time t.

The concept of "Local Risk" minimization is formally defined as follows.

Definition 1. A small perturbation $\Delta = \{\delta_t, \epsilon_t\}$ is bounded variation corresponding to semimartingale decomposition of X_t and $\delta_T = \epsilon_T = 0$. Let π be an arbitrary strategy, $\tau = (t_i)_{(0\leq i\leq n)}$ be a partition of $[0, T]$. We set

$$r^\tau(\pi, \Delta) = \sum_{(t_i,t_{i+1})\in\tau} \frac{R_{t_i}(\pi + \Delta I_{(t_i,t_{i+1})}) - R_{t_i}(\pi)}{E[\int_{t_i}^{t_{i+1}} \sigma^2 X_t^2 dt|\mathcal{F}_i]} I_{(t_i,t_{i+1})}(t). \tag{3.12}$$

An admissible strategy π is called Locally Risk Minimizing if $\liminf_{n\to\infty} r^{\tau_n}(\pi, \Delta) \geq 0$ a.e for every increasing 0-convergent sequence (τ_n) of partitions of [0,T].

The idea of local risk-minimization is equivalent to the following property *Condition : The cost process C is square-integrable martingale orthogonal to M under P.*

Optimal strategies in the case of market incompleteness can now be defined as follows

Definition 2. An admissible strategy (η, ξ) is called optimal, if the associated cost process C satisfies the above condition .

Existence of an optimal strategy is equivalent to the decomposition

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H, \tag{3.13}$$

with $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$ and $L^H = (L_t^H)_{0 \leq t \leq T}$ being a square-integrable martingale orthogonal to M

In order to compute locally Risk minimizing hedging strategy, one needs to determine an EMM, more specifically a minimal EMM P^* such that X is a P^* -martingale. We define minimal martingale measure as follows.

Definition 3. An EMM $P^* \approx P$ is said to be minimal if $P^* = P$ on $\mathcal{F}_0$, and if any square integrable P-martingale which is orthogonal to M under P remains a martingale under P^*.

Remarks

(i) Measure P^* exist and is unique.

(ii) The MMM preserves orthogonality. In the sense that for any square integrable P martingale L that is orthogonal to martingale part M under P is also orthogonal to process X under P^*. Hence the term "minimal" for martingale measure is motivated by the fact that apart from turning X into a martingale, this measure disturbs the overall martingale and orthogonality structure as little as possible.

Hence after obtaining an minimal martingale measure, we proceed to determine an admissible "local" risk minimizing hedging strategy from (3.8)

We now provide an example explaining the minimal martingale measure. The discounted stock price process is a $\mathbb{P}$ semi-martingale given by

$$d\hat{S}(t) = (\mu(t) - r(t))dt + \sigma(t)dB(t).$$

Let $\hat{S}$ be 1-dimensional, $\mathbb{R}$-valued and driven by, say, a 1-dimensional Brownian motions B but with the underlying filtration generated by two possibly independent brownian motions B and $\tilde{B}$(say). The market is expectedly incomplete, and with the assumption of no-arbitrage there exists many equivalent martingale measures to choose from for pricing purpose. Let the time horizon $T < \infty$. The unique EMM is obtained by the following change of measure formula as follows

$$\frac{dQ}{dP}|_{\mathcal{F}_T} \triangleq Z(T), \tag{3.14}$$

where

$$dZ(t) = \gamma(t)Z(t)dB(t). \tag{3.15}$$

$\gamma(t)$ is chosen so as to make $\hat{S}$ a martingale under Q. (3.14) can be rewritten as

$$Z(t) = 1 + \int_0^t \gamma(s)Z(s)dB(s), \tag{3.16}$$

where

$$\gamma(s) = \frac{(\mu(t) - r(t))}{\sigma(t)}. \tag{3.17}$$

We now show that the EMM is in fact the MMM.

Lemma 3. *The martingale measure given by (3.14) is an MMM.*

Proof Under P, $\hat{S}$ satisfies,

$$\begin{aligned} \hat{S}(t) &= \hat{S}(0) + \int_0^t \sigma(s)\hat{S}(s)dB(s) + \int_0^t (\mu(s) - r(s))\hat{S}(s)ds, \\ &= \hat{S}(0) + W(t) + A(t) \end{aligned} \tag{3.18}$$

where $W(t) = \int_0^t \sigma(s)\hat{S}(s)dB(s)$ is a P-martingale and $A(t) = \int_0^t (\mu(s) - r(s))\hat{S}(s)ds$ is a continuous,adapted process. Therefore (3.18) gives a Doob decomposition of $\hat{S}$ under P. Consider now an $L^2(P)$-martingale N which is orthogonal to W so that $< N, W >= 0$. Therefore $< N, Z >= 0$ for Z given by (3.15) i.e N is orthogonal to Z. This obviously implies that N is a Q-local martingale. Now, since by assumption $N, Z \in L^2(P)$, by the Cauchy-Schwartz inequality $NZ \in L^1(P)$. Hence by the result in Proposition 1.23 of Revuz and Yor [32], N is a Q-martingale.

Thus N is an $L^2(P)$-martingale such that $< N, W >= 0 \Rightarrow N$ is a Q-martingale as well.

4 A brief discussion on stochastic stability

Deterministic stability is a branch of the qualitative theory of dynamical systems. The majority of the presently available work that are described as stability results pertain to certain qualitative and quantitative (which do not involve the actual computation of a solution) properties of differential equations. Consider the differential equation $\frac{dx}{dt} = f(x,t)$ with initial condition x_0 belonging to a set R. R may vary but will always be non-empty bounded open set containing the origin. Let P be a set containing R. A typical stability question is as follows: Let P be given. Is there any R such that if $x_0 \in R$, then $x_t \in P$ for all finite t?

Stochastic analogies to the deterministic problem can be easily drawn. For example, let x_t be a right-continuous, strong-Markov process such that its initial value x_0 is deterministic and lies in an open set R that contains the origin. Let $R \subset P$. The analogous question for the finite time horizon would be:
Is there an R such that if $P_x\{x_t \notin P, t < \infty\} \leq \rho < 1$ for some given ρ,P and any x in R?

In an asymptotic sense, we have the following possible candidate criterion for stochastic stability.

Definition 4. Almost-sure exponential stability
The trivial solution of a stochastic differential equation is almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| < 0 \quad a.s. \; \forall x \in \mathbb{R}^r \quad a.s.$$

5 A brief discussion on two player zero sum stochastic games

The following treatment closely follows the seminal article of Fleming and Souganidis [16]. Consider a finite-horizon, stochastic differential game (SDG) with state variable in $\mathbb{R}^d$ and horizon $T > 0$. The state dynamics are given by,

$$\begin{aligned} dX_s &= f(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s, Z_s)dW_s \quad s \in (t,T] \\ X_t &= x \quad x \in \mathbb{R}^d, \end{aligned} \tag{5.1}$$

and the pay-off by,

$$J(x,t,Y.,Z.) = E_{xt}[\int_t^T h(s,X_s,Y_s,Z_s)ds + g(X_T)], \tag{5.2}$$

where $W.$ is a standard M-dimensional Brownian motion, $Y.$, $Z.$ are stochastic processes taking values in some compact subsets $\mathcal{Y}$ and $\mathcal{Z}$.

Assumptions (A1):

The functions f, σ and h are bounded, uniformly continuous and Lipschitiz continuous with respect to (x,t) uniformly in $(y,z) \in \mathcal{Y} \times \mathcal{Z}$. Function g is bounded and Lipschitz continuous.

The idea behind this game is that there are two players I and II. Player I controls $Y.$ and wishes to maximize J over all choice of $Z.$. On the other hand, player II controls $Z.$ and tries to minimize J over all choices of $Y.$. Instantaneous switches of $Y.$ and $Z.$ are possible in continuous time and to avoid this two approximate games namely, lower and upper games are introduced. In the lower game, player II is allowed to know Y_s before choosing Z_s, while the upper game player I chooses Y_s knowing Z_s. Now consider the following Hamilton-Jacobi-Isaacs equation given by

$$\begin{aligned} u_t + H^-(D^2u, Du, x, t) &= 0 \quad \mathbb{R}^d \times [0,T], \\ u &= g \quad on \ \mathbb{R}^d \times \{T\}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} u_t + H^+(D^2u, Du, x, t) &= 0 \quad \mathbb{R}^d \times [0,T], \\ u &= g \quad on \ \mathbb{R}^d \times \{T\}, \end{aligned} \tag{5.4}$$

where (for A being a symmetric $d \times d$ matrix, $p, x \in \mathbb{R}^d$ and $t \in [0,T]$),

$$H^-(A,p,x,t) = \max_{y\in Y}\min_{z\in Z}[trace(\frac{1}{2}a(t,x,y,z)\cdot A + f(t,x,y,z)\cdot p + h(t,x,y,z)], \tag{5.5}$$

$$H^+(A,p,x,t) = \min_{z\in Z}\max_{y\in Y}[trace(\frac{1}{2}a(t,x,y,z)\cdot A + f(t,x,y,z)\cdot p + h(t,x,y,z)], \tag{5.6}$$

with $a = \sigma\sigma'$. Equations (5.3) and (5.4) corresponds to lower and upper value of the game and are well-known not to have global smooth solutions in general. Once the notion of an upper and a lower value function is introduced we pursue the task of proving that these functions are weak(viscosity) solutions of the associated

Bellman-Isaacs equations.

We introduce the concepts of admissible controls and admissible strategies. The sample space for (5.1) is defined as follows. For every $t \in [0, T]$ let

$$\Omega_t^w = \{w \in C([t, T]; \mathbb{R}^d) : w_t = 0\}. \tag{5.7}$$

Let $\mathcal{F}_{t,s}^w$ denote the σ-algebra generated by the paths up to time s in Ω_t^w. When, provided with a Wiener measure P_t^w on $\mathcal{F}_{t,T}^w$, Ω_t^w becomes canonical sample space of (5.1). We also define the space

$$\Omega_{t,s}^w = \{w \in C([t, s]; \mathbb{R}^d) : w_t = 0\}, \tag{5.8}$$

for $0 \leq t < s \leq T$. Define

$$\begin{aligned} w_1 &= w\mid_{[t,\tau]}, \\ w_2 &= w - w\mid_{[t,\tau]}, \\ \pi w &= (w_1, w_2). \end{aligned} \tag{5.9}$$

The map $\pi : \Omega_t^w \to \Omega_{[t,\tau]}^w \otimes \Omega_\tau^w$; induces the identification

$$\Omega_t^w = \Omega_{t,\tau}^w \otimes \Omega_\tau^w. \tag{5.10}$$

Moreover, $w = \pi^{-1}(w_1, w_2)$. As well, $P_t^w = P_{t,\tau}^w \otimes P_\tau^w$, where $P_{t,\tau}^w$ and P_τ^w are Wiener measures on $\Omega_{t,\tau}^w$ and Ω_τ^w respectively. We provide the following definition for control spaces.

Definition 5. An admissible control $Y.$ (resp.$Z.$) for player I(resp. II) on $[t, T]$ is an $\mathcal{F}_{t,s}^w$-progressively measurable process taking values in $\mathcal{Y}$ (resp. $\mathcal{Z}$). The set of all admissible controls for player I (resp. II) on $[t, T]$ is denoted by $M(t)$ (resp. $N(t)$).

Definition 6. An admissible strategy α (resp. β) for player I (resp. player II) on $[t, T]$ is a mapping $\alpha : N(t) \to M(t)$ (resp. $\beta : M(t) \to N(t)$) such that if $P_t^w(Z. = \tilde{Z}.a.e.in[t, s]) = 1$ then $P_t^w(\alpha(Z.) = \alpha(\tilde{Z}.)$ a.e. in $[t, s]$)=1. (resp. for Player II) on $[t, s]$ for every $s \in [t, T]$.

The set of admissible strategies of player I (resp. II) on $[t, T]$ is denoted by $\Gamma(t)$ (resp. $\Delta(t)$). In brief, controls should depend on the past of the Brownian motion and the strategies should only depends on the past of the controls. Note that for $Y. = \alpha[Z.]$, where $Z. \in \Delta(t)$ and $\alpha \in \Gamma(t)$, there exists a unique pathwise solution for X based on assumptions (A1) on f and σ and on controls Y and Z.

We now define the notions of the lower and upper value for the stochastic differential game related to objective (5.2). The intuitive idea being, choosing the controls at time s, the player who moves first (maximizing player for the lower game, minimizing player for the upper game) is allowed to use the past of the Brownian motion $W.$ driving $X.$ up to time s, while the player with the advantage (player II for the lower game, player I for the upper game) is allowed to use the past of both $W.$ and the other player's control.

Definition 7. The lower value of SDG (5.1)-(5.2) with initial data (x, t) is given by,

$$V(x,t) = \inf_{\beta \in \Delta(t)} \sup_{Y. \in M(t)} J(x,t;Y.,\beta). \tag{5.11}$$

The upper value of the game is given by

$$U(x,t) = \inf_{\alpha \in \Gamma(t)} \sup_{Z. \in N(t)} J(x,t;\alpha,Z.). \tag{5.12}$$

We are really interested in coming up with value functions which satisfy the dynamic programming principle. It turns out that the functions U and V do so for a restricted class of strategies called r-strategies. For the same we define for $\bar{t} < t < \tau$ and $Y. \in M(\bar{t})$ and $P^w_{\bar{t},t}$-a.e. $w_1 \in \Omega^w_{\bar{t},t}$, the map $Y(w_1) : [t,T] \times \Omega^w_t \to \mathcal{Y}$ given by $Y(w_1)(w_2)_r = Y(w_1, w_2)_r$ is an admissible control for player I, i.e. $Y(w_1) \in M(t)$. Similar observation holds for player II.

Definition 8. An r-strategy β for player II on $[t, T]$ is an admissible strategy with the following properties: For every $\bar{t} < t < \tau$ and $Y. \in M(\bar{t})$ the map $(r, w) \to \beta[Y(w_1)](w_2)_r$ is $(\mathcal{F}^0_{t,\tau} \otimes \mathcal{F}^w_{\bar{t},\tau}, \mathcal{F}^0_Z)$ measurable. The set of r-strategies of player II on $[t, T]$ is denoted by $\Delta_1(t)$. Similarly r-strategies for player I with their collection denoted by $\Gamma_1(t)$.

Definition 9. The r-lower and the r-upper value of the SDG with initial data (x, t) are given by,

$$V_1(x,t) = \inf_{\beta \in \Delta_1(t)} \sup_{Y. \in M(t)} J(x,t;Y.,\beta), \tag{5.13}$$

and

$$U_1(x,t) = \sup_{\alpha \in \Gamma_1(t)} \inf_{Z. \in N(t)} J(x,t;\alpha,Z.). \tag{5.14}$$

The r-value functions turn out to satisfy the two following inequalities .

Proposition 4. For $t, \tau \in [0, T]$ be such that $t < \tau$. For every $x \in \mathbb{R}^d$ we have

$$V_1(x,t) \leq \inf_{\beta \in \Delta_1(t)} \sup_{Y. \in M(t)} E_{xt}[\int_t^\tau h(s, X_s, Y_s, \beta[Y.]_s)ds + V_1(X_\tau, \tau)]. \quad (5.15)$$

$$U_1(x,t) \geq \sup_{\alpha \in \Gamma_1(t)} \inf_{Z. \in N(t)} E_{xt}[\int_t^\tau h(s, X_s, Y_s, \beta[Y.]_s)ds + V_1(X_\tau, \tau)]. \quad (5.16)$$

We refer to this inequality (5.15) as sub-optimal dynamic programming principle and to the inequality (5.16) as super-optimal dynamic programming principle. These inequalities coupled with a semi-discretization argument developed by Fleming and Souganidis [16] yield that the lower and upper value functions are the unique viscosity solutions of the game and that they satisfy the principle of dynamic programming. We direct the interested reader to the proofs in this article.

6 Semi-Markov modulated jump diffusions

The finite state semi-Markov process is a generalization of the Markov chain in which the sojourn time distribution is any general distribution. We consider a semi-Markov modulated jump-diffusion process in which the drift, diffusion and the jump kernel of the jump-diffusion process is modulated by a semi-Markov process. We introduce some notation before we describe the semi-Markov modulated jump diffusion process.

- Let $\mathbb{R}$: be the reals
- r, M: be any positive integers greater than 1.
- $\mathcal{X} = \{1, ..., M\}$.
- v', A': the transpose of the vector v and matrix A respectively.
- $||v||$: Euclidean norm of a vector v.
- $|A|$: norm of a matrix A.
- $tr(A)$: trace of a square matrix A.

We assume that the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, \mathbb{P})$ is complete with filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ and is right-continuous and $\mathcal{F}(0)$ contains all $\mathbb{P}$ null sets. Let $\{\theta(t)\}_{t\geq 0}$ be a semi-Markov process adapted to filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ taking values in $\mathcal{X}$ with transition probability p_{ij} and conditional holding time distribution $F^h(t|i)$. Thus if $0 \leq t_0 \leq t_1 \leq ...$ are times when jumps occur, then

$$P(\theta(t_{n+1}) = j, t_{n+1} - t_n \leq t | \theta(t_n) = i) = p_{ij} F^h(t|i). \quad (6.1)$$

Matrix $[p_{ij}]_{\{i,j=1,\ldots,M\}}$ is irreducible and for each i, $F^h(\cdot|i)$ has continuously differentiable and bounded density $f^h(\cdot|i)$. For a fixed t, let $n(t) \triangleq \max\{n : t_n \leq t\}$ and $Y(t) \triangleq t - t_{n(t)}$. Thus $Y(t)$ represents the amount of time the process $\theta(t)$ is at the current state after the last jump. The process $(\theta(t), Y(t))$defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is jointly Markov and the differential generator $\mathcal{L}$ given as follows Chap.2 of [22]

$$\mathcal{L}\phi(i,y) = \frac{d}{dy}\phi(i,y) + \frac{f^h(y|i)}{1 - F^h(y|i)} \sum_{j\neq i, j\in\mathcal{X}} p_{ij}[\phi(j,0) - \phi(i,y)]. \tag{6.2}$$

for $\phi : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$ a C^1 function.

We first represent the semi-Markov process $\theta(t)$ as a stochastic integral with respect to a Poisson random measure. With that perspective in mind, embed $\mathcal{X}$ in $\mathbb{R}^M$ by identifying i with $e_i \in \mathbb{R}^M$. For $y \in [0, \infty)$ $i, j \in \mathcal{X}$, define

$$\begin{aligned} \lambda_{ij}(y) &= p_{ij}\frac{f^h(y/i)}{1 - F^h(y/i)} \geq 0 \text{ and } \forall \;\; i \neq j, \\ \lambda_{ii}(y) &= -\sum_{j\in\mathcal{X}, j\neq i}^{M} \lambda_{ij}(y) \;\; \forall \;\; i \;\; \in \mathcal{X}. \end{aligned}$$

For $i \neq j \in \mathcal{X}$, $y \in \mathbb{R}_+$ let $\Lambda_{ij}(y)$ be consecutive (with respect to lexicographic ordering on $\mathcal{X}\times\mathcal{X}$) left-closed, right-open intervals of the real line, each having length $\lambda_{ij}(y)$. Define the functions $\bar{h} : \mathcal{X}\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}^r$ and $\bar{g} : \mathcal{X}\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$ by

$$\bar{h}(i,y,z) = \begin{cases} j - i & \text{if } z \in \Lambda_{ij}(y) \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}(i,y,z) = \begin{cases} y & \text{if } z \in \Lambda_{ij}(y), j \neq i \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{M}(\mathbb{R}_+\times\mathbb{R})$ be the set of all nonnegative integer-valued σ-finite measures on Borel σ-field of $(\mathbb{R}_+ \times \mathbb{R})$. The process $\{\tilde{\theta}(t), Y(t)\}$ is defined by the following stochastic integral equations:

$$\begin{aligned} \tilde{\theta}(t) &= \tilde{\theta}(0) + \int_0^t \int_{\mathbb{R}} \bar{h}(\tilde{\theta}(u-), Y(u-), z) N_1(du, dz), \\ Y(t) &= t - \int_0^t \int_{\mathbb{R}} \bar{g}(\tilde{\theta}(u-), Y(u-), z) N_1(du, dz), \end{aligned} \tag{6.3}$$

where $N_1(dt, dz)$ is an $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$-valued Poisson random measure with intensity $dtm(dz)$ independent of the $\mathcal{X}$-valued random variable $\tilde{\theta}(0)$, where $m(\cdot)$ is a

Lebesgue measure on $\mathbb{R}$. As usual, by definition $Y(t)$ represents the amount of time process $\tilde{\theta}(t)$ is at the current state after the last jump. We define the corresponding compensated or centered one- dimensional Poisson measure as $\tilde{N}_1(ds, dz) = N_1(ds, dz) - dsm(dz)$. It was shown in Theorem 2.1 of Ghosh and Goswami [20] that $\tilde{\theta}(t)$ is a semi-Markov process with transition probability matrix $[p_{ij}]_{\{i,j=1,...,M\}}$ with conditional holding time distributions $F^h(y|i)$. Since, by definition $\theta(t)$ is also a semi-Markov process with transition probability matrix $[p_{ij}]_{\{i,j=1,...,M\}}$ with conditional holding time distributions $F^h(y|i)$ defined on the same underlying probability space, by equivalence, $\tilde{\theta}(t) = \theta(t)$ for $t \geq 0$.

Remark The semi-Markov process with conditional density $f^h(y|i) = \tilde{\lambda}_i e^{-\tilde{\lambda}_i y}$ for some $\tilde{\lambda}_i > 0$, $i = 1, 2..., M$, is in fact a Markov chain.

We now formally introduce the semi-Markov modulated jump-diffusion process. Let $\mathcal{U} \subset \mathbb{R}^r$ be a closed subset. Let $\mathbb{B}_0$ be the family of Borel sets $\Gamma \subset \mathbb{R}^r$ whose closure $\bar{\Gamma}$ does not contain 0. For an Borel set $B \subset \Gamma$, the one-dimensional Poisson random measure $N(t, B)$ counts the number of jumps on $[0, t]$ with values in B. For a predictable process $u : [0, T] \times \Omega \to \mathcal{U}$ with left-continuous, right-limited paths, consider the controlled process X with given initial condition $X(0) = x \in \mathbb{R}^r$ given by

$$\begin{aligned} dX(t) &= b(t, X(t), u(t), \theta(t))dt + \sigma(t, X(t), u(t), \theta(t))dW(t) \\ &+ \int_\Gamma g(t, X(t), u(t), \theta(t)), \gamma)N(dt, d\gamma), \end{aligned} \quad (6.4)$$

where $X(t) \in \mathbb{R}^r$ and $W(t) = (W_1(t), ..., W_r(t))$ is r-dimensional standard Brownian motion. The coefficients $b(\cdot, \cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \to \mathbb{R}^r$,$\sigma(\cdot, \cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \to \mathbb{R}^r \times \mathbb{R}^r$ and $g(\cdot, \cdot, \cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \times \Gamma \to \mathbb{R}^r$ are assumed to satisfy the usual assumptions that guarantee existence and uniqueness of solution to (6.4).

7 Our Contribution

The thesis consists of the following three topics.

Q.1 The first chapter is based on an article titled as "Game-theoretic approach to risk-sensitive benchmarked asset management." Refer [9].

In this chapter we consider a game theoretic approach to the Risk-Sensitive Benchmarked Asset Management problem (RSBAM) of Davis and Lleo [8]. In particular, we consider a stochastic differential game between two players, namely, the investor who has a power utility while the second player represents the market which tries to minimize the expected payoff of the investor. The market does this by modulating a stochastic benchmark that the investor needs to outperform. We obtain an explicit expression for the optimal pair of strategies for both the players under certain conditions.

Q.2 The second chapter is based on an article titled “On the role of Föllmer-Schweizer minimal martingale measure in Risk Sensitive control Asset Management.” Refer [10].

Kuroda and Nagai [26] state that the factor process in risk-sensitive control asset management (RSCAM) is stable under the Föllmer-Schweizer minimal martingale measure . Fleming and Sheu [15] and more recently Föllmer and Schweizer [18] have observed that the role of the minimal martingale measure in this portfolio optimization is yet to be established. We aim to address this question by explicitly connecting the optimal wealth allocation to the minimal martingale measure. We achieve this by using a “trick” of observing this problem in the context of model uncertainty via a two-person zero-sum stochastic differential game between the investor and an antagonistic market that provides a probability measure.

Q.3 The third chapter is based on an article titled as “Sufficient stochastic maximum principle for the optimal control of semi-Markov modulated jump-diffusion with application to Financial optimization.” Refer [11].

In this topic we provide a sufficient stochastic maximum principle for the optimal control of a semi-Markov modulated jump-diffusion process in which the drift, diffusion and the jump kernel of the jump-diffusion process is modulated by a semi-Markov process. We also connect the sufficient stochastic maximum principle with the dynamic programming equation. We apply our results to finite horizon risk-sensitive control portfolio optimization problem and to a quadratic loss minimization problem.

Chapter 2

Game-theoretic approach to risk-sensitive bench marked asset management.

Abstract
In this chapter we consider a game theoretic approach to the Risk-Sensitive Benchmarked Asset Management problem (RSBAM) of Davis and Lleo [8]. In particular, we consider a stochastic differential game between two players, namely, the investor who has a power utility while the second player represents the market which tries to minimize the expected payoff of the investor. The market does this by modulating a stochastic benchmark that the investor needs to outperform. We obtain an explicit expression for the optimal pair of strategies as for both the players.

1 Introduction

In this chapter we shall develop a game theoretic version of a continuous time optimization model with risk-sensitive control approach more specifically termed as Risk-sensitive control portfolio optimization (RSCPO). The RSCPO balances an investor's interest in maximizing the expected growth rate of wealth against his aversion to risk due to deviations of the realized rate from the expectation. The subjective notion of investor's risk aversion is parameterized by a single variable, say θ. More formally,we write the finite horizon risk-sensitive optimization criterion as :

$$\max J_{T,h} := -\frac{1}{\theta} \log E[e^{-\theta F(T,h)}],$$

where $F(T, h)$ is the time-T value reward function corresponding to control h. In the optimal investment problem we take $F(T, h) = \log V(T)$ where $V(t)$ is the time t-value of the portfolio corresponding to portfolio asset allocation h. An asymptotic expansion around $\theta = 0$ for the above criterion yields

$$J_{T,h} = E[F(T, h)] - \frac{\theta}{2} Var(F(T, h)) + O(\theta^2).$$

From this expression it is clear this criterion compromises between maximizing the portfolio return while penalizing the riskiness . The optimal expected utility function depends on θ and is a generalization of the traditional stochastic control approach to utility optimization in the sense that now the degree of risk aversion of the investor is explicitly parameterized through θ rather than importing it in the problem via an exogenous utility function. Values of $\theta > 0$ correspond to a risk-averse investor, $\theta < 0$ to a risk-seeking investor and $\theta = 0$ to a risk-neutral investor who maximizes

$$J_{T,h} := E[F(T, h)].$$

There has been a substantial amount of research on the infinite-time horizon ergodic problem:

$$\begin{aligned} &\max \quad \bar{J}_\infty \quad \text{where,} \\ &\bar{J}_\infty = \liminf_{t\to\infty} -\frac{1}{\theta} t^{-1} \log E[e^{-\theta F(t,h)}]. \end{aligned}$$

Though these type of problems are interesting in their own right, they are not readily applicable to practical asset management because of non-uniqueness of optimal controls.

In the past decade, applications of risk-sensitive control to asset management have proliferated. Risk-sensitive control was first applied to solve financial problems by Lefebvre and Montulet [27] in a corporate finance context. Fleming [14] was the first to show that some investment optimization models could be reformulated as risk-sensitive control problems. Bielecki and Pliska [2] considered a model with n securities and m economic factors with no transaction cost. They were the first to apply continuous-time risk-sensitive control as a practical tool that could be used to solve “real-world” portfolio selection problems. They considered

a long-term asset allocation problem and proposed the logarithm of the investor's wealth as a reward function, so that the investor's objective is to maximize the risk-sensitive (log) return of his/her portfolio. They derived the optimal control and solved the associated Hamilton-Jacobi-Bellman (HJB) PDE under the restrictive assumption that the securities and economic factors have independent noise. In [3], Bielecki and Pliska went on to study the economic properties of the risk-sensitive asset management criterion and then extended the asset management model into an intertemporal CAPM in [4]. Fleming and Sheu [15] analyzed an investment model similar to that of Bielecki and Pliska [2]. In their model, however, the factor process and the security price process were assumed correlated. A major contribution was made by Kuroda and Nagai [26] who introduced an elegant solution method based on a change of measure argument which transforms the risk sensitive control problem into a linear exponential of a quadratic regulator. They solved the associated HJB PDE over a finite time horizon and then studied the properties of the ergodic HJB PDE related to $\bar{J}_\infty$. Recently, Davis and Lleo [8] applied this change of measure technique to solve, for both the finite and an infinite horizon, a risk-sensitive benchmark investment problem (RSBAM) in which an investor selects an asset allocation to outperform a given financial benchmark. In the Kuroda and Nagai set-up θ represents the sensitivity of an investor to total risk, whereas in the RSBAM, θ represents the investors sensitivity to active risk i.e. additional risk the investor is willing to take in order to outperform the benchmark.
It is obvious that for outperforming a stochastic benchmark, an investor will have to modify his or her optimal trading strategy. Then the question of interest to us is: "What is the investor's worst case strategy for an opposing stochastic benchmark"?. In particular, one can even take the jaundiced point of view that the benchmark will be set retrospective to the worst case. For example, if a portfolio fund manager outperforms the set benchmark, the principal may remark this out-performance either as best achieved or poorly achieved with respect to the underlying worst-case scenario. So, in this chapter we consider a game-theoretic version of the problem within the benchmark framework of Davis and Lleo [8]. In it, we consider a stochastic differential game between two players, namely, the investor (who has a power utility) and a second player, representing the market, who tries to minimize the expected payoff of the investor. We explicitly characterize the optimal allocation of assets and the optimal choice of benchmark index.

In this chapter, we consider the benchmark process ex-ante that evolves according to a controlled diffusion process. We contrast this approach to the one of Heath and Platen [23]. In their methodology, they use the growth optimal portfolio itself as a benchmark which is closer to the concept of the numeraire portfolio. Although there has been a long history of applying risk-sensitive optimal control to problems in finance, a game-theoretic version of such problems in finite horizon is missing from the literature. We intend to elaborate further on this now.

In the next section we briefly describe the framework of the risk-sensitive zero sum stochastic differential game corresponding to the desired game (**P1**)(refer 2.8). In the third section we reformulate the objective criterion under evaluation as a linear exponential of quadratic regulator problem (**P2**) (refer 3.11). In the fourth section we provide a verification lemma that will help us solve this game problem. In the fifth section we derive the optimal controls and obtain an explicit expression for the associated value of the game. The chapter as usual concludes with remarks and pointers to future direction of work.

Broadly speaking our aim is to derive the saddle-point equilibrium pair for the game (**P1**). To achieve this, we first obtain saddle point strategy for the game (**P2**). We then show that the saddle point equilibrium for (**P2**) is also saddle point equilibrium for (**P1**).

2 Risk-sensitive zero sum stochastic differential game

We consider a market consisting of $m+1 \geq 2$ securities with $n \geq 1$ factors. We assume that the set of securities includes one bond whose price is governed by the ODE

$$dS_t^0 = r_t S_t^0 dt, \qquad S_0^0 = s^0, \tag{2.1}$$

where r_t is a deterministic function of t. The other security prices and factors are assumed to satisfy the following SDE's

$$dS_t^i = S_t^i \{(a + AX_t)^i dt + \sum_{k=1}^{n+m} \sigma_k^i dW_t^k\}, \qquad S_0^i = s^i, i = 1, ..., m, \tag{2.2}$$

where the factor process X_t satisfies,

$$dX_t = \{(b + BX_t)dt + \Lambda dW_t\}, X_0 = x \in \mathbb{R}^n. \tag{2.3}$$

Here $W_t = (W_t)_{k=1,...,n+m}$ is an $n+m$ dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$.

The factor process can represent macro-economic indicators such as GDP, inflation and market index data. The stock price dynamics are modulated by the factor process. Hence one can incorporate the effect of macro-economic indicators into the investment optimization problem by using the stock price process modulated by the factor process X_t.

The model parameters A, B, Λ are respectively $m \times n, n \times n, n \times (m+n)$ constant matrices and $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$. The constant matrix $(\sigma_k^i)_{\{i=1,2....,m;k=1,2,...,(n+m)\}}$ will be denoted by Σ in what follows.

In Kuroda and Nagai [26] it is assumed that the factor process and the stock price process do not have independent noise i.e. $\Sigma\Lambda' \neq 0$. This assumption is in sharp contrast to Bielecki and Pliska [2] who conversely assume that $\Sigma\Lambda' = 0$. We will assume that $\Sigma\Lambda' \neq 0$.

Let $\mathcal{G}_t = \sigma(W_u; u \leq t)$ be the sigma-field generated by the standard Brownian motions up to time t. For given $h \in \mathcal{H}(T)$, the process $V_t = V_t^h$ represents the investor's wealth at time t, under the control h, and satisfies the following SDE dynamics,

$$\frac{dV_t^h}{V_t^h} = (r_t + h_t'((a + AX_t) - r_t 1))dt + h_t'\Sigma dW_t; V_0^h = v,$$

which can be rewritten as,

$$\frac{dV_t^h}{V_t^h} = (r_t + h_t' d_t)dt + h_t'\Sigma dW_t; V_0^h = v, \tag{2.4}$$

where $d_t \triangleq a + AX_t - r_t 1$. From equation (2.4) it can be seen that if $a + AX_t = r_t 1$ i.e. $d_t = 0$, then the portfolio wealth process evolves with drift equal to the risk less interest rate r_t. We make an assumption here that the securities price volatility matrix Σ is a full rank matrix. If it is not full-rank then $h'\Sigma = 0$ for some $h \neq 0$. Hence the market contains redundant asset(s) and the portfolio value process V_t^h will grow at a rate different than the risk-less interest rate r_t when $h'd \neq 0$ resulting in an arbitrage. This is the case if the portfolio contains two or more redundant assets for example a stock and an option on the same stock. Hence we remove redundancy till the resultant matrix Σ is of full rank thereby ensuring that there exist no further possibility of arbitrage by trading in the resultant portfolio.

In our benchmark model we express the objective through a new optimization criterion corresponding to a reward function F which represents the log excess return of the asset portfolio over its benchmark and is given as

$$F(t; h, \gamma) = \log \frac{V_t^h}{L_t^\gamma} \quad F(0; h, \gamma) = \log f.$$

We now formally state the Risk-sensitive Benchmarked Asset management problem (RSBAM) that we solve.

Problem : Risk-sensitive Benchmarked Asset Management (RSBAM)

We first define the objective criterion J as,

$$\begin{aligned} J(f, x, h, \gamma; T) &\triangleq \frac{-2}{\theta} \log E[\exp[\frac{-\theta}{2} F(T, h, \gamma)]], \\ &= \frac{-2}{\theta} \log E[\left(\frac{V_T^h}{L_T^\gamma}\right)^{-\theta/2}], \\ &= \frac{-2}{\theta} \log E[U(\frac{V_T^h}{L_T^\gamma})]. \end{aligned} \tag{2.5}$$

where the utility function $U(\cdot)$ is $U : x \to x^{-\frac{\theta}{2}}$. The dynamics of the benchmark process is a diffusion process L^γ modulated by a (Markovian) control γ given by

$$\frac{dL_t^\gamma}{L_t^\gamma} = (\alpha_t + \beta_t X_t)dt + \gamma_t^{'} dW_t. \tag{2.6}$$

where $\alpha_t \in \mathbb{R}$ and $\beta \in \mathbb{R}^{1\times n}$. The space of controls $\Gamma(T)$ consists of the market control represented by γ that is $\mathbb{R}^{n+m}$-valued. $\Gamma(T)$ consists of progressively measurable controls measurable w.r.t to $\{\mathcal{B}[0,T] \otimes \mathcal{G}_t\}_{t\geq 0}$ and where $P(\int_0^T |\gamma_s|^2 ds < \infty) = 1 \ \ \forall \, T < \infty$ and $E[e^{\theta^2 \int_0^T \gamma_s^{'} \gamma_s ds}]^{\frac{1}{2}} < \infty$.

By a simple application of Ito's formula we have:

$$\begin{aligned} dF(t, h, \gamma) &= d\log(\frac{V_t^h}{L_t^\gamma}) \\ &= \{[r_t + h_t{}'(a + AX_t - r_t 1) - (\alpha_t + \beta_t X_t) - \frac{1}{2} h_t^{'} \Sigma\Sigma^{'} h_t + \frac{1}{2}\gamma_t^{'}\gamma_t]dt \\ &+ (h_t^{'}\Sigma - \gamma_t^{'})dW_t\}. \end{aligned} \tag{2.7}$$

We are now in a position to formally state the game-theoretic version of the game. For a given $\theta > 0$, we consider a stochastic differential game between two players, namely, the investor (who has a power utility) U and who modulates the payoff for given $\gamma \in \Gamma(T)$ via control $h \in \mathcal{H}(T)$. On the other hand the second player, say the market, behaves antagonistically to the investor by setting a benchmark for the investor to outperform by modulating the control γ for a given control h. This can be conceptualized as a risk-sensitive zero sum stochastic differential game between the investor on one side and the market on the other and is formalized as follows

Problem (P1) Obtain $\hat{h} \in \mathcal{H}(T)$ and $\hat{\gamma} \in \Gamma(T)$ such that,

$$J(f, x, \hat{h}, \hat{\gamma}; T) = \sup_{h\in\mathcal{H}(T)} \inf_{\gamma\in\Gamma(T)} \frac{-2}{\theta} \log E[(\frac{V_T^h}{L_T^\gamma})^{-\frac{\theta}{2}}] = \inf_{\gamma\in\Gamma(T)} \sup_{h\in\mathcal{H}(T)} \frac{-2}{\theta} \log E[(\frac{V_T^h}{L_T^\gamma})^{-\frac{\theta}{2}}]. \tag{2.8a}$$

This can be construed as a game-theoretic version of the RSBAM problem.

Remark 1:

The problem set up (**P1**) is an extension of Kuroda and Nagai [26] and Davis and Lleo [8]. However the former does not consider the benchmarked version i.e. the benchmark index is identically one in [26] while in Davis and Lleo [8] though have a benchmarked portfolio criterion, they solve the one player optimization problem and not the two player saddle point problem.

In light of the mathematical preliminaries just discussed, we formally elaborate the plan to solve the zero sum stochastic differential game (**P1**).

Step 1 We reformulate the original objective criterion as a power utility function to an exponential of an integral function.

Step 2 Define a new path functional $I(f, x, h, \gamma, t; T)$ (refer equation (3.9)) related to the exponential of the integral function. Define $\bar{u}(t, x)$ to be the upper-value function while $\underline{u}(t, x)$ be the lower-value function for the game associated with I. Denote the game related to this objective functional as (**P2**).

Step 3 Deduce the HJBI PDE corresponding to game (**P2**)(refer (3.11).

Step 4 Formulate the conditions that a candidate value function should satisfy for the game with regards to objective function I to have a value. This constitutes the verification lemma.

Step 5 Solve the HJBI PDE derived in step 3 while obtaining the expression for optimal controls. This optimal control pair will constitute a saddle point equilibrium for (**P2**). The candidate value function satisfying all the conditions of the verification lemma is our desired value function for (**P2**).

Step 6 Reverting back to the original problem (**P1)**, show using facts derived in Step 4, that the game with objective criterion J now has a value as well, and is in fact $u(0, x)$.

In the next section we reformulate the objective criterion and formalize our game problem.

3 Problem Reformulation

Step 1

We will first transform the utility optimization problem (2.5) into optimizing the exponential-of-integral performance criterion.

Criterion under the expectation

Our first aim is to write the objective criterion J only in terms of the factor process. Towards that end we define the function $g(x, h, \gamma, r; \theta)$ as follows:

$$\begin{aligned} g(x,h,\gamma,r;\theta) &= \frac{1}{2}(\frac{\theta}{2}+1)h^{'}\Sigma\Sigma^{'}h - r - h^{'}(a+Ax-r1) + (\alpha+\beta x) - \frac{1}{2}\frac{\theta}{2}(h^{'}\Sigma\gamma + \gamma^{'}\Sigma^{'}h) \\ &+ \frac{1}{2}(\frac{\theta}{2}-1)\gamma^{'}\gamma. \end{aligned} \tag{3.1}$$

From (2.7) and (3.1) we therefore have,

$$\begin{aligned} d\exp(\frac{-\theta}{2}F(t;h,\gamma)) &= \frac{\theta}{2}\Big(g(X_t,h_t,\gamma_t,r_t;\theta) - (h_t^{'}\Sigma - \gamma_t^{'})\Sigma dW_t\Big) \\ &- \frac{\theta^2}{8}(h_t^{'}\Sigma - \gamma_t^{'})\Sigma\Sigma^{'}(\Sigma^{'}h_t - \gamma_t)dt. \end{aligned} \tag{3.2}$$

Thus we have,

$$\begin{aligned} \exp(\frac{-\theta}{2}F(t;h,\gamma)) &= f^{-\theta/2}\exp\{\frac{\theta}{2}\int_0^t g(X_s,h_s,\gamma_s,r;\theta)ds \\ &- \frac{\theta}{2}\int_0^t (h_s^{'}\Sigma - \gamma_s^{'})dW_s - \frac{1}{2}(\frac{\theta}{2})^2\int_0^t (h_s^{'}\Sigma - \gamma_s^{'})(h_s^{'}\Sigma - \gamma_s^{'})^{'}ds\}, \end{aligned} \tag{3.3}$$

where $V_0^h = v, L_0^\gamma = l$ and $f = \frac{V_0^h}{L_0^\gamma} = \frac{v}{l}$.

Change of measure

Let $\mathbb{P}^{h,\gamma}$ be the measure on $(\Omega, \mathcal{F})$ defined by,

$$\frac{d\mathbb{P}^{h,\gamma}}{d\mathbb{P}}|_{\mathcal{F}_t} = \bar{\mathcal{X}}_t, \tag{3.4}$$

where $\bar{\mathcal{X}}_t$ is given by

$$\bar{\mathcal{X}}_t = \mathcal{E}(\frac{\theta}{2}\int_0 (h^{'}\Sigma - \gamma^{'})dW)_t, \tag{3.5}$$

and where $\mathcal{E}(\cdot)$ denotes the Doleans-Dade or martingale exponential. From the assumption made on the space of admissible controls $\mathcal{H}(T)$ and $\Gamma(T)$ it is clear that the Kazamaki condition $E[e^{\int_0^t \theta \frac{h_s^{'}\Sigma - \gamma_s^{'}}{2} dW_s}] < \infty \; \forall t \in [0,T]$ is satisfied so that $\mathbb{P}^{h,\gamma}$ to be a probability measure. i.e.

$$E[\mathcal{E}(\frac{\theta}{2}\int_0 (h^{'}\Sigma - \gamma^{'})dW)_T] = 1. \tag{3.6}$$

We note that,

$$W_t^{h,\gamma} \triangleq W_t + \frac{\theta}{2}\int_0^t (h_s^{'}\Sigma - \gamma_s^{'})ds, \tag{3.7}$$

by Girsanov's formula, is a standard Brownian motion under $\mathbb{P}^{h,\gamma}$ and the factor process X_t satisfies,

$$dX_t = (b + BX_t - \frac{\theta}{2}(\Sigma^{'} h_t - \gamma_t))^{'} dt + \Lambda dW_t^{h,\gamma}. \tag{3.8}$$

Step 2

The HJB equation

Taking expectation w.r.t to the physical measure $\mathbb{P}$ and multiplying both sides of equation (3.3) by $\frac{-2}{\theta}$ followed by the change of measure argument of (3.4-3.5) one considers the new path functional I defined as

$$I(f,x,h,\gamma,t,T) = \log f - \frac{2}{\theta}\log E^{h,\gamma}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,h_s,\gamma_s,r_{s+t};\theta)ds\}], \tag{3.9}$$

and then the upper-value function and lower-value function $\bar{u}$ and $\underline{u}$ respectively for the game corresponding to the new path functional I are given by :

$$\bar{u}(t,x) = \sup_{h\in\mathcal{H}(T)} \inf_{\gamma\in\Gamma(T)} I(f,x,h,\gamma,t,T), \tag{3.10a}$$

$$\underline{u}(t,x) = \inf_{\gamma\in\Gamma(T)} \sup_{h\in\mathcal{H}(T)} I(f,x,h,\gamma,t,T), \tag{3.10b}$$

$$u(t,x) = \bar{u}(t,x) = \underline{u}(t,x). \tag{3.10c}$$

If a pair of controls satisfy (3.10c), then the game corresponding to the new path functional I has the value u and the pair of controls constitutes saddle point strategies for the game with regards to I. Let the *exponentially transformed* function $\tilde{I}$ be defined as $\tilde{I} = \exp(-\frac{\theta}{2}I)$ and $\tilde{u}(t,x) := \exp(-\frac{\theta}{2}u(t,x))$. We now consider the problem of determining the saddle-point equilibrium for the game corresponding to the new path functional $\tilde{I}$. We call this problem (**P2**) and it is formally stated as follows:

Problem P2 Obtain $\hat{h} \in \mathcal{H}(T)$ and $\hat{\gamma} \in \Gamma(T)$ such that,

$$\begin{aligned}
\tilde{u}(t,x) &= \inf_{h\in\mathcal{H}(T)} \sup_{\gamma\in\Gamma(T)} \tilde{I}(f,x,h,\gamma,t,T), \\
&= \sup_{\gamma\in\Gamma(T)} \inf_{h\in\mathcal{H}(T)} \tilde{I}(f,x,h,\gamma,t,T), \\
&= E^{\hat{h},\hat{\gamma}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,\hat{h}_s,\hat{\gamma}_s,r_{s+t};\theta)ds\}f^{-\theta/2}]. \qquad (3.11)
\end{aligned}$$

We now provide a verification lemma for this game. Let us first define the process $Y^{h,\gamma}(t)$ by

$$dY^{h,\gamma}(t) = \begin{pmatrix} dt \\ dX_t \end{pmatrix} = \begin{pmatrix} dt \\ (b+BX_t - \frac{\theta}{2}(h_t'\Sigma - \gamma_t'))dt + \Lambda dW_t^{h,\gamma} \end{pmatrix}$$

Let $y \triangleq (t,x)$. The control process $h(t) = h(t,\omega)$ and $\gamma(t) = \gamma(t,\omega)$ for $\omega \in \Omega$ can be assumed to be Markovian. Let $\mathcal{O} = (0,T) \times \mathbb{R}^n$. Then the process $Y^{h,\gamma}(t)$ is a Markov process whose generator $\tilde{\mathcal{A}}^{h,\gamma}$ acting on a function $\tilde{u}(t,x) \in C_0^2([0,T]\times\mathbb{R}^n)$ is given by,

$$\tilde{\mathcal{A}}^{h,\gamma}\tilde{u}(t,x) = \frac{\partial \tilde{u}(t,x)}{\partial t} + (b + Bx - \frac{\theta}{2}\Lambda(\Sigma' h - \gamma))' D\tilde{u}(t,x) + \frac{1}{2}tr(\Lambda\Lambda^* D^2\tilde{u}(t,x)). \qquad (3.12)$$

in which $D\tilde{u}(t,x) \triangleq (\frac{\partial \tilde{u}(t,x)}{\partial x^1}, ..., \frac{\partial \tilde{u}(t,x)}{\partial x^n})'$ and $D^2\tilde{u}(t,x)$ is the matrix defined by $D^2\tilde{u}(t,x) \triangleq [\frac{\partial^2 \tilde{u}(t,x)}{\partial x^i x^j}], i,j = 1,2,...,n$.

Step 3

By an application of the Feynman-Kac formula, it can be deduced from (3.11) that the HJBI PDE for $\tilde{u}(t,x)$ is given by

$$\left(\tilde{\mathcal{A}}^{\hat{h},\hat{\gamma}} + \frac{\theta}{2}g(x,\hat{h},\hat{\gamma},r;\theta)\right)\tilde{u}(t,x) = 0. \qquad (3.13)$$

Reversing the exponential transformation , dividing by $-(\theta/2)\tilde{u}(t,x)$, we can deduce from (3.13) that the HJBI PDE for $u(t,x)$ is given for $h \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^{(m+n)}$ by

$$\mathcal{A}^{\hat{h},\hat{\gamma}}u(t,x) = 0, \qquad (3.14)$$

where the operator $\mathcal{A}^{h,\gamma}$ is given by,

$$\begin{aligned}\mathcal{A}^{h,\gamma}u(t,x) &= \frac{\partial u(t,x)}{\partial t} + (b + Bx - \frac{\theta}{2}\Lambda(\Sigma' h - \gamma))' Du(t,x) + \frac{1}{2}tr(\Lambda\Lambda' D^2 u(t,x)) \\ &- \frac{\theta}{4}(Du(t,x))'\Lambda\Lambda' Du(t,x) - g(x,h,\gamma,r;\theta). \end{aligned} \tag{3.15}$$

In the next section we provide a verification lemma for the game based on the criterion function I.

4 Verification lemma for the game PII

Step 4

We now provide a verification lemma related to the game (**PII**).

Proposition 1. *Suppose $\tilde{w} \in \mathcal{C}^{1,2}(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$ (is the space of twice differentiable functions on $\mathcal{O}$ with respect to x, once continuously differentiable on $\mathcal{O}$ with respect to t and which are continuous on $\bar{\mathcal{O}}$). Suppose there exists a (Markov) control $h(y), \gamma(y)$ that satisfy*

1. $(\tilde{\mathcal{A}}^{h,\hat{\gamma}(y)} + \frac{\theta}{2}g(x,h,\hat{\gamma}(y),r;\theta))[(\tilde{w}(y))] \geq 0 \ \forall \ h \in \mathbb{R}^m$;

2. $(\tilde{\mathcal{A}}^{\hat{h}(y),\gamma} + \frac{\theta}{2}g(x,\hat{h}(y),\gamma,r;\theta))[(\tilde{w}(y))] \leq 0 \ \forall \ \gamma \in \mathbb{R}^{m+n}$;

3. $(\tilde{\mathcal{A}}^{\hat{h}(y),\hat{\gamma}(y)} + \frac{\theta}{2}g(x,\hat{h}(y),\hat{\gamma}(y),r;\theta))[(\tilde{w}(y))] = 0 \ \forall \ y \in \mathcal{O}$;

4. $(\tilde{w}(T,X_T)) = f^{-\theta/2}$.

Define,

$$\tilde{Z}(s) = \tilde{Z}_{(s)}(h,\gamma) = \frac{\theta}{2}\bigg\{ \int_0^s g(X_\tau, h_\tau, \gamma_\tau, r_{t+\tau};\theta)d\tau \bigg\}. \tag{4.1}$$

5. $E^{h,\gamma}[\int_0^{T-t} D\tilde{w}'(t+s,X_s)\Lambda e^{\tilde{Z}_s} dW_s^{h,\gamma}] = 0 \ \forall \ h \in \mathbb{R}^m, \forall \ \gamma \in \mathbb{R}^{m+n}$

Now, define for each $y \in \mathcal{O}$ and $h \in \mathcal{H}(T)$ and $\gamma \in \Gamma(T)$,

$$\begin{aligned}\tilde{I}(f,x,h,\gamma,t,T) &= \exp(-\frac{\theta}{2}I(f,x,h,\gamma,t,T)) \\ &= E^{h,\gamma}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,h_s,\gamma_s,r_{s+t};\theta)ds\}f^{-\theta/2}].\end{aligned}$$

Then $(\hat{h}(y), \hat{\gamma}(y))$ is an optimal (Markov) control i.e.,

$$
\begin{aligned}
\tilde{w}(0,x) = \tilde{u}(0,x) = \tilde{I}(f,x,\hat{h},\gamma,0,T) &= \inf_{h\in\mathcal{H}(T)}\{\sup_{\gamma\in\Gamma(T)}[\tilde{I}(f,x,h,\gamma,0,T)]\} \\
&= \sup_{\gamma\in\Gamma(T)}\{\inf_{h\in\mathcal{H}(T)}[\tilde{I}(f,x,h,\gamma,0,T)]\} \\
&= \sup_{\gamma\in\Gamma(T)} \tilde{I}(f,x,\hat{h},\gamma,0,T) \\
&= \inf_{h\in\mathcal{H}(T)} \tilde{I}(f,x,h,\hat{\gamma},0,T) = \tilde{I}(f,x,\hat{h},\hat{\gamma},0,T).
\end{aligned}
$$

Proof Apply Ito's formula to $\tilde{w}(s,X_s)e^{\tilde{Z}_s}$ to obtain

$$
\begin{aligned}
d(\tilde{w}(t+s,X_s)e^{\tilde{Z}_s}) &= \left[e^{\tilde{Z}_s}(\tilde{\mathcal{A}}^{h,\gamma} + \frac{\theta}{2}g(X_s,h_s,\gamma_s,r_{s+t};\theta))\right][(\tilde{w}(t+s,X_s))]ds \\
&+ e^{\tilde{Z}_s}(D\tilde{w}(t+s,X_s))dW_s^{h,\gamma}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{w}(T,X_{T-t})e^{\tilde{Z}_{T-t}} &= \tilde{w}(t,x) + \int_0^{T-t}((\tilde{\mathcal{A}}^{h,\gamma} + \frac{\theta}{2}g(X_s,h_s,\gamma_s,r_{s+t};\theta))\tilde{w}(t+s,X_s))e^{\tilde{Z}_s}ds \\
&+ \int_0^{T-t}(D\tilde{w}^{'}(t+s,X_s)\Lambda)e^{\tilde{Z}_s}dW_s^{h,\gamma}. \qquad (4.2)
\end{aligned}
$$

From condition(4) of statement of the Proposition 1, we have $\tilde{w}(T,X_T) = f^{-\theta/2}$. Taking expectation with respect to $\mathbb{P}^{h,\gamma}$, setting $t=0$ and using conditions (1) and (5) of the Proposition 1 we get

$$
E^{h,\Gamma}[\tilde{w}(T,X_T)e^{\tilde{Z}_T}] \geq \tilde{w}(0,x)
$$

Since this inequality is true for all $h \in \mathcal{H}(T)$ we have

$$
\inf_{h\in\mathcal{H}(T)} E^{h,\Gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \geq \tilde{w}(0,x).
$$

Hence we have,

$$
\sup_{\gamma\in\Gamma(T)}\inf_{h\in\mathcal{H}(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \geq \inf_{h\in\mathcal{H}(T)} E^{h,\Gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \geq \tilde{w}(0,x). \qquad (4.3)
$$

Similarly, setting $t=0$ we get, using condition (2) of the Proposition 1, we get the following lower bound,

$$E^{\hat{h},\gamma}[\tilde{w}(T,X_T)e^{\tilde{Z}_T}] \leq \tilde{w}(0,x).$$

Since this inequality is true for all $\gamma \in \Gamma(T)$ we have

$$\sup_{\gamma\in\Gamma(T)} E^{\hat{h},\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \leq \tilde{w}(0,x).$$

Hence we have,

$$\inf_{h\in\mathcal{H}(T)} \sup_{\gamma\in\Gamma(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \leq \sup_{\gamma\in\Gamma(T)} E^{\hat{h},\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \leq \tilde{w}(0,x). \tag{4.4}$$

Also , setting $t = 0$ and using condition (3) of the Proposition 1 and using the definition of $\tilde{u}$ in (3.11) we get,

$$\begin{aligned} E^{\hat{h},\hat{\gamma}}[\tilde{w}(T,X_T)e^{\tilde{Z}_T}] &= \tilde{w}(0,x) \\ &= E^{\hat{h},\hat{\gamma}}[\exp\{\frac{\theta}{2}\int_0^T g(X_s,\hat{h}_s,\hat{\gamma}_s,r_{s+t};\theta)ds\}f^{-\theta/2}]. \end{aligned} \tag{4.5}$$

It is automaticaly true that

$$\sup_{\gamma\in\Gamma(T)} \inf_{h\in\mathcal{H}(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \leq \inf_{h\in\mathcal{H}(T)} \sup_{\gamma\in\Gamma(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}]. \tag{4.6}$$

Conversely, from (4.3), (4.4) and (4.5) we have,

$$\inf_{h\in\mathcal{H}(T)} \sup_{\gamma\in\Gamma(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \quad \leq \quad \tilde{w}(0,x) \leq \sup_{\gamma\in\Gamma(T)} \inf_{h\in\mathcal{H}(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}]. \tag{4.7}$$

Hence from (4.6) and (4.7) we have,

$$\begin{aligned} \sup_{\gamma\in\Gamma(T)} \inf_{h\in\mathcal{H}(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] &= \inf_{h\in\mathcal{H}(T)} \sup_{\gamma\in\Gamma(T)} E^{h,\gamma}[f^{-\theta/2}e^{\tilde{Z}_T}] \\ &= \tilde{w}(0,x) = E^{\hat{h},\hat{\gamma}}[f^{-\theta/2}e^{\tilde{Z}_T}]. \end{aligned} \tag{4.8}$$

This completes the proof.

Corollary 2. *Admissible(optimal) strategies for the exponentially transformed prob-*

lem given by (3.11) are also admissible(optimal) for the problem (3.10c). Formally,

$$\begin{aligned} u(0,x) &= \sup_{h\in\mathcal{H}(T)}\{\inf_{\gamma\in\Gamma(T)}[I(f,x,h,\gamma,0,T)]\}, \\ &= \inf_{\gamma\in\Gamma(T)}\{\sup_{h\in\mathcal{H}(T)}[I(f,x,h,\gamma,0,T)]\}, \\ &= \inf_{\gamma\in\Gamma(T)} I(f,x,\hat{h},\gamma,0,T), \\ &= \sup_{h\in\mathcal{H}(T)} I(f,x,h,\hat{\gamma},0,T) = I(f,x,\hat{h},\hat{\gamma},0,T). \end{aligned}$$

Proof The value function u and $\tilde{u}$ are related through the strictly monotone continuous transformation $\tilde{u}(t,x) = \exp(-\frac{\theta}{2}u(t,x))$. Thus admissible (optimal) strategies for the exponentially transformed problem are also admissible(optimal) for the problem (3.10).

5 Solving the risk-sensitive zero sum stochastic differential game

Step 5

We seek to find the value function u for the game defined in (3.12). We guess a solution assuming that it belongs to the class $C^{1,2}((0,T)\times\mathbb{R}^n)$ and show that the guess satisfies all the conditions of our verification lemma given by Proposition. Conditions (1)-(4) of the verification lemma that forms our Proposition 1 can be written in a compact form as

$$\sup_{h\in\mathcal{H}(T)}\inf_{\gamma\in\Gamma(T)} \mathcal{A}^{h,\gamma}u(t,x) = 0; \quad u(T,x) = \log f. \tag{5.1}$$

Motivated by the results in Kuroda and Nagai [26], we will look for a u given by $u(t,x) = \frac{1}{2}x'Q_t x + q_t' x + k_t$ where Q is an $n\times n$ symmetric matrix, $q\in\mathbb{R}^n$ and k is a scalar. Substituting this form in (3.15) we get

$$\begin{aligned} \mathcal{A}^{h,\gamma}u(t,x) &= \frac{1}{2}x'\frac{dQ_t}{dt}x + \frac{dq_t}{dt}'x + \frac{dk_t}{dt} + \left(b + Bx - \frac{\theta}{2}\Lambda(\Sigma' h_t - \gamma(t))\right)'(Q_t x + q_t), \\ &+ \frac{1}{2}(\Lambda\Lambda' Q_t Q_t' \Lambda'\Lambda) - \frac{\theta}{4}(Q_t x + k_t)'\Lambda\Lambda'(Q_t x + k_t), \\ &- \frac{1}{2}(\frac{\theta}{2}+1)h_t'\Sigma\Sigma' h_t + r_t - (\alpha_t + \beta_t x) + h_t'(a + Ax - r_t 1) \\ &+ \frac{1}{2}\frac{\theta}{2}(h_t'\Sigma\gamma + \gamma'\Sigma' h_t) - \frac{1}{2}(\frac{\theta}{2}-1)\gamma_t'\gamma_t. \end{aligned} \tag{5.2}$$

We now solve the first order condition for $\hat{\gamma}$ to minimize $\mathcal{A}^{\hat{h},\gamma}u(t,x)$ over all $\gamma \in \mathbb{R}^{n+m}$:

$$(2-\theta)\hat{\gamma}_t - \theta(\Sigma^{'}\hat{h}_t - \hat{\gamma}^{'})Du(t,x) = 0. \tag{5.3}$$

The first order condition for $\hat{h}$ that maximizes $\mathcal{A}^{h,\hat{\gamma}(y)}\tilde{u}(t,x)$ over all $h \in \mathbb{R}^m$ in terms of $u(t,x)$ is,

$$\hat{h}_t = \frac{2}{(\theta+2)}(\Sigma\Sigma^{'})^{-1}[d_t + \frac{\theta}{2}\Sigma\hat{\gamma}_t - \frac{\theta}{2}\Sigma\Lambda^{'}Du(t,x)]. \tag{5.4}$$

Substituting back $\hat{h}$ obtained in (5.4) into (5.3) we get

$$\hat{\gamma}_t = \frac{\theta}{2-\theta}[\Sigma^{'}\hat{h}_t - \Lambda^{'}Du(t,x)] \tag{5.5}$$

Comment on the valid range of θ.
Since the game considered is for the risk-averse investor $\theta > 0$. Moreover based in the expression for $\hat{\gamma}$ in (5.5), $\theta \neq 2$. This leaves for two possibilities: $\theta \in (0,2)$ or $\theta \in (2,\infty)$. For the optimal strategies $(\hat{h},\hat{\gamma})$ to be a saddle-point equilibrium for the game, we would desire that the equation with the quadratic term in h be negative definite while the quadratic term in γ be positive definite. In fact for the choice $\theta > 0$, the quadratic term in h desirably is negative definite while for $\theta < 2$, the quadratic term in γ is positive definite . Hence for our case, the valid range of θ is between 0 and 2.
The optimal control $\hat{h}_t$ is a global maximum while $\hat{\gamma}_t$ is a global minimum for $t \leq [0,T]$. We re-write $\hat{h}_t$ and $\hat{\gamma}_t$ only in terms of d and u and are given as,

$$\begin{aligned}
\hat{h}_t &= \frac{2-\theta}{2-\theta^2}(\Sigma\Sigma^{'})^{-1}d_t - \frac{\theta}{2-\theta^2}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}Du(t,x). \\
\hat{\gamma}_t &= \frac{\theta}{2-\theta^2}\Sigma^{'}(\Sigma\Sigma^{'})^{-1}d_t - \frac{\theta^2}{(2-\theta)(2-\theta^2)}\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}Du \\
&- \frac{\theta}{2-\theta}\Lambda^{'}Du.
\end{aligned}$$

Based on the above expressions of $\hat{h}.$ and $\hat{\gamma}.$, one can say rather explicitly how strategies compete vis-a-vis when $\Sigma\Lambda^{'} = 0$. In that situation, a quick calculation follows that the investor invests less and less in the stocks when $\theta \to 1$ for the valid range $\theta \in (0,1)$. On the contrary investor's allocation in the risky stocks increases for the valid range $\theta \in (-2,0) - \{-\sqrt{2}\}$ as θ gets more negative. From the perspective of risk-sensitive optimization, this is expected. Likewise we have the following relation between the volatility coefficient of the portfolio wealth process and the benchmark process namely $\Sigma^{'}\hat{h}$ and $\hat{\gamma}$ respectively which is

$$\Sigma\hat{h}_t = \frac{2-\theta}{\theta}\hat{\gamma}_t.$$

For $-2 < \theta < 1$, volatility of the portfolio wealth process is higher than the benchmark process in magnitude. Contrary effect is seen for $1 < \theta < 2$.

We substitute $\hat{h}$ from (5.4) and $\hat{\gamma}$ from (5.5) in (5.1) to obtain

$$\mathcal{A}^{\hat{h},\hat{\gamma}}u(t,x) = 0; \qquad u(T,x) = \log f \tag{5.6}$$

We then group all the resulting quadratic terms in x, linear terms in x and constants together to conclude that the choice of $u(t,x) = \frac{1}{2}x^{'}Q_t x + q_t^{'}x + k_t$ is indeed the solution to the HJBI PDE (5.1) provided that Q, q and k satisfy the following system of differential equations:

• a matrix Ricatti equation related to the coefficient of the quadratic term and used to determine the symmetric non-negative matrix Q_t, given as

$$\begin{aligned} \frac{dQ_t}{dt} &= Q_tK_0Q_t + K_1^{'}Q_t + Q_tK_1 + 2\frac{2-\theta}{(2-\theta^2)^2}A^{'}(\Sigma\Sigma^{-1})^{-1}A = 0 \quad 0 \le t \le T, \\ Q_T &= 0 \end{aligned} \tag{5.7}$$

where

$K_0 = \frac{-\theta^2}{2(2-\theta)}\Lambda\Lambda^{'} + \frac{2\theta^2}{(2-\theta)(2-\theta^2)^2}\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}$

$K_1 = B - \frac{2\theta}{(2-\theta^2)^2}A^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}$

• The following linear ordinary differential equation satisfied by the n element column vector $q(t)$

$$\begin{aligned} \frac{dq_t}{dt} &+ (K_1^{'} + Q_tK_0)q_t + Q_t^{'}b + (a - r_t1)^{'}(\Sigma\Sigma^{'})^{-1}[\frac{-2\theta}{(2-\theta^2)^2}\Sigma\Lambda^{'}Q(t) + \frac{(2-\theta)}{(2-\theta^2)^2}A] \\ &- \beta_t = 0 \\ &q_T = 0 \end{aligned} \tag{5.8}$$

- The following linear ordinary differential equation satisfied by the constant k_t

$$\begin{aligned}
& \frac{dk_t}{dt}+\frac{1}{2}tr(\Lambda\Lambda^{'}Q_t)+r_t-\alpha_t-\frac{2\theta}{(2-\theta^2)^2}(a-r_t1)^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t)\\
+ & \frac{2-\theta}{(2-\theta^2)^2}(a-r_t1)^{-1}(\Sigma\Sigma^{'})^{-1}(a-r_t1)\\
+ & \frac{\theta^2}{(2-\theta)(2-\theta^2)^2}q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t)\\
- & \frac{\theta^2}{4(2-\theta)}q^{'}(t)\Lambda\Lambda^{'}q(t)=0\\
k_T = & \log f
\end{aligned} \tag{5.9}$$

Condition 4 of Proposition 1 in terms of u imposes the terminal condition in (5.9).

If K_0 is positive definite then a unique solution to the Riccati equation (5.7), Q_t , exists for all $t \leq T$. This property of positive definiteness follows from interpretation of the solution Q_t as the covariance matrix of observations from a Kalman filter used to estimate the state of a dynamical system (see Theorem 4.4.1 in Davis [7]) for details. The uniqueness property of Q_t follows from the standard existence-uniqueness theorem for first order differential equations (see Proposition 4.4.2 in Davis [7]).

It remains to be seen if $\tilde{u}=\exp(-\frac{\theta}{2}u)$ for the choice of u satisfies condition (5) of the Proposition .

Lemma 3. $E^{h,\gamma}[\int_0^{T-t} e^{\tilde{Z}_s}(D\tilde{u}^{'}(t+s,X_s)\Lambda)dW_s^{h,\gamma}]=0.$

Proof From the definition of $\tilde{u}$ in (3.11), for any optimal control belonging to $\Gamma(T)$, the strategy $\hat{h}\equiv 0$ is sub-optimal, and hence will provide an upper bound on $\tilde{u}$. Further for the zero-benchmark case namely, $\hat{\gamma}\equiv 0$, we would obtain now an upper bound on $\tilde{u}$

$$\begin{aligned}
\tilde{u}(t,x) &= \inf_{h\in\mathcal{H}(T)} E^{h,\hat{\gamma}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,h_s,\hat{\gamma}_s,r_{s+t};\theta)ds\}f^{-\theta/2}],\\
&\leq E^{0,\hat{\gamma}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,0,\hat{\gamma}_s,r_{s+t};\theta)ds\}f^{-\theta/2}],\\
\therefore \tilde{u}(t,x) &\leq E^{0,0}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X_s,0,0,r_{s+t};\theta)ds\}f^{-\theta/2}],\\
&= \exp(-\frac{\theta}{2}\int_0^{T-t} r_{s+t}ds)f^{-\theta/2}.
\end{aligned}$$

Now Q and q are solutions to the system of o.d.e, and hence are integrals of bounded functions . Hence Q and q are continuous functions of time $t\in[0,T]$ and hence

bounded on $[0,T]$. The matrix Λ is a known constant. From standard existence-uniqueness result of stochastic differential equation (refer Oksendal ([30])) we have $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}^{h,\gamma})$. Hence from the upper bound on $\tilde{u}$ and the fact that $Du(t, X_t) = Q_tX_t + q_t$ is in $L^2(\Omega, \mathcal{F}, \mathbb{P}^{h,\gamma})$, we have that $E^{h,\gamma}([D\tilde{u}\ \Lambda e^Z, D\tilde{u}\ \Lambda e^Z]_t) < \infty\ \forall t \in [0,T]$. Hence we have $E^{h,\gamma}[\int_0^{T-t} D\tilde{u}'(t+s, X_s)\Lambda e^{\tilde{Z}_s} dW_s^{h,\gamma}] = 0$. .
It is clear that our guess for $\tilde{u} = \exp(-\frac{\theta}{2}u)$ satisfies conditions (1)-(5) of Proposition 1. Hence our choice of $\tilde{u}$ indeed is the value of the game (**P2**) and controls $\hat{h}, \hat{\gamma}$ are the saddle point equilibrium of this game.

Corollary 4. *For the choice of space of controls $\mathcal{H}(T)$ and $\Gamma(T)$, we have*

$$E[\mathcal{E}\left(-\frac{\theta}{2}\int_0 [(Q_tX_t + q_t)\Lambda + (h_t'\Sigma - \gamma_t')]dW_t\right)_T] = 1. \tag{5.10}$$

Proof: From the Kazamaki condition, refer (Oksendal [30]), (5.10) holds if $E[\exp(\int_0^t \theta(\frac{(Q_sX_s+q_s)\Lambda + (h_s'\Sigma - \gamma_s')}{2})dW_s)] < \infty\ \forall\ t \in [0,T]$. Hence by application of Cauchy-Schwartz inequality we have,

$$\begin{aligned} & E[\exp(\int_0^t \theta(\frac{(Q_sX_s + q_s)\Lambda + (h_s'\Sigma - \gamma_s')}{2})dW_s)], \\ \leq\ & (E[e^{\int_0^t \theta(Q_sX_s+q_s)\Lambda dW_s}])^{1/2}(E[e^{\int_0^t \theta(h_s'\Sigma - \gamma_s')dW_s}])^{1/2}. \end{aligned}$$

However for $E[e^{\int_0^t \theta(Q_sX_s+q_s)\Lambda dW_s}] < \infty$ to hold , it is enough to show that the Novikov condition given by $E[e^{\int_0^T \theta^2(Q_sX_s+q_s)\Lambda\Lambda'(Q_sX_s+q_s)ds}] < \infty$ hold; refer (Oksendal [30]). Since X is Gaussian process and Q_t and q_t are deterministic, $(Q_tX_t + q_t)\Lambda$ is Gaussian and hence by completion of squares argument detailed in Theorem 4 below we have $E[e^{\int_0^T \theta^2(Q_sX_s+q_s)\Lambda\Lambda'(Q_sX_s+q_s)ds}] < \infty$ holds and hence $E[e^{\int_0^t \theta(Q_sX_s+q_s)\Lambda dW_s}] < \infty\ \forall t \in [0,T]$ is validated. $(E[e^{\int_0^t \theta(h_s'\Sigma - \gamma_s')dW_s}])^{1/2} < \infty$ is validated from similar application of Cauchy-Schwartz inequality followed by the assumption made earlier in the definition of the space of controls $\mathcal{H}(T)$ and $\Gamma(T)$. Thus the Kazamaki condition holds and the conclusion follows.

Theorem 5. *If there exist a solution Q to (5.7), then the strategies $(\hat{h}, \hat{\gamma})$ defined by*

$$\hat{h}_t = \frac{2}{(\theta+2)}(\Sigma\Sigma')^{-1}[d_t + \frac{\theta}{2}\Sigma\gamma_t - \frac{\theta}{2}\Sigma\Lambda'(Q_tX_t + q_t)]. \tag{5.11}$$

$$\hat{\gamma}_t = \frac{\theta}{2-\theta}[\Sigma'\hat{h}_t - \Lambda'(Q_tX_t + q_t)], \tag{5.12}$$

*where q is a solution of (5.8) are admissible i.e. $h \in \mathcal{H}(T)$ and $\gamma \in \Gamma(T)$ and are optimal for the finite horizon game problem (**P1**), namely,*

$$\begin{aligned}
u(0,x) &= \sup_{h\in\mathcal{H}(T)} \inf_{\gamma\in\Gamma(T)} J(f,x,h,\gamma,T;\theta),\\
&= \inf_{\gamma\in\Gamma(T)} \sup_{h\in\mathcal{H}(T)} J(f,x,h,\gamma,T;\theta),\\
&= \inf_{\gamma\in\Gamma(T)} J(f,x,\hat{h},\gamma,T;\theta),\\
&= \sup_{h\in\mathcal{H}(T)} J(f,x,h,\hat{\gamma},T;\theta),\\
&= J(f,x,\hat{h},\hat{\gamma},T;\theta),\\
&= \frac{1}{2}x^{'}Q_0x + q_0^{'}x + k_0.
\end{aligned}$$

Proof The controls derived in section 5, $(\hat{h},\hat{\gamma})$ forms the saddle point equilibrium for the (**P2**) game . We aim to show that these controls are in fact admissible and optimal for the problem (**P1**) as well.

Proof of admissibility From the expression for $\hat{h}$ and $\hat{\gamma}$ in (5.4) and (5.5) respectively we note that $-\frac{\theta}{2}\Bigg((Q_tX_t+q_t)\Lambda+(\hat{h}_t^{'}\Sigma-\hat{\gamma}_t^{'})\Bigg)$ can be written linearly in X_t as $X_t^{'}v_t^1+v_t^2$ where, constants v_t^1 and v_t^2 are given by,

$$\begin{aligned}
v_t^1 &= -\frac{\theta}{2}Q^{'}(t)\Lambda + \frac{\theta(\theta-1)}{(2-\theta^2)}A^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'} + \frac{\theta(\theta-1)}{2-\theta^2}Q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}(a-r1)\\
&- \frac{2\theta^2(\theta-1)}{(2-\theta)(2-\theta^2)}Q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t) - \frac{\theta^2}{(2-\theta)}Q^{'}(t)\Lambda\Lambda^{'}q(t).\\
v_t^2 &= -\frac{\theta}{2}q^{'}(t)\Lambda + \frac{\theta(\theta-1)}{(2-\theta^2)}(a-r1)^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t)\\
&- \frac{\theta^2(\theta-1)}{(2-\theta)(2-\theta^2)}q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t) - \frac{\theta^2}{(2-\theta)}q^{'}(t)\Lambda\Lambda^{'}q(t)
\end{aligned}$$

Since X satisfies the SDE , $dX_t = (b+BX_t)dt + \Lambda dW_t, so E|X_t| \leq E|X(0)| + |b|T + |B|\int_0^t E|X_s|ds$. By Gronwall's inequality, therefore $E|X_t| \leq (E|X(0)| + |b|T)\exp(|B|t)$ and $Cov(X_t) = \Lambda^{'}\Lambda t$. Let $\phi(t) \triangleq v_t^1X_t + v_t^2$. We now explicitly calculate $E[e^{\delta|\phi_t|^2}]$ for some $\delta > 0$ since from Remark 2 in Lemma 2, of section 12 (Gihman and Skorokhod [22]) would imply that the Novikov's condition holds true. Let $R_t = e^{-Bt}X_t + e^{-bt}$. Hence $dR_t = e^{-Bt}\Lambda dW_t$. Therefore R_t is a Gaussian process and hence ϕ_t is Gaussian process with drift. Also $\mu_t = E[|\phi_t|] \leq \sup_{0\leq t\leq T}|v_t^1|(E|X_0| + |b|T)\exp(|B|t) + \sup_{0\leq t\leq T}|v_t^2|$ and $\tilde{\Sigma}_t = Cov(\phi_t) \leq {v^1}_t^{'}\Lambda^{'}\Lambda v_t^1$. Thus mean μ_t and co-variance $\tilde{\Sigma}_t$ are bounded above by t. We use the following completion of squares argument: $\frac{1}{2}z^{'}Az+b^{'}z+c = \frac{1}{2}(z+A^{-1}b)^{'}A(z+A^{-1}b)+c-\frac{1}{2}b^{'}A^{-1}b$

.

$$
\begin{aligned}
E[e^{\delta|\phi_t|^2}] &= \int_{\mathbb{R}^n} \frac{1}{2\pi^{n/2}|\tilde{\Sigma}_t\tilde{\Sigma}_t^{'}|^{1/2}} e^{\delta|\phi|_t^2} e^{-\frac{1}{2}(\phi-\mu_t)^{'}(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}(\phi-\mu_t)dx^1dx^2...dx^n}, \\
&= \frac{1}{2\pi^{n/2}|\tilde{\Sigma}_t\tilde{\Sigma}_t^{'}|^{1/2}} \int_{\mathbb{R}^n} e^{\frac{-\phi^{'}(-2\delta I+(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1})^{-1}\phi+2\mu^{'}(t)(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}\phi-\mu_t^{'}(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}\mu_t}{2}} dx^1....dx^n, \\
&= \frac{|(\tilde{\Sigma}_t^{'}\tilde{\Sigma}_t)|^{-1/2}}{|(-2\delta I+(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1})^{-1}|^{-1/2}} \times, \\
& \quad e^{\frac{-\mu_t^{'}(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}\mu_t+4\mu_t^{'}(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}(-2\delta I+(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1})^{-1}(\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1}\mu_t}{2}}.
\end{aligned}
$$

Matrix $(\tilde{\Sigma}_t\tilde{\Sigma}_t)^{-1}$ is symmetric positive definite with lowest eigenvalue say λ_{min}. Then it is easy to show that for $\delta < \frac{\lambda_{min}}{2}$, matrix $(-2\delta I + (\tilde{\Sigma}_t\tilde{\Sigma}_t^{'})^{-1})^{-1}$ is positive definite . Along with the derived fact that μ_t and $\tilde{\Sigma}_t$ is bounded above by $t \leq T$, hence there exists some constant C such that $E[e^{\delta|\phi_t|^2}] \leq C$. Hence the optimal controls $\hat{h}, \gamma$ belong to their respective admissible class viz. $\mathcal{H}(T)$ and $\Gamma(T)$ respectively.

Proof of optimality Define,

$$
\begin{aligned}
Z_s &= Z_s(h,\gamma) = \frac{\theta}{2}\bigg\{ \int_0^s g(X_\tau, h_\tau, \gamma_\tau, r_{t+\tau}; \theta)d\tau - (h_\tau^{'}\Sigma - \gamma_\tau^{'})dW_\tau, \\
&- \frac{\theta}{4}(h_\tau^{'}\Sigma - \gamma_\tau^{'})^{'}(h_\tau^{'}\Sigma - \gamma_\tau^{'})d\tau\bigg\}. \qquad (5.13)
\end{aligned}
$$

Also define, $\chi(t,x) = -\frac{\theta}{2}(u(t,x) - \log f)$ and $Lu(t,x) = \frac{1}{2}tr(\Lambda\Lambda^{'}D^2u(t,x)) + (b + Bx)^{'}Du(t,x)$

Hence, we have

$$
d\chi(t+s, X_s) = -\frac{\theta}{2}(\frac{\partial u}{\partial t} + Lu)(t+s, X_s)ds - \frac{\theta}{2}Du(t+s, X_s)^{'}\Lambda dW_s.
$$

Hence,

$$
\begin{aligned}
\frac{d\exp\{\chi(t+s, X_s)\}}{\exp\{\chi(t+s, X_s)\}} &= -\frac{\theta}{2}(\frac{\partial u}{\partial t}(t,x) + Lu)(t+s, X_s) - \frac{\theta}{2}Du(t+s, X_s)^{'}\Lambda dW_s \\
&+ \frac{\theta^2}{8}Du^{'}\Lambda\Lambda^{'}Du(t+s, X_s)ds.
\end{aligned}
$$

and so,

$$\frac{d\exp\{\chi(t+s,X_s)\}\exp\{Z(s)\}}{\exp\{\chi(t+s,X_s)\}\exp\{Z(s)\}} = -\frac{\theta}{2}(\frac{\partial u}{\partial t}(t,x)+Lu)(t+s,X_s)-\frac{\theta}{2}Du(t+s,X_s)^{'}\Lambda dW_s,$$
$$+ \frac{\theta^2}{8}Du^{'}\Lambda\Lambda^{'}Du(t+s,X_s)ds+\frac{\theta}{2}g(X_s,h_s,\gamma_s,r_s+t;\theta)ds,$$
$$- \frac{\theta}{2}(h^{'}(s)\Sigma-\gamma^{'}(s))dW_s+\frac{\theta^2}{4}(h^{'}(s)\Sigma-\gamma^{'}(s))\Lambda^{'}Du(t+s,X_s)ds.$$

Hence from (3.15), we have,

$$\exp\{\chi(T,X(T-t))+Z(T-t)\} = \exp(\chi(t,x))\exp\Bigg[\int_0^{T-t}-\frac{\theta}{2}(\mathcal{A}^{h,\gamma}u(t+s,X_s))ds,$$
$$- \int_0^{T-t}\frac{\theta}{2}[Du(t+s,X_s)^{'}\Lambda+(h_t^{'}\Sigma-\gamma_t^{'})]dW_t,$$
$$- \int_0^{T-t}\frac{\theta^2}{8}[Du(t+s,X_s)^{'}+(h_t^{'}\Sigma-\gamma_t^{'})][Du(t+s,X_s)^{'}+(h_t^{'}\Sigma-\gamma_t^{'})]^{'}ds\Bigg]. \tag{5.14}$$

We have shown that u satisfies conditions (1)-(5) of Proposition 1. Hence from condition(4) of the Proposition 1, we have $\chi(T,x)=0$. Now setting $t=0$ and taking condition (1) of the Proposition 1 into account for $\gamma=\hat{\gamma}$, and for any $h\in\hat{\mathcal{H}}(T)$ we see from (5.14) that

$$(\frac{V_T^h}{L_T^\Gamma})^{-\theta/2} \geq e^{-\frac{\theta}{2}u(0,x)}\exp\Bigg[-\int_0^T\frac{\theta}{2}[Du(s,X_s)^{'}\Lambda+(h_s^{'}\Sigma-\Gamma_s^{'})]dW_s$$
$$-\int_0^T\frac{\theta^2}{8}[Du(s,X_s)^{'}+(h_s^{'}\Sigma-\gamma_s^{'})][Du(s,X_s)^{'}+(h_s^{'}\Sigma-\Gamma_s^{'})]^{'}ds\Bigg].$$

Now by taking expectations w.r.t to the physical probability measure $\mathbb{P}$ on both sides of above equation and using corollary , we obtain

$$J(f,x,h,\gamma,T)\leq u(0,x).$$

This inequality is true for all $h\in\mathcal{H}(T)$ so we have,

$$\sup_{h\in\mathcal{H}(T)}J(f,x,h,\gamma,T)\leq u(0,x).$$

Hence we have,

$$\inf_{\gamma\in\Gamma(T)}\sup_{h\in\mathcal{H}(T)}J(f,x,h,\gamma,T)\leq\sup_{h\in\mathcal{H}(T)}J(f,x,h,\gamma,T)\leq u(0,x). \tag{5.15}$$

Likewise, setting $t = 0$ and taking condition (2) of the Proposition 1 into account for $h = \hat{h}$, and for any $\gamma \in \Gamma(T)$ we see that

$$J(f, x, \hat{h}, \gamma, T) \geq u(0, x).$$

This inequality is true for all $h \in \mathcal{H}(T)$ so:

$$\inf_{\gamma \in \Gamma(T)} J(f, x, \hat{h}, \gamma, T) \geq u(0, x).$$

Hence we have,

$$\sup_{h \in \mathcal{H}(T)} \inf_{\gamma \in \Gamma(T)} J(f, x, h, \gamma, T) \geq \inf_{\gamma \in \Gamma(T)} J(f, x, \hat{h}, \gamma, T) \geq u(0, x). \tag{5.16}$$

Hence from (5.15) and (5.16) we have,

$$\sup_{h \in \mathcal{H}(T)} \inf_{\gamma \in \Gamma(T)} J(f, x, h, \gamma, T) \geq u(0, x) \geq \inf_{\gamma \in \Gamma(T)} \sup_{h \in \mathcal{H}(T)} J(f, x, h, \gamma, T). \tag{5.17}$$

Moreover, setting $t = 0$ and taking condition (3) of the Proposition 1 into account for $h = \hat{h}, \gamma = \hat{\gamma}$ (such that $\hat{h} \in \mathcal{H}(T)$ and $\hat{\gamma} \in \Gamma(T)$) we see that

$$J(f, x, \hat{h}, \hat{\gamma}, T) = u(0, x). \tag{5.18}$$

It is always true that

$$\sup_{h \in \mathcal{H}(T)} \left(\inf_{\gamma \in \Gamma(T)} J(f, x, h, \gamma, T) \right) \leq \inf_{\gamma \in \Gamma(T)} \left(\sup_{h \in \mathcal{H}(T)} J(f, x, h, \gamma, T) \right). \tag{5.19}$$

Hence combining (5.17) and (5.19) we deduce the final conclusion that the game (**P1**) has a value and is $u(0, x)$.

6 Conclusion

In this chapter we provide a two player zero sum stochastic differential game in the context of the risk-sensitive benchmark asset management problem. We obtain an explicit expression for the optimal strategies for both the players. Future work could be directed towards considering a game theoretic benchmark problem with infinite horizon risk sensitive criterion.

Chapter 3

On the role of the Föllmer-Schweizer minimal martingale measure in risk sensitive control asset management.

Abstract

Kuroda and Nagai [26] state that the factor process in the Risk Sensitive control Asset Management (RSCAM) is stable under the Föllmer and Schweizer minimal martingale measure (MMM) . Fleming and Sheu [15] and more recently Föllmer and Schweizer [18] have observed that the role of the minimal martingale measure in this portfolio optimization is yet to be established. In this note we aim to address this question by explicitly connecting the optimal wealth allocation to the minimal martingale measure. We achieve this by using a "trick" of setting this problem in the context of model uncertainty via a two-person, zero-sum, stochastic differential game between the investor and an antagonistic market that provides a probability measure. We obtain some startling insights. Firstly, if short-selling is not permitted and if the factor process evolves under the minimal martingale measure then the investor's optimal strategy can only be to invest in the riskless asset (i.e. the no-regret strategy). Secondly, if the factor process and the stock price process have independent noise, then even if the market allows short selling, the optimal strategy for the investor must be the no-regret strategy while the factor process will evolve under the minimal martingale measure.

1 Introduction

Risk sensitive control Asset Management (RSCAM) balances the investors interest in maximizing the expected growth rate of wealth against his aversion to risk due to deviations of the actually realized rate from the expectation for a finite time horizon. The subjective notion of investor's risk aversion is parameterized by a single variable, say θ. In RSCAM we consider the following criterion to be maximized. For a given $\theta > -2, \theta \neq 0$ and for time horizon $T < \infty$, with wealth allocation control denoted by $h(t)$, the risk-sensitive expected growth rate up to time horizon T criterion $J(v, h, T; \theta)$ defined by,

$$J(v, h, T; \theta) \triangleq \frac{-2}{\theta} \log E[\exp[\frac{-\theta}{2} \log V^h(T)]], \tag{1.1}$$

where $V^h(T)$ is time-T portfolio value. An asymptotic expansion around $\theta = 0$ for the above criterion yields

$$J(v, h, T; \theta) = E[V^h(T)] - \frac{\theta}{2} Var(V^h(T)) + O(\theta^2); \quad V^h(0) = v.$$

As is obvious from the preceding equation, $\theta > 0$ corresponds to a risk-averse investor, $\theta < 0$ to a risk seeking investor and $\theta = 0$ to a risk-neutral investor. Hence the optimal expected utility function depends on θ and is a generalization of the traditional stochastic control in the sense that now the degree of risk aversion of the investor is explicitly parameterized through θ rather than importing it in the problem via an externally defined utility function. For this reason investment optimization models have been popularly reformulated as risk-sensitive control problems. For a general reference on risk-sensitive control, refer Whittle [33].

Risk-sensitive control was first applied to solve financial problems by Lefebvre and Montulet [27] in a corporate finance context and by Fleming [14] in a portfolio selection context. A RSCAM problem with m securities and n (economic) factors was introduced by Bielecki and Pliska [2]. Their factor model however made a rather strong assumption that the factor process and the securities price process in their financial optimization model had independent noise. A generalization to this model, relaxing this assumption was made by Kuroda and Nagai [26], who introduced an elegant solution method based on a change of measure argument which transforms the risk sensitive control problem into a linear exponential of a quadratic regulator. They solved the associated HJB PDE over a finite time horizon and then studied the properties of the ergodic HJB PDE. We go about formally stating the problem by first describing the factor model for a risk averse investor.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be the filtered probability space. Consider a market of $m+1 \geq 2$ securities and $n \geq 1$ factors. We assume that the set of securities includes one bond whose price is governed by the ODE

$$dS^0(t) = r(t)S^0(t)dt, \quad S^0(0) = s^0, \tag{1.2}$$

where $r(t)$ is a deterministic function of t. The other security prices are assumed to satisfy the following SDE's,

$$dS^i(t) = S^i(t)\{(a + AX(t))^i dt + \sum_{k=1}^{n+m} \sigma^i_k dW^k(t)\}, S^i(0) = s^i, i = 1, ..., m., \tag{1.3}$$

where the component-wise factor process satisfies,

$$dX^i(t) \quad = \quad (b + BX(t))^i dt + \sum_{k=1}^{n+m} \lambda^i_k dW^k(t), \quad X^i(0) = x^i, i = 1, ..., n.$$

$X(t) = (X^1(t), ..., X^n(t))'$ (where the symbol $'$ signifies transpose) satisfies the following dynamics,

$$dX(t) \quad = \quad (b + BX(t))dt + \Lambda dW(t), X(0) = x \in \mathbb{R}^n. \tag{1.4}$$

Here, $W(t) = (W^k(t))_{k=1,...,n+m}$ is an $n+m$ dimensional standard Brownian motion defined on the filtered probability space. The model parameters A, B are respectively $m \times n, n \times n, n \times (m+n)$ constant matrices and $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$. The constant matrix $[\sigma^i_k] \triangleq \Sigma, i = 1, 2...., m; k = 1, 2, ..., (n+m)$. Similarly, $[\lambda^i_k] \triangleq \Lambda, i = 1, 2...., n; k = 1, 2, ..., (n+m)$. We denote the transpose of l by l'. Likewise let $|v|$ be a suitable vector norm for any vector v and let $||M||$ symbolize a suitable matrix norm for any matrix M. As discussed earlier, as part of generalizing the Bielecki and Pliska factor model [2], Kuroda and Nagai [26] assume that the factor process and the securities price process are correlated i.e. $\Sigma\Lambda' \neq 0$. The investment strategy which represents proportional allocation of total wealth in the i^{th} security $S^i(t)$ is denoted by $h^i(t)$ for $i = 0, 1, ..., m$ and we set, $S(t) := (S^1(t), S^2(t), ..., S^m(t))', h(t) := (h^1(t), ..., h^m(t))'$ and let $\mathcal{G}_t = \sigma(W(u); u \leq t)$ be the filtration generated by the standard Brownian motions that drives the stock price process and the factor process. Let $\mathcal{H}(T)$ be a space of $\mathbb{R}^m$ valued controls for the investor meaning we say that $h(t) \in \mathcal{H}(T)$ where $h(t)$ is $\mathcal{G}_t$-progressively measurable stochastic processes such that $\sum_{i=1}^{m} h^i(t) + h^0(t) = 1$, $P(\int_0^T |h(t)|^2 dt < \infty) = 1$ and $E[e^{\frac{\theta^2}{2}\int_0^T h_t' \Sigma\Sigma' h_t dt}] < \infty$. For given $h(t) \in \mathcal{H}(T)$ the process $V(t) = V^h(t)$ is determined by the SDE,

$$\frac{dV^h(t)}{V^h(t)} = h^0(t)r(t)dt + \sum_{i=1}^{m} h^i(t)\{(a + AX(t))^i dt + \sum_{k=1}^{m+n} \sigma_k^i dW^k(t)\}; \quad V^h(0) = v.$$

which can be written as,

$$\frac{dV^h(t)}{V^h(t)} = (r(t) + h^{'}(t)\delta(t))dt + h^{'}(t)\Sigma dW(t); \quad V^h(0) = v. \tag{1.5}$$

where $\delta(t) \triangleq a + AX(t) - r(t)1$. From the expression of security/stock price dynamics $S(t)$ (1.3), it is obvious that the market is incomplete (as it has m securities and $n+m$ Brownian drivers) and hence there exist many equivalent martingale measures or EMM's. We refer the reader to Karatzas and Shreve [25] for a general treatment on market incompleteness. One such candidate equivalent martingale measure is the Föllmer-Schweizer minimal martingale measure. For the continuous adapted stock price process $S = (S(t))_{0 \le t \le T}$, the minimal martingale measure $\mathbb{P}^*$ (say) is the unique equivalent local martingale measure with the property that the local $\mathbb{P}$-martingale parts of S are also local $\mathbb{P}^*$-martingales. For the Föllmer-Schweizer minimal martingale measure $\mathbb{P}^*$, the density process is given by the following dynamics,

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}(-\int_0 ((\Sigma^{'}(\Sigma\Sigma^{'})^{-1})\ \delta)^{'}\ dW)_T. \tag{1.6}$$

Kuroda and Nagai [26] observe that the condition of stability of the matrix $B - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}A$ induces stability on the factor process $X = (X(t))_{0 \le t \le T}$ under the minimal martingale measure. Fleming and Sheu [15] and more recently Föllmer and Schweizer [18] have observed that this observation and more significantly the role of the minimal martingale measure in this portfolio asset management problem is yet to be established. In this chapter we address these questions. We do so by conceptualizing the RSCAM as a zero-sum, stochastic differential game between (a market) that provides a probability measure that works antagonistically against another player (the investor) who otherwise wants to maximize the risk-sensitive criterion. We call this game (**GI**)(refer (2.4)). We need to determine the controls that forms the saddle point equilibrium to this game. This will then illuminate the explicit dependence between controls $h(t)$ and the probability measure which would then lead us to connect the role played by the minimal martingale measure. We achieve this objective through the following road map:

Key Steps

Step 1: We re-formulate the game (**GI**) into an auxiliary game characterized by the exponential of integral criterion that involves just the factor process X. We call this game (**GII**)(refer equation (2.11)).

Step 2: We then provide a verification lemma for (**GII**).

Step 3: We then obtain the optimal controls and deduce the connection between the minimal martingale measure and investor's optimal strategy.

Step 4: To complete the analysis we end by showing that the controls hence obtained while solving game (**GII**) in Step 3 in fact also constitute a saddle-point equilibrium strategy for the original game (**GI**).

2 Worst-Case risk sensitive zero-sum stochastic differential game

As discussed in the introduction, the Kuroda and Nagai investment market model is incomplete. We are interested in understanding the influence the minimal martingale measure has on this portfolio optimization problem. We conjure an approach, whereby we can explicitly characterize the dependence between the minimal martingale measure and the control variable h. Formally the way we do this is to define a "market world". The market world is a space of probability measures defined as

$$\mathcal{P} \triangleq \{\mathbb{P}^{\eta,\xi} : (\eta, \xi) = (\eta(t), \xi(t))_{T\geq t\geq 0} \in \mathcal{O}(T)\},$$

on $(\Omega, \mathcal{F})$, where $\mathcal{O}(T)$ denotes the set of deterministic controls $\eta(t) \in \mathbb{R}^{n\times(n+m)}$ and $\xi(t) \in \mathbb{R}^{1\times(n+m)}$ which are continuous over the compact set $[0, T]$ and hence bounded. $\mathcal{E}(\cdot)$ is the Doleáns-Dade exponential. For $(\eta(t), \xi(t)) \in \mathcal{O}(T)$, for fixed time horizon T, the restriction of $\mathbb{P}^{\eta,\xi}$ to the $\sigma-$ field $\mathcal{F}_T$ is given by the Radon-Nikodym density

$$D^{\eta,\xi}(T) \triangleq \frac{d\mathbb{P}^{\eta,\xi}}{d\mathbb{P}}|_{\mathcal{F}_T} \triangleq \mathcal{E}\bigg(\int_0 (\eta(t)^{'}X(t) + \xi^{'}(t))^{'}dW(t)\bigg)_T. \tag{2.1}$$

with respect to the reference measure $\mathbb{P}$. We now show that for $(\eta, \xi) \in \mathcal{O}(T)$, $\mathbb{P}^{\eta,\xi}$ is a probability measure.

Lemma 1. $E[D^{\eta,\xi}(T)] = 1$ for all $(\eta, \xi) \in \mathcal{O}(T)$.

Proof The process $X(t)$ in (1.4) is a Gaussian process. From (1.4) and Gronwall's inequality we have $E|X(t)| \leq (E|X(0)| + |b|T)\exp(||B||t)$ and $Cov(X(t)) = \Lambda^{'}\Lambda t$ where Cov is the covariance function. As $\eta(t), \xi(t)$ are deterministic controls and are bounded, $\phi(t) \triangleq X^{'}(t)\eta(t) + \xi(t)$ is also a Gaussian process with bounded mean and covariance on a finite time interval $[0, T]$. Hence by an application of Novikov's condition, the Doleáns-Dade exponential in (2.1) is a $\mathbb{P}$- martingale. A standard proof of this fact can be seen in Lemma 3.1.1 in Bensoussan [1].
We now re-evaluate the optimization criterion J under the new probability measure $\mathbb{P}^{\eta,\xi}$ and call it $\tilde{J}$ which is defined as

$$\tilde{J}(v, h, \eta, \xi, T; \theta) = \frac{-2}{\theta} \log E^{\eta,\xi}[\exp[\frac{-\theta}{2} \log V^{h,\eta,\xi}(T)]].$$

where the portfolio value under the new probability measure $\mathbb{P}^{\eta,\xi}$ is given by

$$\begin{aligned} \frac{dV^{h,\eta,\xi}(t)}{V^{h,\eta,\xi}(t)} &= \Big[r(t) + h^{'}(t)(\delta(t) - \Sigma(\eta^{'}(t)X(t) + \xi^{'}(t)))\Big]dt + h^{'}(t)\Sigma dW^{\eta,\xi}(t), \\ V^{h,\eta,\xi}(0) &= v. \end{aligned} \tag{2.2}$$

From Lemma 1 we have that $\mathbb{P}^{\eta,\xi}$ is a probability measure for $(\eta, \xi) \in \mathcal{O}(T)$.
From the standard result in Girsanov [21], under the probability measure $\mathbb{P}^{\eta,\xi}$,

$$W^{\eta,\xi}(t) \triangleq W(t) + \int_0^t (\eta^{'}(s)X(s) + \xi^{'}(s))ds,$$

is a standard Brownian motion process and therefore the factor process $X(t)$, satisfies the following SDE

$$dX(t) = (b + BX(t) - \Lambda(\eta^{'}(t)X(t) + \xi^{'}(t)))dt + \Lambda dW^{\eta,\xi}(t), \tag{2.3}$$

Remark 1 From equations (1.6) and (2.1) , it is clear that $\mathbb{P}^{\eta,\xi}$ is a minimal martingale measure for $\hat{\eta}(t) \triangleq \eta(t) = A^{'}(\Sigma\Sigma^{'})^{-1}\Sigma$ and $\hat{\xi}(t) \triangleq \xi(t) = (a - r(t)1)^{'}(\Sigma\Sigma^{'})^{-1}\Sigma$.

Kuroda and Nagai [26] have stated that under the condition of stability of the matrix $B - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}A$, the factor process $X(t)$ is stable under the minimal martingale measure. In light of our Remark 1, we validate this statement now.
Remark 2 As $\eta(t) = \hat{\eta}(t)$ and $\xi(t) = \hat{\xi}(t)$ corresponds to the minimal martingale measure, the dynamics of $X(t)$ under the minimal martingale measure can be re-written as

$$dX(t) = \Big(b - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}(a - r(t)1) + (B - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}A)X(t)\Big)dt + \Lambda dW^{\hat{\eta},\hat{\xi}}(t).$$

We are interested in finding the behavior of the solution $X(t)$ as $t \to \infty$. The coefficient of the $X(t)$ term in the drift part of the above equation is $B - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}A$. Since, by assumption, this coefficient term is a stable matrix, $X(t)$ is hence stable under the minimal martingale measure.

We need to now pin down the influence the minimal martingale measure has on this portfolio optimization problem to further resolve the inquiry posed by Fleming and Sheu [15].

To do so, as stated earlier, we conceptualize this problem as a game between a player termed as the *market* against the *investor*. We denote this game as (**GI**).

Game **GI** *Obtain* $\hat{h} \in \mathcal{H}(T)$ *and* $(\hat{\eta}, \hat{\xi}) \in \mathcal{O}(T)$ *such that,*

$$\begin{aligned}\tilde{J}(v, \hat{h}, \hat{\eta}, \hat{\xi}, T; \theta) &= \sup_{h\in\mathcal{H}(T)} \inf_{(\eta,\xi)\in\mathcal{O}(T)} \frac{-2}{\theta} \log E^{\eta,\xi}[\exp[\frac{-\theta}{2} \log V^{h,\eta,\xi}(T)]], \\ &= \inf_{(\eta,\xi)\in\mathcal{O}(T)} \sup_{h\in\mathcal{H}(T)} \frac{-2}{\theta} \log E^{\eta,\xi}[\exp[\frac{-\theta}{2} \log V^{h,\eta,\xi}(T)]]. \quad (2.4)\end{aligned}$$

Our intention is to re-write the objective function $\tilde{J}$ purely in terms of the factor process X. We set to achieve this by defining,

$$g(x, h, \eta, \xi, r; \theta) \triangleq \frac{1}{2}(\frac{\theta}{2} + 1)h^{'}\Sigma\Sigma^{'}h - r - h^{'}\left(\delta - \Sigma(\eta^{'}x + \xi^{'})\right). \quad (2.5)$$

From (2.5), we have

$$\begin{aligned}-\frac{\theta}{2}d\log V^{h,\eta,\xi}(t) &= \left(\frac{\theta}{2}g(X(t), h(t), \eta(t), \xi(t), r(t); \theta) - \frac{\theta^2}{8}h^{'}(t)\Sigma\Sigma^{'}h(t)\right)dt \\ &- \frac{\theta}{2}h^{'}(t)\Sigma dW^{\eta,\xi}(t). \quad (2.6)\end{aligned}$$

We next define the following stochastic exponential and measure:

$$\frac{d\mathbb{P}^{h,\eta,\xi}}{d\mathbb{P}^{\eta,\xi}}|_{\mathcal{F}_T} = \mathcal{E}(-\frac{\theta}{2}\int_0 h^{'}(t)\Sigma dW^{\eta,\xi}(t))_T. \quad (2.7)$$

From the definition of the class of controls $\mathcal{H}(T)$, it is clear from an application of Novikov's condition that $\mathbb{P}^{h,\eta,\xi}$ is a probability measure. Under $\mathbb{P}^{h,\eta,\xi}$, the standard result of Girsanov [21] yields that

$$W^{h,\eta,\xi}(t) \triangleq W^{\eta,\xi}(t) + \int_0^t \frac{\theta}{2}\Sigma^{'}h(s)ds,$$

is a standard $\mathbb{P}^{h,\eta,\xi}$- Brownian motion and the factor process $X(t)$ satisfies the following dynamics

$$dX(t) = (b + BX(t) - \Lambda(\eta^{'}(t)X(t) + \xi^{'}(t)) - \frac{\theta}{2}\Lambda\Sigma^{'}h(t))dt + \Lambda dW^{h,\eta,\xi}(t). \quad (2.8)$$

Now, under the new probability measure $\mathbb{P}^{h,\eta,\xi}$, and using (2.4)-(2.6) and (2.8) we define an auxiliary optimization criterion $I(v, x, h, \eta, \xi, t, T)$ given as

$$I(v, x, h, \eta, \xi, t, T; \theta) = \log v - \frac{2}{\theta} \log E^{h,\eta,\xi}\left[\exp\left(\frac{\theta}{2}\int_0^{T-t} g(X(s), h(s), \eta(s), \xi(s), r(s+t); \theta)ds\right)\right]. \quad (2.9)$$

This will lead us to frame the auxiliary game **GII** that constitutes our first step in the road map in the Introduction.

Step 1:

In a worst-case risk-sensitive asset management scenario, the investor chooses a portfolio process h so as to maximize the expected exponential-of-integral performance index I. Then the response of the market to this choice is to select (η, ξ) (and hence a probability measure)that minimizes the maximum expected exponential-of-integral performance index. Formally,

The upper value of this game is given by

$$\bar{u}(t, x) = \sup_{h\in\mathcal{H}(T)} \inf_{(\eta,\xi)\in\mathcal{O}(T)} I(v, x, h, \eta, \xi, t, T; \theta),$$

while the lower value of the game is given by

$$\underline{u}(t, x) = \inf_{(\eta,\xi)\in\mathcal{O}(T)} \sup_{h\in\mathcal{H}(T)} I(v, x, h, \eta, \xi, t, T; \theta),$$

The game has a value provided,

$$\bar{u}(t, x) = \underline{u}(t, x) = u(t, x) = I(v, x, \hat{h}, \hat{\eta}, \hat{\xi}, t, T; \theta). \quad (2.10)$$

and hence $\hat{h}, (\hat{\eta}, \hat{\xi})$ is a saddle-point equilibrium. We aim to provide a verification lemma for which (2.10) is satisfied. In that spirit, consider the *exponentially transformed* criterion which is simply obtained via the transformation $\tilde{u}(t, x) = \exp(-\frac{\theta}{2}u(t, x))$. This transformation defines what we call game **GII**.

Game (**GII**)

Obtain $\hat{h} \in \mathcal{H}(T)$ and $(\hat{\eta}, \hat{\xi}) \in \mathcal{O}(T)$ such that,

$$\begin{aligned}
\tilde{u}(t,x) &= \inf_{h\in\mathcal{H}(T)} \sup_{(\eta,\xi)\in\mathcal{O}(T)} E^{h,\eta,\xi}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s),h(s),\eta(s),\xi(s),r(s+t);\theta)ds\}v^{-\theta/2}], \\
&= \sup_{(\eta,\xi)\in\mathcal{O}(T)} \inf_{h\in\mathcal{H}(T)} E^{h,\eta,\xi}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s),h(s),\eta(s),\xi(s),r(s+t);\theta)ds\}v^{-\theta/2}], \\
&= E^{\hat{h},\hat{\eta},\hat{\xi}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s),\hat{h}(s),\hat{\eta}(s),\hat{\xi}(s),r(s+t);\theta)ds\}v^{-\theta/2}]. \qquad (2.11)
\end{aligned}$$

We now deduce the values of $(\hat{h}, (\hat{\eta}, \hat{\xi}))$ for game (**GII**).

3 An HJBI equation for game GII.

Step 2:

Let us now define a couplet process $Y^{h,(\eta,\xi)}(t)$ as

$$dY^{h,(\eta,\xi)}(s) = \begin{pmatrix} dY_0(s) \\ dY_1(s) \end{pmatrix} = \begin{pmatrix} ds \\ dX(s) \end{pmatrix} = \begin{pmatrix} ds \\ (b+BX(s)-\Lambda(\eta'(s)X(s)+\xi'(s))-\frac{\theta}{2}\Lambda\Sigma' h(s))dt+\Lambda dW^{h,\eta,\xi}(s) \end{pmatrix}$$

$Y_0(0) = s \in [0,T], Y_1(0) = y = (y^1, ...y^n)$. The control process $h(s) = h(s,\omega)$ is assumed to be Markovian. Then the process $Y^{h,(\eta,\xi)}(s)$ is a Markov process whose generator acting on a function $\tilde{u}(y) \in C_0^{1,2}((0,T)\times\mathbb{R}^n)$ where ($C_0^{1,2}$ is the space of functions with compact support on $(0,T)\times\mathbb{R}^n$ such that it is once continuously differentiable in time and twice continuously differentiable in space variable x) is given by,

$$\tilde{\mathcal{A}}^{h,(\eta,\xi)}\tilde{u}(y) = \frac{\partial \tilde{u}(y)}{\partial s} + (b+Bx-\Lambda(\eta' x+\xi')-\frac{\theta}{2}\Lambda\Sigma' h)' D\tilde{u}(y) + \frac{1}{2}tr(\Lambda\Lambda^* D^2\tilde{u}(y)). \qquad (3.1)$$

in which $D\tilde{u}(y) \triangleq (\frac{\partial \tilde{u}(y)}{\partial y_1^1}, ..., \frac{\partial \tilde{u}(y)}{\partial y_1^n})'$ and $D^2\tilde{u}(y)$ is the matrix defined as $D^2\tilde{u}(y) \triangleq [\frac{\partial^2 \tilde{u}(y)}{\partial y_1^i \partial y_1^j}], i,j = 1,2,...,n.$

By an application of the Feynman-Kac formula, it can be deduced that the HJB PDE for $\tilde{u}(y)$ is given by

$$\left(\tilde{\mathcal{A}}^{\hat{h},(\hat{\eta},\hat{\xi})} + \frac{\theta}{2} g(x,\hat{h}(y),\hat{\eta},\hat{\xi},r;\theta)\right)\tilde{u}(y) = 0. \qquad (3.2)$$

The following proposition presents a diagnostic to identify a solution to the game (**GII**).

Proposition 2. *Define $\mathcal{S} = (0,T)\times\mathbb{R}^n$. Let there exist a function $\tilde{w} \in \mathcal{C}^{1,2}(\mathcal{S})\cap\mathcal{C}(\bar{\mathcal{S}})$ and a (Markov) control $\hat{h} \in \mathcal{H}(T)$ and $(\hat{\eta},\hat{\xi}) \in \mathcal{O}(T)$ such that for each $y \in \mathcal{S}$ satisfy,*
1. $(\tilde{\mathcal{A}}^{h,(\hat{\eta},\hat{\xi})} + \frac{\theta}{2}g(x,h,\hat{\eta},\hat{\xi},r;\theta))[(\tilde{w}(y))] \geq 0 \ \forall\ h \in \mathbb{R}^m$;
2. $(\tilde{\mathcal{A}}^{\hat{h}(y),(\eta,\xi)} + \frac{\theta}{2}g(x,\hat{h}(y),\eta,\xi,r;\theta))[(\tilde{w}(y))] \leq 0 \ \forall\ \eta \in \mathbb{R}^{n\times(n+m)}, \xi \in \mathbb{R}^{1\times(n+m)}$;
3. $(\tilde{\mathcal{A}}^{\hat{h}(y),(\hat{\eta},\hat{\xi})} + \frac{\theta}{2}g(x,\hat{h}(y),\hat{\eta},\hat{\xi},r;\theta))[(\tilde{w}(y))] = 0$;
4. $[(\tilde{w}(T,X_T))] = v^{-\theta/2}$.
5. $E^{h,\eta,\xi}[\int_0^{T-t} D\tilde{w}'(t+s,X(s))\Lambda e^{\tilde{Z}_s} dW_s^{h,\eta,\xi}] = 0 \ \forall\ h \in \mathbb{R}^m, \forall\ \eta \in \mathbb{R}^{n\times(n+m)}, \xi \in \mathbb{R}^{1\times(n+m)}$;
where,

$$\tilde{Z}(s) = \tilde{Z}_s(h,\eta,\xi) := \frac{\theta}{2}\left\{ \int_0^s g(X(\tau),h(\tau),\eta(\tau),\xi(\tau),r(t+\tau);\theta)d\tau \right\}. \tag{3.3}$$

Define ,

$$\begin{aligned}\tilde{I}(v,x,h,\eta,\xi,t,T) &= \exp(-\frac{\theta}{2}I(v,x,h,\eta,\xi,t,T)),\\ &= E^{h,\eta,\xi}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s),h(s),\eta(s),\xi(s),r(s+t);\theta)ds\}v^{-\theta/2}].\end{aligned}$$

Then,

$$\begin{aligned}\tilde{u}(0,x) = \tilde{w}(0,x) = \tilde{I}(v,x,\hat{h},\hat{\eta},\hat{\xi},0,T) &= \inf_{h\in\mathcal{H}(T)}\{\sup_{(\eta,\xi)\in\mathcal{O}(T)}[\tilde{I}(v,x,h,\eta,\xi,0,T)]\},\\ &= \sup_{(\eta,\xi)\in\mathcal{O}(T)}\{\inf_{h\in\mathcal{H}(T)}[\tilde{I}(v,x,h,\eta,\xi,0,T)]\},\\ &= \sup_{(\eta,\xi)\in\mathcal{O}(T)} \tilde{I}(v,x,\hat{h},(\eta,\xi),0,T),\\ &= \inf_{h\in\mathcal{H}(T)} \tilde{I}(v,x,h,\hat{\eta},\hat{\xi},0,T) = \tilde{I}(v,x,\hat{h},\hat{\eta},\hat{\xi},0,T).\end{aligned}$$

and $(\hat{h},(\hat{\eta},\hat{\xi}))$ is a saddle point equilibrium.

Proof Apply Ito's formula to $\tilde{w}(s,X(s))e^{\tilde{Z}(s)}$ to obtain

$$\begin{aligned}&\tilde{w}(T,X(T-t))e^{\tilde{Z}(T-t)} = \tilde{w}(t,x),\\ +&\int_0^{T-t}((\tilde{\mathcal{A}}^{h,\eta,\xi} + \frac{\theta}{2}g(X(s),h(X(s)),\eta(s),\xi(s),r(s+t);\theta))\tilde{w}(t+s,X(s)))e^{\tilde{Z}_s})ds,\\ +&\int_0^{T-t}(D\tilde{w}'(t+s,X(s))\Lambda)e^{\tilde{Z}(s)}dW^{h,\eta,\xi}(s).\end{aligned} \tag{3.4}$$

Taking expectation with respect to $\mathbb{P}^{h,\eta,\xi}$, from condition (5) of the Proposition 2 , the stochastic integral in (3.4) vanishes. Now setting $t = 0$ and further applying condition (1) and (4) of the proposition 2 again , we get

$$E^{h,\eta,\xi}[\tilde{w}(T, X_T)e^{\tilde{Z}_T}] \geq \tilde{w}(0, x).$$

Since this inequality is true for all $h \in \mathcal{H}(T)$ we have

$$\inf_{h\in\mathcal{H}(T)} E^{h,\eta,\xi}[v^{-\theta/2}e^{\tilde{Z}_T}] \geq \tilde{w}(0, x).$$

Hence we have,

$$\sup_{(\eta,\xi)\in\mathcal{O}(T)} \inf_{h\in\mathcal{H}(T)} E^{h,\eta,\xi}[v^{-\theta/2}e^{\tilde{Z}_T}] \geq \inf_{h\in\mathcal{H}(T)} E^{h,\eta,\xi}[v^{-\theta/2}e^{\tilde{Z}_T}] \geq \tilde{w}(0, x). \qquad (3.5)$$

Similarly, setting $t = 0$ and using conditions (5), (2) and (4) of the proposition 2, we get the following upper value of the game:

$$\inf_{h\in\mathcal{H}(T)} \sup_{(\eta,\xi)\in\mathcal{O}(T)} E^{h,\eta,\xi}[v^{-\theta/2}e^{\tilde{Z}_T}] \leq \sup_{(\eta,\xi)\in\mathcal{O}(T)} E^{h,\eta,\xi}[v^{-\theta/2}e^{\tilde{Z}_T}] \leq \tilde{w}(0, x). \qquad (3.6)$$

Also , setting $t = 0$ and using conditions (5), (3) and (4) of the proposition 2 we get:

$$\begin{aligned} E^{\hat{h},(\hat{\eta},\hat{\xi})}[\tilde{w}(T, X_T)e^{\tilde{Z}_T}] &= \tilde{w}(0, x) \\ &= E^{\hat{h},(\hat{\eta},\hat{\xi})}[\exp\{\frac{\theta}{2}\int_0^T g(X(s), \hat{h}(X(s)), \hat{\eta}(s), \hat{\xi}(s), r(s); \theta)ds\}v^{-\theta/2}]. \end{aligned} \qquad (3.7)$$

From (3.5), (3.6) and (3.7), and the fact that $\sup_{(\eta,\xi)\in\mathcal{O}(T)} \inf_{h\in\mathcal{H}(T)}[v^{-\theta/2}e^{\tilde{Z}_T}] \leq \inf_{h\in\mathcal{H}(T)} \sup_{(\eta,\xi)\in\mathcal{O}(T)}[v^{-\theta/2}e^{\tilde{Z}_T}]$ automatically holds, the conclusion now follows.

We now return to the game problem involving u as the payoff function.

Corollary 3. $\underline{u}(0, x) = \bar{u}(0, x) = u(0, x)$.

Proof The value functions u and $\tilde{u}$ are related through the strictly monotone continuous transformation $\tilde{u}(t, x) = \exp(-\frac{\theta}{2}u(t, x))$. Thus admissible (optimal) strategies for the exponentially transformed problem $\tilde{u}$ obtained via Proposition 2 are also admissible(optimal) for the problem u. In other words,

$$
\begin{aligned}
u(0,x) &= \sup_{h\in\mathcal{H}(T)} \inf_{(\eta,\xi)\in\mathcal{O}(T)} \{[I(v,x,h,\eta,\xi,0,T)]\},\\
&= \inf_{(\eta,\xi)\in\mathcal{O}(T)} \{ \sup_{h\in\mathcal{H}(T)} [I(v,x,h,\eta,\xi,0,T)]\},\\
&= \inf_{(\eta,\xi)\in\mathcal{O}(T)} I(v,x,\hat{h},\eta,\xi,0,T),\\
&= \sup_{h\in\mathcal{H}(T)} I(v,x,h,\hat{\eta},\hat{\xi},0,T) = I(v,x,\hat{h},\hat{\eta},\hat{\xi},0,T).
\end{aligned}
$$

Hence $\underline{u}(0,x) = \bar{u}(0,x) = u(0,x)$.

4 Solving game GII.

Step 3 :

We seek to find a function u that would satisfy all the conditions of our verification lemma given by Proposition 2. Conditions (1)-(4) of the verification lemma can be written in a compact form in terms of $u(t,x)$ as

$$
\begin{aligned}
\mathcal{A}^{\hat{h},\hat{\eta},\hat{\xi}}u(t,x) &= 0,\\
u(T,x) &= \log v.
\end{aligned} \tag{4.1}
$$

where the operator $\mathcal{A}^{h,\eta,\xi}u(t,x)$ for any $h\in\mathbb{R}^m$ and $\eta\in\mathbb{R}^{n\times(n+m)}$, $\xi\in\mathbb{R}^{1\times(n+m)}$ is given by,

$$
\begin{aligned}
\mathcal{A}^{h,\eta,\xi}u(t,x) &= \frac{\partial u(t,x)}{\partial t} + (b + Bx - \Lambda(\eta^{'}(s)X(s) + \xi^{'}(s))) - \frac{\theta}{2}\Lambda(\Sigma^{'}h))^{'}Du(t,x)\\
&+ \frac{1}{2}tr(\Lambda\Lambda^{'}D^2u(t,x))\\
&- \frac{\theta}{4}(Du(t,x))^{'}\Lambda\Lambda^{'}Du(t,x) - g(x,h,\eta,\xi,r;\theta).
\end{aligned} \tag{4.2}
$$

The first order condition for $\hat{h}$ that maximizes $\mathcal{A}^{h,\hat{\eta},\hat{\xi}}$ over all $\mathcal{H}(T)$ is given by,

$$
\hat{h}(t) = \frac{2}{(\theta+2)}(\Sigma\Sigma^{'})^{-1}[\delta(t) - \Sigma(\hat{\eta}^{'}(t)X(t) + \hat{\xi}^{'}) - \frac{\theta}{2}\Sigma\Lambda^{'}Du(t,x)]. \tag{4.3}
$$

Substituting (2.5) in (4.2) we obtain an expression for the operator $\mathcal{A}^{h,\eta,\xi}$ in $\eta^{'}(t)$ and $\xi^{'}(t)$. We minimize $\mathcal{A}^{h,\eta,\xi}$ over the set of controls $\mathcal{O}(T)$. As this operator is linear in $\eta^{'}(t)$ and $\xi^{'}(t)$, we guess that the coefficient of the terms $\eta^{'}(t)$ and $\xi^{'}(t)$

vanish[1] leading to

$$\hat{h}(t) = -(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}Du(t,x).$$

Motivated by Kuroda and Nagai [26], we will try the functional form for u given by $u(t,x) = \frac{1}{2}x^T Q(t)x + q^T(t)x + k(t)$ where Q is an $n \times n$ symmetric matrix, q is a n-element column vector and k is a scalar. Hence,

$$\hat{h}(t) = -(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}(Q(t)X(t) + q(t)). \tag{4.4}$$

This when substituted in (4.3) yields,

$$-\Sigma\Lambda^{'}(Q(t)X(t) + q(t)) = \delta(t) - \Sigma(\hat{\eta}^{'}(t)X(t) + \hat{\xi}^{'}(t)). \tag{4.5}$$

which further yields,

$$\left.\begin{aligned} \hat{\eta}(t) &= (Q^{'}(t)\Lambda\Sigma^{'} + A^{'})(\Sigma\Sigma^{'})^{-1}\Sigma, \\ \hat{\xi}(t) &= \Big((a - r(t)1)^{'} + q^{'}(t)\Lambda\Sigma^{'}\Big)(\Sigma\Sigma^{'})^{-1}\Sigma. \end{aligned}\right\} \tag{4.6}$$

Thus $\hat{h}$ is a local maximizing control and $(\hat{\eta}, \hat{\xi})$ is a local minimizer control that constitutes the saddle-point equilibrium for game (**GII**).

Remark 3 From Remark 2 and equation (4.6), it can be seen that $\mathbb{P}^{\hat{\eta},\hat{\xi}}$ is a minimal martingale measure provided $Q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma = 0$ and $q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma = 0$ for $t \leq T$.

Remark 4 From Remark 3, and equation (4.4) it is clear that if the game equilibrium measure corresponds to the minimal martingale measure then the optimal investor strategy satisfies $\hat{h}^{'}(t)\Sigma = 0$. Hence if the portfolio model does not permit short selling then the optimal investor strategy at game equilibrium is the no-regret strategy i.e ($\hat{h}(t)$=0).

Remark 5 In the case where the factor process and the security(stock) price process has independent noise i.e $\Sigma\Lambda^{'}$=0 , then from Remarks 3-4, it is obvious that at

[1] This is a situation typical in bang-bang type stochastic control problems. The optimal control value depends on the the sign that their corresponding coefficient take or whether these coefficients altogether vanish from the expression (4.2). Note that controls ξ, η are bounded and hence cannot blow-up in finite time. Moreover, from Remark 2 we require that X be stable under the MMM $\mathbb{P}^{\eta=\hat{\eta},\xi=\hat{\xi}}$. These insights makes us guess that the optimal controls $\hat{\eta}(t), \hat{\xi}(t)$ can be chosen such that their corresponding coefficients in (4.2) vanish. See also [28]. Hence the conditions that subsequently guarantee that $\mathbb{P}^{\hat{\eta},\hat{\xi}}$ is a MMM also infact, from Remark 3 guarantee that the factor process X is stable under the MMM. This satisfies our objective of determining $\hat{\eta}, \hat{\xi}$ such that X is stable under $\mathbb{P}^{\hat{\eta},\hat{\xi}}$.

optimality, the worst case strategy is the no-regret strategy and the factor process always evolves under the minimal martingale measure since the game equilibrium measure is the minimal martingale measure.

As in Kuroda and Nagai [26], we can verify that $u(t,x) = \frac{1}{2}x^{'}Q(t)x + q^{'}(t)x + k(t)$ satisfies the HJB PDE i.e conditions (1)-(4) of the Proposition provided
• an $n \times n$ symmetric non-negative matrix Q satisfies the following matrix Riccati equation given as

$$\frac{dQ(t)}{dt} - Q(t)K_0Q(t) + K_1^{'}Q(t) + Q(t)K_1 = 0 \quad 0 \leq t \leq T, \quad Q(T) = 0. \tag{4.7}$$

where

$$\begin{aligned} K_0 &= \frac{\theta}{2}\Lambda\Big(I - \frac{\theta-2}{\theta}\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Big)\Lambda^{'}, \\ K_1 &= B - \Lambda\eta^{'}(t) - \Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}A. \end{aligned}$$

• The n element column vector $q(t)$satisfies the following linear ordinary differential equation
for $0 \leq t \leq T$.

$$\begin{aligned} \frac{dq(t)}{dt} &+ (K_1^{'} - Q(t)K_0)q(t) + Q(t)b - Q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}(a - r(t)1) \\ &+ Q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma(\eta^{'}(t) + \xi^{'}(t)) - \xi(t)\Lambda^{'}Q(t) = 0, \\ q(T) &= 0. \end{aligned} \tag{4.8}$$

• and the constant $k(t)$ is a solution to

$$\begin{aligned} \frac{dk(t)}{dt} &+ b^{'}q(t) + \frac{\theta-2}{4}q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma\Lambda^{'}q(t), \\ &+ r - q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}(a - r(t)1) + q^{'}(t)\Lambda\Sigma^{'}(\Sigma\Sigma^{'})^{-1}\Sigma(\eta^{'}(t) + \xi^{'}(t)), \\ &- \xi(t)\Lambda^{'}q(t) + \frac{2-\theta}{4}q^{'}(t)\Lambda\Lambda^{'}q(t) = 0, \end{aligned}$$

$, \forall 0 \leq t \leq T,$

$$k(T) = \log(v). \tag{4.9}$$

The fourth condition of the proposition is obvious from the terminal conditions of Q, q and k. To show that condition (5) of the proposition is satisfied by the choice of our payoff function, we need to show that $E^{h,(\eta,\xi)}(< D\tilde{u}\ \Lambda e^{Z}, D\tilde{u}\ \Lambda e^{Z} >_t) < \infty$ $\forall t \in [0,T]$. To show this we argue as follows. Processes $Q \triangleq (Q(t))_{0 \leq t \leq T}$ and

$q \triangleq (q(t))_{0 \le t \le T}$ are bounded since they are continuous on the compact support $[0, T]$. By a standard existence-uniqueness argument for stochastic differential equation (refer Gihman and Skorokhod [22]), $X \in L^2(\mathbb{P}^{h,(\eta,\xi)})$. Since $D\tilde{u}$ is linear in X with controls (η, ξ) assumed bounded, we also have that $D\tilde{u} \in L^2(\mathbb{P}^{h,\eta,\xi})$. To complete the argument it remains to be shown that $\tilde{u}$ is bounded. We show this now.

Lemma 4. $0 < \tilde{u} < \exp(-\frac{\theta}{2}\int_0^{T-t} r(s+t)ds)v^{-\theta/2}$.

Proof From the definition of $\tilde{u}$ in (2.11), for any optimal control $\mathcal{O}(T)$, the strategy $\hat{h}(t) = 0$ for $t \le T$ is sub-optimal, and hence will provide an upper bound on $\tilde{u}$. Formally,

$$\begin{aligned}
\tilde{u}(t,x) &= \inf_{h \in \mathcal{H}(T)} E^{h,\hat{\eta},\hat{\xi}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s), h(s), \hat{\eta}(s), \hat{\xi}(s), r(s+t); \theta)ds\}v^{-\theta/2}], \\
&\le E^{0,\hat{\eta},\hat{\xi}}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s), 0, \hat{\eta}(s), \hat{\xi}(s), r(s+t); \theta)ds\}v^{-\theta/2}], \\
&\le E^{0,0,0}[\exp\{\frac{\theta}{2}\int_0^{T-t} g(X(s), 0, 0, 0, r(s+t); \theta)ds\}v^{-\theta/2}], \\
&= \exp(-\frac{\theta}{2}\int_0^{T-t} r(s+t)ds)v^{-\theta/2}.
\end{aligned}$$

Hence the conclusion follows.

We now formalize the solution to this game (**GI**).

Step 4:

We first show that the controls belonging to $\mathcal{H}(T)$ and $\mathcal{O}(T)$ satisfy the following change of measure criterion.

Lemma 5. *From the choice of space of controls $h \in \mathcal{H}(T)$ and $(\eta, \xi) \in \mathcal{O}(T)$, we have*

$$E[\mathcal{E}\Big(-\frac{\theta}{2}\int_0 [(Q(t)X(t) + q(t))\Lambda + h'(t)\Sigma]dW^{\eta,\xi}(t)\Big)_T] = 1. \tag{4.10}$$

Proof Above result holds if the following Kazamaki condition, $E[\exp(\int_0^t \theta(\frac{(Q(s)X(s)+q(s))\Lambda+h'(s)\Sigma}{2})dW^{\eta,\xi}(s))] < \infty \ \forall \ t \in [0,T]$ is satisfied. By an application of Cauchy-Schwartz inequality we have $\forall \ t \in [0,T]$,

$$\begin{aligned}
& E[\exp(\int_0^t \theta(\frac{(Q(s)X(s)+q(s))\Lambda + (h'(s)\Sigma)}{2})dW^{\eta,\xi}(s))] \\
\le \ & (E[e^{\int_0^t \theta(Q(s)X(s)+q(s))\Lambda dW^{\eta,\xi}(s)}])^{1/2} \times (E[e^{\int_0^t \theta(h'(s)\Sigma)dW^{\eta,\xi}(s)}])^{1/2}
\end{aligned}$$

Since X is a Gaussian process, mimicking arguments similar to Lemma 1, we have that

$(E[e^{\int_0^t \theta(Q(s)X(s)+q(s))\Lambda dW_s^{\eta,\xi}}])^{1/2} < \infty \ \forall \ t \in [0,T]$. From assumption on the space of controls $\mathcal{H}(T)$, one can conclude that $(E[e^{\int_0^t \theta(h^{'}(s)\Sigma)dW^{\eta,\xi}(s)}])^{1/2} < \infty$ for $t \in [0,T]$. Hence the Kazamaki condition holds true and the conclusion follows.

We now show that the saddle-point equilibrium controls obtained by solving game (**GII**) is in fact also a saddle-point equilibrium for the original game problem (**GI**).

Lemma 6. *If there exist a solution Q to the matrix Ricatti equation (4.7) , then the saddle point equilibrium strategies $\hat{h}$ and $(\hat{\eta},\hat{\xi})$ obtained from (4.4) and (4.6) respectively as a result of solving the auxiliary game (***GII***) where q is a solution to (4.8) and and k is a solution of (4.9) is in fact also the saddle-point equilibrium for the finite horizon game (***GI***), namely,*

$$\begin{aligned}\sup_{h\in\mathcal{H}(T)}\inf_{(\eta,\xi)\in\mathcal{O}(T)}\tilde{J}(v,h,\eta,\xi,T;\theta) &= \inf_{(\eta,\xi)\in\mathcal{O}(T)}\sup_{h\in\mathcal{H}(T)}\tilde{J}(v,h,\eta,\xi,T;\theta),\\ &= \tilde{J}(v,\hat{h},\hat{\eta},\hat{\xi},T;\theta,)\\ &= \frac{1}{2}x^{'}Q(0)x+q^{'}(0)x+k(0).\end{aligned}$$

where,

$$\tilde{J}(v,h,\eta,\xi,T;\theta) \triangleq \frac{-2}{\theta}\log E^{\eta,\xi}[\exp[\frac{-\theta}{2}\log V^{h,\eta,\xi}(T)]].$$

Proof

Define,

$$\begin{aligned}\bar{Z}_s &= \bar{Z}_s(h,\eta,\xi) = \frac{\theta}{2}\Big\{\int_0^s g(X(\tau),h(\tau),\eta(\tau),\xi(\tau),r(t+\tau);\theta)d\tau - (h^{'}(\tau)\Sigma)dW^{\eta,\xi}(\tau)\\ &- \frac{\theta}{4}(h^{'}(\tau)\Sigma)^{'}(h^{'}(\tau)\Sigma)d\tau\Big\}.\end{aligned} \tag{4.11}$$

Also define, $\chi(t,x) = -\frac{\theta}{2}(u(t,x)-\log v)$. From some straightforward calculations provided in the Appendix we obtain the following relation,

$$\begin{aligned}\exp\{\chi(T,X(T-t))+\bar{Z}(T-t)\} &= \exp(\chi(t,x))\exp\Big[\int_0^{T-t}-\frac{\theta}{2}(\mathcal{A}^{h,\eta,\xi}u(t+s,X_s))ds\\ &- \int_0^{T-t}\frac{\theta}{2}[Du(t+s,X_s)^{'}\Lambda+(h^{'}(t)\Sigma)]dW_t^{\eta,\xi}\\ &- \int_0^{T-t}\frac{\theta^2}{8}[Du(t+s,X_s)^{'}+(h^{'}(t)\Sigma)][Du(t+s,X_s)^{'}+h^{'}(t)\Sigma]^{'}ds\Big].\end{aligned} \tag{4.12}$$

We have shown that the saddle-point equilibrium strategies $\hat{h}$ and $(\hat{\eta}, \hat{\xi})$ deduced by solving game (**GI**) with corresponding game payoff function u satisfies conditions (1)-(5) of the Proposition 2 . Therefore from condition(4) of the Proposition 2 , we have $\chi(T, x) = 0$. Moreover $(V^{h,\eta,\xi}(T))^{-\theta/2} = v^{-\theta/2} e^{\bar{Z}_T}$. Setting $t = 0$ and taking condition (1) of the Proposition 2 into account for $\eta = \hat{\eta}, \xi = \hat{\xi}$, and for any $h \in \mathcal{H}(T)$ we see from (4.11) that

$$\begin{aligned}(V^{h,\eta,\xi}(T))^{-\theta/2} \quad &\geq \quad e^{-\frac{\theta}{2}u(0,x)} \exp\Big[- \int_0^T \frac{\theta}{2}[Du(s, X(s))^{'}\Lambda + h^{'}(s)\Sigma]dW^{\eta,\xi}(s) \\ &- \quad \int_0^T \frac{\theta^2}{8}[Du(s, X(s))^{'} + h^{'}(s)\Sigma][Du(s, X(s))^{'} + h^{'}(s)\Sigma]^{'} ds\Big].\end{aligned}$$

Now by taking expectations w.r.t to the physical probability measure $\mathbb{P}^{\eta,\xi}$ on both sides of above equation and using Lemma 5, we obtain

$$\tilde{J}(v, h, \eta, \xi, T) \leq u(0, x).$$

This inequality is true for all $h \in \mathcal{H}(T)$. Hence we have,

$$\sup_{h\in\mathcal{H}(T)} \tilde{J}(v, h, \eta, \xi, T) \leq u(0, x).$$

Hence we have,

$$\inf_{(\eta,\xi)\in\mathcal{O}(T)} \sup_{h\in\mathcal{H}(T)} \tilde{J}(v, h, \eta, \xi, T) \leq \sup_{h\in\mathcal{H}(T)} \tilde{J}(v, h, \eta, \xi, T) \leq u(0, x). \tag{4.13}$$

Likewise, setting $t = 0$ and taking condition (2) and condition (5) of the Proposition 2 into account we see that

$$\sup_{h\in\mathcal{H}(T)} \inf_{(\eta,\xi)\in\mathcal{O}(T)} \tilde{J}(v, h, \eta, \xi, T) \geq u(0, x) \geq \inf_{(\eta,\xi)\in\mathcal{O}(T)} \sup_{h\in\mathcal{H}(T)} \tilde{J}(v, h, \eta, \xi, T). \tag{4.14}$$

Similarly ,setting $t = 0$ and taking condition (3) and (5) of the Proposition 2 into account for $h = \hat{h}, \gamma = \hat{\gamma}$ such that $\hat{h} \in \mathcal{H}(T)$ and $(\hat{\eta}, \hat{\xi}) \in \mathcal{O}(T)$ we see that

$$\tilde{J}(v, \hat{h}, \hat{\eta}, \hat{\xi}, T) = u(0, x). \tag{4.15}$$

From (4.13)-(4.15) and the fact that
$\sup_{h\in\mathcal{H}(T)} \inf_{(\eta,\xi)\in\mathcal{O}(T)} \tilde{J}(v, h, \eta, \xi, T) \leq \inf_{(\eta,\xi)\in\mathcal{O}(T)} \sup_{h\in\mathcal{H}(T)} \tilde{J}(v, h, \eta, \xi, T)$ is au-

tomatically true, we conclude that the saddle-point equilibrium controls obtained by solving game (**GII**) in fact also constitutes saddle-point strategy for the original game (**GI**).

APPENDIX

As part of the proof of Lemma 5

Let $\chi(t,x) = -\frac{\theta}{2}(u(t,x) - \log v)$ and $Lu(t,x) = \frac{1}{2}tr(\Lambda\Lambda' D^2 u(t,x)) + (b + Bx - \Lambda(\eta' x + \xi'))' Du(t,x)$

Then we have

$$
\begin{aligned}
d\chi(t+s, X(s)) &= -\frac{\theta}{2}(\frac{\partial u}{\partial t} + Lu)(t+s, X(s))ds - \frac{\theta}{2} Du(t+s, X(s))' \Lambda dW^{\eta,\xi}(s). \\
\frac{d\exp\{\chi(t+s, X(s))\}}{\exp\{\chi(t+s, X(s))\}} &= -\frac{\theta}{2}(\frac{\partial u}{\partial t}(t,x) + Lu)(t+s, X(s)) - \frac{\theta}{2} Du(t+s, X(s))' \Lambda dW^{\eta,\xi}(s) \\
&+ \frac{\theta^2}{8} Du' \Lambda\Lambda' Du(t+s, X(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\exp\{\chi(t+s, X(s))\}\exp\{Z(s)\}}{\exp\{\chi(t+s, X(s))\}\exp\{Z(s)\}} &= -\frac{\theta}{2}(\frac{\partial u}{\partial t}(t,x) + Lu)(t+s, X(s)) \\
&- \frac{\theta}{2} Du(t+s, X(s))' \Lambda dW^{\eta,\xi}(s) \\
&+ \frac{\theta^2}{8} Du' \Lambda\Lambda' Du(t+s, X(s))ds \\
&+ \frac{\theta}{2} g(X(t), h(t), \eta(t), \xi(t), r(s+t); \theta)ds \\
&- \frac{\theta}{2} h'(s)\Sigma dW^{\eta,\xi}(s) + \frac{\theta^2}{4} h'(s)\Sigma\Lambda' Du(t+s, X(s))ds
\end{aligned}
$$

Integrating the above equation yields (4.12).

Chapter 4

Sufficient stochastic maximum principle for the optimal control of semi-Markov modulated jump-diffusion with an application to financial optimization.

Abstract
The finite state semi-Markov process is a generalization of the Markov chain in which the sojourn time distribution is any general distribution. In this chapter we provide a sufficient stochastic maximum principle for the optimal control of a semi-Markov modulated jump-diffusion process in which the drift, diffusion and the jump kernel of the jump-diffusion process is modulated by a semi-Markov process. We also connect the sufficient stochastic maximum principle with the dynamic programming equation. We apply our results to a finite-horizon, risk-sensitive control portfolio optimization problem and to a quadratic loss-minimization problem.

1 Introduction

The stochastic maximum principle is a stochastic version of the Pontryagin maximum principle which states that the any optimal control must satisfy a system of forward-backward stochastic differential equations, called the optimality system, and should maximize a functional, called the Hamiltonian. The converse indeed is true and gives the sufficient stochastic maximum principle. In this chapter we will derive sufficient stochastic maximum principle for a class of process called as the semi-Markov modulated jump-diffusion process. In this process the drift, the diffusion and the jump kernel term is modulated by an semi-Markov process.

An early investigation of stochastic maximum principle and its application to finance has been credited to Cadenillas and Karatzas [5]. Framstadt et al. [19] formulated the stochastic maximum principle for jump-diffusion process and applied it to a quadratic portfolio optimization problem. Their work has been partly generalized by Donnelly [13] who considered a Markov chain modulated diffusion process in which the drift and the diffusion term is modulated by a Markov chain. Zhang et al. [34] studied sufficient maximum principle of a process similar to that studied by Donnelly additionally with a jump term whose kernel is also modulated by a Markov chain. It can be noted that the Markov modulated process has been quite popular with its recent applications to finance for example options pricing (Deshpande and Ghosh [12]) and references therein and to portfolio optimization refer Xhou and Yin [35]. However application of semi-Markov modulated process to portfolio optimization in which the portfolio wealth process is a semi-Markov modulated diffusion are not many, see for example Ghosh and Goswami [20]. Even so it appears that the sufficient maximum principle has not been formulated for the case of a semi-Markov modulated diffusion process with jumps and studied further in the context of quadratic portfolio optimization. Moreover, application of the sufficient stochastic maximum principle in the context of risk-sensitive control portfolio optimization with the portfolio wealth process following a semi-Markov modulated diffusion process has not been studied. This chapter aims to provide these missing dots and connect them together. For the same reasons, along with providing a popular application of the sufficient stochastic maximum principle to a quadratic- loss minimization problem when the portfolio wealth process follows a semi-Markov modulated jump-diffusion, we also provide an example of risk-sensitive portfolio optimization for the diffusion part of the said dynamics.

The chapter is organized as follows. In the next section we formally describe basic terminologies used in the chapter. In section 3 we detail the control problem that we are going to study. The sufficient maximum principle is proven in Section 4. This is followed by establishing its connection with the dynamic programming. We conclude the chapter by illustrating its applications to risk-sensitive control optimization and to a quadratic loss minimization problem.

2 Mathematical Preliminaries

We adopt the following notations that are valid for the whole paper:
• Let $\mathbb{R}$: be the reals
• r, M: be any positive integers greater than 1.
• $\mathcal{X} = \{1, ..., M\}$.
• $\mathcal{C}^{1,2,1}([0,T]\times\mathbb{R}^r\times\mathcal{X}\times\mathbb{R}_+)$: denote the family of all functions on $[0,T]\times\mathbb{R}^r\times\mathcal{X}\times\mathbb{R}_+$ which are twice continuously differentiable in x and continuously differentiable in t and y.
• $v^{'}$, $A^{'}$: the transpose of the vector v and matrix A respectively.
• $||v||$: Euclidean norm of a vector v.
• $|A|$: norm of a matrix A.
• $tr(A)$: trace of a square matrix A.
• $C_b^m(\mathbb{R}^r)$: Set of real, m-times continuously-differentiable functions which are bounded together with their derivatives up to the m^{th} order.

We assume that the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, \mathbb{P})$ is complete with filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ and is right-continuous and $\mathcal{F}(0)$ contains all $\mathbb{P}$ null sets. Let $\{\theta(t)\}_{t\geq 0}$ be a semi-Markov process adapted to the filtration and takes values in $\mathcal{X}$ with transition probability p_{ij} with conditional holding time distribution $F^h(t|i)$. Thus if $0 \leq t_0 \leq t_1 \leq ...$ are the times when jumps occur, then

$$P(\theta(t_{n+1}) = j, t_{n+1} - t_n \leq t | \theta(t_n) = i) = p_{ij} F^h(t|i). \tag{2.1}$$

Further, we assume the matrix $[p_{ij}]_{\{i,j=1,...,M\}}$ is irreducible and for each i, $F^h(\cdot|i)$ has continuously differentiable and bounded density $f^h(\cdot|i)$. For a fixed t, let $n(t) \triangleq \max\{n : t_n \leq t\}$ and $Y(t) \triangleq t - t_{n(t)}$. Thus $Y(t)$ represents the amount of time the proess $\theta(t)$ is at the current state after the last jump. The process $(\theta(t), Y(t))$defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is jointly Markov and the differential generator $\mathcal{L}$ given as follows (Chap.2, [22])

$$\mathcal{L}\phi(i,y) = \frac{d}{dy}\phi(i,y) + \frac{f^h(y|i)}{1 - F^h(y|i)} \sum_{j\neq i, j\in\mathcal{X}} p_{ij}[\phi(j,0) - \phi(i,y)]. \tag{2.2}$$

for $\phi : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$ a C^1 function.

We first represent the semi-Markov process $\theta(t)$ as a stochastic integral with respect to a Poisson random measure. With that aim in mind, embed $\mathcal{X}$ in $\mathbb{R}^M$ by identifying i with $e_i \in \mathbb{R}^M$. For $y \in [0, \infty)$ $i, j \in \mathcal{X}$, define

$$\begin{aligned}\lambda_{ij}(y) &= p_{ij}\frac{f^h(y/i)}{1 - F^h(y/i)} \geq 0 \text{ and } \forall \ i \neq j,\\ \lambda_{ii}(y) &= -\sum_{j\in\mathcal{X}, j\neq i}^{M} \lambda_{ij}(y) \ \forall \ i \ \in \mathcal{X}.\end{aligned}$$

For $i \neq j \in \mathcal{X}$, $y \in \mathbb{R}_+$ let $\Lambda_{ij}(y)$ be consecutive (with respect to lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left-closed, right-open intervals of the real line, each having length $\lambda_{ij}(y)$. Define the functions $\bar{h} : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^r$ and $\bar{g} : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$\bar{h}(i, y, z) = \begin{cases} j - i & \text{if } z \in \Lambda_{ij}(y) \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}(i, y, z) = \begin{cases} y & \text{if } z \in \Lambda_{ij}(y), j \neq i \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$ be the set of all nonnegative integer-valued σ-finite measures on Borel σ-field of $(\mathbb{R}_+ \times \mathbb{R})$. The process $\{\tilde{\theta}(t), Y(t)\}$ is defined by the following stochastic integral equations:

$$\begin{aligned}\tilde{\theta}(t) &= \tilde{\theta}(0) + \int_0^t \int_{\mathbb{R}} \bar{h}(\tilde{\theta}(u-), Y(u-), z) N_1(du, dz),\\ Y(t) &= t - \int_0^t \int_{\mathbb{R}} \bar{g}(\tilde{\theta}(u-), Y(u-), z) N_1(du, dz),\end{aligned} \tag{2.3}$$

where $N_1(dt, dz)$ is an $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$-valued Poisson random measure with intensity $dt \quad m(dz)$ independent of the $\mathcal{X}$-valued random variable $\tilde{\theta}(0)$, where $m(\cdot)$ is Lebesgue measure on $\mathbb{R}$. As usual, by definition, $Y(t)$ represents the amount of time process $\tilde{\theta}(t)$ is at the current state after the last jump. We define the corresponding compensated or centered one- dimensional Poisson measure as $\tilde{N}_1(ds, dz) = N_1(ds, dz) - dsm(dz)$. It was shown in Theorem 2.1 of Ghosh and Goswami [20] that $\tilde{\theta}(t)$ is a semi-Markov process with transition probability matrix $[p_{ij}]_{\{i,j=1,\ldots,M\}}$ with conditional holding time distributions $F^h(y|i)$. Since by definition, $\theta(t)$ is also a semi-Markov process with transition probability matrix $[p_{ij}]_{\{i,j=1,\ldots,M\}}$ with conditional holding time distributions $F^h(y|i)$ defined on the same underlying probability

space, by equivalence, $\tilde{\theta}(t) =^d \theta(t)$ for $t \geq 0$.
Remark 1 The semi-Markov process with conditional density $f^h(y|i) = \tilde{\lambda}_i e^{-\tilde{\lambda}_i y}$ for some $\tilde{\lambda}_i > 0$, $i = 1, 2..., M$, is in fact a Markov chain.

3 The control problem

Let $\mathcal{U} \subset \mathbb{R}^r$ be a closed subset. Let $\mathbb{B}_0$ be the family of Borel sets $\Gamma \subset \mathbb{R}^r$ whose closure $\bar{\Gamma}$ does not contain 0. For and Borel set $B \subset \Gamma$, one dimensional poisson random measure $N(t, B)$ counts the number of jumps on $[0, t]$ with values in B. For a predictable process $u : [0, T] \times \Omega \to \mathcal{U}$ with left continuous right limit paths, consider the controlled process X with given initial condition $X(0) = x \in \mathbb{R}^r$ given by

$$\begin{aligned} dX(t) &= b(t, X(t), u(t), \theta(t))dt + \sigma(t, X(t), u(t), \theta(t))dW(t) \\ &+ \int_\Gamma g(t, X(t), u(t), \theta(t)), \gamma)N(dt, d\gamma), \end{aligned} \tag{3.1}$$

where $X(t) \in \mathbb{R}^r$ and $W(t) = (W_1(t), ..., W_r(t))$ is r-dimensional standard Brownian motion. The coefficients $b(\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \to \mathbb{R}^r$,$\sigma(\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \to \mathbb{R}^r \times \mathbb{R}^r$ and $g(\cdot,\cdot,\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \times \Gamma \to \mathbb{R}^r$ and satisfy the following conditions,
Assumption (A1)
(At most linear growth) *There exists a constant* $C_1 < \infty$ *for any* $i \in \mathcal{X}$ *such that* $|\sigma(t, x, u, i)|^2 + ||b(t, x, u, i)||^2 + \int_\mathbb{R} ||g(t, x, u, i, \gamma)||^2 \lambda(d\gamma) \leq C_1(1 + ||x||^2)$
(Lipschitz continuity) *There exists a constant* $C_2 < \infty$ *for any* $i \in \mathcal{X}$ *such that*

$$\begin{aligned} |\sigma(t, x, u, i) - \sigma(t, y, u, i)|^2 &+ ||b(t, x, u, i) - b(t, y, u, i)||^2 \\ &+ \int_\Gamma ||g(t, x, u, i, \gamma) - g(t, y, u, i, \gamma)||^2 \lambda(d\gamma) \leq C_2||x - y||^2 \end{aligned}$$

$\forall x, y \in \mathbb{R}^r$.
Then $X(t)$ is a unique cadlag adapted solution given by (3.1)[refer Theorem 1.19 of [30]].

Define $a(t, x, u, i) = \sigma(t, x, u, i)\sigma'(t, x, u, i)$ is a $\mathbb{R}^{r \times r}$ matrix and $a_{kl}(t, x, u, i)$ is the $(k, l)^{th}$ element of the matrix a while $b_k(t, x, u, i)$ is the k^{th} element of the vector $b(t, x, u, i)$. We assume that $N(\cdot,\cdot), N_1(\cdot,\cdot)$ and θ_0, W_t, X_0 defined on $(\Omega, \mathcal{F}, \mathbb{P})$ are independent. For future use we define the compensated Poisson measure $\tilde{N}(dt, d\gamma) = N(dt, d\gamma) - \lambda(d\gamma)dt$, where $\lambda(\cdot)$ is the jump distribution (so a probability measure) and $0 < \lambda < \infty$ is the jump rate such that $\int_\Gamma \min(||\gamma||^2, 1)\lambda(d\gamma) < \infty$.

Consider the performance criterion

$$J^u(x,i,y) = E^{x,i,y}[\int_0^T f_1(t,X(t),u(t),\theta(t),Y(t))dt + f_2(X(T),\theta(T),Y(T))], \quad (3.2)$$

where $f_1 : [0,T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and $f_2 : \mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$ is concave. We say that $u \in \mathcal{A}(T)$ is the admissible class of controls, if

$$E^{x,i,y}\left[\int_0^T |f_1(t,X(t),u(t),\theta(t),Y(t))|dt + f_2(X(T),\theta(T),Y(T))]\right] < \infty.$$

The problem is to maximize J^u over all $u \in \mathcal{A}(T)$ so we seek $\hat{u} \in \mathcal{A}(T)$ such that

$$J^{\hat{u}}(x,i,y) = \sup_{u \in \mathcal{A}(T)} J^u(x,i,y), \quad (3.3)$$

where $\hat{u}$ is an optimal control.
Define a Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^r \times \mathcal{U} \times \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R}^r \times \mathbb{R}^{r\times r} \times \mathbb{R}^r \to \mathbb{R}$ by,

$$\begin{aligned}\mathcal{H}(t,x,u,i,y,p,q,\eta) \quad &:= \quad f_1(t,x,u,i,y) + \Big(b^{'}(t,x,u,i) - \int_\Gamma g^{'}(t,x,u,i,\gamma)\pi(d\gamma)\Big)p \\ &+ \quad tr(\sigma^{'}(t,x,u,i)q) + \Big(\int_\Gamma g^{'}(t,x,u,i,\gamma)\pi(d\gamma)\Big)\eta. \quad (3.4)\end{aligned}$$

We assume that the Hamiltonian $\mathcal{H}$ is differentiable with respect to x. The adjoint equation corresponding to u and X^u in the unknown adapted processes $p(t) \in \mathbb{R}^r$,$q(t) \in \mathbb{R}^{r\times r}$, $\eta : \mathbb{R}_+ \times \mathbb{R}^r - \{0\} \to \mathbb{R}^r$ and $\tilde{\eta}(t,z) = (\eta^{(1)}(t,z),...,\eta^{(r)}(t,z))'$, where $\tilde{\eta}^{(n)}(t,z) \in \mathbb{R}^{r\times r}$ for each $n = 1,2,...,r$, is the backward stochastic differential equation (BSDE),

$$\begin{aligned}dp(t) \quad &= \quad -\nabla_x\mathcal{H}(t,X(t),u(t),\theta(t),p(t),q(t),\eta(t,\gamma))dt + q^{'}(t)dW(t) + \int_\Gamma \eta(t,\gamma)\tilde{N}(dt,d\gamma) \\ &+ \quad \int_{\mathbb{R}} \tilde{\eta}(t,z)\tilde{N}_1(dt,dz), \\ p(T) \quad &= \quad \nabla_x f_2(X(T),\theta(T),Y(T)). \ \ a.s. \quad (3.5)\end{aligned}$$

We have assumed that $\mathcal{H}$ is differentiable with respect to $x = X(t)$ and its derivative is denoted as $\nabla_x\mathcal{H}(t,X(t),u(t),\theta(t),p(t),q(t),\eta(t,\gamma))$.

As per Remark 1, for the special case where the semi-Markov process has exponential holding time distribution, we would have (3.5) to be a BSDE with Markov chain switching. For this special case, Cohen and Elliott [6] have provided conditions for uniqueness of the solution. However, the corresponding uniqueness result for the semi-Markov modulated BSDE as in (3.5) seems not available in the literature. Since this paper concerns sufficient conditions, we will assume ad hoc that a solution to this BSDE exists and is unique.

Remark 2 Notice that there are jumps in the adjoint equation (3.5) attributed to jumps in the semi-Markov process $\theta(t)$. This is because the drift, the diffusion and the jump kernel of the process $X(t)$ is modulated by a semi-Markov process. Also note that the unknown process $\tilde{\eta}(t,z)$ in the adjoint equations (3.5) does not appear in the Hamiltonian (3.4).

4 Sufficient Stochastic Maximum principle

In this section we state and prove the sufficient stochastic maximum principle.

Theorem 1(Sufficient Maximum principle) Let $\hat{u} \in \mathcal{A}(T)$ with corresponding solution $\hat{X} \triangleq X^{\hat{u}}$. Suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{\eta}(t,\gamma), \hat{\tilde{\eta}}(t,z))$of the adjoint equation (3.5) satisfying

$$E\int_0^T ||\Big(\sigma(t,\hat{X}(t),\theta(t)) - \sigma(t,X^u(t),\theta(t))\Big)^{'}\hat{p}(t)||^2 dt < \infty \tag{4.1}$$

$$E\int_0^T ||\hat{q}^{'}(t)\Big(\hat{X}(t) - X^u(t)\Big)||^2 dt < \infty \tag{4.2}$$

$$E\int_0^T ||(\hat{X}(t) - X^u(t))^{'}\hat{\eta}(t,\gamma)||^2 \pi(d\gamma)dt < \infty \tag{4.3}$$

$$E\int_0^T |\Big(\hat{X}(t) - X^u(t)\Big)^{'}\hat{\tilde{\eta}}(t,z)|^2 m(dz)dt < \infty. \tag{4.4}$$

for all admissible controls $u \in \mathcal{A}(T)$. If we further suppose that

1.

$$\mathcal{H}(t,\hat{X}(t),\hat{u}(t),\theta(t),Y(t),\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\cdot)) = \sup_{u\in\mathcal{A}(T)} \mathcal{H}(t,\hat{X}(t),u(t),\theta(t),Y(t),\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\cdot)). \tag{4.5}$$

2. for each fixed $(t,i,y) \in ([0,T]\times\mathcal{X}\times\mathbb{R}_+)$, $\hat{\mathcal{H}}(x) := \sup_{u\in\mathcal{A}(T)} \mathcal{H}(t,x,u,i,y,\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\cdot))$ exists and is a concave function of x. Then $\hat{u}$ is an optimal control.

Proof Fix $u \in \mathcal{A}(T)$ with corresponding solution $X = X^u$. For sake of brevity we will henceforth represent $(t,\hat{X}(t-),\hat{u}(t-),\theta(t-),Y(t-))$ by

$(t, \hat{X}(t-))$ and $(t, X(t-), u(t-), \theta(t-), Y(t-))$ by $(t, X(t-))$. Then,

$$
\begin{aligned}
J(\hat{u}) - J(u) &= E\Big(\int_0^T \Big(f_1(t, \hat{X}(t)) - f_1(t, X(t)) \Big) dt \\
&+ f_2(\hat{X}(T), \theta(T), Y(T)) - f_2(X(T), \theta(T), Y(T)) \Big).
\end{aligned}
$$

By concavity of $f_2(\cdot, i, y)$ we have for each $i \in \mathcal{X},\ y \in \mathbb{R}_+$ and (3.5) the inequalities,

$$
\begin{aligned}
E\Big(f_2(\hat{X}(T), \theta(T), Y(T)) &- f_2(X(T), \theta(T), Y(T)) \Big) \\
&\geq E\Big((\hat{X}(T) - X(T))' \nabla_x f_2(\hat{X}(T), \theta(T), Y(T)) \Big) \\
&\geq E\Big((\hat{X}(T) - X(T))' \hat{p}(T) \Big).
\end{aligned}
$$

which gives

$$
J(\hat{u}) - J(u) \geq E\int_0^T \Big(f_1(t, \hat{X}(t)) - f_1(t, X(t)) \Big) dt + E\Big((\hat{X}(T) - X(T))' \hat{p}(T) \Big). \tag{4.6}
$$

We now expand the above equation (4.6) term by term. For the first term in this equation we use the definition of $\mathcal{H}$ as in (3.4) to obtain

$$
\begin{aligned}
& E\int_0^T \Big(f_1(t, \hat{X}(t)) - f_1(t, X(t)) \Big) dt \\
= \;& E\int_0^T \Big(\mathcal{H}(t, \hat{X}(t), \hat{u}(t), \theta(t), \hat{p}(t), \hat{q}(t), \hat{\eta}(t, \gamma)) \\
- \;& \mathcal{H}(t, X(t), u(t), \theta(t), p(t), q(t), \eta(t, \gamma)) \Big) dt \\
- \;& E\int_0^T \Big[\Big(b(t, \hat{X}(t)) - b(t, X(t)) \\
- \;& \int_\Gamma \Big(g(t, \hat{X}(t-), \hat{u}(t-), \theta(t-), \gamma) - g(t, X(t-), u(t-), \theta(t-), \gamma) \Big) \pi(d\gamma) \Big) \hat{p}(t) \\
+ \;& tr\Big((\sigma(t, \hat{X}(t)) - \sigma(t, X(t)))' \hat{q}(t) \Big) \\
+ \;& \int_\Gamma (g(t, \hat{X}(t-), \hat{u}(t-), \theta(t-), \gamma) - g(t, X(t-), u(t-), \theta(t-), \gamma))' \eta(t, \gamma) \pi(d\gamma) \Big] dt.
\end{aligned} \tag{4.7}
$$

To expand the second term on the right hand side of (4.6) we begin by applying the integration by parts formula to get,

$$
\begin{aligned}
(\hat{X}(T) - X(T))^{'}\hat{p}(T) &= \int_0^T (\hat{X}(t) - X(t))^{'} d\hat{p}(t) \\
&+ \int_0^T \hat{p}^{'}(t) d(\hat{X}(t) - X(t)) + [\hat{X} - X, \hat{p}](T).
\end{aligned}
$$

Substitute for X, $\hat{X}$ and $\hat{p}$ from (3.1) and (3.5) respectively to obtain,

$$
\begin{aligned}
& (\hat{X}(T) - X(T))^{'}\hat{p}(T) \\
= & \int_0^T (\hat{X}(t) - X(t))^{'} \Big(- \nabla_x \mathcal{H}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{\eta}(t,\gamma)) dt + \hat{q}^{'}(t) dW(t) \\
+ & \int_\Gamma \hat{\eta}(t,\gamma) \tilde{N}(dt, d\gamma) + \int_\mathbb{R} \hat{\tilde{\eta}}(t,z) \tilde{N}_1(dt, dz) \Big) \\
+ & \int_0^T \hat{p}^{'}(t) \Big\{ \Big(\Big(b(t, \hat{X}(t)) - b(t, X(t)) \Big) - \int_\Gamma \Big(g(t, \hat{X}(t), \hat{u}(t-), \theta(t-), \gamma) \\
- & g(t, X(t-), u(t-), \theta(t-), \gamma) \Big) \pi(d\gamma) \Big) dt \\
+ & \Big(\sigma(t, \hat{X}(t)) - \sigma(t, X(t)) \Big)^{'} dW(t) \\
+ & \int_\Gamma \Big(g(t, \hat{X}(t-), \hat{u}(t-), \theta(t-), \gamma) - g(t, X(t-), u(t-), \theta(t-), \gamma) \Big) \tilde{N}(dt, d\gamma) \Big\} \\
+ & \int_0^T \Big[tr \Big(\hat{q}^{'}(t) \Big(\sigma(t, \hat{X}(t)) - \sigma(t, X(t)) \Big) \Big) \\
+ & \int_\Gamma \Big(g(t, \hat{X}(t), \hat{u}(t-), \theta(t-), \gamma) - g(t, X(t), u(t-), \theta(t-), \gamma) \Big)^{'} \eta(t,\gamma) \pi(d\gamma) \Big] dt.
\end{aligned}
$$

Due to integrability conditions (4.1)-(4.4), the integral with respect to the Brownian motion and the Poisson random measure are square integrable martingales which are null at the origin. Thus taking expectations we obtain

$$
\begin{aligned}
& E\Big((\hat{X}(T)-X(T))^{'}\hat{p}(T)\Big) \\
= & \int_0^T (\hat{X}(t)-X(t))^{'}\Big(-\nabla_x\mathcal{H}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\gamma))\Big)dt \\
+ & \int_0^T \Big[\hat{p}^{'}(t)\Big(b(t,\hat{X}(t))-b(t,X(t))-\int_\Gamma\Big(g(t,\hat{X}(t-),\hat{u}(t-),\theta(t-),\gamma) \\
- & g(t,X(t),u(t-),\theta(t-),\gamma)\Big)\pi(d\gamma)\Big) \\
+ & \int_0^T tr\Big(\hat{q}^{'}(t)(\sigma(t,\hat{X}(t))-\sigma(t,X(t)))\Big) \\
+ & \int_\Gamma\Big(\Big(g(t,\hat{X}(t-),\theta(t-),u(t-),\gamma)-g(t,X(t-),\theta(t-),u(t-),\gamma)\Big)^{'}\eta(t,\gamma))\Big)\pi(d\gamma)\Big]dt.
\end{aligned}
$$

Substitute the last equation and (4.7) into the inequality (4.6) to find after cancellation that

$$
\begin{aligned}
J(\hat{u})-J(u) \geq\ & E\int_0^T\Big(\mathcal{H}(t,\hat{X}(t),\hat{u}(t),\theta(t),\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\gamma)) \\
- & \mathcal{H}(t,X(t),u(t),\theta(t),p(t),q(t),\eta(t,\gamma)) \\
- & (\hat{X}(t)-X(t))^{'}\nabla_x\mathcal{H}(t,\hat{X}(t),\hat{u}(t),\theta(t),\hat{p}(t),\hat{q}(t),\hat{\eta}(t,\gamma))\Big)dt.
\end{aligned}
\tag{4.8}
$$

We can show that the integrand on the RHS of (4.8) is non-negative a.s. for each $t \in [0,T]$ by fixing the state of the semi-Markov process and then using the assumed concavity of $\hat{\mathcal{H}}(x)$, we apply the argument in Framstad et al. [19] . This gives $J(\hat{u}) \geq J(u)$ and so $\hat{u}$ is an optimal control.

5 Connection to dynamic programming

We show the connection between the stochastic maximum principle and dynamic programming principle for the semi-Markov modulated regime switching jump diffusion. This tantamounts to explicitly showing connection between the value function $V(t,x,i,y)$ of the control problem and the adjoint processes $p(t), q(t)$,$\eta(t,\gamma)$ and $\tilde{\eta}(t,z)$. In order to apply the dynamic programming principle we put the problem into a Markovian framework by defining

$$J^u(t,x,i,y) \triangleq E^{X(t)=x,\theta(t)=i,Y(t)=y}[\int_t^T f_1(t,X(t),u(t),\theta(t),Y(t))dt + f_2(X(T),\theta(T),Y(T))], \tag{5.1}$$

and put

$$V(t,x,i,y) = \sup_{u\in\mathcal{A}(T)} J^u(t,x,i,y) \quad \forall \ (t,x,i,y) \in [0,T]\times\mathbb{R}^r\times\mathcal{X}\times\mathbb{R}_+. \tag{5.2}$$

Theorem 2 *Assume that* $V(\cdot,\cdot,i,\cdot) \in \mathcal{C}^{1,3,1}([0,T]\times\mathbb{R}^r\times\mathcal{X}\times\mathbb{R}_+)$ *for each* $i,j \in \mathcal{X}$ *and that there exists an optimal Markov control* $\hat{u}(t,x,i,y)$ *for (5.2), with the corresponding solution* $\hat{X} = X^{(\hat{u})}$. *Define*

$$p_k(t) \triangleq \frac{\partial V}{\partial x_k}(t,\hat{X}(t),\theta(t),Y(t)). \tag{5.3}$$

$$q_{kl}(t) \triangleq \sum_{i=1}^{r} \sigma_{il}(t,\hat{X}(t),\hat{u}(t),\theta(t))\frac{\partial^2 V}{\partial x_i \partial x_k}(t,\hat{X}(t),\theta(t),Y(t)). \tag{5.4}$$

$$\eta^{(k)}(t,\gamma) \triangleq \frac{\partial V}{\partial x_k}(t,\hat{X}(t),j,Y(t)) - \frac{\partial V}{\partial x_k}(t,\hat{X}(t),i,Y(t)). \tag{5.5}$$

$$\begin{aligned}\tilde{\eta}^{(k)}(t,z) &\triangleq \frac{\partial V}{\partial x_k}(t,\hat{X}(t-),\theta(t-)+\bar{h}(\theta(t-),Y(t-),z),Y(t-)-\bar{g}(\theta(t-),Y(t-),z)) \\ &- \frac{\partial V}{\partial x_k}(t,\hat{X}(t-),\theta(t-),Y(t-)).\end{aligned} \tag{5.6}$$

for each $(k,l=1,...,r)$. *Also we assume that the coefficients* $b(t,x,u,i)$, $\sigma(t,x,u,i)$ *and* $g(t,x,u,i,\gamma)$ *belong to* $C_b^1(\mathbb{R}^r)$. *Then* $p(t), q(t), \eta(t,\gamma)$ *and* $\tilde{\eta}(t,z)$ *solves the adjoint equation (3.5).*

We prove this theorem by using the following Ito's formula.
Theorem 3 Suppose r dimensional process $X(t) = (X_1(t),...,X_r(t))$ or $\{X_o(t)\}$

indexed by $(o = 1, 2, ..., r)$ satisfies the following equation,

$$\begin{aligned} dX_o(t) &= b_o(t, X(t), u(t), \theta(t))dt + \sum_{m=1}^{r} \sigma_{om}(t, X(t), u(t), \theta(t))dW_m(t) \\ &+ \int_\Gamma g_o(t, X(t-), u(t), \theta(t-), \gamma)N(dt, d\gamma). \end{aligned}$$

for some $X(0) = x_0 \in \mathbb{R}^r \quad a.s.$. Further let us assume that the coefficients b, σ, g satisfies the conditions of Assumption (A1).
Let $V(\cdot, \cdot, i, \cdot) \in C^{1,3,1}([0, T] \times \mathbb{R}^r \times \mathcal{X} \times \mathbb{R}_+)$. Then the generalized Ito's formula is given by

$$\begin{aligned} & V(t, X(t), \theta(t), Y(t)) - V(t, x, \theta, y) = \int_0^t GV(s, X(s), \theta(s), Y(s))ds \\ + & \int_0^t (\nabla_x V(s, X(s), \theta(s), Y(s)))'\sigma(s, X(s), \theta(s))dW(s) \\ + & \int_0^t \int_\Gamma [V(s, X(s-) + g(s, X(s-), u(s), \theta(s-), \gamma), \theta(s-), Y(s-)) \\ - & V(s, X(s-), \theta(s-), Y(s-))]\tilde{N}(ds, d\gamma) \\ + & \int_0^t \int_\mathbb{R} [V(s, X(s-), \theta(s-) + \bar{h}(\theta(s-), Y(s-), z), Y(s-) - \bar{g}(\theta(s-), Y(s-), z)) \\ - & V(s, X(s-), \theta(s-), Y(s-))]\tilde{N}_1(ds, dz), \end{aligned}$$

where the local martingale terms are explicitly defined as

$$\begin{aligned} dM_1(t) &\triangleq (\nabla_x V(t, X(t), \theta(t), Y(t)))'\sigma(t, X(t), u(t), \theta(t))dW_t, \\ dM_2(t) &\triangleq \int_\Gamma [V(t, X(t-) + g(t, X(t-), u(t), \theta(t-), \gamma), \theta(t-), Y(t-)) \\ &- V(t, X(t-), \theta(t-), Y(t-))]\tilde{N}(dt, d\gamma), \\ dM_3(t) &\triangleq \int_\mathbb{R} [V\Big(t, X(t-), \theta(t-) + \bar{h}(\theta(t-), Y(t-), z), Y(t-) - \bar{g}(\theta(t-), Y(t-), z)\Big) \\ &- V(t, X(t-), \theta(t-), Y(t-))]\tilde{N}_1(dt, dz), \end{aligned}$$

for

$$
\begin{aligned}
GV(t,x,i,y) =& \frac{\partial V(t,x,i,y)}{\partial t} + \frac{1}{2}\sum_{o,l=1}^{r} a_{ol}(t,x,i)\frac{\partial V(t,x,i,y)}{\partial x_o \partial x_l} \\
&+ \sum_{o=1}^{r} b_o(t,x,i)\frac{\partial V(t,x,i,y)}{\partial x_o} + \frac{\partial V(t,x,i,y)}{\partial y} \\
&+ \frac{f^h(y|i)}{1-F^h(y|i)} \sum_{j\neq i, j\in\mathcal{X}, i=1}^{M} p_{ij}[V(t,x,j,0) - V(t,x,i,y)] \\
&+ \lambda\int_{\Gamma}(V(t,x+g(t,x,i,\gamma),i,y) - V(t,x,i,y))\pi(d\gamma), \forall\ t\ \in\ [0,T]\ , x\in\mathbb{R}^r,
\end{aligned}
$$

$(i = 1, ..., M)$, $y \in \mathbb{R}_+$.

Proof For details refer to Theorem 5.1 in Ikeda and Watanabe [24].

Proof of Theorem 2 From the standard theory of dynamic programming the following HJB equation holds:

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,x,i,y) + \sup_{u\in\mathcal{U}}\{f_1(t,x,u,i,y) + \mathcal{A}^u V(t,x,i,y)\} = 0, \\
V(T,x,i,y) = f_2(x,i,y).
\end{aligned}
$$

where $\mathcal{A}^u$ is the infinitesimal generator of X and the supremum is attained by $\hat{u}(t,x,i,y)$. Define

$$F(t,x,u,i,y) = f_1(t,x,u,i,y) + \frac{\partial V}{\partial t}(t,x,i,y) + \mathcal{A}^u V(t,x,i,y).$$

We assume that f_1 is differentiable w.r.t to x. We use the Ito's formula as described in Theorem 3 to get,

$$
\begin{aligned}
F(t,x,u,i,y) &= f_1(t,x,u,i,y) + \frac{\partial V}{\partial t}(t,x,i,y) \\
&+ \sum_{k=1}^{r}\frac{\partial V}{\partial x_k}(t,x,i,y)b_k(t,x,u,i) \\
&+ \frac{1}{2}\sum_{k=1}^{r}\sum_{l=1}^{r}\frac{\partial^2 V}{\partial x_k \partial x_l}(t,x,i,y)\sum_{i=1}^{r}\sigma_{ki}(t,x,u,i)\sigma_{li}(t,x,u,i) \\
&+ \sum_{j\neq i, i=1}^{M}\frac{p_{ij}f^h(y|i)}{1-F^h(y|i)}(V(t,x,j,0) - V(t,x,i,y)) + \frac{\partial V}{\partial y}(t,x,i,y) \\
&+ \lambda\int_{\Gamma}(V(t,x+g(t,x,u,i,\gamma),i,y) - V(t,x,i,y))\pi(d\gamma). \qquad (5.7)
\end{aligned}
$$

Differentiating $F(t,x,\hat{u}(t,x,i,y),i,y)$ with respect to x_o and evaluate at $x=\hat{X}(t)$, $i=\theta(t)$ and $y=Y(t)$, we get,

$$
\begin{aligned}
0 &= \frac{\partial f_1}{\partial x_o}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t),Y(t)) \\
&+ \frac{\partial^2 V}{\partial x_o \partial t}(t,\hat{X}(t),\theta(t),Y(t)) \\
&+ \sum_{k=1}^{r} \frac{\partial^2 V}{\partial x_o \partial x_k}(t,\hat{X}(t),\theta(t),Y(t)) b_k(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \\
&+ \sum_{k=1}^{r} \frac{\partial V}{\partial x_k}(t,\hat{X}(t),\theta(t),Y(t)) \frac{\partial b_k}{\partial x_o}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \\
&+ \frac{1}{2}\sum_{k=1}^{r}\sum_{l=1}^{r} \frac{\partial^3 V}{\partial x_o \partial x_k \partial x_l}(t,\hat{X}(t),\theta(t),Y(t)) \\
&\times \sum_{i=1}^{r} \sigma_{k,i}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \sigma_{l,i}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \\
&+ \frac{1}{2}\sum_{k=1}^{r}\sum_{l=1}^{r} \frac{\partial^2 V}{\partial x_k \partial x_l}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t),Y(t)) \\
&\times \frac{\partial}{\partial x_o}\sum_{i=1}^{r} \sigma_{k,i}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \sigma_{l,i}(t,\hat{X}(t),\hat{u}(t,\hat{X}(t),\theta(t),Y(t)),\theta(t)) \\
&+ \sum_{j\neq i, j\in\mathcal{X}}^{M} \frac{p_{ij} f^h(y|i)}{1-F^h(y|i)} \left(\frac{\partial V}{\partial x_o}(t,\hat{X}(t),j,0) - \frac{\partial V}{\partial x_o}(t,\hat{X}(t),i,y) \right) \\
&+ \lambda \int_{\Gamma} \left(\frac{\partial V}{\partial x_o}(t,\hat{X}(t)+g(t,\hat{X}(t),\theta(t),\gamma),\theta(t),Y(t)) - \frac{\partial V}{\partial x_o}(t,\hat{X}(t),\theta(t),Y(t)) \right) \pi(d\gamma).
\end{aligned}
\tag{5.8}
$$

Next define, $Y_o = \frac{\partial V}{\partial x_o}(t, \hat{X}(t), \theta(t), Y(t))$ for $(o = 1, ..., r)$. By Ito's formula (Theorem 3) we obtain the dynamics of $Y_o(t)$ as follows,

$$
\begin{aligned}
dY_o(t) &= \Bigg\{\frac{\partial^2 V}{\partial x_o \partial t}(t, \hat{X}(t), \theta(t), Y(t)) \\
&+ \sum_{k=1}^{r} \frac{\partial^2 V}{\partial x_o \partial x_k}(t, \hat{X}(t), \theta(t), Y(t)) b_k(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \\
&+ \frac{1}{2} \sum_{k=1}^{r} \sum_{l=1}^{r} \frac{\partial^3 V}{\partial x_o \partial x_k \partial x_l}(t, \hat{X}(t), \theta(t), Y(t)) \\
&\times \sum_{i=1}^{r} \sigma_{ki}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \times \sigma_{li}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \\
&+ \sum_{j \neq i, j=1}^{M} \frac{p_{ij} f^h(y|i)}{1 - F^h(y|i)} \left(\frac{\partial V}{\partial x_o}(t, \hat{X}(t), j, 0) - \frac{\partial V}{\partial x_o}(t, \hat{X}(t), i, y)\right) \\
&+ \lambda \int_{\Gamma} \Bigg(\frac{\partial V}{\partial x_o}(t, \hat{X}(t) + g(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t), \gamma), \theta(t), Y(t)) \\
&- \frac{\partial V}{\partial x_o}(t, \hat{X}(t), \theta(t), Y(t)) \Bigg) \pi(d\gamma) \Bigg\} dt \\
&+ \sum_{k=1}^{r} \frac{\partial^2 V}{\partial x_o \partial x_k}(t, \hat{X}(t), \theta(t), Y(t)) \sum_{j=1}^{r} \sigma_{kj}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) dW_j(t) \\
&+ \int_{\Gamma} \Bigg\{ \frac{\partial V}{\partial x_o}(t, \hat{X}(t-) + g(t, \hat{X}(t-), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t-), \gamma), \theta(t-), Y(t-)) \\
&- \frac{\partial V}{\partial x_o}(t, \hat{X}(t-), \theta(t-), Y(t-)) \Bigg\} \tilde{N}(dt, d\gamma) \\
&+ \int_{\mathbb{R}} \Bigg\{ \frac{\partial V}{\partial x_o}((t, X(t-), \theta(t-) + \bar{h}(\theta(t-), Y(t-), z), Y(t-) - \bar{g}(\theta(t-), Y(t-), z))) \\
&- \frac{\partial V}{\partial x_o}(t, \hat{X}(t-), \theta(t-), Y(t-)) \Bigg\} \tilde{N}_1(dt, dz).
\end{aligned}
$$

We substitute $\frac{\partial^2 V}{\partial x_o \partial t}$ from (5.8) to get,

$$
\begin{aligned}
dY_o(t) &= -\frac{\partial f_1}{\partial x_o}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t), Y(t))) \\
&- \sum_{k=1}^{r} \frac{\partial V}{\partial x_k}(t, \hat{X}(t), \theta(t), Y(t)) \frac{\partial b_k}{\partial x_o}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \\
&- \frac{1}{2} \sum_{k=1}^{r} \sum_{l=1}^{r} \frac{\partial^2 V}{\partial x_k \partial x_l}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t), Y(t)) \\
&\times \frac{\partial}{\partial x_o}(\sum_{k=1}^{r} \sigma_{ki}(t, \hat{X}(t), \theta(t)) \sigma_{li}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t))) \\
&+ \sum_{k=1}^{r} \frac{\partial^2 V}{\partial x_o \partial x_k}(t, \hat{X}(t), \theta(t), Y(t)) \sum_{j=1}^{r} \sigma_{kj}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) dW_j(t) \\
&+ \int_{\Gamma} \Big\{ (\frac{\partial V}{\partial x_o}(t, \hat{X}(t-) + g(t, X(t-), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t-), \gamma), \theta(t-), Y(t-)) \\
&- \frac{\partial V}{\partial x_o}(t, \hat{X}(t-), \theta(t-), Y(t-))) \Big\} \tilde{N}(dt, d\gamma) \\
&+ \int_{\mathbb{R}} \Big\{ \frac{\partial V}{\partial x_o}((t, X(t-), \theta(t-) + \bar{h}(\theta(t-), Y(t-), z), Y(t-) - \bar{g}(\theta(t-), Y(t-), z))) \\
&- \frac{\partial V}{\partial x_o}(t, \hat{X}(t-), \theta(t-), Y(t-)) \Big\} \tilde{N}_1(dt, dz). \qquad (5.9)
\end{aligned}
$$

We have the following identity,

$$
\begin{aligned}
&\frac{1}{2} \sum_{k=1}^{r} \sum_{l=1}^{r} \frac{\partial^2 V}{\partial x_k \partial x_l}(t, \hat{X}(t), \theta(t), Y(t)) \\
\times &\frac{\partial}{\partial x_o}\Big(\sum_{i=1}^{r} \sigma_{ki}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \sigma_{li}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \Big) \\
= &\sum_{k=1}^{r} \sum_{l=1}^{r} \sum_{i=1}^{r} \sigma_{il}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)) \frac{\partial^2 V}{\partial x_i \partial x_k}(t, \hat{X}(t), \theta(t), Y(t)) \\
\times &\frac{\partial \sigma_{kl}}{\partial x_o}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t)). \qquad (5.10)
\end{aligned}
$$

Next, from (3.4) we obtain,

$$
\begin{aligned}
& \frac{\partial \mathcal{H}}{\partial x_o}(t, X(t), u(t), \theta(t), Y(t), p(t), q(t), \eta(t,\gamma)) \\
= \; & \frac{\partial f_1}{\partial x_o}(t, \hat{X}(t), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t), Y(t)) \\
+ \; & \sum_{i=1}^{r} \Bigg(\frac{\partial b_i}{\partial x_o}(t, \hat{X}(t-), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t-)) \\
- \; & \int_\Gamma \frac{\partial g_i}{\partial x_o}(t, X(t-), \hat{u}(t, \hat{X}(t), \theta(t), Y(t)), \theta(t-), \gamma)\pi(d\gamma)\Bigg) p_i(t) + tr(\frac{\partial \sigma'(t, x, \hat{u}, \theta(t))}{\partial x_o} q) \\
+ \; & \sum_{i=1}^{r} \int_\Gamma \frac{\partial g_i}{\partial x_o}(t, X(t-), \theta(t-), \gamma)\pi(d\gamma)(\eta_i^{(o)}(t,\gamma)). \qquad (5.11)
\end{aligned}
$$

We also note that

$$
\begin{aligned}
tr(\frac{\partial \sigma'(t,x,u,i)}{\partial x_o} q) &= \sum_{l=1}^{r} [\frac{\partial \sigma'(t,x,u,i)}{\partial x_o} q]_{ll} \\
&= \sum_{l=1}^{r} \sum_{k=1}^{r} q_{k,l} \frac{\partial \sigma_{kl}}{\partial x_o}(t,x,u,i).
\end{aligned}
$$

Substituting (5.3)-(5.6) and (5.11) gives,

$$
\begin{aligned}
dY_o(t) &= -\frac{\partial \mathcal{H}}{\partial x_o}(t, X(t), u(t), \theta(t), Y(t), p(t), q(t), \eta(t,\gamma))dt + \sum_{j=1}^{r} q_{oj}(t) dW_j(t) \\
&+ \int_\Gamma \eta(t,\gamma)\tilde{N}(dt, d\gamma) + \int_{\mathbb{R}} \tilde{\eta}(t,z)\tilde{N}_1(dt,dz). \qquad (5.12)
\end{aligned}
$$

Since $Y_o(t) = p_o(t)$ for each $o = 1, ..., r$, we have shown that $p(t), q(t), \eta(t,\gamma)$ and $\tilde{\eta}(t,z)$ solve the adjoint equation (3.5).

6 Applications

We illustrate the theory developed by applying it to some key financial wealth optimization problems. For an early motivation on applying sufficient maximum principle, we first consider wealth dynamics to follow semi-Markov modulated diffusion (no jumps case) and apply it towards the risk-sensitive control portfolio optimization problem. We follow it up by illustrating an application of semi-Markov modulated jump-diffusion wealth dynamics to a quadratic loss minimization problem. Unless otherwise stated, all the processes defined in this section are one dimensional.

Risk-sensitive control portfolio optimization Let us consider a financial market consisting of two continuously traded securities; namely the risk-less bond and a stock. The dynamics of the riskless bond are:

$$dS_0(t) = r(t, \theta(t-))S_0(t)dt \quad S_0(0) = 1.$$

where $r(t, \theta(t))$ is the risk-free interest rate at time t and is modulated by an underlying semi-Markov process as described earlier. The dynamics of the stock price are

$$dS_1(t) = S_1(t)[(\mu(t, \theta(t-)))dt + \sigma(t, \theta(t-))dW(t)],$$

where $(\mu(t, \theta(t-)))$ is the instantaneous expected rate of return and as usual $\sigma(t, \theta(t-))$ is the instantaneous volatility rate. The stock price process is thus driven by a 1-d Brownian motion. We denote the wealth of the investor to be $X(t) \in \mathbb{R}$ at time t. He holds $\theta_1(t)$ proportional units of stock and $\theta_0(t) = 1 - \theta_1(t)$ proportional units is held in the riskless bond market. From the self-financing principle (refer Karatzas and Shreve [25]), the wealth process follows the dynamics given as,

$$\begin{aligned} dX(t) &= (r(t, \theta(t-))X(t) + h(t)\sigma(t, \theta(t-))\bar{m}(t, \theta(t-)))dt + h(t)\sigma(t, \theta(t-))dW(t) \\ X(0) &= x, \end{aligned}$$

where $h(t) = \theta_1(t)S_1(t)$, $\bar{m}(t, i) = \frac{\mu(t,i) - r(t,i)}{\sigma(t,i)} \geq 0$ and the variables $r(t, i), b(t, i)$ and $\sigma(t, i)$, and $\sigma^{-1}(t, i)$ for each $i \in \mathcal{X}$ are measurable and uniformly bounded in $t \in [0, T]$. Also $h(\cdot)$ occuring in the drift and diffusion term in the above dynamics satisfies the following conditions
1. $E[\int_0^T h^2(t)dt] < \infty$
2. $E[\int_0^T |r(t, \theta(t-))X(t) + h(t)\sigma(t, \theta(t-))\bar{m}(t, \theta(t-))|dt + \int_0^T h^2(t)\sigma^2(t, \theta(t-))dt] < \infty$
3. The SDE for X has a unique strong solution.

In a classical risk-sensitive control optimization problem, the investor aims to maximize over some admissible class of portfolio $\mathcal{A}(T)$ the following risk-sensitive criterion given by

$$\begin{aligned} J(\hat{h}(\cdot), x) &= \max_{h \in \mathcal{A}(T)} \frac{1}{\gamma}\mathbb{E}[X(T)^\gamma | X(0) = x, \theta(0) = i, Y(0) = y], \quad \gamma \in (1, \infty) \\ &= -\min_{h \in \mathcal{A}(T)} \frac{1}{\gamma}\mathbb{E}[X(T)^\gamma | X(0) = x, \theta(0) = i, Y(0) = y], \end{aligned}$$

where the exogenous parameter γ is the usual risk-sensitive criterion that describes the risk attitude of an investor. Thus the optimal expected utility function depends on γ and is a generalization of the traditional stochastic control approach to utility optimization in the sense that now the degree of risk aversion of the investor is explicitly parameterized through γ rather than importing it in the problem via an exogeneous utility function. See Whittle [33] for a general overview on risk-sensitive control optimization. We now use the sufficient maximum principle (Theorem 1). Set the control problem $u(t) \triangleq h(t)$.
The corresponding Hamiltonian (for the non-jump case)(3.4) becomes,

$$\mathcal{H}(t,x,u,i,p,q) = (r(t,i)x + u\sigma(t,i)\bar{m}(t,i))p + u\sigma(t,i)q.$$

The adjoint process (3.5) is given by

$$\begin{aligned} dp(t) &= -r(t,\theta(t-))p(t)dt + q(t)dW(t) + \int_{\mathbb{R}} \tilde{\eta}(t,z)\tilde{N}_1(dt,dz), \\ p(T) &= X(T)^{\gamma-1} \quad a.s.. \end{aligned} \tag{6.1}$$

We need to determine $p(t), q(t)$ and $\eta(t,z)$ in (6.1). Going by the terminal condition $p(T)$ we observe that the adjoint process p is the first derivative of (x^γ). Hence we assume that $p(t)$,

$$p(t) = (X(t))^{\gamma-1}e^{\phi(t,\theta(t),Y(t))}.$$

where $\phi(T,\theta(T)=i,Y(T)) = 0 \quad a.s.$ for each $i \in \{1,...,M\}$. Using Ito's formula we get,

$$\begin{aligned} \frac{dp(t)}{p(t)} &= \sum_{i=1}^{M} 1_{\theta(t-)=i}\Bigg((\gamma-1)\Bigg(r(t,\theta(t-)) + \frac{u(t)\sigma(t,\theta(t-))\bar{m}(t,\theta(t-))}{X(t)}\Bigg) \\ &+ \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2(t,\theta(t-))\frac{u^2(t)}{X^2(t)} \\ &+ \phi_t(t,\theta(t-),y) + \phi_y(t,\theta(t-),y) + \frac{f^h(y|\theta(t-)=i)}{1-F^h(y|\theta(t-)=i)}\sum_{j\neq i} p_{ij}(\phi(t,j,0) - \phi(t,\theta(t-),y))\Bigg\}dt \\ &+ (\gamma-1)\frac{u(t)}{X(t)}\sigma(t,\theta(t-))dW(t) \\ &+ \int_{\mathbb{R}}\Bigg(\phi(t,X(t-),\theta(t-)+\bar{h}(\theta(t-),Y(t-),z),Y(t-)-\bar{g}(\theta(t-),Y(t-),z)) \\ &- \phi(t,\theta(t-),Y(t-))\Bigg)\tilde{N}_1(dt,dz). \end{aligned} \tag{6.2}$$

Comparing the coefficients of (6.2) with that in (6.1) we get

$$\begin{aligned}
-r(t,\theta(t-)) &= \sum_{i=1}^{M} 1_{\theta(t-)=i}\Bigg((\gamma-1)\Bigg(r(t,\theta(t-)) + \frac{u(t)\sigma(t,\theta(t-))\bar{m}(t,i)}{X(t)}\Bigg) \\
&+ \frac{1}{2}(\gamma-1)(\gamma-2)\frac{u^2(t)}{X^2(t)} \\
&+ \phi_t(t,\theta(t-),y) + \phi_y(t,\theta(t-),y) \\
&+ \frac{f^h(y|i)}{1-F^h(y|\theta(t-)=i)}\sum_{j\neq i} p_{ij}(\phi(t,j,0)-\phi(t,\theta(t-),y))\Bigg).
\end{aligned} \tag{6.3}$$

$$q(t) = (\gamma-1)\frac{u(t)}{X(t)}\sigma(t,\theta(t-))p(t). \tag{6.4}$$

$$\begin{aligned}
\tilde{\eta}(t,z) &= \Big(\phi(t,\theta(t-)+\bar{h}(\theta(t-),Y(t-),z),Y(t-)-\bar{g}(\theta(t-),Y(t-),z)) \\
&- \phi(t,\theta(t-),Y(t-))\Big)p(t).
\end{aligned} \tag{6.5}$$

Let $\hat{u} \in \mathcal{A}(T)$ be a candidate optimal control corresponding to the wealth process $\hat{X}$ and the adjoint triplet $(\hat{p},\hat{q},\hat{\eta})$, then from the Hamiltonian (3.4) for all $u \in \mathbb{R}$ we have

$$\mathcal{H}(t,\hat{X}(t),u,\theta(t),\hat{p}(t),\hat{q}(t)) = \Big(r(t,\theta(t))\hat{X}(t) + u\sigma(t,\theta(t))\bar{m}(t,\theta(t))\Big)\hat{p}(t) + u\sigma(t,\theta(t))\hat{q}(t). \tag{6.6}$$

As this is a linear function of u, we guess that the coefficient of u vanishes at optimality, which results in the equality

$$\bar{m}(t,\theta(t-))\hat{p}(t) + \hat{q}(t) = 0. \tag{6.7}$$

Substitute equation (6.7) in (6.4) to obtain the expression for the control as

$$\hat{u}(t) = \frac{\bar{m}(t,\theta(t-))}{(1-\gamma)\sigma(t,\theta(t-))}\hat{X}(t). \tag{6.8}$$

We now aim to determine the explicit expression for $p(t)$ which is only possible if we can determine what $\phi(t,\theta(t),Y(t))$ is. We substitute $\hat{u}$ from above and input it in equation (6.3) to get

$$0 = \gamma r(t,\theta(t-)) - \bar{m}^2(t,\theta(t-)) + \frac{(2-\gamma)}{(1-\gamma)}\frac{\bar{m}^2(t,\theta(t-))}{2\sigma^2(t,\theta(t-))}$$
$$+ \phi_t(t,\theta(t-),y) + \phi_y(t,\theta(t-),y) + \frac{f^h(y|\theta(t-)=i)}{1-F^h(y|\theta(t-)=i)} \sum_{i=1,j\neq i}^{M} p_{ij}(\phi(t,j,0) - \phi(t,\theta(t-),y)). \tag{6.9}$$

with terminal boundary condition given as $\phi(T,\theta(T),Y(T)) = 0$ a.s. Consider the process

$$\tilde{\phi}(t,\theta(t),Y(t)) \triangleq E\Big[\exp\Big(\int_t^T \Big\{\gamma r(s,\theta(s)) - \bar{m}^2(s,\theta(s)) + \frac{(2-\gamma)}{(1-\gamma)}\frac{\bar{m}^2(s,\theta(s))}{2\sigma^2(s,\theta(s))}\Big\}ds\Big)|\theta(t-)=i, Y(t-)=y\Big]. \tag{6.10}$$

We aim to show that $\phi = \tilde{\phi}$. For the same we define the following martingale,

$$R(t) \triangleq E\Big[\exp\Big(\int_0^T \Big\{\gamma r(s,\theta(s)) - \bar{m}^2(s,\theta(s)) + \frac{(2-\gamma)}{(1-\gamma)}\frac{\bar{m}^2(s,\theta(s))}{2\sigma^2(s,\theta(s))}\Big\}ds\Big)|\mathcal{F}_t^{\theta,y}\Big], \tag{6.11}$$

where $\mathcal{F}_\tau^{\theta,y} \triangleq \sigma\{\theta(\tau), Y(\tau), \tau \in [0,t]\}$ augmented with $\mathbb{P}$ null sets is the filtration generated by the processes $\theta(t)$ and $Y(t)$. From the $\{\mathcal{F}_t^{\theta,y}\}$-martingale representation theorem, there exist $\{\mathcal{F}_t^{\theta,y}\}$-previsible, square integrable process $\nu(t,i,y)$ such that

$$R(t) = R(0) + \int_0^t \int_{\mathbb{R}} \nu(\tau,\theta(\tau-),Y(\tau-))\tilde{N}_1(d\tau,dz). \tag{6.12}$$

By positivity of $R(t)$ we can define $\hat{\nu}(\tau,\theta(\tau-),Y(\tau-)) \triangleq (\nu(\tau,\theta(\tau-),Y(\tau-)))R^{-1}(\tau-)$ so that

$$R(t) = R(0) + \int_0^t \int_{\mathbb{R}} R(\tau-)\hat{\nu}(\tau,\theta(\tau-),Y(\tau-))\tilde{N}_1(d\tau,dz). \tag{6.13}$$

From the definition of $\tilde{\phi}$ in (6.10) and the definition of R in (6.11) it is easy to see that we have the following relationship

$$R(t) = \tilde{\phi}(t,\theta(t),Y(t)) \exp\Big\{\int_0^t (\gamma r(s,\theta(s)) - \bar{m}^2(s,\theta(s)) + \frac{(2-\gamma)}{(1-\gamma)}\frac{\bar{m}^2(s,\theta(s))}{2\sigma^2(s,\theta(s))})ds\Big\}, \quad \forall\ t \in [0,T]. \tag{6.14}$$

Using Ito's expansion of $\tilde{\phi}(t, \theta(t), Y(t))$ to the RHS of (6.14) followed up by comparing it with martingale representation of $R(t)$ in (6.12) we get $\phi := \tilde{\phi}$. We can thus substitute $\hat{q}$ and $\hat{\tilde{\eta}}$ in expression (6.4),(6.5) in lieu of q and $\tilde{\eta}(t, z)$ respectively. With the choice of control $\hat{u}$ given by (6.8) and boundedness condition on the market parameters r, μ and σ, the conditions in Theorem 1 are satisfied and hence $\hat{u}(t)$ is an optimal control process and the explicit representation of $\hat{p}$ is given by

$$\hat{p}(t) = (X(t))^{\gamma-1} e^{E[\exp(\int_t^T \gamma r(s,\theta(s)) - \bar{m}^2(s,\theta(s)) + \frac{(2-\gamma)}{(1-\gamma)} \frac{\bar{m}^2(s,\theta(s))}{2\sigma^2(s,\theta(s))} ds | \theta(t-)=i, Y(t-)=y)]}.$$

Quadratic loss minimization We now provide an example related to quadratic loss minimization where the portfolio wealth process is given by

$$\begin{aligned} dX^h(t) &= \left(r(t,\theta(t))X^h(t) + h(t)\sigma(t,\theta(t))\bar{m}(t,\theta(t)) - h(t)\int_\Gamma g(t, X^h(t), \theta(t), \gamma)\pi(d\gamma) \right) dt \\ &+ h(t)\sigma(t,\theta(t))dW(t) + h(t)\int_\Gamma g(t, X^h(t), \theta(t), \gamma)\tilde{N}(dt, d\gamma), \\ X^h(0) &= x_0 \;\; a.s. \end{aligned} \tag{6.15}$$

where the market price of risk is defined as $\bar{m}(t,i,y) = \sigma^{-1}(t,i)(b(t,i) - r(t,i))$. As in the earlier example , we have that $\bar{m}(t,i) \geq 0$ and that the variables $r(t,i), b(t,i)$, $\sigma(t,i)$, $\sigma^{-1}(t,i)$ and $g(t,x,i,\gamma)$ for each $i \in \mathcal{X}$ are measurable and uniformly bounded in $t \in [0,T]$. We assume that $g(t,x,i,\gamma) > -1$ for each $i \in \mathcal{X}$ and for a.a. t, x, γ. This insures that $X^h(t) > 0$ for each t. We further assume the following conditions for each $i \in \mathcal{X}$

1. $E[\int_0^T h^2(t)dt] < \infty$.
2. $E[\int_0^T |r(t,i)X(t) + h(t)\sigma(t,i)\bar{m}(t,i)|dt + \int_0^T h^2(t)\sigma^2(t,i)dt + \int_0^T h^2(t)g^2(t,X(t),i,\gamma)dt] < \infty$.
3. $t \to \int_{\mathbb{R}} h^2(t)g^2(t,x,i,\gamma)\pi(d\gamma)$ is bounded.
4. the SDE for X has a unique strong solution.

The portfolio process $h(\cdot)$ satisfying the above four conditions is said to be admissible and belongs to $\mathcal{A}(T)$ (say). We consider the problem of finding an admissible portfolio process $h \in \mathcal{A}(T)$ such that

$$\inf_{h \in \mathcal{A}(T)} E[(X^h(T) - d)^2],$$

over all $h \in \mathcal{A}(T)$. Set the control process $u(t) \triangleq h(t)$ and $X(t) \triangleq X^h(t)$. For this example the Hamiltonian (3.4) becomes

$$\begin{aligned}\mathcal{H}(t,x,h,i,y,p,q,\eta) &= \left[r(t,i)x + u\sigma(t,i)\bar{m}(t,i) - u\int_\Gamma g(t,x,i,\gamma)\pi(d\gamma)\right]p + u\sigma(t,i)q \\ &+ \left(u\int_\Gamma g(t,x,i,\gamma)\pi(d\gamma)\right)\eta, \end{aligned} \tag{6.16}$$

and the adjoint equations are for all time $t \in [0,T)$,

$$\begin{aligned} dp(t) &= -r(t,\theta(t-))p(t)dt + q(t)dW(t) + \int_\Gamma \eta(t,\gamma)\tilde{N}(dt,d\gamma) + \int_{\mathbb{R}} \tilde{\eta}(t,z)\tilde{N}_1(dt,dz), \\ p(T) &= -2X(T) + 2d \ \ a.s. \end{aligned} \tag{6.17}$$

We seek to determine $p(t), q(t), \eta(t,\gamma)$ and $\tilde{\eta}(t,z)$ in (6.17). Going by (6.17) we assume that

$$p(t) = \phi(t,\theta(t),Y(t))X(t) + \psi(t,\theta(t),Y(t)). \tag{6.18}$$

with the terminal boundary conditions being

$$\phi(T,i,y) = -2 \qquad \psi(T,i,y) = 2d \quad \forall\ i\ \in\ \mathcal{X}. \tag{6.19}$$

For the sake of convenience we again rewrite the following Ito's formula for a function $f(t,\theta(t),y(t)) \in \mathcal{C}^{1,2,1}$ given as

$$\begin{aligned} df(t,\theta(t),Y(t)) &= \left(\frac{\partial f(t,\theta(t),Y(t))}{\partial t}\right. \\ &+ \frac{(f^h(y/i))}{(1-F^h(y/i))}\sum_{j\neq i, j=1}^{M} p_{\theta(t-)=i,j}[f(t,j,0) - f(t,\theta(t-),y)] + \left.\frac{\partial f(t,\theta(t),Y(t))}{\partial y}\right)dt \\ &+ \int_{\mathbb{R}} [f(t,\theta(t-)+\bar{h}(\theta(t-),Y(t-),z), Y(t-) - \bar{g}(\theta(t-),Y(t-),z)) - f(t,\theta(t-),Y(t-))]\tilde{N}_1(dt,dz). \end{aligned} \tag{6.20}$$

We apply Ito's product rule to (6.18) to obtain

$$\begin{aligned}
dp(t) &= X(t-)d\phi(t,\theta(t-),Y(t)) + \phi(t,\theta(t-),Y(t))dX(t) + d\phi(t,\theta(t-),Y(t))dX(t) + d\psi(t) \\
&= \sum_{i=1}^{M} 1_{\theta_{t-}=i}\Big\{X(t-)\Big(\phi(t,\theta(t-),y)r(t,\theta(t-)) + \phi_t(t,\theta(t-),Y(t)) + \phi_y(t,\theta(t-),Y(t)) \\
&+ \sum_{i=1,j\neq i}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}(\phi(t,j,0) - \phi(t,\theta(t-),Y(t)))\Big) \\
&+ u(t)\phi(t,\theta(t-),Y(t))\sigma(t,\theta(t-))\bar{m}(t,\theta(t-)) \\
&- u(t)\phi(t,\theta(t-),Y(t))\int_\Gamma g(t,X(t),\theta(t-),\gamma)\pi(d\gamma) + \psi_t(t,\theta(t-),Y(t)) + \psi_y(t,\theta(t-),Y(t)) \\
&+ \sum_{i=1,i\neq j}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}[\psi(t,j,0) - \psi(t,\theta(t-)=i,Y(t))]\Big\}dt \\
&+ u(t)\phi(t,\theta(t-),Y(t))\sigma(t,\theta(t-))dW(t) \\
&+ u(t)\phi(t,\theta(t-),Y(t-))\int_\Gamma g(t,X(t-),\theta(t-),\gamma)\tilde{N}(dt,d\gamma) \\
&+ \int_{\mathbb{R}}\Big[X(t-)(\phi(t,\theta(t-)+\bar{h}(\theta(t-),Y(t-),z),Y(t-)-\bar{g}(\theta(t-),Y(t-),z)) \\
&- \phi(t,\theta(t-),Y(t-))) \\
&+ \psi(t,\theta(t-)+\bar{h}(\theta(t-),Y(t-),z),Y(t-)-\bar{g}(\theta(t-),Y(t-),z)) \\
&- \psi(t,\theta(t-),Y(t-))\Big]\tilde{N}_1(dt,dz).
\end{aligned} \tag{6.21}$$

Comparing coefficients with (6.17) we obtain three equations given as

$$\begin{aligned}
&- r(t,\theta(t-))p(t-) \\
&= \sum_{i=1}^{M} 1_{\{}\theta_{t-}=i, Y(t-)=y\}\Big\{X(t-)\Big(\phi(t,\theta(t-),Y(t))r(t,\theta(t-)) \\
&+ \phi_t(t,\theta(t-),Y(t)) + \phi_y(t,\theta(t-),Y(t)) \\
&+ \sum_{i=1,j\neq i}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}(\phi(t,j,0) - \phi(t,\theta(t-),Y(t)))\Big) \\
&+ u(t)\phi(t,\theta(t-),Y(t))\sigma(t,\theta(t-))\bar{m}(t,\theta(t-)) \\
&- u(t)\phi(t,\theta(t-),Y(t))\int_\Gamma g(t,x,\theta(t-),\gamma)\pi(d\gamma) + \psi_t(t,\theta(t-)),Y(t) + \psi_y(t,\theta(t-),Y(t)) \\
&+ \sum_{i\neq j}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}[\psi(t,j,0) - \psi(t,\theta(t-),Y(t))]\Big\}.
\end{aligned} \tag{6.22}$$

$$q(t) = u(t)\phi(t, \theta(t-), Y(t-))\sigma(t, \theta(t-)). \tag{6.23}$$

$$\eta(t, \gamma) = u(t)\phi(t, \theta(t-), Y(t-))g(t, X(t-), \theta(t-), \gamma). \tag{6.24}$$

$$\begin{aligned}\tilde{\eta}(t, z) &= X(t-)(\phi(t, \theta(t-) + \bar{h}(\theta(t-), Y(t-), z), Y(t-) - \bar{g}(\theta(t-), Y(t-), z)) \\ &- \phi(t, \theta(t-), Y(t-))) + \psi(t, \theta(t-) + \bar{h}(\theta(t-), Y(t-), z), Y(t-) \\ &- \bar{g}(\theta(t-), Y(t-), z)) - \psi(t, \theta(t-), Y(t-)).\end{aligned} \tag{6.25}$$

Let $\hat{u} \in \mathcal{A}(T)$ be a candidate optimal control corresponding to the wealth process $\hat{X}(T)$ and the adjoint triplet $(\hat{p}, \hat{q}, \hat{\eta}, \hat{\tilde{\eta}})$. Then from the Hamiltonian (3.4) for all $u \in \mathcal{A}(T)$ we have

$$\begin{aligned}\mathcal{H}(t, \hat{X}(t), u, \theta(t), \hat{p}(t), \hat{q}(t), \hat{\eta}(t)) &= \Big(r(t, \theta(t))\hat{X}(t) + u\sigma(t, \theta(t))\bar{m}(t, \theta(t)) \\ &- u\int_\Gamma g(t, \hat{X}(t-), \theta(t-), \gamma)\pi d(\gamma)\Big)\hat{p}(t) \\ &+ u\sigma(t, \theta(t))\hat{q}(t) + \Big(u\int_\Gamma g(t, \hat{X}(t-), \theta(t-), \gamma)\pi(d\gamma)\Big)\hat{\eta}(t, \gamma).\end{aligned} \tag{6.26}$$

As this is a linear function of u, we guess that the coefficient of u vanishes at optimality, which results in the following equality

$$\begin{aligned}\hat{q}(t) &= \Big(-\bar{m}(t, \theta(t-)) + \frac{1}{\sigma(t, \theta(t-))}\int_\Gamma g(t, \hat{X}(t), \theta(t), \gamma)\pi(d\gamma)\Big)\hat{p}(t) \\ &- \frac{1}{\sigma(t, \theta(t-))}\int_\Gamma (g^{'}(t, \hat{X}(t), \theta(t), \gamma))\pi(d\gamma)\hat{\eta}(t, \gamma).\end{aligned} \tag{6.27}$$

Also substituting (6.27) for $\hat{q}(t)$ in (6.23) and using (6.18) and(6.24) we get,

$$\hat{u}(t) = \frac{\tilde{\Lambda}(t)}{\Lambda(t)}(\hat{X}(t) + \phi^{-1}(t, \theta(t-), y)\psi(t, \theta(t-), y)), \tag{6.28}$$

where

$$\begin{aligned}\Lambda(t) &= \sigma^2(t,\theta(t-)) + \phi(t,\theta(t-),Y(t))\int_\Gamma g^{'}(t,X(t),\theta(t-),\gamma)g(t,X(t),\theta(t-),\gamma)\pi(d\gamma).\\ \tilde{\Lambda}(t) &= -\bar{m}(t,\theta(t-))\sigma(t,\theta(t-)) + \int_\Gamma g(t,X(t),\theta(t-),\gamma)\pi(d\gamma).\end{aligned} \tag{6.29}$$

To find the optimal control it remains to find ϕ and ψ. To do so set $X(t) := \hat{X}(t), u(t) := \hat{u}(t)$ and $p(t) := \hat{p}(t)$ in (6.22) and then substitute for $\hat{p}(t)$ in (6.18) and $\hat{u}(t)$ from (6.28) . As this result is linear in $\hat{X}(t)$ we compare the coefficient on both side of the resulting equation to get following two equations:

$$\begin{aligned}0 &= 2r\phi(t,i,Y(t)) + \phi_t(t,i,Y(t)) + \phi_y(t,i,Y(t)) + \sum_{i\neq j,i=1}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}(\phi(t,j,0)-\phi(t,i,Y(t)))\\ &+ \frac{\tilde{\Lambda}(t)}{\Lambda(t)}\sigma(t,i)\bar{m}(t,i)\phi(t,i,Y(t)) - \frac{\tilde{\Lambda}(t)}{\Lambda(t)}\phi(t,i,Y(t))\int_\Gamma g(t,X(t),i,\gamma)\pi(d\gamma).\end{aligned} \tag{6.30}$$

$$\begin{aligned}0 &= r\psi(t,i,Y(t)) + \psi_t(t,i,Y(t)) + \psi_y(t,i,Y(t)) + \sum_{i\neq j,i=1}^{M} p_{ij}\frac{f^h(y/i)}{1-F^h(y/i)}(\psi(t,j,0)-\psi(t,i,Y(t)))\\ &+ \frac{\tilde{\Lambda}(t)}{\Lambda(t)}\sigma(t,i)\bar{m}(t,i)\psi(t,i,Y(t)) - \frac{\tilde{\Lambda}(t)}{\Lambda(t)}\psi(t,i,y)\int_\Gamma g(t,X(t),i,\gamma)\pi d(\gamma).\end{aligned} \tag{6.31}$$

with terminal boundary conditions given by (6.19). Consider the following process

$$\begin{aligned}\tilde{\phi}(t,i,y) = -2E\Big[\exp\Big\{\int_t^T\Big(&2r(s,\theta(s-)) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-))\\ &-\frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma)\Big)ds\Big\}|(\theta(s-)=i,Y(t)=y)\Big].\end{aligned} \tag{6.32}$$

$$\begin{aligned}\tilde{\psi}(t,i,y) &= 2dE\Big[\exp\Big\{\int_t^T\Big(r(\theta(s-),s) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-))\\ &- \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma)\Big)ds\Big\}\Big|(\theta(s-)=i,Y(s)=y)\Big].\end{aligned} \tag{6.33}$$

We aim to show that $\phi = \tilde{\phi}$ and $\psi = \tilde{\psi}$. We define the following martingales:

$$R(t) = E\left[\exp\left\{\int_0^T \left(2r(s,\theta(s-)) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-)) - \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma)\right)ds\right\}|\mathcal{F}_t^{\theta,y}\right], \tag{6.34}$$

$$S(t) = E\left[\exp\left\{\int_0^T \left(r(s,\theta(s-)) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-)) - \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma)\right)ds\right\}|\mathcal{F}_t^{\theta,y}\right], \tag{6.35}$$

where $\mathcal{F}_t^{\theta,y}$ is defined as usual. We follow steps similar to that as seen earlier in finite horizon risk-sensitive optimization example and conclude that $\phi = \tilde{\phi}$ and $\psi = \tilde{\psi}$ by using joint-Markov property of $(\theta(t), Y(t))$, to obtain the following expression for the control $\hat{u}(t)$ given as

$$\hat{u}(t) = \frac{\tilde{\Lambda}(t)}{\Lambda(t)}\left(\hat{X}(t) - \frac{dE\left[\exp\left\{\int_t^T (r(s,\theta(s-)) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-)) - \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma))ds\right\}|(\theta(t-)=i,Y(t)=y)\right]}{E\left[\exp\left\{\int_t^T (2r(s,\theta(s-)) + \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\sigma(s,\theta(s-))\bar{m}(s,\theta(s-)) - \frac{\tilde{\Lambda}(s)}{\Lambda(s)}\int_\Gamma g(s,X(s),\theta(s-),\gamma)\pi(d\gamma))ds\right\}|(\theta(t)=i,Y(t)=y)\right]}\right).$$

For the choice of the control parameter and the boundedness conditions on the market parameters r, b,σ and g, the conditions of Theorem 1 are satisfied and hence $\hat{u}$ is the optimal control process.

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