NONLINEAR EVOLUTION EQUATIONS AND APPLICATIONS IN OPTIMAL CONTROL THEORY

Andrew T. Plant

Thesis submitted to the Mathematics Institute at Warwick University for the Degree of Doctor of Philosophy.

December 1973

To Joan , Tommie , Katie and Val.

Abstract

This thesis is an attempt to tackle two related problems in nonlinear functional analysis.

The study of abstract evolution equations started in the early 1950's with the development of the theory of linear contraction semigroups and holomorphic semigroups. The power of the Dunford integral made the holomorphic theory the more attractive , and only in the middle 1960's was it realized that the contraction theory could easily be generalized to semigroups with dissipative nonlinear infinitesimal generators.

Since then the corresponding theory for evolution operators has been greatly studied , Kato probably being the first to do so in 1967. A Hölder type continuity assumption on the time dependence of the generators is common to all this work. It is the purpose of Chapters I and IV to weaken this condition to allow a certain amount of discontinuity in the time dependence. A bounded variation condition replaces Lipschitz continuity in

Chapter I. A Riemann integrability condition replaces a continuity condition in Chapter IV. The original motivation to do this came from Control Theory where discontinuous controls play a major role.

The second purpose of this thesis is to give a rigorous derivation of Pontryagin's Maximum Principle with fixed end-point for nonlinear evolution operators in Banach space. Because the unit ball is not compact we replace Pontryagin's elegant use of the Browder Fixed Point Theorem by an abstract controllability condition which seems appropriate for the particular dissipative systems discussed earlier. We have to derive a first order variational theory for these systems 'from scratch'. Finally we have had to show the 'perturbation cone' is convex , a trivial result in finite dimensions.

Acknowledgments

This thesis was prepared whilst the author held a S.R.C. research grant as a doctoral student in the Mathematics Institute at the University of Warwick, and more recently as a S.R.C. Research Fellow in the Control Theory Centre at the same University.

Of all the members of the Mathematics Institute and Engineering Department with whom I have been associated I should particularly like to thank Dr. Nigel Kalton, Professor Lawrence Markus and Dr. Antony Pritchard for many conversations which have always been interesting and often illuminating.

I owe a debt of gratitude to Peter Stefan for his help in translating a paper of Ju.V.Egorov from the Russian.

Above all it gives me great pleasure to thank my Supervisor, Professor Robert J. Elliott, for his good advice, his encouragement and his infectious enthusiasm for analysis.

Finally especial thanks are due to my parents who in many more ways than one made the writing of this thesis possible.

CONTENTS

Chapters I , IV are each selfcontained and can be read independently. Chapters II and III use results from previous Chapters. Chapters I and IV have previously been published , with only slight alteration , as technical reports of the Control Theory Centre at Warwick University (report No.'s 14 , 23 respectively) and have been submitted for publication elsewhere.

Bibliographic references refer to the bibliography at the end of the Chapter in which the reference is made. This is done purely for the convenience of the author who apologises if this causes the reader excessive 'thumbing of pages'.

CHAPTER I

0. INTRODUCTION.

In this paper we generalize a result of T. Kato [2]. Our motivation is partly a remark at the end of [2], and partly the desire to consider optimal control with fixed end points for some partial differential systems. We consider the nonlinear evolution equation

$$du/dt + A(t)u = 0 \qquad 0 \leq t < \infty \tag{0.1}$$

where for almost all t , $A(t)$ is a quasi-maximal-accretive operator (for definition see section 1) on a Banach space X , with uniformly convex dual X^*.

We have generalized the results of [2] in the following three directions (see conditions I , II of section 3) :

a) $A(t)$ need only be quasi-maximal-accretive rather than maximal-accretive , and the constant of quasi-accretion is allowed to vary with t.

b) The maps $t \to A(t)v$ can be of bounded variation (they are strongly continuous in [2]).

c) The value of $A(s)v$ determines a bound for $A(t)v$ when $s < t$, but $A(t)v$ is roughly speaking independent of $A(s)v$ when $s > t$. This means that in the control theory situation the choice of control at time t does not prejudice the control values in future time as far as existence of solutions is concerned.

Our main result of existence and uniqueness for (0.1) is given in Theorem 2 of section 3. The proof involves considering the equations $du/dt + A_n(t)u = 0$, $u(0) = u_o$; where $\{A_n(.)\}_n$ is chosen to approximate in a suitable way to $A(.)$ as $n \to \infty$, and solutions $u_n(t)$ are known to exist.

In [2] $A_n(t) = A(t)(I + n^{-1}A(t))^{-1}$. In the proof of our theorem $A_n(.)$ is a piecewise constant in time q-m-accretive operator (in fact a 'Riemann approximation' to $A(.)$). To establish the existence of approximating solutions, we first consider the time independent case of (0.1). We do this in Theorem 1 of section 2.

In Theorem 2 and its corollaries we have paid particular attention to the continuity properties of the derivative of solutions of (0.1). We shall need these results when we come to consider perturbations of (0.1) in Chapter II.

1. DEFINITIONS AND BASIC RESULTS.

Throughout this paper X is a real or complex Banach space with uniformly convex dual X^*.

$|.|$ is used for the norm on any of the Banach spaces X, X^*, R (reals), C (complex numbers).

$< . , . >$ represents the real part of the pairing between X and X^*.

$F : X \to X^*$ is the duality mapping. Thus F is the unique single valued map with the properties: $< x , Fx > = |Fx|^2 = |x|^2$. In [2] it is proved that F is uniformly continuous on bounded sets.

We use $\to$ (resp. $\overset{w}{\to}$) to represent strong (resp. weak) convergence in Banach space.

R^+ represents the non-negative reals.

The symbols $\overset{*}{=}$ or $\overset{*}{\leqslant}$ are used to denote the fact that $=$ or $\leqslant$ hold almost everywhere; where the measure in question will always be Lebesgue measure on R^+.

<u>Lemma 1.1</u>. If $x(t)$ is an X-valued curve with weak derivative $dx(s)$ (resp. weak right derivative $d^+x(s)$) at $t = s$ then :

$x(s) \neq 0$

$\Rightarrow$ $|x(t)|$ has derivative $D|x(s)|$ (resp. right derivative $D^+|x(s)|$) at $t = s$.

$$\Rightarrow D^{(+)}|x(s)|^2 = 2|x(s)| \, D^{(+)}|x(s)| = 2 < d^{(+)}x(s) \, , \, Fx(s) >$$

<u>Proof</u>. It is sufficient to prove the case for the right derivative. We have

$$< x(s+h) - x(s) \, , \, Fx(s) > \leq |x(s)|(|x(s+h)| - |x(s)|)$$

dividing by $h > 0$, and letting $h \to 0$ we get

$$< d^+x(s) \, , \, Fx(s) > \leq |x(s)| \underline{\lim_{h\to 0}} \, h^{-1}(|x(s+h)| - |x(s)|) \quad (1.$$

Now weak differentiability (on the right) implies strong continuity (on the right) , so $x(s+h) \to x(s)$. Therefore $|x(s+h)| \to |x(s)|$ and $Fx(s+h) \to Fx(s)$. Now we have

$$< x(s+h) - x(s) \, , \, Fx(s+h) > \geq |x(s+h)|(|x(s+h)| - |x(s)|)$$

Dividing by $h > 0$ and letting $h \to 0$

$$< d^+x(s) \; , \; Fx(s) > \geq |x(s)| \overline{\lim_{h\to o}} \, h^{-1}(|x(s+h)| - |x(s)|)$$

The result now follows by combining this with (1.1).

This Lemma generalises [2 ; Lemma 1.3], (when X^* is uniformly convex).

Corollary 1.1. Suppose $x(t)$ is a locally absolutely continuous $(X, |.|)$ - valued curve on R^+.
Then

i) $d/dt|x(t)|^2 \overset{*}{=} 2|x(t)| \; d/dt|x(t)| \overset{*}{=} 2 < dx(t) \; , \; Fx(t) >$

   and all three expressions exist almost everywhere.

ii) If $f : R^+ \times R^+ \to R$ is any map with $f(o,t) \geq 0$ for all $t \in R^+$, and if

$$|x(t)| d/dt|x(t)| \overset{*}{\leqslant} f(|x(t)|,t) \; |x(t)|$$

   Then

$$d/dt|x(t)| \overset{*}{\leqslant} f(|x(t)|,t)$$

Proof Using Lemma 1.1 and the local absolute continuity of $t \to |x(t)|$, to prove i) it is sufficient to show the weak derivative $dx(t)$ exists a.e.
This follows from the much stronger result proved in Komura [4] :

'An absolutely continuous curve in a reflexive Banach space is strongly differentiable a.e. , and is the indefinite Bochner integral of its derivative.'

ii) is essentially proved in [2; p. 515]

Lemma 1.2. Suppose $p(t)$ and $q(t)$ are locally integrable on R^+ , $x(t)$ is absolutely continuous on bounded intervals and $x'(t) \overset{*}{\leq} p(t)x(t) + q(t)$. Let $y(t)$ be the solution of $y'(t) \overset{*}{=} p(t)y(t) + q(t)$, $y(0) = x(0)$. Then $x(t) \leq y(t)$, $t \geq 0$.

Proof. Put $z(t) = x(t) - y(t)$. Then $z(t)$ is absolutely continuous on bounded intervals and $z'(t) \overset{*}{\leq} p(t)z(t)$. Therefore $z(t)\exp.(-\int_0^t p(s)ds) \leq z(0) = 0$ and the result follows.

Definition 1.1. Let A be an operator (nonlinear) with domain $D \subset X$ and range in X. Then A is said to be accretive if

$$< Av - Au , F(v-u) > \geq 0 \quad \text{for all} \quad u,v \in D \qquad (1.2)$$

It is proved in [2] that (1.2) is equivalent to $kA + I$ being non-contractive on D for all $k > 0$.

<u>Definition 1.2</u>. An accretive operator A is said to be m-accretive (m- for maximal) if range$(A + I) = X$.

If A is m-accretive then $A + kI$ is surjective for all $k > 0$. (For proof see [2])

<u>Definition 1.3</u>. Operator A is said to be q-accretive (q- for quasi-), (resp. q-m-accretive) if there exists a real number k such that $A + kI$ is accretive (resp. m-accretive).

If A is q-accretive we can define

$q = q(A) = \text{Inf}\ \{ k : A + kI \text{ is accretive } \}$.

Then $-\infty < q < \infty$ (unless D is a singleton), and if $k \geq q$ then $A + kI$ is accretive. If A is q-m-accretive then $A + kI$ is m-accretive for all $k \geq q$.

The following results are proved in either [1] or [2] for the case A is m-accretive. The extensions to q-m-accretiveness are quite easy. (See also [5]).

<u>Properties</u>. Let A be q-m-accretive with domain $D \subset X$, and $q(A) = q$. Let $q^+ = \max.\{0, q\}$ and $r = 1/q^+$, so that $r = +\infty$ whenever $q \leq 0$. In any case $r > 0$. If $0 < k < r$ and $h(k) = (1 - kq)^{-1}$ then:

A) $R_k = (kA + I)^{-1}$ is everywhere defined and is Lipschitzian and $|R_k|_{Lip} \leq h(k)$.

B) $A_k = AR_k = k^{-1}(I - R_k)$ is everywhere defined and is Lipschitzian and $|A_k|_{Lip} \leq k^{-1}(1 + h(k))$.

C) A_k is q-accretive and $q(A_k) \leq qh(k)$.

D) If $u \in D$ then $|A_k u| \leq h(k)|Au|$.

E) If $u_n \in D$ $n = 1,2,\ldots$, $u_n \to u$, and $|Au_n|$ bounded then $u \in D$ and $Au_n \xrightarrow{w} Au$.

F) If $x_n \in X$ $n = 1,2,\ldots$, $x_n \to u$, $k_n \in (0,r)$, $k_n \to 0$ and $|A_{k_n} x_n|$ bounded, then $u \in D$ and $A_{k_n} x_n \xrightarrow{w} Au$.

G) If there exists $C < \infty$ such that $< Av - w, F(v - u) > \geq -C|v - u|^2$ for all $v \in D$, then $u \in D$ and $w = Au$.

These results will be referred to as prop.A), prop.B), etc.

2. THE TIME INDEPENDENT CASE.

In this section (0.1) is considered with $A(t) = A$. The results obtained in Theorem 1 are not new. However they are not only needed for the proof of Theorem 2, but they also motivate that Theorem. Also it is interesting to compare the two Theorems to see in which respects the weaker hypotheses of Theorem 2 entail weaker conclusions. The proof of Theorem 1 is a modification of the proof of [1 ; Theorem 28], where the case $q(A) = 0$ is considered. The reason why the modification is not completely trivial is explained in [2 ; Section 3 ; Remark 5].

Theorem 1. Let A be q-m-accretive and $q(A) = q$. Then for each $u_o \in \mathring{D}$ there exists a locally uniformly Lipschitz norm continuous $u : R^+ \to D$ such that:

a) $u(0) = u_o$

b) $Au(t)$ is weakly continuous.

c) The weak derivative $u'(t)$ of $u(t)$ exists for all $t \geq 0$ (For $t = 0$ only the right hand derivative is considered.) and

$$u'(t) = -Au(t) \tag{2.0}$$

d) $|Au(t)|\exp(-qt)$ is non-increasing.

e) $u(t) = u(0) - \int_0^t Au(s)ds$ where the integrand is locally Bochner integrable (globally if $q < 0$).

f) If $v(t)$ also satisfies e) then $|u(t) - v(t)|\exp(-qt)$ is non-increasing.

(For some of the basic properties of the Bochner integral we refer the reader to either [3] or [8])

<u>Proof</u>. For $0 < k < r/2$ the integral equation

$$u_k(t) = u_o - \int_0^t A_k u_k(s)ds \tag{2.1}$$

can be solved using prop.B) and the contraction mapping principle. $u_k(t)$ is strongly continuous , so the strong derivative exists and equals $-A_k u_k(t)$.

If $v_k(t)$ satisfies (2.1) , with v_o replacing u_o then by Corollary 1.1 and prop.C)

$$d/dt|u_k(t) - v_k(t)|^2 \overset{*}{=} -2< A_k u_k(t) - A_k v_k(t) , F(u_k(t)-v_k(t)) >$$
$$\overset{*}{\leqslant} 2qh(k)|u_k(t) - v_k(t)|^2$$

Therefore by Lemma 1.2

$$|u_k(t) - v_k(t)| \leq |u_o - v_o| \exp(qh(k)t) \tag{2.2}$$

We can put $v_k(t) = u_k(t + h)$ in (2.2). Dividing by $h > 0$, and letting $h \to 0$ we get

$$\begin{aligned} |A_k u_k(t)| = |u_k'(t)| &\leq |u_k'(0)| \exp(qh(k)t) \\ &= |A_k u_o| \exp(qh(k)t) \\ &\leq |Au_o| h(k) \exp(qh(k)t) \leq |Au_o| 2\exp(2q^+ t) \end{aligned} \tag{2.3}$$

using prop.D) and $h(k) \leq 2$.

Therefore $u_k(t)$ is locally Lipschitz continuous, and the Lipschitz constant may be chosen independently of k, and t in a compact interval. Thus in particular $\{u_k(.)\}_k$ are uniformly bounded on compacta. From (2.3) and prop.B) we get

$$|u_k(t) - R_k u_k(t)| \leq 2k\exp(2q^+ t)|Au_o| \tag{2.4}$$

It follows that if $0 < j < r/2$ then $\{u_k(.) - u_j(.)\}_{k,j}$ is uniformly bounded on compacta and $(R_k u_k(t) - R_j u_j(t)) - (u_k(t) - u_j(t)) \to 0$ uniformly on compacta as $k, j \to 0$.

Thus given a compact interval $0 \leq t \leq T$ and $\varepsilon > 0$, using the uniform continuity of F, we can obtain $\delta > 0$ such that $0 < k, j < \delta \leq r/2$ and

$$|F(R_k u_k(t) - R_j u_j(t)) - F(u_k(t) - u_j(t))| < \varepsilon \qquad 0 \leq t \leq T \tag{2.5}$$

Also , using (2.4) , we may assume that for the same ε and δ.

$$|R_k u_k(t) - R_j u_j(t)| \leqslant |u_k(t) - u_j(t)| + \varepsilon \qquad 0 \leqslant t \leqslant T$$

Then using the inequality $2x \leqslant 1 + x^2$

$$|R_k u_k(t) - R_j u_j(t)|^2 \leqslant (1 + \varepsilon)(|u_k(t) - u_j(t)|^2 + \varepsilon) \qquad (2.6)$$

Using (2.6) and the accretiveness of $A + qI$

$$d/dt|u_k(t) - u_j(t)|^2 \overset{*}{=} -2 < A_k u_k(t) - A_j u_j(t) \;,\; F(u_k(t) - u_j(t)) >$$

$$\overset{*}{\leqslant} 2q(1 + \varepsilon)(|u_k(t) - u_j(t)|^2 + \varepsilon) + R_{k,j}(t)$$

where

$$R_{k,j}(t) = 2 < A_k u_k(t) - A_j u_j(t) \;,\; F(R_k u_k(t) - R_j u_j(t)) - F(u_k(t) - u_j(t)) >$$

Then (2.3) and (2.5) give

$$|R_{k,j}(t)| \leqslant 8\varepsilon \exp(2q^+T)|Au_o| = \varepsilon' \qquad 0 \leqslant t \leqslant T$$

Therefore by solving the differential equation

$$dy/dt = 2q(1 + \varepsilon)(y + \varepsilon) + \varepsilon'$$

and applying Lemma 1.2

$$|u_k(t) - u_j(t)|^2 \leqslant (\varepsilon + \varepsilon'/2q(1+\varepsilon))\{\exp(2q(1+\varepsilon)t) - 1\} \qquad q \neq 0$$

$$\leqslant \varepsilon' t \qquad q = 0$$

Since $\varepsilon' \to 0$ as $\varepsilon \to 0$; $|u_k(t) - u_j(t)| \to 0$ uniformly on compacta as $k, j \to 0$. Therefore $u_k(t) \to u(t)$ and $u(t)$ is locally uniformly Lipschitz continuous.

We now show $u(t)$ has the required properties a) to f) of the theorem.

a): Trivial.

b): From (2.3) and prop.F) ; $u(t) \in D$ and $A_k u_k(t) \xrightarrow{W} Au(t)$. Also

$$|Au(t)| \leq \underline{\lim_{k\to o}} |A_k u_k(t)| \leq |Au_o| \exp(qt) \tag{2.7}$$

since $h(k) \to 1$ as $k \to 0$.

Let $t_i \to t$, so $u(t_i) \to u(t)$, and by (2.7) and prop.E) $Au(t_i) \xrightarrow{W} Au(t)$.

Therefore $Au(t)$ is weakly continuous.

e): This now follows by taking weak limits and using bounded convergence in (2.1). The Bochner integrability follows from weak continuity in b).

c): Follows from b) and e).

d): Since $v(t) = u(t + s)$ is also a solution of (2.0), from (2.7) we get

$$|Au(t + s)| \leq |Au(s)|\exp(qt) \qquad t \geq 0$$

and the result follows.

f): Applying Corollary 1.1 to $u(t) - v(t)$,

$$d/dt|u(t) - v(t)|^2 \overset{*}{=} -2 < Au(t) - Av(t) , F(u(t) - v(t)) >$$

$$\overset{*}{\leq} 2q|u(t) - v(t)|^2$$

and by Lemma 1.2

$$|u(t) - v(t)| \leq |u(s) - v(s)|\exp(q(t-s)) .$$

This completes the proof of Theorem 1.

Applying result f), the following uniqueness condition is obtained.(An alternative condition is given in Section 5)

<u>Corollary 1</u>. (Uniqueness)

If $u(t)$ satisfies a) and e) then it is unique.

If $U(t) : D \to D : u(0) \to u(t)$ then $\{ U(t) : t \geq 0 \}$ is a nonlinear semigroup of class C_o, with infinitesimal generator $-A$, and contraction class $-q$.

<u>Corollary 2</u>. If X is uniformly convex then $Au(t)$ is strongly continuous at all but a countable number of points, and is strongly continuous on the right everywhere. $u(t)$ is strongly differentiable wherever $Au(t)$ is strongly continuous, and is strongly right differentiable everywhere.

<u>Proof</u>. Since $Au(t)$ is weakly continuous, it is strongly continuous whenever $|Au(t)|$ is continuous. The monotonicity condition d) shows that $|Au(t)|$ has only a countable number of discontinuities.

Suppose $t_i \searrow t$, then

$$|Au(t)| \leq \underline{\mathrm{Lim}}|Au(t_i)| \leq \overline{\mathrm{Lim}}|Au(t_i)| \leq \overline{\mathrm{Lim}}\ e^{qt_i}\ |Au(t)|\ e^{-qt} = |Au(t)|$$

Therefore $|Au(t)|$ is continuous on the right.

The results for the strong differentiability of $u(t)$ now follow from e).

As a consequence of corollary 2 we see that $-A$ is the strong derivative of the semigroup $\{ U(t) : t \geq 0 \}$ whenever X is uniformly convex.

3. THE TIME DEPENDENT CASE.

We now consider a 1-parameter family of operators $\{ A(t) : X \to X \}_{0 \leq t}$, with the properties

I. For almost all $t \in R^+$, $A(t)$ is q-m-accretive with domain D independent of t. $q(t) = q(A(t))$ ($= \infty$ if $A(t)$ not q-accretive) is locally integrable.

II. For all $v \in D$ and $s < t$
$|A(t)v - A(s)v| \leq |p(t) - p(s)| L(|v|)(1 + |A(s)v|)$
where $p(t)$ is a real valued function with locally bounded variation (i.e. bounded variation on compact sets). $L(r)$ is a positive function , bounded on bounded sets.

If we take the special case $q(t) = 0$, $p(t) = t$, and II also holds for $s \geq t$ then we obtain the most general conditions considered in [2] .

As might be expected , I and II are not independent.

<u>Proposition 3.1</u>. If $\{A(t)\}$ satifies I and II then $q(t)$ is lower semicontinuous at points of continuity of $p(t)$.

<u>Proof</u>. Suppose $p(s+) = p(s)$, and $t \gtrsim s$. Then from II $A(t)v \to A(s)v$ for all $v \in D$. From I we get

$$< A(t)v - A(t)u , F(v - u) > \geq -q(t)|v-u|^2 \qquad u,v \in D$$

Taking $\overline{\text{Lim}}$ on both sides as $t \gtrsim s$, we see that $A(s) + \underline{\text{Lim}}\, q(t)$. I is accretive. So $q(s) \leq \underline{\text{Lim}}\, q(t)$.

Suppose $p(t-) = p(t)$, and $s \gtrsim t$. Using II ,

$$|A(s)v - A(0)v| \leq |p(s) - p(0)|L(|v|)(1 + |A(0)v|) .$$

Therefore $|A(s)v|$ is bounded , and so again from II , $A(s)v \to A(t)v$ as $s \gtrsim t$. Left lower semicontinuity then follows using the same method as before.

Even though I only requires $A(s)$ to be q-m-accretive for almost all s , prop.E) holds for all but an , at most , countable number of points s. In fact we have:

<u>Proposition 3.2</u>. Suppose $p(s+) = p(s)$, $v_n \in D$, $v_n \to v$ and $A(s)v_n$ bounded as $n \to \infty$. Then $v \in D$ and $A(s)v_n \overset{W}{\to} A(s)v$.

<u>Proof</u>. Choose $t_i \gtrsim s$ such that $A(t_i)$ is q-m-accretive. Then

$$|A(t_i)v_n - A(s)v_n| \leq |p(t_i)-p(s)|L(|v_n|)(1 + |A(s)v_n|) \qquad (3$$

So as $n \to \infty$ $|A(t_i)v_n|$ is bounded. Therefore , using prop.E) , $v \in D$ and $A(t_i)v_n \overset{W}{\to} A(t_i)v$ as $n \to \infty$.

Since $A(s)v_n$ is bounded , it is weakly subconvergent (Eberlein-Shmulyan Theorem [3]). By taking a subsequence if necessary , suppose $A(s)v_n \overset{W}{\to} w$. Then from (3.1) .

$$|A(t_i)v - w| \leq \underline{\lim}_{n\to\infty}|A(t_i)v_n - A(s)v_n| \to 0 \quad \text{as} \quad i \to \infty$$

But using II , $A(t_i)v \to A(s)v$ as $i \to \infty$.
Therefore $w = A(s)v$, so $A(s)v_n \overset{W}{\to} A(s)v$.

Without loss of generality we may (and do) assume in II $p(0) = 0$ and $p(t)$ non-decreasing (just replace p by its total variation); and $L(r)$ is continuous and non-decreasing.

We now define the following subsets of R^+, all of which have full measure:

$N = \{ t : A(t) \text{ q-m-accretive} \}$

N^+ (resp. N^-) the points of right (resp. left) continuity of $p(t)$.

$M = N^- \cap (N^+ \cup N)$

$$L^+ = \{ s : \overline{\text{Lim}}_{h \to 0+} \frac{1}{h} \int_s^{s+h} q(t)dt < \infty \}$$

So Lebesgue points of $q(t) \subset L^+$

With this notation our main theorem is:

<u>Theorem 2.</u> Suppose $\{A(t)\}$ satisfies I and II, and $u_0 \in D$. Then there exists a locally uniformly Lipschitz continuous $u : R^+ \to D$ such that:

a) $u(0) = u_0$

b) $A(t)u(t)$ is weakly left continuous on M.

b)' If $s \in N \cap N^+$, $t_i \gtrsim s$, $t_i \in M$, $\overline{\text{Lim}}\, q(t_i) < \infty$, then $A(t_i)u(t_i) \xrightarrow{w} A(s)u(s)$.

c) $u(t)$ has weak left derivative $-A(t)u(t)$ for every $t \in M$.

c)' $u(t)$ has weak right derivative $-A(t)u(t)$ for every $t \in N \cap N^+ \cap L^+$.

d) Given $T < \infty$ there exists $Q = Q(T) < \infty$ such that if

$$H(t) = |A(t)u(t)|\exp\left(- \int_0^t |q(r)|dr - k(t)p(t) \right) - Qk(t)p(t$$

then $H(t) \leq H(s)$ for $s \leq t \leq T$ and $t \in M$.

($k(t)$ is defined in (3.16). It is continuous, non-negative and non-decreasing).

In particular $H(t)$ is non-increasing on $[0,T] \cap M$.

e) $$u(t) = u(0) - \int_0^t A(s)u(s)ds$$

where the integrand is locally Bochner integrable.

(So, in particular, $u(t)$ has strong derivative $-A(t)u(t)$ almost everywhere).

f) If $v(t)$ satisfies e), then

$$|v(t) - u(t)|\exp\left(-\int_0^t q(s)ds \right) \quad \text{is non-increasing.}$$

Proof. Partition R^+ into intervals of length $1/n$ $n = 1,2,\ldots..$

Let $\bar{q}^n(t)$ be the step function

$$\bar{q}^n(t) = n \int_{(k-1)/n}^{k/n} q(s)ds \qquad (k-1)/n \leq t < k/n$$

Then it is easy to show:

$$\int_0^t |\bar{q}^n(s)|ds \leqslant \int_0^{t+1} |q(s)|ds \qquad (3.2)$$

$$\left|\int_0^t \bar{q}^n(s)-q(s)\ ds\ \right| \leqslant 2 \int_{(k-1)/n}^{k/n} |q(s)|ds \qquad (k-1)/n \leqslant t \leqslant$$

It follows by absolute continuity that

$$\int_0^t \bar{q}^n(s)ds \to \int_0^t q(s)ds \quad \text{as } n \to \infty \text{, uniformly} \qquad (3.3)$$

for t in a compact set.

Put $t_o^n = 0$ and choose $t_k^n \in (\ (k-1)/n\ ,\ k/n\ )$

such that $t_k^n \in N$ and $q(t_k^n) \leqslant \bar{q}^n(t_k^n)$ (3.4)

Let $\theta^n(t)$ be the step function

$$\theta^n(t) = t_k^n \qquad (k-1)/n \leqslant t < k/n$$

So $q(\theta^n(t)) \leqslant \bar{q}^n(t)$

Applying Theorem 1 to the intervals $[(k-1)/n\ ,\ k/n]$, and piecing together the solutions , the equation

$$(d/dt)u_n(t) \stackrel{*}{=} -A(\theta^n(t))u_n(t) \qquad u_n(0) = u_o \qquad (3.5)$$

has unique solution $u_n(t)$. Since $u_n(t)$ is Lipschitz on $[(k-1)/n\ ,\ k/n]$, it is uniformly Lipschitz (and hence absolutely continuous) on bounded intervals.

<u>Lemma 3.1.</u>

$\{u_n(.)\}_n$ is uniformly bounded on compacta, and (3.7) holds.

<u>Proof.</u> By Corollary 1.1

$$
\begin{aligned}
&|u_n(t)-u_o|\, d/dt\, |u_n(t)-u_o| \\
&\quad \overset{*}{=} -< A(\theta^n(t))u_n(t) \;,\; F(u_n(t) - u_o) > \\
&\quad \overset{*}{\leqslant} -< A(\theta^n(t))u_n(t) - A(\theta^n(t))u_o \;,\; F(u_n(t) - u_o) > \\
&\qquad + |A(\theta^n(t))u_o|\,|u_n(t) - u_o| \\
&\quad \overset{*}{\leqslant} \{\, \bar{q}^n(t)\,|u_n(t)-u_o| + |A(\theta^n(t))u_o|\, \}\, |u_n(t)-u_o|
\end{aligned}
$$

since $A(\theta^n(t)) + \bar{q}^n(t)I$ is accretive.

So again by Corollary 1.1

$$d/dt\,|u_n(t)-u_o| \overset{*}{\leqslant} \bar{q}^n(t)\,|u_n(t)-u_o| + |A(\theta^n(t))u_o| \tag{3.6}$$

Now from II we get

$$|A(\theta^n(t))u_o| \leq |A(0)u_o| + p(t+1/n)L(|u_o|)(1+|A(0)u_o|)$$

Combining this with (3.6) and applying Lemma 1.2 :

$$|u_n(t)-u_o| \leq (\exp \int_0^t \bar{q}^n(s)ds) \times$$
$$\int_0^t \{|A(0)u_o|+p(s+1/n)L(|u_o|)(1+|A(0)u_o|)\}\{\exp-\int_0^s \bar{q}^n(r)dr\} \, ds$$

The uniform boundedness now follows using (3.2).

By using (3.3) and dominated convergence we get

$$\overline{\lim_{n\to\infty}}|u_n(t)-u_o| \leq (\exp \int_0^t q(s)ds) \times$$
$$\int_0^t \{|A(0)u_o|+p(s)L(|u_o|)(1+|A(0)u_o|)\}\{\exp-\int_0^s q(r)dr\} \, ds \qquad (3.7$$

Thus we may suppose $L(|u_n(t)|) \leq K(t) < \infty \quad n = 1,2,$ $K(t)$ non-decreasing and continuous.

We put $B^n(t) = |A(\theta^n(t))u_n(t)|$

__Lemma 3.2.__ $\{B^n(.)\}_n$ uniformly bounded on compacta.

__Proof.__ By d) of Theorem 1 , and (3.4) we have

$$|A(t_k^n)u_n(t)|\exp(-\bar{q}^n(t_k^n)t) \quad \text{non-increasing on} \quad (k-1)/n \leq t \leq k/n \qquad (3.8$$

Now put

$$Z_k^n = |p(t_{k+1}^n) - p(t_k^n)|K(k/n)$$

Then using II we get for $k = 1, 2, \ldots.$

$$B^n(k/n) \leq |A(t^n_{k+1})u_n(k/n) - A(t^n_k)u_n(k/n)| + |A(t^n_k)u_n(k/n)|$$

$$\leq (1+Z^n_k)|A(t^n_k)u_n(k/n)| + Z^n_k$$

So using (3.8)

$$B^n(k/n) \leq (1+Z^n_k)(\exp \bar{q}^n(t^n_k)/n)\, B^n((k-1)/n) + Z^n_k \tag{3.9}$$

Also

$$B^n(0) \leq (1+Z^n_o)|A(0)u_o| + Z^n_o$$

Now for each fixed n we can solve the difference equation

$$C^n(k/n) = (1+Z^n_k)(\exp \bar{q}^n(t^n_k)/n)\, C^n((k-1)/n) + Z^n_k$$

$$C^n(0) = B^n(0)$$

Comparing this with (3.9) we see that $B^n(k/n) \leq C^n(k/n)$

Now put

$$S^n(k/n) = \left(\exp -\int_o^{k/n} q(s)ds\right)\left(\prod_{r=o}^{k}(1+Z^n_r)^{-1}\right) C^n(k/n)$$

Therefore

$$S^n(k/n) - S^n((k-1)/n) = \left(\exp -\int_o^{k/n} q(s)ds\right) Z^n_k \prod_{r=o}^{k}(1+Z^n_r)^{-1}$$

$$\leq Z^n_k \left(\exp -\int_o^{k/n} q(s)ds\right)$$

$$S^n(k/n) \leq S^n(0) + \sum_{r=1}^{k} Z^n_r\left(\exp -\int_o^{r/n} q(s)ds\right)$$

$$\leq |A(0)u_o| + \sum_{r=o}^{k} Z^n_r\left(\exp -\int_o^{r/n} q(s)ds\right) \tag{3.10}$$

Now using the inequality $1 + x \leq e^x$ we obtain

$$B^n(k/n) \leq C^n(k/n) \leq S^n(k/n) \exp\left(\int_0^{k/n} q(s)ds + \sum_{r=0}^{k} Z_r^n \right)$$

Also since $K(t)$ and $p(t)$ are non-decreasing

$$\sum_{r=0}^{k} Z_r^n \leq K(k/n)p(t_{k+1}^n) \leq K(k/n)p((k+1)/n)$$

Combining (3.10) and (3.11) :

$$B^n(k/n) \leq \{ |A(0)u_o| + K(k/n)p((k+1)/n)\exp \int_0^{k/n} |q(s)|ds$$
$$\exp(\int_0^{k/n} |q(s)|ds + K(k/n)p((k+1)/n))$$

Given n choose k so that $(k-1)/n \leq t < k/n$.
Using (3.8) we get

$$B^n(t) = |A(t_k^n)u_n(t)| \leq B^n((k-1)/n)\exp\{(nt-k+1) \int_{(k-1)/n}^{k/n}$$
$$\leq B^n((k-1)/n)\exp \int_{(k-1)/n}^{k/n} |q(s)|ds$$

Therefore

$$B^n(t) \leq \{ |A(0)u_o| + K((k-1)/n)p(k/n)\exp \int_0^{(k-1)/n} |q(s)|$$
$$\exp(\int_0^{k/n} |q(s)|ds + K((k-1)/n)p(k/n))$$

This gives the uniform bound.

<u>Lemma 3.3.</u> $u_n(t) \to u(t)$ uniformly on compacta.

<u>Proof.</u> If $G(t) = K(t)(1 + \sup\{ B^n(s) : n \geq 1, s \leq t \})$,

then $G(t)$ is locally bounded and measurable.

Now by Corollary 1.1,

$$|u_n(t)-u_m(t)|d/dt|u_n(t)-u_m(t)| \overset{*}{=}$$

$$- < A(\theta^n(t))u_n(t)-A(\theta^m(t))u_m(t) , F(u_n(t)-u_m(t)) >$$

Suppose $\theta^n(t) \leq \theta^m(t)$, then using II,

$$|A(\theta^n(t))u_n(t)-A(\theta^m(t))u_n(t)| \leq |p(\theta^n(t))-p(\theta^m(t))| G(t)$$

Therefore, if $m(t) = \max\{ \bar{q}^m(t) , \bar{q}^n(t) \}$,

$$|u_n(t)-u_m(t)|d/dt|u_n(t)-u_m(t)| \overset{*}{\leq}$$

$$|u_n(t)-u_m(t)|\{ |p(\theta^n(t))-p(\theta^m(t))|G(t) + m(t)|u_n(t)-u_m(t)$$

By symmetry this also holds for $\theta^n(t) > \theta^m(t)$.

Using Corollary 1.1 and Lemma 1.2,

$$|u_n(t)-u_m(t)| \leq$$

$$(\exp \int_0^t m(s)ds) \int_0^t |p(\theta^n(s))-p(\theta^m(s))|G(s)(\exp -\int_0^s m(r)dr)\, ds$$

But by (3.2), $| \int_0^t m(s)ds | \leq 2 \int_0^{t+1} |q(s)|ds$. Also as $n \to \infty$, $p(\theta^n(s)) \to p(s)$ a.e.. Then by bounded convergence, $|u_n(t)-u_m(t)| \to 0$ uniformly on compacta, and the Lemma follows by completeness of X.

Now using Theorem 1 , we can integrate (3.5) by the Bochner integral , to get

$$u_n(t) = u_o - \int_o^t A(\theta^n(s))u_n(s)ds$$

Therefore

$$|u_n(t) - u_n(s)| \leq \int_s^t B^n(r)dr$$

So by Lemma 3.2 , $\{u_n(.)\}_n$ is uniformly Lipschitz continuous on compacta , therefore so is $u(.)$.

Also , from (3.7) we get the growth condition

$$|u(t)-u_o| \leq (\exp \int_o^t q(s)ds) \times$$
$$\int_o^t \{|A(0)u_o|+p(s)L(|u_o|)(1+|A(0)u_o|)\}\{\exp -\int_o^s q(r)dr\} ds$$

We now define the following subset of $R^+ \times R^+ \times \{\text{Integers} > 0\}$:

$$S = \{ (s,t,n) : 0 \leq s < t < \infty , \theta^n(s) < t \}$$

Note that if $s < t$ then $\theta^n(s) < t$ for all sufficien large n .

From II we get

$$|A(\theta^n(s))u_n(s) - A(t)u_n(s)| \leq |p(\theta^n(s))-p(t)|G(s) \quad \text{for all } (s,t,n) \in S \qquad (3.1$$

where $G(s)$ is as in the proof of Lemma 3.3. It follows that $|A(t)u_n(s)|$ is bounded as t ranges over a bounded set and $(s,t,n) \in S$.

Now choose any $s \in R^+$, then choose $t > s$ and $t \in N$. Letting $n \to \infty$ we see that $u(s) \in D$ and $A(t)u_n(s) \overset{W}{\to} A(t)u(s)$ (using prop.E).

Now fix $t \in M$, $t > 0$. Then $(t-1/n, t, n) \in S$ for sufficiently large n, and $u_n(t-1/n) \to u(t)$. Since $t \in M$, we must have $t \in N$, or $t \in N^+$, so by using prop.E) and proposition 3.2 ;

$$A(t)u_n(t-1/n) \xrightarrow{W} A(t)u(t) \qquad \text{as} \quad n \to \infty$$

Again using (3.13) we get

$$A(\theta^n(t-1/n))u_n(t-1/n) \xrightarrow{W} A(t)u(t) \qquad \text{for } t \in M-\{0\} \qquad (3.$$

Thus, from Lemma 3.2, $A(t)u(t)$ is locally bounded on M.

We are now in a position to verify that $u(t)$ satisfies conditions a) to f) of Theorem 2.

a) : Trivial

b) : Let $t_i \gtrsim t$, $t_i \in M$, $t \in M$. Then $|A(t_i)u(t_i)|$ is bounded as $i \to \infty$. Therefore using II we obtain $A(t_i)u(t_i) - A(t)u(t_i) \to 0$, and $|A(t)u(t_i)|$ is bounded. Then using prop.E) and proposition 3.2 we obtain $A(t)u(t_i) \xrightarrow{W} A(t)u(t)$. Thus $A(t_i)u(t_i) \xrightarrow{W} A(t)u(t)$.

<u>Corollary</u> : $t \to A(t)u(t)$ is almost everywhere seperably-valued [3] .

<u>Proof</u> : Since M has full measure it is sufficient to show $H = \{ A(t)u(t) : t \in M \}$ is strongly separable. Let $\{t_i\}$ be a countable dense subset of M. Then $\overline{co}\{A(t_i)u(t_i)\}$ is strongly separable and weakly closed, and therefore, using b), contains H.

The result follows.

e) : Using this corollary and (3.14) we see that $A(t)u(t)$ is strongly measurable (Pettis' Theorem [3]), and therefore locally Bochner integrable. Now

$$u_n(t) = u_o - \int_0^t A(\theta^n(s))u_n(s)ds$$
$$= u_o - \int_{1/n}^{t+1/n} A(\theta^n(s-1/n))u_n(s-1/n)ds$$

Taking weak limits on both sides and using bounded convergence, we obtain e).

b)' : Suppose s and $\{t_i\}$ are as described in b)'. Since $s \in N^+$ and using II, $A(t_i)v \to A(s)v \quad v \in D$. Since $t_i \in M$, $A(t_i)u(t_i)$ is bounded, and therefore weakly subconvergent. By taking a subsequence if necessary, suppose $A(t_i)u(t_i) \xrightarrow{W} w$. Now

$$< A(t_i)v-A(t_i)u(t_i) , F(v-u(t_i)) > \geq -q(t_i)|v-u(t_i)|^2$$

Taking $\underline{\text{Lim}}$ on both sides and using the uniform continuity of F

$$< A(s)v-w , F(v-u(s)) > \geq -\overline{\text{Lim}}_i\, q(t_i) \ |v-u(s)|^2$$
$$\geq -C|v-u(s)|^2 \qquad \text{where } C < \infty$$

Therefore by prop.G), $w = A(s)u(s)$, so w is unique and $A(t_i)u(t_i) \xrightarrow{W} A(s)u(s)$.

c) : This follows from b), e) and the fact that M has full measure.

c)' : Suppose $s \in N \cap N^+ \cap L^+$. Then using e), it will be sufficient to show

$$D(h) = \frac{1}{h} \int_s^{s+h} A(t)u(t)dt \xrightarrow{w} A(s)u(s) \qquad \text{as } h \gtrsim 0$$

Since $A(t)u(t)$ is locally bounded on M, and M has full measure, D(h) is bounded as $h \to 0$, and so is weakly subconvergent. Suppose $D(h_i) \xrightarrow{w} w$ as $h_i \gtrsim 0$. It will be sufficient if we can show $w = A(s)u(s)$. Now for any $v \in D$

$$< A(t)v-A(t)u(t) \, , F(v-u(t)) > \geq -q(t)|v-u(t)|^2$$

Integrating this expression with respect to t, from s to $s+h_i$ and dividing by $h_i > 0$; and then letting $h_i \to 0$, it is easy to see

$$< A(s)v-w \, , F(v-u(s))> \geq -\overline{\lim_{h\to 0}} \frac{1}{h}\int_s^{s+h} q(t)dt \; |v-u(s)|^2$$

$$\geq -C|v-u(s)|^2 \qquad \text{where } C < \infty$$

So using prop.G), $w = A(s)u(s)$.

f) : By Corollary 1.1 and e),

$$d/dt|u(t)-v(t)|^2 \overset{*}{=} -2 \quad < A(t)u(t)-A(t)v(t) \, , F(u(t)-v(t)) >$$

$$\overset{*}{\leq} 2q(t)|u(t)-v(t)|^2$$

The result now follows by Lemma 1.2 .

d) : Combining (3.14) and (3.12) we get for $t \in M$

$$|A(t)u(t)| \leqslant \underline{\operatorname{Lim}}_{n\to\infty} B^n(t-1/n)$$
$$\leqslant \{ |A(0)u_o| + K(t)p(t)\exp \int_0^t |q(s)|ds \} \times$$
$$\exp(\int_0^t |q(s)|ds + K(t)p(t)) \qquad (3.1$$

Now by continuity of $L(r)$ and Lemma 3.3

$$L(|u_n(t)|) \to L(|u(t)|) \quad \text{uniformly on compacta.}$$

Thus in (3.15) we may take

$$K(t) = k(t) = \sup_{s\leqslant t}\{ L(|u(s)|) \} \qquad (3.1$$

Suppose we fix $s > 0$ and put $\bar{A}(t) = A(t+s)$. Then $\{\bar{A}(t)\}$ satisfies I and II (with translated $p(.)$ and $q(.)$) and the solution, which we can show is unique using f), of

$$v'(t) = -\bar{A}(t)v(t) \qquad v(o) = u(s)$$

is $v(t) = u(t+s)$.

From (3.15) we then get for $t \geq s$, $t \in M$

$$|A(t)u(t)| \leqslant \{ |A(s)u(s)| + k(t)(p(t)-p(s))\exp \int_s^t |q(r)|dr \}$$
$$\exp(\int_s^t |q(r)|dr + k(t)(p(t)-p(s))) \qquad (3.1$$

We have used here the inequality

$$\sup_{s\leqslant r\leqslant t} \{L(|u(r)|)\} \leqslant \sup_{o\leqslant r\leqslant t} \{L(|u(r)|)\}$$

Now since $k(r)$ and $p(r)$ are non-negative non-decreasing ,

$$0 \leq k(t)(p(t)-p(s)) \leq k(t)p(t) - k(s)p(s) \tag{3.18}$$

Multiplying (3.17) by $\exp(-\int_0^t |q(r)|dr - k(t)p(t))$, and using (3.18) ,

$$\begin{aligned} &|A(t)u(t)|\exp(-\int_0^t |q(r)|dr - k(t)p(t)) \\ &\qquad \leq |A(s)u(s)|\exp(-\int_0^s |q(r)|dr - k(s)p(s)) + \\ &\qquad\quad \{k(t)p(t)-k(s)p(s)\}\exp(\int_0^t |q(r)|dr - k(s)p(s)) \end{aligned}$$

Assuming $t \leq T < \infty$, there is a bound $Q(T) < \infty$ for the last exponential term. This gives d) .

Remark : It is quite easy to obtain a considerably stronger global growth condition on $|A(t)u(t)|$.

This completes the proof of Theorem 2.

<u>Corollary 1</u>. (Uniqueness)

If $v(t)$ is absolutely continuous , $v'(t) \overset{*}{=} -A(t)v(t)$, $v(0) = u(0)$, then $v(t) = u(t)$ where $u(t)$ is the solution given in the theorem.

<u>Proof</u>. The result of Komura [4] , mentioned in the proof of Corollary 1.1 , shows $v(t)$ satisfies e) , and therefore f) holds.

Corollary 2. If X is uniformly convex then $-A(s)u(s)$ is the strong left (resp. right) derivative of $u(t)$ at $t = s$ for $s \in \{ M - \text{countable set} \}$ (resp. $s \in N \cap N^+ \cap L^+$).

Proof. Since $H(t)$ has at most a countable number of discontinuities on $[0,T] \cap M$, so does $t \to |A(t)u(t)|$. Therefore by b) and uniform convexity , $t \to A(t)u(t)$ is strongly left continuous on M at all but a countable number of points. Since M has full measure we can use e) to obtain the result for the left derivative.

Suppose $s \in N \cap N^+ \cap L^+$. Using c)' , e) and uniform convexity , it will be sufficient if we show

$$\overline{\lim_{h\to 0+}} \frac{1}{h} \left| \int_s^{s+h} A(t)u(t)dt \right| \leq |A(s)u(s)| \tag{3.19}$$

Now from d): $\frac{1}{h} \int_s^{s+h} H(t)dt \leq H(s)$. Taking $\overline{\lim}$ of the left hand side as $h \searrow 0$ and cancelling terms (using the fact that $p(s+) = p(s)$) we obtain

$$\overline{\lim_{h\to 0+}} \frac{1}{h} \int_s^{s+h} |A(t)u(t)|dt \leq |A(s)u(s)|$$

This now gives (3.19) , and completes the proof.

4. A PRODUCT FORMULA.

In [5] the product formula

$$\lim_{n\to\infty} \prod_{i=0}^{[n(t-s)]} R_{1/n}(s + i/n)\, x \tag{4.1}$$

is used to construct an evolution operator which under certain conditions [5 , Theorems 3.2 , 3.3 , 3.4] is shown to generate the strong solution of

$$u'(t) + A(t)u(t) = 0 \qquad u(s) = x \in D \tag{4.2}$$

If $\{A(t)\}$ satisfies conditions I and II , then there is no guarantee that the resolvent operators in (4.1) exist , since it may happen that $q(s + i/n) \geq n$, even for large n. (In [5] $q(.)$ is assumed to be constant)

The step functions $\theta^n(t)$ constructed at the beginning of the proof of Theorem 2 were used to pick out points in R^+ at which $q(.)$ was not ''too large''. It seems reasonable to modify (4.1) by the same technique , and consider the product formula

$$U(t,s)x= \lim_{n\to\infty} \prod_{i=0}^{[n(t-s)]} R_{1/n}(\theta^n(s + i/n))\, x \ , \qquad x \in D \ , \ t \geq s \quad (4$$

Now $q(\Theta^n(t)) \leq \bar{q}^n(t) = n \int_{(k-1)/n}^{k/n} q(s)\, ds \qquad (k-1)/n \leq t < k/$

Thus $q(\Theta^n(t)) < n/2$ for large n, uniformly for t in a compact set. Therefore the products in (4.3) exist for large n uniformly for $(t,s) \in$ compact triangle.

The first problem is to show the limit in (4.3) exists for $x \in D$. If $q(.)$ were bounded and $p(.)$ continuous then [5, Theorem 2.1] would be applicable, and moreover the limit would be uniform for $(t,s) \in$ compact triangle. We believe these extra conditions on $q(.)$, $p(.)$ are not essential for the result. Rather than demonstrate this here we prefer to postpone the proof to the more general context of multivalued operators on non-reflexive spaces in a seperate paper.

The second problem is to demonstrate under what conditions the operator $U(t,s)x$ does solve the initial value problem (4.2).

Theorem 3. Let the conditions of Theorem 2 be satisfied. Suppose for some fixed $s \geq 0$ the operators $U(t,s)$ defined in (4.3) exist, and the limit in (4.3) is uniform as t varies over a compact set. Suppose $t \to U(t,s)x$ is continuous for $x \in D$. Then $U(t,s)x$ solves the initial value problem (4.2).

The following elementary lemma is required.

Lemma 4.1. Y Banach space. $x(.) : R \to Y \in L^1_{loc}$. $\{E_n\}$ sequence of intervals such that $0 \in E_n$ and $0 \neq$ diameter $E_n = m(E_n) \to 0$ as $n \to \infty$. Put

$$x_n(t) = m(E_n)^{-1} \int_{E_n+t} x(s)\, ds = m(E_n)^{-1} \int_{E_n} x(s+t)\, ds$$

Then $x_n \to x$ in L^1_{loc}.

i.e. $\int_K |x_n(t)-x(t)|\, dt \to 0$ for all compact intervals K.

Proof. By [8, p.217] $x_n(t) \to x(t)$ a.e. Let K be any compact interval, and let I be another compact interval such that $K + \cup E_n \subset I$.

Let E be a measurable subset of K.

$$\int_E |x_n(t)|\, dt \leq m(E_n)^{-1} \int_E \int_{E_n} |x(s+t)|\, ds\, dt$$

$$= m(E_n)^{-1} \int_{E_n} \int_E |x(s+t)|\, dt\, ds$$

$$\leq m(E_n)^{-1} \int_{E_n} \int_I |x(t)|\, dt\, ds = \int_I |x(t)|\, dt < \infty$$

The last inequality validates the interchange in the order of integration.

Now given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $E \subset I$, $m(E) < \delta$ implies

$$\int_E |x(t)|\, dt < \varepsilon$$

Therefore $E \subset K$, $m(E) < \delta$ implies

$$\int_E |x_n(t)|\, dt < \varepsilon$$

The result then follows from the Vitali Convergence Theorem [8, p.150].

<u>Proof of Theorem 3</u>. (This is a modification of a proof given in [5]).

The case for $s > 0$ requires only trivial modification of the proof given below for $s = 0$.

Therefore we show $U(t,0)u_o = u(t)$, where $u(t)$ is the solution of (0.1) given by Theorem 2.

Let $\quad u_n(t) = \prod_{i=0}^{[nt]} R_{1/n}(\theta^n(i/n))\, u_o \qquad t \geq 0$

For convenience define $u(t) = u_n(t) = u_o$ for $t < 0$

Now choose $T < \infty$, and $N = N(T)$ so that if $n \geq N$ and $0 \leq t \leq T$ then $q(\theta^n(t)) < n/2$. Then

$$u_n(t) = R_{1/n}(\theta^n(t))u_n(t-1/n) \tag{4.4}$$

$$|R_{1/n}(\theta^n(t))|_{Lip} \leq 1^n(t) = (1 - q(\theta^n(t))/n)^{-1} < 2 \tag{4.5}$$

Put $\quad g_n(t) = A(\theta^n(t))u(t - 1/n) + n(u(t - 1/n) - u(t - 2/n))$

Therefore

$$u(t - 1/n) = R_{1/n}(\theta^n(t))(u(t - 2/n) + g_n(t)/n) \tag{4.6}$$

Combining (4.4), (4.5), (4.6)

$$w_n(t) \leq 1^n(t)(w_n(t - 1/n) + |g_n(t)|/n) \tag{4.7}$$

where $w_n(t) = |u_n(t) - u(t - 1/n)|$ and

$w_n(t) \to w(t) = |U(t,0)u_o - u(t)|$ uniformly.

Integrate inequality (4.7) from 0 to $t \leq T$ and rearrange

$$\begin{aligned} n\int_{t-1/n}^{t} w_n(s)\, ds &\leq \int_0^t n(1^n(s)-1)w_n(s - 1/n)\, ds + \int_0^t 1^n(s)|g_n(s)|\, ds \\ &\leq 2\int_0^t q^+(\theta^n(s))w_n(s - 1/n)\, ds + 2\int_0^t |g_n(s)|\, ds \end{aligned} \tag{4.8}$$

Now $t - 1/n < \theta^n(t)$, therefore by condition II ,

there exists a constant K such that

$$\int_0^T |A(\theta^n(s))u(s - 1/n) - A(s - 1/n)u(s - 1/n)| \, ds$$

$$\leq K \int_0^T |p(\theta^n(s)) - p(s - 1/n)| \, ds \to 0 \quad \text{as} \quad n \to \infty$$

Also , by Lemma 4.1 ,

$$\int_0^T |u'(s) - n(u(s) - u(s - 1/n))| \, ds \to 0$$

Therefore $\int_0^t |g_n(s)| \, ds \leq \int_0^T |g_n(s)| \, ds \to 0$ as $n \to \infty$.

Now $q^+(\theta^n(s)) \leq n \int_{(k-1)/n}^{k/n} q^+(r) \, dr$ for $(k-1)/n \leq s < k/n$

So putting $q_n(s) = n \int_{s \vee 0}^{s+2/n} q^+(r) \, dr$

$q^+(\theta^n(s)) \leq q_n(s - 1/n)$, and by Lemma 4.1 ,

$q_n(s) \to 2q^+(s)$ in L^1_{loc} . Therefore

$$\int_0^t q^+(\theta^n(s))w_n(s - 1/n) \, ds \leq \int_0^t q_n(s)w_n(s) \, ds \to 2\int_0^t q^+(s)w(s) \, ds$$

Therefore taking limits in (4.8)

$$w(t) \leq 4\int_0^t q^+(s)w(s) \, ds \qquad 0 \leq t \leq T$$

So by Gronwall's Lemma , $w(t) = 0$, and $U(t,0)u_0 = u(t)$.

5. MISCELLANEOUS REMARKS.

Remark 1. An interesting uniqueness result can be given as follows :

We first define a set valued left derivative of a continuous curve in general Banach space Y.

Definition If u(.) maps an open neighbourhood of $s \in R$ continuously into Y , put

$$\partial^- u(s) = \bigcap_{n=1}^{\infty} \overline{co} \{ h^{-1}(u(s+h) - u(s)) \;:\; -n^{-1} < h < 0 \}$$

So $\partial^- u(s)$ is a closed convex set (possibly empty).

The following Lemmas are easy , and we leave the proofs to the reader.

Lemma 5.1. If $x \in \partial^- u(s)$ and $x^* \in Fu(s)$ then $|u(s)| \, \underline{D}^- |u(s)| \leqslant \langle x , x^* \rangle$.

($\underline{D}^-$ is the lower left-hand Dini derivative)

Lemma 5.2. If x(.) is a continuous real valued function on an interval in R , and $\underline{D}^- x(t) \leqslant 0$ for all t in a co-countable set , then x(t) is non-increasing.

([7] has several results of this type)

<u>Proposition 5.1.</u> If $v(t)$ is strongly continuous in X, $v(0) = u(0)$ and $-A(t)v(t) \in \partial^- v(t)$ for all but a countable number of points $t \in R^+$; then $v(t) = u(t)$, where $u(t)$ is the solution of (0.1) given in Theorem 2.

<u>Proof.</u> By part c) of Theorem 2. we have

$$-A(t)u(t) = d^- u(t) \in \partial^- u(t) \qquad t \in M$$

It is then easy to show

$$-(\, A(t)u(t) - A(t)v(t)\,) \in \partial^-(\, u(t) - v(t)\,)$$
$$t \in R^+ - (\text{countable})$$

Thus using Lemma 5.1.

$$|u(t)-v(t)|\ \underline{D}^- |u(t)-v(t)| \leq -< A(t)u(t)-A(t)v(t)\,,\ F(u(t)-v(t)) >$$
$$\leq q(t)\ |u(t)-v(t)|^2$$
$$t \in R^+ - (\text{countable})$$

Suppose for some $T > 0$, $v(T) \neq u(T)$. Let $(\,r\,,\,T\,]$ be the largest open interval in $[\,0\,,\,T\,]$ on which $u(t) \neq v(t)$. By continuity we have $u(r) = v(r)$. Also $\underline{D}^- x(t) \leq q(t)x(t)$, $t \in (\,r\,,\,T\,] - (\text{countable})$ where $x(t) = |u(t)-v(t)|$.

By Proposition 3.1, at the points of continuity of $p(t)$

$$q(t) \leqslant \underline{\mathrm{Lim}}_{h \to 0+} h^{-1} \int_{t-h}^{t} q(s)ds$$

If $y(t) = \exp. - \int_0^t q(s)ds$ then it is easy to see that

$$\overline{\mathrm{Lim}}_{h \to 0+} h^{-1} \{ y(t) - y(t-h) \} \leqslant -q(t)y(t)$$

Then

$$\underline{D}^{-}\{x(t)y(t)\} \leqslant y(t)\, \underline{D}^{-}x(t) - q(t)x(t)y(t) \leqslant 0$$

$$t \in (r, T] - (\text{countable})$$

(We have used here the inequality

$\underline{\mathrm{Lim}}(a_i + b_i) \leqslant \underline{\mathrm{Lim}}\, a_i + \overline{\mathrm{Lim}}\, b_i$)

Therefore by Lemma 5.2,

$$x(T)y(T) \leqslant x(r)y(r) = 0 \qquad \text{so} \quad x(T) = 0$$

This contradiction shows that $u(t) = v(t)$ for all $t \in R^+$.

Remark 2. Consider the following perturbation problem. Suppose $A(t)$ satisfies conditions I , II. What conditions on $B(t)$ guarantee $A(t)+B(t)$ also satisfy I , II ?

Proposition 5.2. Suppose the following hold

i) $A(t)$ satisfies I , II.

ii) For almost all $t \in R^+$ $B(t)$ is q-m-accretive with $q(B(t)) = q'(t)$ locally integrable.

iii) Domain $B(t) \supset D$

iv) $B(t)$ satisfies II (It may be assumed $A(t), B(t)$ both satisfy II for the same $p(.)$ and $L(.)$).

v) For each $T > 0$ there exists $K < 1$ and $G : R^+ \to R^+$ bounded on bounded sets such that

$$|B(t)v| \leqslant G(|v|) + K|A(t)v| \qquad 0 \leqslant t \leqslant T, v \in D \quad (5.1)$$

(so $B(t)$ is $A(t)$ bounded)

Then $A(t)+B(t)$ satisfies conditions I , II.

Proof. Clearly $q(A(t)+B(t)) \leqslant q(t) + q'(t)$, so I holds for $A(t)+B(t)$ if $A(t)+B(t)$ is q-m-accretive whenever $A(t)$ and $B(t)$ are. For such a t an inequality of the type (5.1) (with a different G)

holds with $A(t)$ (resp. $B(t)$) replaced by $A(t)+q(t)I$ (resp. $B(t)+q'(t)I$). Then by [6 , Theorem 10.2] , $A(t)+B(t)+(q(t)+q'(t))I$ is m-accretive ; so $A(t)+B(t)$ is q-m-accretive and has domain D.

Using (5.1) it is easy to see that $|A(s)v|$ and $|B(s)v|$ are both smaller than

$$(1-K)^{-1}(G(|v|) + |A(s)v + B(s)v|)$$

It then follows that

$$|A(t)v+B(t)v - (A(s)v+B(s)v)|$$

$$\leq |p(t)-p(s)| \; L'(|v|) \; (1 + |A(s)v+B(s)v|)$$

$$0 \leq s < t , \quad s \leq T .$$

Where $L'(r) = 2(1-K)^{-1} L(r) \; (1 + G(r))$.

So II holds for $A(t)+B(t)$.

BIBLIOGRAPHY.

[1] F.E. Browder , Existence Theorems for Nonlinear Partial Differential Equations. Symp. A.M.S. (XVI).

[2] T. Kato , Nonlinear Semigroups and Evolution Equations. J. Math. Soc. Japan 19 (1967).

[3] K. Yoshida , Functional Analysis. Springer (1965).

[4] Y. Komura , Nonlinear Semi-groups in Hilbert Space. J. Math. Soc. Japan 19 (1967).

[5] M.G. Crandall and A. Pazy , Nonlinear Evolution Equations in Banach Spaces. Israel J. Math. (1972).

[6] T. Kato , Accretive Operators and Nonlinear Evolution Equations in Banach Spaces. Symp. A.M.S. (XVIII).

[7] Hobson , The Theory of Functions of a Real Variable and the Theory of Fourier's Series. Vol. 1. C.U.P. (1927)

[8] N. Dunford and J.T. Schwartz , Linear Operators (Part 1). Interscience (1964).

CHAPTER II.

0. In this chapter we study the variational equation

$$y'(t) + \frac{\partial}{\partial x}A(t)(x(t))y(t) = 0 \qquad (0.1)$$

where $x(t)$ is a solution of ((0.1) Chapter I). One reason why (0.1) is important is that its solutions should give first order approximations to solutions of ((0.1) Chapter I) under small perturbations of initial data. The usual approach is to assume conditions strong enough to ensure (0.1) has solutions, and then show these solutions satisfy the perturbation property. (See for example [3, Chapter 4]).

In the infinite dimensional case existence of solutions of (0.1) seems difficult without making unrealistically strong assumptions. (However see section 3 of this chapter). We avoid this difficulty by changing the classical argument as follows. We first assume $\frac{\partial}{\partial x}A(t)x$ exists in a rather weak (Gateaux) sense. We then use solutions of ((0.1) Chapter I) to construct a linear evolution operator of first order variations. This operator may be regarded as being the weak solution of (0.1). If (0.1) has strong solutions then the weak and strong solutions coincide. The construction is based on Lemma 1.7 which is of some interest in itself.

1. We assume $(X, |.|)$ is a Banach space with uniformly convex dual X^*.

Let A be an accretive operator with linear dense domain $D \subset X$.

Definition 1.1. $A'(u)$ is said to be the (strong) Gateaux derivative of A at $u \in D$ if

i) $A'(u) : D \to X$ linear (1.1)

ii) If $u, v \in D$ then

$$|A(u + tv) - Au - A'(u)tv| = o(t) \quad \text{as} \quad t \to 0 \tag{1.2}$$

An extensive discussion of Gateaux derivatives is in [4].

Proposition 1.2. If $A'(u)$ exists then it is unique and accretive.

Proof If B is a Gateaux derivative at u then $|A'(u)tv - Btv| = o(t)$. So $A'(u) = B$.

To show $A'(u)$ accretive write

$t^2 < A'(u)v\ ,\ Fv > = < A'(u)tv\ ,\ Ftv >$

$= - < A(u+tv) - Au - A'(u)tv\ ,\ Ftv > + < A(u+tv) - Au\ ,\ Ftv >$

$\geq -|A(u+tv) - Au - A'(u)tv|\ |tv| = o(t^2)$

Dividing by t^2 and letting $t \to 0$ we obtain

$$< A'(u)v\ ,\ Fv > \ \geq 0$$

Corollary 1.3. If $A'(u)$ exists then $q(A'(u)) \leqslant q(A)$.

Proof $A'(u) + qI = (A + qI)'(u)$ which is accretive.

Let $\mathfrak{A}$ be a collection of q-m-accretive operators with the same linear dense domains $D \subset X$.

Definition 1.4. We say $\mathfrak{A}$ has uniform Gateaux derivatives ($\mathfrak{A} \in$ (U.G.D)) if

i) Each $A \in \mathfrak{A}$ has strong Gateaux derivative. (1.3)

ii) If $v_n \to u$, $v_n, u \in D$, $v_n \neq u$, $|Av_n|$ bounded then

$$|v_n - u|^{-1} |Av_n - Au - A'(u)(v_n - u)| \to 0 \text{ as } n \to \infty \tag{1.4}$$

iii) For each compact subset C of D and each $M < \infty$ there exists $K < \infty$ depending only on C and M such that if $u, v \in C$, $A \in \mathfrak{A}$ and $|Av| + |Au| \leqslant M$ then

$$|Av - Au - A'(u)(v - u)| \leqslant K|v - u| \tag{1.5}$$

iv) For each $A,B \in \mathfrak{A}$ and $M < \infty$ there exists $K < \infty$ such that $|x|+|Ax| < M$ implies $|Bx| < K$. (1.6)

<u>Definition 1.5</u>. A section $[0,T] \to \mathfrak{A} : t \to A(t)$ is said to be a <u>regular</u> control if

i) $x'(t) + A(t)x(t) \overset{*}{=} 0 \qquad x(s) = x_0 \in D$ (1.7)

has unique Lipschitz continuous solution $\Phi(t,s)x_0 \in D$.

ii) For each $M < \infty$ there exists $K < \infty$ such that

if $|x_0|+|A(s)x_0| \leqslant M$ then

$$|\Phi(t,s)x_0|+|A(t)\Phi(t,s)x_0| \leqslant K \qquad t \in [s,T] \tag{1.8}$$

iii) $q(A(t)) = q(t)$ and $\int_0^T |q| < \infty$ (1.9)

<u>Remark</u> Theorem 1 , Chapter I shows that constant controls are regular. If for each $A,B \in \mathfrak{A}$ there exists continuous $L(.)$ such that

$$|Ax - Bx| \leqslant L(|x|)(1 + |Ax|) \tag{1.10}$$

then Theorem 2 , Chapter I shows there are 'plenty' of nonconstant regular controls.

Comparing (1.6) with (1.8) we see that if $A_i(t)$ is regular on $[0,T_i]$ $(i=1,2)$ then

$$A(t) = \begin{cases} A_1(t) & 0 \leqslant t < T_1 \\ A_2(t-T_1) & T_1 \leqslant t \leqslant T_1 + T_2 \end{cases} \qquad \text{is regular.}$$

In this chapter we work in the class of regular controls. This class has solutions which satisfy the 'nice' properties given in the conclusion of Theorem 2 , Chapter I. In particular $\Phi(t,s)$ has Lipschitz constant $\exp\int_s^t q$.

Lemma 1.6. Let $A(t)$ be regular on $[0,T]$, $A(t) \in \mathfrak{A} \in$(U.G.I

Let $\varepsilon_0 > 0$, $[0,\varepsilon_0] \to D : \varepsilon \to x_\varepsilon$ strongly continuous and $(d/d\varepsilon)x_\varepsilon|_{\varepsilon=0} = y_0 \in X$ exists. Suppose

$$\{ \, |A(s)x_\varepsilon| : 0 \leqslant \varepsilon \leqslant \varepsilon_0 \} < \infty \tag{1.1}$$

Let $\Phi(t,s)$ be the solution of (1.7) , and set

$y_\varepsilon(t) = \varepsilon^{-1}(\Phi(t,s)x_\varepsilon - \Phi(t,s)x_0)$

$\lambda(t,\varepsilon) = | \, y_\varepsilon'(t) + A(t)'(\Phi(t,s)x_0)y_\varepsilon(t) \, |$

Then $\lambda(t,\varepsilon) \to 0$ a.e. $t \in [s,T]$, $\int_s^T \lambda(t,\varepsilon)dt \to 0$ as $\varepsilon \downarrow 0$.

Proof If $0 \leqslant \varepsilon, \varepsilon' \leqslant \varepsilon_0$, $s \leqslant t,t' \leqslant T$ then

$$|\Phi(t',s)x_{\varepsilon'} - \Phi(t,s)x_\varepsilon| \leqslant |x_{\varepsilon'} - x_\varepsilon|\exp\int_s^{t'} q + |\Phi(t',s)x_\varepsilon - \Phi(t,s)x_\varepsilon|$$

Thus the map $[0,\varepsilon_0]\times[s,T] \to D : (\varepsilon,t) \to \Phi(t,s)x_\varepsilon$ is continuous , so $C = \{\Phi(t,s)x_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0, \; s \leqslant t \leqslant T\} \subset D$ is compact. Also

$$|y_\varepsilon(t)| \leqslant |y_\varepsilon(s)|\exp\int_s^t q \;\to\; |y_0|\exp\int_s^t q \quad \text{as } \varepsilon\downarrow 0.$$

If $0 < \varepsilon \leqslant \varepsilon_0$

$$\lambda(t,\varepsilon) \overset{*}{=} \varepsilon^{-1}|A(t)\Phi(t,s)x_\varepsilon - A(t)\Phi(t,s)x_0 - A(t)'(\Phi(t,s)x_0)\varepsilon y_\varepsilon(t$$

$$= \begin{cases} 0 & \text{if } \Phi(t,s)x_\varepsilon = \Phi(t,s)x_0 \\ |y_\varepsilon(t)|\,|\Phi(t,s)x_\varepsilon - \Phi(t,s)x_0|^{-1}|A(t)\Phi(t,s)x_\varepsilon - \\ \quad A(t)\Phi(t,s)x_0 - A(t)'(\Phi(t,s)x_0)(\Phi(t,s)x_\varepsilon - \Phi(t,s)x_0)| & \text{otherwise} \end{cases}$$

By (1.8), (1.11) $\sup_{\substack{0 \leq \varepsilon \leq \varepsilon_0 \\ s \leq t \leq T}} |A(t)\Phi(t,s)x_\varepsilon| < \infty$.

Then by (1.5), $\lambda(t,\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ a.e. $t \in [s,T]$.

By (1.6) and dominated convergence $\int_s^T \lambda(t,\varepsilon)dt \to 0$.

Lemma 1.7. Let $\{B(t)\}_{s \leq t \leq T}$ be any one- parameter family of $q(t)$-accretive operators such that $\int_s^T |q| < \infty$. For each $\varepsilon \in (0, \varepsilon_0]$ let $y_\varepsilon : [s,T] \to X$ be strongly absolutely continuous. Suppose $y_\varepsilon(s) \to y_0$ and

$$\int_s^T |y_\varepsilon'(t) + B(t)y_\varepsilon(t)|dt \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 \tag{1.12}$$

Then

i) $y_\varepsilon(t) \to y(t)$ uniformly on $[s,T]$, $y(t)$ is continuous and $y(s) = y_0$.

ii) If $\{\bar{y}_\varepsilon(\cdot)\}_{0 < \varepsilon \leq \bar{\varepsilon}_0}$ also satisfies the conditions of this lemma ($\bar{y}_\varepsilon(t) \to \bar{y}(t)$) then

$$|y(t) - \bar{y}(t)| \leq |y(s) - \bar{y}(s)| \exp \int_s^t q \tag{1.13}$$

Proof Set $R(\alpha,\beta,t) = |y_\alpha'(t)+B(t)y_\alpha(t)| + |\bar{y}_\beta'(t)+B(t)\bar{y}_\beta(t)|$.

Then $\int_s^T R(\alpha,\beta,t)dt \to 0$ as $\alpha,\beta \downarrow 0$.

$$d/dt|y_\alpha(t)-\bar{y}_\beta(t)|^2 \overset{*}{=} 2 < y_\alpha'(t)-\bar{y}_\beta'(t) \;,\; F(y_\alpha(t)-\bar{y}_\beta(t)) >$$

$$\leq -2 < B(t)y_\alpha(t)-B(t)\bar{y}_\beta(t), F(y_\alpha(t)-\bar{y}_\beta(t)) > + 2|y_\alpha(t)-\bar{y}_\beta(t)|R(\alpha,\beta,t)$$

$$\leq 2|y_\alpha(t)-\bar{y}_\beta(t)|(q(t)|y_\alpha(t)-\bar{y}_\beta(t)| + R(\alpha,\beta,t))$$

This gives

$$|y_\alpha(t)-\bar{y}_\beta(t)| \leq \left(|y_\alpha(s)-\bar{y}_\beta(s)|+\int_s^T R(\alpha,\beta,t)dt\right)\exp\int_s^t q \tag{1.14}$$

Taking $\bar{y}_\beta \equiv y_\beta$ we obtain $\{y_\alpha(t)\}_\alpha$ is Cauchy as $\alpha \downarrow 0$ uniformly for $t \in [s,T]$. This gives i). To obtain ii) let $\alpha, \beta \to 0$ in (1.14).

It is now easy to prove

<u>Corollary 1.8</u>. Let $D_s = \{y_o \in X : \exists$ family of curves $y_\varepsilon(.)$ satisfying the conditions of Lemma 1.7 $\}$ and define $\psi(t,s)y_o = \lim y_\varepsilon(t)$. Then

i) $\psi(t,s)$ is well defined on D_s and $\psi(t,s)D_s \subset D_t$

ii) ψ is an evolution operator on $[0,T]$.

iii) $|\psi(t,s)u - \psi(t,s)v| \leq |u-v|\exp\int_s^t q$ (1.15

iv) $t \to \psi(t,s)v$ is continuous on $[s,T]$.

<u>Definition</u>. We call ψ the pseudo-solution of

$$x'(t) + B(t)x(t) = 0 \qquad (1.16$$

It follows directly that if $x(t)$ is absolutely continuous and satisfies (1.16) a.e. then $x(t) = \psi(t,s)x(s)$. So strong solutions of (1.16) are pseudo-solutions.

<u>Theorem 1.9</u>. Suppose $A(t)$ is a regular control of $\mathfrak{A} \in$ (U.G.D). Then corresponding to each solution of $u'(t) + A(t)u(t) \overset{*}{=} 0$, $u(0) = u_o \in D$ on $[0,T]$ there exists a unique evolution operator

$\psi(t,s) \in L(X) \quad 0 \leqslant s \leqslant t \leqslant T$ such that

i) $|\psi(t,s)|_{L(X)} \leqslant \exp \int_s^t q$ (1.17)

ii) $t \to \psi(t,s)x$ is continuous for all $x \in X$.

iii) Suppose $[0,\varepsilon_0] \to D : \varepsilon \to u_\varepsilon$ strongly continuous,

$u_0 = u(s)$, $\{|A(s)u_\varepsilon| : 0 \leqslant \varepsilon \leqslant \varepsilon_0\} < \infty$, and

$y_0 = (d/d\varepsilon)u_\varepsilon|_{\varepsilon=0}$. Let $u_\varepsilon(t)$ be the solution

of $u'(t) + A(t)u(t) \overset{*}{=} 0$, $u(s) = u_\varepsilon$. Then

$|u_\varepsilon(t) - u(t) - \varepsilon\psi(t,s)y_0| = o(\varepsilon)$ (1.18)

uniformly for $t \in [s,T]$.

iv) ψ is the pseudo-solution of

$x'(t) + A(t)'(u(t))x(t) = 0$ (1.19)

<u>Proof</u> Set $B(t) = A(t)'(u(t))$ in Lemma 1.7 and let $\psi(t,s)$ be the evolution operator constructed in Corollary 1.8. Let $y_0 \in D$ and set $u_\varepsilon = u(s) + \varepsilon y_0 \in D$.

By (1.2), $|A(s)u_\varepsilon|$ is bounded for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ where ε_0 is sufficiently small > 0. Also $(d/d\varepsilon)u_\varepsilon|_{\varepsilon=0} = y_0$. Then by Lemma 1.6, $y_\varepsilon(t) = \varepsilon^{-1}(u_\varepsilon(t)-u(t))$ satisfies the conditions of Lemma 1.7. So $y_0 \in D_s$ and $D \subset D_s$.

Now let $v^i \in D_s$, α^i scalar $(i=1,2)$. Let $y_\varepsilon^i(t)$ satisfy the conditions of Lemma 1.7 and $y_\varepsilon^i(s) \to v^i$.

Then by linearity of $B(t)$, $\sum_i \alpha^i y_\varepsilon^i(t)$ also satisfies the conditions of Lemma 1.7. Therefore

$$\Sigma\, \alpha^i \psi(t,s)v^i = \lim_{\varepsilon \downarrow 0} \Sigma\, \alpha^i y_\varepsilon^i(t) = \psi(t,s)\Sigma\, \alpha^i v^i$$

So D_s is a vector s/space of X and $\psi(t,s)$ is linear. Moreover , by (1.15) ,

$$|\psi(t,s)v| \leq |v|\exp\int_s^t q \,, \quad \text{for} \quad v \in D_s.$$

$\psi(t,s)$ can now be extended to all X since D , and therefore D_s , is dense in X.

Parts ii),iii) of the theorem now follow directly from Lemma 1.6 and Corollary 1.8.

2. In this section we study the effect of perturbations of a regular control. It is not altogether straightforeward , and we have to assume X is uniformly convex.

It is feasable to write an expression for the general form of a perturbed control. It is rather complicated and not really necessary. To see how complicated it can be we refer the reader to [O,pp.84].

We now must assume the following

i) X (as well as X^*) is uniformly convex (2.1)

ii) $\mathfrak{A} \in (U.G.D)$ (2.2)

iii) $A(t)$ is a regular control. (2.4)

For convenience we make the following definition.

Definition 2.1. Let $y \in X$. Suppose there exist regular controls $A_\varepsilon(.)$ and 'times' t_ε $(0 < \varepsilon < \varepsilon_0)$ such that

i) $t_\varepsilon = t + \alpha\varepsilon$ for some α (2.5)

ii) The solutions $x_\varepsilon(t)$ of $x'(t) + A_\varepsilon(t)x(t) \overset{*}{=} 0$

$x(0) = x_0 \in D$ have the property

$$x_\varepsilon(t_\varepsilon) = x(t) + \varepsilon y + o(\varepsilon) \tag{2.6}$$

iii) $\{ |A_\varepsilon(t)x_\varepsilon(t)| : 0 \leqslant t \leqslant t_\varepsilon , 0 < \varepsilon < \varepsilon_0\} < \infty$ (2.7)

Then we say y is realizable at t (by $\{A_\varepsilon, t_\varepsilon\}$).

If in addition $\exists\, \delta > 0$ such that

$$x_\varepsilon(t_\varepsilon - \lambda) = x(t-\lambda) + \varepsilon y(t-\lambda) + o(\varepsilon) \quad \text{unif. } 0 \leqslant \lambda < \delta \tag{2.8}$$

$\lambda \to y(t-\lambda)$ is strongly continuous; then we say y is locally realizable.

Remark. 0 is locally realizable at $t > 0$.

Lemma 2.2. Let $\lambda > 0$. If y is locally realizable at t and t is Lebesque point of $t \to A(t)x(t)$ then $y + \lambda A(t)x(t)$ is realizable at t. If y is realizable at t then so is $y - \lambda Bx(t)$ $\forall B \in \mathfrak{A}$.

<u>Proof</u> Suppose y locally realizable at t by $\{A_\varepsilon, t_\varepsilon\}$. Then by (2.8)

$$x_\varepsilon(t_\varepsilon - \lambda\varepsilon) = x(t-\lambda\varepsilon) + \varepsilon y(t-\lambda\varepsilon) + o(\varepsilon)$$

$$= x(t-\lambda\varepsilon) + \varepsilon y + o(\varepsilon) \qquad (2.9)$$

But , since t is a Lebesque point

$$x(t-\lambda\varepsilon) = x(t) + \varepsilon\lambda A(t)x(t) + o(\varepsilon) \qquad (2.10)$$

Adding (2.9) , (2.10) we see $y + \lambda A(t)x(t)$ is realizable at t by $\{A_\varepsilon, t_\varepsilon - \lambda\varepsilon\}$.

Now suppose y only realizable. Set $\tilde{t}_\varepsilon = t_\varepsilon + \lambda\varepsilon$

$$\tilde{A}_\varepsilon(t) = \begin{cases} A_\varepsilon(t) & 0 \leqslant t < t_\varepsilon \\ B & t_\varepsilon \leqslant t \leqslant \tilde{t}_\varepsilon \end{cases}$$

Let $\Phi_B(t,s) = \Phi_B(t-s)$ be the semi-group of solutions of $x'(t) + Bx(t) \overset{*}{=} 0$. Then by Corollary 2 , Section 2 , Chapter I.

$$\Phi_B(\lambda\varepsilon)x(t) = x(t) - \lambda\varepsilon Bx(t) + o(\varepsilon)$$

(It is here we need X uniformly convex). Also by Theorem 1.9 there exists continuous $z(s)$ such that

$$\Phi_B(s)x_\varepsilon(t_\varepsilon) = \Phi_B(s)x(t) + \varepsilon z(s) + o(\varepsilon)$$

uniformly for $s \geq 0$, and $z(0) = y$. Therefore

$$\tilde{x}_\varepsilon(\tilde{t}_\varepsilon) = \Phi_B(\lambda\varepsilon)x_\varepsilon(t_\varepsilon) = \Phi_B(\lambda\varepsilon)x(t) + \varepsilon z(\lambda\varepsilon) + o(\varepsilon)$$

$$= x(t) + \varepsilon(y - \lambda Bx(t)) + o(\varepsilon)$$

Therefore $y - \lambda Bx(t)$ is realizable by $\{\tilde{A}_\varepsilon(.), \tilde{t}_\varepsilon\}$.

This completes the proof of the Lemma.

<u>Lemma 2.3</u>. Let $C(t)$ be the convex cone generated by

$$Z(t) = \begin{cases} \{-Bx(t) : B \in \mathfrak{R}\} \cup \{A(t)x(t)\} & t > 0 \quad \text{L-point} \\ \{-Bx(t) : B \in \mathfrak{R}\} & \text{otherwise} \end{cases} \qquad (2.11$$

Suppose y is locally realizable at t (by $\{A_\varepsilon, t_\varepsilon\}$). Then all points in $y + C(t)$ are realizable at t.

<u>Proof</u> Let $z \in y + C(t)$. Then

$$z = y + \lambda A(t)x(t) - \sum_1^n \lambda_i B_i x(t) \qquad \lambda \geq 0 \, , \; \lambda_i > 0 \, , \; B_i \in \mathfrak{R} .$$

Either $\lambda = 0$ or t is L-point of $A(t)x(t)$.

By the previous lemma $y + \lambda A(t)x(t)$ is realizable.

Then again by the same lemma $y + \lambda A(t)x(t) - \lambda_n B_n x(t)$ is realizable , and so on , to give z realizable.

<u>Lemma 2.4</u>. Let y be realizable at $s < t_1$ by $\{A_\varepsilon, s_\varepsilon\}$. Then $\psi(t_1,s)y$ is locally realizable at t_1.

<u>Proof</u> Set $t_\varepsilon = s_\varepsilon + t_1 - s$

$$\tilde{A}_\varepsilon(t) = \begin{cases} A_\varepsilon(t) & 0 \leqslant t < s_\varepsilon \\ A(t+s-s_\varepsilon) & s_\varepsilon \leq t \leqslant t_\varepsilon \end{cases}$$

By hypothesis $\tilde{x}_\varepsilon(s_\varepsilon) = x_\varepsilon(s_\varepsilon) = x(s) + \varepsilon y + o(\varepsilon)$.

By Theorem 1.9 , if $0 \leqslant \lambda \leqslant t_1 - s$

$$\begin{aligned} \tilde{x}_\varepsilon(t_\varepsilon - \lambda) &= \tilde{x}_\varepsilon(s_\varepsilon + t_1 - \lambda - s) = \Phi(t_1-\lambda, s)\tilde{x}_\varepsilon(s_\varepsilon) \\ &= \Phi(t_1-\lambda,s)x(s) + \varepsilon\psi(t_1-\lambda,s)y + o(\varepsilon) \\ &= x(t_1-\lambda) + \varepsilon\psi(t_1-\lambda,s)y + o(\varepsilon) \end{aligned}$$

Since $\lambda \to \psi(t_1-\lambda,s)y$ is continuous , $\psi(t_1,s)y$ is locally realizable at t_1.

Lemmas 2.3 , 2.4 and a simple induction gives

Theorem 2.5. Each element of the convex cone $K(t)$ generated by $\bigcup_{0\leq s\leq t} \psi(t,s)Z(s)$ is realizable at t , where $Z(s)$ is defined in (2.11).

Remark $x(t) + K(t)$ lies in the 'tangent cone of attainability'.

3. In this section we briefly consider the problem of when a pseudo-solution of the variational equation is a strong solution. Since a strong solution is a pseudo-solution , we need only consider the problem of existence of strong solutions. We make the following assumptions.

i) $\{A\} \in (U.G.D)$ (3.1)

ii) For each $u \in D$ either $A'(u)$ is closed , or , more generally , $A'(u)$ is closable and the closure has domain $\tilde{D}$ independent of u. (3.2)

iii) $|A'(u)x - A'(v)x| \leq |u-v|L(|u|+|v|+|Au|+|Av|)|A'(u)x|$ (3.3)

Theorem 3.1. Suppose (3.1),(3.2),(3.3) hold. Let $x(t)$ be the solution of $x'(t) + Ax(t) \stackrel{*}{=} 0$, $x(0) = x_0 \in D$. Then

$$y'(t) + A'(x(t))y(t) \stackrel{*}{=} 0 \quad y(0) = y_0 \in \tilde{D} \tag{3.4}$$

has unique Lipschitz continuous solution.

<u>Proof</u> We show $\{A'(x(t)) : t \geq 0\}$ satisfies the conditions of Theorem 2 , Chapter I.

Since $x(t)$ is Lipschitz continuous , (3.3) shows there exists a constant K such that

$$|A'(x(t))x - A'(x(s))x| \leqslant K|t-s||A'(x(s))x| \quad 0 \leqslant s < t \leqslant T \tag{3.5}$$

It remains to show $A'(u)$ is q-m-accretive. By Corollary 1.3 , $A'(u)$ is q(A)-accretive , so for small enough $\lambda > 0$, $I + \lambda A'(u)$ has a continuous inverse which is closed since $A'(u)$ closed. Therefore (see for example [5,pp178]) $I + \lambda A'(u)$ has closed range.

We now show the range is dense. If not , then by the Hahn-Banach Theorem there exists $x^* \in X^*$, $|x^*| = 1$ such that

$$< (I + \lambda A'(u))x , x^* > = 0 \qquad \text{for all } x \in D \tag{3.6}$$

Since X reflexive there exists $x \in X$, $|x| = 1$ and $< x,x^* > = 1$. Let $0 < \rho < 1$. Then since $I + \lambda A$ is surjective there exists $x_\rho \in D$ such that

$$(I + \lambda A)(u + x_\rho) - (I + \lambda A)u = \rho x \tag{3.7}$$

$$|x_\rho| = |(I+\lambda A)^{-1}(I+\lambda A)(u+x_\rho)-(I+\lambda A)^{-1}(I+\lambda A)u| \leqslant K\rho \tag{3.8}$$

where K is the Lipschitz constant of $(I+\lambda A)^{-1}$

(3.7) shows $|A(u + x_\rho)|$ is bounded, and (3.8) shows $u + x_\rho \to u$ as $\rho \to 0$. So by (1.4) and (3.8)

$$|A(u + x_\rho) - Au - A'(u)x_\rho| = o(\rho) \quad \text{as} \quad \rho \downarrow 0 \tag{3.9}$$

Then by (3.6)

$$\rho = \langle \rho x, x^* \rangle = \langle (I+\lambda A)(u+x_\rho) - (I+\lambda A)u - (I+\lambda A'(u))x_\rho, x^* \rangle$$

$$= \lambda \langle A(u+x_\rho) - Au - A'(u)x_\rho, x^* \rangle \leqslant \lambda o(\rho) \qquad \text{by (3.9)}.$$

Dividing by ρ gives $1 \leqslant o(1)$. So $I + \lambda A'(u)$ is surjective. The proof is complete.

It may be worth noting that Theorem 3.1 doesn't fit the standard conditions which are usually assumed for the existence of linear evolution operators ($A'(u)$ does not generate an analytic semi-group). From an extensive literature see for example [1] or [2].

4. In all this Chapter we have been concerned with the linearization of (1.7). Theorem 1.9 gives conditions under which the 'classical theory' holds in infinite dimensions. Not much research seems to have been done on this problem (in fact we don't have any references), the probable reason being that existence theory for abstract nonlinear partial

differential equations is still in its infancy. However this problem has been studied for particular important equations with rather suprising results. Dr. Pironneau recently communicated to me the following 'non-classical' phenomenon. The formal variational equation of the Navier-Stokes equation has weak solutions , but these solutions do not appear to give first order approximations to solutions of the Navier-Stokes equation (presumably in any 'reasonable' topology). It seems hopeless in this situation to try to obtain any of the classical optimization results in control theory.

We should remark that although we have worked with the strong topology of X throughout this Chapter it is possible to use weaker topologies. We have proved an analogue to Theorem1.9 using the weak topology. The essential difference is that a 'weak version' of (1.2),(1.4),(1.5) is assumed and then a 'weak version' of (1.18) is obtained.

References.

[0] Ju.V.Egorov - Necessary Conditions for Optimal Control in Banach Space - Mat. Sbornik , 64 , pp. 79-101 (1964).

[1] A.Friedman - Partial Differential Equations - Holt , Rinehart and Winston (1969).

[2] S.G.Krein - Linear Differential Equations in Banach Space - A.M.S. Translations V29 (1971)

[3] E.B.Lee and L.Markus - Foundations of Optimal Control Theory - John Wiley and Sons.

[4] R.A.Taipa - The Differentiation and Integration of Nonlinear Operators - Proceedings of an Advanced Seminar on Nonlinear Functional Analysis and Applications - Conducted by Math. Research Center , University Wisconsin (1970).

[5] A.E.Taylor - Introduction to Functional Analysis - John Wiley and Sons (1958).

CHAPTER III

0. It is the purpose of this chapter to apply the results of Chapter II to obtain some maximum principles. It is not our intention to obtain the most generality possible , but rather to demonstrate a method , which , we hope , has wider applicability to nonlinear optimisation problems.

Section 1 demonstrates a rather pleasing controllability property of accretive operators. Section 2 formulates an abstract separation theorem. This contains the 'kernel' of an idea in [2].+ However our argument is much simpler than [2], and in particular we don't require the 'tangent cone of attainability' to have interior point.

Paper [2] demonstrates an abstract maximum principle for evolutionary systems in Banach space. However it seems to contain many obscurities; see for example Avner Friedman's comments in [3]. One proposition which is assumed without proof is the following: If U is the open unit ball in Banach space X , $S : [0,1] \times U \to X : (t,x) \to S_t(x)$. S_t is a homeomorphism from U to $S_t(U)$,

+ I should like to thank my Supervisor for initiating my interest in this paper.

$S_t(U)$ open , $t \to S_t(x)$ is continuous and $S_0 = id_U$. Then there exists $\varepsilon > 0$ such that $0 \in \bigcap_{0 \leq t \leq \varepsilon} S_t(U)$. The Browder fixed point theorem shows this is true if X is finite dimensional. We do not know if it is true in infinite dimensions. It might seem Bessaga's Theorem , see for example [4], would supply a counterexample but we have been unable to show this.*

In Section 3 we prove two maximum principles with fixed end-point. In the first the 'time' at which the end-point is attained is not fixed. In the second it is. Egorov [1], [2] only considered the first case. An elementary but important example is given in Section 4.

* I should like to thank Dr. David Elworthy for suggesting I look at Bessaga's Theorem.

1. In this section we prove a controllability condition for accretive operators. It is based on the following observation. If q(.) in Theorem 2 (f) , Chapter I has integral $-\infty$ on $[0,T]$ then all solutions merge together from whatever their initial point.

<u>Proposition 1.1</u>. Let B be m-accretive with domain D. Let $x_1 \in D$ and suppose $z'(t) + Bz(t) = 0$, $z(0) = x_1$ has solution $z(t)$ on the non-trivial interval $[-\tau, 0]$. (We do not assume backward uniqueness, only local backward existence). Then

$$x'(t) + Bx(t) - t^{-1}(x(t) - z(t)) \overset{*}{=} 0 \tag{1.1}$$

$$x(s) = x_0 \in D, \qquad -\tau \leqslant s < 0 \tag{1.2}$$

has (unique) Lipschitz continuous solution $x(t)$ on $[s, 0]$ such that

i) $$|t|^{-1}|x(t) - z(t)| \leqslant |s|^{-1}|x_0 - z(s)| \tag{1.3}$$

ii) $$|x'(t)| \overset{*}{\leqslant} |s|^{-1}\{\ |t||x'(s)| + K|t-s|\ \} \tag{1.4}$$

where K is defined in (1.5).

<u>Proof</u> First observe that if $s < \varepsilon < 0$ then, since $z(t)$ is Lipschitz continuous, (1.1) satisfies the conditions of Theorem 2, Chapter I on $[s, \varepsilon]$. If $x_0 = z(s)$ then (1.1), (1.2) has solution $z(t)$. Therefore by Theorem 2(f), Chapter I

$$|x(t) - z(t)| \leqslant |x_0 - z(s)| \exp \int_s^t du/u .$$

Integrating gives (1.3). Letting $\varepsilon \uparrow 0$ shows $x(t) \to z(0) = x_1$ as $t \uparrow 0$. Thus $x(t)$ can be defined by continuity on $[s, 0]$.

To obtain an estimate for the Lipschitz constant of $x(t)$ it is sufficient to obtain an essential bound of $x'(t)$. Such an estimate is given in Theorem 2(d), Chapter I, but in this case it turns out to be too weak. We therefore proceed with a direct computation.

Fix small $h > 0$ and let $s \leq t < t+h < 0$. Let K_1 be the Lipschitz constant for $z(t)$ on $[-\tau,0]$ and set $y(t) = x(t+h) - x(t)$. Then using (1.3)

$$(d/dt)|y(t)|^2 = 2 < x'(t+h) - x'(t) \;,\; Fy(t) >$$

$$\overset{*}{=} -2 < Bx(t+h)-Bx(t), Fy(t) > + 2t^{-1} < y(t)-z(t+h)+z(t), Fy(t) >$$

$$+2\{ (t+h)^{-1} - t^{-1} \} < x(t+h) - z(t+h) \;,\; Fy(t) >$$

$$\leq 2|t|^{-1}\{-|y(t)|^2 + hK_1|y(t)|\}$$

$$+2|(t+h)^{-1}-t^{-1}|\,|t+h|\,|s|^{-1}|x_o-z(s)|\,|y(t)|$$

$$(d/dt)|y(t)| \leq |t|^{-1}\{-|y(t)|+hK_1\} + h|st|^{-1}|x_o - z(s)|$$

$$(d/dt)|t|^{-1}|y(t)| \leq hK_1|t|^{-2} + h|st^2|^{-1}|x_o - z(s)|$$

Integrating from s to t

$$|t|^{-1}|y(t)| \leq |s|^{-1}|y(s)|+h\{K_1+|s|^{-1}|x_o-z(s)|\}\{|t|^{-1}-|s|^{-1}\}$$

Dividing by h and letting $h\downarrow 0$ gives (1.4) with

$$K = K_1 + |s|^{-1}|x_o - z(s)| \tag{1.5}$$

<u>Remark 1.2</u>. If B is q-m-accretive and $q(B) > 0$ then (1.3),(1.4),(1.5) need slight modification. This does not affect the result of the next Corollary.

<u>Corollary 1.3</u>. Suppose, in addition to the conditions of Proposition 1.1, $z'(0)$ exists. Let $\delta > 0$. Then there exists open set U in X and open interval $J \subset (-\delta, 0)$ such that if $(x,t) \in U \cap D \times J$ and $0 < \lambda \leqslant 1$ then $x_1 + \lambda x$ is steered by (1.1) along $x(t)$ to x_1 in 'time' interval $[\lambda t, 0]$ and moreover

$$|s^{-1}(x(s) - z(s))| < \delta \qquad \lambda t \leqslant s \leqslant 0 \tag{1.6}$$

$$|x(s) - x_1| < \delta \qquad \text{''} \tag{1.7}$$

<u>Proof</u> (1.6) shows that

$$|x(s) - x_1| < (\delta + K_1)|s|$$

So, by choosing J sufficiently close to 0, (1.7) is automatically satisfied. It remains to find U and J to satisfy (1.6).

Let $B(s)$ be the open ball centre $z(s)$, radius $\delta|s|$. Then by (1.3) each point of $B(s) \cap D$ is steered by (1.1) to x_1 along $x(t) \in B(t)$ $(s \leqslant t < 0)$. Now

$$z(s) = x_1 + sz'(0) + o(|s|)$$

Let $U(s)$ be the open ball centre $sz'(0)$, radius $\delta|s|/2$. Then for some $s_0 \in (-\delta, 0)$,

$$x_1 + U(s) \subset B(s) \quad \text{for all} \quad s \in [s_0, 0).$$

Now choose $s_1 \in (s_0, 0)$ such that $U = U(s_1) \cap U(s_0) \neq \phi$.

Then it is easy to see $U = \bigcap_{s_0 \leq s \leq s_1} U(s)$.

Set $J = (s_0, s_1)$.

It is now a trivial verification to show U , J satisfy our requirements.

2. In this section we use some of the jargon of control theory.

If x_0 , $x_1 \in D \subset X$ we say admissible control $c(t)$ steers x_0 to x_1 (in time interval $[s,t]$) if the corresponding admissible trajectory $x(t)$ (assumed unique) with initial point $x_0 = x(s)$ has end-point $x_1 = x(t)$. If c_i steers x_i to x_{i+1} in time interval $[t_i, t_{i+1}]$ $(i = 0,1)$ then we assume the 'compound' control is admissible and steers x_0 to x_2 (via x_1) in time interval $[t_0, t_2]$.

Suppose to each admissible control $c(t)$ (on $[s,t]$) and corresponding admissible trajectory $x(t)$ there is an associated cost functional which has the form

$$F^0(\, x(.), c(.)\,) = \int_s^t f^0(x(u), c(u))du \tag{2.1}$$

Thus we can define admissible trajectories in $\tilde{X} = R \times X$ by $t \to \left(\int_s^t f^0(x(u),c(u))du \, , \, x(t) \right)$

Fix x_0 , $x_1 \in D \subset X$. Let A (the set of attainability) be the points in D to which x_0 can be steered. Define $\tilde{A}$ in $\tilde{X}$ to be the points to which $(0,x_o)$ can be steered. Let Λ be the set of points in X which can be steered to x_1. If $x \in \Lambda$ define

$$\rho(x) = \inf \{ F^o(x(.),c(.)) : c \text{ steers } x \text{ to } x_1 \} \qquad (2.2)$$

Suppose the following attainability condition holds: There exists an open set U in X such that

$$(x_1 + \lambda U)\cap A \subset \Lambda\ , \ \rho((x_1 + \lambda U)\cap A) \leq \lambda \quad 0 < \lambda < 1 \qquad (2.3)$$

Without loss in generality we can assume U bounded, convex and $0 \notin U$.

We say cone C with vertex 0 in $\tilde{X}$ is open if $C - \{0\}$ is an open set. We say the ray $\{x+\lambda y : \lambda \geq 0\}$ in $\tilde{X}$ is tangent to $\tilde{A}$ if for each open cone C (vertex 0) containing y and each neighbourhood U of 0.

$$(x + C \cap (U - \{0\})) \cap \tilde{A} \neq \phi \qquad (2.4)$$

(this is the geometric interpretation of the usual analytic definition).

<u>Lemma 2.1</u>. Let l be the ray $\{(-\lambda,0): \lambda \geq 0\}$ in $\tilde{X}$. Suppose $((x^o,x_1) + l) \cap \tilde{A} = (x^o,x_1)$ (This is the optimality condition). Let K be a convex cone (vertex 0) in $\tilde{X}$ such that each ray of $(x^o,x_1) + K$ is tangent to $\tilde{A}$. Let U be the open set defined in (2.3) and $W = \bigcup_{0<\lambda<1} \lambda U$.

Let $C = \{(-\lambda, \lambda W) : \lambda \geq 0\}$. So C is an open convex cone and $1 \subset \partial C$. Then $K \cap C = \{0\}$.

Proof Suppose the contrary that $K \cap C$ contains a ray. Then since C is open, we obtain from (2.4) $y = (x^0, x_1) + (-\lambda, \lambda w) \in ((x^0, x_1) + (C - \{0\})) \cap \tilde{A}$ for some $\lambda \in (0,1)$, $w \in W$. But then $w = \mu u$ for some $\mu \in (0,1)$, $u \in U$. Since $y \in \tilde{A}$, $x_1 + \lambda\mu u \in A$. Then by (2.3) $x_1 + \lambda\mu u \in \Lambda$ and $\rho(x_1 + \lambda\mu u) \leqslant \lambda\mu < \lambda$. Therefore $y = (x^0 - \lambda, x_1 + \lambda\mu u)$ is steerable to $(x^0 - \varepsilon, x_1)$ for some $\varepsilon > 0$. This contradicts the optimality assumption.

Proposition 2.2. If the conditions of Lemma 2.1 hold then there exists $y_1^* = (y_0^*, y^*) \in R^* \times X^* \cong \tilde{X}^*$ such that $y_0^* \leqslant 0$ and $< z , y_1^* > \leqslant 0$ for all $z \in K$.

Proof By standard separation theorems (see for example [Nirenberg 7, pp13]) there exists $y_1^* \in \tilde{X}^*$ such that

$$< z , y_1^* > \leqslant \alpha \leqslant < u , y_1^* > \quad \text{for all } z \in K , u \in \bar{C}$$

Since $0 \in K$, $1 \subset \bar{C}$, $0 \leqslant \alpha \leqslant -\lambda y_0^*$ for all $\lambda \geq 0$. Therefore $\alpha = 0$, $y_0^* \leqslant 0$.

Remark It is clear that this result can be proved under more general conditions. In particular

X could be any locally convex space. However by taking a weaker topology on X (for instance the weak topology) assumption (2.3) becomes stronger.

It is also clear that x_1 could be replaced by any closed convex 'target set', and one would obtain the usual transversality condition. If the target set also contained an interior point then condition (2.3) is automatically satisfied.

3. We apply Proposition 2.2 to systems discussed in Chapter II.

Let X, X^* be uniformly convex Banach spaces, $\mathfrak{A} \in$ (U.G.D)

<u>Definition 3.1</u>. We call a section $[0,T] \to \mathfrak{A} : t \to A(t)$ an admissible control if $[0,T'(<T)] \to \mathfrak{A} : t \to A(t)$ is regular.

Notice that a regular control followed by an admissible control is admissible; and that a regular control is admissible, but not conversely.

<u>Definition 3.2</u>. If $A(t)$ is an admissible control on $[0,T]$ we say $x(t)$ is an admissible trajectory if $x'(t) + A(t)x(t) = 0$ a.e. $t \in [0,T]$ and $x(t)$ is Lipschitz continuous on $[0,T]$.

<u>Remark</u> Since $A(t)$ is regular on $[0,T'(<T)]$ we know $x(t)$ is Lipschitz on $[0,T']$. However the Lipschitz constant may blow up to $+\infty$ as $T'\uparrow T$.

Let $f^0 : X \times \mathcal{A} \to R$ and suppose $f^0(x,A)$ and the Fréchet derivative $(\partial/\partial x)f^0(x,A)$ are continuous in the first variable. Consider the system

$$(d/dt)x^0(t) \overset{*}{=} f^0(x(t),A(t)) \qquad x^0(0) = 0 \tag{3.1}$$

$$(d/dt)x(t) \overset{*}{=} -A(t)x(t) \qquad x(0) = x_0 \in D \tag{3.2}$$

in $\tilde{X} = R \times X$, $A(t)$ is regular control.

Let $\psi(t,s)$ be the pseudo-solution of

$$(d/dt)y(t) + A(t)'(x(t))y(t) = 0 \tag{3.3}$$

as shown to exist in Chapter II, Theorem 1.9. Then the variational operator for (3.1),(3.2) has matrix form

$$\psi_1(t,s) = \begin{pmatrix} 1 & \psi^0(t,s) \\ 0 & \psi(t,s) \end{pmatrix} \tag{3.4}$$

$$\psi^0(t,s) = \int_s^t (\partial/\partial x)f^0(x(\lambda),A(\lambda))\circ\psi(\lambda,s)d\lambda . \tag{3.5}$$

<u>Theorem 3.3</u>. (Maximum Principle)

Let X, X^* be uniformly convex Banach spaces, $\mathcal{A} \in$ (U.G.D), x_0, $x_1 \in D$. Suppose there exists $B \in \mathcal{A}$ such that

i) $z'(t) + Bz(t) \overset{*}{=} 0$, $z(0) = x_1$ has solution on $[-\tau,0]$ differentiable at $t = 0$.

ii) There exists $\delta > 0$ such that $B + h$ (i.e. the operator $x \to Bx + h$) $\in \mathfrak{A}$ for all $|h| < \delta$, and $f^0(x,B + h) < M < \infty$ for all $|x - x_1| < \delta$ and $|h| < \delta$. (Without loss in generality we may assume $\delta < M^{-1}$).

Suppose amongst all admissible controls $A(t)$, steering x_0 to x_1 by (3.2) along an admissible trajectory , there is an optimal control $\bar{A}(t)$ defined on $[0,T]$. That is to say $\bar{x}(T) = x_1$ and $\bar{x}^0(T)$ is minimized. Suppose $\bar{A}$ is regular. Then there exists $y_1^* = (y_0^*(\leqslant 0) , y^*) \in R^* \times X^* \cong \tilde{X}^*$ such that if

$$H(z,t) = < z , \Psi_1^*(T,t)y_1^* > \tag{3.6}$$

then

$$H((f^0(\bar{x}(t),A),-A\bar{x}(t)) , t) \leqslant 0 \tag{3.7}$$

for all $t \in [0,T]$, $A \in \mathfrak{A}$.

Moreover equality holds in (3.7) for almost all t if $A = \bar{A}(t)$.

<u>Proof</u> Using Theorem 2.5 in Chapter II set K to be the convex cone generated by $\Psi_1(T,t)\begin{pmatrix} f^0(\bar{x}(t),A) \\ -A\bar{x}(t) \end{pmatrix}$

and if t is Lebesque point of $t \to \begin{pmatrix} f^0(\bar{x}(t),\bar{A}(t)) \\ \bar{A}(t)\bar{x}(t) \end{pmatrix}$ we

also include the vectors

$$\Psi_1(T,t)\begin{pmatrix}-f^0(\bar{x}(t),\bar{A}(t))\\ \bar{A}(t)\bar{x}(t)\end{pmatrix}$$

Then each ray of $(\bar{x}^0(T),x_1)+K$ is tangent to the set $\tilde{A}$ in $\tilde{X}$ which are reachable by regular controls.

Using the δ (given in hypothesis ii)) in Corollary 1.3 . we obtain open set $U\subset X$, such that (1.1) steers $x_1+\lambda x\in x_1+\lambda U\cap D$ along $x(t)$ to x_1 in time interval $[\lambda t_o,0]\subset[-\lambda\delta\ ,0]$, where $t_o\in J$.
Let $u(t+S)=-t^{-1}(x(t)-z(t))$. Then $x'(t)+Bx(t)+u(t)\overset{*}{=}0$ steers $x_1+\lambda x$ to x_1 in time interval $[S+\lambda t_o,S]$.
By (1.6) , $|u(t)|<\delta$ so $B+u(t)\in\Omega$. By (1.4) , $x(t)$ is Lipschitz continuous. Therefore $u(t)$ is Lipschitz continuous on $[S+\lambda t_o,S'(<S)]$, and so , by Theorem 2 Chapter I , $B+u(t)$ is an admissible control (but not necessarily regular) and $x(t)$ is an admissible trajectory. Moreover

$$\int_{S+\lambda t_o}^{S} f^0(x(t),B+u(t))dt<\lambda|t_o|M<\lambda\delta M<\lambda$$

so U satisfies (2.3).

The Theorem now follows from Proposition 2.2 , and the observation that

$$\pm H(\ (f^0(\bar{x}(t),\bar{A}(t)),-\bar{A}(t)\bar{x}(t))\ ,\ t\ )\leqslant 0\qquad\text{a.e.}\quad t\in[0,T].$$

<u>Theorem 3.4</u>. Suppose the assumptions of Theorem 3.3 hold and $\bar{A}(t)$ is optimal amongst controls steering x_0 to x_1 in the given time interval $[0,T]$ (i.e. we now fix T as well as x_1). Then

$$H(\ (f^o(\bar{x}(t),A),-A\bar{x}(t))\ ,\ t\ ) \leq$$

$$H(\ (f^o(\bar{x}(t),\bar{A}(t)),-\bar{A}(t)\bar{x}(t))\ ,\ t\ ) \overset{*}{=} C = \text{const.} \tag{3.8}$$

<u>Proof</u> Adjoin the time coordinate to $\tilde{X}$, so $\tilde{X}$ becomes $R \times X \times R$. The variational operator becomes

$$\Psi_1 = \begin{pmatrix} 1 & \Psi^o & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the set U we take $U \times J$ as constructed in Corollary 1.3. Applying Theorem 3.3 we obtain

$$H(\ (f^o(\bar{x}(t),A),-A\bar{x}(t))\ ,\ t\ ) + t^* \leq 0 \quad \text{for some } t^* \in R$$

$$H(\ (f^o(\bar{x}(t),\bar{A}(t)),-\bar{A}(t)\bar{x}(t))\ ,\ t\ ) + t^* \overset{*}{=} 0$$

<u>4</u>. We apply Theorem 3.4 to the following standard example.

Let X, X^* be uniformly convex Banach spaces. Let A be q-m-accretive on $D \subset X$, and suppose A satisfies conditions (1.3),(1.4),(1.5) of Chapter II. Let Ω be the closed unit ball in X. Then $\{\ A + u : u \in \Omega\ \} \in$ (U.G.D).

Consider the control system

$$x'(t) + Ax(t) \overset{*}{=} u(t) \in \Omega \qquad x(0) = x_0 \in D \tag{4.1}$$

and quadratic cost functional

$$\int_0^T |u(t)|^2 + \alpha|x(t)|^2 \, dt \qquad (\alpha \geq 0) \tag{4.2}$$

Let $x_1 \in D$ be the target point and suppose

$$x'(t) + Ax(t) = 0 \qquad x(0) = x_1$$

has local backward solution differentiable at $t = 0$.

Let $\overline{x}(t)$ be the trajectory for optimal (regular) control $\overline{u}(t)$ on $[0,T]$ (T fixed).

Let ψ be the pseudo-solution of

$$y'(t) + A'(\overline{x}(t))y(t) = 0$$

By direct computation (3.5) becomes

$$\psi^0(t,s)y = \int_s^t < y \, , \, 2\alpha\psi^*(\lambda,s)F\overline{x}(\lambda) > d\lambda \tag{4.3}$$

F is the duality map which is bijective since X, X^* are uniformly convex. All the assumptions of Theorem 3.4 are satisfied so from (3.8) we obtain

$$y_0^*(|u|^2 + \alpha|\overline{x}(t)|^2) + < -A\overline{x}(t) + u \, , \, V^*(t) > \leqslant C \tag{4.4}$$

$$V^*(t) = \psi^*(T,t)y^* + 2\alpha y_0^* \int_t^T \psi^*(\lambda,t)F\overline{x}(\lambda) \, d\lambda \tag{4.5}$$

and equality holds a.e. in (4.4) if $u = \overline{u}(t)$.

If $z \neq 0$ write $z^\wedge = z/|z|$. If $y_0^* = 0$ then (4.4) gives

$$\overline{u}(t) \overset{*}{=} F^{-1}(V^*(t))^\wedge \tag{4.6}$$

$$|V^*(t)| - < A\overline{x}(t), V^*(t) > \overset{*}{=} C = |y^*| - < Ax_1, y^* > \tag{4.7}$$

Now suppose $y_0^* < 0$. Then by homogeneity we may assume $y_0^* = -1/2$. Then by (4.4)

$$\bar{u}(t) \overset{*}{=} \bar{\lambda} F^{-1}(V^*(t))^{\wedge} \qquad 0 \leqslant \bar{\lambda} \leqslant 1$$

and $\bar{\lambda}$ maximizes

$$-1/2\ (\lambda^2 + \alpha|\bar{x}(t)|^2) + \lambda|V^*(t)| - < A\bar{x}(t), V^*(t) >$$

So $\bar{\lambda} = |V^*(t)| \wedge 1$

$$\bar{u}(t) \overset{*}{=} (|V^*(t)| \wedge 1) F^{-1}(V^*(t))^{\wedge} \tag{4.8}$$

The condition that $\bar{u}(t)$ is a boundary control is

$y_0^* = 0$ or

$y_0^* = -1/2$ and

$$|\psi^*(T,t)y^* - \alpha\int_t^T \psi^*(\lambda,t)F\bar{x}(\lambda)d\lambda| \geq 1 \tag{4.9}$$

which implies $|y^*| \geq 1$

5. The maximum theorems in section 3 are not as satisfactory as we might wish. The problem is that we were not able to steer from an open set to the target point by a regular control, only by an admissible control; but Theorem 1.9 Chapter II is only valid for a regular control. Until more powerfull controllability results than Corollary 1.3 are obtained for nonlinear dissipative systems this problem will probably remain unresolved.

So far we have completely ignored the question of existence of optimal controls. This problem has been very sucessfully tackled by Lions in [6]. His technique is standard in that he takes a minimizing sequence of controls and then using sequential compactness shows that a subsequence converges to an optimal control. However it seems hard to topologize the set of controls which generate strong solutions in a suitable way. Lions considered weak solutions, and then completeness of the space of controls is usually self evident.

It may be possible to bring together existence of optimal control and the maximum principle by considering product integral representations of solutions. If u(t) is Riemann integrable then the results of the next Chapter show solutions of (4.1) have a product integral representation

$$x(t) = \lim_{n\to\infty} \prod_{i=1}^{n} (I + (t/n)A)^{-1}(\,\cdot\, + (t/n)u(it/n))\, x_0 \qquad (5.1)$$

Thus it seems worthwhile to consider the variational properties of expressions like (5.1). That is to say, when is the map $x_0 \to x(t)$ in (5.1) differentiable?

References

[1] Ju.V. Egorov - Optimal Control in Banach Space - Soviet Math. Dokl., 4 pp. 630 - 633 (1963).

[2] Ju.V. Egorov - Necessary Conditions for Optimal Control in Banach Space - Mat. Sbornik , 64 pp. 79 - 101 (1964).

[3] A. Friedman - Optimal Control in Banach Spaces - J. Math. Analysis and Applications , 19 pp. 35 - 55 (1967).

[4] N. Kuiper - Notes from the Summer Conference in Global Analysis - Montréal (1969).

[5] E.B. Lee , L. Markus - Foundations of Optimal Control Theory - John Wiley and Sons.

[6] J.L. Lions - Optimisation pour certaines classes d'equations d'evolution non lineaires - Proc. Symp. on Math. Theory of Control , U.S.C. (1967).

[7] L. Nirenberg - Functional Analysis - Lecture Notes - Courant Institute of Mathematical Sciences , New York University.

[8] L.W. Neustadt - A General Theory of Extremals - J. Computer and System Sciences 3 pp. 57 - 92 (1969).

[9] L.S. Pontryagin , V.G. Boltyanskii , R.V. Gamkrelidze , E.F. Mishchenko - The Mathematical Theory of Optimal Processes - John Wiley and Sons.

[10] E. Roxin - A Geometric Interpretation of Pontryagin's Maximum Principle - International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics pp. 303 - 324 (1962).

CHAPTER IV

0. Introduction. In Crandall and Pazy [2] the evolution equation

$$\left.\begin{array}{ll} u'(t) + A(t)u(t) \ni 0 & s \leqslant t \leqslant T \\ u(s) = x & \end{array}\right\} \qquad (0.1)$$

on Banach space X is considered.

We assume the same conditions on the maps $x \to A(t)x$ as [2] (see A1 , A2 , A3 in section 2). In [2] the maps $t \to A(t)x$ are conditioned as follows

C1) $||J(t,\lambda)x - J(s,\lambda)x|| \leqslant \lambda||f(t)-f(s)||L(||x||)$

where $L : [0,\infty) \to [0,\infty)$ is monotone increasing , and f is X valued and continuous.

In this paper we show C1) can be weakened. We give analogous results for the more interesting condition C2) elsewhere.

Although it is often natural , when considering concrete examples , to assume f is X valued ; it is easy to see that all the proofs in [2] still go through without modification if f takes values in any Banach space Y. It is particularly interesting to take $Y = C[0,T]$ and

$$f(t)(s) = \begin{cases} 0 & s \leqslant t \\ |t-s|^{\alpha} & s > t \end{cases}$$

where $\alpha > 0$. Then $||f(t) - f(s)|| \geq |t-s|^{\alpha}$ and f is continuous. Therefore all the results of [2] hold if C1) is replaced by the Hölder continuity condition

$$||J(t,\lambda)x - J(s,\lambda)x|| \leqslant \lambda |t-s|^{\alpha} L(||x||) \qquad \alpha > 0$$

<u>Remark</u>. It is a consequence of the Denjoy-Young-Saks Theorem [7, p.18] that if $\alpha < 1$ then no real valued continuous f satisfies $|f(t)-f(s)| \geq |t-s|^{\alpha}$.

Some while ago we showed (not published) that the proofs in [2] can be adapted to the case f has bounded variation but is not necessarily continuous (however see [5]). We now show Riemann integrability of f is sufficient.

The role of f in C1) is to generate an interval function $I(s,t) = ||f(s)-f(t)||$. Interval functions and their Riemann integrals are discussed in section 1. It might seem that using an interval function I instead of f in C1) would produce further generality. It turns out this is not the case. If interval function I satisfies our hypotheses, then there always exists a Riemann integrable, Banach space valued f such that $I(s,t) \leqslant ||f(s)-f(t)||$ (see Lemma 1.1 and Remark 4.2).

The theorems of this paper are stated in section 4. In section 5 the basic existence result is proved. It is stronger than [2, Theorem 2.1]. The appendix is self contained.

1. Riemann Integrals. There are several possible definitions for the Riemann integral of a Banach space valued curve. The one we use is as follows.

DEFINITION. Let $\sigma = (0=t_0 < t_1 < \ldots\ldots < t_n=T)$ be a partition of $[0,T]$, $|\sigma| = \max|t_i - t_{i-1}|$. Let $\xi_i , \xi'_i \in [t_{i-1} , t_i]$. Then f is said to be Riemann integrable on $[0,T]$ if the directed limit

$$\operatorname*{Lim}_{|\sigma|\to 0} \sum_i ||f(\xi_i) - f(\xi'_i)|| (t_i - t_{i-1}) = 0 \tag{1.1}$$

In which case

$$\int_0^T f(t)dt = \operatorname*{Lim}_{|\sigma|\to 0} \sum_i f(\xi_i)(t_i - t_{i-1})$$

DEFINITION. A (real valued) interval function I on $[0,T]$ is any real valued map with Domain(I) = {subintervals of $[0,T]$ } / $\sim$ where $\sim$ identifies intervals with the same end-points.

If P is a subinterval of $[0,T]$ with

end-points $a < b$, then by abuse of notation we write $I(P) = I(a,b) = I(b,a)$.

DEFINITION. Interval function I is said to be Riemann integrable on $[0,T]$ if the directed limit on partitions σ of $[0,T]$, $\lim_{|\sigma|\to 0} \sum_{P\in\sigma} I(P)$, exists and is finite.

If f is Riemann integrable and $I(t,s) = ||f(t) - f(s)||$ then (1.1) shows the interval function $P \to |P|I(P)$ has Riemann integral zero. This has a converse. Define $MI(P) = \sup \{ I(Q) : Q \subset P \}$ and consider the condition

$$I(s,t) \leqslant I(s,r) + I(r,t) \qquad 0 \leqslant r,s,t \leqslant T \qquad (*)$$

Let $B[0,T]$ be the Banach space of bounded functions on $[0,T]$.

LEMMA 1.1. _If interval function_ I _is positive, satisfies_ (*) _and_ $|P|I(P)$ _has Riemann integral zero then there exists a Riemann integrable_ $B[0,T]$ _valued_ f _such that_

$I(s,t) \leq MI(s,t) = \|f(s) - f(t)\|$.

<u>Proof</u>. To show I bounded choose $\delta > 0$ such that if $|\sigma| \leq \delta$ then $\sum_{P \in \sigma} |P| I(P) \leq 1$. Let Q be any interval and choose a partition σ of Q such that if $P \in \sigma$ then $\delta/2 \wedge |Q| \leq |P| \leq \delta$. Then by (*)

$$|Q| I(Q) \leq |Q| \sum_{P \in \sigma} I(P) \leq |Q| (\delta/2 \wedge |Q|)^{-1} \sum_{P \in \sigma} |P| I(P)$$
$$\leq 2T\delta^{-1} \vee 1 < \infty$$

Therefore if $|Q| \geq T/3$ then $I(Q) \leq M < \infty$. If Q has end- points t,s then at least one of the following hold: (a) $|t-s| \geq T/3$, (b) $|t|$, $|s| > T/3$ (c) $|T-t|$, $|T-s| > T/3$. Therefore by (*) , $I(Q) \leq 2M < \infty$.

Clearly $I \leq MI$ and MI has the same bound as I. It is easy to see MI satisfies (*).

Now $\sum_{P \in \sigma} |P| MI(P)$ decreases under refinement of σ. Therefore, by Darboux Theorem (see for example [4, pp.32]), $|P| MI(P)$ is Riemann integrable.

Let $F(t)$ be the indefinite integral. Then $F(t)$ is Lipschitz continuous, and by [7,pp.23]

$F'(t) = \text{Lim}\, MI(P_t)$ a.e. $t \in [0,T]$ (P_t is any interval containing t, and the limit is taken as $|P_t| \to 0$). Since $|P|I(P)$ has zero indefinite Riemann integral, the same theorem shows

$\text{Lim}\, I(P_t) = 0$ a.e. $t \in [0,T]$.

Let t be any point where both these limits exist, and put $P_n = [t - 1/n, t]$. Then

$$MI(P_n) = \sup \{ I(r,s) : t - 1/n \leq r < s \leq t \}$$

$$\leq 2\sup \{ I(s,t) : t - 1/n \leq s \leq t \} \quad \text{by } (*).$$

Therefore $MI(P_n) \to 0$ as $n \to \infty$. So $F'(t) = 0$ a.e. $t \in [0,T]$. Since $F(0) = 0$, $F(t) \equiv 0$ and $|P|MI(P)$ has Riemann integral zero. To complete the proof put

$$f(t)(s) = \begin{cases} 0 & s \leq t \\ MI(t,s) & s > t \end{cases} \tag{1.2}$$

Then since MI satisfies $(*)$, $||f(t) - f(s)|| = MI(t,s)$, and moreover if $\xi, \xi' \in [s,t]$ then

$||f(\xi) - f(\xi')|| \leq ||f(t) - f(s)||$ so f is Riemann integrable.

The following corollaries are immediate.

COROLLARY 1.2. _Banach space valued_ f _is Riemann integrable iff_ $\lim_{|\sigma|\to 0} \sum_{\sigma} ||f(t_i)-f(t_{i-1})||(t_i-t_{i-1})=0$

COROLLARY 1.3. _If the conditions of Lemma_ 1.1. _hold then_ I , MI , _and_ f _have the same points of continuity , and are continuous_ a.e. $[0,T]$.

Let $\mathcal{S}$ be any non-empty subset of $[0,T]$, and $r > 0$. Then there exists $\rho(r,\mathcal{S}) < \infty$ such that if $\{P_i\}$ is any finite set of disjoint intervals each of length $\leq 4r$, and $P_i \cap \mathcal{S} \neq \phi$ then

$$\sum_i |P_i| MI(P_i) \leq \rho(r,\mathcal{S}) \tag{1.3}$$

(If $P_i \not\subset [0,T]$, then $MI(P_i) = MI(P_i \cap [0,T])$ by definition). Since $|P|MI(P)$ has Riemann integral zero , and MI is bounded , we have

COROLLARY 1.4. *There exists* ρ *which satisfies* (1.3) *and has the following properties.*

(i) $\rho(r,\mathcal{S})$ *is continuous in* r *on* $[0,\infty)$ *and* $\rho(0,\mathcal{S}) = 0$.

(ii) ρ *is monotone increasing in both variables.*

(i.e. If $r \leq r'$, $\mathcal{S} \subset \mathcal{S}'$ *then* $\rho(r,\mathcal{S}) \leq \rho(r',\mathcal{S}')$).

(iii) *If* M *is continuous at* s *then* $\rho(r,\{s\}) = o(r)$.

2. Product Integrals. Let $Op(X)$ be the set of all maps with domains and ranges in Banach space X. Let T, $\lambda_0 > 0$ and

$$S : [0,T] \times (0,\lambda_0) \to Op(X) \quad : \quad (t,\lambda) \to S(t,\lambda)$$

Suppose $0 \leqslant s < t \leqslant T$ and $\sigma = (s=t_0 < t_1 < \ldots < t_n=t)$ a partition of $[s,t]$, $\mu_i = t_i - t_{i-1}$, $|\sigma| = \max \mu_i < \lambda_0$

Let $\xi = \{\xi_i\}_1^n$ be any n-vector with $\xi_i \in [s,t]$ and define $d(\sigma,\xi) = \max\limits_i \sup\limits_t \{ |\xi_i - t| : t_{i-1} \leqslant t \leqslant t_i \}$

$= \max\limits_i |\xi_i - t_{i-1}| \vee |\xi_i - t_i|$

For some $x \in X$ suppose S has the property that

$$PS(\sigma,\xi)x = \prod_{i=1}^{n} S(\xi_i,\mu_i)x \tag{2.1}$$

always exists.

DEFINITION. If the directed limit

$\operatorname*{Lim}\limits_{d\to 0} PS(\sigma,\xi)x$ exists (in norm topology of X) then the limit is written as $\prod\limits_s^t S(u,du)x$, and is called the product integral of S on $[s,t]$ at x.

If the limit is uniform for $(s,t,x) \in \Delta \subset [0,T]^2 \times X$, then we say the product integral is uniform on Δ.

This definition of a product integral is rather strong , and has the unusual feature that the 'sample points' ξ_i may lie outside the intervals $[t_{i-1},t_i]$. Section 4 shows the advantage in this. The definition could be weakened in two directions. One might specify ξ_i (say $\xi_i=t_i$) and then only consider those σ's for which $\mu_i=\mu_j$. We then obtain the product formulae of [2]. Alternatively one might take limits under refinement of σ. This is done in [8] (with $\xi_i \in [t_{i-1},t_i]$).

3. Accretive Operators. For the convenience of the reader we collect together the definition and some properties of accretive set-valued maps. Proofs can be found in [1], [2].

Let $(X,||.||)$ be a Banach space. $A \subset X \times X$ is in the class $\mathcal{A}(\omega)$ if for each $\lambda > 0$, $\lambda\omega < 1$ and $(x_i, y_i) \in A$ $i = 1,2$ we have

$$|| (x_1 + \lambda y_1) - (x_2 + \lambda y_2) || \geq (1 - \lambda\omega) ||x_1 - x_2||.$$

If $A \in \mathcal{A}(\omega)$ $\lambda > 0$, $\lambda\omega < 1$ set $J_\lambda = (I + \lambda A)^{-1}$,

$D_\lambda = D(J_\lambda) = R(I + \lambda A)$, $A_\lambda = \lambda^{-1}(I - J_\lambda)$ then

(a) J_λ, A_λ are functions and

$$\left.\begin{array}{l} ||J_\lambda x - J_\lambda y|| \leq (1-\lambda\omega)^{-1} \; ||x-y|| \\ ||A_\lambda x - A_\lambda y|| \leq \lambda^{-1}(1 + (1-\lambda\omega)^{-1}) \; ||x-y|| \end{array}\right\} \; x, y \in D_\lambda$$

(b) Set $\mathcal{D} = \cup_{\lambda > 0} \cap_{\lambda > \mu > 0} D_\mu$. Then

$|Ax| = \lim\limits_{\lambda \downarrow 0} ||A_\lambda x||$ exists if $x \in \mathcal{D}$, and

$|Ax| \leq \inf \{ \; ||y|| \; : \; y \in Ax \; \}$ if $x \in D(A) \cap \mathcal{D}$.

(c) Set $D^*(A) = \{ \; x \in \mathcal{D} : |Ax| < \infty \; \}$. Then

$D(A) \cap \mathcal{D} \subset D^*(A) \subset \mathcal{D}$ and $D^*(A) \subset D(A)^c$.

(d) $$\left.\begin{aligned} &||J_\lambda x - x|| \leq \lambda(1-\lambda w)^{-1} \, |Ax| \\ &||A_\lambda x|| \leq (1-\lambda w)^{-1} \, |Ax| \end{aligned}\right\} \quad x \in D_\lambda \cap D^*(A)$$

(e) $$\left.\begin{aligned} &J_\lambda x = J_\mu(\alpha x + \beta J_\lambda x) && x \in D_\lambda \\ &(1-\lambda w)||A_\lambda x|| \leq (1-\mu w)||A_\mu x|| && x \in D_\lambda \cap D_\mu \end{aligned}\right\} 0 < \mu \leq \lambda$$

where $\alpha = \mu\lambda^{-1}$, $\beta = 1-\alpha$

(f) $A_\lambda \in \mathfrak{A}(w(1-\lambda w)^{-1})$ and $A_\lambda x \in AJ_\lambda x$ if $x \in D_\lambda$

Properties (a) - (f) will henceforeward be used without specific reference.

4. The Theorems. Let $(X, ||.||)$ be any Banach space, $\{A(t) : 0 \leq t \leq T\}$ a 1-parameter family of operators (set-valued maps) on X such that for some real ω and some $\lambda_0 > 0$, $\lambda_0 \omega < 1$

A1) $A(t) \in \mathcal{A}(\omega) \quad 0 \leq t \leq T$

A2) $D^c = D(A(t))^c$ is independent of t.

A3) $R(I + \lambda A(t)) \supset D^c \quad 0 < \lambda < \lambda_0$

We put $J(t,\lambda) = (I + \lambda A(t))^{-1}$.

Remark.4.1 If $\omega_1 < \omega_2$ then $\mathcal{A}(\omega_1) \subset \mathcal{A}(\omega_2)$. Consequently, without loss in generality, we assume $\omega \geq 0$.

The time dependence of $A(t)$ is conditioned as follows.

C) For each $M > 0$ there exists interval function I_M such that $|P| I_M(P)$ has Riemann integral zero on $[0,T]$ and such that if $x \in D^c$, $||x|| \leq M$ and $0 < \lambda < \lambda_0$ then

$$||J(t,\lambda)x - J(s,\lambda)x|| \leq \lambda I_M(t,s) \quad 0 \leq s,t \leq T \tag{4.1}$$

Remark. 4.2. Clearly I_M must be positive, and without loss in generality we may assume I_M satisfies condition (*) of section 1. Therefore, by Lemma 1.1, an equivalent condition to C) is obtained by replacing $I_M(t,s)$ in (4.1) by $||f_M(t) - f_M(s)||$, where f_M is Riemann integrable.

Let $L'(M) = \sup \{ I_M(P) : P \subset [0,T] \}$ (4.2)

Then by Lemma 1.1, $L'(M) < \infty$. Dividing (4.1) by λ,

$$\big|\, ||A_\lambda(t)x|| - ||A_\lambda(s)x|| \,\big| \leq ||A_\lambda(t)x - A_\lambda(s)x|| \leq L'(||x||)$$

So if $x \in D^c$, $\big|\, |A(t)x| - |A(s)x| \,\big| \leq L'(||x||)$.

Therefore $D^* = D^*(A(t))$ is independent of t, and moreover if $x \in D^*$

$$M(x) = \sup_t |A(t)x| \leq |A(0)x| + L'(||x||) < \infty \tag{4.3}$$

Suppose $\{ A(t) : 0 \leq t \leq T \}$ satisfies A1), A2), A3) and C), then the following theorems hold.

THEOREM 1. $U(t,s)x = \prod_s^t J(u,du)x \in D^c$ exists for $0 \leqslant s \leqslant t \leqslant T$, $x \in D^c$ and is uniform on any set $\Delta = \{ (s,t,x) : 0 \leqslant s \leqslant t \leqslant T,\ ||x|| + |A(0)x|$ bounded $\}$.

THEOREM 2. $U(t,s)$ has the following properties

(a) $||U(t,s)x - U(t,s)y|| \leqslant \exp(\omega(t-s))||x-y||, x,y \in D^c$

(b) $U(s,s)x = x$, $U(t,s)U(s,r)x = U(t,r)x$ $x \in D^c$

$0 \leqslant r \leqslant s \leqslant t \leqslant T$.

(c) $(s,t) \to U(t,s)x$ is continuous on $0 \leqslant s \leqslant t \leqslant T$, and uniformly continuous on Δ.

THEOREM 3. Let

$$s \to S(t,s)x = \prod_0^s J(t,du)x \tag{4.4}$$

represent the semigroup on D^c with infinitesimal generator $A(t)$. Then

(a) Theorem 1 holds with J replaced by S.

(b) For almost all $s \in [0,T]$, and in particular for all s at which I_M is continuous for sufficiently large M

$$||U(s+h,s)x - S(s,h)x|| = o(h) \quad \text{as} \quad h \downarrow 0.$$

PROPOSITION 1. *For any* Δ *in Theorem 1 there exists a constant* K *and a* ρ *with properties* (i),(ii),(iii) *of Corollary 1.4 such that if* $(s,t,x) \in \Delta$ *then for sufficiently large integer* m

$$\left\|U(t,s)x - \prod_{i=1}^{m} J(s+i(t-s)/m\ ,\ (t-s)/m)x\ \right\| \leq$$
$$K(t-s)m^{-\frac{1}{2}} + \rho((t-s)m^{-1/4}\ ,\ (s,t])$$

(This should be compared with [2,Proposition 2.5])

DEFINITION. As in [2], we say $u(t)$ is a strong solution of (0.1) iff $u(t)$ is continuous on $[s,T]$, locally absolutely continuous and strongly differentiable a.e. on (s,T), and satisfies (0.1) a.e..

THEOREM 4. *If* $u(t)$ *is a strong solution of* (0.1) *then* $u(t) = U(t,s)x\ ,\ s \leq t \leq T$.

Conversly suppose for each t, $A(t)$ *is a closed subset of* $X \times X$, $x \in D^c$ *and* $t \to U(t,s)x$ *is locally absolutely continuous and strongly differentiable a.e. on* (s,T). *Then* $t \to U(t,s)x$ *is a strong solution of* (0.1).

Let $PJ(\sigma,\xi)x$ be defined as in (2.1). Then $PJ(\sigma,\xi)x$ exists for $x \in D^c$. The first part of the next lemma follows from the Lipschitz continuity of $J(t,\lambda)$. A slight modification of the proof of [2, Lemma 2.2] gives the second part.

LEMMA 4.1. *If* $C = \exp(\ (t-s)w(1 - |\sigma|w)^{-1}\ ) \leq \exp(\ Tw(1 - \lambda_0 w)^{-1}\ )$ *then*

(i) $||PJ(\sigma,\xi)x - PJ(\sigma,\xi)y|| \leq C\ ||x-y||$ $\qquad x,y \in D^c$

(ii) $||PJ(\sigma,\xi)x - x|| \leq C(t-s)M(x)$ $\qquad x \in D^*$

COROLLARY 4.2. *There exists a continuous increasing* L *such that* $||PJ(\sigma,\xi)x|| \leq L(||x||)$, $x \in D^c$.

Proof. Fix any $y \in D^*$. Then

$$\begin{aligned} ||PJ(\sigma,\xi)x|| &\leq C||x-y|| + ||PJ(\sigma,\xi)y|| \\ &\leq C||x|| + (C+1)||y|| + CTM(y) \end{aligned}$$

Therefore we may take $L(r) = Cr + (C+1)||y|| + CTM(y)$.

LEMMA 4.3. *Suppose* $\prod_s^t J(u,du)x$ *exists uniformly* $(s,t,x) \in \Delta$, *where* Δ *is any set as in Theorem 1. Then Theorems 1* , *2(a) ,2(b), 3(a) hold.*

Proof. By Lemma 4.1 $PJ(\sigma,\xi)$ has Lipschitz bound (on D^c) which converges to $\exp((t-s)\omega)$ as $|\sigma| \to 0$. Since, by hypothesis, $PJ(\sigma,\xi) \to \prod_s^t J(u,du)$ on D^* as $d(\sigma,\xi) \to 0$, D^* dense in D^c and $PJ(\sigma,\xi)x \in D^c$, we obtain Theorems 1, 2(a). Theorem 2(b) is then trivial.

$\{ A_1(s) = A(t) : 0 \leqslant s \leqslant T \}$ satisfies conditions A1),A2),A3) and C), so (4.4) is well defined, and so is $PS(\sigma,\xi)$ on D^c. Given $x \in D^c$, $\varepsilon > 0$, choosing $d(\sigma,\xi)$ sufficiently small and using Theorem 1, $||U(t,s)x - PJ(\sigma,\xi)x|| \leqslant \varepsilon$.

Let $\sigma' = (s=t'_0 < t'_1 < \ldots\ldots\ldots < t'_m=t)$ be any refinement of σ, and $\xi' = \{\xi'_j\}_1^m$ be such that $\xi'_j = \xi_i$ iff $[t'_{j-1}, t'_j] \subset [t_{i-1}, t_i]$. Then $d(\sigma',\xi') = d(\sigma,\xi)$ and moreover as $|\sigma'| \to 0$ $PJ(\sigma',\xi')x \to \prod_i \prod_{t_{i-1}}^{t_i} J(\xi_i,du)x = PS(\sigma,\xi)x$. Therefore $||U(t,s)x - PS(\sigma,\xi)x|| \leqslant \varepsilon$ and Theorem 3(a) follows.

LEMMA 4.4. Theorems 1,2,3 imply Theorem 4.

Proof. The uniqueness part of Theorem 4 has the same proof as [2, Theorem 3.1]. The only difference is that we require $\int_0^t I_M([s/\varepsilon]\varepsilon , s)ds \to 0$ as $\varepsilon \to 0$. But if f_M is taken as in Lemma 1.1 then

$$0 \leq \int_0^t I_M([s/\varepsilon]\varepsilon , s)ds \leq \int_0^t ||f_M([s/\varepsilon]\varepsilon) - f_M(s)||ds$$

$$\leq \varepsilon \sum_{i=0}^{[t/\varepsilon]} ||f_M(i\varepsilon) - f_M((i+1)\varepsilon)|| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

since f_M is Riemann integrable.

To prove the second part of Theorem 4 we only need show for almost all $t \in [s,T]$

$$U(t,s)x \in D(A(t)) \; , \quad d/dtU(t,s)x + A(t)U(t,s)x \ni 0 \tag{4.5}$$

The proof of [2, Theorem 3.3] shows (4.5) holds for those t such that $d/dtU(t,s)x$ exists and $||U(t+h,t)x - S(t,h)x|| = o(h)$. Therefore by Theorem 3(b), (4.5) holds a.e..

To complete the proofs of the Theorems we show the hypothesis of Lemma 4.3, Theorems 2(c),3(b) and Proposition 1 hold. This is done in the next section.

5. Main Proof. Let

$\Delta = \{ (s,t,x) : 0 \le s \le t \le T , ||x|| \le K_1 , |A(0)x| \le K_2 \}$

For the moment we suppose $(s,t,x) \in \Delta$. By Corollary 4.2 , $||PJ(\sigma,\xi)x|| \le L(K_1)$. Set $I = I_{L(K_1)}$, $L' = L'(L(K_1))$ (where $L'(.)$ is defined in (4.2)) and $M = K_2 + L'(K_1)$ (so by (4.3) , $M(x) \le M$).

Suppose $0 < |\sigma| \le \lambda < \lambda_0$, $0 \le s' \le s' + m\lambda \le T$. Set $\alpha_j = \mu_j/\lambda$, $\beta_j = 1 - \alpha_j$ and let

$$P_k = P_{\lambda,k}(s')x = \prod_{i=1}^{k} J(s' + i\lambda , \lambda)x \qquad k \le m$$

$$Q_l = Q_l(\sigma,\xi)x = \prod_{j=1}^{l} J(\xi_j , \mu_j)x$$

Then $P_0 = Q_0 = x$, $Q_n = PJ(\sigma,\xi)x$

Our aim is to compare $PJ(\sigma,\xi)x$ with $PJ(\sigma',\xi')x$ where (σ',ξ') is arbitary. However a simpler recurrence relation is obtained by comparing $PJ(\sigma,\xi)x$ with P_m. (Note that by a suitable choice of (σ',ξ') we obtain $P_m = PJ(\sigma',\xi')x$). This technique is used in [6] for the autonomous case.

Let $a_{k,l} = ||P_k - Q_l||$. Then if $k,l > 0$ we use condition C) to obtain

$$a_{k,l} = ||J(s'+k\lambda , \lambda)P_{k-1} - J(\xi_l , \mu_l)Q_{l-1}||$$

$$\leq ||J(s'+k\lambda, \mu_l)(\alpha_l P_{k-1} + \beta_l P_k) - J(s'+k\lambda, \mu_l)Q_{l-1}||$$

$$+ \mu_l \, I(\xi_l, s'+k\lambda)$$

$$\leq (1 - \mu_l w)^{-1}(\alpha_l a_{k-1,l-1} + \beta_l a_{k,l-1}) + \mu_l I(\xi_l, s'+k\lambda) \qquad (5.1$$

and Lemma 4.1 to obtain

$$a_{k,0} = ||P_k - x|| \leq CM\lambda k \ , \quad a_{0,l} = ||x - Q_l|| \leq CM\lambda\sum_1^l \alpha_i \qquad (5.2$$

where $C = \exp(Tw(1-\lambda_0 w)^{-1})$

By comparing (5.1) , (5.2) with (A.1) in the appendix we estimate the quantities $K, W, M_j, M_j(\kappa)$ which appear on the right-hand side of (A.2).

$$K = CM\lambda \ , \quad W = \prod_1^n (1-\mu_j w)^{-1} \leq C \ , \quad M_j \leq \mu_j L' \qquad (5.3$$

If $|m - \sum_{j+1}^n \alpha_i - i| < \kappa$ then it is easy to see $|s' + \lambda i - \xi_j| < r$, where r is defined by

$$r = \lambda\kappa + |s' + m\lambda - t| + d(\sigma, \xi). \qquad (5.4$$

So $M_j(\kappa) \leq \mu_j MI(B(\xi_j, r))$ where $B(\xi_j, r)$ is the interval centre ξ_j, radius r, and MI is defined in section 1. Let $\{B_q\}_q$ be a linearly ordered covering of the set ξ by a finite number of disjoint intervals B_q each of length $2r$. Now if $\xi_j \in B_q$ then t_j and t_{j-1} have distance at most $d(\sigma,\xi) \leq r$ from B_q, so t_j, $t_{j-1} \in 2B_q$ ($2B_q$ is the interval with the same centre as B_q and twice the length). Therefore $\Sigma_{\xi_j \in B_q} \mu_j \leq 4r = |2B_q|$ and $\Sigma_{\xi_j \in B_q} \mu_j MI(B(\xi_j,r)) \leq |2B_q| MI(2B_q)$. Now $2B_p \cap 2B_q = \phi$ or singleton if p, q are not consecutive, therefore, by Corollary 1.4

$$\sum_1^n M_j(\kappa) \leq \sum_q |2B_q| MI(2B_q) \leq 2\rho(r,\xi) \tag{5.5}$$

Substituting (5.3),(5.4),(5.5) into (A.2), and (for simplicity) setting $\kappa = ((t-s)/\lambda)^{3/4}$, Theorem A gives (after some trivial estimates)

$$||P_{\lambda,m}(s')x - PJ(\sigma,\xi)x|| = a_{m,n} \leq C^2M\{(t-s-m\lambda)^2 + \lambda(t-s)\}^{\frac{1}{2}}$$

$$+ CL'(\lambda(t-s))^{\frac{1}{2}} + 2C\rho(\lambda^{1/4}(t-s)^{3/4} + |t-s'-m\lambda| + d(\sigma,\xi), \xi) \tag{5.6}$$

By choosing σ , ξ in the obvious way we first set $PJ(\sigma,\xi)x = P_{\mu,n}(s)x$, $d(\sigma,\xi) = \mu$, $t = s+n\mu$ in (5.6). So if $\mu \leq \lambda$, $(s,t,x) \in \Delta$

$$||P_{\lambda,m}(s)x - P_{\mu,n}(s)x|| \leq C^2M\{(n\mu - m\lambda)^2 + \lambda n\mu\}^{\frac{1}{2}} +$$
$$CL'(\lambda n\mu)^{\frac{1}{2}} + 2C\rho(\lambda^{1/4}(n\mu)^{3/4} + |n\mu - m\lambda| + \mu , (s,s+n\mu]) \qquad (5.7)$$

Therefore $P_{\mu,n}(s)x$ converges as $n\mu \to t-s \leq T-s$, $\mu \to 0$. Let this limit be $U(t,s)x$. Taking the limit in (5.7)

$$||P_{\lambda,m}(s)x - U(t,s)x|| \leq C^2M\{(t-s-m\lambda)^2 + \lambda(t-s)\}^{\frac{1}{2}}+$$
$$CL'(\lambda(t-s))^{\frac{1}{2}} + 2C\rho(\lambda^{1/4}(t-s)^{3/4} + |t-s-m\lambda| , (s,t]) \qquad (5.8)$$

Proposition 1 follows setting $\lambda = (t-s)/m$ in (5.8).

Suppose σ , ξ are given , and $(s',t',x) \in \Delta$. Choose m so that $|t' - s' - m|\sigma|| < |\sigma|$. Then from (5.6) and (5.8) , using $(a^2+b^2)^{\frac{1}{2}} \leq |a|+|b|$,

$$||PJ(\sigma,\xi)x - U(t',s')x||$$
$$\leq ||PJ(\sigma,\xi)x - P_{|\sigma|,m}(s')x|| + ||P_{|\sigma|,m}(s')x - U(t',s')x||$$
$$\leq C^2M(|(t-s)-(t'-s')| + 2|\sigma|)+C(CM+L')|\sigma|^{\frac{1}{2}}((t-s)^{\frac{1}{2}}+(t'-s')^{\frac{1}{2}})$$
$$+ 2C\rho(|\sigma|^{1/4}(t-s)^{3/4} + |t-t'| + |\sigma| + d(\sigma,\xi) , \xi)$$
$$+ 2C\rho(|\sigma|^{1/4}(t'-s')^{3/4} + |\sigma| , (s',t']) \qquad (5.9)$$

The hypothesis of Lemma 4.3 follows by setting $s' = s$, $t' = t$ in (5.9). Letting $d(\sigma,\xi) \to 0$ in (5.9) ,

$$||U(t,s)x - U(t',s')x|| \leq C^2M|(t-s)-(t'-s')| + 2C\rho(|t-t'| , [s,t]) \tag{5.10}$$

which gives Theorem 2(c).

To prove Theorem 3(b) set $s' = s$, $t' = t$, $\xi = \{s\}$ in (5.9). Let $|\sigma| \to 0$. Then $d(\sigma,\xi) \to t-s = h$ and $PJ(\sigma,\xi)x \to S(s,h)x$. Therefore , using Corollary 1.4 $||S(s,h)x - U(s+h,s)x|| \leq 2C\rho(h,\{s\}) = o(h)$ if MI is continuous at s. Theorem 3(b) now follows from Corollary 1.3.

This completes the proof of the theorems in section 4.

Appendix. We derive an estimate for the solutions of the 2-dimensional recurrence inequality obtained in section 5 . This recurrence inequality is more complicated than those needed in [1] , [2] and [6] , but our estimate (which is in fact superiour by a factor of 2 on the boundary conditions) is derived without recourse to the rather complicated induction arguments employed in the above papers.

Two elementary inequalities from probability theory are needed. The first is only the Cauchy-Schwartz inequality. The second is usually called Chebychev's inequality [3, p.233].

LEMMA A1. Let X be a random variable with finite mean μ and variance σ^2 . Let $E(.)$ be the expectation operator, so $E(X)=\mu$, $E((X-\mu)^2)=\sigma^2$, then

i) $E(\,|X-m|\,) \leq \{ (m-\mu)^2 + \sigma^2 \}^{\frac{1}{2}}$

ii) $P(\,|X-\mu| \geq \kappa\,) \leq \sigma^2/\kappa^2$

From now on we assume $\alpha_i, \beta_i \geq 0$, $\alpha_i + \beta_i = 1$, $W_i \geq 1$, $i = 1,2,\ldots$, $K \geq 0$, $b_{i,j} \geq 0$.

THEOREM A. <u>Suppose for</u> $k, l \geq 0$, $(a_{k,l})$ <u>satisfies the following recurrence inequality</u>

$$\left.\begin{aligned} &a_{k,l} \leq W_l\alpha_l a_{k-1,l-1} + W_l\beta_l a_{k,l-1} + b_{k,l} \qquad k,l > 0 \\ &a_{k,0} \leq Kk \qquad a_{0,l} \leq K \sum_1^l \alpha_i \end{aligned}\right\} \quad (A.1)$$

Let $\kappa > 0$, <u>and set</u>

$$M_j(\kappa) = \max_i.\{\, b_{i,j} \;:\; i \leq m\,,\ |m - \sum_{j+1}^n \alpha_i - i| < \kappa \,\}$$

$M_j = M_j(\infty)$, $W = \prod_1^n W_i$. <u>Then</u>

$$a_{m,n} \leq WK\{\, (m - \sum_1^n \alpha_i)^2 + \sum_1^n \alpha_i\beta_i \,\}^{\frac{1}{2}} + W\kappa^{-2} \sum_{j=1}^n M_j \sum_{j+1}^n \alpha_i\beta_i + W \sum_1^n M_j(\kappa) \qquad (A.2)$$

To prove Theorem A we first make two reductions.

LEMMA A2. <u>It is sufficient to prove Theorem A for the case</u> $W_i = 1$, $i=1,2,\ldots$

<u>Proof</u>. If $W_i \neq 1$ set $c_{k,l} = a_{k,l} \prod_1^l W_i^{-1}$. Then since $\prod_1^l W_i \geq 1$, $(c_{k,l})$ satisfies (A.1) with $W_i=1$. Therefore if Theorem A holds for $W_i=1$ then $c_{m,n}$ satisfies (A.2) with $W=1$. But $a_{m,n} = c_{m,n} \prod_1^n W_i = Wc_{m,n}$.

LEMMA A3. Without loss in generality we may assume the limiting case (A.3) replaces (A.1) where

$$\left.\begin{array}{ll} a_{k,l} = \alpha_l a_{k-1,l-1} + \beta_l a_{k,l-1} + b_{k,l} & k,l > 0 \\ a_{k,0} = Kk \qquad a_{0,l} = K \sum_{1}^{l} \alpha_i & \end{array}\right\} \quad (A.3)$$

Proof. The possibility that $W_l \neq 1$ has already been covered in Lemma A2. Set $Z = \{(a_{k,l}) : (a_{k,l})$ satisfies (A.1)$\}$. $c_{k,l} = \sup \{ a_{k,l} : (a_{k,l}) \in Z \}$. Then $(c_{k,l})$ satisfies (A.3).

Remark. 1. This last statement depended on the assumption that $\gamma_l = W_l \alpha_l$, $\kappa_l = W_l \beta_l$ are both non-negative. In [2 , Lemma A] γ_l , κ_l are independent of l , but in the statement of the Lemma they were not assumed non-negative. However the proof did assume this , and in fact it is easy to show that the estimate given is in general false for negative γ or κ .

Remark. 2. The proof of Lemma A3. also assumed $Z \neq \phi$. It is easy to see by a recursion on l in

(A.1) that this is the case. In fact we show (A.3) has a solution which must of course be in Z.

<u>Proof of Theorem A</u>. We derive (A.2) (with W=1) from (A.3). Rather than solving (A.3) directly we consider the following slightly different boundary value problem (A.4).

Set $b_{k,l} = 0$ for $k \leq 0$

$$\left.\begin{array}{ll} a_{k,l} = \alpha_l a_{k-1,l-1} + \beta_l a_{k,l-1} + b_{k,l} & l > 0 , \\ a_{k,0} = K\,|k| & -\infty < k < \infty \end{array}\right\} \quad (A.4)$$

To solve (A.4) define the following formal Laurent series.

$$A_l(x) = \sum_{k=-\infty}^{\infty} a_{k,l}\, x^k$$

$$B_l(x) = \sum_{k=1}^{\infty} b_{k,l}\, x^k \quad l > 0 , \qquad B_0(x) = \sum_{k=-\infty}^{\infty} K\,|k|\, x^k$$

Then (A.4) is formally equivalent to

$$\left.\begin{array}{ll} A_l(x) = (\beta_l + \alpha_l x)\, A_{l-1}(x) + B_l(x) & l > 0 \\ A_0(x) = B_0(x) & \end{array}\right\} \quad (A.5)$$

Set $Q_j^n(x) = \prod_{i=j}^{n} (\beta_i + \alpha_i x)$

The solution of (A.5) is

$$A_n(x) = \sum_{j=0}^{n} Q_{j+1}^n (x) B_j(x) \tag{A.6}$$

Now let X_j^n be the random variable of the number of successes of n-j+1 Bernoulli trials with probabilities of success $\alpha_j, \alpha_{j+1}, \ldots\ldots \alpha_n$ respectively. Then the generating function of X_j^n is $Q_j^n(x)$, and so

$$E(X_j^n) = \sum_j^n \alpha_i \quad , \quad Var(X_j^n) = \sum_j^n \alpha_i \beta_i \tag{A.7}$$

By equating coefficients of x^m in (A.6)

$$a_{m,n} = K\ E(|X_1^n - m|) + \sum_{j=1}^{n} \sum_i P(X_{j+1}^n = m - i)\ b_{i,j} \tag{A.8}$$

Setting $m = 0$,

$$a_{0,n} = K\ E(|X_1^n|) = K\ E(X_1^n) = K \sum_1^n \alpha_i$$

Therefore (A.8) satisfies the boundary conditions of (A.3), and so gives the solution of (A.3).

By Lemma A1 , and relations (A.7)

$$E(|X_1^{\,n} - m|) \leqslant \left\{ (m - \sum_1^n \alpha_i)^2 + \sum_1^n \alpha_i \beta_i \right\}^{\frac{1}{2}}$$

$$\sum_i P(X_{j+1}^{\,n} = m-i)\; b_{i,j} \leqslant P(|X_{j+1}^{\,n} - \sum_{j+1}^n \alpha_i| \geq \kappa)\; M_j + M_j(\kappa)$$

$$\leqslant M_j \kappa^{-2} \sum_{j+1}^n \alpha_i \beta_i + M_j(\kappa)$$

(A.2) (with W=1) now follows by substituting these estimates into (A.8).

This completes the proof of Theorem A.

REFERENCES.

[1] M.G. Crandall and T.M. Liggett ; Generation of semi-groups of nonlinear transformations on general Banach spaces ; Amer. J. Math. 93 (1971) 265-298.

[2] M.G. Crandall and A. Pazy ; Nonlinear evolution equations in Banach spaces ; Israel J. Math. 11 (1972) , 57-94.

[3] W. Feller ; An introduction to probability theory and its applications ; Vol. 1 (third edition) , John Wiley and Sons.

[4] T.H. Hildebrandt ; Introduction to the theory of integration ; Academic Press. 1963

[5] A.T. Plant ; Nonlinear evolution equations with generators of irregular time dependence ; To appear.

[6] S. Rasmussen ; Non-linear semi-groups , evolution equations and productintegral representations ; Various Publications Series No. 20 , Matematisk Institute , Aarhus Universitet , Danmark.

[7] F. Riesz and B. Sz-Nagy ; Functional analysis ; Frederick Ungar ; New York (1955).

[8] G.F. Webb ; Product integral representation of time dependent nonlinear evolution equations in Banach spaces ; Pacific J. Math. V32 No.1 (1970) 269-281.