Ricci Flow and Metric Geometry

by

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Contents

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Declarations

The large majority of the original content of this thesis (appearing in Chapters 4 and 5) was conducted in collaboration with my supervisor, Peter M. Topping. To complement this work, I include a survey chapter that exposits already-known results, presented together in the present manner for the first time, to my knowledge. Exhaustive references are provided in the text where results are not my own work.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

This thesis considers two separate problems in the field of Ricci flow on surfaces. Firstly, we examine the situation of the Ricci flow on Alexandrov surfaces, which are a class of metric spaces equipped with a notion of curvature. We extend the existence and uniqueness results of Thomas Richard in the closed case to the setting of *non-compact* Alexandrov surfaces that are *uniformly non-collapsed*. We complement these results with an extensive survey that collects together, for the first time, the essential topics in the metric geometry of Alexandrov spaces due to a variety of authors.

Secondly, we consider a problem in the well-posedness theory of the Ricci flow on surfaces. We show that given an appropriate initial Riemannian surface, we may construct a smooth, complete, immortal Ricci flow that takes on the initial surface in a *geometric* sense, in contrast to the traditional *analytic* notions of initial condition. In this way, we challenge the contemporary understanding of well-posedness for geometric equations.

Chapter 1

Introduction

A very popular and successful approach to tackling geometric and topological problems is the study of *parabolic flows*, pioneered by Eells and Sampson with their introduction of the *harmonic map heat flow* in [15]. Following from this work, Hamilton introduced the central object of study in this thesis, the *Ricci flow*, in the ground-breaking article [18]. This is, heuristically-speaking, an analogue for the usual heat equation posed on the space $\Gamma\left(\mathrm{Sym}^2_+ T^*\mathcal{M}\right)$ of positive-definite, symmetric two-tensor fields on a smooth manifold $\mathcal{M}$, given by the equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}$$

where $\mathrm{Ric}_{g(t)}$ denotes the Ricci curvature tensor of a solution $(g(t))_{t\in[a,b]}$. Hamilton set about constructing a programme to use Ricci flow to prove the Thurston Geometrisation Conjecture (see [41]), which classifies three-manifolds, and contains the famous Poincaré Conjecture as a special case. Building upon the foundations laid by Hamilton, the goal of this programme was ultimately realised through the revolutionary work of Perelman in his seminal papers [29], [30] and [31].

A key concern in the development of the theory of Ricci flow is the problem of *well-posedness.* This encompasses three important questions:

- In what class should we look for *solutions* to the Ricci flow equation?
- In what class should *initial data* lie?
- What does it mean for a Ricci flow to *take on its initial data*?

In Hamilton's introductory article [18], it was shown that given any *closed* Riemannian manifold $(\mathcal{M}, g_0)$, there exists a smooth Ricci flow solution $(g(t))_{t\in[0,T)}$ on $\mathcal{M}$, for some $T > 0$, with $g(0) = g_0$. Some years later, Shi proved the corresponding existence result in the case that g_0 is complete and of bounded curvature in [38], with uniqueness in this case not following until nearly two decades later through the work of Chen and Zhu in [12]. In recent years, Giesen and Topping have provided (see [16] and [46]) an exhaustive theory of so-called *instantaneously complete* Ricci flows on arbitrary Riemannian *surfaces*, giving the most natural description of well-posedness in this dimension to date.

On a similar theme, various authors have considered the pressing question of extending Ricci flow initial data to wider classes of metric spaces. Most notably, Miles Simon proved in [39] the existence (in dimensions two and three) of Ricci flows starting at metric spaces which themselves arise as limits of smooth Riemannian manifolds in the Gromov–Hausdorff topology (see §3.2). In [36] and [37], Thomas Richard proved the uniqueness of such flows in dimension two, when the initial data is a compact *Alexandrov space* with curvature bounded below.

1.1 Outline of the thesis

This thesis consists of two principal components: an examination of Ricci flow on Alexandrov surfaces, making up Chapters 3 and 4, and considerations involving well-posedness of the Ricci flow in dimension two, which form Chapter 5.

Chapter 2 fixes notation in what follows, and includes introductory material and a small collection of foundational results in Ricci flow theory and differential geometry more generally. It may be skipped by the experienced reader.

Chapter 3 consists of an extensive survey of Alexandrov spaces with lower curvature bounds, ranging from fundamental concepts in metric geometry to more advanced material concerning dimension and the Riemannian structure of such spaces. To our knowledge, this is the first time such a comprehensive survey of this material has been assembled. We point out that there are several other well-known texts surveying material related to metric geometry and Alexandrov spaces, such as the standard reference [5], which also includes results concerning Alexandrov spaces with *upper* curvature bounds. It does not include the study of the Riemannian structure of Alexandrov surfaces, which we present in §3.5.

We then turn our attention to Ricci flow on *Alexandrov surfaces* (see Definition

3.5.1) in Chapter 4, beginning with a review of the compact case due to Miles Simon and Thomas Richard. We begin by considering what it means for a Ricci flow to take a metric space as initial condition, and restate the following definition, first suggested by Miles Simon:

Definition 4.1.2. We say that a smooth Ricci flow $(\mathcal{M}^n, g(t))_{t\in(0,T)}$ takes the metric space (X, d) *as initial condition* if the Riemannian distances $d_{g(t)}$ converge uniformly on compact subsets of $\mathcal{M} \times \mathcal{M}$ to a metric $\tilde{d}$ on $\mathcal{M}$ such that $\left(\mathcal{M}, \tilde{d}\right)$ and (X, d) are isometric.

We then restate one of the main results of Richard's thesis [37], which shows that the problem of flowing *compact* Alexandrov surfaces, taking Definition 4.1.2 as our notion of 'initial condition', is well-posed:

Theorem 4.2.1. *Let (X, d) be a compact Alexandrov surface with curvature bounded below by -1. Then there exist a $T > 0$ and a smooth Ricci flow $(\mathcal{M}^2, g(t))_{t\in(0,T)}$ such that*

- $K_{g(t)} \geq -1$ *for all* $t \in (0, T)$ *and*
- $(\mathcal{M}, g(t))$ *takes* (X, d) *as initial condition.*

Moreover, $(\mathcal{M}, g(t))$ *is the unique Ricci flow satisfying these conditions up to diffeomorphism.*

The strategy for proving the existence part of this result is as follows: first, it is shown that (X, d) can be approximated in the Gromov–Hausdorff topology by a sequence of *smooth*, closed Riemannian surfaces $(\mathcal{M}_i, g_i)$, such that $K_{g_i} \geq -1$ for each i. This allows us to use classical existence theory for the Ricci flow, as discussed above, to generate a sequence of Ricci flows $(\mathcal{M}_i, g_i(t))$ that converges in the Cheeger–Gromov sense to a Ricci flow $(\mathcal{M}, g(t))$ as a consequence of a compactness argument. A straightforward application of the definitions of these modes of convergence then shows that this flow takes (X, d) as initial condition, in the sense of Definition 4.1.2.

These considerations lead naturally to the question of extending such an existence and uniqueness result to the setting of *non-compact* initial data. We observe at the outset that this is not a trivial matter. There are two immediate stumbling blocks in adapting such an approach to the non-compact case. Whilst we can find an approximating sequence $(\mathcal{M}_i, g_i)$ without much trouble (see Lemma 4.3.7), flowing such surfaces (which may have no upper bound on curvature) was only recently made possible through the work of Giesen and Topping, which we summarise in Theorem 2.2.4. This allows us to find flows $(\mathcal{M}_i, g_i(t))$ with $g_i(0) = g_i$, and we show that these flows necessary exist for all positive time.

Then, in order to apply Hamilton-style compactness arguments to this sequence of flows, we need that, for some $T > 0$,

$$\sup_i \sup_{\substack{x\in\mathcal{M}_i \\ t\in(0,T)}} \left|K_{g_i(t)}\right|(x) < \infty.$$

Such an estimate is provided by the recent pseudolocality results of Miles Simon:

Theorem 2.4.6. *Let $(\mathcal{M}^2, g(t))_{t\in[0,T)}$ be a smooth, complete Ricci flow and let $x_0 \in \mathcal{M}$. Let $\sigma < 1$ and $v_0, r > 0, N > 1$ be given. Suppose that*

- $\mathrm{vol}_{g(0)}\left(B_r^{g(0)}(x_0)\right) \geq v_0 r^2$ *and*
- $\mathrm{R}_{g(0)} \geq -\frac{N}{r^2}$ *on* $B_r^{g(0)}(x_0)$.

Then there exist $\tilde{v}_0 = \tilde{v}_0(v_0, \sigma, N) > 0$ and $\delta_0 = \delta_0(v_0, \sigma, N) > 0$ such that

- $\mathrm{vol}_{g(t)}\left(B_{r(1-\sigma)}^{g(t)}(x_0)\right) \geq \tilde{v}_0 r^2$
- $|\mathrm{R}_{g(t)}| \leq \frac{1}{r^2\delta_0^2 t}$ *on* $B_{r(1-\sigma)}^{g(t)}(x_0)$

as long as $t \leq \delta_0^2 r^2$.

Such pseudolocality estimates were only known (in the complete, non-compact case) for solutions of *bounded curvature* before this recent work of Miles Simon, a property which we certainly do not have *a priori*. Consequently, our extension of Theorem 4.2.1 to the non-compact case depends critically upon contemporary improvements in the understanding of Ricci flow on smooth surfaces.

With these considerations in hand, we prove the following:

Theorem 4.3.9. *Let (X, d) be a non-compact Alexandrov surface with curvature bounded below by -1. Suppose there exists $v_0 > 0$ such that*

$$\inf_{x\in X} \mathscr{H}^2(B_1(x)) \geq v_0.$$

Then there exists $T = T(v_0) > 0$ and a smooth Ricci flow $(\mathcal{M}^2, g(t))_{t\in(0,T)}$ such that:

1. *$g(t)$ is complete for all $t \in (0,T)$,*

2. *$\left|K_{g(t)}\right| \leq \frac{A}{t}$ for some $A = A(v_0) \geq 0$ and all $t \in (0,T)$,*

3. *$(\mathcal{M}, g(t))$ takes (X,d) as its initial condition (in the sense of Definition 4.1.2).*

We also show that this flow is unique up to conformal equivalence (in Theorem 4.3.12), which again relies on a recent result of Topping.

In Chapter 5, we turn our attention towards well-posedness of the Ricci flow equation on smooth surfaces. As we discuss, the recent existence and uniqueness results of Giesen and Topping for *instantaneously complete* Ricci flows beginning with smooth Riemannian surfaces that may be incomplete or of unbounded curvature represent the fullest picture of well-posedness in this dimension to date. Nevertheless, our principal concern is the long-accepted notion of 'initial condition' for a Ricci flow, namely that a Ricci flow $(g(t))_{t\in[0,T)}$ takes the metric g_0 as initial condition if $g(0) = g_0$, i.e. if $g(t) \to g_0$ smoothly, locally as tensors as $t \to 0$. We point out that such a notion does not necessary imply natural geometric expectations, such as Gromov–Hausdorff convergence of the flow to the initial data (consider, for example, instantaneously complete flows starting at incomplete metrics).

In light of this observation, we propose an alternative notion based on Gromov–Hausdorff convergence (which is introduced in §3.2). The central message of Chapter 5 is that given a suitable Riemannian surface, it is possible to construct two distinct, complete Ricci flows that take this surface as initial condition, in different ways, both of which can be claimed to be natural.

We restrict our attention to a particular class of Riemann surfaces, namely those surfaces Ω that can be conformally embedded into a closed Riemann surface $\mathcal{M}$ of genus at least two, such that the image of the embedding is a compactly contained subset of $\mathcal{M}$, and is not equal to $\mathcal{M}$. Within this framework, we make precise the notion of Riemannian metrics on Ω that tend to zero as we approach $\partial\Omega$. We then prove the following result, which says that given such appropriate initial data, we may find a *complete* Ricci flow, existing for all positive time, that takes on the initial data in a geometric sense:

Theorem 5.2.3. *Given appropriate initial data $(\Omega, \hat{g})$, let g_0 be the degenerate* *metric on $\mathcal{M}$ defined by*

$$g_0(p) := \begin{cases} \hat{g}(p) & \text{if } p \in \Omega \\ 0 & \text{if } p \in \mathcal{M} \setminus \Omega. \end{cases}$$

Then there exists a smooth Ricci flow $(g(t))_{t\in(0,\infty)}$ on $\mathcal{M}$ such that:

- *$g(t) \to g_0$ in $C^0(\mathcal{M})$ as $t \downarrow 0$;*
- *$\left(\mathcal{M}, d_{g(t)}\right) \to (\Omega, d_{\hat{g}})$ in the Gromov–Hausdorff sense as $t \downarrow 0$ (as in Definition 5.1.5).*

The aim of this result is to challenge the contemporary understanding of well-posedness: we demonstrate that, given a new, geometric definition of what it means for a smooth Ricci flow to take on its initial data, we can construct a complete flow that does so, provided our initial data belongs to a particular class. We point out that the flow starting from $(\Omega, \hat{g})$ given by the work of Giesen and Topping (Theorem 2.2.4) does *not* satisfy the conclusion of Theorem 5.2.3, yet a case can be argued that this flow is a natural one to consider. Therefore, our result invites the question of just which framework is the 'correct' one to work in, given that both seem natural from different perspectives.

We go on to provide a conjecture for the uniqueness of such a flow, and consider the scope for improvement of such a result.

Finally, we provide three appendices in which we collect a number of complementary results and expand on some arguments presented in the main text.

Chapter 2

Ricci flow fundamentals

In this introductory chapter, we begin by fixing notation and clarifying the basic definitions we use in what follows. We also record a number of 'classical' results in the theory of the Ricci flow, all of which we depend upon in later chapters.

Throughout this thesis, all manifolds and metric spaces are assumed to be connected, unless otherwise stated.

2.1 Notation

Given a smooth manifold $\mathcal{M}$, we denote the sections of a vector bundle $E \to \mathcal{M}$ by $\Gamma(E)$. We write $T^{(*)}\mathcal{M}$ for the (co)tangent bundle of $\mathcal{M}$. Consequently, the space of vector fields on $\mathcal{M}$ is denoted $\Gamma(T\mathcal{M})$. We write $\mathrm{Sym}^2_{(+)}T^*\mathcal{M}$ for the bundle of (positive-definite) symmetric bilinear forms on $\mathcal{M}$ so that, given $g \in \Gamma\left(\mathrm{Sym}^2_+ T^*\mathcal{M}\right)$ and $p \in \mathcal{M}$, we get a symmetric, positive-definite bilinear map

$$g_p \colon T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R},$$

also denoted by $\langle \, , \, \rangle_{g_p}$.

Given a Riemannian metric $g \in \Gamma\left(\mathrm{Sym}^2_+ T^*\mathcal{M}\right)$, its Levi-Civita connection will be denoted ∇^g, or just ∇ if there is no confusion, and we use the same notation to denote the generalisation of this connection to arbitrary tensor fields, i.e.

$$\nabla \colon \Gamma\left(\bigotimes^p T^*\mathcal{M} \otimes \bigotimes^q T\mathcal{M}\right) \to \Gamma\left(\bigotimes^{p+1} T^*\mathcal{M} \otimes \bigotimes^q T\mathcal{M}\right).$$

Associated to this connection, we have the Riemannian curvature tensor $\mathrm{Rm}_g \in$

$\Gamma\left(\bigwedge^2 T^*\mathcal{M}\otimes T^*\mathcal{M}\otimes T\mathcal{M}\right)$, where we adopt the sign convention

$$\mathrm{Rm}_g(X,Y)Z := -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \Gamma(T\mathcal{M})$, and where $[\,,\,]$ is the Lie bracket. We also have the Ricci curvature tensor, $\mathrm{Ric}_g \in \Gamma(\mathrm{Sym}^2 T^*\mathcal{M})$, defined by

$$\mathrm{Ric}_g(X,Y) := \mathrm{tr}\{Z \mapsto \mathrm{Rm}_g(X,Z)Y\},$$

and the scalar curvature $R_g := \mathrm{tr}_g\mathrm{Ric}_g$. The sectional curvature is denoted Sec_g. In the case $(\mathcal{M}, g)$ is a Riemannian surface, the Gauß curvature is $K_g = \frac{1}{2}R_g$.

Given a tensor field $T \in \Gamma\left(\bigotimes^p T^*\mathcal{M}\otimes\bigotimes^q T\mathcal{M}\right)$, we define its C^k-norm by

$$||T||_{C^k(\mathcal{M};g)} := \sum_{j=0}^{k}\sup_{\mathcal{M}}|\nabla^j T|_g.$$

If $(g(t))_{t\in[0,T]}$ is a smooth family of metrics on $\mathcal{M}$, and $T(t)$ is a smooth family of tensor fields on $\mathcal{M}$, we write $|T(t)|$ to mean $|T(t)|_{g(t)}$.

The volume form associated to the metric g will be written $\mathrm{d}\mu_g$. The volume measured with g is vol_g, and the geodesic ball of radius $r > 0$ centred at $p \in \mathcal{M}$ is $B_r^g(p)$. The Laplace–Beltrami operator associated to g is $\Delta_g := \mathrm{tr}_g\nabla^g \circ \mathrm{d}$. Given a smooth curve γ in $\mathcal{M}$, we denote its length measured with respect to g by $\mathscr{L}_g(\gamma)$.

We will largely be working with Riemannian surfaces $(\mathcal{M}^2, g)$. In this setting, given $p \in \mathcal{M}$, we may find an open set $U \subset \mathcal{M}$, a non-negative function $v \in C^\infty(U)$, and a complex coordinate $z = x + iy$ on U such that $g = v|\mathrm{d}z|^2$, where $|\mathrm{d}z|^2 := \mathrm{d}x^2 + \mathrm{d}y^2$. We call v the *conformal factor* of g with respect to these so-called *isothermal* coordinates. We sometimes define a function $u \in C^\infty(U)$ by the relation $v = e^{2u}$. This function u is also often referred to in the literature as the conformal factor of g.

We will make frequent use of the following well-known theorem:

Theorem 2.1.1 (Uniformisation Theorem)**.** *Every Riemannian surface* $(\mathcal{M}^2, g)$ *is conformally equivalent to the quotient of either*

- *the unit sphere* S^2 *with the round metric* $\mathring{g}$ *of Gauß curvature* $+1$*, or*
- *the complex plane* $\mathbb{C}$ *with the flat metric* $|\mathrm{d}z|^2$*, or*

- *the unit disc $\mathbb{D} \subset \mathbb{C}$ with the Poincaré metric $g_{\mathbb{H}}$ of constant Gauß curvature -1*

by a discrete group of isometries which is isomorphic to the fundamental group $\pi_1(\mathcal{M})$. In particular, $\mathcal{M}$ admits a conformally equivalent, complete metric of constant Gauß curvature.

2.2 Ricci flow

Given a smooth Riemannian manifold $(\mathcal{M}^n, g_0)$, by a *Ricci flow* on $\mathcal{M}$ with initial condition g_0, we mean a family $(g(t))_{t\in[0,T)}$ of Riemannian metrics on $\mathcal{M}$, for some $T > 0$, which is a classical solution to the equation

$$\begin{cases} \frac{\partial g}{\partial t} & = -2\mathrm{Ric}_{g(t)} \qquad \text{on } \mathcal{M} \times (0,T) \\ g(0) & = g_0. \end{cases}$$

In arbitrary dimension, the existence and uniqueness theory for this equation is encompassed by the following:

Theorem 2.2.1 ([12, 14, 18, 38])**.** *Let $(\mathcal{M}^n, g_0)$ be a complete Riemannian manifold with bounded curvature $|\mathrm{Rm}_{g_0}| \leq K$. Then there exists a $T = T(n, K) > 0$ and a complete Ricci flow $(g(t))_{t\in[0,T]}$ with $g(0) = g_0$, which is of bounded curvature. Moreover, any other complete Ricci flow of bounded curvature taking g_0 as initial condition agrees with $(g(t))$ for as long as both flows exist.*

It is worth observing that this theorem, collecting together numerous contributions of individuals, is such an achievement because, in dimension higher than two, the Ricci flow equation is *not* parabolic, and so the standard theory of quasilinear equations cannot be applied. This difficulty arises from the diffeomorphism invariance of the equation, and is overcome in a particularly elegant way in [14], by adjusting the equation in an appropriate way to make it parabolic, applying the standard theory (as found in, for instance, [24]), and then drawing conclusions about the original problem. This is the so-called *DeTurck trick*.

2.2.1 Surfaces and the Logarithmic Fast Diffusion Equation

Restricting once again to the setting of a Riemannian surface $(\mathcal{M}^2, g)$, where $\mathrm{Ric}_g = K_g \cdot g$, the Ricci flow equation now reads

$$\frac{\partial g}{\partial t} = -2K_{g(t)} \cdot g(t)$$

and so, in dimension two, we see that the Ricci flow is a *conformal* flow, i.e. the metrics $g(t)$ are conformally equivalent at each t for which the flow exists. Appealing to Theorem 2.1.1, we may hence write any Ricci flow as $g(t) = v(t)h$ for some fixed metric h on $\mathcal{M}$ (which we can take to be of constant curvature), and for some smooth family of functions $v(t) \in C^\infty(\mathcal{M})$. It is then easy to verify the following:

Proposition 2.2.2. *Let $(g(t))_{t\in(0,T)}$ be a smooth Ricci flow on a smooth surface $\mathcal{M}^2$. Write $g(t) = v(t)h$, where h is a fixed metric on $\mathcal{M}$ of constant curvature and $v \in C^\infty(\mathcal{M} \times (0,T))$ is non-negative. Then v solves the equation*

$$\frac{\partial v}{\partial t} = \Delta_h \log v - 2K_h, \tag{2.1}$$

where Δ_h is the Laplace-Beltrami operator associated to h.

We note that this equation is then genuinely parabolic, in contrast to the higher-dimensional case. If we define a function u by $v(t) = e^{2u(t)}$, the equation then reads

$$\frac{\partial u}{\partial t} = e^{-2u(t)}(\Delta_h u - K_h),$$

which greater exposes the Ricci flow as a damped heat equation (particularly in the case that $\mathcal{M}$ is conformally flat).

Indeed, when $\mathcal{M}$ is conformally flat and we choose h to be the flat metric, equation (2.1) is called the *Logarithmic Fast Diffusion Equation* (LFDE), which is of independent interest. The equation has been well-studied in the physics literature as a model for various physical phenomena. For instance, [26] deduces that the equation models the expansion of an electron cloud in a vacuum.

The LFDE has also been given as a model for thin-film dynamics. When a very thin (approximately 10-100nm) film of fluid is spread on a surface, a structural instability begins to occur, driven by molecular van der Waals attractions. This instability is only present when the film is sufficiently thin. In this context, Williams and Davis (see [49, Equation 38a]) derive the equation

$$\frac{\partial h}{\partial t} + \Delta \log h + \operatorname{div}[h^2 \nabla \Delta h] = 0,$$

where h is the distance between the surface and the interface between the film and the air. When the van der Waals forces are instead *repulsive*, the sign of the $\Delta \log h$ term changes, and we arrive at the LFDE with an extra fourth-order term. Numerical methods show that with appropriate initial conditions, h becomes zero in finite time, corresponding to a rupture of the film and exposure of the underlying surface. It was later shown (see [7, Equation 7.9]) that under certain physical assumptions (namely, that surface tension is negligible), the fourth-order term can be disregarded.

Returning to the mathematical theory in the context of surfaces, a much more satisfactory existence and uniqueness theory than Theorem 2.2.1 has been developed in recent years by Giesen and Topping, which requires neither that the initial metric is complete, nor that it is of bounded curvature. Before recording this result, we first need a definition:

Definition 2.2.3. We call a family $(g(t))_{t\in[0,T]}$ of Riemannian metrics *instantaneously complete* if $g(t)$ is complete for all $t \in (0,T]$.

Theorem 2.2.4 (Flowing possibly incomplete surfaces, [16, 46])**.** *Let $(\mathcal{M}^2, g_0)$ be a smooth Riemannian surface that need not be complete, and which could have unbounded curvature. Depending on the conformal type, we define $T \in (0,\infty]$ by*

$$T := \begin{cases} \frac{\mathrm{vol}_{g_0}\mathcal{M}}{4\pi\chi(\mathcal{M})} & \text{if } (\mathcal{M}, g_0) \cong S^2,\ \mathbb{R}\mathrm{P}^2 \text{ or } \mathbb{C}, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a unique smooth Ricci flow $(g(t))_{t\in[0,T)}$ such that

- $g(0) = g_0$
- *$g(t)$ is instantaneously complete.*

Moreover, $(g(t))$ is maximally stretched, in the sense that if $(h(t))_{t\in[0,\tilde{T})}$ is any other Ricci flow on $\mathcal{M}$ with $h(0) \leq g_0$, then we must have $h(t) \leq g(t)$ as long as both flows exist.

Inspection of the proof of this theorem reveals that the only way in which the flow $(g(t))$ stops is if the area of the surface (i.e. $\mathrm{vol}_{g(t)}\mathcal{M}$) becomes zero in finite time. In particular, provided that area has not become zero, this flow continues to exist even if curvature blows up, which is *not* the case for the flow provided by Theorem 2.2.1.

An example to consider is to take as initial condition the flat disc $\mathbb{D} \subset \mathbb{C}$. Clearly, the constant family $g(t) = |\mathrm{d}z|^2$ is a solution to the Ricci flow equation, which is of course incomplete. However, Theorem 2.2.4 provides a solution which is instantaneously complete and exists for all time. This solution immediately 'lifts off' from the boundary of the disc in order to ensure this completeness at all positive times.

An important result used in the proof of Theorem 2.2.4, and which we rely on later, provides us with C^0 estimates for flows on hyperbolic surfaces (i.e. surfaces that admit a metric of constant curvature -1):

Theorem 2.2.5 ([16, Lemma 2.1])**.** *Let $(\mathcal{M}^2, h)$ be a complete hyperbolic surface and let $(g(t))_{t\in[0,T]}$ be a Ricci flow on $\mathcal{M}$ that is conformally equivalent to h.*

- *If $(g(t))$ is instantaneously complete, then*

$$2th \leq g(t)$$

  for all $t \in (0, T]$.

- *If there exists a constant $M > 0$ such that $g(0) \leq Mh$, then*

$$g(t) \leq (2t + M)h$$

  for all $t \in [0, T]$.

2.3 Compactness of flows and other classical results

In this section, we discuss some classical results in the study of Ricci flow that serve as a foundation of the subject. Firstly, as is common in analysis, compactness results are of great importance, and have played a fundamental role in the development of the theory. In particular, their power can be seen in so-called *singularity analysis*. Suppose a Ricci flow $(\mathcal{M}, g(t))$ develops a singularity in finite time (consider, for instance, a *neck-pinch singularity*, where two round spheres are connected by a thin neck, which becomes thinner and thinner over time, eventually 'pinching off'). An approach to studying this scenario is to zoom in around the singularity ('blow up'), and to somehow take a limit to expose the geometry of the singularity. For this, we need a good notion of 'convergence of flows':

Definition 2.3.1 (Cheeger–Gromov convergence for flows)**.** Let $(\mathcal{M}_i, g_i(t))$ be a sequence of smooth families of Riemannian manifolds for $t \in (a, b)$ where $-\infty \leq a < b \leq \infty$. Let $p_i \in \mathcal{M}_i$ for each i. Let $(\mathcal{M}, g(t))$ be a smooth family of Riemannian manifolds for $t \in (a, b)$ and let $p \in \mathcal{M}$. We say that

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{M}, g(t), p)$$

as $i \to \infty$ *in the Cheeger–Gromov sense* if there exist a nested sequence of compact $\Omega_i \subset \mathcal{M}$ exhausting $\mathcal{M}$ with $p \in \operatorname{int}(\Omega_i)$ for each i, and a sequence of smooth maps $\phi_i \colon \Omega_i \to \mathcal{M}_i$, diffeomorphisms onto their images with $\phi_i(p) = p_i$, such that $\phi_i^*(g_i(t)) \to g(t)$ smoothly locally on $\mathcal{M} \times (a, b)$ as $i \to \infty$.

Remark 2.3.2. One may question why the above definition is phrased in terms of *pointed* spaces, i.e. why we have to choose points p_i and p. This becomes clear upon considering even simple examples. Indeed, consider the cylinder $S^1 \times [0, \infty)$ capped off with a unit hemisphere, and call this manifold with the obvious metric $(\mathcal{M}, g)$. If we let the points $p_i \in \mathcal{M}$ be situated a distance i from the join between the hemisphere and the cylinder, then the sequence $(\mathcal{M}, g, p_i)$ will converge in the Cheeger–Gromov sense to an infinite cylinder. If instead we take p_i to be the point at the 'tip' of the hemisphere for each i, then the limit is the same manifold we started with. Thus, to get a well-defined notion of convergence, we must use pointed manifolds.

With this in hand, we can state the compactness theorem for flows, due to Hamilton:

Theorem 2.3.3 (Compactness of Ricci flows, [19]). *Let $(\mathcal{M}_i^n, g_i(t), p_i)$ be a sequence of complete, pointed Ricci flows for $t \in (a, b)$ with $-\infty \leq a < 0 < b \leq \infty$. Suppose that:*

1. $$\sup_i \sup_{\substack{x \in \mathcal{M}_i \\ t \in (a,b)}} |\mathrm{Rm}(g_i(t))|(x) < \infty;$$

   and

2. $$\inf_i \operatorname{inj}(\mathcal{M}_i, g_i(0), p_i) > 0,$$

   where inj *denotes the injectivity radius.*

Then there exist a smooth manifold $\mathcal{M}^n$, a complete Ricci flow $(g(t))$ on $\mathcal{M}$ for $t \in (a, b)$, and a point $p \in \mathcal{M}$ such that, after passing to a subsequence in i, we have

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{M}, g(t), p)$$

as $i \to \infty$, in the Cheeger–Gromov sense (as in Definition 2.3.1).

Remark 2.3.4. There are many variants of Theorem 2.3.3, some of which are listed in [23, Appendix E]. For instance, hypothesis (1) above can be relaxed to requiring only local curvature bounds over compact time intervals in (a, b). As one relaxes such restraints however, one must also pay special attention to the conclusion. As demonstrated in [45], completeness of the limit is not guaranteed.

Remark 2.3.5. We note that conditions (1) and (2) in Theorem 2.3.3 are necessary: if the sequence $(\mathcal{M}_i, g_i, p_i)$ is not of uniformly bounded curvature, then we could get convergence of smooth Riemannian manifolds to a Euclidean cone, which is not even a smooth manifold. This convergence would, however, satisfy the definition of *Gromov–Hausdorff convergence*, which we discuss later (see Definitions 3.2.2 and 3.2.5). If the sequence does not have uniformly positive injectivity radius at time $t = 0$, then we could have 'collapse' situations, such as a sequence of cylinders of decreasing radii converging to a line. Once again, this would be allowed under Gromov–Hausdorff convergence.

We now state another classical result, which gives us a relationship between injectivity radius and volume:

Theorem 2.3.6 ([10, Theorem 4.7])**.** *Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold and let $p \in \mathcal{M}$. Suppose there exist $K, v_0 > 0$ such that $|\mathrm{Rm}_g| \leq K$ and $\mathrm{vol}_g(B_1^g(p)) \geq v_0$. Then there exists $\iota_0 = \iota_0(v_0, K)$ such that $\mathrm{inj}(\mathcal{M}, g, p) \geq \iota_0$.*

Finally, we will need the following theorem of Hamilton, which gives estimates on how distances change under the Ricci flow:

Theorem 2.3.7 ([19, Theorem 17.1 variant])**.** *Let $(\mathcal{M}^2, g(t))_{t \in (0,T)}$ be a Ricci flow with*

- *$K_{g(t)} \geq -1$ for all $t \in (0, T)$;*
- *$\left|K_{g(t)}\right| \leq \frac{A}{t}$ for all $t \in (0, T)$, for some constant A.*

Denote by d_t the distance induced by the metric $g(t)$. Then there exists $C = C(A)$ such that

$$e^{-C(t-s)} d_t \leq d_s \leq d_t + C(\sqrt{t} - \sqrt{s})$$

on any compact subset of $\mathcal{M} \times \mathcal{M}$ and for any $0 < s < t < T$.

2.4 Pseudolocality

An extremely powerful tool introduced by Perelman in the first of his three seminal Ricci flow papers [29] is the phenomenon of *pseudolocality*. Pseudolocality is a property of solutions that is not shared with the usual heat equation, from which we often gain intuition for the Ricci flow. The result tells us that the effect of the Ricci flow is principally local; in other words, regions of large curvature cannot instantaneously affect 'nice enough' regions elsewhere in the manifold where curvature is controlled. The proof is very technical, and so we omit it, but a thorough walkthrough of the theorem can be found in [23, §30-34].

Theorem 2.4.1 (Pseudolocality Theorem, [29, Theorem 10.1])**.** *For every* $\alpha > 0$*, there exist* $\delta, \varepsilon > 0$ *satisfying the following property. Suppose we have a smooth, pointed Ricci flow* $(\mathcal{M}, g(t), x_0)$*, defined for* $t \in [0, (\varepsilon r_0)^2]$*, where* $\mathcal{M}$ *is closed. Suppose further that for* $x \in B^{g(0)}_{r_0}(x_0)$ *and any* $\Omega \subset B^{g(0)}_{r_0}(x_0)$ *we have:*

1. $R_{g(0)}(x) \geq -\frac{1}{r_0^2}$ *and*
2. $\mathrm{vol}_{g(0)}(\partial\Omega)^n \geq (1-\delta)\theta_n \mathrm{vol}_{g(0)}(\Omega)^{n-1}$*, where* $\theta_n = n^n \omega_n$ *is the Euclidean isoperimetric constant, with* ω_n *the volume of the unit* n*-ball in* $\mathbb{R}^{n+1}$*.*

Then we have

$$|\mathrm{Rm}_{g(t)}|(x) < \frac{\alpha}{t} + \frac{1}{(\varepsilon r_0)^2}$$

provided that $t \in (0, (\varepsilon r_0)^2]$ *and* $d(x,t) := d_{g(t)}(x, x_0) \leq \varepsilon r_0$*.*

This theorem was extended to the setting of complete manifolds of bounded curvature in [8].

Remark 2.4.2. We now attempt to get a feel for the ways in which conditions (1) and (2) are restrictive. In a sense, they ensure that the region in question has not-too-wild curvature at initial time. Indeed, condition (1) prevents curvature being too negative, since a manifold of constant negative curvature will automatically satisfy condition (2). On the other hand, condition (2), which demands that initially the region is 'almost isoperimetrically Euclidean', safeguards against curvature being too positive, since for instance a manifold of very large constant positive curvature will not satisfy the condition. Thus, together, the conditions could be said to demand that the region is 'almost Euclidean'.

Remark 2.4.3. The requirement that $g(t)$ be complete is necessary. Indeed, as shown in [17], we can construct a Ricci flow on the unit disc $\mathbb{D} \subset \mathbb{C}$ such that $g(0)$ is the (incomplete) flat metric, but such that the Gauß curvature blows up as quickly as we like, which would contradict the conclusion of Theorem 2.4.1, provided we chose this blow-up time to be earlier than the $(\varepsilon r_0)^2$ given by the theorem.

We point out that the conditions of Theorem 2.4.1 are not the only way of ensuring a region is 'almost Euclidean'. Indeed, [42, Proposition 3.1] instead assumes the

Ricci curvature and volume of balls in the initial metric are 'not too far' from being Euclidean, and arrives at a similar conclusion. Yet another version of the same phenomenon is encapsulated by a result of Chen:

Theorem 2.4.4 ([11, Proposition 3.9])**.** *Let $(g(t))_{t\in[0,T]}$ be a smooth Ricci flow with initial metric g_0 on a smooth surface $\mathcal{M}^2$. Let $x_0 \in \mathcal{M}$ and assume, for some $r_0 > 0$, that $B_{r_0}^{g(t)}(x_0)$ is compactly contained in $\mathcal{M}$ for any $t \in [0,T]$. Suppose further that there exists $v_0 > 0$ such that*

- $|R_{g_0}| \leq \frac{1}{r_0^2}$ *on* $B_{r_0}^{g_0}(x_0)$ *and*
- $\mathrm{vol}_{g_0}(B_{r_0}^{g_0}(x_0)) \geq v_0 r_0^2$.

Then there exists a constant K, depending only on v_0, such that

$$|R_{g(t)}| \leq \frac{2}{r_0^2} \qquad \text{on } B^{g(t)}_{\frac{r_0}{2}}(x_0)$$

for $0 \leq t \leq \min\left\{T, \frac{r_0^2}{K}\right\}$.

An immediate corollary of this result is the following, since metric balls in complete metrics are always compactly contained in the given surface:

Corollary 2.4.5. *Let $(g(t))_{t\in[0,T]}$ be a complete Ricci flow on a surface $\mathcal{M}^2$, and suppose there exist $\kappa < \infty$ and $v_0 > 0$ such that $|K_{g(0)}| \leq \kappa$ and $\mathrm{vol}_{g(0)}\left(B_1^{g(0)}(p)\right) \geq v_0$ for all $p \in \mathcal{M}$. Then there exists a constant $C = C(v_0, \kappa) < \infty$ such that*

$$|K_{g(t)}| \leq C$$

for $t \in [0, \min\{T, \kappa^{-1}\}]$.

Finally, we provide somewhat of a generalisation of the pseudolocality theorem, due to Miles Simon:

Theorem 2.4.6 (Local smoothing of Ricci flows on surfaces, [40, Theorem 1.1])**.** *Let $(\mathcal{M}^2, g(t))_{t\in[0,T)}$ be a smooth, complete Ricci flow and let $x_0 \in \mathcal{M}$. Let $\sigma < 1$ and $v_0, r > 0, N > 1$ be given. Suppose that*

- $\mathrm{vol}_{g(0)}\left(B_r^{g(0)}(x_0)\right) \geq v_0 r^2$ *and*
- $\mathrm{R}_{g(0)} \geq -\frac{N}{r^2}$ *on* $B_r^{g(0)}(x_0)$.

Then there exist $\tilde{v}_0 = \tilde{v}_0(v_0, \sigma, N) > 0$ and $\delta_0 = \delta_0(v_0, \sigma, N) > 0$ such that

- $\mathrm{vol}_{g(t)}\left(B^{g(t)}_{r(1-\sigma)}(x_0)\right) \geq \tilde{v}_0 r^2$
- $|\mathrm{R}_{g(t)}| \leq \frac{1}{r^2\delta_0^2 t}$ *on* $B^{g(t)}_{r(1-\sigma)}(x_0)$

as long as $t \leq \delta_0^2 r^2$.

Remark 2.4.7. Notice that Theorem 2.4.6 is still valid if the Ricci flow $(g(t))$ is merely *instantaneously* complete, rather than complete at $t = 0$.

Chapter 3

Survey: Alexandrov spaces with lower curvature bounds

Beginning in earnest in the mid-twentieth century, the study of the geometry of metric spaces with notions of curvature was undertaken by a school of Russian mathematicians, led principally by A.D. Alexandrov. Throughout the subsequent decades, great progress was made in the study of so-called *Alexandrov spaces*, which are a particular kind of metric space endowed with upper or lower curvature bounds, where *curvature* in this setting is defined entirely through the metric properties of the space in question. This chapter collects a broad range of spectacular results pertaining to such spaces. Surprisingly, imposing only a lower curvature bound results in particularly strong geometric phenomena. For instance, such spaces come with a well-defined dimension, which is either an integer or infinite, and also admit the structure of a Riemann surface in dimension two. The results presented in this chapter have been distributed throughout the literature, and we believe they are presented in a unified way for the first time in the present work. Important references for this material are [5], which covers most of the theory presented here except that appearing in §3.5, and [6], which is a more advanced paper, but which also omits the results of §3.5.

We begin by discussing the *intrinsic geometry of metric spaces*, before going on to consider curvature in this setting.

3.1 Length spaces

We begin by clarifying notation and some definitions:

Definition 3.1.1. Let (X, d) be a metric space. A *path* in X is any continuous map $\gamma\colon [-1, 1] \to X$. Let $\mathscr{P}(X)$ denote the collection of all paths in X. The *length structure associated to* d, $L\colon \mathscr{P}(X) \to [0, \infty]$, is defined as follows. Given a path $\gamma \in \mathscr{P}(X)$, let $Y = \{y_0, y_1, \ldots, y_N\}$ be a partition of $[-1, 1]$, i.e.

$$-1 = y_0 \leq y_1 \leq \cdots \leq y_N = 1,$$

and let Φ denote the set of all such partitions. Then

$$L(\gamma) := \sup_{Y \in \Phi} \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)).$$

If $L(\gamma) < \infty$, we call γ a *rectifiable* path in X.

Proposition 3.1.2 ([5, Proposition 2.3.4])**.** *Let (X, d) be a metric space, and let L be the length structure associated to d. Then L is a lower semi-continuous functional on $\mathscr{P}(X)$ with respect to the topology of pointwise convergence. That is, if $\{\gamma_i\}$ is a sequence of rectifiable paths in X such that*

$$\lim_{i \to \infty} \gamma_i(t) = \gamma(t)$$

for every $t \in [-1, 1]$, for some rectifiable path γ in X, then

$$\liminf_{i \to \infty} L(\gamma_i) \geq L(\gamma).$$

Proof. Given a partition Y of $[-1, 1]$, define

$$\Sigma(Y) := \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i)).$$

Now fix $\varepsilon > 0$ and fix a partition Y of $[-1, 1]$ such that $L(\gamma) - \Sigma(Y) < \varepsilon$. Let $\Sigma_j(Y)$ be the corresponding sums for the paths γ_j with respect to the same partition Y. Choose j sufficiently large so that $d(\gamma_j(y_i), \gamma(y_i)) < \varepsilon$ for all $y_i \in Y$. Then

$$L(\gamma) \leq \Sigma(Y) + \varepsilon \leq \Sigma_j(Y) + \varepsilon + 2N\varepsilon \leq L(\gamma_j) + (2N + 1)\varepsilon,$$

from which the result follows. □

We will occasionally make use of *ε-nets*, which provide a useful technique for discussing compactness of metric spaces:

Definition 3.1.3. Let (X, d) be a metric space, and let $\varepsilon > 0$ be given. A subset $\mathscr{S} \subset X$ is called an *ε-net* in X if, for any point $p \in X$ there exists a point $q \in \mathscr{S}$ such that $d(p, q) < \varepsilon$.

We call X *totally bounded* if there exists a finite ε-net in X, for any $\varepsilon > 0$.

The following is well-known:

Proposition 3.1.4 ([5, Theorem 1.6.5])**.** *A metric space X is compact if and only if it is complete and totally bounded.*

Consequently, to show that a complete metric space X is compact, a possible approach is to construct a finite ε-net in X for each $\varepsilon > 0$.

Definition 3.1.5. Let X be a metric space. We say that X is:

- *locally compact* if every point $p \in X$ admits a pre-compact neighbourhood. That is, given $p \in X$, there exists an open set $U \ni p$ such that $\overline{U}$ is compact;
- *boundedly compact* if every closed and bounded subset of X is compact.

Definition 3.1.6. Let (X, d) be a metric space. Call the metric d *intrinsic* if for any points x,$y \in X$, we have

$$d(x, y) = \inf \left\{ L(\gamma) \,\middle|\, \gamma\colon [-1, 1] \to X \text{ is a rectifiable path with } \gamma(-1) = x \text{ and } \gamma(1) = y \right\}.$$

We call such a curve γ realising this infimum a *shortest path* between x and y. If every pair of points can be joined by a shortest path, we call the metric d *strictly intrinsic.*

If d is intrinsic, we call the pair (X, d) a *length space.*

Suppose (X, d) is a length space and let $G \subset X$. The *induced intrinsic metric* on G is defined by

$$d_G(x, y)\colon = \inf \left\{ L_d(\gamma) \,\middle|\, \gamma \text{ is a rectifiable path in } G \text{ joining } x \text{ and } y \right\},$$

where L_d is the length structure associated to d. Notice that in general we have $d_G(x, y) \geq d(x, y)$ for any $x, y \in G$.

Call a subset $A \subset X$ *convex* if the restriction of d to A is strictly intrinsic and finite.

Remark 3.1.7. If $\gamma\colon [-1,1] \to X$ is a shortest path, we often also refer to the *image* of γ as a shortest path.

Remark 3.1.8. If (X,d) is a strictly intrinsic length space, it is readily seen that a subset $A \subset X$ is convex if and only if, given points $x, y \in A$, there exists a shortest path joining x and y that lies in A. In this case, $d|_A = d_A$, the induced intrinsic metric on A as in Definition 3.1.6.

With these definitions in hand, we move now towards some fundamental results that we rely upon later. In particular, a primary concern for us in what follows is the existence of shortest paths between two given points of a length space. Shortest paths need not exist in general - consider, for instance, antipodal points on the boundary of the punctured disc in the plane. However, the following provides us with necessary and sufficient conditions for the existence of such paths:

Theorem 3.1.9 ([5, Theorem 2.5.23])**.** *Let (X,d) be a complete, locally compact length space. Then d is strictly intrinsic.*

Example 3.1.10. Both completeness and local compactness are essential in Theorem 3.1.9. Observe that $\mathbb{R}^2 \setminus \{0\}$ is locally compact, but incomplete, and shortest paths do not exist between all points. Also consider the space

$$T := \bigcup_{n=1}^{\infty} \left\{ \left(x, \frac{1}{n}(1-|x|) \right) \,\middle|\, x \in [-1,1] \right\} \subset \mathbb{R}^2$$

with distance d_T, the induced intrinsic metric from $\mathbb{R}^2$. Then (T, d_T) is a complete length space, but is not locally compact. Notice that $d_T((-1,0),(1,0)) = 2$, but that all paths in T joining these points have length strictly greater than 2.

In light of Theorem 3.1.9, from now on, we will work exclusively with complete, locally compact length spaces, as it will be important for us to be able to join any given pair of points by a shortest path. We approach the proof of Theorem 3.1.9 in a sequence of steps:

Proposition 3.1.11 ([5, Proposition 2.5.19])**.** *Let (X,d) be a compact metric space, and let $x, y \in X$ be two points that can be connected by at least one rectifiable path in X. Then there exists a shortest path joining x and y.*

Proof. Let $\mathscr{L}$ denote the infimum of the lengths of rectifiable paths joining x and y. Let $\{\gamma_i\}$ be a sequence of rectifiable paths joining x and y, such that

$$L(\gamma_i) \to \mathscr{L}$$

as $i \to \infty$, where L is as defined in Definition 3.1.1. By the Arzelà–Ascoli Theorem (see Corollary B.2.2) there exists a subsequence $\{\gamma_{i_j}\}$ that converges uniformly to some path γ joining x and y. Then, by the lower semi-continuity of length structures (Proposition 3.1.2), we have that

$$L(\gamma) \leq \lim_{j\to\infty} L(\gamma_{i_j}) = \mathscr{L},$$

and so $L(\gamma) = \mathscr{L}$. Hence γ is a shortest path joining x and y. □

Remark 3.1.12. We note that shortest paths are not necessarily unique, even in the compact case. For example, antipodal points on the sphere S^2 with the usual spherical metric can be joined by infinitely many distinct shortest paths.

Corollary 3.1.13. *Proposition 3.1.11 holds when X is boundedly compact, instead of compact.*

Proof. Any rectifiable curve joining x and y is contained in a closed metric ball, which is compact by assumption. □

An important result that we will use independently in Theorem 4.3.6 gives a sufficient condition for closed metric balls to be compact:

Proposition 3.1.14 ([5, Proposition 2.5.22])**.** *Every complete, locally compact length space (X, d) is boundedly compact.*

Proof. Take $x \in X$ and define

$$\mathcal{R} := \sup\left\{ r > 0 \;\middle|\; \overline{B_r(x)} \text{ is compact} \right\}.$$

Since X is locally compact, $\mathcal{R} > 0$. Suppose that $\mathcal{R} < \infty$, and let $B := \overline{B_{\mathcal{R}}(x)}$. We now prove that B is compact. Since B is closed and X is complete, by Proposition 3.1.4 we need only show that for any $\varepsilon > 0$, B contains a finite ε-net (as in Definition 3.1.3).

We may assume that $\varepsilon < \mathcal{R}$. Then, by the definition of $\mathcal{R}$,

$$B' := \overline{B_{\mathcal{R}-\frac{\varepsilon}{3}}(x)}$$

is compact, and so contains a finite $\frac{\varepsilon}{3}$-net, say $\mathscr{S}$. Now let $y \in B$. Then, since X is a length space, $d(y, B') \leq \frac{\varepsilon}{3}$, and so there exists a point $y' \in B'$ with $d(y, y') < \frac{\varepsilon}{2}$. Now we know that $d(y', \mathscr{S}) \leq \frac{\varepsilon}{2}$, and thus by the triangle inequality, $d(y, \mathscr{S}) \leq \varepsilon$.

So $\mathscr{S}$ is an ε-net in B, and hence B is compact.

Now let $y \in B$ and let U_y be a pre-compact neighbourhood of y, which exists by the assumption that X is locally compact. Let

$$\mathcal{U} := \bigcup_{y \in Y} U_y$$

be the union of finitely many such neighbourhoods that covers B. Using the compactness of B again, we may find $\varepsilon > 0$ such that the ε-neighbourhood of B is contained in $\mathcal{U}$, which is pre-compact. But since X is a length space, this says that $\overline{B_{\mathcal{R}+\varepsilon}(x)}$ is contained in a compact set (namely $\overline{\mathcal{U}}$), and is thus compact itself, contradicting the maximality of $\mathcal{R}$. So $\mathcal{R} = \infty$ and X is boundedly compact. □

Combining Corollary 3.1.13 and Proposition 3.1.14, we get Theorem 3.1.9.

3.2 The Gromov–Hausdorff topology

A natural question to consider is whether one can define a useful notion of 'convergence' of a sequence of metric spaces. This question arises, for example, when attempting to generalise well-understood properties of Riemannian manifolds to the setting of metric spaces. An approach to treating this problem is to construct a sequence of Riemannian manifolds that 'converges' to a given metric space, and prove that certain properties are preserved in the limit. Indeed, this will be a key technique in later chapters. The natural candidate for such a notion of convergence is *Gromov–Hausdorff convergence*, which we discuss in this section.

Definition 3.2.1. Let $(X, d_X), (Y, d_Y)$ be compact metric spaces. We say that a map $f\colon X \to Y$ is an *ε-isometry* (or an *ε-Gromov–Hausdorff approximation*) if

1. $|d_Y(f(x), f(y)) - d_X(x, y)| \leq \varepsilon$ for all $x, y \in X$
2. $\bigcup_{x \in X} B_Y(f(x), \varepsilon) = Y$.

Definition 3.2.2 (Gromov–Hausdorff distance)**.** The *Gromov-Hausdorff distance* between (X, d_X) and (Y, d_Y) is the least $\varepsilon \geq 0$ such that there exist ε-isometries $X \to Y$ and $Y \to X$, and we denote this distance on the set of compact metric spaces by d_{GH}.

We say that a sequence of compact metric spaces (X_i, d_i) converges to the compact metric space (X, d) *in the Gromov–Hausdorff topology* as $i \to \infty$ if $d_{GH}(X_i, X) \to 0$ as $i \to \infty$.

Remark 3.2.3. Compact metric spaces X and Y are isometric if and only if $d_{GH}(X, Y) = 0$.

Remark 3.2.4. We can define an equivalent metric on the family of compact metric spaces as a generalisation of the usual Hausdorff distance (between two subsets of a given metric space). Indeed, given compact metric spaces (X, d_X) and (Y, d_Y), define a distance $\tilde{d}_{GH}$ by

$$\tilde{d}_{GH}((X, d_X), (Y, d_Y)) := \inf_{Z,i,j} d^Z_{\mathscr{H}}(i(X), j(Y)),$$

where $i \colon X \to Z, j \colon Y \to Z$ are isometric embeddings into a compact metric space Z and $d^Z_{\mathscr{H}}$ is the Hausdorff distance in Z. Then $\tilde{d}_{GH}$ and d_{GH} are equivalent metrics (see, for instance, [5, Corollary 7.3.28]). It is clear, however, that attempting to work with $\tilde{d}_{GH}$ would likely be cumbersome, and this is the last time we mention it.

There is also a notion of a Gromov–Hausdorff topology on the collection of all *non-compact* metric spaces. For this, however, we need to consider *pointed* spaces:

Definition 3.2.5. Let (X_i, d_i, p_i) be a sequence of non-compact pointed metric spaces, i.e. $p_i \in X_i$ for each i. Given a pointed metric space (X, d, p), we say $(X_i, d_i, p_i) \to (X, d, p)$ *in the Gromov–Hausdorff sense* as $i \to \infty$ if the following holds: for every $r > 0$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for every $i > n_0$ there is a (not necessarily continuous) map $f_i \colon B_{d_i}(p_i, r) \to X$ with the properties that:

1. $f_i(p_i) = p$ for each i;
2. $$\sup_{x,y \in B_{d_i}(p_i,r)} |d(f_i(x), f_i(y)) - d_i(x, y)| < \varepsilon;$$
3. the ε-neighbourhood of the set $f_i(B_{d_i}(p_i, r))$ contains the ball $B_d(p, r - \varepsilon)$.

Remark 3.2.6. In the case that (X, d) is a length space, it is true (see [5, Exercise 8.1.3]) that the Gromov–Hausdorff convergence defined in Definition 3.2.5 implies that

$$\left(\overline{B_{d_i}(p_i, r)}, d_i\right) \to \left(\overline{B_d(p, r)}, d\right)$$

in the Gromov–Hausdorff sense defined in Definition 3.2.2, for every $r > 0$.

Remark 3.2.7. Let $(\mathcal{M}_i^n, g_i, p_i)$ be a sequence of pointed Riemannian manifolds. There is also a notion of *Cheeger–Gromov convergence* of such a sequence to another pointed Riemannian manifold $(\mathcal{M}^n, g, p)$, very similar to Definition 2.3.1 (see, for example, [43, §7.1]). This involves exhausting $\mathcal{M}$ with nested open sets Ω_i containing p so that we have smooth maps $\varphi_i \colon \Omega_i \to \mathcal{M}_i$, diffeomorphisms onto their images with $\varphi_i(p) = p_i$, such that $\varphi_i^* g_i \to g$ smoothly locally on $\mathcal{M}$. We can of course view the sequence $(\mathcal{M}_i, g_i)$ as a sequence of metric spaces $(\mathcal{M}_i, d_{g_i})$ where d_{g_i} is the Riemannian distance induced by the metric g_i. Thus, in this context, we have two topologies on the collection of (pointed) Riemannian manifolds, namely the Gromov–Hausdorff topology, and the Cheeger–Gromov topology. The former is much weaker than the latter, since for example we can have collapsing (i.e. loss of dimension), as demonstrated in the following example:

Example 3.2.8. Consider a sequence of cylinders $S^1(r) \times (0, \infty)$, where $S^1(r)$ denotes the circle of length $2\pi r$. Then as $r \to 0$, the sequence converges in the Gromov–Hausdorff sense to the ray $(0, \infty)$. This is an example of a sequence of two-dimensional smooth manifolds converging to a one-dimensional one. The sequence *does not*, however, converge in the Cheeger–Gromov sense. For this reason, we can think of the Gromov–Hausdorff topology as the 'weak' topology in this setting.

A crucial result relating volume and the Gromov–Hausdorff topology is the following:

Theorem 3.2.9 ([13, Theorem 0.1])**.** *Given $r > 0$, let $\mathcal{B}_r$ denote the topological space of all metric balls of radius r in all complete, n-dimensional Riemannian manifolds with Ricci curvature bounded below by $-(n-1)$, equipped with the Gromov–Hausdorff topology. Then the volume function*

$$\mathrm{vol} \colon \mathcal{B}_r \to \mathbb{R}_{\geq 0}$$

is continuous.

This theorem will be useful to us later in Lemma 4.3.7, where we have control on the volume of metric balls in a sequence of Riemannian manifolds converging to a certain metric space in the Gromov–Hausdorff sense. As a consequence of the theorem, we will be able to deduce information about the volume of balls in the limiting metric space.

3.3 Alexandrov spaces

We now turn towards the central objects of this chapter: Alexandrov spaces with lower curvature bounds. Roughly speaking, these are length spaces that come with a very natural notion of curvature. Instead of defining a curvature function or tensor on these spaces, as is the case in Riemannian geometry, we instead add just enough structure to allow us to make sense of a curvature *bound*. Consider, for example, the unit sphere S^2. When considering this idea of curvature heuristically, we certainly want to come up with a definition that tells us that the sphere is non-negatively curved. The approach we take is the following: inspired by our intuition from Riemannian geometry, we know that increasing curvature makes objects 'fatter'. A way of seeing this is to draw a triangle on the sphere, and compare this triangle with a corresponding triangle on the flat plane, the side lengths of which are equal to those of the spherical triangle. We will then see that the distance from a vertex to the midpoint of the opposite side in the spherical triangle is greater than the corresponding distance in the flat triangle. This is essentially the definition of S^2 being an *Alexandrov space with curvature bounded below by zero.* We now make sense of the more general definition.

Definition 3.3.1. By a *k-plane*, we mean

$$\begin{cases} \mathbb{H}^2(k) & \text{if } k < 0 \\ \mathbb{R}^2 & \text{if } k = 0 \\ S^2\left(\frac{1}{\sqrt{k}}\right) & \text{if } k > 0. \end{cases}$$

Here, $\mathbb{H}^2(k)$ is the hyperbolic plane of constant (Gauß) curvature k, and $S^2\left(\frac{1}{\sqrt{k}}\right)$ is the sphere of constant curvature k.

Definition 3.3.2. Let (X, d) be a length space. By a *triangle* in X, we mean three points (the *vertices*) joined to each other by shortest paths (the *sides*). Given a triangle T in X, we write $T = \triangle xyz$ to identify its vertex points.

Definition 3.3.3 (Admissible triangles)**.** Let (X, d) be a length space and $k \in \mathbb{R}$. Denote by $\mathscr{T}_k(X)$ the set

$$\mathscr{T}_k(X) := \begin{cases} \{\text{all triangles in } X\} & \text{if } k \leq 0 \\ \left\{\text{all triangles in } X \text{ with perimeter strictly less than } \frac{2\pi}{\sqrt{k}}\right\} & \text{if } k > 0. \end{cases}$$

We call a triangle belonging to $\mathscr{T}_k(X)$ an *admissible triangle.*

Definition 3.3.4 (Comparison triangle)**.** Let (X, d) be a length space, let $k \in \mathbb{R}$ be given, and let $\triangle xyz \in \mathscr{T}_k(X)$. By the *comparison triangle* $\widetilde{\triangle} xyz$ we mean the triangle $\triangle \bar{x}\bar{y}\bar{z}$ in the k-plane, where

$$|\bar{x}\bar{y}| = |xy|, \quad |\bar{x}\bar{z}| = |xz|, \quad |\bar{y}\bar{z}| = |yz|,$$

where we are now abusing the notation $|\cdot\cdot|$ to denote the distances in the appropriate spaces.

Remark 3.3.5. We defined the set $\mathscr{T}_k(X)$ so that, given any triangle in the set, its comparison triangle exists and is unique up to rigid motions of the k-plane.

With these ideas in hand, we can now define Alexandrov spaces with curvature bounded below. There are a number of equivalent definitions:

3.3.1 Definition by length

Definition 3.3.6 (Alexandrov space)**.** Let (X, d) be a complete length space and $k \in \mathbb{R}$, such that if $k > 0$, X has diameter not greater than $\frac{\pi}{\sqrt{k}}$. We say that (X, d) is an *Alexandrov space with curvature bounded below by* $k \in \mathbb{R}$ if, in some neighbourhood U_q of each point $q \in X$, we have that for every triangle $\triangle xyz \in \mathscr{T}_k(U_q)$ and every point p on the side of $\triangle xyz$ joining x and z, the inequality

$$|py| \geq |\bar{p}\bar{y}|$$

holds, where $\bar{p}$ is the point on the side of $\widetilde{\triangle} xyz$ joining $\bar{x}$ and $\bar{z}$ such that $|\bar{x}\bar{p}| = |xp|$.

Remark 3.3.7. Notice that if $k > 0$, we stipulate that the diameter of X is not greater than $\frac{\pi}{\sqrt{k}}$. As shown in [5, Theorem 10.4.1], this is equivalent to excluding some exceptional examples that we do not wish to consider as belonging to the class of Alexandrov spaces with curvature bounded below. Namely, the exceptions are the real line, the half-line, segments of length greater than $\frac{\pi}{\sqrt{k}}$, and circles of length greater than $\frac{2\pi}{\sqrt{k}}$.

Remark 3.3.8. Intuitively, Definition 3.3.6 tells us that in an Alexandrov space with curvature bounded below by $k \in \mathbb{R}$, triangles are 'fatter' than the comparison triangles drawn in the k-plane.

Remark 3.3.9. Let X be an Alexandrov space of curvature bounded below by $k > 0$. As shown in [5, Corollary 10.4.2], when $k > 0$, no triangle in X has perimeter greater than $\frac{2\pi}{\sqrt{k}}$. Thus, the definition of $\mathscr{T}_k(X)$ only excludes those triangles of perimeter exactly $\frac{2\pi}{\sqrt{k}}$. Notice that we certainly wish to exclude such triangles from consideration, since their comparison triangles are not unique.

Remark 3.3.10. As discussed in [6, Definition 2.3], if X is merely locally complete, it is possible to interpret it as an Alexandrov space with curvature bounded below. Indeed, even if X is not locally complete, but still satisfies the other requirements of Definition 3.3.6, it may be made into an Alexandrov space with curvature bounded below by local completion. However, we will exclusively consider spaces that are (globally) complete. This is because, by Theorem 3.1.9, in complete, locally compact length spaces, any two points can be joined by a shortest path. Later (Theorem 3.6.2), we will see that all finite-dimensional Alexandrov spaces with curvature bounded below are locally compact, and so working with complete spaces is sufficient to allow us to join any two points by shortest paths.

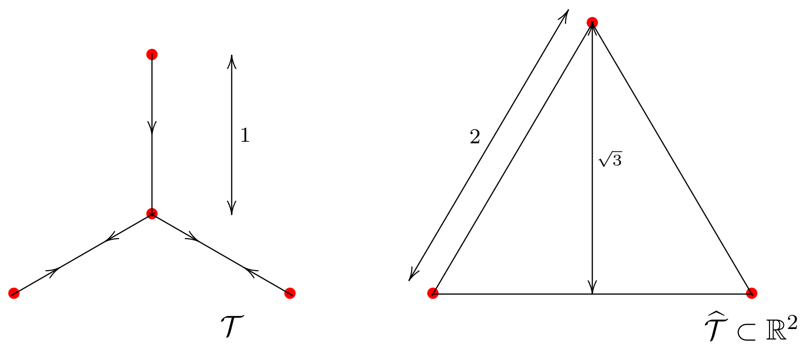
Definition 3.3.11. For brevity, given $k \in \mathbb{R}$, define $\mathscr{A}(k)$ to be the class of all Alexandrov spaces with curvature bounded from below by k.

It is useful to have to hand a selection of examples:

Example 3.3.12 ([6, §2.9], [5, Theorem 6.5.6])**.** A Riemannian manifold $(\mathcal{M}, g)$ with $\mathrm{Sec}_g \geq k$ belongs to $\mathscr{A}(k)$. Moreover, $(\mathcal{M}, g) \in \mathscr{A}(0)$ if and only if $\mathrm{Sec}_g \geq 0$.

A good source of non-examples arises by considering spaces with 'triple junctions':

Example 3.3.13. Any space containing a 'triple junction' will fail to belong to $\mathscr{A}(k)$ for any $k \in \mathbb{R}$. Consider the example in the following diagram:[1]



Here, it is obvious how to define the intrinsic metric to make the triple junction into a length space. Call the space $\mathcal{T}$ and consider the triangle in $\mathcal{T}$ whose vertices are points on each of the three 'legs' of $\mathcal{T}$, each a distance 1 away from the midpoint. To show that $\mathcal{T} \notin \mathscr{A}(0)$, we draw the comparison triangle $\widehat{\mathcal{T}}$ in $\mathbb{R}^2$, which is an equilateral triangle of side length 2. Here, the distance from any vertex to the midpoint of its opposite side is $\sqrt{3}$, but in $\mathcal{T}$ itself, the corresponding distance is 1. Thus $\mathcal{T}$ fails to satisfy the requirements of Definition 3.3.6, and so does not belong to $\mathscr{A}(0)$. To show that $\mathcal{T}$ does not belong to $\mathscr{A}(k)$ for any k, first observe that this is certainly the case if $k > 0$, since $\mathscr{A}(k) \subset \mathscr{A}(0)$ for any $k > 0$. Then, if

[1] Figure credit: Sam Derbyshire.

$k < 0$, simply make the side lengths of the triangle in $\mathcal{T}$ sufficiently small so that the comparison triangle fits inside a chart on $\mathbb{H}^2(k)$, in which it looks flat, and the same argument holds as above.

We now move on to consider alternative definitions of $\mathscr{A}(k)$, which can be useful in practice when working with lengths may be cumbersome.

3.3.2 Definition by angle

An alternative to Definition 3.3.6 is to compare angles in triangles in our given space with the corresponding angles in comparison triangles. However, the notion of angle in metric spaces is a delicate topic, and we must take care to ensure that it is well-defined in our setting.

Definition 3.3.14 (Comparison angle)**.** Let (X, d) be a length space, and let $k \in \mathbb{R}$ be given. Let $\triangle xyz \in \mathscr{T}_k(X)$. The *comparison angle* $\widetilde{\angle}^k xyz$ is the angle at $\bar{y}$ in the comparison triangle $\widetilde{\triangle} xyz$ (as in Definition 3.3.4), measured in the k-plane.

An important tool that we will need later on is the following:

Lemma 3.3.15 (Alexandrov's Lemma, [2, §3.2.1], [5, Lemma 4.3.3])**.** *Let (X, d) be a length space, let $k \in \mathbb{R}$, let $\triangle pqr \in \mathscr{T}_k(X)$ be given, and let z be a point on the side of $\triangle pqr$ joining p and r. Then we have*

$$\widetilde{\angle}^k pqr \geq \widetilde{\angle}^k pqz + \widetilde{\angle}^k zqr,$$

with equality if and only if

$$\widetilde{\angle}^k qpz = \widetilde{\angle}^k qpr$$

and

$$\widetilde{\angle}^k qzp + \widetilde{\angle}^k qzr = \pi.$$

We can now give a definition of angle between shortest paths in length spaces:

Definition 3.3.16 (Angle)**.** Let (X, d) be a length space, and let $\gamma, \sigma \colon [0, 1] \to X$ be two shortest paths in X emanating from the same point $p = \gamma(0) = \sigma(0)$. The *angle* between γ and σ, denoted $\angle\gamma\sigma$, is defined by

$$\angle\gamma\sigma := \lim_{t,s \to 0} \widetilde{\angle}^0 \gamma(t) p \sigma(s),$$

whenever this limit exists.

We define the *upper angle* between γ and σ to be

$$\overline{\angle}\gamma\sigma := \limsup_{t,s\to 0} \widetilde{\angle}^0\gamma(t)p\sigma(s),$$

and the *lower angle* between γ and σ to be

$$\underline{\angle}\gamma\sigma := \liminf_{t,s\to 0} \widetilde{\angle}^0\gamma(t)p\sigma(s).$$

Given a triangle $\triangle xyz \subset X$, by the *angle* $\angle xyz$, we mean the angle $\angle\gamma\sigma$, where $\gamma, \sigma\colon [0,1] \to X$ are parameterisations of the sides of $\triangle xyz$ joining y to x and y to z respectively (with $\gamma(0) = \sigma(0) = y$).

Remark 3.3.17. In the above definition, we used $\widetilde{\angle}^0$, but we could just as well have used $\widetilde{\angle}^k$ for any $k \in \mathbb{R}$, as it is clear that the limit is independent of k (because for t and s small enough, all points under consideration will lie within a chart in the k-plane, in which the picture looks Euclidean).

We now state a familiar result that also holds in the setting of length spaces:

Proposition 3.3.18 (Triangle inequality, [2, §3.4.2])**.** *Let (X, d) be a length space and let $\gamma_1, \gamma_2, \gamma_3\colon [0,1] \to X$ be three shortest paths in X emanating from the same point $p = \gamma_1(0) = \gamma_2(0) = \gamma_3(0)$. If all of the angles $\alpha_{ij} := \angle\gamma_i\gamma_j$ are defined, then*

$$\alpha_{13} \leq \alpha_{12} + \alpha_{23}.$$

As mentioned previously, we must take care as to whether or not angles are well-defined. As shown in Theorem 3.3.23, angles between shortest paths always exist in Alexandrov spaces with lower curvature bounds. However, in a more general setting, this is not the case, even in simple examples:

Example 3.3.19 ([4, §4.1.3])**.** Consider $\mathbb{R}^2$ with the norm $||(x,y)|| := |x| + |y|$. Consider paths

$$\gamma_1(t) := (t,t) \qquad \gamma_2(t) := (t,0) \qquad \gamma_3(t) := (0,t)$$

for $t \geq 0$. Then a quick computation shows that $\angle\gamma_2\gamma_3 = \pi$. However, $\angle\gamma_1\gamma_2$ does not exist, since the *upper angle*

$$\limsup_{t,s\to 0} \widetilde{\angle}^0\gamma_1(t)0\gamma_2(s) = \frac{\pi}{2},$$

while the *lower angle*

$$\liminf_{t,s\to 0} \widetilde{\angle}^0 \gamma_1(t) 0 \gamma_2(s) = 0.$$

Consequently, when working with angles in general length spaces, authors typically deal exclusively with upper angles, which always exist.

We can now formulate another definition of Alexandrov space, which is equivalent to Definition 3.3.6 (see Theorem 3.3.23):

Definition 3.3.20. Let (X, d) be a complete length space and $k \in \mathbb{R}$, such that if $k > 0$, X has diameter not greater than $\frac{\pi}{\sqrt{k}}$. We say (X, d) is an *Alexandrov space with curvature bounded below by* k if, in some neighbourhood U_q of each point $q \in X$, we have the following: given any triangle $\triangle xyz \in \mathscr{T}_k(U_q)$, the angles $\angle xyz$, $\angle yxz$ and $\angle xzy$ exist and we have the inequalities

$$\angle xyz \geq \widetilde{\angle}^k xyz \qquad \angle yxz \geq \widetilde{\angle}^k yxz \qquad \angle xzy \geq \widetilde{\angle}^k xzy,$$

and moreover, for any point w on the side of $\triangle xyz$ joining y and z, and any shortest path σ joining w and x, we have the equality

$$\angle xwz + \angle xwy = \pi,$$

where the angles are computed using σ.

3.3.3 The quadruple condition and equivalence of definitions

We now introduce one more definition of an Alexandrov space with a lower curvature bound, which seems somewhat less geometrically motivated than Definitions 3.3.6 and 3.3.20, but which benefits from being rather easy to work with in practice. We refer to the following as the *quadruple condition*:

Definition 3.3.21. Let (X, d) be a complete length space and $k \in \mathbb{R}$, such that if $k > 0$, X has diameter not greater than $\frac{\pi}{\sqrt{k}}$. We say (X, d) is an *Alexandrov space with curvature bounded below by* k if, in some neighbourhood U_q of each point $q \in X$, we have the following: for any collection of four distinct points $a, b, c, d \in U_q$, we have that either

$$\widetilde{\angle}^k bac + \widetilde{\angle}^k cad + \widetilde{\angle}^k dab \leq 2\pi,$$

or that at least one of these angles is not defined.

Remark 3.3.22. We point out that in the above definition, if $k \leq 0$, then all of the stated angles will exist. If $k > 0$, it is possible that some or all of the stated angles do not exist, which would be the case if the triangle with vertices at the given points were not admissible (see Definition 3.3.3).

The equivalence of Definitions 3.3.6, 3.3.20 and 3.3.21 then follows from the following theorem:

Theorem 3.3.23 ([2, Theorem 8.2.1, variant])**.** *Let (X, d) be a complete length space and $k \in \mathbb{R}$, such that if $k > 0$, X has diameter not greater than $\frac{\pi}{\sqrt{k}}$. Let $p, x, y \in X$, and let*

- *γ be a shortest path joining x and y;*
- *η be a shortest path joining x and p;*
- *ι be a shortest path joining p and y,*

such that $\triangle pxy \in \mathscr{T}_k(X)$, where $\triangle pxy$ has sides γ, η and ι. Then the following are equivalent:

1. *For any $z \in X$, we have that either*

$$\tilde{\angle}^k xpz + \tilde{\angle}^k ypz + \tilde{\angle}^k xpy \leq 2\pi,$$

or that at least one of these angles is not defined;

2. *For any point z on γ with $x \neq z \neq y$, we have the inequality*

$$\tilde{\angle}^k pzx + \tilde{\angle}^k pzy \leq \pi;$$

3. *For any point z on γ with $x \neq z \neq y$, we have*

$$|pz| \geq |\bar{p}\bar{z}|,$$

where $\bar{z}$ is determined as in Definition 3.3.6;

4. *The angle $\angle pxy$ is defined, with*

$$\angle pxy \geq \tilde{\angle}^k pxy,$$

and moreover, for any point z on γ, and any shortest path σ joining z and p, we have

$$\angle pzy + \angle pzx = \pi,$$

where both angles are computed using σ.

Proof. (1) $\implies$ (2): Since z lies on γ, we have that $\tilde{\angle}^k xzy = \pi$. Since by (1) we have

$$\tilde{\angle}^k xzy + \tilde{\angle}^k pzx + \tilde{\angle}^k pzy \leq 2\pi,$$

it follows that

$$\tilde{\angle}^k pzx + \tilde{\angle}^k pzy \leq \pi.$$

(2) $\iff$ (3): This follows immediately from Alexandrov's Lemma (Lemma 3.3.15), after observing that (3) is equivalent to having that for any z on γ with $x \neq z \neq y$, $\tilde{\angle}^k pxy \leq \tilde{\angle}^k pxz$.

(2) + (3) $\implies$ (4): As a consequence of the above observation, we have that (3) implies the following: for any $\hat{p}$ on η and any $\hat{y}$ on γ, the map

$$(|x\hat{p}|, |x\hat{y}|) \mapsto \tilde{\angle}^k \hat{p}x\hat{y}$$

is weakly decreasing in each argument. Thus,

$$\angle pxy = \sup_{\hat{p},\hat{y}} \tilde{\angle}^k \hat{p}x\hat{y} \geq \tilde{\angle}^k pxy.$$

Thus the first part of (4) is proved. For the equality, first notice that the above, (2), and Alexandrov's Lemma (Lemma 3.3.15) together imply that

$$\angle pzy + \angle pzx \leq \pi$$

(the full argument appears in [5, Lemma 4.3.7]). Then, by the triangle inequality (Proposition 3.3.18) and the above, we have

$$\angle pzy + \angle pzx \geq \angle yzx \geq \tilde{\angle}^k yzx = \pi,$$

and so we have the desired equality.

(4) $\implies$ (1): Consider a point w on σ close to p. From (4), we have that

$$\angle xwz + \angle xwp \leq \pi \qquad \text{and} \qquad \angle ywz + \angle ywp \leq \pi.$$

Applying the triangle inequality again, it follows that

$$\angle xwz + \angle ywz + \angle xwy \leq 2\pi.$$

Now applying the inequality in (4), we get

$$\widetilde{\angle}^k xwz + \widetilde{\angle}^k ywz + \widetilde{\angle}^k xwy \leq 2\pi,$$

and (1) now follows by taking limits as $w \to p$. □

Now that we have shown that all of our definitions of Alexandrov spaces with lower curvature bounds are equivalent, we have the following immediate corollary of Definition 3.3.20:

Corollary 3.3.24 (Non-branching lemma, [2, §8.5.1])**.** *Let (X, d) be a complete Alexandrov space with curvature bounded below. Let $\gamma, \sigma\colon [0,1] \to X$ be two shortest paths in X with $\gamma(0) = \sigma(0) = p$. Then:*

- *if there exists $\varepsilon > 0$ such that $\gamma(t) = \sigma(t)$ for all $t \in [0, \varepsilon)$, then either $\gamma([0,1]) \subseteq \sigma([0,1])$ or $\sigma([0,1]) \subseteq \gamma([0,1])$;*
- *if $\angle\gamma\sigma = 0$ then either $\gamma([0,1]) \subseteq \sigma([0,1])$ or $\sigma([0,1]) \subseteq \gamma([0,1])$.*

3.3.4 Space of directions

A common theme of what lies ahead is the generalisation of concepts from Riemannian geometry to the setting of Alexandrov spaces with curvature bounded below. We have now developed enough groundwork to discuss the *space of directions* at a point, which is a generalisation of the tangent space to a smooth manifold:

Definition 3.3.25. Let (X, d) be a complete Alexandrov space with curvature bounded below. Given $p \in X$, let $\mathscr{C}_p(X)$ denote the collection of all shortest paths in X emanating from p. Define an equivalence relation $\sim$ on $\mathscr{C}_p(X)$ by

$$\gamma \sim \sigma \iff \angle\gamma\sigma = 0.$$

Let $\Sigma_p^\star(X) := \mathscr{C}_p(X)/\sim$, with the metric $d_\star([\gamma_1], [\gamma_2]) := \angle\gamma_1\gamma_2$. The *space of directions* to X at p, denoted $\Sigma_p(X)$, is the completion of $\Sigma_p^\star(X)$ with respect to the metric $d_\star$.

Remark 3.3.26. Notice that in order to define the space of directions, we are implicitly using that the angle between two shortest paths is defined (which it is in our setting, by Theorem 3.3.23). In general length spaces, where this is not the case, a similar construction can be made using the upper angle, as given by Definition 3.3.16.

3.3.5 Other notions of curvature

In this subsection, we briefly mention alternative ideas of curvature in spaces that are more general than Riemannian manifolds. Firstly, whilst we have defined Alexandrov spaces with curvature bounded *below*, changing the sign of the inequality in Definition 3.3.6 gives the definition of an Alexandrov space with curvature bounded *above*. Such spaces have also been well-studied and exhibit their own interesting properties, surveys of which can be found in [5, Chapter 9] and [4]. As we will see in Theorem 3.6.3, Gromov–Hausdorff limits of sequences of Alexandrov spaces with curvature bounded below are also spaces with curvature bounded below, but the same is not true for spaces with only an upper curvature bound.

As we discuss in §3.4, Alexandrov spaces with curvature bounded below come with a well-defined dimension (which is either an integer or infinite), which coincides with their Hausdorff dimension. In dimension two, these spaces are in fact topological manifolds, possibly with boundary. In the setting of topological surfaces with intrinsic metrics, a less restrictive notion of curvature may be formulated, as we now discuss.

Let $(\mathcal{M}, g)$ be a Riemannian surface and let K_g denote the Gauß curvature of the metric g. It is well-known that for a sufficiently small triangle T bounding a disc in $\mathcal{M}$, we have that

$$\int_T K_g \, \mathrm{d}\mu_g = \alpha + \beta + \gamma - \pi,$$

where α, β and γ are the interior angles of T. We can generalise this idea, but first we need to restrict our attention to a particular type of triangle:

Definition 3.3.27 (Simple triangle)**.** Let (X, d) be a length space that is topologically a surface. Let $T \subset X$ be a triangle whose sides form a simple curve in X, and hence bound a region G. Suppose G is homeomorphic to a disc. Suppose further that no two points on the sides of T can be joined by a curve lying outside of G that is shorter than the portion of the sides joining those points. Then T is called a *simple triangle* in X.

Definition 3.3.28. We say that a length space (X, d) is a *space of bounded integral curvature* if:

1. X is topologically a surface;

2. Given $p \in X$, there exists a neighbourhood $U \ni p$ such that the following holds: for any finite collection $\mathscr{T}$ of non-overlapping simple triangles in U,

there exists a $C = C(U)$ such that

$$\sum_{T \in \mathscr{T}} \delta(T) \leq C < \infty,$$

where $\delta(T)$ is the *excess* of the triangle T, given by $\delta(T) := \overline{\alpha} + \overline{\beta} + \overline{\gamma} - \pi$, where $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$ are the interior upper angles of T (see Definition 3.3.16).

We note here that Alexandrov surfaces (see Definition 3.5.1) are also spaces of bounded integral curvature, proof of which can be found in [37, Proposition 3.2.2] or [27, Theorem 1.8].

Now let (X, d) be a space of bounded integral curvature. Given an open set $G \subset X$, define

$$\omega^+(G) := \sup_{\mathscr{T}} \sum_{T \in \mathscr{T}} \delta^+(T),$$

where the supremum is taken over all finite collections of non-overlapping simple triangles in G, and δ^+ is the positive part of δ. Similarly, define

$$\omega^-(G) := \sup_{\mathscr{T}} \sum_{T \in \mathscr{T}} \delta^-(T),$$

where $\delta^- \geq 0$ is the negative part of δ.

For an arbitrary set $M \subset X$, put

$$\omega^+(M) := \inf_{\substack{G \text{ open} \\ G \supset M}} \omega^+(G) \quad \text{and} \quad \omega^-(M) := \inf_{\substack{G \text{ open} \\ G \supset M}} \omega^-(G).$$

Definition 3.3.29 ([1])**.** The *curvature measure* of X is the signed measure $\omega := \omega^+ - \omega^-$.

We will give examples of this measure in §3.5, but for now we note that in the case of a Riemannian manifold $(\mathcal{M}, g)$, we have that $\omega = K_g \, \mathrm{d}\mu_g$. For Alexandrov spaces with lower curvature bounds, ω need not be absolutely continuous with respect to volume.

3.4 Dimension

Whilst Alexandrov spaces can have many singular points (for example, points with neighbourhoods homeomorphic to neighbourhoods of the vertex of a Euclidean cone), those with curvature bounded below exhibit the stunning property of having a well-defined dimension, which is either an integer or infinite, and coincides with their Hausdorff dimension $\dim_{\mathscr{H}}$. In this section, we give the main results leading to this fact, following [6]. In the process, it is shown that any Alexandrov space with curvature bounded below contains a dense subset that is a topological manifold, and in particular that, in dimension two, such an Alexandrov space must itself be a topological manifold, possibly with boundary. As a consequence, it makes sense to discuss 'Alexandrov surfaces', by which we mean Alexandrov spaces with curvature bounded below that are homeomorphic to a surface *without boundary.* These objects are the main focus of study in Chapter 4, where we show the existence and uniqueness of a Ricci flow taking such a surface as initial condition in a particular sense.

Remark 3.4.1. The results contained in this subsection are valid for all Alexandrov spaces with curvature bounded below by some $k \in \mathbb{R}$. However, *we will only consider the case where* $k = 0$, as it vastly simplifies the exposition. Also, without explicitly saying so, all local arguments take place within some open set in which the requirements of the definitions of Alexandrov spaces are met.

The first step towards proving the dimensionality result is the idea of a *burst point* and their corresponding *explosions*, which are an attempt to adorn a given Alexandrov space with coordinate axes:

Definition 3.4.2 (Burst points)**.** Let X be a complete Alexandrov space with curvature bounded below. A point $p \in X$ is called an (n, δ)-*burst point* if there are n pairs of points $\{(a_i, b_i)\}$ distinct from p such that the following inequalities hold:

$$\widetilde{\angle} a_i p b_i > \pi - \delta \qquad \widetilde{\angle} a_i p a_j > \frac{\pi}{2} - \delta$$

$$\widetilde{\angle} a_i p b_j > \frac{\pi}{2} - \delta \qquad \widetilde{\angle} b_i p b_j > \frac{\pi}{2} - \delta,$$

where $i \neq j$ and $\widetilde{\angle} = \widetilde{\angle}^0$, as defined in Definition 3.3.14. The collection of these points a_i, b_i is called an *explosion* at p.

Remark 3.4.3. In our exposition, recall that we are always working with spaces that are complete (and hence locally compact by Theorem 3.6.2). Consequently, that $p \in X$ is an (n, δ)-burst point is equivalent to the existence of $2n$ shortest paths γ_i (joining p to a_i) and σ_i (joining p to b_i), the angles between which satisfy analogous inequalities to those in Definition 3.4.2. Notice that this means that if p is such a burst point, we can choose the explosion points to be arbitrarily close to p.

We now use burst points to find 'coordinate charts' on X:

Theorem 3.4.4 ([6, Theorem 5.4]). *Let X be a complete Alexandrov space with curvature bounded below. Suppose $p \in X$ is an (n, δ)-burst point with explosion $\{(a_i, b_i)\}$, with $\delta < \frac{1}{2n}$, and such that there exists a neighbourhood $U \ni p$ that contains no $(n+1, 4\delta)$-burst points. Then there exists a neighbourhood $V \ni p$ and a domain $\Omega \subset \mathbb{R}^n$ such that the map $\varphi \colon V \to \Omega$ defined by*

$$\varphi(q) := (|a_1 q|, \ldots, |a_n q|)$$

is a bi-Lipschitz homeomorphism. We call φ an explosion map.

To prove that the Hausdorff dimension of X, $\dim_{\mathscr{H}}(X)$, is an integer or infinite, we need the notion of a *burst index*, which will coincide with this dimension:

Definition 3.4.5 (Burst index). Let X be a complete Alexandrov space of curvature bounded below. The *burst index* of X near $p \in X$, denoted $\mathrm{Burst}_p(X)$, is defined to be the largest $n \in \mathbb{N}$ such that every neighbourhood of p contains (n, δ)-burst points for any $\delta > 0$, but such that no neighbourhood of p contains $(n+1, \delta)$-burst points. If no such n exists, we say $\mathrm{Burst}_p(X) = \infty$.

In what follows, we will show that the burst index $\mathrm{Burst}_p(X)$ is constant as we vary p. The first step towards this is to consider the so-called *rough dimension* of a space:

Definition 3.4.6 (Rough volume and dimension). Let X be a metric space and let $U \subset X$ be a bounded subset. The *rough a-dimensional volume* of U is

$$Vr_a(U) := \limsup_{\varepsilon \to 0} \varepsilon^a \beta_U(\varepsilon),$$

where $\beta_U(\varepsilon)$ is the largest possible number of points in U that are at least ε pairwise-distant from each other (which could be infinite).

The *rough dimension of* U is then

$$\dim_r(U) := \inf\{a \mid Vr_a(U) = 0\}.$$

Clearly we have $\dim_r(U) \geq \dim_{\mathscr{H}}(U)$. We note another obvious fact about rough and Hausdorff dimensions:

Proposition 3.4.7. *Let X and Y be metric spaces, and suppose $f \colon X \to Y$ is bi-Lipschitz. Then*

$$\dim_r(X) = \dim_r(Y)$$

and

$$\dim_{\mathscr{H}}(X) = \dim_{\mathscr{H}}(Y).$$

Lemma 3.4.8 ([6, Lemma 6.3])**.** *Let (X, d) be a complete Alexandrov space with curvature bounded below. Let $u, v \in X$ be two points, and let $U \ni u, V \ni v$ be neighbourhoods. Then for U and V sufficiently small,* $\dim_r(U) = \dim_r(V)$.

Proof. The 'sufficiently small' requirement is to ensure that we are working in domains for which the conditions of Definition 3.3.6 hold. Let $\dim_r(V) > a \geq 0$. Then by definition,

$$\limsup_{\varepsilon \to 0} \varepsilon^a \beta_V(\varepsilon) = \infty,$$

and so there exists a sequence $\{\varepsilon_i\} \to 0$ and a constant $c > 0$ such that $\varepsilon_i^a \beta_V(\varepsilon_i) \geq c$ for each i. Consequently, for each i we may find a set of points $c_1, \ldots, c_{N_i} \in V$ that are at least ε_i pairwise distant from each other, and such that $\varepsilon_i^a N_i \geq c$.

Now fix a ball $B := B_R(u)$ in U, and consider shortest paths γ_j joining u to the points c_j. Choose points b_j on γ_j such that

$$|ub_j| = \frac{R}{D}|uc_j|,$$

where $D = \sup\{|ux| \mid x \in V\}$. Clearly this condition ensures that $b_j \in B$ for each j, and by Definition 3.3.6, the points b_j are at least $\varepsilon_i' := \dfrac{\varepsilon_i R}{D}$ distant from each other. We hence have that

$$(\varepsilon_i')^a \beta_U(\varepsilon_i') \geq \left(\frac{R}{D}\right)^a \varepsilon_i^a N_i \geq c \left(\frac{R}{D}\right)^a,$$

and so $Vr_a(U) > 0$ and $\dim_r(U) \geq \dim_r(V)$. Exchanging U and V and re-running the argument completes the proof. □

The following lemma pieces together the notions of burst index (Definition 3.4.5), rough dimension, and Hausdorff dimension:

Lemma 3.4.9 ([6, Lemma 6.4])**.** *Let (X, d) be a complete Alexandrov space with curvature bounded below, and let $p \in X$. Then for a sufficiently small neighbourhood $U \ni p$, we have*

$$\mathrm{Burst}_p(U) = \dim_r(U) = \dim_{\mathscr{H}}(U).$$

Proof. Suppose $\mathrm{Burst}_p(X) = n \in \mathbb{N}$ and let $U \ni p$ be sufficiently small. Then by the definition of burst index, there are no $(n+1, \frac{1}{8}(n+1))$-burst points in U. Thus

Theorem 3.4.4 gives a bi-Lipschitz homeomorphism $\varphi\colon U_1 \subset U \to V \subset \mathbb{R}^n$. This immediately implies that

$$\dim_r(U_1) = \dim_{\mathscr{H}}(U_1)$$

by Proposition 3.4.7. By Lemma 3.4.8, it then follows that $\dim_r(U) = \dim_r(U_1) = n$. Finally, $\dim_{\mathscr{H}}(U) = n$ since

$$n = \dim_{\mathscr{H}}(U_1) \leq \dim_{\mathscr{H}}(U) \leq \dim_r(U) = n.$$

The argument for $\mathrm{Burst}_p(X) = \infty$ is deduced similarly. □

We hence arrive at the main result of this section, which tells us that Alexandrov spaces with lower curvature bounds have a well-defined dimension:

Theorem 3.4.10 (Dimension Theorem, [6, Corollary 6.5])**.** *Let (X, d) be a complete Alexandrov space with curvature bounded below. The burst indices* $\mathrm{Burst}_p(X)$ *are independent of p. They coincide with the Hausdorff dimension of the space. If* $\mathrm{Burst}_p(X) < \infty$*, they coincide with the topological dimension of the space.*

Proof. The first two assertions follow from Lemmata 3.4.8 and 3.4.9. The third follows from Theorem 3.4.4, and from the fact that the Hausdorff dimension of a space does not exceed its topological dimension. □

We can now make the following definition:

Definition 3.4.11 (Dimension)**.** Let (X, d) be a complete Alexandrov space with curvature bounded below. The *dimension* of X is

$$\dim(X) := \mathrm{Burst}_p(X) \in \mathbb{N} \cup \{\infty\},$$

for any choice of $p \in X$.

Remark 3.4.12. Notice that if $p \in X$ is not a burst point with the properties required by Theorem 3.4.4, we are not guaranteed to find a homeomorphism from a neighbourhood of p onto a domain in $\mathbb{R}^n$. Consequently, it is not true that all Alexandrov spaces with curvature bounded below are topological manifolds - merely that they contain a dense subset that is such a manifold (the set of (n, δ)-burst points are dense in X for any $\delta > 0$ when X is n-dimensional by [6, Corollary 6.7]). An example of an Alexandrov space that is not a topological manifold is $\mathbb{R}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by symmetries.

Whilst it is possible to provide examples of Alexandrov spaces with lower curvature bounds that are *not* topological manifolds, the two-dimensional case is simpler, more discussion of which can be found following Theorem 3.6.7.

Theorem 3.4.13 ([5, Corollary 10.10.3])**.** *Let X be a complete Alexandrov space with curvature bounded below* of dimension two*. Then X is a topological manifold, possibly with boundary.*

Remark 3.4.14. An alternative exposition of the above results on dimension can be found in [5, §10.8], which uses slightly different nomenclature (notably referring to burst points as 'strained points').

3.5 Reshetnyak metrics

In this section, we provide theorems that show that Alexandrov spaces *of dimension two* with curvature bounded below are in fact 'almost Riemannian', in the sense that they admit a Riemannian metric which, whilst not smooth, is nevertheless very useful. The results in this section were originally provided in [21], [33] and [34], none of which are in English. A summary of the main results, which we closely follow, is to be found in English in [35]. The motivation comes directly from Riemannian geometry. First, we give a definition:

Definition 3.5.1. A complete Alexandrov space (X, d) with curvature bounded below of dimension 2 (as in Definition 3.4.11) that is topologically a surface *without boundary* is called an *Alexandrov surface.*

Now let $(\mathcal{M}, g)$ be a smooth Riemannian surface, and suppose that g may be written as

$$g = \lambda(x, y)|\mathrm{d}z|^2$$

with respect to a local complex coordinate $z = x + iy$ on an open set $G \subset \mathbb{C}$ (i.e. in some chart). That this is always possible is a well-known result. A standard formula gives the Gauß curvature of g as

$$K_g = -\frac{1}{2\lambda}\Delta \log \lambda,$$

where Δ is the Laplacian with respect to the coordinates x and y. Solving by means of the standard formula for the solution to the Poisson equation, we arrive at the formula

$$\log \lambda(z) = \frac{1}{\pi}\int_G \log \frac{1}{|z-\xi|} K_g(\xi)\lambda(\xi)\,\mathrm{d}\xi + h(z),$$

where h is some harmonic function. Recall that for a Riemannian manifold, the curvature measure (see Definition 3.3.29) is $\omega = K_g \, \mathrm{d}\mu_g$, and we hence have

$$\log \lambda(z) = \frac{1}{\pi} \int_G \log \frac{1}{|z - \xi|} \, \mathrm{d}\omega(\xi) + h(z).$$

The idea now is that on Alexandrov surfaces, we also have Riemannian metrics taking this form, but with ω a signed measure that is not necessarily absolutely continuous with respect to area (and which is, in fact, the curvature measure defined in Definition 3.3.29).

From now on, given an open set $G \subset \mathbb{C}$, a signed measure ω on the Borel σ-algebra $\mathscr{B}(G)$, and a harmonic function h on G, set

$$\lambda(z) \equiv \lambda(z, \omega, h) := \exp\left(\frac{1}{\pi} \int_G \log \frac{1}{|z - \xi|} \, \mathrm{d}\omega(\xi) + h(z) \right).$$

As discussed in [35], λ is defined and finite ω-almost everywhere in G.

Remark 3.5.2. We can say a little more about the regularity of such functions λ. Indeed, by [20, Theorem 3.9], a function u on G is subharmonic if and only if it takes the form

$$u(z) \equiv u(z, \mu, h) = \frac{1}{\pi} \int_G \log |z - \xi| \, \mathrm{d}\mu(\xi) + h(z)$$

for some (non-negative) measure μ and some harmonic function h. In this notation,

$$\log \lambda(z, \omega, h) = u_1(z, \omega^+, h) - u_2(z, \omega^-, 0)$$

for some subharmonic functions u_1 and u_2, and where $\omega = \omega^+ - \omega^-$ is the Jordan decomposition of the signed measure ω. In other words, $\log \lambda$ arises as the difference of subharmonic functions on G.

Definition 3.5.3. Given λ and $G \subset \mathbb{C}$ as above, we call $g_\lambda := \lambda |\mathrm{d}z|^2$ a *Reshetnyak metric* on G.

As is the case in Riemannian geometry, given a path γ in G, we can use g_λ to measure its length:

Definition 3.5.4. Let $\gamma \colon [0, \ell] \to G$ be a rectifiable curve in $G \subset \mathbb{C}$ parameterised by arc-length, and let λ be as above. The *length of γ with respect to g_λ* is

$$L_\lambda(\gamma) := \int_0^\ell \sqrt{\lambda(\gamma(s))}\,\mathrm{d}s.$$

We then define the *distance induced by g_λ* by

$$d_\lambda(x,y) := \inf_{\gamma\in\Phi} L_\lambda(\gamma),$$

where Φ is the collection of all rectifiable paths joining x and y parameterised by arc-length.

Clearly this is precisely the same approach taken in the definition of an intrinsic metric on a metric space (see Definition 3.1.6). However, a possible issue of concern here is that the function λ is defined only almost-everywhere, and since a path has two-dimensional Hausdorff measure zero, it is possible that $\lambda(\gamma(s))$ could only be defined on a null set. This concern is avoided by the following lemma:

Lemma 3.5.5 ([33, 34])**.** *Let u be a subharmonic function on an open set $G \subset \mathbb{C}$. Then $-\infty \le u(z) < \infty$. Moreover, let E be the set of those z for which $u(z) = -\infty$. Then for any $\varepsilon > 0$ and any $\alpha > 0$, there exists a sequence of open discs $B_m := B_{r_m}(c_m)$ such that*

$$E \subset \bigcup_{m=1}^{\infty} B_m$$

and

$$\sum_{m=1}^{\infty} r_m^\alpha < \varepsilon.$$

In our setting, we can use this lemma as follows: for $\lambda(z)$ to be undefined, we would need that $u_1(z) = -\infty$, or that both $u_1(z)$ and $u_2(z)$ are $-\infty$, where u_1 and u_2 are subharmonic functions as in Remark 3.5.2. But by Lemma 3.5.5, given $\varepsilon > 0$, these points can be covered by a sequence of discs, the sum of whose radii is less than ε. Consequently, the length $L_\lambda(\gamma)$ is well-defined.

Remark 3.5.6. The following results were originally stated for spaces of bounded integral curvature, as in Definition 3.3.28. However, as we have already noted, Alexandrov surfaces satisfy this definition.

We are now ready to provide the main theorem of this section, which tells us that Alexandrov surfaces come equipped with Reshetnyak metrics:

Theorem 3.5.7 ([35, Theorem 7.1.2])**.** *Let (X, d) be an Alexandrov surface, and let $p \in X$. Then there exist*

- *an open set $U \ni p$;*
- *an open set $G \subset \mathbb{C}$, a signed measure ω on the Borel σ-algebra $\mathscr{B}(G)$, and a harmonic function h on G;*
- *an isometry $\mathscr{I}: (U, d_U) \to (G, d_\lambda)$,*

where $\lambda(z) = \lambda(z, \omega, h)$, d_U is the induced intrinsic metric on U (Definition 3.1.6), and d_λ is the distance induced by the Reshetnyak metric g_λ (Definitions 3.5.3 and 3.5.4).

Definition 3.5.8. The isometry $\mathscr{I}$ in Theorem 3.5.7 is called an *isothermal coordinate chart for X.*

An alternative phrasing of this theorem, and an accompanying exposition, can be found in French in [47]. In that paper, all discussion takes place on the Alexandrov surface itself, rather than working in a chart as we have done, and the corresponding theorem is as follows:

Definition 3.5.9. Given a Riemannian surface (S, g), let $V(S, g)$ denote the collection of all functions $u \in L^1_{\text{loc}}(S, \mathrm{d}\mu_g)$ such that $\Delta_g u$ defines a signed measure, where the Laplacian is computed in the sense of distributions.

Given $u \in V(S, h)$, define

$$d_{g,u}(x, y) := \inf \left\{ \int_0^1 e^{u(\gamma(t))} |\gamma'(t)|_g \, \mathrm{d}t \right\},$$

where the infimum is taken over all C^1 paths $\gamma\colon [0, 1] \to S$ joining x and y.

Theorem 3.5.10 ([47, Theorem 7.1 variant])**.** *Let (X, d) be an Alexandrov surface. Then there exists a Riemannian metric g and a function $u \in V(X, g)$ such that $d = d_{g,u}$.*

One can easily reconcile Theorems 3.5.7 and 3.5.10. Indeed, write the Reshetnyak metric as $g_\lambda = e^\rho e^h |\mathrm{d}z|^2$, where

$$\rho(z) := \frac{1}{\pi} \int_G \log \frac{1}{|z - \xi|} \, \mathrm{d}\omega(\xi).$$

A quick computation shows that $\Delta\rho = -\omega$. In Theorem 3.5.10, the metric g comes from pulling back the metric $e^h|\mathrm{d}z|^2$ using the chart in which we have been working, and the function u corresponds to ρ in the same way. Another quick computation shows that $\tilde{\omega} = K_g \, \mathrm{d}\mu_g + \Delta_g u$, where $\tilde{\omega}$ is the measure ω after pulling back by the chart (i.e. $\tilde{\omega}(A) = \omega(\varphi(A))$, where φ is the chart and A is a Borel set in X).

In the language of Theorem 3.5.10, we state another useful result:

Theorem 3.5.11 ([47, Theorem 6.4]). *Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be two smooth Riemannian surfaces, and let $u \in V(\mathcal{M}, g)$ and $w \in V(\mathcal{N}, h)$. If*

$$f \colon (\mathcal{M}, d_{g,u}) \to (\mathcal{N}, d_{h,w})$$

is an isometry, then f is also a conformal diffeomorphism $(\mathcal{M}, g) \to (\mathcal{N}, h)$.

We also state the following technical result that we use in Chapter 4:

Theorem 3.5.12 ([47, Theorem 6.2], [33, Theorem III]). *Let (S, g) be a smooth Riemannian surface. Let $\{\mu_n^+\}$ and $\{\mu_n^-\}$ be two sequences of Radon measures on S, which converge weakly to measures μ^+ and μ^- respectively. Suppose that $\mu^+(\{p\}) < 2\pi$ for all $p \in S$. Define functions u_n and u on S by the relations $\Delta_g u_n = \mu_n := \mu_n^+ - \mu_n^-$ and $\Delta_g u = \mu := \mu^+ - \mu^-$, where the Laplacian is meant in the sense of distributions. Then*

$$d_{g,u_n} \to d_{g,u}$$

uniformly on $\mathcal{M} \times \mathcal{M}$, where the distances are as defined in Definition 3.5.9.

Remark 3.5.13. Let (X, d) be an Alexandrov surface with curvature bounded below, and let g and u be as in Theorem 3.5.10. Let $\mu := \omega_d - K_g \mathrm{d}\mu_g$, where ω_d is the curvature measure of d as given in Definition 3.3.29. We then have that $\Delta_g u = \mu$ in the sense of distributions. As pointed out by Richard in [37, Remark 3.2.11], we necessarily have that $\mu^+(\{x\}) < 2\pi$ for all $x \in X$. Indeed, at points where $\mu^+(\{x\}) \geq 2\pi$, the space of directions $\Sigma_x(X)$ at x (see Definition 3.3.25) is a point, corresponding geometrically to so-called 'cusp points' on X. However, by [22, Theorem 1.3], all Alexandrov surfaces with curvature bounded below that can be approximated in the Gromov–Hausdorff topology by a sequence of *smooth* Riemannian surfaces have the property that $\Sigma_x(X) \cong S^1$ for all $x \in X$. Later, we prove Lemma 4.3.7, which shows that Alexandrov surfaces can be approximated in this way. Consequently we must have that $\mu^+(\{x\}) < 2\pi$ for all $x \in X$.

The final theorem we give in this section tells us that an Alexandrov surface admits the structure of a Riemann surface:

Theorem 3.5.14 (Conformal structure of Alexandrov surfaces, [35, Theorem 7.1.3], [21]). *Let (X, d) be an Alexandrov surface. Let $\varphi \colon U \to \mathbb{C}$ and $\psi \colon V \to \mathbb{C}$ be* *isothermal coordinate charts on X, with $U \cap V \neq \varnothing$. Let $G := \varphi(U \cap V)$ and let $H := \psi(U \cap V)$. Then G and H are open sets in $\mathbb{C}$, and the map $\theta\colon z \mapsto \psi \circ \varphi^{-1}(z)$ is conformal.*

Moreover, if $\lambda(z)|\mathrm{d}z|^2$ and $\mu(w)|\mathrm{d}w|^2$ are the Reshetnyak metrics associated to φ and ψ respectively, then for all $z \in G$,

$$\lambda(z) = \mu(\theta(z))|\theta'(z)|^2.$$

Corollary 3.5.15. *Let (X, d) be an Alexandrov surface. Then taking all possible isothermal coordinate charts on X as an atlas, X is a Riemann surface.*

Example 3.5.16. An easy example that demonstrates the preceding theorems is the Euclidean cone with cone angle θ. This space is isometric to $\mathbb{C}$ with the Reshetnyak metric

$$g = |z|^{2\beta}\,|\mathrm{d}z|^2,$$

where $\beta = \frac{\theta}{2\pi} - 1$. The curvature measure of this cone is $\omega = -2\pi\beta\delta_0$, where δ_0 denotes the Dirac mass at the origin. Indeed:

$$\frac{1}{\pi}\int_{\mathbb{C}} \log\frac{1}{|z-\xi|}\,\mathrm{d}\omega(\xi) = 2\beta\int_{\mathbb{C}} \log|z-\xi|\,\mathrm{d}\delta_0 = 2\beta\log|z|,$$

and so, in the language of the previous results, $\log\lambda = 2\beta\log|z|$, as expected.

3.5.1 Technique: smoothing Reshetnyak metrics

We take a brief detour to discuss a useful technique we appeal to later. When attempting to approximate Alexandrov surfaces in the Gromov–Hausdorff topology, for example, a possible line of attack would be to 'smooth out' a Reshetnyak metric on the space. Here, we present a method for doing this for the Euclidean cone, which also carries over to other situations.

Returning to Example 3.5.16, it is useful to change notation and let $\alpha := 2\pi - \theta$, where θ is again the cone angle of a Euclidean cone. The Reshetnyak metric of this cone is then given by $g := |z|^{-\frac{\alpha}{\pi}}\,|\mathrm{d}z|^2$. We now smooth out this metric by spreading the curvature in a neighbourhood of the vertex, i.e. the origin.

Define $\psi \in C^\infty(\mathbb{R})$ by

$$\psi(t) := \begin{cases} \exp(-\frac{1}{t}) & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

Also define $H(z) := C\psi\left(\frac{1}{4} - |z|^2\right)$, for a constant C determined by the relation

$$\int_{\mathbb{R}^2} H(x,y)\,\mathrm{d}\mu = 1,$$

where μ is Lebesgue measure on $\mathbb{R}^2$. Notice that $H \in C^\infty(\mathbb{C})$ and that $H(z) = 0$ whenever $|z| > \frac{1}{2}$.

Given $h > 0$, we then define the function λ_h by

$$\log \lambda_h(z) := \frac{\alpha}{\pi} \int_{\mathbb{C}} \log \frac{1}{|z-\xi|} \cdot \frac{1}{h^2} H\left(\frac{\xi}{h}\right) \mathrm{d}\xi.$$

It is easy to check that $\lambda_h \in C^\infty(\mathbb{C})$ for any $h > 0$. Moreover, we have that $\lambda_h(z) \to |z|^{-\frac{\alpha}{\pi}}$ uniformly, locally as $h \downarrow 0$, which follows from the representation formula for Reshetnyak metrics and the observation that

$$\frac{\alpha}{h^2} H\left(\frac{\xi}{h}\right) \mathrm{d}\xi \to \alpha\delta_0 = -2\pi\beta\delta_0$$

in the weak-$\star$ sense of measures as $h \downarrow 0$, where β is as in Example 3.5.16.

A computation shows that, if we write $g_h := \lambda_h(z)|\mathrm{d}z|^2$, then the Gauß curvature of g_h is

$$K_{g_h}(z) = \frac{\alpha}{h^2} H\left(\frac{z}{h}\right) \cdot \frac{1}{\lambda_h(z)} \ge 0.$$

In this way, we have found a sequence of smooth Riemannian metrics with curvature bounded below that converges uniformly, locally to the Reshetnyak metric g, and hence certainly in the Gromov–Hausdorff sense.

Remark 3.5.17. Whilst this example deals with a Euclidean cone, the same technique can be used to smooth the Reshetnyak metrics of any Alexandrov surface with curvature bounded below. Indeed, for any space of bounded integral curvature (see Definition 3.3.28), and hence for any Alexandrov surface, we can perform such a smoothing by following the discussion in [35, pp. 115–118].

3.6 Some other big theorems

In this final section, we collect some other key theorems in the study of Alexandrov spaces with curvature bounded below. Once again, a clear theme is that classical results of Riemannian geometry carry over, at least in some form, to this setting.

3.6.1 Globalisation

The first result we present here is of a very technical nature, and so we do not discuss the proof. Recall from §3.3 that our definitions of Alexandrov spaces were always of a local nature, i.e. for every point in the space, there had to exist a neighbourhood of that point in which a certain condition involving lengths or angles had to hold. The following theorem tells us that, in fact, these conditions hold 'in the large':

Theorem 3.6.1 (Globalisation Theorem [5, Theorem 10.3.1], [6, Theorem 3.2]). *Let (X, d) be a complete Alexandrov space with curvature bounded below by k. Then for* any *four points $a, b, c, d \in X$, we have the inequality*

$$\tilde{\angle}^k bac + \tilde{\angle}^k bad + \tilde{\angle}^k cad \leq 2\pi.$$

It is worth pointing out that completeness is necessary here, even if we can make sense of Alexandrov spaces that are merely locally complete. For example, the plane with a closed disc removed satisfies the local conditions of Definitions 3.3.6, 3.3.20 and 3.3.21 with $k = 0$, but clearly the conclusion of the Globalisation Theorem does not hold.

3.6.2 Local compactness

Now we recall Remark 3.3.10, where we noted that throughout this chapter, we work with complete Alexandrov spaces, which is a sufficient condition to ensure the metric is strictly intrinsic, i.e. that every pair of points can be joined by a shortest path. From Theorem 3.1.9, we recall that both completeness *and* local compactness are required for a metric in a length space to be strictly intrinsic. Indeed, the following result shows that for finite-dimensional Alexandrov spaces, local compactness comes for free:

Theorem 3.6.2 ([5, Corollary 10.8.20]). *All complete, finite-dimensional Alexandrov spaces with curvature bounded below are locally compact.*

Sketch proof. Let (X, d) be a complete Alexandrov space of dimension n, in the sense of Definition 3.4.11. Let $p \in X$ be an (n, δ)-burst point, for δ sufficiently small (the set of such points is dense in X). Then by Theorem 3.4.4, some neighbourhood U of p is bi-Lipschitz homeomorphic to a domain in $\mathbb{R}^n$. Thus, U is locally compact, and so we may find $r > 0$ such that $B_r(p)$ is pre-compact.

Now we claim that for any $R > 0$, $B_R(p)$ is pre-compact. If not, there exists an $\varepsilon > 0$ and an infinite set $\mathcal{S}$ that is an ε-net for $B_R(p)$. One can construct a homothety map $f\colon B_R(p) \to B_r(p)$ such that

$$d(f(x), f(y)) > \frac{r}{2R} d(x, y)$$

for any x, y belonging to any finite subset $\mathcal{S}' \subset \mathcal{S}$. Consequently, the ball $B_r(p)$ contains an $\frac{r}{2R}$-net $f(\mathcal{S}')$ of an arbitrarily large number of points, which contradicts the pre-compactness of $B_r(p)$. So any ball in X is pre-compact, and so X is locally compact. □

We have now justified the assertion of Remark 3.3.10 that in complete Alexandrov spaces with curvature bounded below, the metric is always *strictly* intrinsic.

3.6.3 Gromov–Hausdorff limits

We mentioned earlier that Gromov–Hausdorff limits of sequences of Alexandrov spaces with lower curvature bounds are themselves Alexandrov spaces with lower curvature bounds. We discuss this somewhat more in this subsection. Indeed, we can employ the Globalisation Theorem (Theorem 3.6.1) to this end:

Theorem 3.6.3 ([5, Proposition 10.7.1])**.** *Let (X_i, d_i) be a sequence of Alexandrov spaces with curvature bounded below by k. Suppose that there exist points $p_i \in X_i$, and a pointed metric space (X, d, p) such that*

$$(X_i, d_i, p_i) \to (X, d, p)$$

in the Gromov–Hausdorff sense (see Definition 3.2.5) as $i \to \infty$. Then (X, d) is also an Alexandrov space with curvature bounded below by k.

Sketch proof. By Globalisation (Theorem 3.6.1), the quadruple condition holds for any four points in (X_i, d_i). Under the stated convergence, if $a, b, c, d \in X$, we may find $a_i, b_i, c_i, d_i \in X_i$ such that $d_i(a_i, b_i) \to d(a, b)$, and analogously for the other pairings of points, as $i \to \infty$. The quadruple condition for the points a, b, c, d then follows from the quadruple condition on the points a_i, b_i, c_i, d_i and the continuity of comparison angles. □

Recall the Gromov Compactness Theorem from Riemannian geometry:

Theorem 3.6.4 (Gromov Compactness Theorem for Riemannian manifolds)**.** *The class of n-dimensional Riemannian manifolds, with Ricci curvature bounded below by some $k \in \mathbb{R}$ and with diameter bounded above by some $D > 0$, is pre-compact in the Gromov–Hausdorff topology.*

This theorem does not give us information on the structure of the limit space of such a sequence. The reason for this is exposed in the following result. Loosely speaking, the above theorem is true because the Bishop–Gromov inequality holds for lower bounds on the *Ricci* curvature. If instead we restrict this condition, and require lower bounds on *sectional* curvature, we find that the limit space is in fact an Alexandrov space:

Theorem 3.6.5 (Gromov Compactness for Alexandrov spaces, [5, Theorem 10.7.2])**.** *Given $n \in \mathbb{N}, k > 0$ and $D > 0$, let $\mathfrak{M}(n, k, D)$ denote the class of all Alexandrov spaces with curvature bounded below by k, of dimension at most n, and with diameter bounded above by D. Then $\mathfrak{M}(n, k, D)$ is compact in the Gromov–Hausdorff topology.*

This theorem is in fact a major motivation for the study of Alexandrov spaces: it tells us in particular that given a sequence of Riemannian manifolds of bounded diameter, and with a lower bound on sectional curvature, there exists (passing to a subsequence if necessary), a Gromov–Hausdorff limit space *that is an Alexandrov space*. Consequently, the study of Alexandrov spaces with lower curvature bounds could be viewed as a study of the limit spaces obtained from such sequences of Riemannian manifolds.

3.6.4 Unpublished results of G. Perelman

Whilst the paper [6] is a fundamental resource to the student of Alexandrov spaces, Perelman wrote an unpublished sequel [32] that contains some further spectacular results.

A very useful theorem tells us that Alexandrov spaces with lower curvature bounds can be 'stratified' into topological manifolds:

Definition 3.6.6. A collection of subsets $\{X_i\}_{i=1}^{N}$ of a topological space X is a *stratification* of X into topological manifolds if

- The sets X_i are disjoint, and $\bigcup_i X_i = X$;

- For each i, X_i is a topological manifold *without boundary*;
- For each $1 \leq i \leq N-1$, $\dim(X_i) > \dim(X_{i+1})$;
- For each $1 \leq k \leq N$, the set
$$\bigcup_{i=k}^{N} X_i$$
is closed in X.

The sets X_i are called *strata*.

Theorem 3.6.7 (Stratification Theorem, [32]). *Every finite-dimensional Alexandrov space with curvature bounded below admits a stratification into topological manifolds.*

The sets X_i can be described loosely as follows: given an Alexandrov space X with curvature bounded below of dimension n, X_1 is the set of points that admit a neighbourhood homeomorphic to a domain in $\mathbb{R}^n$. As discussed in §3.4, X_1 will hence consist of all (n, δ)-burst points in X, for δ sufficiently small (see Definition 3.4.2). Next, X_2 will consist of all points in X that admit neighbourhoods homeomorphic to domains in $\mathbb{R}^{n-1}$, provided this set is non-empty. Continuing in this way, we arrive at a stratification into topological manifolds.

An example of such a stratification would be to consider the cone over $\mathbb{RP}^2$, which is stratified into the complement of the origin (which is three-dimensional), and the origin itself. Another use of this result is to prove Theorem 3.4.13, which says that a two-dimensional Alexandrov space with curvature bounded below is a topological manifold, possibly with boundary. In this case, the space is stratified into a two-dimensional manifold without boundary, together with its boundary.

Another striking result of Perelman gives us some insight into the question of whether or not an Alexandrov space may be approximated by manifolds of the same dimension in the Gromov–Hausdorff topology. Indeed, the so-called Stability Theorem tells us (in the compact case) that this is not possible if the Alexandrov space in question is not itself a manifold:

Theorem 3.6.8 (Stability Theorem, [32]). *Let (X, d) be a compact Alexandrov space of dimension n with curvature bounded below by $k \in \mathbb{R}$. Then there exists an $\varepsilon > 0$ such that for any other n-dimensional, compact Alexandrov space $\left(Y, \tilde{d}\right)$ with* *curvature bounded below by k, we have that X and Y are homeomorphic whenever*

$$d_{GH}(X, Y) < \varepsilon.$$

Here d_{GH} is the Gromov–Hausdorff distance, as in Definition 3.2.2.

Remark 3.6.9. If (X, d) is not topologically a manifold, then we have no hope of finding a sequence of n-dimensional compact manifolds that converges to X in the Gromov–Hausdorff sense. This observation has important implications for the study of Ricci flow later on in Chapter 4. There, we prove that *Alexandrov surfaces* (with particular properties) can be used as 'initial conditions' for the Ricci flow. The line of attack is to first approximate the surface by smooth Riemannian manifolds. Theorem 3.6.8 shows us that this approach will not be possible for arbitrary Alexandrov spaces of dimension higher than two. In dimension two, however, this obstacle is not present, thanks to Theorem 3.4.13.

The final result we present in this section is another analogue of a much-celebrated theorem in Riemannian geometry:

Theorem 3.6.10 (Soul Theorem, [9, Theorem 1.11])**.** *Let $(\mathcal{M}, g)$ be a complete, connected Riemannian manifold with sectional curvature $K_g \geq 0$. Then there exists a submanifold $\mathcal{S} \subset \mathcal{M}$ (the* soul*) without boundary, which is compact, totally convex, and totally geodesic, such that $\mathcal{M}$ is diffeomorphic to the normal bundle of $\mathcal{S}$.*

This theorem largely simplifies the study of complete Riemannian manifolds of non-negative sectional curvature to the compact case. It carries over in a similar form to the setting of Alexandrov spaces, but before stating this version, we recall a standard topological notion:

Definition 3.6.11. Let X be a topological space and $A \subset X$. By a *deformation retract* of X onto A, we mean a continuous map $F\colon X \times [0, 1] \to X$ such that, given $x \in X$ and $a \in A$, we have

- $F(x, 0) = x$;
- $F(x, 1) \in A$ and
- $F(a, t) = a$ for all $t \in [0, 1]$.

Theorem 3.6.12 (Soul Theorem for Alexandrov spaces, [32])**.** *Let X be a non-compact, finite-dimensional Alexandrov space with curvature bounded below by $k = 0$. Then there exists a convex, compact subset $\mathcal{S} \subset X$ without boundary, such that there exists a deformation retract of X onto $\mathcal{S}$.*

Chapter 4

Ricci flow of Alexandrov surfaces

Throughout our survey in Chapter 3, a common theme we observed was that several properties of Riemannian manifolds, which are typically thought of as *inherently Riemannian*, in fact carry over (at least in some form) to the setting of Alexandrov spaces with lower curvature bounds. We also remarked, in light of Theorem 3.6.5, that such Alexandrov spaces arise naturally as the Gromov–Hausdorff limits of smooth Riemannian manifolds with lower bounds on sectional curvature. Consequently, given our discussion of the Ricci flow in Chapter 2 where initial conditions are smooth Riemannian manifolds, it is natural to ask if we can make sense of Ricci flows taking Alexandrov spaces as initial condition. Indeed, this question was first tackled by Miles Simon in [39], as we discuss in what follows. This result was utilised by Thomas Richard in [37] to prove the existence and uniquess of Ricci flows taking *compact* Alexandrov surfaces as initial condition in a certain sense. In this chapter, we extend these results to the setting of *non-compact* Alexandrov surfaces.

4.1 Introductory results

Before examining these results, however, we must consider what it means for a Ricci flow to take a metric space as initial condition. One possibility that immediately springs to mind is the following: let $(\mathcal{M}, g(t), p)_{t\in(0,T)}$ be a smooth, pointed Ricci flow, and (X, d, x) a pointed metric space. Say $(\mathcal{M}, g(t), p)$ takes (X, d, x) as initial condition *in the Gromov–Hausdorff sense* if

$$(\mathcal{M}, g(t), p) \to (X, d, x)$$

as $t \downarrow 0$, in the Gromov–Hausdorff sense of Definition 3.2.5. At first glance, this may seem like a natural choice. However, upon consideration of relatively simple examples, we see that it is too weak a definition without further hypotheses, as we have non-uniqueness:

Example 4.1.1. Consider the torus T^2 and, given $p \in T^2$, define $T^2_p := T^2 \setminus \{p\}$. Let h be the complete hyperbolic metric on T^2_p. A solution to the Ricci flow equation on T^2_p beginning at h is the standard dilating flow $h(t) := (1 + 2t)h$. However, in [44], Topping proves the existence of a unique flow $\hat{h}(t)$ on the (non-punctured) torus T^2 such that $(T^2, \hat{h}(t)) \to (T^2_p, h)$ in the Cheeger–Gromov sense as $t \downarrow 0$ for an appropriate choice of points. Consequently, we have two distinct flows $h(t)$ and $\hat{h}(t)$, both of which take (T^2_p, h) as initial condition in the Gromov–Hausdorff topology.

The alternative we use is the following:

Definition 4.1.2. We say that a smooth Ricci flow $(\mathcal{M}^n, g(t))_{t\in(0,T)}$ takes the metric space (X, d) *as initial condition* if the Riemannian distances $d_{g(t)}$ converge uniformly on compact subsets of $\mathcal{M} \times \mathcal{M}$ to a metric $\tilde{d}$ on $\mathcal{M}$ such that $\left(\mathcal{M}, \tilde{d}\right)$ and (X, d) are isometric.

We are now ready to state the following result of Miles Simon, which gives existence of Ricci flows taking certain metric spaces as initial condition:

Theorem 4.1.3 ([39, Theorem 9.2])**.** *Given $k, v_0 > 0$, let (X, d) be a compact metric space such that there exists a sequence of smooth, compact Riemannian manifolds $(\mathcal{M}^n_i, g_i)$, with $n = 2$ or 3, converging to (X, d) in the Gromov–Hausdorff sense (Definition 3.2.2), and such that:*

- *for each i, $\mathrm{Ric}_{g_i} \geq -kg_i$ and*
- *for each $x \in \mathcal{M}_i$, $\mathrm{vol}_{g_i}(B^{g_i}_1(x)) \geq v_0$ for each i.*

Then there exist a $T > 0$ and a smooth Ricci flow $(\mathcal{M}^n, g(t))_{t\in(0,T)}$ taking (X, d) as initial condition (Definition 4.1.2).

Remark 4.1.4. As discussed in [39], this theorem also holds if the metric space (X, d) and the approximating sequence $(\mathcal{M}_i, g_i)$ are non-compact, but we require the extra assumption that the approximating sequence has controlled geometry at infinity. For instance, one option would be to require that the metrics g_i be of bounded curvature.

We observe the following immediate corollary that is to date the best result in this direction in dimension three:

Corollary 4.1.5. *Let (X, d) be a compact Alexandrov space with curvature bounded below of dimension three, which can be approximated in the Gromov–Hausdorff topology by a sequence of smooth, compact Riemannian manifolds satisfying the conditions in Theorem 4.1.3. Then there exist a $T > 0$ and a smooth Ricci flow $(\mathcal{M}^3, g(t))_{t\in(0,T)}$ taking (X, d) as initial condition.*

Remark 4.1.6. It seems unlikely that Corollary 4.1.5 can be improved. A major issue is that in typical scenarios, three-dimensional Alexandrov spaces cannot be approximated in the required way - indeed, as a consequence of Theorem 3.6.8, the Alexandrov space in question would necessarily have to be a topological manifold, which is not always the case (recall the example of $\mathbb{R}^3/\mathbb{Z}_2$). Then, even if the space were to be a manifold, finding an appropriate approximating sequence in this dimension appears to be tricky. In [25], it is shown that three-dimensional compact *polyhedral manifolds*, which are manifolds that can be triangulated in such a way that each simplex is isometric to a Euclidean simplex, and which are also Alexandrov spaces with curvature bounded below by 0 in the sense of Definition 3.3.6, can be approximated by smooth Riemannian manifolds in the Gromov–Hausdorff topology.

4.2 The compact case

In his thesis [37], Thomas Richard considered the problem of finding Ricci flows taking as initial condition (in the sense of Definition 4.1.2) *compact* Alexandrov surfaces. In this section, we sketch the proof of this result, which is also the content of the article [36].

Theorem 4.2.1 ([36] , [37, Theorem 3.1.1])**.** *Let (X, d) be a compact Alexandrov surface with curvature bounded below by -1. Then there exist a $T > 0$ and a smooth Ricci flow $(\mathcal{M}^2, g(t))_{t\in(0,T)}$ such that*

- $K_{g(t)} \geq -1$ *for all* $t \in (0, T)$ *and*
- $(\mathcal{M}, g(t))$ *takes* (X, d) *as initial condition.*

Moreover, $(\mathcal{M}, g(t))$ is the unique Ricci flow satisfying these conditions up to diffeomorphism.

The proof of this theorem marries the results we have discussed in Chapter 3 on metric geometry of Alexandrov spaces, and the smooth theory of the Ricci flow. To prove the existence part of the theorem, Richard uses Theorem 4.1.3. Consequently, existence is proved provided a compact Alexandrov surface (X, d) can be approximated in the Gromov–Hausdorff sense by a sequence of smooth Riemannian manifolds satisfying the conditions of Theorem 4.1.3. This is indeed the case:

Theorem 4.2.2 (Smoothing of compact Alexandrov surfaces, [37, Theorem 3.2.4])**.** *Let (X, d) be a compact Alexandrov surface with curvature bounded below by -1. Then there exists a sequence of smooth, compact Riemannian surfaces $(\mathcal{M}_i^2, g_i)$ such that*

- $(\mathcal{M}_i, d_{g_i})$ *converges to* (X, d) *in the Gromov–Hausdorff sense (Definition 3.2.2),*
- $K_{g_i} \geq -1$ *for each* $i \in \mathbb{N}$,
- *for each* i, *we have*

$$\frac{1}{2}\operatorname{Diam}(X, d) \leq \operatorname{Diam}(\mathcal{M}_i, g_i) \leq 2\operatorname{Diam}(X, d)$$

  and

- *for each* i, *we have*

$$\frac{1}{2}\mathscr{H}^2(X, d) \leq \operatorname{vol}_{g_i}(\mathcal{M}_i, g_i) \leq 2\mathscr{H}^2(X, d),$$

  where $\mathscr{H}^2$ *is the two-dimensional Hausdorff measure.*

We prove a similar result in the non-compact case using essentially the same arguments as Richard (see Lemma 4.3.7), and so we do not dwell on the details here. With this result in hand, the existence portion of Theorem 4.2.1 follows from an application of Theorem 4.1.3. We now concentrate on the uniqueness claim of Theorem 4.2.1. The first step towards this is to show uniqueness of the conformal class of Ricci flows taking a compact Alexandrov surface as initial condition:

Proposition 4.2.3 ([37, Proposition 3.4.1])**.** *Let (X, d) be a compact Alexandrov surface with curvature bounded below by -1. Suppose that $(\mathcal{M}^2, g(t))_{t\in(0,T)}$ and $(\mathcal{N}^2, h(t))_{t\in(0,T)}$ are two smooth Ricci flows taking (X, d) as initial condition. Suppose further that there exists $C > 0$ such that*

$$-1 \leq K_{g(t)}, K_{h(t)} \leq \frac{C}{t}$$

for all $t \in (0, T)$. If we write $g(t) = e^{2u(t)}g$ and $h(t) = e^{2w(t)}h$ for smooth Riemannian metrics g and h on $\mathcal{M}$ and $\mathcal{N}$ respectively, then there exists a conformal diffeomorphism

$$\varphi\colon (\mathcal{M}, g) \to (\mathcal{N}, h).$$

Sketch proof. By Definition 4.1.2, there exist distances η and ρ on $\mathcal{M}$ and $\mathcal{N}$ respectively such that $d_{g(t)} \to \eta$ and $d_{h(t)} \to \rho$ uniformly as $t \downarrow 0$. The first step in proving the result is then to show that the functions $u(\cdot, t)$ and $w(\cdot, t)$ converge in $L^1(\mathrm{d}\mu_g)$ and $L^1(\mathrm{d}\mu_h)$ respectively to integrable functions u_0 and w_0. This follows almost immediately from the Ricci flow equation, the lower curvature bound of -1, and an application of Jensen's Inequality. Next, it is shown that the functions u_0 and w_0 actually belong to the spaces $V(\mathcal{M}, g)$ and $V(\mathcal{N}, h)$ respectively, as given in Definition 3.5.9. In other words, these functions are such that e^{2u_0} and e^{2w_0} are of the correct regularity to define Reshetnyak metrics on $\mathcal{M}$ and $\mathcal{N}$ respectively (see Definition 3.5.3). We then show that in fact these functions *do* define bona-fide Reshetnyak metrics on their corresponding surfaces. To do this, it is shown that the distance induced by the metric $e^{2u_0}g$ coincides with the distance η, and also the corresponding statement for the metric $e^{2w_0}h$. Consequently, we have an isometry $\mathscr{I}\colon (\mathcal{M}, \eta) \to (\mathcal{N}, \rho)$, which is hence a conformal map $(\mathcal{M}, g) \to (\mathcal{N}, h)$ by Theorem 3.5.11. □

We may now, without loss of generality, consider two smooth Ricci flows $g_1(t) = v_1(t)h$ and $g_2(t) = v_2(t)h$ for $t \in (0, T)$, on a smooth Riemannian surface $(\mathcal{M}^2, h)$, which take (X, d), an Alexandrov surface with curvature bounded below by -1, as initial condition. To prove the uniqueness statement of Theorem 4.2.1, it hence suffices to show that $v_1(t) = v_2(t)$ for all $t \in (0, T)$. The first step towards this is the following, which is a consequence of Definition 4.1.2:

Lemma 4.2.4 ([37, Lemma 3.5.1])**.** *For v_1 and v_2 as above, there exists a function $v \in L^1(\mathrm{d}\mu_h)$ such that*

$$v = \lim_{t \downarrow 0} v_1(\cdot, t) = \lim_{t \downarrow 0} v_2(\cdot, t)$$

in $L^1(\mathrm{d}\mu_h)$.

With this in hand, uniqueness is then a consequence of the following result, bearing in mind the equation solved by conformal factors of Ricci flows on smooth surfaces (Proposition 2.2.2):

Proposition 4.2.5 ([37, Proposition 3.5.2])**.** *Let $(\mathcal{M}^2, h)$ be a smooth, closed Riemannian surface. Let $v_1, v_2\colon \mathcal{M} \times (0, T) \to \mathbb{R}_{>0}$ be two solutions to the equation*

$$\frac{\partial \phi}{\partial t} = \Delta_h \log \phi - 2K_h.$$

Suppose further that

$$\lim_{t\downarrow 0}\int_{\mathcal{M}} |v_1(x,t)-v_2(x,t)|\,\mathrm{d}\mu_h(x)=0.$$

Then $v_1 = v_2$ *on* $\mathcal{M}\times(0,T)$.

Proof. It suffices to show that for every smooth, positive function $\varrho\colon \mathcal{M}\to\mathbb{R}$ and for every $t_0\in(0,T)$, we have

$$\int_{\mathcal{M}}(v_1(x,t_0)-v_2(x,t_0))\varrho(x)\,\mathrm{d}\mu_h(x)=0.$$

To this end, define a function $A\in C^\infty(\mathcal{M}\times(0,T))$ by

$$A(x,t):=\begin{cases}\frac{\log v_1(x,t)-\log v_2(x,t)}{v_1(x,t)-v_2(x,t)} & \text{if } v_1(x,t)\neq v_2(x,t)\\ \frac{1}{v_1(x,t)} & \text{otherwise.}\end{cases}$$

Now let ψ be the unique, positive, smooth solution to the backwards heat equation

$$\begin{cases}\frac{\partial\psi(x,t)}{\partial t} & = -A(x,t)\Delta_h\psi(x,t) \qquad \text{for } (x,t)\in\mathcal{M}\times(0,t_0)\\ \psi(x,t_0) & =\varrho(x) \qquad \text{for } x\in\mathcal{M}.\end{cases}$$

Then, multiplying by ψ the equation satisfied by v_1-v_2 and integrating by parts, we arrive at the equality

$$\int_{\mathcal{M}}(v_1(x,t_0)-v_2(x,t_0))\varrho(x)\,\mathrm{d}\mu_h(x)=\int_{\mathcal{M}}(v_1(x,\tau)-v_2(x,\tau))\psi(x,\tau)\,\mathrm{d}\mu_h(x),$$

for any $\tau\in(0,t_0)$.

Now observe that by the maximum principle,

$$\sup_{x\in\mathcal{M}}\psi(x,t)\le\sup_{x\in\mathcal{M}}\varrho(x),$$

for any $t\in(0,t_0)$, i.e. ψ is bounded above independently of t. Combining this with Lemma 4.2.4 and taking the limit as $\tau\downarrow 0$, we find that $v_1(x,t_0)=v_2(x,t_0)$ for all $x\in\mathcal{M}$. Since t_0 was arbitrary, it follows that $v_1=v_2$ on $\mathcal{M}\times(0,T)$, as required. □

Remark 4.2.6. Let X be a compact Kähler manifold (of any finite dimension), let $\alpha_0\in H^{1,1}(X;\mathbb{R})$ be a Kähler class, and let T_0 be a closed, positive $(1,1)$-current in the class α_0. In [28], it is shown that the Kähler–Ricci flow starting from T_0 is unique, and smooth outside a particular analytic subset of X. This result serves as a complement to Theorem 4.2.1 in the Kähler case.

4.3 The non-compact case

In this section, we extend Theorem 4.2.1 to the setting of non-compact Alexandrov surfaces with curvature bounded below by -1, with the extra assumption of *non-collapsedness*, i.e. that the volume of unit balls is controlled uniformly from below. Before doing this, however, we must take a pause to consider triangulations of such spaces:

4.3.1 Triangulation of Alexandrov surfaces

A common theme in what follows is the approximation of Alexandrov surfaces by *polyhedral surfaces.* To do this, we need some more definitions, and some results of Alexandrov collected in [3]. Alexandrov's results were originally written in the context of topological surfaces endowed with an intrinsic metric, and the results stated here also require that shortest paths do not branch, in other words that if two shortest paths have common points, then either they are continuations of their intersection, or the common points are the ends of both shortest paths. By Corollary 3.3.24, Alexandrov spaces with curvature bounded below exhibit this property. Consequently, we reformulate these results in our context in what follows.

Definition 4.3.1. Let $n \geq 3$. Given a strictly intrinsic length space (X, d), an *Alexandrov polygon with n sides* in X is a map $t\colon \overline{\mathbb{D}} \to X$ that is a homeomorphism onto its image, such that $t(\partial\mathbb{D})$ is the union of n shortest paths in X, and such that the image of t is convex. We call the image $t\left(\overline{\mathbb{D}}\right)$ a *convex polygon* in X. The *sides* of $t\left(\overline{\mathbb{D}}\right)$ are the shortest paths making up $t(\partial\mathbb{D})$, and the *vertices* of $t\left(\overline{\mathbb{D}}\right)$ are the endpoints of these shortest paths. An *Alexandrov triangle* is an Alexandrov polygon with 3 sides, and a *convex triangle* is its image.

Definition 4.3.2. Let (X, d) be an Alexandrov surface. A *triangulation of* X is a locally-finite collection $\mathscr{T}$ of convex triangles that covers X, and such that no two triangles in $\mathscr{T}$ have common interior points.

Lemma 4.3.3 ([3, Chapter 2, §4, Theorem 2])**.** *Let (X, d) be an Alexandrov surface. Then given $p \in X$ and $\varepsilon > 0$, there exists a convex polygon $\mathcal{P}_p$ in X, containing p, with diameter at most ε.*

Lemma 4.3.4 ([3, Chapter 2, §6, Lemma 4]). *Let (X, d) be an Alexandrov surface. Let $P_1, \dots, P_N$ be convex polygons in X. Then the $\{P_i\}$ can be divided into convex polygons $Q_1, \dots, Q_M$* without common interior points.

Lemma 4.3.5 ([3, Chapter 2, §6, Theorem]). *Let (X, d) be an Alexandrov surface. Let $P \subset X$ be a convex polygon. Then, given $\varepsilon > 0$, P can be divided into finitely many convex* triangles *of diameter at most ε.*

With these foundational results in hand, we are ready to prove that Alexandrov surfaces can be triangulated with convex triangles in the sense of Definition 4.3.2:

Lemma 4.3.6. *Let (X, d) be a non-compact Alexandrov surface with curvature bounded below by -1. Then there exists a triangulation of (X, d) by convex triangles of arbitrarily small diameter.*

Proof. Fix $\varepsilon > 0$. By Lemma 4.3.3, given $p \in X$ we can find a convex polygon $\mathcal{P}_p$ containing p in its interior, of diameter at most ε. In this way, cover X by convex polygons of diameter at most ε, and call the cover $\mathcal{P}$. By Proposition 3.1.14, X is boundedly compact, and so $\mathcal{P}$ may be chosen to be locally finite. Fix a point $x \in X$ and $r > 0$. Define the set $\mathcal{F}_r$ by

$$\mathcal{F}_r := \left\{ \mathcal{P}_p \in \mathcal{P} \,\middle|\, \mathcal{P}_p \subseteq \overline{B_d(x, r)} \right\}.$$

In other words, $\mathcal{F}_r$ is the set of convex polygons obtained by taking those covering $\overline{B_d(x, r)}$ and excluding those that pass across the boundary of the ball.

Write the elements of $\mathcal{F}_r$ as $P_1, \dots, P_J$, and perform the following procedure: use Lemma 4.3.4 to divide the $\{P_i\}$ into convex polygons $Q_1, \dots, Q_K$ that have pairwise disjoint interiors. Then apply Lemma 4.3.5 to divide each polygon Q_i into finitely many convex triangles $T_1, \dots, T_L$ of diameter at most ε. Denote by $\mathcal{T}$ the collection of these triangles.

For $s > r$, let $\mathcal{G}_s$ be the set $\mathcal{F}_s$ with the polygons covered by elements of $\mathcal{T}$ removed. Perform the same procedure as above on the convex polygons in $\mathcal{G}_s$. Taking $s \to \infty$ gives the required triangulation. □

4.3.2 Ricci flow of non-compact Alexandrov surfaces

In this subsection, we tackle the problem of extending Theorem 4.2.1 to the non-compact setting. The arguments presented here are in the same spirit as those of Richard in the compact case (see [37]), but we must employ new technology to overcome difficulties posed by the lack of compactness. Indeed, this is not a trivial extension of the compact situation. For one thing, in order to use a compactness argument, we will need estimates of the form $\left|K_{g_i(t)}\right| \leq C$ uniformly on compact time intervals, where $(g_i(t))$ are Ricci flows starting from smooth approximations of the initial data's distance, in the same vein as Theorem 4.2.2. For this, we utilise the recent estimates of Miles Simon, given in Theorem 2.4.6. We also point out that the recent existence results of Giesen and Topping (see Theorem 2.2.4) are required to flow the smooth approximating sequence, which may not have bounded curvature.

We now state a comparable result to Theorem 4.2.2 for *non-compact* Alexandrov surfaces:

Lemma 4.3.7. *Let (X,d) be a non-compact Alexandrov surface with curvature bounded below by -1 that is non-collapsed, i.e. suppose there exists $\tau > 0$ such that $\mathscr{H}^2(B_1(x)) \geq 2\tau$ for any $x \in X$. Then there exists a sequence $(\mathcal{M}_i, g_i)$ of complete, non-compact, smooth Riemannian surfaces and points $p \in X$, $p_i \in \mathcal{M}_i$ such that:*

1. *$(\mathcal{M}_i, g_i, p_i) \to (X, d, p)$ in the Gromov-Hausdorff topology as $i \to \infty$ (in the sense of Definition 3.2.5),*
2. *$K_{g_i} \geq -1$ for each i,*
3. *We have that*
$$\inf_{x \in \mathcal{M}_i} \mathrm{vol}_{g_i}\left(B_2^{g_i}(x)\right) \geq \tau$$
for each i sufficiently large.

Proof. For each $i \in \mathbb{N}$, let $\mathcal{T} = \mathcal{T}(i)$ be a triangulation of X by convex triangles of diameter at most $\frac{1}{i}$ as given by Lemma 4.3.6. It is possible to choose a point $p \in X$ so that p is a vertex point in $\mathcal{T}$ for any i (since, by construction, decreasing the diameter of triangles in the triangulation only adds more vertices through subdivision of already-existing convex triangles into smaller ones). Now construct a polyhedral surface $\mathcal{M}_i$ as follows: given a convex triangle $T \in \mathcal{T}$, let $\widehat{T}$ be the comparison triangle of T in the hyperbolic plane $\mathbb{H}^2$. Let $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(i)$ denote the collection of these comparison triangles. Glue together the elements of $\widehat{\mathcal{T}}$ in the same configuration as the original triangles appear in X. Denote the length space so-obtained by $(\mathcal{M}_i, d_i)$, where d_i is the intrinsic metric of $\mathcal{M}_i$[1]. Moreover, let p_i be the point in $\mathcal{M}_i$ corresponding to the point $p \in X$ chosen earlier. By utilising the same arguments as in [37, Lemma A.2.1], which we provide in Proposition A.1, it is easy to see that $\mathcal{M}_i$ is an Alexandrov surface with curvature bounded below by -1.

Now notice that the vertex points of $\mathcal{T}$ and $\widehat{\mathcal{T}}$ form $\frac{1}{i}$-nets of X and $\mathcal{M}_i$ respectively (in the sense of Definition 3.1.3). As such, to show that $(\mathcal{M}_i, d_i, p_i) \to (X, d, p)$ in the Gromov–Hausdorff sense (as in Definition 3.2.5) as $i \to \infty$, it suffices to show that given vertex points v, w of the triangulation $\mathcal{T}(i)$ and their corresponding points $\hat{v}, \hat{w}$ in $\widehat{\mathcal{T}}(i)$, we have that

$$|d(v, w) - d_i(\hat{v}, \hat{w})| \leq \varrho(i),$$

where $\varrho(i) \downarrow 0$ as $i \to \infty$. Thus we are in precisely the same scenario as the local arguments of Richard in [37, §A.2], which we reproduce in Proposition A.2, from which we deduce the convergence claimed above.

We may now smooth the rough Reshetnyak metrics (see Theorem 3.5.7) locally on the Alexandrov surfaces $(\mathcal{M}_i, d_i)$ to obtain smooth, non-compact, pointed Riemannian surfaces $(\mathcal{M}_i, g_i, p_i)$ that converge to (X, d, p) in the Gromov–Hausdorff sense as $i \to \infty$, and such that $K_{g_i} \geq -1$. We do this using precisely the same techniques presented in §3.5.1. It now remains to establish the uniform volume bounds of point (3.).

Indeed, notice that the choice of points p and p_i above was not forced upon us. In fact, we could have chosen any vertex point of the triangulation $\mathcal{T}$ and the corresponding vertex points of $\widehat{\mathcal{T}}$ (this observation is expanded upon in Remark 4.3.8).

Thus, construct a sequence $\{x_i^j\}$ as follows: fix $i = 2$ and let $\{x_2^j\}_{j=1}^{\infty}$ be the vertex points of the triangulation used to construct $\mathcal{M}_2$. Then, for $i > 2$, let $\{x_i^j\}_{j=1}^{\infty}$ be the same sequence of points, viewed as points in $\mathcal{M}_i$ (notice that this sequence will no longer contain *all* the vertex points of $\mathcal{M}_i$, but this does not matter).

---

[1] The metric d_i is the unique maximal intrinsic metric on $\mathcal{M}_i$ such that, given any $\widehat{T} \in \widehat{\mathcal{T}}$ and any $x, y \in \widehat{T}$, we have that

$$d_i(x, y) \leq d_{\widehat{T}}(x, y),$$

where $d_{\widehat{T}}$ is the intrinsic metric induced on $\widehat{T}$ by the hyperbolic metric of $\mathbb{H}^2$, as in Definition 3.1.6. See [5, Corollary 3.1.24] for proof of the existence of such a metric.

Now let $x^j \in X$ be the vertex point in the triangulation of X corresponding to the points x_i^j. Then by precisely the same arguments as above, $(\mathcal{M}_i, g_i, x_i^j) \to (X, d, x^j)$ in the Gromov–Hausdorff sense (Definition 3.2.5), as $i \to \infty$, for each j.

Now notice that the points $\{x_i^j\}_j$ form a $\frac{1}{2}$-net of $\mathcal{M}_i$ for each $i \geq 2$, and so the balls $\left\{B_1^{g_i}(x_i^j)\right\}_j$ cover $\mathcal{M}_i$ for each $i \geq 2$. By the Gromov–Hausdorff convergence, we have that

$$\left(\overline{B_1^{g_i}(x_i^j)}, d_{g_i}\right) \to \left(\overline{B_d(x^j, 1)}, d\right)$$

in the sense of compact spaces (see Definition 3.2.2 and Remark 3.2.6) as $i \to \infty$, for each j. Hence, by Theorem 3.2.9, we have that

$$\mathrm{vol}_{g_i}(B_1^{g_i}(x_i^j)) \to \mathscr{H}^2(B_d(x^j, 1)) \geq 2\tau$$

as $i \to \infty$, and for each j, by the non-collapsed hypothesis on X. Hence, for sufficiently large i (independent of j), we have

$$\mathrm{vol}_{g_i}(B_1^{g_i}(x_i^j)) \geq \tau$$

for all j.

Then, given $x \in \mathcal{M}_i$, for i sufficiently large, $B_2^{g_i}(x)$ contains $B_1^{g_i}(x_i^j)$ for some j, and so

$$\mathrm{vol}_{g_i}(B_2^{g_i}(x)) \geq \tau,$$

as required. □

Remark 4.3.8. It is possible to overestimate the importance of the choice of points p and p_i in the above discussion. They are, in fact, somewhat of a red-herring. The point p corresponds to the point p_i in such a way that p_i is the *same point* in $\mathcal{M}_i$ for each $i \in \mathbb{N}$. So p serves as an 'anchor', preventing the sequence $\{p_i\}$ from diverging to infinity as $i \to \infty$, which would prevent the Gromov–Hausdorff convergence from taking place. Thus, whilst choosing a completely arbitrary sequence $\{p_i\}$ would not work in the above proof, carefully choosing a sequence of vertex points presents us with infinitely many options for the point p and the sequence $\{p_i\}$.

With this in hand, we are now ready to prove the following:

Theorem 4.3.9 (Ricci flow of non-compact Alexandrov surfaces)**.** *Let (X, d) be a non-compact Alexandrov surface with curvature bounded below by -1. Suppose there* *exists* $v_0 > 0$ *such that*

$$\inf_{x \in X} \mathscr{H}^2(B_1(x)) \geq v_0.$$

Then there exists $T = T(v_0) > 0$ *and a smooth Ricci flow* $(\mathcal{M}^2, g(t))_{t \in (0,T)}$ *such that:*

1. $g(t)$ *is complete for all* $t \in (0, T)$,
2. $\left|K_{g(t)}\right| \leq \frac{A}{t}$ *for some* $A = A(v_0) \geq 0$ *and all* $t \in (0, T)$,
3. $(\mathcal{M}, g(t))$ *takes* (X, d) *as its initial condition (in the sense of Definition 4.1.2).*

Proof. Let $(\mathcal{M}_i, g_i, p_i)$ be a sequence of smooth, pointed Riemannian manifolds approximating (X, d, p) in the Gromov–Hausdorff topology, where the sequence $(\mathcal{M}_i, g_i)$ and the points $p_i \in \mathcal{M}_i$, $p \in X$ are as given by Lemma 4.3.7. Appealing to Theorem 2.2.4, we can flow each element of the sequence to get a unique Ricci flow $(\mathcal{M}_i, g_i(t), p_i)_{t \in [0, S_i)}$, for some maximal time S_i, with $g_i(0) = g_i$ and with the property that $g_i(t)$ is instantaneously complete, i.e. complete for each $t > 0$.

We claim that $S_i = \infty$ for each i. Indeed, if this were not the case, we would have (for some i) that $(\mathcal{M}_i, g_i) \cong (\mathbb{C}, g)$ where g is some complete, finite-area metric. Now notice that we can find a sequence $\{z_j\} \subset \mathbb{C}$ such that $d_g(z_j, z_k) > 5$ for all $j \neq k$. If this were not possible, all points of $\mathbb{C}$ would lie within a ball $B_r^g(0)$ for some $0 < r < \infty$, contradicting our assumption of non-compactness. Now we have

$$\mathrm{vol}_g(\mathbb{C}) \geq \sum_{j=1}^{\infty} \mathrm{vol}_g(B_2^g(z_j)) = \infty$$

where we have used the non-collapsed property of the smooth approximations $(\mathcal{M}_i, g_i)$ (point (3.) of Lemma 4.3.7). But this contradicts that g is a finite-volume metric on $\mathbb{C}$, and thus $S_i = \infty$ for each i.

By construction, we have that

- $K_{g_i} \geq -1$ for each i and
- $\inf_{x \in \mathcal{M}_i} \mathrm{vol}_{g_i}(B_2^{g_i}(x)) \geq \dfrac{v_0}{2}$ for all i sufficiently large.

Consequently, employing Theorem 2.4.6, we can find $T > 0$ such that

$$\left|K_{g_i(t)}\right| \leq \frac{A}{t} \tag{4.1}$$

on $\mathcal{M}_i \times (0,T)$, where both A and T depend only on v_0 thanks to the uniform estimates above. In other words, on any compact time interval $[a,b] \subset (0,T)$, we have a *global* i-independent bound on the curvature of the metric $g_i(t)$, provided that i is sufficiently large.

Moreover, again by Theorem 2.4.6, there exists an $\tilde{\eta} > 0$, independent of i, such that

$$\inf_{x\in\mathcal{M}_i} \mathrm{vol}_{g_i(t_0)} \left(B_2^{g_i(t_0)}(x)\right) \geq \tilde{\eta}$$

for some $t_0 > 0$ sufficiently small, provided that i is sufficiently large. Together with the above global curvature bounds, this volume estimate guarantees us positive, uniform, global, lower injectivity radius bounds at time t_0 using Theorem 2.3.6. Thus, appealing to Theorem 2.3.3, we may find a smooth, complete, pointed Ricci flow $(\mathcal{M}, g(t), \bar{p})_{t\in(0,T)}$ such that, passing to a subsequence if necessary,

$$(\mathcal{M}_i, g_i(t), p_i) \to (\mathcal{M}, g(t), \bar{p})$$

in the Cheeger–Gromov sense (see Definition 2.3.1) as $i \to \infty$.

It remains to show that $(\mathcal{M}, g(t))$ takes (X,d) as its initial condition, the proof of which mimics the argument in [39, Theorem 9.2]. Indeed, define $d_t := d_{g(t)}$. Since the curvature bound (4.1) survives in the limit as $i \to \infty$, we may use Theorem 2.3.7 to obtain the estimate

$$e^{-C(t-s)} d_t \leq d_s \leq d_t + C(\sqrt{t} - \sqrt{s})$$

on compact subsets of $\mathcal{M} \times \mathcal{M}$, for $0 < s < t < T$, where C is a constant depending only on A. Thus the family $\{d_t\}_{t\in(0,T)}$ is uniformly Cauchy on every compact subset of $\mathcal{M} \times \mathcal{M}$ as $t \downarrow 0$, and consequently we have that $d_t \to \tilde{d}$ uniformly, locally on $\mathcal{M} \times \mathcal{M}$ as $t \downarrow 0$, where $\tilde{d}$ is some continuous function $\mathcal{M} \times \mathcal{M} \to \mathbb{R}$. It is easy to see that in fact $\tilde{d}$ is a distance on $\mathcal{M}$.

Finally, we show that $\left(\mathcal{M}, \tilde{d}\right)$ and (X,d) are isometric, for which it suffices that for each $r > 0$, we have

$$d_{GH}\left(\left(\overline{B_d(p,r)}, d\right), \left(\overline{B_{\tilde{d}}(\bar{p},r)}, \tilde{d}\right)\right) = 0.$$

Let $d_{i,t}$ denote the distance induced by the metric $g_i(t)$ above. Using Theorem 2.3.7

once more, we obtain the similar estimate

$$e^{-C(t-s)}d_{i,t} \leq d_{i,s} \leq d_{i,t} + C(\sqrt{t} - \sqrt{s})$$

for each i and $0 < s < t < T$, where C again depends only on A. Thus, taking $s \downarrow 0$, we may find a function μ such that

$$d_{GH}\left(\left(\overline{B_{d_{i,t}}(p_i, r)}, d_{i,t}\right), \left(\overline{B_{d_{i,0}}(p_i, r)}, d_{i,0}\right)\right) \leq \mu(t),$$

where $\mu(t) \downarrow 0$ as $t \downarrow 0$.

Now given $\varepsilon > 0$, find $t_\varepsilon > 0$ such that $\mu(t) \leq \varepsilon$ whenever $t \leq t_\varepsilon$. Recall that by Lemma 4.3.7, $(\mathcal{M}_i, g_i(0), p_i) \to (X, d, p)$ in the Gromov–Hausdorff sense as $i \to \infty$, and so we can find i_ε such that

$$i \geq i_\varepsilon \implies d_{GH}\left(\left(\overline{B_d(p, r)}, d\right), \left(\overline{B_{d_{i,0}}(p_i, r)}, d_{i,0}\right)\right) \leq \varepsilon.$$

Now apply the triangle inequality to find that

$$d_{GH}\left(\left(\overline{B_d(p, r)}, d\right), \left(\overline{B_{d_{i,t}}(p_i, r)}, d_{i,t}\right)\right) \leq 2\varepsilon \tag{4.2}$$

whenever $i \geq i_\varepsilon$ and $t \leq t_\varepsilon$.

From earlier in the proof, we have that $d_t \to \tilde{d}$ uniformly, locally as $t \downarrow 0$, and so we may find t'_ε such that if $t \leq t'_\varepsilon$ then

$$d_{GH}\left(\left(\overline{B_{\tilde{d}}(\bar{p}, r)}, \tilde{d}\right), \left(\overline{B_{d_t}(\bar{p}, r)}, d_t\right)\right) \leq \varepsilon.$$

Also, since $(\mathcal{M}_i, g_i(t'_\varepsilon), p_i) \to (\mathcal{M}, g(t'_\varepsilon), \bar{p})$ in the Cheeger–Gromov sense as $i \to \infty$, the convergence also takes place in the Gromov–Hausdorff sense, and so we can find an $i'_\varepsilon > 0$ such that

$$d_{GH}\left(\left(\overline{B_{d_{i,t}}(p_i, r)}, d_{i,t}\right), \left(\overline{B_{d_t}(\bar{p}, r)}, d_t\right)\right) \leq \varepsilon$$

whenever $t \leq t'_\varepsilon$ and $i \geq i'_\varepsilon$. Thus, applying the triangle inequality again, we get that

$$d_{GH}\left(\left(\overline{B_{d_{i,t}}(p_i, r)}, d_{i,t}\right), \left(\overline{B_{\tilde{d}}(\bar{p}, r)}, \tilde{d}\right)\right) \leq 2\varepsilon \tag{4.3}$$

whenever $t \leq t'_\varepsilon$ and $i \geq i'_\varepsilon$.

Combining (4.2) with (4.3), we have that

$$d_{GH}\left(\left(\overline{B_d(p,r)},d\right),\left(\overline{B_{\tilde{d}}(\bar{p},r)},\tilde{d}\right)\right)\leq 4\varepsilon$$

and hence $\left(\mathcal{M},\tilde{d}\right)\cong(X,d)$ as required, since $\varepsilon>0$ was arbitrary.

□

We also have uniqueness statements, which depend on the following recent result:

Theorem 4.3.10 ([46, Theorem 5.2])**.** *Suppose that $g(t)$ and $\hat{g}(t)$ are two conformally equivalent, complete Ricci flows on a surface $\mathcal{M}$, defined for $t\in(0,T]$, both with curvature uniformly bounded from below, and with*

$$\lim_{t\downarrow 0}\mathrm{vol}_{g(t)}(\Omega)=\lim_{t\downarrow 0}\mathrm{vol}_{\hat{g}(t)}(\Omega)<\infty$$

for all $\Omega\subset\subset\mathcal{M}$. Then $g(t)=\hat{g}(t)$ for all $t\in(0,T]$.

Theorem 4.3.11. *Let (X,d) be a non-compact Alexandrov surface with curvature bounded below by -1. Let $\mathcal{M}^2$ be a smooth surface, let $T>0$, and suppose that $g(t)$ and $\hat{g}(t)$ are two smooth, conformally equivalent, complete Ricci flows defined on $\mathcal{M}$ for $t\in(0,T)$, both of which take (X,d) as initial condition (see Definition 4.1.2). Suppose further that*

$$K_{g(t)},K_{\hat{g}(t)}\geq -1$$

for all $t\in(0,T)$. Then $g(t)=\hat{g}(t)$ for all $t\in(0,T)$.

Proof. Since uniform local convergence of the induced distances $d_{g(t)}$ and $d_{\hat{g}(t)}$ implies Gromov–Hausdorff convergence for a suitable choice of points, we may apply Theorem 3.2.9 to see that given any $\Omega\subset\subset\mathcal{M}$ we have

$$\lim_{t\downarrow 0}\mathrm{vol}_{g(t)}(\Omega)=\lim_{t\downarrow 0}\mathrm{vol}_{\hat{g}(t)}(\Omega).$$

This puts us precisely in the scenario of Theorem 4.3.10 above, and thus the result follows. □

Theorem 4.3.12. *Let (X,d) be a non-compact Alexandrov surface with curvature bounded below by -1. Let $(\mathcal{M}^2,g(t))$ and $\left(\widehat{\mathcal{M}}^2,\hat{g}(t)\right)$ be smooth, complete Ricci flows, defined for $t\in(0,T)$, such that*

$$K_{g(t)},K_{\hat{g}(t)}\geq -1$$

for all $t \in (0,T)$, and which are both uniformly non-collapsed, i.e. such that there exists $\eta > 0$ such that

$$\inf_{p\in\mathcal{M}} \mathrm{vol}_{g(t)}\left(B_1^{g(t)}(p)\right), \inf_{\hat{p}\in\widehat{\mathcal{M}}} \mathrm{vol}_{\hat{g}(t)}\left(B_1^{\hat{g}(t)}(\hat{p})\right) \geq \eta$$

for all $t \in (0,T)$. Suppose further that both $(\mathcal{M}, g(t))$ and $\left(\widehat{\mathcal{M}}, \hat{g}(t)\right)$ take the Alexandrov surface (X,d) as their initial condition (see Definition 4.1.2). Then both flows are conformally equivalent, and under this conformal equivalence, $g(t) = \hat{g}(t)$ for all $t \in (0,T)$.

Proof. By assumption, there exist distances ρ and $\hat{\rho}$ on $\mathcal{M}$ and $\widehat{\mathcal{M}}$ respectively, and an isometry $f\colon (\mathcal{M},\rho) \to \left(\widehat{\mathcal{M}}, \hat{\rho}\right)$. Let $p \in \mathcal{M}$ and $r > 0$ be given, and denote the restriction of f to the closed ball $\overline{B_\rho(p,r)}$ by

$$f_r\colon \left(\overline{B_\rho(p,r)}, \rho\right) \to \left(\overline{B_{\hat{\rho}}(f(p),r)}, \hat{\rho}\right).$$

Since the Ricci flow is a conformal flow in two dimensions, we may write $g(t) = e^{2u(t)}h$ and $\hat{g}(t) = e^{2\hat{u}(t)}\hat{h}$ where h and $\hat{h}$ are fixed metrics on $\mathcal{M}$ and $\widehat{\mathcal{M}}$ respectively. To finish the proof, we follow the same arguments as in [37, Lemmata 3.4.2, 3.4.3, 3.4.4], with the aim of showing that the distances ρ and $\hat{\rho}$ are induced by appropriate Reshetnyak metrics, in order to apply Theorem 3.5.11. Indeed, observe that by applying Theorem 2.4.6 again, we may find $A = A(\eta) > 0$ such that

$$-1 \leq K_{g(t)}, K_{\hat{g}(t)} \leq \frac{A}{t} \tag{4.4}$$

for all $t \in (0,T)$. We now claim that there exists a function $u_0 \in L^1_{\mathrm{loc}}(\mathcal{M})$ such that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathcal{M})$ as $t \downarrow 0$. Since

$$\frac{\partial u}{\partial t} = -K_{g(t)} \leq 1,$$

we have, for every $x \in \mathcal{M}$, that $u(x,t) - t$ increases as $t \downarrow 0$. Consequently, there exists a pointwise limit

$$u_0 := \lim_{t\to 0} u(t).$$

Let $\mathcal{K} \subset \mathcal{M}$ be compact. To show that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathcal{M})$, it suffices by the Monotone Convergence Theorem to show that $u(t) - t$ is uniformly bounded below on $\mathcal{K}$, and is bounded in $L^1(\mathcal{K})$.

Fix $t_0 > 0$. Then $u(t) - t \geq u(t_0) - t_0$ for $t \in (0, t_0)$. Since $u(t_0)$ is smooth on $\mathcal{M}$

and $\mathcal{K}$ is compact, $u(t)-t$ is hence uniformly bounded below on $\mathcal{K}$ for all $t \in (0, t_0)$. Then notice that by Jensen's Inequality,

$$\exp\left(\int_{\mathcal{K}} \frac{2u(x,t)}{\mathrm{vol}_h(\mathcal{K})}\,\mathrm{d}\mu_h(x)\right) \le \int_{\mathcal{K}} \frac{e^{2u(x,t)}}{\mathrm{vol}_h(\mathcal{K})}\,\mathrm{d}\mu_h(x) = \frac{\mathrm{vol}_{g(t)}(\mathcal{K})}{\mathrm{vol}_h(\mathcal{K})}.$$

Since $d_{g(t)}$ converges uniformly, locally as $t \downarrow 0$, the right-hand-side above is bounded as $t \downarrow 0$. Thus $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathcal{M})$.

Next, we claim that $u_0 \in V(\mathcal{M}, h)$, where the space $V(\mathcal{M}, h)$ is as defined in Definition 3.5.9. We have already shown that $u_0 \in L^1_{\mathrm{loc}}(\mathcal{M})$, and thus it remains to show that $\Delta_h u_0$ defines a signed measure on $\mathcal{M}$. Indeed, notice that

$$\Delta_h u(t) = K_h - K_{g(t)} e^{2u(t)}.$$

Let $\tau \in C_c^\infty(\mathcal{M})$. Then

$$\begin{aligned}\int_{\mathcal{M}} \tau(x)\Delta_h u(x,t)\,\mathrm{d}\mu_h(x) &= \int_{\mathcal{M}} \tau(x)\left[K_h - K_{g(t)}e^{2u(x,t)}\right]\mathrm{d}\mu_h(x) \\ &= \int_{\mathcal{M}} \tau(x) K_h\,\mathrm{d}\mu_h(x) - \int_{\mathcal{M}} \tau(x)\,\mathrm{d}\omega_{g(t)},\end{aligned}$$

where $\omega_{g(t)}$ is the curvature measure of $g(t)$. Since $d_{g(t)} \to \rho$ uniformly, locally on $\mathcal{M} \times \mathcal{M}$ as $t \downarrow 0$, it follows by [1, Ch. VII, §4, Theorem 6] that $\omega_{g(t)} \to \omega_\rho$ weakly, locally as measures as $t \downarrow 0$. Thus, integrating the left-hand-side by parts and taking the limit as $t \downarrow 0$, we find that

$$\int_{\mathcal{M}} u_0(x)\Delta_h \tau(x)\,\mathrm{d}\mu_h(x) = \int_{\mathcal{M}} \tau(x) K_h\,\mathrm{d}\mu_h(x) - \int_{\mathcal{M}} \tau(x)\,\mathrm{d}\omega_\rho(x).$$

Thus, $\Delta_h u_0$ is the signed measure $K_h\,\mathrm{d}\mu_h - \omega_\rho$, as we wanted to show.

Now let $d_0 := d_{h,u_0}$, with notation as in Definition 3.5.9. We claim that $d_0 = \rho$. Recall once again that the curvature measure of $g(t)$ is

$$\omega_{g(t)} = K_{g(t)} e^{2u(t)}\,\mathrm{d}\mu_h.$$

Write $\mu_t := \omega_{g(t)} - K_h\,\mathrm{d}\mu_h$. Then, using [1, Ch. VII, §4, Theorem 6] again, we find that

$$\mu_t \to \mu := \omega_\rho - K_h\,\mathrm{d}\mu_h$$

weakly, locally as $t \downarrow 0$. Let $\omega^+_{g(t)} := K^+_{g(t)} e^{2u(t)}\,\mathrm{d}\mu_h$ and $\omega^-_{g(t)} := K^-_{g(t)} e^{2u(t)}\,\mathrm{d}\mu_h$, which are both positive measures. By the curvature bounds (4.4) and the Gauß–Bonnet formula, the masses of $\omega^+_{g(t)}$ and $\omega^-_{g(t)}$ are uniformly bounded above in t on any compact subset $\mathcal{K} \subset \mathcal{M}$.

Let $\mu^+_t := \omega^+_{g(t)} + K^-_h \, \mathrm{d}\mu_h$ and $\mu^-_t := \omega^-_{g(t)} + K^+_h \, \mathrm{d}\mu_h$. Then, by the above observations, we have that $\mu^+_{t_i}$ and $\mu^-_{t_i}$ converge weakly, locally to measures μ^+ and μ^- respectively, for some sequence $\{t_i\} \to 0$, where $\mu = \mu^+ - \mu^-$.

Since $-\Delta_h u(t_i) = \mu_{t_i}$ and $-\Delta_h u_0 = \mu$, it follows from Theorem 3.5.12 that $d_{g(t_i)} \to d_0$ uniformly, locally on $\mathcal{M} \times \mathcal{M}$ as $i \to \infty$. Since $g_{g(t_i)} \to \rho$ uniformly, locally on $\mathcal{M} \times \mathcal{M}$ as $i \to \infty$, we have that $d_0 = \rho$, as claimed.

Performing the same arguments with $\hat{u}$ and $\hat{h}$, we can then appeal to Theorem 3.5.11 to conclude that the maps

$$f_r \colon \left(\overline{B_\rho(p,r)}, h\right) \to \left(\overline{B_{\hat{\rho}}(f(p),r)}, \hat{h}\right)$$

are conformal for all $r > 0$. Consequently, $(\mathcal{M}, h)$ and $\left(\widehat{\mathcal{M}}, \hat{h}\right)$ are conformally equivalent.

Without loss of generality, we may now assume that $g(t)$ and $\hat{g}(t)$ are both Ricci flows on the same smooth surface $\mathcal{M}$, and the result then follows from Theorem 4.3.11 above.

□

Chapter 5

A problem in well-posedness theory for the Ricci flow

A stimulating area of research in recent years has involved considerations of well-posedness for the Ricci flow initial value problem. In other words, answers to the question 'in what class is it most natural to look for solutions to the Ricci flow equation'? Other aspects of the well-posedness problem are determining in which class initial data ought to lie, and moreover, what the notion of 'attaining initial data' actually means, which is the problem we focus on in this chapter. In Chapter 4, we proved that the problem of finding Ricci flows starting at non-compact, uniformly non-collapsed Alexandrov surfaces is well-posed, where the notion of 'initial data' is provided by Definition 4.1.2. Theorem 2.2.1 tells us that the Ricci flow initial value problem starting at smooth Riemannian manifolds of bounded curvature is well-posed in arbitrary dimension in the class of complete solutions of bounded curvature. Furthermore, in dimension two, the work of Giesen and Topping, culminating in Theorem 2.2.4, shows that a much broader class of solutions is natural, namely that of *instantaneously complete* flows (see Definition 2.2.3). We may hence flow any smooth metric on any smooth surface, regardless of completeness or curvature bounds, and find a unique, instantaneously complete solution. Whilst this result is overwhelmingly positive, there are still questions to answer in this arena.

5.1 Geometric notions of initial data

Traditionally, we say that a Ricci flow $(g(t))_{t\in[0,T)}$ on a smooth manifold $\mathcal{M}$ takes the metric g_0 as initial data if $g(0) = g_0$, i.e. if $g(t) \to g_0$ smoothly, locally as tensors as $t \to 0$. Of course, under this scheme, we often have that geometric quantities such as volume and diameter converge as well. However, it seems natural to ask that a Ricci flow $(g(t))$ and its initial data are metrically related for small t. For instance, we could impose that $(\mathcal{M}, g(t)) \to (\mathcal{M}, g_0)$ in the Gromov–Hausdorff sense as $t \downarrow 0$ (perhaps for a choice of points if $\mathcal{M}$ is non-compact). This is certainly not always the case: consider, for example, instantaneously complete flows starting from an incomplete metric, which can never satisfy this condition. This observation prompts us to consider the case where the flow $(g(t))$ exists on a different underlying manifold than the initial data.

Remark 5.1.1. We point out that convergence of tensors is well-known to be at odds with geometric intuition. Consider, for instance, the Cheeger–Gromov convergence of a sequence of smooth Riemannian manifolds to another smooth Riemannian manifold. In the definition of this convergence, we adjust by diffeomorphisms to ensure that the notion is *diffeomorphism invariant*, which is a natural expectation in Riemannian geometry. Without doing this, i.e. by simply claiming that the sequence $(\mathcal{M}, g_i) \to (\mathcal{M}, g_0)$ as $i \to \infty$ if $g_i \to g_0$ smoothly, locally as tensors as $i \to \infty$, we sacrifice the geometric interpretation of convergence: altering the metrics g_i by i-dependent diffeomorphisms will produce a different limiting metric in general.

In [44], Topping proposed an alternative notion for attaining initial data, namely that a complete Ricci flow $(\mathcal{M}, g(t))_{t\in(0,T]}$ takes the complete Riemannian manifold $(\mathcal{N}, g_0)$ as initial data if there exists a smooth map $\varphi\colon \mathcal{N} \to \mathcal{M}$, which is a diffeomorphism onto its image, such that $\varphi^*(g(t)) \to g_0$ smoothly, locally on $\mathcal{N}$ as $t \downarrow 0$. Within this framework, examples such as the following were considered: take the torus T^2, and let $p \in T^2$. Defining $T^2_p := T^2 \setminus \{p\}$, let h be the conformal, complete hyperbolic metric on T^2_p. It is possible to flow this surface keeping the hyperbolic cusp in place, namely by taking $h(t) := (1 + 2t)h$. However, the central idea of [44] is that we can find a unique flow $\left(T^2, \hat{h}(t)\right)$ that takes (T^2_p, h) as initial data as described above. This is often referred to as the 'contracting cusp' flow, as we can imagine the flow developing a hyperbolic cusp at p as $t \downarrow 0$.

In light of these considerations, we propose another alternative notion of *initial data* for the Ricci flow starting from a particular class of Riemannian surfaces, based on Gromov–Hausdorff convergence of the flow to a specified surface. Given an appropriate initial surface (Ω, g_0), we construct a *complete* Ricci flow $(\mathcal{M}, g(t))$ on some Riemann surface $\mathcal{M}$, existing for all $t \in (0, \infty)$, which can be viewed as taking (Ω, g_0) as initial data.

Before proceeding, we clarify some notions of convergence that we use in what follows:

Remark 5.1.2. Let $(\mathcal{M}, h)$ be a smooth Riemannian surface, and let $(g(t))_{t\in(0,T)}$ be a smooth Ricci flow on $\mathcal{M}$ that is conformally equivalent to h for all $t \in (0, T)$. Write $g(t) = v(t)h$, and let $g_0 = v_0 h$ be another metric on $\mathcal{M}$. Let $\Sigma \subseteq \mathcal{M}$ be open. We point out that the statement

$$g(t) \to g_0 \text{ in } L^1_{\text{loc}}(\Sigma) \text{ as } t \to 0$$

is equivalent to the statement

$$\lim_{t\downarrow 0} \int_{\mathcal{K}} |v(t) - v_0| \,\mathrm{d}\mu_h = 0$$

for any compact subset $\mathcal{K} \subset \Sigma$. Likewise, that

$$g(t) \to g_0 \text{ in } C^0_{\text{loc}}(\Sigma) \text{ as } t \to 0$$

is equivalent to having that

$$\sup_{p\in\mathcal{K}} |v(t)(p) - v_0(p)| \to 0 \text{ as } t \to 0$$

for any compact subset $\mathcal{K} \subset \Sigma$.

Definition 5.1.3. Let $\mathcal{M}$ be a smooth Riemann surface, and let $\Omega \subset\subset \mathcal{M}$ be an open subset such that $\Omega \neq \mathcal{M}$. Let g be a smooth Riemannian metric on Ω that is compatible with the conformal structure of $\mathcal{M}$. We say that

$$g(p) \to 0 \text{ as } p \to \partial\Omega \text{ within } \mathcal{M}$$

if for a (equivalently, any) Riemannian metric h on $\mathcal{M}$, we have that $|g(p)|_h \to 0$ as $p \to \partial\Omega$. Denote by $\mathcal{O}(\Omega, \mathcal{M})$ the collection of all such metrics.

Geometrically, metrics belonging to $\mathcal{O}(\Omega, \mathcal{M})$ are those for which points near the boundary of Ω are very close together. In the subsequent section, we construct complete Ricci flows that take such metrics as initial data in a certain, natural, geometric sense, as outlined below. In this way, we construct complete Ricci flows that differ from those provided by theorems such as Theorem 2.2.4. In other words, we can demonstrate two complete Ricci flows that take the same Riemannian surface as initial condition, both in a natural way. Consequently, we have a marked *non-*

uniqueness for the Ricci flow starting at a surface belonging to a certain class, challenging perceptions of what the 'correct' notion of initial data should be.

Remark 5.1.4. For simplicity of exposition, we assume in what follows that Ω has only one boundary component. A minor adjustment of the following definitions and arguments shows that our results are also true when Ω has more than one boundary component.

We point out that metrics belonging to $\mathcal{O}(\Omega, \mathcal{M})$ naturally extend to *pseudometrics* on $\overline{\Omega}$ (i.e. metrics for which distinct points can be distance zero apart). Indeed, given $g_0 \in \mathcal{O}(\Omega, \mathcal{M})$, let h be any smooth Riemannian metric on $\mathcal{M}$ and write $g_0 = v_0 h\big|_{\Omega}$, for some $v_0 \in C^\infty(\Omega)$. Define

$$v(p) := \begin{cases} v_0(p) & \text{if } p \in \Omega \\ 0 & \text{if } p \in \partial\Omega. \end{cases}$$

Then, by virtue of the fact that $g(p) \to 0$ as $p \to \partial\Omega$ within $\mathcal{M}$, we have that $v \in C^0\left(\overline{\Omega}\right)$. It is hence readily seen that

$$\bar{d}_{g_0}(p,q) := \inf_{\gamma \subset \overline{\Omega}} \int_0^1 \sqrt{v(\gamma(t))} |\gamma'(t)|_h \, \mathrm{d}t,$$

where the infimum is taken, as usual, over all smooth curves γ with $\gamma(0) = p$ and $\gamma(1) = q$, defines a pseudometric on $\overline{\Omega}$. Moreover, $\bar{d}_{g_0}$ coincides with d_{g_0} when restricted to Ω. This construction is also clearly independent of the choice of h.

By construction, we have that $\bar{d}_{g_0}(p,q) = 0$ if and only if either $p = q$ or $p, q \in \partial\Omega$. Consequently, defining an equivalence relation $\sim$ on $\overline{\Omega}$ by

$$p \sim q \iff p, q \in \partial\Omega,$$

we arrive at a compact metric space $\Omega^\star := \overline{\Omega}/\sim$, with the quotient metric $d^\star_{g_0}$ of $\bar{d}_{g_0}$.

Definition 5.1.5. Let $\mathcal{M}$ be a closed Riemann surface, let $\Omega \subset\subset \mathcal{M}$ be an open subset such that $\Omega \neq \mathcal{M}$, and let $g_0 \in \mathcal{O}(\Omega, \mathcal{M})$. Let $\{g_i\}$ be a sequence of smooth Riemannian metrics on $\mathcal{M}$. We say that $(\mathcal{M}, d_{g_i}) \to (\Omega, d_{g_0})$ *in the Gromov–Hausdorff sense* if $(\mathcal{M}, d_{g_i}) \to (\Omega^\star, d^\star_{g_0})$ in the sense of Definition 3.2.2.

5.2 Ricci flows taking initial data in a geometric sense

In this section, we prove the existence of Ricci flows taking particular Riemannian surfaces as initial data in a geometric sense. First, we need the following definition:

Definition 5.2.1. By an *initial data triple*, we mean a triple $(\Omega, \mathcal{M}, g)$ where:

- $\mathcal{M}$ is a closed Riemann surface *of genus at least two*;
- Ω is a Riemann surface such that there exists a conformal embedding $i\colon \Omega \hookrightarrow \mathcal{M}$ with $i(\Omega) \subset\subset \mathcal{M}$, $i(\Omega) \neq \mathcal{M}$, and such that $\partial\left(i(\Omega)\right)$ is smooth;
- $g \in \mathcal{O}(\Omega, \mathcal{M})$.

Remark 5.2.2. Given an initial data triple $(\Omega, \mathcal{M}, g)$ as above, and a conformal embedding $i\colon \Omega \hookrightarrow \mathcal{M}$, we will abuse notation and write $\Omega \equiv i(\Omega)$, viewing Ω as a subset of $\mathcal{M}$.

We can now state the main theorem of this chapter:

Theorem 5.2.3. *Let $(\Omega, \mathcal{M}, \hat{g})$ be an initial data triple. Let g_0 be the degenerate metric on $\mathcal{M}$ defined by*

$$g_0(p) := \begin{cases} \hat{g}(p) & \text{if } p \in \Omega \\ 0 & \text{if } p \in \mathcal{M} \setminus \Omega. \end{cases}$$

Then there exists a smooth Ricci flow $(g(t))_{t \in (0,\infty)}$ on $\mathcal{M}$ such that:

- *$g(t) \to g_0$ in $C^0(\mathcal{M})$ as $t \downarrow 0$;*
- *$\left(\mathcal{M}, d_{g(t)}\right) \to (\Omega, d_{\hat{g}})$ in the Gromov–Hausdorff sense as $t \downarrow 0$ (as in Definition 5.1.5).*

Remark 5.2.4. We point out that Ω need not be an especially esoteric surface. Indeed, even taking Ω as a disc that embeds conformally into a suitable surface $\mathcal{M}$, and taking a metric $g \in \mathcal{O}(\Omega, \mathcal{M})$ demonstrates the interesting nature of this problem: Theorem 2.2.4 will give an instantaneously complete Ricci flow on Ω starting at g. Theorem 5.2.3 will then give an alternative complete flow which can likewise be claimed to take (Ω, g) as initial data in a natural way.

Before proving Theorem 5.2.3, we prove some technical results we need in what follows. First, some notation:

Definition 5.2.5. Let $(\Omega, \mathcal{M}, \hat{g})$ be an initial data triple. Given $p, q \in \Omega$, define

- $\Gamma_{\mathcal{M}}(p,q)$ to be the collection of all smooth curves $\gamma\colon [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$;
- $\Gamma_\Omega(p,q)$ to be the collection of all smooth curves $\gamma\colon [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

When there is no confusion, we write $\Gamma_{\mathcal{M}} \equiv \Gamma_{\mathcal{M}}(p,q)$ and $\Gamma_\Omega \equiv \Gamma_\Omega(p,q)$.

Lemma 5.2.6. *Let $(\Omega, \mathcal{M}, \hat{g})$ be an initial data triple, and let g_0 be defined as in Theorem 5.2.3. Then $d_{g_0}\big|_\Omega = d_{\hat{g}}$.*

Proof. Let $p, q \in \Omega$ and let h be a smooth Riemannian metric on $\mathcal{M}$. Since $g_0\big|_\Omega = \hat{g}$, it suffices to show that

$$\inf_{\gamma\in\Gamma_{\mathcal{M}}} \mathscr{L}_{g_0}(\gamma) = \inf_{\gamma\in\Gamma_\Omega} \mathscr{L}_{g_0}(\gamma).$$

It is clear that

$$\inf_{\gamma\in\Gamma_{\mathcal{M}}} \mathscr{L}_{g_0}(\gamma) \leq \inf_{\gamma\in\Gamma_\Omega} \mathscr{L}_{g_0}(\gamma),$$

and so we show the reverse inequality. To this end, fix $\varepsilon > 0$ and let $\bar{\gamma} \in \Gamma_{\mathcal{M}}$ be such that

$$\left| \mathscr{L}_{g_0}(\bar{\gamma}) - \inf_{\gamma\in\Gamma_{\mathcal{M}}} \mathscr{L}_{g_0}(\gamma) \right| < \varepsilon.$$

Since $g_0 = 0$ on $\partial\Omega$, we may assume, without loss of generality, that $\bar{\gamma} \subset \overline{\Omega}$. Since $\hat{g}(p) \to 0$ as $p \to \partial\Omega$ within $\mathcal{M}$, we may find $\delta > 0$ such that $|g_0(p)|_h < \varepsilon$ whenever $d_h(p, \partial\Omega) < \delta$. Let N_δ be the tubular neighbourhood of $\partial\Omega$ of diameter δ measured with respect to h.

We may then find a curve $\hat{\gamma} \in \Gamma_\Omega$ such that $\hat{\gamma} = \bar{\gamma}$ on $\Omega \setminus N_\delta$, and such that

$$\mathscr{L}_{g_0}(\hat{\gamma}) \leq \mathscr{L}_{g_0}(\bar{\gamma}) + C(\varepsilon),$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Such a curve can be found since $|g_0(p)|_h < \varepsilon$ on N_δ and $\partial\Omega$ is smooth, and is hence of finite length.

Then

$$\begin{aligned}\inf_{\gamma\in\Gamma_\Omega} \mathscr{L}_{g_0}(\gamma) \leq \mathscr{L}_{g_0}(\hat{\gamma}) &\leq \mathscr{L}_{g_0}(\bar{\gamma}) + C(\varepsilon) \\ &\leq \inf_{\gamma\in\Gamma_{\mathcal{M}}} \mathscr{L}_{g_0}(\gamma) + \varepsilon + C(\varepsilon).\end{aligned}$$

Taking $\varepsilon \to 0$ proves the statement. □

We now make an elementary observation:

Lemma 5.2.7. *Let $\mathcal{M}$ be a closed Riemann surface. Let $\{g_i(t)\}_{t\geq 0}$ be a sequence of Ricci flows that are compatible with the conformal structure of $\mathcal{M}$, and define $g_i := g_i(0)$. Let $\Omega \subset\subset \mathcal{M}$ be an open subset such that $\Omega \neq \mathcal{M}$. Let g be a smooth, conformal Riemannian metric on Ω. Suppose that:*

$$\left\{ g_i\big|_{\Omega} \right\} \to g \text{ in } C^\infty_{\text{loc}}(\Omega) \text{ as } i \to \infty.$$

Then, given $x_0 \in \Omega$, we may find $v_0, r_0 > 0$ such that:

- $B^{g_i}_{r_0}(x_0) \subset\subset \Omega$;
- $|K_{g_i}| \leq \frac{2}{r_0^2}$ *on* $B^{g_i}_{r_0}(x_0)$;
- $\text{vol}_{g_i}\left(B^{g_i}_{r_0}(x_0)\right) \geq v_0 r_0^2$

for all i sufficiently large.

Proof. Let $\tilde{r_0}$ be such that $B^g_{\tilde{r_0}}(x_0) \subset\subset \Omega$. Appealing to the smoothness of g on $\mathcal{M}$, we may reduce $\tilde{r_0}$ if necessary to find $0 < r_0 \leq \tilde{r_0}$ and $v_0 > 0$ such that:

- $|K_g| \leq \frac{1}{r_0^2}$ on $B^g_{r_0}(x_0)$;
- $\text{vol}_g\left(B^g_{r_0}(x_0)\right) \geq 2v_0 r_0^2$.

Since $\left\{ g_i\big|_{\Omega} \right\} \to g$ in $C^\infty_{\text{loc}}(\Omega)$, we hence have that for i sufficiently large, $B^{g_i}_{r_0} \subset\subset \Omega$, and that

- $|K_{g_i}| \leq \frac{2}{r_0^2}$ on $B^{g_i}_{r_0}(x_0)$;
- $\text{vol}_{g_i}\left(B^{g_i}_{r_0}(x_0)\right) \geq v_0 r_0^2$,

as required. □

We will rely on the following pseudolocality result in order to show that the initial data is attained by the Ricci flow we construct:

Lemma 5.2.8. *Let $\mathcal{M}$ be a closed Riemann surface. Let $\{g_i(t)\}_{t\geq 0}$ be a sequence of Ricci flows that are compatible with the conformal structure of $\mathcal{M}$, and define $g_i := g_i(0)$. Let $\Omega \subset\subset \mathcal{M}$ be an open subset such that $\Omega \neq \mathcal{M}$. Let g be a smooth, conformal Riemannian metric on Ω. Suppose that:*

$$\left\{ g_i\big|_{\Omega} \right\} \to g \text{ in } C^\infty_{\text{loc}}(\Omega) \text{ as } i \to \infty.$$

Given $x_0 \in \Omega$, let $v_0, r_0 > 0$ be given by Lemma 5.2.7 so that

$$|K_{g_i}| \leq \frac{2}{r_0^2} \qquad \text{on } B^{g_i}_{r_0}(x_0) \subset\subset \Omega$$

and

$$\mathrm{vol}_{g_i}\left(B^{g_i}_{r_0}(x_0)\right) \geq v_0 r_0^2$$

for each $i \in \mathbb{N}$ sufficiently large. Then there exists $T = T(v_0, r_0)$ and $C = \frac{4}{r_0^2}$ such that

$$|K_{g_i(t)}| \leq C$$

on $B^{g_i(t)}_{\frac{r_0}{2}}(x_0)$ for $0 < t \leq T$ and i sufficiently large.

Proof. The result follows immediately from an application of Theorem 2.4.4. □

Proof of Theorem 5.2.3. By assumption, $\mathcal{M}$ is a Riemann surface of genus at least two, and hence supports a conformal hyperbolic metric, say h. Let $g_i = v_i h$ be a sequence of *smooth* metrics on $\mathcal{M}$, decreasing in i, with $v_i > 0$, such that:

- $\{g_i\} \to g_0$ in $C^0(\mathcal{M})$ as $i \to \infty$ and
- $\left\{g_i\big|_\Omega\right\} \to \hat{g}$ in $C^\infty_{\mathrm{loc}}(\Omega)$ as $i \to \infty$.

Such convergence is possible due to the fact that $\hat{g} \in \mathcal{O}(\Omega, \mathcal{M})$, and so g_0 is continuous. We may hence mollify any continuous approximation to g_0 on the interior of Ω to arrive at such a sequence $\{g_i\}$.

Now let $g_i(t) = v_i(t)h$ be the unique, smooth Ricci flow on $\mathcal{M}$ with $g_i(0) = g_i$ given by Theorem 2.2.4. Note that since the metrics g_i are conformally hyperbolic for each i, these flows exist for all positive time.

We may find $M > 0$, independent of i, such that $g_i \leq Mh$ on $\mathcal{M}$ for all i. Hence, appealing to Theorem 2.2.5, we find that

$$2th \leq g_i(t) \leq (2t + M)h$$

on $\mathcal{M}$, for all $t > 0$ and each $i \in \mathbb{N}$. Applying standard parabolic regularity theory (see, for example, Theorem B.1.2), we deduce that for each $t > 0$ and $k \in \mathbb{N}$, the quantity $||g_i(t)||_{C^k(\mathcal{M})}$ is controlled uniformly in i on compact time intervals in $(0, \infty)$. Consequently, by Corollary B.2.3, there exists a Ricci flow $(g(t))_{t>0}$ on $\mathcal{M}$ such that, passing to a subsequence if necessary, $g_i(t) \to g(t)$ smoothly, locally on

$\mathcal{M} \times (0, \infty)$ as $i \to \infty$.

It remains to show that $(g(t))$ takes g_0 as initial condition in the required sense. Recall that, by assumption, the sequence $\{g_i\}$ is decreasing in i. Thus, since for $i \leq j$ we have $g_i \geq g_j$, the maximal-stretchedness property of the flows given by Theorem 2.2.4 tells us that $g_i(t) \geq g_j(t)$ for any $t \geq 0$. So $\{g_i(t)\}$ is decreasing in i for any $t \geq 0$, and hence $g(t) \leq g_i(t)$ for any $t > 0$ and any $i \in \mathbb{N}$.

Firstly, we claim that $g(t)\big|_{\mathcal{M}\setminus\overline{\Omega}} \to 0$ in $C^0_{\text{loc}}\left(\mathcal{M} \setminus \overline{\Omega}\right)$ as $t \downarrow 0$. Indeed, given $z \in \mathcal{M}\setminus\overline{\Omega}$ and $\varepsilon > 0$, let $r > 0$ be such that $B_r(z) := B^h_r(z)$ is compactly contained in $\mathcal{M} \setminus \overline{\Omega}$. Since $\{g_i\} \to 0$ in $C^0\left(\mathcal{M} \setminus \overline{\Omega}\right)$ as $i \to \infty$, choose i' such that

$$g_{i'} \leq \varepsilon h_z$$

on $B_r(z)$, where h_z denotes the conformal, complete hyperbolic metric on $B_r(z)$. Thus, appealing once again to Theorem 2.2.5, we get that

$$g_{i'}(t) \leq (2t + \varepsilon)h_z$$

on $B_r(z)$ for all $t \geq 0$. Consequently,

$$g(t) \leq (2t + \varepsilon)h_z$$

on $B_r(z)$ for all $t > 0$. Since $\varepsilon > 0$ is arbitrary, we hence have that

$$g(t) \leq 2th_z$$

on $B_r(z)$ for all $t > 0$. Taking t to zero shows that $g(t) \to 0$ in $C^0_{\text{loc}}\left(\mathcal{M} \setminus \overline{\Omega}\right)$ as $t \downarrow 0$.

Secondly, we claim that $g(t)\big|_{\Omega} \to \hat{g}$ in $C^0_{\text{loc}}(\Omega)$ as $t \downarrow 0$. Indeed, let $x_0 \in \Omega$ and let $v_0, r_0 > 0$ be given by Lemma 5.2.7, so that the conditions of Lemma 5.2.8 are satisfied for the flows $(g_i(t))$. Then, applying that lemma, we find constants $C, T > 0$ so that

$$|K_{g_i(t)}| \leq C \tag{5.1}$$

on $B^{g_i(t)}_{\frac{r_0}{2}}(x_0)$ for all $t \in (0, T]$ and all i sufficiently large. Taking the limit as $i \to \infty$, we hence also have that

$$|K_{g(t)}| \leq C \tag{5.2}$$

on $B^{g(t)}_{\frac{r_0}{2}}(x_0)$ for all $t \in (0, T]$

Write $\hat{g} = \hat{v}h\big|_\Omega$ for some $\hat{v} \in C^\infty(\Omega)$. For each $t > 0$, define a positive function $v(t)$ on $\mathcal{M}$ by $g(t) = v(t)h$. Define $u(t) := \frac{1}{2}\log v(t)$, $u_i(t) := \frac{1}{2}\log v_i(t)$, $\hat{u} := \frac{1}{2}\log \hat{v}$, $u_i := \frac{1}{2}\log v_i$, and fix $\varepsilon > 0$, assuming without loss of generality that $\varepsilon < T$. Observe that $g_i(\varepsilon) \to g(\varepsilon)$ and $g_i \to \hat{g}$ uniformly on Ω as $i \to \infty$. Consequently, we may find $j = j(\varepsilon)$ such that the estimate (5.1) holds for $g_j(t)$, and such that

$$|u(\varepsilon)(x_0) - u_j(\varepsilon)(x_0)| < \varepsilon$$

and

$$|u_j(x_0) - \hat{u}(x_0)| < \varepsilon.$$

Now we claim that in fact

$$|u(t)(x_0) - u_j(t)(x_0)| < (1 + 2C)\varepsilon$$

for all $0 < t \leq \varepsilon$. Indeed, using that

$$\frac{\partial u_j}{\partial t} = -K_{g_j(t)} \qquad \text{and} \qquad \frac{\partial u}{\partial t} = -K_{g(t)},$$

and applying the estimates (5.1) and (5.2), we have that

$$\begin{aligned} |u(t)(x_0) - u_j(t)(x_0)| &\leq |u(\varepsilon)(x_0) - u_j(\varepsilon)(x_0)| + 2C(\varepsilon - t) \\ &< (1 + 2C)\varepsilon, \end{aligned}$$

as claimed.

Now we use (for j fixed as above) that $g_j(0) = g_j$ on $\mathcal{M}$ to find $\hat{T} > 0$ such that

$$|u_j(t)(x_0) - u_j(x_0)| < \varepsilon$$

whenever $0 \leq t \leq \hat{T}$. Then, for $t \leq T' := \min\{\varepsilon, \hat{T}\}$, it follows from the triangle inequality that

$$\begin{aligned} |u(t)(x_0) - \hat{u}(x_0)| &\leq |u(t)(x_0) - u_j(t)(x_0)| + |u_j(t)(x_0) - u_j(x_0)| + |u_j(x_0) - \hat{u}(x_0)| \\ &< 2\varepsilon + (1 + 2C)\varepsilon = (3 + 2C)\varepsilon. \end{aligned}$$

Applying Corollary B.2.5, we find that $g(t) \to \hat{g}$ in $C^0_{\text{loc}}(\Omega)$ as $t \downarrow 0$.

By Corollary B.2.5, to show that $g(t) \to g_0$ in $C^0(\mathcal{M})$ as $t \downarrow 0$, it suffices to show that the convergence takes place *pointwise*. Thus, following from what has already been shown, we need only prove that $g(t)(z) \to 0$ as $t \to 0$ for any $z \in \partial\Omega$. Indeed, let $z \in \partial\Omega$ and fix $\varepsilon > 0$. Then, using that $\hat{g} \in \mathcal{O}(\Omega, \mathcal{M})$, we may find $\delta > 0$ such that $|\hat{g}(q)|_h < \frac{\varepsilon}{2}$ whenever $q \in \Omega$ and $d_h(z, q) < \delta$. Let $\mathcal{B} := B^h_\delta(z)$. Then, using what we have shown so far, we may find $S > 0$ such that $|g(t)(p)|_h < \varepsilon$ whenever $t \leq S$ and for p such that $p \in \mathcal{B}$ and $p \notin \partial\Omega$. Since $g(t)$ is continuous on $\mathcal{M}$, it follows that $g(t)(z) \to 0$ as $t \downarrow 0$, and we hence conclude that $g(t) \to g_0$ in $C^0(\mathcal{M})$ as $t \to 0$.

It remains to show that $\big(\mathcal{M}, d_{g(t)}\big) \to (\Omega, d_{\hat{g}})$ in the Gromov–Hausdorff sense as $t \downarrow 0$. First, we observe that $d_{g(t)}\big|_\Omega \to d_{\hat{g}}$ uniformly on Ω as $t \downarrow 0$. Indeed, by Lemma 5.2.6, we have that $d_{g_0}\big|_\Omega = d_{\hat{g}}$, and consequently it suffices to show that $d_{g(t)} \to d_{g_0}$ uniformly on Ω as $t \downarrow 0$. But this follows from the fact that $g(t) \to g_0$ in $C^0(\mathcal{M})$ as $t \downarrow 0$.

We may then construct Gromov–Hausdorff approximations as follows: given $\varepsilon > 0$, let $f\colon (\Omega, d_{\hat{g}}) \to \big(\mathcal{M}, d_{g(t)}\big)$ be inclusion, and let $g\colon \big(\mathcal{M}, d_{g(t)}\big) \to (\Omega, d_{\hat{g}})$ be such that $g\big|_\Omega$ is the identity, and such that g sends any $p \in \mathcal{M} \setminus \Omega$ to a fixed point $q \in \Omega$ with $\bar{d}_{\hat{g}}(q, \partial\Omega) < \varepsilon$. Then, by the above observations, both f and g are ε-Gromov–Hausdorff approximations for t sufficiently small. □

Remark 5.2.9. An alternative in the above would be to construct the approximating sequence $\{g_i\}$ so that $g_i = \hat{g}$ on Ω, except for a neighbourhood of $\partial\Omega$ that shrinks as $i \to \infty$.

Remark 5.2.10. We point out that obtaining a complete solution $(g(t))$ on Ω, starting from $\hat{g}$ in the sense of Theorem 5.2.3 is impossible, since $(\Omega, \hat{g})$ is of finite-diameter, whereas any complete metric on Ω is of infinite diameter. For instance, the instantaneously complete flow provided by Theorem 2.2.4 does *not* satisfy the conclusion of Theorem 5.2.3.

Remark 5.2.11. It may be asked whether the space $\mathcal{O}(\Omega, \mathcal{M})$ is a natural one to consider. Indeed, it is the broadest class of metrics for which a result like Theorem 5.2.3 is possible. This is because, as one can observe by inspection of the proof, we need initial metrics $\hat{g}$ on Ω such that, when embedded in the larger surface $\mathcal{M}$ and surrounded by the 'zero metric', curves joining points in Ω cannot decrease their length by going outside of Ω. In other words, we need that $d_{g_0}\big|_\Omega = d_{\hat{g}}$, which is only possible if $\hat{g} \in \mathcal{O}(\Omega, \mathcal{M})$.

The preceding remark exposes the key obstacle in the geometric interpretation of the taking on of initial data: it is possible to construct complete Ricci flows that take initial conditions in an analytic sense, such as L^1_{loc}-convergence, but which fail to realise any geometric notion of convergence. For instance, by using essentially the same methods, we can prove the following theorem:

Theorem 5.2.12. *Let $\hat{g}$ be a smooth, finite-area metric on $\mathbb{D}$. Let g_0 be the degenerate metric on $\mathbb{D}_2$ (the disc of radius 2 in $\mathbb{C}$) defined by*

$$g_0(z) := \begin{cases} \hat{g} & \text{if } z \in \mathbb{D} \\ 0 & \text{if } z \in \mathbb{D}_2 \setminus \mathbb{D}. \end{cases}$$

Then there exists a smooth, complete Ricci flow $(g(t))_{t\in(0,\infty)}$ on $\mathbb{D}_2$ such that

$$g(t) \to g_0$$

in $L^1_{\mathrm{loc}}(\mathbb{D}_2)$ as $t \downarrow 0$.

The only extra tools we need in order to prove the above theorem are the local L^∞-bounds for solutions to the Ricci flow equation starting from L^1_{loc}-initial data, provided by [48, Theorem 8.8]. Nevertheless, whilst we can use the techniques of Theorem 5.2.3 to construct the flow in Theorem 5.2.12, it is not geometrically significant (unless the initial metric is zero at $\partial\mathbb{D}$): curves in $\mathbb{D}$, when measured with $g(t)$, can reduce their length by leaving $\mathbb{D}$ as t gets small. Consequently, for small t, the metrics $g(t)$ and $\hat{g}$ are not geometrically close. For instance, consider the points $p = \frac{3}{4}$ and $q = -\frac{3}{4}$ in $\mathbb{D}$, taking $\hat{g}$ as the flat metric. Then $d_{\hat{g}}(p,q) = \frac{3}{2}$, whereas $d_{g(t)}(p,q) \to \frac{1}{2}$ as $t \downarrow 0$.

Remark 5.2.13. A possible future avenue of investigation is to consider to what extent the flow constructed in Theorem 5.2.3 is unique. A conjecture would be that once the initial data triple $(\Omega, \mathcal{M}, \hat{g})$ and the conformal embedding $i\colon \Omega \hookrightarrow \mathcal{M}$ are specified, the flow given by Theorem 5.2.3 is unique (up to diffeomorphism). An open problem is whether Theorem 5.2.3 is true in the case where $\mathcal{M}$ has genus zero or one (and hence does not support a conformal hyperbolic metric). The difficulty in this instance is showing that the constructed flow $(g(t))$ is non-degenerate, since we cannot bound the sequence $\{g_i(t)\}$ from below by the 'big-bang' Ricci flow $2th$ as in the hyperbolic case.

Appendix A

Local arguments in Lemma 4.3.7

In the proof of Lemma 4.3.7, we made two claims that we justify in this appendix. These arguments are a reproduction of elements of [37, Appendix A], modified and translated from the original French. In essence, they follow the proofs provided in [3, Chapter VII] and [1, Chapter III, §6] of analogous statements in the case of *non-negative* curvature due to Alexandrov and Zalgaller. In our setting, curvature is bounded below by -1, and consequently there are points in the proofs where the Euclidean character of the spaces in question arises, which we cannot use verbatim. Richard has carefully identified these portions, and has replaced them with hyperbolic versions of the same statements, so that, all things considered, the same overall results also apply in our scenario.

Proposition A.1 ([37, Lemma A.2.1])**.** *With all objects as defined in Lemma 4.3.7, the metric space $(\mathcal{M}_i, d_i)$ is an Alexandrov surface of curvature bounded below by -1.*

Proof. By the Globalisation Theorem (Theorem 3.6.1), it suffices to show that every point of $\mathcal{M}_i$ admits a neighbourhood of curvature at least -1. If $q \in \mathcal{M}_i$ is *not* a vertex point of $\widehat{\mathcal{T}}$, then it admits a neighbourhood that is isometric to a domain in $\mathbb{H}^2$. Consequently, we need only check that vertex points of $\widehat{\mathcal{T}}$ admit appropriate neighbourhoods.

Indeed, let $p \in \mathcal{M}_i$ be a vertex point of $\widehat{\mathcal{T}}$, and let $T_1, T_2, \ldots, T_n$ be the triangles of $\widehat{\mathcal{T}}$ containing p (so p is either a vertex point or a point on a side of these triangles). Let α denote the sum of the angles at p in the triangles T_i, with the convention that if p is on the side of a triangle, the contribution to α from this triangle is π. It is well-known that a sufficiently small neighbourhood of p is isometric to a neighbourhood of the vertex of a hyperbolic cone over a circle of diameter α. By [5, Theorem 10.2.3], for this neighbourhood to be of curvature at least -1, it suffices that $\alpha \leq 2\pi$. But this follows immediately from the quadruple condition (given in Definition 3.3.21). □

Now we proceed to consider the following result, which is of a highly technical nature, and which we used to justify the Gromov–Hausdorff convergence of our polyhedral approximations to the original Alexandrov surface (X, d) in Lemma 4.3.7:

Proposition A.2. *For $i \in \mathbb{N}$, let $\mathcal{T}$ and $\widehat{\mathcal{T}}$ be as in Lemma 4.3.7. Given vertex points $v, w \in \mathcal{T}$ and their corresponding vertex points $\hat{v}, \hat{w} \in \widehat{\mathcal{T}}$, we have that*

$$|d(v, w) - d_i(\hat{v}, \hat{w})| \leq \varrho(i),$$

where $\varrho(i) \to 0$ as $i \to \infty$.

To prove this, we need the following, which is a hyperbolic analogue of the result used by Alexandrov to prove the corresponding proposition in the case of non-negative curvature:

Lemma A.3 ([37, Lemma A.2.2], hyperbolic variant of [3, Lemma 1, p. 259])**.** *Let (X, d) be a complete Alexandrov surface of curvature bounded below by -1. Let T be a convex triangle in X with vertices a, b, c, let x be a point on the side joining a and b, and let y be a point on the side joining a and c. Let $D := \mathrm{Diam}(T)$. Let $\widehat{T}$ be the comparison triangle of T in $\mathbb{H}^2$ (see Definition 3.3.4), with $\bar{a}$ denoting the vertex point of $\widehat{T}$ corresponding to a, and so on. Then there exists a constant C such that*

$$|\bar{x}\bar{y}| \leq |xy| \leq |\bar{x}\bar{y}| + CD(\delta(T) + \mathrm{Area}(T)),$$

where $\delta(T)$ is the excess of T as in Definition 3.3.28.

With this in hand, we can now show that $d_i(\hat{v}, \hat{w}) \leq d(v, w)$. To this end, let $\gamma\colon [0, 1] \to X$ be a shortest path joining v and w. Construct a path in $\mathcal{M}_i$ joining $\hat{v}$ and $\hat{w}$ as follows: each time γ passes into the interior of a triangle $T \in \mathcal{T}$, take note of the points in the comparison triangle $\widehat{T}$ corresponding to the entry and exit points of γ in T, and join them by the segment of the sides of $\widehat{T}$ between them. Since all triangles in $\mathcal{T}$ are convex, we may assume this occurs at most once in each triangle. In this way, we construct a path joining $\hat{v}$ and $\hat{w}$ in $\mathcal{M}_i$ whose length is shorter than that of γ thanks to Lemma A.3.

It remains to show that $d(v, w)$ cannot be too much larger than $d_i(\hat{v}, \hat{w})$. For this, we consider a shortest path $\hat{\gamma}$ joining $\hat{v}$ and $\hat{w}$ in $\mathcal{M}_i$. The idea is that to each segment of the path passing into the interior of a triangle $\widehat{T} \in \widehat{\mathcal{T}}$ we assign a shortest path joining the corresponding entry and exit points of the corresponding triangle $T \in \mathcal{T}$. There is, however, a problem: each time we perform this operation, we make an error of $CD(\delta(T) + \text{Area}(T))$ by Lemma A.3. Since the triangles in $\widehat{\mathcal{T}}$ are not necessarily convex, we have no way *a priori* of controlling the number of times we must perform this operation, and hence have no control over the error. Nevertheless, the arguments of [1, §3, Lemma 19, p.82] show that by slightly modifying $\hat{\gamma}$, we can ensure that the path enters the interior of each triangle in $\widehat{\mathcal{T}}$ at most twice. For this to work in our case, we rely upon [37, Lemma A.2.4], which is again a hyperbolic fix for a situation in which the original argument depends upon the non-negatively curved nature of the space under investigation.

The proof of Proposition A.2 is completed by using the arguments of [1, §3, Theorem 10], where it is concluded (see Remark 3 following that theorem) that every compact set of X can be embedded in a convex polygon $P \subset X$ such that the induced intrinsic metric d_P on P is the limit of a uniformly converging sequence of polyhedral metrics of bounded curvature. We may hence deduce that $(\mathcal{M}_i, d_i, p_i) \to (X, d, p)$ in the Gromov–Hausdorff sense as $i \to \infty$ in Lemma 4.3.7.

Appendix B

Analytical preliminaries

In this appendix, we collect a handful of useful results from real analysis that are employed in the main text.

B.1 Estimates for solutions to parabolic equations

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $I \subset [0, \infty)$ and let $\Omega_T := \Omega \times I$. We define *parabolic* Hölder spaces as follows:

Definition B.1.1. Define a distance ρ on Ω_T by

$$\rho((x_1, t_1), (x_2, t_2)) := |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}.$$

Fix $\delta \in (0, 1)$. Define the *parabolic Hölder space* $C^{\delta, \frac{\delta}{2}}(\Omega_T)$ to be the space of continuous functions on $\overline{\Omega_T}$ such that

$$[u]_{C^{\delta, \frac{\delta}{2}}(\Omega_T)} := \sup_{\substack{z_1 \neq z_2 \\ z_1, z_2 \in \Omega_T}} \frac{|u(z_1) - u(z_2)|}{\rho(z_1, z_2)^\delta} < \infty.$$

The $C^{\delta, \frac{\delta}{2}}(\Omega_T)$-norm is then

$$||u||_{C^{\delta, \frac{\delta}{2}}(\Omega_T)} := ||u||_{C^0(\Omega_T)} + [u]_{C^{\delta, \frac{\delta}{2}}(\Omega_T)}.$$

This norm gives $C^{\delta, \frac{\delta}{2}}(\Omega_T)$ the structure of a Banach space.

The spaces $C^{2k+\delta, k+\frac{\delta}{2}}(\Omega_T)$ are defined as follows: let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex, and let $|\alpha| := \sum_i \alpha_i$. Define

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{B.1}$$

Then

$$u \in C^{2k+\delta, k+\frac{\delta}{2}}(\Omega_T) \iff \frac{\partial^r}{\partial t^r}\partial^\alpha u \in C^{\delta, \frac{\delta}{2}}(\Omega_T)$$

for $2r + |\alpha| \leq 2k$.

We then have the following theorem, which shows that a smooth, bounded solution of a quasi-linear parabolic partial differential equation has its Hölder norms controlled by its L^∞ norm:

Theorem B.1.2 (Simplified version of [24, Theorem IV.10.1])**.** *Let $\delta \in (0,1]$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $u \in C^{(2k+2)+\delta,(k+1)+\frac{\delta}{2}}(\Omega_T)$ is a solution of the linear parabolic equation*

$$\frac{\partial}{\partial t}u(t,p) = \big\langle a(t,p), \nabla^2 u(t,p)\big\rangle + b(t,p),$$

where we make the assumptions that:

- $a \in C^{2k+\delta,k+\frac{\delta}{2}}(\Omega_T; \mathrm{Sym}^2\mathbb{R}^n)$,
- $b \in C^{2k+\delta,k+\frac{\delta}{2}}(\Omega_T)$ *and*
- *there exists $\lambda > 0$ such that for any pair $(p,t) \in \Omega_T$, we have the uniform parabolicity condition*

  $$\langle a(p,t), \eta \otimes \eta\rangle \geq \lambda|\eta|^2 > 0$$

  for all $\eta \in \mathbb{R}^n \setminus \{0\}$.

Then for any $\Sigma \subset\subset \Omega$ with $\beta := d(\Sigma, \partial\Omega) > 0$, there exists $C > 0$, depending only on $n, k, \lambda, \delta, \beta, ||a||_{C^{2k+\delta,k+\frac{\delta}{2}}(\Omega_T;\mathrm{Sym}^2\mathbb{R}^n)}$ and $||b||_{C^{2k+\delta,k+\frac{\delta}{2}}(\Omega_T)}$ such that

$$||u||_{C^{(2k+2)+\delta,(k+1)+\frac{\delta}{2}}(\Sigma\times I)} \leq C||u||_{L^\infty(\Omega_T)}.$$

Remark B.1.3. If instead u solves the *quasi*-linear equation

$$\frac{\partial}{\partial t}u(t,p) = \langle A(t,p,u(t,p),\nabla u(t,p),\nabla^2 u(t,p)), \nabla^2 u(t,p)\rangle + B(t,p,u(t,p),\nabla u(t,p)), \tag{B.2}$$

we may recover the same conclusion as Theorem B.1.2 by defining

$$a(t,p) := A(t,p,u(t,p),\nabla u(t,p)) \qquad \text{and} \qquad b(t,p) := B(t,p,u(t,p),\nabla u(t,p)).$$

B.2 The Arzelà–Ascoli Theorem and consequences

We present here the well-known theorem of Arzelà and Ascoli, together with some of its corollaries.

Theorem B.2.1 (Arzelà–Ascoli)**.** *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions defined on an open subset $\Omega \subseteq \mathbb{R}^n$. Suppose that*

- *there exists $C > 0$ such that*

$$\sup_n ||u_n||_{C^0(\Omega)} \leq C$$

  and

- *the sequence $\{u_n\}$ is* equicontinuous*, i.e. given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_n |u_n(x) - u_n(y)| < \varepsilon$$

  whenever $x, y \in \Omega$ and $|x - y| < \delta$.

Then there exists a continuous function $u \in C^0(\Omega)$ and a subsequence $\{u_{n_j}\}$ such that

$$\{u_{n_j}\} \to u$$

as $j \to \infty$, uniformly on compact subsets of Ω.

We have the following corollary in the context of paths in metric spaces:

Corollary B.2.2 ([5, Theorem 2.5.14])**.** *In a compact metric space, any sequence of paths $\{\gamma_i\}$ with uniformly bounded lengths admits a uniformly converging subsequence $\{\gamma_{i_j}\}$, in the sense that the γ_{i_j} admit parameterisations (with the same domain) that converge uniformly to a parameterisation of another path γ.*

Now let $\{u_n\} \subset C^{k+1}(\Omega)$ and suppose there exists $C > 0$ such that $||\nabla^{k+1} u_n||_{C^0(\Omega)} \leq C$. Then we observe that

$$|\nabla^k u_n(x) - \nabla^k u_n(y)| \leq C|x - y|,$$

and so the sequence $\{\nabla^k u_n\}_n$ is automatically equicontinuous on Ω. Consequently, we have the following:

Corollary B.2.3. *Let $\{u_n\} \subset C^{k+1}(\Omega)$ be a sequence such that there exists $C > 0$ with $||\nabla^l u_n|| \leq C$ for all $n \in \mathbb{N}$ and all $0 \leq l \leq k+1$. Then there exists a function $u \in C^k(\Omega)$ and a subsequence $\{u_{n_j}\}$ such that*

$$\{u_{n_j}\} \to u$$

in $C^k_{\text{loc}}(\Omega)$ as $j \to \infty$.

Now we give the result that under appropriate conditions, pointwise convergence of a sequence is actually uniform:

Theorem B.2.4. *For Ω as above, suppose that $\{u_n\} \subset C^0(\Omega)$ is an equicontinuous sequence. Suppose further that there exists a function $u\colon \Omega \to \mathbb{R}$ such that $\{u_n\} \to u$ pointwise on Ω as $n \to \infty$. Then $\{u_n\} \to u$* uniformly *on compact subsets of Ω as $n \to \infty$.*

Proof. Let $K \subset \Omega$ be compact and fix $\varepsilon > 0$. By the definition of equicontinuity, there exists $\delta > 0$ such that

$$\sup_n |u_n(x) - u_n(y)| < \frac{\varepsilon}{3}$$

whenever $|x - y| < \delta$. Consequently, by the assumption on pointwise convergence, we have that

$$|u(x) - u(y)| < \frac{\varepsilon}{3}$$

whenever $|x - y| < \delta$.

Now since K is compact, we may find finitely many points $p_1, \ldots, p_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B_\delta(p_i).$$

Since $u_n(p_j) \to u(p_j)$ for each $j = 1, \ldots, k$, we may find N such that

$$|u_n(p_j) - u(p_j)| < \frac{\varepsilon}{3}$$

whenever $n \geq N$, for any $j \in \{1, \ldots, k\}$.

Then given $x \in K$, we have that $x \in B_\delta(p_j)$ for some j, and so

$$\begin{aligned}|u_n(x) - u(x)| &\leq |u_n(x) - u_n(p_j)| + |u_n(p_j) - u(p_j)| + |u(p_j) - u(x)| \\ &< \varepsilon\end{aligned}$$

whenever $n \geq N$, as required. □

Finally, we have an application to PDEs:

Corollary B.2.5. *Let Ω be as above, and let $u(t)_{t\in(0,T)}$ be a smooth, bounded solution to a quasi-linear PDE of the form* (B.2)*. Suppose there exists some function $u_0 \colon \Omega \to \mathbb{R}$ such that $u(t) \to u_0$ pointwise on Ω as $t \downarrow 0$. Then $u(t) \to u_0$ uniformly on compact subsets of Ω.*

Proof. By Theorem B.1.2, given a compact subset $\mathcal{K} \subset \Omega$, we have that for any $k \in \mathbb{N}$ there exists a $C > 0$ such that $||u(t)||_{C^k(\mathcal{K})} \leq C$ for all $t \in (0, T)$. Thus by an earlier observation, the sequence $\{u(t)\}_{t\in(0,T)}$ is equicontinuous on $\mathcal{K}$. The result then follows from Theorem B.2.4. □

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