COMPLEMENTATION IN FINITE GROUPS

by

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ABSTRACT

This thesis is a study of the restrictions which are imposed on the structure of a finite group by some conditions on its lattice of subgroups. The conditions considered fall into two categories: either (1) the demand is made that certain of the subgroups of the group should have complements, or (2) it is specified that all subgroups should have supplements of a particular kind.

There are three chapters. Chapter 1 develops some techniques and results about complements and pronormality which are used later, mainly in Chapter 2. A problem from category (1) above is the subject of Chapter 2, which is an investigation of finite groups with the property that all the pronormal subgroups have complements. Necessary and sufficient conditions are given for a soluble group of derived length at most 3 to have that property. Chapter 3 is concerned with category (2); the basic theme is that of a finite group G in which each subgroup H has a supplement S such that $H \cap S$ belongs to some prescribed class $\mathfrak{X}$.

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CHAPTER 1

1.1 Introduction

In (H1) and (H2), Hall showed that a necessary and sufficient condition for a finite group to be soluble is that every Sylow subgroup of the group should have a complement. Much of the subsequent development in the theory of finite soluble groups stems from this and related results in (H1) and (H2), and from the ideas in (H3) and (H4) which consequently arise.

A different emphasis can be put on the theorem quoted above, by expressing it as follows: the class of finite groups in which every Sylow subgroup has a complement coincides with the class of finite soluble groups. This leads naturally to an interest in classes of groups in which certain kinds of subgroups have complements; such questions may loosely be described as "complementation problems". The most obvious complementation problem to consider is that of groups in which all subgroups have complements; this class was studied by Hall in (H5), and was shown to have a simple structure: it coincides with the class of super-soluble groups in which all the Sylow subgroups are elementary abelian.

The idea of examining complementation problems has been taken up by many authors, although nothing rivalling the importance of Hall's original result on soluble groups has emerged. Many papers on such topics, in both finite and infinite groups, have appeared in Russian journals, and perhaps the most deserving of mention is the work of Cernikov (C1) and Cernikova (C2), in which the results of (H5) are extended to infinite groups. Further reference to Russian papers in this area can be found in (S3), which is itself on groups in which all the non-normal subgroups have complements. A more recent contribution is (Z1), on groups in which all the non-cyclic subgroups have complements.

A natural complementation problem to consider is that of groups in which all the normal subgroups have complements. Such groups have been examined in (C3), (C4) and (D1); they are mentioned further in 1.4, and appear frequently in the investigations of Chapter 2.

In the present work, attention is confined entirely to finite groups, and there are two main themes, neither of which is considered in the papers mentioned above. Chapter 2 is an investigation of a complementation problem, namely the question of finite groups with the property that all the pronormal subgroups have complements. Necessary and sufficient conditions are obtained for a soluble group of derived length at most 3 to have this property. The second theme (the subject of Chapter 3) is an attempt to find some interesting problems similar to the question of groups in which all the subgroups have complements. The idea which is most extensively explored is that of finite groups in which every subgroup has an $\mathfrak{X}$-intersection supplement, where $\mathfrak{X}$ is some prescribed class of groups. (An $\mathfrak{X}$-intersection supplement to a subgroup H of a group G is a supplement S such that $H \cap S$ belongs to $\mathfrak{X}$). It is shown that, if every subgroup of a finite group G has a cyclic intersection supplement, then G is soluble and has rank at most 2. Also, if G is a finite soluble group in which every subgroup has an abelian intersection supplement, then G has derived length at most 4. If π is a set of prime numbers, then the study of finite groups in which all the subgroups have π-intersection supplements (i.e. the intersection is a π-group) leads to a complementation problem, namely the question of finite groups in which all the π-subgroups have complements. It is shown that a finite π-soluble group G has all its π-subgroups complemented if and only if, for each prime number p in the set π, G is p-supersoluble and has elementary abelian Sylow p-subgroups.

The subject matter of Chapter 1 lies in well-explored territory, so all the results have probably been noted before, possibly in a different guise: most are well known, or are simple deductions from well known theorems. The results of Chapters 2 and 3 are, to the best of my knowledge, original, except where an explicit reference is given.

1.2 Notation, assumed results

The notation used is fairly standard in contemporary group theory, and corresponds to that given in 1.1 of (G2), so it will not be necessary to give an exhaustive list. Some further notations, which do not appear in (G2), but which are used here, are listed below:

$H \leqslant G$: H is a subgroup of G ;

$H < G$: H is a proper subgroup of G ;

g^h : $h^{-1}g\,h$ (where g and h are elements of some group G).

The language of classes and closure operations, first introduced by P.Hall (H6, p.533) is used throughout, since it provides a convenient and economical way of expressing many results. The usual conventions are adopted that, if $\mathfrak{X}$ is a class of groups then $\mathfrak{X}$ contains all groups of order 1, and if $G \in \mathfrak{X}$ then $\mathfrak{X}$ contains all groups isomorphic to G . If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups then $\mathfrak{X}\mathfrak{Y}$ is the class defined by: $G \in \mathfrak{X}\mathfrak{Y}$ if and only if G has a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. The only closure operations which are used are S, S_n, Q, R_0, D_0, E and E_Φ, the definitions of which are now given:

$G \in S\mathfrak{X} \iff G$ is isomorphic to a subgroup of an $\mathfrak{X}$-group;

$G \in S_n\mathfrak{X} \iff$ G is isomorphic to a subnormal subgroup of an $\mathfrak{X}$-group;

$G \in Q\mathfrak{X} \iff$ G is isomorphic to a quotient group of an $\mathfrak{X}$-group;

$G \in R_0\mathfrak{X} \iff$ G has normal subgroups $N_1, \dots, N_r$ (where r is finite) such that $N_1 \cap \dots \cap N_r = 1$ and $G/N_i \in \mathfrak{X}$ ($i = 1,\dots,r$);

$G \in D_0\mathfrak{X} \iff$ G is isomorphic to a direct product of $\mathfrak{X}$-groups;

$G \in E\mathfrak{X} \iff$ G has a finite series $1 = G_1 \leqslant G_2 \leqslant \dots \leqslant G_n = G$ such that for each $i = 1,\dots,n-1$, G_i is normal in G_{i+1} and $G_{i+1}/G_i \in \mathfrak{X}$;

$G \in E_\Phi\mathfrak{X} \iff$ G has a normal subgroup $N \leqslant \Phi(G)$ (where $\Phi(G)$ denotes the Frattini subgroup of G) such that $G/N \in \mathfrak{X}$.

If A and B are closure operations, then $\{A,B\}\mathfrak{X}$ is the smallest class containing $\mathfrak{X}$ which is both A-closed and B-closed. Often $\{A,B\}$ will coincide with one of the naturally-defined products AB, BA; e.g. $\{Q,R_0\} = QR_0$, $\{S,D_0\} = SD_0$, $\{S,Q\} = QS$.

If C is a unary closure operation, i.e. if $C\mathfrak{X} = \bigcup_{G\in\mathfrak{X}} C(G)$ for every class of groups $\mathfrak{X}$, then there is a unique largest C-closed class contained in $\mathfrak{X}$. This class is denoted by $\mathfrak{X}^C$, is given by

$$\mathfrak{X}^C = \{G : C(G) \subseteq \mathfrak{X}\} ,$$

and is called the C-interior of $\mathfrak{X}$. The only closure operation for which this concept is used here is S.

Two important classes of groups are $\mathfrak{A}$ and $\mathfrak{N}$: $\mathfrak{A}$ denotes the class of finite abelian groups, and $\mathfrak{N}$ the class of finite nilpotent groups. Thus $E\mathfrak{A}$ is the class of finite soluble groups. If n is a positive integer, then $\mathfrak{A}^n$ denotes the class of finite soluble groups of derived length at most n .

Whenever the word "group" appears,it can be taken to mean "finite group". Many basic results of finite group theory (e.g. the contents of the first three chapters of (G2)) may be used without explicit reference. The phrase "elementary abelian group" will refer to an abelian group, the Sylow subgroups of which are of prime exponent: it will not be restricted to apply only to groups of prime-power order.

Most of the arguments are concerned with finite soluble groups. The fact that a p-chief factor of a finite soluble group G can be regarded as an irreducible $Z_p[G]$-module permits the introduction of representation theory techniques; one result which is particularly valuable in this context is Clifford's Theorem (H8,$\underline{V}$,17.3) which gives detailed information about the restriction of an irreducible representation to a normal subgroup. An important special case in which these techniques are useful is that of a soluble group G with a unique minimal normal subgroup V which has a complement in G. Such a group G is called a primitive soluble group (because, if H is a maximal subgroup of G which complements V in G, then the permutation representation of G on the cosets of H is a faithful representation of G as a primitive permutation group). The significant properties of primitive soluble groups are given in (H8, II, 3.2 and 3.3).

The following standard notation is used in constructing particular examples of groups:

C_n : the cyclic group of order n ;

S_n : the symmetric group of degree n ;

A_n : the alternating group of degree n ;

$GL(n,F)$: the general linear group of degree n over F ;

$SL(n,F)$: the special linear group of degree n over F .

Statements of standard definitions and results are sprinkled throughout the text, on the principle that it is better to give these when they are needed rather than list them all in one long and tedious introductory section.

1.3 Complements

Definitions Let H be a subgroup of a group G . A supplement to H in G is a subgroup K of G such that $HK = G$. A complement to H in G is a subgroup K of G such that $HK = G$ and $H \cap K = 1$.

Clearly, a complement in G to a subgroup H of G is a set of coset representatives of H in G which happens to form a group. As only finite groups are under consideration, either of the following alternative criteria can be used to show that K is a complement to H in G :

(a) $HK = G$ and $|H||K| = |G|$;

(b) $H \cap K = 1$ and $|H||K| = |G|$.

1.3.1 If K is a complement to H in G , then for any $a, b \in G$, K^a is a complement to H^b in G .

A similar statement holds with "supplement" substituted everywhere for "complement". 1.3.1, which is well-known and easily proved, shows that, in a sense, it would be more natural to consider complementary conjugacy classes of subgroups, rather than subgroups which complement each other.

One of the most useful tools for dealing with questions involving complements or supplements is the so-called "Dedekind modular law" for subgroups, which can be stated as follows:

If H, K and L are subgroups of a group G and $K \leqslant H$, then $H \cap KL = K(H \cap L)$.

(It is not necessary to assume that KL or $K(H \cap L)$ is a subgroup, although this is almost always the case when the result is applied). This Dedekind law will be used frequently in the sequel, probably without further explicit reference. It makes its first appearance in the proofs of some of the following results.

<u>1.3.2</u> Let G be a group and let H be a subgroup of G which has a complement C in G.

(a) If $H \leqslant K \leqslant G$ then $C \cap K$ is a complement to H in K.

(b) If N is a normal subgroup of G and $N \leqslant H$, then CN/N is a complement to H/N in G/N.

<u>Proof</u> (a) $H(C \cap K) = HC \cap K = G \cap K = K$,

and $H \cap (C \cap K) = (H \cap C) \cap K = 1$.

(b) $H(CN) = (HC)N = G$,

and $H \cap CN = (H \cap C)N = N$.

<u>1.3.3</u> (a) If H is a subgroup of G and there exists a normal subgroup N of G such that, HN/N has a complement in G/N and $H \cap N$ has a complement in G, then H has a complement in G.

(b) Suppose $G = KN$, where N is normal in G, $K \leqslant G$, and $K \cap N = 1$, and let H be a subgroup of G. If C_1 is a complement to $HN \cap K$ in K, C_2 is a complement to $H \cap N$ in N, and C_1 and C_2 permute (i.e. C_1C_2 is a group), then C_1C_2 is a complement to H in G.

(c) Suppose $G = G_nG_{n-1}\ldots G_2G_1$, where for each $i = 1,\ldots,n-1$, $G_iG_{i-1}\ldots G_1$ is normal in G and $G_{i+1} \cap G_iG_{i-1}\ldots G_1 = 1$. Let H be a subgroup of G. If C_1 is a complement to $H \cap G_1$ in G_1 and for each $i \geqslant 1$, C_{i+1} is a complement to $H\, G_iG_{i-1}\ldots G_1 \cap G_{i+1}$ in G_{i+1} and C_{i+1} permutes with ~~normalizes~~ $C_iC_{i-1}\ldots C_1$, then $C_nC_{n-1}\ldots C_1$ is a complement to H in G.

<u>Proof</u> (a) Let K/N be a complement to HN/N in G/N. By 1.3.2(a), $H \cap N$ has a complement, C say, in K. Then

$$HC = H(H\cap N)C = HK = G,$$

and $H \cap C = H \cap HN \cap K \cap C = H \cap N \cap C = 1$.

Therefore C is a complement to H in G.

(b) $H\,C_1C_2 = H(H\cap N)C_2C_1 = HNC_1 = HN(HN\cap K)C_1 = HNK = G$.

Also
$$\begin{aligned} H \cap C_1C_2 &= H \cap HN \cap C_1C_2 \\ &= H \cap (HN \cap C_1)C_2 \qquad (\text{as } C_2 \leqslant N \leqslant HN) \\ &= H \cap (HN\cap K\cap C_1)C_2 = H \cap C_2 = 1. \end{aligned}$$

(c) It is enough to prove by induction on i that for each i, $C_iC_{i-1}\ldots C_1$ is a complement to $H \cap G_iG_{i-1}\ldots G_1$ in $G_iG_{i-1}\ldots G_1$. This is certainly true when $i = 1$. Suppose it is true for a particular i; then an application of (b), with $G_{i+1}G_i \ldots G_1$, G_{i+1} , $G_iG_{i-1} \ldots G_1$, $H \cap G_{i+1}G_i \ldots G_1$, C_{i+1} and $C_iC_{i-1} \ldots C_1$ in place of G , K , N , H , C_1 and C_2 respectively, shows immediately that it is also true for $i+1$. Hence the result holds.

In some ways, complements are not "well-behaved", as is illustrated by the following examples; this often hampers the investigation of complementation problems.

<u>1.3.4 Example</u> Let $G = S_4$

(a) The complements of a given subgroup of G are not necessarily all isomorphic:

Let $H = \langle\langle (123),(12) \rangle\rangle$, $V = \langle\langle (12)(34),(13)(24) \rangle\rangle$, $K = \langle\langle (1234) \rangle\rangle$; then V and K are both complements to H in G , but are not isomorphic.

(b) If a subgroup H of G has a complement in G , and $K \leqslant G$, then it is not necessarily true that $H \cap K$ has a complement in K :

Let $H = \langle\langle (1234) \rangle\rangle$, $K = A_4$; then $\langle\langle (123),(12) \rangle\rangle$ is a complement to H in G , but $H \cap K$ (= $\langle\langle (13)(24) \rangle\rangle$) is a proper, non-trivial subgroup of the minimal normal subgroup of K , and therefore cannot have a complement in K .

(c) A supplement to a complemented subgroup of G need not contain a complement:

Let $H = \langle\langle (1234) \rangle\rangle$, $K = A_4$ as in (b); then H is a supplement to K in G , and K has a complement in G . If H contained a complement to K in G , then that complement would be of order 2 ; but $\langle\langle (13)(24) \rangle\rangle$ is the only subgroup of H of order 2, and $(13)(24) \in K$.

<u>1.3.5 Example</u> If H has a complement in a group G and N is normal in G, it does not necessarily follow that HN/N has a complement in G/N. (Cf. 1.3.2(b)):

Let $V_i = \langle v_i, w_i \rangle \cong C_2 \times C_2$ $(i = 1,2)$, and let G be the split extension of $V_1 \times V_2$ by $\langle x \rangle \cong C_3$, where

$$v_i^{\,x} = w_i \ , \ w_i^{\,x} = v_i w_i \qquad (i = 1,2).$$

Thus $\langle x \rangle V_i \cong A_4$ $(i = 1,2)$. Let $H = \langle v_1, w_2 \rangle$; then $\langle x, v_1 v_2, w_1 w_2 \rangle$ is a complement to H in G, but HV_2/V_2 $(= \langle v_1 \rangle V_2/V_2)$ has no complement in G/V_2 $(\cong A_4)$, because it is a proper non-trivial subgroup of the unique minimal normal subgroup of G/V_2.

<u>1.3.6 Example</u> A subgroup H of G such that $H \leqslant K \leqslant G$, which has a complement in G, can have a complement in K which does not extend to a complement in G :

Let $G = \langle a,b \rangle \times \langle c \rangle$, where $\langle a,b \rangle \cong S_3$, with $a^3 = b^2 = 1$ and $a^b = a^{-1}$, and $\langle c \rangle \cong C_3$. Let $H = \langle a \rangle$, $K = \langle a,c \rangle$, $C = \langle ac \rangle$. Then H has a complement in G, and C is a complement to H in K, but C is not contained in any complement to H in G, because no 2-element of G normalizes C.

(This example is used in (D2) for a different purpose, namely to show that a normal subgroup H of a group G which has a unique conjugacy class of complements in G can have more than one conjugacy class of complements in a (normal) subgroup K of G which contains H).

1.4 Groups with complemented normal subgroups

Definitions Let H/K be a normal factor of a group G (i.e. H and K are normal subgroups of G and $K \leqslant H$).

(1) H/K is a complemented factor of G if H/K has a complement in G/K.

(2) H/K is a Frattini factor of G if $H/K \leqslant \Phi(G/K)$.

(3) A subgroup L of G is said to cover H/K if $(L \cap H)K = H$; L is said to avoid H/K if $(L \cap H)K = K$.

(4) A subgroup L of G is said to have the cover-avoidance property if, for each chief factor H/K of G, L either covers H/K or avoids H/K.

1.4.1 Theorem Suppose G has a chief series

$$1 = G_0 < G_1 < \ldots < G_n = G ,$$

and $H \leqslant G$ covers or avoids each of the chief factors in this series. If the factors covered by H are all complemented, then H has a complement in G.

Proof: By induction on n. If $n = 1$, then $H = 1$ or $H = G$, so H has a complement in G. Suppose that the result holds for groups with fewer than n chief factors in a chief series, and let $\overline{G} = G/G_1$. The "bar convention" will be used, i.e. the image of a subgroup K of G under the natural epimorphism $G \to \overline{G}$ will be denoted by $\overline{K}$. Then $\overline{1} = \overline{G}_1 < \overline{G}_2 < \ldots < \overline{G}_n = \overline{G}$ is a chief series of

$\overline{G}$ and $\overline{H}$ covers or avoids each factor in this series. For $i \geqslant 2$, $\overline{H}$ covers $\overline{G}_i / \overline{G}_{i-1}$ if and only if H covers G_i / G_{i-1}, and $\overline{G}_i / \overline{G}_{i-1}$ is a complemented chief factor of $\overline{G}$ if and only if G_i / G_{i-1} is a complemented chief factor of G. Hence by the induction hypothesis, $\overline{H}$ has a complement in $\overline{G}$, i.e. HG_1 / G_1 has a complement in G/G_1.

Now consider $H \cap G_1$: if $H \cap G_1 = 1$ then $H \cap G_1$ certainly has a complement in G ; if $H \cap G_1 > 1$ then, by hypothesis, H must cover $G_1 / 1$ and so $H \cap G_1 = G_1$. The hypotheses then imply that G_1 has a complement in G. Thus, in every case, $H \cap G_1$ has a complement in G. Therefore, by 1.3.3(a), H has a complement in G.

<u>Notation</u> Let $\mathscr{C}_n$ denote the class of finite groups in which every normal subgroup has a complement.

<u>1.4.2</u> $\mathscr{C}_n$ is Q-closed.

<u>Proof</u> Let $G \in \mathscr{C}_n$ and let N be a normal subgroup of G. Let H/N be a normal subgroup of G/N ; then H is normal in G, and so H has a complement in G. Then by 1.3.2(b), H/N has a complement in G/N. Therefore $G/N \in \mathscr{C}_n$.

<u>1.4.3 Theorem</u> The following are equivalent:

(1) $G \in \mathcal{C}_n$;

(2) all the chief factors of G are complemented;

(3) G has a chief series in which all the chief factors are complemented;

(4) every subgroup of G which has the cover-avoidance property has a complement in G .

<u>Proof</u> (1) $\Rightarrow$ (2) : immediate from 1.4.2.

(2) $\Rightarrow$ (3) : trivial.

(3) $\Rightarrow$ (4) : follows at once from 1.4.1.

(4) $\Rightarrow$ (1) : normal subgroups have the cover-avoidance property, so this is obvious.

In a soluble group, every chief factor is either a Frattini factor or is complemented, hence:

<u>1.4.4 Corollary</u> If G is soluble then $G \in \mathcal{C}_n$ if and only if G has no Frattini chief factors in a given chief series.

Since the praefrattini subgroups of a soluble group cover Frattini chief factors and avoid complemented ones, 1.4.4 is equivalent to Theorem 6.6 in (Gl), in which Gaschütz observes that a soluble group has trivial praefrattini subgroups if and only if every normal subgroup has a complement.

<u>Notation</u> Let $\mathcal{C}$ denote the class of groups in which every subgroup has a complement.

The following corollary to 1.4.4 is used in 3.1 :

<u>1.4.5</u> $G \in \mathcal{C}$ if and only if G is supersoluble and $G \in \mathcal{C}_n$.

<u>Proof</u> Suppose $G \in \mathcal{C}$; then obviously $G \in \mathcal{C}_n$. Also it is shown in (H5) that $\mathcal{C}$-groups are supersoluble.

Now suppose G is a supersoluble group which belongs to $\mathcal{C}_n$, and proceed by induction on $|G|$. Let N be a minimal normal subgroup of G ; then $G/N \in \mathcal{C}_n$ by 1.4.2, and G/N is supersoluble, so by induction, $G/N \in \mathcal{C}$. Thus, given any subgroup H of G , HN/N has a complement in G/N . Since G is supersoluble, $H \cap N$ is either 1 or N , and so, because $G \in \mathcal{C}_n$, $H \cap N$ has a complement in G . Therefore, by 1.3.3(a), H has a complement in G. Hence $G \in \mathcal{C}$.

For the sake of the investigations in Chapter 2, it is useful to explore further the closure properties of $\mathcal{C}_n$.

<u>1.4.6</u> $\mathcal{C}_n$ is R_0-closed.

<u>Proof</u> Suppose G is a group which has normal subgroups N_1 and N_2 such that $N_1 \cap N_2 = 1$ and both G/N_1 and G/N_2 belong to $\mathcal{C}_n$. To show that $\mathcal{C}_n$ is R_0-closed, it will be enough to show that $G \in \mathcal{C}_n$.

Let H be a normal subgroup of G ; then HN_1/N_1 is normal in G/N_1 , so HN_1/N_1 has a complement in G/N_1 .

Also $(H \cap N_1)N_2$ is normal in G, so $(H \cap N_1)N_2 / N_2$ has a complement in G/N_2. Since $(H \cap N_1) \cap N_2 = 1$, it follows by 1.3.3(a) that $H \cap N_1$ has a complement in G. Another application of 1.3.3(a) then shows that H has a complement in G. Therefore $G \in \mathscr{C}_n$. Q.e.d.

In (D1), it is shown that, if a group G has the minimum condition on subgroups and all its characteristic subgroups have complements, then all its normal subgroups have complements (of course, the first condition always holds for finite groups). It is easily deduced from this that $\mathscr{C}_n$ is S_n-closed. A short alternative proof that $\mathscr{C}_n \cap E\mathfrak{A}$ is S_n-closed (which is all that is needed for the purposes of the present work) is now given.

<u>1.4.7</u> A soluble normal subgroup of a $\mathscr{C}_n$-group is itself in $\mathscr{C}_n$. In particular, $\mathscr{C}_n \cap E\mathfrak{A}$ is S_n-closed.

<u>Proof</u> Let $G \in \mathscr{C}_n$, and let H be a soluble normal subgroup of G. Let N be a minimal normal subgroup of G contained in H. Using induction on $|G|$, it can be assumed that $H/N \in \mathscr{C}_n$. Thus for each normal subgroup K of H, KN/N has a complement in H/N. Consider $K \cap N$: N is elementary abelian, so $K \cap N$ is an abelian normal subgroup of H. Hence, by III, 4.4 of (H8), $K \cap N$ will have a complement in H provided $K \cap N \cap \Phi(H) = 1$. But $\Phi(H) \leqslant \Phi(G)$, by (H8, III, 3.3(b)), and $\Phi(G) = 1$, because $G \in \mathscr{C}_n$; thus $K \cap N$ does have a complement in H.

Therefore, by 1.3.3(a), K has a complement in H . Hence $H \in \mathcal{C}_n$. Q.e.d.

With the help of the following result, which is an immediate consequence of an important theorem of Gaschütz on complements of abelian normal subgroups, a description of the S-interior of $\mathcal{C}_n$ is obtained in 1.4.9.

<u>1.4.8</u> Let p be a prime number and suppose that the group G has elementary abelian Sylow p-subgroups. Then every normal p-subgroup of G has a complement in G .

<u>Proof</u> Let N be a normal p-subgroup of G , and let G_p be a Sylow p-subgroup of G . G_p is elementary abelian, so N is abelian and has a complement in G_p ; further, $(|N|, |G:G_p|) = 1$. Therefore, by (H8, I, 17.4), N has a complement in G .

<u>1.4.9 Theorem</u> The following conditions are equivalent:

(1) $G \in \mathcal{C}_n{}^S$;

(2) the Sylow subgroups of G are all elementary abelian;

(3) every subgroup of G has trivial Frattini subgroup.

<u>Proof</u> That (1) implies (2) is clear, for if $G \in \mathcal{C}_n{}^S$, p is a prime number and G_p is a Sylow p-subgroup of G , then $G_p \in \mathcal{C}_n$, so $\Phi(G_p) = 1$ and hence G_p is elementary abelian.

(2) $\Rightarrow$ (3) : Let $\mathcal{L}$ denote the class of groups with elementary abelian Sylow subgroups. $\mathcal{L}$ is clearly S-closed,

so it will be enough to show that a group in $\mathcal{L}$ must have trivial Frattini subgroup. Let $G \in \mathcal{L}$, and for a contradiction suppose that $\Phi(G) > 1$. Let N be a minimal normal subgroup of G such that $N \leqslant \Phi(G)$. $\Phi(G)$ is nilpotent, so N is a p-group for some prime number p ; but then, by 1.4.8, N has a complement in G , which contradicts $N \leqslant \Phi(G)$.

(3) $\Rightarrow$ (1) : Suppose that (3) holds, and let H be a subgroup of G . Let N be a normal subgroup of H , and let S be a minimal supplement to N in H . It is well known that in this situation, $N \cap S \leqslant \Phi(S)$ (otherwise, if M is a maximal subgroup of S such that $N \cap S \nleqslant M$, then M is a supplement to N in H , contradicting the choice of S). But by hypothesis, $\Phi(S) = 1$, so $N \cap S = 1$ and S is a complement to N in H . Therefore $G \in \mathcal{C}_n^S$. Q.e.d.

1.4.10 gives an elementary proof, which does not rely on Gaschütz's sophisticated theorem, of the implication (2) $\Rightarrow$ (1) in 1.4.9, in the case of a <u>soluble</u> group.

<u>1.4.10</u> $\mathcal{L} \cap E\mathfrak{A} \subseteq \mathcal{C}_n^S$, where $\mathcal{L}$ is the class of groups with elementary abelian Sylow subgroups.

<u>Proof</u> Let $G \in \mathcal{L} \cap E\mathfrak{A}$. Since $\mathcal{L} \cap E\mathfrak{A}$ is S-closed, it will be enough to show $G \in \mathcal{C}_n$. Let N be a minimal normal subgroup of G , and let p be the prime of which $|N|$ is a power.

$G/N \in \mathfrak{L} \cap E\mathfrak{A}$, so, using induction on $|G|$, it can be assumed that $G/N \in \mathscr{C}_n$. Hence, by 1.4.3, it will be enough to show that N has a complement in G . If $N = C_G(N)$, then G is primitive soluble, and so N has a complement in G . Thus it can be assumed that $N < C_G(N)$. Let M/N be a chief factor of G with $M \leqslant C_G(N)$, and let q be the prime number of which $|M/N|$ is a power.

If $q \neq p$ it follows that, if M_q is a Sylow q-subgroup of M , then $M = M_qN$, $M_q \cap N = 1$ and $[M_q, N] = 1$. Thus M_q is characteristic in M and therefore normal in G . By induction, $G/M_q \in \mathscr{C}_n$, so (since $G/N \in \mathscr{C}_n$ also) $G \in R_0\mathscr{C}_n = \mathscr{C}_n$.

If $q = p$, then M is a p-group and so (by hypothesis) is elementary abelian; thus M can be regarded as a $Z_p[G/C_G(M)]$-module. $C_G(M)$ contains the Sylow p-subgroups of G , so $G/C_G(M)$ is a p'-group; hence, using Maschke's Theorem, $M = N \times L$, where L is normal in G . By induction, $G/L \in \mathscr{C}_n$, so, as before, $G \in R_0\mathscr{C}_n = \mathscr{C}_n$.

1.5 Pronormal subgroups

Well-known results about pronormal subgroups, which are used in the study (in Chapter 2) of groups in which all the pronormal subgroups have complements, are collected together in this section.

Definitions A subgroup L of a group G is said to be pronormal in G if for every $g \in G$, L and L^g are conjugate in their join $\langle L, L^g \rangle$. L is said to be abnormal in G if $g \in \langle L, L^g \rangle$ for all $g \in G$.

Perhaps the most obvious examples of pronormal subgroups of a group G are the Sylow subgroups of G, and also, if G is soluble, the Hall subgroups of G. Normal subgroups of G are clearly pronormal, and any maximal subgroup obviously must be either normal or abnormal in G, and hence is pronormal in G.

1.5.1 Let $H \leqslant G$. Then H is abnormal in G if and only if H is both pronormal and self-normalizing in G.

Proof Suppose H is abnormal in G; then H is certainly pronormal in G. Suppose $g \in N_G(H)$; then $\langle H, H^g \rangle = H$, so $g \in H$. Hence $N_G(H) = H$.

Now suppose H is both pronormal and self-normalizing in G. Let g be any element of G. Then there is an element x of $\langle H,H^g\rangle$ such that $H^x = H^g$, and hence $gx^{-1} \in N_G(H) = H$. Therefore $g \in Hx \leqslant \langle H,H^g\rangle$. Thus H is abnormal in G.

Several basic properties of pronormal subgroups are given in (R1): those which will be needed in Chapter 2 are recorded below, in 1.5.2 - 1.5.5.

<u>1.5.2</u> (R1, 1.3) If N is a normal subgroup of G and $N \leqslant L \leqslant G$, then L is pronormal in G if and only if L/N is pronormal in G/N.

<u>1.5.3</u> (R1, 1.4) If L is pronormal in G and N is normal in G, then LN is pronormal in G and

$$N_G(LN) = N_G(L)N .$$

<u>1.5.4</u> (R1, 1.5) If $H \leqslant G$, then H is both pronormal and subnormal in G if and only if H is normal in G.

<u>1.5.5</u> (R1, 1.6) If L is pronormal in G then $N_G(L)$ is abnormal in G.

<u>1.5.6</u> Suppose $G = HN$, where N is normal in G and $H \cap N = 1$. If L is a pronormal subgroup of G then $LN \cap H$ is pronormal in H.

<u>Proof</u> LN is pronormal in G by 1.5.3, so, by replacing L by LN, it can be assumed that $L \geqslant N$. Let $h \in H$. Then

$$\langle L,L^h\rangle = \langle (L\cap H)N, (L\cap H)^hN\rangle = \langle L\cap H, (L\cap H)^h\rangle N \quad \ldots(1)$$

As L is pronormal in G, there exists $x \in \langle L,L^h\rangle$ such that $L^h = L^x$. By (1), there exists $y \in \langle L \cap H, (L \cap H)^h \rangle$ such that $x \in yN \leqslant yL$. Thus $L^x = L^y$, and so

$$(L \cap H)^y = L^y \cap H = L^x \cap H = L^h \cap H = (L \cap H)^h .$$

Therefore $L \cap H$ is pronormal in H .

A fundamental fact about system normalizers (which are defined in (H4)) is used in a later proof (2.6.16), and is recorded here for convenience.

<u>1.5.7</u> Every abnormal subgroup of a soluble group G contains a system normalizer of G .

<u>Proof</u> The system normalizers of G are the minimal subabnormal subgroups of G (H8, VI, 11.21).

A rich source of pronormal subgroups of soluble groups arises from the theory of "formations", a brief summary of the elements of which is now given. A lucid exposition of the basic theory can be found in (H8, VI, Section 7).

A <u>formation</u> is a QR_0-closed class of groups; a <u>saturated formation</u> is a $\{Q, R_0, E_\Phi\}$-closed class of groups. A <u>formation function</u> f is a function which assigns to each prime number p either a formation $f(p)$ or the empty set $\emptyset$. Given a formation function f , a saturated formation $\mathfrak{F}$ can be defined by specifying that G belongs to $\mathfrak{F}$ if and only if, for each prime number p and each p-chief factor H/K of G , $G / C_G(H/K) \in f(p)$.
(If $f(p) = \emptyset$ this is interpreted to mean that p does

not divide $|G|$). $\mathfrak{F}$ is said to be <u>locally defined</u> by f . A result of fundamental importance is that every saturated formation of soluble groups has a local definition (H8, VI, 7.25).

If $\mathfrak{F}$ is a formation and G is a group, then the <u>$\mathfrak{F}$-residual</u> of G , denoted by $G^{\mathfrak{F}}$, is the intersection of all the normal subgroups N of G such that $G/N \in \mathfrak{F}$. An <u>$\mathfrak{F}$-projector</u> of G is a subgroup F of G such that $F \in \mathfrak{F}$ and, whenever $F \leqslant H \leqslant G$, then $FH^{\mathfrak{F}} = H$. The important property of $\mathfrak{F}$-projectors is that, if $\mathfrak{F}$ is a saturated formation then every soluble group has a unique conjugacy class of $\mathfrak{F}$-projectors . In the case of the saturated formation $\mathfrak{N}$, the $\mathfrak{N}$-projectors of a soluble group G coincide with the nilpotent self-normalizing subgroups of G discovered by R.W.Carter (and consequently known as "Carter subgroups").

The promised rich source of pronormal subgroups can now be revealed:

<u>1.5.8</u> Let $\mathfrak{F}$ be a saturated formation, and N a soluble normal subgroup of a group G . Then the $\mathfrak{F}$-projectors of N are pronormal in G .

<u>Proof</u> Let F be an $\mathfrak{F}$-projector of N , and let $g \in G$. It is easily checked that F^g is an $\mathfrak{F}$-projector of N , and hence that F and F^g are both $\mathfrak{F}$-projectors of $\langle F, F^g \rangle$. Thus F and F^g are conjugate in $\langle F, F^g \rangle$.

1.5.9 If G is a soluble group and $\mathfrak{F}$ is a saturated formation containing $\mathfrak{N}$, then the $\mathfrak{F}$-projectors of G are abnormal in G.

Proof Let F be an $\mathfrak{F}$-projector of G. By 1.5.8, F is pronormal in G, so by 1.5.1, it will be enough to show that F is self-normalizing in G. Let $H = N_G(F)$, suppose $H > F$ and let K be a maximal normal subgroup of H containing F. Then H/K is of prime order, and so $H/K \in \mathfrak{N} \subseteq \mathfrak{F}$. But then, by definition of an $\mathfrak{F}$-projector, $FK = H$, a contradiction. Therefore $F = N_G(F)$.

In determining whether a given subgroup of a group is pronormal, the following criterion (1.5.10) is often helpful; 1.5.11, which is deduced from it, is put to use in 2.6 and 2.7.

1.5.10 (Gaschütz (?)) Let H be a subgroup of G and suppose N is normal in G. Then H is pronormal in G if and only if

(1) HN is pronormal in G;

(2) H is pronormal in $N_G(HN)$.

Proof That H being pronormal implies (1) and (2) is obvious. Conversely, suppose that (1) and (2) hold, and let $g \in G$. Let $J = \langle H, H^g \rangle$; then by (1), there exists $x \in \langle HN, (HN)^g \rangle = JN$ such that $(HN)^g = (HN)^x$. Writing $x = ny$ with $n \in N$, $y \in J$, then $(HN)^g = (HN)^y$, and so

$gy^{-1} \in N_G(HN)$. Thus, by (2), there exists $z \in \langle H, H^{gy^{-1}} \rangle \leqslant J$ such that $H^{gy^{-1}} = H^z$; then $zy \in J$ and $H^g = H^{zy}$. Therefore H is pronormal in G .

<u>1.5.11</u> Let $G = HV$, where V is normal in G , and $H \cap V = 1$. Let L be a pronormal subgroup of H , and suppose that $W \leqslant V$ is such that

(1) V and $N_H(L)$ both normalize W , and

(2) $[V,L] \leqslant W$.

Then LW is pronormal in G .

<u>Proof</u> (i) <u>LV is pronormal in G</u>

To prove this, suppose $g \in G$; then $g = vh$ for some $v \in V$ and $h \in H$, so $(LV)^g = (LV)^h$. As L is pronormal in H , there exists $x \in \langle L, L^h \rangle \leqslant \langle LV, (LV)^h \rangle$ such that $L^h = L^x$, and hence $(LV)^h = (LV)^x$.

(ii) <u>LW is pronormal in $N_G(LV)$</u> .

In fact LW is normal in $N_G(LV)$; for, let $g \in N_G(LV)$. Then $g = hv$, with $h \in H$, $v \in V$, from which it follows that $h \in N_H(LV \cap H) = N_H(L)$. Thus

$$\begin{aligned} (LW)^g &= L^{hv}W^{hv} \\ &= L^v W && \text{(by (1))} \\ &\leqslant L[V,L]W \\ &= LW && \text{(by (2))} \end{aligned}$$

Applying 1.5.10, (i) and (ii) imply that LW is pronormal in G .

<u>1.5.12</u> Let L be a pronormal subgroup of G and let N be an abelian normal subgroup of G. Then $N_G(LN)$ normalizes $L \cap N$.

<u>Proof</u> Let $g \in N_G(LN) = N_G(L)N$ (using 1.5.3); then $g = xn$ with $x \in N_G(L)$ and $n \in N$, and therefore

$$(L \cap N)^g = L^{xn} \cap N = L^n \cap N$$
$$= (L \cap N)^n = L \cap N \quad \text{(as } N \text{ is abelian)}.$$

CHAPTER 2

2.1 Metabelian groups with complemented pronormal subgroups

Let $\mathcal{C}_p$ denote the class of finite groups in which every pronormal subgroup has a complement. Some important observations about $\mathcal{C}_p$ can be made immediately:

2.1.1 $\mathcal{C}_p \subseteq \mathcal{C}_n \cap E\mathfrak{A}$.

Proof The Sylow subgroups of a group G are all pronormal; but if the Sylow subgroups of G all have complements in G, then G is soluble. Hence $\mathcal{C}_p$-groups are soluble. Also the normal subgroups of a group are all pronormal, so $\mathcal{C}_p \subseteq \mathcal{C}_n$.

2.1.2 $\mathcal{C}_p$ is Q-closed.

Proof Let $G \in \mathcal{C}_p$ and let N be a normal subgroup of G. If H/N is a pronormal subgroup of G/N, then by 1.5.2, H is pronormal in G, so H has a complement in G and hence, by 1.3.2(b), H/N has a complement in G/N. Therefore $G/N \in \mathcal{C}_p$.

Unlike $\mathcal{C}_n$, $\mathcal{C}_p$ is neither S_n-closed nor R_0-closed (in fact $\mathcal{C}_p$ is not even D_0-closed); this is shown in Examples 2.7.5 and 2.7.6. The remainder of 2.1 is devoted to showing that when attention is confined to metabelian groups, $\mathcal{C}_p$ coincides with $\mathcal{C}_n$.

<u>2.1.3</u> $\mathcal{C}_n \cap \mathfrak{A}^2$ is S-closed.

<u>Proof</u> Suppose that the result is false, and let G be a group of minimal order such that, $G \in \mathcal{C}_n \cap \mathfrak{A}^2$ and G possesses a subgroup H not in $\mathcal{C}_n$. Then H is not contained in any proper normal subgroup of G , because $\mathcal{C}_n \cap E\mathfrak{A}$ is S_n-closed (1.4.7). Since HG' is normal in G, it follows that $HG' = G$. Now G is metabelian, so G' is abelian; hence $H \cap G'$ is normal in both H and G' , and therefore in G .

If $H \cap G' = 1$, then

$H \cong H/(H \cap G') \cong HG'/G' = G/G' \in Q\mathcal{C}_n = \mathcal{C}_n$,

i.e. $H \in \mathcal{C}_n$, a contradiction. Hence $H \cap G' > 1$, and therefore $H \cap G'$ contains a minimal normal subgroup, N say, of G . Now any normal subgroup of H contained in $H \cap G'$ is normal in G (because $HG' = G$ and $G' \in \mathfrak{A}$); consequently, N is also a minimal normal subgroup of H . $|G/N| < |G|$, so, because of the way in which G was chosen, $H/N \in \mathcal{C}_n$. But N has a complement in G , because $G \in \mathcal{C}_n$, so N has a complement in H (by 1.3.2(a)). Therefore, by 1.4.3, $H \in \mathcal{C}_n$, a contradiction. Therefore a group such as G cannot exist, and thus the theorem is proved.

2.1.3 and 1.4.9 together yield the following immediate corollary:

<u>2.1.4 Corollary</u> (Cf. (C3), Theorem 5.4)

$$\mathscr{C}_n \cap \mathfrak{A}^2 = \mathscr{L} \cap \mathfrak{A}^2 ,$$

where $\mathscr{L}$ is the class of groups in which all the Sylow subgroups are elementary abelian.

<u>2.1.5</u> Suppose L is pronormal in G, and $L \leqslant N$, where N is a metabelian normal subgroup of G. Let M be a normal subgroup of G contained in N such that both M and N/M are abelian. Then LM and $L \cap M$ are both normal in G.

<u>Proof</u> $LM/M \leqslant N/M$, and N/M is an abelian normal subgroup of G/M, so LM/M is subnormal in G/M. But LM/M is pronormal in G/M, by 1.5.3 and 1.5.2, so by 1.5.4, LM/M is normal in G/M; thus LM is normal in G.

By 1.5.12, $N_G(LM)$ normalizes $L \cap M$; hence $L \cap M$ is normal in G.

<u>2.1.6 Theorem</u> If $G \in \mathscr{C}_n$, then every pronormal subgroup of G which is contained in a metabelian normal subgroup of G has a complement in G.

<u>Proof</u> Let L be a pronormal subgroup of G, where $G \in \mathscr{C}_n$, and suppose $L \leqslant N$, where N is a metabelian normal subgroup of G; let $M = N'$. Then by 2.1.5, $L \cap M$ and LM are both normal in G. Since G and G/M both belong to $\mathscr{C}_n$, it follows that LM/M has a complement in G/M and $L \cap M$ has a complement in G. Therefore, by 1.3.3(a), L has a complement in G.

Together, 2.1.4 and 2.1.6 yield the following characterisation of metabelian $\mathcal{C}_p$-groups:

<u>2.1.7 Corollary</u> $\mathcal{C}_p \cap \mathfrak{A}^2 = \mathcal{C}_n \cap \mathfrak{A}^2 = \mathcal{L} \cap \mathfrak{A}^2$,
i.e. , if G is metabelian then $G \in \mathcal{C}_p$ if and only if the Sylow subgroups of G are elementary abelian.

2.2 Further properties of groups in $\mathscr{C}_n \cap \mathfrak{A}^2$.

Several lemmas, which will be used later in investigating $\mathscr{C}_p \cap \mathfrak{A}^3$, are collected together here. The main lemmas, 2.2.4 and 2.2.5, are rather "technical", and the reader might prefer to omit this section, returning to it only when it becomes necessary.

2.2.1 Let $H \in \mathscr{C}_n \cap \mathfrak{A}^2$, let $F = F(H)$, and let $F_0 \leqslant F$ be a normal subgroup of H. Then for any subgroup S of H,

(a) $[F_0, S]$ and $C_{F_0}(S)$ are both normal subgroups of H;

(b) $F_0 = [F_0, S] \times C_{F_0}(S)$.

(Hence $[F_0, S, S] = [F_0, S]$ and $C_{[F_0,S]}(S) = 1$).

Proof Let B be a complement to F in H, and let $B_0 = SF \cap B$. Then, as F is abelian,

$$[F_0, S] = [F_0, SF] = [F_0, SF \cap B] = [F_0, B_0].$$

Suppose now that $a_0 \in C_{F_0}(S)$, and let $b \in B_0$. Then $b = sa$ for some $s \in S$, $a \in F$, and so

$$[b, a_0] = [sa, a_0] = [s, a_0]^a [a, a_0] = 1.$$

Therefore $C_{F_0}(S) \subseteq C_{F_0}(B_0)$.

Conversely, suppose $a_0 \in C_{F_0}(B_0)$, and let $s \in S$. Then $s \in SF = B_0F$, so $s = b_0a$ for some $b_0 \in B_0$ and $a \in F$. Hence

$$[s, a_0] = [b_0a, a_0] = [b_0, a_0]^a [a, a_0] = 1.$$

Therefore $C_{F_0}(B_0) \subseteq C_{F_0}(S)$.

Thus $[F_o, S] = [F_o, B_o]$ and $C_{F_o}(S) = C_{F_o}(B_o)$, so it can be assumed that $S = B_o$.

(a) Given any $h \in H$, $h = ba$ for some $b \in B$ and $a \in F$, and so

$$[F_o, B_o]^h = [F_o{}^b, B_o{}^b]^a$$
$$= [F_o, B_o]^a \quad (B_o \text{ is normal in } B \text{ as } B \text{ is abelian})$$
$$= [F_o, B_o] \quad (\text{as } F \text{ is abelian}).$$

Hence $[F_o, B_o]$ is normal in H .

Let $a_o \in C_{F_o}(B_o)$ and let $b \in B$; then for any $b_o \in B_o$,

$$[a_o{}^b, b_o] = [a_o, b_o]^b \quad (\text{as } B \text{ is abelian})$$
$$= 1 .$$

Hence $a_o{}^b \in C_{F_o}(B_o)$. Therefore B normalizes $C_{F_o}(B_o)$, and so (since F is abelian) $C_{F_o}(B_o)$ is normal in H .

(b) Let $H_o = B_oF_o$. Then $H_o \in \mathcal{E}_n \cap \mathfrak{A}^2$, because $\mathcal{E}_n \cap \mathfrak{A}^2$ is S-closed (by 2.1.3), so in particular, $\Phi(H_o) = 1$. Now $F_o \leqslant F(H_o)$ and $[F_o, B_o]$ is normal in H_o , so there is a normal subgroup N of H_o such that $F_o = [F_o, B_o] \times N$. Then $[N, B_o] \leqslant N \cap [F_o, B_o] = 1$, so $N \leqslant C_{F_o}(B_o)$. On the other hand,

$$[F_o, B_o] \cap C_{F_o}(B_o) \leqslant H_o{}' \cap Z(H_o) \leqslant \Phi(H_o) = 1 ,$$

so $N = C_{F_o}(B_o)$.

2.2.2 Let G be a group and N a normal subgroup of G, and suppose B is a complement to N in G. If $N_0 \leqslant N$ is normal in G and $B_0 \leqslant B$ is normal in B then B_0N_0 is normal in G if and only if $[N,B_0] \leqslant N_0$.

Proof If B_0N_0 is normal in G then

$$[N,B_0] \leqslant N \cap B_0N_0 = N_0$$

Now suppose $[N,B_0] \leqslant N_0$, and let $g \in G$. Then $g = bn$ for some $b \in B$, $n \in N$, and so

$$(B_0N_0)^g = B_0{}^nN_0 \leqslant B_0[N,B_0]N_0 = B_0N_0 .$$

2.2.3 Suppose that $F(G)$ is abelian, and N is an abelian normal subgroup of G. If N has a complement B in G and $C_B(N) = 1$, then $N = F(G)$.

Proof $B \cap F(G)$ centralizes N, so $B \cap F(G) = 1$.

2.2.4 Let $H \in \mathscr{C}_n \cap \mathfrak{A}^2$, let $A = H'$ and $Z = Z(H)$. Suppose that $S > 1$ is a subgroup of H such that $S \cap AZ = 1$, and let Z_0 be a subgroup of Z. Let $A_0 = [A,S]$ (thus $A_0 > 1$), and let $H_0 = SA_0 \times Z_0$. If N is a normal subgroup of H_0, but N contains no non-trivial normal subgroup of H contained in A, then

(a) $N \leqslant F(H_0) = A_0 \times Z_0$;

(b) $F(H_0/N) = F(H_0)/N$.

Proof (a) $H_0' = [A_0,S] = A_0$ and $C_{A_0}(S) = 1$.

$C_S(A_0Z_0) = C_S(A_0) = S_0$ (say).

Then $[A,S_o] \leqslant [A,S] = A_o \leqslant C_A(S_o)$.
But by 2.2.1(b), $[A,S_o] \cap C_A(S_o) = 1$, so $[A,S_o] = 1$, i.e. $S_o \leqslant C_H(A) = AZ$. Therefore $S_o \leqslant S \cap AZ = 1$, so $C_S(A_oZ_o) = 1$, and hence, by 2.2.3, $\underline{A_oZ_o = F(H_o)}$.

Since N is normal in H_o, $[A_o, N] \leqslant N$. But $[A_o,N]$ is normal in H, by 2.2.1(a), and by hypothesis, N contains no non-trivial normal subgroups of H which are contained in A. Therefore $[A_o, N] = 1$, and so

$$N \leqslant C_{H_o}(A_o) = A_oZ_o = F(H_o) .$$

(b) Let $A_o = M_1 \times \ldots \times M_t$, where the M_i are minimal normal subgroups of H. By hypothesis, $N \cap M_i < M_i$, so for each i there is a minimal normal subgroup N_i of H_o such that

$$N_i \leqslant M_i \quad \text{and} \quad N_i \cap N = 1 .$$

Therefore H_o has a factor group H_o/N_o such that $N \leqslant N_o$ and, assuming without loss of generality that each H_o-isomorphism class of groups in $\{N_1, \ldots, N_t\}$ has exactly one representative amongst $N_1, \ldots, N_r$,

$$A_oZ_o/N_o \underset{H_o}{\cong} N_1 \times \ldots \times N_r.$$

Now for a fixed i in $\{1,..,t\}$, let M_i be a p_i-group, let $\bar{H} = H/C_H(M_i)$, and consider M_i as an irreducible $Z_{p_i}[\bar{H}]$-module. Since $H' = A \leqslant C_H(M_i)$, $\bar{H}$ is abelian and hence, as a $Z_{p_i}[\bar{S}]$-module (using the "bar convention"), M_i is homogeneous (i.e the irreducible $Z_{p_i}[\bar{S}]$-submodules of M_i are all isomorphic).

Therefore $C_S(N_i) = C_S(M_i)$, and so

$$C_S(N_1 \times \ldots \times N_t) = \bigcap_{i=1}^{t} C_S(N_i) = \bigcap_{i=1}^{t} C_S(M_i)$$

$$= C_S(M_1 \times \ldots \times M_t) = C_S(A_o) = 1 .$$

Applying 2.2.3 to H_o/N_o, it follows that

$F(H_o/N_o) = A_oZ_o/N_o$, and therefore, as $N \leqslant N_o$,

$$F(H_o/N) = A_oZ_o/N = F(H_o)/N .$$

<u>2.2.5</u> Let $H \in \mathcal{C}_n \cap \mathfrak{A}^2$, let $A = H'$ and $Z = Z(H)$ as in 2.2.4, and let B be a complement to $F(H) = A \times Z$ in H ; thus $H = BA \times Z$. Let B_o be a subgroup of B and let C be a complement to B_o in H . Then:

(a) $[A, C \cap BZ] \leqslant C'$.

(b) If $z \in Z$ then there exists $c \in C$ and $b \in B_o$ such that $|b|$ divides $|z|$ and $c = bz$. It then follows that $\langle c \rangle = \langle bz_1 \rangle \times \langle z_2 \rangle$, where $\langle z \rangle = \langle z_1 \rangle \times \langle z_2 \rangle$ and $(|b|, |z_2|) = 1$, $(|b|, |z_1|) = |z_1|$.

<u>Proof</u> (a) Take any $a \in A$ and any $c \in C \cap BZ$. Since $H = B_oC$, there exist $b_o \in B_o$ and $c_o \in C$ such that $a = b_o^{-1}c_o$. Then

$$[c_o, c] = [b_oa, c] = [b_o, c]^a[a, c]$$
$$= [a, c] \qquad (c \in BZ \text{ so } [b_o, c] = 1)$$

Hence $[a, c] = [c_o, c] \in C'$, and therefore

$$[A, C \cap BZ] \leqslant C' .$$

(b) Let $z \in Z$. Then $z = b_o^{-1}c_o$ for some $b_o \in B_o$, $c_o \in C$; i.e. $c_o = b_oz$. Let π denote the set of prime divisors of $|z|$, and write $b_o = b_\pi b_{\pi'}$, where b_π is a

π-element and $b_{\pi'}$ is a π'-element. Then

$$\langle b_o z\rangle = \langle b_{\pi'} b_\pi z\rangle = \langle b_{\pi'}\rangle \times \langle b_\pi z\rangle .$$

Let $m = |b_{\pi'}|$; then $(b_o z)^m = b_\pi{}^m z^m$, i.e. $c_o{}^m = b_\pi{}^m z^m$. Now $(m,|z|) = 1$, so there is an integer n such that $mn \equiv 1 \pmod{|z|}$. Thus

$$c_o{}^{mn} = b_\pi{}^{mn} z .$$

Therefore, taking $c = c_o{}^{mn}$ and $b = b_\pi{}^{mn}$, the result is established.

2.3 The invariant $d_F(A)$

Throughout 2.3, let A denote an abelian group and let $F = GF(p^f)$, where p is a prime number and f is a natural number.

Definition The number $d_F(A)$ is defined by:

$$d_F(A) = \max\{\dim_F V : V \text{ an irreducible } F[A]\text{-module}\} .$$

The nature of the invariant $d_F(A)$ can be elucidated by making use of the fundamental result (H8, II, 3.10) that if A has a faithful irreducible representation over F, then A is cyclic, and the degree of the representation is determined as the smallest natural number n such that

$$p^{fn} \equiv 1 \pmod{|A|} .$$

(The essence of this theorem is that the only kind of situation in which an abelian group can have a faithful irreducible representation over $GF(p^f)$ is one in which a subgroup of the multiplicative group of $GF(p^{fn})$ acts (by multiplication) on the additive group of $GF(p^{fn})$, the latter being regarded as a vector space over $GF(p^f)$).

2.3.1 Suppose $q_1, q_2, \ldots, q_s$ are all the prime divisors of $|A_{p'}|$ (where $A_{p'}$ is the p-complement of A), and $q_i^{e_i}$ is the exponent of A_{q_i} (the Sylow q_i-subgroup of A), for each $i \in \{1,\ldots,s\}$. Let d be the smallest natural number such that $p^{fd} \equiv 1 \pmod{q_1^{e_1} q_2^{e_2} \ldots q_s^{e_s}}$

Then $d = d_F(A)$.

Proof A has a cyclic factor group of order $q_1^{e_1}\dots q_s^{e_s}$, and hence has an irreducible representation over F of degree d (corresponding to a faithful irreducible representation of the factor group). Hence $d_F(A) \geqslant d$.

Let V be an irreducible $F[A]$-module, and let $C = C_A(V)$. Then V is a faithful irreducible $F[A/C]$-module, so A/C is cyclic and $\dim_F V$ is the smallest natural number n such that $p^{fn} \equiv 1 \pmod{|A/C|}$. But $|A/C|$ must be a divisor of $q_1^{e_1}q_2^{e_2}\dots q_s^{e_s}$ (as A/C is a cyclic p'-group), so $p^{fd} \equiv 1 \pmod{|A/C|}$. Therefore $n \mid d$. Hence $d_F(A) = d$.

Corollary The F-dimension of any irreducible $F[A]$-module is a divisor of $d_F(A)$.

2.3.2 Let $d = d_F(A)$, and let $E = GF(p^{fd})$. Then E is a splitting field for A.

Proof Let V be an irreducible $E[A]$-module, and let $\dim_E V = n$. Let $C = C_A(V)$. Then $p^{fdn} \equiv 1 \pmod{|A/C|}$, and this congruence holds (given f and d) for no natural number smaller than n. But 2.3.1 shows that $p^{fd} \equiv 1 \pmod{|A/A_0|}$ for any cyclic factor group A/A_0 of A. Hence $n = 1$, i.e. the E-dimension of any irreducible $E[A]$-module is 1. Q.e.d.

Here and later lcm(m,n) denotes the lowest common multiple of m and n.

<u>2.3.3</u> (a) If $A_o \leqslant A$ then $d_F(A_o) \mid d_F(A)$.

(b) If A_1 , A_2 are subgroups of A and $A = A_1A_2$, then

$$d_F(A) = \mathrm{lcm}(d_F(A_1), d_F(A_2)) .$$

Hence, if $d_F(A_2) = 1$ then $d_F(A) = d_F(A_1)$.

<u>Proof</u> (a) This is immediate from 2.3.1 : let $d = d_F(A)$ and $d_o = d_F(A_o)$. Then d is the smallest natural number such that $p^{fd} \equiv 1 \pmod{q_1^{e_1} \ldots q_s^{e_s}}$ (with notation as in 2.3.1), and d_o is the smallest natural number such that $p^{fd_o} \equiv 1 \pmod{q_1^{f_1} \ldots q_s^{f_s}}$, where $q_i^{f_i}$ is the exponent of the Sylow q_i-subgroup of A_o . Since $f_i \leqslant e_i$ for each i , it follows that $p^{fd} \equiv 1 \pmod{q_1^{f_1} \ldots q_s^{f_s}}$, and hence $d_o \mid d$.

(b) Let $d_i = d_F(A_i)$, $(i = 1,2)$, and let $d' = \mathrm{lcm}(d_1, d_2)$. By (a), both d_1 and d_2 divide d , so $d' \mid d$.

Now for each $i \in \{1, \ldots, s\}$, either $p^{fd_1} \equiv 1 \pmod{q_i^{e_i}}$ or $p^{fd_2} \equiv 1 \pmod{q_i^{e_i}}$, as $e_i = \max\{e_{i1}, e_{i2}\}$, where $q_i^{e_{ij}}$ denotes the exponent of the Sylow q_i-subgroup of A_j . Therefore $p^{fd'} \equiv 1 \pmod{q_i^{e_i}}$ for each i , and thus $p^{fd'} \equiv 1 \pmod{q_1^{e_1} \ldots q_s^{e_s}}$, which implies that $d \mid d'$. Therefore $d = d'$.

<u>2.3.4 Theorem</u> Let G be a group, let N be an abelian normal subgroup of G, and let $F = GF(p^f)$. If V is a faithful irreducible $F[G]$-module, then the dimension of the irreducible $F[N]$-submodules of V is precisely $d_F(N)$.

<u>Proof</u> Let W be an irreducible $F[N]$-submodule of V, and let $C = C_N(W)$. Then by Clifford's Theorem,

$$V = \sum_{g\in G} Wg \ ,$$

and $C_N(Wg) = C^g$ for each $g \in G$. Thus $C_N(V) = \bigcap_{g\in G} C^g$. Since V is faithful, it follows that C contains no non-trivial normal subgroup of G.

$\dim_F W$ is determined as the smallest natural number d such that $p^{fd} \equiv 1 \pmod{|N/C|}$, so, by 2.3.1, to establish the result it will be enough to show that $|N/C| = q_1^{e_1}\cdots q_s^{e_s}$, where $q_1,\ldots,q_s$ are the distinct prime numbers different from p, dividing $|N|$, and $q_i^{e_i}$ is the exponent of the Sylow q_i-subgroup of N.

Let N_i denote the Sylow q_i-subgroup of N and let

$$\mho_j(N_i) = \left\{x^{(q_i^{\,j})} : x \in N_i\right\} .$$

Then for each $j = 1,\ldots,e_i-1$, $\mho_j(N_i)$ is a characteristic subgroup of N_i, and therefore a normal subgroup of G. In particular, it follows that either $e_i = 1$ ~~$\mho_{e_i-1}(N_i) = 1$~~ or $\mho_{e_i-1}(N_i) \nleq C$. In the ~~The~~ first case, ~~can only arise if~~ N_i is of exponent q_i; then certainly $N_i \nleq C$, so there is an element x of N_i of order q_i such that $x \notin C$; thus $\langle x\rangle \cap C = 1$. In the second case, there is an element x

of N_i such that $x^{\left(q_i^{e_i-1}\right)} \notin C$; then x has order $q_i^{e_i}$ and $\langle x\rangle \cap C = 1$. Thus in both cases, N/C has an element of order $q_i^{e_i}$. The result now follows.

2.4 Metabelian groups with faithful irreducible representations.

The results obtained in this section are needed in 2.6 to construct some useful pronormal subgroups of the group considered there. Two more theorems from (H8), namely $\underline{V}$, 13.2 and $\underline{V}$, 13.3 , are used in the proofs.

Throughout 2.4, H denotes a metabelian group, N is a normal subgroup of H such that both N and H/N are abelian, $F = GF(p^f)$, and V is a faithful irreducible $F[H]$-module.

2.4.1 Suppose F is a splitting field for N . Then:
(a) the homogeneous components of V_N all have stabilizer $C_H(N)$;
(b) if $N = C_H(N)$ then $\dim_F V = |H:N|$.

Proof (a) Let $V_N = V_1 \oplus \ldots \oplus V_t$ be the decomposition of V_N into homogeneous components, and let S be the stabilizer of V_1 . Then $N \leqslant S$, and hence S is normal in H (as H/N is abelian). Since the stabilizers of $V_2,\ldots,V_t$ are all conjugates of S , it follows that all the homogeneous components have stabilizer S .

Let $C_i = C_S(V_i)$ $(i = 1,\ldots,t)$. Since F is a splitting field for N , and N is abelian, N acts on each V_i as a group of scalar matrices, so

$$NC_i/C_i \leqslant Z(S/C_i) \quad \text{for each } i ,$$

i.e. $[N,S] \leqslant C_i$. But V is a faithful $F[H]$-module, so $C_1 \cap C_2 \cap \ldots \cap C_t = 1$, and therefore $[N,S] = 1$, i.e. $S \leqslant C_H(N)$. Since it is always true that $S \geqslant C_H(N)$, it follows that $S = C_H(N)$.

(b) If $N = C_H(N)$ then by (a) and Clifford's Theorem, each V_i is an irreducible $F[N]$-module, and therefore V_i has dimension 1 (since N is abelian and F is a splitting field for N). Hence $\dim_F V = |H:S| \dim_F V_1 = |H:N|$.

2.4.2 Suppose that F is a splitting field for N, ~~$N = C_H(N)$,~~ and N has a complement B in H. Let $N_o \leqslant N$ be a normal subgroup of H and let $B_o \leqslant B$. Then the dimension of an irreducible $F[B_oN_o]$-submodule U of V is at least $|B_o : C_{B_o}(N_o)|$; if $C_{B_o}(N_o) = 1$ then

$$\dim_F U = |B_o| .$$

Proof Let W be an irreducible $F[N_o]$-submodule of V. Then

$$b \in B , \quad W \underset{F[N_o]}{\cong} Wb \iff b \in C_B(N_o) \qquad \ldots\ldots\ldots(*)$$

The implication from right to left is obvious. For the converse, suppose that $b \in B$, and that there is an $F[N_o]$-isomorphism $\phi : W \to Wb$. F is a splitting field for N, and therefore for N_o, so W is one-dimensional : let $W = \langle w \rangle$, and let $\xi \in F$ be such that $w\phi = \xi(wb)$. Then for any $n_o \in N_o$,

ϕ is an $F[N_0]$-isomorphism

$\Rightarrow$ $(w\phi)n_0 = (wn_0)\phi$

$\Rightarrow$ $(\xi(wb))n_0 = \xi((wn_0)b)$

$\Rightarrow$ $\xi(wbn_0) = \xi(wn_0b)$

$\Rightarrow$ $bn_0b^{-1}n_0^{-1} \in C_{N_0}(W)$.

Therefore $[b,N_0] \leqslant C_{N_0}(W)$. Since $[b,N_0]$ is a normal subgroup of H , it follows, by Clifford's Theorem, that $[b,N_0] \leqslant C_{N_0}(V)$. But V is a faithful module, so $[b,N_0] = 1$, i.e. $b \in C_B(N_0)$. This establishes (*) .

Now consider U , an irreducible $F[B_0N_0]$-submodule of V , and let W be an irreducible $F[N_0]$-submodule of U . Let $r = |B_0:C_{B_0}(N_0)|$, and let $\{1,b_2,\ldots,b_r\}$ be a set of coset representatives of $C_{B_0}(N_0)$ in B_0 . Then by (*), W, Wb_2, ... , Wb_r are pairwise inequivalent irreducible $F[N_0]$-submodules of U , so

$$U \geqslant W \oplus Wb_2 \oplus \ldots \oplus Wb_r \ .$$

Hence $\dim_F U \geqslant r = |B_0:C_{B_0}(N_0)|$.

If $C_{B_0}(N_0) = 1$, then $B_0 = \{1,b_2,\ldots,b_r\}$, so $W \oplus Wb_2 \oplus \ldots \oplus Wb_r$ is an $F[B_0N_0]$-submodule of U , and therefore must be the whole of U . Hence $\dim_F U = |B_0|$.

<u>2.4.3</u> If $N = C_H(N)$ then $\dim_F V$ is divisible by $\mathrm{lcm}(|H:N|, d_F(N))$.

<u>Proof</u> It follows from 2.3.4 that the dimension of the irreducible $F[N]$-submodules of V is $d_F(N)$, so

$d_F(N) \mid \dim_F V$. It remains to show that $|H:N|$ divides $\dim_F V$.

Let $d = d_F(N)$ and let $E = GF(p^{fd})$; then E is a Galois extension of F and also a splitting field for N (by 2.3.2). Let $V^* = E \otimes_F V$; then by (H8, $\underline{V}$, 13.3), V^* decomposes into a direct sum of irreducible "algebraically conjugate" $E[H]$-modules, $V_1^*, V_2^*, \ldots, V_n^*$ say. Each V_i^* is also a faithful $E[H]$-module, since V^* is faithful, so by 2.4.1(b), $\dim_E V_i^* = |H:N|$. Therefore $\dim_E V^*$ is divisible by $|H:N|$, and so, since $\dim_E V^* = \dim_F V$, the proof is complete.

<u>2.4.4</u> Suppose that $N = C_H(N)$ and N has a complement B in H . Then:

(a) $C_V(B) > 0$ and $[V,B] < V$:

(b) if $N_o \leqslant N$ is normal in H and $B_o = C_B(N_o)$, then V has an $F[B_oN_o]$-submodule W of codimension $d_F(N_o)$, such that B_o centralizes V/W .

<u>Proof</u> (a) As in the proof of 2.4.3, let $d = d_F(N)$, let $E = GF(p^{fd})$, and consider $V^* = E \otimes_F V$. Then, as before, $V^* = V_1^* \oplus \ldots \oplus V_n^*$, where each V_i^* is a faithful irreducible $E[H]$-module; consider V_1^* . The situation is as in 2.4.2 (with E , V_1^* , N , B playing the roles of F , V , N_o , B_o respectively), and since $C_B(N) = 1$, it can be seen that, as in the last part of the proof of 2.4.2, if

W_1^* is an irreducible $E[N]$-submodule of V_1^* and $B = \{1, b_2, \ldots, b_r\}$, then $W_1^*, W_1^*b_2, \ldots, W_1^*b_r$ are pairwise inequivalent $E[N]$-modules and

$$V_1^* = W_1^* \oplus W_1^*b_2 \oplus \ldots \oplus W_1^*b_r .$$

It now follows that if $w_1 \in W_1^* \setminus 0$, then $w_1 + w_1b_2 + \ldots + w_1b_r \neq 0$, and so $C_{V_1^*}(B) > 0$.

Also $W^* = \left\{ \sum_{i=1}^{r} \xi_i (w_1 b_i) : \xi_i \in E, \ \sum_{i=1}^{r} \xi_i = 0 \right\}$ is a proper $E[B]$-submodule of V_1^*, and B centralizes V_1^*/W^*. Therefore $C_{V^*}(B) > 0$ and $[V^*, B] < V^*$.

Let G be the Galois group of E over F; then V^* is a G-module, with the action described in (H8, $\overline{\underline{V}}$, 13.2). Straightforward calculations show that $C_{V^*}(B)$ and $[V^*, B]$ are G-submodules of V^*; therefore, by (H8, $\overline{\underline{V}}$, 13.2), there are $F[B]$-submodules U_1 and U_2 of V such that $C_{V^*}(B) = E \otimes_F U_1$, $[V^*, B] = E \otimes_F U_2$. Now

$$\begin{aligned} u \in U_1 &\iff 1 \otimes u \in C_{V^*}(B) \\ &\iff \forall b \in B, \ 1 \otimes ub = 1 \otimes u \\ &\iff \forall b \in B, \ ub = u \\ &\iff u \in C_V(B) . \end{aligned}$$

Therefore $C_V(B) = U_1 > 0$.

Also, given $v \in V$ and $b \in B$, $1 \otimes [v,b] \in [V^*, B]$, and therefore $[v,b] \in U_2$. Hence $[V,B] \leqslant U_2 < V$.

(b) Let $d_o = d_F(N_o)$, $E_o = GF(p^{fd_o})$, and $V^* = E_o \otimes_F V$. In (a) it was shown that $[V,B] < V$; hence

$[V^*, B] < V^*$, and therefore $[V^*, B_o] < V^*$. Because N_o centralizes B_o , $[V^*, B_o]$ is an $E_o[N_o]$-submodule of V^* , and therefore $\overline{V} = V^*/[V^*, B_o]$ is an $E_o[B_oN_o]$-module. E_o is a splitting field for N_o (by 2.3.2), so $\overline{V}$ is a direct sum of one-dimensional $E_o[N_o]$-submodules. In particular, $\overline{V}$ has an $E_o[N_o]$-submodule $\overline{W}_o$ of codimension 1 , $\overline{W}_o = W_o^*/[V^*, B_o]$, say. Since B_o acts trivially on $\overline{V}$, W_o^* is in fact an $E_o[B_oN_o]$-submodule of V^* of codimension 1 , such that B_o acts trivially on V^*/W_o^* .

Let G_o be the Galois group of E_o over F , and, with the action of G_o on V^* as described in (H8, $\underline{V}$, 13.2), let $W^* = \bigcap_{g \in G_o} W_o^* g$. $[V^*, B_o]$ is a G_o-submodule of V^* , so $W^* \geqslant [V^*, B_o]$, and hence B_o centralizes V^*/W^* . Also W^* is both an $E_o[B_oN_o]$-submodule, and a G_o-submodule, of V^* , and therefore (by H8, $\underline{V}$, 13.2), V has an $F[B_oN_o]$-submodule W such that $W^* = E_o \otimes_F W$.

Now G_o has order d_o , so

$$\operatorname{codim}_{E_o} W^* \leqslant |G_o| \operatorname{codim}_{E_o} W_o^* = d_o .$$

Hence $\operatorname{codim}_F W \leqslant d_o$. On the other hand, W is a proper $F[N_o]$-submodule of V , so $\operatorname{codim}_F W \geqslant d_F(N_o) = d_o$. Therefore $\operatorname{codim}_F W = d_o$.

Since $W^* \geqslant [V^*, B_o]$, and $W^* = E_o \otimes_F W$, it follows, as in the proof of (a), that $W \geqslant [V, B_o]$, and therefore B_o centralizes V/W .

2.5 A representation theorem

As a last preliminary to the study of $\mathcal{C}_p \cap \mathfrak{N}^3$, another result from representation theory is developed. The result (2.5.2) is based on a theorem in (H8), from the statement and proof of which the following information is collected:

2.5.1 (H8, II, 3.11) Let G be a group, let $F = GF(p^f)$, and suppose that V is a faithful $F[G]$-module. Let N be an abelian normal subgroup of G, and suppose that, as an $F[N]$-module, V is homogeneous. Let k be the dimension of the irreducible $F[N]$-submodules of V. Then N is cyclic, say $N = \langle x \rangle$, and k is determined as the smallest natural number such that $p^{fk} \equiv 1 \pmod{|N|}$. (Thus $k = d_F(N)$).

Let V^* be the direct sum of $\dim_F V / k$ copies of $E = GF(p^{fk})$. Then there is a monomorphism $x^i \mapsto \xi^i$ of N into the multiplicative group of E such that $F[\xi] = E$, and a linear isomorphism of F-spaces $\delta : V \to V^*$ such that

$$(vx)^\delta = v^\delta \xi \quad \text{for all} \quad v \in V .$$

A semi-linear action of G on V^* (regarded as an E-space) is defined by $v^\delta g = (vg)^\delta$. The subgroup of G of elements whose action is linear is precisely $C_G(N)$.

Thus if $N \leqslant Z(G)$, the action of G on V^* is linear, and V^* becomes a faithful $E[G]$-module.

<u>2.5.2</u> Let G, N, F, E, V and V^* be as in 2.5.1, and suppose that $N \leqslant Z(G)$. Further, suppose that N has a complement K in G (thus $G = K \times N$). Let K_o be a subgroup of K, let W be an F-subspace of V, and let $W^* = W^\delta$. Then:

(a) W is an $F[K_oN]$-submodule of V if and only if W^* is an $E[K_o]$-submodule of V^*;

(b) if W is an $F[K_oN]$-submodule of V then

$$C_{K_o}(W) = C_{K_o}(W^*) .$$

((a) shows that there is a one-one correspondence between the $F[K_oN]$-submodules of V and the $E[K_o]$-submodules of V^*, in which irreducible submodules correspond to irreducible submodules).

<u>Proof</u> (a) Suppose that W is an $F[K_oN]$-submodule of V. W^* is an F-subspace of V^*, and $W^*\xi \subseteq W^*$, because

$$\begin{aligned} v^* \in W^*\xi &\implies v^* = w^\delta \xi \quad \text{for some } w \in W \\ &\implies v^* = (wx)^\delta \quad \text{for some } w \in W \\ &\implies v^* \in W^* . \end{aligned}$$

Therefore, as $E = F[\xi]$, W^* is an E-subspace of V^*. Also, $k_o \in K_o \implies W^*k_o = (Wk_o)^\delta \subseteq W^\delta = W^*$. Therefore W^* is an $E[K_o]$-submodule of V^*.

Conversely, suppose that W^* is an $E[K_o]$-submodule of V^*; then W is clearly an $F[K_o]$-submodule of V. Also, if $w \in W$ then $(wx)^\delta = w^\delta\xi \in W^\delta = W^*$, and therefore $wx \in W$. Hence W is an $F[K_oN]$-module.

(b) $k_o \in C_{K_o}(W) \iff wk_o = w$ for all $w \in W$

$\iff (wk_o)^\delta = w^\delta$ for all $w \in W$

$\iff w^\delta k_o = w^\delta$ for all $w \in W$

$\iff k_o \in C_{K_o}(W^\delta)$.

2.6 Primitive soluble groups in $\mathcal{C}_p \cap \mathfrak{A}^3$

The ultimate aim of this section is to give necessary and sufficient conditions for a primitive soluble group of derived length 3 to lie in $\mathcal{C}_p$; this aim is achieved in Theorem 2.6.19 .

Throughout 2.6, G denotes a primitive soluble group of derived length 3 ; V is the unique minimal normal subgroup of G , and p is the prime number of which the order of V is a power. H is a complement to V in G ; thus $H \in \mathfrak{A}^2$, and V can be regarded as a faithful irreducible $Z_p[H]$-module. The invariant $d_{Z_p}(X)$ (where X is an abelian group) introduced in 2.3 will be referred to frequently, and will always be abbreviated to d(X) .

2.6.1 $G \in \mathcal{C}_n$ if and only if $H \in \mathcal{C}_n$.

Proof This is an immediate consequence of Theorem 1.4.3 .

Thus, by 2.1.4 , $G \in \mathcal{C}_n$ if and only if H has elementary abelian Sylow subgroups. It will be assumed from now on that $H \in \mathcal{C}_n$. Let $A = H'$, $Z = Z(H)$, and let B be a complement to AZ ($= F(H)$) in H . Thus

$$H = BA \times Z .$$

2.6.2 (a) A , Z , and B are all elementary abelian; Z is cyclic.

(b) $C_A(B) = 1$ and $C_B(A) = 1$.

(c) If $B_0 \leqslant B$ then $N_H(B_0) = B\,C_A(B_0)Z$.

(d) AZ is a p'-group.

<u>Proof</u> (a),(b) and (c) follow from the fact that $H \in \mathcal{E}_n \cap \mathfrak{A}^2$ and the definitions of A , Z and B .

(d) V is a faithful irreducible $Z_p[H]$-module, so $O_p(H) = 1$, and therefore AZ (a direct product of minimal normal subgroups of H) must be a p'-group.

<u>2.6.3</u> If N is a non-trivial normal subgroup of H then $C_V(N) = 1$.

<u>Proof</u> It follows immediately from Clifford's Theorem that if N centralizes some non-trivial element of V then N must centralize the whole of V ; this cannot happen, because V is faithful.

Consider the action of Z on V : since $C_H(Z) = H$, it follows from Clifford's Theorem that V_Z is homogeneous; thus the situation is precisely that discussed in 2.5, with H , Z , Z_p in place of G , N , F respectively. Hence, by 2.5.1 , writing $F = GF(p^{d(Z)})$, to V there corresponds a faithful irreducible $F[H]$-module V^* of F-dimension $\dim_{Z_p} V / d(Z)$. By 2.5.2(a) (with BA in place of K), V^* is in fact an irreducible $F[BA]$-module, and there is a bijection $W \mapsto W^*$ between the Z_p-subspaces of V and the F-subspaces of V^* such that, for any $K_o \leq BA$, if W is a $Z_p[K_oZ]$-submodule of V of Z_p-dimension n , then W^* is an $F[K_o]$-submodule of V^*

of F-dimension $n / d(Z)$. Also, W is irreducible if and only if W^* is irreducible, and $C_{K_o}(W) = C_{K_o}(W^*)$.

<u>2.6.4</u> $\dim_{Z_p} V$ is divisible by $d(Z)\ \mathrm{lcm}(|B|, d_F(A))$, where $F = GF(p^{d(Z)})$.

<u>Proof</u> 2.4.3 can be applied to the metabelian group H and faithful irreducible $F[H]$-module V^* , with AZ playing the part of the self-centralizing normal subgroup N of H : this shows that $\dim_F V^*$ is divisible by $\mathrm{lcm}(|H:AZ|, d_F(AZ))$. Now $d_F(Z) = 1$, so by 2.3.3(b), $d_F(AZ) = d_F(A)$; also $|H:AZ| = |B|$. Therefore $\dim_F V^*$ is divisible by $\mathrm{lcm}(|B|, d_F(A))$. The result now follows, as $\dim_F V^* = \dim_{Z_p} V / d(Z)$.

<u>2.6.5</u> V has a $Z_p[BZ]$-submodule W of codimension $d(Z)$ such that B centralizes V/W .

<u>Proof</u> This is a straightforward application of 2.4.4(b), with AZ , Z and Z_p for N , N_o and F respectively.

<u>2.6.6</u> For any $B_o \leqslant B$, B_o is pronormal in H .

<u>Proof</u> By 1.5.8 , it will be enough to show that B_o is a Carter subgroup of $B_o[A,B_o]$, since the latter subgroup is normal in H , by 2.2.1(a) and 2.2.2 . Write $A_o = [A,B_o]$; then by 2.2.1(b), $C_{A_o}(B_o) = 1$, so B_o is self-normalizing in B_oA_o . Since B_o is abelian, it follows that B_o is a Carter subgroup of B_oA_o .

2.6.7 Let W be the $Z_p[BZ]$-submodule of V described in 2.6.5 , and suppose that $B_o \leqslant B$ is such that $C_A(B_o) = 1$. Then B_oW is pronormal in G .

Proof B_o is pronormal in H , by 2.6.6 , and $N_H(B_o) = B\,C_A(B_o)\,Z = BZ$: thus $N_H(B_o)$ normalizes W . The result follows, by applying 1.5.11 (with B_o playing the part of L) .

2.6.8 Suppose $B_o \leqslant B$, $W < V$ is normalized by B_o , and S is a supplement to B_oW in G . Then $S \cap V > 1$.

Proof The result holds for S if and only if it holds for some conjugate of S ; this justifies the following manoeuvres.

The p-complement, H^p say, of H is a p-complement of G , so it can be arranged, by replacing S by a conjugate if necessary, that H^p contains a p-complement, S^p say, of S . Let S_p be a Sylow p-subgroup of S , and let B_p be the Sylow p-subgroup of B . Since AZ is a p'-group, every Sylow p-subgroup of G has the form ${B_p}^h V$ for some $h \in H$; hence, by replacing S by S^h , it can be arranged that $S_p \leqslant B_pV$. (Note that the new choice of S retains the property that $S^p \leqslant H^p$).

Let x be an element of $S_p \setminus B_p$: such an element must exist, because otherwise $S = S_pS^p \leqslant H$, whence S could not be a supplement to BW in G . Then $x = vb$ for some $v \in V$ and $b \in B_p$, with $v \neq 1$. Thus $b = v^{-1}x \in SV$. Now SV is a supplement to B_o (and therefore to B) in G, so $SV \cap H$ is a supplement to B in H. Hence if a is an arbitrary element of A, there exist $b_1 \in B$ and $x_1 \in SV \cap H$ such that $a = b_1x_1$. Consequently

$$[a,b] = [b_1x_1, b] = [b_1, b]^{x_1}[x_1, b] = [x_1, b] \in SV .$$

Therefore

$$\begin{aligned}[A,b] &\leqslant SV \cap A \\ &= S^pS_pV \cap H^p \cap A \\ &= S^p(S_pV \cap H^p) \cap A \\ &= S^p \cap A \\ &\leqslant S .\end{aligned}$$

If $b = 1$ then $1 \neq x = v \in S \cap V$; thus it can be assumed that $b \neq 1$. Then $[A,b]$ is a non-trivial normal subgroup of H, so $C_V([A,b]) = 1$. Hence there exists $a_1 \in [A,b]$ such that $[a_1, v] \neq 1$. But $[a_1, vb] = [a_1, b][a_1, v]^b$, and both $[a_1, vb]$ and $[a_1, b]$ belong to S; therefore $[a_1, v]^b$ is a non-trivial element of $S \cap V$. I.e. $S \cap V > 1$. Q.e.d.

It is now possible to give a necessary condition for G to lie in $\mathcal{C}_p$.

<u>2.6.9 Theorem</u> If $G \in \mathscr{C}_p$ then $d(A) \mid d(Z)$.

<u>Proof</u> Suppose $G \in \mathscr{C}_p$, and consider the pronormal subgroup BW of G , where W is the $Z_p[BZ]$-submodule of V , of codimension $d(Z)$, described in 2.6.5 . It is first shown that:

BW has a complement C in G such that $C \cap H$ complements B in H and $C \cap V$ complements W in V . ...(1)

If C is a complement to BW in G , then

$$|C| = \frac{|G|}{|BW|} = \frac{|H||V|}{|B||W|} = |H:B||V:W| = |AZ|p^{d(Z)} .$$

Since AZ is a p'-group, it follows that a p-complement of C has order $|AZ|$. The p-complement of H is a p-complement of G , so it can be arranged, by replacing C by a conjugate if necessary, that H contains a p-complement , C^p say, of C . Then by order considerations,

C^p is a complement to B in H ....(2)

Let $Z = \langle z_0 \rangle$; then $z_0 = c_0 b_0$ for some $c_0 \in C^p$, $b_0 \in B$. Hence

$$(C \cap V)^{z_0} = (C \cap V)^{c_0 b_0} = (C \cap V)^{b_0} \leqslant (C \cap V)W$$

(the final inclusion holds because B centralizes V/W). Therefore $(C \cap V)W$ is a $Z_p[Z]$-submodule of V ; but $C \cap V > 1$, by 2.6.8, and codim $W = d(Z)$, so it follows that $(C \cap V)W = V$. Therefore

$C \cap V$ is a complement to W in V ....(3)

Hence, by (2) and (3), the complement C is of the form specified in (1).

The next objective is to show that

$$Z \leqslant C^p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Again let $Z = \langle z_o \rangle$; by 2.2.5(b), there exist $c_o \in C^p$, $b_o \in B$ such that $|b_o|$ divides $|z_o|$, $c_o = b_o z_o$, and $\langle c_o \rangle = \langle b_o z_1 \rangle \times \langle z_2 \rangle$, where $\langle z_o \rangle = \langle z_1 \rangle \times \langle z_2 \rangle$ and $(|b_o|, |z_2|) = 1$, $(|b_o|, |z_1|) = |z_1|$. Then both of $b_o z_1$ and z_2 lie in C^p; also $|b_o| = |b_o z_1| = |z_1|$.

Suppose $Z \nleqslant C^p$; then $b_o \neq 1$. Let

$$H_1 = \langle b_o z_1 \rangle [A, b_o z_1] \times \langle z_2 \rangle .$$

By 2.2.5(a), $[A, b_o z_1] \leqslant [A, BZ \cap C^p] \leqslant C^p$, and so $H_1 \leqslant C^p$. Let V_1 be an irreducible $Z_p[H_1]$-submodule of V, and let $N_1 = C_{H_1}(V_1)$. It is easily checked that H_1 is a normal subgroup of H, from which it follows, by an argument like that at the beginning of the proof of 2.3.4, that N_1 contains no non-trivial normal subgroups of H. Thus the situation is precisely like that in 2.2.4, with $\langle b_o z_1 \rangle$, $\langle z_2 \rangle$, H_1, N_1 in place of S, Z_o, H_o, N respectively. Hence

$$N_1 \leqslant F(H_1) = [A, b_o z_1] \times \langle z_2 \rangle ,$$

and $$F(H_1/N_1) = F(H_1)/N_1 .$$

Let $\overline{H}_1 = H_1/N_1$; then V_1 is a faithful irreducible $Z_p[\overline{H}_1]$-module, and the split extension $\overline{H}_1 V_1$ is a primitive soluble group in $\mathscr{C}_n \cap \mathfrak{A}^3$. Therefore, by 2.6.4, $\dim_{Z_p} V_1$ is divisible by

$$d(\langle \overline{z}_2 \rangle) \operatorname{lcm}(|\overline{b_o z_1}|, d_F(\overline{A}_1)) ,$$

where the "bar convention" (alluded to already in the proofs of 1.4.1 and 2.2.4) is used, $A_1 = [A, b_o z_1]$, and

$F = GF(p^{d(\langle\overline{z_2}\rangle)})$. As $N_1 \leqslant F(H_1)$ and $|z_1| = |b_0|$, it is clear that $N_1 \cap \langle b_0 z_1\rangle = 1$, and hence that $|\overline{b_0 z_1}| = |b_0 z_1| = |z_1|$. Also $N_1 \cap \langle z_2\rangle$ is normal in H , so $N_1 \cap \langle z_2\rangle = 1$, hence $|\overline{z_2}| = |z_2|$, and therefore $d(\langle\overline{z_2}\rangle) = d(\langle z_2\rangle)$. Hence the remark about $\dim_{Z_p} V_1$ yields the information that

$$|z_1|\, d(\langle z_2\rangle) \;\Big|\; \dim_{Z_p} V_1 .$$

It is clear that $d(\langle z_1\rangle) \leqslant |z_1| - 1$ ($|z_1|$ and p are coprime, so the congruence $p^k \equiv 1 \pmod{|z_1|}$ holds when $k = \phi(|z_1|)$, where ϕ is Euler's function; but $\phi(|z_1|)$ divides $(|z_1| - 1)$), and thus

$$\begin{aligned} \dim_{Z_p} V_1 &\geqslant |z_1|\, d(\langle z_2\rangle) \\ &> d(\langle z_1\rangle)\, d(\langle z_2\rangle) \\ &\geqslant d(\langle z_0\rangle) \qquad \text{(by 2.3.3(b))} \\ &= d(Z) . \end{aligned}$$

Hence the dimension of the irreducible $Z_p[H_1]$-submodules of V exceeds $d(Z)$. Since $H_1 \leqslant C^p$, it follows that the dimension of any non-trivial $Z_p[C^p]$-submodule of V exceeds $d(Z)$; but this gives a contradiction, because (3) implies that $C \cap V$ has Z_p-dimension $d(Z)$. Therefore $Z \leqslant C^p$, i.e. (4) is established.

The last major step is to prove that

$$A \leqslant C^p \qquad \text{..................(5)}$$

Suppose $A \nleqslant C^p$; then there exists a minimal normal subgroup N of H, contained in A, such that $N \nleqslant C^p$. Suppose $n \in N \setminus C^p$; then there exist $b_1 \in B$, $c_1 \in C^p$ such that $n = b_1^{-1}c_1$, and $b_1 \neq 1$. In fact

$$[n, b_1] = 1 \qquad \text{.................(6)}$$

To prove this, suppose for a contradiction that $[n, b_1] \neq 1$. Let $\langle b^* \rangle = C_{\langle b_1 \rangle}(n)$; then (since B is elementary abelian) there is an element $b' \neq 1$ in B such that $b_1 = b^* b'$ and $\langle b_1 \rangle = \langle b^* \rangle \times \langle b' \rangle$. Now $\langle b' \rangle$ acts fixed-point-free on N, for, given $x \neq 1$ in $\langle b' \rangle$, x acts non-trivially on N, so $C_N(x) < N$; but $C_N(x)$ is normal in H by 2.2.1(a); therefore $C_N(x) = 1$. Hence, by (H8, $\underline{V}$, 8.5), $\langle b' \rangle N$ is a Frobenius group, with Frobenius kernel N. It follows, because $b'n \notin N$, that $b'n$ belongs to a conjugate in $\langle b' \rangle N$ of $\langle b' \rangle$. One consequence of this is that $|b'n|$ divides $|b'|$. Now $(|b^*|, |b'|) = 1$, since $\langle b_1 \rangle = \langle b^* \rangle \times \langle b' \rangle$, and so $(|b^*|, |b'n|) = 1$ also. Therefore

$$\langle (b'n)^{|b^*|} \rangle = \langle b'n \rangle .$$

Recall that $c_1 = b_1 n = b^* b' n$. $[b^*, b'n] = 1$, so

$$c_1^{|b^*|} = (b'n)^{|b^*|}, \text{ hence } \langle b'n \rangle = \langle (b'n)^{|b^*|} \rangle \leqslant C^p .$$

But $b'n$ belongs to a conjugate in $\langle b' \rangle N$ of $\langle b' \rangle \leqslant B$, and $C^p \cap B^h = 1$ for any $h \in H$. This implies $b'n = 1$, hence $b' = 1$ and $n = 1$, a contradiction. Thus (6) holds.

Suppose that q is the prime of which $|N|$ is a power. Then $n^q = 1$, so $b_1{}^q = b_1{}^q n^q = (b_1 n)^q = c_1{}^q \in C^p$. It follows that b_1 and c_1 both have order q.

It is now possible to show that

$$[A, c_1] \leqslant C^p \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Let M be a minimal normal subgroup of H contained in $[A, c_1]$; then $M = [M, c_1] \times C_M(c_1) = [M, c_1]$.

Let $m \in M$; then $m = b_2^{-1}c_2$ for some $b_2 \in B$, $c_2 \in C^p$.

By the usual commutator manipulations,

$$[c_2, c_1] = [b_2 m, b_1 n] = [b_2, b_1 n]^m [m, b_1 n]$$

$$= [b_2, n]^m [b_2, b_1]^{nm} [m, c_1] ,$$

i.e. $[c_2, c_1] = [b_2, n][m, c_1]$. Hence

$$[m, c_1]^q = [c_2, c_1]^q \in C^p .$$

But M is a q'-group, because otherwise the q-element c_1 would centralize some non-trivial element of M , so it follows that $M = [M, c_1] \leqslant C^p$. It is immediate from this that (7) holds, as $[A, c_1]$ is a direct product of minimal normal subgroups of H .

Let $$H_2 = \langle c_1 \rangle [A, c_1] \times Z .$$

H_2 is easily seen to be normal in H , and $H_2 \leqslant C^p$ by (4) and (7). An argument similar to that applied to H_1 earlier is now employed. Let V_2 be an irreducible $Z_p[H_2]$-submodule of V , and let $N_2 = C_{H_2}(V_2)$. Then, as in the previous argument, N_2 contains no non-trivial normal subgroups of H , and so $N_2 \leqslant F(H_2) = [A, c_1] \times Z$, and $F(H_2/N_2) = [A, c_1]Z / N_2$. Now 2.6.4 , applied to the split extension $\overline{H}_2 V_2$, where $\overline{H}_2 = H_2/N_2$, shows that

$\dim_{Z_p} V_2$ is divisible by $d(\overline{Z})\ \mathrm{lcm}(|\overline{c_1}|, d_F(\overline{A}_2))$, where the bar convention is used, $A_2 = [A, c_1]$, and $F = GF(p^{d(\overline{Z})})$. As in the case of H_1 , it is easily shown that $d(\overline{Z}) = d(Z)$ and $|\overline{c_1}| = |c_1| = q$. Therefore

$$\dim_{Z_p} V_2 \geqslant d(Z)|c_1| > d(Z) ,$$

i.e. the dimension of the irreducible $Z_p[H_2]$-submodules of V_2 exceeds $d(Z)$; but $H_2 \leqslant C^p$ and $C \cap V$ has Z_p-dimension $d(Z)$. Thus the assumption that $A \nleqslant C^p$ has at last given rise to a contradiction, and consequently (5) is established.

Hence, by (4) and (5), $AZ \leqslant C^p$, and so $C \cap V$ is a $Z_p[AZ]$-submodule of V , from which it follows that $d(AZ) = d(Z)$. Therefore $d(A) \mid d(Z)$, and the theorem is proved.

Theorem 2.6.9 gives a condition on the Fitting subgroup of H which must necessarily hold if $G \in \mathcal{C}_p$. The next important result (2.6.13) is another necessary condition that $G \in \mathcal{C}_p$, involving the action of the complement B of $F(H)$ on $F(H)$. The condition is that the action of B on $F(H)$ should be <u>absolutely faithful</u> in the sense defined in (R2), i.e. that for every subgroup B_1 of B ,

$$C_B(C_{F(H)}(B_1)) = B_1 .$$

It is clear that B acts absolutely faithfully on $F(H)$ $(= A \times Z)$ if and only if B acts absolutely faithfully on A.

<u>2.6.10</u> Suppose $d(A) \mid d(Z)$, let B_1 be a subgroup of B, and let $B_o = C_B(C_A(B_1))$. Then V has a proper $Z_p[B\ C_A(B_1)\ Z]$-submodule U of codimension at most $|B:B_o|d(Z)$ such that B_o centralizes V/U.

<u>Proof</u> By 2.4.4(b), with AZ, $C_A(B_1)Z$, Z_p in place of N, N_o, F respectively, V has a $Z_p[B_oC_A(B_1)Z]$-submodule W, of codimension $d(C_A(B_1)Z)$, such that B_o centralizes V/W. Since $d(A) \mid d(Z)$, codim $W = d(Z)$.

Let B_2 be a complement to B_o in B, and let $U = \bigcap_{b \in B_2} W^b$. Then U is a $Z_p[B\ C_A(B_1)\ Z]$-submodule of V, and codim $U \leqslant |B_2|$codim $W = |B:B_o|d(Z)$. Also $[V, B_o] \leqslant W$, and $[V, B_o]$ is a B-submodule of V, so $[V, B_o] \leqslant U$. Hence U has all the desired properties.

<u>2.6.11</u> Suppose $d(A) \mid d(Z)$, let B_1 be a subgroup of B, and let U be as in 2.6.10. Then B_1U is pronormal in G.

<u>Proof</u> B_1 is pronormal in H by 2.6.6, $[V, B_1] \leqslant U$, and $N_H(B_1)$ $(= B\ C_A(B_1)\ Z)$ normalizes U, so the result is immediate from 1.5.11.

<u>2.6.12</u> (a) If $d(A) \mid d(Z)$ then $\dim_{Z_p} V = |B| d(Z)$.

(b) Suppose $d(A) \mid d(Z)$, let $A_o \leqslant A$ be a normal subgroup of H , and let $B_o \leqslant B$. If V_o is an irreducible $Z_p[B_oA_oZ]$-submodule of V then $\dim_{Z_p} V_o \geqslant |B_o : C_{B_o}(A_o)| d(Z)$. If $C_{B_o}(A_o) = 1$ then $\dim_{Z_p} V_o = |B_o| d(Z)$.

<u>Proof</u> (a) Consider the faithful irreducible $F[BAZ]$-module V^* , where $F = GF(p^{d(Z)})$, introduced in the remarks preceding 2.6.4 . Since $d(A) \mid d(Z)$, F is a splitting field for AZ , and therefore, by 2.4.1(b), with AZ , V^* in place of N , V respectively, $\dim_F V^* = |H:AZ| = |B|$. Therefore $\dim_{Z_p} V = d(Z).\dim_F V^* = |B| d(Z)$.

(b) Consider V_o^* , the irreducible $F[B_oA_o]$-submodule of V^* corresponding to V_o . By 2.4.2 , with V^*, AZ, A_o, V_o^* in place of V, N, N_o, U respectively, $\dim_F V_o^* \geqslant |B_o : C_{B_o}(A_o)|$, and $\dim_F V_o^* = |B_o|$ if $C_{B_o}(A_o) = 1$. The desired results now follow from the fact that $\dim_{Z_p} V_o = d(Z).\dim_F V_o^*$.

<u>2.6.13 Theorem</u> If $G \in \mathcal{C}_p$ then B acts absolutely faithfully on A .

<u>Proof</u> Let $G \in \mathcal{C}_p$, and let B_1 be a subgroup of B . To prove the theorem, it must be shown that

$$C_B(C_A(B_1)) = B_1 .$$

Let $B_o = C_B(C_A(B_1))$. By 2.6.9 , $d(A) \mid d(Z)$, so 2.6.10 , 2.6.11 and 2.6.12 can be applied. In particular, G has a pronormal subgroup B_1U , where U is the $Z_p[B\ C_A(B_1)\ Z]$-submodule of V described in 2.6.10 , such that $d(Z) \leqslant \operatorname{codim} U \leqslant |B:B_o|\, d(Z)$. Let C be a complement to B_1U in G . Then by 2.6.8 ,

$$C \cap V > 1 \qquad \text{.....................(1)}$$

$CV \cap H$ is a supplement in H to B_1 ; also $CV \cap B_1$ has a complement in H (every subgroup of B has a complement in H), and therefore has a complement, C_1 say, in $CV \cap H$. It then follows that:

C_1 is a complement to B_1 in H , and C_1 normalizes $C \cap V$ ....(2)

An argument similar to one in the proof of 2.6.9 is now employed to show that

$$Z \leqslant C_1 \qquad \text{.....................(3)}$$

Assume $Z \nleqslant C_1$, and let $Z = \langle z_o \rangle$. By 2.2.5(b), with B_1 , C_1 in place of B_o , C respectively, there exist $c_1 \in C_1 , b_1 \in B_1$ such that $|b_1|$ divides $|z_o|$ and $c_1 = b_1 z_o$, and consequently, if $z_o = z_1 z_2$ with $(|b_1| , |z_2|) = 1$, $(|b_1| , |z_1|) = |z_1|$, then $\langle c_1 \rangle = \langle b_1 z_1 \rangle \times \langle z_2 \rangle$.

Let $B_2 = C_1 \cap B$ (hence $B = B_1 \times B_2$); then $B_2 \langle b_1 z_1 \rangle$ is a group, and by 2.2.5(a), $[A , B_2 \langle b_1 z_1 \rangle] \leqslant C_1$. It follows, defining

$$H_1 = \langle b_1 z_1 \rangle [A , B_2 \langle b_1 z_1 \rangle] \times \langle z_2 \rangle ,$$

that

$$H_1 \leqslant C_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

It is clear also that H_1 is a normal subgroup of H . Let V_1 be an irreducible $Z_p[H_1]$-submodule of V , and let $N_1 = C_{H_1}(V_1)$. Then 2.2.4 can be applied, as in the proof of 2.6.9 , to show that

$$N_1 \leqslant F(H_1) = [A , B_2\langle b_1 z_1\rangle] \times \langle z_2\rangle$$

and $F(H_1/N_1) = F(H_1)/N_1$.

Let $\overline{H}_1 = H_1/N_1$; then 2.6.4 , applied to the primitive soluble $\mathcal{B}_n \cap \mathfrak{A}^3$-group $\overline{H}_1 V_1$ (split extension) shows that

$d(\langle\overline{z_2}\rangle)\ \mathrm{lcm}(|\overline{B_2\langle b_1 z_1\rangle}| , d_F(\overline{A_1}))$ divides $\dim_{Z_p} V_1$,

where $F = GF(p^{d(\langle\overline{z_2}\rangle)})$, $A_1 = [A , B_2\langle b_1 z_1\rangle]$, and the bar convention is used. The same kind of argument as in the corresponding part of the proof of 2.6.9 now shows that

$$\begin{aligned} \dim_{Z_p} V_1 &\geqslant |B_2\langle b_1 z_1\rangle|\, d(\langle z_2\rangle) \\ &= |B_2|\,|z_1|\, d(\langle z_2\rangle) \\ &> |B_2|\, d(Z) . \end{aligned}$$

Since $C \cap V$ is a $Z_p[C_1]$-module (by (2)), and therefore a $Z_p[H_1]$-module (by (4)), and $C \cap V$ is non-trivial (by (1)), this shows that $\dim_{Z_p}(C \cap V) > |B_2|\, d(Z)$. But

$|B_2| = |C_1 \cap B| = |B:B_1|$ and

$$\dim_{Z_p}(C\cap V) \leqslant \operatorname{codim} U \leqslant |B:B_0|\, d(Z) \leqslant |B:B_1|\, d(Z) \qquad \ldots\ldots(5)$$

This gives a contradiction; hence (3) holds.

Let $H_2 = B_2[A, B_2] \times Z$. By 2.6.12, as $C_{B_2}([A, B_2]) = 1$, the Z_p-dimension of any irreducible $Z_p[H_2]$-submodule of V is $|B_2|d(Z)$, i.e. $|B:B_1|d(Z)$. Now by 2.2.5(a), $[A, B_2] \leqslant C_1$, and hence, using (3) as well, $H_2 \leqslant C_1$. Therefore $C \cap V$ is a (non-trivial) $Z_p[H_2]$-submodule of V , and so

$$\dim_{Z_p}(C \cap V) \geqslant |B:B_1|\,d(Z) \qquad \ldots\ldots\ldots\ldots(6)$$

(5) and (6) together imply that $|B:B_1| = |B:B_o|$, and thus (since $B_1 \leqslant B_o$) the desired result, i.e. that $B_1 = B_o$, follows.

Although it is not necessary in proving the theorem, it is useful for future purposes to notice that the above argument implies that $C \cap V$ is a complement to U in V , and hence that $C_1(C \cap V)$ is a complement to B_1U in V . Thus from the proof of 2.6.13 can be extracted the following result:

<u>2.6.14</u> If $G \in \mathcal{C}_p$, B_1 and U are as in the proof of 2.6.13 , and C is a complement to B_1U in G such that $C = (C \cap H)(C \cap V)$, then $Z \leqslant C \cap H$.

2.6.9 and 2.6.13 give necessary conditions for G to lie in $\mathcal{C}_p$; the remainder of 2.6 is devoted to showing that these conditions together are sufficient for G to be a $\mathcal{C}_p$-group.

2.6.15 Suppose $d(A) \mid d(Z)$ and B acts absolutely faithfully on A . Let B_1 be a subgroup of B and let B_2 be a complement to B_1 in B ; thus $B = B_1 \times B_2$. Then any irreducible $Z_p[B_2AZ]$-submodule of V is also irreducible as a $Z_p[B_2C_A(B_1)Z]$-module.

Proof Let V_1 be an irreducible $Z_p[B_2AZ]$-submodule of V . Then by 2.6.12(b),

$$\dim_{Z_p} V_1 = |B_2|d(Z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By hypothesis, $C_B(C_A(B_1)) = B_1$, so $C_{B_2}(C_A(B_1)) = B_2 \cap C_B(C_A(B_1)) = B_2 \cap B_1 = 1$. Thus by 2.6.12(b) again, if V_2 is an irreducible $Z_p[B_2C_A(B_1)Z]$-submodule of V , then

$$\dim_{Z_p} V_2 = |B_2|d(Z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The result now follows from (1) and (2).

The wording of the next lemma is dictated by the needs of Section 2.7 . The notation $F_2(G)$ denotes the second term in the upper Fitting series of G ; i.e. $F_2(G)$ is given by: $\qquad F_2(G) / F(G) = F(G/F(G))$.

<u>2.6.16</u> Suppose that whenever B is decomposed as $B = B_1 \times B_2$, each irreducible $Z_p[B_2AZ]$-submodule of V is an irreducible $Z_p[B_2C_A(B_1)Z]$-module. Let $F = F_2(G) = AZV$. If L is a pronormal subgroup of G , $W \leqslant V$ is normalized by $N_G(LV)$, and C/F is a complement to LF/F in G/F , then W has a complement in V which is normalized by C .

<u>Proof</u> $LV \cap H$ is a pronormal subgroup of H , by 1.5.6, So $N_H(LV \cap H)$ is abnormal in H , by 1.5.5 . Therefore, by 1.5.7 , $N_H(LV \cap H)$ contains a system normalizer of H . Now $H \in \mathfrak{N}^2$, so by (H8, VI, 12.4) , the system normalizers of H coincide with the Carter subgroups of H , and hence $N_H(LV \cap H)$ contains a Carter subgroup of H . But BZ is clearly a Carter subgroup of H , so $N_H(LV \cap H) \geqslant B^hZ$ for some $h \in H$. It is easily seen that the result holds for L provided it holds for some conjugate of L ; thus it is legitimate to replace L by $L^{h^{-1}}$ and hence arrange that $N_H(LV \cap H) \geqslant BZ$.

Let $B_1 = LF \cap B$, $B_2 = C \cap B$; thus $B = B_1 \times B_2$. $LV \cap H \leqslant LF \cap H = B_1AZ$, so $C_A(B_1) \leqslant C_H(LV \cap H)$, and therefore $N_H(LV \cap H) \geqslant B\, C_A(B_1)\, Z$. Hence the hypothesis that W is normalized by $N_G(LV)$ implies that:

W is a $Z_p[B\, C_A(B_1)\, Z]$-submodule of V ....(1)

B_2AZ is a normal subgroup of H , so, writing V additively and using Clifford's Theorem, there is a decomposition

$$V = V_1 \oplus \dots \oplus V_r \quad \text{(for some } r \geqslant 1)$$

of V into irreducible $Z_p[B_2AZ]$-modules. Clearly, it can be assumed that $W < V$. Hence the indexing of the V_i can be chosen so that, for some $s \geqslant 1$,

$W \cap (V_1 \oplus \dots \oplus V_s) = 0$, but

$$W \cap (V_1 \oplus \dots \oplus V_s \oplus V_i) > 0 \quad \text{for any } i > s \quad \dots\dots(2)$$

Let $U = V_1 \oplus \dots \oplus V_s$; then

$$U \text{ is a } Z_p[B_2AZ]\text{-module} \quad \dots\dots\dots\dots\dots\dots(3)$$

If $U \oplus W < V$, then without loss of generality it can be assumed that $V_{s+1} \nleqslant U \oplus W$. By (1) and (3), $U \oplus W$ is a $Z_p[B_2C_A(B_1)Z]$-module, and, by the main hypothesis, V_{s+1} is an irreducible $Z_p[B_2C_A(B_1)Z]$-module; therefore $(U \oplus W) \cap V_{s+1} = 0$, and so $W \cap (U \oplus V_{s+1}) = 0$. But this contradicts (2). Hence $U \oplus W = V$. U is a $Z_p[B_2AZ]$-submodule of V, and hence U is normalized by B_2AZV, i.e. by C; this completes the proof.

<u>2.6.17 Corollary</u> If for every decomposition of B as $B = B_1 \times B_2$, each irreducible $Z_p[B_2AZ]$-submodule of V is an irreducible $Z_p[B_2C_A(B_1)Z]$-module, then $G \in \mathscr{C}_p$.

<u>Proof</u> Let L be a pronormal subgroup of G, let C/F be a complement to LF/F in G/F, and let $B_2 = C \cap B$. By 1.5.12, $L \cap V$ is normalized by $N_G(LV)$, so by 2.6.16, $L \cap V$ has a complement, U say, in V which is normalized

by B_2AZ $(\leqslant C)$

$LV \cap H$ is pronormal in H by 1.5.6 , and $H \in \mathfrak{A}^2$, so it follows from 2.1.5 that $LV \cap AZ$ is normal in H . Consequently there is a normal subgroup N of H such that $AZ = (LV \cap AZ) \times N$. Also B_2 is a complement to $LVAZ \cap B$ $(= LF \cap B)$ in B , and therefore, by 1.3.3(b) B_2N is a complement to $LV \cap H$ in H . Hence, again by 1.3.3(b), B_2NU is a complement to L in G . Therefore $G \in \mathcal{C}_p$.

<u>2.6.18 Theorem</u> The following are equivalent:

(1) $G \in \mathcal{C}_p$;

(2) $d(A) \mid d(Z)$, and B acts absolutely faithfully on A ;

(3) whenever B is expressed as $B = B_1 \times B_2$, each irreducible $Z_p[B_2AZ]$-submodule of V is irreducible as a $Z_p[B_2C_A(B_1)Z]$-module.

<u>Proof</u> (1) $\Rightarrow$ (2) : Theorems 2.6.9 and 2.6.13;

(2) $\Rightarrow$ (3) : 2.6.15;

(3) $\Rightarrow$ (1) : Corollary 2.6.17 .

To sum up the main investigation of this section, a "self-contained" version of 2.6.18 is given in 2.6.19 .

<u>2.6.19 Theorem</u> Let G be a primitive soluble group of derived length at most 3, let V be the unique minimal normal subgroup of G, where V is a p-group, say, and let H be a complement to V in G. Then $G \in \mathscr{E}_p$ if and only if the following three conditions hold:

(a) H has elementary abelian Sylow subgroups;

(b) $d_{Z_p}(F(H)) = d_{Z_p}(Z(H))$;

(c) if B is a complement to $F(H)$ in H then B acts absolutely faithfully on $F(H)$.

The last result of 2.6 shows that the condition, in 2.6.18, that B should act absolutely faithfully on A, can be expressed in different ways.

<u>2.6.20</u> Let B be an elementary abelian group which acts fixed-point-free ~~faithfully~~ on a group A (i.e. $C_A(B) = 1$); then the following are equivalent:

(1) B acts absolutely faithfully on A ;

(2) the mapping $X \mapsto C_A(X)$, from the set of subgroups of B into the set of subgroups of A, is injective;

(3) $C_A(B_o) > 1$ for every maximal subgroup B_o of B .

<u>Proof</u> (1) $\Leftrightarrow$ (2) : See (R2, 1.1). This is true even when B is a non-abelian group.

(1) $\Rightarrow$ (3) : If for some maximal subgroup B_o of B, $C_A(B_o) = 1$, then $C_B(C_A(B_o)) = B > B_o$.

(3) $\Rightarrow$ (2) : Suppose (2) does not hold. Then B has subgroups B_1 and B_2, with $B_1 \neq B_2$, such that $C_A(B_1) = C_A(B_2)$. Hence

$$C_A(B_1B_2) = C_A(B_1) \cap C_A(B_2) = C_A(B_1),$$

so B_2 can be replaced by B_1B_2, and thus it can be assumed that $B_1 < B_2$. Since B is elementary abelian, B has a subgroup B_3 such that $B = B_2 \times B_3$. Hence

$$\begin{aligned} 1 = C_A(B) &= C_A(B_2B_1B_3) \\ &= C_A(B_2) \cap C_A(B_1B_3) \\ &= C_A(B_1) \cap C_A(B_1B_3) \\ &= C_A(B_1B_3) . \end{aligned}$$

Thus if B_o is a maximal subgroup of B containing B_1B_3, then $C_A(B_o) = 1$, and so (3) does not hold.

2.7 Groups in $\mathfrak{A}^3$ with complemented pronormal subgroups

In this section necessary and sufficient conditions are found for a soluble group of derived length at most 3 to belong to $\mathscr{C}_p$.

Let $G \in \mathfrak{A}^3$. If G is to lie in $\mathscr{C}_p$, then clearly the following two conditions must be met:

(a) $\Phi(G) = 1$;

(b) the primitive soluble quotient groups of G all belong to $\mathscr{C}_p$.

If (a) and (b) hold then $G \in \mathscr{C}_n$ (because, a soluble group with trivial Frattini subgroup is in the residual closure of its set of primitive soluble quotient groups, $\mathscr{C}_p \subseteq \mathscr{C}_n$, and $\mathscr{C}_n$ is R_o-closed).

It will be assumed throughout 2.7 that (a) and (b) hold, and hence that $G \in \mathscr{C}_n \cap \mathfrak{A}^3$. It follows that there are minimal normal subgroups $V_1, V_2, \ldots, V_t$ of G , for some $t \geqslant 1$, such that

$$F(G) = V_1 \times V_2 \times \ldots \times V_t .$$

Let H be a complement to $F(G)$ in G , and for each $i \in \{1,\ldots,t\}$ let

$$N_i = C_H(V_i)\, V_1 \ldots \hat{V}_i \ldots V_t ,$$

where the "hat" has the usual meaning, i.e. that V_i is to be omitted from the product. Then N_i is a normal subgroup of G and $V_iN_i = C_G(V_i)$; G/N_i is a primitive soluble group with unique minimal normal subgroup V_iN_i/N_i , and HN_i/N_i is a complement to V_iN_i/N_i in G/N_i .

For each $i \in \{1,\dots,t\}$, define the subgroup $F^{(i)}(G)$ of G by

$$F^{(i)}(G) \,/\, C_G(V_i) = F(\, G/C_G(V_i)\,)\ .$$

<u>2.7.1</u> Let $s \in \{1,\dots,t\}$, let N be a normal subgroup of G contained in $N_1 \cap \dots \cap N_s$, and let $\overline{G} = G/N$. Then, using the bar convention, for each $i \in \{1,\dots,s\}$:

(a) $C_{\overline{G}}(\overline{V}_i) = \overline{C_G(V_i)} = \overline{V_iN_i}$;

(b) $F^{(i)}(\overline{G}) = \overline{F^{(i)}(G)}$.

(c) If $N = N_1 \cap \dots \cap N_s$, then $F(\overline{G}) = \overline{V_1 \dots V_s}$, and $F^{(1)}(\overline{G}) \cap \dots \cap F^{(s)}(\overline{G}) = F_2(\overline{G})$

(where $F_2(G)$ is the second term of the upper Fitting series of G , i.e. $F_2(G) \,/\, F(G) = F(\, G/F(G)\,)$).

<u>Proof</u> (a)

$$\begin{aligned} \overline{g} \in C_{\overline{G}}(\overline{V}_i) &\iff [V_i , g] \leqslant N \\ &\iff [V_i , g] \leqslant V_i \cap N_i = 1 \\ &\iff g \in C_G(V_i)\ . \end{aligned}$$

(b) Abbreviate $F^{(i)}(G)$ to F_i . In a $\mathscr{C}_n$-group, there are no Frattini factors, so the Fitting subgroup is determined as the unique abelian self-centralizing normal subgroup. Now $\overline{F}_i/C_{\overline{G}}(\overline{V}_i)$ $(= \overline{F}_i/\overline{C_G(V_i)}\ )$ is abelian, from the definition of F_i , so to establish (b) it will be enough to show that $\overline{F_i}/C_{\overline{G}}(\overline{V_i})$ is self-centralizing in $\overline{G}/C_{\overline{G}}(\overline{V_i})$:

$$\begin{aligned}
& \overline{g} \in C_{\overline{G}}(\ \overline{F_i}/C_{\overline{G}}(\overline{V_i})\ ) \\
\Longleftrightarrow\ & \left[\overline{F_i}\,,\,\overline{g}\right] \leqslant C_{\overline{G}}(\overline{V_i}) = \overline{C_G(V_i)} \\
\Longleftrightarrow\ & \left[F_i\,,\,g\right] \leqslant C_G(V_i)N_i = C_G(V_i) \\
\Longleftrightarrow\ & g \in C_G(\ F_i/C_G(V_i)\ ) \\
\Longleftrightarrow\ & g \in F_i\ .
\end{aligned}$$

(c) Now suppose $N = N_1 \cap \ldots \cap N_s$. Then $\overline{V_1}\,,\overline{V_2}\,,\ldots,\overline{V_s}$ are minimal normal subgroups of $\overline{G}$ and $\overline{V_{s+1}} = \ldots = \overline{V_t} = \overline{1}$, so $\overline{F(G)} = \overline{V_1 \ldots V_s} \leqslant F(\overline{G})$. Also

$$F(\overline{G}) \leqslant \bigcap_{i=1}^{s} C_{\overline{G}}(\overline{V_i}) = \bigcap_{i=1}^{s} \overline{C_G(V_i)} = \bigcap_{i=1}^{s} \overline{N_iV_i} \qquad \text{(by (a))}$$

$$= \overline{\bigcap_{i=1}^{s} N_iV_i} \qquad (\text{as } N_i \geqslant N \text{ for each } i \in \{1,\ldots,s\})$$

Now $$\bigcap_{i=1}^{s} N_iV_i = \bigcap_{i=1}^{s} C_H(V_i)F(G) = (\bigcap_{i=1}^{s} C_H(V_i)\ )\ F(G) \leqslant NF(G)\ ,$$

so $F(\overline{G}) \leqslant \overline{NF(G)} = \overline{F(G)}$.

Hence $F(\overline{G}) = \overline{V_1 \ldots V_s} = \overline{F(G)}$.

Let F be the subgroup of G containing N such that $\overline{F} = F_2(\overline{G})$. It is clear that $\overline{F}/\overline{N_i} \leqslant F_2(\overline{G}/\overline{N_i})$ for each $i = 1,\ldots,s$, i.e. $\overline{F} \leqslant \overline{F_i}$ for each $i = 1,\ldots,s$.

Thus $\overline{F} \leqslant \overline{F_1} \cap \ldots \cap \overline{F_s}$.

Conversely,

$$\bar{g} \in \overline{F_1} \cap \ldots \cap \overline{F_s}$$

$$\Rightarrow \quad \bar{g} \in C_{\bar{G}}(\ \overline{F_i}/C_{\bar{G}}(\overline{V_i})\ ) \qquad (i = 1,\ldots,s)$$

$$\Rightarrow \quad \left[\overline{F_i}\,,\bar{g}\right] \leqslant C_{\bar{G}}(\overline{V_i}) \qquad (i = 1,\ldots,s)$$

$$\Rightarrow \quad \left[\bar{F}\,,\bar{g}\right] \leqslant \bigcap_{i=1}^{s}\left[\overline{F_i}\,,\bar{g}\right] \leqslant \bigcap_{i=1}^{s} C_{\bar{G}}(\overline{V_i}) = F(\bar{G})$$

$$\Rightarrow \quad \bar{g} \in C_{\bar{G}}(\ \bar{F}/F(\bar{G})\ ) = \bar{F}\ .$$

Therefore $\quad \bar{F} = \overline{F_1} \cap \ldots \cap \overline{F_s}$.

<u>2.7.2 Theorem</u> If $G \in \mathcal{C}_p$ then

$$(\,|F^{(i)}(G)|\,,\ |F^{(j)}(G)|\,) = |F^{(i)}(G) \cap F^{(j)}(G)|$$

for all $i\,,j \in \{1,\ldots,t\}$.

<u>Proof</u> Suppose G is a counterexample, i.e., $G \in \mathcal{C}_p$ but the condition on the $F^{(i)}(G)$ does not hold. Then without loss of generality it can be assumed that

$$(\,|F^{(1)}(G)|\,,\ |F^{(2)}(G)|\,) > |F^{(1)}(G) \cap F^{(2)}(G)| \qquad \ldots.(1)$$

Consider $\bar{G} = G/N_1 \cap N_2$. $\bar{G} \in Q\mathcal{C}_p = \mathcal{C}_p$; also $N_1 \cap N_2 \leqslant F^{(1)}(G) \cap F^{(2)}(G)$, so

$$(\,|\overline{F^{(1)}(G)}|\,,\ |\overline{F^{(2)}(G)}|\,) > |\overline{F^{(1)}(G)} \cap \overline{F^{(2)}(G)}| \qquad \ldots.(2)$$

But by 2.7.1(b), $\overline{F^{(i)}(G)} = F^{(i)}(\bar{G})$ for $i = 1\,,2$, so (2) shows that $\bar{G}$ is also a counterexample to the theorem. Hence, replacing G by $\bar{G}$, it can be assumed that $t = 2$, so that G has just two minimal normal subgroups, V_1 and V_2 (note that V_1 and V_2 cannot be G-isomorphic, because otherwise $C_G(V_1) = C_G(V_2) = F(G)$, and hence $F^{(1)}(G) = F^{(2)}(G) = F_2(G)$, which contradicts (1)).

From now on write F_i for $F^{(i)}(G)$, let $V = V_1V_2 = F(G)$, and let $F = F_2(G) = F_1 \cap F_2$ (using 2.7.1(c)). Recall that H is a complement to V in G ; let B be a complement to $F \cap H$ $(= F(H))$ in H . Thus, as $H \in \mathcal{E}_n \cap \mathfrak{A}^2$, B is elementary abelian; also B is a complement to F in G . Further $F_i \cap B$ is a complement to F in F_i $(i = 1,2)$, and thus

$$
\begin{aligned}
(|F_1 \cap B|, |F_2 \cap B|) &= (|F_1 : F|, |F_2 : F|) \\
&= \left(\frac{|F_1|}{|F_1 \cap F_2|}, \frac{|F_2|}{|F_1 \cap F_2|}\right) \\
&> \quad 1 \qquad\qquad \text{(by (1)).}
\end{aligned}
$$

Let q be a prime number which divides both $|F_1 \cap B|$ and $|F_2 \cap B|$, and for each i let b_i be an element of order q in $F_i \cap B$. Let $b = b_1b_2$; then $b \notin F_1$ and $b \notin F_2$.

Since B is elementary abelian, a complement B_1 to $F_1 \cap B$ in B , such that $\langle b\rangle \leqslant B_1$, can be constructed. Now let $\overline{G} = G/N_1$ (where N_1 is the normal subgroup of G defined in the remarks preceding 2.7.1). Then $\overline{G} \in Q\mathcal{E}_p = \mathcal{E}_p$, and $\overline{G}$ is a primitive soluble group in $\mathfrak{A}^3$, with unique minimal normal subgroup $\overline{V_1}$. $\overline{\langle b\rangle}$ is a subgroup of a complement $\overline{B_1}$ to $F(\overline{H})$ in $\overline{H}$, so, applying 2.6.10 to $\overline{G}$

(with $\overline{\langle b\rangle}$ playing the role of "B_1" in that result), there is a subgroup U_1 of G, with $N_1 < U_1 < V_1N_1$, such that $\overline{U_1}$ is normalized by $N_{\overline{H}}(\overline{\langle b\rangle})$, $\overline{V_1}/\overline{U_1}$ is centralized by $\overline{\langle b\rangle}$, and (since $C_{\overline{B_1}}(C_{F(\overline{H})}(\overline{\langle b\rangle})) = \overline{\langle b\rangle}$ by 2.6.19), the codimension of $\overline{U_1}$ in $\overline{V_1}$ is $|\overline{B_1}:\overline{\langle b\rangle}|d_{Z_{p_1}}(Z(\overline{H}))$,

where p_1 is the prime number of which $|V_1|$ is a power. Let $W_1 = U_1 \cap V_1$; then $1 < W_1 < V_1$, W_1 is normalized by $N_H(\langle b\rangle)$, and $\langle b\rangle$ centralizes V_1/W_1. To establish the last assertion, note that, since $\overline{\langle b\rangle}$ centralizes $\overline{V_1}/\overline{U_1}$, $[V_1, b] \leq U_1 \cap V_1 = W_1$.

The same argument applied to G/N_2 demonstrates the existence of a subgroup W_2 of V_2 such that $1 < W_2 < V_2$, $N_H(\langle b\rangle)$ normalizes W_2, and $\langle b\rangle$ centralizes V_2/W_2. Hence $N_H(\langle b\rangle)$ normalizes W_1W_2 and $\langle b\rangle$ centralizes V/W_1W_2. Also $\langle b\rangle$ is pronormal in H, as $\langle b\rangle$ is a Carter subgroup of the normal subgroup $\langle b\rangle[F(H), b]$ of H (cf. 2.6.6). Hence by 1.5.11, $\langle b\rangle W_1W_2$ is pronormal in G.

Now $G \in \mathcal{C}_p$, and hence there exists a complement, C say, to $\langle b\rangle W_1W_2$ in G. CV is then a supplement to $\langle b\rangle$ in G, and therefore (as b is of prime order q) either $CV = G$ or $|G : CV| = q$. If $CV = G$ then $C \cap V$ is a normal subgroup of G, and so $C \cap V$ is one of 1, V_1, V_2, V (recall that $V_1 \not\cong_G V_2$); hence $C \cap V = 1$, because any other possibility would contradict $C \cap W_1W_2 = 1$. Then C is a complement to V in G, and therefore

$|C| = |G|/|V|$. But

$|C| = |G|/|\langle b\rangle W_1W_2| = |G|/q|W_1||W_2|$; therefore

$|V| = q|W_1||W_2|$, and hence $q = |V_1/W_1||V_2/W_2|$, which is impossible, as q is a prime number and $|V_i/W_i| > 1$ for each i . Hence $CV \neq G$, and so $|G : CV| = q$, whence CV is a complement to $\langle b\rangle$ in G ; thus $CV \cap H$ is a complement to $\langle b\rangle$ in H . Also

$|C \cap V| = \dfrac{|C||V|}{|CV|} = \dfrac{|G|}{q|W_1||W_2|}|V|\dfrac{q}{|G|} = |V : W_1W_2|$, so $C \cap V$

is a complement to W_1W_2 in V . It follows that $(CV \cap H)(C \cap V)$ is a complement to $\langle b\rangle W_1W_2$ in G , and hence, replacing C by $(CV \cap H)(C \cap V)$, it can be assumed that:

$\langle b\rangle W_1W_2$ has a complement C in G such that
$C = (C \cap H)(C \cap V)$ ....(3)

It is useful to show further that $C \cap V = (C \cap V_1)(C \cap V_2)$, and as a first step it is necessary to show

$$H' \cap C_H(V_1) > 1 \qquad \text{....................(4)}$$

Suppose that (4) does not hold. Then, since $C_H(V_1)$ is normal in H , it follows that

$$[H , C_H(V_1)] \leqslant H' \cap C_H(V_1) = 1 ,$$

and therefore $C_H(V_1) \leqslant Z(H)$. Let $\bar{H} = H/C_H(V_1)$. Then

$$\bar{h} \in Z(\bar{H}) \iff [H , h] \leqslant C_H(V_1) \cap H' = 1$$
$$\iff h \in Z(H) ,$$

so $Z(\bar{H}) = \overline{Z(H)}$.

Therefore $F(\overline{H}) = \overline{H}' \times Z(\overline{H}) = \overline{H'} \times \overline{Z(H)} = \overline{F(H)}$.
But $F(\ H/C_H(V_1)\ ) = F_1 \cap H \ / \ C_H(V_1)$, so it can be deduced that $F_1 = F \leqslant F_2$. This contradicts (1). Therefore (4) holds.

Let N be a non-trivial normal subgroup of H contained in $H' \cap C_H(V_1)$; thus

$[T_1 , N] = 1$ for each subgroup T_1 of V_1 ....(5)

Now V_2 is an irreducible $Z_{p_2}[H]$-module (where p_2 is the prime number of which $|V_2|$ is a power), so Clifford's Theorem can be applied to $V_2|_N$, to show that, if $C_{V_2}(N) > 1$ then $C_{V_2}(N) = V_2$, i.e. $N \leqslant C_H(V_2)$. But $N \nleqslant C_H(V_2)$, for otherwise $N \leqslant C_H(V_1) \cap C_H(V_2) = C_H(V) = 1$. Therefore

$[T_2 , N] > 1$ for each non-trivial subgroup T_2 of V_2 ....(6)

Thus, by (5) and (6), a non-trivial subgroup of V_1 cannot be H'-isomorphic to a subgroup of V_2 , and hence, if $T \leqslant V$ is normalized by H' then $T = (T \cap V_1)(T \cap V_2)$ ....(7)

Suppose that M is a minimal normal subgroup of H such that $M \nleqslant C \cap H$. Then $(C \cap H)M = H$ (as $C \cap H$ has prime index q in H), so M is a q-group. Now $H \in \mathcal{E}_n \cap \mathfrak{A}^2$, so by 2.1.4 , H has elementary abelian Sylow q-subgroups; hence

$[M , b] = 1$ ....(8)

By 2.2.1(a), $[M, C \cap B]$ is normal in H, so $[M, C \cap B] = 1$ or M. An argument like the one used to establish 2.2.5(a) shows that $[M, C \cap B] \leqslant C \cap H$, and, since $M \nleqslant C \cap H$, it follows that

$$[M, C \cap B] = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

But $B = \langle b \rangle \times (C \cap B)$, so (8) and (9) imply that B is centralized by M. Therefore $M \leqslant Z(H)$.

Hence all the minimal normal subgroups of H which are contained in H' are also contained in $C \cap H$, and therefore

$$H' \leqslant C \cap H \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

From (7) and (10) it follows that $C \cap V = (C \cap V_1)(C \cap V_2)$, and thus

$$C = (C \cap H)(C \cap V_1)(C \cap V_2) \qquad \ldots\ldots\ldots\ldots(11)$$

Given $i \in \{1,2\}$, suppose that $N_i \nleqslant CV$; then $CVN_i = G$, and so $C \cap V_i$, being normalized by C and centralized by V and N_i, must be a normal subgroup of G. Then $C \cap V_i = 1$ or V_i, which gives a contradiction, because a particular consequence of (11) is that $1 < C \cap V_i < V_i$ for $i = 1,2$. Hence

$$N_i \leqslant CV \quad (i = 1,2) \qquad \ldots\ldots\ldots\ldots\ldots(12)$$

In particular,

$$C_H(V_i) \leqslant CV \cap H = C \cap H \quad (i = 1,2) \qquad \ldots\ldots(13)$$

Let $\bar{G} = G/N_1$. Then:

$\bar{C} = (\bar{C} \cap \bar{H})(\bar{C} \cap \bar{V})$, and $\bar{C}$ is a complement to $\overline{\langle b \rangle W_1}$ in $\bar{G}$ ....(14)

For,

$$CN_1 = (C \cap H)(C \cap V_1)(C \cap V_2)\ C_H(V_1)\ V_2 \qquad \text{(by (11))}$$

$$= (C \cap H)(C \cap V_1)\ V_2 \qquad \text{(using (13))},$$

and thus

$$CN_1 \cap HN_1 = (CN_1 \cap H)N_1 = (C \cap H)N_1 \text{ , and}$$

$$CN_1 \cap V_1N_1 = (CN_1 \cap V_1)N_1 = (C \cap V_1)N_1 \qquad \text{.........(15)}$$

This proves $\bar{C} = (\bar{C} \cap \bar{H})(\bar{C} \cap \bar{V})$. Also,

$$\begin{aligned}
& CN_1 \cap \langle b \rangle W_1N_1 \\
= \ & CN_1 \cap CV \cap \langle b \rangle W_1N_1 && (CN_1 \leqslant CV \text{ , by (12))} \\
= \ & CN_1 \cap (CV \cap \langle b \rangle)W_1N_1 && \text{(using (12) again)} \\
= \ & CN_1 \cap W_1N_1 \\
= \ & CN_1 \cap V_1N_1 \cap W_1N_1 \\
= \ & (C \cap V_1)N_1 \cap W_1N_1 && \text{(by (15))} \\
= \ & (C \cap V_1 \cap W_1N_1)N_1 \\
= \ & (C \cap (V_1 \cap N_1)W_1)\ N_1 && (\text{as } W_1 \leqslant V_1) \\
= \ & (C \cap W_1)N_1 \\
= \ & N_1 \ .
\end{aligned}$$

Hence $\bar{C} \cap \overline{\langle b \rangle W_1} = \bar{1}$. Further, as $W_2 \leqslant N_1$, it is clear that $\bar{C}\overline{\langle b \rangle W_1} = \bar{G}$, so $\bar{C}$ is a complement to $\overline{\langle b \rangle W_1}$ in $\bar{G}$, and therefore (14) is established.

Now 2.6.14 can be applied to the primitive soluble group $\overline{G}$, with $\overline{\langle b\rangle}$, $\overline{W_1}$, $\overline{C}$, $\overline{H}$, $\overline{V_1}$ in place of B_1 , U , C , H , V respectively (recall that $\overline{W_1}$ corresponds to the subgroup U constructed in 2.6.10) : this shows that $Z(\overline{H}) \leqslant \overline{C} \cap \overline{H}$. Therefore (invoking (10)),

$$F(\overline{H}) = \overline{H'} \times Z(\overline{H}) \leqslant \overline{C} \cap \overline{H} .$$

Since $\overline{F_1 \cap H} = F_2(\overline{G}) \cap \overline{H} = F(\overline{H})$, it follows that

$$\begin{aligned} F_1 \cap H &\leqslant CN_1 \cap H \\ &\leqslant CV \cap H \qquad \text{(by (12))} \\ &= C \cap H . \end{aligned}$$

Similarly, by considering G/N_2 , it can be shown that $F_2 \cap H \leqslant C \cap H$. Therefore

$$b = b_1 b_2 \in (F_1 \cap B)(F_2 \cap B) \leqslant C .$$

This is a contradiction, as C is a complement to $\langle b\rangle W_1 W_2$ in G . Therefore no such group as G can exist, and so the theorem is proved.

<u>2.7.3 Theorem</u> If

$$(|F^{(i)}(G)| , |F^{(j)}(G)|) = |F^{(i)}(G) \cap F^{(j)}(G)|$$

for all $i , j \in \{1,\ldots,t\}$, then $G \in \mathscr{C}_p$.

<u>Proof</u> As before, $F^{(i)}(G)$ will be abbreviated to F_i . Suppose that the condition on the F_i holds, and let L be an arbitrary pronormal subgroup of G ; to prove the

theorem, it will be enough to produce a complement to L in G.

Let $F = F_2(G)$; then $F = F_1 \cap \ldots \cap F_t$, by 2.7.1(c), and G/F is elementary abelian. Let $q_1, \ldots, q_r$ be the distinct prime divisors of $|G/F|$. Then
$G/F = S_1/F \times \ldots \times S_r/F$, where S_j/F is the Sylow q_j-subgroup of G/F $(j = 1,\ldots,r)$. Let

$$F_{ij} = F_i \cap S_j \quad (i = 1,\ldots,t \; ; \; j = 1,\ldots,r) \; ;$$

thus F_{ij}/F is the Sylow q_j-subgroup of F_i/F . For every $j \in \{1,\ldots,r\}$ and all $i, k \in \{1,\ldots,t\}$, either
$F_{ij} \leqslant F_{kj}$ or $F_{kj} \leqslant F_{ij}$ (for otherwise,
$(|F_{ij}/F|, |F_{kj}/F|) > |(F_{ij} \cap F_{kj})/F|$, which leads to a contradiction of the hypothesis of the theorem). Therefore, for each $j \in \{1,\ldots,r\}$ there is an ordering

$$F_{i_1,j} \leqslant F_{i_2,j} \leqslant \cdots \leqslant F_{i_t,j}$$

of $F_{1j}, \ldots, F_{tj}$. This fact is now used to construct a sequence of subgroups of S_j :

Let $C_{i_1,j}$ be such that $C_{i_1,j}/F$ is a complement to $(LF \cap F_{i_1,j})/F$ in $F_{i_1,j}/F$. Then, because S_j/F is elementary abelian, a complement $C_{i_2,j}/F$ in $F_{i_2,j}/F$ to $(LF \cap F_{i_2,j})/F$ can be constructed such that $C_{i_2,j}/F \cap F_{i_1,j}/F = C_{i_1,j}/F$. Continuing this process, subgroups $C_{i_1,j} \leqslant C_{i_2,j} \leqslant \ldots \leqslant C_{i_t,j}$ of S_j can be constructed such that, for each $k \in \{1,\ldots,t\}$, $C_{i_k,j}/F$ is a complement to $(LF \cap F_{i_k,j})/F$ in $F_{i_k,j}/F$, and

$$C_{i_k,j} \cap F_{i_\ell,j} = C_{i_\ell,j} \quad \text{for each } \ell \leqslant k .$$

The process can be carried one step farther, to produce a subgroup C_j of S_j such that C_j/F is a complement to $(LF \cap S_j)/F$ in S_j/F, and $C_j \cap F_{i_t,j} = C_{i_t,j}$. Hence

$$C_j \cap F_{ij} = C_{ij} \quad \text{for each } i \in \{1,\ldots,t\} .$$

Having constructed C_j for each $j \in \{1,\ldots,r\}$, let $C = C_1C_2\ldots C_r$, so that $C/F = C_1/F \times \ldots \times C_r/F$. Since $LF/F = LF \cap S_1 / F \times \ldots \times LF \cap S_r / F$, it follows that

C/F is a complement to LF/F in G/F ....(1)

Further, it will be shown that, for each $i \in \{1,\ldots,t\}$,

CF_i/F_i is a complement to LF_i/F_i in G/F_i ....(2)

Consider an arbitrary $i \in \{1,\ldots,t\}$. For each $j \in \{1,\ldots,r\}$, S_jF_i/F_i is the Sylow q_j-subgroup of G/F_i, $(LF \cap S_j)F_i / F_i$ is the Sylow q_j-subgroup of LF_i/F_i, and C_jF_i/F_i is the Sylow q_j-subgroup of CF_i/F_i ; hence, to prove (2), it is enough to show that:

C_jF_i/F_i is a complement to $(LF \cap S_j)F_i/F_i$ in S_jF_i/F_i
.....................(3)

It is clear that the two subgroups supplement each other in S_jF_i/F_i. Also,

$$\begin{aligned}
& C_jF_i \cap (LF \cap S_j)F_i \\
= \; & (C_j \cap (LF \cap S_j)F_i)\, F_i \\
= \; & (C_j \cap S_j \cap (LF \cap S_j)F_i)\, F_i \\
= \; & (C_j \cap (S_j \cap F_i)(LF \cap S_j))\, F_i \\
= \; & (C_j \cap F_{ij}(LF \cap S_j))\, F_i
\end{aligned}$$

$= (C_j \cap C_{ij}(LF \cap F_{ij})(LF \cap S_j))\ F_i$ (as C_{ij}/F is a complement to $(LF \cap F_{ij})/F$ in F_{ij}/F)

$= (C_j \cap C_{ij}(LF \cap S_j))\ F_i$

$= ((C_j \cap LF \cap S_j)C_{ij})\ F_i$ (as $C_{ij} \leqslant C_j$)

$= F\ C_{ij}\ F_i$ (as C_j/F is a complement to $(LF \cap S_j)/F$ in S_j/F)

$= F_i$.

Therefore (3) holds, and consequently (2) is proved.

Continuing with an arbitrary $i \in \{1,\dots,t\}$, let $\overline{G} = G/N_i$. $\overline{G}$ is a primitive soluble quotient group of G , so by one of the two overall assumptions of this section, $\overline{G} \in \mathcal{C}_p$. Now 2.6.16 is applied to $\overline{G}$. 2.6.18 shows that the first hypothesis of 2.6.16 is equivalent to the condition that the primitive soluble group which is the subject of the lemma is a $\mathcal{C}_p$-group, so it holds for $\overline{G}$. $F_2(\overline{G}) = \overline{F_i}$, $\overline{L}$ is pronormal in $\overline{G}$ (by 1.5.3 and 1.5.2), and $\overline{CF_i}/\overline{F_i}$ is a complement to $\overline{LF_i}/\overline{F_i}$ in $\overline{G}/\overline{F_i}$ (by (2) and the fact that $N_i \leqslant F_i$). By 1.5.12, $N_G(LV)$ normalizes $L \cap V$; therefore $N_G(LV)$ normalizes $(L \cap V)V_1 \dots V_{i-1} \cap V_i$. By 1.5.3, $N_{\overline{G}}(\overline{LV_i}) = N_{\overline{G}}(\overline{LV}) = \overline{N_G(LV)}$, so $N_{\overline{G}}(\overline{LV_i})$ normalizes $\overline{(L \cap V)V_1 \dots V_{i-1} \cap V_i}$. Therefore, by 2.6.16, $\overline{(L \cap V)V_1 \dots V_{i-1} \cap V_i}$ has a complement, $\overline{W_i}$ say (where $W_i \geqslant N_i$), in $\overline{V_i}$, such that $\overline{W_i}$ is normalized by $\overline{CF_i}$. Therefore CF_i normalizes $W_i \cap V_i$, which is a complement to $(L \cap V)V_1 \dots V_{i-1} \cap V_i$ in V_i . Let $U_i = W_i \cap V_i$.

Then, by 1.3.3(c), applied to $V = V_1V_2\ldots V_t$,

$U_1\ldots U_t$ is a complement to $L \cap V$ in V , and $U_1\ldots U_t$ is normalized by C ....(4)

Let B be a complement to $F \cap H$ (= $F(H)$) in H ; then $C = (C \cap B)F = (C \cap B)\, F(H)\, V$, and

$C \cap B$ is a complement to $LF \cap B$ in B ....(5)

$LV \cap H$ is pronormal in H , by 1.5.6, and H is metabelian, so by 2.1.5, $LV \cap F(H)$ is normal in H . Therefore there is a normal subgroup, N say, of H such that

$$F(H) = (LV \cap F(H)) \times N \text{ .} \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

Now $(LV \cap H)\, F(H) \cap B = LF \cap B$, so by (5), (6) and 1.3.3(b),

$(C \cap B)N$ is a complement to $LV \cap H$ in H ......(7)

$(C \cap B)N \leqslant C$, so $(C \cap B)N$ normalizes $U_1\ldots U_t$; therefore, by (4), (7) and 1.3.3(b) again, $(C \cap B)NU_1\ldots U_t$ is a complement to L in G . Therefore $G \in \mathcal{C}_p$. Q.e.d.

The results of the section are summarised in 2.7.4 , which, together with 2.6.19 , yields a complete description of $\mathcal{C}_p \cap \mathfrak{A}^3$.

<u>2.7.4 Theorem</u> Let G be a soluble group of derived length at most 3. Then $G \in \mathcal{C}_p$ if and only if all the following conditions are satisfied :

(a) $\Phi(G) = 1$;

(b) the primitive soluble quotient groups of G all belong to $\mathcal{C}_p$;

(c) if $F(G) = V_1 \times \dots \times V_t$, where the V_i are minimal normal subgroups of G , and $F_1, F_2, \dots, F_t \leqslant G$ are defined by

$$F_i/C_G(V_i) \;=\; F(\,G/C_G(V_i)\,) \qquad (i = 1,\dots,t),$$

then $(|F_i|, |F_j|) = |F_i \cap F_j|$ for each $i, j \in \{1,\dots,t\}$.

The question of further investigation of $\mathcal{C}_p$ seems to be difficult : it appears to be reasonable to conjecture that $\mathcal{C}_p \subseteq \mathfrak{A}^3$, in which case 2.7.4 would give a complete description of $\mathcal{C}_p$. However it is apparent that an investigation of that conjecture, using techniques analogous to those developed in this chapter, would be a formidable task.

Chapter 2 ends with two examples which substantiate the statement, made in section 2.1, that $\mathcal{C}_p$ is neither S_n-closed nor D_o-closed.

<u>2.7.5 Example</u> Let $\omega \in GF(5^2)$ be such that $\omega^3 = 1$, and define $H \leqslant GL(2, 5^2)$ by

$$H = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \right\rangle .$$

Then $H \cong S_3 \times C_3$. Let V be a $GF(5^2)$-space of dimension 2, and let G be the semidirect product HV (with a natural action of H on V). Then G is a primitive soluble group, and 2.6.18 can be applied to show that $G \in \mathscr{C}_p$.

Let $H_1 = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \cong S_3$; then H_1V is a normal subgroup of G, and it is easily seen that a primitive soluble quotient group of H_1V violates condition (b) of 2.6.19. Thus $H_1V \notin \mathscr{C}_p$, and hence $\mathscr{C}_p$ is not S_n-closed.

<u>2.7.6 Example</u> Define $H \leqslant GL(2,7)$ by

$$H = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \cong S_3 ,$$

let V be a $GF(7)$-space of dimension 2, and let G be the semidirect product HV (with a natural action); then G is a primitive soluble group. It is easy to check that G satisfies the conditions of 2.6.19, so $G \in \mathscr{C}_p$. However $G \times G$ violates condition (c) of 2.7.4, so $G \times G \notin \mathscr{C}_p$. Hence $\mathscr{C}_p$ is not D_o-closed. (The principle behind this example can of course be applied to <u>any</u> group G of derived length 3, to show that $G \times G \notin \mathscr{C}_p$).

CHAPTER 3

3.1 Groups with Frattini intersection supplements

Recall that $\mathcal{C}$ denotes the class of groups in which every subgroup has a complement (studied in H5). Chapter 3 is concerned with classes of groups of which $\mathcal{C}$ may be regarded as the archetype. The class considered in this section turns out to be no more than the "saturation" of $\mathcal{C}$, i.e. the smallest saturated formation containing $\mathcal{C}$. For convenience, Hall's description of $\mathcal{C}$ will first be set down. Let $\mathcal{R}$ denote the class of groups of square-free order, i.e. $G \in \mathcal{R}$ if and only if, for every prime number p, $p^2 \nmid |G|$.

3.1.1 Theorem (H5) The following are equivalent:

(1) $G \in \mathcal{C}$;

(2) $G \in SD_0\mathcal{R}$;

(3) G is supersoluble and the Sylow subgroups of G are all elementary abelian.

Definition A supplement S in a group G to a subgroup H of G is called a Frattini intersection supplement if $H \cap S \leqslant \Phi(G)$.

Throughout 3.1, $\mathfrak{F}$ will denote the class of groups in which every subgroup has a Frattini intersection supplement.

<u>3.1.2</u> $\mathfrak{F}$ is Q-closed.

<u>Proof</u> Let $G \in \mathfrak{F}$ and let N be a normal subgroup of G. Let H/N be a subgroup of G/N. Then H has a Frattini intersection supplement S in G. SN/N is a supplement to H/N in G/N, and

$$H/N \cap SN/N = (H \cap S)N/N \leqslant \Phi(G)N/N \leqslant \Phi(G/N) .$$

Therefore $G/N \in \mathfrak{F}$.

<u>3.1.3</u> If $G \in \mathfrak{F}$ and $\Phi(G) = 1$ then $G \in \mathcal{C}$.

<u>Proof</u> This is immediate from the definition of $\mathfrak{F}$.

<u>3.1.4</u> $G \in \mathfrak{F}$ if and only if $G/\Phi(G) \in \mathcal{C}$.

<u>Proof</u> Let $G \in \mathfrak{F}$. Then $G/\Phi(G) \in \mathfrak{F}$ by 3.1.2, and $G/\Phi(G)$ has trivial Frattini subgroup, so by 3.1.3, $G/\Phi(G) \in \mathcal{C}$.

Conversely, suppose $G/\Phi(G) \in \mathcal{C}$. Let $H \leqslant G$; then $H\Phi(G)/\Phi(G)$ has a complement, $S/\Phi(G)$ say, in $G/\Phi(G)$. Thus $HS = G$ and $H \cap S \leqslant H\Phi(G) \cap S = \Phi(G)$. Therefore $G \in \mathfrak{F}$.

Thus $\mathfrak{F} = E_{\Phi}\mathcal{C}$. The next theorem shows that $\mathfrak{F}$ is a saturated formation, and gives a local definition for $\mathfrak{F}$.

Note that the fact that $\mathfrak{C}$ is a formation does not immediately imply that $E_{\Phi}\mathfrak{C}$ is a formation : 3.1.6 exhibits an example of a formation $\mathfrak{X}$ such that $E_{\Phi}\mathfrak{X}$ is not R_0-closed.

<u>3.1.5 Theorem</u> $\mathfrak{F}$ is a saturated formation and is locally defined by the formation function f ,where for each prime number p , $G \in f(p)$ if and only if G is an elementary abelian group of exponent dividing $p-1$.

<u>Proof</u> Let $\mathfrak{F}^*$ be the saturated formation locally defined by the formation function f . It is well known that the class of supersoluble groups (which will be denoted hereafter by $\mathfrak{J}$) is a saturated formation, locally defined by the formation function f_0 , where for each prime number p , $G \in f_0(p)$ if and only if G is abelian and has exponent dividing $p-1$. It is clear that

$$f(p) = f_0(p) \cap \mathfrak{C} \quad \text{(for all } p) \qquad ....(1),$$

so in particular, $\mathfrak{F}^* \subseteq \mathfrak{J}$. Since $\mathfrak{C}$-groups are supersoluble, it is also true that $\mathfrak{F} = E_{\Phi}\mathfrak{C} \subseteq \mathfrak{J}$.

(i) $\mathfrak{F} \subseteq \mathfrak{F}^*$: let $G \in \mathfrak{F}$, let p be a prime number, and let H/K be a p-chief factor of G . $G \in \mathfrak{J}$, so $G/C_G(H/K) \in f_0(p)$. $F(G)$ is the intersection of the centralizers of the chief factors of G (H8, III, 4.3), so in particular, $\Phi(G) \leqslant F(G) \leqslant C_G(H/K)$. Therefore (using (1)) $G/C_G(H/K) \in f_0(p) \cap \mathcal{C} = f(p)$. Hence $G \in \mathfrak{F}^*$.

(ii) $\mathfrak{F}^* \subseteq \mathfrak{F}$: let $G \in \mathfrak{F}^*$; it will be enough to show $G/\Phi(G) \in \mathcal{C}$, so it is permissible to assume that $\Phi(G) = 1$, and show that $G \in \mathcal{C}$. $G \in \mathfrak{J}$, so by 1.4.5 , it is enough to show $G \in \mathcal{C}_n$.

$F(G) = \bigcap C_G(H/K)$, the intersection being taken over all the chief factors of G , and $G/C_G(H/K) \in f(p) \subseteq \mathcal{C}$ for each prime number p and each p-chief factor H/K of G . Therefore

$$G/F(G) \in R_0\mathcal{C} = \mathcal{C} \qquad \text{....................(2)}$$

By assumption, $\Phi(G) = 1$, so by (H8, III, 4.4),

every normal subgroup of G contained in $F(G)$ has a complement in G ....(3)

Let H be a normal subgroup of G ; then $HF(G)/F(G)$ has a complement in $G/\Phi(G)$, by (2), and $H \cap F(G)$ has a complement in G , by (3). Therefore, by 1.3.3(a), H has a complement in G . Hence $G \in \mathcal{C}_n$, and the proof is complete.

A similar study can be made of groups in which each <u>normal</u> subgroup has a Frattini intersection supplement. The class of all such groups coincides with $E_\Phi\mathcal{C}_n$, and is a saturated formation (though not consisting of soluble groups). In fact it can easily be proved that :

$$E_{\Phi}\mathcal{C}_n = \{G : G/F(G) \in \mathcal{C}_n\} = \mathfrak{N}\mathcal{C}_n .$$

It follows immediately that $E_{\Phi}\mathcal{C}_n$ is a saturated formation, as it is well known that, if $\mathfrak{X}$ is a formation then $\mathfrak{N}\mathfrak{X}$ is a saturated formation.

3.1.6 Example In (S1), Schunck gives an example which shows that a "saturated homomorph" (or "Schunck class") is not necessarily a formation. The example can also be used to show that, if $\mathfrak{X}$ is a formation, it does not necessarily follow that $E_{\Phi}\mathfrak{X}$ is a saturated formation.

Let $\mathfrak{X}$ be the class of groups defined by : $G \in \mathfrak{X}$ if and only if G is 2-perfect and has abelian Sylow 2-subgroups. (To say that a group is "π-perfect", for some set π of prime numbers, means that it has no non-trivial π-quotient groups). It follows from (H9), Lemma 1.6, that $\mathfrak{X}$ is a formation. $A_4 \in \mathfrak{X}$, and so $SL(2,3) \in E_{\Phi}\mathfrak{X}$ (the Frattini quotient group of $SL(2,3)$ is isomorphic to A_4).

Let $G = SL(2,3) \times \langle b\rangle$, where $\langle b\rangle \cong C_2$, and let $\langle a\rangle = \Phi(SL(2,3))$ ($\cong C_2$). Then $N_1 = \langle ab\rangle$ and $N_2 = \langle b\rangle$ are normal subgroups of G, $N_1 \cap N_2 = 1$, and $G/N_i \cong SL(2,3)$ for each i ; hence $G \in R_0(E_{\Phi}\mathfrak{X})$.

If $G \in E_{\Phi}\mathfrak{X}$ then, since $\Phi(G) = \langle a\rangle \cong C_2$, and so $G/\Phi(G) \cong A_4 \times C_2$, it must follow that either $G \in \mathfrak{X}$ or $A_4 \times C_2 \in \mathfrak{X}$. But neither of these groups is 2-perfect, therefore $G \notin E_{\Phi}\mathfrak{X}$. Hence $R_0(E_{\Phi}\mathfrak{X}) \neq E_{\Phi}\mathfrak{X}$, and so $E_{\Phi}\mathfrak{X}$ is not a formation.

3.2 Groups with $\mathfrak{X}$-intersection supplements.

In the last section an attempt was made to find a generalisation of the concept of a $\mathscr{C}$-group, but this did not lead very far from $\mathscr{C}$. A more general approach is tried in this section, although the basic idea, of putting restrictions on the intersection of a subgroup and a supplement, is retained.

Definition Let $\mathfrak{X}$ be a class of groups. If S is a supplement in a group G to a subgroup H of G, such that $H \cap S \in \mathfrak{X}$, then S is called an $\mathfrak{X}$-intersection supplement to H in G.

Notation Let $\mathscr{S}(\mathfrak{X})$ denote the class of groups in which every subgroup has an $\mathfrak{X}$-intersection supplement. I.e. $G \in \mathscr{S}(\mathfrak{X})$ if and only if for all $H \leqslant G$ there exists $S \leqslant G$ such that $HS = G$ and $H \cap S \in \mathfrak{X}$.

As an example, let $\mathfrak{X}$ be the class of groups of order 1 ; then $\mathscr{S}(\mathfrak{X}) = \mathscr{C}$. It is clear that, for any class $\mathfrak{X}$, $\mathscr{C} \subseteq \mathscr{S}(\mathfrak{X})$, and in fact $\mathfrak{X}^S\mathscr{C} \subseteq \mathscr{S}(\mathfrak{X})$.

Definition A subgroup H of a group G is said to be unsupplemented in G if H has no proper supplement in G, i.e. the only subgroup of G which is a supplement to H is G itself.

3.2.1 Any subgroup H of a group G has a supplement S in G such that $H \cap S$ is unsupplemented in G.

Proof Let S be a minimal supplement to H in G (i.e. no proper subgroup of S is a supplement to H in G). Suppose $H \cap S$ has a proper supplement, T say, in G. Then

$$\begin{aligned} H(T \cap S) &= H(H \cap S)(T \cap S) = H((H \cap S)T \cap S) \\ &= H(G \cap S) = HS = G, \end{aligned}$$

so by choice of S, $T \cap S = S$, i.e. $S \leqslant T$. But then $H \cap S \leqslant T$, so $T = (H \cap S)T = G$, a contradiction. Therefore $H \cap S$ is an unsupplemented subgroup of G.

3.2.2 Theorem For any group G and any class $\mathfrak{X}$, $G \in \mathfrak{S}(\mathfrak{X})$ if and only if the unsupplemented subgroups of G are all $\mathfrak{X}$-groups.

Proof (1) Let $G \in \mathfrak{S}(\mathfrak{X})$ and suppose that H is an unsupplemented subgroup of G. Since G is the only supplement to H in G, it must follow that $H = H \cap G \in \mathfrak{X}$.

(2) Suppose that the unsupplemented subgroups of G are all $\mathfrak{X}$-groups, and let H be any subgroup of G. By 3.2.1, H has a supplement S in G such that $H \cap S$ is unsupplemented in G. Hence $H \cap S \in \mathfrak{X}$, and therefore $G \in \mathfrak{S}(\mathfrak{X})$.

Theorem 3.2.2 effectively reduces the study of $\mathcal{S}(\mathfrak{X})$ to the question of which subgroups of a given group are unsupplemented. However this question is one to which a complete answer is difficult to find; the following series of results gives some partial information on the subject. One interesting consequence of 3.2.2 is that (taking $\mathfrak{X}$ to be the class of groups of order 1), if every non-trivial subgroup of G has a proper supplement in G then $G \in \mathcal{C}$.

3.2.3 (a) If H is unsupplemented in G then the same is true of any subgroup of H .

(b) If $H \leqslant G$ and K is unsupplemented in H , then K is unsupplemented in G .

(c) If H is unsupplemented in G and N is normal in G then HN/N is unsupplemented in G/N .

Proof (a) A supplement in G to a subgroup of H is also a supplement in G to H .

(b) Suppose S is a supplement to K in G . Then $K(S \cap H) = H$, so $S \cap H = H$ and hence $K \leqslant H \leqslant S$. Therefore $S = G$.

(c) Suppose S/N is a supplement to HN/N in G/N . Then $HS = HNS = G$, and so $S = G$. Thus HN/N is unsupplemented in G/N .

From 3.2.3(a) it follows that, if all the unsupplemented subgroups of G are $\mathfrak{X}$-groups, then they are in fact all $\mathfrak{X}^S$-groups. Hence:

3.2.4 For any class of groups $\mathfrak{X}$, $\mathcal{S}(\mathfrak{X}) = \mathcal{S}(\mathfrak{X}^S)$.

3.2.5 (a) (H8, III, 3.2(b)) If N is normal in G then N is unsupplemented in G if and only if $N \leqslant \Phi(G)$.
(b) If M is an abelian minimal normal subgroup of G then every proper subgroup of M is unsupplemented in G.

Proof (a) If there exists a maximal subgroup H of G such that $N \nleqslant H$, then $HN = G$, and so N has a proper supplement in G.

(b) Suppose $H < M$. A supplement S to H in G is also a supplement to M in G. Thus $M \cap S$ is normal in G, so $M \cap S = 1$ or M. But $M \cap S > 1$, because $M \cap S$ is a supplement to H in M and $H < M$; therefore $M \cap S = M$. Thus $H \leqslant S$, and hence $S = G$.

3.2.6 If H is unsupplemented in G then $H \leqslant G^{\mathfrak{C}}$.
Proof Let $N = G^{\mathfrak{C}}$, and let $H \leqslant G$. Then HN/N has a complement, C/N say, in G/N. If $H \nleqslant N$ then $HN > N$, so $C < G$, and therefore H has a proper supplement (namely C) in G.

3.2.7 Let G be a group and let $\mathfrak{M}_1, \ldots, \mathfrak{M}_r$ be the distinct conjugacy classes of maximal subgroups of G. For each i, choose $M_i \in \mathfrak{M}_i$; then every subgroup of G contained in $M_1 \cap \ldots \cap M_r$ is unsupplemented in G.

Proof Suppose $H \leqslant M_1 \cap \ldots \cap M_r$, and H has a proper supplement S in G. Let M be a maximal subgroup of G which contains S; thus $HM = G$. Without loss of generality it can be assumed that $M = M_1{}^g$ for some $g \in G$; but then $HM_1 = G$, a contradiction. Hence H is unsupplemented in G.

In (G1), Gaschütz defined the praefrattini subgroups of a soluble group, and showed that a subgroup W of a soluble group G is a praefrattini subgroup of G if and only if, (1) W covers each Frattini chief factor of G, and (2) W is contained in some conjugate in G of each maximal subgroup of G. From (2) it follows, by 3.2.7, that the praefrattini subgroups of G are unsupplemented in G.

Praefrattini subgroups, as well as covering Frattini chief factors, avoid complemented chief factors. It is not true that every subgroup of a soluble group G which avoids all the complemented chief factors of G, is contained in a praefrattini subgroup of G, but it will be shown that any such subgroup is unsupplemented in G: this is the result of 3.2.9.

<u>3.2.8</u> (H7, Theorems 13, 14) If G is soluble and has a faithful irreducible representation of degree n over $GF(p)$, then

$$|G|_p \leqslant p^{n-1}.$$

($|G|_p$ denotes the order of the Sylow p-subgroups of G).

<u>3.2.9 Theorem</u> If G is soluble and $H \leqslant G$ avoids all the complemented chief factors of G, then H is unsupplemented in G.

<u>Proof</u> Suppose that the result is false, and let G be a minimal counterexample. Then G has a subgroup H which avoids all the complemented chief factors of G, and yet has a proper supplement in G. Let M be a maximal subgroup of G which supplements H, and suppose that M contains a minimal normal subgroup N of G. Then M/N is a proper supplement to HN/N in G/N, and HN/N avoids all the complemented chief factors of G/N ; but this contradicts the choice of G. Therefore M has trivial core, and so G is a primitive soluble group.

Let V be the unique minimal normal subgroup of G, and suppose $|V| = p^n$, p being a prime number. The maximal subgroup M will be a complement to V in G, and hence $|G:M| = |V| = p^n$. Now $HM = G$, so $|H:H \cap M| = |HM|/|M| = |G:M| = p^n$, and therefore $|H|_p \geqslant p^n$. Since H avoids all the complemented chief factors of G, $H \cap V = 1$, and therefore

$p^n \leqslant |H|_p = |HV/V|_p \leqslant |G/V|_p = |M|_p$, i.e.

$$|M|_p \geqslant p^n \qquad \text{.......................(1)}$$

But V can be regarded as a faithful irreducible $Z_p[M]$-module, of dimension n , and hence, by 3.2.8 ,

$$|M|_p \leqslant p^{n-1} \qquad \text{......................(2)}$$

Inequalities (1) and (2) are incompatible, so no such group as G can exist.

<u>3.2.10</u> If $\Phi(G) = 1$ then each normal subgroup of G contained in $F(G)$ has a complement in G . If in addition G is supersoluble, then every subgroup of $F(G)$ has a complement in G .

<u>Proof</u> The first result has been observed previously (in the proof of 3.1.5), and is a direct consequence of (H8, III, 4.4). Now suppose that G is supersoluble, and let $H \leqslant F(G)$. Let K be a complement to $F(G)$ in G , and write $F(G) = V_1 \times \ldots \times V_t$, where the V_i are minimal normal subgroups of G : each V_i is of prime order. It can be assumed that the indexing is chosen so that, for some $s \in \{1,\ldots,t\}$, $H \cap (V_1 \times \ldots \times V_s) = 1$, but $H \cap (V_1 \times \ldots \times V_s \times V_i) > 1$ for any $i > s$. Write $V = V_1 \times \ldots \times V_s$, and suppose that $HV < F(G)$. Then $V_i \nleqslant HV$ for some $i \in \{1,\ldots,t\}$. Since V_i has prime order, it follows that $V_i \cap HV = 1$, and hence $H \cap (V \times V_i) = 1$, a contradiction. Hence $HV = F(G)$, and

therefore KV is a complement to H in G .

The next theorem gives complete information about the unsupplemented subgroups of G when G is either a nilpotent or a supersoluble group.

<u>3.2.11 Theorem</u> (a) If G is nilpotent then H is unsupplemented in G if and only if $H \leqslant \Phi(G)$.

(b) If G is supersoluble then H is unsupplemented in G if and only if $H \leqslant W$ for some praefrattini subgroup W of G .

<u>Proof</u> (a) In a nilpotent group G , $G/\Phi(G)$ is elementary abelian, so $G^{\mathcal{C}} \leqslant \Phi(G)$ (in fact $G^{\mathcal{C}} = \Phi(G)$). The result follows, by 3.2.6 .

(b) That subgroups of praefrattini subgroups are unsupplemented is immediate from 3.2.3(a) and the remarks following 3.2.7 .

For the converse, suppose that H is unsupplemented in G , and proceed by induction on $|G|$. If $T = \Phi(G) > 1$, then by induction there is a praefrattini subgroup W/T of G/T such that $HT/T \leqslant W/T$ (HT/T is unsupplemented in G/T , by 3.2.3(c)); then W is a praefrattini subgroup of G and $H \leqslant W$. Thus it can be assumed that $\Phi(G) = 1$; hence, by 3.2.10, every subgroup of $F(G)$ has a complement in G . Therefore $H \cap F(G) = 1$, since otherwise a complement to $H \cap F(G)$ in G would be a proper supplement to H in G, contradicting the hypothesis that H is unsupplemented in G.

Let p be the largest prime divisor of $|G|$. Since G is supersoluble, G has a normal Sylow p-subgroup, which is consequently contained in $F(G)$. Thus $p \nmid |G/F(G)|$, and in particular, $p \nmid |H|$.

Let V be a minimal normal subgroup of G, of order p. By induction, $HV/V \leqslant WV/V$ for some praefrattini subgroup W of G, and hence $H \leqslant WV$. Every complement to V in G contains a p-complement of G, and hence contains a conjugate of H; therefore $H \leqslant K$ for some complement K to V in G. K is a maximal subgroup of G, so W, being a praefrattini subgroup of G, is contained in some conjugate of K. Thus, since $G = KV$, there exists an element x in V with $W^x \leqslant K$. Then $WV = W^xV$, and so

$$H \leqslant WV \cap K = W^xV \cap K = W^x(V \cap K) = W^x ,$$

i.e. H is contained in W^x, a praefrattini subgroup of G. Q.e.d.

The result of 3.2.11(b) is false for any soluble group G which is not supersoluble. For, if G is such a group, then by (H8, VI, 9.9) there is a non-cyclic chief factor of G between $\Phi(G)$ and $F(G)$, and therefore, since

$F(G)/\Phi(G)$ can be expressed as a direct product of minimal normal subgroups of $G/\Phi(G)$, G has a non-cyclic chief factor of the form $H/\Phi(G)$. Choosing U such that $\Phi(G) < U < H$, it follows from 3.2.5(b) that $U/\Phi(G)$ is unsupplemented in $G/\Phi(G)$, and hence (by 3.3.1(a) in the next section) U is unsupplemented in G . U does not avoid the complemented chief factor $H/\Phi(G)$ of G , so U cannot be contained in a praefrattini subgroup of G .

The last of this series of results on unsupplemented subgroups is concerned with the unsupplemented subgroups of a direct product.

<u>3.2.12</u> Let G_1 and G_2 be groups. Then U is an unsupplemented subgroup of $G_1 \times G_2$ if and only if there exist unsupplemented subgroups U_i of G_i $(i = 1,2)$ such that $U \leq U_1 \times U_2$.

<u>Proof</u> Let $G = G_1 \times G_2$. Suppose U is unsupplemented in G , and let π_i be the projection $G \to G_i$ $(i = 1,2)$. Then by 3.2.3(c), $U\pi_i$ is unsupplemented in G_i for each i . Also $U \leq U\pi_1 \times U\pi_2$.

For the converse, by 3.2.3(a) it will be enough to prove that, if U_i is unsupplemented in G_i $(i = 1,2)$ then $U_1 \times U_2$ is unsupplemented in G . Let $U = U_1 \times U_2$, and suppose that S is a supplement to U in G. Let $S_1 = S \cap G_1U_2$. Then S_1U_2 is a group, and is thus a supplement to U_1 in G_1U_2. Therefore $S_1U_2 \cap G_1$ is a supplement to U_1 in G_1, and hence $S_1U_2 \cap G_1 = G_1$, i.e. $S_1U_2 \geqslant G_1$. A fortiori, $G_1 \subseteq SU_2$. Therefore

$$G = SU = SU_2U_1 \leqslant SU_2G_1 = SU_2,$$

and so S is a supplement to U_2 in G. But U_2 is unsupplemented in G, by 3.2.3(b); hence $S = G$. Therefore U is unsupplemented in G. Q.e.d.

3.2.12 suggests the conjecture that, if N_1 and N_2 are normal subgroups of G, $N_1 \cap N_2 = 1$, and $H \leqslant G$ is such that HN_i/N_i is unsupplemented in G/N_i $(i = 1,2)$, then H is unsupplemented in G. However Example 1.3.5 can be used to refute this conjecture: with the notation of that example, V_i is normal in G $(i = 1,2)$, $V_1 \cap V_2 = 1$, and HV_i/V_i is unsupplemented in G/V_i $(i = 1,2)$, but H has a complement in G.

Some of the information which has been obtained about unsupplemented subgroups is now put to use in making further observations about $\mathcal{S}(\mathfrak{X})$.

3.2.13 For any class of groups $\mathfrak{X}$, $\mathcal{S}(\mathfrak{X})$ is S-closed. (Since it has already been seen that $\mathcal{S}(\mathfrak{X}) = \mathcal{S}(\mathfrak{X}^S)$, this result is not surprising).

Proof Let $G \in \mathcal{S}(\mathfrak{X})$ and let $H \leqslant G$. If U is an unsupplemented subgroup of H, then by 3.2.3(b), U is unsupplemented in G, and therefore, by 3.2.2, $U \in \mathfrak{X}$. Hence, applying 3.2.2 again, it follows that $H \in \mathcal{S}(\mathfrak{X})$.

3.2.14 Theorem Let $\mathfrak{X}$ be any class of groups.

(a) If G is nilpotent then $G \in \mathcal{S}(\mathfrak{X})$ if and only if $\Phi(G) \in \mathfrak{X}^S$.

(b) If G is supersoluble then $G \in \mathcal{S}(\mathfrak{X})$ if and only if the praefrattini subgroups of G belong to $\mathfrak{X}^S$.

Proof (a) is immediate from 3.2.2 and 3.2.11(a); (b) is immediate from 3.2.2 and 3.2.11(b).

3.2.13 shows that $\mathcal{S}(\mathfrak{X})$ is always S-closed; in 3.2.15 the closure properties of $\mathcal{S}(\mathfrak{X})$ are examined further.

3.2.15 (a) If $\mathfrak{X}$ is Q-closed then so is $\mathcal{S}(\mathfrak{X})$.

(b) If $\mathfrak{X}$ is R_0-closed then so is $\mathcal{S}(\mathfrak{X})$.

Proof Let $\mathfrak{X} = Q\mathfrak{X}$, let $G \in \mathcal{S}(\mathfrak{X})$, and let N be a normal subgroup of G. Let $H/N \leqslant G/N$. Then H has a supplement S in G such that $H \cap S \in \mathfrak{X}$. It follows that SN/N is a supplement to H/N in G/N and $H/N \cap SN/N = (H \cap S)N/N \in Q\mathfrak{X} = \mathfrak{X}$. Therefore $G/N \in \mathcal{S}(\mathfrak{X})$.

(b) Suppose $\mathfrak{X} = R_0\mathfrak{X}$, and suppose that G is a group with normal subgroups N_1 and N_2 such that $N_1 \cap N_2 = 1$ and G/N_1, G/N_2 both belong to $\mathcal{S}(\mathfrak{X})$. Suppose that U is an unsupplemented subgroup of G; then by 3.2.3(c), UN_i/N_i is unsupplemented in G/N_i for each i. Hence, by 3.2.2, $UN_i/N_i \in \mathfrak{X}$. But then $U/(U \cap N_1)$ and $U/(U \cap N_2)$ both belong to $\mathfrak{X}$, and therefore $U \in R_0\mathfrak{X} = \mathfrak{X}$. Therefore, by 3.2.2 again, $G \in \mathcal{S}(\mathfrak{X})$, and so $\mathcal{S}(\mathfrak{X})$ is R_0-closed.

3.3 Nilpotent, abelian, and cyclic intersection supplements

The properties of $\mathcal{S}(\mathfrak{X})$ are now considered for particular choices of $\mathfrak{X}$. In analogy with the archetype $\mathfrak{C} = \mathcal{S}(1)$, the kind of properties sought are, e.g., solubility, bounds on the derived length of soluble $\mathcal{S}(\mathfrak{X})$-groups, and bounds on the rank of chief factors.

First consider $\mathcal{S}(\mathfrak{N})$, where $\mathfrak{N}$ is the class of nilpotent groups. The results of 3.2 show immediately that

$$\mathfrak{N} \subseteq \mathcal{S}(\mathfrak{N}) = \{Q, S, R_0\}\mathcal{S}(\mathfrak{N}) .$$

It seems very reasonable to conjecture that $\mathcal{S}(\mathfrak{N})$-groups are necessarily soluble, although this question has not been settled. Only soluble groups in $\mathcal{S}(\mathfrak{N})$ will be considered here, so for convenience the following notation is introduced: for any class of groups $\mathfrak{X}$, let

$$\mathcal{S}^*(\mathfrak{X}) = \mathcal{S}(\mathfrak{X}) \cap E\mathfrak{A}.$$

3.3.1 (a) Let N be a normal subgroup of a group G, and let S be a minimal supplement to N in G (hence $N \cap S$ is unsupplemented in G). Suppose that $U \geqslant N$ and U/N is unsupplemented in G/N. Then $U \cap S$ is an unsupplemented subgroup of G.

(b) Suppose that G has an abelian minimal normal subgroup V which has a unique conjugacy class of complements in G. Let H be a complement to V in G, and suppose that U is an unsupplemented subgroup of H. If U normalizes a proper subgroup W of V, then UW is unsupplemented in G.

<u>Proof</u> (a) Suppose that T is a supplement to $U \cap S$ in G. Then $T \cap S$ is a supplement to $U \cap S$ in S, and so $(T \cap S)N/N$ is a supplement to $(U \cap S)N/N$ ($= U/N$) in G/N. Therefore $(T \cap S)N = G$, because U/N is unsuppl-mented in G/N. Hence

$$(N \cap S)(T \cap S) = N(T \cap S) \cap S = S ,$$

so

$$(N \cap S)T = (N \cap S)(T \cap S)T = ST \geqslant (U \cap S)T = G .$$

Thus $T = G$, because $N \cap S$ is unsupplemented in G. Therefore $U \cap S$ is unsupplemented in G.

(b) Suppose that S is a supplement to UW in G. Then SV/V is a supplement to UV/V in G/V. By 3.2.3(b), U is unsupplemented in G, so by 3.2.3(c), UV/V is unsupplemented in G/V. Therefore $SV = G$. Hence, because V is abelian and normal in G, $S \cap V$ is normal in G, and so, since V is a minimal normal subgroup of G, either (1) $S \cap V = 1$ or (2) $S \cap V = V$.

If (1) holds then S is a complement to V in G, so by hypothesis $S = H^g$ for some $g \in G$. Therefore, since a conjugate of a supplement is itself a supplement, H is a supplement to UW in G. But this is impossible, because

$$|HUW| = |HW| = |H||W| < |G| .$$

Therefore (2) holds, hence $V \leqslant S$, and therefore $S = G$. Thus UW is unsupplemented in G Q.e.d.

If in 3.3.1(b) the condition that the complements of V are all conjugate is omitted, then the result fails. Example 1.3.5 can be used to illustrate this: in the notation of that example, the minimal normal subgroup V_1 has a complement $H_1 = \langle x \rangle V_2$, and H_1 has an unsupplemented subgroup $U = \langle w_2 \rangle$. $\langle w_2 \rangle$ normalizes the proper subgroup $W = \langle v_1 \rangle$ of V_1, but $UW = \langle w_2 \rangle \langle v_1 \rangle$ is not an unsupplemented subgroup of G, because $\langle x, v_1 v_2, w_1 w_2 \rangle$ is a complement to UW in G.

<u>3.3.2</u> Let p be a prime number and n a positive integer. Let $\pi = \pi(p,n)$ be the set of prime numbers q such that $q \mid (p^n - 1)$, but $q \nmid (p^r - 1)$ for $r < n$. Then for all $q \in \pi$, the Sylow q-subgroups of $GL(n,p)$ are cyclic.

<u>Proof</u> Let $q \in \pi$ and let q^a be the order of the Sylow q-subgroups of $GL(n,p)$. It is well known that

$$|GL(n,p)| = p^{\frac{1}{2}n(n-1)}(p^n - 1)(p^{n-1} - 1)\ldots(p-1) ,$$

and therefore $q^a \mid (p^n - 1)$ (since q does not divide any other factor in the above expression for $|GL(n,p)|$). $GL(n,p)$ has a cyclic subgroup of order $(p^n - 1)$, corresponding to the multiplicative group of $GF(p^n)$, and therefore $GL(n,p)$ has a cyclic subgroup of order q^a .

<u>3.3.3 Theorem</u> Let G be a primitive soluble group in $\mathcal{S}(\mathfrak{N})$, with unique minimal normal subgroup V (of order p^n , say). Let H be a complement to V in G . Then H is supersoluble and metabelian, and thus $G \in \mathfrak{A}^3$.

<u>Proof</u> Regard V as a faithful irreducible $Z_p[H]$-module. Suppose that U is a non-trivial unsupplemented p'-subgroup of H. $H \in Q\mathcal{S}(\mathfrak{N}) = \mathcal{S}(\mathfrak{N})$, so U is nilpotent. Let x be a non-trivial element of U ; x is a p'-element, so by Maschke's Theorem,

$$V = V_1 \oplus \ldots \oplus V_r ,$$

where the V_i are irreducible $Z_p[\langle x\rangle]$-modules. x cannot act trivially on all the V_i , since V is a faithful $\langle x\rangle$-module, so assume without loss of generality that x acts non-trivially on V_1 . Then $\langle x\rangle V_1$ is a non-nilpotent subgroup of G . If $V_1 < V$, then $\langle x\rangle V_1$ is unsupplemented in G , by 3.3.1(b); but $G \in \mathcal{S}(\mathfrak{N})$, so this cannot be so, and therefore $V_1 = V$, i.e. $\langle x\rangle$ acts irreducibly on V .

Thus V is a faithful irreducible $Z_p[\langle x\rangle]$-module, of degree n, so, by (H8, II, 3.10), x is a π-element, where $\pi = \pi(p,n)$ is as defined in 3.3.2 . Hence U is a nilpotent π-group, and is therefore cyclic, by 3.3.2 . Thus :

All the non-trivial unsupplemented p'-subgroups of H are cyclic π-groups, and act irreducibly on V ......(1)

Let N be a minimal normal subgroup of H . Because H has a faithful irreducible representation over Z_p, $O_p(H)$ must be trivial (H8, $\underline{\overline{V}}$, 5.17), and so N is a p'-group. Suppose that N is not cyclic, and choose U such that $1 < U < N$. Then, by 3.2.5(b), U is unsupplemented in H, so, by (1), U acts irreducibly on V, and hence, a fortiori, N acts (faithfully and) irreducibly on V . But N is non-cyclic, so this contradicts (H8, II, 3.10). Therefore :

The minimal normal subgroups of H are all cyclic ....(2)

Another consequence of the fact that $O_p(H) = 1$ is that $\Phi(H)$ is a p'-group, and hence, by (1),

$\Phi(H)$ is a cyclic π-group ....(3)

To show that H is supersoluble, it will be enough, by (H8, VI, 9.9), to show that the chief factors of H between $\Phi(H)$ and $F(H)$ are cyclic. Since $F(H)/\Phi(H)$ is a direct product of minimal normal subgroups of $H/\Phi(H)$, it will therefore be enough to show that the minimal normal subgroups of $H/\Phi(H)$ are cyclic. Let $M/\Phi(H)$ be a minimal normal subgroup of $H/\Phi(H)$. If $M/\Phi(H)$ is a π'-group, then (because $\Phi(H)$ is a π-group, by (3)) $M = \Phi(H) \times N$, where N is a minimal normal subgroup of H ; N is cyclic, by (2), and so $M/\Phi(H)$ is cyclic. If, on the other hand, $M/\Phi(H)$ is a π-group, then $M/\Phi(H)$ is a q-group for some $q \in \pi$; the Sylow q-subgroups of H are cyclic, by 3.3.2 , so $M/\Phi(H)$ is cyclic . Therefore H is supersoluble.

Let F_π and $F_{\pi'}$ be the Hall π-subgroup and Hall π'-subgroup respectively of $F(H)$; thus $F(H) = F_\pi \times F_{\pi'}$. F_π has cyclic Sylow subgroups, and hence is itself cyclic; $F_{\pi'}$ has trivial Frattini subgroup, and is therefore abelian. Because H is supersoluble, $H/F(H)$ is also abelian (by H8, VI, 9.1(b)). Hence H is metabelian.

<u>3.3.4 Theorem</u> Let $G \in \mathcal{S}^*(\mathfrak{N})$. Then $G/F(G)$ is supersoluble and metabelian. Hence $G \in E_\Phi \mathfrak{A}^3$.

<u>Proof</u> Since $F(G/\Phi(G)) = F(G)/\Phi(G)$, (by H8, III, 4.2(d)), it will be enough to prove the result for $G/\Phi(G)$, and hence it can be assumed that $\Phi(G) = 1$. Then $F(G)$ can be decomposed, as $F(G) = V_1 \times \dots \times V_t$ say, into a direct product of minimal normal subgroups of G , and $F(G)$ has a complement, H say, in G . Also

$$F(G) = C_G(F(G)) = \bigcap_{i=1}^{t} C_G(V_i) .$$

As in the preliminaries of Section 2.7, let

$$N_i = C_H(V_i)\, V_1 \dots \hat{V}_i \dots V_t \qquad (i = 1, \dots, t).$$

Then N_i is normal in G, and G/N_i is a primitive soluble group, with unique minimal normal subgroup $C_G(V_i)/N_i$. Also $G/N_i \in Q\mathcal{S}^*(\mathfrak{N}) = \mathcal{S}^*(\mathfrak{N})$, so, by 3.3.3, $G/C_G(V_i)$ is supersoluble and metabelian. Since the class of supersoluble groups and the class of metabelian groups are both R_0-closed, it follows that $G/\bigcap_{i=1}^{t} C_G(V_i)$, i.e. $G/F(G)$, is supersoluble and metabelian. Q.e.d.

A conjecture which might be considered as a possible converse to 3.3.4 is that all supersoluble metabelian groups belong to $\mathcal{S}(\mathfrak{N})$. But this is not the case; e.g. let G be the split extension of $\langle a \rangle = C_{25}$ by $\langle x \rangle = C_4$, with action defined by $a^x = a^7$. Then $U = \langle a^5, x^2 \rangle$ avoids the complemented chief factors of G, and so, by 3.2.9, U is unsupplemented in G. But U is not nilpotent, so $G \notin \mathcal{S}(\mathfrak{N})$.

Now consider $\mathfrak{S}(\mathfrak{A})$, where $\mathfrak{A}$ denotes the class of abelian groups. From the results of 3.2,

$$\mathfrak{S}(\mathfrak{A}) = \{Q, S, R_0\}\mathfrak{S}(\mathfrak{A}) .$$

The question of solubility of $\mathfrak{S}(\mathfrak{A})$-groups remains open. Nevertheless, as in the case of $\mathfrak{S}(\mathfrak{N})$, only soluble $\mathfrak{S}(\mathfrak{A})$-groups will be investigated. The only results given here are immediate consequences of Theorem 3.3.4 .

<u>3.3.5 Theorem</u> Let $G \in \mathfrak{S}^*(\mathfrak{A})$. Then $G/F(G)$ is metabelian and supersoluble, and $G \in \mathfrak{A}^4$.

<u>Proof</u> It follows at once from 3.3.4 that $G/F(G)$ is metabelian and supersoluble, and $G/\Phi(G) \in \mathfrak{A}^3$. Also, $\Phi(G)$ unsupplemented in G , so $\Phi(G)$ is abelian. Therefore $G \in \mathfrak{A}^4$.

<u>3.3.6 Example</u> The unsupplemented subgroups of $GL(2,3)$ are all cyclic. Therefore $GL(2,3) \in \mathfrak{S}^*(\mathfrak{A})$, and so the bound of 4 on the derived length of $\mathfrak{S}^*(\mathfrak{A})$-groups cannot be improved.

<u>Proof</u> Let $G = GL(2,3)$ and let $N = F(G)$. Thus N is a quaternion group, and $G/N \cong S_3$. Therefore $G/N \in \mathfrak{C}$, and so, by 3.2.6 , all the unsupplemented subgroups of G are contained in N . N itself has a complement in G , so

all the unsupplemented subgroups of G are proper subgroups of N , and are therefore cyclic.

$\mathfrak{S}(\mathfrak{N})$ and $\mathfrak{S}(\mathfrak{A})$ may be far removed from the archetype $\mathfrak{C}$, because it may be that neither class consists solely of soluble groups. However, 3.3.7 gives a guarantee of the solubility of $\mathfrak{S}(\mathfrak{X})$-groups, provided that a severe restriction is placed on the class $\mathfrak{X}$. The proof of 3.3.7 requires the use of two well-known, but very deep, results, namely the theorem of Feit and Thompson, that a group of odd order is soluble, and the results of Brauer and Suzuki, which show that a group with a quaternion or generalized quaternion Sylow 2-subgroup cannot be simple.

<u>3.3.7 Theorem</u> If $\mathfrak{X}$ is a class of groups such that $C_2 \times C_2 \notin Q\mathfrak{X}$, then $\mathfrak{S}(\mathfrak{X}) \subseteq E\mathfrak{A}$.

<u>Proof</u> It will be enough to prove that $\mathfrak{S}(Q\mathfrak{X}) \subseteq E\mathfrak{A}$, and hence it can be assumed that $\mathfrak{X}$ is Q-closed.

Suppose that the result is false, and let G be a minimal non-soluble group belonging to $\mathfrak{S}(\mathfrak{X})$. By 3.2.13 , 3.2.15(a) , and the added hypothesis that $\mathfrak{X}$ is Q-closed,

$\mathfrak{S}(\mathfrak{X})$ is QS-closed, so G must be a simple group.

Let G_2 be a Sylow 2-subgroup of G, and suppose that G_2 has a subgroup $H \cong C_2 \times C_2$. Now $G \in \mathfrak{S}(\mathfrak{X})$, so H has a supplement S in G such that $H \cap S \in \mathfrak{X}$; since $C_2 \times C_2 \notin \mathfrak{X}$, it follows that $H \cap S < H$, and so $|H \cap S|$ is either 1 or 2. Then $|G : S| = |H : H \cap S| = 2$ or 4; therefore, considering the permutation representation of G on the cosets of S, and recalling that G is simple, it follows that G is isomorphic to a subgroup of the soluble group S_4. This is of course a contradiction, so G_2 cannot contain a subgroup isomorphic to $C_2 \times C_2$. Therefore (since $G_2 > 1$, by the Feit-Thompson Theorem) G_2 contains precisely one element of order 2, so, by (SL, 9.7.3), G_2 is either cyclic or quaternion or generalised quaternion. The work of Brauer and Suzuki precludes the last two possibilities. If the first arises, then, by the well-known "Burnside Transfer Theorem", G has a normal 2-complement, so G is not simple, a contradiction. Therefore such a group as G cannot exist, and the theorem is proved.

Perhaps the most natural class to choose which has the property of $\mathfrak{X}$ in 3.3.7 is the class of cyclic groups, which will be denoted here by $\mathfrak{L}$. The remainder of this section is devoted to deriving some information about $\mathfrak{S}(\mathfrak{L})$.

<u>3.3.8</u> Suppose V_1 and V_2 are abelian minimal normal subgroups of a group G, which are not G-isomorphic. If $U_1 < V_1$ and $U_2 < V_2$ then U_1U_2 is unsupplemented in G.

<u>Proof</u> By 3.2.5(b), it can be assumed that both U_1 and U_2 are non-trivial. Suppose that S is a proper supplement to U_1U_2 in G. Then $SV_1V_2 = G$, and hence, since V_1V_2 is abelian, $S \cap V_1V_2$ is normal in G. Thus, because $V_1 \not\cong_G V_2$, $S \cap V_1V_2 = V_1$ or V_2. (If $S \cap V_1V_2 = V_1V_2$ then $S = G$). But then $S \cap V_1V_2$ is not a supplement to U_1U_2 in V_1V_2, a contradiction. Therefore U_1U_2 is unsupplemented in G.

Two standard definitions are now needed:

<u>Definitions</u> Let G be a soluble group and H/K a chief factor of G; suppose that H/K is a p-group and $|H/K| = p^n$. Then n is called the <u>rank</u> of the chief factor H/K.

If $N \leqslant M$ and N and M are normal subgroups of G, then $r_G(M/N)$ denotes the maximum of the ranks of the chief factors of G between N and M. $r_G(G/1)$ is abbreviated to $r(G)$, and called the <u>rank of G</u>.

<u>3.3.9 Theorem</u> Suppose $G \in \mathcal{S}(\mathfrak{L})$. Then

(a) G is soluble;

(b) $G/F(G)$ is supersoluble and metabelian,

$r_G(\ F(G)/\Phi(G)\ ) \leqslant 2$, and $\Phi(G)$ is cyclic;

(c) the praefrattini subgroups of G are cyclic;

(d) for any given prime number p , any two p-chief factors of G' of rank 2 are G-isomorphic.

<u>Proof</u> (a) is immediate from 3.3.7 .

(b) The first statement of (b) follows from 3.3.4 , and the last is obvious (and follows from (c), in any case). For the remaining assertion, suppose H/K is a chief factor of G of rank at least 3, and let p be the prime number of which $|H/K|$ is a power. Then H/K has a proper subgroup $U/K \cong C_p \times C_p$. By 3.2.5(b), U/K is unsupplemented in G/K , so $G/K \notin \mathcal{S}(\mathfrak{L})$. This gives a contradiction, because $\mathcal{S}(\mathfrak{L})$ is Q-closed, by 3.2.15(a).

(c) is immediate from 3.2.7 and the remarks which follow it.

(d) From (b), chief factors of G of rank 2, if any exist, must be G-isomorphic to factors ~~lie~~ between $\Phi(G)$ and $F(G)$. Let $N = \Phi(G)$ and write $F(G)/N = V_1/N \times \ldots \times V_t/N$, where each V_i is a minimal normal subgroup of G/N . Suppose $|V_i/N| = |V_j/N| = p^2$ for some prime number p , where $i, j \in \{1,\ldots,t\}$ and $i \neq j$. Let U_i/N and U_j/N be subgroups of V_i/N and V_j/N respectively, each of order p . Then, since U_iU_j/N is not cyclic, and

$G/N \in Q\mathcal{S}(\mathfrak{L}) = \mathcal{S}(\mathfrak{L})$, U_iU_j/N cannot be unsupplemented in G/N . Therefore, by 3.3.8, $V_i/N \underset{G}{\cong} V_j/N$. Q.e.d.

$GL(2,3) \in \mathcal{S}(\mathfrak{L})$, by 3.3.6, so, as in the case of $\mathcal{S}^*(\mathfrak{A})$-groups, the bound of 4 on the derived length of $\mathcal{S}(\mathfrak{L})$-groups is "best-possible".

Conditions (a)-(d) of 3.3.9 are not sufficient to ensure that $G \in \mathcal{S}(\mathfrak{L})$. For example, $C_4 \times A_4$ satisfies all of these conditions, and it is easily shown, with the help of 3.2.12, that $C_4 \times A_4$ has an unsupplemented subgroup isomorphic to $C_2 \times C_2$, so that $C_4 \times A_4 \notin \mathcal{S}(\mathfrak{L})$.

Unlike $\mathcal{S}(\mathfrak{N})$ and $\mathcal{S}(\mathfrak{A})$, $\mathcal{S}(\mathfrak{L})$ does not have the useful property of being R_0-closed (the above example, $C_4 \times A_4$, illustrates this), so there arises the problem of determining whether or not a given subdirect product of $\mathcal{S}(\mathfrak{L})$-groups is itself an $\mathcal{S}(\mathfrak{L})$-group. In the last result of this section, an answer is given to the corresponding question in the easier case of a direct product of $\mathcal{S}(\mathfrak{L})$-groups.

<u>3.3.10 Theorem</u> Suppose $G_1 , G_2 \in \mathcal{S}(\mathfrak{L})$, and for each i let $\sigma_i = \{p :$ p is a prime divisor of the order of an unsupplemented subgroup of $G_i\}$.

Then $G_1 \times G_2 \in \mathcal{S}(\mathfrak{L})$ if and only if $\sigma_1 \cap \sigma_2 = \emptyset$.

<u>Proof</u> If $p \in \sigma_1 \cap \sigma_2$, then for each i, G_i contains an unsupplemented subgroup U_i of order p. Then $U_1 \times U_2$ is not cyclic, and, by 3.2.12, $U_1 \times U_2$ is unsupplemented in $G_1 \times G_2$. Thus $G_1 \times G_2 \notin \mathcal{S}(\mathfrak{L})$.

Now suppose $\sigma_1 \cap \sigma_2$ is empty, and let U be an unsupplemented subgroup of $G_1 \times G_2$. By 3.2.12, there are unsupplemented subgroups U_1 of G_1 and U_2 of G_2 such that $U \leqslant U_1 \times U_2$. $G_i \in \mathcal{S}(\mathfrak{L})$, so U_i is cyclic $(i = 1,2)$. Also, U_i is a σ_i-group $(i = 1,2)$, so $(|U_1|, |U_2|) = 1$, and therefore $U_1 \times U_2$ is cyclic. Hence $G_1 \times G_2 \in \mathcal{S}(\mathfrak{L})$.

3.4 Groups with complemented π-subgroups.

To obtain another example of a class of groups of the form $\mathcal{S}(\mathfrak{X})$, let $\mathfrak{X}$ be the class of all π-groups for some fixed set π of prime numbers. An appropriate notation for $\mathcal{S}(\mathfrak{X})$ in this case is $\mathcal{S}(\pi)$. By 3.2.13 and 3.2.15, $\mathcal{S}(\pi)$ is $\{Q, S, R_0\}$-closed.

Notation Let $\mathcal{C}(\pi)$ denote the class of groups defined by: $G \in \mathcal{C}(\pi)$ if and only if every π-subgroup of G has a complement in G. If π consists of a single prime number p, then $\mathcal{C}(\{p\})$ will be abbreviated to $\mathcal{C}(p)$. ($\mathcal{C}(p)$ should not be confused with the class $\mathcal{C}_p$, investigated in Chapter 2).

3.4.1 $\mathcal{S}(\pi) = \mathcal{C}(\pi')$.

Proof A π-intersection supplement to a π'-subgroup of G is clearly a complement to H in G. Therefore

$$\mathcal{S}(\pi) \subseteq \mathcal{C}(\pi').$$

Suppose $G \in \mathcal{C}(\pi')$; then every non-trivial π'-subgroup of G has a complement in G, which is in particular a proper supplement in G. Hence the unsupplemented subgroups of G must all be π-groups, and therefore, by 3.2.2, $G \in \mathcal{S}(\pi)$. Therefore

$$\mathcal{C}(\pi') \subseteq \mathcal{S}(\pi).$$

In the light of 3.4.1, it is natural to consider $\mathcal{C}(\pi)$ rather than $\mathcal{S}(\pi)$; results about the one class lead to dual theorems about the other.

<u>3.4.2</u> $\mathcal{C}(\pi)$ is $\{Q,S,R_0\}$-closed.

<u>Proof</u> This is just 3.2.13 and 3.2.15 applied to the class $\mathcal{S}(\pi')$.

<u>3.4.3 Theorem</u> For any group G , $G \in \mathcal{C}(\pi)$ if and only if the π-subgroups of G of prime order have complements in G .

<u>Proof</u> It is clear that the first condition implies the second. For the converse, suppose that the second condition holds, i.e. :

Every π-subgroup of G of prime order has a complement in G ....(1)

Proceed by induction on $|G|$. Let H be a π-subgroup of G . If $H = 1$ then H certainly has a complement in G , so it is safe to assume that $H > 1$. Let P be a subgroup of H , of prime order. By hypothesis, P has a complement, K say, in G . Hypothesis (1) carries over to subgroups of G , by 1.3.2(a), so by induction, $K \in \mathcal{C}(\pi)$. Thus the π-subgroup $H \cap K$ of K has a complement, C say, in K . C is then a complement to H in G , because

$HC = H(H \cap K)C = HK = G$ and $H \cap C = H \cap K \cap C = 1$.

Therefore $G \in \mathcal{C}(\pi)$.

<u>3.4.4 Corollary</u> $\mathscr{C}(\pi) = \bigcap_{p\in\pi} \mathscr{C}(p)$.

<u>Proof</u> This is immediate from 3.4.3 .

Thus some knowledge of the class $\mathscr{C}(\pi)$, for an arbitrary set π of prime numbers, can be gained from information about $\mathscr{C}(p)$-groups, for an arbitrary prime number p . It is obvious that (for $p \neq 2$) $\mathscr{C}(p)$-groups are not necessarily soluble, since every p'-group will belong to $\mathscr{C}(p)$. It would be reasonable to hope that $\mathscr{C}(p)$-groups should necessarily be p-soluble, but even this is not true in general, as is shown in 3.4.5 . Before that result, two standard definitions are recalled.

<u>Definitions</u> Let π be a set of primes.

(1) A group is <u>π-soluble</u> if each of its chief factors is either an abelian π-group or a π'-group.

(2) A group is <u>π-supersoluble</u> if each of its chief factors is either a cyclic π-group or a π'-group.

<u>3.4.5 Theorem</u> (a) If $G \in \mathscr{C}(2)$ then G is 2-soluble (and therefore soluble, by the Feit-Thompson Theorem).

(b) If $G \in \mathscr{C}(3)$ then G is 3-soluble.

(c) For $p \geq 5$, $\mathscr{C}(p)$-groups are not necessarily p-soluble.

Proof Let p be any prime number, and suppose that G is a group of minimal order such that, $G \in \mathscr{C}(p)$ and G is not p-soluble. Since $\mathscr{C}(p)$ is QS-closed, it follows that G is simple. $p \mid |G|$ (otherwise G would certainly be p-soluble), so G contains a subgroup H of order p. A complement K to H in G has index p in G, and thus, considering the permutation representation of G on the cosets of K, G is isomorphic to a subgroup of S_p (the symmetric group of degree p). If $p \leqslant 3$ then it follows that G is soluble and a fortiori p-soluble, a contradiction. For general p, the argument shows that if G is a simple group in $\mathscr{C}(p)$, and $p \mid |G|$, then p is the largest prime divisor of $|G|$, and $p^2 \nmid |G|$. For $p \geqslant 5$, A_p has these properties, and indeed $A_p \in \mathscr{C}(p)$: any cycle of length p in A_p is complemented in A_p by the stabilizer of any symbol.

The next result should be compared with Theorem 3.1.1.

3.4.6 Theorem Let p be a prime number and let G be a p-soluble group. Then $G \in \mathscr{C}(p)$ if and only if G is p-supersoluble and has elementary abelian Sylow p-subgroups.

Proof (1) Suppose that $G \in \mathscr{C}(p)$, and let H/K be a p-chief factor of G. Suppose that H/K is not cyclic, and let L be such that $K < L < H$. Then by 3.2.5(b), L/K is unsupplemented in G/K , which contradicts the fact that $G/K \in Q\mathcal{C}(p) = \mathcal{C}(p) = \mathcal{S}(p')$. Hence G is p-supersoluble. If G_p is a Sylow p-subgroup of G , then $G_p \in S\mathcal{C}(p) = \mathcal{C}(p)$, so in particular $\Phi(G_p) = 1$, and hence G_p is elementary abelian.

(2) Suppose now that G is p-supersoluble and has elementary abelian Sylow p-subgroups, and use induction on $|G|$. Let N be a minimal normal subgroup of G , and let P be a p-subgroup of G . The hypotheses on G are clearly inherited by quotients of G , so by induction, $G/N \in \mathcal{C}(p)$, and hence PN/N has a complement in G/N . Thus, if $P \cap N = 1$ then P has a complement in G , by 1.3.3(a); so assume $P \cap N > 1$. Then, since G is p-supersoluble, it follows that $P \cap N = N$ and $|N| = p$. But, by 1.4.8, all the normal p-subgroups of G have complements in G ; therefore $P \cap N$ has a complement in G , and hence, by 1.3.3(a) again, P has a complement in G . Therefore $G \in \mathcal{C}(p)$. Q.e.d.

There is no corresponding "local" version of the other part of 3.1.1, i.e. it is <u>not</u> true that, if G is p-soluble then $G \in \mathcal{C}(p)$ if and only if $G \in SD_o\mathcal{R}(p)$ (where $\mathcal{R}(p)$ denotes the class of p-soluble groups whose order is not divisible by p^2). To demonstrate this, it is useful to first make the following observation:

Lemma If $\mathfrak{X} = S\mathfrak{X}$, $G \in SD_0\mathfrak{X}$ and G has a unique minimal normal subgroup, then $G \in \mathfrak{X}$.

Proof Suppose $G \leqslant G_1 \times \ldots \times G_n$, with $G_i \in \mathfrak{X}$ $(i = 1,\ldots,n)$, and use induction on n. If $n = 1$ then $G \in S\mathfrak{X} = \mathfrak{X}$. Suppose that $n > 1$. $G \cap G_i$ is a normal subgroup of G for each i, so, because G has a unique minimal normal subgroup, $G \cap G_i = 1$ for all but (at most) one i. Thus it can be assumed that $G \cap G_n = 1$. Then $G \cong GG_n/G_n$, hence G is isomorphic to a subgroup of $G_1 \times \ldots \times G_{n-1}$, and therefore, by induction, $G \in \mathfrak{X}$.

Suppose that G_1 and G_2 belong to $\mathcal{R}(p)$, and p divides both $|G_1|$ and $|G_2|$. Then $G_1 \times G_2 \notin \mathcal{R}(p)$, but $G_1 \times G_2 \in \mathcal{B}(p)$. Suppose that G is a group with a complemented unique minimal normal subgroup V, which is a p'-group, such that $G/N \cong G_1 \times G_2$. Then it is easily seen (using 3.4.6) that $G \in \mathcal{B}(p)$. But $G \notin \mathcal{R}(p)$, so, by the lemma, $G \notin SD_0\mathcal{R}(p)$, i.e. $G \in \mathcal{B}(p) \setminus SD_0\mathcal{R}(p)$.

An example of such a group G is easily constructed: $GL(2,7)$ has subgroups

$$G_1 = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle \cong S_3$$

and
$$G_2 = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle \cong C_3 .$$

Thus $G_1, G_2 \in \mathcal{R}(3)$ and $\langle G_1, G_2 \rangle \cong G_1 \times G_2$. Let V be a GF(7)-space of dimension 2 ; then the split extension $G = (G_1 \times G_2)V$ (with a natural action of $G_1 \times G_2$ on V) is a group of the desired form, and so $G \in \mathcal{C}(3) \setminus SD_0\mathcal{R}(\overset{3}{\cancel{2}})$.

In view of 3.4.4 , Theorem 3.4.6 has the following immediate corollary:

<u>3.4.7 Corollary</u> If π is a set of prime numbers and G is a π-soluble group, then $G \in \mathcal{C}(\pi)$ if and only if G is π-supersoluble and, for all $p \in \pi$, G has elementary abelian Sylow p-subgroups.

Notice that $A_5 \in \mathcal{C}(\{2,3\}')$, so that $\mathcal{C}(\pi)$-groups are not necessarily soluble even if $|\pi'| = 2$. However, 3.4.8 shows that, if $|\pi'| = 1$ then $\mathcal{C}(\pi)$-groups are soluble.

<u>3.4.8 Theorem</u> For any prime number p , $\mathcal{C}(p')$-groups are soluble.
(Of course, if $p \neq 2$, the result follows from 3.4.5(a), but the proof given here does not appeal to the Feit-Thompson Theorem).

<u>Proof</u> Suppose that the result is false, and let G be a minimal non-soluble $\mathcal{C}(p')$-group. Then, as in the proof of

3.4.5, G is simple. Let q be a prime divisor of $|G|$, different from p : such a q exists, of course, for otherwise G would be nilpotent. Then $G \in \mathcal{C}(q)$, so the argument of the proof of 3.4.5 shows that G is isomorphic to a subgroup of S_q . Therefore q is the largest prime divisor of $|G|$. This shows that G is a $\{p,q\}$-group, because q was chosen arbitrarily from amongst the prime divisors of $|G|$ distinct from p . It now follows, by a well-known theorem of Burnside, that G is soluble. This is a contradiction, and so the result must be true.

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