ON THE IRREDUCIBLE CHARACTERS

OF THE WEYL GROUPS

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ABSTRACT

In this thesis we study the irreducible characters of the Weyl groups of the simple Lie algebras, in order to give a unified approach to this problem.

Chapter one sets up notation. In chapter two we give some known results on the character theory of Weyl groups of type A (the symmetric group) using Weyl subgroups. These are a common feature of Weyl groups and allow us, in chapter three, to generalize to type C. Chapter four deals with type D which presents a more difficult problem; chapter five is a brief study of the Weyl groups of type B, and finally, chapter six deals with the calculations in the exceptional types G_2, F_4 and E_6.

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Chapter five WEYL GROUPS OF TYPE B

Chapter six WEYL GROUPS OF EXCEPTIONAL TYPE

INTRODUCTION

The Weyl groups of the simple Lie algebras were classified many years ago and their conjugacy classes and irreducible characters were individually determined by many people (Frobenius, Schur, Young, Specht, Frame and Kondo, to name but a few) in many different ways. However, up till recently no unified approach had been obtained, using the common structure of the Weyl groups as reflection groups. It is desirable to do this in view of the importance of Weyl groups in many branches of mathematics; for example, immediate applications can be envisaged in the theories of algebraic groups and Chevalley groups.

Carter [5] has given such a unified approach to the problem of determining the conjugacy classes, and this thesis is directed towards solving the same problem for the irreducible characters. The fundamental idea in Carter's paper is that of a Weyl subgroup. He gives a correspondence (which is in general not a bijection) between the conjugacy classes and certain admissible diagrams. Some of these diagrams correspond to the Dynkin diagrams of Weyl subgroups and the others to, what we shall call, semi-Coxeter types.

As the numbers of conjugacy classes and irreducible characters are equal, one would hope that a similar association could be obtained between the irreducible characters and Weyl subgroups or semi-Coxeter types.

In the Weyl group of type A, the symmetric group, we reformulate some of the known results in order to exhibit this association (which in this case is a bijection). We then go on to consider Weyl groups of type C and show that these results generalize very well. The situation in Weyl groups of type D is rather more complicated and the association is not so easy to find. However, we are able to give an algorithm which allows us to calculate the irreducible constituents of the principal character of a Weyl subgroup induced up to the Weyl group. This generalizes an algorithm introduced in type C which further extends the usual partial ordering on partitions in type A. A discussion in §4.7 shows how the results in type D should lead to the required association.

We also give a short chapter, mainly for completeness sake, on Weyl groups of type B, giving a similar algorithm for this case. We conclude with a chapter on the exceptional Weyl groups of types G_2, F_4 and E_6 and calculate the association that we want.

Parabolic subgroups are the usual tools for attacking problems of this kind, but methods using them are often unsatisfactory. For example, Solomon [17] has given a decomposition of the group algebra of a finite Coxeter group, which is far from complete; it would appear that Weyl subgroups may well lead to a refinement of the decomposition. It is with this idea in mind that we examine Solomon's results in the case of Weyl groups of types A, C and D.

Unless otherwise stated the results in this thesis are believed to be new.

I take great pleasure in thanking my supervisor, Professor R.W. Carter, for his encouragement, inspiration and unfailing patience. I would also like to thank Professor J.A. Green for the interest he has shown in my work, and Dr. G.B. Elkington and Mr. P.C. Gager for their fruitful conversations. Finally, I express my appreciation to the Science Research Council for their grant.

Chapter one NOTATION AND TERMINOLOGY

In this chapter we introduce the necessary notation and terminology, and state, and in some cases prove, a few elementary character theoretic results.

§1.1 Weyl groups

All groups considered in this thesis will be finite and all Lie algebras finite-dimensional, semi-simple and over the complex field.

Much of the terminology in this section may be found in Jacobson [13].

Let V be a Euclidean space of dimension l. For each non-zero vector r in V, let w_r be the reflection in the hyperplane orthogonal to r.

Thus $w_r(x) = x - \frac{2(r,x)r}{(r,r)}$

Let Φ be a subset of V satisfying the following axioms:

(i) Φ is a finite subset of non-zero vectors which span V;

(ii) if $r,s \in \Phi$ then $w_r(s) \in \Phi$;

(iii) if $r,s \in \Phi$ then $\frac{2(r,s)}{(r,r)}$ is a rational integer;

(iv) if $r, \xi r \in \Phi$ where ξ is real, then $\xi = \pm 1$.

Then Φ is a root system of some semi-simple Lie algebra, whose Weyl group is isomorphic to the group W of orthogonal transformations of V generated by the reflections w_r for all $r \in \Phi$. The dimension l of V is called the rank of W.

Definitions

(i) A sub-root system of a root system Φ is a subset of Φ

which is itself a root system in the space which it spans.
(ii) If W is the Weyl group of Φ, a <u>Weyl subgroup</u> of W is the subgroup generated by the reflections w_r corresponding to the roots $r \in \Phi'$, where Φ' is a sub-root system of Φ.

The graphs which are Dynkin diagrams of Weyl subgroups of a Weyl group W may be obtained by a standard algorithm ([2], [7]). To the Dynkin diagram of W is added a node corresponding to the negative of the highest root, forming the extended Dynkin diagram. The Dynkin diagrams of all possible Weyl subgroups may be obtained as follows. Take the extended Dynkin diagram of Φ (the root system whose Weyl group is W) and remove one or more nodes in all possible ways. Take also the duals of the diagrams obtained in the same way from the dual system $\tilde{\Phi}$ (which is obtained from Φ by interchanging long and short roots). Then repeat the process with the diagrams obtained, and continue any number of times.

It is then easy to determine the <u>maximal Weyl subgroups</u> of W - the proper Weyl subgroups of W not contained in any other proper Weyl subgroup of W. These have rank equal to rank W or rank $W - 1$. So the Dynkin diagrams of the maximal Weyl subgroups are those obtained by leaving out a node from the extended Dynkin diagram of W and also by leaving out a node from the Dynkin diagram of W, and eliminating those of rank equal to rank $W - 1$ or rank W contained inside those whose rank is rank W.

The Weyl subgroups which are obtained by leaving out any number of nodes from the Dynkin diagram of W, are generated by a subset of the generating set of W and are called <u>parabolic subgroups</u> of W.

So much for the general theory. The simple Lie algebras have been classified [13] and their Weyl groups are:

$W(A_l)$ $l \geqslant 1$

$W(B_l) \cong W(C_l)$ $l \geqslant 2$

$W(D_l)$ $l \geqslant 3$

$W(G_2)$

$W(F_4)$

$W(E_6)$

$W(E_7)$

$W(E_8)$

It will occassionally be convenient to add to this list two more Weyl groups

$W(C_1)$ - the cyclic group of order 2 generated by a sign change (see chapter three). The underlying Lie algebra is of type A_1 so $W(C_1) \cong W(A_1)$.

$W(D_2)$ - the non-cyclic group of order 4 generated by a transposition and a product of 2 sign changes (see chapter four). In this case the underlying Lie algebra $A_1 + A_1$ is not simple.

The Weyl group $W(A_l)$ is isomorphic to the symmetric group S_{l+1} on $l+1$ letters;

$W(B_l)$ and $W(C_l)$ are both isomorphic to the hyper-octahedral group of order $2^l . l!$;

$W(D_l)$ is a subgroup of $W(C_l)$ of index 2 ;

$W(G_2)$ is isomorphic to the dihedral group of order 12 ;

$W(F_4)$ is a soluble group of order 1152, isomorphic to the orthogonal group $O_4(3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the Galois field of 3 elements.

We shall mainly be interested in the four infinite families, and their Weyl subgroups are given in the relevant chapters. We can also obtain the maximal Weyl subgroups in each case, which again are listed in the sections where we use them. Notice that $W(B_l)$ and $W(C_l)$, although isomorphic, have different Weyl subgroups because the underlying root systems are different.

A fundamental distinction between $W(A_l)$ and Weyl groups of other types is that in $W(A_l)$ a Weyl subgroup is always conjugate to a parabolic subgroup, so that in the symmetric group the two ideas are equivalent; it is only in the other cases that a distinction arises.

§1.2 Some character theoretic results

We shall be assuming a background of (ordinary) character theory, but we give here a few of the important results, many of which appear in Curtis and Reiner [6].

If I,J are 2 sets $J \subsetneq I$ will mean J is a proper subset of I ($J \subseteq I$ and $J \neq I$).

Let G be a group (assumed to be finite), then its order is denoted by $|G|$. We adopt the convention that $x^y = yxy^{-1}$ where $x,y \in G$, so that $H^g = gHg^{-1}$ where H is a subgroup of G ($H \leqslant G$) and $g \in G$. We use $\langle \ \ \rangle$ to mean the group generated by the elements inside the diamond brackets.

All characters and representations (unless otherwise stated) will be assumed to be over the complex field $\mathbb{C}$, so that all tensor products are also over $\mathbb{C}$. A representation module of a group G will be called, interchangeably, a

$\mathbb{C}G$- or G-module.

$\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ denote the reals, rationals and rational integers respectively.

$(\ ,\ )$ will denote the scalar product of characters, and where we need to specify the group we shall write e.g. $(\ ,\ )_G$.

Let χ be a character of a group H and $K \leqslant H \leqslant G$. Then χ^G denotes the induced character of G, χ_K the restricted character of K. We also write ${}^g\chi$ for the character of $N = H^{g^{-1}}$ defined by ${}^g\chi(n) = \chi(gng^{-1})$ for all $n \in N$.

If $H \lhd G$, we define the centralizer of χ in G to be

$$C_G(\chi) = \{g \in G : {}^g\chi = \chi\}$$

It is easy to see that $C_G({}^x\chi) = C_G(\chi)^x$ for all $x \in G$.

The Weyl groups W admit a homomorphism $\varepsilon : W \to \{+1, -1\}$ defined by $\varepsilon(r_i) = -1$ for $r_i \in I$, where I is the generating set of involutions of W. Thus ε is a linear character of W and will be called the <u>sign character</u> of W. 1 (or 1_W) will always denote the principal character of W.

A result that is fundamental to our work is a theorem in character theory due to Mackey

<u>Theorem 1.2.1</u> (Mackey's Formula)

Let $H, K \leqslant G$ and suppose $\{y_i\}$ is a set of (H,K)-double coset representatives in G. Suppose also that χ is a character of H, θ a character of K. Then

$$(\chi^G, \theta^G) = \sum_{y \in \{y_i\}} ({}^y\chi_{H^y \cap K}\ ,\ \theta_{H^y \cap K})$$

Because the scalar product is symmetric, which character is conjugated is unimportant. In applying this theorem we shall always assume that $y = 1$.

An equivalent result, which we shall only use once, is also due to Mackey

<u>Theorem 1.2.2</u> (Mackey's Subgroup Formula)

With the notation of 1.2.1

$$(\chi^G)_K = \sum_{y \in \{y_i\}} (({}^y\chi)_{H^y \cap K})^K$$

A particular case of these results (when $H = G$) is

<u>Theorem 1.2.3</u> (Frobenius' Reciprocity Formula)

With the notation of 1.2.1

$$(\chi, \theta^G) = (\chi_K, \theta)$$

The application of this theorem will invariably be indicated by the phrase 'by Frobenius'.

A useful result (which we state in a restricted form) is

<u>Lemma 1.2.4</u>

Let $H \leqslant G$, χ a character of G, θ a character of H. Then

$$\chi . \theta^G = (\chi_H . \theta)^G$$

<u>Lemma 1.2.5</u>

(i) Let $H, K \leqslant G$ such that $G = HK$ and $H \cap K = 1$. Suppose χ is a character of G such that $\chi(hk) = \theta(h)\phi(k)$, for all $h \in H$, $k \in K$, where θ is a character of H, ϕ a character of K. Then

$$(\chi, \chi) = (\theta, \theta)(\phi, \phi)$$

(ii) Suppose $G = H \times K$ and $H_1 \leqslant H$, $K_1 \leqslant K$ and θ is a character of H_1, ϕ a character of K_1. Then

$$(\theta.\phi)^{H\times K} = \theta^H.\phi^K$$

(iii) If $H \leqslant K \leqslant G$, χ a character of H and $g \in G$, then

$$^g(\chi^K) = (^g\chi)^{K^{g^{-1}}}$$

Proof

(i) is trivial to check using

$$(\chi,\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g)\chi(g^{-1})$$

(ii) and (iii) follow immediately from the formula (χ a character of $H \leqslant G$)

$$\chi^G(y) = \frac{1}{|H|}\sum_{x\in G}\hat{\chi}(xyx^{-1})$$

where $\hat{\chi}(y) = 0$ if $y \in G\setminus H$

and $\hat{\chi}(y) = \chi(y)$ if $y \in H$.

Lemma 1.2.6

Suppose $H \leqslant G$, χ,θ both characters of G. Then

$$(\chi,\theta) \neq 0 \Rightarrow (\chi_H, \theta_H) \neq 0$$

Proof

$(\chi, \theta) \neq 0 \Rightarrow \chi,\theta$ have an irreducible constituent, ϕ say, in common. Hence χ_H, θ_H have the character ϕ_H of H in common, so $(\chi_H, \theta_H) \neq 0$

We conclude this chapter with a couple of results about representation modules.

Let G be a group and $A = \mathbb{C}G$, its complex group algebra. Let * be the unique $\mathbb{C}$-linear map $A \to A$ such that $g^* = g^{-1}$ for all $g \in G$. Then we see that $*$ is an involutory anti-automorphism of A. The map $*$ was introduced by Solomon [17] , and he proved

Theorem 1.2.7 ([17] lemma 6)

If $x \in A$ then Ax and Ax^* are isomorphic A-modules.

Note that if χ is a character of G and e is an idempotent of A defined by

$$e = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$$

then $e^* = e$.

If B, C are two A-modules such that B is isomorphic to a submodule of C, we write $B \lesssim C$.

Lemma 1.2.8

Let e_1, e_2, e be idempotents of A and suppose Ae_1, Ae_2, Ae afford the characters χ_1, χ_2, χ of G respectively. Suppose also that the left A-module Ae_3, where $e_3 = e_1 e_2$, affords the character χ_3 of G. Then

$$(\chi, \chi_3) \neq 0 \Rightarrow (\chi, \chi_1) \neq 0 \text{ and } (\chi, \chi_2) \neq 0$$

Proof

Suppose that θ is an irreducible constituent of χ such that $(\theta, \chi_3) \neq 0$; let Ae' afford θ. Then $Ae' \lesssim Ae_3 = Ae_1 e_2 \leqslant Ae_2$ so $(\theta, \chi_2) \neq 0$.

However,

$$\begin{aligned} Ae' \lesssim Ae_3 = Ae_1 e_2 &\cong A(e_1 e_2)^* \quad \text{by 1.2.7} \\ &= Ae_2^* e_1^* \\ &\leqslant Ae_1^* \\ &\cong Ae_1 \quad \text{by 1.2.7 again} \end{aligned}$$

So $Ae' \lesssim Ae_1$ and therefore $(\theta, \chi_2) \neq 0$. Because $(\theta, \chi) \neq 0$ we have that $(\chi, \chi_1) \neq 0$ and $(\chi, \chi_2) \neq 0$.

Lemma 1.2.9

Let $H \leqslant G$ and $A' = \mathbb{C}H$. Suppose $A'e$ is an A'-module affording the character χ of H. Then Ae affords the character χ^G of G.

Proof

This follows from the definition of the induced representation, since

$$Ae = A \otimes_{A'} A'e = (A'e)^G .$$

Chapter two THE SYMMETRIC GROUP

Frobenius, Specht, Young and many others have contributed much to the character theory of the symmetric group. However, we shall be presenting their results here in a new light, occasionally with new proofs, as we shall be viewing the symmetric group as the Weyl group of type A. This will enable us to apply the methods to other Weyl groups of simple Lie algebras.

§2.1 Some classical results

In this chapter only, we write $W = W(A_l) \cong S_{l+1}$. It might be more natural to use l instead of $l+1$ for the symmetric group, but we shall stick to a notation more in keeping with our overall view.

Many of the assumed results appear in [6] (pp 190-197), and in [1] (chapter IV).

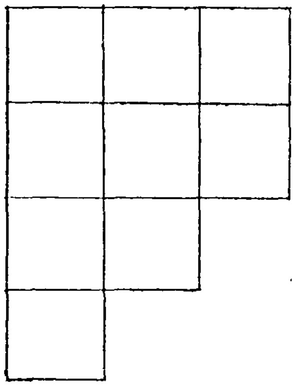
Definition

A partition λ of $l+1$ (written $\lambda \vdash l+1$ or $|\lambda| = l+1$), is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_r = l+1$. $\lambda_1, \ldots, \lambda_r$ are called the parts of λ.

Young ([18] and [19]) introduced the idea of frames and tableaux.

Suppose $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash l+1$. Then the frame associated with λ consists of λ_1 squares in the first row, λ_2 squares in the second row, ... , and λ_r squares in the last row.

e.g. if $l+1 = 9$ then the frame corresponding to $(3,3,2,1)$, which we shall often write as $(3^2 21)$, is



A tableau (or diagram) D_λ corresponding to λ is obtained by filling the squares of the frame with the symbols $1, \ldots, l+1$ in any order.

The dual (tableau is obtained from the original
(frame

(tableau by interchanging the rows and columns.
(frame

The dual frame gives rise to a partition of $l+1$ which is denoted by λ' and is called the dual of λ.

The row stabilizer $R(D_\lambda)$ of a tableaux D_λ is the group of row permutations of D_λ.

i.e. $R(D_\lambda) = \{ p \in S_{l+1} : p \text{ permutes the symbols in each row of } D_\lambda \}$

Similarly, the column stabilizer $C(D_\lambda)$ is the group of column permutations and so is the row stabilizer of the dual tableau $D_{\lambda'}$.

Now $R(D_\lambda) \cong S_{\lambda_1} \times \ldots \times S_{\lambda_r}$ and this is a Weyl subgroup of W of type $A_{\lambda_1 - 1} + \ldots + A_{\lambda_r - 1}$. In fact all Weyl subgroups of W can be considered in this way as the row stabilizer of some diagram. Thus the Weyl subgroups can be parameterized by the partitions of $l+1$, so that a Weyl subgroup isomorphic to $S_{\lambda_1} \times \dots \times S_{\lambda_r}$ will be written W_λ; in particular $W = W_{(l+1)}$.

Thus $W_\lambda = R(D_\lambda)$, $W_{\lambda'} = C(D_\lambda)$.

The group W acts on a diagram D_λ by defining wD_λ for $w \in W$, to be the diagram obtained by applying w to the symbols in D_λ.

We then have the following easy, but fundamental, result

<u>Lemma 2.1.1</u> ([6] 28.10)

If $w \in W$, $\lambda \vdash l+1$ then $R(wD_\lambda) = wR(D_\lambda)w^{-1}$ and $C(wD_\lambda) = wC(D_\lambda)w^{-1}$.

It follows that any two isomorphic Weyl subgroups of W are conjugate via the element of W that transforms one associated diagram into the other.

<u>Definition</u>

Two symbols which lie in the same row (resp. column) of a diagram are said to be <u>collinear</u> (resp. <u>co-columnar</u>).

<u>Lemma 2.1.2</u> ([6] 28.11)

An element $w \in W$ is expressible in the form $w = pq$, where $p \in W_\lambda$, $q \in W_{\lambda'}$, if and only if no two collinear symbols of D_λ are co-columnar in wD_λ.

Let $A = \mathbb{C}W$ - the group algebra of W over $\mathbb{C}$. We define two essential idempotents of A (an <u>essential idempotent</u> being a scalar multiple of an idempotent) :

$$\xi_\lambda = \sum_{p \in W_\lambda} p \quad , \quad \eta_\lambda = \sum_{q \in W_{\lambda'}} \varepsilon(q) q$$

where $\mathcal{E}$ is the sign character of W.
Thus $A\xi_\lambda$, $A\eta_\lambda$ afford the characters $1_{W_\lambda}^{\ W}$ and $\mathcal{E}_{W_{\lambda'}}^{\ W}$ respectively of W considered as A-modules.
Let $e_\lambda = \xi_\lambda \eta_\lambda$. Notice that e_λ depends on W_λ ,$W_{\lambda'}$ and hence on the particular arrangement of the symbols in D_λ.
However a different arrangement only gives rise to $we_\lambda w^{-1}$, for some $w \in W$, by 2.1.2, and hence to an A-module isomorphic to Ae_λ .

The following result appears in [6] (28.15)

Theorem 2.1.3

Let $\lambda \vdash l+1$. For each diagram D_λ, e_λ is essentially idempotent and Ae_λ is a minimal left ideal of A, hence an irreducible A-module. Further, ideals coming from different diagrams with the same frame are isomorphic, but ideals from diagrams with different frames are not. Thus the ideals $\{Ae_\lambda\}$ where λ ranges over all the partitions of l+1, gives a full set of non-isomorphic irreducible A-modules.

Notation

The irreducible character of W afforded by Ae_λ will be denoted by χ^λ.

Thus the irreducible characters of W may be parameterized by partitions of l+1; we shall be giving an alternative characterization of χ^λ in §2.2 .

The above results hold if we replace $\mathbb{C}$ by $\mathbb{Q}$. Hence (with respect to some basis depending on the representation) the matrix entries of any representation of W lie in $\mathbb{Q}$. However, by a result in [6] (75.4), they are also algebraic integers and so are rational integers.

Thus we have

Theorem 2.1.4

Any complex representation of W may be afforded by a basis with respect to which the matrix entries consist of rational integers. In particular, the characters of W are (rational) integral-valued.

One can obtain a decomposition of the group algebra A into minimal left ideals by using the notion of standard tableaux.

Definition

A standard tableau is a tableau in which the numbers increase in every row from left to right and in every column downwards.

Now A splits up into a number of simple rings A_i, $1 \leqslant i \leqslant r$ i.e. $A = A_1 \oplus \ldots \oplus A_r$ and each A_i consists of a direct sum of isomorphic minimal left ideals of A, which are not isomorphic to any that occur in an $A_j, j \neq i$.

Theorem 2.1.5 ([1] IV,4.6)

The minimal left ideals which arise from the standard tableaux belonging to one frame in the way indicated in 2.1.3, are linearly independent and span a simple ring A_i. Thus A is the direct sum of the minimal left ideals which arise from the standard tableaux belonging to any frame associated with a partition of $l+1$.

It follows that the degree of χ^λ is equal to the number of standard tableaux belonging to a frame associated with λ. This leads to a formula for the degree.

Definition

Let $\lambda \vdash l+1$ and F_λ its associated frame. The square in the i^{th} row and j^{th} column is called the ij-node. The number of squares to the right and below this node (including the ij-node) is called the hook length of the ij-node. The hook product H_λ is the product of the $l+1$ hook lengths.

A hook graph is a partition of the form $(i, 1^{l+1-i})$ for some $i \in \{1, \ldots, l+1\}$. Thus the frame of a hook graph is a hook.

Theorem 2.1.6 ([10] theorem 1)

$$\chi^\lambda(1) = \frac{(l+1)!}{H_\lambda}$$

Finally, we state a further formula (which is used in proving 2.1.5) relating the degree of χ^λ $(\lambda \vdash l)$ to degrees of characters of partitions of $l+1$.

Lemma 2.1.7

Let $\lambda \vdash l$. Then

$$(l+1)\chi^\lambda(1) = \sum_\mu \chi^\mu(1)$$

summed over all partitions μ of $l+1$ whose frame may be obtained by adding a square to the end of a row of the frame of λ.

§2.2 Decomposition of induced principal character

Let $\lambda \vdash l+1$ and fix a diagram D_λ and we let $W_\lambda = R(D_\lambda)$. The aim of this section is to decompose $1_{W_\lambda}^{W}$ into its irreducible components.

First we obtain an alternative characterization of χ^λ. We shall need:

Lemma 2.2.1

If $y \in W$, then $W_\lambda \cap yW_{\lambda'}y^{-1}$ contains only even permutations if and only if $y \in W_\lambda W_{\lambda'}$.

Proof

Suppose $W_\lambda \cap yW_{\lambda'}y^{-1}$ contains only even permutations and that there exist two symbols $a, b \in \{1, \ldots, l+1\}$ such that a, b are collinear in D_λ and co-columnar in yD_λ. Let t be the transposition (ab).

Hence $t \in R(D_\lambda) \cap C(yD_\lambda)$

$$= R(D_\lambda) \cap yC(D_\lambda)y^{-1} \qquad \text{by } 2.1.1$$

$$= W_\lambda \cap yW_{\lambda'}y^{-1}$$

which is a contradiction since t is an odd permutation. Thus no two collinear symbols of D_λ are co-columnar in yD_λ and so by 2.1.2, $y \in W_\lambda W_{\lambda'}$.

Conversely, let $y = pq$ where $p \in W_\lambda$, $q \in W_{\lambda'}$.

Then
$$\begin{aligned} W_\lambda \cap pqW_{\lambda'}(pq)^{-1} &= W_\lambda \cap pW_{\lambda'}p^{-1} \\ &= p(p^{-1}W_\lambda p \cap W_{\lambda'})p^{-1} \\ &= p(W_\lambda \cap W_{\lambda'})p^{-1} \\ &= p(R(D_\lambda) \cap C(D_\lambda))p^{-1} \\ &= p.1.p^{-1} \\ &= 1 \end{aligned}$$

so certainly $W_\lambda \cap yW_{\lambda'}y^{-1}$ only contains even permutations.

Lemma 2.2.2

$$(1_{W_\lambda}^W, \varepsilon_{W_{\lambda'}}^W) = 1$$

Proof

By Mackey's formula

$$(1_{W_\lambda}^W, \varepsilon_{W_{\lambda'}}^W) = \sum_{y \in \{y_i\}} (1_{W_\lambda \cap y W_{\lambda'} y^{-1}}, {}^y\varepsilon_{W_\lambda \cap y W_{\lambda'} y^{-1}})$$

where $\{y_i\}$ is a set of $(W_\lambda, W_{\lambda'})$-double coset representatives.

Now $(1_{W_\lambda \cap y W_{\lambda'} y^{-1}}, {}^y\varepsilon_{W_\lambda \cap y W_{\lambda'} y^{-1}}) \neq 0$

$\Leftrightarrow$ $1_{W_\lambda \cap y W_{\lambda'} y^{-1}} = {}^y\varepsilon_{W_\lambda \cap y W_{\lambda'} y^{-1}}$ since both characters are linear

$\Leftrightarrow$ $W_\lambda \cap y W_{\lambda'} y^{-1}$ contains only even permutations

$\Leftrightarrow$ $y \in W_\lambda W_{\lambda'}$ by 2.2.1

$\Leftrightarrow$ $y = y_1 = 1$

Thus only the first term is non-zero and is

$$(1_{W_\lambda \cap W_{\lambda'}}, \varepsilon_{W_\lambda \cap W_{\lambda'}}) = (1_{\{1\}}, \varepsilon_{\{1\}}) = 1$$

which proves the lemma.

It follows from 2.2.2 that $1_{W_\lambda}^W$ and ${}_{W_{\lambda'}}^W$ contain a unique common irreducible constituent; we shall show that this is χ^λ.

$1_{W_\lambda}^W$ is afforded by the A-module $A\xi_\lambda$, $\varepsilon_{W_{\lambda'}}^W$ by the A-module $A\eta_\lambda$ and χ^λ by the irreducible A-module $A\xi_\lambda\eta_\lambda$. It is clear that $A\xi_\lambda\eta_\lambda \leqslant A\eta_\lambda$. It follows, using 1.2.7, that $A\xi_\lambda\eta_\lambda \cong A(\xi_\lambda\eta_\lambda)^* = A\eta_\lambda\xi_\lambda \leqslant A\xi_\lambda$. Thus $A\xi_\lambda\eta_\lambda$ is isomorphic both to a submodule of $A\xi_\lambda$ and of $A\eta_\lambda$. Hence χ^λ is an irreducible component of both $1_{W_\lambda}^W$ and $\varepsilon_{W_{\lambda'}}^W$ and by 2.2.2 the result follows. We have thus proved :

Theorem 2.2.3

χ^λ is the unique common irreducible constituent of

$1_{W_\lambda}^{W}$ and $\varepsilon_{W_{\lambda'}}^{W}$ and occurs with multiplicity one.

We now define a partial ordering on the partitions of l+1; this ordering is weaker than the lexicographic ordering which is often used (see e.g. [6] p 191) but is much more natural for our purposes as will become apparent in later sections.

Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash l+1$ and $\mu = (\mu_1, \ldots, \mu_s) \vdash l+1$. Then $\lambda \leqslant \mu$ if and only if $\sum_{i=1}^{m} \lambda_i \leqslant \sum_{i=1}^{m} \mu_i$, for $m = 1, \ldots, \min(r,s)$.

This is not a total ordering (e.g. (3^2) and (41^2) are not comparable) and we shall be investigating the partial ordering further in §2.3 .

However, we can now utilize this ordering to decompose $1_{W_\lambda}^{W}$.

Lemma 2.2.4

Let $\lambda, \mu \vdash l+1$ and suppose $\lambda \nleqslant \mu$. Then if D_λ, D_μ are corresponding diagrams, then there exist two symbols collinear in D_λ and co-columnar in D_μ .

Proof

Put $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$. Suppose that any 2 symbols collinear in D_λ are not co-columnar in D_μ . Therefore, the λ_1 entries in the first row of D_λ must occur in different columns of D_μ . Since D_μ has μ_1 columns we have $\lambda_1 \leqslant \mu_1$. Apply a column permutation to D_μ to obtain a new diagram D'_μ so that the entries in the first row of D_λ appear in the first row of D'_μ .

Now, inductively assume $\sum_{i=1}^{m-1} \lambda_i \leq \sum_{i=1}^{m-1} \mu_i$, (we have $\lambda_1 \leq \mu_1$ above) and that the entries in the first m-1 rows of D_λ lie in the first m-1 rows of D_μ'', and no two symbols collinear in D_λ are co-columnar in D_μ''. Then the λ_m entries in the m^{th} row of D_λ lie in different columns of D_μ'', and we can bring them up, via a column permutation, to occupy squares in the first m rows of D_μ''.

It follows that $\sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \mu_i$. Hence by induction, this holds for all m, so that $\lambda \leq \mu$, contradicting our hypothesis, which proves the lemma.

Corollary 2.2.5

Let $\lambda, \mu \vdash l+1$. Then

$$\lambda \nleq \mu \Rightarrow (1_{W_\lambda}^{W}, \varepsilon_{W_{\mu'}}^{W}) = 0 .$$

Proof

As in the proof of 2.2.2, if $\{y_i\}$ is a set of $(W_\lambda, W_{\mu'})$-double coset representatives

$(1_{W_\lambda}^{W}, \varepsilon_{W_{\mu'}}^{W})$ = the number of y's $\in \{y_i\}$ such that $W_\lambda \cap yW_{\mu'}y^{-1}$ contains only even permutations.

By 2.2.4, there exist 2 symbols, a,b say, collinear in D_λ and co-columnar in yD_μ (where $W_\lambda = R(D_\lambda)$, $W_\mu = R(D_\mu)$) for any $y \in W$.

Hence the transposition $t = (ab) \in R(D_\lambda) \cap C(yD_\mu)$

$$= W_\lambda \cap yW_{\mu'}y^{-1}$$

Since t is an odd permutation it follows that

$$(1_{W_\lambda}^{W}, \varepsilon_{W_{\mu'}}^{W}) = 0 .$$

The previous corollary allows us to give an alternative proof of a well-known result

Corollary 2.2.6

$$\lambda \neq \mu \quad \Rightarrow \quad \chi^\lambda \neq \chi^\mu$$

Proof

Suppose $\chi^\lambda = \chi^\mu$. Then by 2.2.3 χ^μ occurs as a common irreducible constituent of $1_{W_\lambda}^W$ and $\varepsilon_{W_{\mu'}}^W$ and χ^λ occurs as a common irreducible constituent of $1_{W_\mu}^W$ and $\varepsilon_{W_{\lambda'}}^W$. Thus

$$(1_{W_\lambda}^W , \varepsilon_{W_{\mu'}}^W) \neq 0 \text{ and } (1_{W_\mu}^W , \varepsilon_{W_{\lambda'}}^W) \neq 0.$$

But $\lambda \neq \mu \Rightarrow \lambda \not\leq \mu$ or $\mu \not\leq \lambda$. It follows from 2.2.5 that one of the above multiplicities is zero, contradicting our assumption that $\chi^\lambda = \chi^\mu$.

Since the conjugacy classes of W are parameterized by partitions of l+1, we have that all irreducible characters of W have the form χ^λ where $\lambda \vdash l+1$.

We are now in the position to give the main theorem of this section, which was originally proved by Frobenius.

Theorem 2.2.7

Let $\lambda, \mu \vdash l+1$. Then

$$1_{W_\lambda}^W = \chi^\lambda + \sum_{\mu > \lambda} a_\mu \chi^\mu$$

and

$$\varepsilon_{W_{\lambda'}}^W = \chi^\lambda + \sum_{\mu < \lambda} b_\mu \chi^\mu$$

where a_μ, b_μ are non-negative integers.

Proof

Suppose $(1_{W_\lambda}^W , \chi^\mu) \neq 0$, then by 2.2.3 $(\varepsilon_{W_{\mu'}}^W , \chi^\mu) \neq 0$, so that $(1_{W_\lambda}^W , \varepsilon_{W_{\mu'}}^W) \neq 0$ and hence, by 2.2.5, $\lambda \leq \mu$. $(1_{W_\lambda}^W , \chi^\lambda) = 1$ by 2.2.3 proving the first equation. The second equation follows similarly.

In §2.3 we shall strengthen 2.2.7 and show that both a_μ and b_μ are non-zero.

This theorem allows us to define a bijection between the Weyl subgroups and irreducible characters of W in a manner which will generalize to other Weyl groups.

Define a map

X : set of Weyl subgroups $\longrightarrow$ set of irreducible characters by

$$X(W_\lambda) = \left\{ \chi \text{ irred. character} : (\chi, 1_{W_\lambda}^{W}) \neq 0 \text{ and } (\chi, 1_{W'}^{W}) = 0 \text{ for all Weyl subgroups } W' = W_\mu \text{ such that } \mu > \lambda \right\}$$

<u>Theorem 2.2.8</u>

$X(W_\lambda) = \{\chi^\lambda\}$ for all partitions λ of $l+1$

<u>Proof</u>

2.2.7 shows $(\chi^\mu, 1_{W_\lambda}^{W}) \neq 0 \Rightarrow \lambda \leq \mu$

Suppose $\mu > \lambda$. By 2.2.3 $(\chi^\mu, 1_{W_\mu}^{W}) \neq 0$, so putting $W' = W_\mu$ we see that $\chi^\mu \notin X(W_\lambda)$.

Also $(\chi^\lambda, 1_{W_\lambda}^{W}) \neq 0$ and by 2.2.7 $\mu > \lambda \Rightarrow (\chi^\lambda, 1_{W_\mu}^{W}) = 0$ so $\chi^\lambda \in X(W_\lambda)$.

Thus $X(W_\lambda) = \{\chi^\lambda\}$

§2.3 <u>The partial ordering on partitions</u>

In this section we shall give a more convenient definition of the partial ordering defined in §2.2, which will simplify some of the proofs.

In the rest of this section we shall assume that $\lambda, \mu \vdash l+1$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$.

It will often be convenient to abuse notation by referring to a diagram or frame of a partition λ simply as λ itself. It will be clear from the context, when not specifically stated, what is meant e.g. in 2.3.1 we are dealing with the frames.

Theorem 2.3.1

$\lambda \leqslant \mu$ if and only if μ may be obtained from λ by repeating as many times as is necessary the operation of taking a square from the end of a row of λ and adding it onto the end of a row higher up so as to obtain another partition.

This process will often be referred to as 'moving (squares) up'.

Proof

Suppose μ may be obtained from λ by the given algorithm. If we move a square up from the j^{th} row of λ to the i^{th} row ($i<j$) to obtain a partition $\nu = (\nu_1, \nu_2, \dots)$ then

$$\sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \nu_k \qquad \text{for} \quad m \geqslant j \quad \text{or} \quad m<i$$

and $$\sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \nu_k - 1 \leqslant \sum_{k=1}^{m} \nu_k \qquad \text{for } i \leqslant m < j$$

Thus $\lambda \leqslant \nu$. Since $\leqslant$ is a partial ordering, repeating the process gives $\lambda \leqslant \mu$.

Conversely suppose $\lambda \leqslant \mu$. We have

$\sum_{i=1}^{m} \lambda_i \leqslant \sum_{i=1}^{m} \mu_i$ for all m, and we may suppose $\lambda < \mu$.

We choose k to be the first row in which λ_k differs from μ_k i.e. $\lambda_i = \mu_i$ for $i<k$

and $\lambda_k < \mu_k$

Let j be the last row in which λ_j differs from μ_j

i.e. $\lambda_i = \mu_i$ for $i>j$

and $\lambda_j > \mu_j$

Since $\lambda < \mu$ and $|\lambda| = |\mu|$, k and j exist. Now move a square from the j^{th} row up to the k^{th} row to obtain a partition ν. It follows from the first part that $\lambda < \nu$. If $\nu = (\nu_1, \ldots, \nu_r)$ then

$$\sum_{i=1}^{m} \nu_i = \sum_{i=1}^{m} \mu_i \qquad \text{for } m<k \text{ or } m \geqslant j$$

$$\sum_{i=1}^{m} \nu_i = \sum_{i=1}^{m} \lambda_i + 1 \leqslant \sum_{i=1}^{m} \mu_i \qquad \text{for } k \geqslant m > j$$

Hence $\nu \leqslant \mu$, so we may repeat the operation of moving one square up in ν. Eventually we will reach μ, proving the theorem.

We can now prove a fundamental property of this ordering

Lemma 2.3.2 (Duality Relation)

$$\lambda \leqslant \mu \quad \Leftrightarrow \quad \mu' \leqslant \lambda'$$

Proof

It will be sufficient to prove the implication in one direction. So suppose $\lambda \leqslant \mu$. By 2.3.1 we may move squares up inside λ to obtain μ. But this means that we are moving down inside μ' to obtain λ'. Hence, by 2.3.1, $\mu' \leqslant \lambda'$.

The rest of this section will be devoted to showing that all the irreducible characters χ^{μ} which may occur in the decomposition of $1_{W_\lambda}^{W}$ given in 2.2.7 actually do occur. This is a special case of the Littlewood-Richardson rule (see [15]) which gives a method of calculating the multiplicity $a_\mu = (1_{W_\lambda}^{W}, \chi^{\mu})$. However, as we shall not need the full power of this rule, it is worth giving an alternative proof that a_μ is non-zero.

We first prove the converse of 2.2.4

Lemma 2.3.3

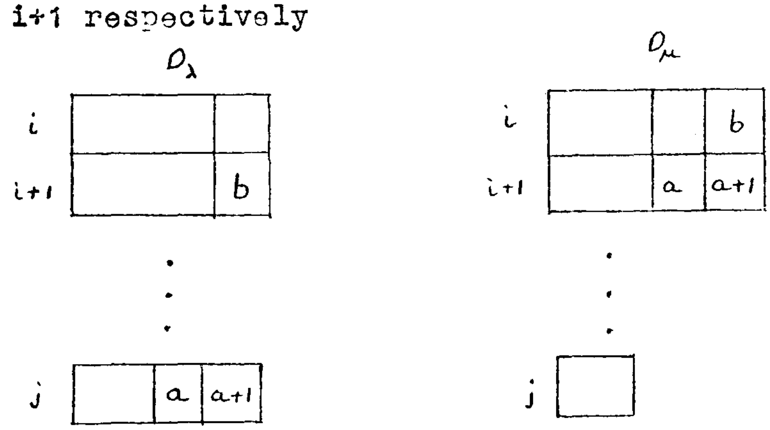
Let D_λ be a diagram corresponding to λ and suppose $\lambda \leqslant \mu$. Then there exists a diagram D_μ corresponding to μ such that no two collinear symbols in D_λ are co-columnar in D_μ.

Proof

By possibly renumbering the symbols in D_μ we may assume that the symbols in D_λ are given by numbering the squares consecutively from the top left-hand corner moving across each row and then onto the next row; this will be called the natural ordering of the symbols in D_λ.

Since $\lambda \leqslant \mu$, we may move squares up in the frame for λ to obtain the frame for μ. Thus we may move up the squares in D_λ in the same way to obtain a diagram for μ (by keeping the symbols in their squares). To obtain the required D_μ we move the squares up in D_λ in this way, except for the following case:

Suppose $\lambda_i = \lambda_{i+1}$, and $j > i+1$ and we are required to move 2 consecutive squares in row j of D_λ containing the symbols $a, a+1$ and put them onto the end of row i and row $i+1$ respectively



Let b be the symbol occurring at the end of row $i+1$ of D_λ. Then move this square up to row i (even though this may not be allowed in the definition of moving squares up in 2.3.1) and then move the squares containing the symbols $a, a+1$ onto the end of row $i+1$ of the resulting diagram.

By the transitivity of the ordering we may then repeat the process, on moving squares up, to obtain D_μ. It is clear from the construction that no 2 symbols are collinear in D_λ and co-columnar in D_μ.

The proof of the next theorem was suggested to me by J.A. Green

Theorem 2.3.4

Suppose $\lambda \leqslant \mu$ and that D_λ, D_μ are corresponding diagrams such that no 2 collinear symbols of D_λ are co-columnar in D_μ. Then, with the notation of §2.1,

$$\xi_\lambda e_\mu \neq 0$$

Proof

Let $W_\lambda = R(D_\lambda)$ and $W_\mu = R(D_\mu)$; the condition in the statement of the theorem becomes $R(D_\lambda) \cap C(D_\mu) = 1$.

We have that $\lambda = (\lambda_1, \dots, \lambda_r)$ where $\lambda_1 \geqslant \dots \geqslant \lambda_r > 0$, and we shall use induction on the number of parts n_λ say, of λ not equal to 1.

If $n_\lambda = 0$ then $\lambda = (1^r)$ and the result is trivial because $\xi_\lambda = 1$.

However, it will be necessary to prove the case in which $n_\lambda = 1$. Thus $\lambda = (\lambda_1, 1^{l+1-\lambda_1})$ with $\lambda_1 > 1$.

To show $\xi_\lambda e_\mu \neq 0$ it will be sufficient to show that the coefficient of the unit element 1 of W in $\xi_\lambda e_\mu$ is non-zero. This coefficient is $\sum \mathcal{E}(q_\mu)$ summed over those elements q_μ of W_μ such that there exist elements p_μ of W_μ

and p_λ of W_λ such that $p_\lambda p_\mu q_\mu = 1$.

Suppose that the symbols in the first row of D_λ are $\{a_1, \dots, a_{\lambda_1}\}$ and let $b \notin \{a_1, \dots, a_{\lambda_1}\}$. Then because $p_\lambda(b) = b$ we have that $p_\mu(b) = q_\mu^{-1}(b) = c$, say. Hence b and c are collinear in D_μ and co-columnar in D_μ, so we must have $b = c$ i.e. $q_\mu(b) = b$. Thus in the cycle decomposition of q_μ only the symbols $\{a_1, \dots, a_{\lambda_1}\}$ can occur, i.e. $q_\mu \in W_\lambda$, so if $q_\mu \neq 1$ it contains two distinct symbols which are collinear in D_λ and co-columnar in D_μ, an impossibility. Hence $q_\mu = 1$ and therefore $\sum \varepsilon(q_\mu) = \sum \varepsilon(1) > 0$. So we have shown that $\xi_\lambda e_\mu \neq 0$ for $n_\lambda = 1$.

Now suppose $n_\lambda > 1$ and that if $\nu \vdash l+1$, $\nu \leqslant \mu$ and $R(D_\nu) \cap C(D_\mu) = 1$ then $n_\nu < n_\lambda \Rightarrow \xi_\nu e_\mu \neq 0$.

We let $\tilde\lambda = (\lambda_1, \dots, \lambda_{s-1}, 1^{t})$ and $\bar\lambda = (\lambda_s, 1^{l+1-\lambda_s})$
where $s = n_\lambda$, so $\lambda_s > 1$, $\lambda_{s+1} = 1$, and $t = (l+1) - (\lambda_1 + \dots + \lambda_{s-1})$
which are both partitions of $l+1$. Notice that, because $n_\lambda > 1$,

$$\bar\lambda \leqslant \tilde\lambda < \lambda \leqslant \mu, \quad n_{\tilde\lambda} < n_\lambda \quad \text{and} \quad n_{\bar\lambda} \leqslant 1$$

So by induction, if $\tilde\lambda \leqslant \nu$, $R(D_{\tilde\lambda}) \cap C(D_\nu) = 1$ then $\xi_{\tilde\lambda} e_\nu \neq 0$. However, e_ν is a multiple of a primitive idempotent (2.1.3) so

$$\xi_{\tilde\lambda} = \sum_{\tilde\lambda \leqslant \nu} x_\nu e_\nu \qquad \dots \quad (1)$$

where $R(D_{\tilde\lambda}) \cap C(D_\nu) = 1$ and x_ν are positive non-zero integers. Similarly, because $n_{\bar\lambda} \leqslant 1 < n_\lambda$

$$\xi_{\bar\lambda} = \sum_{\bar\lambda \leqslant \rho} y_\rho e_\rho \qquad \dots \quad (2)$$

where $R(D_{\bar\lambda}) \cap C(D_\rho) = 1$ and y_ρ are positive non-zero integers.

We are at liberty to choose $D_{\tilde\lambda}$ and $D_{\bar\lambda}$ as we please. So order the symbols in the first $s-1$ rows of $D_{\tilde\lambda}$ in the same way as in the first $r-1$ rows of D_λ and order the symbols in the first row of $D_{\bar\lambda}$ in the same way as the r^{th} row of D_λ.

It follows that with these orderings, $\xi_\lambda = \xi_{\tilde\lambda}\xi_{\bar\lambda}$. Thus, from (1) and (2),

$$\xi_\lambda = \sum_{\tilde\lambda \leqslant \nu} \sum_{\bar\lambda \leqslant \rho} x_\nu y_\rho e_\nu e_\rho$$

summed over the appropriate ν, ρ. But if $\nu \neq \rho$, e_ν and e_ρ are orthogonal primitive idempotents which afford distinct irreducible characters of W (2.1.3). Hence, as $\bar\lambda \leqslant \tilde\lambda$,

$$\xi_\lambda = \sum_{\tilde\lambda \leqslant \nu} x_\nu y_\nu e_\nu \qquad \ldots \quad (3)$$

where $R(D_{\tilde\lambda}) \cap C(D_\nu) = 1$ and $R(D_{\bar\lambda}) \cap C(D_\nu) = 1$.

Hence because $x_\nu y_\nu \neq 0$ for such partitions ν of $l+1$, $\xi_\lambda e_\nu \neq 0$.

Returning to μ, as we have arranged $R(D_{\tilde\lambda}) \leqslant R(D_\lambda)$, $R(D_{\bar\lambda}) \leqslant R(D_\lambda)$, we know that $\mu \geqslant \lambda > \tilde\lambda$ and $R(D_{\tilde\lambda}) \cap C(D_\mu) = 1$ and $R(D_{\bar\lambda}) \cap C(D_\mu) = 1$.

Hence $\xi_\lambda e_\mu \neq 0$, which, by induction, completes the theorem.

<u>Remark</u>

In (3), for ν to satisfy the required conditions it is easy to see that, in fact, $\lambda \leqslant \nu$; this verifies part of 2.2.7.

<u>Lemma 2.3.5</u>

$$\varepsilon \chi^\lambda = \chi^{\lambda'}$$

<u>Proof</u>

By 2.2.3, $\chi^{\lambda'}$ is the unique common irreducible

constituent of $1_{W_{\lambda'}}^{W}$ and $\varepsilon_{W_\lambda}^{W}$.

But $(\varepsilon\chi^\lambda, 1_{W_{\lambda'}}^{W}) = (\chi^\lambda, \varepsilon . 1_{W_{\lambda'}}^{W})$ since $\varepsilon^2 = 1$

$$= (\chi^\lambda, \varepsilon_{W_{\lambda'}}^{W})$$

$$= 1 \text{ by } 2.2.3$$

and $(\varepsilon\chi^\lambda, \varepsilon_{W_\lambda}^{W}) = (\chi^\lambda, \varepsilon . \varepsilon_{W_\lambda}^{W})$

$$= (\chi^\lambda, 1_{W_\lambda}^{W})$$

$$= 1 \text{ by } 2.2.3$$

Hence $\varepsilon\chi^\lambda$ is a common constituent of $1_{W_{\lambda'}}^{W}$ and $\varepsilon_{W_\lambda}^{W}$ and is also irreducible since $(\varepsilon\chi^\lambda, \varepsilon\chi^\lambda) = (\chi^\lambda, \chi^\lambda) = 1$.

So $\varepsilon\chi^\lambda = \chi^{\lambda'}$.

Corollary 2.3.6

$$(1_{W_\lambda}^{W}, \chi^\mu) \neq 0 \iff \lambda \leqslant \mu$$

$$(\varepsilon_{W_{\lambda'}}^{W}, \chi^\mu) \neq 0 \iff \lambda \geqslant \mu$$

Proof

If $(1_{W_\lambda}^{W}, \chi^\mu) \neq 0$ then $\lambda \leqslant \mu$ by 2.2.7. Conversely, let $\lambda \leqslant \mu$. Therefore, by 2.3.3, there exist diagrams D_λ and D_μ satisfying the conditions of 2.3.4. Hence, by 2.3.4, $\xi_\lambda e_\mu \neq 0$, so that $Ae_\mu \leqslant A\xi_\lambda$ since Ae_μ is irreducible. But $A\xi_\lambda$ affords the character $1_{W_\lambda}^{W}$, and Ae_μ affords χ^μ, so $(1_{W_\lambda}^{W}, \chi^\mu) \neq 0$.

The second half of the result follows from 2.2.7 and the fact that

$$\lambda \geqslant \mu \implies \lambda' \leqslant \mu' \quad (2.3.2)$$

$$\implies (1_{W_{\lambda'}}^{W}, \chi^{\mu'}) \neq 0 \text{ by the first part}$$

$$\implies (\varepsilon . 1_{W_{\lambda'}}^{W}, \varepsilon\chi^{\mu'}) \neq 0$$

$$\Rightarrow (\varepsilon_{W_{\chi'}}^{W}, \chi^{\mu}) \neq 0 \qquad \text{by } 2.3.5$$

§2.4 A decomposition of the group algebra of $W(A_l)$

Solomon [17] has given a decomposition of the group algebra of an arbitrary finite Coxeter group, and in this section we interpret his results as applied to the symmetric group. In later chapters we look at the decompositio for other Weyl groups.

By tensoring with $\mathbb{C}$, we shall assume that all modules, representations and characters are over the field of complex numbers. In particular, $A = \mathbb{C}W$. Otherwise we shall use the same notation as in [17].

The generating set I for W is the set of l transpositions $\{(12),(23), \ldots ,(l\ l+1)\}$. Let $J \subseteq I$, then W_J is the parabolic subgroup of W generated by the elements of J. Now, W_J is also a Weyl subgroup of W, and hence is of the form W_ρ for some partition ρ of $l+1$. Thus each subset J of I defines a unique partition ρ of $l+1$ and we write $p(J) = \rho$.

We fix an arbitrary subset J of I. Let $p(J) = \rho$, and since $\hat{J}$ - the complement of J in I - is also a subset of I, we can put $p(\hat{J}) = \mu'$, where $\mu \vdash l+1$ (we use the dual of μ for convenience only).

Then define

$$\xi_J = \sum_{w \in W_J} w \quad , \quad \eta_{\hat{J}} = \sum_{w \in W_{\hat{J}}} \varepsilon(w) w$$

(these differ from [17] only by a scalar multiple, but the module $A\xi_J\eta_{\hat{J}}$ is the same in both cases), so that

$\xi_J = \xi_\rho$, $\eta_{\hat{J}} = \eta_\mu$ as defined in §2.1.

Solomon [17] shows that the module $A\xi_J\eta_{\hat{J}}$ affords the character Ψ_J of W where

$$\Psi_J = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \, 1_{W_K}{}^{W}$$

We shall be investigating the irreducible submodules of $A\xi_J\eta_{\hat{J}}$.

Theorem 2.4.1

Let $\lambda \vdash l+1$. Then $(\Psi_J , \chi^\lambda) \neq 0 \Rightarrow \rho \leq \lambda \leq \mu$ i.e. $A\xi_J\eta_{\hat{J}}$ only contains irreducible submodules isomorphic to some $A\xi_\lambda\eta_\lambda$, where $\rho \leq \lambda \leq \mu$.

Proof

By 1.2.8, since $A\xi_J = A\xi_\rho$ affords $1_{W_\rho}{}^{W}$ and $A\eta_{\hat{J}} = A\eta_\mu$ affords $\mathcal{E}_{W_{\mu'}}{}^{W}$ (1.2.9)

$(\Psi_J , \chi^\lambda) \neq 0 \;\Rightarrow\; (1_{W_\rho}{}^{W} , \chi^\lambda) \neq 0$ and $(\mathcal{E}_{W_{\mu'}}{}^{W} , \chi^\lambda) = 0$

$\Rightarrow \rho \leq \lambda \leq \mu$ by 2.3.6

Lemma 2.4.2

$$(\Psi_J , \chi^\rho) = (\Psi_J , \chi^\mu) = 1$$

Hence $\rho \leq \mu$.

Proof

Suppose $J \subsetneq K \subseteq I$. Then if $p(K) = \sigma$, σ consists of ρ with complete rows moved up. In particular $\sigma > \rho$.
Hence, by 2.3.6, $(1_{W_\sigma}{}^{W} , \chi^\rho) = 0$ i.e. $(1_{W_K}{}^{W} , \chi^\rho) = 0$.

Thus, $$(\Psi_J , \chi^\rho) = \sum_{J \subseteq K \subseteq I} (1_{W_K}{}^{W} , \chi^\rho)$$

$$= (1_{W_J}{}^{W} , \chi^\rho)$$

$= 1$ by 2.2.7

Similarly, $(\psi_{\hat{J}}, \chi^{\mu'}) = 1$ since $p(J) = \mu'$.

Now by [17] lemma 7, $\varepsilon\psi_J = \psi_{\hat{J}}$

Thus

$$(\psi_J , \chi^{\mu}) = (\varepsilon\psi_J , \varepsilon\chi^{\mu}) = (\psi_{\hat{J}} , \chi^{\mu'}) \text{ by 2.3.5}$$
$$= 1$$

It follows immediately from above and 2.4.1 that $\rho \leqslant \mu$.

Solomon [17] theorem 4, also shows that if $|\hat{J}| = p$ then $A\xi_J\eta_{\hat{J}}$ has a unique irreducible submodule isomorphic to $\wedge^p V$ of dimension $\binom{l}{p}$, where V is the Euclidean space of dimension l which affords the Witt representation of W as a reflection group.

In our case V is the hyperplane of $\mathbb{R}^{l+1}$ consisting of the points whose sum of the coordinates is zero ([3], table I). We shall now identify $\wedge^p V$ and the irreducible character it affords.

Suppose $|\hat{J}| = p$

Definition

Let β be the partition of l+1 given by $\beta = (l-p+1, 1^p)$. Then we call β the hook graph for J and χ^{β} the hook character for J.

Notice that the hook graph depends only on the order of J, and that $\chi^{\beta}(1) = \binom{l}{p}$ by 2.1.6.

If $\lambda \vdash l+1$ then let $r(\lambda)$ = the number of rows of (the frame of) λ.

Lemma 2.4.3

(i) $r(\rho) = p+1$

(ii) $(\psi_J, \chi^{\delta}) = 1$

Proof

(i) Let D_ρ be the diagram corresponding to ρ which is defined by W_J. Then there exists an element x of W such that xD_ρ is a diagram corresponding to ρ whose symbols are naturally ordered.

Hence, $R(xD_\rho) = xW_Jx^{-1}$ (2.1.1)

$$= W_{xJx^{-1}}$$

$$= W_{J^x}$$

By construction

$$J^x = \bigcup_{i=0}^{r-1} \{(a_i+1\ a_i+2), (a_i+2\ a_i+3), \ldots, (a_{i+1}-1\ a_{i+1})\}$$

where $\rho = (\rho_1, \ldots, \rho_r)$ so that $r(\rho) = r$

and
$$a_0 = 0$$
$$a_1 = \rho_1$$
$$a_2 = \rho_1 + \rho_2$$
$$\vdots$$
$$a_{r-1} = \rho_1 + \ldots + \rho_{r-1}$$
$$a_r = \rho_1 + \ldots + \rho_r = 1+1$$

Hence $\hat{J}^x = \{(a_1\ a_1+1), (a_2\ a_2+1), \ldots, (a_{r-1}\ a_{r-1}+1)\}$

so that
$$r-1 = |\hat{J}^x| = |I| - |J^x|$$
$$= 1 - |J|$$
$$= p \quad \text{since } |\hat{J}| = p$$

Hence $r(\rho) = r = p+1$

(ii) Move up all the squares of ρ that do not lie in the first column, up to the first row. This gives us a frame whose first column has length equal to the length of the first column of ρ which is $r(\rho) = p+1$. Since this frame is a hook by definition, it represents the partition $(l-p+1, 1^p) = \beta$. Thus, by 2.3.1 $\rho \leqslant \beta$.

Now suppose $J \subsetneq K \subseteq I$ and $p(K) = \alpha$. Then α is obtained from ρ by moving up whole rows.

i.e. $r(\alpha) < r(\rho) = p+1 = r(\beta)$

But if $\alpha \leqslant \beta$ then it is clear from 2.3.1 that $r(\alpha) \geqslant r(\beta)$.

Thus $\alpha \nleqslant \beta$.

Therefore, by 2.3.6, $(1_{W_K}{}^W, \chi^\beta) = 0$

Hence

$$
\begin{aligned}
(\Psi_J, \chi^\beta) &= \sum_{J \subseteq K \subseteq I} (1_{W_K}{}^W, \chi^\beta) \\
&= (1_{W_J}{}^W, \chi^\beta) \\
&= (1_{W_\rho}{}^W, \chi^\beta) \neq 0 \text{ by 2.3.6 since } \rho \leqslant \beta
\end{aligned}
$$

Thus we have shown $|\hat{J}| = p \Rightarrow (\Psi_J, \chi^\beta) \neq 0$

Now the fundamental result in [17] is that

$$A = \sum_{J \subseteq I} A\xi_J \eta_{\hat{J}}$$

so that

$$\chi^{reg} = \sum_{J \subseteq I} \Psi_J \quad \text{, where } \chi^{reg} \text{ is the regular}$$

character of W.

Hence $\binom{l}{p} = \chi^\beta(1) = (\chi^{reg}, \chi^\beta) = \sum_{J \subseteq I} (\Psi_J, \chi^\beta)$

But there are $\binom{l}{p}$ subsets J of I such that $|\hat{J}| = p$, and for each of these $(\Psi_J, \chi^\beta) \neq 0$. It follows immediately that $(\Psi_J, \chi^\beta) = 1$; and, incidentally, that $(\Psi_K, \chi^\beta) = 0$ if $|\hat{K}| \neq p$.

Theorem 2.4.4

Let χ be the irreducible character of W afforded by $\Lambda^p V$. Then $\chi = \chi^\beta$. Thus $\Lambda^p V \cong A\xi_\beta \eta_\beta$

Proof

χ is irreducible so $\chi = \chi^{\lambda}$ for some $\lambda \vdash l+1$.

Let $J = \{(12),(23), \ldots ,(l-p \; l-p+1)\}$

hence $\hat{J} = \{(l-p+1 \; l-p+2), \ldots ,(l \; l+1)\}$

so that $|\hat{J}| = p$.

Then $\rho = p(J) = (l-p+1,1^p) = \beta$

and $\mu' = p(J) = (p+1,1^{l-p}) = \beta'$ i.e. $\mu = \beta$

By [17] $\Lambda^p V$ is an irreducible submodule of $A\xi_J\eta_{\hat{J}}$ and therefore $(\Psi_J , \chi^{\lambda}) \neq 0$. Hence, by 2.4.1, $\rho \leq \lambda \leq \mu$ i.e. $\beta \leq \lambda \leq \beta$ so that $\lambda = \beta$ as required.

It will be of interest to determine for which J, the module $A\xi_J\eta_{\hat{J}}$ is irreducible. We show that this happens for only a few subsets J of I, so that the decomposition given in [17] (theorem 2) is far from being a complete decomposition of A.

Definition

Let J be a subset of I. Then J is decomposable if $J = J_1 \cup J_2$ such that all the elements of J_1 commute with all the elements of J_2. Otherwise J is indecomposable

It is easy to see that J is indecomposable if and only if J consists only of consecutive generating involutions.

Theorem 2.4.5

$A\xi_J\eta_{\hat{J}}$ is irreducible if and only if both J and $\hat{J}$ are indecomposable.

Proof

Suppose $A\xi_J\eta_{\hat{J}}$ is irreducible so that Ψ_J is irreducible. Let $|\hat{J}| = p$, then by 2.4.2, 2.4.3, $\rho = \beta = \mu$, so that

ρ, μ and therefore μ', are all hook graphs. Thus the generating sets J, $\hat{J}$ consist of consecutive generators and so are indecomposable.

Conversely, suppose that both J, $\hat{J}$ are indecomposable. Then it is easy to see that $\rho = \mu$. Hence $A\xi_J \eta_{\hat{J}} = A\xi_\rho \eta_\mu = A\xi_\rho \eta_\rho$ which is an irreducible A-module affording the character χ^ρ.

§2.5 <u>The maximal Weyl subgroups of $W(A_l)$</u>

In the final section of this chapter we deal with the maximal Weyl subgroups of W, which can be determined by the algorithm in §1.1.

They are the Weyl groups of type A_{l-1} and $A_i + A_{l-i-1}$ for $1 \leqslant i \leqslant l-2$.

In 2.2.8 we defined a bijection X from the set of Weyl subgroups of W to the set of irreducible characters of W. So if W' is a Weyl subgroup of W we define $\chi_{W'}(W)$ to be the irreducible character of W associated in this way with W'.

We shall be particularly interested in the case $W' = W(A_{l-1})$. Suppose W'' is a Weyl subgroup of W' then it has associated with it an irreducible character $\chi_{W''}(W')$ of W'. However, W'' is also a Weyl subgroup of W to which the irreducible character $\chi_{W''}(W)$ is associated. The next result will show that these associations are consistent in the sense that

$$\left[\chi_{W''}(W')\right]^W = \chi_{W''}(W) + \text{higher terms} \qquad \ldots \quad (1)$$

where we order the irreducible characters by their corresponding partitions:

if $\lambda, \mu \vdash l+1$ then $\chi^\lambda \leq \chi^\mu \Leftrightarrow \lambda \leq \mu$

Now suppose $\lambda \vdash l$ and $\chi_{W''}(W') = \chi^\lambda$, so that by our construction $W'' = W_\lambda$.

We let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda^* = (\lambda_1, \ldots, \lambda_r, 1)$ which we can write as $\lambda^* = (\lambda 1)$. Then $\lambda^* \vdash l+1$ and

$W_{\lambda^*} \cong S_{\lambda_1} \times \ldots \times S_{\lambda_r} \times S_1 \cong S_{\lambda_1} \times \ldots \times S_{\lambda_r} \cong W_\lambda = W''$.

Hence $\chi_{W''}(W) = \chi^{\lambda^*}$ since $W'' \cong W_{\lambda^*}$ as a Weyl subgroup of W. Thus (1) becomes

$$(\chi^\lambda)^W = \chi^{\lambda^*} + \sum_{\mu > \lambda^*} a_\mu \chi^\mu$$

for some non-negative integers a_μ.

The theorem we prove is slightly stronger than is required above, and is a special case of the Murnaghan-Nakayama rule ([1] VI,3.1)

Theorem 2.5.1

Let $\lambda \vdash l$ and $\lambda^* = (\lambda 1)$. Then

$$(\chi^\lambda)^W = \chi^{\lambda^*} + \sum_\mu \chi^\mu$$

summed over all those partitions $\mu (\neq \lambda^*)$ of $l+1$ such that the frame for μ consists of that of λ with one square added to the end of a row.

In particular, $\mu > \lambda^*$.

Proof

Let μ be an arbitrary partition of $l+1$. Define a partition $\bar{\lambda}$ of $l+1$ by $\bar{\lambda} = (\lambda_1+1, \lambda_2, \ldots, \lambda_r)$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$. Hence $\bar{\lambda}' = (\lambda' 1) = (\lambda')^*$.

Thus $W_{\bar{\lambda}'} \cong W_{\lambda'}$ and $W_{\lambda^*} \cong W_\lambda$.

$((\chi^\lambda)^W, \chi^\mu) \neq 0 \Rightarrow (\chi^\lambda, (\chi^\mu)_{W'}) \neq 0$ by Frobenius

$\Rightarrow (1_{W_\lambda}^{W'}, (\chi^\mu)_{W'}) \neq 0$ and $(\varepsilon_{W_{\lambda'}}^{W'}, (\chi^\mu)_{W'}) \neq 0$ by 2.2.3

$\Rightarrow (1_{W_\lambda}, (\chi^\mu)_{W_\lambda}) \neq 0$ and $(\varepsilon_{W_{\lambda'}}, (\chi^\mu)_{W_{\lambda'}}) \neq 0$ by Frobenius again

$\Rightarrow (1_{W_{\lambda^*}}, (\chi^\mu)_{W_{\lambda^*}}) \neq 0$ and $(\varepsilon_{W_{\bar\lambda'}}, (\chi^\mu)_{W_{\bar\lambda'}}) \neq 0$

$\Rightarrow (1_{W_{\lambda^*}}^W, \chi^\mu) \neq 0$ and $(\varepsilon_{W_{\bar\lambda'}}^W, \chi^\mu) \neq 0$ once more by Frobenius

$\Rightarrow \lambda^* \leq \mu \leq \bar\lambda$ by 2.3.6

i.e. $\mu = (\lambda_1, \ldots, \lambda_i+1, \ldots, \lambda_r)$ for some i such that $\lambda_{i-1} > \lambda_i$ so that μ has the form required.

We have left to show that $\lambda^* \leq \mu \leq \bar\lambda \Rightarrow ((\chi^\lambda)^W, \chi^\mu) = 1$. So suppose $\lambda^* \leq \mu \leq \bar\lambda$ so that μ consists of λ with a square added to the i^{th} row for some i.

χ^μ is afforded by the minimal left ideal Ae_μ of $A = \mathbb{C}W$, and χ^λ is afforded by the minimal left ideal $A'e_\lambda$ of $A' = \mathbb{C}W'$.

Hence $(\chi^\lambda)^W$ is afforded by the (no longer minimal) left ideal Ae_λ of A.

We shall show $Ae_\mu \lesssim Ae_\lambda$; it will be sufficient to prove $e_\mu e_\lambda^* \neq 0$, (1.2.7), for then $Ae_\mu \leq Ae_\lambda^* \cong Ae_\lambda$.

Let D_λ be a diagram corresponding to λ then let D_μ be the diagram of μ given by adding a square containing the symbol l+1 to the i^{th} row of D_λ.

Thus $R(D_\lambda) \leq R(D_\mu)$ and $C(D_\lambda) \leq C(D_\mu)$ so that $W_\lambda \leq W_\mu$ and $W_{\lambda'} \leq W_{\mu'}$.

It follows easily from this fact, that $\eta_\mu\eta_\lambda = \eta_\mu$. Hence $e = e_\mu e_\lambda^* = \xi_\mu\eta_\mu\eta_\lambda\xi_\lambda = \xi_\mu\eta_\mu\xi_\lambda$. Therefore the coefficient of 1 in e is given by $\sum \varepsilon(q_\mu)$ summed over all elements

q_μ of W_μ, such that there exist elements p_μ of W_μ and p_λ of W_λ such that $p_\mu q_\mu p_\lambda = 1$.
Hence $q_\mu = p_\mu^{-1} p_\lambda^{-1} \in W_\mu$ $(\geqslant W_\lambda)$ so $q_\mu \in R(D_\mu) \cap C(D_\mu) = 1$.
Thus the coefficient of 1 in e is non-zero so $e_\mu e_\lambda^* \neq 0$.
Hence

$$(\chi^\lambda)^W = \sum_{\lambda^* \leqslant \mu \leqslant \bar{\lambda}} a_\mu \chi^\mu$$

where the a_μ's are non-zero positive integers. By considering the degrees of the characters in this equation, it follows from 2.1.7 that $a_\mu = 1$, proving the theorem.

In 2.2.7 we have only given the decomposition of the linear characters $1, \varepsilon$ on inducing up to W from a Weyl subgroup. It is of interest to note what happens when we induce up an arbitrary irreducible character from a Weyl subgroup; since all the Weyl subgroups of W are direct products of Weyl groups of type A, it will be sufficient to consider inducing irreducible characters up from maximal Weyl subgroups of W, as any Weyl subgroup is contained in a maximal one.

We have already dealt with the maximal Weyl subgroup $W(A_{l-1})$ in 2.5.1; the result for the ones of type $A_i + A_{l-i-1}$ $(1 \leqslant i \leqslant l-2)$ is given in chapter three (3.6.4) where the notation and proof properly belong.

Chapter three WEYL GROUPS OF TYPE C

The Weyl group of type C has also been extensively studied (sometimes under the guise of the 'hyper-octahedral group'); Young [20] determined the conjugacy classes and irreducible characters and Osima [16] considered the group as an example of a generalized symmetric group.

Again, we shall be considering this group as the Weyl group of the simple Lie algebra C_l in much the same way as we studied the Weyl group of A_l.

We shall not be assuming (apart, of course, from the definition) any known results about this group, as nearly all the proofs we give are new (as far as is known).

In particular, we generalize the partial ordering on partitions given in §2.2, to one on pairs of partitions.

The results in this chapter certainly do justify Osima's idea of considering this group as a generalization of the symmetric group.

§3.1 The conjugacy classes and irreducible characters

We shall give some notation which will be used in this and the next two chapters.

Let $G = W(C_l)$ - the Weyl group of rank l of type C. Then G is the group of permutations of the symbols $\{1, \dots, l, -1, \dots, -l\}$ generated by the involutions $\{(12), (23), \dots, (l-1\ l), (l, -l)\}$ where

$$(ab) : \begin{array}{lcl} a & \mapsto & b \\ b & \mapsto & a \\ -a & \mapsto & -b \\ -b & \mapsto & -a \end{array} \qquad \text{and} \qquad (l,-l) : \begin{array}{lcl} l & \mapsto & -l \\ -l & \mapsto & l \end{array}$$

We shall express the elements of G as products of cycles of the following form:

(a) positive n-cycles $(a_1\ a_2\ \dots\ a_n)$ for $1 \leqslant n \leqslant l$ and $\pm a_i \in \{1, \dots, l\}$ which maps

$$a_1 \mapsto a_2 \mapsto a_3 \mapsto \dots \mapsto a_n \mapsto a_1$$

and $-a_1 \mapsto -a_2 \mapsto -a_3 \mapsto \dots \mapsto -a_n \mapsto -a_1$

(b) negative n-cycles $\overline{(a_1\ a_2\ \dots\ a_n)}$ for $2 \leqslant n \leqslant l$ and $\pm a_i \in \{1, \dots, l\}$ which maps

$$a_1 \mapsto a_2 \mapsto \dots \mapsto a_n \mapsto -a_1 \mapsto -a_2 \mapsto \dots \mapsto -a_n \mapsto a_1$$

(c) negative 1-cycles $(i, -i)$ for $1 \leqslant i \leqslant l$, called <u>sign changes</u> which maps $i \mapsto -i \mapsto i$

The cycles are multiplied together in much the same way as those in the symmetric group, remembering the fact that $\overline{(a_1\ a_2\ \dots\ a_n)}$ is shorthand for

$$(a_1\ a_2\ \dots\ a_n\ -a_1\ -a_2\ \dots\ -a_n) = (a_1\ a_2\ \dots\ a_n)(a_n, -a_n)$$

Thus G is the split extension of N by H, where $N \cong \underbrace{C_2 \times \dots \times C_2}_{l}$ is the subgroup of G generated by the sign changes, and $H = S_l$ - the symmetric group on l letters, and H acts on N in the obvious way viz. H permutes the l cyclic groups of order 2.

Hence $|G| = |N|\,|H| = 2^l . l!$

Notation: we let $W(C_1) = \{(1), (1,-1)\}$

As in the symmetric group we may express any element of G as the product of disjoint (positive and negative) cycles.

Definition

Let $g \in G$. Suppose g is the product of disjoint cycles $c_1 \dots c_r d_1 \dots d_s$, where, for $1 \leqslant i \leqslant r$, c_i is a positive m_i-cycle, and for $1 \leqslant j \leqslant s$, d_j is a negative n_j-cycle. Then the signed cycle-type of g is the set of integers $(m_1, \dots, m_r ; n_1, \dots, n_r)$.

Note

The signed cycle-type is ordered in the sense that $(m_1, \dots, m_r ; n_1, \dots, n_s)$ is not the same as $(n_1, \dots, n_s ; m_1, \dots, m_r)$ since the first set corresponds to positive cycles and the second to negative cycles.

Lemma 3.1.1

Two elements of G are conjugate if and only if they have the same signed cycle-type.

Proof

Let $g \in G$ and let $g = c_1 \dots c_r d_1 \dots d_s$ be the decomposition of g into disjoint cycles, where c_i $(1 \leqslant i \leqslant r)$ are positive cycles, d_j $(1 \leqslant j \leqslant s)$ are negative cycles.

Fix $c = c_i = (a_1 \dots a_m)$ say where $a_1, \dots, a_m \in \{\pm 1, \dots, \pm l\}$. Then if $x \in G$,

$$xcx^{-1} = (x(a_1) \dots x(a_m))$$

a positive cycle of the same length as c.

Similarly, if $d = d_j = \overline{(b_1 \dots b_n)} = (b_1 \dots b_n \; -b_1 \dots -b_n)$ then

$$\begin{aligned} xdx^{-1} &= (x(b_1) \dots x(b_n) \; -x(b_1) \dots -x(b_n)) \\ &= \overline{(x(b_1) \dots x(b_n))} \end{aligned}$$

a negative cycle of the same length as d.

Thus $xgx^{-1} = xc_1x^{-1}. \dots .xc_rx^{-1}.xd_1x^{-1}. \dots .xd_sx^{-1}$ has the same signed cycle-type as g.

Conversely, suppose g is as above and that $g' = c'_1 \dots c'_r d'_1 \dots d'_s$ has the same signed-cycle type as g. If

$$c = (a_1 \dots a_m) \quad \text{and} \quad c' = (a'_1 \dots a'_m)$$

then c and c' are conjugate via an element $x \in G$ such that $x(a_i) = a'_i$ for $1 \leq i \leq m$.

Similarly, if

$$d = \overline{(b_1 \dots b_n)} \quad \text{and} \quad d' = \overline{(b'_1 \dots b'_n)}$$

are conjugate via an element $x \in G$ such that $x(b_j) = x(b'_j)$ $(1 \leq j \leq n)$.

Thus, since all the cycles are disjoint, we can choose an element $x \in G$ such that $g' = xgx^{-1}$.

Definition

A pair of partitions $(\lambda;\mu)$ of l consists of partitions λ, μ such that $|\lambda| + |\mu| = l$.

Let $g \in G$ have signed cycle-type $(\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_s)$ where we arrange the cycles so that $\lambda_1 \geq \dots \geq \lambda_r > 0$, $\mu_1 \geq \dots \geq \mu_s > 0$. Then this defines a pair of partitions $(\lambda;\mu)$ of l where $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$.

Hence, by 3.1.1, we have shown that the conjugacy classes of G are parameterized by pairs of partitions of l.

We turn now to the irreducible characters of G. Since G has a fairly large normal subgroup N, we can use the methods of Clifford (see [11] and [12] (17.11)).

Theorem 3.1.2

Let ς be an irreducible character of N, $C = C_H(\varsigma)$ and ψ an irreducible character of C. Then $C \cong S_m \times S_n$ where $m + n = l$, and where m is the number of generating sign changes of N on which ς takes the value 1, and n is the number on which ς takes the value -1.

Define a map $\phi : NC \to \mathbb{C}$ by $\phi(nc) = \varsigma(n)\psi(c)$. Then ϕ is an irreducible character of NC, and we write $\phi = \varsigma\psi$. Also

(a) ϕ^G is an irreducible character of G;

(b) if $\phi_1 = \varsigma_1\psi_1$, $\phi_2 = \varsigma_2\psi_2$ then $\phi_1{}^G = \phi_2{}^G$ if and only if both $\varsigma_1 = {}^h\varsigma_2$ and $\psi_1 = {}^h\psi_2$ for some $h \in H$;

(c) every irreducible character of G may be obtained in this way i.e. has the form ϕ^G for some ϕ.

Proof

Since N is abelian, ς is a linear character. Thus if $(i,-i)$ is a sign change, which therefore has order 2, $\varsigma(i,-i) = \pm 1$. Relabel the symbols $\{1,\ldots,l\}$ so that

$$\varsigma(1,-1) = \ldots = \varsigma(m,-m) \qquad \text{(some m)}$$

$$\varsigma(m+1,-(m+1)) = \ldots = \varsigma(l,-l)$$

and write

$$N_1 = \langle (1,-1), \ldots ,(m,-m) \rangle$$

$$N_2 = \langle (m+1,-(m+1)), \ldots ,(l,-l) \rangle$$

so that $N = N_1 \times N_2$.

Let $c \in C$ then ${}^c\varsigma(i,-i) = \varsigma(c(i),-c(i))$. Thus

$${}^c\varsigma = \varsigma \text{ if and only if } (i,-i) \in N_1 \Rightarrow (c(i),-c(i)) \in N_1$$
$$\text{and } (i,-i) \in N_2 \Rightarrow (c(i),-c(i)) \in N_2$$

Thus the elements of C are precisely those which permute the symbols $\{1,\ldots,m\}$ and $\{m+1,\ldots,l\}$ independently. Hence $C \cong S_m \times S_n$ where $n = l-m$.

The symbols $\{1, \ldots, m\}$ will be called <u>symbols of the first type</u> and $\{(m+1), \ldots, l\}$ <u>symbols of the second type</u>.

Every element of NC is uniquely expressible in the form nc where $n \in N$, $c \in C$, because $N \cap C = 1$.

Let V_1 be the N-module affording ς and V_2 the C-module affording . Then $V_1 \times V_2$ is an NC-module with character ϕ. For, the module axioms are easy to check, with the one exception which we now prove:

suppose $n, n' \in N$, $c, c' \in C$, $v_1 \in V_1$, $v_2 \in V_2$ then we must show

$$(v_1 \otimes v_2)(nc.n'c') = [(v_1 \otimes v_2)nc]\, n'c'$$

$$(v_1 \otimes v_2)(nc.n'c') = (v_1 \otimes v_2)(nn'^{c^{\blacktriangleleft}}.cc')$$

$$= v_1(nn'^{c^{\blacktriangleleft}}) \otimes v_2(cc') \quad \text{by definition of}$$

the tensor product of modules

$$= (v_1 n)n'^{c^{\blacktriangleleft}} \otimes (v_2 c)c' \quad \ldots \quad (1)$$

since V_1, V_2 are modules.

But $c^{\blacktriangleleft} \in C = C_H(\varsigma)$ so that $\varsigma(cnc^{-1}) = \varsigma(n)$ for all $n \in N$.

Because ς is linear, ς is the representation of N afforded by V_1 i.e. $v_1 n = v_1 \varsigma(n)$ for all $n \in N$.

It follows that

$$v_1 n' = v_1 \varsigma(n') = v_1 \varsigma(cn'c^{-1}) = v_1(cn'c^{-1}) = v_1 n'^{c^{\blacktriangleleft}}$$

and therefore

$$(v_1 n)n'^{c^{\blacktriangleleft}} = (v_1 n)n' \quad \text{since } v_1 n \in V_1 .$$

Returning to (1)

$$(v_1 \otimes v_2)(nc.n'c') = (v_1 n)n' \otimes (v_2 c)c'$$

$$= (v_1 n \otimes v_2 c)n'c'$$

since $v_1 n \in V_1$, $v_2 \in V_2$

$$= \left[(v_1 \otimes v_2)nc\right](n'c')$$

as required.

The character afforded by $V_1 \otimes V_2$ is the character ϕ as defined in the theorem. ϕ is irreducible since $\phi(1) = \varsigma(1)\psi(1) > 0$ and $(\phi,\phi)_{NC} = (\varsigma,\varsigma)_N(\psi,\psi)_C$ as $N \cap C =$

$= 1$ as ς,ψ are irreducibl

(a) We show ϕ^G is irreducible.

Firstly, $\phi^G(1) = |G:NC|\phi(1) > 0$.

Secondly, by Mackey's formula

$$(\phi^G , \phi^G) = \sum_{y \in y_i} (\phi_{NC\cap(NC)^y} , ({}^y\phi)_{NC\cap(NC)^y})$$

where $\{y_i\}$ is a set of (NC,NC)-double coset representatives.

Let $L = NC \cap (NC)^y = NC \cap NC^y$ since $N \lhd G$

$$= N(C \cap C^y)$$

$$\geqslant N$$

and suppose $(\phi_L,({}^y\phi)_L) \neq 0$. Then, by 1.2.6,

$(\phi_N,({}^y\phi)_N) \neq 0$ so that $(\varsigma,{}^y\varsigma) \neq 0$ because $\phi_N = \psi(1).\varsigma$, and therefore $\varsigma = {}^y\varsigma$, so $y \in C_G(\varsigma)$.

Now $NC = NC_H(\varsigma) = N(C_G(\varsigma) \cap H) = C_G(\varsigma) \cap NH$ by the modular la

$$= C_G(\varsigma) \cap G$$

$$= C_G(\varsigma)$$

Thus $y \in C_G(\varsigma) = NC$ i.e. $y = 1$.

Hence $(\phi^G , \phi^G) = (\phi , \phi) = 1$, so ϕ^G is an irreducible character of G.

(b) Suppose $\phi_1{}^G = \phi_2{}^G$ and let $C_1 = C_H(\varsigma_1)$, $C_2 = C_H(\varsigma_2)$

Then, if $n \in N$

$$\phi_1{}^G(n) = \frac{|G:NC_1|}{|C(n)|} \sum_{x \in C(n)\cap NC_1} \phi_1(x)$$

(where C(n) is the conjugacy class of G containing n, and since $N \lhd G$, $N \cap C_1 = 1$, $C(n) \cap NC_1 = C(n) \cap N$)

$$= \frac{\Psi_1(1)}{|C_1|} \frac{|G:N|}{|C(n)|} \sum_{x \in C(n) \cap N} \varsigma_1(x)$$

$$= |C_1|^{-1} \Psi_1(1) \varsigma_1^{\,G}(n)$$

Hence $\quad |C_2| \Psi_1(1) (\varsigma_1^{\,G})_N = |C_1| \Psi_2(1) (\varsigma_2^{\,G})_N \quad \ldots \quad (2)$

Evaluating the degrees of both sides

$$|C_2| \Psi_1(1)\, |G:N| = |C_1| \Psi_2(1)\, |G:N|$$

Thus by (2)

$$(\varsigma_1^{\,G})_N = (\varsigma_2^{\,G})_N \quad \ldots \quad (3)$$

Suppose, for a contradiction, that for all $h \in H$, $\varsigma_1 \neq {}^h\varsigma_2$. Because $G = NH$ and ς_i $(i=1,2)$ are characters of N, we have that $\varsigma_1 \neq {}^g\varsigma_2$ for all $g \in G$. These are irreducible characters so $(\varsigma_1, {}^g\varsigma_2) = 0$ for all $g \in G$. Now, by Mackey's formula, letting $\{y_i\}$ be a set of (N,N)-double coset representatives

$$(\varsigma_1^{\,G}, \varsigma_2^{\,G}) = \sum_{y \in \{y_i\}} ((\varsigma_1)_{N \cap N^y}, ({}^y\varsigma_2)_{N \cap N^y})$$

$$= \sum_{y \in \{y_i\}} (\varsigma_1, {}^y\varsigma_2)_N$$

$$= 0 \quad \text{by above}$$

So $\quad (\varsigma_1^{\,G}, \varsigma_1^{\,G}) = ((\varsigma_1^{\,G})_N, \varsigma_1^{\,G})$ by Frobenius

$$= ((\varsigma_2^{\,G})_N, \varsigma_1^{\,G}) \quad \text{by (3)}$$

$$= (\varsigma_2^{\,G}, \varsigma_1^{\,G}) \quad \text{by Frobenius}$$

$$= 0 \quad \text{by above}$$

which is a contradiction, since $\varsigma_1^{\,G}$ is a character of G.

Thus there exists an $h \in H$ such that $\varsigma_1 = {}^h\varsigma_2$.
So $C_1 = C_H(\varsigma_1) = C_H({}^h\varsigma_2) = C_H(\varsigma_2)^h = C_2{}^h$.

Therefore, $|C_1| = |C_2|$ so by (2) and (3), $\Psi_1(1) = \Psi_2(1)$.

It will be sufficient to prove the result for $h = 1$
i.e. that $\varsigma_1 = \varsigma_2$ implies $\Psi_1 = \Psi_2$. For,

$${}^h\phi_2 = {}^h\varsigma_2{}^h\Psi_2 = \varsigma_1{}^h\Psi_2 . \quad \text{Also } ({}^h\phi_2)^G = \phi_2{}^G = \phi_1{}^G \text{ by}$$
assumption.

But $\phi_1 = \varsigma_1\Psi_1$, so by the result for $h=1$, we have
$\Psi_1 = {}^h\Psi_2$ as required.

Therefore we let $\varsigma = \varsigma_1 = \varsigma_2$, $C = C_1 = C_2$, $T=NC$.
Suppose that $\Psi_1 \neq \Psi_2$ and $\Psi_1, \Psi_2, \ldots, \Psi_k$ are all the distinct irreducible characters of C and let $\phi_i = \varsigma_i\Psi_i$ $(1 \leqslant i \leqslant k)$ - irreducible characters of T.

$$\begin{aligned}\text{Now, } (\varsigma^T, \phi_i) &= (\varsigma, (\phi_i)_N) \quad \text{by Frobenius} \\ &= (\varsigma, \Psi_i(1)\varsigma) \\ &= \Psi_i(1) = \phi_i(1)\end{aligned}$$

Thus $\varsigma^T = a_1\phi_1 + \ldots + a_k\phi_k + \lambda$, where λ is a character of T such that $(\lambda, \phi_i) = 0$ for $i \in \{1, \ldots, k\}$ and $a_i = \Psi_i(1) = \phi_i(1) \neq 0$.

Now because $\{\Psi_i\}_{i=1}^k$ is a complete set of irreducible characters of C,

$$\begin{aligned}|C| + \lambda(1) &= a_1{}^2 + \ldots + a_k{}^2 + \lambda(1) \\ &= a_1\phi_1(1) + \ldots + a_k\phi_k(1) + \lambda(1) \\ &= \varsigma^T(1) \\ &= |T:N| = |C|\end{aligned}$$

Hence $\lambda(1) = 0$ so $\lambda = 0$. Therefore $\varsigma^T = \sum_{i=1}^{k} a_i\phi_i$

and it follows by the transitivity of induction that

$$\varsigma^G = \sum_{i=1}^{k} a_i\phi_i^G \qquad \ldots \quad (4)$$

We now compute $(\varsigma^G, \varsigma^G)$.

Let $|G:T| = t$ and the set $G/T = \{g_1T, \ldots, g_tT\}$

Hence, if $n \in N$

$$\varsigma^G(n) = \frac{1}{|N|} \sum_{g\in G} ({}^g\varsigma)(n)$$

$$= \frac{|T|}{|N|} \sum_{i=1}^{t} ({}^{g_i}\varsigma)(n) \quad \text{as } T = C_G(\varsigma)$$

Thus $(\varsigma^G)_N = |T:N| \sum_{i=1}^{t} {}^{g_i}\varsigma$.

Therefore,

$$(\varsigma^G, \varsigma^G) = ((\varsigma^G)_N, \varsigma) \quad \text{by Frobenius}$$

$$= |T:N| \sum_{i=1}^{t} ({}^{g_i}\varsigma, \varsigma)$$

$$= |T:N|$$

because $({}^{g_i}\varsigma, \varsigma) \neq 0 \iff {}^{g_i}\varsigma = \varsigma \iff g_i \in C_G(\varsigma) = T$.

Our contradiction now follows, since by (4) and above

$$|T:N| = (\varsigma^G, \varsigma^G) = \left(\sum_{i=1}^{k} a_i\phi_i^G, \sum_{i=1}^{k} a_i\phi_i^G\right)$$

$$\geqslant (a_1+a_2)^2 + a_3^2 + \ldots + a_k^2$$

$$> a_1^2 + \ldots + a_k^2$$

$= |C| = |T:N|$, a transparent impossibility.

Thus we have shown $\Psi_1 = \Psi_2$, completing this part of the theorem.

Now suppose $\varsigma_1 = {}^h\varsigma_2$ and $\Psi_1 = {}^h\Psi_2$ for some $h \in H$.

Then $\phi_1 = {}^h\varsigma_2 . {}^h\Psi_2 = {}^h\phi_2$ so $\phi_1^G = \phi_2^G$, completing (b).

(c) We use a combinatorial argument to show that all the irreducible characters of G may be obtained in the manner described.

By (b), ϕ^G detemines, up to conjugation by an element of H, an irreducible character ς of N, which in turn determines integers m,n such that m+n=1, and $C = C_H(\varsigma) \cong S_m \times S_n$. But if ϕ^G also gives ${}^h\varsigma$ then $C_H({}^h\varsigma) = C_H(\varsigma)^h \cong C_H(\varsigma)$ so gives rise to the same integers. Thus ϕ^G determines uniquely integers m,n such that m+n=1 .

Also, ϕ^G determines, up to conjugation by an element of H, an irreducible character ψ of $S_m \times S_n$, which is therefore a product of two irreducible characters χ^λ, χ^μ of S_m, S_n respectively, where $\lambda \vdash m, \mu \vdash n$. Because ${}^h\psi$ determines the same partitions λ, μ we see that ϕ^G determines, in a unique way, a pair of partitions $(\lambda;\mu)$ of 1 i.e. given $(\lambda;\mu)$ we can construct, uniquely, ϕ^G.

However, the number of irreducible characters of G is equal to the number of conjugacy classes of G, which by p 42 is the number of pairs of partitions of 1. Thus we have all the irreducible characters of G.

<u>Notation</u>

In 3.1.2(c), we showed how to associate with a given ϕ^G a unique pair of partitions $(\lambda;\mu)$. We therefore write ϕ^G as $\chi^{(\lambda;\mu)}$.

Hence the irreducible characters of G are also parameterized by pairs of partitions of 1.

We shall always use the notation of 3.1.2.

We note here, for reference, a technical lemma

Lemma 3.1.3

(i) $\quad (\phi^G)_N = |C|^{-1}\psi(1)(\varsigma^G)_N$

(ii) $\quad (\phi^G)_H = \psi^H$

Proof

(i) was proved in 3.1.2

(ii) if $h \in H$ let $C^H(h)$ be the conjugacy class in H containing h, and $C^G(h)$ the conjugacy class in G containing h. Fix $h \in H$, then

$$\phi^G(h) = \frac{|G:NC|}{|C^G(h)|} \sum_{x \in C^G(h) \cap NC} \phi(x)$$

Suppose $x = ghg^{-1} \in NC$ where $g = nh_1$, $n \in N$, $h_1 \in H$.

Then $x = nh_1hh_1^{-1}n^{-1} = n(h_1hh_1^{-1}n^{-1}h_1h^{-1}h_1^{-1})h_1hh_1^{-1}$

Since $x \in NC$ and $N \triangleleft G$ we have that $h_1hh_1^{-1} \in C = C_H(\varsigma)$, so $h_1hh_1^{-1}$ centralizes ς.

Now $\phi(x) = \varsigma\left[n(h_1hh_1^{-1}n^{-1}h_1h^{-1}h_1^{-1})\right]\psi(h_1hh_1^{-1})$ by definition

$= \varsigma(n)^{h_1hh_1^{-1}}\varsigma(n^{-1})\psi(h_1hh_1^{-1})$ since ς is linear

$= \varsigma(n)\varsigma(n^{-1})\psi(h_1hh_1^{-1})$ since $h_1hh_1^{-1} \in C$

$= \psi(h_1hh_1^{-1})$ again since ς is linear

Thus

$$\phi^G(h) = \frac{|G:NC|}{|C^G(h)|} \sum_{nh_1hh_1^{-1}n^{-1} \in NC} \psi(h_1hh_1^{-1})$$

$$= \frac{|H:C|}{|C^G(h)|} \cdot \frac{|C^G(h)|}{|C^H(h)|} \sum_{h_1hh_1^{-1} \in C} \psi(h_1hh_1^{-1})$$

$= \psi^H(h)$ proving the lemma.

We conclude with the following well-known result

Theorem 3.1.4

Any complex representation of G may be afforded by a basis with respect to which the matrix entries consist of rational integers. In particular, the characters of G are rational integral-valued.

Proof

From 3.1.2, we see that the irreducible representations of G may be obtained from those of the symmetric group by

(i) tensoring these representations together and with representations which only take the values ± 1;

(ii) inducing up the representations obtained in (i).

The theorem then follows from 2.1.4, since the operations in (i), (ii) clearly preserve the required properties.

§3.2 Two linear characters of $W(C_l)$

G has four linear characters. Let $w_i = (i\ i{+}1)$, $1 \leqslant i \leqslant l-1$, and $w_l = (l,-l)$. Then G is generated by $\{w_1, \dots, w_l\}$ subject only to the defining relations ([3] p 279)

$$(w_1w_2)^3 = (w_2w_3)^3 = \dots = (w_{l-2}w_{l-1})^3 = (w_{l-1}w_l)^4 =$$

It follows that G can only have the following linear characters: $(1 \leqslant i \leqslant l-1)$

(a) the principal character 1 where $1(w_i)=1$, $1(w_l)=1$

(b) the sign character ε where $\varepsilon(w_i)=-1$, $\varepsilon(w_l)=-1$

(c) the long sign character ξ where $\xi(w_i)=1$, $\xi(w_l)=-1$

(d) the short sign character η where $\eta(w_i)=-1$, $\eta(w_l)=1$

The last two names were chosen because w_1 corresponds to the long root in the Dynkin diagram for $W(C_1)$.
Thus

(a) $1(g) = 1$ for all $g \in G$

(b) ε(permutation) = sign of permutation, ε(sign change) = -1

(c) ξ(permutation) = 1, ξ(sign change) = -1

(d) η(permutation) = sign of permutation, η(sign change) = 1

Lemma 3.2.1

(i) $\varepsilon \xi = \eta$ so $\varepsilon . \xi_W{}^G = \eta_W{}^G$ and $\xi . \varepsilon_W{}^G = \eta_W{}^G$

(ii) $\varepsilon . \chi^{(\lambda;\mu)} = \chi^{(\mu';\lambda')}$

(iii) $\xi . \chi^{(\lambda;\mu)} = \chi^{(\mu;\lambda)}$

(iv) $\eta . \chi^{(\lambda;\mu)} = \chi^{(\lambda';\mu')}$

Proof

(i) $\varepsilon\xi = \eta$ trivially. The rest follows from 1.2.4

(ii) Let $\chi^{(\lambda;\mu)} = \phi^G$, so $\varepsilon\chi^{(\lambda;\mu)} = \varepsilon\phi^G$

$$= (\varepsilon_{NC}\phi)^G \quad \text{by } 1.2.4$$

Now $\varepsilon_{NC} . \phi = (\varepsilon_N \varsigma) . (\varepsilon_C \psi)$. Because ε_N takes the value -1 on sign changes, it interchanges the symbols of the first and second type so that $C_H(\varepsilon_N \varsigma) \cong S_n \times S_m$.
$\psi = \chi^\lambda . \chi^\mu$ by definition so

$$\varepsilon_C \psi = \varepsilon_{S_m} \chi^\lambda . \varepsilon_{S_n} \chi^\mu = \chi^{\lambda'} . \chi^{\mu'} \text{ (by 2.3.5)} = \chi^{\mu'} . \chi^{\lambda'}$$

Thus $\varepsilon . \chi^{(\lambda;\mu)} = (\varepsilon_{NC}\phi)^G = \chi^{(\mu';\lambda')}$

(iii) $\xi . \chi^{(\lambda;\mu)} = (\xi_{NC}\phi)^G$ and

$\xi_{NC}\phi = (\xi_N \varsigma) . (\xi_C \psi) = (\xi_N \varsigma) . \psi$ by definition of ξ.
Because ξ_N takes the value -1 on sign changes it interchanges the symbols of the first and second type.

Also $\psi = \chi^{\lambda}_{\bullet} \chi^{\mu}$, so $\xi \chi^{(\lambda;\mu)} = \chi^{(\mu;\lambda)}$

(iv) follows from the first three parts.

The two linear characters we shall be interested in are ξ and η rather than 1 and $\mathcal{E}$ as in the symmetric group. However, the previous lemma shows that the distinction is more notational than anything else, as we shall point out when we have proved, for G, a result corresponding to that of 2.2.7 for S_{l+1}.

Remark

We shall only be interested in the Weyl subgroups of G which are Weyl groups of regular root systems (i.e. root systems which are additively closed). This is because, in $W(C_l)$, any Coxeter element of a Weyl subgroup is conjugate to a Coxeter element of one of these regular Weyl subgroups (see [5]), and so, for our purposes, may be ignored.

Thus in the rest of this chapter Weyl subgroups will always be assumed to be regular.

The Weyl subgroups of G are of the form

$$S_{\lambda_1} \times \dots \times S_{\lambda_r} \times W(C_{\mu_1}) \times \dots \times W(C_{\mu_s})$$

where $\sum \lambda_i + \sum \mu_i = l$.

We shall write this subgroup as $W_{(\lambda;\mu)}$ putting $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ and we may assume that $\lambda_1 \geqslant \dots \geqslant \lambda_r > 0$, $\mu_1 \geqslant \dots \geqslant \mu_s > 0$. Thus the Weyl subgroups of G may be parameterized by pairs of partitions of l.

Let $D_{(\lambda;\mu)}$ be a pair of diagrams for λ and μ obtained by filling the frames associated with λ and μ

with the symbols $\{1,\ldots,1,-1,\ldots,-1\}$ (in any order) such that the moduli of all the numbers appearing are distinct. We often write $D_{(\lambda;\mu)} = D_\lambda \cup D_\mu$.

Definition

A row permutation of a diagram $D_{(\lambda;\mu)}$ is an elenent p of G such that p permutes the symbols in each row of D_λ and in each row of D_μ and changes the sign of the symbols in D_μ.

The row stabilizer $R(D_{(\lambda;\mu)})$ is the group of row permutations of $D_{(\lambda;\mu)}$.

$$\text{Now } R(D_{(\lambda;\mu)}) \cong S_{\lambda_1} \times \ldots \times S_{\lambda_r} \times W(C_{\mu_1}) \times \ldots \times W(C_{\mu_s}) = W_{(\lambda;\mu)}$$

Thus all the Weyl subgroups of G can be considered as the row stabilizer of some diagram $D_{(\lambda;\mu)}$. As in the symmetric group, G acts on a diagram $D_{(\lambda;\mu)}$ by defining $gD_{(\lambda;\mu)}$ for $g \in G$, to be the diagram obtained by applying g to the symbols in $D_{(\lambda;\mu)}$.

It is then easy to see that $R(gD_{(\lambda;\mu)}) = gR(D_{(\lambda;\mu)})g^{-1}$ so that any two isomorphic Weyl subgroups are conjugate via the element of G that transforms one associated diagram into the other.

Comparing 2.3.5 with 3.2.1(ii), it is natural to make the following definition

Definition

$(\mu';\lambda')$ is called the dual of $(\lambda;\mu)$. Similarly define the dual of a frame, diagram or Weyl subgroup.

The reason for considering the characters ξ, $\bar{\eta}$ is contained in the next few results.

We let $(\lambda;\mu)$ be a pair of partitions of l such that $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ and $|\lambda| = m$, $|\mu| = n$.

Theorem 3.2.2

$$(\xi_{W_{(\lambda;\mu)}}{}^{G}, \chi^{(\lambda;\mu)}) = 1$$

Proof

Let $W = W_{(\lambda;\mu)}$ and adopt the notation of §3.1.

Thus

$$(\xi_W{}^G, \chi^{(\lambda;\mu)}) = (\xi_W{}^G, \phi^G)$$

$$= \sum_{y \in \{y_i\}} (\xi_{W \cap (NC)^y}, ({}^y\phi)_{W \cap (NC)^y})$$

by Mackey's formula, where $\{y_i\}$ is a set of (W,NC)-double coset representatives.

Suppose $(\xi_{W \cap (NC)^y}, ({}^y\phi)_{W \cap (NC)^y}) \neq 0$, then because

$W \cap (NC)^y = W \cap NC^y \geqslant W \cap N$ as $N \triangleleft G$, we have

$(\xi_{W \cap N}, ({}^y\phi)_{W \cap N}) \neq 0$.

Now $N = N_1 \times N_2$ as in 3.1.2, and we choose $W_{(\lambda;\mu)}$ so that $D_{(\lambda;\mu)}$ is filled with the symbols $\{1, \dots, l\}$ where $\{1, \dots, m\}$ occur in D_λ and $\{m+1, \dots, l\}$ in D_μ. It is then immediate that $W \cap N = N_2$. Since $N_2 \leqslant N$, $({}^y\phi)_{N_2} = ({}^y\varsigma)_{N_2}\Psi(1)$. Thus $(\xi_{N_1}, ({}^y\varsigma)_{N_2}\Psi(1)) \neq 0$

so $(\xi_{N_2}, ({}^y\varsigma)_{N_2}) \neq 0$ and because the characters are linear $\xi_{N_2} = ({}^y\varsigma)_{N_2}$. But by construction of ξ, $\xi_{N_2} = \varsigma_{N_2}$. Thus $\varsigma_{N_2} = ({}^y\varsigma)_{N_2}$ and therefore $N_2{}^y = N_2$, by definition of ς. It follows that $\varsigma = {}^y\varsigma$ because ς

takes the value 1 on N_1. Therefore $y \in C_G(\varsigma) = NC$, and so $y = 1$.

Hence $(\xi_W{}^G, \chi^{(\lambda;\mu)}) = (\xi_{W \cap NC}, \phi_{W \cap NC})$.

But $W = S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_r} \times W(C_{\mu_1}) \times \dots \times W(C_{\mu_s})$

$= S_{\lambda_1} \times \dots \times S_{\lambda_r} \times N_{\mu_1} S_{\mu_1} \times \dots \times N_{\mu_s} S_{\mu_s}$

with the obvious notation

$= (N_{\mu_1} \times \dots \times N_{\mu_s})(S_{\lambda_1} \times \dots \times S_{\lambda_r} \times S_{\mu_1} \times \dots \times S_{\mu_s})$

since the direct factors commute

$\leqslant N(S_m \times S_n) = NC$

Thus $W \leqslant NC$ and $W \cap N = N_{\mu_1} \times \dots \times N_{\mu_s} = N_2$,

$W \cap H = S_{\lambda_1} \times \dots \times S_{\lambda_r} \times S_{\mu_1} \times \dots \times S_{\mu_s} = W_\lambda \times W_\mu$,

where W_λ and W_μ are the appropriate Weyl subgroups of S_m and S_n respectively.

So we see that $W = (W \cap N)(W \cap H)$. Hence

$$(\xi_W{}^G, \chi^{(\lambda;\mu)}) = (\xi_W, \phi_W)$$

$$= (\xi_{W \cap N}, \varsigma_{W \cap N})(\xi_{W \cap H}, \Psi_{W \cap H})$$

$$= (\xi_{N_2}, \varsigma_{N_2})(1_{W_\lambda \times W_\mu}, \Psi_{W_\lambda \times W_\mu})$$

$$= 1.(1_{W_\lambda}, (\chi^\lambda)_{W_\lambda})(1_{W_\mu}, (\chi^\mu)_{W_\mu})$$

since $\xi_{N_2} = \varsigma_{N_2}$

$$= ((1_{W_\lambda})^{S_m}, \chi^\lambda)((1_{W_\mu})^{S_n}, \chi^\mu)$$

$= 1$ by 2.2.7, completing the proof of the theorem.

<u>Corollary 3.2.3</u>

$$(\eta_{W_{(\mu';\lambda')}}{}^G, \chi^{(\lambda;\mu)}) = 1$$

Proof

$$\left(\eta_{W_{(\mu';\lambda')}}{}^{G}, \chi^{(\lambda;\mu)}\right) = \left(\varepsilon\xi_{W_{(\mu';\lambda')}}{}^{G}, \chi^{(\lambda;\mu)}\right) \quad \text{by } 3.2.1$$

$$= \left(\xi_{W_{(\mu';\lambda')}}{}^{G}, \varepsilon\chi^{(\lambda;\mu)}\right)$$

$$= \left(\xi_{W_{(\mu';\lambda')}}{}^{G}, \chi^{(\mu';\lambda')}\right) \quad \text{by } 3.2.1$$

$$= 1 \quad \text{by } 3.2.2$$

Theorem 3.2.4

$$\left(\xi_{W_{(\lambda;\mu)}}{}^{G}, \eta_{W_{(\mu';\lambda')}}{}^{G}\right) = 1$$

Proof

Write $W = W_{(\lambda;\mu)}$, $W' = W_{(\mu';\lambda')}$ and suppose D_λ is filled with the symbols $\{1,\ldots,m\}$ and D_μ with $\{m+1,\ldots,l\}$ where $|\lambda| = m$. Then by Mackey's formula

$$\left(\xi_W{}^G, \eta_{W'}{}^G\right) = \sum_{y\in\{y_i\}} \left(\xi_{W\cap W'^y}, ({}^y\eta)_{W\cap W'^y}\right)$$

where $\{y_i\}$ is a set of (W,W')-double coset representatives. Now $W \cap N = N_2$ and similarly $W' \cap N = N_1$ so $N = N_1 \times N_2 \leqslant WW'$. Therefore we may assume that, because $G = NH$, each $y_i \in H$.

Suppose $\left(\xi_{W\cap W'^y}, ({}^y\eta)_{W\cap W'^y}\right) \neq 0$, then since the characters are linear $\xi_{W\cap W'^y} = ({}^y\eta)_{W\cap W'^y}$, so by definition of ξ, η, $W \cap W'^y$ does not contain a transposition or a sign change.

W'^y is the row stabilizer of $yD_{(\mu';\lambda')} = yD_{\mu'} \cup yD_{\lambda'}$. We claim that $yD_{\lambda'}$ contains the same symbols as those in D_λ.

For,

suppose not; then there exists a symbol a such that a appears in $yD_{\lambda'}$ but not in D_λ. We write $a \in yD_{\lambda'}$, $a \notin D_\lambda$. Since $a \notin D_\lambda$, we have that $a \in D_\mu$ and hence $(a,-a) \in W$. Similarly $a \in yD_{\lambda'}$ implies $(a,-a) \in W'^y$ so $(a,-a) \in W \cap W'^y$, a contradiction. The fact that D_λ and $yD_{\lambda'}$ both contain m squares proves the claim.

Because $W \cap W'^y$ does not contain any transpositions, no two collinear symbols of D_λ are co-columnar in $yD_{\lambda'}$. Hence, by 2.1.2, $y|_{S_m} = pq$, where $p \in W_\lambda$, $q \in W_{\lambda'}$.

Similarly, $y|_{S_n} = p_1q_1$, where $p_1 \in W_\mu$, $q_1 \in W_{\mu'}$.

Hence $y = pqp_1q_1 = (pp_1)(qq_1)$ since the diagrams D_λ, D_μ are disjoint and therefore $W_\lambda \cap W_\mu = 1$

$$\begin{aligned} &= (pp_1)(q_1q) \\ &\in (W_\lambda \times W_\mu)(W_{\mu'} \times W_{\lambda'}) \\ &\leqslant WW' \end{aligned}$$

i.e. $y = 1$.

So $(\xi_W{}^G, \eta_{W'}{}^G) = (\xi_{W\cap W'}, \eta_{W\cap W'})$. But it is clear that $W \cap W' = R(D_{(\lambda;\mu)}) \cap R(D_{(\mu';\lambda')}) = 1$. Hence $(\xi_W{}^G, \eta_{W'}{}^G) = 1$ as required.

3.2.2, 3.2.3 and 3.2.4 together show that $\chi^{(\lambda;\mu)}$ is the unique common irreducible constituent of $\xi_{W_{(\lambda;\mu)}}{}^G$ and $\eta_{W_{(\mu';\lambda')}}{}^G$.

§3.3 An algorithm for $W(C_1)$

In this section we generalize 2.2.7 (and 2.3.6) to G, and in so doing define a partial ordering on the pairs of partitions of l. We first define a reflexive, anti-symmetric relation on the pairs of partitions of l, which will give us an algorithm for determining exactly which irreducible characters occur in $\xi_W^{\ G}$, for a given Weyl subgroup W of G.

Let $(\alpha;\beta)$ and $(\lambda;\mu)$ be pairs of partitions of l. By the usual abuse of notation we shall refer to the frames also as $(\alpha;\beta)$ and $(\lambda;\mu)$ respectively.

We write $(\lambda;\mu) \rightarrow (\alpha;\beta)$ (and in later chapters, where we introduce further algorithms, we shall write $\underset{C}{\rightarrow}$), if $(\alpha;\beta)$ may be obtained from $(\lambda;\mu)$ by

first (a) removing connected squares from the end of a row of λ and placing them, in the same order, at the bottom of μ ;

then (b) repeating (a) with squares from <u>different</u> rows of λ ;

then (c) reordering the resulting rows so as to give frames of a pair of partitions $(\gamma;\delta)$, say;

finally (d) moving up inside γ and δ, according to the usual partial ordering on partitions, so as to obtain α and β respectively (so $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$).

<u>Remark</u>

It is easy to see that $\rightarrow$ is reflexive and anti-

symmetric but is not transitive because e.g.
$(2,0) \to (1,1)$ and $(1,1) \to (0,1^2)$ but $(2,0) \not\to (0,1^2)$.
Later on we shall extend $\to$ to a partial ordering.

We can now state the first main result of this section

<u>Theorem 3.3.1</u>

Let $(\alpha;\beta)$ and $(\lambda;\mu)$ be pairs of partitions of l. Then, with the usual notation,

$$(\xi_{W_{(\lambda;\mu)}}{}^{G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\mu) \to (\alpha;\beta)$$

Before proving this we need a lemma

<u>Lemma 3.3.2</u>

Let $W = R(D_{(\lambda;\mu)})$. Then

(a) $W = (N \cap W)(H \cap W)$ and $(N \cap W) \cap (H \cap W) = 1$

If also $g \in H$ and $C = C_H(\varsigma)$ for some irreducible character ς of N

(b) $W^g = (N \cap W^g)(H \cap W^g)$ and $(N \cap W^g) \cap (H \cap W^g) = 1$

(c) $NC \cap W^g = (N \cap W^g)(C \cap W^g)$ and $(N \cap W^g) \cap (H \cap W^g) = 1$

<u>Proof</u>

(a) $W = S_{\lambda_1} \times \dots \times S_{\lambda_r} \times W(C_{\mu_1}) \times \dots \times W(C_{\mu_s})$

$= S_{\lambda_1} \times \dots \times S_{\lambda_r} \times N_{\mu_1} S_{\mu_1} \times \dots \times N_{\mu_s} S_{\mu_s}$, with the obvious notation

$= (N_{\mu_1} \times \dots \times N_{\mu_s})(S_{\lambda_1} \times \dots \times S_{\lambda_r} \times S_{\mu_1} \times \dots \times S_{\mu_s})$

$= (W \cap N)(W \cap H)$

(b) $W^g = (N \cap W)^g (H \cap W)^g$ by (a)

$\leq (N \cap W^g)(H \cap W^g)$ since $N \lhd G$, $g \in H$

$\leq W^g$ hence equality

(c) Let $x \in NC \cap W^g$ and by (b) $x = nc = n_1 h$ for some $n \in N$, $c \in C$, $n_1 \in N \cap W^g$, $h \in H \cap W^g$.

Hence $n_1^{-1}n = hc^{-1} \in N \cap H = 1$, so $n = n_1$ and $c = c_1$ and therefore $c \in C \cap H \cap W^g = C \cap W^g$ and $n \in N \cap W^g$. So $x = nc \in (N \cap W^g)(C \cap W^g)$ which implies $W^g \leqslant (N \cap W^g)(C \cap W^g) \leqslant W^g$ proving (c).

The trivial intersections all follow from the fact that $N \cap H = 1$.

<u>Proof of 3.3.1</u>

Suppose first that $(\xi_{W_{(\lambda;\mu)}}{}^G, \chi^{(\alpha;\beta)}) \neq 0$.
We use the notation of §3.1 and also let $W = W_{(\lambda;\mu)}$.
Hence

$$(\xi_W{}^G, \chi^{(\alpha;\beta)}) = (\xi_W{}^G, \phi^G) = \sum_{g \in \{g_1\}} ({}^g\xi_{NC \cap W^g}, \phi_{NC \cap W^g})$$

where $\{g_1\}$ is a set of (W,NC)-double coset representatives and since $G = NH$ we may suppose each $g_1 \in H$.

Thus there exists $g \in \{g_1\}$ such that $({}^g\xi_{NC \cap W^g}, \phi_{NC \cap W^g}) \neq 0$

We let $|\alpha| = m$, $|\beta| = n$ and let $N = N_1 \times N_2$ as in 3.1.2 so that $\varsigma(a,-a) = 1$ for $(a,-a) \in N_1$ and $\varsigma(a,-a) = -1$ for $(a,-a) \in N_2$. Now by 3.3.2(c)

$$0 \neq ({}^g\xi_{NC \cap W^g}, \phi_{NC \cap W^g}) = ({}^g\xi_{N \cap W^g}, \varsigma_{N \cap W^g})({}^g\xi_{C \cap W^g}, \Psi_{C \cap W^g})$$

Hence

$$({}^g\xi_{N \cap W^g}, \varsigma_{N \cap W^g}) \neq 0 \text{ and } ({}^g\xi_{C \cap W^g}, \Psi_{C \cap W^g}) \neq 0 \quad \ldots \text{(A)}$$

Since ξ, ς are linear, ${}^g\xi_{N \cap W^g} = \varsigma_{N \cap W^g}$. But ξ takes the value -1 on all sign changes in G and hence on those in $N \cap W^g$. Thus $N \cap W^g \leqslant N_2$.

Now W^g defines a diagram $D_{(\lambda;\mu)}$, so since $N \cap W^g \leqslant N_2$, all the symbols in D_μ are of the second type. We may also assume that the symbols of the second type in D_λ lie at the ends of the rows, since W^g only defines $D_{(\lambda;\mu)}$ up to row permutations (and sign changes in D_μ). Hence we may remove squares from D_λ and put them on the bottom of D_μ (so that moved squares in the same row remain in the same row) and then reorder the rows to obtain a diagram $D_{(\gamma;\delta)}$ of a pair of partitions $(\gamma;\delta)$ of 1 such that D_γ contains all the symbols of the first type and D_δ the symbols of the second type. This corresponds to the operations (a), (b) and (c) on p 59. So to show $(\lambda;\mu) \longrightarrow (\alpha;\beta)$ we only have to show $\gamma \leqslant \alpha$, $\delta \leqslant \beta$.

By construction $|\gamma| = m = |\alpha|$, $|\delta| = n = |\beta|$

By (A) $({}^g\xi_{C\cap W^g} , \Psi_{C\cap W^g}) \neq 0$. However, ξ takes the value 1 on the elements of H, hence $(1_{C\cap W^g} , \Psi_{C\cap W^g}) \neq 0$.

Now $C \cap W^g \cong (S_m \cap W^g) \times (S_n \cap W^g)$ since $C \cong S_m \times S_n$ and so $C \cap W^g$ permutes the symbols of each type independently, and therefore these actions commute. Hence, by definition of Ψ ,

$$(1_{S_m\cap W^g} , \chi^\alpha_{S_m\cap W^g})(1_{S_n\cap W^g} , \chi^\beta_{S_n\cap W^g}) \neq 0 .$$

But $S_m \cap W^g$ is the group of row permutations of the symbols of the first type in $D_{(\lambda;\mu)}$ and thus the group of row permutations of D_γ . Therefore $S_m \cap W^g = R(D_\gamma) = W_\gamma$ - a Weyl subgroup of S_m . Similarly, $S_n \cap W^g = R(D_\delta) = W_\delta$

Hence,

$$(1_{W_\gamma} , \chi^\alpha_{W_\gamma})(1_{W_\delta} , \chi^\beta_{W_\delta}) \neq 0$$

and by Frobenius $(1_{W_\gamma}^{S_m} , \chi^\alpha) \neq 0$ and $(1_{W_\delta}^{S_n} , \chi^\beta) \neq 0$

from which it follows by 2.3.6 that $\gamma \leqslant \alpha$ and

Thus by the above remarks $(\lambda;\mu) \to (\alpha;\beta)$.

Conversely, suppose $(\lambda;\mu) \to (\alpha;\beta)$. Therefore we may move parts of rows of λ across to μ to obtain a pair of partitions $(\gamma;\delta)$ of l such that $\gamma \leqslant \alpha$, $\delta \leqslant \beta$. Hence $|\lambda| \geqslant |\gamma| = |\alpha| = m$ and $|\mu| \leqslant |\delta| = |\beta| = n$.
So define $D_{(\lambda;\mu)}$ to be a diagram of $(\lambda;\mu)$ filled with the symbols $\{1,\ldots,l\}$ such that $\{1,\ldots,m\}$ all occur in D_λ .
Let $W = W_{(\lambda;\mu)} = R(D_{(\lambda;\mu)})$. Then
$N \cap W = N_{\mu_1} \times \ldots \times N_{\mu_s} \leqslant N_2$ by construction. Hence $\xi_{N\cap W} = \varsigma_{N\cap W}$ and therefore $(\xi_{N\cap W} , \varsigma_{N\cap W}) \neq 0$. Also, by 2.3.6,

$$\gamma \leqslant \alpha \quad \Rightarrow ((1_{W_\gamma})^{S_m} , \chi^\alpha) \neq 0$$

$$\delta \leqslant \beta \quad \Rightarrow ((1_{W_\delta})^{S_n} , \chi^\beta) \neq 0$$

So $(\xi_{N\cap W} , \varsigma_{N\cap W})(1_{W_\gamma}^{S_m} , \chi^\alpha)(1_{W_\delta}^{S_n} , \chi^\beta) \neq 0$ and this is, by the proof of the first part of the theorem, the first summand in the Mackey formula for $(\xi_W^{\ G} , \chi^{(\alpha;\beta)})$.
Hence

$(\xi_W^{\ G} , \chi^{(\alpha;\beta)}) \neq 0$ proving the theorem.

We now wish to extend $\to$ to a partial ordering on the pairs of partitions of l.

The reason why $\to$ is not transitive is that we are not allowed to split up a row when moving it across so that e.g. $(2,0) \not\to (0,1^2)$. This gives us a hint as to how to define a partial ordering.

<u>Definition</u>

Let $(\alpha;\beta)$, $(\lambda;\mu)$ be pairs of partitions of l.
Then $(\lambda;\mu) \leqslant (\alpha;\beta)$ if we may obtain $(\alpha;\beta)$ from $(\lambda;\mu)$ by

(a) removing a square from the end of a row of λ and putting it at the bottom of μ ;

(b) repeating (a) as many times as is necessary to obtain a pair of partitions $(\gamma;\delta)$ of 1;

(c) moving up inside γ and δ to obtain α and β respectively (so that $\gamma \leq \alpha$, $\delta \leq \beta$).

It is clear that $(\lambda;\mu) \rightarrow (\alpha;\beta) \Rightarrow (\lambda;\mu) \leq (\alpha;\beta)$ and that $(\lambda;\mu) \leq (\alpha;\beta)$ if and only if there exist pairs of partitions $(\rho_i;\sigma_i)$ of 1 such that

$(\lambda;\mu) \rightarrow (\rho_1;\sigma_1) \rightarrow (\rho_2;\sigma_2) \rightarrow \dots \rightarrow (\rho_n;\sigma_n) \rightarrow (\alpha;\beta)$

($(\rho_i;\sigma_i)$ is obtained from $(\rho_{i-1};\sigma_{i-1})$ by moving across one square at a time and letting $(\rho_n;\sigma_n) = (\gamma;\delta)$).

<u>Lemma 3.3.3</u>

$\leq$ is a partial ordering

<u>Proof</u>

This is clear

<u>Lemma 3.3.4</u> (Duality Relation for $\leq$)

$$(\lambda;\mu) \leq (\alpha;\beta) \iff (\beta';\alpha') \leq (\mu';\lambda')$$

<u>Proof</u>

It will be sufficient to prove the implication in one direction. We may also suppose $(\alpha;\beta)$ is obtained from $(\lambda;\mu)$ by moving only one square from λ to μ . For, we may write

$(\lambda;\mu) \leq (\rho_1;\sigma_1) \leq \dots \leq (\rho_n;\sigma_n) \leq (\alpha;\beta)$

where each term is obtained from the previous one by moving one square across except that $\rho_n \leq \alpha$ and $\sigma_n \leq \beta$.

By assumption,

$(\sigma_n' ; \rho_n') \leqslant (\sigma_{n-1}' ; \rho_{n-1}') \leqslant \dots \leqslant (\sigma_1' ; \rho_1') \leqslant (\mu' ; \lambda')$

Now by 2.3.2 , $\alpha' \leqslant \rho_n'$ and $\beta' \leqslant \sigma_n'$

so $(\beta' ; \alpha') \leqslant (\sigma_n' ; \rho_n') \leqslant (\mu' ; \lambda')$ i.e. $(\beta' ; \alpha') \leqslant (\mu' ; \lambda')$

So suppose we have moved one square from λ to μ to obtain $(\alpha ; \beta)$. Hence we may move one square from β to α to obtain $(\mu ; \lambda)$ from $(\beta ; \alpha)$. Therefore, we may move one square from β' to α' to obtain $(\mu' ; \lambda')$ from $(\beta' ; \alpha')$ i.e. $(\beta' ; \alpha') \leqslant (\mu' ; \lambda')$ as required.

This enables us to prove the same result for $\rightarrow$

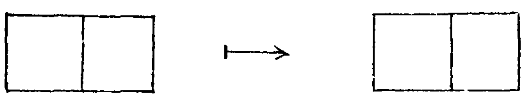
<u>Lemma 3.3.5</u> (Duality Relation for $\rightarrow$)

$$(\lambda ; \mu) \rightarrow (\alpha ; \beta) \iff (\beta' ; \alpha') \rightarrow (\mu' ; \lambda')$$

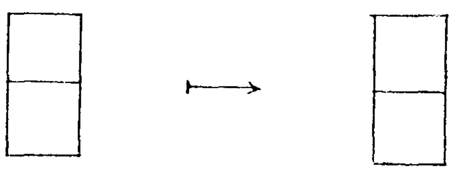
<u>Proof</u>

Suppose $(\lambda ; \mu) \rightarrow (\alpha ; \beta)$ then $(\lambda ; \mu) \leqslant (\alpha ; \beta)$ so $(\beta' ; \alpha') \leqslant (\mu' ; \lambda')$ by 3.3.4.

We must show $(\beta' ; \alpha') \rightarrow (\mu' ; \lambda')$, so by definition of the operations defined by $\leqslant$ and $\rightarrow$ it will be enough to show that when we move rows from β' to α' we do not split these rows up. It will be easier to prove this diagramatically. We must show that



in moving across from β' to α' . But since $(\lambda ; \mu) \rightarrow (\alpha ; \beta)$ we have that



in moving across from λ to μ , and in doing the reverse operation to obtain $(\mu' ; \lambda')$ from $(\beta' ; \alpha')$ we see that the first diagram must indeed be the case.

Theorem 3.3.6

$$\xi_{W_{(\lambda;\mu)}}^{\;G} = \chi^{(\lambda;\mu)} + \sum_{(\lambda;\mu)<(\alpha;\beta)} a_{(\alpha;\beta)}\, \chi^{(\alpha;\beta)}$$

$$\eta_{W_{(\mu';\lambda')}}^{\;G} = \chi^{(\lambda;\mu)} + \sum_{(\lambda;\mu)>(\alpha;\beta)} b_{(\alpha;\beta)}\, \chi^{(\alpha;\beta)}$$

where $a_{(\alpha;\beta)}$ and $b_{(\alpha;\beta)}$ are non-negative integers

Proof

The first equation follows from 3.2.2 and 3.3.1. The second equation comes from the first by multiplying it by ε and using 3.2.1 and 3.3.4 (after relabelling).

Remark

If in 3.3.6 we replace $\leqslant$ by $\rightarrow$, using 3.3.5, we will then have non-zero coefficients by 3.3.7 below.

As promised in §3.2, we shall show that, by a change of notation, we could use the linear characters $1, \varepsilon$ instead of ξ, η.

Theorem 3.3.7

$$(\xi_{W_{(\lambda;\mu)}}^{\;G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\mu) \rightarrow (\alpha;\beta)$$

$$(\eta_{W_{(\mu';\lambda')}}^{\;G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\beta) \rightarrow (\alpha;\beta)$$

Proof

The first part is 3.3.1. The second follows from the first by 3.2.1 from which we obtain

$$(\eta_{W_{(\mu';\lambda')}}{}^{G}, \chi^{(\alpha;\beta)}) \iff (\mu';\lambda') \rightarrow (\beta';\alpha')$$

$$\iff (\alpha;\beta) \rightarrow (\lambda;\mu) \quad \text{by } 3.3.5$$

So by multiplying the results in 3.3.7 by ξ and using 3.2.1 we have

<u>Theorem 3.3.8</u>

$$(1_{W_{(\lambda;\mu)}}{}^{G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\mu) \rightarrow (\beta;\alpha)$$

$$(\varepsilon_{W_{(\mu';\lambda')}}{}^{G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\beta;\alpha) \rightarrow (\lambda;\mu)$$

Similarly, using 3.3.6 we obtain

<u>Theorem 3.3.9</u>

$$1_{W_{(\lambda;\mu)}}{}^{G} = \chi^{(\mu;\lambda)} + \sum_{(\lambda;\mu)<(\beta;\alpha)} a_{(\beta;\alpha)} \chi^{(\beta;\alpha)}$$

$$\varepsilon_{W_{(\mu';\lambda')}}{}^{G} = \chi^{(\mu;\lambda)} + \sum_{(\lambda;\mu)>(\beta;\alpha)} b_{(\beta;\alpha)} \chi^{(\beta;\alpha)}$$

where $a_{(\beta;\alpha)}$ and $b_{(\beta;\alpha)}$ are non-negative integers.

So we may replace ξ, η by 1, ε if we write $\chi^{(\beta;\alpha)}$ instead of $\chi^{(\alpha;\beta)}$ as defined in §3.1.

We now define a bijection between the Weyl subgroups and irreducible characters of G.

Define a map

X : set of Weyl subgroups $\longrightarrow$ set of irreducible characters

by

$$X(W_{(\lambda;\mu)}) = \left\{ \begin{array}{l} \chi \text{ irred. character} : (\chi, \xi_{W_{(\lambda;\mu)}}{}^{G}) \neq 0 \text{ and} \\ (\chi, \xi_{W'}{}^{G}) = 0 \text{ for all Weyl subgroups} \end{array} \right\}$$

Theorem 3.3.10

$X(W_{(\lambda;\mu)}) = \{\chi^{(\lambda;\mu)}\}$ for all pairs of partitions $(\lambda;\mu)$ of 1.

Proof

This follows from 3.3.6 with the same proof as in 2.2.8.

§3.4 Decomposition of the group algebra into minimal left ideals

This section is a generalization to G of some of the results in §2.1, especially 2.1.3 and 2.1.5.

Let $A = \mathbb{C}G$ - the complex group algebra of G. Let $(\lambda;\mu)$ be a pair of partitions of 1 and $W_{(\lambda;\mu)}$ a Weyl subgroup of G. We define two essential idempotents of G

$$p_{(\lambda;\mu)} = \sum_{w \in W_{(\lambda;\mu)}} w\xi(w)$$

$$q_{(\lambda;\mu)} = \sum_{w \in W_{(\mu';\lambda')}} w\eta(w)$$

and let $e_{(\lambda;\mu)} = p_{(\lambda;\mu)} q_{(\lambda;\mu)}$

(note that $W_{(\lambda;\mu)} \cap W_{(\mu';\lambda')} = 1$)

Then $Ap_{(\lambda;\mu)}$ affords the character $\xi_{W_{(\lambda;\mu)}}^{G}$ of G, and $Aq_{(\lambda;\mu)}$ affords $\eta_{W_{(\mu';\lambda')}}^{G}$.

Theorem 3.4.1

$Ae_{(\lambda;\mu)}$ is a minimal left ideal of A affording $\chi^{(\lambda;\mu)}$

Proof

Let $e = e_{(\lambda;\mu)}$. To show that e is a multiple of a primitive idempotent we may follow the proof in the symmetric group for e_λ ([6] 28.15); this is purely routine.

Alternatively, we may use the first two lemmas in [4] and 3.3.6, from which the result immediately follows.

Hence Ae is a minimal left ideal and is isomorphic (using the *-map) to a submodule of both $Ap_{(\lambda;\mu)}$ and $Aq_{(\lambda;\mu)}$. Hence Ae affords an irreducible character which is a component of both $\xi_{W_{(\lambda;\mu)}}^{G}$ and $\eta_{W_{(\mu';\lambda')}}^{G}$, so by §3.2 affords $\chi^{(\lambda;\mu)}$.

Because $\chi^{(\lambda;\mu)} = \chi^{(\alpha;\beta)}$ implies $\lambda = \alpha$ and $\mu = \beta$ we see that ideals of the form $Ae_{(\lambda;\mu)}$ coming from different diagrams with the same frame are isomorphic but ideals from diagrams with different frames are not; so the ideals $\{Ae_{(\lambda;\mu)}\}$ where $(\lambda;\mu)$ ranges over all pairs of partitions of l, gives a full set of non-isomorphic irreducible A-modules.

Frame [8] has already introduced standard tableaux for G and given the formula for the number of standard tableaux of a given frame.

Definition

A standard tableau is a diagram $D_{(\lambda;\mu)}$ filled with the symbols $\{1,\ldots,l\}$ such that both D_λ and D_μ are standard tableaux for the appropriate symmetric groups.

Let H_λ be the hook product of a frame of λ. Define the hook product of $(\lambda;\mu)$ to be $H_{(\lambda;\mu)} = H_\lambda H_\mu$.

Lemma 3.4.2

The number of standard tableaux of the frame associated with $(\lambda;\mu)$ is given by $\dfrac{l!}{H_{(\lambda;\mu)}}$

Proof

Let $|\lambda| = m$, $|\mu| = n$ and let $D_{(\lambda;\mu)}$ be a standard tableau. Then there are $\binom{l}{m}$ ways of assigning the symbols $\{1,\ldots,l\}$ to each half of $D_{(\lambda;\mu)}$. Now by 2.1.6 there are $\dfrac{m!}{H_\lambda}$ ways of ordering the symbols in D_λ to give a standard tableau and similarly $\dfrac{n!}{H_\mu}$ ways of obtaining a standard tableau D_μ. Hence the number of standard tableau corresponding to $(\lambda;\mu)$ is

$$\binom{l}{m}\frac{m!}{H_\lambda}\frac{n!}{H_\mu} = \frac{l!}{H_{(\lambda;\mu)}}$$

Lemma 3.4.3

$$\chi^{(\lambda;\mu)}(1) = \frac{l!}{H_{(\lambda;\mu)}} = \text{number of standard tableaux}$$

Proof

With the usual notation $\chi^{(\lambda;\mu)} = \phi^G$ where $\phi = \varsigma\psi$ and $\psi = \chi^\lambda . \chi^\mu$. Let $|\lambda| = m$, $|\mu| = n$. Thus

$$\chi^{(\lambda;\mu)}(1) = |G:NC|\,\varsigma(1)\chi^\lambda(1)\chi^\mu(1)$$

$$= |H:C|\frac{m!}{H_\lambda}\frac{n!}{H_\mu} \quad \text{by 2.1.6}$$

$$= \frac{l!}{m!\,n!}\frac{m!}{H_\lambda}\frac{n!}{H_\mu} \qquad \text{since } C \cong S_m \times S_n$$

$$= \frac{l!}{H_{(\lambda;\mu)}}$$

A splits up into a number of simple rings A_i, $A = A_1 + A_2 + \dots + A_r$, where each A_i consists of a direct sum of isomorphic minimal left ideals of A, which are not isomorphic to any that occur in A_j, $j \neq i$.

The next theorem is proved in exactly the same way as that in the symmetric group ([1] IV,4.6) utilizing the previous two lemmas, and is routine so we shall not give the proof

Theorem 3.4.4

The minimal left ideals which arise from the standard tableaux belonging to a given frame are linearly independent and span a simple ring A_i. Thus A is the direct sum of the minimal left ideals which arise from the standard tableaux belonging to any frame associated with a pair of partitions of l.

§3.5 Solomon's decomposition of the group algebra of $W(C_l)$

As in §2.4 we interpret Solomon [17] for the Weyl group $W(C_l)$. Again we may assume that all modules, representations and characters are over the field of complex numbers.

The main feature distinguishing G from the symmetric group is that not all Weyl subgroups of G are conjugate to a parabolic subgroup. Indeed it is easy to see that the parabolic subgroups of G are the Weyl subgroups $W_{(\alpha;\beta)}$ such that β has only 1 or 0 parts (since $W_{(\alpha;\beta)}$ must include sign changes (a,-a) for every symbol a occurring in D_β).

The generating set I for G is the set of l-1 transpositions and one sign change, $\{(12),(23),\ldots,(l-1\ l),(l,-l)\}$ Let $J \subseteq I$, then the parabolic subgroup $W_J = W_{(\rho;\sigma)}$ for some pair of partitions $(\rho;\sigma)$ of l such that σ has only 1 or 0 parts. We therefore write $p(J) = (\rho;\sigma)$.

We fix an arbitrary subset J of I. Let $\hat{J}$ be the complement of J in I, and $p(J) = (\rho;\sigma)$, $p(\hat{J}) = (\beta';\alpha')$ (again, we use the dual for convenience only).

Define

$$\xi_J = \sum_{w \in W_J} w \quad , \quad \eta_{\hat{J}} = \sum_{w \in W_{\hat{J}}} \varepsilon(w) w$$

as in §2.4 (which should not be confused with the linear characters ξ, η of G). Then $A\xi_J\eta_{\hat{J}}$ affords the character

$$\Psi_J = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \, 1_{W_K}{}^G \quad \text{of } G, \text{ by } [17] \; .$$

Theorem 3.5.1

Let $(\lambda;\mu)$ be a pair of partitions of l. Then

$$(\Psi_J , \chi^{(\mu;\lambda)}) \neq 0 \quad \Rightarrow \quad (\rho;\sigma) \rightarrow (\lambda;\mu) \rightarrow (\alpha;\beta)$$

Proof

Since $A\xi_J$ affords $1_{W_J}{}^G = 1_{W_{(\rho;\sigma)}}{}^G$ and $A\eta_{\hat{J}}$ affords $\varepsilon_{W_{(\beta';\alpha')}}{}^G$ we have (1.2.8)

$$(\Psi_J , \chi^{(\mu;\lambda)}) \neq 0 \;\Rightarrow\; (1_{W_{(\rho;\sigma)}}{}^G , \chi^{(\mu;\lambda)}) \neq 0 \text{ and } (\varepsilon_{W_{(\beta';\alpha')}}{}^G , \chi^{(\mu;\lambda)}) \neq 0$$

$$\Rightarrow (\rho;\sigma) \rightarrow (\lambda;\mu) \rightarrow (\alpha;\beta) \text{ by } 3.3.8$$

Lemma 3.5.2

$$(\psi_J, \chi^{(\sigma;\rho)}) = (\psi_J, \chi^{(\beta;\alpha)}) = 1$$

Hence $(\rho;\sigma) \rightarrow (\alpha;\beta)$

Proof

Suppose $J \subsetneq K \subseteq I$ and let $p(K) = (\gamma;\delta)$, so that $(\gamma;\delta)$ is obtained from $(\rho;\sigma)$ by moving whole rows up inside ρ, and moving whole rows of ρ across to the end of σ. In particular, $(\rho;\sigma) \underset{\neq}{\rightarrow} (\gamma;\delta)$ so $(\gamma;\delta) \not\rightarrow (\rho;\sigma)$ since $\rightarrow$ is anti-symmetric. Hence by 3.3.8

$(1_{W_{(\gamma;\delta)}}^{G}, \chi^{(\sigma;\rho)}) = 0$ i.e. $(1_{W_K}^{G}, \chi^{(\sigma;\rho)}) = 0.$

$$\text{Thus } (\psi_J, \chi^{(\sigma;\rho)}) = \sum_{J \subseteq K \subseteq I} (1_{W_K}^{G}, \chi^{(\sigma;\rho)})$$

$$= (1_{W_J}^{G}, \chi^{(\sigma;\rho)})$$

$$= 1 \text{ by } 3.3.9$$

Similarly, $(\psi_{\hat{J}}, \chi^{(\alpha';\beta')}) = 1$ since $p(\hat{J}) = (\beta';\alpha')$.

Now by [17] lemma 7, $\varepsilon\psi_J = \psi_{\hat{J}}$. Hence

$$(\psi_J, \chi^{(\beta;\alpha)}) = (\varepsilon\psi_J, \varepsilon\chi^{(\beta;\alpha)}) = (\psi_{\hat{J}}, \chi^{(\alpha';\beta')}) \text{ by } 3.2.1$$

$$= 1$$

It follows from 3.5.1 that $(\rho;\sigma) \rightarrow (\alpha;\beta)$.

We now identify the irreducible module $\Lambda^p V$ defined in [17] , and in this case $V = \mathbb{R}^l$ ([3], table III).

Suppose $|\hat{J}| = p$.

Definition

Let $(\lambda;\mu)$ be the pair of partitions of l given by

$(\lambda;\mu) = (1^p\ ;\ 1-p)$. We call $(\lambda;\mu)$ the <u>hook graph for J</u> and $\chi^{(\mu;\lambda)}$ the <u>hook character of J</u>.

Notice that the hook graph $(\lambda;\mu)$ depends only on the order of J and that $\chi^{(\mu;\lambda)}(1) = \binom{1}{p}$ by 3.4.3.

As in §2.4, let $r(\nu)$ = the number of rows of (the frame of) a partition ν.

<u>Lemma 3.5.3</u>

(i) $r(\rho) = p$

(ii) $(\psi_J\ ,\ \chi^{(\mu;\lambda)}) = 1$

<u>Proof</u>

(i) Since $p(J) = (\rho\ ;\sigma)$ we have that $J = J_1 \cup J_2$ where $p(J_1) = \rho$ and $p(J_2) = \sigma$, (writing $p(\emptyset) = (0)$). Let $|\rho| = m$, $|\sigma| = n$. Then $|\hat{J}| = 1 - |J|$. But if $(1,-1) \in J$ then $(1,-1) \notin \hat{J}$, hence $\sigma \neq 0 \Rightarrow \alpha = 0$ $(p(\hat{J}) = (\beta';\alpha'))$ and conversely, $\alpha \neq 0 = \sigma = 0$. So $|\hat{J}| = m - |J_1|$. By 2.4.3, because $p(J_1) = \rho$, $r(\rho) = m - |J_1| = |\hat{J}| = p$ as required.

(ii) Move across to σ all but the squares which do not lie in the first column of ρ and then move up to the first row of σ. Since $r(\rho) = p$, we obtain $(1^p;1-p) = (\lambda;\mu)$. Thus $(\rho\ ;\sigma) \rightarrow (\lambda;\mu)$.

Now suppose $J \subsetneq K \subseteq I$ and $p(K) = (\gamma;\delta)$. Then $(\gamma;\delta)$ is obtained from $(\rho\ ;\sigma)$ by moving whole rows up in ρ and also across to σ. Hence $r(\gamma) < r(\rho) = p = r(\lambda)$, so $(\gamma;\delta) \not\rightarrow (\lambda;\mu)$ and therefore by 3.3.8 $(1_{W_K}{}^G\ ,\chi^{(\mu;\lambda)}) = 0$. Hence

$(\psi_J\ ,\ \chi^{(\mu;\lambda)}) = (1_{W_J}{}^G\ ,\ \chi^{(\mu;\lambda)}) \neq 0$ by 3.3.8 since

$(\rho\ ;\sigma) \rightarrow (\lambda;\mu)$.

So $|\hat{J}| = p \Rightarrow (\Psi_J, \chi^{(\mu;\lambda)}) \neq 0$. Because there are $\binom{l}{p}$ subsets of I of order p and $\chi^{(\mu;\lambda)}(1) = \binom{l}{p}$, we have as in 2.4.3 $(\Psi_J, \chi^{(\mu;\lambda)}) = 1$ (and $(\Psi_K, \chi^{(\mu;\lambda)}) = 0$ if $|\hat{K}| \neq p$).

Theorem 3.5.4

Let χ be the irreducible character of G afforded by $\Lambda^p V$. Then $\chi = \chi^{(\mu;\lambda)}$.

Proof

χ is irreducible so $\chi = \chi^{(\delta;\gamma)}$ for some pair of partitions $(\gamma;\delta)$ of l.

Let $J = \{(p+1\ p+2), \ldots, (l-1\ l), (l,-l)\}$

hence $\hat{J} = \{(12), (23), \ldots, (p\ p+1)\}$

so that $|\hat{J}| = p$.

Then $(\rho;\sigma) = p(J) = (1^p; l-p) = (\lambda;\mu)$.

By [17] $\Lambda^p V$ is an irreducible submodule of $A\xi_J\eta_{\hat{J}}$ and therefore $(\Psi_J, \chi^{(\delta;\gamma)}) \neq 0$. So by 3.5.1 $(\rho;\sigma) \rightarrow (\gamma;\delta)$ i.e. $(\lambda;\mu) \rightarrow (\gamma;\delta)$.

Now let $J = \{(12), \ldots, (l-p\ l-p+1)\}$

so $\hat{J} = \{(l-p+1\ l-p+2), \ldots, (l-1\ l), (l,-l)\}$, $|\hat{J}| = p$.

Then $(\beta';\alpha') = p(J) = (1^{l-p}; p) = (\mu';\lambda')$. Hence $(\alpha;\beta) = (\lambda;\mu)$. Again $(\Psi_J, \chi^{(\delta;\gamma)}) \neq 0$ so by 3.5.1, $(\gamma;\delta) \rightarrow (\alpha;\beta)$, i.e. $(\gamma;\delta) \rightarrow (\lambda;\mu)$.

So $(\lambda;\mu) \rightarrow (\gamma;\delta) \rightarrow (\lambda;\mu)$ and since $\rightarrow$ is anti-symmetric $(\lambda;\mu) = (\gamma;\delta)$ as required.

We now show that there are only two subsets J of I such that $A\xi_J\eta_{\hat{J}}$ is irreducible, so that Solomon's decomposition is a long way from being a complete decomposition of A.

Theorem 3.5.5

$A\xi_J\eta_{\hat{J}}$ is irreducible if and only if $J = \emptyset$ or $J = I$

Proof

Suppose $A\xi_J\eta_{\hat{J}}$ (and therefore ψ_J) is irreducible. Let $|\hat{J}| = p$, then by 3.5.2, 3.5.3

$(\rho;\sigma) = (\alpha;\beta) = (\lambda;\mu) = (1^p ; 1-p)$

therefore $\sigma = (1-p)$ and $\alpha' = (p)$. But $J \cap \hat{J} = \emptyset$, so either $\sigma = 0$ or $\alpha = 0$. Hence $p = 0$ or $p = 1$. Therefore $(\rho;\sigma) = (- ; 1)$ or $(\rho;\sigma) = (1^1 ; -)$ so $J = I$ or $J = \emptyset$.

Conversely, suppose $J = I$, then $A\xi_J\eta_{\hat{J}} = A.1$ which affords the unit character of G. If $J = \emptyset$, $A\xi_J\eta_{\hat{J}} = A\varepsilon$ which affords the sign character ε of G. In both cases, therefore, $A\xi_J\eta_{\hat{J}}$ is irreducible.

§3.6 Maximal and other Weyl subgroups of $W(C_l)$

In §3.3 we defined a bijection X from the set of Weyl subgroups of G to the set of irreducible characters of G. We want to prove this is consistent in much the same way as in §2.5, and this is done in 3.6.1.

The maximal Weyl subgroups of G are of type A_{l-1} and $C_i + C_{l-i}$ for $1 \leqslant i \leqslant l-1$. Thus $W(C_{l-1})$ is not a maximal Weyl subgroup of G, and we consider the maximal ones later on in this section.

Define (as in §2.5) $\lambda^* = (\lambda 1)$ where λ is a partition

Theorem 3.6.1

Let $(\lambda;\mu)$ be a pair of partitions of $l-1$ and let $(\lambda;\mu)^* = (\lambda^*;\mu)$ - a pair of partitions of l. Then

$$(\chi^{(\lambda;\mu)})^G = \chi^{(\lambda;\mu)^*} + \sum \chi^{(\alpha;\beta)}$$

summed over all those pairs of partitions $(\alpha;\beta)$ $(\neq (\lambda;\mu)^*)$ of l obtained from $(\lambda;\mu)$ by adding a square to the end of a row of λ or by adding a square to the end of a row of μ. In particular, $(\alpha;\beta) > (\lambda;\mu)^*$.

Proof

Notation: $G' = W(C_{l-1})$ so $G' = N'H'$, $H' = S_{l-1}$. $\chi^{(\alpha;\beta)} = \phi^G$ with the usual notation, and $\chi^{(\lambda;\mu)} = \phi'^{G'}$ with the notation as in §3.1, except that we dash the appropriate symbols. We shall also assume H' is the symmetric group on the letters $\{1,\dots,l-1\}$.

Let $\Gamma = ((\chi^{(\lambda;\mu)})^G, \chi^{(\alpha;\beta)})$. Then

$$\Gamma \neq 0 \Rightarrow 0 \neq \sum_{y \in \{y_i\}} (\phi'_{N'C' \cap (NC)^y}, {}^y\phi_{N'C' \cap (NC)^y})$$

where $\{y_i\}$ is a set of $(N'C', NC)$-double coset representatives and each $y_i \in H$. Hence for some $y \in \{y_i\}$,

$$(\phi'_{N'C' \cap (NC)^y}, {}^y\phi_{N'C' \cap (NC)^y}) \neq 0.$$

It is easy to see that

$$N'C' \cap (NC)^y = N'C' \cap NC^y = N'(C' \cap C^y)$$

Hence

$$(\varsigma', ({}^y\varsigma)_{N'})(\psi'_{C' \cap C^y}, ({}^y\psi)_{C' \cap C^y}) \neq 0$$

So $(\varsigma', ({}^y\varsigma)_{N'}) \neq 0$ and therefore $\varsigma' = ({}^y\varsigma)_{N'}$.

Let $|\lambda| = m'$, $|\mu| = n'$, $|\alpha| = m$, $|\beta| = n$.

Now ς' takes the value 1 (resp. -1) on the sign changes given by the m' (resp. n') symbols of the first (resp. second) type. Similarly for ς. Thus ${}^y\varsigma$ takes the value

1 (resp. -1) on m (resp. n) sign changes.

Since $\varsigma' = ({}^{y}\varsigma)_{N'}$ we have $m' \leqslant m$ and $n' \leqslant n$. But $m + n = l$, $m' + n' = l-1$ so $m = m'$ or $m' + 1$ and $n = n' + 1$ or n'. Therefore we may assume that in G' $\{1,\ldots,m'\}$ are the symbols of the first type and $\{m'+1,\ldots,l-1\}$ are the symbols of the second type. So we have that in G, by rearranging the symbols, $\{1,\ldots,m'\}$ are also of the first type and $\{m'+1,\ldots,l-1\}$ are also of the second type and the symbol l is undetermined.

It follows immediately that $\varsigma' = \varsigma_{N'}$ so

$$({}^{y}\varsigma)_{N'} = \varsigma_{N'} \qquad \ldots \quad (A)$$

We now show that $y = 1$.

Let $(b,-b) \in N$ and $y^{-1}(b) \neq l$ so $(y^{-1}(b),-y^{-1}(b)) \in N'$

Then

$$\begin{aligned} {}^{y^{-1}}\varsigma(b,-b) = \varsigma(y^{-1}(b),-y^{-1}(b)) &= {}^{y}\varsigma(y^{-1}(b),-y^{-1}(b)) \text{ by (A)} \\ &= \varsigma(b,-b) \end{aligned}$$

Now consider $(l,-l) \notin N'$. Then if

(i) $y^{-1}l = l$ then ${}^{y^{-1}}\varsigma(l,-l) = \varsigma(y^{-1}l,-y^{-1}l) = \varsigma(l,-l)$

(ii) $y^{-1}l \neq l$ then $(y^{-1}l,-y^{-1}l) \in N'$ so as for b above

$${}^{y^{-1}}\varsigma(l,-l) = \varsigma(l,-l)$$

Finally,

suppose $y(l) = a \neq l$, so $(a,-a) \in N'$. Then

$$\begin{aligned} \varsigma(a,-a) &= {}^{y}\varsigma(a,-a) \quad \text{by (A)} \\ &= \varsigma(ya,-ya) \\ &= \ldots \\ &= \varsigma(y^{r-1}a,-y^{r-1}a) \quad \text{by applying (A)} \end{aligned}$$

where y includes the r-cycle $(l\ a\ ya\ \ldots\ y^{r-1}a)$

$$\begin{aligned} &= \varsigma(l,-l) \text{ by applying (A) again} \\ &= \varsigma(y^{-1}a,-y^{-1}a) \\ &= {}^{y^{-1}}\varsigma(a,-a) \end{aligned}$$

Hence for all symbols $d \in \{1,\dots,l\}$,
$y^{-1}\varsigma(d,-d) = \varsigma(d,-d)$ i.e. $y^{-1} \in C_H(\varsigma) = C$, so $y \in C$ and therefore y is in the first double coset $N'C'NC$ so $y = 1$.

Hence $\Gamma \neq 0 \Rightarrow \Gamma = (\phi'_{N'C' \cap NC}, \phi_{N'C' \cap NC})$

$$= (\phi'_{N'C'}, \phi_{N'C'})$$

since by construction $C' \leq C$

$$= (\varsigma', \varsigma_{N'})(\psi', \psi_{C'})$$

$$= 1.(\psi', \psi_{C'})$$

$$= (\chi^{\lambda}, \chi^{\alpha}{}_{S_{m'}})(\chi^{\mu}, \chi^{\beta}{}_{S_{n'}})$$

because $C' \cong S_{m'} \times S_{n'}$ and $S_{m'} \leq S_m$, $S_{n'} \leq S_n$.

So $((\chi^{\lambda})^{S_m}, \chi^{\alpha}) \neq 0$ and $((\chi^{\mu})^{S_n}, \chi^{\beta}) \neq 0$ by Frobenius

If (a) $m = m' + 1$ and $n = n'$, then $\mu = \beta$ and by 2.5.1, α is obtained by adding a square to the end of a row of λ; or if (b) $m = m'$ and $n = n' + 1$ then $\lambda = \alpha$ and by 2.5.1, β is obtained by adding a square to the end of a row of μ. Hence $(\alpha;\beta)$ is obtained by adding a square to the end of a row of λ or μ. In either case, by 2.5.1, $((\chi^{\lambda})^{S_m}, \chi^{\alpha}) = 1 = ((\chi^{\mu})^{S_n}, \chi^{\beta})$, so $\Gamma = 1$.

Finally, if $(\alpha;\beta)$ is obtained by adding a square to the end of a row of λ or μ we see by 2.5.1 that $((\chi^{\lambda})^{S_m}, \chi^{\alpha}) = 1 = ((\chi^{\mu})^{S_n}, \chi^{\beta})$ and $\varsigma' = \varsigma_{N'}$ so the first term in the summand of Γ is non-zero i.e. $\Gamma \neq 0$. Hence $\chi^{(\alpha;\beta)}$ occurs in the decomposition of $(\chi^{(\lambda;\mu)})^G$.

We now give the decomposition for inducing an irreducible character up from a maximal Weyl subgroup of G.

<u>Theorem 3.6.2</u> (Inducing up from A_{l-1})

Let $\lambda \vdash l$ and $(\alpha;\beta)$ a pair of partitions of l. Then

$$((\chi^\lambda)^G, \chi^{(\alpha;\beta)}) \neq 0 \quad \Rightarrow \quad (\lambda;-) \to (\alpha;\beta) \to (-;\lambda)$$

and

$$((\chi^\lambda)^G, \chi^{(\lambda;-)}) = 1$$

<u>Proof</u>

Suppose $0 \neq ((\chi^\lambda)^G, \chi^{(\alpha;\beta)}) = (\chi^\lambda, \chi^{(\alpha;\beta)}_H)$ by Frobenius. Hence by 2.2.7

$$(1_{W_\lambda}^H, \chi_H^{(\alpha;\beta)}) \neq 0 \quad \text{and} \quad (\varepsilon_{W_{\lambda'}}^H, \chi_H^{(\alpha;\beta)}) \neq 0.$$

Now $W_\lambda = W_{(\lambda;-)}$ and $W_{\lambda'} = W_{(\lambda';-)}$ as Weyl subgroups of G, so using Frobenius again

$$(1_{W_{(\lambda;-)}}^G, \chi^{(\alpha;\beta)}) \neq 0 \quad \text{and} \quad (\varepsilon_{W_{(\lambda';-)}}^G, \chi^{(\alpha;\beta)}) \neq 0$$

and by 3.3.8 $(\lambda;-) \to (\beta;\alpha) \to (-;\lambda)$. It follows that $(\lambda;-) \to (\alpha;\beta) \to (-;\lambda)$ by moving across a complementary set of squares.

Also $((\chi^\lambda)^G, \chi^{(\lambda;-)}) = (\chi^\lambda, \chi_H^{(\lambda;-)})$ by Frobenius

$= (\chi^\lambda, \psi^H)$ by 3.1.3

and by definition $C = H$, $\psi = \chi^\lambda$

$= (\chi^\lambda, \chi^\lambda)$

$= 1$ since χ^λ is irreducible.

<u>Theorem 3.6.3</u> (Inducing up from $C_i + C_{l-i}$)

Let $(\lambda;\mu)$ be a pair of partitions of i and $(\rho;\sigma)$ a pair of partitions of j, where $i + j = l$; let $(\alpha;\beta)$ be a pair of partitions of l. Then

$((\chi^{(\lambda;\mu)} . \chi^{(\rho;\sigma)})^G , \chi^{(\alpha;\beta)}) \neq 0$ implies

$(\alpha;-) \to (\lambda;\rho) \to (-;\alpha)$ and $(\beta;-) \to (\mu;\sigma) \to (-;\beta)$

<u>Proof</u>

We let $G_i = W(C_i)$, $G_j = W(C_j)$ and $G_i = N_iH_i$, $G_j = N_jH_j$ and use the obvious notation for characters. Let $Z = H_i \times H_j$ and $Y = G_i \times G_j$. Then $N = N_i \times N_j$ and $Y = NZ$.

Let $\Delta = ((\chi^{(\lambda;\mu)} . \chi^{(\rho;\sigma)})^G , \chi^{(\alpha;\beta)})$

$= ((\phi_i{}^{G_i} . \phi_j{}^{G_j})^G , \phi^G)$

$= ((((\phi_i . \phi_j)^{G_i \times G_j})^G , \phi^G)$ by 1.2.5(ii)

$= ((\phi_i . \phi_j)^G , \phi^G)$ by transitivity of induction

$$= \sum_{g \in \{g_i\}} ((\phi_i . \phi_j)_{NC' \cap NC^g} , {}^g\phi_{NC' \cap NC^g})$$

where $\{g_i\}$ is a set of (NC',NC)-double coset representatives $g_i \in H$ and $C' = C_i \times C_j = C_H(\varsigma_i) \times C_H(\varsigma_j)$.

Hence $\Delta \neq 0$ implies that there exists $g \in \{g_i\}$ such that

$$((\phi_i . \phi_j)_{NC' \cap NC^g} , {}^g\phi_{NC' \cap NC^g}) \neq 0 .$$

But $NC' \cap NC^g = N(C' \cap C^g)$ since $g \in H$. Thus

$0 \neq ((\phi_i . \phi_j)_N , ({}^g\phi)_N) = ((\phi_i)_{N_i} . (\phi_j)_{N_j} , ({}^g\phi)_N)$

$\Rightarrow (\varsigma_i . \varsigma_j , \varsigma) \neq 0$

$\Rightarrow (\varsigma_i , ({}^g\varsigma)_{N_i})(\varsigma_j , ({}^g\varsigma)_{N_j}) \neq 0$, since $N = N_i \times N_j$ and ${}^g\varsigma$ is linear

$\Rightarrow \varsigma_i = ({}^g\varsigma)_{N_i}$ and $\varsigma_j = ({}^g\varsigma)_{N_j}$.

Let $|\lambda| = m_i$, $|\mu| = n_i$, $|\rho| = m_j$, $|\sigma| = n_j$, $|\alpha| = m$, $|\beta| = n$.

It follows, as in 3.6.1, that $m_i + m_j = m$, $n_i + n_j = n$, so by ordering the symbols correctly we have

$\varsigma_i = \varsigma_{N_i}$, $\varsigma_j = \varsigma_{N_j}$. Hence

$$g_\varsigma = (g_\varsigma)_{N_i} \cdot (g_\varsigma)_{N_j} = \varsigma_i \cdot \varsigma_j = \varsigma_{N_i} \cdot \varsigma_{N_j} = \varsigma .$$

So $g \in C_H(\varsigma) = C$ which is in the first double coset, i.e. $g = 1$.

Therefore $\Delta = ((\phi_i \cdot \phi_j)_{NC' \cap NC} , \phi_{NC' \cap NC})$.

Now we have ensured that
$$\begin{aligned} C' &= C_i \times C_j \\ &= S_{m_i} \times S_{n_i} \times S_{m_j} \times S_{n_j} \\ &= S_{m_i} \times S_{m_j} \times S_{n_i} \times S_{n_j} \\ &\leq S_m \times S_n = C \end{aligned}$$

Therefore $NC' \leq NC$. So

$$\begin{aligned}
\Delta &= ((\phi_i \cdot \phi_j) , \phi_{NC'}) \\
&= (\varsigma_i \cdot \varsigma_j , \varsigma)(\psi_i \cdot \psi_j , \psi_{C'}) \\
&= (\varsigma , \varsigma)(\psi_i \cdot \psi_j , \psi_{C'}) \quad \text{by above} \\
&= ((\chi^\lambda \cdot \chi^\mu).(\chi^\rho \cdot \chi^\sigma) , (\chi^\alpha \cdot \chi^\beta)_{C'}) \quad \text{since } \varsigma \text{ is irreducible} \\
&= ((\chi^\lambda \cdot \chi^\rho).(\chi^\mu \cdot \chi^\sigma) , (\chi^\alpha \cdot \chi^\beta)_{(S_{m_i} \times S_{m_j}) \times (S_{n_i} \times S_{n_j})}) \\
&= (\chi^\lambda \cdot \chi^\rho , \chi^\alpha_{S_{m_i} \times S_{m_j}})(\chi^\mu \cdot \chi^\sigma , \chi^\beta_{S_{n_i} \times S_{n_j}}) \\
&= ((\chi^\lambda \cdot \chi^\rho)^{S_m} , \chi^\alpha)((\chi^\mu \cdot \chi^\sigma)^{S_n} , \chi^\beta) \\
&= (\chi^{(\lambda;\rho)}{}_{S_m} , \chi^\alpha)(\chi^{(\mu;\sigma)}{}_{S_n} , \chi^\beta) \quad \text{by 3.1.3(ii)}
\end{aligned}$$

$$= (\chi^{(\lambda;\rho)}, (\chi^{\alpha})^{G_m})(\chi^{(\mu;\sigma)}, (\chi^{\beta})^{G_n})$$

where $G_m = W(C_m)$, $G_n = W(C_n)$.

So $\Delta \neq 0 \Rightarrow ((\chi^{\alpha})^{G_m}, \chi^{(\lambda;\rho)}) \neq 0$ and $((\chi^{\beta})^{G_n}, \chi^{(\mu;\sigma)}) \neq 0$

and therefore by 3.6.2

$(\alpha;-) \to (\lambda;\rho) \to (-;\alpha)$ and $(\beta;-) \to (\mu;\sigma) \to (-;\beta)$,

proving the theorem.

We shall now give the theorem, mentioned at the end of chapter two, about inducing up the irreducible characters from the maximal Weyl subgroup $A_i + A_{l-i-1}$ of $W(A_l)$.

<u>Theorem 3.6.4</u>

Suppose $\lambda \vdash l+1$, $\alpha \vdash i+1$, $\beta \vdash l-i$. Let $W = S_{l+1}$.

Then

$((\chi^{\alpha}.\chi^{\beta})^{W}, \chi^{\lambda}) \neq 0$ implies $(\lambda;-) \to (\alpha;\beta) \to (-;\lambda)$

<u>Proof</u>

Regard $W \leqslant G' = W(C_{l+1})$.

$$((\chi^{\alpha}.\chi^{\beta})^{W}, \chi^{\lambda}) \neq 0 \quad \Rightarrow \quad (\chi^{(\alpha;\beta)}_{W}, \chi^{\lambda}) \neq 0 \text{ by } 3.1.3(ii)$$

$$\Rightarrow \quad (\chi^{(\alpha;\beta)}, (\chi^{\lambda})^{G_1}) \neq 0$$

by Frobenius

$$\Rightarrow \quad (\lambda;-) \to (\alpha;\beta) \to (-;\lambda)$$

by 3.6.2 .

Chapter four   WEYL GROUPS OF TYPE D

The Weyl group of type D has been rather less well studied, and poses problems that do not occur in either the symmetric group or Weyl groups of type C.

Young [20] determined the conjugacy classes and irreducible characters. We shall be considering this group in the same manner as the groups in the previous two chapters, although we cannot expect to get such 'nice' results. However, we can give an algorithm to determine the decomposition of $1_W^{W(D_l)}$, where W is a Weyl subgroup of $W(D_l)$.

§4.1   The conjugacy classes and irreducible characters

Throughout this chapter we shall be using the notation of chapter three.

Let $K = W(D_l)$ - the Weyl group of rank l of type D. Then K is a subgroup of $G = W(C_l)$ of index 2, hence $K \lhd G$. We can describe K by considering it as a subgroup of G; viz. an element $g \in G$ lies in K if and only if the cycle decomposition of g into disjoint cycles contains an even number of negative cycles.

It is then clear that $|G:K| = 2$ so $|K| = 2^{l-1}.l!$

$K \cap N$ is the subgroup of index 2 of N, generated by pairs of sign changes. If we remember that a negative cycle is a positive cycle multiplied by a sign change (p 40) we see that $K = (K \cap N)H$.

Notation: we let $W(D_2) = \{(1),(12),(1,-1)(2,-2),(1,-2)\}$ which is isomorphic to the non-cyclic group of order 4.

The conjugacy classes of K were given by Carter [5]

Lemma 4.1.1

Two elements of K are conjugate if and only if they have the same signed cycle-type, except that if all the cycles are even and positive there are two conjugacy classes.

In the latter case, the conjugacy classes consist of elements in which the total number of negative signs appearing in the cycles is even or in which the total number is odd.

We turn now to the irreducible characters, where we find a similar situation to that in 4.1.1.

Theorem 4.1.2

With the usual notation, let $(\lambda;\mu)$ be a pair of partitions of l. Then

(i) $\chi_K^{(\lambda;\mu)}$ is an irreducible character of K if $\lambda \neq \mu$;

(ii) $\chi_K^{(\lambda;\mu)} = \chi_K^{(\mu;\lambda)}$;

(iii) $\chi_K^{(\lambda;\lambda)}$ is the sum of 2 distinct irreducible characters of K of the same degree;

(iv) every irreducible character of K has the form $\chi_K^{(\lambda;\mu)}$ $(\lambda\neq\mu)$ or is a component of $\chi_K^{(\lambda;\lambda)}$ for some λ, μ ;

(v) all the irreducible characters of K mentioned in (iv) are distinct.

Before proving 4.1.2, we prove the following, more general, result

Lemma 4.1.3

For the purposes of this lemma only, let G , K be arbitrary finite groups such that K is a subgroup of G of index 2.

(a) Let θ be an irreducible character of K. Then either (i) θ^G is irreducible and $(\theta^G)_K = \theta + \theta'$, where θ' is an irreducible character of K such that $\theta \neq \theta'$ and $\theta^G = \theta'^G$;

or (ii) $\theta^G = \chi_1 + \chi_2$ where χ_1, χ_2 are distinct irreducible characters of G such that $(\chi_1)_K = \theta = (\chi_2)_K$.

(b) Let χ be an irreducible character of G. Then either (i) χ_K is irreducible and $(\chi_K)^G = \chi + \chi'$ where χ' is an irreducible character of G, $\chi \neq \chi'$ and $\chi_K = \chi'_K$;

or (ii) $\chi_K = \theta_1 + \theta_2$ where θ_1, θ_2 are distinct irreducible characters of K such that $\theta_1{}^G = \chi = \theta_2{}^G$.

<u>Proof</u>

(a) Let $T = C_G(\theta)$ so $K \leqslant T \leqslant G$ (θ is a class function on K) hence either (i) $T = K$ or (ii) $T = G$.

(i) $T = K$

Therefore $$(\theta^G,\theta^G) = \sum_{y \in \{y_i\}} (\theta_{K \cap K^y} , ({}^y\theta)_{K \cap K^y})$$

where $\{y_i\}$ is a set of (K,K)-double coset representatives.

Hence $(\theta_{K \cap K^y} , ({}^y\theta)_{K \cap K^y}) \neq 0 \quad \Rightarrow \quad (\theta , {}^y\theta) \neq 0 \quad (K \lhd G)$

$\Rightarrow \quad \theta = {}^y\theta$

$\Rightarrow \quad y \in T \quad \Rightarrow y = 1$

Therefore $(\theta^G , \theta^G) = (\theta , \theta) = 1$, hence θ^G is irreducible.

So $((\theta^G)_K , \theta) = (\theta^G , \theta^G)$ by Frobenius

$= 1$

Let $(\theta^G)_K = \theta + \theta'$ where θ' is a character of K such that $(\theta , \theta') = 0$. Thus

$(\theta^G , \theta'^G) = ((\theta^G)_K , \theta') = (\theta+\theta' , \theta') = (\theta' , \theta') \neq 0$.

So since θ^G is irreducible, $\theta'^G = (\theta',\theta')\theta^G + \chi$ where χ is a character of G such that $(\chi,\theta^G) = 0$.

Now $\theta'(1) = (\theta^G)_K(1) - \theta(1) = 2\theta(1) - \theta(1) = \theta(1)$. So

$2\theta(1) = 2\theta'(1) = \theta'^G(1) = (\theta',\theta')\theta^G(1) + \chi(1)$

i.e. $2\theta(1) = 2(\theta',\theta')\theta(1) + \chi(1)$. Hence $\chi(1) = 0$, so $\chi = 0$, and $(\theta',\theta') = 1$ and so θ' is irreducible and $\theta'^G = \theta^G$ and $(\theta,\theta') = 0$ implies $\theta \neq \theta'$, which proves (i).

(ii) $T = G$

Let $\theta^G = \sum_{i=1}^{r} n_i\chi_i$ where χ_i are distinct irreducible characters of G. Since $G = C_G(\theta)$ it follows that for all $k \in K$

$$\theta^G(k) = \frac{1}{|K|} \sum_{g\in G} \theta(gkg^{-1}) = \frac{1}{|K|} \sum_{g\in G} \theta(k)$$

$$= 2\theta(k)$$

i.e. $(\theta^G)_K = 2\theta$. Hence

$$\sum_{i=1}^{r} n_i(\chi_i)_K = 2\theta \quad \ldots \quad (1)$$

Also, by Frobenius, $(\theta^G,\theta^G) = ((\theta^G)_K , \theta) = (2\theta,\theta) = 2$ since θ is irreducible. Thus $r = 2$ and $n_1 = n_2 = 1$, so $\theta^G = \chi_1 + \chi_2$ and from (1), $(\chi_1)_K + (\chi_2)_K = 2\theta$. Because

θ is irreducible we see that $(\chi_1)_K = \theta = (\chi_2)_K$ proving (ii).

(b) Let $\chi_K = \sum_{i=1}^{s} m_i\theta_i$ where θ_i are distinct irreducible characters of K. By (a), θ_i^G is either irreducible or the sum of two distinct irreducibles. Hence $m_i = (\chi_K , \theta_i) = (\chi , \theta_i^G)$ by Frobenius

$= 0$ or 1.

Therefore we may write $\chi_K = \sum_{i=1}^{t} \theta_i$ where θ_i are distinct irreducible characters of K such that $(\theta_i{}^G, \chi) = 1$. So $\theta_i{}^G = \chi + \chi_i$ where <u>either</u> χ_i is an irreducible character of G such that $\chi_i \neq \chi$ and $\chi_K = (\chi_i)_K$ <u>or</u> $\chi_i = 0$. Hence

$$(\chi_K)^G = \sum_{i=1}^{t} \theta_i{}^G = \sum_{i=1}^{t} (\chi + \chi_i)$$

So $2\chi(1) = t\chi(1) + \sum_{i=1}^{t} \chi_i(1)$ and therefore

either (i) $t = 1$ and $\chi_K = \theta_1$ which is irreducible and

$$(\chi_K)^G = \chi + \chi_1 \ , \ \chi_K = (\chi_1)_K$$

or (ii) $t = 2$ and $\chi_K = \theta_1 + \theta_2$ and $\theta_1{}^G = \chi = \theta_2{}^G$ completing the lemma.

We revert to the notation in chapter three

<u>Lemma 4.1.4</u>

Let $\chi = \phi^G$ be an irreducible character of G, then $\chi_K = (\phi_L)^K$ where $L = (K \cap N)C$.

<u>Proof</u>

$NC.K = NK$ since $C = C_H(\varsigma) \leqslant H \leqslant K$

$\geqslant NH = G$

So $G = NC.K$. Since ϕ is an irreducible character of NC, it follows, by Mackey's subgroup formula 1.2.2, that $(\phi^G)_K = (\phi_L)^K$.

The following combinatorial result is of independent interest and was proved by Young ([20] §8)

<u>Lemma 4.1.5</u>

Let

A be the number of ordered pairs of partitions $(\lambda;\mu)$ of l such that the number of parts of μ are even;

B be the number of partitions λ of l such that all the parts of λ are even;

C be the number of unordered pairs $(\lambda;\mu)$ of partitions of l;

D be the number of partitions λ of $l/2$ (define D = 0 if l is odd).

Then $\quad A + B = C + D$

This will turn out to be the statement that the number of conjugacy classes of K is equal to the number of irreducible characters of K. Indeed, from 4.1.1, we see that the number of conjugacy classes of K is precisely A + B.

We are now in a position to prove 4.1.2

<u>Proof of 4.1.2</u>

We first prove (ii)

With the usual notation let $\chi^{(\lambda;\mu)} = \phi_1{}^G$, $\chi^{(\mu;\lambda)} = \phi_2{}^G$ where $\phi_1 = \varsigma_1\Psi_1$, $\phi_2 = \varsigma_2\Psi_2$.

By definition $\varsigma_1(a,-a) = -\varsigma_2(a,-a)$ for all $a \in \{1,\ldots,l\}$.

Since $K \cap N$ is generated by pairs of sign changes

$$(\varsigma_1)_{K\cap N} = (\varsigma_2)_{K\cap N}$$

Also $C = C_H(\varsigma_1) = C_H(\varsigma_2) \cong S_m \times S_n$, and

$\Psi_1 = \chi^\lambda . \chi^\mu = \chi^\mu . \chi^\lambda = \Psi_2 = \Psi$, say. Thus letting

$L = (K\cap N)C$, $(\phi_1)_L = (\varsigma_1)_{K\cap N}\Psi = (\varsigma_2)_{K\cap N}\Psi = (\phi_2)_L$

Hence by 4.1.4

$$\chi^{(\lambda;\mu)}_K = ((\phi_1)_L)^K = ((\phi_2)_L)^K = \chi^{(\mu;\lambda)}_K .$$

(i) By 4.1.3, $\chi^{(\lambda;\mu)}_K$ $(\lambda \neq \mu)$ is either irreducible or is the sum of 2 irreducibles.

Suppose the latter is the case; then $\chi_K^{(\lambda;\mu)} = \theta_1 + \theta_2$ where θ_1, θ_2 are distinct irreducible characters of K, and

$$\theta_1^{\ G} = \chi^{(\lambda;\mu)} = \theta_2^{\ G} \quad . \text{ But by (ii)}$$

$\chi_K^{(\mu;\lambda)} = \theta_1 + \theta_2$ so that $\theta_1^{\ G} = \chi^{(\mu;\lambda)} = \theta_2^{\ G}$. Hence $\chi^{(\lambda;\mu)} = \chi^{(\mu;\lambda)}$ and therefore $(\lambda;\mu) = (\mu;\lambda)$, a contradiction since $\lambda \neq \mu$.

Therefore $\chi_K^{(\lambda;\mu)}$ $(\lambda \neq \mu)$ is irreducible. It follows from 4.1.3 that if $\theta = \chi_K^{(\lambda;\mu)}$ $(\lambda \neq \mu)$

$$= \chi_K^{(\mu;\lambda)}$$

then $\theta^G = \chi^{(\lambda;\mu)} + \chi^{(\mu;\lambda)}$

(iii),(iv),(v) We use the notation in 4.1.5. The irreducible characters $\chi_K^{(\lambda;\mu)}$ $(\lambda \neq \mu)$ have not been shown to be distinct, but there are at most C - D of them (by (ii)). Also the number of irreducible characters of K = the number of conjugacy classes of K

$$= A + B$$

$$= C + D \text{ by 4.1.5}$$

Hence we have unaccounted for at least (C + D) - (C - D) = 2D irreducible characters of K. The only case we have not considered is that of $\chi_K^{(\lambda;\lambda)}$, of which there can be at most D of them. By 4.1.3, $\chi_K^{(\lambda;\lambda)}$ is a sum of one or two irreducible characters of K.

The only way we can reconcile all these inequalities is for $\chi_K^{(\lambda;\lambda)}$ to be the sum of two irreducible characters of K for all pairs of partitions $(\lambda;\lambda)$ of 1; for all the irreducible characters so far obtained to be distinct; and for all the irreducible characters of K to be of the form $\chi_K^{(\lambda;\mu)}$ $(\lambda \neq \mu)$ or the component of some $\chi_K^{(\lambda;\lambda)}$.

We shall return to an investigation of the irreducible components of $\chi_{\kappa}^{(\lambda;\lambda)}$ (which only occur when l is even) in a later section.

§4.2 An algorithm for $W(D_l)$

The Weyl subgroups of K have the form
$S_{\lambda_1} \times \ldots \times S_{\lambda_r} \times W(D_{\mu_1}) \times \ldots \times W(D_{\mu_s})$ where
$\sum \lambda_i + \sum \mu_i = l$ and $\mu_i \neq 1$.

We shall write this subgroup as $W_{(\lambda;\mu)}$ putting $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ and we may assume that $\lambda_1 \geqslant \ldots \geqslant \lambda_r > 0$, $\mu_1 \geqslant \ldots \geqslant \mu_s > 1$. Thus the Weyl subgroups may be parameterized by pairs of partitions $(\lambda;\mu)$ of l such that no part of μ is 1.

Just as in §3.2, we may consider $W_{(\lambda;\mu)}$ as the row stabilizer of a diagram $D_{(\lambda;\mu)}$, where in this case a row permutation of $D_{(\lambda;\mu)}$ is an element of K which permutes the symbols in each row of D_λ and in each row of D_μ and also changes the signs of an <u>even</u> number of symbols in D_μ.

<u>Definition</u>

A pair of partitions $(\lambda;\mu)$ of l is called <u>bad</u> if $\mu = 0$ and all the parts of λ are even.
Otherwise $(\lambda;\mu)$ is called <u>good</u>.

It is evident from 4.1.1 and the fact that $R(gD_{(\lambda;\mu)}) = gR(D_{(\lambda;\mu)})g^{-1}$ for all $g \in G$ that (see [5])

<u>Lemma 4.2.1</u>

(a) If $(\lambda;\mu)$ is good, Weyl subgroups isomorphic to $W_{(\lambda;\mu)}$ are conjugate to it in K. In particular, if $x = (1,-1) \in G \setminus K$ then $W^x_{(\lambda;\mu)}$ is conjugate, in K, to $W_{(\lambda;\mu)}$.

(b) If $(\lambda;\mu)$ is bad, then the set of Weyl subgroups isomorphic to $W_{(\lambda;\mu)}$ splits up into two conjugacy classes. In particular, with x as above, $W^x_{(\lambda;\mu)}$ is not conjugate in K to $W_{(\lambda;\mu)}$.

We now wish to describe an algorithm for determining for a given pair of partitions $(\lambda;\mu)$ of 1, which pair of partitions $(\alpha;\beta)$ of 1 satisfy

$$(1_{W_{(\lambda;\mu)}}^{K}, \chi_K^{(\alpha;\beta)}) \neq 0 .$$

However, since the Weyl subgroups of K are parameterized by ordered pairs of partitions $(\lambda;\mu)$ such that no part of μ is 1, and the characters of K of the form $\chi_K^{(\alpha;\beta)}$ by unordered pairs of partitions (4.1.2), we cannot expect to get any sort of relation.

<u>Definition</u>

Let $(\lambda;\mu)$ be an ordered pair of partitions of 1 such that no part of μ is 1, and $(\alpha;\beta)$ an unordered pair. Write $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\beta)$ if $(\alpha;\beta)$ may be obtained from $(\lambda;\mu)$ by

(a) removing connected squares from the end of a row of λ and placing them, in the same order, at the bottom of μ ;

(b) repeating (a) with squares from different rows of λ ;

and at the same time, but independently, (so no square is moved twice)

(c) transferring complete rows of μ and placing them at the bottom of λ ;

then

(d) reordering the resulting rows so as to give frames of a pair of partitions $(\gamma;\delta)$ say;

and finally

(e) moving up inside γ and δ , according to the usual partial ordering on partitions, so as to obtain α and β respectively (so $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$).

By moving across a complementary set of squares between λ and μ we see that

$$(\lambda;\mu) \xrightarrow[D]{} (\alpha;\beta) \quad \Longleftrightarrow \quad (\lambda;\mu) \xrightarrow[D]{} (\beta;\alpha)$$

which is consistent with our choice of $(\alpha;\beta)$ to be unordered.

The algorithm introduced in chapter three for G will from now on be written as $\xrightarrow[C]{}$. It is clear that (provided no part of μ is 1)

$$(\lambda;\mu) \xrightarrow[C]{} (\alpha;\beta) \quad \Rightarrow \quad (\lambda;\mu) \xrightarrow[D]{} (\alpha;\beta)$$

We can now state

<u>Theorem 4.2.2</u>

Let $(\lambda;\mu)$, $(\alpha;\beta)$ be ordered (resp. unordered) pairs of partitions of l such that no part of μ is 1. Then

$$(1_{W_{(\lambda;\mu)}}^{K}, \chi_{K}^{(\alpha;\beta)}) \neq 0 \Longleftrightarrow (\lambda;\mu) \xrightarrow[D]{} (\alpha;\beta)$$

The following lemma is proved in precisely the same way as 3.3.2

Lemma 4.2.3

Let $W = R(D_{(\lambda;\mu)})$. Then

(a) $W = (N \cap W)(H \cap W)$ and $(N \cap W) \cap (H \cap W) = 1$

If also $y \in H$, $C = C_H(\varsigma)$ for some irreducible character ς of N and $L = (K \cap N)C$

(b) $W^y = (N \cap W^y)(H \cap W^y)$ and $(N \cap W^y) \cap (H \cap W^y) = 1$

(c) $L \cap W^y = (N \cap W^y)(C \cap W^y)$ and $(N \cap W^y) \cap (C \cap W^y) = 1$

Proof of 4.2.2

Suppose first that $(1_{W_{(\lambda;\mu)}}^{K}, \chi_K^{(\alpha;\beta)}) \neq 0$ and let $W = W_{(\lambda;\mu)}$. Then by 4.1.4 and Mackey's formula,

$$\neq (1_W^{K}, \chi_K^{(\alpha;\beta)}) = (1_W^{K}, (\phi_L)^K) = \sum_{y \in \{y_1\}} ({}^{y}1_{W^y \cap L}, \phi_{W^y \cap L})$$

here $\{y_1\}$ is a set of (W,L)-double coset representatives nd we may assume $y_1 \in H$. Thus there exists $y \in \{y_1\}$ uch that $({}^{y}1_{W^y \cap L}, \phi_{W^y \cap L}) \neq 0$.

We let $|\alpha| = m$, $|\beta| = n$, $N = N_1 \times N_2$ (as in 3.1.2) o that $\varsigma(a,-a) = 1$ for $(a,-a) \in N_1$ and $\varsigma(a,-a) = -1$ or $(a,-a) \in N_2$. Now by 4.2.3(c)

$$\neq ({}^{y}1_{W^y \cap L}, \phi_{W^y \cap L}) = ({}^{y}1_{N \cap W^y}, \varsigma_{N \cap W^y})({}^{y}1_{C \cap W^y}, \psi_{C \cap W^y})$$

ence

$$({}^{y}1_{N \cap W^y}, \varsigma_{N \cap W^y}) \neq 0 \quad \text{and} \quad ({}^{y}1_{C \cap W^y}, \psi_{C \cap W^y}) \neq 0 \qquad \ldots \text{(A)}$$

ince ς is linear, ${}^{y}1_{N \cap W^y} = \varsigma_{N \cap W^y}$ i.e. $1_{N \cap W^y} = \varsigma_{N \cap W^y}$

o ς takes the value 1 on the pairs of sign changes in $\cap\, W^y$ ($W \leq K$, so $N \cap W^y \leq N \cap K$).

Now W^y defines a diagram $D_{(\lambda;\mu)}$ and W^y only changes the signs of symbols in D_μ. Thus in any one row of D_μ the symbols must either all be of the first type or all of the second type (otherwise $\varsigma_{N\cap W^y} \neq 1$). Hence we may transfer those complete rows of D_μ which contain symbols of the first type to D_λ, and independently move the squares of D_λ (so that moved squares in the same row stay in the same row) containing the symbols of the second type to D_μ. On reordering the rows we obtain a diagram $D_{(\gamma;\delta)}$ of a pair of partitions $(\gamma;\delta)$ of l such that D_γ contains all the symbols of the first type and D_δ contains all the symbols of the second type. This corresponds to operations (a), (b), (c), (d) on p 92-3. So to show $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\beta)$ we have only to show $\gamma \leq \alpha$, $\delta \leq \beta$.

By construction $|\gamma| = m = |\alpha|$, $|\delta| = n = |\beta|$.

By (A) above

$$(1_{C\cap W^y}, \psi_{C\cap W^y}) \neq 0$$

But this is exactly the same stage that we reached in the proof of 3.3.1. So by precisely the same argument

$$0 \neq (1_{C\cap W^y}, \psi_{C\cap W^y}) = ((1_{W_\gamma})^{S_m}, \chi^\alpha)((1_{W_\delta})^{S_n}, \chi^\beta)$$

and therefore by 2.3.6, $\gamma \leq \alpha$ and $\delta \leq \beta$.

So $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\beta)$.

Conversely, suppose $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\beta)$. Therefore we may move parts of rows of λ across to μ and complete rows of μ across to λ to obtain a pair of partitions $(\gamma;\delta)$ of l such that $\gamma \leq \alpha$, $\delta \leq \beta$. Hence we may define a diagram $D_{(\lambda;\mu)}$ filled with the symbols $\{1,\ldots,l\}$ such that each row of D_μ contains only symbols of one type.

Then let $W = W_{(\lambda;\mu)} = R(D_{(\lambda;\mu)})$, so all pairs of sign changes in $N \cap W$ consist of symbols which are of the same type i.e. $\varsigma_{N\cap W} = 1$. So $(\varsigma_{N\cap W}, 1_{N\cap W}) \neq 0$.

Also by 2.3.6, since $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$

$$(1_{N\cap W}, \varsigma_{N\cap W})((1_{W_\gamma})^{S_m}, \chi^\alpha)((1_{W_\delta})^{S_n}, \chi^\delta) \neq 0$$

and this is, by the proof of the first part of the theorem, the first summand in the Mackey formula for $(1_W{}^K, \chi_K^{(\alpha;\beta)})$. Hence $(1_W{}^K, \chi_K^{(\alpha;\beta)}) \neq 0$, proving the theorem.

<u>Remark</u>

If $\mu = 0$ then $W_{(\lambda;-)}$ is a Weyl subgroup of G and as such is also written $W_{(\lambda;-)}$. Now

$$(1_{W_{(\lambda;-)}}{}^K, \chi_K^{(\alpha;\beta)}) = (1_{W_{(\lambda;-)}}{}^K, \chi_K^{(\beta;\alpha)}) = (1_{W_{(\lambda;-)}}{}^G, \chi^{(\beta;\alpha)})$$

So by 3.3.8 and 4.2.2

$$(\lambda;-) \underset{D}{\rightarrow} (\alpha;\beta) \iff (\lambda;-) \underset{C}{\rightarrow} (\alpha;\beta)$$

a result which can be seen to be true from the definitions of $\underset{C}{\rightarrow}$ and $\underset{D}{\rightarrow}$.

Before we can strengthen 4.2.2 and find which irreducible components of $\chi_K^{(\alpha;\alpha)}$ occur in $1_{W_{(\lambda;\mu)}}{}^K$ where $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\alpha)$ we shall need to study these components more carefully.

§4.3 <u>The remaining irreducible characters</u>

In this section we shall assume that l is even, so that characters of the form $\chi_K^{(\lambda;\lambda)}$ do occur.

Let $x = (1,-1)$ - a single sign change, so $x \in G \setminus K$.
Hence $G/K = \{K, xK\} = \{K, Kx\}$

For the whole of this section $\lambda \vdash l/2$.

<u>Lemma 4.3.1</u>

$$\chi_K^{(\lambda;\lambda)} = \theta_\lambda + {}^x\theta_\lambda$$

where θ_λ, ${}^x\theta_\lambda$ are distinct irreducible characters of K.

<u>Proof</u>

By 4.1.3, $\chi_K^{(\lambda;\lambda)} = \theta_\lambda + \theta'_\lambda$ where $\theta_\lambda \neq \theta'_\lambda$ and from the proof of 4.1.3(a) we see that $C_G(\theta_\lambda) = K$.
Because $\chi^{(\lambda;\lambda)}$ is a class function on G

$$\theta_\lambda + \theta'_\lambda = \chi_K^{(\lambda;\lambda)} = {}^x\chi_K^{(\lambda;\lambda)} = {}^x\theta_\lambda + {}^x\theta'_\lambda$$

Now θ_λ, θ'_λ, ${}^x\theta_\lambda$, ${}^x\theta'_\lambda$ are all irreducible so either $\theta_\lambda = {}^x\theta_\lambda$ or $\theta'_\lambda = {}^x\theta_\lambda$.
But x generates G/K so, since θ_λ is a class function on K, $\theta_\lambda = {}^x\theta_\lambda \Rightarrow \theta_\lambda = {}^g\theta_\lambda$ for all $g \in G$

$$\Rightarrow C_G(\theta_\lambda) = G, \text{ a contradiction.}$$

Hence $\theta'_\lambda = {}^x\theta_\lambda$ proving the lemma.

We would like to obtain θ_λ and ${}^x\theta_\lambda$ in the form of induced characters in much the same way as we did for $\chi^{(\lambda;\mu)}$.

By definition of $\chi^{(\lambda;\lambda)}$ the number of symbols of the first type is the same as the number of symbols of the second type viz. $l/2$. So we arrange the symbols so that

$\varsigma(a,-a) = 1$ for $a \in \{1,\ldots,l/2\}$
and $\varsigma(a,-a) = -1$ for $a \in \{l/2+1,\ldots,l\}$

We now define an involution in H which interchanges the symbols of the first type into those of the second ype and vice-versa.

Let $y = (1\ 1/_2{+}1)(2\ 1/_2{+}2)\ \dots\ (1/_2\ 1)$ and note hat $y \in K$.

<u>emma 4.3.2</u>

Let $T = C_K(\varsigma_{K\cap N})$. Then $T = L\langle y\rangle$ and $L \cap \langle y\rangle = 1$

<u>roof</u>

Let $t \in L$ then $t \in (K \cap N)C$

$= K \cap NC$ by the modular law

$= K \cap C_G(\varsigma) = C_K(\varsigma)$

o ${}^t\varsigma = \varsigma$ and hence ${}^t\varsigma_{K\cap N} = \varsigma_{K\cap N}$ and therefore $t \in T$. ence $L \leqslant T$.

By definition of y, ${}^y\varsigma(a,-a) = -\varsigma(a,-a)$ for all $\in \{1,\dots,l\}$ and so ${}^y\varsigma(a,-a)(b,-b) = \varsigma(a,-a)(b,-b)$ or all $a,b \in \{1,\dots,l\}$. Since $K \cap N$ is generated y sign changes ${}^y\varsigma_{K\cap N} = \varsigma_{K\cap N}$ and therefore $y \in T$. Thus $\langle y\rangle \leqslant T$.

Also $y \notin C_K(\varsigma) = L$ so $L \cap \langle y\rangle = 1$.

Conversely, let $t \in T$ so that ${}^t\varsigma_{K\cap N} = \varsigma_{K\cap N}$. Suppose $\notin L$, then there exists $(a,-a) \in N$ such that $\varsigma(a,-a) = -\varsigma(a,-a)$. Let $\{a_1,\dots,a_r\}$ be the subset of $1,\dots,l\}$ such that ${}^t\varsigma(a_i,-a_i) = -\varsigma(a_i,-a_i)$ for $1 \leqslant i \leqslant r$, nd ${}^t\varsigma(b,-b) = \varsigma(b,-b)$ for $b \notin \{a_1,\dots,a_r\}$.

Then ${}^t\varsigma(b,-b)(a_i,-a_i) = -\varsigma(b,-b)(a_i,-a_i)$, a ontradiction, since ${}^t\varsigma_{K\cap N} = \varsigma_{K\cap N}$. Thus ${}^t\varsigma(a,-a) = -\varsigma(a,-a)$ or all $a \in \{1,\dots,l\}$ so ${}^{ty}\varsigma(a,-a) = \varsigma(a,-a)$ and erefore $ty \in C_K(\varsigma) = L$. Hence $t \in L\langle y\rangle$.

<u>emma 4.3.3</u>

ϕ_L is irreducible

Proof

$\phi = \varsigma\psi$ so $\phi_L = \varsigma_{K\cap N}\psi$. Therefore $(\phi_L , \phi_L) = (\varsigma_{K\cap N} , \varsigma_{K\cap N})(\psi , \psi) = 1$ since $\varsigma_{K\cap N}$ is linear and ψ is irreducible.

The group $<y>$ has two irreducible characters $1, \tau$ say where $\tau(y) = -1$.

Define maps $\omega_i : T \longrightarrow \mathbb{C}$ $(i=1,2)$ by $\omega_1(ly) = \phi(l)$ and $\omega_2(ly) = \phi(l)\tau(y) = -\phi(l)$ for all $l \in L$.

We can write $\omega_i = \phi_L \tau_i$ where $\tau_1 = 1$, $\tau_2 = \tau$

Lemma 4.3.4

ω_1 , ω_2 are irreducible characters of T

Proof

Let V_i be the $<y>$-module affording τ_i where $\tau_1 = 1$ $\tau_2 = \tau$, and let U be the L-module affording ϕ_L .
Then $U \otimes V_i$ are T-modules affording characters ω_i $(i=1,2)$
For, the module axioms are easy to check with the one exception which we now prove.
Suppose l_1y' , $l_2y'' \in T$ $(l_1, l_2 \in L$ and $y', y'' = 1$ or $y)$ and $u \in U$, $v \in V_i$. Then we must show

$$(u \otimes v_i)(l_1y' . l_2y'') = [(u \otimes v)l_1y'](l_2y'')$$

Let U afford the representation R of L, P the representation of $K\cap N$ affording $\varsigma_{K\cap N}$ (so $P = \varsigma_{K\cap N}$) and Q the representation of C affording ψ . Then by definition of ϕ , $R = P \otimes Q$. Hence $u_1 l_2{}^y = u_1 R(l_2{}^y)$ for all $u_1 \in U$.
Let $l_2 = nc$ $(n \in K\cap N$, $c \in C)$. But by definition of y, y interchanges the symbols of each type so that $y \in C_H(C)$ i.e. $c^y = c$ for all $c \in C$. Therefore $l_2{}^y = n^y c^y = n^y c$.

So

$$u_1 l_2{}^y = u_1 R(l_2{}^y) = u_1 R(n^y c) = u_1 P(n^y)Q(c)$$

$$= u_1 \varsigma_{K\cap N}(n^y)Q(c)$$

$$= u_1 \varsigma_{K\cap N}(n)Q(c) \text{ since } y \in T$$

$$= u_1 P(n)Q(c)$$

$$= u_1 R(l_2)$$

$$= u_1 l_2$$

But $ul_1 \in U$, so $(ul_1)l_2{}^y = (ul_1)l_2$

Hence

$$(u \otimes v_i)(l_1 y' . l_2 y'') = (u \otimes v_i)(l_1 l_2{}^{y'} y' y'')$$
$$= u(l_1 l_2{}^{y'}) \otimes v_i(y' y'')$$
$$= (ul_1)l_2{}^{y'} \otimes (v_i y')y''$$
$$= (ul_1)l_2 \otimes (v_i y')y'' \quad \text{by above}$$
$$= (ul_1 \otimes v_i y')(l_2 y'')$$
$$= [(u \times v_i)l_1 y'](l_2 y'')$$

as required.

It is clear that $U \otimes V_i$ affords ω_i, therefore ω_1, ω_2 are characters of T.

Finally, $(\omega_i, \omega_i) = (\phi_L \tau_i \, , \phi_L \tau_i)$

$$= (\phi_L \, , \phi_L)(\tau_i, \tau_i)$$
$$= 1 \quad \text{by } 4.3.3$$

Thus ω_1, ω_2 are irreducible characters of T.

Lemma 4.3.5

$\omega_i{}^K$ are irreducible characters of K, i = 1,2

Proof

Let $\{k_i\}$ be a set of (T,T)-double coset representatives then by Mackey's formula

$$(\omega_i^{\kappa}, \omega_i^{\kappa}) = \sum_{k \in \{k_i\}} ((\omega_i)_{T \cap T^k}, ({}^k\omega_i)_{T \cap T^k})$$

Suppose $((\omega_i)_{T \cap T^k}, ({}^k\omega_i)_{T \cap T^k}) \neq 0$ for some $k \in \{k_i\}$

Then $T \cap T^k = L\langle y \rangle \cap L^k{\langle y \rangle}^k$

$= (K \cap N)C\langle y \rangle \cap (K \cap N)C^k{\langle y \rangle}^k \qquad (K \cap N \triangleleft K)$

$\geqslant K \cap N$

Therefore $((\omega_i)_{K \cap N}, ({}^k\omega_i)_{K \cap N}) \neq 0$ by 1.2.6

But $(\omega_i)_{K \cap N} = \varsigma_{K \cap N}\tau_i(1) = \varsigma_{K \cap N}$. Hence

$(\varsigma_{K \cap N}, {}^k\varsigma_{K \cap N}) \neq 0$ which implies $\varsigma_{K \cap N} = {}^k\varsigma_{K \cap N}$, so

$k \in T$ i.e. $k = 1$

Thus $(\omega_i^{\kappa}, \omega_i^{\kappa}) = (\omega_i, \omega_i) = 1$ by 4.3.4

We can now prove the result we are after

<u>Theorem 4.3.6</u>

$\theta_\lambda = \omega_1^{\kappa}$ and ${}^x\theta_\lambda = \omega_2^{\kappa}$ or vice-versa

<u>Proof</u>

Let $\chi = \chi^{(\lambda;\lambda)}$. Then

$$(\chi_\kappa, \omega_i^{\kappa}) = ((\phi_L)^K, \omega_i^{\kappa}) = \sum_{k \in \{k_i\}} (\phi_{L \cap T^k}, ({}^k\omega_i)_{L \cap T^k})$$

where $\{k_i\}$ is a set of (L,T)-double coset representatives.

Now

$(\phi_{L \cap T^k}, ({}^k\omega_i)_{L \cap T^k}) \neq 0 \Rightarrow (\phi_{K \cap N}, ({}^k\omega_i)_{K \cap N}) \neq 0$

since $L \cap T^k \geqslant K \cap N$

$\Rightarrow (\varsigma_{K \cap N}, {}^k\varsigma_{K \cap N}) \neq 0$

$\Rightarrow k \in T \Rightarrow k = 1$

Thus

$$(\chi_\kappa, \omega_i^{\kappa}) = (\phi_{L \cap T}, (\omega_i)_{L \cap T}) = (\phi_L, (\omega_i)_L)$$
$$= (\phi_L, \phi_L) = 1 \text{ by } 4.3.3$$

Thus $\chi_K = \omega_1^K + \omega_2^K + \theta$ where θ is a character of K such that $(\theta, \omega_i) = 0, i = 1,2$.

But $\chi_K(1) = (\phi_L)^K(1) = |K:L|\phi(1)$

and $(\omega_1^K + \omega_2^K)(1) = |K:T|(\omega_1(1) + \omega_2(1))$

$$= |K:T| 2\phi(1) = |K:L|\phi(1)$$

since $|T:L| = |\langle y\rangle| = 2$.

Hence $\theta(1) = 0$ so $\theta = 0$. Therefore $\chi_K = \omega_1^K + \omega_2^K$

But $\omega_i^K (i=1,2)$ are irreducible and also $\chi_K = \theta_\lambda + {}^x\theta_\lambda$ is a decomposition into irreducible characters of K.

So $\omega_1^K = \theta_\lambda$ and $\omega_2^K = {}^x\theta_\lambda$ or vice-versa

Notation

Since our choice of θ_λ , ${}^x\theta_\lambda$ is completely arbitrary $(x^2 = 1)$ we shall assume from now on that $\theta_\lambda = \omega_1^K$ and ${}^x\theta_\lambda = \omega_2^K$.

The following is well-known, but it will be convenient to prove it here

Corollary 4.3.7

Any complex representation of K may be afforded by a basis with respect to which the matrix entries consist of rational integers. In particular, the characters of K are rational integral-valued.

Proof

From 4.1.2 and 4.3.6 we see that the irreducible representations of K may be obtained from those of the symmetric group by

(i) tensoring these, and various restrictions of these, representations together and with representations which take the values ± 1;

(ii) inducing up representations in (i).

The theorem then follows by 2.1.4, since the operations in (i), (ii) clearly preserve the required properties.

§4.4 Completion of the decomposition of the induced principal character

We now return to the problem of determining which of θ_α and ${}^x\theta_\alpha$ occur in $1_{W_{(\lambda;\mu)}}{}^K$. Of course these may only occur if $(\chi_k^{(\alpha;\alpha)}, 1_{W_{(\lambda;\mu)}}{}^K) \neq 0$ so that $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\alpha)$ by 4.2.2.

So throughout this section assume that l is even, that $(\lambda;\mu)$ and $(\alpha;\alpha)$ are pairs of partitions of l (therefore $\alpha \vdash l/2$) such that no part of μ is 1, and $(\lambda;\mu) \underset{D}{\rightarrow} (\alpha;\alpha)$.

There will be two cases: $(\lambda;\mu)$ good or bad.

Theorem 4.4.1

Suppose $(\lambda;\mu)$ is good. Then

$$(1_{W_{(\lambda;\mu)}}{}^K, \theta_\alpha) = (1_{W_{(\lambda;\mu)}}{}^K, {}^x\theta_\alpha) = \tfrac{1}{2}(1_{W_{(\lambda;\mu)}}{}^K, \chi_k^{(\alpha;\alpha)}) \neq 0$$

Proof

Let $W = W_{(\lambda;\mu)}$. Since $(1_W{}^K, \chi_k^{(\alpha;\alpha)}) \neq 0$, θ_α or ${}^x\theta_\alpha$ occur in $1_W{}^K$.

We shall assume without loss of generality that $(1_W{}^K, \theta_\alpha) = a_\alpha \neq 0$. By 4.2.1, $W^x = W^k$ for some $k \in K$.

Hence

$$ {}^x(1_W)^K = ({}^x1_W)^K \quad (K \triangleleft G) \quad = (1_{W^x})^K \quad (x^2 = 1)$$

$$= (1_{W^k})^K = {}^k(1_W{}^K) \quad \text{similarly}$$

$= 1_W^{\ K}$ since $1_W^{\ K}$ is a class function of K.

Thus $1_W^{\ K} = a_\alpha \theta_\alpha + \ldots$ so $1_W^{\ K} = {}^x 1_W^{\ K} = a_\alpha {}^x\theta_\alpha + \ldots$

i.e. $(1_W^{\ K}, {}^x\theta_\alpha) = a_\alpha \neq 0$.

Since $\chi_k^{(\alpha;\alpha)} = \theta_\alpha + {}^x\theta_\alpha$, the theorem follows immediately.

Theorem 4.4.2

$$\lambda \neq \mu \quad \Rightarrow \quad (1_{W_{(\lambda;\mu)}}^{\ K}, \chi_k^{(\lambda;\mu)}) = 1$$

$$\lambda = \mu \quad \Rightarrow \quad (1_{W_{(\lambda;\lambda)}}^{\ K}, \chi_k^{(\lambda;\lambda)}) = 2$$

and $(1_{W_{(\lambda;\lambda)}}^{\ K}, \theta_\lambda) = 1 = (1_{W_{(\lambda;\lambda)}}^{\ K}, {}^x\theta_\lambda)$

Proof

Let $W = W_{(\lambda;\mu)}$, $\chi = \chi^{(\lambda;\mu)}$, $|\lambda| = m$, $|\mu| = n$.
We shall assume that $W = R(D_{(\lambda;\mu)})$ where $D_{(\lambda;\mu)}$ is a diagram, where D_λ is filled with the symbols $\{1,\ldots,m\}$ which are of the first type and D_μ is filled with $\{m+1,\ldots,l\}$ which are of the second type. Hence $H \cap W \leq C$. So $W = (N \cap W)(H \cap W)$ (4.2.3)

$$\leq (N \cap K)C \qquad (W \leq K)$$
$$= L$$

Also we have that $\varsigma_{W\cap N} = 1$ so $(1_{W\cap N}, \varsigma_{W\cap N}) = 1$.
Now

$$(1_{W\cap L}, \phi_{W\cap L}) = (1_{W\cap N}, \varsigma_{W\cap N})(1_{W\cap C}, \psi_{W\cap C}) \quad (4.2.3)$$

$$= (1_{W\cap N}, \varsigma_{W\cap N})(1_{W\cap S_m}, \chi^\lambda{}_{W\cap S_m})(1_{W\cap S_n}, \chi^\mu{}_{W\cap S_n})$$

But $W \cap S_m = W_\lambda$ and $W \cap S_n = W_\mu$.
So

$$(1_{W\cap L}, \phi_{W\cap L}) = (1_{W\cap N}, \varsigma_{W\cap N})((1_{W_\lambda})^{S_m}, \chi^\lambda)((1_{W_\mu})^{S_n}, \chi^\mu)$$

by Frobenius

$$= 1 \text{ by } 2.2.7$$

By Mackey's formula, if $\{k_i\}$ is a set of (L,W)-double coset representatives, where each $k_i \in H$,

$$(1_W{}^K, \chi_K) = (1_W{}^K, (\phi_L)^K) = \sum_{k\in\{k_i\}} (1_{W^k\cap L}, \phi_{W^k\cap L})$$

Thus the first summand is $(1_{W\cap L}, \phi_{W\cap L}) = 1$, by above

Suppose now that $(1_{W^k\cap L}, \phi_{W^k\cap L}) \neq 0$ for some $k \in \{k_i\}$.

Because $W^k \cap L \geqslant W^k \cap N$ (4.2.3) we have that

$(1_{W^k\cap N}, \phi_{W^k\cap N}) \neq 0$ and hence $(1_{W^k\cap N}, \varsigma_{W^k\cap N}) \neq 0$,

therefore $\varsigma_{W^k\cap N} = 1_{W^k\cap N}$ i.e. ${}^k\varsigma_{W\cap N} = 1$ since $N \lhd G$

But we know that $N = N_1 \times N_2$ (defined by $(\lambda;\mu)$) and by construction of W, $W \cap N = K \cap N_2$. Hence ${}^k\varsigma_{W\cap N_2} = 1$

Thus if $(a,-a), (b,-b) \in N_2$ we have that $\varsigma[(a,-a)(b,-b)]^k = 1$.

Therefore

$$\begin{array}{llll} \text{either} & (a,-a)^k \in N_1 & \text{and} & (b,-b)^k \in N_1 \\ \text{or} & (a,-a)^k \in N_2 & \text{and} & (b,-b)^k \in N_2 \end{array}$$

It follows that $N_1{}^k = N_1$ or $N_1{}^k = N_2$

(a) Suppose $|\lambda| \neq |\mu|$ i.e. $m \neq n$

If $N_1{}^k = N_2$ then $|N_1| = |N_2|$. But $|N_1| = 2^m$, $|N_2| = 2^n$ so $m = n$, a contradiction. Therefore $N_1{}^k = N_1$ and so $N_2{}^k = N_2$ i.e. $k \in C = C_H(\varsigma) \leqslant L$. So $k = 1$ and

$$(1_W{}^K, \chi_K) = (1_{W\cap L}, \phi_{W\cap L}) = 1$$

(b) Suppose $|\lambda| = |\mu|$.

If $N_1{}^k = N_2$ then $N_2{}^k = N_1$. Therefore $N_1{}^{ky} = N_1$ and

$N_2^{ky} = N_2$, so $ky \in C$ which implies $k \in C\langle y\rangle \leqslant L\langle y\rangle$. Thus k is in the same (L,W)- double coset as y, and so we may assume that $k = y$.

So we have shown that at most two summands in the Mackey formula are non-zero and are given by the double coset representatives 1 and y. By 4.2.3

$$0 \neq (1_{W^k \cap L}, \phi_{W^k \cap L}) = (1_{W^k \cap N}, \varsigma_{W^k \cap N})(1_{W^k \cap C}, \Psi_{W^k \cap C})$$

$$= (1_{W^k \cap C}, \Psi_{W^k \cap C}) \quad \text{for } k = 1 \text{ or } y.$$

But $W^k \cap C = (W^k \cap S_m) \times (W^k \cap S_m)$ and $|\lambda| = |\mu|$ so y just interchanges the symbols in D_λ and D_μ. It follows that $W^y \cap S_m = W_\mu$ and $W^y \cap S_n = W_\lambda$. Therefore

$$0 \neq (1_{W^y \cap C}, \Psi_{W^y \cap C}) = (1_{W_\mu}, \chi^\lambda_{W_\mu})(1_{W_\lambda}, \chi^\mu_{W_\lambda})$$

$$= ((1_{W_\mu})^{S_m}, \chi^\lambda)((1_{W_\lambda})^{S_m}, \chi^\mu)$$

by Frobenius.

Therefore, by 2.3.6, $\mu \leqslant \lambda \leqslant \mu$ so $\lambda = \mu$

Hence

(i) $\lambda \neq \mu$ implies that the summand with $k = y$ is zero so that only the first summand is non-zero and as in (a), $(1_W^K, \chi_\kappa) = 1$

(ii) $\lambda = \mu$, the summand with $k = y$ is

$$((1_{W_\lambda})^{S_m}, \chi^\lambda)((1_{W_\lambda})^{S_m}, \chi^\lambda) = 1 \quad \text{by 2.2.7}$$

Thus both the summands with $k = 1$ and $k = y$ contribute the value 1 i.e. $(1_W^K, \chi_\kappa) = 2$.

(N.B. the double cosets LW and LyW are not equal, for, if they were then $y \in LW \leqslant L$ (p 104) $= C_K(\varsigma)$, a contradiction).

Finally, $(1_W^{\ K}, \theta_\lambda) = 1 = (1_W^{\ K}, {}^x\theta_\lambda)$ by 4.4.1, since $\lambda = \mu$ implies $(\lambda;\mu)$ is good.

We now deal with the cases in which $(\lambda;\mu)$ is bad. So for the rest of this section we suppose that $\mu = \emptyset$ and all parts of λ are even, and $\lambda \vdash l$. Hence $\lambda = (2\nu_1, \ldots, 2\nu_r)$ for some partition $\nu = (\nu_1, \ldots, \nu_r)$ of $l/2$. We shall write $\nu = \frac{1}{2}\lambda$ and $\lambda = 2\nu$.

We shall continue to suppose that $\alpha \vdash l/2$ and

$$(\lambda;-) \xrightarrow[D]{} (\alpha;\alpha)$$

<u>Theorem 4.4.3</u>

With the above notation and the remark below

$$(1_{W_{(\lambda;-)}}^{\ K}, \theta_\alpha) = (1_{W_{(\lambda;-)}}^{\ K}, {}^x\theta_\alpha) = (1_{W_\nu}^{\ S_{l/2}}, \chi^\alpha)^2 \neq 0$$

<u>Proof</u>

Let $W = W_{(\lambda;-)}$; $C = C_H(\varsigma)$ corresponds to $\chi = \chi_K^{(\alpha;\alpha)}$ We choose $W = R(D_{(\lambda;-)}) = R(D_\lambda)$ where D_λ is filled with the symbols $\{1,\ldots,l\}$ in the following way :
because $\lambda = 2\nu$, we may write $D_\lambda = D_\nu + D_\nu{}'$, D_ν corresponding to the left-half of D and $D_\nu{}'$ to the right.
Fill D_ν with the symbols $\{1,\ldots,l/2\}$ in the natural ordering and then fill $D_\nu{}'$ with the symbols $\{l/_2+1,\ldots,l\}$ in the natural ordering.

It follows that $W \leqslant H$ and $y \in W$.

<u>Remark</u>

We have two choices for $W_{(\lambda;-)}$ (4.2.1), either W as defined above or W^x $(x = (1,-1))$. But if we use

W^x, then the only effect on the theorem is to interchange θ_α and ${}^x\theta_\alpha$, giving the negative of the left-hand side of the equation in the statement of the theorem. The proof of the theorem, using W^x, will be exactly the same as the proof we give below for W, and so we might as well suppose $W_{(\lambda;-)} = W$. In fact as using W^x only leads to a change in notation, we will in future assume $W_{(\lambda;-)} = W \leqslant H$ the symmetric group on $\{1,\ldots,l\}$.

Before continuing with the proof of the theorem, we will prove a couple of preliminary lemmas

Lemma 4.4.4 (compare with 2.1.2)

Let $z \in H$, $c \in C$, $w \in W$. Then $cy = zwz^{-1}$ implies $z \in CW$.

Proof

Since all the elements in the statement of the lemma are inside H, we can work in the symmetric group. Now $W = R(D_\lambda)$, so by 2.1.1, $zwz^{-1} \in R(zD_\lambda)$. Also cy does not have a fixed point in $\{1,\ldots,l\}$ because $cy(D_\nu) = c(D_\nu') = D_\nu'$.

Consider first, the top row of zD_λ. Let $(a_1 \ldots a_r)$ be a cycle in the decomposition of cy such that $a_1, \ldots, a_r$ occur in the top row of zD_λ. As $cy(D_\nu) = D_\nu'$, either a_1 or $a_2 \in D_\nu$ and, by writing $(a_2 \ldots a_r\, a_1)$ if necessary, we may suppose $a_1 \in D_\nu$. Hence $a_1 \in D_\nu$, $a_2 \in D_\nu'$, $a_3 \in D_\nu$, ... and because $cy(a_r) = a_1$, we have $a_r \in D_\nu'$ so that r is even. Thus

$$a_1, a_3, \ldots, a_{r-1} \in D_\nu$$
$$\text{and } a_2, a_4, \ldots, a_r \in D_\nu'$$

Now we also have, by construction,

$$1\ ,\ 2\ ,\ \dots\ ,\ r/2\ \in D_\nu$$

$$1/2+1\ ,\ 1/2+1\ ,\ \dots\ ,\ 1/2+r/2\ \in D_\nu:$$

Set $c_1 = (1\ a_1)(2\ a_3)\dots(r/2\ a_{r-1})(1/2+1\ a_2)(1/2+2\ a_4)\dots(1/2+r/2$

Then $c_1 \in C$.

So the top row of $c_1 z D_\lambda$ contains the symbols $\{1,2,\dots,r/2,1/2+1,1/2+2,\dots,1/2+r/2\}$ in some order .

Let $z_1 = c_1 z$ then $R(z_1 D_\lambda) = c_1 R(zD_\lambda)c_1^{-1}$, so $c_1(cy)c_1^{-1} \in R(z_1 D_\lambda)$. But $c_1(cy)c_1^{-1} = (c_1 c y c_1^{-1} y)y$. Then set $c_2 = c_1 c y c_1^{-1} y \in C$ (as $C^y = C$) so $c_2 y \in R(z_1 D_\lambda)$, and therefore

$$c_2 y = z_1 w' z_1^{-1} \quad \text{for some } w' \in W.$$

But $c_2 y$ is easily seen to contain the cycle $(1\ 1/2+1\ \dots\ r/2\ 1/2+r/2)$ and therefore we may apply the same process as before to the rest of the elements in the top row.

Repeating this process enough times we obtain a diagram $z_2 D_\lambda$ with $z_2 = c_3 z$, $c_3 \in C$, and such that $z_2 D_\lambda$ has the same symbols in its top row (in some order) as D_λ . Remembering that cy has no fixed points, we may repeat the process with the other rows to obtain a diagram $z^* D_\lambda$ such that $z^* = c^* z$, $c^* \in C$ and $z^* D_\lambda$ has the same symbols (in some order) in each of its rows as D_λ . Therefore there exists $w^* \in W$ such that $w^* z^* D_\lambda = D_\lambda$ i.e. $w^* z^* = 1$ which implies $z^* \in W$. Finally, $z = c^{*-1} z^* \in CW$ as required.

We let $T = C_K(\epsilon_{K\cap N}) = L\langle y\rangle$ as usual.

<u>Lemma 4.4.5</u>

If $z \in H$ then

$$T \cap zWz^{-1} = (C \cap zWz^{-1})(\langle y\rangle \cap zWz^{-1})$$
$$= (L \cap zWz^{-1})(\langle y\rangle \cap zWz^{-1})$$

Proof

Firstly, $L \cap zWz^{-1} = (K \cap N)C \cap zWz^{-1}$

$\leqslant (K \cap N)C \cap H$ as $W \leqslant H$, $z \in H$

$= C$

Because also $C \leqslant L$, $L \cap zWz^{-1} = C \cap zWz^{-1}$. Thus it is sufficient to prove the first equality. Trivially $(C \cap zWz^{-1})(\langle y\rangle \cap zWz^{-1}) \leqslant T \cap zWz^{-1}$.

Conversely, let $t \in T \cap zWz^{-1} = L\langle y\rangle \cap zWz^{-1}$

Therefore $t = ly' = zwz^{-1}$, where $y' = y$ or 1, $w \in W$, $l \in L$

But $L = (K\cap N)C$ so $l = nc$, $n \in N$, $c \in C$.

Hence $ncy' = zwz^{-1} \Rightarrow n = (zwz^{-1})y'^{-1}c^{-1} \in H$

$\Rightarrow n \in N\cap H = 1$

Thus $cy' = zwz^{-1}$. If

(a) $y' = 1$ then $c = zwz^{-1}$ so that $t = c = zwz^{-1} \in C \cap zWz^{-1}$ which is a subgroup of $(C\cap zWz^{-1})(\langle y\rangle\cap zWz^{-1})$

(b) $y' = y$ then $cy = zwz^{-1}$, so by 4.4.4, $z \in CW$.

Hence $z = c_1w_1$, $c_1 \in C$, $w_1 \in W$. Therefore

$cy = c_1w_1ww_1^{-1}c_1^{-1} \Rightarrow c = yc_1w_1ww_1^{-1}c_1^{-1}$

$= c_1(yw_1ww_1^{-1})c_1^{-1}$ as $y \in C_H(C)$

$\in c_1Wc_1^{-1}$

$= zWz^{-1}$

Thus $c \in C \cap zWz^{-1}$. As $cy \in zWz^{-1}$ and $c \in zWz^{-1}$ we have $y \in \langle y\rangle \cap zWz^{-1}$ so that

$t = cy \in (C \cap zWz^{-1})(\langle y\rangle \cap zWz^{-1})$

proving the lemma.

We return now to the proof of the theorem.

Let $\theta = \theta_\alpha$ or ${}^x\theta_\alpha$ and $\tau_1 = 1$, $\tau_2 = \tau$, therefore

$\omega_i = \phi_L \tau_i \quad (i=1,2)$

$$(1_W^{\ K}, \theta) = (1_W^{\ K}, \omega_i^{\ K}) \quad (4.3.6)$$

$$= \sum_{z \in \{z_i\}} ({}^z 1_{T \cap W^z}, (\omega_i)_{T \cap W^z})$$

where $\{z_i\}$ is a set of (T,W)-double coset representatives and each $z_i \in H$. So by 4.4.5

$$(1_W^{\ K}, \theta) = \sum_{z \in \{z_i\}} ({}^z 1_{L \cap W^z}, \phi_{L \cap W^z})({}^z 1_{y \cap W^z}, (\tau_i)_{y \cap W^z})$$

... (A)

by definition of ω.

But $\langle y \rangle \cap zWz^{-1} = 1 \quad \Rightarrow \quad y \in zWz^{-1}$

$\Rightarrow \quad y = zwz^{-1}$ some $w \in W$

$\Rightarrow \quad z \in CW \leqslant TW$ by 4.4.4

$\Rightarrow \quad z = 1$

Conversely, as $y \in W$, $z = 1 \Rightarrow \langle y \rangle \cap zWz^{-1} \neq 1$

Now

$$(\tau_1)_{\langle y \rangle \cap W^z} = {}^z 1_{\langle y \rangle \cap W^z} \qquad \text{for all } z$$

$$(\tau_2)_{\langle y \rangle \cap W^z} = {}^z 1_{\langle y \rangle \cap W^z} \iff \langle y \rangle \cap W^z = 1 \iff z = 1$$

Hence

$$({}^z 1_{\langle y \rangle \cap W^z}, (\tau_i)_{\langle y \rangle \cap W^z}) = \begin{cases} 0 & \text{if } z = 1 \text{ and } i = 2 \\ 1 & \text{otherwise} \end{cases}$$

So from (A)

$$(1_W^{\ K}, \theta_\alpha) - (1_W^{\ K}, {}^x\theta_\alpha) = (1_{L \cap W}, \phi_{L \cap W})$$

(i.e. the decompositions of the Mackey formula only differ in the first summand)

However, as in the proof of 4.4.5, $L \cap W = C \cap W$.

We let $B = (1_W{}^K, \theta_\alpha) - (1_W{}^K, {}^x\theta_\alpha) = (1_{C\cap W}, \phi_{C\cap W})$

Then we only have to show $B = (1_{W_\nu}{}^{S_{1/2}}, \chi^\alpha)^2 \neq 0$.

$$B = (1_{C\cap W}, \phi_{C\cap W})$$

$$= (1_{S_m\cap W}, \chi^\alpha{}_{S_m\cap W})(1_{S_m\cap W}, \chi^\alpha{}_{S_m\cap W})$$

as $C \cap W = (S_m \cap W) \times (S_m \cap W)$ where $m = \frac{1}{2}$

$$= (1_{S_m\cap W}{}^{S_m}, \chi^\alpha)^2$$

But by the construction of W, $S_m \cap W = W_\nu$, so

$B = (1_{W_\nu}{}^{S_m}, \chi^\alpha)^2$.

Finally, in order to show $B \neq 0$ it is sufficient, by 2.3.6, to show that $\nu \leqslant \alpha$.

By assumption, $(\lambda;-) \underset{D}{\rightarrow} (\alpha;\alpha)$ so $(\lambda;-) \underset{C}{\rightarrow} (\alpha;\alpha)$ as on p 96. Therefore $(\lambda;-) \underset{C}{\rightarrow} (\alpha;\alpha) \underset{C}{\rightarrow} (-;2\alpha)$ so that $(\lambda;-) \leqslant (-;2\alpha)$ which implies $\lambda \leqslant 2\alpha$ by moving the whole of λ across to the right-hand side.

Now $2\alpha = (2\alpha_1, \ldots, 2\alpha_s)$ and $\lambda = (2\nu_1, \ldots, 2\nu_r)$ so that

$$\lambda \leqslant 2\alpha \quad \Rightarrow \quad \sum_{i=1}^{m} 2\nu_i \leqslant \sum_{i=1}^{m} 2\alpha_i \quad \text{for all } m$$

$$\Rightarrow \quad \sum_{i=1}^{m} \nu_i \leqslant \sum_{i=1}^{m} \alpha_i \quad \text{for all } m$$

$$\Rightarrow \quad \nu \leqslant \alpha \quad \text{, completing the theorem.}$$

Finally, we prove

<u>Theorem 4.4.6</u>

With the notation of 4.4.3

$$(1_{W_{(\lambda;-)}}{}^K, \theta_\alpha) \neq 0 \iff \nu \leqslant \alpha \iff (1_{W_{(\lambda;-)}}{}^K, \chi_K^{(\alpha;\alpha)}) \neq 0$$

and $(1_{W_{(\lambda;-)}}{}^K, {}^x\theta_\alpha) \neq 0 \iff \nu < \alpha$

Again we need a preliminary lemma, which uses the same notation as the theorem

Lemma 4.4.7

$$(1_{W_{(\lambda;-)}}{}^{K}, \chi_{K}^{(\nu;\nu)}) = 1$$

Proof

Let $W = W_{(\lambda;-)}$ be the Weyl subgroup of K defined in 4.4.3 so that $W \leqslant H$.

$$(1_W{}^{K}, \chi_K^{(\nu;\nu)}) = \sum_{z\in\{z_i\}} (1_{L\cap W^z}, \phi_{L\cap W^z})$$

where $\{z_i\}$ is a set of (L,W)-double coset representatives, with $z_i \in H$. Hence, as in 4.4.5, $L \cap W^z = C \cap W^z$.

Thus

$$(1_W{}^{K}, \chi_K^{(\nu;\nu)}) = \sum_{z\in\{z_i\}} (1_{C\cap W^z}, \phi_{C\cap W^z})$$

$$= \sum_{z\in\{z_i\}} (1_{C\cap W^z}, \psi_{C\cap W^z})$$

Suppose $(1_{C\cap W^z}, \psi_{C\cap W^z}) \neq 0$ for some $z \in \{z_i\}$.

We may as well assume that in $zD_{(\lambda;-)}$ (where $W = R(D_{(\lambda;-)})$) all the symbols of the second type lie at the ends of rows of $zD_{(\lambda;-)}$ as this only has the effect of multiplying z by an element $w \in W$, which is in the same (L,W)-double coset as z.

Thus $C \cap W^z = W_\gamma \times W_\delta$ where $(\gamma;\delta)$ is a pair of partitions of 1 with $\gamma, \delta \vdash \frac{1}{2}$, and $D_{(\gamma;\delta)}$ is obtained by moving the squares in $D_{(\lambda;-)}$ containing symbols of the second type over to the right-hand side, and reordering the two resulting diagrams. Therefore

$$0 \neq (1_{C\cap W^z}, \psi_{C\cap W^z}) = (1_{W_\gamma}, \chi^\nu_{W_\gamma})(1_{W_\delta}, \chi^\nu_{W_\delta})$$

$$= ((1_{W_\gamma})^{S_{1/2}}, \chi^\nu)((1_{W_\delta})^{S_{1/2}}, \chi^\nu)$$

so by 2.3.6, $\gamma \leq \nu$ and $\delta \leq \nu$.

We shall show that $\gamma = \nu$ and $\delta = \nu$. Hence $zD_{(\lambda;-)} = D_\gamma + D_\delta = D_\nu + D_\nu' = D_{(\lambda;-)}$. So $z=1$.

i.e. $(1_W{}^K, \chi_k^{(\nu;\nu)}) = (1_{CnW}, \Psi_{CnW})$

$$= ((1_{W_\nu})^{S_{1/2}}, \chi^\nu)((1_{W_\nu})^{S_{1/2}}, \chi^\nu)$$

$= 1$ by 2.2.7, as required.

So we have only left to show $\gamma = \delta = \nu$.

By construction of γ, δ, for all k there exist i_k, j_k such that $\lambda_k = 2\nu_k = \gamma_{i_k} + \delta_{j_k}$ where $\gamma = (\gamma_1, \ldots, \gamma_s)$ and $\delta = (\delta_1, \ldots, \delta_s)$ (add zeros to ensure that γ and δ have the same number of parts) and $\lambda = (2\nu_1, \ldots, 2\nu_s)$ (automatically λ has s parts).

Putting $k = 1$, $\gamma_{i_1} + \delta_{j_1} = 2\nu_1$.

But $\gamma_{i_1} \leq \gamma_1 \leq \nu_1$ since $\gamma \leq \nu$ and similarly

$\delta_{i_1} \leq \delta_1 \leq \nu_1$ since $\delta \leq \nu$.

Therefore $\gamma_{i_1} = \delta_{j_1} = \nu_1$ and $\gamma_1 = \delta_1 = \nu_1$. This starts off the induction.

Suppose, for $k < r$, we have $\gamma_k = \delta_k = \nu_k$. Since $\gamma \leq \nu$

$\sum_{i=1}^{r} \gamma_i \leq \sum_{i=1}^{r} \nu_i$ we have $\gamma_r \leq \nu_r$ and similarly $\delta_r \leq \nu_r$.

Now $\gamma_{i_r} + \delta_{i_r} = 2\nu_r$ and we already have $\gamma_k = \delta_k = \nu_k$ for $k < r$. So $i_r \geq r$ and $j_r \geq r$. Hence

$\gamma_{i_r} \leq \gamma_r \leq \nu_r$ and $\delta_{j_r} \leq \delta_r \leq \nu_r$ and so $\gamma_r = \delta_r = \nu_r$.

Therefore by induction, $\gamma_k = \delta_k = \nu_k$ for all k

i.e. $\gamma = \delta = \nu$, proving the lemma.

Proof of 4.4.6

$$(1_W{}^K, \chi_K^{(\alpha;\alpha)}) \neq 0 \iff (\lambda;-) \xrightarrow[D]{} (\alpha;\alpha) \quad (4.2.2)$$

$$\iff \nu \leq \alpha \text{ as in the proof of 4.4.3}$$

$$\iff ((1_{W_\nu})^{S_m}, \chi^\alpha)^2 \neq 0 \text{ by 2.3.6}$$

$$\iff (1_W{}^K, \theta_\alpha) - (1_W{}^K, {}^x\theta_\alpha) \neq 0 \text{ by 4.4.3}$$

$$\iff (1_W{}^K, \theta_\alpha) \neq 0$$

proving the first part of the theorem.

Now let $\nu = \alpha$. Then

$$(1_W{}^K, \theta_\nu) + (1_W{}^K, {}^x\theta_\nu) = (1_W{}^K, \chi_K^{(\nu;\nu)}) = 1 \text{ by 4.4.7}$$

By the first part $(1_W{}^K, \theta_\nu) \neq 0$. Hence

$(1_W{}^K, \theta_\nu) = 1$ and $(1_W{}^K, {}^x\theta_\nu) = 0$. Therefore

$$(1_W{}^K, {}^x\theta_\alpha) \neq 0 \Rightarrow \nu \neq \alpha \text{ and } (1_W{}^K, \chi_K^{(\alpha;\alpha)}) \neq 0$$

$$\Rightarrow \nu \neq \alpha \text{ and } \nu \leq \alpha \text{ as above}$$

$$\Rightarrow \nu < \alpha$$

Finally, suppose $\nu < \alpha$ then we show $(1_W{}^K, {}^x\theta_\alpha) \neq 0$ which will finish the theorem.

By the proof of 4.4.3 (p 96)

$$(1_W{}^K, {}^x\theta_\alpha) = \sum_{z \in \{z_i\}} ({}^z1_{L \cap W^z}, \phi_{L \cap W^z})({}^z1_{\langle y \rangle \cap W^z}, (\tau_i)_{\langle y \rangle \cap W^z})$$

where $\{z_i\}$ is a set of (T,W)-double coset representatives, $z_i \in H$, and

$$({}^z1_{\langle y \rangle \cap W^z}, (\tau_i)_{\langle y \rangle \cap W^z}) = \begin{cases} 1 & \text{if } z \neq 1 \\ 0 & \text{if } z = 1 \end{cases}$$

Also

$$(1_{L \cap W^z}, \phi_{L \cap W^z}) = (1_{C \cap W^z}, \Psi_{C \cap W^z}) \quad \text{by 4.4.5}$$

$$= ((1_{W_\gamma})^{S_m}, \chi^\alpha)((1_{W_\delta})^{S_n}, \chi^\alpha)$$

where $S_m \cap zWz^{-1} = W_\gamma$, $S_n \cap zWz^{-1} = W_\delta$ and $m = l/2 = n$. We shall choose a $z \notin TW$ (below) such that $\gamma \leq \alpha$ and $\delta \leq \alpha$ (γ, δ depend on z). Then by 2.3.6, $(1_W{}^K, x_{\theta_\alpha}) \neq 0$ as $z \neq 1$.

It will be sufficient to choose $z \notin CW$. We have that $\nu < \alpha$. Let $\nu = \mu^{(0)} < \mu^{(1)} < \dots < \mu^{(r)} = \alpha$ where $\mu^{(i)}$ is obtained from $\mu^{(i-1)}$ by moving up one square. Let $\nu = (\nu_1, \dots, \nu_r)$. Then

$$\mu^{(1)} = (\nu_1, \dots, \nu_i+1, \dots, \nu_j-1, \dots, \nu_r)$$

some $i < j$ (rearranged to give a partition).
Let $\beta = (\nu_1, \dots, \nu_i-1, \dots, \nu_j+1, \dots, \nu_r)$ rearranged to give a partition of $l/2$.
It is easy to see that $\beta \leq \mu^{(1)} < \alpha$. Thus $\mu^{(1)} < \alpha$ and $\beta < \alpha$.

Now $D_{(\lambda;-)} = D_\lambda = D_\nu + D_\nu{}'$ where D_ν is filled with $\{1, \dots, l/2\}$ in the natural order, and $D_\nu{}'$ is filled with $\{l/2+1, \dots, l\}$ in the natural order. We may therefore obtain a diagram $D_{\mu^{(1)}}$ from D_ν by moving a square containing the symbol $a \in \{1, \dots, l/2\}$ and D_β may be obtained from $D_\nu{}'$ by moving a square containing the symbol $b \in \{l/2+1, \dots, l\}$. Then a, b lie in rows j and i respectively, of D_λ.

Let $z = (ab) \in H$. Then to form zD_λ we just swap the symbols a and b. It follows then that $\gamma = \mu^{(1)}$ and $\delta = \beta$, so $\gamma < \alpha$ and $\delta < \alpha$, and therefore $\gamma \leq \alpha$ and $\delta \leq \alpha$. We have left to show $z \notin CW$.

Suppose, for a contradiction, $z \in CW$. Then $z = c_1 w$ with $c_1 \in C$, $w \in W$, and so $c(ab) = w$ where

$c = c_1^{-1} \in C$. Express c as a product of disjoint cycles. One of these cycles must contain b, otherwise $w(a) = b$, an impossibility, as we have chosen a and b to lie in different rows of D_λ. Therefore c contains a cycle $(b\ d_1 \ldots d_t)$, and since the cycles are disjoint $c = x(b\ d_1 \ldots d_t)$ where x does not contain any of the symbols $b, d_1, \ldots, d_t$.

Suppose $a \neq d_k$, $1 \leq k \leq t$. Then $w = x(b\ d_1 \ldots d_t)(ab) \in W$. Thus $w(a) = d_1$, $w(d_1) = d_2$, ..., $w(d_t) = b$, and so all the symbols $a, d_1, d_2, \ldots, d_t, b$ are collinear in D_λ, again an impossibility.

Thus for some k, $a = d_k$. However, $C \cong S_{l/2} \times S_{l/2}$, so we can assume each cycle lies in one of the symmetric groups and is therefore in C. Thus $(b\ d_1 \ldots d_t) \in C$ and because $a = d_k$ some k, $z = (ab) \in C$, a contradiction since $a \in \{1, \ldots, l/2\}$ and $b \in \{l/2+1, \ldots, l\}$. This contradiction shows that $z \in CW$ and completes the theorem.

§4.5 Solomon's decomposition of the group algebra of $W(D_l)$

We interpret Solomon [17] for the Weyl group $W(D_l)$. As usual, we may assume that all modules, representations and characters are over the field of complex numbers.

The generating set I for $K = W(D_l)$ is $\{(12), (23), \ldots, (l-1\ l), (l-1, -l)\}$ and the parabolic subgroups of K are the Weyl subgroups $W_{(\alpha;\beta)}$ such that β has only 1 or 0 parts.

The results for K are more complicated than those for G, as will be illustrated in the examples below.

We shall therefore confine ourselves to determining $\Lambda^p V$ of [17] where $V = R^l$ ([3], table IV).

Let $J \subseteq I$, then the parabolic subgroup $W_J = W_{(\rho;\sigma)}$ for some pair of partitions $(\rho;\sigma)$ of l such that σ has only 1 or 0 parts and $\sigma \neq (1)$. We can then write $p(J) = (\rho;\sigma)$.

Fix an arbitrary subset J of I, let $\hat{J}$ be the complement of J in I, and $p(J) = (\rho;\sigma)$, $p(\hat{J}) = (\beta';\alpha')$.

Define

$$\xi_J = \sum_{w \in W_J} w \quad \text{and} \quad \eta_J = \sum_{w \in W_{\hat{J}}} \varepsilon(w) w$$

so that $A\xi_J \eta_{\hat{J}}$ affords the character

$$\psi_J = \sum_{J \subseteq M \subseteq I} (-1)^{|M-J|} 1_{W_M}^K \qquad (17)$$

<u>Theorem 4.5.1</u>

Let $(\lambda;\mu)$ be a pair of partitions of l. Then

$$(\psi_J, \chi_K^{(\lambda;\mu)}) \neq 0 \quad \Rightarrow \quad (\rho;\sigma) \xrightarrow[D]{} (\lambda;\mu) \text{ and } (\beta';\alpha') \xrightarrow[D]{} (\mu';\lambda')$$

<u>Proof</u>

As in previous chapters

$$(\psi_J, \chi_K^{(\lambda;\mu)}) \neq 0 \Rightarrow (1_{W_{(\rho;\sigma)}}^K, \chi_K^{(\lambda;\mu)}) \neq 0 \text{ and } (\varepsilon_{W_{(\beta';\alpha')}}^K, \chi_K^{(\lambda;\mu)}) \neq 0$$

$$\Rightarrow (1_{W_{(\rho;\sigma)}}^K, \chi_K^{(\lambda;\mu)}) \neq 0 \text{ and } (1_{W_{(\beta';\alpha')}}^K, \chi_K^{(\mu';\lambda')}) \neq 0 \quad \text{by 3.2.1}$$

$$= (\rho;\sigma) \xrightarrow[D]{} (\lambda;\mu) \quad \text{and} \quad (\beta';\alpha') \xrightarrow[D]{} (\mu';\lambda') \quad \text{by 4.2.2}$$

<u>Examples</u>

(a) It is possible that $(\psi_J, \chi_K^{(\rho;\sigma)}) = 0$ (cf. 3.5.2)

Let $J = \{(12),(23),\ \dots\ ,(l-1\ l)\}$ so $(\rho;\sigma) = p(J) = (l;-)$.
Thus $M \supseteq J$ implies $M = J$ or $M = I$, and $W_J \leqslant H \leqslant G$.
Hence

$$(\psi_J , \chi_K^{(\rho;\sigma)}) = (1_{W_J}^{K} , \chi_K^{(\rho;\sigma)}) - (1_{W_I}^{K} , \chi_K^{(\rho;\sigma)})$$

$$= (1_{W_J}^{K} , \chi_K^{(\sigma;\rho)}) - (1_K , \chi_K^{(\sigma;\rho)}) \qquad (4.1.2)$$

$$= (1_{W_{(l;-)}}^{G} , \chi_K^{(-;l)}) - (1_K , \chi_K^{(-;l)})$$

by Frobenius

$= 1 - (1_K , 1_K)$ by 3.3.9 and the fact that $\chi^{(-;l)} = 1_G$ from the definition in §3.1

$= 1 - 1 = 0$

(b) Similarly, it is possible that $(\psi_J , \chi_K^{(\alpha;\beta)}) = 0$ (cf. 3.5.2)

Let $\hat{J} = \{(12),(23),\ \dots\ ,(l-1\ l)\}$ so $(\beta';\alpha') = p(J) = (l;-)$.
As for (a), $(\psi_{\hat{J}} , \chi_K^{(\beta';\alpha')}) = 0$. Now by [17] lemma 7, $\psi_{\hat{J}} = \varepsilon\psi_J$, and so by 3.2.1 $(\psi_J , \chi_K^{(\alpha;\beta)}) = 0$.

We now wish to identify $\Lambda^p V$ so we suppose $|\hat{J}| = p$.

<u>Definition</u>

Let $(\lambda;\mu)$ be the pair of partitions of l given by $(\lambda;\mu) = (1^p ; l-p)$. We call $(\lambda;\mu)$ the <u>hook graph for J</u> and $\chi_K^{(\lambda;\mu)}$ the <u>hook character of J</u>.

The hook graph $(\lambda;\mu)$ depends only on the order of J and $\chi_K^{(\lambda;\mu)}(1) = \binom{l}{p}$ by 3.4.3.

Now $\lambda \neq \mu$ and hence $\chi_K^{(\lambda;\mu)}$ is irreducible, unless $l = 2$ and $p = 1$. However, when $l = 2$, K is a decomposable Coxeter group and therefore excluded from Solomon's consideration ([17] theorem 4), and in this case $\Lambda^1 V = V$

is reducible. We shall therefore assume for the purposes of this section that $l \geqslant 3$.

The following lemma may be proved in precisely the same way as 3.5.3

Lemma 4.5.2

(i) The number of rows of ρ = $r(\rho)$ = p

(ii) $(\psi_J , \chi_K^{(\lambda;\mu)}) = 1$

Theorem 4.5.3

Let χ be the irreducible character of K afforded by $\Lambda^p V$. Then $\chi = \chi_K^{(\lambda;\mu)}$

Proof

The proof is somewhat more complex than that for G.

χ is irreducible, so $\chi = \chi_K^{(\gamma;\delta)}$ for some pair of partitions $(\gamma;\delta)$ of K such that $\gamma \neq \delta$, or $\chi = \theta_\alpha$ or ${}^x\theta_\alpha$ for some partition α of $l/2$.

Let $J = \{(p+1\ p+2), \dots ,(l-1\ l),(l-1,-l)\}$

hence $\hat{J} = \{(12),(23), \dots ,(p\ p+1)\}$ so that $|\hat{J}| = p$.

Then $(\rho;\sigma) = p(J) = (1^p ; l-p) = (\lambda;\mu)$. By [17] $\Lambda^p V$ is an irreducible submodule of $A\xi_J\eta_{\hat{J}}$ and therefore $(\psi_J , \chi) \neq 0$. So $(\psi_J , \chi_K^{(\gamma;\delta)}) \neq 0$ or $(\psi_J , \theta_\alpha) \neq 0$ or $(\psi_J , {}^x\theta_\alpha) \neq 0$. In the last two cases $(\psi_J , \chi_K^{(\alpha;\alpha)}) \neq 0$.

Therefore by 4.5.1, $(\rho;\sigma) \xrightarrow[D]{} (\gamma;\delta)$ or $(\rho;\sigma) \xrightarrow[D]{} (\alpha;\alpha)$.

If we allow $\gamma = \delta = \alpha$ then we can put these results together as $(\rho;\sigma) \xrightarrow[D]{} (\gamma;\delta)$

i.e. $(1^p ; l-p) \xrightarrow[D]{} (\gamma;\delta)$.

Now let $J_1 = \{(12), \dots ,(l-p\ l-p+1)\}$

so $\hat{J}_1 = \{(l-p+1\ l-p+2), \dots ,(l-1\ l),(l-1,-l)\}$

Then $(\beta';\alpha') = (1^{1-p} ; p) = (\mu';\lambda')$.

Again $(\psi_{J_1}, \chi) \neq 0$ so (allowing $\gamma = \delta = \alpha$) by 4.5.1 $(\beta';\alpha') \xrightarrow[D]{} (\delta';\gamma')$. Thus $(1^{1-p} ; p) \xrightarrow[D]{} (\delta';\gamma')$

Hence

$$(1^p ; 1-p) \xrightarrow[D]{} (\gamma;\delta) \quad \text{and} \quad (1^{1-p} ; p) \xrightarrow[D]{} (\delta';\gamma')$$

We break the proof up into four cases:

(a) Suppose $(1^p ; 1-p) \xrightarrow[C]{} (\gamma;\delta)$ and $(1^{1-p} ; p) \xrightarrow[C]{} (\delta';\gamma')$. Then by 3.3.5, $(\gamma;\delta) \xrightarrow[C]{} (1^p ; 1-p)$ and since $\xrightarrow[C]{}$ is anti-symmetric

$$(\gamma;\delta) = (1^p ; 1-p) \quad (\text{so } \gamma \neq \delta)$$

and $\chi = \chi_K^{(\lambda:\mu)}$ as required.

(b) Suppose $(1^p ; 1-p) \not\xrightarrow[C]{} (\gamma;\delta)$ and $(1^{1-p} ; p) \not\xrightarrow[C]{} (\delta';\gamma')$. Then the right-hand row must be moved to the left in both cases. Therefore $|\delta| \leqslant p$ and $|\gamma| = |\gamma'| \leqslant 1-p$. However $|\gamma| + |\delta| = 1$, therefore $|\delta| = p$, $|\gamma| = 1-p$. It follows that $\delta = 1^p$, $\gamma' = 1^{1-p}$. Therefore $(\gamma;\delta) = (1-p ; 1^p)$. So $\chi = \chi_K^{(1-p:1^p)} = \chi_K^{(1^p:1-p)} = \chi_K^{(\lambda:\mu)}$

(c) Suppose $(1^p ; 1-p) \xrightarrow[C]{} (\; ; \;)$ but $(1^{1-p} ; p) \not\xrightarrow[C]{} (\delta';\gamma')$. Therefore by 3.3.5, $(\delta';\gamma') \xrightarrow[C]{} (1^{1-p} ; p)$ so that $r(\delta') \geqslant 1-p$.

Also $(1^{1-p} ; p) \not\xrightarrow[C]{} (\delta';\gamma')$ means we have to move the row of length p over to the left-hand side. Thus

<u>either</u> $\gamma' = 1$ and $\delta' = (p,1^{1-p-1})$

<u>or</u> $\gamma' = 0$ and $\delta' \geqslant (p,1^{1-p})$ and because $r(\delta') \geqslant 1-p$

$$\delta' = (p,1^{1-p}) \text{ or } (p+1,1^{1-p-1}) \text{ or } (p,2,1^{1-p-2}).$$

Hence

$$\begin{aligned}(\gamma;\delta) &= (1 ; (1-p),1^{p-1}) \\ &\text{or } (- ; (1-p+1),1^{p-1}) \\ &\text{or } (- ; (1-p),1^p) \\ &\text{or } (- ; (1-p),2,1^{1-p-2})\end{aligned}$$

We see from this that $\gamma \neq \delta$ therefore $\chi = \chi_K^{(\gamma;\delta)}$.

Now $(\psi_{J_1}, \chi_K^{(\gamma;\delta)}) \neq 0$ so $(1_{W_{J_1}}^{K}, \chi_K^{(\gamma;\delta)}) \neq 0$ (1.2.8)

But $W_{J_1} \leqslant H$ so $W_{J_1} = W_{(l-p+1,1^{p-1})}$ as a Weyl subgroup of H. Suppose that $\gamma = 0$. Therefore

$$0 \neq (1_{W_{J_1}}^{K}, \chi_K^{(-;\delta)}) = (1_{W_{(l-p+1,1^{p-1})}}^{H}, \chi_H^{(-;\delta)})$$

by Frobenius

$$= (1_{W_{(l-p+1,1^{p-1})}}^{H}, \chi^{\delta})$$

using 3.1.3(ii).

Therefore by 2.2.7, $\delta \geqslant (l-p+1,1^{p-1})$. But we have already restricted δ above.

Thus $\gamma = 0 \Rightarrow \delta = (l-p+1,1^{p-1})$.

So $(\gamma;\delta) = (1 ; l-p,1^{p-1})$ or $(- ; l-p+1,1^{p-1})$.

But χ is afforded by $\Lambda^p V$ so $\chi(1) = \dim \Lambda^p V = \binom{l}{p}$.

i.e. $\chi^{(\gamma;\delta)}(1) = \binom{l}{p}$.

If $\gamma = 0$ then $\chi^{(\gamma;\delta)}(1) = \dfrac{l!}{l(l-p)!(p-1)!}$ using 3.4.3.

Equating this with $\binom{l}{p}$, we see that $p = l$, so $(\gamma;\delta) = (-;1^l)$ and therefore $\chi = \chi_K^{(\gamma;\delta)} = \chi_K^{(\delta;\gamma)} = \chi_K^{(1^l;-)}$

$= \chi_K^{(\lambda;\mu)}$ for $p = l$

If $\gamma = 1$ then $\chi^{(\gamma;\delta)}(1) = \dfrac{l!}{(l-1)(l-p-1)!(p-1)!}$

and equating this with $\binom{l}{p}$, we find $p = l-1$ or $p = 1$.

Hence $(\gamma;\delta) = (1 ; 1^{l-1})$ or $(1 ; l-1)$

$= (\lambda;\mu)$ or $(\mu;\lambda)$ respectively

and again

$$\chi = \chi_K^{(\gamma;\delta)} = \chi_K^{(\delta;\gamma)} = \chi_K^{(\lambda;\mu)}$$

Finally,

(d) Suppose $(1^p ; l-p) \not\xrightarrow[C]{} (\gamma;\delta)$ but $(1^{l-p} ; p) \xrightarrow[C]{} (\delta';\gamma')$.

Therefore by 3.3.5, $(\gamma;\delta) \xrightarrow[C]{} (1^p ; l-p)$ so that $r(\gamma) \geqslant p$.

Also $(1^p\ ;\ l-p) \underset{C}{\not\to} (\gamma;\delta)$ means we have to move the row of length $l-p$ over to the left-hand side. Thus

either $\delta = 1$ and $\gamma = (l-p, 1^{p-1})$

or $\delta = 0$ and $\gamma = (l-p, 1^p)$ or $(l-p+1, 1^{p-1})$

or $(l-p, 2, 1^{p-2})$

and so $\gamma \neq \delta$.

But $\chi = \chi_K^{(\gamma;\delta)} = \chi_K^{(\delta;\gamma)}$ and these were exactly the cases covered in (c). So the same argument shows $\chi = \chi_K^{(\lambda;\mu)}$.

§4.6 The maximal Weyl subgroups of $W(D_l)$

The maximal Weyl subgroups of K are of type D_{l-1}, A_{l-1} and $D_i + D_{l-i}$ $(2 \leq i \leq l-2)$.

In this section we give the decomposition for inducing an irreducible character up from a maximal Weyl subgroup of K. We can usually reduce the problem to considering G by using Frobenius reciprocity.

Theorem 4.6.1 (Inducing up from D_{l-1})

Let $(\lambda;\mu)$ be a pair of partitions of $l-1$ and $(\alpha;\beta)$ a pair of partitions of l. Then if $K' = W(D_{l-1})$

$((\chi_{K'}^{(\lambda;\mu)})^K, \chi_K^{(\alpha;\beta)}) \neq 0 \iff (\alpha;\beta)$ may be obtained

from $(\lambda;\mu)$ or $(\mu;\lambda)$ by adding a square to the end of a row of λ or μ; i.e. $(\alpha;\beta) \in Y_{(\lambda;\mu)}$, say.

Furthermore,

(i) l odd:

Suppose $(\alpha;\beta) \in Y_{(\lambda;\mu)}$, then

$$\lambda \neq \mu \quad \Rightarrow \quad ((\chi_{K'}^{(\lambda;\mu)})^K, \chi_K^{(\alpha;\beta)}) = 1$$

$$\lambda = \mu \qquad \Rightarrow \qquad ((\chi^{(\lambda;\lambda)}_{K'})^K , \chi^{(\alpha;\beta)}_{K}) = 2$$

and if $\theta = \theta_\lambda$ or ${}^{x'}\theta_\lambda$ where $x' = ((1-1),-(1-1))$, then

$(\theta^K , \chi^{(\alpha;\beta)}_{K}) \neq 0 \iff (\alpha;\beta) \in Y_{(\lambda;\lambda)}$

in which case

$$(\theta^K , \chi^{(\alpha;\beta)}_{K}) = 1$$

(ii) l even:

Suppose $(\alpha;\beta) \in Y_{(\lambda;\mu)}$, then

$$\alpha \neq \beta \qquad \Rightarrow \qquad ((\chi^{(\lambda;\mu)}_{K'})^K , \chi^{(\alpha;\beta)}_{K}) = 1$$

$$\alpha = \beta \qquad \Rightarrow \qquad ((\chi^{(\lambda;\mu)}_{K'})^K , \chi^{(\alpha;\alpha)}_{K}) = 2$$

and in this case

$$((\chi^{(\lambda;\mu)}_{K'})^K , \theta) = 1$$

where $\theta = \theta_\alpha$ or ${}^{x}\theta_\alpha$, $x = (1,-1)$.

<u>Proof</u>

Let $G' = W(C_{l-1})$.

$((\chi^{(\lambda;\mu)}_{K'})^K , \chi^{(\alpha;\beta)}_{K}) = ([(\chi^{(\lambda;\mu)}_{K'})^K]^G , \chi^{(\alpha;\beta)})$

by Frobenius

$$= ((\chi^{(\lambda;\mu)}_{K'})^G , \chi^{(\alpha;\beta)})$$

by transitivity of induction

$$= ([(\chi^{(\lambda;\mu)}_{K'})^{G'}]^G , \chi^{(\alpha;\beta)})$$

as $K' \leqslant G' \leqslant G$

$$= ((\chi^{(\lambda;\mu)} + \chi^{(\mu;\lambda)})^G , \chi^{(\alpha;\beta)}) \quad \text{by 4.1.3}$$

$$= ((\chi^{(\lambda;\mu)})^G , \chi^{(\alpha;\beta)}) + ((\chi^{(\mu;\lambda)})^G , \chi^{(\alpha;\beta)}) \qquad \ldots \quad (A)$$

Thus the first part of the theorem follows from 3.6.1.

(i) l odd:

If $\lambda \neq \mu$ then $(\alpha;\beta)$ cannot be obtained by adding a square,

from both $(\lambda;\mu)$ and $(\mu;\lambda)$ (as l odd implies $|\alpha| \neq |\beta|$)
Therefore one of the terms in (A) is zero and the other takes the value 1, by 3.6.1
i.e. $((\chi_{K'}^{(\lambda;\mu)})^K, \chi^{(\alpha;\beta)}) = 1$
If $\lambda = \mu$ then $(\alpha;\beta)$ can be obtained by adding a square to both $(\lambda;\mu)$ and $(\mu;\lambda)$, so both terms in (A) take the value 1 i.e. $((\chi_{K'}^{(\lambda;\lambda)})^K, \chi^{(\alpha;\beta)}) = 2.$
Let $\theta = \theta_\lambda$ or ${}^{x'}\theta_\lambda$ so

$$\begin{aligned}(\theta^K, \chi_K^{(\alpha;\beta)}) &= (\theta^G, \chi^{(\alpha;\beta)}) \quad \text{by Frobenius}\\ &= ((\theta^{G'})^G, \chi^{(\alpha;\beta)}) \quad \text{as } K' \leqslant G' \leqslant G\\ &= ((\chi^{(\lambda;\lambda)})^G, \chi^{(\alpha;\beta)}) \quad \text{by 4.1.3}\end{aligned}$$

which takes the value 1 if and only if $(\alpha;\beta) \in Y_{(\lambda;\lambda)}$ by 3.6.1.

(ii) l even:

If $\alpha \neq \beta$ then $(\alpha;\beta)$ cannot be obtained by adding a square, from both $(\lambda;\mu)$ and $(\mu;\lambda)$ (as l-1 odd implies $|\lambda| \neq |\mu|$). Therefore, as in (i),
$((\chi_{K'}^{(\lambda;\mu)})^K, \chi^{(\alpha;\beta)}) = 1.$
If $\alpha = \beta$ then $(\alpha;\beta)$ can be obtained by adding a square, from both $(\lambda;\mu)$ and $(\mu;\lambda)$ so, as in (i),
$((\chi_{K'}^{(\lambda;\mu)})^K, \chi^{(\alpha;\alpha)}) = 2.$
Finally, since the elements of K' can be chosen so as not to involve the symbol l , $({}^x\theta_\alpha)_{K'} = (\theta_\alpha)_{K'}$ $(x = (l,-l))$
Therefore

$$((\chi_{k'}^{(\lambda;\mu)})^K, \theta_\alpha) = (\chi_{k'}^{(\lambda;\mu)}, (\theta_\alpha)_{K'})$$

by Frobenius

$$\begin{aligned}&= (\chi_{k'}^{(\lambda;\mu)}, ({}^x\theta_\alpha)_{K'})\\ &= ((\chi_{k'}^{(\lambda;\mu)})^K, {}^x\theta_\alpha)\end{aligned}$$

by Frobenius

and $((\chi^{(\lambda;\mu)}_{K'})^{K}, \theta) = \frac{1}{2}((\chi^{(\lambda;\mu)}_{K'})^{K}, \chi^{(\alpha;\alpha)}_{K})$

$= 1$ by above,

where $\theta = \theta_\alpha$ or ${}^{x}\theta_\alpha$.

<u>Theorem 4.6.2</u> (Inducing up from A_{l-1})

Let $\lambda \vdash l$ and $(\alpha;\beta)$ a pair of partitions of l.

Then

$$((\chi^{\lambda})^{K}, \chi^{(\alpha;\beta)}_{K}) \neq 0 \Rightarrow (\lambda;-) \xrightarrow[C]{} (\alpha;\beta) \xrightarrow[C]{} (-;\lambda)$$

and

$$((\chi^{\lambda})^{K}, \chi^{(\lambda;-)}_{K}) = 1$$

<u>Proof</u>

This follows immediately from 3.6.2 using Frobenius reciprocity.

<u>Theorem 4.6.3</u> (Inducing up from $D_i + D_{l-i}$)

Let $(\lambda;\mu)$ be a pair of partitions of i and $(\rho;\sigma)$ a pair of partitions of j, where $i + j = l$; let $(\alpha;\beta)$ be a pair of partitions of l. Let $K_i = W(D_i)$, $K_j = W(D_j)$. Then

$((\chi^{(\lambda;\mu)}_{K_i} \cdot \chi^{(\rho;\sigma)}_{K_j})^{K}, \chi^{(\alpha;\beta)}_{K}) \neq 0$ implies one of the following holds:

(i) $(\alpha;-) \xrightarrow[C]{} (\lambda;\rho) \rightarrow (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\mu;\sigma) \xrightarrow[C]{} (-;\beta)$

(ii) $(\alpha;-) \xrightarrow[C]{} (\lambda;\sigma) \xrightarrow[C]{} (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\mu;\rho) \xrightarrow[C]{} (-;\beta)$

(iii) $(\alpha;-) \xrightarrow[C]{} (\mu;\sigma) \xrightarrow[C]{} (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\lambda;\rho) \xrightarrow[C]{} (-;\beta)$

(iv) $(\alpha;-) \xrightarrow[C]{} (\mu;\rho) \xrightarrow[C]{} (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\lambda;\sigma) \xrightarrow[C]{} (-;\beta)$

<u>Proof</u>

Let $G_i = W(C_i)$, $G_j = W(C_j)$. Then

$$\Gamma = ((\chi^{(\lambda;\mu)}_{K_i} \cdot \chi^{(\rho;\sigma)}_{K_j})^K, \chi^{(\alpha;\beta)}_K)$$

$$= ((\chi^{(\lambda;\mu)}_{K_i} \cdot \chi^{(\rho;\sigma)}_{K_j})^G, \chi^{(\alpha;\beta)}) \quad \text{by Frobenius}$$

$$= ([(\chi^{(\lambda;\mu)}_{K_i} \cdot \chi^{(\rho;\sigma)}_{K_j})^{G_i \times G_j}]^G, \chi^{(\alpha;\beta)})$$

as $K_i \times K_j \leqslant G_i \times G_j \leqslant G$

$$= ([(\chi^{(\lambda;\mu)}_{K_i})^{G_i} \cdot (\chi^{(\rho;\sigma)}_{K_j})^{G_j}]^G, \chi^{(\alpha;\beta)}) \quad \text{by 1.2.5(ii)}$$

$$= (([\chi^{(\lambda;\mu)} + \chi^{(\mu;\lambda)}] \cdot [\chi^{(\rho;\sigma)} + \chi^{(\sigma;\rho)}])^G, \chi^{(\alpha;\beta)}) \quad \text{by 4.1.3}$$

$$= ((\chi^{(\lambda;\mu)} \cdot \chi^{(\rho;\sigma)})^G, \chi^{(\alpha;\beta)}) + ((\chi^{(\lambda;\mu)} \cdot \chi^{(\sigma;\rho)})^G, \chi^{(\alpha;\beta)})$$

$$+ ((\chi^{(\mu;\lambda)} \cdot \chi^{(\sigma;\rho)})^G, \chi^{(\alpha;\beta)}) + ((\chi^{(\mu;\lambda)} \cdot \chi^{(\rho;\sigma)})^G, \chi^{(\alpha;\beta)})$$

Thus if $\Gamma \neq 0$ then one of the summands is non-zero. The theorem then follows from 3.6.3.

§4.7 Some remarks on Weyl groups of type D

The situation in $W(D_l)$ is not quite so good as in $W(A_l)$ and $W(C_l)$. In both of the latter cases we were able to find a bijection between the irreducible characters and the Weyl subgroups, and gave a partial ordering on partitions or pairs of partitions which parameterized both of these sets. In other words we were able to give a partial ordering on the Weyl subgroups and then defined, where $W = W(A_l)$ or $W(C_l)$ and W_1 is a Weyl subgroup of W,

$$X(W_1) = \left\{ \begin{array}{l} \text{irred. character } \chi : (1_{W_1}{}^W, \chi) \neq 0 \text{ but } (1_{W_2}{}^W, \chi) = 0 \\ \text{for all Weyl subgroups } W_2 \text{ such that } W_2 > W_1 \end{array} \right\}$$

The map X turned out to be a bijection.

We would like to find an ordering of the Weyl subgroups and/or irreducible characters of $W(D_l)$ so that if we were to define X as above, then X would be almost a bijection. We certainly could not expect X to be a bijection as the number of Weyl subgroups of $W(D_l)$ is, in general, less than the number of irreducible characters. Thus the set $X(W_1)$ will sometimes contain more than one irreducible character. However, if we could also find a partial ordering on the irreducible characters, then we would choose to associate with W_1, the (we hope) unique character which is the lowest in $X(W_1)$ with respect to the ordering, and call this a <u>dominant</u> character.

This leaves us with a set of non-dominant characters. We would then like to associate each of these with a semi-Coxeter type $D_l(a_j)$ or $D_l(b_j)$ (see [5]) in a consistent way. Indeed, we would hope that the resulting bijection between irreducible characters and Weyl subgroups or semi-Coxeter types is consistent in the following manner (cf. §2.5 and 3.6.1) :
let χ be an irreducible character of $W(D_l)$ associated with a Weyl subgroup or semi-Coxeter type W, and suppose

$$\chi^{W(D_{l+1})} = \sum_{i=1}^{r} a_i \chi_i$$

(χ_i irreducible characters of $W(D_{l+1})$).
Then we would like there to be a unique lowest character χ_1 (say) of the set $\{\chi_1, \dots, \chi_r\}$, with respect to the partial ordering on the irreducible characters, such that $a_1 = 1$ and χ_1 is associated with W inside $W(D_{l+1})$.

It is for this reason that we have included the section §4.6 on maximal Weyl subgroups.

It turns out that it is possible to give such a bijection in Weyl groups of type D of low rank (i.e. $l \leq 7$) and we list the results for $l = 4$ and $l = 5$ in §4.8.

A study of these low rank groups reveals the following facts:
suppose $W_{(\lambda;\mu)}$ is a Weyl subgroup of $K = W(D_l)$ $(l \leq 7)$ and $\chi_K^{(\alpha;\beta)}$ is an irreducible character associated with $W_{(\lambda;\mu)}$. It seems that we may obtain $(\alpha;\beta)$ (an unordered pair) from $(\lambda;\mu)$ (which is ordered and no part of μ is 1) by the map Θ where

$$\Theta(\lambda;\mu) = (\lambda^*,\mu \,;\, \lambda^{**})$$

where λ^*, λ^{**} are obtained by splitting each of the parts of λ almost evenly (depending on μ). Note that $(\lambda;\mu) \xrightarrow[D]{} (\lambda^*,\mu \,;\, \lambda^{**})$ but no moving up is required in this operation.

If λ has all its parts even so that $\lambda = 2\nu$ then $\Theta(\lambda;-) = (\nu;\nu)$ and the two Weyl subgroups $W_{(\lambda;-)}$ (see 4.2.1 and remark p 107) seem to be associated with the two irreducible components θ_ν and ${}^x\theta_\nu$ of $\chi_K^{(\nu;\nu)}$.

Also it seems that $\chi_K^{(l-j;j)}$ should be associated with $D_l(a_j)$ in $W(D_l)$ $(1 \leq j < l/2)$.

If $(\alpha;\beta)$ and $(\rho;\sigma)$ are two pairs of partitions of l such that $|\alpha| = |\rho|$ and $|\beta| = |\sigma|$ and $\alpha \leq \rho$, $\beta \leq \sigma$ then it appears that the ordering of the characters satisfies $\chi_K^{(\alpha;\beta)} \leq \chi_K^{(\rho;\sigma)}$.

To show that the problem is not solely due to the fact that, with the characters of $W(D_l)$, we are dealing with unordered pairs of partitions, we have included a chapter on $W(B_l)$, which contains $W(D_l)$ as a regular Weyl subgroup. It will be seen that here, although the characters are parameterized by ordered pairs of partitions, the problem seems to be equivalent to that for $W(D_l)$, as the operation $\underset{B}{\rightarrow}$ defined in that chapter is very similar to $\underset{D}{\rightarrow}$.

§4.8 The groups $W(D_4)$ and $W(D_5)$

We list the bijection, found by direct calculation, between the irreducible characters of $W(D_4)$ and $W(D_5)$ and their Weyl subgroups and semi-Coxeter types. The tables were used for the calculations for $W(F_4)$ and $W(E_6)$ in chapter six.

The notation is as follows :
the first column gives the type of the Weyl subgroup or semi-Coxeter type; the second column gives the pair of partitions $(\lambda;\mu)$ parameterizing the Weyl subgroup $W_{(\lambda;\mu)}$ (where appropriate); the last column gives the pair of partitions $(\alpha;\beta)$ parameterizing the character $\chi_k^{(\alpha;\beta)}$, (we shall write this so that $|\alpha| \geqslant |\beta|$).

TABLE 1

$K = W(D_4)$

Type	$W_{(\lambda;\mu)}$	$\chi_{\kappa}^{(\alpha;\beta)}$
D_4	$(- ; 4)$	$(4 ; -)$
$D_4(a_1)$	$-$	$(3 ; 1)$
D_3	$(1 ; 3)$	$(31 ; -)$
A_3	$(4 ; -)$	$(2 ; 2)$
$D_2 + D_2$	$(- ; 2^2)$	$(2^2 ; -)$
$A_1 + D_2$	$(2 ; 2)$	$(21 ; 1)$
A_2	$(31 ; -)$	$(2 ; 1^2)$
D_2	$(1^2 ; 2)$	$(21^2 ; -)$
$A_1 + A_1$	$(2^2 ; -)$	$(1^2 ; 1^2)$
A_1	$(21^2 ; -)$	$(1^3 ; 1)$
$\emptyset$	$(1^4 ; -)$	$(1^4 ; -)$

TABLE 2

$K = W(D_5)$

Type	$W_{(\lambda;\mu)}$	$\chi_{K}^{(\alpha;\beta)}$
D_5	$(- ; 5)$	$(5 ; -)$
$D_5(a_1)$	$-$	$(4 ; 1)$
D_4	$(1 ; 4)$	$(41 ; -)$
A_4	$(5 ; -)$	$(3 ; 2)$
$D_3 + D_2$	$(- ; 32)$	$(32 ; -)$
$A_1 + D_3$	$(2 ; 3)$	$(31 ; 1)$
$D_4(a_1)$	$-$	$(3 ; 1^2)$
D_3	$(1^2 ; 3)$	$(31^2 ; -)$
$A_2 + D_2$	$(3 ; 2)$	$(2^2 ; 1)$
A_3	$(41 ; -)$	$(31 ; 2)$
$D_2 + D_2$	$(1 ; 2^2)$	$(2^21 ; -)$
$A_2 + A_1$	$(32 ; -)$	$(21 ; 1^2)$
$A_1 + D_2$	$(21 ; 2)$	$(21^2 ; 1)$
A_2	$(31^2 ; -)$	$(1^3 ; 2)$
$A_1 + A_1$	$(2^21 ; -)$	$(1^3 ; 1^2)$
D_2	$(1^3 ; 2)$	$(21^3 ; -)$
A_1	$(21^3 ; -)$	$(1^4 ; 1)$
$\emptyset$	$(1^5 ; -)$	$(1^5 ; -)$

Chapter five WEYL GROUPS OF TYPE B

For the sake of completeness, we give an algorithm for Weyl groups of type B, similar to ones in types C and D (§3.3 and §4.2), and include some results on inducing up irreducible characters from maximal Weyl subgroups of this group.

$W(B_l)$ is isomorphic to $W(C_l)$ and hence has the same characters. However the Weyl subgroups are different, which would lead to a different association of irreducible characters to Weyl subgroups (cf. §4.7).

We let $G = W(B_l)$ and, as far as the character theory goes, use the same notation as in chapters three and four.

§5.1 An algorithm for $W(B_l)$

Remark

As in chapter three, we shall only be interested in the regular Weyl subgroups, although in this case they do not form a complete set of conjugates. For example in $W(B_4)$, the Weyl subgroup of type $B_2 + B_2$ is not conjugate to any regular one. In the rest of this chapter we shall assume all Weyl subgroups are regular.

The Weyl subgroups of G have the form

$$S_{\lambda_1} \times \dots \times S_{\lambda_r} \times W(D_{\mu_1}) \times \dots \times W(D_{\mu_s}) \times W(B_t)$$

where $\sum \lambda_i + \sum \mu_i + t = l$ and $\mu_i \neq 1$, $t \geq 0$.

We shall write this subgroup as $W_{(\lambda;\mu;t)}$ where

$\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, and we may assume that $\lambda_1 \geqslant \ldots \geqslant \lambda_r > 0$, $\mu_1 \geqslant \ldots \geqslant \mu_s > 1$. Thus the Weyl subgroups may be parameterized by triples of partitions $(\lambda;\mu;t)$ where no part of μ is 1 and $t \geqslant 0$ (we shall write t for the partition (t), and interpret $W(B_t) = 1$ when $t = 0$).

As in previous chapters, $W_{(\lambda;\mu;t)}$ may be regarded as the row stabilizer of a diagram $D_{(\lambda;\mu;t)}$, where a row permutation of $D_{(\lambda;\mu;t)}$ permutes the symbols in each row of D_λ, D_μ, D_t (a single row), changes the sign of an even number of symbols in D_μ and changes the sign of any number of symbols in D_t.

We shall be interested in giving an algorithm which determines which pair of partitions $(\alpha;\beta)$ of l satisfy

$$(1_{W_{(\lambda;\mu;t)}}^{G}, \chi^{(\alpha;\beta)}) \neq 0$$

<u>Definition</u>

Let $(\lambda;\mu;t)$ be a triple of partitions of l such that no part of μ is 1 and $t \geqslant 0$, and let $(\alpha;\beta)$ be an (ordered) pair of partitions of l. Write $(\lambda;\mu;t) \xrightarrow[B]{} (\alpha;\beta)$ if $(\alpha;\beta)$ may be obtained from $(\lambda;\mu;t)$ by

(a) removing connected squares from the end of a row of λ and placing them, in the same order, at the bottom of μ;

(b) repeating (a) with squares from different rows of λ;

and at the same time, but independently, (so no square is moved twice)

(c) transferring complete rows of μ and placing them at the bottom of λ ;

and again at the same time, but independently,

(d) transferring the whole of t across to the bottom of λ ;

then

(e) reordering the resulting rows so as to give a pair of partitions $(\gamma;\delta)$ say;

and finally

(f) moving up inside γ and δ , according to the usual partial ordering on partitions, so as to obtain α and β respectively (so $\gamma \leq \alpha$ and $\delta \leq \beta$).

Remark

If $t = 0$ then $(\lambda;\mu;-) \xrightarrow[B]{} (\alpha;\beta) \iff (\lambda;\mu) \xrightarrow[D]{} (\alpha;\beta)$.
Indeed, for $t = 0$, $W_{(\lambda;\mu;-)}$ is a Weyl subgroup of type $W_{(\lambda;\mu)}$ of $W(D_l)$ and as

$$(1_{W_{(\lambda;\mu)}}^{G}, \chi^{(\alpha;\beta)}) = (1_{W_{(\lambda;\mu)}}^{K}, \chi_K^{(\alpha;\beta)}) \quad \text{by Frobenius}$$

we would expect to get the same algorithm, in this case, as in $W(D_l)$.

It is for this reason that it appears that the problem of associating irreducible characters to Weyl subgroups in $W(B_l)$ seems to be equivalent to that for $W(D_l)$ (see §4.7).

Theorem 5.1.1

$$(1_{W_{(\lambda;\mu;t)}}^{G}, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\mu;t) \xrightarrow[B]{} (\alpha;\beta)$$

The following lemma is proved in precisely the same way as 3.3.2

Lemma 5.1.2

Let $W = R(D_{(\lambda;\mu;t)})$. Then

(a) $W = (N \cap W)(H \cap W)$ and $(N \cap W) \cap (H \cap W) = 1$

If also $g \in H$, $C = C_H(\varsigma)$ for some irreducible character ς of N

(b) $W^g = (N \cap W^g)(H \cap W^g)$ and $(N \cap W^g) \cap (H \cap W^g) = 1$

(c) $NC \cap W^g = (N \cap W^g)(C \cap W^g)$ and $(N \cap W^g) \cap (C' \cap W^g) = 1$

Proof of 5.1.1

Let $W = W_{(\lambda;\mu;t)}$. Then we suppose $(1_W^{\;G}, \chi^{(\alpha;\beta)}) \neq 0$. Hence, with the usual notation,

$$0 = (1_W^{\;G}, \chi^{(\alpha;\beta)}) = (1_W^{\;G}, \phi^G)$$

$$= \sum_{g \in [g_i]} ({}^{g}1_{W^g \cap NC}, \phi_{W^g \cap NC})$$

where $[g_i]$ is a set of (W,NC)-double coset representatives and each $g_i \in H$. Thus there exists $g \in [g_i]$ such that

$$0 \neq (1_{W^g \cap NC}, \phi_{W^g \cap NC}) = (1_{W^g \cap N}, \varsigma_{W^g \cap N})(1_{W^g \cap C}, \psi_{W^g \cap C})$$

by 5.1.2, and so $1_{W^g \cap N} = \varsigma_{W^g \cap N}$.

Let $|\alpha| = m$, $|\beta| = n$ and we have that ς takes the value 1 on all sign changes in W^g. Now W^g defines a diagram $D_{(\lambda;\mu;t)}$, and therefore in any row of D_μ all the symbols are of the same type, and all the symbols in D_t are of the same type. Hence we may transfer those complete rows of D_μ which contain symbols of the first type to D_λ , independently move the squares of D_λ (so that moved squares in the same row stay in the same row) containing the symbols of the second type to D_μ and, again independently, move the whole of D_t across to D_λ . On reordering we obtain a diagram $D_{(\gamma;\delta)}$ of a pair of partitions $(\gamma;\delta)$ of 1 such that D_γ contains all the symbols of the first type and D_δ all the symbols of the second type. This corresponds to operations (a), (b), (c), (d) and (e) on p 134-5. So to show $(\lambda;\mu;t) \underset{B}{\rightarrow} (\alpha;\beta)$ we only have to show $\gamma \leqslant \alpha$, $\delta \leqslant \beta$.

By construction, $|\gamma| = m = |\alpha|$, $|\delta| = n = |\beta|$ and $(1_{C\cap W^g}, \Psi_{C\cap W^g}) \neq 0$. Again, just as in 3.3.1, we obtain

$$0 \neq (1_{C\cap W^g}, \Psi_{C\cap W^g}) = ((1_{W_\gamma})^{S_m}, \chi^\alpha)((1_{W_\delta})^{S_n}, \chi^\beta)$$

so $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$.

Hence $(\lambda;\mu;t) \underset{B}{\rightarrow} (\alpha;\beta)$.

Conversely, suppose $(\lambda;\mu;t) \underset{B}{\rightarrow} (\alpha;\beta)$. Therefore we may move parts of rows of λ across to μ , complete rows of μ across to λ , and the whole of t across to λ , to obtain a pair of partitions $(\gamma;\delta)$ of 1 such that $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$. Hence we may define a diagram $D_{(\lambda;\mu;t)}$ filled with the symbols $\{1,\ldots,1\}$ such that each row of D_μ contains only symbols of one type, and D_t only contains symbols of the first type.

Let $W = R(D_{(\lambda,\mu;t)})$ so all pairs of sign changes in $N \cap W$ consist of symbols which are of the same type i.e. $\varsigma_{N\cap W} = 1$. Also, by 2.3.6, since $\gamma \leqslant \alpha$ and $\delta \leqslant \beta$

$$(1_{N\cap W}, \varsigma_{N\cap W})((1_W)^{S_m}, \chi^\alpha)((1_W)^{S_n}, \chi^\beta) \neq 0$$

and this is by the proof of the first part of the theorem, the first summand in the Mackey formula for $(1_W{}^G, \chi^{(\alpha;\beta)})$. Hence $(1_W{}^G, \chi^{(\alpha;\beta)}) \neq 0$.

§5.2 The maximal Weyl subgroups of $W(B_l)$

The maximal Weyl subgroups of G are of type B_{l-1}, D_l, and $D_{l-i} + B_i$ for $1 \leqslant i \leqslant l-2$.

Inducing up irreducible characters from the maximal Weyl subgroups we obtain the following results. All the theorems follow almost straight-away from those for $W(C_l)$ (§3.6) in the same manner as we proved them for $W(D_l)$ (§4.6) ; thus we shall omit the proofs.

Theorem 5.2.1 (Inducing up from B_{l-1})

Let $(\lambda;\mu)$ be a pair of partitions of $l-1$ and let $(\lambda;\mu)^* = (\lambda^*;\mu) = (\lambda 1;\mu)$. Then

$$(\chi^{(\lambda;\mu)})^G \quad = \quad \chi^{(\lambda;\mu)*} \quad + \quad \sum \chi^{(\alpha;\beta)}$$

summed over all those pairs of partitions $(\alpha;\beta)$ $(\neq (\lambda;\mu)^*)$ of l obtained from $(\lambda;\mu)$ by adding a square to the end of a row of λ or by adding a square to the end of a row of μ.

Theorem 5.2.2 (Inducing up from D_l)

Let $(\lambda;\mu)$ and $(\alpha;\beta)$ be pairs of partitions of l and $K = W(D_l)$. Then

$$((\chi_K^{(\lambda;\mu)})^G, \chi^{(\alpha;\beta)}) \neq 0 \iff (\lambda;\mu) = (\alpha;\beta) \text{ or } (\mu;\lambda) = (\alpha;\beta)$$

If l is even, $(\theta^G, \chi^{(\alpha;\beta)}) \neq 0 \iff \alpha = \lambda = \beta$ where $\theta = \theta_\lambda$ or ${}^x\theta_\lambda$.

In particular, all non-zero multiplicities are 1.

<u>Theorem 5.2.3</u> (Inducing up from $B_i + D_{l-i}$)

Let $(\lambda;\mu)$ be a pair of partitions of i, $(\rho;\sigma)$ a pair of partitions of j, where $i + j = l$. Let $K_j = W(D_j)$, and $(\alpha;\beta)$ be a pair of partitions of l. Then

$((\chi^{(\lambda;\mu)}\cdot\chi^{(\rho;\sigma)}_{K_j})^G , \chi^{(\alpha;\beta)}) \neq 0$ implies

<u>either</u> $(\alpha;-) \xrightarrow[C]{} (\lambda;\rho) \xrightarrow[C]{} (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\mu;\sigma) \xrightarrow[C]{} (-;\beta)$

<u>or</u> $(\alpha;-) \xrightarrow[C]{} (\lambda;\sigma) \xrightarrow[C]{} (-;\alpha)$ and $(\beta;-) \xrightarrow[C]{} (\mu;\rho) \xrightarrow[C]{} (-;\beta)$

Chapter six WEYL GROUPS OF EXCEPTIONAL TYPE

In this chapter we give an association between the irreducible characters and the Weyl subgroups of the Weyl groups of type G_2 , F_4 and E_6 .

Using a computer, similar results ought to be obtainable for Weyl groups of type E_7 and E_8 .

As the number of Weyl subgroups differs from the number of irreducible characters in each case, we could not expect this association to be a bijection.

We shall use the notation in [5] .

§6.1 Construction of the mapping Y

The details given in this section are similar to those in §4.7.

Let W be a Weyl group of type G_2 , F_4 or E_6 , and suppose W' is a Weyl subgroup of W. We first calculate the irreducible characters occurring in $1_{W'}^{W}$, using the information on the conjugacy classes given in [5] , and the character tables in [9] and [14] (the Weyl group of type G_2 is the dihedral group of order 12 and so is easy to work with).

From this we wish to associate a set of irreducible characters to the Weyl subgroup W' using a partial ordering $\leqslant$ on the Weyl subgroups

i.e. $X(W') = \left\{ \chi \text{ irred. character of } W : (1_{W'}^{W}, \chi) \neq 0 \text{ but } (1_{W''}^{W}, \chi) = 0 \text{ for all Weyl subgroups } W'' > W \right\}$

In defining the partial ordering we work from the highest Weyl subgroup downwards (highest means with respect to the ordering). We let W be the highest Weyl subgroup so 1_W^W is the principal character. Inductively, suppose $W=W_1, \ldots, W_r$ have been ordered and so $X(W_1), \ldots, X(W_r)$ determined. Let $\bigcup_{i=1}^{r} X(W_i) = \{\chi_1, \ldots, \chi_s\}$. Then we look at those Weyl subgroups W' of W for which $1_{W'}^W$ contains the minimal number of irreducible characters not in the set $\{\chi_1, \ldots, \chi_s\}$. Then these Weyl subgroups are defined to be the next in the partial ordering and X(W') as the set of irreducible characters occurring in $1_{W'}^W$ but not in $\{\chi_1, \ldots, \chi_s\}$. The unique lowest (with respect to the ordering) Weyl subgroup is 1 since inducing up to W from it gives the regular character, which contains all the irreducible characters of W.

Thus for each Weyl subgroup W' we have defined X(W'). We are then able to give a partial ordering $\leqslant$ on the irreducible characters of W. Let χ', χ'' be irreducible characters of W and suppose $\chi' \in X(W')$, $\chi'' \in X(W'')$. Define

$$\chi' \leqslant \chi'' \quad \Leftrightarrow \quad W' \leqslant W''$$

By construction of X, if $X(W') \cap X(W'') \neq \emptyset$, then W' and W'' are not comparable with respect to $\leqslant$, but $W' \leqslant W''' \Leftrightarrow W'' \leqslant W'''$, so that the above definition is well-defined.

The Weyl groups of each type then have their own particular problems, so we deal with each separately.

(a) $W(G_2)$

It turns out that $|X(W')| = 1$ for each Weyl subgroup W' of $W(G_2)$, and we define a reverse mapping from the set of irreducible characters to the set of Weyl subgroups of $W(G_2)$:

$$Y(\chi) = \{ W' : \chi \in X(W') \}$$

The results are given in table 3, along with the ordering on the Weyl subgroups.

(b) $W(E_6)$

In this case, the number of irreducible characters of $W(E_6)$ (i.e. 25) equals the number of Weyl subgroups (i.e. 21) plus the number of semi-Coxeter types (i.e. 4). It is therefore desirable to obtain a bijection between these sets.

Now the semi-Coxeter types in E_6 are $E_6(a_1)$, $E_6(a_2)$, $D_5(a_1)$, $D_4(a_1)$ (see [5]) and the last two lie inside the maximal Weyl subgroup $W(D_5)$ of $W(E_6)$.

Inside $W(D_5)$ we have associated to $D_5(a_1)$ and $D_4(a_1)$ irreducible characters of $W(D_5)$ (see table 2), call them χ_1, χ_2 respectively. In order to obtain a consistent association of irreducible characters to Weyl subgroups and semi-Coxeter types (as in §4.7), we calculate $\chi_1^{W(E_6)}$ and $\chi_2^{W(E_6)}$. Then, inside $W(E_6)$, we associate to $D_5(a_1)$ and $D_4(a_1)$ the lowest irreducible character of $W(E_6)$ occurring in $\chi_1^{W(E_6)}$ and $\chi_2^{W(E_6)}$.

Similarly, for those Weyl subgroups W' for which $|X(W')| > 1$, we associate to W' the lowest irreducible character in $X(W')$ (which is unique except for one case).

Finally, to $E_6(a_1)$ and $E_6(a_2)$, we associate (arbitrarily) the remaining two irreducible characters.

We thus obtain a bijection X_1 between the Weyl subgroups and semi-Coxeter types and the irreducible characters. Note that the final result is not unique i.e. there are two ways of defining X_1 satisfying the given conditions (see table 4).

The reverse mapping

$$Y(\chi) = \{W' : \chi \in X_1(W')\}$$

is just $Y = X_1^{-1}$ since X_1 is a bijection (W' may be a semi-Coxeter type here).

The result is given in table 4.

(c) $W(F_4)$

In $W(F_4)$, the number of Weyl subgroups is 37, the number of semi-Coxeter types is 3 (given by $F_4(a_1)$, $D_4(a_1)$ and $\tilde{D}_4(a_1)$, where $\sim$ denotes a short root system), but the number of irreducible characters is 25. Thus we cannot hope to get anything like a bijection.

As in $W(E_6)$, to each Weyl subgroup W' we associate the set of lowest characters in $X(W')$. Using table 1, we induce up to $W(F_4)$ the irreducible characters χ_1, χ_2 of D_4 , $\tilde{D}_4$ respectively, which correspond to $D_4(a_1)$, $\tilde{D}_4(a_1)$ respectively. Then, in $W(F_4)$, we associate to each of $D_4(a_1)$ and $\tilde{D}_4(a_1)$ the set of lowest irreducible characters of $W(F_4)$ in $\chi_1^{W(F_4)}$ and $\chi_2^{W(F_4)}$ respectively.

This still leaves some choice, so the final criterion applied is the idea of duality between long and short roots.

Let W' be any Weyl subgroup or semi-Coxeter type in $W(F_4)$ and $\tilde{W}'$ its dual (possibly W' and $\tilde{W}'$ are conjugate inside $W(F_4)$). Then given any irreducible character χ

of $W(F_4)$, we define the <u>dual character</u> $\tilde{\chi}$ to be that irreducible character of $W(F_4)$ which satisfies

$$(1_{W'}{}^{W(F_4)}, \chi) \neq 0 \iff (1_{\tilde{W}'}{}^{W(F_4)}, \tilde{\chi}) \neq 0$$

(such duals exist by inspecting $1_{W'}{}^{W(F_4)}$, $1_{\tilde{W}'}{}^{W(F_4)}$ and are unique). A character is often self-dual i.e. $\chi = \tilde{\chi}$

We then demand that in the association X_1 of characters to Weyl subgroups and semi-Coxeter types,

$$\chi \in X_1(W') \iff \tilde{\chi} \in X_1(\tilde{W}')$$

where W' is a Weyl subgroup or semi-Coxeter type.

It then follows that $|X_1(W')| = 1$, and we associate the one remaining irreducible character to $F_4(a_1)$.

In table 5 we give the unique result, using the reverse mapping

$$Y(\chi) = \{W' : \chi \in X_1(W')\}$$

§6.2 <u>Some further remarks</u>

In $W(G_2)$ and $W(F_4)$, because of the existence of roots of different lengths, two Weyl subgroups may have the same Coxeter element and so be conjugate; similarly, semi-Coxeter classes may be representable in various ways. Thus we have equivalent Weyl subgroups or semi-Coxeter types which represent the same conjugacy class.

These are listed below; types are equivalent if and only if they are written on the same line.

$W(G_2)$:

A_2 $\qquad\qquad$ $\tilde{A}_2$

$W(F_4)$:

$2A_1$	$2\tilde{A}_1$		
$3A_1$	$2\tilde{A}_1 + A_1$		
$2A_1 + \tilde{A}_1$	$3\tilde{A}_1$		
A_3	$B_2 + \tilde{A}_1$		
$B_2 + A_1$	$\tilde{A}_3$		
$4A_1$	$2A_1 + 2\tilde{A}_1$	$4\tilde{A}_1$	
$A_3 + \tilde{A}_1$	$B_2 + 2A_1$	$B_2 + 2\tilde{A}_1$	$\tilde{A}_3 + A_1$
D_4	$B_3 + \tilde{A}_1$		
$\tilde{D}_4$	$C_3 + A_1$		
$D_4(a_1)$	$2B_2$	$\tilde{D}_4(a_1)$	
B_4	C_4		

However, a different sort of equivalence may be defined using the characters :
W' and W'' are equivalent if and only if there exists an irreducible character χ such that W' , $W'' \in Y(\chi)$ (W',W'' are Weyl subgroups or semi-Coxeter types).

The form this equivalence takes is evident in the tables, and in both $W(G_2)$ and $W(F_4)$ we get a completely different equivalence from that defined using the conjugacy classes.

§6.3 The tables

The notation used in the tables is as follows :
In $W(G_2)$ and $W(F_4)$ ~ denotes a system of short roots, without ~ the system consists of long roots.

The first column of each of the tables gives the irreducible characters of the Weyl group; the second column gives the Weyl subgroups or semi-Coxeter types given by the mapping Y defined in §6.1.

In $W(G_2)$, χ_1, χ_2, χ_3, χ_4 are the characters of degree 1 (χ_1 the principal character, χ_2 the sign character) and χ_5, χ_6 the characters of degree 2.

In $W(E_6)$ we give in the third column Frame's notation for the characters in [9] .

In $W(F_4)$, the characters are numbered consecutively on p 152 of [14] (χ_1 is the principal character etc.).

TABLE 3

$W(G_2)$

χ	$Y(\chi)$
χ_1	G_2
χ_2	$\emptyset$
χ_3	A_2
χ_4	$\tilde{A}_2$
χ_5	$A_1 + \tilde{A}_1$
χ_6	A_1 , $\tilde{A}_1$

The ordering of the Weyl subgroups in $W(G_2)$:

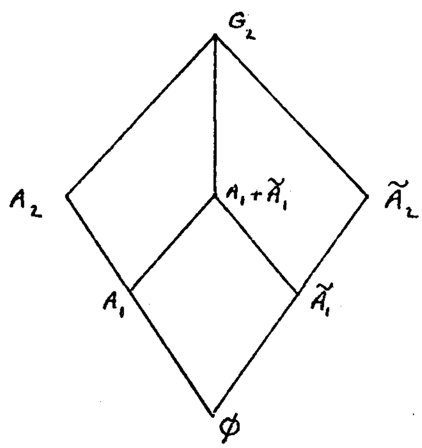


TABLE 4

$W(E_6)$

χ	$Y(\chi)$	Frame's notation for χ
χ_1	E_6	1_p
χ_2	D_5	6_p
χ_3	$D_5(a_1)$	15_p
χ_4	$E_6(a_1)$	20_p
χ_5	A_5	30_p
χ_6	$A_4 + A_1$ or $E_6(a_2)$	64_p
χ_7	A_4	81_p
χ_8	$A_5 + A_1$	15_q
χ_9	D_4	24_p
χ_{10}	$E_6(a_2)$ or $A_4 + A_1$	60_p
χ_{11}	$D_4(a_1)$	20_s
χ_{12}	$A_3 + A_1$	90_s
χ_{13}	$2A_2 + A_1$	80_s
χ_{14}	$A_3 + 2A_1$	60_s
χ_{15}	$3A_2$	10_s
χ_{16}	$\emptyset$	1_n
χ_{17}	A_1	6_n
χ_{18}	A_2	15_n
χ_{19}	$2A_1$	20_n
χ_{20}	$3A_1$	30_n
χ_{21}	$A_2 + A_1$	64_n
χ_{22}	A_3	81_n
χ_{23}	$4A_1$	15_m
χ_{24}	$2A_2$	24_n
χ_{25}	$A_2 + 2A_1$	60_n

TABLE 5

$W(F_4)$

χ	$Y(\chi)$
χ_1	F_4
χ_2	D_4
χ_3	$\tilde{D}_4$
χ_4	$\emptyset$
χ_5	B_4
χ_6	$4\tilde{A}_1$
χ_7	C_4
χ_8	$4A_1$
χ_9	$2B_2$
χ_{10}	$B_3 + \tilde{A}_1$, $C_3 + A_1$
χ_{11}	$B_2 + 2A_1$
χ_{12}	$B_2 + 2\tilde{A}_1$
χ_{13}	$2\tilde{A}_1$, $A_1 + \tilde{A}_1$, $2A_1$
χ_{14}	$A_2 + \tilde{A}_2$
χ_{15}	A_2 , $\tilde{A}_2$
χ_{16}	$2A_1 + 2\tilde{A}_1$
χ_{17}	$F_4(a_1)$
χ_{18}	$A_3 + A_1$, A_3 , $D_4(a_1)$
χ_{19}	$\tilde{A}_3 + A_1$, $\tilde{A}_3$, $\tilde{D}_4(a_1)$
χ_{20}	A_1 , $\tilde{A}_1$
χ_{21}	B_3
χ_{22}	$2\tilde{A}_1 + A_1$, $3\tilde{A}_1$
χ_{23}	C_3
χ_{24}	$2A_1 + \tilde{A}_1$, $3A_1$
χ_{25}	$B_2 + A_1$, B_2 , $\tilde{A}_1 + A_2$, $B_2 + \tilde{A}_1$, $A_1 + \tilde{A}_2$

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