DYNAMICS OF MODEL-REFERENCE

AND

HILL CLIMBING SYSTEMS

by

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STATEMENT

The work presented in this thesis is original with the exceptions stated below, and has not been submitted for another degree of this or any other University. The exceptions are :

(i) Throughout the emphasis has been that of systematically applying known mathematical theories to solve the problems at hand. This is clearly indicated by the work described in chapter 1 where various known mathematical results are brought together in procedures for investigating the stability of a system of linear differential equations with periodic coefficients.

(ii) The work in chapter 2 is a review of and commentary on the current state of the theory of stability of linear stochastic systems.

(iii) The actual systems considered in chapters 3-6 have been discussed previously in the literature cited and it is only their mathematical analysis that is new. Throughout the analysis results deduced by other workers are clearly indicated in the references.

(iv) The idea of representing the input, in section 4.5., as a sequence of impulses was first suggested by Dr. P.C. Parks.

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ABSTRACT

In this thesis the theory of both linear differential equations with periodic coefficients and linear differential equations with random coefficients is applied to investigate the stability and accuracy of parameter adaptation of sinusoidal perturbation and model reference adaptive control systems. Throughout dimensional analysis is applied so that all the results are presented in a non-dimensional form.

The first part of the thesis is devoted to investigating the stability of such differential equations. In chapter 1 a system of linear homogeneous differential equations with periodic coefficients is considered and a numerical procedure, based on Floquet theory and well suited for use on a digital computer, is presented for obtaining necessary and sufficient conditions for asymptotic stability of the null solution. Also considered in this chapter is the so called infinite determinant method of obtaining the stability boundaries for a restricted class of linear differential equations with periodic coefficients. Chapter 2 is devoted to reviewing the current state of the stability theory of linear differential equations with random coefficients.

In chapter 3 a theoretical analysis of the stability and accuracy of parameter adaptation of a single input, sinusoidal perturbation, extremum control system with output lag is considered. Using the principle of harmonic balance it is shown that various stable harmonic and sub-harmonic *steady state* solutions are possible in certain regions of the parameter space. By examining the domains of attraction, corresponding to the stable solutions, regions in three dimensional space are obtained within which initial conditions will lead to a given *steady state* stable oscillation. It is also shown that the subharmonic *steady state* solutions do not correspond to the optimum solution, so that, for certain initial conditions and parameter values, it is possible

for the system to reach a *steady state* solution which is not the optimum solution. All the theoretical results are verified by direct analogue computer simulation of the system.

The remainder of the thesis is devoted to investigating the stability and accuracy of parameter adaptation of model reference adaptive control systems. In order to develop a mathematical analysis, and to illustrate the difficulties involved, a stability analysis of a first order M.I.T. type system with controllable gain, when the input varies with time in both a periodic and random manner, is first carried out. Also considered are the effects of

(a) random disturbances at the system output

and (b) periodic and random variations, with time, of the controlled process environmental parameters,

on the stability of the system and the accuracy of its parameter adaptation.

When the input varies sinusoidally with time stability boundaries are obtained using both a numerical implementation of Floquet theory and the infinite determinant method; the relative merits of the two methods is discussed. The theoretical results are compared with stability boundaries obtained by analogue computer simulation of the system. It is shown that the stability boundaries are complex in nature and that some knowledge of such boundaries is desirable before embarking on an analogue computer investigation of the system.

When the input varies randomly with time the stability problem reduces to one of investigating the stability of a system of linear differential equations with random coefficients. Both the theory of Markov processes, involving use of the Fokker-Planck equation, and the second method of Liapunov are used to investigate the problem; limitations and difficulty of applications of the theory is discussed. The theoretical results obtained are compared with those obtained by digital simulation of the system.

If the controlled process environmental parameter is allowed to become time varying then it is shown that this effects both the stability of the system and the accuracy of its parameter adaptation. Theoretical results are obtained for the cases of the parameter varying both sinusoidally and randomly with time; some of the results are compared with those obtained by digital simulation of the system. It is also shown that noise disturbance at the system output has no effect on the system stability but does effect the accuracy of the parameter adaptation.

The doubts concerning the stability and the difficulty of analysis of the M.I.T.,type system have led researchers to think about redesigning the model reference system from the point of view of stability. In particular we have the Liapunov synthesis method where the resulting system is guaranteed stable for all possible inputs. However, in designing such systems the controlled process environmental parameters are assumed constant and, by considering the Liapunov redesign scheme of the first order M.I.T. system previously discussed, it will be shown that the effect of making such parameters time varying is to introduce a stability problem.

In chapter 6 the methods developed for analysing the first order system are extended to examine the stability of a higher order M.I.T. type system. The system considered has a third order process and a second order model and a stability analysis is presented for both sinusoidal and random input. *Steady state* values of the adapting parameters are first obtained and the linearized variational equations, for small disturbances about such *steady states*, examined to answer the stability problem. Theoretical results are compared with those obtained by direct analogue computer simulation of the system. The effect, on the mathematical analysis, of replacing the system multipliers by diode switching units is also considered in this chapter. The chapter concludes by presenting a method of obtaining a Liapunov redesign scheme for the system under discussion.

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INTRODUCTION

In the last twenty years or so a great deal of work has been published on so called *adaptive* or *self-adaptive* control systems. Although many different systems have been described as adaptive there is still no general agreement on a definition of an adaptive control system which would embrace them all. However, it is generally accepted that such systems must be capable of monitoring their own performance during operation and also possess the ability to adjust their own parameters, in response to a changing environment (time variations in environment may be in the form of input signals, disturbances, changing performance objectives or a process with changing parameters), in order to achieve a satisfactory performance. The parameter adjustments are performed in a closed loop fashion in accordance with some predetermined index of performance P; this being a mathematical measure of the quality of a particular system response and is a means by which an engineer may impose his definition of optimum operation upon the system. The index of performance is measured and compared with its optimum or desired value; the resulting error actuates the parameter adjustment loops which in turn change the value of P as measured - the adapting loops being so designed that the measured value of P is forced to agree with its optimum value in the steady state, thus achieving the required satisfactory performance.

A number of types of adaptive control systems have been proposed and the reader is referred to some of the works written on the subject[88-92]. In practical applications to devices such as autopilots where rapid adaption is required, two methods which have been given considerable attention in the literature are :

(i) extremum control or hill climbing systems

and (ii) model-reference systems.

Extremum control or hill climbing systems are a well defined class of adaptive control systems for which a formulation has been given elsewhere [93]. This method of adaption was first discussed in a paper by Draper and Li [63] who applied the idea to the control of a throttle in a gasoline engine. Basically the idea is to give the particular parameter of the system which is under control a small displacement or perturbation and to measure the effect of this perturbation on the index of performance. The information is then used to adjust automatically the parameter to the value which optimizes the index of performance. Since it requires the injection of a signal from outside in order to perturbe the parameter, this type of system is often referred to as a parameter perturbation adaptive control system.

In general the index of performance is a function of the system parameters and input so that if the input signal characteristics change or if the system exercises disturbances or variations in its parameters then the index of performance changes, thus causing the optimum values of the adapting parameters to change. It is therefore desirable that the index of performance is continuously optimized and this may be achieved by applying a periodically varying perturbation; for this reason extremum control systems employing periodic perturbations have received much attention in the literature. Although it is possible to use any periodic waveform as a perturbation signal, for example, Douce uses a square wave [94], it is systems employing sinusoidal perturbations that have received most popularity [61-62].

In a model reference adaptive control system all the desired dynamic characteristics of the controlled system are incorporated into a model so that the model output is proportional to the desired system response. The problem then reduces to one of making the system behave like the model by minimizing some function of the error between the response of the controlled system and that of the model when fed with the same input signal. Since this error signal is to be zero when the system is in its optimum state it is used as a demand signal for the adaptive loops which adjusts the variable parameters in the controlled system to their desired values.

Various methods of synthesizing the adaptive loops have been proposed [95-96] but the one that has proved most popular is that developed by Whitaker et al [75] at the Massachusetts' Institute of Technology and referred to as the M.I.T. rule. Here the performance criterion is taken as the integral of error squared and a heuristical argument is given for reducing this over an unspecified period of time. This leads to the rule that a particular parameter should be adjusted according to the rule :

$$\text{Rate of change of parameter} = -\text{Gain} \times \text{error} \times \frac{\partial(\text{error})}{\partial(\text{parameter})}$$

Although many sinusoidal perturbation and model reference adaptive control systems have been proposed their design has generally been carried out by much analogue computer simulation and a detailed mathematical analysis of the stability and the accuracy of parameter adaptation is still lacking. The reason for this is undoubtedly due to the fact that the system equations are both nonlinear and nonautonomous and so their analysis has proven to be very difficult. However, since any successful system design requires a basic understanding of the influence design parameters have on the over-all system performance, the development of such an analysis would be an important asset.

In this thesis a detailed mathematical analysis of both sinusoidal perturbation and model reference type adaptive control systems will be carried out and throughout dimensional analysis will be employed in order that all the results may be presented in a non-dimensional form. It is appropriate that such systems be considered in a particular project for, although the structure of an M.I.T. type model reference system differs widely from that of a sinusoidal perturbation system, some of the model reference systems proposed in the literature have adaptive controllers which have been structured using sinusoidal perturbation signals [96-97].

It will be seen that in all the problems considered the stability problem reduces to one of investigating the stability of either a system of linear differential equations with periodic coefficients or a system of linear differential equations with random coefficients. The earlier part of the thesis is therefore devoted to investigating these two problems.

In chapter 3 the stability and accuracy of parameter adaptation of a single input, sinusoidal perturbation adaptive control system with output lag [60-61] is considered. By considering an equivalent circuit Eveleigh [62] has carried out an approximate stability investigation for such a system but to date no satisfactory stability analysis exists. Here a detailed analysis will be presented; various harmonic and sub-harmonic *steady-state* solutions of the system equations are first obtained, using the principle of harmonic balance, and then the stability of each solution is investigated using the theory developed in chapter 1. The effect of initial conditions on the system behaviour will be obtained by plotting the *domains of attraction* corresponding to the stable *steady-state* solutions. A knowledge of such domains is essential if the system is to be subjected to input disturbance and measurement noise. All the theoretical results will be verified by direct analogue computer simulation of the system.

The remainder of the thesis is devoted to model reference adaptive control systems. Bongiorno [98] obtained necessary conditions for the stability of a particular type of model reference system when the input varied sinusoidally with time; the results, however, are only applicable to systems which are stable when the time varying terms are equated to zero and are therefore not applicable to the type of problem considered in this work. White [2] investigated the stability of an M.I.T. type system, when the input varied sinusoidally with time, by time averaging the coefficients, of the system equations, over a period - a method well known to be unreliable. For the case of random input Bell [82] carried out an approximate stability investigation, for an M.I.T. type system, by replacing the set of nonautonomous differential equations, representing the system, by a set of autonomous equations and then applying Liapunov's Direct Method.

In order to develop a mathematical analysis, and to illustrate the difficulties involved, we shall first carry out a theoretical stability analysis for a first order M.I.T. type system with controllable gain when the input varies with time in both a sinusoidal and random manner. Also considered will be the effects of

(a) random disturbances at the system output

and (b) periodic and random variations, with time, of the controlled process environmental parameter,

on the system stability and the accuracy of the parameter adaptation. All the theoretical results will be compared with results obtained by either analogue or digital computer simulation of the system.

The doubts concerning the stability and the difficulty of analysis of the M.I.T. type system have led researchers to think about redesigning the model-reference system from the point of view of stability. In particular we have the Liapunov synthesis method [46,76,77]; in this approach a Liapunov function is proposed and control signals are chosen such that its time derivative is negative definite. Should this be possible then the resulting system can be guaranteed stable for all possible inputs. However, in designing such systems the controlled process environmental parameters are assumed constant; by considering the Liapunov redesign scheme of the first order M.I.T. system discussed earlier it will be shown that the effect of making the environmental process parameters time varying is to introduce a stability problem.

The thesis is concluded by extending the analysis developed for analysing the first order system to investigate the stability of a higher order M.I.T. type system. The system considered is that developed by White [12] and has a third order controlled process and a second order model. As pointed out by Horrocks [86] such a system, where the controlled process and model are not of the same order, is most likely to be of the type used in practice. This is due to the fact that the order of the model is almost exclusively determined by bandwidth requirements whilst the order of the system is usually determined by the unavoidability of including components and subsidiary loops to perform specific subsidiary functions and by these components etc. introducing lags intrinsic to their structure.

A stability analysis for both sinusoidal and random inputs will be presented. *Steady-state* values of the adapting parameters are first obtained and then the linearized variational equations, for small disturbances about such *steady-states*, examined in order to answer the stability problem. Also considered will be the effect, on the mathematical analysis, of replacing the system multipliers by diode switching units.

Since the model and controlled process are not of the same order it is no longer possible to have perfect correspondence between the two. It follows that for such systems it is not possible to obtain a Liapunov function that will guarantee asymptotic stability. By introducing adjustable parameters around the controlled process a method will be presented of obtaining a Liapunov redesign scheme for the M.I.T. system under discussion.

CHAPTER 1

LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

1.1. Introduction

Linear differential equations with periodic coefficients form a most important sub-class of linear differential equations with variable coefficients. The equations, which occur frequently in practice, may arise directly from the equations of motion of a dynamic system, for example, the flapping of a helicopter rotor blade [1], but more frequently arise from an examination of the stability of oscillations in non-linear systems.

Surprisingly the problem of investigating the stability of such a system of equations is one of extreme difficulty and even the relatively simple scalar equation

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t)x = 0,$$

the undamped Mathieu equation, poses major difficulties and has essentially a theory of its own. A procedure frequently used by authors [2, 3] is to time average the coefficients over a period and replace the system of differential equations with periodic coefficients by a system of differential equations having constant coefficients. This is a dangerous procedure and gives rise to serious doubts regarding the validity of the consequent stability analysis. It is seen to fail for comparatively simple equations such as the equation

$$\frac{d^2x}{dt^2} + 0.2\frac{dx}{dt} + (4.5 - 4 \cos 2t)x = 0$$

which is unstable [4].

Various authors have employed Liapunov's second (or direct) method to obtain sufficient conditions for the stability of linear differential equations with time varying coefficients whilst Bonginoro[5] applied an extension of the Nyquist-Barkhausen stability criterion to obtain similar conditions when the coefficients are periodic. These methods, however, can only be applied to examine systems of the form

$$\dot{\underline{x}}(t) = \underline{B}\ \underline{x}\ (t) + \underline{A}\ (t)\ \underline{x}\ (t),$$

where dot denotes differentiation with respect to time t, $\underline{x}(t)$ is an n-column vector, $\underline{B}$ is a constant nxn matrix such that the system $\dot{\underline{y}} = \underline{B}\ \underline{y}$ is asymptotically stable and $\underline{A}(t)$ is an nxn matrix whose non-identically zero elements are time-varying. Furthermore, the results obtained by these methods are usually rather conservative.

In this chapter a rigorous numerical implementation of Floquet theory, well suited for digital computation, will be presented for obtaining necessary and sufficient conditions to guarantee the stability of a system of linear differential equations with periodic coefficients.

The chapter will be concluded by considering the so called infinite determinant method of obtaining the stability boundaries in parameter space for a restricted class of linear differential equations with periodic coefficients.

1.2. Stability theorem

Writing the system of equations as a set of first order differential equations we have the vector matrix differential equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}\ (t), \qquad (1.2.1.)$$

where $\underline{x}(t)$ is an n-vector and $\underline{A}(t)$ an nxn matrix satisfying the condition.

$$\underline{A}\ (t + T) = \underline{A}\ (t),\ t\ \varepsilon\ [0, \infty)$$

For a linear system such as that defined in equation (1.2.1.) it is possible, using Floquet analysis, to prove the following theorem.

Theorem 1.2.1.

For the system of differential equations (1.2.1.) there exists a constant nxn matrix $\underline{C}$, known as the monodromy matrix [6] of the system, such that

$$\underline{x}\,(t_0 + T) \;=\; \underline{C}\,\underline{x}\,(t_0),\; t_0 \in [0, \infty);$$

and a necessary and sufficient condition for the null solution of (1.2.1.) to be uniformly asymptotically stable is that all the eigenvalues of the matrix $\underline{C}$ lie within the unit circle $|z|<|$. If the eigenvalues of the monodromy matrix $\underline{C}$ lie in the circle $|z|\leqslant|$, and the eigenvalues on $|z| = 1$ correspond to unidimensional Jordan cells, then the null solution of (1.2.1.) is uniformly stable.

Proof

The solution of system (1.2.1.) may be written [7] as

$$\underline{x}(t) \;=\; \Phi(t)\,\underline{x}\,(t_0) \qquad (1.2.2.)$$

where $\underline{x}(t_0)$ is arbitary, $\underline{\Phi}(t_0) \;=\; \underline{I}$ the identity matrix, and

$$\underline{\Phi}(t) \;=\; \underline{P}(t)\exp\{\,\underline{R}(t - t_0)\,\}, \qquad (1.2.3.)$$

where $\underline{P}(t)$ is a non-singular periodic matrix with period T and $\underline{R}$ is an nxn constant matrix.

From equations (1.2.2.) and (1.2.3.) we have that

$$\begin{aligned}\underline{x}\,(t_0 + T) \;&=\; \underline{\Phi}\,(t_0 + T)\,\underline{x}\,(t_0)\\ &= \underline{P}\,(t_0 + T)\exp\{\,\underline{R}(t_0 + T - t_0)\,\}\,\underline{x}\,(t_0)\\ &= \exp\{\,\underline{R}(T)\,\}\,\underline{x}\,(t_0)\end{aligned}$$

since $\underline{P}(t_0 + T) \;=\; \underline{P}(t_0) \;=\; \underline{I}$.

Hence,

$$\underline{x}\,(t_0 + T) = \underline{C}\,\underline{x}\,(t_0) \qquad (1.2.4.)$$

where $\underline{C}$ is the constant nxn matrix defined by

$$\underline{C} = \exp\{\,\underline{R}\,T\,\} \qquad (1.2.5.)$$

Every linear system of differential equations with periodic coefficients, such as (1.2.1.), is reducible, in the sense of Liapunov [8], by means of the Liapunov transformation

$$\underline{x} = \underline{P}\,(t)\,\underline{y},$$

where $\underline{P}(t)$ is defined as in equation (1.2.3.). This transformation carries equations (1.2.1.) into the form

$$\dot{\underline{y}} = \underline{R}\,\underline{y} \qquad (1.2.6.)$$

where $\underline{R}$ is the constant nxn matrix defined in equation (1.2.3.).

An important property of a Liapunov transformation, which makes it attractive in stability investigations, is that it does not alter the character of the zero or null solution as regards stability; so that, the null solution of system (1.2.1.) is uniformly asymptotically stable (or uniformly stable) if and only if the null solution of system (1.2.5.) is uniformly asymptotically stable (or uniformly stable).

Since $\underline{C} = \exp\{\,\underline{R}\,T\,\}$ it follows that the eigenvalues λ_i, μ_i (i = 1,2, ---, n) of $\underline{R}$ and $\underline{C}$ respectively are related by the formulae

$$\lambda_i = \frac{1}{T}\log \mu_i, \quad i = 1,2 ----, n$$

From the stability theory of linear differential equations with constant coefficients [6, 9] the results of the theorem follow.

1.3. Evaluation of the monodromy matrix

In order to compute the monodromy matrix $\underline{C}$, of equation (1.2.4.), the system of equations (1.2.1.) are integrated numerically over a period T. To do this it is convenient to employ a numerical procedure that may be reformulated in such a way as to give $\underline{C}$ directly. Since it is completely self contained and requires no pre determination of a set of starting values the Runge-Kutta method [10] is given preference over the various predictor-corrector methods of integration. An alternative, as pointed out by Davison [11], is the Crank-Nicolson procedure [12]. A formulation of $\underline{C}$, based on these two procedures, will now be presented.

1.3.1. Runge-Kutta procedure

In order to compute the monodromy matrix $\underline{C}$ the fourth order Runge-Kutta procedure of solving a system of first order linear differential equations is reformulated as follows :

If

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}\ (t), \quad t \ \varepsilon \ [0, \infty) \tag{1.3.1.}$$

where $\underline{x}(t)$ is an n-column vector and $\underline{A}(t)$ an nxn periodic matrix of period T, then the period is split into a large number of intervals N, each of duration Δt (= T/N), and the following finite difference relationship employed

$$\underline{x}(\overline{m+1\ \Delta t}) = \underline{x}\ (m\Delta t) + \frac{1}{6}\ (\underline{\alpha}_1 + 2\underline{\alpha}_2 + 2\underline{\alpha}_3 + \underline{\alpha}_4) \tag{1.3.2.}$$

where

$$\underline{\alpha}_1 = \Delta t\ \underline{A}\ (m\Delta t)\ \underline{x}\ (m\Delta t) = \underline{K}_1\ \underline{x}\ (m\Delta t)$$

$$\underline{\alpha}_2 = \Delta t\ \underline{A}\ (\overline{m + \tfrac{1}{2}\ \Delta t})\ [\ \underline{x}\ (m\Delta t) + \tfrac{1}{2}\ \underline{\alpha}_1\ ]$$

$$= \Delta t\ \underline{A}\ (\overline{m + \tfrac{1}{2}\ \Delta t})\ [\ \underline{I} + \tfrac{1}{2}\ \underline{K}_1\ ]\ \underline{x}\ (m\Delta t),$$ where $\underline{I}$ is the nxn identity matrix

$$= \underline{K}_2\ \underline{x}\ (m\Delta t)$$

$$\underline{\alpha}_3 = \Delta t \, \underline{A} \, (\overline{m + \tfrac{1}{2} \, \Delta t}) \, [\underline{x}(m\Delta t) + \tfrac{1}{2} \, \underline{\alpha}_2]$$

$$= \Delta t \, \underline{A} \, (\overline{m + \tfrac{1}{2} \, \Delta t}) \, [\underline{I} + \tfrac{1}{2} \, \underline{K}_2] \, \underline{x}(m\Delta t)$$

$$= \underline{K}_4 \, \underline{x} \, (m\Delta t)$$

$$\underline{\alpha}_4 = \Delta t \, \underline{A} \, (\overline{m + 1 \, \Delta t}) \, [\underline{x}(m\Delta t) + \underline{\alpha}_3]$$

$$= \Delta t \, \underline{A} \, (\overline{m + 1 \, \Delta t}) \, [\underline{I} + \underline{K}_3] \, \underline{x}(m\Delta t)$$

$$= \underline{K}_4 \, \underline{x}(m\Delta t)$$

Substituting in equation (1.3.2.) for $\underline{\alpha}_1$, $\underline{\alpha}_2$, $\underline{\alpha}_3$ and $\underline{\alpha}_4$ gives

$$\underline{x}(\overline{m + 1 \, \Delta t}) = [\, \underline{I} + \frac{1}{6} \underline{K}_1 + \frac{1}{3} \underline{K}_2 + \frac{1}{3} \underline{K}_3 + \frac{1}{6} \underline{K}_4 \,] \, \underline{x}(m\Delta t),$$

that is,

$$\underline{x}(\overline{m + 1 \, \Delta t}) = \underline{B} \, (m\Delta t) \, \underline{x} \, (m\Delta t) \qquad (1.3.3.)$$

where

$$\underline{B}(m\Delta t) = \underline{I} + \frac{1}{6} \underline{K}_1 + \frac{1}{3} \underline{K}_2 + \frac{1}{3} \underline{K}_3 + \frac{1}{6} \underline{K}_4 \qquad (1.3.4.)$$

By repeated application of (1.3.3.) the solution at the end of a period in terms of that at the beginning of the period becomes

$$\underline{x}(T) = \underline{x}(N \, \Delta t) = \underline{B} \, (\overline{N - 1 \, \Delta t}) \, \underline{B} \, (\overline{N - 2 \, \Delta t}) \, ---- \, \underline{B}(o) \, \underline{x}(o)$$

$$= \prod_{r = o}^{N - 1} \underline{B} \, (r \, \Delta t) \, \underline{x}(o)$$

$$= \underline{C} \, \underline{x}(o),$$

where

$$\underline{C} = \prod_{r = o}^{N - 1} \underline{B} \, (r \, \Delta t) \qquad (1.3.5.)$$

is the required monodromy matrix.

1.3.2. Crank-Nicolson procedure

The Crank Nicolson method of numerical integration for system (1.3.1.) leads to the following finite difference relationship

$$\underline{x}(\overline{m + 1\ \Delta t}) = [\ \underline{I} - \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ]^{-1}\ [\ \underline{I} + \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ]\ \underline{x}(m\Delta t) \tag{1.3.6.}$$

where $\underline{I}$ is the nxn identity matrix and Δt as defined in Section (1.3.1.)

that is,

$$\underline{x}(\overline{m + 1\ \Delta t}) = \underline{B}^*\ (m\Delta t)\ \underline{x}(m\Delta t) \tag{1.3.7.}$$

where

$$\underline{B}^*\ (m\Delta t) = [\ \underline{I} - \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ]^{-1}\ [\ \underline{I} + \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ] \tag{1.3.8.}$$

By repeated application of relationship (1.3.7.) the solution at the end of a period in terms of that at the beginning of the period becomes

$$\begin{aligned} \underline{x}(T) &= \underline{x}(N\Delta t) \\ &= \underline{B}^*\ (\overline{N - 1\ \Delta t})\ \underline{B}^*\ (\overline{N - 2\ \Delta t})\ \text{------}\ \underline{B}^*\ (o)\ \underline{x}\ (o) \\ &= \prod_{r = o}^{N - 1} \underline{B}^*\ (r\ \Delta t)\ \underline{x}\ (o) \\ &= \underline{C}\ \underline{x}\ (o) \end{aligned}$$

where $$C = \prod_{r = o}^{N - 1} \underline{B}^*\ (r\ \Delta t) \tag{1.3.9.}$$

is the required monodromy matrix.

As indicated by Davison [11] this method of determining $\underline{C}$ is inherently stable for any choice of N since the eigenvalues of the matrices

$$[\ \underline{I} - \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ]^{-1},\ [\ \underline{I} + \frac{\Delta t}{2}\ \underline{A}\ (m\Delta t)\ ];\ m = 0,1,\ ---,\ N - 1$$

are all less than unity in absolute value. Davison found that at least four significant figure accuracy in the calculation of $\underline{C}$ was obtained if N was chosen as

$$N = 30\,\iota \max_{t} \left| \lambda_{dom}\, \underline{A}(t) \right| ,$$

where $\lambda_{dom}\, \underline{A}(t)$ is the largest dominant eigenvalue of $\underline{A}(t)$ which is non zero.

Both the Runge-Kutta and Crank-Nicolson procedures have been employed in the work described in this text and it was found that although a smaller step length Δt had to be used when employing the Crank-Nicolson procedure the computer running time was usually less.

1.3.3. Numerical check on the value of the monodromy matrix

A check on the value of the monodromy matrix $\underline{C}$ may be made using the following result.

Consider the determinant D(t) of the monodromy matrix at time t, then

$$D\,(t + \Delta t) = \left| \underline{I} + \underline{A}\,(t)\,\delta t \right| D(t)$$

$$= (1 + \text{trace}\, \underline{A}(t)\, \delta t)\; D(t)$$

to first order in δt; so that

$$\dot{D}\,(t) = \text{trace}\, \underline{A}(t)\; D(t)$$

Thus, since $D(o) = 1$, we have that

$$D(t) = \exp \{ \int_0^t \text{trace}\, \underline{A}(t)\, dt \} \qquad (1.3.10.)$$

Hence, over a period T equation (1.3.10) gives

$$\det \underline{C} = D(T) = \exp \{ \int_0^T \text{trace}\, \underline{A}(t)\, dt \} \qquad (1.3.11)$$

1.4. Eigenvalues of the monodromy matrix

Having computed the monodromy matrix $\underline{C}$ the next step is to examine its eigenvalues. Although the matrix $\underline{C}$ is itself real some of its eigenvalues may occur as complex conjugates and this sometimes causes difficulties regarding time of convergence when employing standard numerical methods of evaluating the eigenvalues. A good discussion of these numerical methods, together with the difficulties involved when the eigenvalues occur as complex conjugates, may be found in the work of Wilkinson [13]. Davison [11] in his paper evaluated the eigenvalues of $\underline{C}$ using the Q-R procedure [13].

Bearing in mind that in the problem at hand it is not necessary to know the exact values of the eigenvalues of $\underline{C}$, but rather it is only required to show that their modulii are less than unity, a very elegant procedure, based on the works of Faddeev [14] and Jury [15, 16], has been introduced to deal with this problem. This procedure, which only involves matrix multiplication and the evaluation of determinants of order two, has many advantages over any of the numerical methods available for evaluating eigenvalues. It is a comparatively simple procedure and requires far fewer arithmetic operations; it is readily programmed and the running time is comparatively small. It has the distinct advantage in that it is not an iterative procedure, so that the question of convergence does not arise.

In this work therefore the method employed to examine the eigenvalues of $\underline{C}$ (except for matrices of order two where the eigenvalues were obtained by direct solution of the quadratic) is to first obtain the characteristic polynomial using the Faddeev algorithm and then determine whether or not the roots of this polynomial lie inside the unit circle using the determinant method of Jury.

1.4.1. The Faddeev algorithm

If the characteristic polynomial of an nxn matrix $\underline{A}$ is represented in the form

$$\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} \text{-------} - p_{n-1}\lambda - p_n$$

then the Faddeev algorithm states that the coefficient $p_i (i = 1,2, ---, n)$ may be computed in the following manner :-

$$p_r = \frac{1}{r} \text{ trace } \underline{A}_r, \; r = 1,2, ---, n$$

where

$$\underline{A}_r = \underline{A} \text{ when } r = 1$$

$$= \underline{A}\,\underline{B}_{r-1} \quad \text{when } r = 2,3, ---, n$$

and $\underline{B}_r = \underline{A}_r - p_r\underline{I}$, where $\underline{I}$ is the identity matrix of order n.
A check may be employed since

$$\underline{B}_n = \underline{A}_n - p_n\underline{I}$$

must be the null matrix.

NOTE By the trace of a matrix is meant the sum of the terms in the leading diagonal; that is,

$$\text{trace } \underline{A} = \sum_{i=1}^{n} (a_{ii})$$

An algol program for this procedure is given in appendix (1.1).

1.4.2. The determinant method of Jury

This procedure gives necessary and sufficient conditions for the polynomial

$$F(x) \equiv x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ----- a_1x + a_0 \tag{1.4.1.}$$

to have all its roots inside the unit circle. It only requires the evaluation of second order determinants and can be easily programmed on a digital computer.

The conditions are obtained by forming table (1.4.1.) (note that the elements of row 2K + 2 consists of the elements of row 2K + 1 written in reverse order; K = 0,1,2, ---, n).

TABLE 1.4.1.

A Stability Procedure Table

Row	x^n	x^{n-1}	x^{n-2}	-----	x^{n-K}	-----	x^2	x^1	x^0
1	1	a_{n-1}	a_{n-2}		a_{n-K}		a_2	a_1	a_o
2	a_o	a_1	a_2		a_K		a_{n-2}	a_{n-1}	1
3	b_o	b_1	b_2		b_K		b_{n-2}	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}		b_{n-2-K}		b_1	b_o	
5	c_o	c_1	c_2		c_K		c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}		c_{n-2-K}		c_o		
7	d_o	d_1	d_2		d_K				
8	d_{n-3}	d_{n-4}	d_{n-5}		d_{n-3-K}				
'									
'									
'									
'									
'									
'									
'									
2n-5	s_o	s_1	s_2	s_3					
2n-4	s_3	s_2	s_1	s_o					
2n-3	r_o	r_1	r_2						
2n-2	r_2	r_1	r_o						
2n-1	t_o								

where $b_K = \begin{vmatrix} 1 & a_{n-K} \\ a_o & a_K \end{vmatrix}$, $c_K = \begin{vmatrix} b_o & b_{n-2-K} \\ b_{n-1} & b_K \end{vmatrix}$, $d_K = \begin{vmatrix} c_o & c_{n-2-K} \\ c_{n-2} & c_K \end{vmatrix}$,

-------------, $t_o = \begin{vmatrix} r_o & r_2 \\ r_2 & r_o \end{vmatrix}$

Necessary and sufficient conditions for the polynomial (1.4.1.) to have all its roots inside the unit circle (that is, stability conditions) are :

(i) $F(1) > 0$, $(-1)^n F(-1) > 0$

(ii) $b_o > 0$, $c_o > 0$, $d_o > 0$, ------ $s_o > 0$, $r_o > 0$, $t_o > 0$

(n-1) constraints

An algol program for this procedure is given in appendix (1.2).

1.5. Infinite determinant method

Since it requires that the form of the solution on the transition boundary between stable and unstable solutions (henceforth referred to as the transition boundaries) be known this method is only suitable for obtaining the stability boundaries in parameter space for a restricted class of linear, periodic coefficient ordinary homogeneous differential equations. The method has been used extensively by Bolotin [17], in studying problems of elastic stability of structures under parametric excitation, and in this section we shall restrict ourselves to a brief discussion.

System (1.2.1.) does not necessarily have periodic non zero solutions of period T. However, a basic consequence of the work of Floquet [18, 9] is the conclusion that the system has at least one solution of the form

$$\underline{x}_i(t) = \exp \{ (\log\lambda_i)t/T \} \underline{p}_i(t) \qquad (1.5.1)$$

where $\underline{p}_i(t + T) = \underline{p}_i(t)$ and λ_i is an eigenvalue of the monodromy matrix of the system. Furthermore, if the eigenvalues $\lambda_i (i = 1,2,----, m)$, $1 \leqslant m \leqslant n$, of the corresponding monodromy matrix are all distinct then there are m independent solutions described by (1.5.1), and if

$m = n$ these form a fundamental system of solutions.

Equation (1.5.1) may be written in the form

$$\underline{x}_i(t) = \exp\{\log|\lambda_i|t/T\}\exp\{j\arg(\lambda_i)t/T\}\,\underline{p}_i(t),\ j = \sqrt{-1} \tag{1.5.2}$$

Since, from theorem 1.2.1, there must always be an eigenvalue with modulus exceeding unity for instability, and there can be no such eigenvalue for stability, it follows from the assumed continuous dependence of stability on parameter values that there must exist an eigenvalue of modulus unity for parameter values on any boundary between stable and unstable regions. Thus, on these transition boundaries there must exist an almost periodic solution of the form

$$\underline{x}_i(t) = \exp\{j(\arg\lambda_i)t/T\}\,\underline{p}_i(t) \tag{1.5.3}$$

(Note that in general such solutions may also exist inside stable and unstable regions but not within regions of uniform asymptotic stability).

If the monodromy matrix of the system is symplectic† then its characteristic equation is reciprocal [6] (canonical systems†† are examples of such systems). It follows from theorem 1.2.1. that asymptotic stability is impossible for such systems since if λ_i, a root of the characteristic equation of the monodromy matrix, is in $|z|<|$ then $1/\lambda_i$ (also a root of the characteristic equation) is in $|z|>|$. In the case when the eigenvalues are on the unit circle

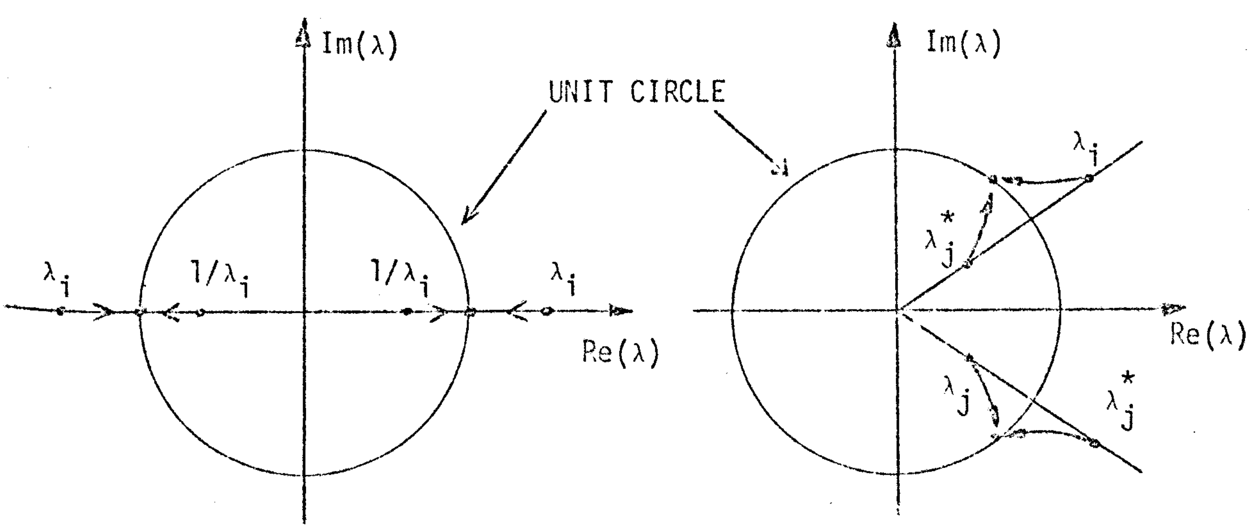
---

† A matrix $\underline{A}$ is said to be symplectic if it satisfies the property $\underline{A}^*\,\underline{E}\,\underline{A} = \underline{E}$ where $\underline{E} = \begin{bmatrix} 0 & \underline{I} \\ -\underline{I} & 0 \end{bmatrix}$, $\underline{I}$ being the identity matrix.

†† A differential equation system is termed canonical if it can be written in the form of Hamiltons canonical equations $\dot{p}_i = \frac{\partial H}{\partial q_i}$, $\dot{q}_i = -\frac{\partial H}{\partial p_i}$. If $\underline{H}$ is a quadratic form in the variables p_i, q_i then the system will be linear.

and the normal Jordan form is diagonal all solutions are bounded. (This is the case in the stable region for the undamped Mathieu equation). The remainder of this section will be devoted to systems having a symplectic monodromy matrix.

It follows that if the transition boundary is characterised by the presence of a real root λ_i then λ_i and its reciprocal must each have the value +1, or the value -1, giving a root of multiplicity two on the boundary. Figure 1.5.1(a) illustrates the locus of points in the complex plane occupied by the real numbers λ_i and $1/\lambda_i$ as transition is made in the parameter space from the indicated unstable region to a stability boundary. However, in general, the transition boundary may be characterised by complex roots. If λ_i, $|\lambda_i|>1$, is a complex root then, due to the characteristic equation having real coefficients, there must exist, in addition to the reciprocal λ_j of λ_i, the complex conjugates of both λ_i and λ_j (denoted by λ_i^*, λ_j^* respectively). Thus, as illustrated in Fig. 1.5.1(b), the transition to a stability boundary from a region made unstable by a complex root of modulus exceeding unit is marked by the presence of two roots of multiplicity two.



(a) (b)

FIG. 1.5.1. Locus of characteristic roots in transition from unstable region to stability boundary.

If $\lambda_i = +1$ is substituted into equation (1.5.3) the result is a periodic solution of period T

$$\underline{x}_i(t) = \underline{p}_i(t) \tag{1.5.4}$$

On the other hand substituting $\lambda_i = -1$ into equation (1.5.3) gives

$$\underline{x}_i(t) = \exp \{ j\pi t/T \} \underline{p}_i(t) \tag{1.5.5}$$

which is a function of period 2T. Thus, if one can be assured a priori that for a given system all transition boundaries are marked by no more than one multiple root then such boundaries are characterised by the existence of a periodic solution of period T or 2T (Note that equation (1.5.5.) is a restricted class of functions of period 2T, e.g. a constant does not qualify).

Thus, if it is known a priori that one pair of multiple roots marks a transition boundary then the path is clear to search for such boundaries in the form of a search for solutions of known period. This information is known for second order systems since then there are only two eigenvalues and instability due to complex roots is impossible. This argument also applies to uncoupled canonical systems of dimension n = 2N, with N an integer representing the number of degrees of freedom of the dynamical system. In other words if equation (1.2.1) may be written in the form

$$\underline{I}(t)\ddot{\underline{q}} + \underline{K}(t)\underline{q} = 0,$$

with the NxN matrices $\underline{I}(t)$, $\underline{K}(t)$ diagonal, then the system is no more than a collection of independent second order scalar equations and the previous argument applies to each of the scalar equations individually. The several unstable regions can then be superimposed without ambiguity, even though it may happen that a stability boundary from one scalar equation crosses into a region which is unstable by virtue of another scalar equation. It is clear however that more work is needed to permit the identification of those differential equations (1.2.1) which meet the requirement of having solutions of period T or 2T on all transition boundaries and nowhere else in parameter space.

For systems whose transition boundaries are characterised by periodic solutions of period T or 2T the numerical procedure employed to find the transition boundaries is as follows. If there exists a solution of period T, it must be representable by a Fourier series of the form

$$\underline{x} = \underline{b}_0 + \sum_{K=2,4,6}^{\infty} \left[\underline{a}_K \sin \frac{K\pi t}{T} + \underline{b}_K \cos \frac{K\pi t}{T} \right]$$

where $\underline{b}_0$, $\underline{a}_K$ and $\underline{b}_K$ are real, constant n column vectors, of values as yet undetermined. This series solution is then substituted into the system equations (1.2.1) and the principle of harmonic balance employed to obtain an infinite system of simultaneous, linear, homogeneous algebraic equations for the coefficients. For those values of the parameters which admit the assumed periodic solution the homogeneous algebraic equations must have a non-trivial solution and this is the case only if the infinite determinant (Hill determinant) of the coefficients is zero. In practice the Fourier series is truncated and the corresponding Hill determinant, if finite order, solved to give lines in parameter space (which correspond to zeros of the determinant). If the truncation point of the Fourier series is extended and the zeros of the corresponding Hill determinants of increasing order converge to some limit set of lines then the infinite determinant procedure is said to be convergent; the convergent set of lines in parameter space being the required transition boundary between stable and unstable regions. To determine the region of instability bounded by the periodic solution, with period 2T, we represent the solution by a Fourier series of the form

$$\underline{x} = \sum_{K=1,3,5}^{\infty} \left[\underline{a}_K \sin \frac{K\pi t}{T} + \underline{b}_K \cos \frac{K\pi t}{T} \right]$$

and then proceed in an analogous manner to that employed for the solution of period T.

In a recent paper Lindh and Likins [19] extended the method discussed above to completely damped mechanical systems of the form

$$\underline{I}(t)\ddot{\underline{q}} + \underline{G}(t)\dot{\underline{q}} + \underline{D}(t)\underline{q} + \underline{K}(t)\underline{q} = 0,$$

where $\underline{I}(t)$ and $\underline{D}(t)$ are symmetric, $\underline{G}(t)$ is skew symmetric and all the matrices are periodic with period T. Their method involved a search for an almost periodic solution, of the form defined by equation (1.5.3), which by the nature of the system must exist on all stability boundaries, and cannot exist within regions of stability. The method was illustrated by application to an attitude stability problem.

CHAPTER 2

LINEAR DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

2.1. Introduction

In recent years the problem of investigating the stability of the solutions of differential equations with randomly varying coefficients has been studied by many authors and a recent survey of this work may be found in a paper by Kozin [20]. The most successful of these investigations made white noise assumptions for the coefficients as then the methods of the theory of Markov processes may be utilized. Investigation of the stability under non white excitation has proven to be much more difficult and for this reason most authors have limited their investigations to systems of some particular type.

Definitions for various types of stability have been proposed for systems with stochastic coefficients [20, 21]. Which of these stability concepts is most useful, or most significant is still undecided and Kushner [22] has stated that the proper concept of stochastic stability remains to be settled as the subject develops. Kozin [20, 23], maintains the view that when studying real systems that are subjected to random variations in their parameters, or are operating within randomly perturbed environmental conditions then one desires stability properties as close to deterministic stability as possible so that conditions that will guarantee almost sure symptotic stability is the goal to aim for.

In this work we shall be concerned mainly with three types of stability namely : stability in mean, stability in mean square and almost sure asymptotic stability and we shall investigate such stability under two random coefficients variations viz :

(i) Gaussian white noise processes

and (ii) Gaussian non-white processes

2.2. White and physical white noises

When dealing with white noise one must be careful to distinguish between two distinctly different types of systems that appear in the literature.

In practice when one is dealing with noise corrupted systems the state equations representing the system are written

$$\frac{d\underline{x}(t)}{dt} = \underline{a}(\underline{x}, t) + \underline{B}(\underline{x}, t)\underline{X}(t), \ t \in [\ 0, \infty\ ), \qquad (2.2.1)$$

where $\underline{x}(t)$ is the n-state vector, $\underline{a}$ an n-vector, $\underline{B}$ an nxn matrix and $\underline{X}(t)$ an n-vector whose elements $X_i(i = 1,2, ----, n)$ are regarded as Gaussian white noise processes satisfying the conditions

$$E\{X_i(t)\} = 0 \qquad i,j = 1,2, ----, n \qquad (2.2.2)$$
$$E\{X_i(t)\ X_j(t + \tau)\} = 2D_{ij}\ \delta(\tau)$$

where E denotes the mathematical expectation, δ denotes the Dirac delta function, $4D_{ij}$ is the cross spectral density of $X_i(t)$ and $X_j(t)$ and when $i = j$, $4D_{ii}$ is the spectral density (self) of $X_i(t)$.

Formally the integral of Gaussian white noise is the Wiener or Brownian process [24] so that equation (2.2.1) is apparently identical with the stochastic differential equation or Itô equation [25]

$$d\underline{x}(t) = \underline{a}(\underline{x}, t)dt + \underline{B}(\underline{x}, t)d\ \underline{z}(t)\ , \ t \in [\ 0, \infty\ ), \qquad (2.2.3)$$

where $\underline{z}(t)$ is an n-vector process whose components are assumed to be Wiener or Brownian processes. However, this is not the case since white noise is not integrable (since it has an infinite mean square or power) and a Wiener process is not differentiable so that the relationship between equations (2.2.1) and (2.2.3) is strictly a formal one.

To analyse the solutions of equation (2.2.1) all that is required are the ideas of the ordinary calculus but in the case of equation (2.2.3) the situation is quite different. Equation (2.2.3) is strictly symbolic and one cannot divide throughout by dt to give an equation that makes sense in the ordinary meaning of a differential equation. This equation, however, has been given a precise definition by Itô [25] who represents it by the stochastic relation

$$\underline{x}(t) = \underline{x}(t_0) + \int_{t_0}^{t} \underline{a}(\underline{x}(\tau),\tau)\, d\tau + \int_{t_0}^{t} \underline{B}(\underline{x}(\tau),\tau) d\underline{z}(\tau) \;, \qquad (2.2.4)$$

where the last integral in equation (2.2.4) is the so called *stochastic integral* introduced by Itô for studying such systems. It is in the introduction of this stochastic integral that the difference between equations (2.2.1) and (2.2.3) begin; considered as a normal integral it does not exist and as demonstrated by Doob [24] it differs quite often from formal integrals. For a more detailed discussion of the relationship between equations (2.2.1) and (2.2.3) the reader is referred to a paper by Wong and Zakai [26].

In practice the elements $x_i(t)$, i = 1,2 ----, n, of $\underline{x}(t)$ are usually an approximation for Gaussian white noise and sometimes referred to as *physical white noise*; that is, they are Gaussian processes with a very small correlation time τ ($\tau = \frac{1}{\omega}$ where ω is the cut-off frequency of the process) which is not identically zero as for a δ-correlated process (theoretical white noise). Thus, in practice a wideband noise process whose cut off frequency is high but finite is regarded as physical white noise if $\omega >> \frac{1}{T}$, where T is the integrating constant of the physical system (such as a low pass filter). It follows therefore that in practice the elements $dz_i(t)$ of equation (2.2.3) do not truly represent Wiener processes but rather ${}^{dz_i(t)}/dt$ are mathematical approximations to Gaussian white noise processes with small correlation times. In this case the elements $z_i(t)$, $i = 1,2, ----, n$, of $\underline{z}(t)$ are not such that they require the use of a different concept of integration and for this reason engineers usually use a formal integral in equation (2.2.4). However, it should be clearly understood that the representation of white noise processes by physical white noise processes is only a convenient mathematical abstraction and should be treated as such; in the strict mathematical sense the last integral of equation (2.2.4) should be taken as the stochastic integral and equation (2.2.3) defined as by Itô. This last paragraph serves to illustrate why the solutions of equations (2.2.1) and (2.2.3) are seen in a different light by mathematicians and engineers.

2.3. Use of the Fokker-Planck equation

The work of this section follows closely that of Ariaratnam and Graeffe [27, 28, 29, 30], Caughey, Dienes and Gray [31, 32] and Kozin [33].

We shall consider the linear system

$$\frac{d\underline{x}(t)}{dt} + \underline{A}\,\underline{x}(t) + \underline{B}(t)\,\underline{x}(t) = \underline{\beta}(t)\ , \ t \in [0, \infty), \tag{2.3.1}$$

where $\underline{x}(t)$ is an n-column vector representing the state of the system, $\underline{A}$ an nxn constant matrix, $\underline{B} \equiv (\beta_{ij}(t))$, $i, j = 1,2, ----, n$, an nxn matrix and $\underline{\beta} \equiv (\beta_{io}(t))$, $i = 1,2, ----, n$, an n-column vector of Gaussian white noise processes with the following statistical properties

$$\begin{array}{ll} E\{\beta_{ij}(t)\} = 0 & i, r = 1,2, ----, n \\ E\{\beta_{ij}(t)\beta_{rs}(t+\tau)\} = 2D_{ijrs}\ \delta(\tau) & j, s = 0,1,2, ----, n \end{array} , \tag{2.3.2}$$

the symbols being defined as in section 2.2.

It may be shown (Ariaratnam [34], Wang and Ublenbeck [35]) that the state vector $\underline{x} = [\, x_1, x_2, ----, x_n \,]^1$ of system (2.3.1) is an example of a continuous n-dimensional Markov process (that is, a process whose future state depends only on its present state and is independent of how the process attained its present state). Such a process is completely described by its transition probability law which may be obtained as the fundamental solution of the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (A_i p) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 (B_{ij} p)}{\partial x_i \partial x_j} \qquad (2.3.3)$$

appropriate to the system, where $p = p(\underline{x}, t/\underline{x}_o, t_o)$ denotes the probability that the state point lies in the differential element $(\underline{x} + d\underline{x})$ of state space at time t given that it was at the point $\underline{x}_o$ at time t_o and the coefficients A_i and B_{ij} are given by

$$A_i = \underset{\delta t \to o}{\text{limit}} \frac{E\{\delta x_i\}}{\delta t}$$

$$i,j = 1,2, ----, n \qquad (2.3.4)$$

$$B_{ij} = \underset{\delta t \to o}{\text{limit}} \frac{E\{\delta x_i \delta x_j\}}{\delta t}$$

[*Note 1* When the theoretical white noise processes of equation (2.2.5) are replaced by the practical physical white processes then the state vector $\underline{x}(t)$ does not truly represent an n-dimensional Markov process, but if one is only interested in behaviour that takes place in "macroscopic" time intervals (those larger than the correlation time) then only the first conditional probability density is needed to describe $\underline{x}(t)$ and the process is effectively Markovian [32].

Note 2 The Itô or stochastic differential equation

$$d\underline{x}(t) + \underline{A}\,\underline{x}\,dt + d\underline{B}(t)x = d\underline{\beta}(t),$$

where $d\underline{B}(t) \equiv (d\beta_{ij}(t))$ and $d\beta(t) \equiv (d\beta_{io}(t))$ are nxn and nx1 matrices respectively whose elements are increments of Wiener or Brownian processes, will give rise to a different Fokker-Planck equation to that representing system of equations (2.3.1) [30, 32]]

Various methods have been presented for evaluating these coefficients [28-34] and for the system of equations (2.3.1) subject to conditions (2.3.2) the coefficients have been evaluated by Ariaratnam and Graeffe [39] as

$$A_i = -\sum_{r=1}^{n} \left[a_{ir}\,x_r + D_{irro} - \sum_{s=1}^{n} D_{irrs}\,x_s \right] \qquad (2.3.5)$$

$$B_{ij} = 2\{D_{iojo} + \sum_{r=1}^{n} \left[-D_{irjo}\,x_r + \sum_{s=1}^{n} (-D_{iojs}\,x_s + D_{irjs}\,x_r x_s) \right]\}$$

(2.3.6)

The working details of obtaining these results are easily understood by studying the problems considered in chapter 4.

The general solution of the Fokker-Planck equation presents great difficulties [36] and explicit solutions have been obtained only for certain linear systems [31]. However, in stability investigations a knowledge of the moments of the system response is usually sufficient; differential equations governing these moments are conveniently obtained from the Fokker-Planck equation and these then may be solved recursively for the various moments.

Denote the mixed moments of order N by

$$m_N\,(K_1, K_2, ---, K_n) = E\{x_1^{K_1}\, x_2^{K_2} \,-----\, x_n^{K_n}\}, \; N = 1,2, \text{ etc.}$$

(2.3.7)

where $K_1, K_2, ---, K_n$ are positive integers satisfying

$$K_1 + K_2 + ---- + K_n = N$$

Multiplying equation (2.3.3) throughout by $x_1^{K_1} x_2^{K_2} --- x_n^{K_n}$ and integrating by parts over the entire state-space $-\infty < \underline{x} < \infty$ leads to the following moment equations [29].

$$\frac{d}{dt} m_N(K_1,K_2,---,K_n) = -\sum_{i=1}^{n}\sum_{r=1}^{n} a_{ir} K_i m_N(K_1,---, K_i-1,---, K_r+1,---K_n)$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \left\{\begin{matrix} K_iK_j \ (i \neq j) \\ K_i(K_i-1) \ (i = j) \end{matrix}\right\} D_{irjs} m_N(K_1,---, K_i-1,---,K_j-1,---, K_r+1,---,K_s+1,---, K_n)$$

$$-2 \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} \left\{\begin{matrix} K_iK_j \ (i \neq j) \\ K_i(K_i-1) \ (i = j) \end{matrix}\right\} D_{irjo} m_{N-1}(K_1,---, K_i-1,---K_j-1,--- K_r+1,---, K_n)$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{\begin{matrix} K_iK_j \ (i \neq j) \\ K_i(K_i-1) \ (i = j) \end{matrix}\right\} D_{io}D_{jo} m_{N-2} (K_1,---, K_i-1,---,K_j-1,---,K_n)$$

$$- \sum_{i=1}^{n}\sum_{r=1}^{n} D_{irro} K_i m_{N-1} (K_1,---, K_i-1, ---, K_n)$$

$$+ \sum_{i=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} D_{irrs} K_i m_N (K_1,---, K_i-1,---, K_s+1, ---, K_n)$$

(2.3.8)

[*Note* Care must be exercised when evaluating the last term in each of the summations as it may happen that more than one K with the same subscript ℓ arises in the argument of m. In these cases, the appropriate power of the corresponding x_ℓ in the expression for the moment should be taken equal to K_ℓ plus the algebraic sum of the numbers added to all the K_ℓ

appearing in the moment term. For example, the fourth summation in (2.3.8) for the case $i = j = n$ which appears as

$$K_n(K_n-1)\, D_{nono}\, m_{N-2}\, (K_1, ---, K_n-1, ---, K_n-1, ---, K_n)$$

is to be regarded as

$$K_n(K_n-1)\, D_{nono}\, m_{N-2}\, (K_1, ---, K_n-2). \quad]$$

From equation (2.3.8) it is seen that the moments of order N are only related to the moments of order less than N, and for a particular N there are as many equations as there are mixed moments of that order. Thus, all the higher moments can be found recursively once the first moments are found.

When using the Fokker Planck equation approach in stability theory we are interested in two types of stability, namely stability in mean and stability in mean square. The system, represented by equation (2.2.3), will be said to be *stable in the mean* if

$$\lim_{t\to\infty} E\{x_i(t)\} < K_1 \ , \ K_1 = \text{constant}, \ (i = 1,2, ---, n)$$

and *stable in the mean square* if

$$\lim_{t\to\infty} E\{x_i(t)x_j(t)\} < K_2, \ K_2 = \text{constant}, \ (i,j = 1,2, ---, n)$$

It follows from equation (2.3.8) that a necessary condition for mean square stability is the existence of mean stability.

Necessary and sufficient conditions for stability in the mean and stability in the mean square are readily obtained by applying the Routh-Hurwitz criterion to the sets of differential equations governing first and second moments respectively.

If the coefficient variations of the system are Gaussian but non-white, then the response of the system no longer forms a Markov process[35]. It is possible however to construct time invariant linear filters which, with Gaussian white noise as input, will have the required non-white Gaussian variations as ~~in~~output. Introducing the filters produces a system whose response is a Markov process and therefore its corresponding Fokker-Planck equation may be obtained by the procedure described above. However, as illustrated through an example by Kozin [33] (see also an example in section (4.6.1) of this text), the additional state variables introduced by the filters enter the system equations in such a way as to make them non-linear. The differential equations governing the moments of the response can no longer be solved recursively, since it is found that they are interrelated; that is, when solving for the N^{th} moments one requires knowledge of moments higher than N. To talk of mean square stability in this case would therefore mean neglecting higher order moments than the second and, in general, no justification can be given for doing this. It therefore follows that the stability problem, in the case of Gaussian non-white coefficient variations, using the moments approach reduces to a problem of examining the eigenvalues of an infinite matrix.

In conclusion it is noted that another important property of the Fokker-Planck equation is that it gives a convenient method of obtaining the correlation functions and the power spectra of the system response[30,31].

2.4. Work of Samuels

Samuels and Eringen [37] developed specific criteria of determining the mean squared stability of random systems when a single parameter varies as a white noise process while the others remain constant. A general theory of mean squared stability of random linear systems was developed by Samuels [38] under certain assumptions, but specific results were only given for systems with a single random parameter.

In a later paper Samuels [39] considered the n^{th} order linear system

$$\sum_{i=0}^{n} (a_i + \beta_i(t)) \frac{d^i x(t)}{dt^i} = f(t) \qquad (2.4.1)$$

where f(t) is a prescribed stationary ergodic random function independent of the $\beta_i(t)$, $(i = 1,2, ---, n)$, and bounded in the mean square, the $\beta_i(t)$, $(i = 1,2, ---, n)$, assumed to be white noise processes having the following statistical properties

$$\begin{aligned} &E\{\beta_i(t)\} = 0 \\ &E\{\beta_i(t)\ \beta_j(t + \tau)\} = 2D_{ij}\ \delta(\tau) \end{aligned} \qquad i, j = 1,2, ---, n \qquad (2.4.2)$$

the symbols being defined as in section 2.2. Samuels did not make use of the fact that the state vector is a Markov process and he avoided the pathological properties of the white noise processes by considering the integral equation associated with equation (2.4.1). A solution of the associated integral equation was obtained by the method of successive approximations and, by multiplying the solutions at two points t_1 and t_2 and averaging, a set of sufficient conditions were obtained that guaranteed asymptotic boundedness of the second moment $E\{x^2(t)\}$ and referred to by Samuels as mean square stability (in fact the condition guarantees that the second moment decays to zero exponentially) Samuels applies his theory to obtain sufficient conditions for the mean square stability of an LCR circuit in which the resistance and capacitance have purely random fluctuations; however, as pointed out by Caughey [40] this part of the paper contains a number of errors which invalidate many of the results. The results of this paper are rather complicated and are not readily extended to a system of linear differential equations.

Samuels in a fourth paper [41] extended his concept of mean squared stability to consider the stability of the linear system (2.4.1) where the parameter processes $\beta_i(t)$, $(i = 1,2, ---, n)$, are no longer white noise processes but Gaussian processes having statistical properties

$$E\{\beta_i(t)\} = 0 \qquad\qquad i,j = 1,2, ---, n \tag{2.4.3}$$
$$E\{\beta_i(t)\beta_j(t + \tau)\} = \rho_{ij}(\tau)$$

where $\rho_{ij}(\tau)$ is the cross correlation function of $\beta_i(t)$ and $\beta_j(t)$ and, when $i = j$, ρ_{ii} is the autocorrelation function of $\beta_i(t)$. It is further assumed that the Gaussian processes are obtained by linear filtering of white noise, that is,

$$\beta_i(t) = \int_{-\infty}^{\infty} W_i\ (t - \xi)\chi(\xi)d\xi\ ,\ i = 1,2, ---, n, \tag{2.4.4}$$

where $W_i(t)$, $i = 1,2, ---, n$, are known weighing functions and $\chi(t)$ is a white noise process which is represented as a sequence of independent delta functions as follows:

$$\chi(t) = \sum_{K=1}^{N} A_K\ \delta(t - \tau_K)\ ,\ E\{A_K\} = 0 \tag{2.4.5}$$

By considering the integral equation associated with equation (2.4.1) Samuels obtained a system of linear integral equations for determining the various second order moments of the system. This system of integral equations were then solved by using a double Fourier transform to obtain sufficient conditions to guarantee mean square stability. Again the procedure is very complicated and not readily extended to a system of linear differential equations with random coefficients.

2.5. Liapunov Concepts of Stability

Since it gives rise to stability conditions without actually solving the differential equations representing the systen the second method of Liapunov [42, 43] seems to be a natural tool for the study of dynamic systems with stochastic coefficients and a large portion of the literature has been devoted to this approach.

Stability is essentially a question of convergence so that when dealing with systems having stochastic coefficients, where the question of convergence deals with limits involving random variables, it is necessary to speak of convergence in a stochastic sense. In probability theory there are three common modes of convergence, namely :

(i) convergence in probability

(ii) convergence in the mean

and (iii) almost sure convergence.

The stochastic versions of Liapunov stability relative to these three modes of convergence are accomplished by changing the modes of convergence as they appear in the concepts of Liapunov stability for deterministic systems. These definitions of stability appear in a paper by Bertram and Sarachik [44] (a brief discussion of these is given in appendix 2.1) who appear to be the first in the U.S.A. to apply the second method of Liapunov to the study of stability of stochastic systems.

In reference (44) the authors consider the stability of the equilibrium solution $\underline{x}(t) \equiv 0$ of the general system

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), t, \underline{y}(t)) , \; t \in [0, \infty) \qquad (2.5.1)$$

where $\underline{x}(t)$ is an n-vector describing the state of the system, $\underline{f}$ is a continuous n-vector function of the stochastic process $\underline{y}(t)$, $\underline{f}$ satisfies a Lipschitz condition and is such that $\underline{f}(0, t, \underline{y}(t)) = 0$ for all $t \in [0, \infty)$. Although the results of the paper give sufficient conditions for stability in the mean for the general system (2.5.1) no method is given for obtaining the necessary expectations and the criteria obtained are effective only in the cases when the solution of the system is a Markov process. A particular example considered in the paper is

$$\dot{\underline{x}}(t) = \underline{A}(t) \, \underline{x}(t), \; \underline{x}(t_0) = \underline{x}_0 \; , \; t \in [t_0, \infty), \qquad (2.5.2)$$

where $\underline{A}(t)$ is an nxn matrix with time varying random elements. In the particular case where $\underline{A}(t)$ is the diagonal matrix

$$[a_{ij}(t)\ \delta_{ij}]\ ,\ \left.\begin{array}{l}\delta_{ij} = 1,\ i = j \\ \delta_{ij} = 0,\ i \neq j\end{array}\right\}\ i, j = 1,2, ---, n$$

and the parameter variations are stationary Gaussian processes with

$$a_i(t) = -b_i + c_i(t)\ ,\ \text{where } E\{a_i(t)\} = -\ b_i$$

a sufficient condition for global stability in the mean is

$$\frac{d}{dt}\ \{\ \exp\ [\ -\ b_i + \int_{t_o}^{t} \int_{t_o}^{t} \phi_{ii}(\tau - u)d\tau du\ \} < 0,\ i = 1,2,---, n,$$

where $\phi_{ii}(\tau - u)$ is the correlation function $E\{c_i(\tau)c_i(u)\}$. However, it is only in the case where

$$\phi_{ii}(\tau - u) = a_i^{\,2}\delta(\tau - u) \qquad \text{(theoretical white noise)}$$

that the condition for global stability in the mean may be written explicitly as

$$b_i > \frac{a_i^{\,2}}{2}\ ,\ i = 1,2,\ ---,\ n.$$

Almost simultaneously Kats and Krasvoskii (who appear to be the first workers in the U.S.S.R. to apply Liapunov techniques to stochastic systems) published a paper [45] similar to that of Bertram and Sarachik. Although the authors in this paper present a more comprehensive study of the subject they restrict themselves to considering the particular case when $\underline{y}(t)$ is a homogeneous Markov chain with a finite number of states $\{\underline{y}_1, \underline{y}_2, ---, \underline{y}_K\}$. The probability $p_{ij}(\Delta t)$ of the change from state $\underline{y}_i$ to state $\underline{y}_j$ during the time Δt satisfying the condition

$$p_{ij}(\Delta t) = \beta_{ij}\Delta t + o(\Delta t)\ ,\ i = j,\ i\ j = 1,2,\ ---\ K$$
$$\beta_{ij} = \text{constant}\ ,$$

where $o(\Delta t)$ denotes an infinitesimal of higher order than Δt. In this particular case they were able to obtain a specific formula for the quantity

$$\frac{d}{dt} E\{V(\underline{x},t,\underline{y})\} ,$$

where $V(x,t,y)$ is a Liapunov function for the stochastic system (2.5.1), in the form

$$\frac{d}{dt} E\{V/\underline{x}, \underline{y}_j, t\} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\underline{x},t,\underline{y}_j) + \sum_{\ell \neq j}^{K} \beta_{j\ell} \left[V(\underline{x},t,\underline{y}_k)\right] - V(\underline{x}, t, \underline{y}_j), \tag{2.5.3}$$

where $E\{V/\underline{x}, \underline{y}_j, t\}$ denotes the conditional mean of V with $\underline{x}(t) = \underline{x}, \underline{y}(t) = \underline{y}_j(t)$.

It is worth noting that for computing the derivative, just as in the case of ordinary deterministic systems, it is not necessary to integrate equation (2.5.1) but it is sufficient to know only the right hand sides of the equation and the probability characteristics of the random process $\underline{y}(t)$.

Restricting themselves to the linear system

$$\dot{\underline{x}}(t) = \underline{A}(\underline{y})\, \underline{x} \tag{2.5.4}$$

Kats and Krasvoskii established that if the equilibrium solution $\underline{x} \equiv 0$ posses asymptotic stability of the mean, then, for any positive definite form $W(\underline{x}, \underline{y})$ there exists a unique Liapunov function V for which

$$\frac{d}{dt} E\{V(\underline{x}, \underline{y})\} = -W(\underline{x}, \underline{y})$$

Thus by evaluating (2.5.3) for the form

$$V(\underline{x}, \underline{y}_j) = \sum_{i=1}^{n} b_i(\underline{y}_j)\, x_i^2$$

they obtain

$$\sum_{s=1}^{n} \left[a_{s1}(\underline{y}_j)x_1 + \text{-----} + a_{sn}(\underline{y}_j)x_n\right] \frac{\partial V}{\partial x_s} (\underline{x}, \underline{y}_j)$$

$$+ \sum_{K \neq j} \beta_{jK} \left[V(\underline{x}, \underline{y}_K) - V(\underline{x}, \underline{y}_j)\right] = -W(\underline{x}, \underline{y}_j) \qquad (2.5.5)$$

They are then able to obtain conditions on the coefficients of V by equating them with the coefficients of the positive definite form W via equation (2.5.5).

As will be mentioned in section (5.3) Shackcloth and Butchart [46] applied these results, with little success, to investigating the stability of a model reference adaptive control system with time varying environmental parameters.

2.6. Piecewise constant systems

Due to the difficulties involved in obtaining explicit results for continuous linear systems many of the early researchers turned their attention to investigating the stability of linear piecewise constant systems of the form

$$\dot{\underline{x}}(t) = \underline{A}(t)\, \underline{x}(t), \qquad (2.6.1)$$

where $\underline{A}(t) = \underline{A}_K,\ t_K \leqslant t \leqslant t_{K+1}$, $K = 0, 1,$ etc. (2.6.2)

$\underline{A}_K$ constant matrices.

Using a direct method of solution Bertram and Sarachik [46] obtained the solution

$$\underline{x}(t) = \underline{\phi}_K(t - t_K) \prod_{i=1}^{K} \underline{\phi}_{i-1}(t_i - t_{i-1})\, \underline{x}(t_o),\ t_K \leqslant t \leqslant t_{K+1},$$

where $\underline{x}(t_o)$ is the initial state and

$$\underline{\phi}_K(t - t_K) = \exp\left[\underline{A}_K(t - t_K)\right]$$

In the case where the $\underline{A}_K$ are statistically independent on successive intervals a sufficient condition for global stability in the mean of the system is that

$$E\{\underline{\phi}_K^1\ (t - t_K)\ (\underline{A}_K^1\, \underline{Q}\ +\ \underline{Q}\underline{A}_K)\ \underline{\phi}_K\ (t - t_K)\}$$

be negative definite in each interval for a positive definite $\underline{Q}$.

Using the Kronecker product of matrices [47] Barachua [48] examined the stability of the discrete linear system (2.6.1). Although his results allowed him to treat a wide class of systems it required explicit integration and is also quite cumbersome due to the introduction of the Kronecker products.

2.7. Linear systems of the form $\dot{\underline{x}} = (\underline{A} + \underline{F}(t))\underline{x}$

In order to obtain results for higher order continuous systems without being restricted to the class of systems that can be explicitly solved in closed form Kozin [49] considered linear systems of the form

$$\dot{\underline{x}}(t) = [\underline{A} + \underline{F}(t)]\underline{x}(t) \quad , \; t \in [t_0, \infty) \tag{2.7.1}$$

where (i) $\underline{x}(t)$ is an n-vector representing the state of the system

(ii) $\underline{A}$ is a constant nxn matrix such that the system $\dot{\underline{y}} = \underline{A}y$ is asymptotically stable in the large; that is, $\underline{y}(t) \to 0$ as $t \to \infty$ for all $\underline{y}(t_0)$. This is guaranteed if all the eigenvalues of $\underline{A}$ have negative real parts; that is, $\underline{A}$ is what is termed a stability matrix.

(iii) $\underline{F}(t)$ is an nxn matrix whose non-identically zero elements are stochastic processes

$$\{f_{ij}(t) \; , \; t \in [t_0, \infty)\}$$

satisfying (a) the processes are continuous on $[t_0, \infty)$ with probability one

(b) the processes are strictly stationary

(c) the processes satisfy an ergodic property guaranteeing the equality of time averages and process expectations with probability one.

The trivial or stationary solution $\underline{x}(t) \equiv 0$ of system (2.7.1) is said to be almost surely asymptotically stable in the large relative to a region R if for all solutions $\underline{x}(t; \underline{x}_0, t_0)$, $\underline{x}_0 \varepsilon R$, of (2.7.1) we have the property that

$$\lim_{t\to\infty} || \underline{x}(t; \underline{x}_0, t_0)|| = 0 \tag{2.7.2}$$

holds with probability one. (Note that this definition is equivalent to almost sure Liapunov asymptotic stability, since (2.7.2) coupled with the continuity of the solutions guarantees the boundedness of the solutions of (2.7.1), which in turn implies Liapunov stability for linear homogeneous systems, this equivalence however does not hold for systems in general [23]).

By applying the Gronwall-Bellmann lemma [50] to the integral equation equivalence of equation (2.7.1) Kozin showed that a sufficient condition for system (2.7.1), subject to conditions (i), (ii), (iii), to be almost surely asymptotic stable in the large is that

$$E\{||\underline{F}(t)||\} < {}^{a}/_{b} \quad ^{\dagger} \tag{2.7.3}$$

where -a is an upper bound on the real parts of the eigenvalues of the matrix $\underline{A}$ and b is the upper bound of $|h(t)|$, where h(t) is the impulse response of the deterministic system $\dot{\underline{y}} = \underline{A}\ \underline{y}$; that is, since $\underline{A}$ is a stability matrix a and b are positive constants such that

$$|h(t)| \leqslant be^{-at}. \tag{2.7.4}$$

Kozin considered as an example the second order system

$$\ddot{x}(t) + 2\xi\ \dot{x}(t) + [1 + f(t)]\,x(t) = 0 \tag{2.7.5}$$

which is of great importance in the study of physical systems. His results however are too conservative and predicts that the standard deviation of the parameter f(t) should be zero when the system is critically damped ($\xi = 1$). This peculiar dip in the stability boundary at $\xi = 1$ is due entirely to Kozin's choice of b. Ariaratnam and Graefe [27] improved on Kozin's stability boundary and eliminated the sharp decrease

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$\dagger$ $||x|| = \sum_{i=1}^{n} |x_i|$, $||\underline{A}|| = \sum_{ij=1}^{n} |a_{ij}|$

at $\xi = 1$ by introducing three different bounds on the impulse function for three separate ranges of the damping coefficient ξ.

Using a Liapunov-type approach Caughey and Gray [51] obtained sharper sufficient conditions, guaranteeing almost surely asymptotic stability, than those obtained by Kozin through application of the Gronwall-Bellman approximation but not sharper than those obtained in reference (27). Since, by condition (i), $\underline{A}$ is a stability matrix it follows [43, 52] that there exists a symmetric, positive definite matrix $\underline{P}$, such that

$$\underline{A}'\underline{P} + \underline{P}\underline{A} = -\underline{C} , \tag{2.7.6}$$

where $\underline{C}$ is a positive definite matrix. Taking $\underline{C}$ to be identity matrix and $V = \underline{x}'\underline{P}\,\underline{x}$ (where $\underline{P}$ is the corresponding solution of equation (2.7.6)) as a Liapunov function for system (2.7.1) Caughey and Gray, by making use of the Schwarz inequality, obtained the following inequality relating V and $\dot{V}$ (the derivative of V along the trajectory of (2.7.1))

$$\dot{V} \leqslant \frac{-V}{\lambda_{maxP}} + V||\underline{Q}(t)|| , \tag{2.7.7}$$

where

$$\underline{Q}(t) = \underline{P}^{-\frac{1}{2}}\,\underline{F}(t)\underline{P}^{\frac{1}{2}} + \underline{P}^{\frac{1}{2}}\,\underline{F}(t)\underline{P}^{-\frac{1}{2}} \quad {}^{\dagger\dagger} \tag{2.7.8}$$

and λ_{maxP} is the largest eigenvalue of $\underline{P}$. Integrating (2.7.7) and using conditions (ii) and (iii) they obtain the sufficient condition

$$E\{||\,\underline{Q}(t)\,||\,\} < \frac{1}{\lambda_{max^P}} \tag{2.7.9}$$

for system (2.7.1) to be almost surely asymptotically stable in the large.

---

†† $\underline{P}^{\frac{1}{2}}$ and $\underline{P}^{-\frac{1}{2}}$ are unique matrices and given by

$$\underline{P}^{\frac{1}{2}} = \underline{M}\begin{bmatrix} \lambda_1^{\frac{1}{2}} & & \\ & \ddots\,\lambda_i^{\frac{1}{2}}\,\ddots & \\ & & \lambda_n^{\frac{1}{2}} \end{bmatrix}\underline{M}^{1} , \quad \underline{P}^{-\frac{1}{2}} = \underline{M}\begin{bmatrix} \lambda_1^{-\frac{1}{2}} & & \\ & \ddots\,\lambda_i^{-\frac{1}{2}}\,\ddots & \\ & & \lambda_n^{\frac{1}{2}} \end{bmatrix}\underline{M}^{1}$$

where $\underline{M}$ is the orthogonal transformation such that

$$\underline{M}\,\underline{M}^{1} = \underline{I} , \quad \underline{M}^{1}\,\underline{P}\,\underline{M} = \begin{bmatrix} \lambda_1 & & \\ & \ddots\,\lambda_i\,\ddots & \\ & & \lambda_n \end{bmatrix}$$

For the particular case where the matrix $\underline{F}(t)$ may be written in the form

$$\underline{F}(t) = \sum_{i=1}^{R} f_i(t)\underline{G}_i \quad , \quad R < N^2 \tag{2.7.10}$$

where $f_i(t)$ are scalar functions of time and $\underline{G}_i$ constant matrices then a sufficient condition for system (2.7.1) to be almost surely asymptotically stable in the large is that

$$\sum_{i=1}^{R} |\mu_i|_{max} \, E\{|f_i(t)|\} \tag{2.7.11}$$

exists and is less than $\dfrac{1}{\lambda_{maxP}}$, where $|\mu_i|_{max}$ is the numerically largest eigenvalue of the matrix.

$$\underline{B}_i = \underline{P}^{-\frac{1}{2}} \underline{G}_i^{1} \underline{P}^{\frac{1}{2}} + \underline{P}^{\frac{1}{2}} \underline{G}_i \underline{P}^{-\frac{1}{2}} \tag{2.7.12}$$

Caughey and Gray applied their results to system (2.7.5) and obtained sharper results than Kozin and Ariaratnam. However, as indicated by Mehr and Wang [53], their results do not indicate that $E\{|f(t)|\}$ goes to infinity as $\xi \to \infty$ as seems should happen from the physical standpoint.

The choice of $\underline{C}$ as the identity matrix obviously restricts the above results. If $V = \underline{x}'\underline{P}\,\underline{x}$, where $\underline{P}$ is the solution of (2.7.6), is taken as a Liapunov function for system (2.7.1) then Caughey and Gray's method would lead to the following sufficient conditions for almost surely asymptotically stability in the large for system (2.7.1)

$$E\{||\underline{Q}(t)||\} < \frac{\lambda_{minC}}{\lambda_{maxP}} \tag{2.7.13}$$

$$\sum_{i=1}^{R} |\mu_i|_{max} \, E\{|f_i(t)|\} < \frac{\lambda_{minC}}{\lambda_{maxP}} \tag{2.7.14}$$

[These results correspond to (2.7.9) and (2.7.11) respectively and λ_{minC} is the smallest eigenvalue of the matrix $\underline{C}$]

An obvious outstanding problem is the choice of $\underline{C}$ to give optimum results. Unfortunately, even for the second order system (2.7.5), taking $\underline{C}$ to be the general quadratic form

$$\underline{C} = \begin{bmatrix} \alpha_1^2 + \alpha_2 & \alpha_1 \\ \alpha_1 & 1 \end{bmatrix}, \quad \alpha_2 > 0 \tag{2.7.15}$$

leads to such complicated inequalities that optimum values of α_1 and α_2 (that is, values to maximize $E\{|f(t)|\}$) are not easily obtained. It can be shown that for this particular example the identity matrix is the optimum diagonal matrix.

Results similar to those of Caughey and Gray may be found in a paper by Khasminskii [54].

Another extension on the results for system (2.7.1), subject to conditions (i)-(iii), is that due to Infante [55]. Taking $\underline{x}'\underline{B}\,\underline{x}$, where $\underline{B}$ is some positive definite matrix, as the Liapunov function for system (2.7.1) and applying the results of the external properties of the eigenvalues of pencils of quadratic form [8] he was able to obtain the following sufficient condition for almost surely asymptotic stability in the large of system (2.7.1)

$$E\{\lambda_{max}\,[(\underline{A}^1 + \underline{F}^1) + \underline{B}(\underline{A} + \underline{F})\underline{B}^{-1}]\} \leqslant -\varepsilon \text{ for some } \varepsilon > 0 \tag{2.7.16}$$

For computational reasons this result was simplified to the form

$$\sum_{i=1}^{R} \frac{1}{2} E\{|f_i(t)|\}[\lambda_{max}\,(G_i^1 + \underline{B}\,\underline{G}_i\,\underline{B}^{-1}) - \lambda_{min}\,(G_i^1 + \underline{B}\,\underline{G}_i\,\underline{B}^{-1})] \leqslant$$
$$- \lambda_{max}\,(\underline{A}' + \underline{B}\,\underline{A}\,\underline{B}^{-1}) - \varepsilon \tag{2.7.17}$$

where $\underline{G}_i$ and f_i are defined as in equation (2.7.10).

Taking $\underline{B}$ as the general quadratic form of equation (2.7.15) Infante applied inequality (2.7.17) to system (2.7.5). He then chose optimum values of α_1 and α_2 in order to maximize $E\{f^2(t)\}$, to obtain the following sufficient condition for a.s.a.s. of the system

$$E\{f^2(t)\} < 4\,\xi^2 - \varepsilon\ , \quad \varepsilon > 0 \tag{2.7.18}$$

This is a dramatic extension of the region of stability over previous work and moreover it answers the conjecture raised by Mehr and Wang mentioned earlier (i.e. the variance of the coefficient process can approach infinity as the damping coefficient ξ approaches infinity). The results of Infante together with those of the other authors discussed are shown in Fig. 2.7.1a.

Infante obtained the optimum $\underline{B}$ by maximizing a function of two independent variables α_1 and α_2. This procedure is very time consuming and is limited in application to second order systems. Ideally one would like to be able to prove a theorem to give optimum $\underline{B}$. However, Infante showed via an example that if the matrix $\underline{F}$ contains more than one time varying coefficient then an optimum $\underline{B}$ does not exist but it is felt that it should be possible to obtain an optimum norm for the case of $\underline{F}$ having only one time varying element. The problem of making Infante's results practically applicable to higher order systems remains an outstanding problem.

The most recent development on the results for system (2.7.1), subject to conditions (i)-(iii), is that due to Man [56]. By simultaneously reducing two quadratic forms to diagonal form Man extends the development of Infante to obtain a sharper stability criteria; namely, that a sufficient condition for the system to be almost surely asymptotically stable in the large is that

$$E\{\lambda_{max}[(\underline{F}'\underline{P} + \underline{P}\,\underline{F})\,\underline{Q}^{-1}]\} < 1, \tag{2.7.19}$$

where $\underline{P}$ and $\underline{Q}$ are positive definite constant symmetric matrices satisfying the Liapunov matrix equation

$$\underline{A}'\underline{P} + \underline{P}\,\underline{A} = -\,\underline{Q} \tag{2.7.20}$$

Taking $\underline{Q}$ to be the identity matrix Man applied criterion (2.7.19)

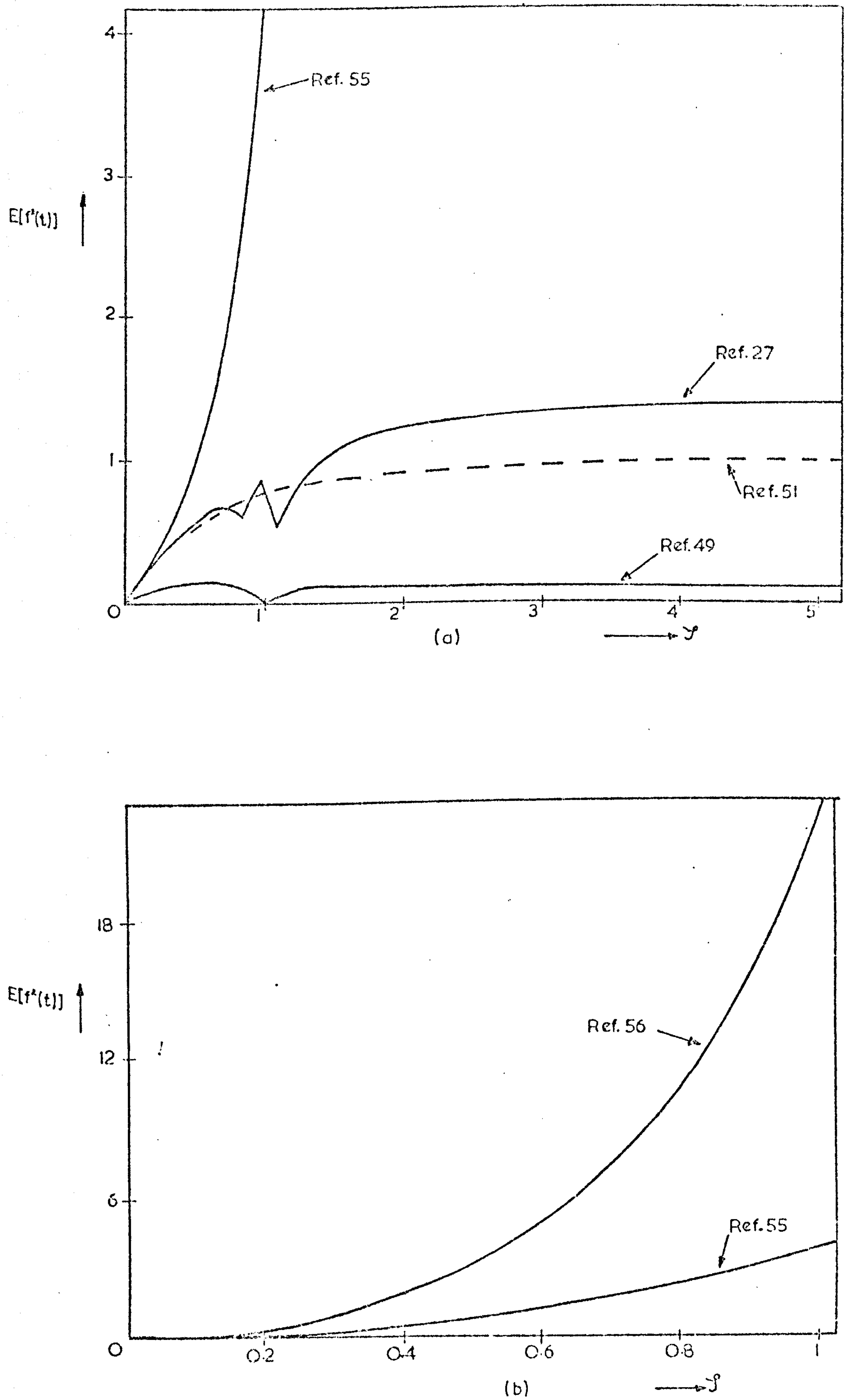

FIG. 2.7.1. Stability conditions for equation (2.7.5)

to equations (2.7.5) to obtain the following sufficient condition for a.s.a.s. in the large

$$E\{f^2(t)\} < 4\zeta^2/[2\zeta^2 - 2\zeta\sqrt{\zeta^2+1} + 1] \tag{2.7.21}$$

The improvement over condition (2.7.18), obtained by Infante, is illustrated in fig (2.7.1b).

Man obtained condition (2.7.21) by taking $\underline{Q}$ to be the identity matrix. One would expect to obtain an even sharper criterion if $\underline{Q}$ were taken to be the most general symmetric positive definite form $\underline{C}$, defined by equation (2.7.15), and optimum values of α_1 and α_2 then obtained in order to maximize $E\{f^2(t)\}$. Taking $\underline{Q}$ to be such a matrix and applying the stability criterion (2.7.19) we have that a sufficient condition for system (2.7.5) to be a.s.a.s. in the large is that

$$E\{f^2(t)\} < 16\zeta^2/[8\zeta^2X^2 + (1+X)^2 - 4\zeta X\sqrt{4\zeta^2X^2 + (1+X)^2}\,], \tag{2.7.22}$$

where $X = \alpha_1{}^2 + \alpha_2$. (Since the right hand side of (2.7.22) is a function of $(\alpha_1{}^2 + \alpha_2)$ we can take $\alpha_1 = 0$ and optimize for $X = \alpha_2 > 0$).

By differentiating with respect to X we have that the right hand side of criterion (2.7.22) has a stationary value provided

$$(4\zeta^2 + 1)X^4 + (6\zeta^2 + 4)X^3 + 6X^2 + (4 - 8\zeta^2)X + (6 - 4\zeta^2) = 0 \tag{2.7.23}$$

Since equation (2.7.23) has no positive root when $\zeta < 1/\sqrt{2}$ it follows that there is no optimum X for all ζ but equation (2.7.23) clearly indicates that the identity matrix is not an optimum choice for $\underline{Q}$. A possible method of attack is to solve equation (2.7.23) numerically for different values of ζ and then use criterion (2.7.19) to construct the stability boundary. Clearly the choice of the optimum $\underline{Q}$ remains an outstanding problem.

If the matrix $\underline{F}$, of equation (2.7.1), contains more than one time varying element then due to computational reasons criterion (2.7.19) will have to be simplified to the form

$$\sum_{i=1}^{R} E\{|f_i(t)|\} \left[\lambda_{max} \left(\underline{G}_i \underline{P} + \underline{P} \underline{G}_i\right) \underline{Q}^{-1}\right] < 1, \qquad (2.7.24)$$

where $\underline{G}_i$ and f_i are defined as in equation (2.7.10).

2.8. Use of differential generator

In this section we shall be concerned with systems of the form

$$\dot{\underline{x}}(t) = \underline{m}(\underline{x}, t) + \underline{\sigma}(\underline{x}, t)\, \underline{W}(t)^{\dagger} \qquad (2.8.1)$$

where $\underline{x}(t)$ and $\underline{m}(\underline{x}, t)$ are n-vectors, $\underline{\sigma}(\underline{x}, t)$ an nxn matrix and $\underline{W}(t)$ an n-vector with Gaussian white noise components.

As mentioned earlier the solution process $\underline{x}(t)$ is a Markov process and furthermore associated with it is the operator

$$L = \frac{1}{2} \sum_{j=1}^{n} b_{ij}(\underline{x}, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \underline{m}_i(\underline{x}, t) \frac{\partial}{\partial x_i} \qquad (2.8.2)$$

where $b_{ij} = \underline{B} = \sigma\sigma^1$, referred to as the differential generator of process (2.8.1), which is of fundamental importance in the application of Liapunov's second method to studying process (2.8.1). The salient feature that allows the ideas of Liapunov's second method to be applied to Itô equations is the fact that for twice continuously differentiable functions $V(\underline{x}, t)$ the expected value of the derivative of this function along the trajectory of the process defined by (2.8.1) with initial condition $(\underline{x}, t)$ is given via the differential generator (2.8.2) as $LV(\underline{x}, t)$ [57].

---

† This equation is usually written as equation in differentials

$$d\underline{x} = \underline{m}(\underline{x}, t)dt + \underline{\sigma}(\underline{x}, t)d\underline{B}$$

where $\underline{B}$ is a Brownian motion process.

The first application of these results from Markov process theory to the stability problem appears to be due to Khas'minskii [57] who examined the stability in probability of the equilibrium solution of process (2.8.1) under the condition that there exists a continuous function $m(\underline{x})$, which is positive when $\underline{x} \neq 0$, such that for real λ_i the inequality

$$\sum_{i,j=1}^{n} b_{ij} \lambda_i \lambda_j \geqslant m(\underline{x}) \sum_{i=1}^{n} \lambda_i^2$$

is valid (this guarantees that every component equation of the system (2.8.1) possesses noise coefficients).

Further developments may be found in a paper by Nevelson and Khas'minskii [58]; the pattern of this paper follows closely that of Kats and Krasvoskii [45] except that Itô type equations are being considered. In particular they prove that for asymptotic stability in the mean square of a stationary linear stochastic system with Gaussian white noise coefficients it is necessary and sufficient that for any positive definite quadratic form $W(\underline{x})$ another positive definite quadratic form $V(\underline{x})$ may be found for which $LV(\underline{x}) = -W(\underline{x})$. They show further that if such a system is asymptotically stable in the mean square then the deterministic system obtained by setting the noise terms equal to zero posses an asymptotically stable equilibrium solution.

In a later paper Nevelson and Khasminskii [59] use the results of reference 58 to obtain necessary and sufficient conditions for asymptotic stability in the mean square for the linear system defined by

$$y^{(n)} + [a_1 + \eta_1(t)]y^{(n-1)} + \text{--------} + [a_n + \eta_n(t)]y = 0 \tag{2.8.3}$$

where a_i, $i = 1,2,$ ----, n, are constants and $\eta_i(t)$, $i = 1,2,$ ---, n, Gaussian white noise processes with the following statistical properties :

$$\begin{aligned} &E\{n_i(t)\} = 0 \\ &E\{n_i(t)n_j(t-\tau)\} = 2D_{ij}\delta(\tau) \end{aligned} \qquad i,j = 1,2, ----, n \qquad (2.8.4)$$

The conditions are that

$$\Delta_1 > 0,\ \Delta_2 > 0,\ ----,\ \Delta_n > 0,\ \Delta_n > \Delta \qquad (2.8.5)$$

where Δ_i $(i = 1,2, ---, n)$ are the Routh Hurwitz determinants of the constant coefficients $a_i (i = 1,2, ---, n)$ of the deterministic part of (2.8.3) and Δ differs from Δ_n only in the first row; that is :

$$\Delta = \begin{vmatrix} q_{nn}^{(0)} & q_{nn}^{(1)} & q_{nn}^{(2)} & \text{---------------} & q_{nn}^{(n-1)} \\ 1 & a_2 & a_4 & \text{---------------} & 0 \\ 0 & a_1 & a_3 & \text{---------------} & 0 \\ \text{-----} & \text{-----} & \text{-----} & \text{---------------} & \text{-----} \\ \text{-----} & \text{-----} & \text{-----} & \text{---------------} & \text{-----} \\ 0 & 0 & 0 & \text{---------------} & a_n \end{vmatrix}, \qquad (2.8.6)$$

where the quantities $q_{nn}^{(r)}$ $(r = 0, 1, ---, n - 1)$ are expressed through the correlation coefficients D_{ij}, defined in (2.8.4), according to

$$q_{nn}^{n-K-1} = \sum_{i+j=2(n-K)} (-1)^{i+1} D_{ij}, \quad K = 0,1, ---, n-1 \qquad (2.8.7)$$

Note that when all $D_{ij} = 0$ then criteria (2.8.5) transfers into the Routh-Hurwitz criteria for the deterministic system

$$y^{(n)} + a_1 y^{(n-1)} + ------- + a_n y = 0$$

For the case where the white noise processes $n_i(t)$ $(i = 1,2,---, n)$ are independent, that is

$$a_{ij} = 0 \quad \text{for } i \neq j,\ i, j = 1,2, ---, n,$$

Δ assumes the simple form

$$\Delta = \begin{vmatrix} D_{11}-D_{22} & D_{33} & \cdots\cdots\cdots & (-1)^{n+1}\, D_{nn} \\ 1 & a_2 & \cdots\cdots\cdots & 0 \\ 0 & a_1 & \cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ 0 & 0 & \cdots\cdots\cdots & a_n \end{vmatrix} \tag{2.8.8}$$

CHAPTER 3

STABILITY OF A HILL CLIMBING SYSTEM

3.1. Introduction

Although extremum control or *hill-climbing* systems are a well defined class of adaptive control systems the important problem of analysing their stability has often been ignored. In this chapter a theoretical stability analysis for a single input, sinusoidal perturbation, extremal control system with output lag is presented, the results of which have been verified by analogue computer simulation.

The system equations, which are forced, non-linear and non-autonomous are first nondimensionalized using dimensional analysis, and periodic solutions of the resulting equations obtained by the principle of harmonic balance. The stability of these equilibrium states is then investigated by setting up variational equations, which, for small disturbances about the equilibrium state, form a set of linear differential equations with periodic coefficients. It will be shown that various stable harmonic and subharmonic *steady-state* solutions are possible in certain regions of the parameter space.

The steady state finally reached depends on the prescribed initial conditions. By plotting the *domains of attraction* of *fixed points,* which are invariant under a certain mapping, regions in state space are obtained within which initial conditions will lead to a given stable *steady state* oscillation.

3.2. The sinusoidal perturbation adaptive control system

The block diagram of a typical single dimensional, sinusoidal perturbation, adaptive control system is shown in fig. 3.2.1. [60, 61].

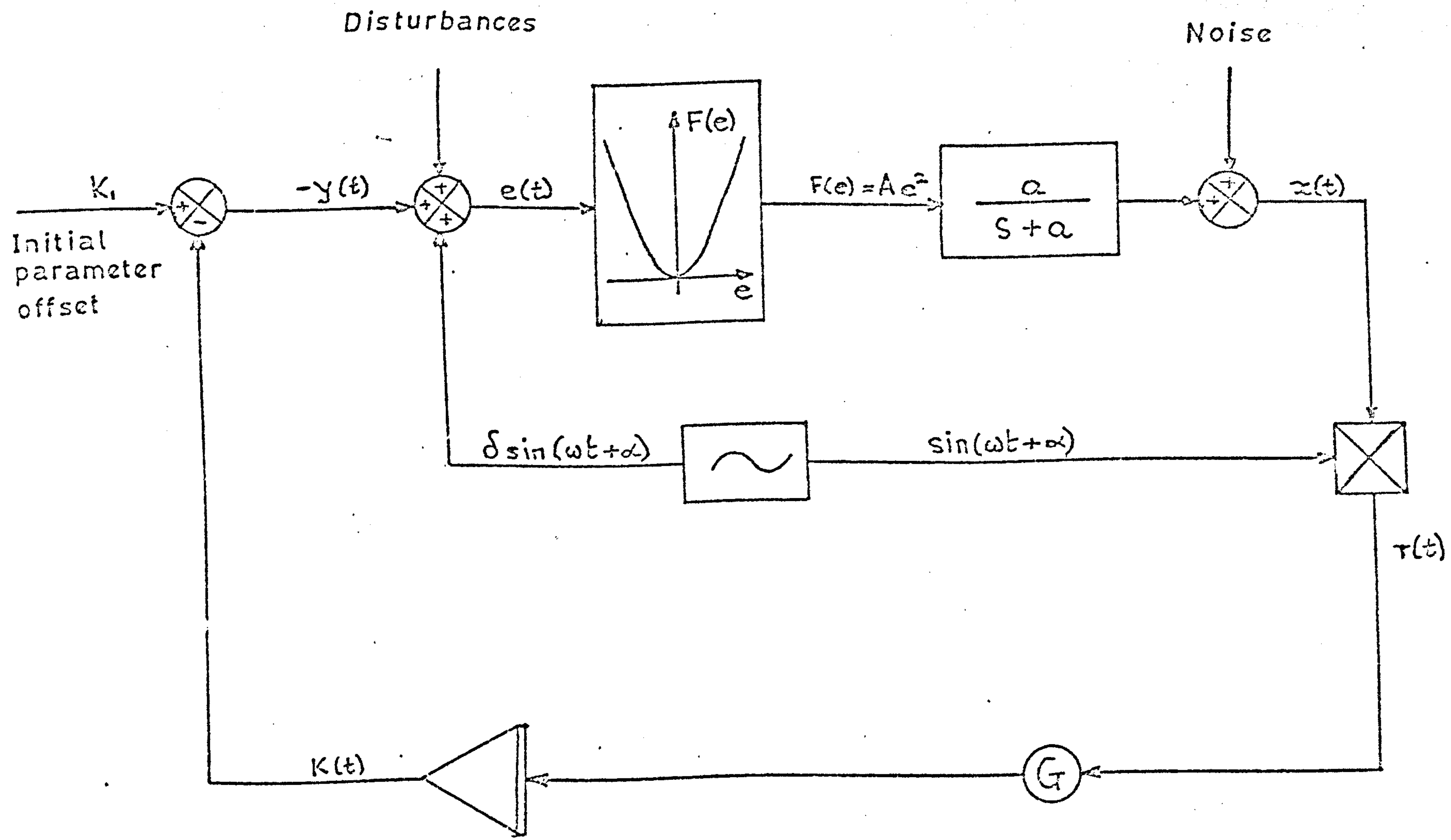

FIG. 3.2.1. Sinusoidal perturbation extremal control system.

The index of performance I.P. [62] is $F[e(t)] = A[e(t)]^2$ and the output lag is represented by a low pass filter with time constant $\frac{1}{a}$.

The adaptive controller employed is similar to that previously discussed in the literature [63, 64]. Briefly, a sinusoidal perturbation $\delta \sin(\omega t + \alpha)$ is added to the input and by demodulating the corresponding perturbation in the output a signal $r(t)$ that varies with the slope $\frac{\partial F}{\partial e}$ of the index of performance is obtained. The signal $r(t)$ is then passed through a smoothing integrator, with gain G, to develop a correction signal which tends to reduce $\frac{\partial F}{\partial e}$ to zero.

3.3. Dimensional Analysis

In the absence of the disturbances and noise the differential equations representing the system of fig. 3.2.1. are :

$$\frac{dx}{dt} + ax = Aa\,[\,y - \delta \sin(\omega t + \alpha)\,]^2 \qquad (3.3.1)$$

$$\frac{dy}{dt} = Gx \sin(\omega t + \alpha)$$

The performance of the system depends on the values of the five parameters. A, a, δ, ω and G which are expressed in three sorts of units (input units, output units and time) as follows :

A : (output units) (input units)$^{-2}$

a : (time)$^{-1}$

δ : (input units)

ω : (time)$^{-1}$

G : (input units) (output units)$^{-1}$ (time)$^{-1}$

By Buckingham's π theorem [65] non-dimensional parameters can be defined so as to reduce the number of parameters that need be considered by the number of units. Thus, in this case the number of non-dimensional parameters that need be considered are two and these are taken as

$$\Pi_1 = GA\delta/a \qquad (3.3.2)$$

$$\Pi_2 = {}^{\omega}/_{a}$$

If, in addition to (3.3.2), we introduce the dimensionless variables

$$\xi_1 = x/(A\delta^2)$$

$$\xi_2 = y/\delta \qquad (3.3.3)$$

$$\tau = \omega t + \alpha$$

then the system equations (2.3.1) may be written in the non-dimensional form

$$\dot{\xi}_1 = -\frac{1}{\Pi_2}\xi_1 + \frac{1}{\Pi_2}(\xi_2 - \sin\tau)^2 \qquad (3.3.4a)$$

$$\dot{\xi}_2 = \frac{\Pi_1}{\Pi_2}\xi_1 \sin\tau \qquad (3.3.4b)$$

Where dots denote differentiation with respect to τ.

3.4. Periodic solutions using the principle of harmonic balance

A distinctive feature of a system of nonlinear differential equations, such as (3.3.4), is that various types of *steady-state* periodic oscillations may exist depending on the initial values of the variables. In this work the method of solution employed, for obtaining the *steady-state* solutions, is to assume for ξ_1, ξ_2 Fourier series developments with undetermined coefficients and then determine these coefficients by the principle of harmonic balance [66], a method widely used for the analysis of nonlinear control systems. Periodic solutions whose fundamental frequencies are equal to that of the applied perturbation frequency will be termed *harmonic solutions*, whilst solutions whose fundamental frequencies are a fraction $\frac{1}{n}$ (n = 2,3, etc.) of the applied perturbation frequency will be termed *subharmonic solutions of order* $\frac{1}{n}$.

3.4.1. Harmonic solutions

When the system has reached a *steady-state* there will be no constant or "d.c." component out of the multiplier (that is, $\xi_1(\tau)$ contains no term in sin τ), so that as a first approximation we assume solutions of the form

$$\xi_1(\tau) = a_o + a_2\cos\tau \tag{3.4.1}$$

$$\xi_2(\tau) = b_o + b_1\sin\tau + b_2\cos\tau$$

Substituting equations (3.4.1) in (3.3.4b) gives

$$b_1\cos\tau - b_2\sin\tau = \frac{\Pi_1}{\Pi_2}\left[a_o\sin\tau + \frac{a_2}{2}\sin 2\tau \right]$$

Since the first approximation contains only the terms of the fundamental frequency we ignore second haromonic components and equate coefficients of the $\sin\tau$ and $\cos\tau$ terms to give

$$b_1 = o, \quad b_2 = -\frac{\Pi_1}{\Pi_2} a_o \tag{3.4.2}$$

Substituting equations (3.4.1) in (3.3.4a) gives

$$-\Pi_2 a_2\sin\tau = \left[-a_o + b_o^2 + \frac{(b_1 - 1)^2}{2} + \frac{b_2^2}{2} \right]$$

$$+ \left[2b_ob_2 - a_2 \right]\cos\tau + 2b_o(b_1 - 1)\sin\tau$$

$$+ \left[\frac{b_2^2}{2} - \frac{(b_1 - 1)^2}{2} \right]\cos 2\tau + (b_1 - 1)b_2\sin 2\tau$$

Balancing the coefficients gives, on using results (3.4.2),

$$\Pi_2 a_2 = 2b_o \tag{3.4.3a}$$

$$a_o = b_o^2 + \frac{\Pi_1^2}{2\Pi_2^2} a_o^2 + \frac{1}{2} \tag{3.4.3b}$$

$$\Pi_2 a_2 = -2\Pi_1 b_o a_o \tag{3.4.3c}$$

Since, from equation (3.4.3b), a_o is to be non-negative it follows from equations (3.4.3a) and (3.4.3c) that

$$b_o = a_2 = 0$$

Equation (3.4.3b) then becomes

$$\frac{\Pi_1^{\,2}}{2\Pi_2^{\,2}} a_o^{\,2} - a_o + \frac{1}{2} = 0 \qquad (3.4.4)$$

Equation (3.4.4) has real solutions for $\Pi_2 > \Pi_1$, when the solutions are

$$a_o = r^2 \pm r\sqrt{r^2 - 1} \,, \qquad (3.4.5)$$

where $r = \Pi_2/\Pi_1$.

Thus, to a first approximation harmonic solutions exist in the region

$$R_1 \;:\; \Pi_2 > \Pi_1 \qquad (3.4.6)$$

and in this region the harmonic solutions

$$\xi_1 = r^2 \pm r\sqrt{r^2 - 1}$$
$$\xi_2 = (-r \mp \sqrt{r^2 - 1})\cos\tau \qquad (3.4.7)$$

are possible.

These results agree with analogue computer simulation where it is found that harmonic solutions for $\xi_2(\tau)$ contains no d.c. component, thus implying that $\xi_1(\tau)$ has no component in $\cos\tau$.

A closer approximation to the harmonic solutions may be obtained if more terms of the Fourier series developments for $\xi_1(\tau)$ and $\xi_2(\tau)$ are taken into account; however, numerical computation will become too unwieldy. The method, employed in this work, of improving the approximation is an extension of the method due to Hayashi [66]; this method is particularly useful when the amplitude of each harmonic component decreases with increasing order of the harmonics. An alternative method, well suited for digital computation, is that developed by Urabe and Reiter [67] and based on the Galerkin procedure [68].

On substituting Fourier series developments for $\xi_1(\tau)$ and $\xi_2(\tau)$ in equations (3.3.4) and balancing the coefficients of like terms it is readily seen that the series representing $\xi_1(\tau)$ will consist only of odd harmonic terms whilst the series representing $\xi_2(\tau)$ consists only of even harmonic terms. Thus, a second approximation is now assumed in the form

$$\begin{aligned}\xi_1 &= (a_0 + \varepsilon_{a_0}) + (a_3 + \varepsilon_{a_3})\sin 2\tau + (a_4 + \varepsilon_{a_4})\cos 2\tau \\ \xi_2 &= (b_1 + \varepsilon_{b_1})\sin\tau + (b_2 + \varepsilon_{b_2})\cos\tau + (b_3 + \varepsilon_{b_3})\sin 3\tau + (b_4 + \varepsilon_{b_4})\cos 3\tau\end{aligned} \tag{3.4.8}$$

where the terms containing ε represent the correction terms. Substituting developments (3.4.8) in equation (3.3.4b) and balancing the coefficients of like terms gives

$$\begin{aligned}\varepsilon_{b_1} &= \frac{\Pi_1}{2\Pi_2}(a_3 + \varepsilon_{a_3}) - b_1 \\ \varepsilon_{b_2} &= -\frac{\Pi_1}{\Pi_2}\left[(a_0 + \varepsilon_{a_0}) - \frac{1}{2}(a_4 + \varepsilon_{a_4})\right] - b_2 \\ \varepsilon_{b_3} &= -\frac{\Pi_1}{6\Pi_2}(a_3 + \varepsilon_{a_3}) - b_3 \\ \varepsilon_{b_4} &= -\frac{\Pi_1}{6\Pi_2}(a_4 + \varepsilon_{a_4}) - b_4\end{aligned} \tag{3.4.9}$$

Substituting developments (3.4.8) in equation (3.3.4a) and balancing the coefficients of like terms lead, on using results (3.4.9) and neglecting terms of order higher than the first in ε, to the following set of linear simultaneous algebraic equations in the correction terms ε_{a_o}, ε_{a_3}, ε_{a_4}.

$$6(\Pi_2 + \Pi_1 b_2)\varepsilon_{a_o} + \Pi_1(3 + b_3 - 3b_1)\varepsilon_{a_3} + \Pi_1(b_4 - 3b_2)\varepsilon_{a_4}$$

$$= \Pi_1(3b_1a_3 - 6a_ob_2 + 3b_2a_4 - a_3b_3 - a_4b_4 - 3a_3) + 3\Pi_2(1 - 2a_o - b_1^2 - b_2^2 - b_3^2 - b_4^2).$$

$$6\Pi_1(b_1 + b_3 - 1)\varepsilon_{a_o} + (6\Pi_2 - 2\Pi_1b_2 + 3\Pi_1b_4)\varepsilon_{a_3} + (4\Pi_1 - 12\Pi_2^2 - 4\Pi_1b_1 - 3\Pi_1b_3)\varepsilon_{a_4}$$

$$= \Pi_1(3a_3b_2 - 6a_ob_1 + 4b_1a_4 - 3b_4a_3 - 6b_3a_o + 3a_4b_3 - b_2a_3 + 6a_o - 4a_4) + 6\Pi_2(2\Pi_2a_4 - a_3 - b_2b_3 + b_1b_4 - b_1b_2)$$

$$6\Pi_1(b_2 + b_4)\varepsilon_{a_o} - (4\Pi_1 - 12\Pi_2^2 - 4\Pi_1b_1 + 3\Pi_1b_3)\varepsilon_{a_3} + (6\Pi_2 - 2\Pi_1b_2 - 3\Pi_1b_4)\varepsilon_{a_4}$$

$$= 3\Pi_2(b_1^2 - b_2^2 - 1 - 2b_1b_3 - 2b_2b_4 - 2a_4 - 4\Pi_2a_3) - \Pi_1(3b_1a_3 + 6b_2a_o - 3b_2a_4 - 3b_3a_3 + b_1a_3 - 3a_3 + 6b_4a_o - 3b_4a_4 + b_2a_4 - a_3)$$

$$(3.4.10)$$

The system of equations (3.4.10) are now solved, using digital computation, with initial values $a_3 = a_4 = b_1 = b_3 = b_4 = 0$ and b_2, a_o given by equations (3.4.2) and (3.4.5) respectively; corresponding values for the correction terms ε_{b_i} $(i = 1,2,3,4)$ are then obtained from equations (3.4.9). $a_i + \varepsilon_{a_i}$ $(i = 0,3,4)$ and $b_i + \varepsilon_{b_i}$ $(i = 1,2,3,4)$ are then taken as the new values of the coefficients a_i $(i = 0,3,4)$ and b_i $(i = 1,2,3,4)$ respectively and equations (3.4.10) and (3.4.9) solved to find the new values of the correction terms. This process is repeated until values of the coefficients, which give, on solution of (3.4.10) and (3.4.9), sufficiently small correction terms, are obtained. Coefficients of higher order harmonics are then obtained in a similar way.

3.4.2. Subharmonic solutions

To a first approximation we assume solutions

$$\xi_1 = a_o + a_1 \sin\frac{\tau}{n} + a_2 \cos\frac{\tau}{n} + a_3 \cos\tau$$
$$\xi_2 = b_o + b_1 \sin\frac{\tau}{n} + b_2 \cos\frac{\tau}{n} + b_3 \sin\tau + b_4 \cos\tau \qquad (3.4.11)$$

On substituion equations (3.4.11) in equations (3.3.4) and balancing the coefficients of like terms it is seen (see appendix 3.1) that eight subharmonic solutions, of order 2, are possible in a region R_2 of parameter space defined by the system of inequalities:

$$N = \Pi_2^4 (289 - 105\,\Pi_2^2 - 420\,\Pi_1) + 196\,\Pi_1^2 + 308\,\Pi_2^2\Pi_1 - 588\,\Pi_2^2\Pi_1^2 > 0$$

$$L^2 = \frac{2}{3}\Pi_2^2 - \Pi_1^2 - \frac{1}{12}\Pi_2^4 - \frac{1}{3}\Pi_1\,\Pi_2^2 > 0$$

$$|X| = \left|\frac{1}{42\Pi_2}(\sqrt{N} - (11\,\Pi_2^2 + 14\,\Pi_1))\right| < L \qquad (3.4.12)$$

$$F(X) = 7\Pi_2 X^3 + \left(\frac{11}{2}\Pi_2^2 + 7\,\Pi_1\right)X^2 + \left(\frac{5}{4}\Pi_2^5 + 5\,\Pi_2^3\,\Pi_1 + 7\,\Pi_2\,\Pi_1^2 - 2\,\Pi_2^3\right)X$$
$$+ \left(\frac{9}{8}\Pi_2^6 + \frac{23}{4}\Pi_2^4\,\Pi_1 - \Pi_2^4 + \frac{21}{2}\Pi_1^2\,\Pi_2^2 - 2\,\Pi_2^2\,\Pi_1 + 7\,\Pi_1^3\right) > 0$$

The regions R_1 and R_2 of parameter space, in which different types of *steady-state* oscillations are sustained are shown in Fig. 3.4.1.; the region R_2 having been plotted using digital computation (see appendix 3.1. for flowchart). Outside region R_1 there are, to a first approximation, no periodic solutions so that, for parameter values in this region, the system will be totally unstable.

An improvement in the accuracy of the subharmonic solutions (3.4.11) may be obtained using the same procedure as described in section 3.4.1. for harmonic solutions. An important feature of the subharmonic

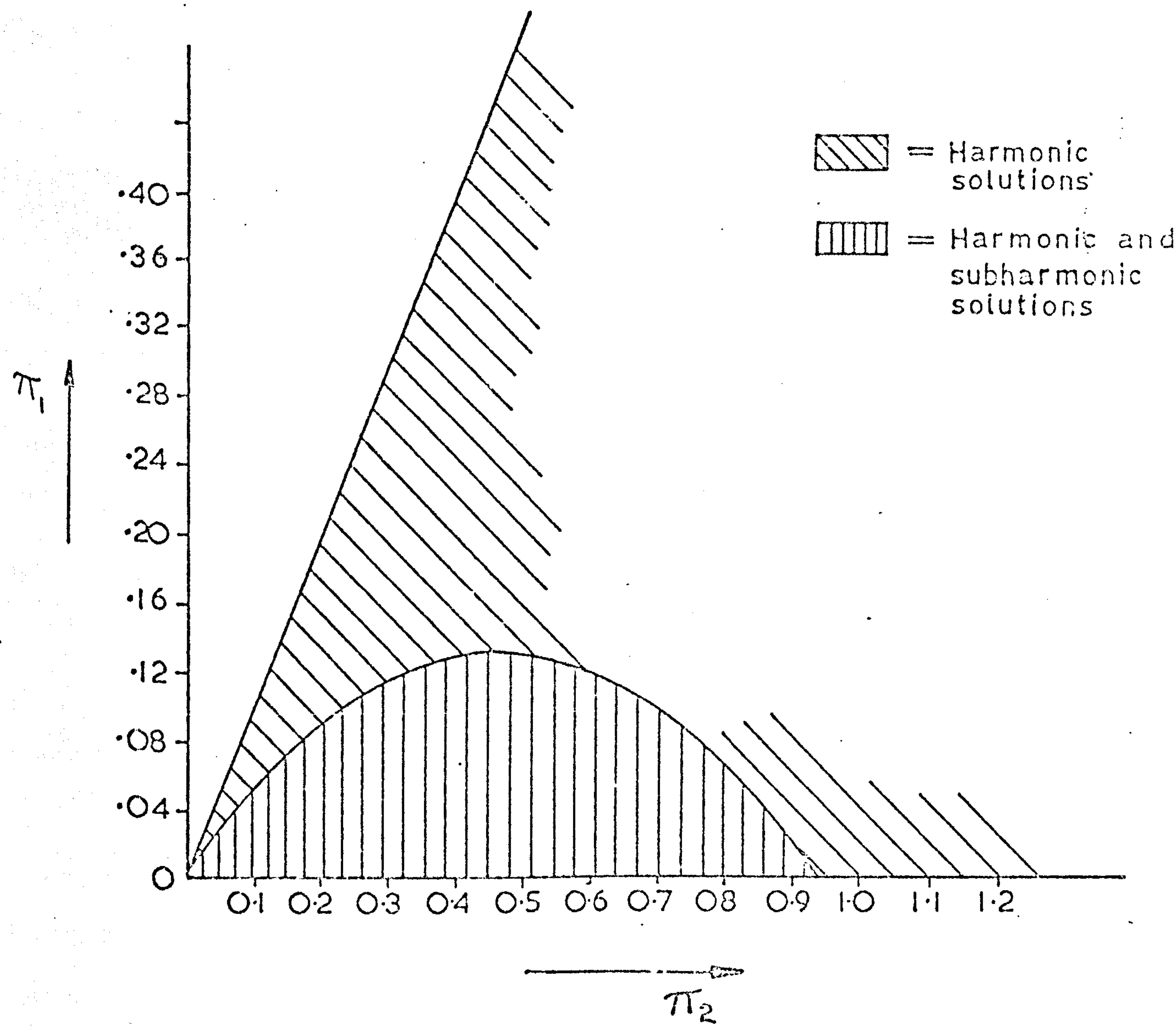

FIG. 3.4.1. Regions where different types of oscillations are sustained.

solutions is that the solution for $\xi_2(\tau)$ contains a constant or "d.c." component so that for certain initial conditions the system, with parameter values in region R_2, will adapt to an oscillation about an offset position; this phenomenon has been verified using analogue computer simulation.

3.5. Stability of *steady-state* solutions

A periodic solution obtained by the method of harmonic balance merely represents a state of equilibrium; this equilibrium state is actually realisable only if it is stable so that its actual existence must be confirmed by a stability investigation.

Let $\hat{\underline{\xi}}(\tau) = [\, \hat{\xi}_1(\tau),\, \hat{\xi}_2(\tau)\,]'$, where prime denotes the transpose, having period T, represent a particular state of equilibrium then in order to investigate its stability we consider small variations $\underline{\eta}(\tau) = [\, \eta_1(\tau), \eta_2(\tau)\,]'$ from this equilibrium state; if in the ensuing motion $\underline{\eta}(\tau)$ tends to zero then the original undisturbed equilibrium state is said to be asymptotically stable.

From equations (3.3.4) we set up the variational equations

$$\begin{aligned} \dot{\eta}_1(\tau) &= -\frac{1}{\Pi_1}\,\eta_1(\tau) + \frac{2}{\Pi_1}\,(\hat{\xi}_2(\tau) - \sin\tau)\,\eta_2(\tau) \\ \dot{\eta}_2(\tau) &= \frac{\Pi_1}{\Pi_2}\,\sin\tau\,\eta_1(\tau) \end{aligned} \tag{3.5.1}$$

in $\eta_1(\tau)$ and $\eta_2(\tau)$. These variational equations form a set of linear ordinary homogeneous differential equations with periodic coefficients, of period T, in τ; that is, they are of the form

$$\dot{\underline{\eta}}(\tau) = \underline{P}(\tau)\underline{\eta}(\tau)$$

with $\quad \underline{P}(\tau) = \underline{P}(\tau + T)$.

The asymptotic stability of the null solution of such equations has been discussed in chapter 1. If the eigenvalues of the monodromy matrix are both greater than unity in absolute value the solution will be termed completely unstable, whilst it will be termed unstable if the monodromy matrix has one eigenvalue greater than, and one less than, unity in absolute value. If both the eigenvalues of the monodromy matrix are less than unity in absolute value the solution will be termed completely stable.

A stability investigation of the periodic solutions of section 3.4. shows that in region R_1 we have one stable and one unstable harmonic solution whilst in region R_2 we have, in addition to the stable and unstable harmonic solutions, four stable and four unstable subharmonic solutions of order 2.

3.6. Effect of initial conditions

In the absence of the output lag the system will be represented by the single non-dimensional equation

$$\dot{\xi} = \Pi (\xi - \sin\tau)^2 \sin\tau \tag{3.6.1}$$

where $\Pi = GA\delta/\omega$, $\xi = y/\delta$ and $\tau = \omega t + \alpha$.

Equation (3.6.1) may be solved in the $\xi - \tau$ plane, for a particular value of the parameter Π, by the method of isoclines [69]. A graphical solution obtained by this method, for $\Pi = 0.2$, is shown in Fig. 3.6.1 and it shows clearly the effect on stability of the initial conditions in this case. Solutions are shown for the initial values $\tau = 0$ and $\tau = \pi/2$.

When the output lag is included, however, we can no longer solve the system equations by the method of isoclines. In order to examine the relationship between the initial conditions and the different types of periodic solutions we examine the *domains of attraction* of the stable periodic solutions; a concept employed by Blair and Loud [70] when examining the solutions of a second order nonlinear differential equation with a periodic forcing term. A brief discussion of the theory of fixed points and domains of attraction will now be given.

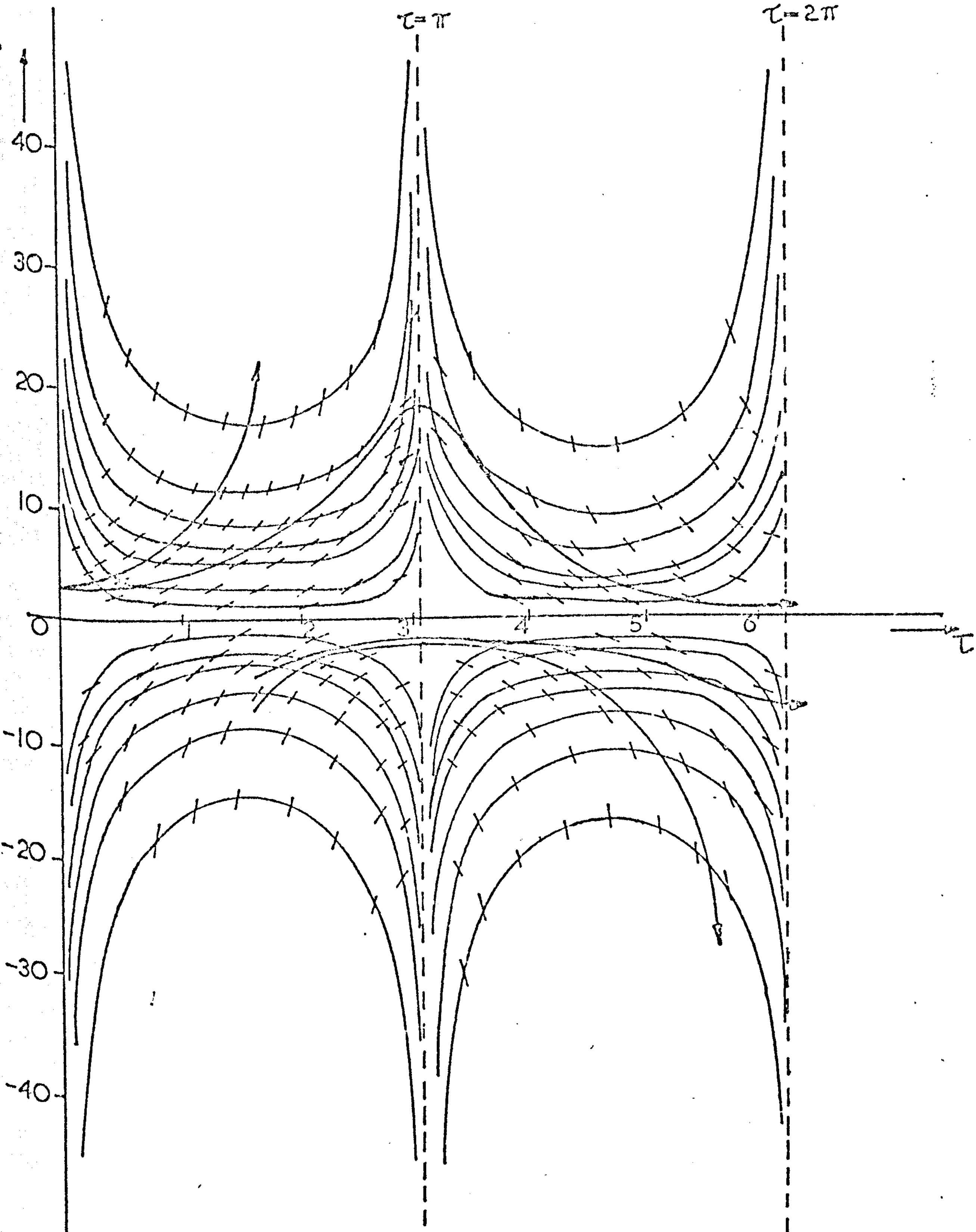

FIG. 3.6.1. Graphical solution of equation (3.6.1) for Π = 0.2.

3.7. Fixed points and domains of attraction

Let us consider the solution of the general system of equations

$$\begin{aligned} \dot{\xi}_1 &= f(\xi_1, \xi_2, \tau) \\ \dot{\xi}_2 &= g(\xi_1, \xi_2, \tau) \end{aligned} \tag{3.7.1}$$

where f and g have period T in τ. Provided f, g and their partial derivative with respect to ξ_1 and ξ_2 are continuous in ξ_1, ξ_2 and τ it follows [71] that there exists a mapping

$$M : \underline{\xi}(\tau_1 + nT) \rightarrow \underline{\xi}(\tau_1 + \overline{n+1}\,T),\ n = 0, 1, 2, \text{ etc.} \tag{3.7.2}$$

where $\underline{\xi} = [\xi_1, \xi_2]^1$, which is a one to one, continuous and orientation preserving mapping of the ξ_1-ξ_2 plane into itself (that is, M defines a homeomorphism in the ξ_1-ξ_2 plane). If $\hat{\underline{\xi}}(\tau) = [\hat{\xi}_1(\tau), \hat{\xi}_2(\tau)]^1$ is a harmonic solution of equations (3.7.1) then there exists, for each τ_1, $0 < \tau_1 < T$, a corresponding point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$, in the ξ_1-ξ_2 plane, which is invariant under the mapping M; that is, a *fixed point* of the mapping M corresponds to a harmonic solution of equations (3.7.1).

Defining iterates of the mapping M by

$$M^2(P) \doteq M(M(P)) \quad \text{etc.,}$$

it follows that if $\hat{\underline{\xi}}(\tau) = [\hat{\xi}_1(\tau), \hat{\xi}_2(\tau)]^1$ is a subharmonic solution, of order n, of equations (3.7.1), then there exists, for each τ_1, $0 < \tau_1 < nT$, a corresponding point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$ in the ξ_1-ξ_2 plane, which is invariant under the nth iterate M^n of the mapping M; that is, a fixed point of the nth iterate of the mapping M corresponds to a subharmonic solution, of order n, of the system of equations (3.7.1).

There are certain standard types of fixed points, most of them corresponding closely to standard types of singular points for differential equations of order one. We shall now discuss the three most significant types; for a fuller discussion see the works of Cartwright [71] and Levinson [72, 73]. In the following $P_0(\hat{\xi}_1(\tau_1),\hat{\xi}_2(\tau_1))$ is taken as a fixed point, in the ξ_1-ξ_2 plane at $\tau = \tau_1$, of the mapping M.

(a) stable fixed point

This is a fixed point P_o such that if P be any point in the neighbourhood of P_o then $M^n(P) \to P_o$ as $n \to \infty$; that is, successive images M(P), $M^2(P)$, etc., of the point P approach the fixed point P_o. This point is analagous to a node or focus in the theory of singular points, since the definition is true whether the loci of successive images move towards P radially (fig. 3.7.1a) as in the case of nodes, or in a spiral fashion (fig. 3.7.1b) as in the case of foci in the theory of singular points (remember that in this case P moves in jumps M(P), $M^2(P)$, etc., and not along a continuous curve). By definition, this fixed point corresponds to a stable solution of the system of equations (3.7.1.)

Note that a stable subharmonic solution, of order n, of equations (3.7.1) corresponds to a stable fixed point under the mapping M^n.

(b) unstable fixed point

This is a fixed point P_o which is stable under the inverse mapping of M and corresponds to a completely unstable solution of the system of equations (3.7.1). Such points are illustrated in figs. (3.7.2a) and (3.7.2b).

(c) saddle point

This is a fixed point P_o through which there passes two curves or directions γ_1, γ_2, see fig. (3.7.3a), which are invariant under the mapping M. Points on γ_1 approach P_o under iterations of the mapping M, while points on γ_2 approach P_o under iterations of the inverse

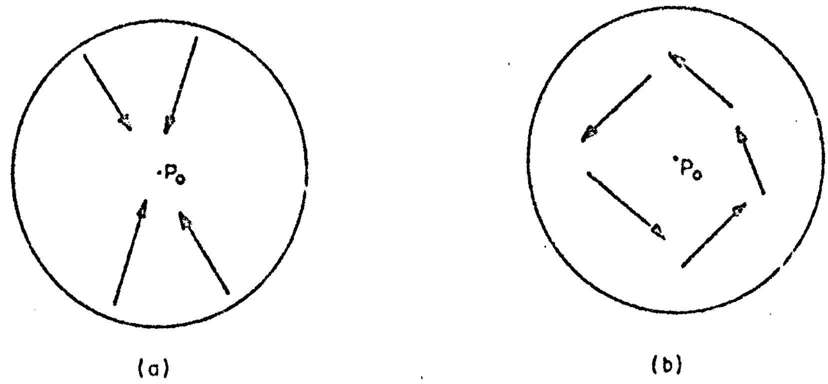

FIG. 3.7.1. Stable fixed points.

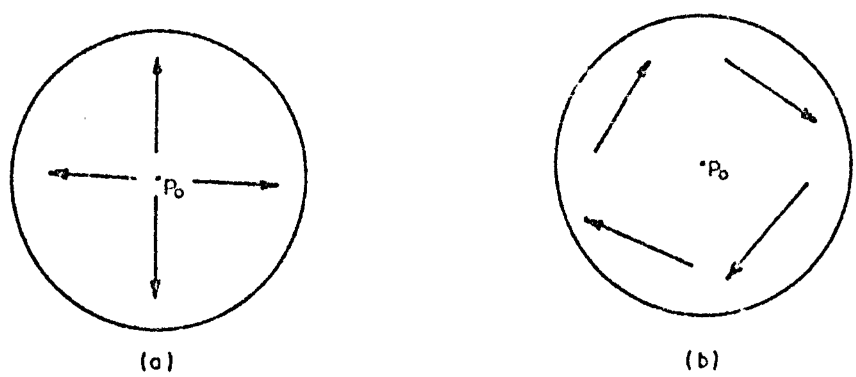

FIG. 3.7.2. Unstable fixed points.

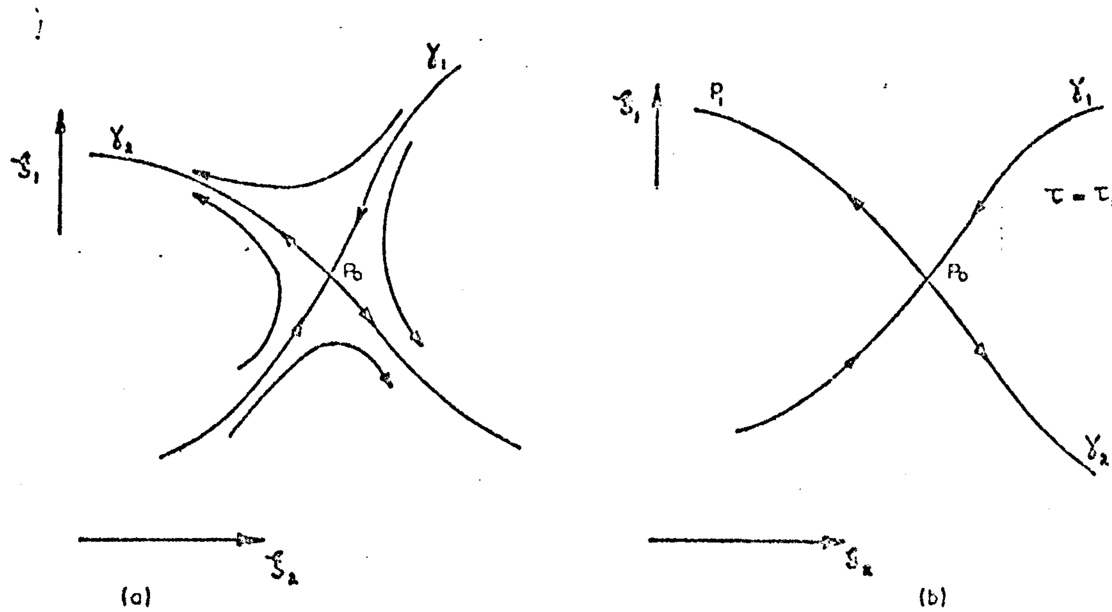

FIG. 3.7.3.

mapping. In this case the loci of successive images is analogous to that of the integral curves in the neighbourhood of a saddle point in the theory of singular points. A saddle point corresponds to an unstable solution of the system of equations (3.7.1).

Defining $\xi_1(\tau, \xi_1(\tau_1), \xi_2(\tau_1))$, $\xi_2(\tau, \xi_1(\tau_1), \xi_2(\tau_1))$ as a solution of the system of equations (3.7.1), for which the initial conditions, at $\tau = \tau_1$, are $\xi_1(\tau_1), \xi_2(\tau_1)$, we define the *domain of attraction* of a stable fixed point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$ in the ξ_1-ξ_2 plane, at $\tau = \tau_1$ as the set of points $(\xi_1(\tau_1), \xi_2(\tau_1))$ for which the solution $\underline{\xi}\ (\tau, \xi_1(\tau_1), \xi_2(\tau_1))$ converges to the asymptotic stable periodic solution corresponding to the fixed point P_0.

As pointed out by Blair and Loud [70], the general question of finding the shape of a domain of attraction is quite difficult and studies by Hayashi [7] show that for comparatively simple equations the domains of attraction can be highly complicated. In this section we shall confine ourselves to discussing the domains of attraction of two fixed points, one being stable and the other a saddle point; this being the case that arises for the harmonic solutions of section 3.4.1.

Suppose the two fixed points are represented by P_1 and P_2, see fig. (3.7.3b); P_1 being the stable fixed point and P_2 the saddle point. As indicated previously, through P_2 there pass two curves γ_1, γ_2 which are invariant under the mapping M, with points on γ_1 approaching P_2 under iterations of the mapping whilst points on γ_2 approach P_2 under iterations of the inverse mapping. Hence the successive images of an initial point $(\xi_1(\tau_1), \xi_2(\tau_1))$ will tend either to P_1 or to infinity, depending on which side of γ_1 the initial point is. Thus the invariant curve γ_1 is of great importance in any stability investigations for it is the boundary between two regions in each of which initial conditions will lead to a particular type of oscillation; that is, it is the boundary between *domains of* *attraction*. (For a more mathematical treatment see the work of Blair and Loud [70]). In the particular case illustrated in fig. (3.7.3b) the invariant curve γ_1 is the boundary between the domain of attraction of the fixed point P_1 and the domain of attraction of the point at infinity (that is, initial conditions in this domain will lead to a solution that grows indefinitely with time).

If the saddle point P_2 corresponds to the unstable periodic solution $\hat{\underline{\xi}}(\tau)$ of the system of equations (3.7.1) then the corresponding monodromy matrix $\underline{C}$ in the stability investigation, will have one eigenvalue greater than, and one less than, unity in absolute values. At the point P_2 the curve γ_1 will be in the direction of the eigenvector of $\underline{C}$ which corresponds to the eigenvalue that is less than unity in absolute value; thus the slope α of γ_1 at P_2 may be found. Theoretically therefore the invariant curve γ_1 may be obtained by starting just on either side of the fixed point P_2 (in the direction α) and integrating equations (3.7.1) numerically for decreasing τ; the curve γ_1 then being the loci of the successive images of the starting point under iterations of the mapping M (or M^n for a subharmonic solution of order n). In practice however it was found that if the unstable fixed points are not known accurately enough the image points of the numerical integration deviate from the desired boundary after a few cycles, so that the loci obtained by the numerical procedure may only be used as a guide and a more accurate boundary must be obtained by analogue computer studies.

3.8. Numerical example

3.8.1. Parameter values in region R_1

In the region R_1 of parameter space the parameter values $\Pi_1 = 0.1$, $\Pi_2 = 1$ were taken. For the first solution the following sequence of results shows the convergence of the numerical procedure, for obtaining the steady state solutions, as more and more terms are taken in the Fourier series approximations of the system variables ξ_1 and ξ_2.

SOLUTION I

(i)
$$\xi_1 = 273.29 + 88.71 \sin 2\tau + 84.08 \cos 2\tau$$
$$\xi_2 = 4.44 \sin\tau - 23.13 \cos\tau$$

(ii)
$$\xi_1 = 258.39 + 95.92 \sin 2\tau + 70.26 \cos 2\tau$$
$$\xi_2 = 4.80 \sin\tau - 22.33 \cos\tau - 1.60 \sin 3\tau - 1.17 \cos 3\tau$$

(iii)
$$\xi_1 = 260.73 + 97.15 \sin 2\tau \pm 73.00 \cos 2\tau + 9.76 \sin 4\tau - 4.48 \cos 4\tau$$
$$\xi_2 = 4.86 \sin\tau - 22.42 \cos\tau - 1.46 \sin 3\tau - 1.29 \cos 3\tau$$

(iv)
$$\xi_1 = 260.62 + 97.22 \sin 2\tau + 72.90 \cos 2\tau + 9.57 \sin 4\tau - 5.03 \cos 4\tau$$
$$\xi_2 = 4.86 \sin\tau - 22.42 \cos\tau - 1.46 \sin 3\tau - 1.30 \cos 3\tau - 0.10 \sin 5\tau - 0.05 \cos 5\tau$$

(v)
$$\xi_1 = 260.63 + 97.24 \sin 2\tau + 72.93 \cos 2\tau + 9.62 \sin 4\tau - 5.06 \cos 4\tau - 0.66 \sin 6\tau - 0.84 \cos 6\tau$$
$$\xi_2 = 4.86 \sin\tau - 22.42 \cos\tau - 1.46 \sin 3\tau - 1.30 \cos 3\tau - 0.10 \sin 5\tau + 0.04 \cos 5\tau$$

The numerical procedure was then terminated and (v) taken as a sufficiently accurate approximation to the first soltuion of equations (3.3.4), for the particular parameter values chosen.

Similarly, a sufficiently accurate approximation to the second solution of equations (3.3.4), for the particular parameter values chosen, is

SOLUTION 2

$$\xi_1 = 0.512 - 0.192 \sin 2\tau - 0.126 \cos 2\tau + 0.0004 \sin 4\tau + 0.0008 \cos 4\tau$$
$$\xi_2 = -0.009 \sin\tau - 0.057 \cos\tau + 0.004 \sin 3\tau + 0.002 \cos 3\tau + 0.0000 \sin 5\tau + 0.0000 \cos 5\tau$$

Using the procedure described in section 3.5 solution 1 was found to be unstable while soltuion 2 was found to be stable. Thus, for each τ,

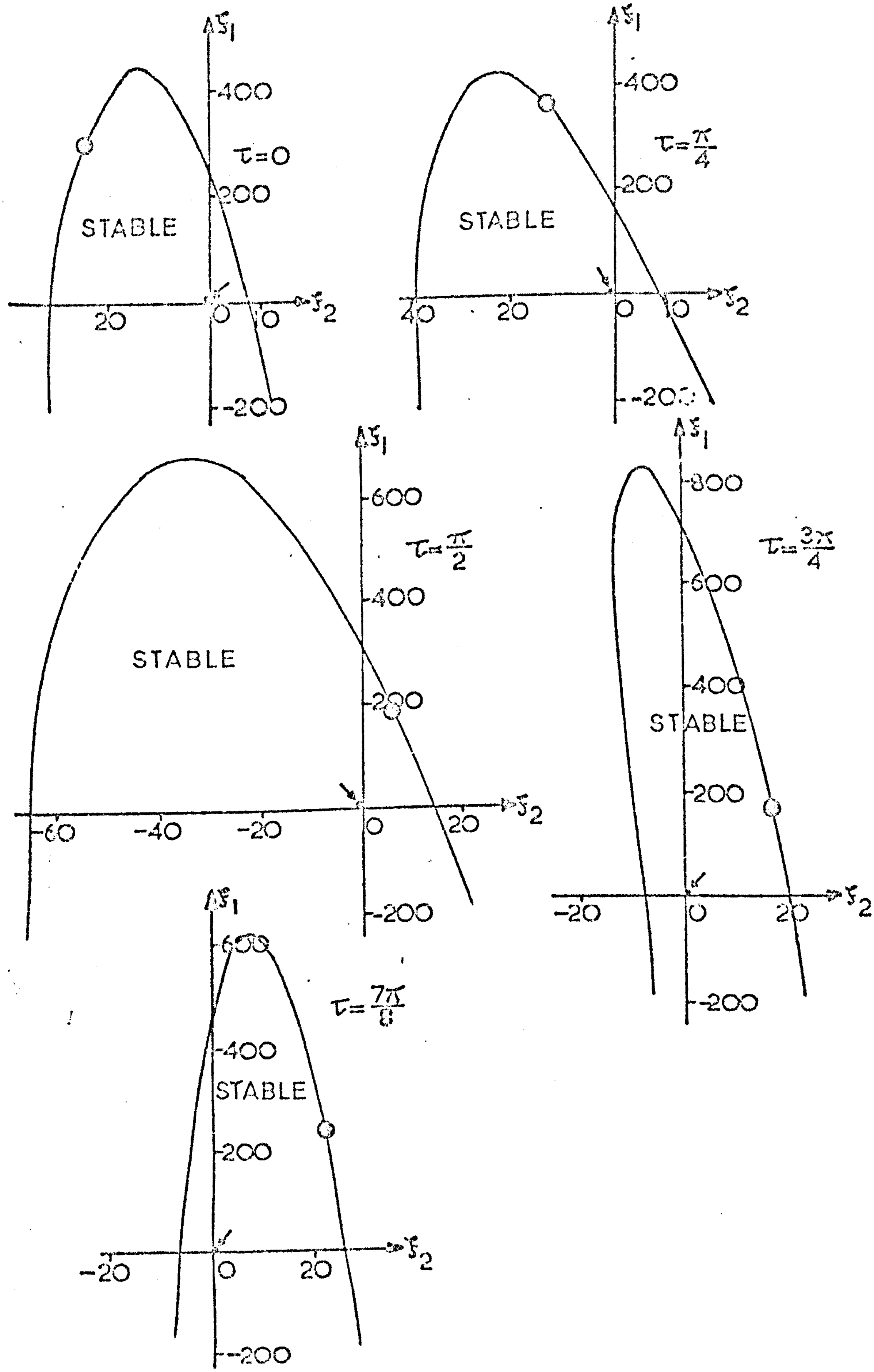

FIG. 3.8.1. Domains of attraction and corresponding fixed points for $\Pi_1 = 0.1$, $\Pi_2 = 1$.

(⊙ unstable fixed points, ↙ stable fixed points)

$0 < \tau < 2\pi$, there exists a domain of attraction in the ξ_1-ξ_2 plane so that for all initial conditions within the domain the system will be stable while for initial conditions outside the domain the system will be unstable. Further in this case, the solution for ξ_2 has no d.c. component so that in the stable region the system will adapt to an oscillation about zero error e(t). The stability boundaries, in the state space, together with their corresponding fixed points, are shown in fig. (3.8.1) for various values of τ in the range $(0, \pi)$; corresponding boundaries for τ in the range $(\pi, 2\pi)$ being the mirror images about the ξ_1 axis of those given. These stability boundaries have been verified using analogue computer simulation and also agree with experimental work carried out by Jacobs and Shering [74].

3.8.2. Parameter values in region R_2

In the region R_2 of parameter space the parameter values $\Pi_1 = 0.04$, $\Pi_2 = 0.2\pi = 0.628$ were taken.

The two harmonic solutions were obtained as in section 3.8.1 and found to be :

SOLUTION I

$$\xi_1 = 860.6 + 351 \sin 2\tau + 470.7 \cos 2\tau + 87.4 \sin 4\tau - 1.7 \cos 4\tau$$

$$\xi_2 = 11.175 \sin\tau - 39.81 \cos\tau + 2.8 \sin 3\tau - 4.975 \cos 3\tau - 0.555 \sin 5\tau - 0.01 \cos 5\tau$$

SOLUTION II

$$\xi_1 = 0.5 - 0.025 \sin 2\tau - 0.25 \cos 2\tau$$

$$\xi_2 = -0.005 \sin\tau - 0.04 \cos\tau + 0.0025 \sin 3\tau - 0.0025 \cos 3\tau$$

Solution I is found to be unstable whilst solution II is found to be stable.

Solving cubic (A3.1.15) and using $X = \Pi_1 z$ gives the three values -3.33, -15.715, 5.11 for z. However, on substituting in equation (A3.1.12), only the values -3.33 and 5.11, for z, give real values of r. Using the procedure described in appendix 3.1 eight solutions of the form

$$\xi_1 = a_o + a_1 \sin\frac{\tau}{2} + a_2 \cos\frac{\tau}{2} + a_3 \cos\tau$$

$$\xi_2 = b_o + b_1 \sin\frac{\tau}{2} + b_2 \cos\frac{\tau}{2} + b_4 \cos\tau$$

are obtained with the coefficients $a_i (i = 0,1,2,3)$ and $b_j (j = 0,1,2,4)$ having values as shown in table (3.8.1).

Solution Number	a_o	a_1	a_2	a_3	b_o	b_1	b_2	b_4
1	194.5	197.2	-115.3	-136	3.4	12.6	7.35	-12.4
2	194.5	-197.2	115.3	-136	3.4	-12.6	-7.35	-12.4
3	194.5	-115.3	197.2	136	-3.4	-7.35	12.6	-12.4
4	194.5	115.3	-197.2	136	-3.4	7.35	-12.6	-12.4
5	327.1	90	-196.4	-101.3	3.9	5.72	12.5	-20.8
6	327.1	-90	196.4	-101.3	3.9	-5.72	-12.5	-20.8
7	327.1	-196.4	-90	101.3	-3.9	-12.5	5.72	-20.8
8	327.1	196.4	90	101.3	-3.9	12.5	-5.72	-20.8

Table (3.8.1)

Coefficients of subharmonic solutions for $\Pi_1 = 0.04$, $\Pi_2 = 0.628$

Solutions containing higher harmonics are readily obtained using the procedure described in section (3.4.1). Solutions 3,4,7 and 8 were found to be stable whilst the other four solutions were found to be unstable. The trajectories of the stable subharmonic solutions, that is, the loci of the point $(\xi_1(\tau).\xi_2(\tau))$ in the ξ_1-ξ_2 plane, are shown in figs. (3.8.2) and (3.8.3). The location of the stable fixed points are also shown in the figures. It is noted that the fixed points corresponding to the subharmonic solutions 3 and 4 (similarly for solutions 7 and 8) lie on the same trajectory and that, under iterations of the mapping M,

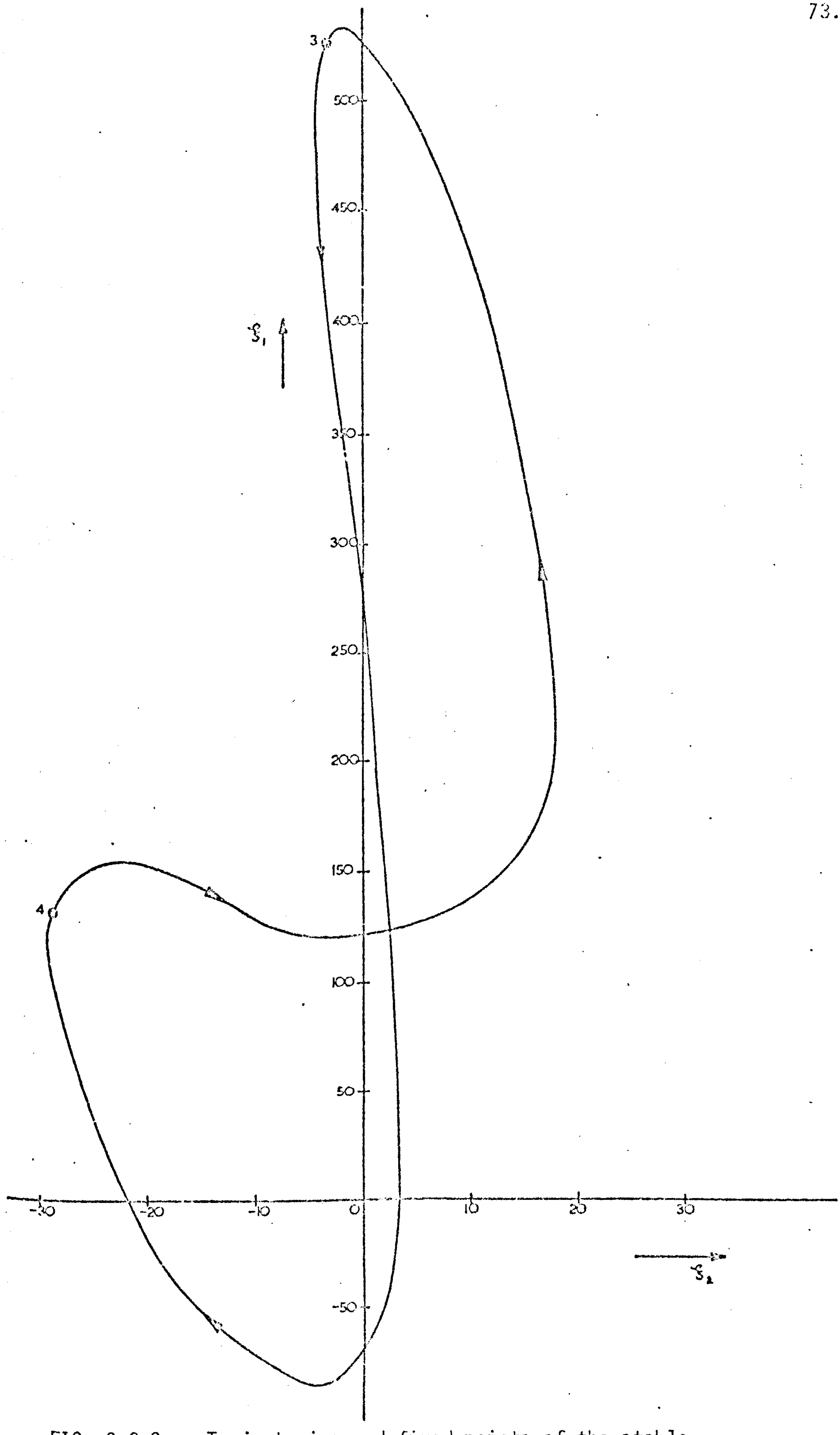

FIG. 3.8.2. Trajectories and fixed points of the stable subharmonic solutions 3 and 4.

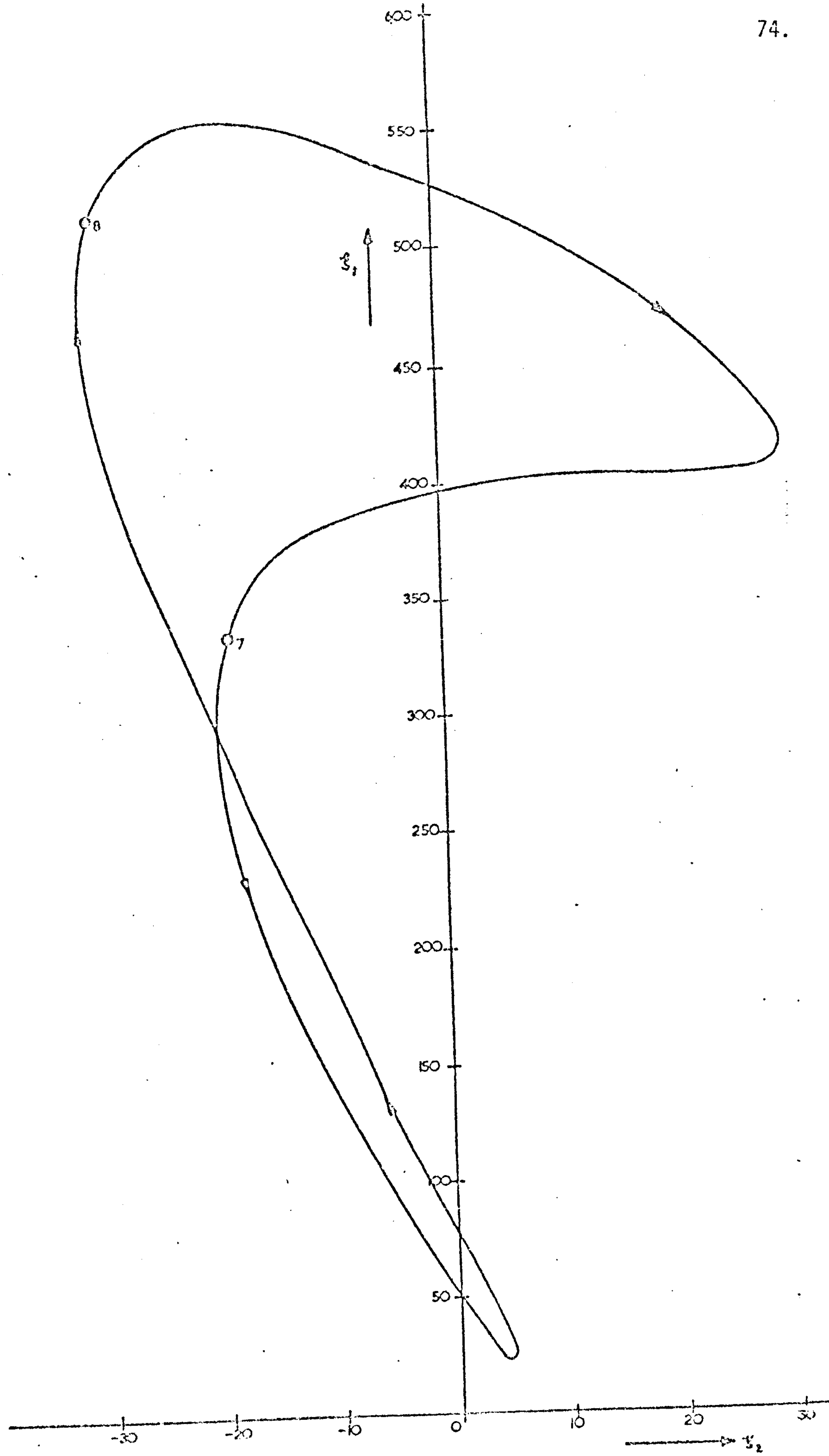

FIG. 3.8.3. Trajectories and fixed points of the stable subharmonic solutions 7 and 8.

these fixed points are transferred successively to the points that follow in the direction of the arrows. By following the procedure described in section 3.7 the whole diagram of the domains of attractions leading to the harmonic and subharmonic response may be plotted. However, such a diagram would be highly complicated and its computation would require extensive use of both analogue and digital computers. It is therefore felt that, since parameter values in this regions are unsuitable for practical purposes, complete computation of the domains of attraction is unnecessary.

3.9. Analogue computer layout

Due to overloading for certain values of the parameters magnitude scaling has to be introduced

Writing

$$mx \rightarrow X$$

$$ny \rightarrow Y$$

the system equations (3.3.1) become

$$\left(\frac{m\dot{x}}{m} \right) + a\left(\frac{mx}{m} \right) = Aa \left[\left(\frac{ny}{n} \right)^2 - 2\left(\frac{ny}{n} \right) \delta \sin(\omega t + \alpha) + \delta^2 \sin^2(\omega t + \alpha) \right]^2$$

i.e. $$\dot{X} + a X = \frac{mAa}{n^2} \left[Y - (n\delta)\sin(\omega t + \alpha) \right]^2$$

and $$\left(\frac{n\dot{y}}{n} \right) = G\left(\frac{mx}{m} \right) \sin(\omega t + \alpha)$$

i.e. $$\dot{Y} = \left(\frac{nG}{m} \right) X \sin(\omega t + \alpha)$$

The simul~~ar~~ to layout then employed is as shown in fig. (3.9.1)

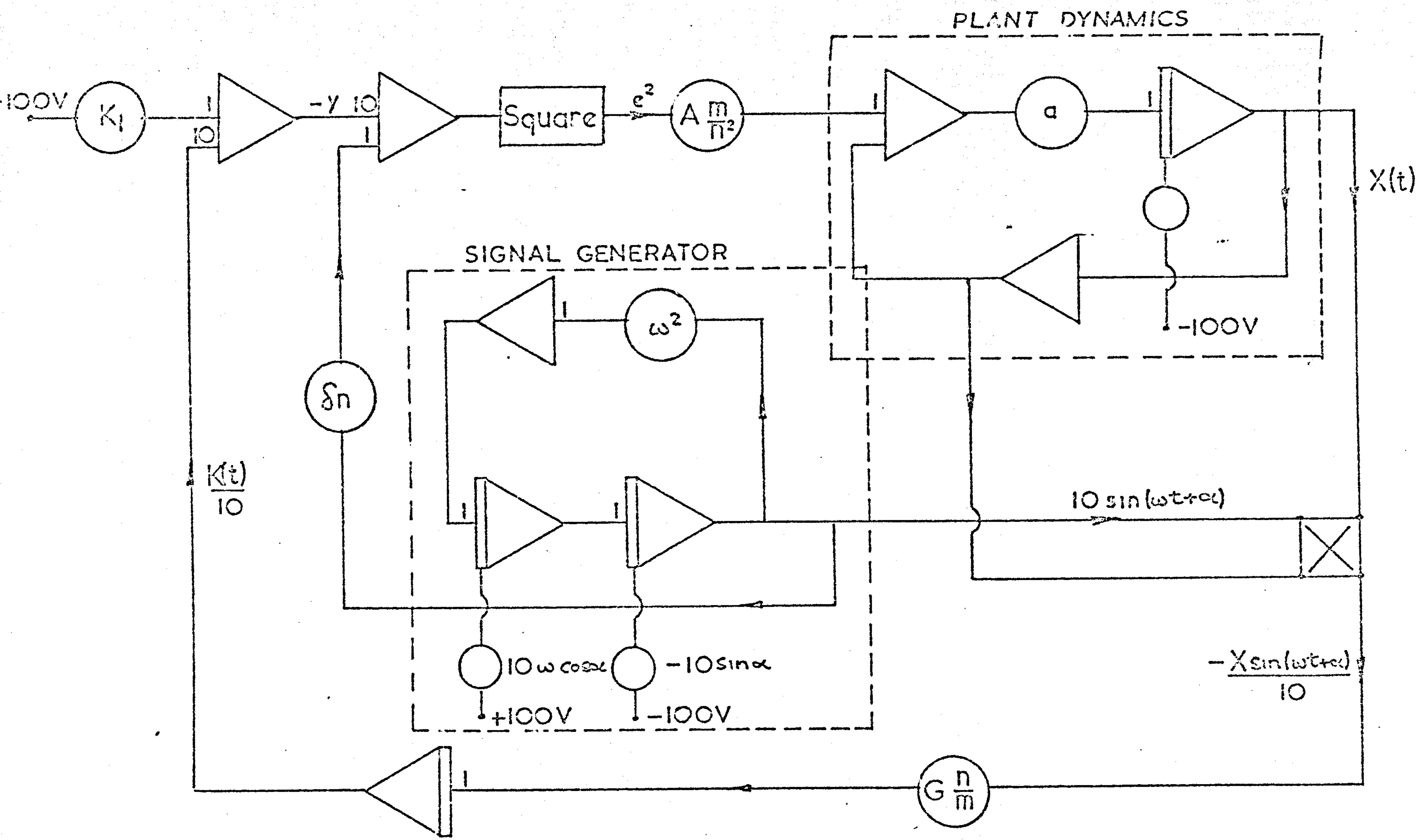

FIG. 3.9.1. Analogue simulation layout.

CHAPTER 4

STABILITY OF A FIRST ORDER MODEL-REFERENCE ADAPTIVE CONTROL SYSTEM

4.1. Introduction

In recent years model reference adaptive control systems have proved very popular, particularly for practical applications to devices such as autopilots where rapid adaptation is required. The basic idea is shown in fig. (4.1.1). The input $\theta_i(t)$ to the controlled system or process is also fed to a reference model, the output of which is proportional to the desired response; the outputs of the model and process are then differenced to form an error

$$e(t) = \theta_m(t) - \theta_s(t) \tag{4.1.1}$$

Since this error is to be zero when the process is in its optimum state it is used as a demand signal for the adaptive loops which adjusts the variable parameters in the process to their desired values.

Various methods of synthesizing the adaptive loops have been proposed but the one that has proved most popular is that developed by Whitaker et al [75] at the Massachusetts' Institute of Technology and referred to as the *sensitivity* or M.I.T. rule. Here the performance criterion is taken as the integral of error squared and a heuristical argument is given for reducing this over an unspecified period of time. This leads to a rule that a particular parameter K_i should be adjusted so that

$$\dot{K}_i = -Ge \frac{\partial e}{\partial K_i} \tag{4.1.2}$$

where G is the constant gain and dot denotes differentiation with respect to time t.

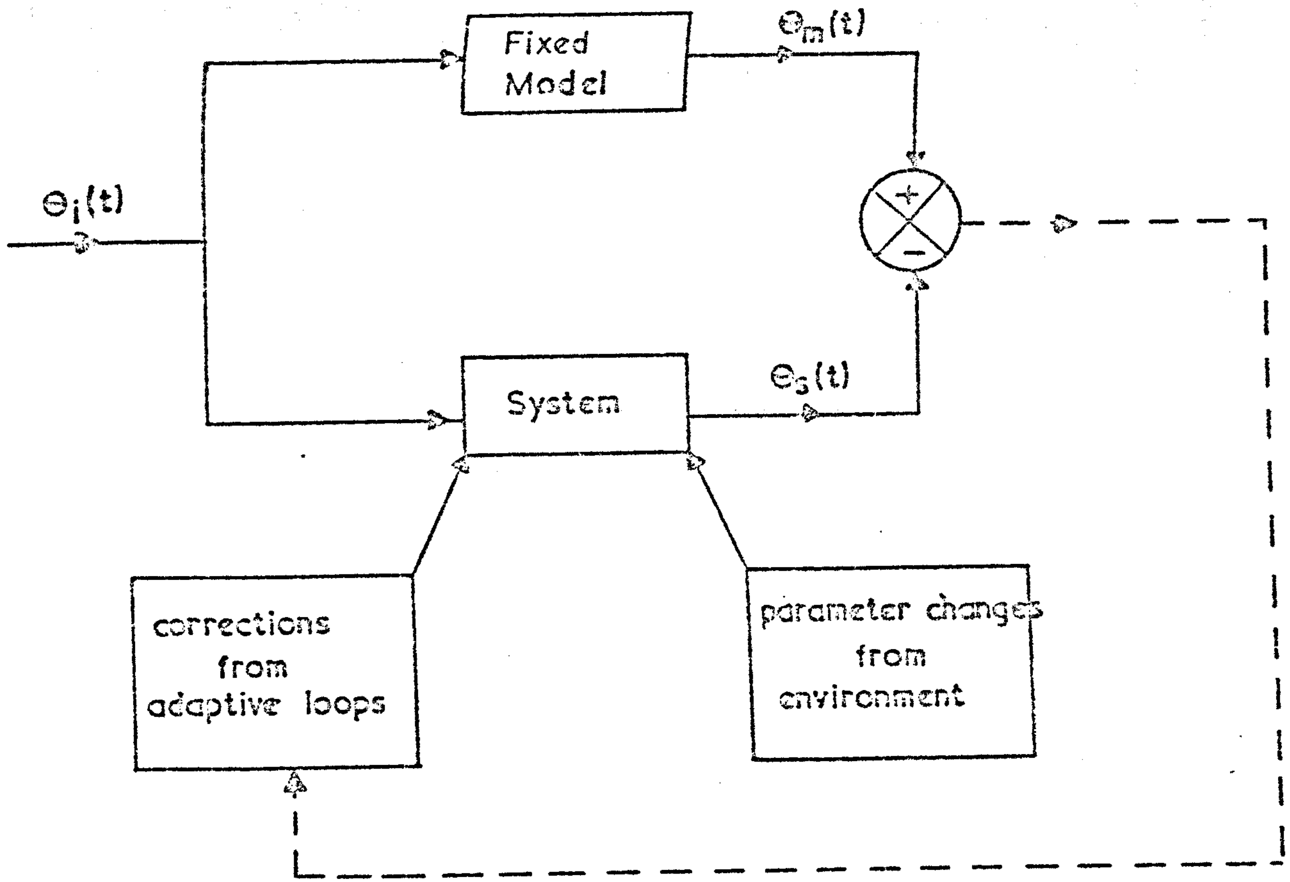

FIG. 4.1.1. Model-reference scheme.

The popularity of this method is undoubtedly due to the practical convenience with which the $\frac{\partial e}{\partial K_i}$ signals are generated by easily realizable filters.

Although the M.I.T. rule results in practically realizable systems mathematical analysis of the adaptive loops, even for simple inputs, prove to be very difficult and it is usual in the design process to carry out much analogue computer simulation. The system equations are nonlinear and nonautonomous and since the nonlinearity is of the multiplicative kind the mass of theory on instantaneous nonlinearities associated with the names of Luré and Popov, in particular, is not applicable.

In this chapter we shall apply the theory developed in chapters 1 and 2 to examine the stability of a first order M.I.T. type model reference system with controllable gain when the input varies both sinusoidally and randomly with time and the process environmental parameter assumed constant. In chapter 5 the effect of making the environmental parameter time varying, on the system stability and accuracy of its parameter adaptation, will be considered.

The doubts concerning the stability and the difficulty of analysis of the M.I.T. type system have led researchers to think about redesigning the model reference system from the point of view of stability. In particular we have the Liapunov synthesis method[46,76,77]; in this approach a Liapunov function is proposed and control signals are chosen such that its time derivative is negative definite. Should this be possible then the resulting system can be guaranteed stable for all possible inputs.

4.2. Adaptive control system

In order to develop a mathematical analysis, and to illustrate the difficulties involved, we shall first consider a first order system with controllable gain.

Consider the model and controlled system to be governed respectively by the equations

$$T\,\dot{\theta}_m(t) \;+\; \theta_m(t) \;=\; K\theta_i(t) \qquad (4.2.1)$$

$$T\,\dot{\theta}_s(t) \;+\; \theta_s(t) \;=\; K_v K_c \theta_i(t) \qquad (4.2.2)$$

where a dot denoted differentiation with respect to time t, the time constant T and the model gain K are constant and known but the process gain K_v is unknown and possibly time varying. The problem here is to determine a suitable adaptive locp to control K_c so that K_vK_c eventually equals the model gain K. The M.I.T. rule gives

$$\dot{K}_c \;=\; -Ge\,\frac{\partial e}{\partial K_c} \qquad (G,\ \text{constant} > 0)$$

where

$$\frac{\partial e}{\partial K_c} \;=\; \frac{-K_v\theta_i(t)}{1 + Ts} \quad ,$$

s being the Laplacian operator, and is found by differentiating partially the transfer function

$$e \;=\; \frac{(K - K_vK_c)\theta_i(t)}{1 + Ts}$$

with respect to K_c. The signal $(\partial e/\partial K_c)$ is usually generated by additional circuitry but here the signal $-\theta_m(t)$ is effectively all that is required leading to the scheme shown in fig. (4.2.1), where

$$\dot{K}_c \;=\; Be\theta_m(t) \qquad (4.2.3)$$

The equations of the system illustrated in fig.(4.2.1) are:

$$\begin{aligned} T\dot{e} \;+\; e \;&=\; (K - K_vK_c)\theta_i(t) \\ T\dot{\theta}_m \;+\; \theta_m \;&=\; K\theta_i(t) \\ \dot{K}_c \;&=\; Be\theta_m \end{aligned} \qquad (4.2.4)$$

The Liapunov redesign of the system in fig. (4.2.1) is shown in fig. (4.2.2) [77] and is represented by the equations :

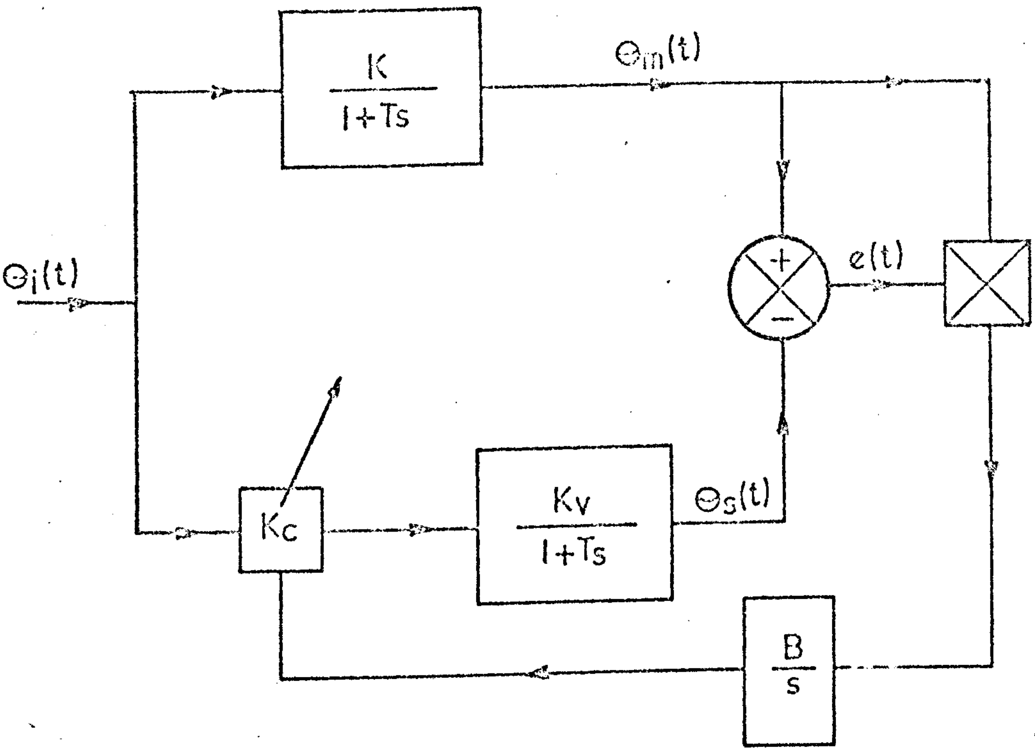

FIG. 4.2.1. First order system-M.I.T. gain adaption

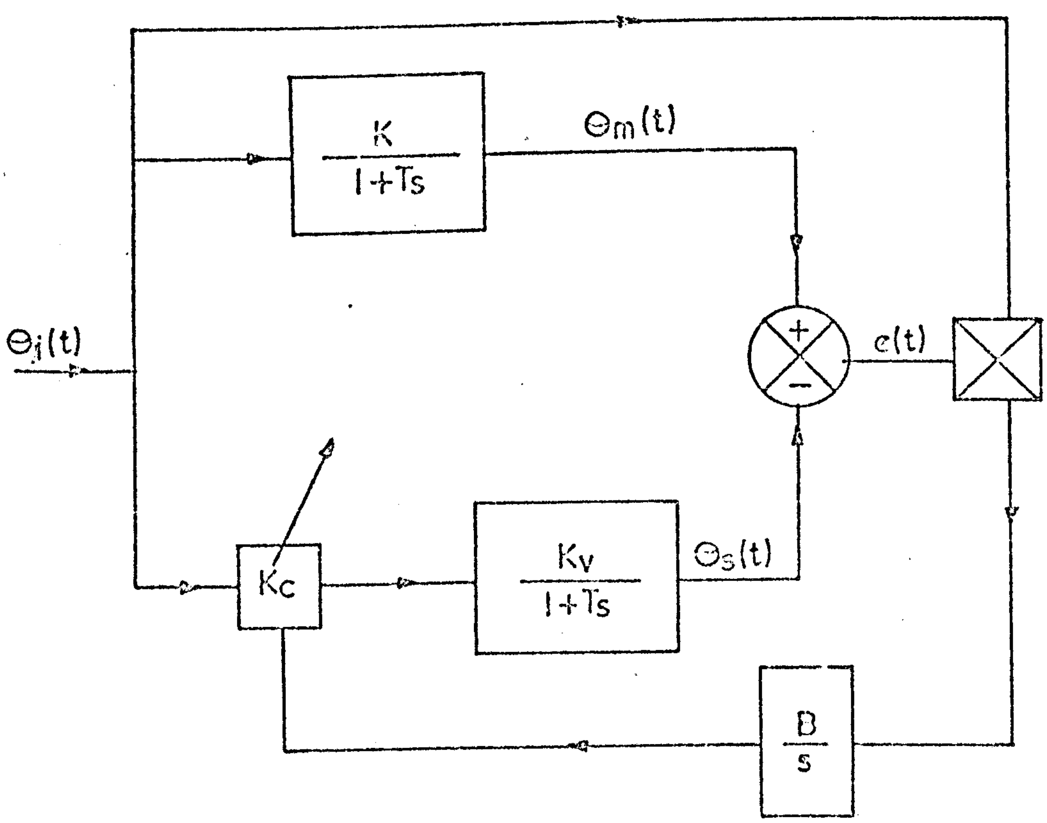

FIG. 4.2.2. Liapunov redesign of fig. 4.2.1.

$$T\dot{e} + e = (K - K_v K_c)\theta_i(t)$$
$$\dot{K}_c = Be\theta_i(t) \qquad (4.2.5)$$

4.3. Stability of adaptive control system

It is readily shown that for constant K_v the system illustrated in fig. (4.2.2) is stable for all inputs $\theta_i(t)$ and for all values of the system parameters. Writing $x_1 = e$ and $x_2 = (K - K_v K_c)$ equations (4.2.5) become

$$\dot{x}_i = -\frac{1}{T} x_1 + \frac{x_2}{T} \theta_i(t)$$
$$\dot{x}_2 = -K_v B x_1 \theta_i(t) \qquad (4.3.1)$$

Consider a tentative Liapunov function

$$V = x_1^2 + \lambda x_2^2$$

Differentiating along the trajectory of (4.3.1) gives

$$\frac{dV}{dt} = 2x_1 \left(-\frac{1}{T} x_1 + \frac{\theta_i}{T} x_2\right) + 2\lambda x_2 \left(-K_v B\theta_i x_1\right)$$

$$= -\frac{2}{T} x_1^2 + 2\left(\frac{1}{T} - B\lambda K_v\right) \theta_i x_1 x_2$$

If K_v is constant then taking $\lambda = 1/(BTK_v)$ gives

$$\frac{dV}{dt} = -\frac{2}{T} x_1^2$$

Thus, provided $\theta_i(t) \neq 0$, the scheme illustrated in fig. (4.2.2) is guaranteed to be asymptotically stable for all inputs $\theta_i(t)$ and for all values of the system parameters.

For the M.I.T. system of fig. (4.2.1) the stability analysis is not so readily carried out. Using an extension of the Dini-Hukuara theorem Parks [77] showed that if a step input of magnitude r is applied at time t = 0, when θ_m, θ_s are zero and $K_v K_c \neq K$, and if subsequently K_v remains constant but K_c is adjusted according to equation (4.2.3) then the system will be asymptotically stable for all values of the gain B and, as $t \to \infty$, $e \to 0$ and $K_vK_c \to K$ as required. We shall now proceed to carry out a stability analysis, of the system illustrated in fig. (4.2.1), when the input $\theta_i(t)$ varies both sinusoidally and randomly with time.

4.4. Sinusoidal input

If to the system of fig. (4.2.1) a sinusoidal input of magnitude $R\sin\omega t$ is applied at time $t = 0$, when θ_m, θ_s are zero and $K_vK_c \neq K$, and if subsequently K_v remains constant but K_c is adjusted according to equation (4.2.3) then the system equations (4.2.4) become

$$\begin{aligned} T\dot{e} + e &= (K - K_vK_c)R\sin\omega t \\ T\dot{\theta}_m + \theta_m &= KR\sin\omega t \\ \dot{K}_c &= Be\theta_m \end{aligned} \qquad (4.4.1)$$

Now consider that the adaption is switched on when the model response $\theta_m(t)$ has reached its steady state value $\theta_{ms}(t)$ given by

$$\theta_{ms}(t) = \frac{KR}{1 + T^2\omega^2}(\sin\omega t - T\omega\cos\omega t),$$

then equations (4.4.1) become

$$\begin{aligned} T\dot{e} + e &= x\sin\omega t \\ \dot{x} &= \frac{-BKK_vR}{1 + T^2\omega^2}(\sin\omega t - T\omega\cos\omega t)e \end{aligned} \qquad (4.4.2)$$

where $x = K - K_vK_c$.

By Buckingham's π theorem the number of non-dimensional parameters that need be considered are two and these are taken as

$$\begin{aligned} \Pi_1 &= \omega T \\ \Pi_2 &= BKK_vR^2T \end{aligned} \qquad (4.4.3)$$

If, in addition to (4.4.3), we introduce the dimensionless variables

$$\xi_1 = e/(KR)$$
$$\xi_2 = -x/K \qquad (4.4.4)$$
$$\tau = \omega t$$

then the system equations (4.4.2) may be written in the non-deminsional form

$$\frac{d}{d\tau}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix} -\frac{1}{\Pi_1} & -\frac{1}{\Pi_1}\sin\tau \\ \frac{\Pi_2}{\Pi_1(1+\Pi_1^2)}(\sin\tau - \Pi_1\cos\tau) & 0 \end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$$

(4.4.5)

which is a linear matrix differential equation of the form $\underline{\xi}^1 = \underline{A}(\tau)\underline{\xi}$, where a prime denotes differentiation with respect to τ and $\underline{A}(\tau)$ is periodic in τ with period 2π.

We are interested in obtaining the domains of the parameter space for which the null solution of the system of first order differential equations (4.4.5) is stable and such a problem was discussed in some detail in chapter 1.

4.4.1. Floquet theory analysis

Applying the numerical implementation of Floquet theory to the system of equations (4.4.5) stability boundaries in the parameter space Π_1-Π_2 are obtained. These stability boundaries are rather complicated and are shown in fig. (4.4.1).

As was pointed out in chapter 1 the main disadvantage with this method is that it requires the assessment of the system stability at each nodal point of a gridwork in the parameter space. However, in this case, it is found that for a given value of Π_1, whilst the product $\lambda_1\lambda_2$ of the eigenvalues of the system monodromy matrix remains constant, the value of the eigenvalue λ_1, such that $|\lambda_1| > |\lambda_2|$, follows a definite

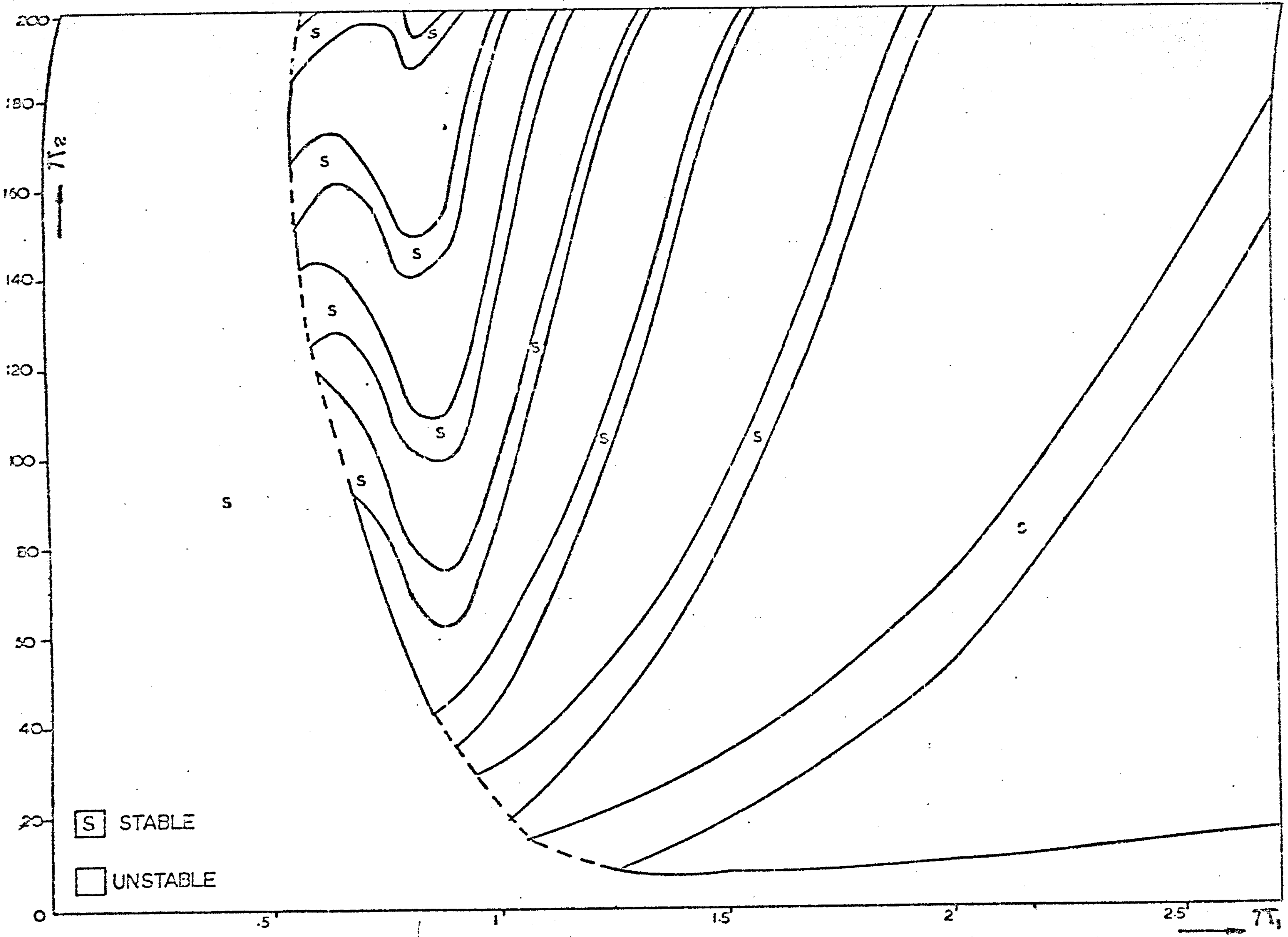

FIG. 4.4.1. Stability boundaries using Floquet analysis.

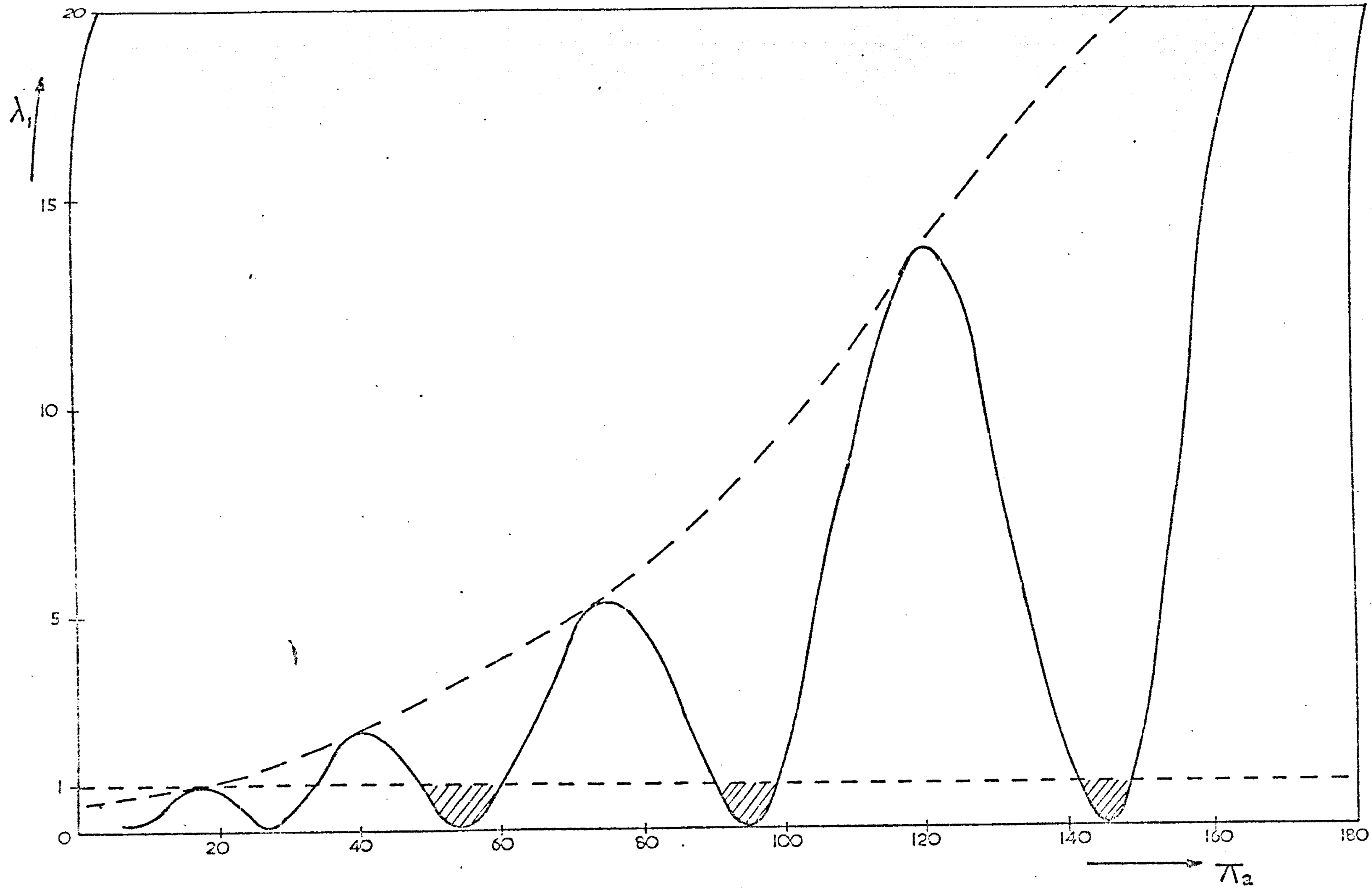

FIG. 4.4.2. Relationship between λ_1 and Π_2, when $\Pi_1 = 1$, for equations (4.4.5).

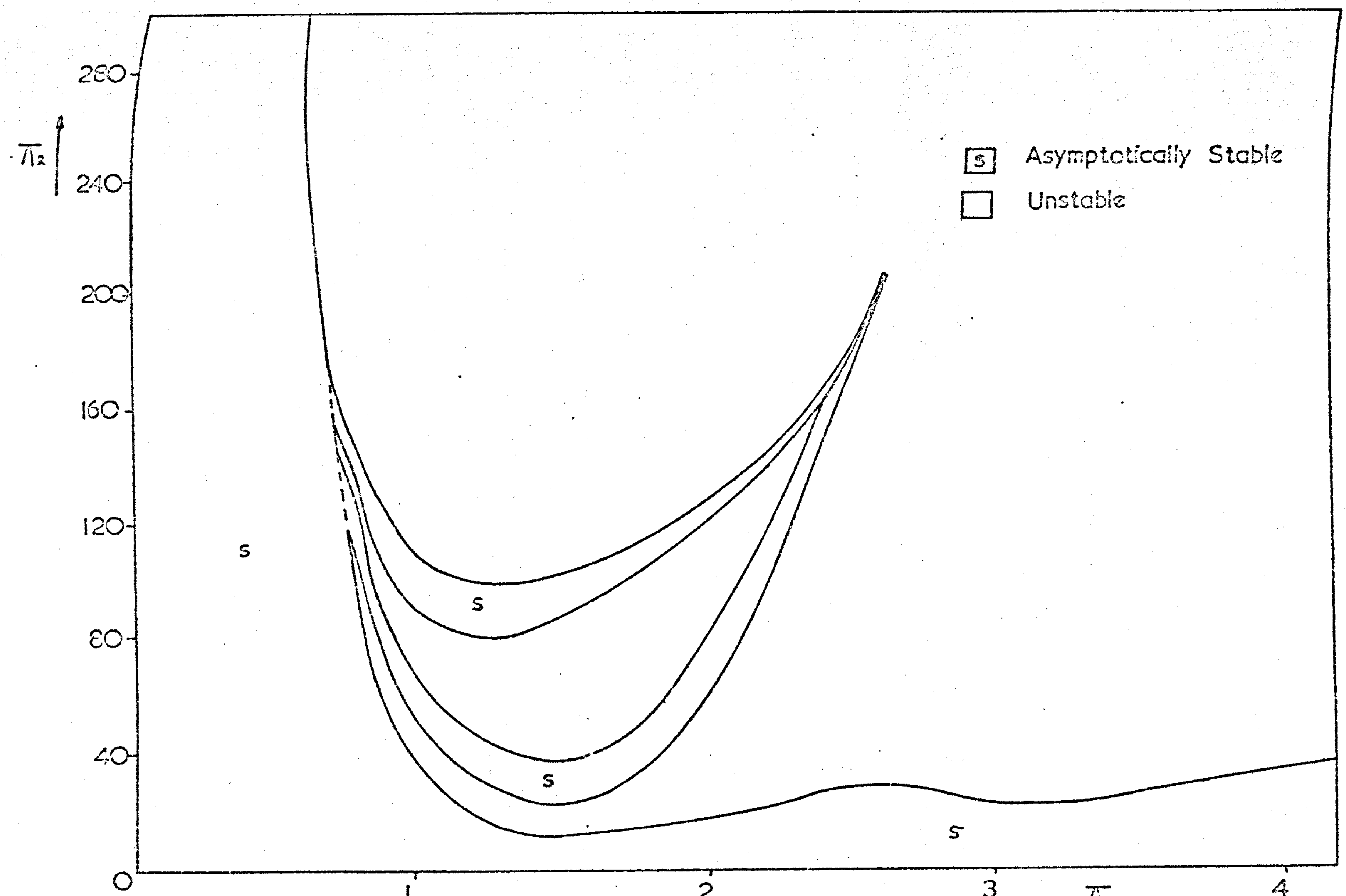

FIG. 4.4.3. Stability boundaries using Analogue computer simulation.

oscillation. A plot of λ_1 against Π_2 for the case $\Pi_1 = 1$ is shown in fig. (4.4.2) (corresponding value of λ_2 are such that $\lambda_1\lambda_2$ = constant). This continuous relationship between λ_1 and Π_2 enables us to predict approximate values of Π_2 for which the system changes in character from being stable to unstable and vice-versa. Further, fig. (4.4.2) suggests that it may be possible to obtain a mathematical expression for λ_1 but to date no such expression has been found.

4.4.2. Analogue computer simulation

In order to provide a check on the accuracy of the above numerical method the results obtained are compared to those obtained from an analogue computer study. Unfortunately analogue computer simulation of model reference systems gives rise to difficult scaling problems and continual overloading of the amplifiers often prevents the correct solution from being observed. Evans and Underdown [78] carried out an analogue computer study of the system shown in fig. (4.2.1) and assumed that persistent overloading of the computer meant instability of the system. Their stability boundaries are shown in fig. (4.4.3). It is seen that their main stability boundaries correspond closely to the theoretical results but they were unable to detect the narrow stability regions that exist in the predominantly unstable region of parameter space. An analogue computer study of the Liapunov redesign system of fig. (4.2.2) showed the system to be unconditionally stable for values of Π_2 up to 10,000.

4.4.3. Infinite determinant method

An alternative approach to the stability analysis, of the system of equations (4.4.5), is the infinite determinant method discussed in chapter 1.

Although the system of equations (4.4.5) is not canonical it can be shown that the only solutions corresponding to transition boundaries from stable to unstable regions, and vice-versa, are those of period T and 2T (2π and 4π in this particular case). From equation (1.3.10) the monodromy matrix $\underline{C}$ of system (4.4.5) is such that

$$\det\underline{C} = \exp\left\{ \int_0^{2\pi} \text{trace } \underline{A}(\tau)d\tau \right\} = \exp\{ -2\pi/\Pi_1 \}$$

so that if $\lambda_1\lambda_2$ are the eigenvalues of $\underline{C}$ then

$$\lambda_1\lambda_2 = \exp\{ -2\pi/\Pi_1 \} = \rho^2 (\text{say}) \qquad (4.4.6)$$

If λ_1, λ_2 are complex conjugates then it follows from equation (4.4.6) that they lie on a circle of radius ρ so that complex roots cannot have modulii unity since this would imply $\rho = 1$ which is only possible if Π_1 is infinite. Thus on the transition boundary there must exist a real root having the value +1 or -1 (note that real roots λ_1, λ_2 are inverse points with respect to the circle of radius ρ). It follows from equation (1.5.3) that the transition boundary is characterized by a solution of period T or 2T (i.e. 2π or 4π in this case).

On substituting a Fourier series, with undetermined coefficients, of period 4π in equations (4.4.5) and balancing like terms it can be shown, see appendix 4.1, that the corresponding Hill determinants are sums of squares and therefore cannot be zero for any value of the parameters. This has been verified by analogue computer simulation and by the results of the Floquet theory analysis.

In the case of the harmonic solution, of period 2π, substituting the Fourier series

$$\xi_1 = a_o + \sum_{n=1}^{\infty} (a_n \cos n\tau + b_n \sin n\tau)$$

into equations (4.4.5) (the corresponding series for ξ_2 being obtained by integrating the second equation of pair (4.4.5)) and balancing like terms, see appendix 4.2, leads to two distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n}, b_{2n}), $(a_{2n+1},$

b_{2n+1}), ($n = 0.1,2$, etc.), respectively. By truncating the Fourier series it is found that the corresponding Hill determinants of order r, in each case, are polynomials of order r in Π_2 having coefficients which are functions of Π_1. For a particular r these polynomials are solved, using the procedure described in appendix 4.3., for a range of values of Π_2 and the zeros plotted to obtain the transition boundaries in the parameter space. The value of r is then increased and the corresponding Hill determinants solved until a convergent set of boundaries is obtained. It is found that for $\Pi_1 > 1.5$, where the enveloping boundary is continuous, consideration of fifth order Hill determinants is sufficient but for $\Pi_1 < 1.5$, where the enveloping boundary is discontinuous, the method is not found to be very satisfactory. Hill determinants of order eleven have to be considered before a true picture begins to emerge and the order has to be increased still further before an enveloping boundary is obtained to a satisfactory degree of accuracy. Plots of the stability boundaries when r is 5 and 9 are shown in figs. (4.4.4) and (4.4.5) respectively and these illustrate the difficulty in interpretating the results using this method.

4.5. Random impulsive input

If the input, to system of fig. (4.2.1), is a general random signal $\alpha(t)$ then the system equations (4.2.4) become

$$T\dot{\theta}_m(t) + \theta_m(t) = K\alpha(t) \tag{4.5.1}$$

$$T\dot{e}(t) + e(t) = x\alpha(t) \tag{4.5.2}$$

$$\dot{x}(t) = -K_v Be(t)\theta_m(t) \tag{4.5.3}$$

where, as before, $x = K - K_v K_c$.

Despite the recent progress in stochastic stability theory, as surveyed in chapter 2, methods of investigating the stability of the above system of equations, where the system is not asymptotically stable when the noise terms are equated to zero, are not forthcoming. In order to have a first look at the problem, we shall assume that $\alpha(t)$ is a

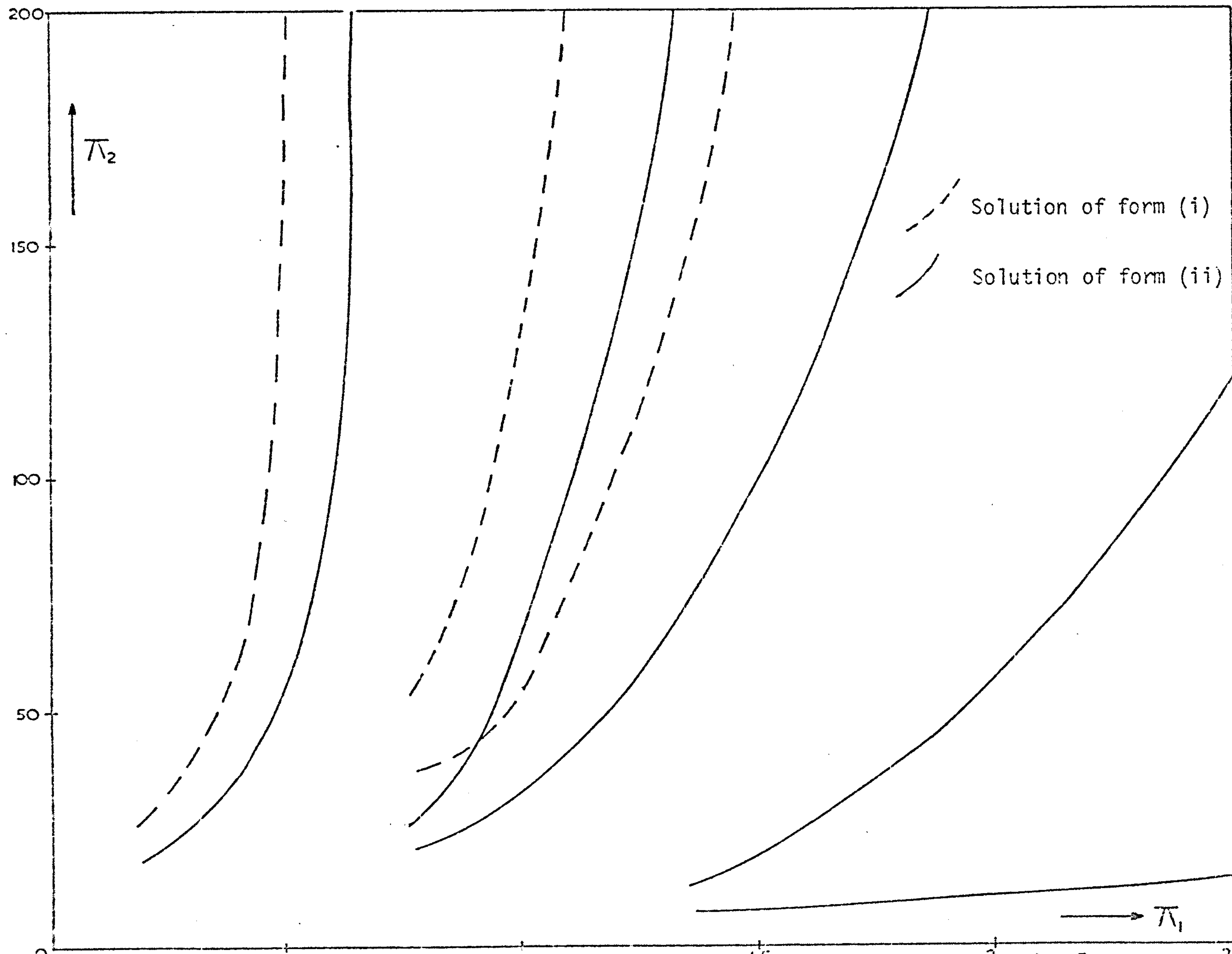

FIG. 4.4.4. Stability boundaries using Hill determinants order 5.

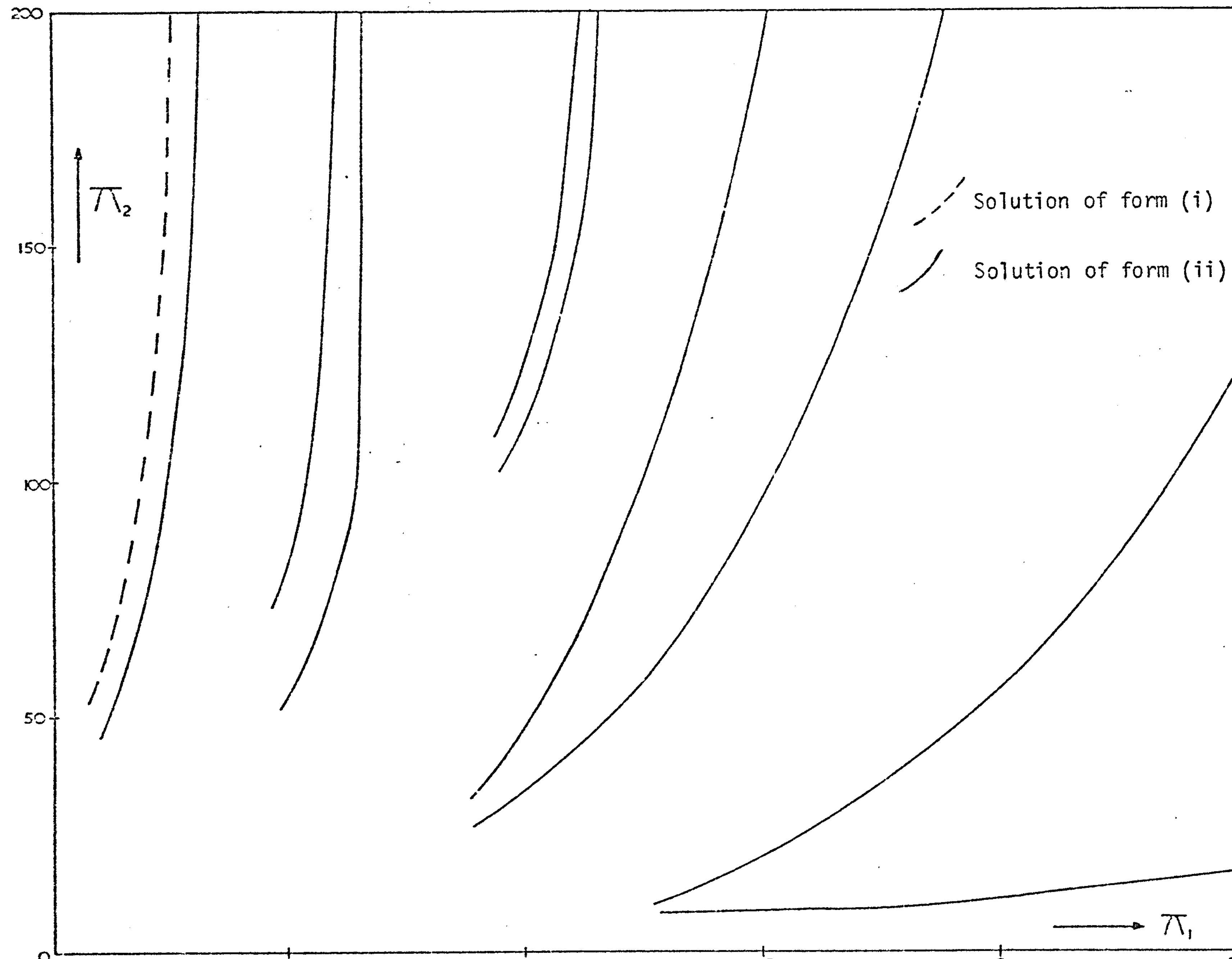

FIG. 4.4.5. Stability boundaries using Hill determinants order 9.

sequence of random impulses spaced sufficiently far apart in time (compared with the system time constant T) that the transient effects from a particular impulse have died out before the next impulse arrives; that is,

$$\alpha(t) = \sum_{r=0}^{N} A_r \, \delta(t - \tau r) \qquad (4.5.4)$$

where $\tau >> T$, N is the total number of impulses occuring and A_r (r = 0,1,2, ---, N) are random variables drawn from an amplitude probability distribution p(y), say.

During the time interval $r\tau < t < (r + 1)\tau$ equations (4.5.1) and (4.5.2) may be solved to give

$$\theta_m(t) = \frac{KA_r}{T} \exp \{-(t - r\tau)/T\} \qquad (4.5.5)$$

and

$$e(t) = \frac{A_r}{T} x(r\tau) \exp \{-(t - r\tau)/T\} \qquad (4.5.6)$$

Substituting equations (4.5.5) and (4.5.6) in equation (4.5.3) gives

$$\dot{x}(t) = \frac{-KK_vBA_r^2 \ x(r\tau)}{T^2} \exp \{-2(t - r\tau)/T\} \qquad (4.5.7)$$

Integrating equation (4.5.7) and substituting back for the constant of integration gives

$$x(t) = x(r\tau) \left[1 - \frac{KK_vBA_r^2}{2T} \right] + \frac{KK_vBA_r^2}{2T} \quad x(rt) \exp \{-2(t - r\tau)/T\}$$

(4.5.8)

Since $\tau >> T$ equation (4.5.8) furnishes the following recurrence relationship for $x(r\tau)$

$$x(\overline{r + 1} \ \tau) = \left[1 - \frac{KK_vBA_r^2}{2T} \right] x(r\tau) \qquad (4.5.9)$$

Successive application of relationship (4.5.9) leads to

$$x(\overline{r+1}\,\tau) = x(o) \prod_{s=0}^{r} \left[1 - \frac{KK_v BA_s^2}{2T} \right] \tag{4.5.10}$$

Thus, in order to investigate the stability of the system we must examine, by letting $r \to \infty$, the convergence of the infinite product

$$\prod_{r=0}^{\infty} \left[1 - aA_r^2 \right] , \tag{4.5.11}$$

where $a = \dfrac{KK_v B}{2T} > 0$. By considering a large number of terms and their distribution and by considering the logarithm of the product of these terms we are led to consider the integral

$$I = \int_0^\infty \log|1 - ay^2|\ p(y)dy \tag{4.5.12}$$

If this integral is positive, the infinite product diverges and the system is unstable; if this integral is negative, the infinite produce 'diverges to zero' [79], $x(r\tau) \to 0$ as $r \to \infty$ and the system is stable.

If y has a Gaussian distribution with zero mean and variance σ^2 the integrand of integral (4.5.12) takes the form sketched in fig. (4.5.1). It has not been found possible, in this particular case, to evaluate the integral in closed form so that the integration has to be carried out numerically. In order to take account of the singularity at $y = 1/\sqrt{a}$ the numerical integration was crosschecked using both the trapezioidal rule and Simpson's rule and a critical value of $a\sigma^2$ found for which the positive and negative areas in fig. (4.5.1) balance. This yields the stability criterion

$$a\sigma^2 < 2.5 \tag{4.5.13}$$

For y having a uniform distribution between $y = \pm b$, the integral of equation (4.5.12) can be evaluated in the closed form

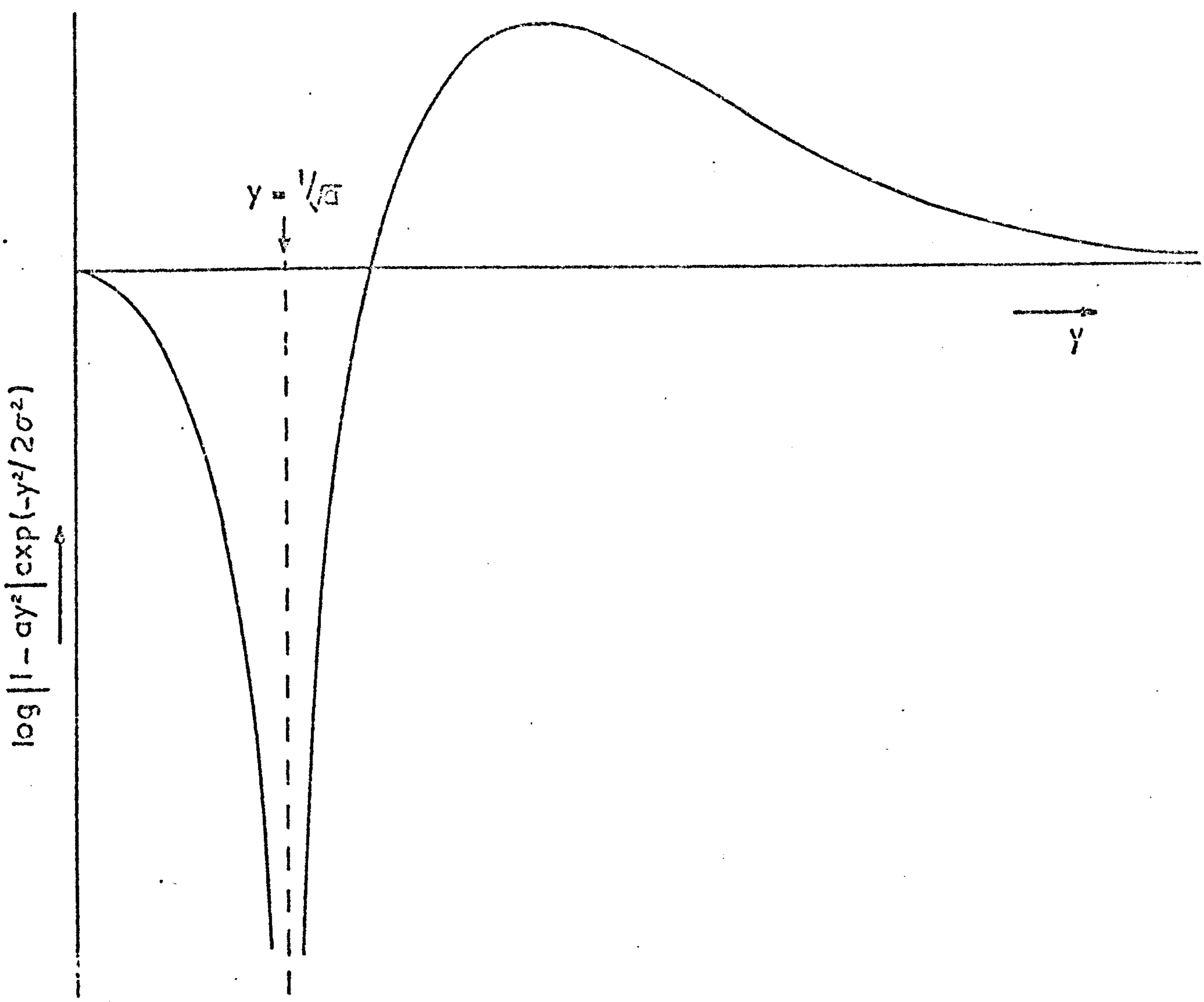

FIG. 4.5.1. Form of integrand for Gaussian distribution.

$$I = \frac{1}{b\sqrt{a}} \{ b\sqrt{a} \log|ab^2 - 1| - 2b\sqrt{a} + \log \left| \frac{b\sqrt{a} + 1}{b\sqrt{a} - 1} \right| \}$$

Equating I to zero yields the stability criterion

$$\begin{aligned} & ab^2 < 6.25 \\ \text{or} \quad & a\sigma^2 < 2.08 \end{aligned} \qquad (4.5.14)$$

since, in this case

$$a^2 = b^2/3$$

Results (4.5.13) and (4.5.14) have been confirmed by direct digital simulation of the infinite product (4.5.11) using appropriate random-number generation procedures (these are discussed in appendix (4.4)).

If one attempts to develop the above approach a step further and allow the effects of the impulses to overlap (that is, the r^{th} impulse arrives before the transient effects of the previous impulses have died out so that we do not make the assumption $\tau >> T$), then, by solving equations (4.5.1) - (4.5.3) during the time interval $r\tau < t < (r + 1)\tau$ we arrive at the following matrix recurrence relationship.

$$\begin{bmatrix} \overline{e(r + 1\tau)} \\ \\ \overline{x(r + 1\tau)} \end{bmatrix} = \begin{bmatrix} e^{-\tau/T} & \frac{A_r}{T} e^{-\tau/T} \\ -\frac{K_v BT}{2} \left(\frac{KA_r}{T} + \theta_m(r\tau)\right)(1 - \alpha) & \left[1 - \frac{K_v BA_r}{T} \left(\frac{KA_r}{T} + \theta_m(r\tau)\right)\right](1-\alpha) \end{bmatrix} \begin{bmatrix} e(r\tau) \\ \\ x(r\tau) \end{bmatrix} \qquad (4.5.15)$$

where $\alpha = \exp\{-2(t - \tau r)/T\}$ and $\theta_m(r\tau)$ is given (by solving equation (4.5.1) by

$$\theta_m(r\tau) = \frac{K}{T} \sum_{j=0}^{r} A_{r-j} \exp\{-(j + 1)\tau/T\}.$$

If we assume that when the r^{th} impulse arrives the transient effects of all the previous impulses, apart from that of the $(r - 1)^{th}$ impulse, have died out (that is, neglect terms in $\exp(-\beta\tau/T)$, $\beta \geqslant 2$ in the recurrence relationship (4.5.15)) then the question of stability, reduces to one of examining the convergence of an infinite product of matrices and to date no progress has been made with this problem.

Although it has not been possible to obtain theoretical stability boundaries for the system of fig. (4.2.1) when the input is purely random digital simulation of the system for such inputs suggests that such boundaries exist. It is clear therefore that a search for theoretical methods of investigating the stability of a linear system of equations of the form $\dot{\underline{x}} = \underline{A}(t)\underline{x}$, where the time varying elements of $\underline{A}(t)$ are stochastic processes and the system is such that it is not asymptotically stable when the stochastic terms are equated to zero, is a field for further research. A more detailed discussion on the digital simulation of the system is given in section 4.7.

4.6. Step plus random input

In order to avoid the theoretical difficulties involved when the input is purely random we shall take the input $\theta_i(t)$ as consisting of a constant step R together with a random variable $\alpha(t)$; that is

$$\theta_i(t) = R + \alpha(t) \tag{4.6.1}$$

This assures, from section 4.3., that when the random term $\alpha(t)$ is identically zero the system is asymptotically stable for all values of the system parameters. It follows that, in this case, the theoretical results discussed in chapter 2 are applicable to investigating the system stability.

When $\theta_i(t)$ is defined as in equation (4.6.1) the system equations (4.2.4) become

$$\begin{aligned} T\dot{e}(t) + e(t) &= x(t)\ (R + \alpha(t)) \\ \dot{x}(t) &= -K_vBe(t)\theta_m(t) \\ T\dot{\theta}_m(t) + \theta_m(t) &= K\ (R + \alpha(t)) \end{aligned} \tag{4.6.2}$$

Assuming, as before, that the adaptation is switched on when the model response $\theta_m(t)$ has reached its steady state value $\theta_{ms}(t)$ equations (4.6.2) become

$$\begin{aligned} T\dot{e}(t) + e(t) &= x(t)\ (R + \alpha(t)) \\ \dot{x}(t) &= -K_vBe(t)\theta_{ms}(t) \end{aligned} \tag{4.6.3}$$

Since the model is linear we have, by the principle of superposition, that

$$\theta_{ms}(t) = KR + \beta(t) \tag{4.6.4}$$

where KR and $\beta(t)$ are the steady state response of the model to the step and random term respectively.

Substituting equation (4.6.4) in equation (4.6.3) gives

$$\frac{d}{dt}\begin{bmatrix} e(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & \frac{R}{T} \\ -K_vBKR & 0 \end{bmatrix}\begin{bmatrix} e(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{\alpha(t)}{T} \\ -K_vB\beta(t) & 0 \end{bmatrix}\begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \tag{4.6.5}$$

Equation (4.6.5) is of the form

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{F}(t)\ \underline{x}(t)\ ,$$

where $\underline{A}$ is a constant matrix such that the system $\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t)$ is asymptotically stable for all parameter values and $\underline{F}(t)$ is a matrix whose non identically zero elements are stochastic processes. The stability of such a system was discussed in some detail in chapter 2.

4.6.1. $\alpha(t)$ Gaussian white noise

If the input $\alpha(t)$ is taken to be a gaussian white noise process, with statistics

$$\begin{aligned} < \alpha(t) > &= 0 \\ < \alpha(t)\alpha(t+\tau) > &= 2D\delta(\tau) \end{aligned} \qquad (4.6.6)$$

then the filtered noise $\beta(t)$ will be gaussian but no longer white. It follows [35] that the process $\{e(t)\ x(t)\}$, defined by equations (4.6.5), is not a Markov process. If we introduce a third variable $x_3(t)$, defined by $x_3(t) = \beta(t)$, then the system equations may be written

$$\begin{aligned} \dot{x}_1(t) &= -\frac{1}{T}\,x_1(t) + \left[\frac{R}{T} + \frac{\alpha(t)}{T}\right] x_2(t) \\ \dot{x}_2(t) &= -BKK_vRx_1(t) - BK_vx_1(t)\,x_3(t) \\ \dot{x}_3(t) &= -\frac{1}{T}\,x_3(t) + \frac{K}{T}\,\alpha(t) \end{aligned} \qquad (4.6.7)$$

where $x_1(t) = e(t)$ and $x_2(t) = x(t) = K - K_vK_c(t)$. The process $\{x_1(t)\ x_2(t)\ x_3(t)\}$, defined by equations (4.6.7), will now be Markov (Note that the effect of making the process Markov is to make the system equations nonlinear) so that its conditional probability function p satisfies the Fokker-Planck equation.

Using the theory developed in section (2.3) we have that the Fokker-Planck equation corresponding to the system of equations (4.6.7) is (see appendix 4.4)

$$\begin{aligned} \frac{\partial p}{\partial t} = {} & \frac{1}{T}\frac{\partial}{\partial x_1}(x_1p) - \frac{R}{T}x_2\frac{\partial p}{\partial x_1} + (BRKK_vx_1 + BK_vx_1x_3)\frac{\partial p}{\partial x_2} + \frac{1}{T}\frac{\partial}{\partial x_3}(x_3p) \\ & + \frac{Dx_2^{\,2}}{T}\frac{\partial^2 p}{\partial x_1^{\,2}} + \frac{2DK}{T^2}x_2\frac{\partial^2 p}{\partial x_1\partial x_3} + \frac{K^2D}{T^2}\frac{\partial^2 p}{\partial x_3^{\,2}} \end{aligned} \qquad (4.6.8)$$

The first order moment equations are :

$$\begin{aligned} \dot{m}_{1,0,0} &= -\frac{1}{T}m_{1,0,0} + \frac{R}{T}m_{0,1,0} \\ \dot{m}_{0,1,0} &= -BRKK_v\,m_{1,0,0} - BK_v\,m_{1,0,1} \\ \dot{m}_{0,0,1} &= -\frac{1}{T}\,m_{0,0,1} \end{aligned} \qquad (4.6.9a)$$

and the second order moment equations are :

$$\dot{m}_{2,0,0} = -\frac{2}{T} m_{2,0,0} + \frac{2R}{T} m_{1,1,0} + \frac{2D}{T^2} m_{0,2,0}$$

$$\dot{m}_{1,1,0} = -\frac{1}{T} m_{1,1,0} + \frac{R}{T} m_{0,2,0} - BRKK_V m_{2,0,0} - BK_V m_{2,0,1}$$

$$\dot{m}_{0,2,0} = -2BRKK_V m_{1,1,0} - 2BK_V m_{1,1,1}$$

$$\dot{m}_{1,0,1} = -\frac{2}{T} m_{1,0,1} + \frac{R}{T} m_{0,1,1} + \frac{2DK}{T^2} m_{0,1,0}$$

$$\dot{m}_{0,1,1} = -BRKK_V m_{1,0,1} - \frac{1}{T} m_{0,1,1} - BK_V m_{1,0,2}$$

$$\dot{m}_{0,0,2} = -\frac{2}{T} m_{0,0,2} + \frac{2K^2D}{T^2} \qquad (4.6.9b)$$

and the general order moment equations may be obtained from the relationship

$$\dot{m}_{K_1,K_2,K_3} = -\frac{K_1}{T} m_{K_1,K_2,K_3} + \frac{R}{T} K_1 m_{K_1-1,K_2+1,K_3} - BRKK_V K_2 m_{K_1+1,K_2-1,K_3}$$
$$- BK_V K_2 m_{K_1+1,K_2-1,K_3+1} - \frac{K_3}{T} m_{K_1,K_2,K_3} + \frac{D}{T^2} K_1(K_1-1) m_{K_1-2,K_2+2,K_3}$$
$$+ \frac{DK^2}{T^2} K_3(K_3-1) m_{K_1,K_2,K_3-2} - \frac{2DK}{T^2} K_1K_3 m_{K_1-1,\, K_2+1,\, K_3-1} \qquad (4.6.9c)$$

We easily find from the equations for $m_{0,0,K}$ that $x_3(t)$ is Gaussian with steady state mean and variance equal to zero and $\frac{KD}{T}$ respectively. However, all the remaining moment equations in (4.6.8) are interrelated; the first order moments depend upon the second; the second order moments depend upon the first and third order moments; the third order moments depend upon the first, second and fourth; etc.

Due to the inter-dependence of the moment equations it is difficult to see how, in this case, it is possible to speak of *mean* *square stability* since this would imply that only second order moments are considered in the determination of stability. This could only be achieved if moments higher than the second are neglected but, unfortunately, no justification can be given for doing this. It appears that, in this case one must examine all the moment equations in order to answer the stability problem; that is, the stability problem can only be answered by examining the behaviour, regarding stability, of an infinite system of first order differential equations (i.e. the moment equations of all order) with constant coefficients [80].

It follows, from the above, that the stability problem is one of examining the roots of an infinite determinant. Although some such determinants have been examined successfully [4] by the use of recurrence relationships no progress has been made, to date, with examining the determinant representing the infinite system of equations (4.6.9).

4.6.2. α(t) Gaussian

If $\alpha(t)$ is an ergodic Gaussian process then it follows that $\beta(t)$, which is obtained by passing $\alpha(t)$ through a linear filter, is also an ergodic Gaussian process. Thus, in this case, we can apply the theory discussed in section (2.7) to obtain sufficient conditions for almost sure asymptotic stability of the system defined by equation (4.6.5) which, for convenience, is rewritten in the form

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & -\frac{R}{T} \\ BKK_VR & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \alpha(t)\begin{bmatrix} 0 & -\frac{1}{T} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \beta(t)\begin{bmatrix} 0 & 0 \\ BK_V & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.6.10}$$

where $x_1(t) = e(t)$ and $x_2(t) = K_VK_c(t) - K$.

Introducing the non-dimensional parameters

$$\Pi_2 = BKK_VR^2T \;,\; \Pi_3 = {}^{\sigma_i}/_R \;\text{ and }\; \Pi_4 = {}^{\sigma_o}/_{KR} \;, \tag{4.6.11}$$

where $\sigma_i^{\,2}$ and $\sigma_0^{\,2}$ are the variances of $\alpha(t)$ and $\beta(t)$ respectively, and the dimensionless variables

$$\xi_1 = x_1/(KR) \ , \ \xi_2 = x_2/K \ , \ \tau = {}^{t}/_{T} \tag{4.6.12}$$

then equation (4.6.10) may be written in the non-dimensional form

$$\frac{d}{d\tau}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} = \begin{bmatrix}-1 & -1\\ \Pi_2 & 0\end{bmatrix}\begin{vmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{vmatrix} + \alpha_1(\tau)\begin{bmatrix}0 & -\Pi_3\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} + \beta_1(\tau)\begin{bmatrix}0 & 0\\ \Pi_2\Pi_4 & 0\end{bmatrix}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix}$$

$$= \underline{A}\ \underline{\xi}\ (\tau) + \alpha_1(\tau)\ \underline{G}_1\ \underline{\xi}(\tau) \ + \ \beta_1(\tau)\ \underline{G}_2\ \underline{\xi}(\tau) \tag{4.6.13}$$

where $\underline{\xi}(\tau) = [\ \xi_1(\tau)\ \xi_2(\tau)\ ]^{T}$ and $\alpha_1(\tau)$, $\beta_1(\tau)$ are the normalized form of $\alpha(\tau)$ and $\beta(\tau)$ respectively (that is, both $\alpha_1(\tau)$ and $\beta_1(\tau)$ have unit variance).

Equation (4.6.13) is of the form

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + (\ \sum_i f_i(t)\ \underline{G}_i)\ \underline{x}(t) \ ,$$

a type of equation that was discussed in detail in section (2.7). It can be shown that the stability boundaries obtained using the criteria developed by Infante [55] and Man [56] are much sharper than those obtained using the other criteria cited in section (2.7). Thus, we shall confine ourselves to obtaining stability boundaries, for equation (4.6.13), using only these two criteria. However, before doing so we shall first obtain a relationship between Π_3 and Π_4.

$\beta(t)$ is the steady-state response of the model to the random input $\alpha(t)$. Thus, if $G_i(\omega)$ and $G_0(\omega)$ are the power spectral densities of $\alpha(t)$ and $\beta(t)$ respectively, then it is well known that

$$G_0(\omega) = |\ F(j\omega)\ |^2\ G_i(\omega) \ , \tag{4.6.14}$$

where $F(s)$ is the transfer function of the model. Since, in this case,

$$F(s) = \frac{K}{1 + Ts}$$

it follows from equation (4.6.11) that

$$G_o(\omega) = \frac{K^2}{1 + \omega^2 T^2} \; G_i(\omega) \qquad (4.6.15)$$

Since, in the digital simulation of the system, the numerical procedure commits us to a small step length h we shall assume that $G_i(\omega)$ is given by equation (A4.6.6), namely

$$G_i(\omega) = \frac{\sigma_N^2 h}{\pi},$$

where σ_N and h are defined as in appendix (4.6). Thus, from equation (4.6.12)

$$G_o(\omega) = \frac{\sigma_N^2 h K^2}{\pi(1 + \omega^2 T^2)}$$

Since, in our notation, $G_o(\omega)$ is measured in power/radians/sec and defined for positive frequencies only, it follows from equation (A4.6.3) that the variance σ_o^2, of the output signal $\beta(t)$, is given by

$$\sigma_o^2 = \int_0^\infty G_o(\omega)\,d\omega$$

i.e.

$$\sigma_o^2 = \frac{\sigma_N^2 h K^2}{2T} \qquad (4.6.16)$$

Also, if σ_i^2 is the variance of the input signal $\alpha(t)$ then, from equation (A4.6.3)

$$\sigma_i^2 = \frac{2}{3}\sigma_N^2 \qquad (4.6.17)$$

Introducing the non-dimensional parameters

$$\Pi_5 = \frac{\sigma_N}{R} \quad \text{and} \quad \Pi_6 = \frac{h}{T} \qquad (4.6.18)$$

we have, from equations (4.6.16) and (4.6.17), that

$$\Pi_3^{\,2} = \frac{2}{3}\,\Pi_5^{\,2}$$

$$\text{and} \quad \Pi_4^{\,2} = \frac{1}{2}\,\Pi_6\Pi_5^{\,2} = \frac{3}{4}\,\Pi_6\Pi_3^{\,2} \qquad (4.6.19)$$

4.6.2.1 Infante type analysis

In Infante's work, see section (2.7), consider for the symmetric positive definite matrix $\underline{B}$ the most general form

$$\underline{B} = \begin{bmatrix} a_1^2 + a_2 & a_1 \\ a_1 & 1 \end{bmatrix}, \quad a_2 > 0 \qquad (4.6.20)$$

Simple computation yields :

$$\underline{B}^{-1} = \frac{1}{a_2}\begin{bmatrix} 1 & -a_1 \\ -a_1 & a_1^2 + a_2 \end{bmatrix}$$

Taking the matrices $\underline{G}_1$ and $\underline{G}_2$, as defined in equation (4.6.13), we have that

$$\text{(i)} \quad \underline{G}_1^{\,1} + \underline{B}\underline{G}_1\underline{B}^{-1} = \frac{\Pi_3}{a_2}\begin{bmatrix} a_1(a_1^2 + a_2) & -(a_1^2 + a_2)^2 \\ (a_1^2 - a_2) & -a_1(a_1^2 + a_2) \end{bmatrix}$$

giving

$$\lambda_{max}(\underline{G}_1^{\,1} + \underline{B}\,\underline{G}_1\,\underline{B}^{-1}) - \lambda_{min}(\underline{G}_1^{\,1} + \underline{B}\,\underline{G}_1\,\underline{B}^{-1}) = \frac{2\Pi_3(a_1^2 + a_2)}{\sqrt{a_2}} \qquad (4.6.21)$$

and

$$\text{(ii)} \quad \underline{G}_2^{\,1} + \underline{B}\,\underline{G}_2\,\underline{B}^{-1} = \frac{\Pi_2\Pi_4}{a_2}\begin{bmatrix} a_1 & a_2 - a_1^2 \\ 1 & -a_1 \end{bmatrix}$$

giving

$$\lambda_{max}(\underline{G}_2^{\,1} + \underline{B}\,\underline{G}_2\,\underline{B}^{-1}) - \lambda_{min}(\underline{G}_2^{\,1} + \underline{B}\,\underline{G}_2\,\underline{B}^{-1}) = \frac{2\Pi_2\Pi_4}{\sqrt{a_2}} \qquad (4.6.22)$$

Taking the matrix $\underline{A}$, as defined in equation (4.6.13), and computing the matrix $\underline{A}^1 + \underline{B}\ \underline{A}\ \underline{B}^{-1}$ then further computation yields :

$$\lambda_{max}(\underline{A}^1 + \underline{B}\ \underline{A}\ \underline{B}^{-1}) = -1 + \{(1 - 2a_1)^2 + \frac{1}{a_2}\left[(a_1^{\ 2} + a_2) - \Pi_2 + a_1(1 - 2a_1)\right]^2\}^{\frac{1}{2}} \tag{4.6.23}$$

Using results (4.6.21) - (4.6.23) it follows from the stability criterion (2.7.17) that the system, represented by equation (4.6.13), is almost surely asymptotically stable in the large if

$$\frac{1}{2}\left[<|\alpha_1(\tau)|> \frac{2\Pi_3(a_1^{\ 2} + a_2)}{\sqrt{a_2}} + <|\beta_1(\tau)|> \frac{2\Pi_2\Pi_4}{\sqrt{a_2}}\right] < -\lambda_{max}(\underline{A}^1 + \underline{B}\ \underline{A}\ \underline{B}^{-1}) \tag{4.6.24}$$

where $\lambda_{max}(\underline{A}^1 + \underline{B}\ \underline{A}\ \underline{B}^{-1})$ is given by equation (4.6.23).

If $\gamma(t)$ is a Gaussian random signal with variance σ^2 then

$$<|\gamma(t)|> = \sqrt{\frac{2}{\pi}}\ \sigma$$

Thus, since $\alpha_1(\tau)$ and $\beta_1(\tau)$ have unit variance, criterion (4.6.24) becomes

$$\Pi_3\sqrt{\frac{2}{\pi a_2}}\left[(a_1^{\ 2} + a_2) + \Pi_2(\tfrac{3}{4}\Pi_6)^{\frac{1}{2}}\right] < 1 - \{(1 - 2a_1)^2 + \frac{1}{a_2}\left[(a_1^{\ 2} + a_2) - \Pi_2 + a_1(1 - 2a_1)\right]^2\}^{\frac{1}{2}} \tag{4.6.25}$$

Optimum values of a_1 and a_2 are now obtained by finding the condition for Π_3, regarded as a function of the two variables a_1 and a_2, to be a maximum. Such values are found to be :

$$a_1 = \frac{1}{2}\ , \quad a_2 = \frac{4\,\Pi_2 - 1}{4} \tag{4.6.26}$$

Using values of a_1 and a_2 given by equations (4.6.26) in (4.6.25) gives that the system is almost surely asymptotically stable if

$$\Pi_3^{\ 2} < \frac{\pi}{8} \cdot \frac{4\,\Pi_2 - 1}{\Pi_2^{\ 2}\left[1 + \sqrt{.75\Pi_6}\right]^2}$$

or using equations (4.6.19),

$$\Pi_5^{\,2} \quad < \quad \frac{3\pi}{16\,[1 + \sqrt{.75\Pi_6}]^2} \cdot \frac{(4\,\Pi_2 - 1)}{\Pi_2^{\,2}} \qquad (4.6.27)$$

4.6.2.2. Man type analysis

In Man's work, see section (2.7), take for $\underline{Q}$ the identity matrix $\underline{I}$; solving the Liapunov matrix equation

$$\underline{A}^1\underline{P} + \underline{P}\,\underline{A} = -\underline{Q} \quad , \qquad (4.6.28)$$

where $\underline{A}$ is given by equation (4.6.13), then gives the matrix $\underline{P}$ in the form

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \frac{(1 + \Pi_2)}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{(2 + \Pi_2)}{2\Pi_2} \end{bmatrix}$$

Taking the matrices $\underline{G}_1$ and $\underline{G}_2$, as defined in equations (4.6.13), simple computation yields :

$$\text{(i)}\ \underline{G}_1^{\,1}\underline{P} + \underline{P}\,\underline{G}_1 = \Pi_3 \begin{bmatrix} 0 & -p_{11} \\ -p_{11} & -2p_{12} \end{bmatrix}$$

giving

$$\lambda_{max}(\underline{G}_1^{\,1}\underline{P} + \underline{P}\,\underline{G}_1) = \frac{\Pi_3}{2}\,[-1 + \sqrt{1 + (1 + \Pi_2)^2}\,] \qquad (4.6.29)$$

and $\quad \text{(ii)}\ \underline{G}_2^{\,1}\underline{P} + \underline{P}\,\underline{G}_2 = \Pi_2\Pi_4 \begin{bmatrix} 2p_{12} & p_{22} \\ p_{22} & 0 \end{bmatrix}$

giving

$$\lambda_{max}(\underline{G}_2^{\,1}\underline{P} + \underline{P}\,\underline{G}_2) = \frac{\Pi_4}{2}\,[\,\Pi_2 + \sqrt{\Pi_2^{\,2} + (2 + \Pi_2)^2}\,] \qquad (4.6.30)$$

Using results (4.6.29) and (4.6.30) it follows from the stability criterion (2.7.24) that the system, represented by equation (4.6.13), is almost surely asymptotically stable if.

$$<|\alpha_1(\tau)|> \cdot \frac{\Pi_3}{2}\left[-1 + \sqrt{1 + (1 + \Pi_2)^2}\right] + <|\beta_1(\tau)|> \frac{\Pi_4}{2}\left[\Pi_2 + \sqrt{\Pi_2^{\,2} + (2 + \Pi_2)^2}\right] < 1 \tag{4.6.31}$$

Again, since $\alpha_1(\tau)$ and $\beta_1(\tau)$ are Gaussian with unit variance,

$$<|\alpha_1(\tau)|> \;=\; <|\beta_1(\tau)|> \;= \sqrt{\frac{2}{\pi}};$$

so that criterion (4.6.31) becomes, on using equations (4.6.19),

$$\Pi_5 < \frac{\sqrt{3\pi}}{-1 + \sqrt{1 + (1 + \Pi_2)^2} + \sqrt{.75\Pi_6}\left[\Pi_2 + \sqrt{\Pi_2^{\,2} + (2 + \Pi_2)^2}\right]} \tag{4.6.32}$$

Note that in this analysis the matrix $\underline{Q}$ was taken to be the identity matrix. One would expect to obtain a sharper criterion if $\underline{Q}$ were taken to be the most general symmetric positive definite form $\underline{B}$, defined by equation (4.6.20), and then optimum values of a_1 and a_2 obtained in order to maximize Π_5. However, taking $\underline{Q}$ to be such a matrix it is readily shown that no such optimum values exist for a_1 and a_2.

4.7. Digital Simulation

The random variables are provided by a subroutine using a procedure for generating pseudo random numbers (see appendix 4.5). Sequences of numbers generated by this numerical procedure are assumed to have the same statistical properties as sequences of independent random numbers although they are in fact completely determined by the first number in the sequence. This means that the sequence is regenerable which is an essential requirement for program development and comparison of different numerical methods for solving the problem at hand. The occurrence of identical sequences can be avoided by taking a different first number for each sequence that is generated.

The basic numerical procedure generates a sequence of numbers uniformly distributed over the finite range (0,1). From a table of values for erf(x) this sequence of numbers is transferred into a sequence of normally distributed random numbers (0,1) which are then mapped onto (μ, σ_N). Thus, the complete subroutine produces a sequence of random numbers which are normally distributed with mean μ and variance σ_N.

By spacing these gaussian distributed numbers at intervals h and joining them by straight lines, a segmentally linear function is obtained that approximates white noise; the degree of approximation being dependent on the spacing h. It can be shown from the auto-correlation function (see appendix 4.6) that, for small values of ωh, the power spectral density of the function is given approximately by

$$G(\omega) = \frac{\sigma_N^2 h}{\pi} \left[1 - \frac{(\omega h)^2}{6} \right] \qquad (4.7.1)$$

White noise is defined as a stationary random signal having a gaussian probability amplitude distribution and a spectral density which is constant for all frequencies. As was pointed out in section (2.2) true white noise is physically impossible to produce but what is produceable is 'band-limited' or physical white noise; that is, a function whose spectral density is essentially constant over the range of interest but falls to zero as $\omega \to \infty$. Thus, by making h sufficiently small, it follows, from equation (4.7.1), that the function produced by the numerical procedure outlined above approximates white noise $\alpha(t)$ having

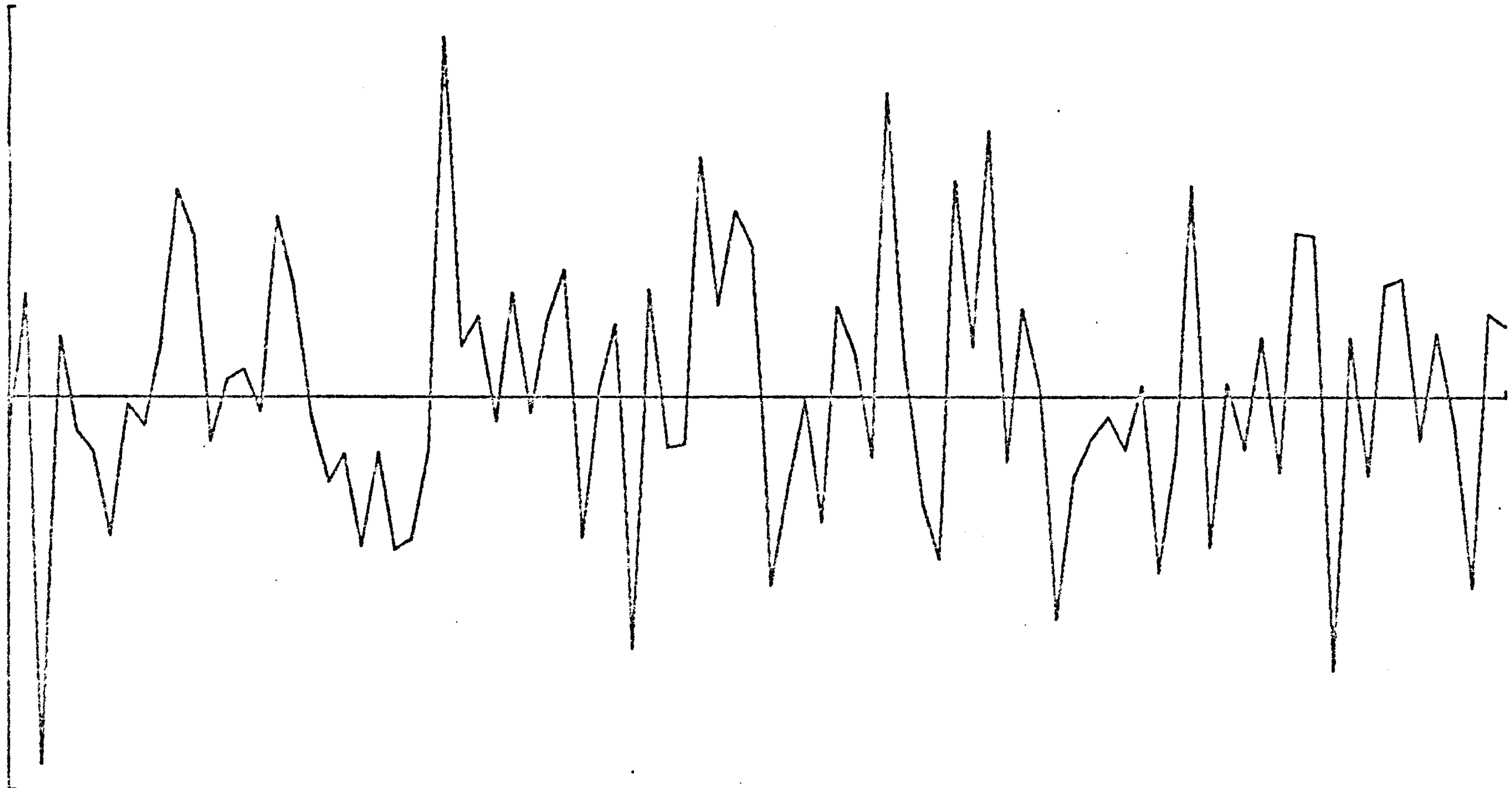

FIG. 4.7.1. Sample function of random process.

constant spectral density $\frac{\sigma_N^2 h}{\pi}$ and time domain statistical properties

$$\begin{aligned} <\alpha(t)> &= 0 \\ <\alpha(t)\alpha(t+\tau)> &= \sigma_N^2 h\delta(\tau) \end{aligned} \qquad (4.7.2)$$

A sample function of the process α (t) is shown in fig. (4.7.1).

Using the above numerical subroutine to generate the random variable $\alpha(t)$ the system of equations (4.6.2), representing the M.I.T. system of fig. (4.2.1), are simulated on a digital computer. Two methods of numerical integration are employed, using in turn

(i) the Runge-Kutta procedure

and (ii) the Crank-Nicolson procedure

The Runge-Kutta procedure for solving a system of first order differential equations is given in section (1.3.1); whilst the Crank Nicolson procedure may be stated as follows [81] :

For the system of first order differential equations

$$\dot{x}_i = f_i(x_1, x_2, ---, x_n, t) = f_i(\underline{x}, t), \; i = 1,2, --- n,$$

the following finite difference relationship is employed

$$x_i(\overline{m+1}\,\Delta t) = x_i(m\,\Delta t) + K_i$$

where

$$K_i = \Delta t\Big[f_i(\underline{x}, t) + \frac{1}{2}\sum_{j=1}^{n} K_j \frac{\partial f_i}{\partial x_j}(\underline{x},t)\Big] \qquad (4.7.3)$$

In order to obtain the variations of the ensemble moments with time the system of equations (4.6.2) are solved m times simultaneously, with each set having a different $\alpha(t)$. The mean square values, over the m outputs, of both e(t) and $K - K_v K_c(t)$ are then printed out at each instant of time and a decision taken as to whether the system is stable or unstable in the mean square.

4.7.1. Step Plus random input

With parameter values R = 5, K = 1, T = 0.25, K_v = 0.5, initial K_c = 0.5, h = 0.01 and m = 3 the system of equations (4.6.2) are solved for various values of σ_N. For each σ_N the equations are solved for a range of values of the gain B until a value is obtained for which the system is unstable. Regions of stability in the Π_5-Π_2 plane are shown in fig. (4.7.2) for both the Runge-Kutta and Crank-Nicolson procedures. In both cases the graphs indicate the lowest value of Π_2 (for each Π_5) for which the system is unstable. Numerical results indicate that there are other narrow regions of stability above these boundaries (possibley similar to those shown in fig. (4.4.1) for sinusoidal input). Detailed calculation of these narrow regions of stability would require a great deal of computer time and since also they are of no practical interest the exercise has not been carried out.

Of the two numerical procedures it is felt that the Crank-Nicolson procedure is more suited, than the Runge-Kutta procedure, for simulating random processes since it is less effected by the discontinuous nature of the signal $\alpha(t)$ at each sampling point.

Also shown in fig. (4.7.2) is the theoretical stability boundary represented by criterion (4.6.27); the boundary represented by criterion (4.6.32) is slightly inferior to this one and is therefore not included in the figure. It is seen that the theoretical results are very conservative; this however is not totally unexpected as both criteria (4.6.27) and (4.6.32) are only sufficient, but not necessary, conditions for almost surely asymptotic stability; further, they are both obtained using the second method of Liapunov which is known, even for deterministic systems, to give rather conservative stability boundaries.

4.7.2. Randon input

Although we have been unable to obtain theoretical results when the input is purely random nevertheless the system was simulated digitally under such conditions. For parameter values R = 0, K = 1, T = 0.25, K_v = 0.5, initial K_c = 0.5, h = 0.01 and m = 3 the stability boundaries, obtained using both the Runge-Kutta and Crank-Nicolson procedures, in the Π_7-Π_8 plane, where Π_7 and Π_8 are the non-dimensional parameters

$$\Pi_8 = BKK_v^{\,3}T \quad , \quad \Pi_7 = \sigma_N^{\,2}/K_v^{\,2} \qquad (4.7.4)$$

are shown in fig. (4.7.3). Also plotted in fig (4.7.3) is the rectangular hyperbola $\Pi_7\Pi_8$ = 175 and approximately this represents the mean of the two boundaries obtained using the numerical procedures. Thus a form of stability criteria one would like to obtain theoretically is

$$\Pi_7\Pi_8 < 175 \qquad (4.7.5)$$

Substituting for Π_7 and Π_8 and using equation (A4.6.6) condition (4.7.5) suggests, on proceeding to the limit $h \to 0$, that the system is unstable, for all values of the system parameters, when the input is pure white noise.

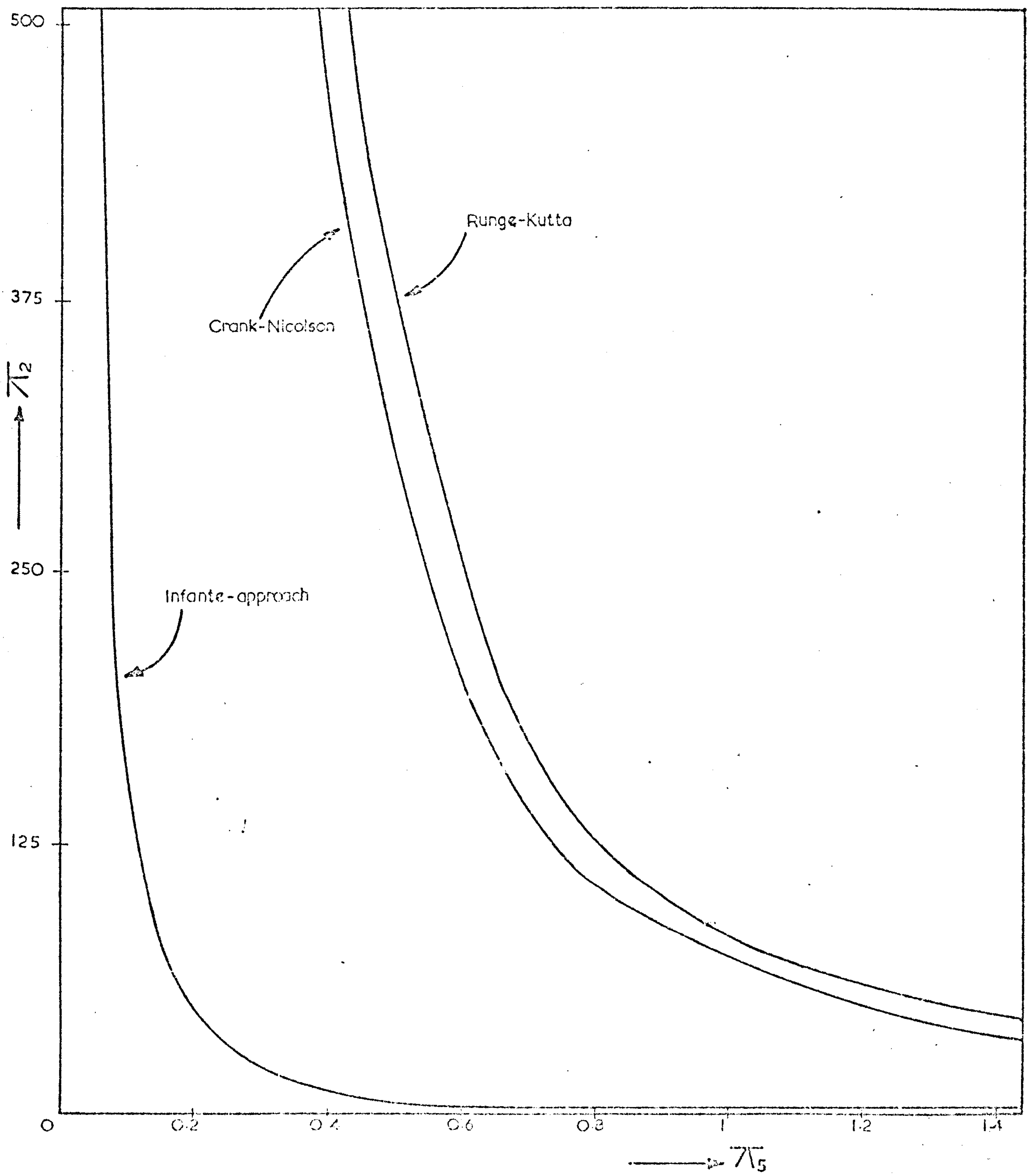

FIG. 4.7.2. Stability boundaries for step plus random input.

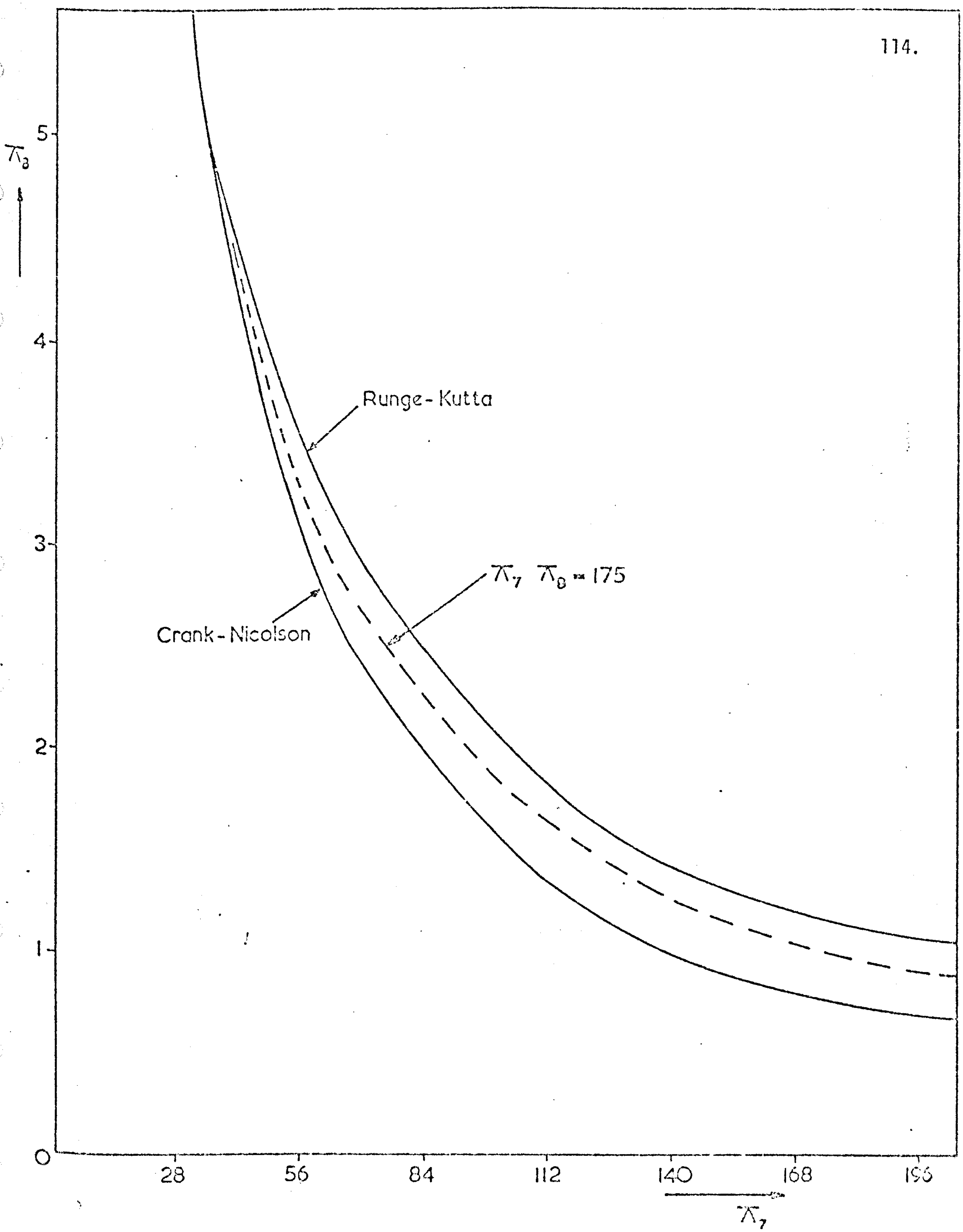

FIG. 4.7.3. Stability boundaries for random input

CHAPTER 5

STABILITY OF A FIRST ORDER MODEL-REFERENCE SYSTEM WHEN ENVIRONMENTAL PARAMETER TIME VARYING

5.1. Introduction

In chapter 4 the stability of a first order model reference adaptive control system, for various inputs, was investigated and throughout the environmental process parameter was assumed to be constant. In practice, however, the adapting parameters may be required to change in response not only to the system input signal but also to changes in the environmental parameters so that it is important that the stability of the system under such conditions be investigated. Very little theoretical work has been done on this problem as most of the literature assumes that the environmental process parameters are constant or only varying slowly with time [82], and then analogue computer studies are made to see how the system behaviour is affected if these variations become rapid. In this chapter a theoretical analysis of the stability and accuracy of the parameter adaptation, for the system illustrated in fig. (4.2.1), is presented for the cases of K_v varying both sinusoidally and randomly with time.

In designing the Liapunov redesign system of fig. (4.2.2.) the environmental parameter K_v is assumed constant and only under such a condition will the system be stable for all inputs $\theta_i(t)$ and all values of the system parameters. If K_v is time varying then it will be shown that the system illustrated in fig. (4.2.2) also gives rise to stability problems.

The chapter is concluded by considering the effect, on the system stability and accuracy of its parameter adaptation, of noise disturbances at the system output.

5.2. $\underline{K_v}$ varying sinusoidally with time

5.2.1. The M.I.T. system

Let the M.I.T. system of fig. (4.7.1) be subjected to a step input of magnitude R at time t = 0, when $\theta_m(t)$ and $\theta_s(t)$ are zero and $K_vK_c \neq K$. Subsequently let K_v vary with time, according to

$$K_v = Z \sin\omega t \; , \tag{5.2.1}$$

and K_c be adjusted according to equation (4.2.3). Considering, as before, that the adaption is switched on after the model response $\theta_m(t)$ has reached its steady state value KR the system equations (4.2.4) become

$$\begin{aligned} T\dot{e} + e &= (K - K_c \, Z \sin\omega t)R \\ \dot{K}_c &= BKRe \end{aligned} \tag{5.2.2}$$

By Buckingham's π theorem the number of non-dimensional parameters that need be considered are two and these are taken as

$$\Pi_1 = \omega T \; , \quad \Pi_9 = BKR^2TZ \tag{5.2.3}$$

If, in addition to (5.2.3), we introduce the dimensionless variables

$$\xi_1 = e/(KR) \; , \quad \xi_2 = ZK_c/K \; , \; \tau = \omega t \tag{5.2.4}$$

then the system equations (5.2.2) may be written in the non-dimensional form

$$\xi_1{}' = -\frac{1}{\Pi_1}\xi_1 - \frac{1}{\Pi_1}\sin\tau\,\xi_2 + \frac{1}{\Pi_1} \tag{5.2.5a}$$

$$\xi_2{}' = \frac{\Pi_9}{\Pi_1}\xi_1 \tag{5.2.5b}$$

where a prime denotes differentiation with respect to τ.

Substituting for ξ_1, from equation (5.2.5b), in equation (5.2.5a) gives

$$\xi_2{}'' + \frac{1}{\Pi_1}\xi_2{}' + \frac{\Pi_9}{\Pi_1{}^2}\sin\tau\,\xi_2 = \frac{\Pi_9}{\Pi_1{}^2} \tag{5.2.6}$$

which is a forced linear differential equation with periodic coefficients.

Equilibrium or *steady state* solutions of equation (5.2.6) may be found by the principle of harmonic balance. To a first approximation it is readily shown that the steady state harmonic solution is

$$\hat{\xi}_2 = \frac{2(1 + \Pi_1^2)}{\Pi_9} + 2\sin\tau + \frac{2}{\Pi_1}\cos\tau \qquad (5.2.7)$$

(Note. It is readily shown that the only subharmonic solutions possible are those of order 2 and that these can only occur on the transition boundary between stable and unstable regions)

The corresponding solution for ξ_1 is found, from equation (5.2.5b) to be

$$\hat{\xi}_1 = \frac{2\Pi_1}{\Pi_9}\cos\tau - \frac{2}{\Pi_9}\sin\tau \qquad (5.2.8)$$

Also,

$$\begin{aligned}(K - K_V K_C) &= (K - K_V Z\sin\omega t)\\ &= K(1 - \xi_2\sin\tau)\\ &= \frac{-2(1 + \Pi_1^2)}{\Pi_9}\sin\tau + \frac{1}{\Pi_1}\sin 2\tau + \cos 2\tau\end{aligned} \qquad (5.2.9)$$

Equations (5.2.8) and (5.2.9) show that the *steady state* solutions for both the error and $K - K_V K_C$ contain no constant or d.c. term; that is, both e(t) and $K - K_V K_C$ adapt to oscillations about zero as required; the amplitude of the oscillations depending on the parameters Π_1 and Π_9.

Having found the equilibrium solution $\hat{\xi}_1$, $\hat{\xi}_2$ the next step is to investigate its stability. Considering small variations η_1, η_2, in ξ_1 and ξ_2 respectively, about the equilibrium state we have, from equation (5.2.6), that

$$\eta_2^{11} + \frac{1}{\Pi_1}\eta_1^{1} + \frac{\Pi_9}{\Pi_1^2}\sin\tau\,\eta_1 = 0$$

or a matrix form

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ -\frac{\Pi_9}{\Pi_1{}^2}\sin\tau & -\frac{1}{\Pi_1} \end{bmatrix}\begin{bmatrix} x_2 \\ \\ x_2 \end{bmatrix} , \qquad (5.2.10)$$

where $x_1 = \eta_2$ and $x_2 = \eta_2{}^1$.

Equation (5.2.10) is of the form $\dot{\underline{x}} = \underline{A}(t)\underline{x}$, $\underline{A}(t) = \underline{A}(t + T)$ and its stability is investigated using the numerical implementation of Floquet theory as described in chapter 1. Examining the eigenvalues of the monodromy matrix at a network of points in parameter space leads to the stability diagram shown in fig. (5.2.1).

5.2.2. The Liapunov redesign system

Applying a step input of magnitude R to the Liapunov redesign system of fig. (4.2.2) and allowing K_v to vary according to equation (5.2.1) the system equations (4.2.5) become

$$\begin{aligned} T\dot{e} + e &= (K - K_v Z\sin\omega t)R \\ \dot{K}_c &= BRe \end{aligned} \qquad (5.2.11)$$

Introducing the non-dimensional parameters

$$\Pi_1 = \omega T \ , \ \Pi_{10} = TZBR^2 \qquad (5.2.12)$$

and the dimensionless variables ξ_1, ξ_2 and τ, as defined in equations (5.2.4), equations (5.2.11) may be written in the non-dimensional form

$$\begin{aligned} \xi_1{}^1 &= -\frac{1}{\Pi_1}\xi_1 - \frac{1}{\Pi_1}\sin\tau\ \xi_2 + \frac{1}{\Pi_1} \\ \xi_2{}^1 &= \frac{\Pi_{10}}{\Pi_1}\ \xi_1 \end{aligned} \qquad (5.2.13)$$

that is, the system equations are the same as for the M.I.T. system, except that Π_9 is replaced by Π_{10}, so that the stability problem is identical. This is as expected for the case of a step input since the

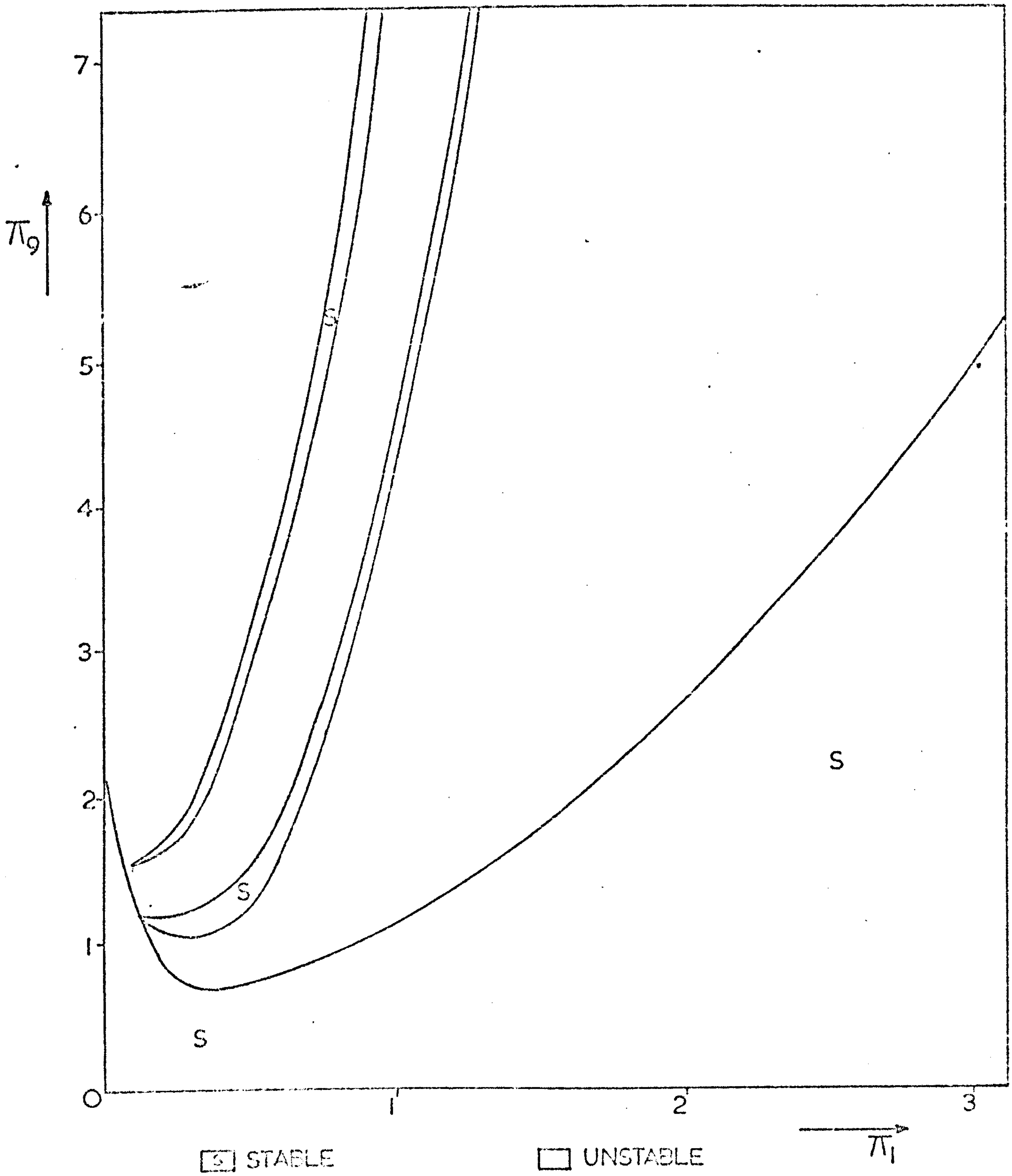

FIG. 5.2.1. Stability boundaries when K_v sinusoidal.

steady state response of the model is given by

$$(\theta_m(t))_{ss} = KR = \frac{\theta_i}{R}$$

and the adaptive loop is essentially the same. If both θ_i and K_v are time varying then the Liapunov redesign system will give rise to a stability problem but it will not be identical to that of the M.I.T. system.

5.2.3. $K_v = S + Z\sin\omega t$

If, instead of regarding K_v as being purely sinusoidal, we take the sinusoidal term as being a disturbance on an initial given value S (constant); that is,

$$K_v(t) = S + Z\sin\omega t \tag{5.2.14}$$

then the M.I.T. system equations become

$$\xi_1{}' = -\frac{1}{\Pi_1}\xi_1 + \frac{1}{\Pi_1} - \frac{\Pi_{11}}{\Pi_1}\xi_2 - \frac{1}{\Pi_1}\sin\tau\,\xi_2 \tag{5.2.15a}$$

$$\xi_2{}' = \frac{\Pi_9}{\Pi_1}\xi_1 \tag{5.2.15b}$$

where Π_1, Π_9, ξ_1, ξ_2 and τ are defined as in equations (5.2.3) and (5.2.4) and

$$\Pi_{11} = \frac{S}{Z} \tag{5.2.16}$$

Substituting for ξ_1 from equation (5.2.15b) in (5.2.15a) gives

$$\xi_2{}'' + \frac{1}{\Pi_1}\xi_2{}' + \left(\frac{\Pi_9\Pi_{11}}{\Pi_1{}^2} + \frac{\Pi_9}{\Pi_1{}^2}\sin\tau\right)\xi_2 = \frac{\Pi_9}{\Pi_1{}^2} \tag{5.2.17}$$

As before, equilibrium solutions, for equation (5.2.17), may be found using the principle of harmonic balance but the stability problem may be solved by considering the unforced differential equation

$$\xi_2{}'' + \frac{1}{\Pi_1}\xi_2{}' + \left(\frac{\Pi_9\Pi_{11}}{\Pi_1{}^2} + \frac{\Pi_9}{\Pi_1{}^2}\sin\tau\right)\xi_2 = 0 \tag{5.2.18}$$

Exact stability boundaries, for equation (5.2.18), in parameter space $\Pi_1 - \Pi_9$, for various values of Π_{11}, may be obtained using the numerical procedure employed for equation (5.2.10). However, approximate stability boundaries may be obtained using other techniques.

Writing $\tau = 2t - \pi/2$ equation (5.2.18) becomes

$$\ddot{\xi}_2 + \frac{2}{\Pi_1}\mathring{\xi}_2 + 4\left[\frac{\Pi_9\Pi_{11}}{\Pi_1^2} - \frac{\Pi_9}{\Pi_1^2}\cos 2t\right]\xi_2 = 0 \qquad (5.2.19)$$

which is of the form of the damped Mathieu equation.

$$\ddot{x} + 2\zeta\dot{x} + (a - 2q\cos 2t)x = 0\ , \qquad (5.2.20)$$

$$\text{with } x = \xi,\ \zeta = \frac{1}{\Pi_1}\ ,\ a = 4\Pi_9\Pi_{11}/\Pi_1^2 \text{ and } q = 2\Pi_9/\Pi_1^2 \qquad (5.2.21)$$

Equation (5.2.20) has been dealt with extensively by various researchers and we shall confine ourselves here to considering some of the sharper stability criteria that have been presented. These criteria are sufficient but not necessary conditions for stability.

(i) By generating a suitable Liapunov function Pritchard[83] obtained the stability criterion

$$(1 + 4\zeta^2)^{\frac{1}{2}}\,|q| < \zeta\, a \qquad (5.2.22)$$

Substituting for ζ, a and q from equations (5.2.21) the stability criterion for equation (5.2.19) becomes.

$$\Pi_{11}^2 > 1 + \frac{\Pi_1^2}{4}\ ,\ \Pi_1 > 0 \qquad (5.2.23)$$

(ii) If in Infante's work, see section (2.7), we take

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -a & -2\zeta \end{bmatrix},\ \underline{F}(t) = f(t)\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},\ \underline{B} = \begin{bmatrix} \alpha_1^2 + \alpha_2 & \alpha_1 \\ \alpha_1 & 1 \end{bmatrix}, \qquad (5.2.24)$$

where $f(t) = -2q\cos 2t$, then condition (2.7.16) for asymptotic stability becomes:

$$E\{-2\zeta + [4(\zeta - \alpha_1)^2 + \frac{1}{\alpha_2}(\alpha_2 + \alpha_1^2 - a - f(t) + 2\alpha_1(\zeta - \alpha_1))^2]^{\frac{1}{2}}\} < 0$$

or

$$E\{f^2(t)\} < \alpha_2[4\zeta^2 - 4(\zeta - \alpha_1)^2] - [\alpha_2 + \alpha_1^2 - a + 2\alpha_1(\zeta - \alpha_1)]^2 \qquad (5.2.25)$$

Optimum values of α_1 and α_2. in order to maximize $E\{f^2(t)\}$, are found to be

$$\alpha_1 = \zeta \ , \ \alpha_2 = \zeta^2 + 1$$

With these values of α_1 and α_2 condition (5.2.25) becomes

$$E\{f^2(t)\} = \frac{4q^2}{2} < 4a\zeta^2$$

$$\text{i.e.} \quad q^2 < 2a\zeta^2 \qquad (5.2.26)$$

Substituting for ζ, a and q from equations (5.2.21) we have that the criterion for equation (5.2.19) to be asymptotically stable is that

$$\Pi_9 < 2\Pi_{11} \qquad (5.2.27)$$

(iii) If in Man's work, see section (2.7), we take the matrices $\underline{A}$ and $\underline{F}(t)$ to be defined as in equations (5.2.24) and take the matrix $\underline{Q}$ to be the identity matrix the condition (2.7.19) for asymptotic stability becomes :

$$E\{f^2(t)\} = 2q^2 < \frac{16a^2\zeta^2}{8\zeta^2 + (1 + a)^2 - 4\zeta\sqrt{4\zeta^2 + (1 + a)^2}}$$

$$\text{or} \quad q^2 < \frac{8a^2\zeta^2}{8\zeta^2 + (1 + a)^2 - 4\zeta\sqrt{4\zeta^2 + (1 + a)^2}} \qquad (5.2.28)$$

Substituting for ζ, a and q from equations (5.2.21) we have that the criteria for equation (5.2.19) to be asymptotically stable is that

$$8\Pi_1^2 + (\Pi_1^2 + 4\Pi_9\Pi_{11})^2 - 4\Pi_1(4\Pi_1^2 + (\Pi_1^2 + 4\Pi_9\Pi_{11})^2)^{\frac{1}{2}} < 32\Pi_{11}^2\Pi_1^2 \qquad (5.2.29)$$

The stability boundaries, in the a-q plane, corresponding to criteria

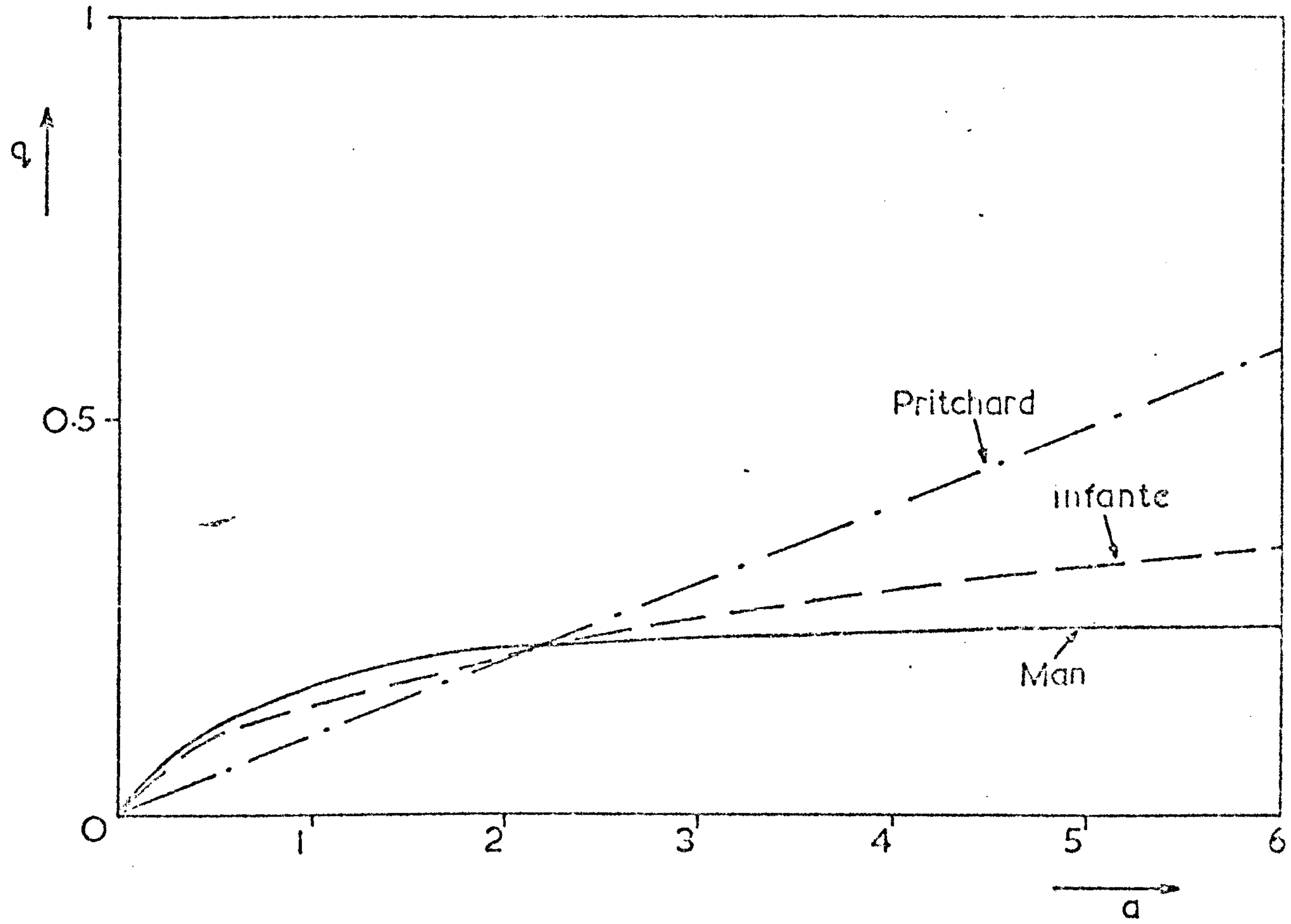

FIG. 5.2.2. Stability boundaries when $\zeta = 0.1$

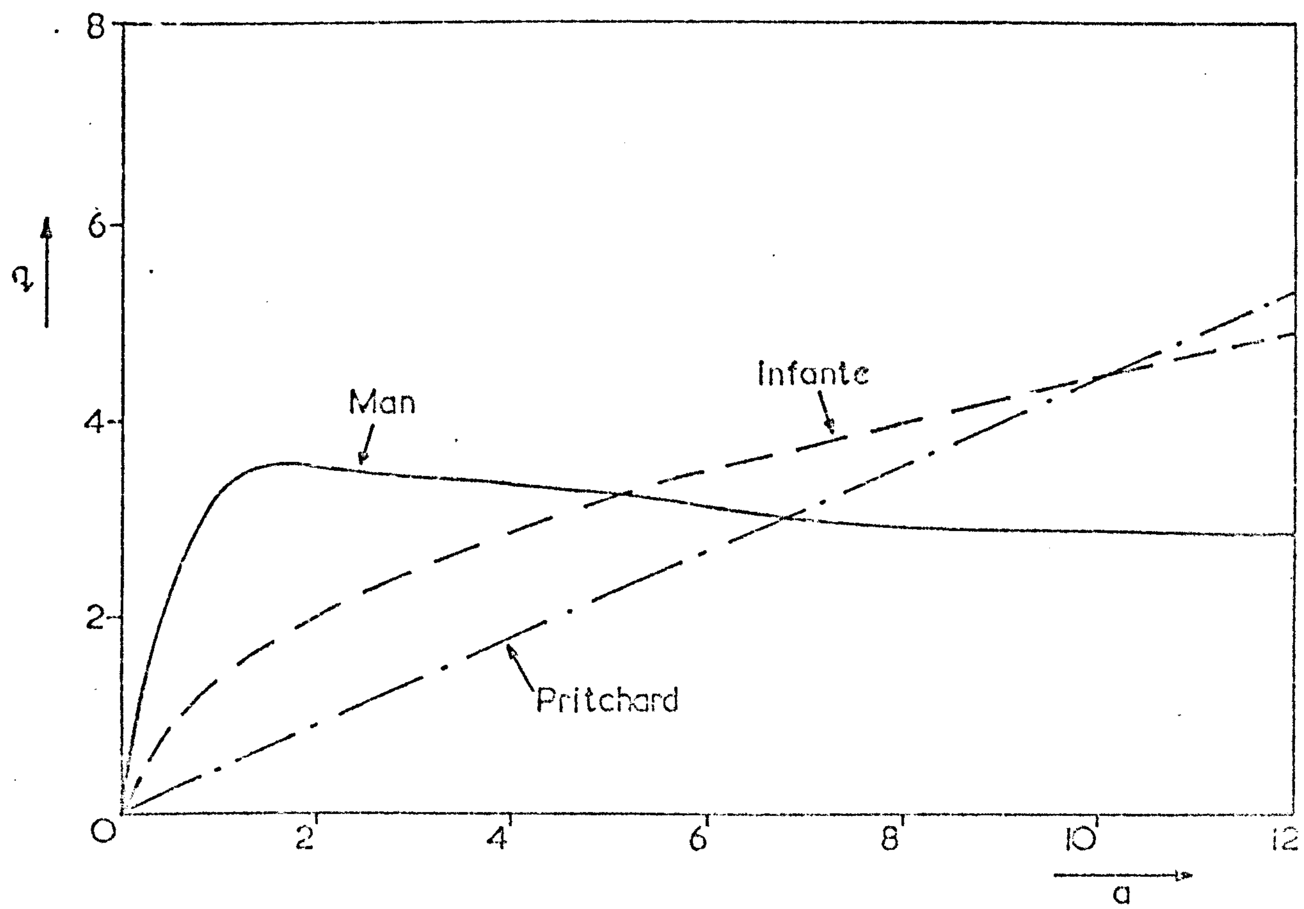

FIG. 5.2.3. Stability boundaries when $\zeta = 1$

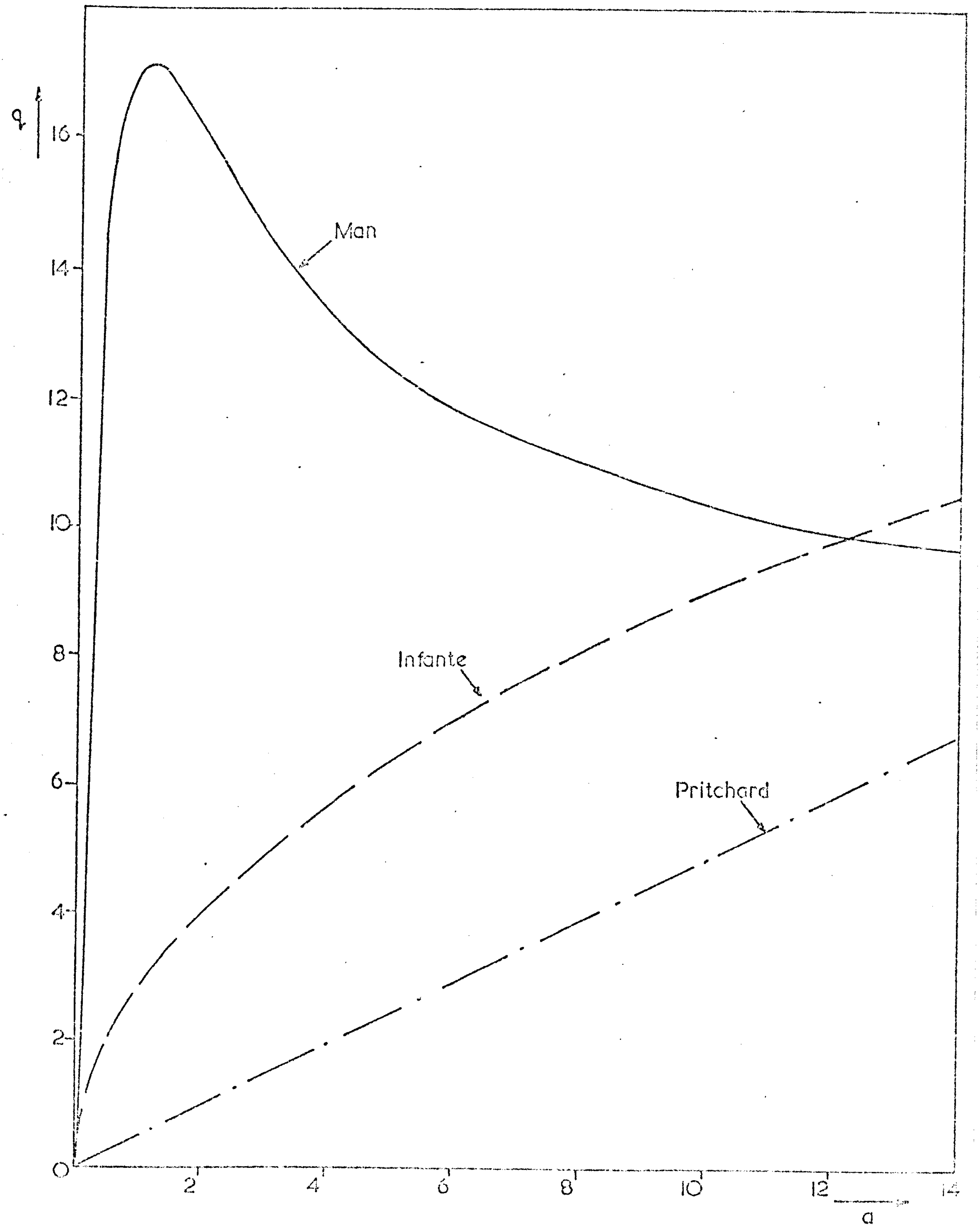

FIG. 5.2.4. Stability boundaries when $\zeta = 2$

(5.2.22), (5.2.26) and (5.2.28) are shown, for three different values of ζ, in figs. (5.2.2) - (5.2.4). It is readily seen, from these figures, that an optimum stability boundary is obtained by using a combination of all three criteria.

5.3. K_v varying randomly with time

Let the M.I.T. system of fig. (4.2.1) be subjected to a step input of magnitude R at time t = 0 and subsequently let K_v vary with time according to

$$K_v(t) \quad = \quad S + \alpha(t) \ , \qquad (5.3.1)$$

where $\alpha(t)$ may be regarded as a random disturbance on the initial value S. Considering, as before, that the adaption is switched on when the model response has reached its steady state value KR the system equations (4.2.4) become

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & -\frac{SR}{T} - \frac{R}{T}\,\alpha(t) \\ BKR & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{KR}{T} \\ 0 \end{bmatrix} , \qquad (5.3.2)$$

where $x_1 = e(t)$ and $x_2 = K_c(t)$.

As in the case of K_v varying sinusoidally with time the stability problem for the Liapunov redesign system of fig. (4.2.2), when the input is a step of magnitude R and K_v varies according to equation (5.3.1), will be the same as that for the system of equations (5.3.2).

Shackcloth and Butchart [46] attempted to obtain stability boundaries for system (5.3.2) using the theorems of Kats and Krasvoskii (see section (2.5)). They introduced the variable

$$Z = K_c - K/K_v$$

and wrote the system equations in the form

$$\dot{e}(t) = -\frac{1}{T}\,e(t) - \frac{K_v(t)}{T}\;RZ(t)$$

$$\dot{Z}(t) = \;Be(t)R - \frac{d}{dt}\,(K/K_v(t)) \qquad (5.3.3)$$

In order to satisfy the requirements of the theorem $K_v(t)$ must be a stationary Markov process with a finite number of states; it was therefore assumed to have the simplest possible form in that it was only allowed to have two possible values a_1 and a_2 with β_{12} being the probability of a transfer from a_1 to a_2 in time δt and β_{21} that from a_2 to a_1. In the absence of the disturbance term $\frac{d}{dt}\,(K/K_v)$ the first theorem of Kats and Krasvoskii may be employed to obtain a sufficient condition for asymptotically stability in the mean as

$$(a_1 - a_2)^2\,\beta^2 < \left[4a_1a_2\,(2\beta + 1/T)\right]/T\;,$$

where β is the probability of a change.

From the second theorem of Kats and Krasvoskii the system of equations (5.3.3) will not be made unstable if the disturbance term $d(K/K_v)/dt$ is a Gaussian process. Unfortunately it is not a Gaussian process, so that the second theorem is not valid. However, the disturbance term will become a closer approximation to a Gaussian process as the number of finite states K_v is allowed to have increases. Although the method may be extended to deal with such cases computation becomes a major problem.

5.3.1. $\alpha(t)$ Gaussian white noise

If $\alpha(t)$ is taken to be a Gaussian white noise process, with statistics

$$\langle\alpha(t)\rangle = 0$$
$$\langle\alpha(t)\alpha(t+\tau)\rangle = 2D\delta(\tau) \qquad (5.3.4)$$

then the process $\{x_1, x_2\}$, defined by equations (5.3.2), forms a Markov process so that its conditional probability function p satisfies the Fokker-Planck equation. Using the procedure, as described in appendix

(4.4), used for the system of equations (4.6.6) the Fokker-Planck equation corresponding to the system of equations (5.3.2) is obtained in the form

$$\frac{\partial p}{\partial t} = \frac{1}{T}\frac{\partial}{\partial x_1}(x_1 p) - \frac{KR}{T}\frac{\partial p}{\partial x_1} - BKR\, x_1 \frac{\partial p}{\partial x_2} + \frac{R^2 x_2{}^2 D}{T^2}\frac{\partial^2 p}{\partial x_1{}^2} + \frac{SR}{T}\, x_2 \frac{\partial p}{\partial x_1} \tag{5.3.5}$$

Introducing the non-dimensional parameters

$$\begin{aligned} \Pi_{12} &= BKR^2ST \\ \Pi_{13} &= D/(TS^2) \end{aligned} \tag{5.3.6}$$

and the dimensionless variables

$$\xi_1 = x_1/(KR)\ ,\ \xi_2 = Sx_2/K\ ,\ \tau = t/T \tag{5.3.7}$$

equation (5.3.5) may be written in the non-dimensional from

$$\frac{\partial p}{\partial \tau} = \frac{\partial}{\partial \xi_1}(\xi_1 p) - \frac{\partial p}{\partial \xi_1} - \Pi_{12}\,\xi_1 \frac{\partial p}{\partial \xi_2} + \Pi_{13}\,\xi_2{}^2 \frac{\partial^2 p}{\partial \xi_1{}^2} + \xi_2 \frac{\partial p}{\partial \xi_1} \tag{5.3.8}$$

By multiplying equation (5.3.7) throughout by ξ_i $(i = 1,2)$ and integrating over all ξ_1, ξ_2 the first order moment equations become :

$$\frac{d}{d\tau}\begin{bmatrix} m_{1,0} \\ \\ m_{0,1} \end{bmatrix} = \frac{d}{d\tau}\begin{bmatrix} \langle\xi_1\rangle \\ \\ \langle\xi_2\rangle \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ \\ \Pi_{12} & 0 \end{bmatrix}\begin{bmatrix} \langle\xi_1\rangle \\ \\ \langle\xi_2\rangle \end{bmatrix} + \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \tag{5.3.9}$$

A necessary and sufficient condition that $\langle\xi_1\rangle$ and $\langle\xi_2\rangle$ converge is that the eigenvalues of the coefficient matrix have negative real parts. Since the eigenvalues of the coefficient matrix are given by

$$2\lambda = -1 \pm \sqrt{1 - 4\Pi_{12}}\ ,$$

it follows that $<\xi_1>$ and $<\xi_2>$ converge for all Π_{12} and Π_{13}; that is, the system is stable in the mean for all values of the system parameters Also, for all values of Π_{12} and Π_{13}, the asymptotic values, as $\tau \to \infty$, of $<\xi_1>$ and $<\xi_2>$ are given by

$$-<\xi_1> - <\xi_2> + 1 = 0$$

$$\Pi_{12} <\xi_1> = 0$$

that is,

$$\lim_{\tau \to \infty} <\xi_1> = 0 \quad \text{and} \quad \lim_{\tau \to \infty} <\xi_2> = 1 \tag{5.3.10}$$

Using relationships (5.3.7) it follows from (5.3.10) that, as $t \to \infty$, both $e(t)$ and $K - K_v(t)K_c(t)$ tend to zero in the mean, which is as required.

By multiplying equation (5.3.8) throughout by $\xi_i\xi_j$ $(i,j = 1,2)$ and integrating over all ξ_1, ξ_2 the second order moment equations become:

$$\frac{d}{dt}\begin{bmatrix} m_{2,0} \\ m_{1,1} \\ m_{0,2} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} <\xi_1^2> \\ <\xi_1\xi_2> \\ <\xi_2^2> \end{bmatrix} = \begin{bmatrix} -2 & -2 & 2\Pi_{13} \\ \Pi_{12} & -1 & -1 \\ 0 & 2\Pi_{12} & 0 \end{bmatrix}\begin{bmatrix} <\xi_1^2> \\ <\xi_1\xi_2> \\ <\xi_2^2> \end{bmatrix} + \begin{bmatrix} 2<\xi_1> \\ <\xi_2> \\ 0 \end{bmatrix} \tag{5.3.11}$$

Note that in this case the second order moment equations depend only on the first and second order moments and do not depend on higher order moments as in the case of the nonlinear system (4.6.6). It is therefore possible, in this case, to speak of stability in the mean square. The eigenvalues of the coefficient matrix, of equations (5.3.11), are given by the cubic

$$\lambda^3 + 3\lambda^2 + (2 + 4\Pi_{12})\lambda + 4\Pi_{12}(1 - \Pi_{12}\Pi_{13}) = 0$$

It follows that the system is stable in the mean square if it is stable in the mean and if

(i) $(1 - \Pi_{12}\Pi_{13}) > 0$

and (ii) $3(2 + 4\Pi_{12}) > 4\Pi_{12}(1 - \Pi_{12}\Pi_{13})$

Since the system is stable in the mean for all Π_{12} and Π_{13} and condition (ii) holds for all Π_{12} and Π_{13} it follows that the system is stable in the mean square if

$$\Pi_{13} < \frac{1}{\Pi_{12}} \tag{5.3.12}$$

When condition (5.3.12) holds the time derivative of the second order moments will become vanishingly small after a long time, thus making it possible to obtain the asymptotic solutions of the second moments. Solving

$$\frac{d}{d\tau}\langle\xi_1{}^2\rangle = \frac{d}{d\tau}\langle\xi_1\xi_2\rangle = \frac{d}{d\tau}\langle\xi_2{}^2\rangle = 0$$

gives :

$$\begin{aligned}
&\lim_{\tau\to\infty}\langle\xi_1{}^2\rangle = \frac{\Pi_{13}}{1 - \Pi_{12}\Pi_{13}} \\
&\lim_{\tau\to\infty}\langle\xi_1\xi_2\rangle = 0 \\
&\lim_{\tau\to\infty}\langle\xi_2{}^2\rangle = \frac{1}{1 - \Pi_{12}\Pi_{13}}
\end{aligned} \tag{5.3.13}$$

Another item of interest is the probability that the system error exceeds a specific value Y. If the Fokker-Planck equation (5.3.8) can not be solved for p, then this probability cannot be determined. However, as pointed out by Ariaratnam [27], by using the Chebyshev inequality [84] we are able to obtain an upper bound on this probability. Chebyshev inequality is given by :

$$P\{|\xi_1 - \langle\xi_1\rangle| > Y\sigma\} \leq \frac{1}{Y^2} ,$$

where $\sigma = \{\langle [\xi_1 - \langle\xi_1\rangle]^2 \rangle\}^{\frac{1}{2}}$

Using this and assuming that the first and second order moments of ξ_1 are bounded we find, from equations (5.3.10) and (5.3.13), that for sufficiently large τ

$$\sigma^2 = <\xi_1^2> - <\xi_1>^2$$

$$= \frac{\Pi_{13}}{1 - \Pi_{12}\Pi_{13}}$$

and that

$$P\{|\xi_1| < Y\} \leq \frac{1}{Y^2} \{ \frac{\Pi_{13}}{1 - \Pi_{12}\Pi_{13}} \} \qquad (5.3.14)$$

If the spectral densities and correlation functions of ξ_1 and ξ_2 are required then they are readily obtained from the Fokker-Planck equation using the procedure described by Ariaratnam and Graeffe [30].

5.3.2. <u>$\alpha(t)$ Gaussian</u>

If $\alpha(t)$ is Gaussian but non white then the response $\{x_1\ x_2\}$ of the system, represented by equations (5.3.2), no longer forms a Markov process. Since the effect of making the process Markov, by introducing a third variable, is to make the system equations non-linear it follows, using the same argument as in section (4.6.1), that it is not possible to use the Fokker-Planck equation to obtain conditions for stability in the mean square. We shall therefore, in this section, apply the theory developed in section (2.7) to obtain stability criteria which are sufficient conditions for almost sure asymptotic stability of the system. As we are concerned only with the question of stability we may restrict our attention to the unforced system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & -\frac{SR}{T} - \frac{R}{T}\alpha(t) \\ BKR & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (5.3.15)$$

where $\alpha(t)$ is taken to be an ergodic Gaussian process, with zero mean and variance σ^2.

Introducing the non-dimensional parameters

$$\Pi_{12} = BKR^2ST \quad , \quad \Pi_{14} = \sigma/S \tag{5.3.16}$$

and the dimensionless variables ξ_1, ξ_2, τ, defined by equations (5.3.6), equation (5.3.15) may be written in the non-dimensional form

$$\frac{d}{d\tau}\begin{bmatrix}\xi_2(\tau)\\ \\ \xi_1(\tau)\end{bmatrix} = \begin{bmatrix}0 & \Pi_{12}\\ \\ -1 & -1\end{bmatrix}\begin{bmatrix}\xi_2(\tau)\\ \\ \xi_1(\tau)\end{bmatrix} + \beta(\tau)\begin{bmatrix}0 & 0\\ \\ -\Pi_{14} & 0\end{bmatrix}\begin{bmatrix}\xi_2(\tau)\\ \\ \xi_1(\tau)\end{bmatrix} \tag{5.3.17}$$

where $\beta(\tau)$ is an ergodic, Gaussian process with zero mean and variance unity.

Equation (5.3.17) is of the form

$$\dot{\underline{x}} = [\,\underline{A} + \underline{F}(t)\,]\,\underline{x}\ ,$$

a type of equation that was discussed in detail in section (2.7). Again the stability boundaries obtained by using the criteria developed by Infante [55] and Man [56] are much sharper than those obtained using the other criteria cited in section (2.7). Thus, we shall present a stability analysis for equation (5.3.17) using only these two criteria.

5.3.2.1. Infante type analysis

In Infante's work, see section (2.7), take the matrix $\underline{B}$ to be the most general symmetric positive definite form, defined as in equation (4.6.20). Simple computation then yields

$$\underline{A}^1 + \underline{F}^1 + \underline{B}(\underline{A} + \underline{F})\underline{B}^{-1} =$$

$$\frac{1}{a_2}\begin{bmatrix}-a_1(1+\Pi_{14})+a_1^2-a_1\Pi_{12}(a_1^2+a_2) & (1+\Pi_{14})(a_1^2-a_2)+\Pi_{12}(a_1^2+a_2)^2 - a_1(a_1^2+a_2)\\ a_1-\Pi_{12}(a_1^2-a_2)-(1+\Pi_{14}) & a_1(1+\Pi_{14})+(\Pi_{12}a_1-1)(a_1^2+a_2) - a_2\end{bmatrix}$$

(5.3.18)

where,

$$\underline{A} = \begin{bmatrix} 0 & \Pi_{12} \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \underline{F} = \begin{bmatrix} 0 & 0 \\ -\Pi_{14}\beta(\tau) & 0 \end{bmatrix} \tag{5.3.19}$$

The maximum eigenvalue of the matrix given in equation (5.3.18) is computed as

$$\lambda_{max}[\underline{A}^1 + \underline{F}^1 + \underline{B}(\underline{A} + \underline{F})\underline{B}^{-1}] = -1 + \{(1 - 2\Pi_{12}a_1)^2 + \frac{1}{a_2}[\Pi_{12}(a_2 + a_1^2) -$$

$$(1 + \Pi_{14}\beta(\tau)) + a_1(1 - 2\Pi_{12}a_1)]^2\}^{\frac{1}{2}} \tag{5.3.20}$$

Thus, from criterion (2.7.16), it follows that a sufficient condition for almost sure asymptotic stability, of the system represented by equation (5.3.17), is that

$$E\{-1 + [(1 - 2\Pi_{12}a_1)^2 + \frac{1}{a_2}[\Pi_{12}(a_2 + a_1^2) - (1 + \Pi_{14}\beta(\tau))$$

$$+ a_1(1 - 2\Pi_{12}a_1)]^2]^{\frac{1}{2}} \} \leqslant - \varepsilon$$

$$\tag{5.3.21}$$

Since $\beta(\tau)$ has zero mean and variance unity condition (5.3.21) may be written in the form

$$\Pi_{14}^2\ E\{\beta^2(\tau)\} = \Pi_{14}^2 \leqslant a_2[1 - (1 - 2\Pi_{12}a_1)^2] - [\Pi_{12}(a_2 + a_1^2) - 1 +$$

$$a_1(1 - 2\Pi_{12}a_1))]^2$$

$$\tag{5.3.22}$$

Optimum values of a_1 and a_2, in order to maximize Π_{14}^2, are found to be

$$a_1 = \frac{1}{2\Pi_{12}} \quad , \quad a_2 = \frac{1 + 4\Pi_{12}}{4\Pi_{12}^2} \tag{5.3.23}$$

Substituting these values of a_1 and a_2 in (5.3.22) gives the stability condition as

$$\Pi_{14}^{2} < \frac{1}{\Pi_{12}} \tag{5.3.24}$$

Defining the non-dimensional parameter Π_{15} as

$$\Pi_{15} = \Pi_{14}^{2} = \frac{\sigma^2}{s^2} \tag{5.3.25}$$

condition (5.3.24), which is a sufficient condition for almost sure asymptotic stability of the system represented by equation (5.3.17), becomes

$$\Pi_{15} < \frac{1}{\Pi_{12}} \tag{5.3.26}$$

5.3.2.2. <u>Man type analysis</u>

In Man's work, see section (2.7), let the matrices $\underline{A}$ and $\underline{F}$ be defined as in equations (5.3.19) and take $\underline{Q}$ to be the identity matrix $\underline{I}$. Solving the Liapunov matrix equation (4.6.28) gives the matrix $\underline{P}$ in the form

$$\underline{P} = \begin{bmatrix} \dfrac{2+\Pi_{12}}{2\Pi_{12}} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1+\Pi_{12}}{2} \end{bmatrix}$$

Simple computation then gives

$$(\underline{F}^{1}P + \underline{PF})\underline{Q}^{-1} = \Pi_{14}\beta(\tau) \begin{bmatrix} -1 & -1(1+\Pi_{12})/2 \\ -(1+\Pi_{12})/2 & 0 \end{bmatrix}$$

and

$$\lambda_{max}[(\underline{F}^{1}\underline{P} + \underline{P}\ \underline{F})\underline{Q}^{-1}] = \frac{\Pi_{14}\beta(\tau)}{2}\{-1 + [1 + (1+\Pi_{12})^{2}]^{\frac{1}{2}}\}$$

Thus, from criterion (2.7.19), it follows that a sufficient condition for almost sure asymptotic stability, of the system represented by equation (5.3.17), is that

$$E\{\Pi_{14}\beta(\tau)\ \{-1 + [(1 + \Pi_{12})^2]^{\frac{1}{2}}\}\} < 2 \qquad (5.3.27)$$

Since $\beta(\tau)$ has zero mean and variance unity condition (5.3.27) may be written in the form

$$\Pi_{14}{}^2 < \frac{4}{2 + (1 + \Pi_{12})^2 - 2\sqrt{1 + (1 + \Pi_{12})^2}} \qquad (5.3.28)$$

or

$$\Pi_{15} < \frac{4}{2 + (1 + \Pi_{12})^2 - 2\sqrt{1 + (1 + \Pi_{12})^2}} \qquad (5.3.29)$$

where Π_{15} is defined as in equation (5.3.25).

5.4. Digital Simulation

Using the same procedure as employed in section 4.7. the system of equations (5.3.2) are simulated on a digital computer with parameter values $R = 5$, $T = 0.25$, $K = 1$, $S = 0.5$, $h = 0.01$, $m = 3$ and initial $K_c = 0.5$. Since, in this case, $e(t)$ and $K - K_v K_c$ do not adapt to zero in the mean square it is difficult to decide, for certain parameter values, whether the solution represents a stable or unstable situation. A particular solution is taken to represent an unstable situation when there is no question about it being a possible stable oscillation about zero. Stability boundaries obtained using both the Runge-Kutta and Crank-Nicolson procedures are shown in fig. (5.4.1) together with the theoretical boundary obtained using the Fokker-Planck equation; it is seen that the theoretical results correspond closely to the simulated

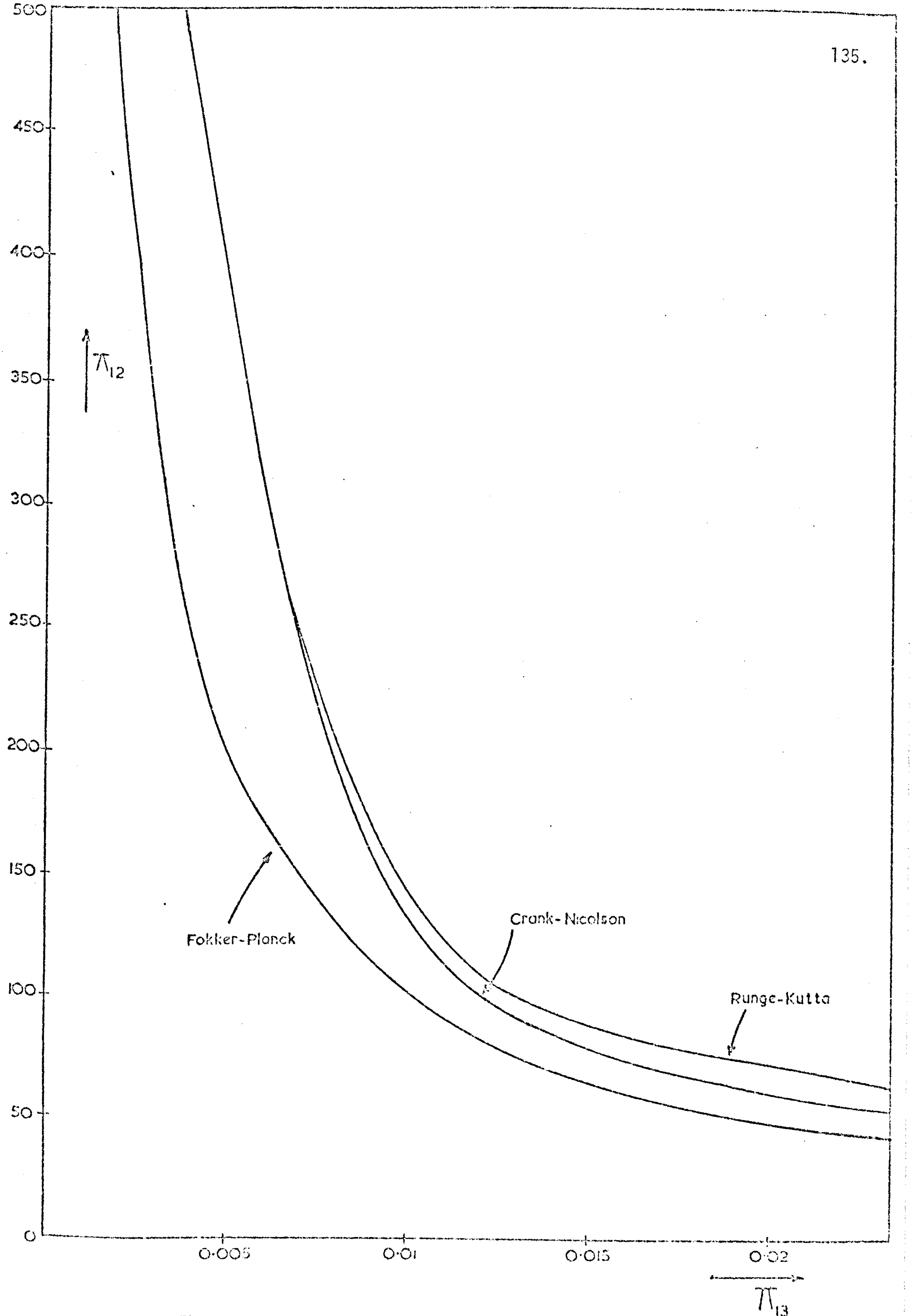

FIG. 5.4.1. Stability boundaries for system (5.3.17)

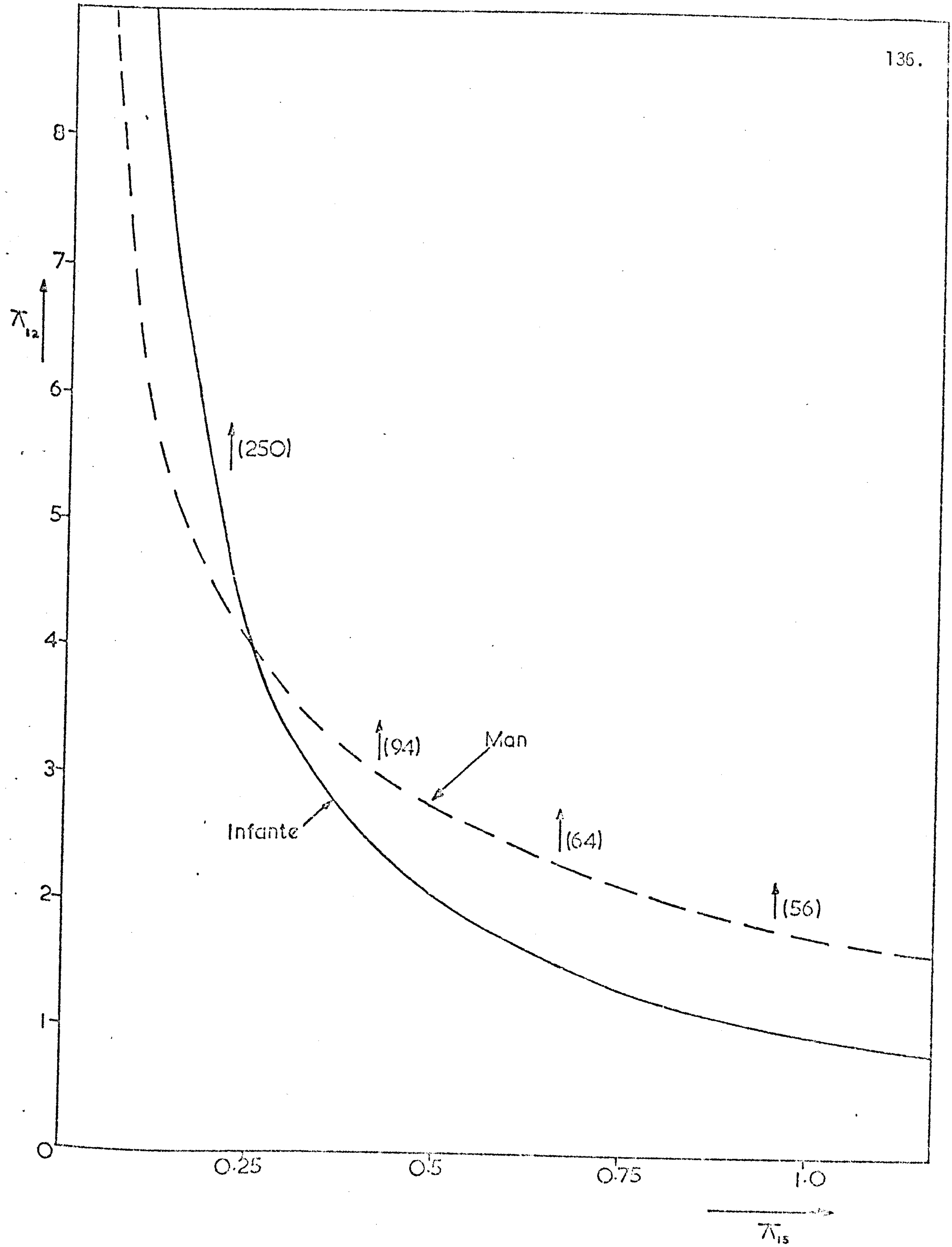

FIG. 5.4.2. Stability boundaries for system (5.3.17)

results. However, the theoretical results obtained using the Liapunov type analysis are very conservative; stability boundaries corresponding to criteria (5.3.26) and (5.3.29) are shown in fig. (5.4.2) and in order to illustrate how weak the criteria are values of Π_{12}, corresponding to instability, obtained using the Crank-Nicolson procedure are indicated in the figure, for certain values of Π_{15}, alongside the vertical arrows.

5.5. Effect of random disturbances at system output

Since any disturbance applied at the system output will be directly superimposed on the error signal e(t), and hence on the adaptive parameter, its effect must be considered in any stability analysis.

Consider the M.I.T. system of fig. (4.2.1) to be subjected to a step input of magnitude R at time t = 0, when $\theta_m(t)$ and $\theta_s(t)$ are zero and $K_vK_c \neq K$. Subsequently let K_v remain constant and K_c be adjusted according to equation (4.2.3). Considering, as before, that the adaption is switched on after the model response $\theta_m(t)$ has reached its steady state value KR the system equations (4.2.4) become

$$\begin{aligned} T\,\dot{e}(t) + e(t) &= (K - K_vK_c)R \\ \dot{K}_c(t) &= BKR\,e(t) \end{aligned} \tag{5.5.1}$$

Consider further that the system is subjected to a random disturbance $\alpha(t)$ at the system output so that equations (5.5.1) become

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & \frac{R}{T} \\ -K_vBRK & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -BK_vKR\,\alpha(t) \end{bmatrix} \tag{5.5.2}$$

where $x_1(t) = e(t)$ and $x_2(t) = K - K_vK_c(t)$.

If $\alpha(t)$ is regarded as Gaussian white noise with statistics as defined in equations (5.3.3) then, as before, the process $\underline{x}(t) = \{x_1(t)\ x_2(t)\}$, defined by equations (5.5.2), is a Markov process whose conditional probability density function p satisfies the Fokker-Planck equation. Using the procedure described in appendix 4.4 the Fokker-

Planck equation corresponding to the system of equations (5.5.2) is

$$\frac{\partial p}{\partial t} = \frac{1}{T}\frac{\partial}{\partial x_1}(x_1 p) - \frac{R}{T} x_2 \frac{\partial p}{\partial x_1} + BKK_V R x_1 \frac{\partial p}{\partial x_2} + (BRKK_V)^2 D \frac{\partial^2 p}{\partial {x_2}^2} \tag{5.5.3}$$

Introducing the non dimensional parameters

$$\Pi_{12} = BKR^2ST \quad , \quad \Pi_{16} = \frac{D}{TR^2K^2} \tag{5.5.4}$$

and the dimensionless variables

$$\xi_1 = \frac{x_1}{KR} \quad , \quad \xi_2 = \frac{x_2}{K} \quad , \quad \tau = \frac{t}{T} \tag{5.5.5}$$

equation (5.5.3) may be written in the non-dimensional form

$$\frac{\partial p}{\partial \tau} = \frac{\partial}{\partial \xi_1}(\xi_1 p) - \xi_2 \frac{\partial p}{\partial \xi_1} + \Pi_{12}\xi_1 \frac{\partial p}{\partial \xi_2} + \Pi^2{}_{12}\Pi_{16} \frac{\partial^2 p}{\partial \xi^2{}_2} \tag{5.5.6}$$

By multiplying equation (5.5.6) throught by $\xi_i (i = 1,2)$ and integrating over all ξ_1, ξ_2 the first order moment equations become :

$$\frac{d}{d\tau}\begin{bmatrix} <\xi_1(\tau)> \\ <\xi_2(\tau)> \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -\Pi_{12} & 0 \end{bmatrix}\begin{bmatrix} <\xi_1(\tau)> \\ <\xi_2(\tau)> \end{bmatrix} \tag{5.5.7}$$

Since the eigenvalues of the coefficient matrix, for system (5.5.7), is given by

$$\lambda = \frac{-1 \pm \sqrt{1 - 4\Pi_{12}}}{2}$$

it follows that $<\xi_1(\tau)>$ and $<\xi_2(\tau)>$ converges for all Π_{12} and Π_{16}; that is, the system is stable in the mean for all values of the system parameters. Further, for all values of Π_{12} and Π_{16}, $<\xi_1(\tau)>$ and $<\xi_2(\tau)>$ tend to zero asymptotically as $t \to \infty$ which is as required.

By multiplying equation (5.5.6) throughout by $\xi_i\xi_j$ (i, j = 1,2) and integrating over all ξ_1, ξ_2 the second order moment equations become :

$$\frac{d}{d\tau}\begin{bmatrix} <\xi_1^2(\tau)> \\ <\xi_1(\tau)\xi_2(\tau)> \\ <\xi_2^2(\tau)> \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ -\Pi_{12} & -1 & 1 \\ 0 & -2\Pi_{12} & 0 \end{bmatrix} \begin{bmatrix} <\xi_1^2(\tau)> \\ <\xi_1(\tau)\xi_2(\tau)> \\ <\xi_2^2(\tau)> \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2\Pi^2_{12}\Pi_{16} \end{bmatrix} \tag{5.5.8}$$

Since the eigenvalues of the coefficient matrix, for system (5.5.8), is given by the cubic

$$\lambda^3 + 3\lambda^2 + 2(2\Pi_{12} + 1)\lambda + 4\Pi_{12} = 0$$

it follows that the system is stable in the mean square if

(i) $4\Pi_{12} > 0$

and (ii) $6(2\Pi_{12} + 1) > 4\Pi_{12}$

As conditions (i) and (ii) hold for all Π_{12} and Π_{16} it follows that the system is stable in the mean square for all values of the system parameters.

By solving the equations

$$\frac{d}{d\tau} <\xi_1^2(\tau)> = \frac{d}{d\tau} <\xi_1(\tau)\xi_2(\tau)> = \frac{d}{d\tau}<\xi_2^2(\tau)> = 0$$

the asymptotic solutions, as $\tau \to \infty$, of the second moments are found to be :

$$\begin{aligned} \lim_{\tau\to\infty} <\xi_1^2(\tau)> &= \Pi_{12}\Pi_{16} \\ \lim_{\tau\to\infty} <\xi_1(\tau)\xi_2(\tau)> &= \Pi_{12}\Pi_{16} \\ \lim_{\tau\to\infty} <\xi_2^2(\tau)> &= \Pi_{12}\Pi_{16}(1 + \Pi_{12}) \end{aligned} \tag{5.5.9}$$

Thus, although noise disturbance at the system output has no effect on the stability of the system in the mean square it does have an effect on the accuracy of the parameter adaptation.

CHAPTER 6

STABILITY OF A HIGHER ORDER MODEL-REFERENCE SYSTEM

6.1. Introduction

In this chapter the methods developed in the previous two chapters, for investigating the stability of a first order model reference adaptive control system, are extended to examine the stability of a higher order system. The system considered is that developed by White [2] and is shown in fig. (6.1.1). The basic equation of the process is :

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\theta_s = A_1A_2A_3K_2\theta_i \ , \qquad (6.1.1)$$

where D is the differential operator d/dt and $A_1 = 16.78$, $A_2 = 14.88$, $A_3 = 4.57$, $K_1 = K_2 = 0.5$ (nominal). The reference model is second order and has transfer function

$$Y_m = \frac{40}{s^2 + 6.32s + 40} \ , \qquad (6.1.2)$$

where s is the Laplacian operator.

The input signal $\theta_i(t)$ is common to the process and model and the adaptive error is

$$e(t) = \theta_s(t) - \theta_m(t) \qquad (6.1.3)$$

where $\theta_s(t)$ and $\theta_m(t)$ are the outputs from the process and model respectively.

The self-adaptive performance criterion employed to adjust the parameters K_i $(i = 1,2)$ is, as before, the M.I.T. rule; that is, the parameters are varied according to the adaptive control law

$$\frac{\partial K_i}{\partial t} \ \alpha \ e(t) \ \frac{\partial e}{\partial K_i} \qquad (i = 1,2)$$

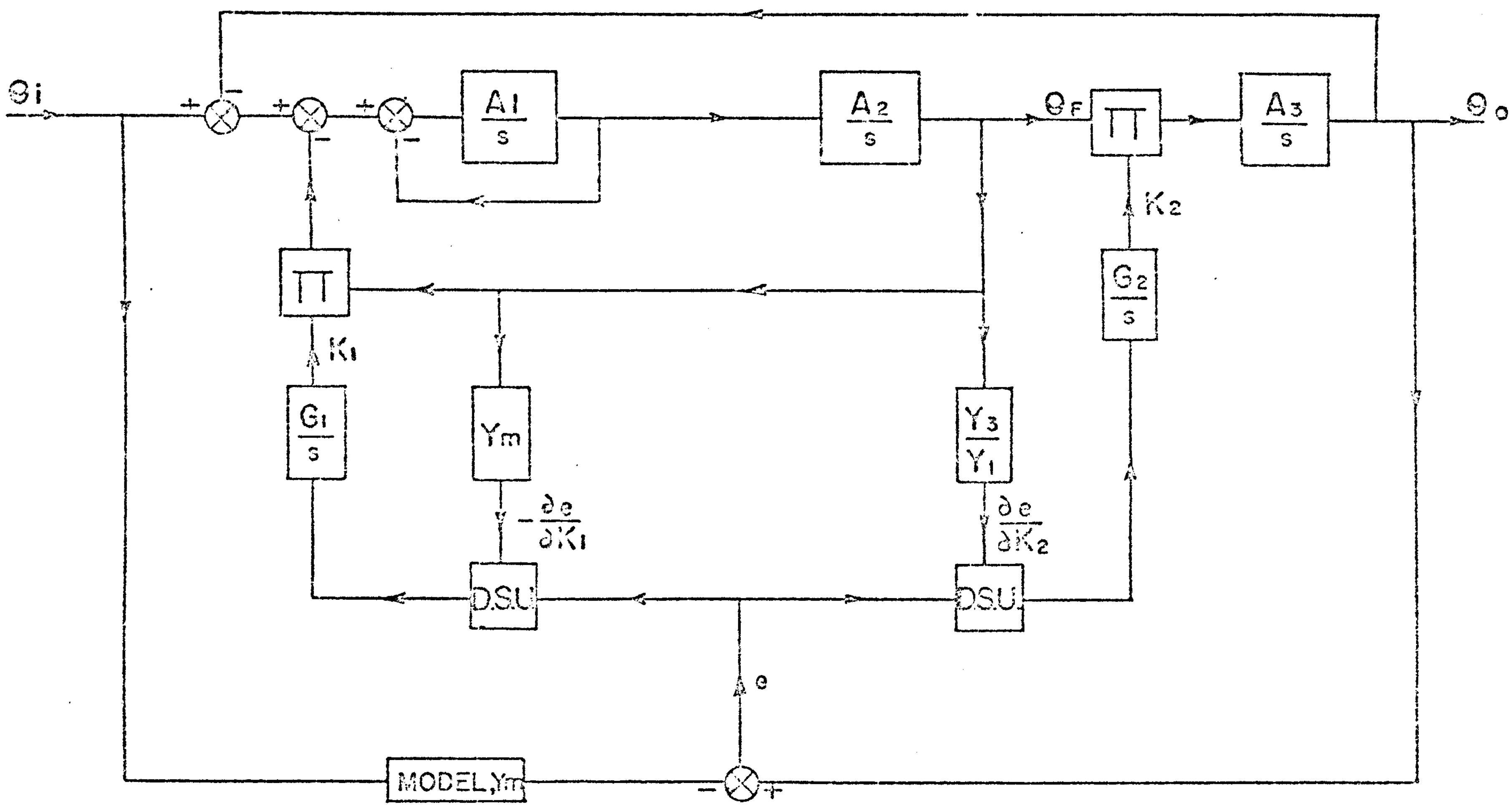

FIG. 6.1.1. Block diagram of adaptive control system

It was shown by Whitaker et al [75] that his law could be taken in the form :

$$\frac{\partial K_i}{\partial t} = -G_i e(t) \text{ sign } \frac{\partial e}{\partial K_i} \qquad (i = 1,2) \qquad (6.1.4)$$

where only the sign of $\frac{\partial e}{\partial K_i}$ is taken to ensure that the sign of the product $e(t)(\partial e/\partial K_i)$ is correct. The product

$$e(t) \text{ sign } \left(\frac{\partial e}{\partial K_i} \right), \quad (i = 1,2)$$

is formed by passing the two signals into a diode switching unit (d.s.u.) the output of which is $\pm e(t)$ depending on the sign of the signal $\partial e/\partial K_i$. The approximations to $\frac{\partial e}{\partial K_i}$ and $\frac{\partial e}{\partial K_2}$ are obtained by feeding the signal θ_F, given by

$$(D^3 + A_1 D^2 + A_1 A_2 K_1 D + A_1 A_2 A_3 K_2)\theta_F = A_1 A_2 D \theta_i \quad , \qquad (6.1.5)$$

through filters which can be identified as follows :

$$\frac{\partial e}{\partial K_1} = \frac{\partial}{\partial K_1} (\theta_s - \theta_m) = - \frac{A_1^{\,2} A_2^{\,2} A_3 K_2 s}{[s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2]^2} \theta_i$$

$$= - Y_s \theta_F$$

$$\simeq - Y_m \theta_F \qquad (6.1.6)$$

since the model is a good approximation of the system around the correct value of K_1; that is, the signal obtained by passing θ_F through a filter identical with the model is $-(\partial e/\partial K_1)$ and not $(\partial e/\partial K_1)$ as indicated by White, as the latter signal would lead to a negative gain G_1.

$$\frac{\partial e}{\partial K_2} = \frac{\partial \theta_s}{\partial K_2} = \frac{A_1 A_2 A_3 (s^2 + A_1 s + A_1 A_2 K_1) s}{[s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2]^2} \theta_i$$

$$= \frac{A_3\,(s^2 + A_1 s + A_1 A_2 K_1)}{(s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2} \qquad \theta_F$$

$$= \frac{Y_3}{Y_1}\ \theta_F\ , \tag{6.1.7}$$

where Y_3 is the transfer function of the system with the parameters fixed at their normal values and Y_1 is the transfer function from the system input to the parameter disturbance summation point; that is,

$$Y_1 = \frac{A_1 A_2 K_2}{s^2 + A_1 s + A_1 A_2 K_1}$$

giving

$$\frac{Y_3}{Y_1} = \frac{A_3(s^2 + A_1 s + A_1 A_2 K_1)}{s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2}$$

$$= \frac{4.57(s^2 + 16.78s + 124.8)}{s^3 + 16.78s^2 + 124.8s + 570} \tag{6.1.8}$$

The method of approach in a stability analysis is to first obtain the *steady-state* values of the adapting parameters K_1 and K_2 and then set up variational equations for small perturbations about those *steady-state* values. The resulting variational equations are linear differential equations with time varying coefficients and their stability characteristics are investigated using the theory developed in chapters 1 and 2.

6.2. Sinusoidal input

Consider the adaptive control system to be adapting on a steady sinusoidal input $\&\sin\omega t$.

6.2.1. Steady state values of the parameters

Suppose that the K_2 parameter is held constant and that the K_1

parameter is adapting alone. When K_1 has reached its *steady-state* value the output of the switching unit (in the K_1 adapting loop) must be such that it contains no d.c. component (otherwise the d.c. term will integrate up to change the value of K_1). It can be shown [2] that this occurs when the $\frac{\partial e}{\partial K_1}$ and e signals are in quadrature. Using this property White showed that the relationship between the *steady-state* values of K_1 and the frequency ω of the input is

$$K_1 = 0.004\,\omega^2 + 0.72\,K_2 \qquad (6.2.1)$$

Since K_2 is considered fixed at 0.5 the relationship becomes

$$K_1 = 0.004\,\omega^2 + 0.36 \qquad (6.2.2)$$

A similar procedure for the other loop gives a relationship between the *steady-state* value of K_2 and the input frequency ω when the parameter K_1 is fixed. This relationship was found by White to be

$$K_2 = \frac{6.32\left[16.78\omega^2 + \tan\emptyset(\omega^3 - 250K_1\omega)\right] - (40-\omega^2)\left[(\omega^2 - 250K_1) - 16.78\omega\tan\emptyset\right]}{28.5(252.8 - \omega^3\tan\emptyset)} \qquad (6.2.3)$$

Since K_1 is fixed at 0.5 the relationship becomes

$$K_2 = \frac{(\omega^4 - 58.7\omega^2 + 4990) - \omega(10.46\omega^2 + 117)\tan\emptyset}{28.5(252.8 - \omega^3\tan\emptyset)} \qquad (6.2.4)$$

where $\emptyset$ is the phase change across the filter Y_3/Y_1 and is given by :

$$\tan\emptyset = \frac{16.78\omega(570 - 16.78\omega^2) - \omega(124.8 - \omega^2)^2}{570(124.8 - \omega^2)} \qquad (6.2.5)$$

By solving equations (6.2.1) and (6.2.3) for K_1 and K_2 *steady-state* values of the parameters are found for the case when they are adapting together. These values are found to be

$$K_1 = 0.004\,\omega^2 + 0.424$$
$$K_2 = 0.588 \qquad (6.2.6)$$

The *steady-state* values of the parameters given by equations (6.2.2), (6.2.4) and (6.2.6) are all verified in White's paper using analogue computer simulation.

6.2.2. Stability consideration

The basic equation of the system is as given by equation (6.1.1). Consider a small perturbation of the adapting parameters about their *steady-state* values; so that, in the perturbed state

$$K_1 \rightarrow K_1 + \delta K_1 \ , \ K_2 \rightarrow K_2 + \delta K_2 \ , \ \theta_s \rightarrow \theta_s + \delta\theta_s$$

Subtracting equation (6.1.1) from the perturbed state equation gives

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta\theta_s + A_1A_2\delta K_1 D\theta_s - A_1A_2A_3\delta K_2 e_s = 0 \qquad (6.2.7)$$

where e_s is the system error $\theta_i - \theta_s$.

Since the model is unaffected by perturbations in the parameters we have from equation (6.1.3) that

$$\delta e = \delta\theta_s$$

so that the perturbed equation for the error is

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta e + A_1A_2\delta K_1 D\theta_s - A_1A_2A_3\delta K_2 e_s = 0 \qquad (6.2.8)$$

The terms $D\theta_s$ and e_s vary periodically with time, but, in his analysis, White replaced these time varying terms, by time averaging over a period, with constant terms. He then examined the resulting linearized differential equations with constant coefficients using the Routh-Hurwitz criterion; the disadvantages of this method are discussed in chapter 1. In this chapter a more rigorous analysis, based on the Floquet analysis of chapter 1, is presented.

From equation (6.1.4) we have that in the perturbed state :

$$\dot{K}_i + \delta\dot{K}_i = -G_i(e + \delta e)\text{sign}\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right] \ , (i = 1,2) \qquad (6.2.9)$$

Subtracting equations (6.1.4) and (6.2.9) gives :

$$\delta\dot{K}_i = -G_i\delta e \text{ sign}\,\frac{\partial e}{\partial K_i} \ - \ G_i e \left[\text{sign}\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right] - \text{sign}\,\frac{\partial e}{\partial K_i}\right] ,$$

$$(6.2.10)$$

the term $\delta e\text{sign}\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right]$ having been replaced by $\delta e\text{sign}\,\frac{\partial e}{\partial K_i}$ since we are dealing with the linearized equations in the perturbation terms.

$\partial e/\partial K_i$ is a sine wave and $\partial e/\partial K_i + \delta(\partial e/\partial K_i)$ a perturbed wave, being approximately sinusoidal as shown in fig. (6.2.1(a)). The terms sign $\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right]$ and sign $\frac{\partial e}{\partial K_i}$ are represented by the square waves of fig. (6.2.1(b)), and the term

$$\left[\text{sign}\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right] - \text{sign}\,\frac{\partial e}{\partial K_i}\right]$$

by the pulses of fig. (6.2.1(c)). The last term of equation (6.2.10) had been omitted by Parks [85] in his discussion on White's paper. The omission of this term was found to have a considerable effect on the final results.

Using the condition that the e and $\partial e/\partial K_i$ signals are in quadrature when the system has adapted, the term

$$e\left[\text{sign}\left[\frac{\partial e}{\partial K_i} + \delta\,\frac{\partial e}{\partial K_i}\right] - \text{sign}\,\frac{\partial e}{\partial K_i}\right]$$

can be represented by the pulses of fig. (6.2.1e). Two of these pulses

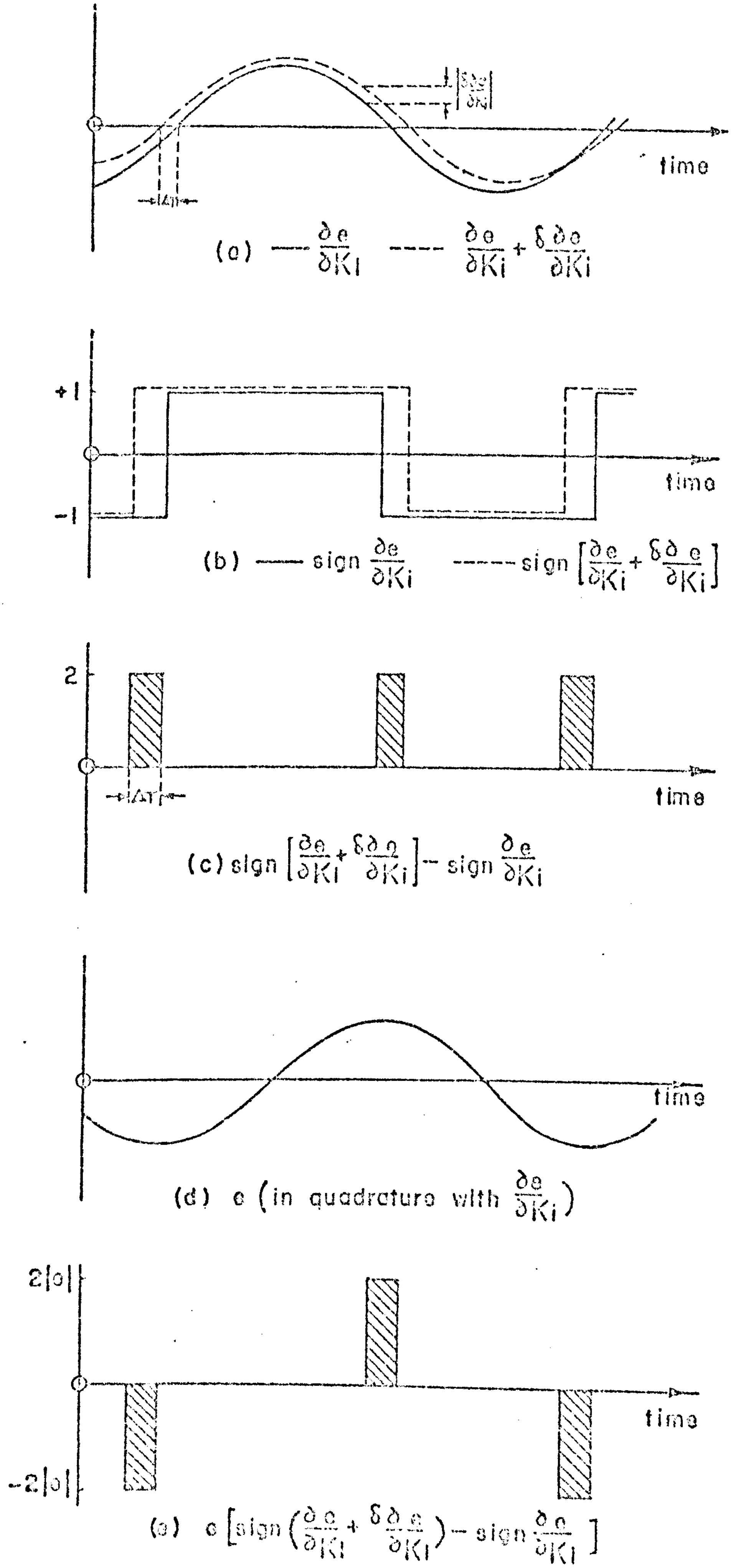
time
$|\Delta\tau|$
(a) —— $\frac{\partial e}{\partial K_i}$ ---- $\frac{\partial e}{\partial K_i} + \delta\frac{\partial e}{\partial K_i}$
+1
-1
time
(b) —— sign $\frac{\partial e}{\partial K_i}$ ---- sign $\left[\frac{\partial e}{\partial K_i} + \delta\frac{\partial e}{\partial K_i}\right]$
2
$|\Delta\tau|$
time
(c) sign $\left[\frac{\partial e}{\partial K_i} + \delta\frac{\partial e}{\partial K_i}\right]$ − sign $\frac{\partial e}{\partial K_i}$
time
(d) e (in quadrature with $\frac{\partial e}{\partial K_i}$)
$2|e|$
time
$-2|e|$
(e) e $\left[\text{sign}\left(\frac{\partial e}{\partial K_i} + \delta\frac{\partial e}{\partial K_i}\right) - \text{sign}\frac{\partial e}{\partial K_i}\right]$

FIG. 6.2.1.

occurring within one period T, at times T_1 and $T_1 + T/2$ respectively, where T_1 is the time when the $\frac{\partial e}{\partial K_i}$ signal first changes sign.

The integrated values of these pulses is $\pm 2|e|\Delta T$, where $|e|$ is the peak amplitude of the error signal and ΔT the duration of the pulse and given by

$$\Delta T = \left| \delta \left(\frac{\partial e}{\partial K_i}\right) \right| / \left| \text{ slope of } \frac{\partial e}{\partial K_i} \text{ signal at the points where it changes sign} \right|$$

$$= \left| \delta \left(\frac{\partial e}{\partial K_i}\right) \right| / \left[\omega \left| \frac{\partial e}{\partial K_i} \right| \right] \qquad (6.2.11)$$

where $|\partial e/\partial K_i|$ is the peak amplitude of the $\partial e/\partial K_i$ signal and ω the frequency of the input signal.

It was shown earlier that the signal $\partial e/\partial K_i$ is obtained by passing the signal θ_F through a suitable filter, so that $\delta(\partial e/\partial K_i)$ is a filtered version of $\delta\theta_F$.

6.2.2.1. $\underline{K_1}$ <u>adapting alone</u>

When K_1 is adapting alone δK_2 is zero and equation (6.2.8) becomes :

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta e + A_1A_2\delta K_1 D\theta_s = 0 \qquad (6.2.12)$$

From fig. (6.1.1)

$$\theta_F = \frac{1}{A_3K_2} D\theta_s$$

so that

$$\delta\theta_F = \frac{1}{A_3K_2} \delta(D\theta_s)$$

Since $\theta_s = e + \theta_m$ and $\delta\dot{\theta}_m = 0$ we have that

$$\delta\theta_F = \frac{1}{A_3K_2} \delta\dot{e}$$

Hence,

$$\delta \frac{\partial e}{\partial K_1} = - \frac{1}{A_3 K_2} \delta \dot{e}_F \quad ,$$

where $\delta \dot{e}_F = Y_m \delta \dot{e}$ or

$$(D^2 + 6.32D + 40)\delta \dot{e}_F = 40\delta \dot{e} \qquad (6.2.13)$$

From equation (6.2.10)

$$\delta \dot{K}_1 = - G_1 [\delta e \text{ sign} \frac{\partial e}{\partial K_1} + P_1(t)] , \qquad (6.2.14)$$

where $P_1(t)$ are pulses, with integrated values :

$$\pm \frac{2|e|\delta \dot{e}_F}{A_3 K_2 \text{ amp} \frac{\partial e}{\partial K_1}} = \pm P_1{}^1(t)\delta \dot{e}_F \quad ,$$

occurring at times T_1 and $T_1 + \frac{T}{2}$, T being the period and T_1 the time when the $\frac{\partial e}{\partial K_1}$ signal first changes sign. Care must be taken (by examination of the phase angles) to ensure that the positive and negative pulses are inserted in the correct order.

Equations (6.2.12), (6.2.13), (6.2.14) lead to the following system of linear differential equations :

$$\frac{d}{dt}\begin{bmatrix} \delta e \\ \delta \dot{e} \\ \delta \ddot{e} \\ \delta \dot{e}_F \\ \delta \ddot{e}_F \\ \delta K_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -A_1A_2A_3K_2 & -A_1A_2K_1 & -A_1 & 0 & 0 & -A_1A_2D\theta_s \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 40 & 0 & -40 & -6.32 & 0 \\ -G_1 \text{sign} \frac{\partial e}{\partial K_1} & 0 & 0 & -G_1P_1{}^1(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta e \\ \delta \dot{e} \\ \delta \ddot{e} \\ \delta \dot{e}_F \\ \delta \ddot{e}_F \\ \delta K_1 \end{bmatrix}$$

(6.2.15)

where K_2 has the fixed value 0.5 and K_1 is given by equation (6.2.2.). Equation (6.2.15) is of the form

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}(t)\ ,\ \underline{A}(t) = \underline{A}(t + T)$$

and the stability of such a system of equations is discussed in detail in chapter 1. Note that the 6 x 6 matrix of equation (6.2.15) differs from the 4 x 4 matrix suggested by Parks [85]. The increase in the order of the matrix is due to the fact that Parks had neglected the effect of the filter Y_m and taken $\delta\theta_F$ as an approximation for $\delta(\partial e/\partial K_1)$. This approximation was found to be inadequate and the effect of the filter must be taken into account.

The eigenvalues of the monodromy matrix $\underline{C}$ of the matrix differential equation (6.2.15) are examined, using the procedure described in section 1.4., for various values of the adaptive gain G_1 and the critical values of the gain calculated. The results are illustrated in fig. (6.2.2.) together with the results obtained by White using simulator and analytical studies. Since, in this case, the critical gain is inversely proportional to the amplitude ℓ of the input signal the product of ℓ with critical gain is plotted against the frequency of the input signal so that the resulting curve is independent of ℓ (In order to obtain the graph of reference (2) divide the ordinates by 7/50).

6.2.2.2. $\underline{K_2}$ <u>adapting alone</u>

In this case $\delta K_1 = 0$ and equation (6.2.8) becomes :

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta e - A_1A_2A_3e_s\delta K_2 = 0 \qquad (6.2.16)$$

From fig. (6.1.1)

$$\theta_F = \frac{1}{A_3K_2}\ D\theta_s \qquad (6.2.17)$$

When the perturbation is imposed this becomes :

$$\begin{aligned} A_3(K_2 + \delta K_2)(\theta_F + \delta\theta_F) &= D\theta_s + \delta(D\theta_s) \\ &= D\theta_s + \dot{\delta e} \qquad (6.2.18) \end{aligned}$$

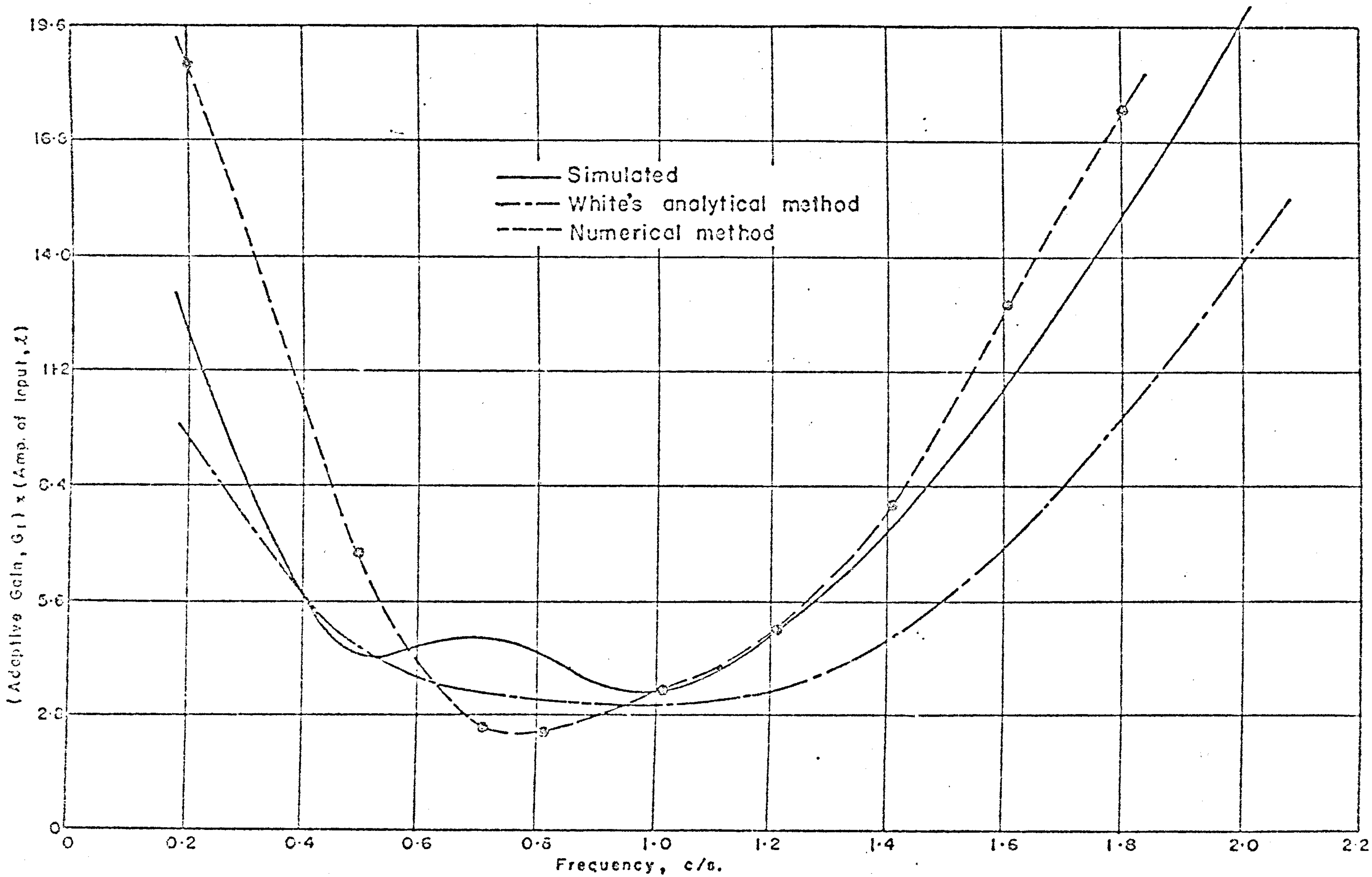

FIG. 6.2.2. Critical adaptive gain with K_1 loop adapting alone

Subtracting equation (6.2.17) from equation (6.2.18) gives

$$A_3K_2\delta\theta_F \;+\; A_3\theta_F\delta K_2 \;=\; \delta\dot{e} \quad ,$$

that is,

$$\delta\theta_F \;=\; \frac{1}{A_3K_2}\;\delta\dot{e} - \frac{\theta_F}{K_2}\;\delta K_2 = X$$

Hence,

$$\delta\left(\frac{\partial e}{\partial K_2}\right) \;=\; X_F \;,\ \text{where } X_F = \frac{Y_3}{Y_1}\,(X)$$

From equation (6.2.10)

$$\delta\dot{K}_2 \;=\; -G_2\left[\,\delta e \text{ sign}\,\frac{\partial e}{\partial K_2} \;+\; P_2(t)\right] \qquad (6.2.19)$$

where $P_2(t)$ are pulses, with integrated values

$$\pm\,\frac{2|e|X_F}{\omega \text{ amp}\,\frac{\partial e}{\partial K_2}} \;=\; \pm\,P_2{}^1(t)\;X_F \;,$$

occurring at times T_2 and $T_2 + \frac{T}{2}$, T being the period and T_2 the time when the signal $\frac{\partial e}{\partial K_2}$ first changes sign; care again being taken to ensure that the sign of the pulses are taken in the correct order.

Equation (6.2.16) and (6.2.19) may be written (see appendix 6.1) in the form

$$\frac{d}{dt}\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ Y_1\\ Y_2\\ Y_3\\ \delta K_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -A_1A_2A_3K_2 & -A_1A_2K_1 & -A_1 & 0 & 0 & 0 & -A_1A_2A_3e_s\\ 0 & 1/K_2 & 0 & 0 & 1 & 0 & -\frac{A_3}{K_2}\,\theta_F\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -570 & -124.8 & -16.78 & 0\\ -G_2\text{sign}\,\frac{\partial e}{\partial K_2} & 0 & 0 & -G_2P_2{}^1(t) & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ Y_1\\ Y_2\\ Y_3\\ \delta K_2\end{bmatrix}$$

(6.2.20)

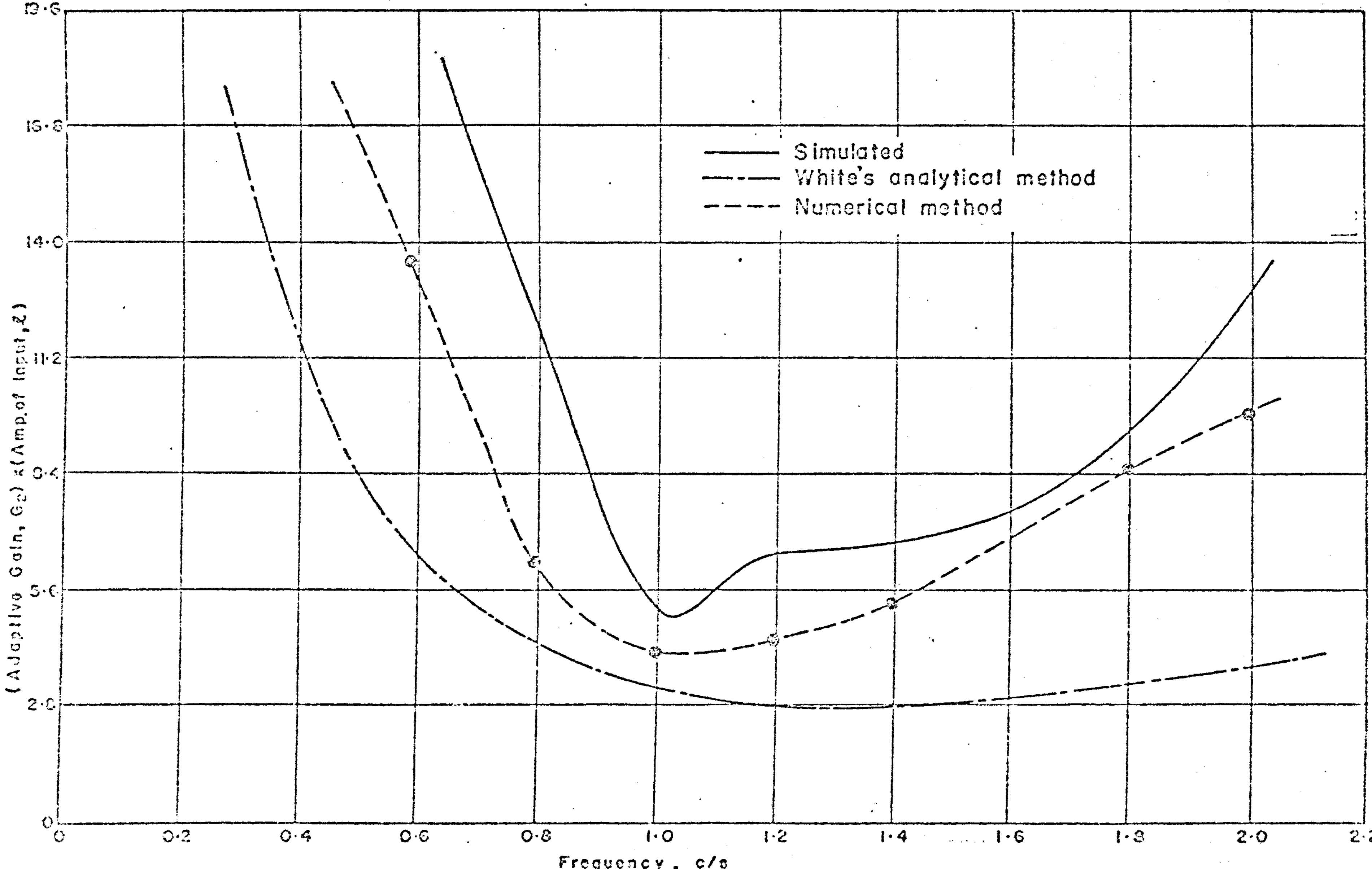

FIG. 6.2.3. Critical adaptive gain with K_2 locp adapting alone

where $Y_1 = X_F$, $Y_2 = \dot{X}_F - 4.57X$ and $Y_3 = \ddot{X}_F - 4.57X$; K_1 has the fixed value 0.5 and K_2 is given by equation (6.2.4).

The eigenvalues of the monodromy matrix, of the system of equations (6.2.20), are again examined using the procedure described in section 1.4. Critical values of the gain G_2 are calculated for varying input frequencies and the results are illustrated in fig. (6.2.3). The method is readily extended to cover the case when both K_1 and K_2 are adapting simultaneously.

6.3. Random input

6.3.1. Steady-state values of the parameters

In order to simplify the analysis the diode switching units are replaced by multipliers. If the input $\theta_i(t)$ to the system is a random variable of time then the output of the multiplier M_j $(j = 1,2)$ will be the product of two random signals $e(t)$ and $\frac{\partial e(t)}{\partial K_j}$, $j = 1,2$, which are both filtered forms of $\theta_i(t)$.

A block diagram showing the output of a typical multiplier M_j is shown in fig. (6.3.1), where $Y_1(s)$ and $Y_2(s)$ are the transfer functions relating $e(t)$ and $\frac{\partial e(t)}{\partial K_j}$ respectively to the input $\theta_i(t)$. If $h_1(\tau)$ and $h_2(\tau)$ are the impulse response or weighting functions corresponding to the transfer functions $Y_1(s)$ and $Y_2(s)$ respectively then $e(t)$ and $\frac{\partial e(t)}{\partial K_j}$ are given, in the time domain, by the convolution integrals.

$$e(t) = \int_0^\infty h_1(\tau_1)\, \theta_i\,(t - \tau_1)\, d\tau_1 \tag{6.3.1}$$

$$\frac{\partial e(t)}{\partial K_j} = \int_0^\infty h_2(\tau_2)\, \theta_i\,(t - \tau_2)\, d\tau_2 \tag{6.3.2}$$

so that the output of the multiplier M_j is

$$x_0(t) = e(t)\,\frac{\partial e(t)}{\partial K_j} = \int_0^\infty h_1(\tau_1)\theta_i(t - \tau_1)d\tau_1 \int_0^\infty h_2(\tau_2)\theta_i(t - \tau_2)d\tau_2 \tag{6.3.3}$$

Taking time averages we have that the time average of the output is

$$\overline{x_o(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_o(t)dt$$

$$= \int_0^\infty h_1(\tau_1) \int_0^\infty h_2(\tau_2) \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \theta_i(t - \tau_1)\theta_i(t - \tau_2)dt \right] d\tau_2 d\tau_1 \qquad (6.3.4)$$

Thus, provided $\theta_i(t)$ is a stationary process, equation (6.3.4) becomes

$$\overline{x_o(t)} = \int_0^\infty h_1(\tau_1) \int_0^\infty h_2(\tau_2)\, \phi(\tau_2 - \tau_1)d\tau_2 d\tau_1 \quad , \qquad (6.3.5)$$

where $\phi(\tau)$ is the autocorrelation function of the input signal $\theta_i(t)$.

If the input signal $\theta_i(t)$ is regarded as white noise with constant spectral density K then, by the Wiener-Khinchine relationship,

$$\phi(\tau) = K\pi\delta(\tau) \quad ,$$

so that equation (6.3.5) becomes

$$\overline{x_o(t)} = \int_0^\infty h_1(\tau_1) \int_0^\infty h_2(\tau_2)\pi K\delta(\tau_2 - \tau_1)\, d\tau_2 d\tau_1 \quad , \qquad (6.3.6)$$

Since this expression is zero except when $\tau_1 = \tau_2$ we may write

$$\tau = \tau_1 = \tau_2$$

so that equation (6.3.6) becomes

$$\overline{x_o(t)} = \pi K \int_0^\infty h_1(\tau)\, h_2(\tau)\, d\tau \qquad (6.3.7)$$

The value of $\overline{x_o(t)}$ may therefore be obtained by calculating $h_1(\tau)$ and $h_2(\tau)$ and integrating the integral of equation (6.3.7) term by term. When the system has reached its steady state value $\overline{x_0(t)}$ will be zero so that equating the value of $\overline{x_0(t)}$, obtained from equation (6.3.7), to zero gives the steady state value of the parameter K_i. However, even for the third order system considered here, evaluating the integral of equation (6.3.7) is a laborious procedure and for this reason the calculations are transferred to the frequency domain. The output of the multiplier, instead of being the product of two time functions, becomes a convolution of two frequency functions. For white noise input with spectral density K Horrocks [86] showed that

$$\overline{x_0(t)} = K \int_0^\infty |Y_i(\omega)||Y_2(\omega)| \cos (\emptyset - \emptyset_2)\, d\omega \qquad (6.3.8)$$

where $Y_i(\omega) = |Y_i(\omega)| \exp (j\, \emptyset_i)$, $i = 1,2$.

The integral of equation (6.3.8) may be evaluated numerically, using a digital computer, for various values of the parameter K_j and a graph of $\overline{x_0(t)}$ against K_j plotted to obtain the steady state value of K_j.

If, as in the system of fig. (6.1.1), the multipliers M_j ($j = 1,2$) be replaced by a diode switching unit, the output of which is $\pm e(t)$ depending on the sign of $\frac{\partial e(t)}{\partial K_j}$, then the way in which this change will effect the value of $\overline{x_0(t)}$ given by equations (6.3.7) and (6.3.8) has been calculated by Jackson [86]. He showed that, in this case, the resultant d.c. output will be

$$\overline{x_0(t)}_{d.s.u} = \sqrt{\frac{2}{\pi}}\ \frac{\overline{x_0(t)}}{\sigma_j} \qquad (6.3.9)$$

where σ_j is the r.m.s. value of $\frac{\partial e}{\partial K_j}$.

In order to illustrate the theory we shall consider the case of K_1 adapting alone. From equations (6.1.1) - (6.1.3) we have that

$$e(s) = \left[\frac{A_1A_2A_3K_2}{s^3 + A_1s^2 + A_1A_2K_1s + A_1A_2A_3K_2} - \frac{40}{s^2 + 6.32s + 40} \right] \theta_i(s) ,$$

where $e(s)$ and $\theta_i(s)$ denote the Laplace transforms of $e(t)$ and $\theta_i(t)$ respectively. Thus

$$Y_1(s) = \frac{A_1A_2A_3K_2}{s^3 + A_1s^2 + A_1A_2K_1s + A_1A_2A_3K_2} - \frac{40}{s^2 + 6.32s + 40} \tag{6.3.10}$$

Also, from equations (6.1.5) and (6.1.6),

$$\frac{\partial e(s)}{\partial K_1} = -Y_m(s)\ \theta_F(s)$$

$$= - \frac{40}{s^2 + 6.32s + 40} \cdot \frac{A_1A_2}{s^3 + A_1s^2 + A_1A_2K_1s + A_1A_2A_3K_2}\ \theta_i(s) ,$$

so that ,

$$Y_2(s) = - \frac{40\ A_1A_2}{(s^2 + 6.32s + 40)(s^3 + A_1s^2 + A_1A_2K_1s + A_1A_2A_3K_2)} \tag{6.3.11}$$

Substituting $s = j\omega$ in equations (6.3.10) and (6.3.11) and rationalizing the values of $|Y_i(\omega)|$ and ϕ_i, $i = 1,2$, are readily obtained. These values are then substituted into the integral of equation (6.3.8) and the integral evaluated numerically for a range of values of K_1, and K_2 fixed at its nominal value 0.5. The integral is evaluated, using Simpson's rule (and cross checked using the trapezoidal rule), for limits 0 to N and the values of N increased until a convergent value is obtained to required degree of accuracy. The form of the integrand when $K_1 = 0.5$ is shown in fig. (6.3.2) and a plot of $\overline{x_o(t)}$

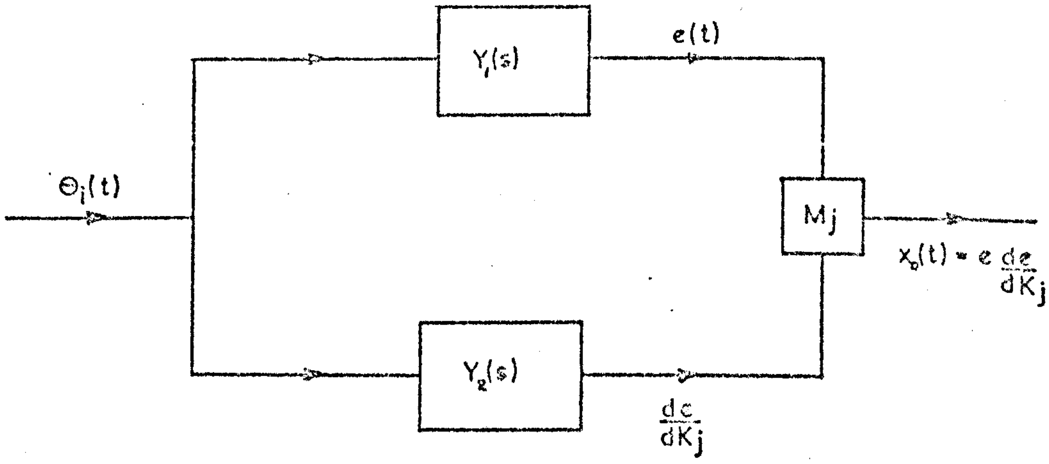

FIG. 6.3.1. Output of typical multiplier

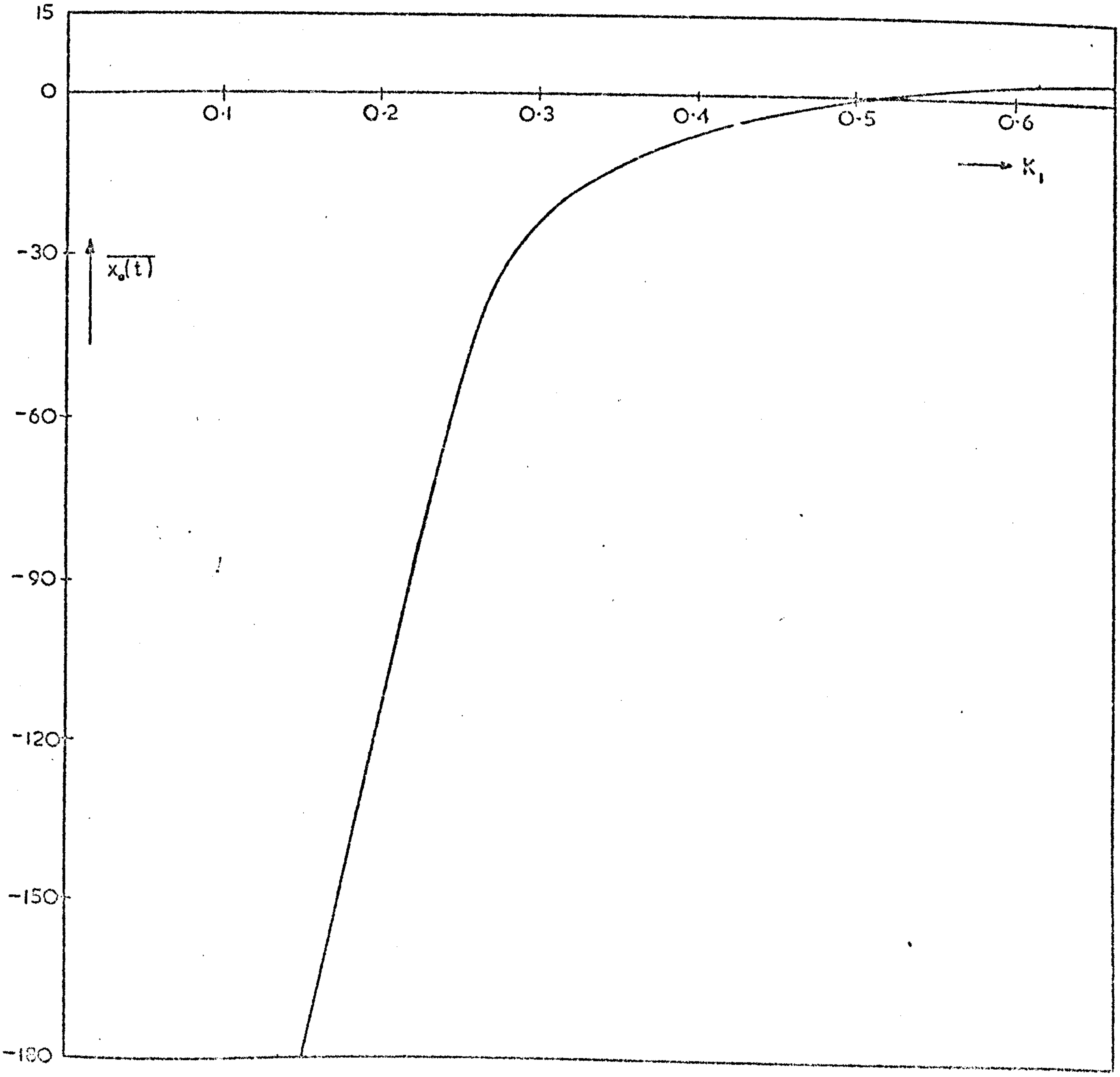

FIG. 6.3.3. Steady-state value of K_1

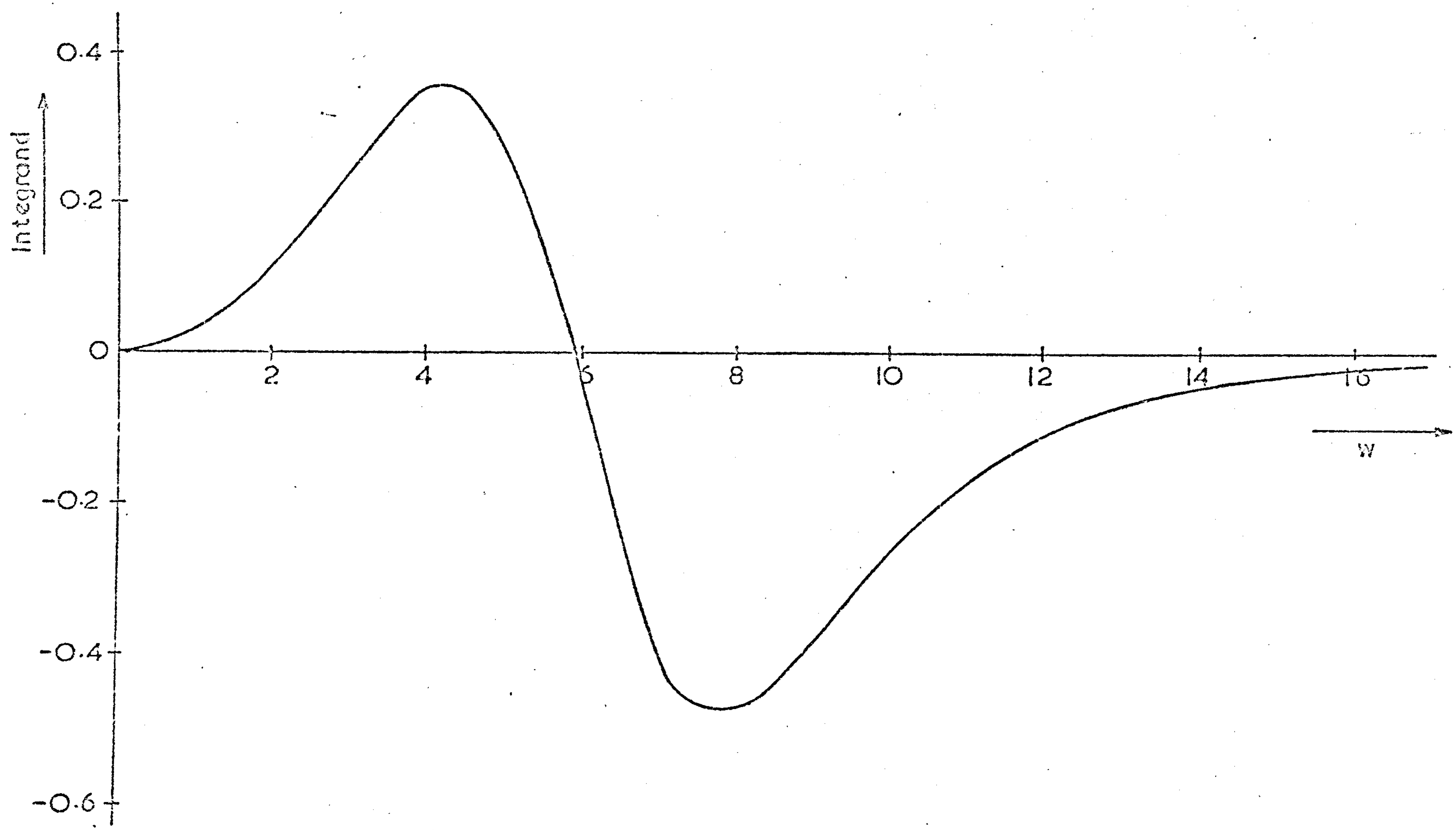

FIG. 6.3.2. Integrand of integral (6.3.8) when $K_1 = 0.5$

against K_1 is shown in fig. (6.3.3). It is seen from fig. (6.3.3) that $x_0(t)$ is zero when

$$K_1 = 0.513 \qquad (6.3.12)$$

this then being the required steady state value of the parameter K_1

6.3.2 Stability consideration

Having found the steady state value of the parameter K_j stability is examined in a manner analogous to that employed in section (6.2.2) for the case of the input varying sinusoidally with time. We shall again illustrate by considering the case of K_1 adapting alone.

As for the case of sinusoidal input the perturbed equation of the system is as equation (6.2.8), namely

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta e + A_1A_2D\theta_s\ \delta K_1 - A_1A_2A_3e_s\ \delta K_2 = 0 \qquad (6.3.13)$$

where, in this case, $D\theta_s$ and e_s are random variables of time. Since we are considering K_1 adapting alone and K_2 fixed at its nominal value (i.e. $\delta K_2 = 0$) equation (6.3.13) becomes

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\delta e + A_1A_2D\theta_s\ \delta K_1 = 0 \qquad (6.3.14)$$

Since we are considering the switching units as being replaced by multipliers the parameter K_1 is given by the adaptive control law

$$\dot{K}_1(t) = \frac{\partial K_1(t)}{\partial t} = -G_1 e(t)\frac{\partial e(t)}{\partial K_1} \qquad (6.3.15)$$

In the perturbed state this becomes

$$\dot{K}_1(t) + \delta\dot{K}_1(t) = -G_1[e(t) + \delta e(t)]\left[\frac{\partial e(t)}{\partial K_1} + \delta\frac{\partial e(t)}{\partial K_1}\right] \qquad (6.3.16)$$

Subtracting equation (6.3.15) from equation (6.3.16) gives

$$\dot{\delta K_1}(t) = -G_1 \frac{\partial e(t)}{\partial K_1} \delta e(t) - G_1 e(t) \delta\left(\frac{\partial e(t)}{\partial K_1} \right) \tag{6.3.17}$$

Now $\frac{\partial e(t)}{\partial K_1}$ is a filtered form of $\theta_F(t)$ so that $\delta\left(\frac{\partial e}{\partial K_1} \right)$ is a filtered form of $\delta\theta_F(t)$. From fig. (6.1.1)

$$\theta_F = \frac{1}{A_3 K_2} D\theta_s$$

so that $$\delta\theta_F = \frac{1}{A_3 K_2} \delta(D\theta_s).$$

Since $\theta_s = e + \theta_m$ and $\dot{\delta\theta_m} = 0$ we have that

$$\delta\theta_F = \frac{1}{A_3 K_2} \dot{\delta e}$$

Hence,

$$\delta\left(\frac{\partial e}{\partial K_1} \right) = - \frac{1}{A_3 K_2} \dot{\delta e}_F \tag{6.3.18}$$

where δe_F is given by equation (6.2.13).

Substituting equation (6.3.18) in equation (6.3.17) gives

$$\dot{\delta K_1} = -G_1 \frac{\partial e}{\partial K_1} \delta e + \frac{G_1 e(t)}{A_3 K_2} \dot{\delta e}_F \tag{6.3.19}$$

Equations (6.3.14), (6.3.19) and (6.2.13) lead to the following system of linear differential equations

$$\frac{d}{dt}\begin{bmatrix} \delta e \\ \dot{\delta e} \\ \ddot{\delta e} \\ \dot{\delta e}_F \\ \ddot{\delta e}_F \\ \delta K_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -A_1A_2A_3K_2 & -A_1A_2K_1 & -A_1 & 0 & 0 & -A_1A_2D\theta_s(t) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 40 & 0 & -40 & -6.32 & 0 \\ -G_1 \frac{\partial e(t)}{\partial K_1} & 0 & 0 & \frac{G_1 e(t)}{A_3 K_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta e \\ \dot{\delta e} \\ \ddot{\delta e} \\ \dot{\delta e}_F \\ \ddot{\delta e}_F \\ \delta K_1 \end{bmatrix} \tag{6.3.20}$$

where K_2 has the fixed value 0.5 and K_1 given by equation (6.3.12).

Note that the terms $D\theta_s(t)$, $e(t)$ and $\frac{\partial e(t)}{\partial K_1}$ are all random variables of time and since they are all filtered forms of the input signal $\theta_i(t)$ they are correlated; further white noise assumptions for the coefficients of system (6.3.20) are not justifiable. Equation (6.3.20) is of the form

$$\underline{\dot{x}}(t) = \underline{A}(t)\ \underline{x}(t) \tag{6.3.21}$$

where the non constant elements of $\underline{A}(t)$ are random functions of time. As was pointed out in section 4.5. methods of investigating the stability of system (6.3.21), where the system is not asymptotically zero when the noise terms are equated to zero, are not forthcoming. Thus, for the case of $\theta_i(t)$ being purely random, stability boundaries for the system of equations (6.3.20) are not readily obtained. However, the effect on stability, of random disturbances at the system input may be examined in a manner analogous to that employed in section 4.5. for the first order system. Since the intention of this section is to illustrate how the ideas of chapter 4 may be extended to higher order systems detailed calculations for the stability of the system of equations (6.3.20), when the input signal consists of a step function plus a random variable, will not be presented.

6.4. Liapunov redesign system

In the paper presented by Parks [77] the model and system are of the same order, so that, when equilibrium is achieved the model-system error and all parameter differences are zero. However, in this case, the model is of the second order whilst the system is of the third order and under such conditions it is no longer possible to have perfect correspondence between the two. It follows therefore that for such systems it is not possible to obtain a Liapunov function that will guarantee asymptotic stability.

One way of dealing with this problem is to use the approach described by Shackcloth [87]. Adjustable parameters are introduced around the system (or plant), as shown in fig. (6.4.1), so that the

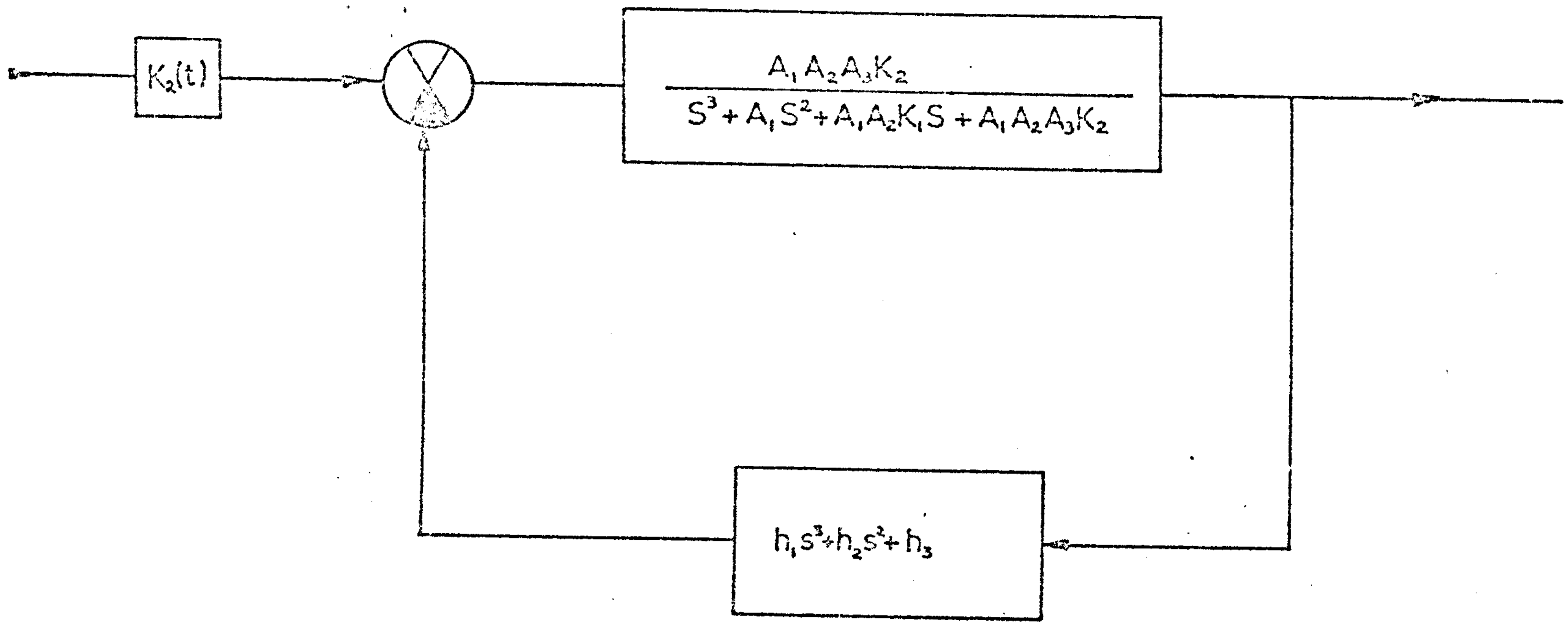

FIG. 6.4.1. Adjustable parameters around the process

equation of the controlled system or plant is

$$[(1 + h_1(t)A_1A_2A_3)D^3 + (A_1 + h_2(t)A_1A_2A_3)D^2 + A_1A_2K_1(t)D + (A_1A_2A_3K_2(t) + h_3(t)A_1A_2A_3)]\theta_s(t) = A_1A_2A_3K_2(t)\theta_i(t) \tag{6.4.1}$$

Since control is exercised over each plant parameter some are adjusted to zero thus enabling the plant to be made the same order as the model of a lower order.

Taking the system response $\theta_s(t)$ and model response $\theta_m(t)$ to be given by equations (6.4.1) and (6.1.2) respectively the error equation (6.1.3) may be written in the form

$$\begin{aligned}(D^2 + 6.32D + 40)e(t) &= 40\ \theta_i(t) - (D^2 + 6.32D + 40)\theta_s(t) \\ &= (40 - A_1A_2A_3K_2(t))\theta_i(t) + (1 + h_1{}^1(t))D^3\theta_s(t) - (1 - A_1 - h_2{}^1(t))D^2\theta_s(t) \\ &\quad - (6.32 - A_1A_2K_1(t))D\theta_s(t) - (40 - A_1A_2A_3K_2(t) - h_3{}^1(t))\theta_s(t)\end{aligned} \tag{6.4.2}$$

where,

$$h_i{}^1(t) = A_1A_2A_3h_i(t)\ ,\ i = 1,2,3\ , \tag{6.4.3}$$

that is,

$$\begin{aligned}(D^2 + 6.32D + 40)e(t) &= x_1(t)\theta_i(t) + x_2(t)D^3\theta_s(t) - x_3(t)D^2\theta_s(t) \\ &\quad - x_4(t)D\theta_s(t) - x_5(t)\theta_s(t)\end{aligned} \tag{6.4.4}$$

where $x_i(t)$, $i = 1,2,3,4,5$, are the coefficients in equation (6.4.2).

By assuming that any variations of the process parameters, the input signal, or the derivatives of the input signal, is continuous and bounded Liapunov's second may be applied. Following Shackcloth we take as Liapunov function

$$V = \underline{y}^1 \, \underline{H} \, \underline{y} + \frac{x_1^2}{B_1} + \frac{x_2^2}{B_2} + \frac{x_3^2}{B_3} + \frac{x_4^2}{B_4} + \frac{x_5^2}{B_5} \tag{6.4.5}$$

where $\underline{y}^1 = \{e(t) \; \dot{e}(t)\}$ and $\underline{H}$ is the Hermite matrix of the equation

$$(D^2 + 6.32D + 40) \, e(t) = 0 \tag{6.4.6}$$

and B_i, $(i = 1,2,3,4,5)$, are positive constants.

Since, in this case,

$$\underline{H} = \begin{bmatrix} 252.8 & 0 \\ 0 & 6.32 \end{bmatrix}$$

the Liapunov function V, defined by equation (6.4.5), becomes

$$V = 252.8 \, e^2(t) + 6.32 \, \dot{e}^2(t) + \frac{x_1^2}{B_1} + \frac{x_2^2}{B_2} + \frac{x_3^2}{B_3} + \frac{x_4^2}{B_4} + \frac{x_5^2}{B_5} \tag{6.4.7}$$

The derivative of V, with respect to time, is

$$\dot{V} = 2(252.8)e\dot{e} + 2(6.32)\dot{e}\ddot{e} + 2\left[\frac{x_1\dot{x}_1}{B_1} + \frac{x_2\dot{x}_2}{B_2} + \frac{x_3\dot{x}_3}{B_3} + \frac{x_4\dot{x}_4}{B_4} + \frac{x_5\dot{x}_5}{B_5}\right] \tag{6.4.8}$$

Substituting for $\ddot{e}(t)$ from equation (6.4.4) equation (6.4.8) becomes

$$\dot{V} = 2(6.32)\dot{e}(t) \left[x_1\theta_i + x_2D^3\theta_s - x_3D^2\theta_s - x_4D\theta_s - x_5\theta_s - 6.32\dot{e}(t)\right]$$

$$+ 2\left[\frac{x_1\dot{x}_1}{B_1} + \frac{x_2\dot{x}_2}{B_2} + \frac{x_3\dot{x}_3}{B_3} + \frac{x_4\dot{x}_4}{B_4} + \frac{x_5\dot{x}_5}{B_5}\right], \tag{6.4.9}$$

that is,

$$\dot{V} = -2(6.32)^2 \, \dot{e}^2(t) \tag{6.4.10}$$

provided

$$\begin{aligned}
\dot{x}_1(t) &= -6.32\dot{e}(t)B_1\theta_i(t) \\
\dot{x}_2(t) &= -6.32\dot{e}(t)B_2D^3\theta_s(t) \\
\dot{x}_3(t) &= 6.32\dot{e}(t)B_3D^2\theta_s(t) \\
\dot{x}_4(t) &= 6.32\dot{e}(t)B_4D\theta_s(t) \\
\dot{x}_5(t) &= 6.32\dot{e}(t)B_5\theta_s(t)
\end{aligned} \qquad (6.4.11)$$

Substituting for $x_i(t)$, i = 1,2,3,4,5, from equation (6.4.2) we have that

$$\begin{aligned}
\dot{K}_2(t) &= C_1\dot{e}(t)\ \theta_i(t) \\
\dot{h}_1(t) &= -C_2\dot{e}(t)D^3\theta_s(t) \\
\dot{h}_2(t) &= -C_3\dot{e}(t)D^2\theta_s(t) \\
\dot{K}_1(t) &= -C_4\dot{e}(t)D\theta_s(t) \\
\dot{K}_2(t) + \dot{h}_3(t) &= -C_5\dot{e}(t)\theta_s(t)
\end{aligned} \qquad (6.4.12)$$

where C_i, (i = 1,2,3,4,5) are positive constants given by

$$C_i = 6.32\ B_i/(A_1A_2A_3)\ ,\ i = 1,2,4,5$$

and $$C_4 = 6.32\ B_4/(A_1A_2)$$

Thus, provided that the model is stable, which it is, use of adaptive loops, defined by equations (6.4.12), will ensure that the system (or plant) and model outputs will eventually become identical. Using these adaptive loops the complete layout of the model reference system will be as shown in fig. (6.4.2). The most serious objection to the form of these adaptive loops is that differentiators have to be employed, and consequently, in a practical system, serious noise difficulties could occur.

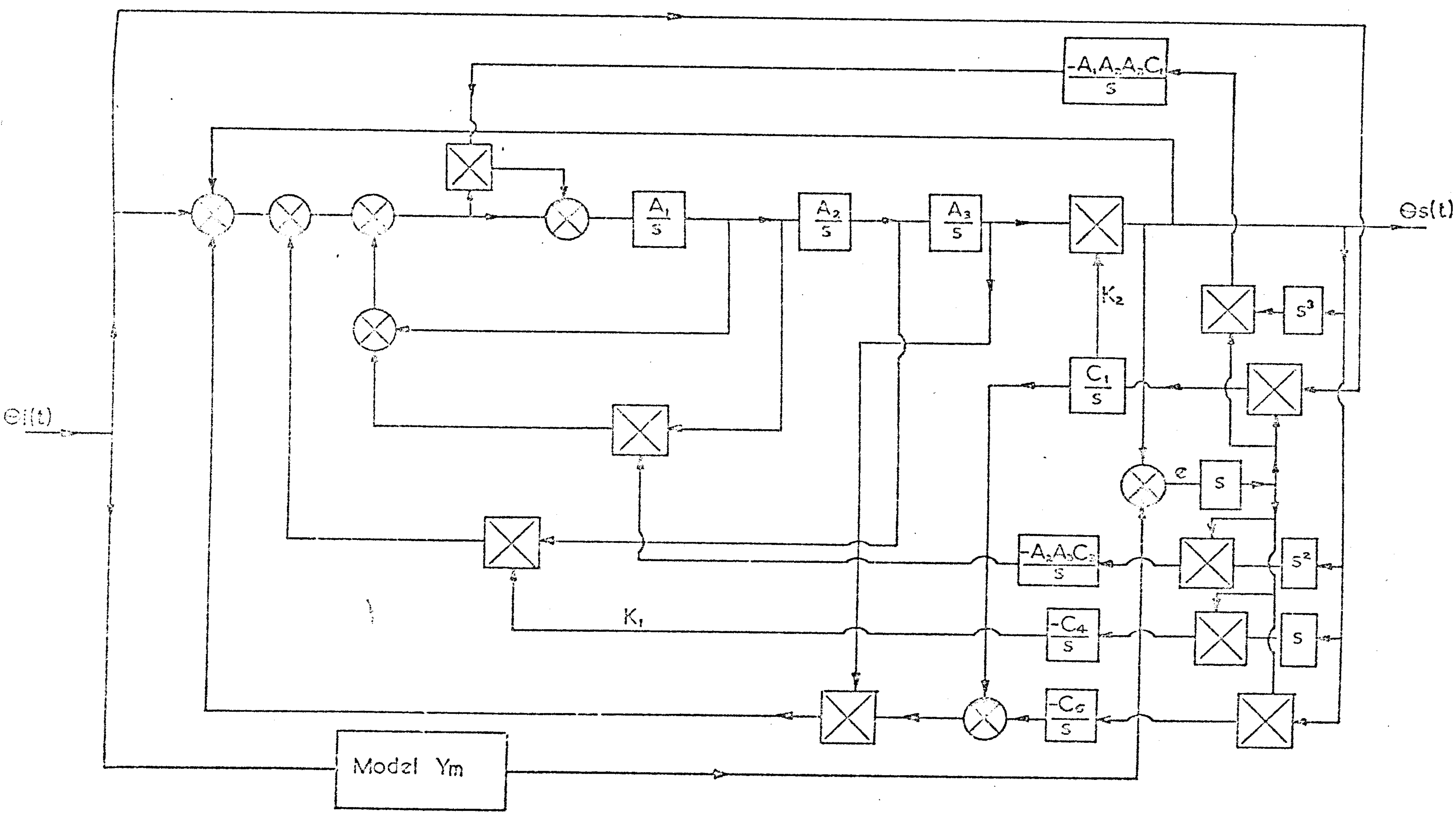

FIG. 6.4.2. Liapunov redesign of fig. (6.1.1)

CHAPTER 7

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this thesis the stability and accuracy of parameter adaptation of hill-climbing and model reference adaptive control systems has been considered. The differential equations governing such systems are both nonlinear and non-autonomous and, depending on whether the parameters vary with time in a periodic or random manner, the local stability problem reduced to one of investigating the stability of a system of linear differential equations with periodic coefficients or a system of linear differential equations with random coefficients. For periodic variations the accuracy of parameter adaptation was investigated using the principle of harmonic balance whilst the Fokker-Planck equation was used for the case of the time varying terms being white noise Gaussian processes.

When dealing with linear differential equations with periodic coefficients both a numerical implementation of Floquet theory and the infinite determinant method have been employed to investigate stability. The method based on Floquet theory is a rigorous numerical method, well suited for use on a digital computer, for obtaining necessary and sufficient conditions for asymptotic stability. The main disadvantage with this method is that it requires the formation of a gridwork in the parameter space and then make an assessment of stability for each of the nodal points of the gridwork; as a consequence the method could be expensive on computer time. However, for the two dimensional problems considered in this work a plot of the value of the dominant eigenvalue of the monodromy matrix against system parameter followed a definite form, for example, see section 4.4.1., which suggests that it may be possible, in some cases, to obtain mathematical expressions for the eigenvalues of the monodromy matrix. Thus a problem requiring further research is that of investigating the possibility of obtaining theoretical expressions for the eigenvalues of the monodromy matrix of, possibly, a restricted class of linear differential equations with periodic coefficients. For higher order systems the use of the Faddeev algorithm and Jury procedure greatly reduce computer time over a direct evaluation of the eigenvalues of the monodromy matrix. However, it should be pointed out that direct evaluation of the eigenvalues gives more information than simply the existence of a stability boundary; in particular, it yields useful information concerning the character of the solutions in the regions away from the stability boundary.

The infinite determinant method is not as general as the Floquet analysis and is limited in application to a restricted class of linear differential equations. It is clear that much further work is to be done to permit the identification of those systems for which the method is applicable. Even when it is applicable it was not found to be very satisfactory in the region of parameter space where the stability boundaries are complex in nature, for example, see fig. (4.4.5) and since, when dealing with linear differential equations with periodic coefficients, such boundaries frequently occur it throws some doubt on the performance of the method in general. However, due to its computational simplicity, the method, when applicable, may be used to obtain preliminary results and then the Floquet analysis used to obtain an improvement in the accuracy with which the stability boundaries are established.

Although a vast amount of literature exists on the stability of linear differential equations with random coefficients, it is apparent from the problems considered in this thesis that a great deal of further research is to be done before the results are of any significant value in application to practical problems. Which of the stability concepts discussed in chapter 2 is most significant in practice is still an open question although it is becoming more accepted that almost sure asymptotic stability is the ultimate aim. However, almost sure properties are not as immediately obtainable as are mean properties of the system and the relationship between the two is somewhat nonintiutive. Caughey [44] mentioned "that mean square stability is a necessary but not sufficient condition for stability of a system" but in fact, as pointed out by Kozin [20], almost exactly the opposite is the case. For linear systems whose coefficients vary as white noise processes Kozin [23] showed that exponential stability of the second moments implies almost sure asymptotic stability.

If the system is not stable when the random terms are equated to zero [that is, the system is of the form $\dot{\underline{x}} = \underline{A}(t)\underline{x}$, where the non-constant elements of $\underline{A}(t)$ vary randomly with time and $\dot{\underline{x}} = \underline{A}(o)\underline{x}$ does not represent a stable system] then there is, at present, no method available for investigating its stability. However, digital simulation of the first order MIT system, when the input was purely random, indicates that stability boundaries exist for such systems so that obtaining theoretical results for investigating such systems is an obvious field for future research.

If the system equations may be written in the form

$$\dot{\underline{x}} = [\ \underline{A} + \underline{F}(t)\ ]\ \underline{x}\ ,$$

where the non-vanishing elements of $\underline{F}(t)$ are random processes and $\dot{\underline{x}} = \underline{A}\ \underline{x}$ is a stable system, then two different cases have been considered in this work.

(i) When the non-vanishing elements of $\underline{F}(t)$ are Gaussian white noise processes conditions for stability in the mean and stability in the mean square were readily obtained using the Fokker-Planck equation. The results obtained suggest that stability boundaries obtained using this method agree favourably with those obtained by direct simulation of the system.

(ii) When the non-vanishing elements of $\underline{F}(t)$ are Gaussian non-white processes then the differential equations for the moments of a particular order, deduced from the Fokker-Planck equation, contain terms involving higher order moments so that it is not possible, in this case, to speak of mean square stability, since this would mean neglecting moments of order higher than two. Thus, the stability problem reduces to one of investigating the stability of an infinite system of linear differential equations and although some recent published work [100] exists on this problem its solution remains an open problem for further research. The stability criteria based on Liapunov's second method, which constitute sufficient conditions for almost sure asymptotic stability, have proven to be highly conservative, when compared with results obtained by simulation, and an obvious field for further research is that of obtaining optimum Liapunov functions for the criteria cited in the literature or, as the ultimate aim in a particular problem must be, to obtain criteria which are necessary and sufficient for almost sure asymptotic stability. In the author's opinion a serious disadvantage with these criteria based on Liapunov's second method is that they only involve the variances of the random terms and do not utilize the frequency spectrum of such terms.

The work described in chapter 3 emphasizes the importance of the knowledge of periodic solutions in the analysis of a sinusoidal perturbation, extremal control system. It has been shown that for the system considered the parameter space may be divided into three regions, viz :

(i) Regions where no periodic solutions exist so that the system is totally unstable.

(ii) Region R_1, where there exists two harmonic solutions, one stable and one unstable.

(iii) Region R_3, where, in addition to the two harmonic solutions of (ii), there exists four stable and four unstable sub-harmonic solutions of order 2.

In the case of the subharmonic solutions the periodic solution for the adapting variable contains a d.c. term, so that for certain initial conditions the system, with parameter values in R_2, will adapt to an oscillation about an offset position; that is, the bottom of the hill is not reached. It is important therefore in any practical application to employ parameter values in region R_1, but outside region R_2. Thus a knowledge of the boundaries of the regions R_1 and R_2 is essential in any design consideration.

By plotting the domains of attraction, corresponding to the stable steady state solutions, regions in three dimensional space were obtained, for particular parameter values in R_2, within which initial conditions will lead to a stable oscillation. Information about these stability boundaries in the state-space is also highly relevant in any design consideration of a practical system; for if a system is subjected to random disturbances and noise there will be a finite probability of the system entering any region of its state-space. However, no parameter values will make the system stable everywhere, so that, in order that the probability of the system being driven unstable by the random disturbances and noise is negligibly small, it is essential that the normal region of operation of the system is well within the stability boundary. The theoretical results have been verified by analogue computer simulation of the system.

In order to illustrate the complexity of the problem the stability of a first order MIT type model reference system was first considered. For the case of a sinusoidal input stability boundaries were obtained in non-dimensional space using both the numerical implementation of Floquet theory and the infinite determinant method; these stability boundaries proved to be very complex in nature so that it is desirable that a designer should have some knowledge of such boundaries before embarking on a detailed analogue computer study of the system; this being particularly so due to the difficult scaling problems involved with simulating model reference systems thus making it difficult to decide on a practical criterion for stability, a difficulty which is more pronounced at low frequences.

When the input is purely random stability boundaries have been obtained by digital simulation of the system and the results suggest that the system will be unstable, for all parameter values, when the input is white noise. However, to date, it has not been possible to obtain stability boundaries using theoretical methods and this remains an outstanding problem for future research. In an attempt to solve the theoretical problem the input was regarded as consisting of a sequence of impulses of random magnitude. When the impulses were assumed to be spaced sufficiently far apart in time, in comparison to the system time constant, that the transient effects from a particular impulse have died out before the next impulse arrives the stability problem reduces to one of investigating the convergence of an infinite product for which results have been obtained; however, if the effects of the impulses are allowed to overlap the stability problem is one of examining the convergence of an infinite product of matrices, a problem which, as yet, has not been solved. If the input is taken to be a random variable superimposed on a constant step then theoretical boundaries have been obtained, using criteria based on Liapunov's second method, which constitute sufficient conditions for almost sure asymptotic stability; however, compared to stability boundaries obtained by digital simulation, the theoretical results are rather conservative.

Stability boundaries have also been obtained when the process environmental parameter varies with time in both a periodic and random manner. For the case of random variations the results obtained using the Fokker-Planck equation agree very favourably with results obtained by digital simulation of the system; again the stability boundaries obtained using criteria based on Liapunov's second method prove to be highly conservative. In the case of periodic variations steady state solutions for the system error and adapting parameter have been obtained using the principle of harmonic balance whilst in the case of random variations asymptotic values, with time, for their mean square values have been deduced from the Fokker-Planck equation. It has also been shown that allowing the process environmental parameter to become time varying in Liapunov redesign model reference systems, which have been synthesized from the point of view of stability, gives rise to a stability problem. Also considered has been the effects of random disturbances at the system output; it has been shown that such disturbances have no effect on the system stability whilst they do have an effect on the accuracy of the parameter adaptation.

In chapter 6 the ideas developed for analysing the first order system were extended to investigate the stability of a higher order model reference system when the input varied, with time, in both a sinusoidal and random manner. Steady state values of the adapting parameters were first obtained and then linearized variational equations set up for small disturbances about such steady state values. These equations constitute a set of linear differential equations with periodic coefficient or a set of linear differential equations with random coefficients so that their stability properties may be investigated using the same methods as used for the first order system.

For the case of sinusoidal input theoretical boundaries have been compared with results obtained by White [2] using analogue computer simulation. The theoretical boundaries provides necessary and sufficient conditions for the asymptotic stability of the linearized equations with periodic coefficients. The system, however, is a forced non linear system with periodic coefficients; by appealing to the stability theorems of Zubov [101] the asymptotic stability of the linearized system certainly, in the absence of the forcing term, leads to asymptotic stability in the small for the nonlinear system. The effects of the forcing term could however invalidate the neglect of the nonlinear terms and make the stable region of the linearized system unstable for the non-linear system. The forcing term in this case is of the multiplicative kind and its actual effect on the stable region is obviously a field for further research. Instability of the linearized system however gives sufficient conditions for the instability of the nonlinear system so that the results of this work suggest that the results obtained by White using analogue simulation are not very accurate. As was mentioned earlier it is difficult to decide on a practical criterion for instability when simulating model reference systems, a fact that was reiterated in discussions with White; in this case the presence of harmonics which are forcing the system further masks the problem. The results of this work do, however, suggest that the problem at hand may be studied satisfactorily by considering the stability of the linearized system.

Also considered in chapter 6 are the effects, on the mathematical analysis, of replacing the system multipliers by diode switching units and the complexity of the problem of obtaining a Liapunov redesign system for a model reference system having a model whose order is different to that of the process.

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APPENDIX 1.1.

ALGOL PROGRAM FOR FADDEEV ALGORITHM

```
PROCEDURE LATPOLY(A,P)'
ARRAY A,P'
BEGIN COMMENT THIS PROCEDURE EVALUATES THE COEFFICIENTS P(I) OF
              THE CHARACTERISTIC POLYNOMIAL
                 X**N-P(1)X**(N-1)-        -P(N-1)X-P(N)
              OF THE MATRIX A USING THE METHOD OF FADDEEV'
      INTEGER I,J,K,N'
      ARRAY B,C(1:RANGE(A,1),1:RANGE(A,2))'

      PROCEDURE COPY(X,Y)'
      ARRAY X,Y'
      BEGIN COMMENT THIS PROCEDURE COPIES ARRAY Y INTO ARRAY X'
            INTEGER I,J,K,L'
            I:=ADDRESS(X)'     K:=I+SIZE(X)-1'
            J:=ADDRESS(Y)'
            FOR L:=I STEP 1 UNTIL K DO
              BEGIN LOCATION(L):=LOCATION(J)'
                    J:=J+1'
              END'
      END OF COPY'

      PROCEDURE MULT(A,B,C)'
      ARRAY A,B,C'
      BEGIN COMMENT THIS PROCEDURE MULTIPLIES ARRAY B BY ARRAY C
                    AND PUTS THE RESULT IN ARRAY A'
            INTEGER I,J,K'
            FOR I:=1 STEP 1 UNTIL N DO
            FOR J:=1 STEP 1 UNTIL N DO
               BEGIN A(I,J):=0'
                     FOR K:=1 STEP 1 UNTIL N DO
                        A(I,J):=A(I,J)+B(I,K)*C(K,J)'
               END'
      END OF MULT'

      IF RANGE(A,1) NOTEQ RANGE(A,2) THEN
         BEGIN PRINT £CL??,£LATPOLY ERROR.   MATRIX NOT SQUARE?'
               STOP'
         END'
      N:=RANGE(A,1)'
      COPY(C,A)'
      FOR I:=1 STEP 1 UNTIL N DO
BEGIN P(I):=0'
      FOR J:=1 STEP 1 UNTIL N DO P(I):=P(I)+C(J,J)'
          P(I):=P(I)/I'
      COPY(B,C)'
      FOR J:=1 STEP 1 UNTIL N DO B(J,J):=B(J,J)-P(I)'
      MULT(C,A,B)'
  END'
END OF LATPOLY'
```

APPENDIX 1.2.

ALGOL PROGRAM FOR JURY PROCEDURE

```
BOOLEAN PROCEDURE STABILITY(P)'
ARRAY P'
BEGIN
      COMMENT THIS PROCEDURE PRODUCES A STABILITY TABLE Q FROM
      THE COEFFICIENTS P(I) OF THE POLYNOMIAL
                X**N+P(N-1)X**(N-1)+ - - - -+P(0)
      AND THEN TESTS FOR ROOTS WITHIN THE UNIT CIRCLE USING
      JURYS PROCEDURE.THE FINAL VALUE OF STABILITY IS TRUE IF
      ALL THE ROOTS LIE WITHIN THE UNIT CIRCLE AND FALSE OTHERWISE'

      ARRAY Q(0:SIZE(P)-2)'
      INTEGER N,M,K,I,A'
      SWITCH SS:=XX'
      REAL SUM1,SUM2'
      STABILITY:=TRUE'
      SUM1:=SUM2:=0'    A:=1'    N:=SIZE(P)-1'
      FOR I:=0 STEP 1 UNTIL N DO BEGIN SUM1:=SUM1+P(I)'
                                       SUM2:=SUM2+A*P(I)'
                                       A:=-A'
                              END'
      IF SUM1 LESS 0 OR SUM2 LESS 0 THEN BEGIN STABILITY:=FALSE'
                                       GOTO XX'
                              END'
             FOR M:=N-1 STEP -1 UNTIL 1 DO
         BEGIN FOR K:=0 STEP 1 UNTIL M DO Q(K):=P(0)*P(K)
                                              -P(M-K+1)*P(M+1)'
              FOR K:=0 STEP 1 UNTIL M DO P(K):=Q(K)'
              IF CHECKR(P(0)) LESS 0 THEN BEGIN STABILITY:=FALSE'
                                            GOTO XX'
                           END'
        END'
XX:'
END OF STABILITY'
```

APPENDIX 2.1.

LIAPUNOV CONCEPTS OF STABILITY FOR STOCHASTIC SYSTEMS

When dealing with systems of equations with stochastic coefficients it is necessary to speak of convergence in a stochastic sense since the question of convergence deals with limits involving random variables. In this appendix we shall consider the stochastic analogues of the concepts of Liapunov stability; these definitions are accomplished by simply changing the modes of convergence as they appear in the concepts of Liapunov stability for deterministic systems.

For completeness we shall first state the concepts of Liapunov stability for deterministic systems [42, 43]. In this case we will be concerned with the system characterized by

$$\dot{\underline{x}} = \underline{f}\ (\underline{x}, t) \tag{A2.1.1}$$

where $\underline{x}$ is an n-vector describing the state of the system, $\underline{f}$ is a continuous vector function satisfying a Lipschitz condition and such that $\underline{f}\ (0, t) \equiv 0$ for all t. Thus the null solution $\underline{x}(t) \equiv 0$ is an equilibrium solution of (A2.1.1) and its stability is in question. The notation $\underline{x}(t;\ \underline{x}_0,\ t_0)$ will be used to denote the solution, of system (A2.1.1) at time t, having initial state $\underline{x}_0$ at the initial time t_0.

<u>Definition I.</u> <u>Liapunov Stability</u>

The equilibrium solution $\underline{x}(t) \equiv 0$ of system (A2.1.1) is said to be stable, in the sense of Liapunov, if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that for any initial condition whose norm satisfies $||\ \underline{x}_0\ || < \delta$, the norm of the solution satisfies $||\ \underline{x}(t)\ || < \varepsilon$ for all $t \geqslant t_0$; that is

$$\sup_{t \geqslant t_0} ||\ \underline{x}(t;\ \underline{x}_0,\ t_0)|| < \varepsilon \tag{A2.1.2}$$

Definition II. Uniform Liapunov Stability

If in definition I the δ is a function of ε only; that is, (A2.1.2) holds for any t_0, then the equilibrium solution is said to be uniformly stable. (means uniformly in time in this case).

Definition III. Asymptotic Liapunov Stability

The equilibrium solution $\underline{x}(t) \equiv 0$ of system (A2.1.1) is said to be asymptotically stable if it is stable and if there exists a $\delta^1(t_0)$ such that $|| \underline{x}_0 || < \delta^1$ implies that

$$\lim_{t\to\infty} || \underline{x}(t; \underline{x}_0, t_0) || = 0 \tag{A2.1.3}$$

If (A2.1.3) holds for all $\underline{x}_0$ then the equilibrium solution is said to be asymptotically stable in the large.

From a physical point of view definition I states that if the system is initially perturbed only slightly from the steady state the response remains in the neighbourhood of this state. Asymptotic stability, on the other hand, requires even more, not only must the solution remain in the neighbourhood of the steady-state solution, but it must approach the steady state asymptotically as time approaches infinity.

In order to define similar concepts of stability in the case of a system with randomly time-varying parameters, we consider the system characterized by

$$\dot{\underline{x}} = \underline{f}\left[\underline{x}, \underline{\alpha}(t), t\right] \tag{A2.1.4}$$

which is similar to system (A2.1.1) except for $\underline{\alpha}(t)$ which denotes the randomly time varying parameter. As before, we assume that $\underline{f}(0, \underline{\alpha}(t), t) \equiv 0$ for all t so that the null solution $\underline{x}(t) \equiv 0$ is the equilibrium solution whose stability is to be investigated.

In probability theory the three common modes of convergence are :

(i) convergence in probability

(ii) convergence in the mean

and (iii) almost sure convergence (convergence with probability one), and we shall now translate the stability statements for deterministic systems into stability statements relative to each of these modes of convergence.

Definition IV. Liapunov stability in probability

The equilibrium solution of system (A2.1.4) is said to possess Liapunov stability in probability if, given ε, $\varepsilon^1 > 0$, there exists $\delta(\varepsilon, \varepsilon^1, t_0) > 0$ such that $\| \underline{x}_0 \| < \delta$ implies

$$P\{ \sup_{t \geq t_0} \| \underline{x}(t; \underline{x}_0, t_0) \| > \varepsilon^1 \} < \varepsilon \tag{A2.1.5}$$

Definition V. Liapunov stability in the m^{th} mean

The equilibrium solution of system (A2.1.4) is said to possess Liapunov stability in the m^{th} mean if the m^{th} moments of the solution vector exists and, given $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that $\| \underline{x}_0 \|^m < \delta$ implies that

$$E \{ \sup_{t \geq t_0} \| \underline{x}(t; \underline{x}_0, t_0) \|^m \} < \varepsilon \tag{A2.1.6}$$

where E{.} denotes the mathematical expectation and $\| \underline{x} \|^m = \sum_{i=1}^{n} |x_i|^m$.

Definition VI. Almost sure Liapunov stability

The equilibrium solution of system (A2.1.4) is said to be almost surely stable (that is, stable with probability one) if

$$P \{ \lim_{\|\underline{x}_0\| \to 0} \sup_{t \geq t_0} \| \underline{x} (t; \underline{x}_0, t_0) \| = 0 \} = 1 \tag{A2.1.7}$$

This is sometimes referred to as almost sure sample Liapunov stability since it says that the equilibrium solution is stable for almost all sample systems.

In a similar fashion we can define asymptotic stability relative to the three modes of convergence (i) - (iii).

Definition VII. Asymptotic stability in probability

The equilibrium solution of system (A2.1.4) is said to be asymptotically stable in probability if it is stable in probability and, if for $\varepsilon > 0$, there exists $\delta^1 > 0$ such that $\| \underline{x}_0 \| < \delta^1$ implies that

$$P \{ \sup_{t \geq t_0} \| \underline{x}(t; \underline{x}_0, t_0) \| > \varepsilon \} \to 0 \text{ as } t \to \infty$$

that is,

$$\lim_{t \to \infty} P \{ \sup_{t \geq t_0} \| \underline{x}(t; \underline{x}_0, t_0) \| > \varepsilon \} = 0 \tag{A2.1.8}$$

Definition VIII. Asymptotic stability in the m^{th} mean

The equilibrium solution of system A(2.1.4) is said to be asymptotically stable in the m^{th} mean if it is stable in the m^{th} mean and there exists a $\delta^1 > 0$ such that $\| \underline{x}_0 \| < \delta^1$ implies that

$$E \{ \sup_{t \geq t_0} \| \underline{x} (t; \underline{x}_0, t_0) \|^m \} \to 0 \text{ as } t \to \infty$$

that is,

$$\lim_{t \to \infty} E \{ \sup_{t \geq t_0} \| \underline{x} (t; \underline{x}_0, t_0 \|^m \} = 0 \tag{A2.1.9}$$

Definition IX. Almost sure asymptotic Liapunov stability

The equilibrium solution of system A(2.1.4) is said to be almost surely asymptotically stable if it is almost surely stable and there exists a $\delta^1 > 0$ such that $\| \underline{x}_0 \| < \delta^1$ implies that for any $\varepsilon > 0$

$$P \{ \sup_{t \geq t_0} \| \underline{x} (t; \underline{x}_0, t_0) \| > \varepsilon \} \to 0 \text{ as } t \to \infty$$

that is,

$$\lim_{t \to \infty} P \{ \sup_{t \geq t_0} || \underline{x} (t; \underline{x}_0, t_0) || > \varepsilon \} = 0 \tag{A2.1.10}$$

Definitions I - IX, as presented above, are direct analogues of the concept of Liapunov stability for dynamical systems and are concerned with sample behaviour on the interval (t_0, ∞). Examples of other concepts of stability appearing in the literature are :

Definition X. Liapunov stability of the probability

The equilibrium solution of system A(2.1.4) is said to possess stability of the probability if given ε, $\varepsilon^1 > 0$ there exists $\delta(\varepsilon, \varepsilon^1, t_0) > 0$ such that $|| \underline{x}_0 || < \delta$ implies that

$$P \{ || \underline{x} (t; \underline{x}_0, t_0) || > \varepsilon^1 \} < \varepsilon$$

for all $t \geq t_0$; that is,

$$\sup_{t \geq t_0} P \{ || \underline{x} (t; \underline{x}_0, t_0) || > \varepsilon^1 \} < \varepsilon \tag{A2.1.11}$$

Definition XI. Liapunov stability of the m^{th} mean

The equilibrium solution of system (A2.1.4) is said to possess stability of the m^{th} mean if, given $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that $|| \underline{x}_0 || < \delta$ implies

$$E \{ || \underline{x}(t; \underline{x}_0, t_0) ||^m \} < \varepsilon$$

for all $t \geq t_0$; that is,

$$\sup_{t \geq t_0} E \{ || \underline{x}(t; \underline{x}_0, t_0) ||^m \} < \varepsilon \tag{A2.1.12}$$

Definition XII. Asymptotic stability of the probability

The equilibrium solution of system (A2.1.4) is said to possess asymptotic stability of the probability if it possesses stability of the probability and if for $\varepsilon > 0$ there exists $\delta^1 > 0$ such that $|| \underline{x}_0 || < \delta^1$ implies that

$$\sup_{t \geq t_0} \lim_{t \to \infty} P \{ \| \underline{x}(t; \underline{x}_0, t_0) \| > \varepsilon \} = 0 \qquad \text{(A2.1.13)}$$

Definition XIII. Asymptotic stability of the m^{th} mean

The equilibrium solution of system (A2.1.4) is said to possess asymptotic stability of the m^{th} mean if it possesses stability of the m^{th} mean and if there exists a $\delta^1 > 0$ such that $\| \underline{x}_0 \| < \delta^1$ implies that

$$\sup_{t \geq t_0} E \{ \| \underline{x}(t; \underline{x}_0, t_0) \|^m \} \to 0 \text{ as } t \to \infty \qquad \text{(A2.1.14)}$$

The distinction between definitions IV, V, VII, VIII on one hand and X - XIII on the other is that in the former the supremum is taken on the sample and hence is included under the probability statement and under the expectation operator, whereas in the latter the supremum is taken on the probability function and on the expectation operator. Definitions X - XIII are conditions on the first distribution function and its associated moments and therefore cannot be considered as strong stability conditions as IV, V, VII, VIII on the solution process. However, under certain conditions, for example, linear homogeneous systems, these weaker stability criteria do have significant implications for sample stability. A thorough discussion of the implications that exist among the stability concepts discussed in this appendix may be found in references 20 and 21.

A P P E N D I X 3.1.

SUBHARMONIC SOLUTIONS OF SYSTEM EQUATIONS (3.3.4)

Substituting equations (3.4.11) in equation (3.3.4b) and balancing like terms it is readily seen that the only possible value for n is 2 and that

$$b_1 = \frac{\Pi_1}{\Pi_2} a_1, \quad b_2 = -\frac{\Pi_1}{\Pi_2} a_2, \quad b_3 = 0, \quad b_4 = -\frac{\Pi_1}{\Pi_2} a_o \tag{A3.1.1}$$

Substituting equations (3.4.11) in equation (3.3.4a) and balancing like terms gives, on using results (A3.1.1).

$$-\frac{\Pi_2}{\Pi_1} b_4 = b_o^2 + \frac{b_1^2}{2} + \frac{b_2^2}{2} + \frac{b_4^2}{2} + \frac{1}{2} \tag{A.3.1.2}$$

$$\frac{\Pi_2}{\Pi_1} b_1 + \frac{\Pi_2^2}{2\Pi_1} b_2 = 2b_o b_1 - b_1 b_4 - b_2 \tag{A3.1.3}$$

$$-\frac{\Pi_2}{\Pi_1} b_2 + \frac{\Pi_2^2}{2\Pi_1} b_1 = 2b_o b_2 - b_1 + b_2 b_4 \tag{A3.1.4}$$

$$0 = -\frac{\Pi_2}{2} b_1^2 - \frac{\Pi_2}{2} b_2^2 + 2\Pi_2 b_o b_4 - 2b_o + b_1 b_2 \tag{A3.1.5}$$

$$a_3 = -\frac{b_1^2}{2} + \frac{b_2^2}{2} + 2b_o b_4 \tag{A3.1.6}$$

Equations (A3.1.3) and (A3.1.4) are homogeneous in b_1 and b_2 and will have a nontrivial solution for these coefficients (that is, subharmonic solutions exist) if and only if

$$\left(\frac{\Pi_2}{\Pi_1} + b_4 \right)^2 - 4b_o^2 + \left(\frac{\Pi_2^2}{2\Pi_1} + 1 \right)^2 = 0 \tag{A3.1.7}$$

Writing $\dfrac{b_1}{b_2} = \left[\dfrac{\Pi_2}{\Pi_1} + 2b_o + b_4 \right] / \left[\dfrac{\Pi_2^2}{2\Pi_1} + 1 \right] = \mu,$

$r^2 = b_1^2 + b_2^2$ and $R = -b_4$,

we have that $b_1 b_2 = \mu b_2^2$ giving

$$b_1 b_2 = \mu r^2/(1 + \mu^2) \quad \text{and} \quad b_1^2 - b_2^2 = -\left[\frac{1 - \mu^2}{1 + \mu^2}\right] r^2$$

Substituting back, equations (A3.1.3), (A3.1.5) and (A3.1.7) become

$$2b_o^2 + r^2 + \left(R - \frac{\Pi_2}{\Pi_1}\right)^2 = \left(\frac{\Pi_2^2}{\Pi_1^2} - 1\right) \tag{A3.1.8}$$

$$(1 + 2\mu/\Pi_2 - \mu^2)r^2 = 4b_o\,(R + 1/\Pi_2)(1 + \mu^2) \tag{A3.1.9}$$

$$(\Pi_2/\Pi_1 - R)^2 - 4b_o^2 + (\Pi_2^2/(2\Pi_1) + 1)^2 = 0 \tag{A3.1.10}$$

Eliminating b_o^2 from (A3.1.8) and (A3.1.10) gives

$$r^2 + \tfrac{3}{2}\,(R - \Pi_2/\Pi_1)^2 = \lambda^2\,, \tag{A3.1.11}$$

where $\lambda^2 = \frac{1}{2}\left[\frac{2\Pi_2^2}{\Pi_1^2} - 3 - \frac{\Pi_2^4}{4\Pi_1^2} - \frac{\Pi_2^2}{\Pi_1}\right]$; that is, the loci of the moduli of the subharmonics and harmonics are ellipses as shown in fig. (A3.1.1).

Writing $z = R - \frac{\Pi_2}{\Pi_1}$ and $\alpha = \frac{\Pi_2^2}{2\Pi_1} + 1$ (A3.1.12)

equation (A3.1.11) becomes

$$r^2 + \tfrac{3}{2} z^2 = \lambda^2 \tag{A3.1.13}$$

Eliminating r^2 from equations (A3.1.8) and (A3.1.13) gives

$$4b_o^2 - z^2 = \alpha^2 \tag{A3.1.14}$$

while eliminating r^2 from equations (A3.1.9) and (A3.1.13), and substituting for α^2 from (A3.1.14) gives

$$\left(\frac{1}{\Pi_2} + z\right)\left(\lambda^2 - \tfrac{3}{2} z^2\right) = 2(\alpha^2 + z^2)\left(z + \frac{\Pi_2}{\Pi_1} + 1\right)$$

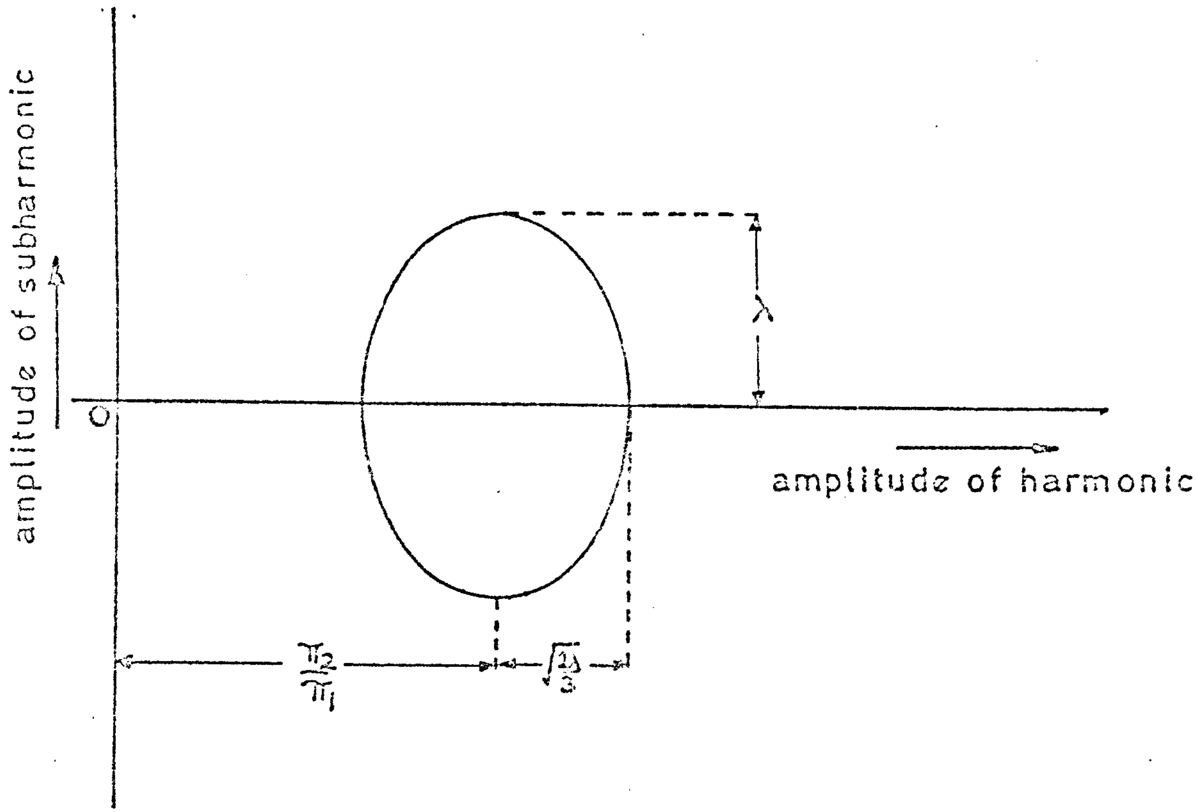


FIG. A3.1.1.   Ratio of amplitudes

Substituting for λ and α and expanding gives

$$F(X) = 7\Pi_2 X^3 + \left(\frac{11}{2}\Pi_2^2 + 7\Pi_1\right)X^2 + \left(\frac{5}{4}\Pi_2^5 + 5\Pi_2^3\Pi_1 + 7\Pi_2\Pi_1^2 - 2\Pi_2^3\right)X$$

$$+ \left(\frac{9}{8}\Pi_2^6 + \frac{23}{4}\Pi_2^4\Pi_1 - \Pi_2^4 + \frac{21}{2}\Pi_1^2\Pi_2^2 - 2\Pi_2^2\Pi_1 + 7\Pi_1^3\right) = 0$$

(A3.1.15)

where $X = \Pi_1 z$.

For a subharmonic solution to exist a real root of equation (A3.1.15) must be such that equation (A3.1.13) gives a real value of r; that is

$$X^2 < \frac{2}{3}\Pi_2^2 - \Pi_1^2 - \frac{1}{12}\Pi_2^4 - \frac{1}{3}\Pi_2^2\Pi_1 = L^2 > 0 \qquad \text{(A3.1.16)}$$

It is readily shown that $F(L) > 0$ and $F(-L) > 0$ so that equation (A3.1.15) cannot have an odd number of roots between $-L$ and L. Thus, equation (A3.1.15) has either no roots or two roots in the interval $-L < X < L$.

Since $F(X) < 0$ for large negative values of X and $F(-L) > 0$ it follows that F(X) has at least one root in the region $X < -L$. Since also $F(L) > 0$ it follows that F(X) will have two roots in the region $-L < X < L$ provided F(X) has a stationary value at $X = X_1$ which is a minimum and such that $-L < X_1 < L$ and $F(X_1) < 0$.

Provided $N > 0$, where

$$N = \Pi_2^4(289 - 105\,\Pi_2^2 - 420\,\Pi_1) + 196\,\Pi_1^2 + 308\,\Pi_2^2\Pi_1 - 588\,\Pi_2^2\Pi_1^2$$

(A3.1.17)

F(X) has a stationary value, which is a minimum, at

$$X_1 = \frac{1}{42\Pi_2}\left[\sqrt{N} - (11\Pi_2^2 + 14\Pi_1)\right]$$

Thus equation (A3.1.15) will have two roots in the region $-L < X < L$, that is, subharmonic solutions exist, provided

(a) $N > 0$

(b) $L > 0$

(c) $-L < X_1 < L$

and (d) $F(X_1) < 0$

Equation (A3.1.16) may be written

$$N = \Pi_2^4 (289 - 105 \Pi_2^2) + \Pi_2^2 (308 - 420 \Pi_2^2)\Pi_1 - (588 \Pi_2^2 - 196)\Pi_1^2 \tag{A3.1.18}$$

From equation (A3.1.18) it is readily seen that :

(i) if $\Pi_2^2 < \frac{196}{588}$ then $N > 0$ all Π_1

(ii) if $\frac{196}{588} < \Pi_2^2 < \frac{308}{420}$ then $N > 0$ if $\Pi_1 < \beta$ where $\beta > \alpha$

and $N = (\alpha + \Pi_1)(\beta - \Pi_1)$

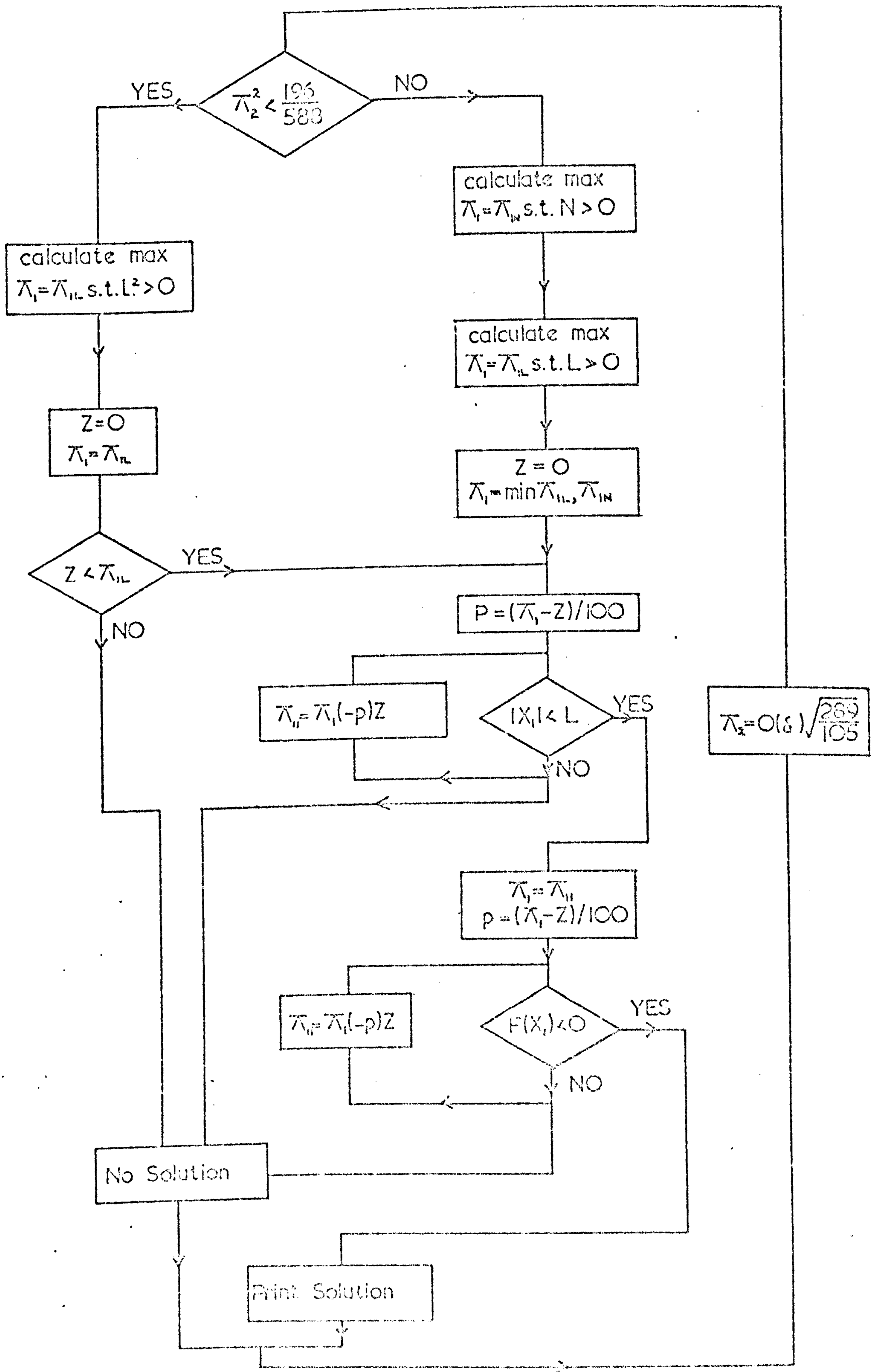
(iii) if $\frac{308}{420} < \Pi_2^2 < \frac{289}{105}$ then $N > 0$ if $\Pi_1 < \alpha$ where $\alpha < \beta$

and $N = (\alpha + \Pi_1)(\beta - \Pi_1)$

(iv) if $\Pi_2^2 > \frac{289}{105}$ then $N < 0$ all Π_1

The region R_2 in parameter space defined by inequalities (a) - (d), that is, the region of parameter space where parameter values give rise to subharmonic solutions, is then plotted using digital computation; the flowchart for the computer program being Flowchart (A3.1.1).

For parameter values taken in region R_2 equation (A3.1.15) may be solved numerically to give two real values of X between -L and L. For each value of X, $|X| < L$, equations (A3.1.12), (A3.1.13) and (A3.1.14) are solved to give the corresponding values of R, r and b_0^2 respectively. Solving $b_1 = \mu b_2$, $b_1^2 + b_2^2 = r^2$, $R = -b_4$ then give the coefficients a_0, a_1 and a_2 whilst equation (A3.1.6) gives the corresponding value of a_3. It is readily seen that there exist eight subharmonic solutions of order two, each with a d.c. component in ξ_1 and ξ_2, for each point in the region R_2.



Flowchart (A3.1.1)
Regions of parameter space where subharmonic solutions exist

APPENDIX 4.1.

HILL DETERMINANTS FOR HARMONIC SOLUTIONS OF EQUATIONS 4.4.5.

Writing, for convenience, $\alpha = \frac{1}{\Pi_1}$, $\beta = \frac{\Pi_2}{\Pi_1(1 + \Pi_1^{\,2})}$ equations (4.4.5) may be written as

$$\xi_1^{\,1}(\tau) = -\alpha\,\xi_1(\tau) - \alpha\sin\tau\,\xi_2(\tau) \tag{A4.1.1}$$

$$\xi_2^{\,1}(\tau) = \beta(\sin\tau - \frac{1}{\alpha}\cos\tau)\xi_1(\tau) \tag{A4.1.2}$$

Assume that $\xi_1(\tau)$ be given by the Fourier series development

$$\xi_1(\tau) = a_o + \sum_{n=1}^{\infty} (a_n \cos n\tau + b_n \sin n\tau) \tag{A4.1.3}$$

where, as yet, the coefficients $a_i (i = 0,1,2$ etc.), b_i $(i = 1,2,$ etc) are undetermined.

Substituting (A4.1.3) in equation (A4.1.2) gives

$$\xi_2^{\,1}(\tau) = \beta[a_o \sin\tau - \frac{a_o}{\alpha}\cos\tau] + \frac{1}{2}\sum_{n=1}^{\infty}[(a_n - \frac{b_n}{\alpha})\sin\overline{n+1}\,\tau$$

$$- (a_n + \frac{b_n}{\alpha})\sin\overline{n-1}\,\tau - (b_n + \frac{a_n}{\alpha})\cos\overline{n+1}\,\tau + (b_n - \frac{a_n}{\alpha})\cos\overline{n-1}\,\tau]$$

$$\text{(A4.1.4)}$$

When the system has adopted the $\xi_2^{\,1}(\tau)$ signal will have no d.c. component so that

$$a_1 = \alpha b_1 \tag{A4.1.5}$$

Integrating equation (A4.1.4), with respect to τ, gives

$$\xi_2(\tau) = A + \beta[-a_o \cos\tau - \frac{a_o}{\alpha}\sin\tau] + \frac{1}{2}\sum_{n=1}^{\infty}[-(a_n - \frac{b_n}{\alpha})\frac{\cos\overline{n+1}\,\tau}{n+1}$$

$$+ (a_n + \frac{b_n}{\alpha})\frac{\sin\overline{n-1}\,\tau}{n-1} - (b_n + \frac{a_n}{\alpha})\frac{\sin\overline{n+1}\,\tau}{n+1} + (b_n - \frac{a_n}{\alpha})\frac{\sin\overline{n-1}\,\tau}{n-1}]$$

$$\text{(A.4.1.6)}$$

where A is a constant, as yet, undetermined.

Substituting equation (A4.1.6) in equation (A4.1.1) and balancing the coefficients of like terms leads to the following two distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n}, b_{2n}) and (a_{2n+1}, b_{2n+1}) respectively.

$$(\alpha - \frac{\beta}{2})\, a_o - \frac{\beta}{4} a_2 + \frac{\alpha\beta}{4} b_2 = 0$$

$$-\frac{\alpha\beta}{2} a_o + (-2 + \frac{\alpha\beta}{3})a_2 + (\alpha + \frac{\beta}{6})b_2 - \frac{\alpha\beta}{12} a_4 - \frac{\beta}{12} b_4 = 0$$

$$\frac{\beta}{2} a_o + (\alpha + \frac{\beta}{6})a_2 + (2 - \frac{\alpha\beta}{3})b_2 - \frac{\beta}{12} a_4 + \frac{\alpha\beta}{12} b_4 = 0$$

$$\frac{-\alpha\beta}{4(2n-1)} a_{2n-2} + \frac{\beta}{4(2n-1)} b_{2n-2} + (-2n + \frac{n\alpha\beta}{(4n^2-1)})a_{2n} + (\alpha + \frac{\beta}{2(4n^2-1)})\, b_{2n}$$

$$\frac{-\alpha\beta}{4(2n+1)} a_{2n+2} - \frac{\beta}{4(2n+1)} b_{2n+2} = 0\ ,\ n > 1$$

$$\frac{\beta}{4(2n-1)} a_{2n-2} + \frac{\alpha\beta}{4(2n-1)} + (\alpha + \frac{\beta}{2(4n^2-1)})a_{2n} + (2n - \frac{n\alpha\beta}{(4n^2-1)})\, b_{2n}$$

$$\frac{-\beta}{4(2n+1)} a_{2n+2} + \frac{\alpha\beta}{4(2n+1)} b_{2n+2} = 0,\ n > 1$$

(A.4.1.7)

and

$$a_1 - \alpha b_1 = 0$$

$$(\alpha - \frac{\beta}{8})a_1 + (1 - \frac{\alpha\beta}{8})b_1 - \frac{\beta}{8} a_3 + \frac{\alpha\beta}{8} b_3 = 0$$

$$-\frac{\alpha\beta}{8n} a_{2n-1} + \frac{\beta}{8n} b_{2n-1} + (2n+1)[-1 + \frac{\alpha\beta}{8n(n+1)}]\, a_{2n+1} + [\alpha + \frac{\beta}{8n(n+1)}]b_{2n+1}$$

$$- \frac{\alpha\beta}{8(n+1)} a_{2n+3} - \frac{\beta}{8(n+1)} b_{2n+3} = 0, \; n > 0$$

$$\frac{\beta}{8n} a_{2n-1} + \frac{\alpha\beta}{8n} b_{2n-1} + \left[\alpha + \frac{\beta}{8n(n+1)}\right] a_{2n+1} + (2n+1)\left[1 - \frac{\alpha\beta}{8n(n+1)}\right] b_{2n+1}$$

$$- \frac{\beta}{8(n+1)} a_{2n+3} + \frac{\alpha\beta}{8(n+1)} b_{2n+3} = 0, \; n > 0$$

(A4.1.8)

The value of the constant A is then given by the equation

$$\alpha A + (-1 + \frac{\alpha\beta}{8})a_1 + (\alpha - \frac{\beta}{8})b_1 - \frac{\alpha\beta}{8} a_3 - \frac{\beta}{8} b_3 = 0 \qquad \text{(A4.1.9)}$$

Thus, two types of harmonic solutions are possible for equations (4.4.5), namely

(i) $\xi_1(\tau) = a_o + \sum_{n=1}^{\infty} [a_{2n} \cos 2n\,\tau + b_{2n} \sin 2n\,\tau]$, where the coefficients (a_{2n}, b_{2n}) are given by the homogeneous algebraic equations (A4.1.7).

and

(ii) $\xi_1(\tau) = \sum_{n=0}^{\infty} [a_{2n+1} \cos(2n+1)\tau + b_{2n+1} \sin(2n+1)\tau]$, where the coefficients (a_{2n+1}, b_{2n+1}) are given by the homogeneous algebraic equations (A4.1.8).

In each case the corresponding solution for $\xi_2(\tau)$ is given by equation (A4.1.6), with A given by equation (A4.1.9).

An harmonic solution of the form (i) or (ii) will exist if the corresponding set of simultaneous homogeneous algebraic equations have a non-trivial solution for the Fourier coefficients. It is well known that this is the case provided the determinant of the coefficients, in the equations, is zero. Thus, a solution of the form (i) exists if the determinant of the coefficients of (a_{2n}, b_{2n}) in equations (A4.1.7) - often referred to as a Hill determinant-vanishes. Similarly a solution of the form (ii) exists if the determinant of the coefficients of (a_{2n+1}, b_{2n+1}) in equations (A4.1.8) vanishes.

If a full Fourier series development for $\xi_1(\tau)$ is assumed then, for both solutions (i) and (ii), the Hill determinants will be infinite determinants. However, in practice, the Fourier series is truncated so that only finite order Hill determinants are considered. Since, in both sets of equations (A4.1.7) and (A4.1.8), the coefficients are all linear functions of β it follows that equating an Hill determinant, of order r, to zero results in a polynomial equation, of order r, in β (or Π_2) having coefficients which are functions of α(or Π_1). These polynomials are then solved using the procedure described in appendix (4.3).

APPENDIX 4.2.

HILL DETERMINANTS FOR SUB-HARMONIC SOLUTIONS OF EQUATIONS 4.4.5.

Writing, for convenience, $\alpha = \frac{1}{\Pi_1}$, $\beta = \frac{\Pi^2}{\Pi_1(1 + \Pi_1^2)}$ equations (4.4.5) may be written as

$$\xi_1^1(\tau) = - \alpha \, \xi_1(\tau) - \alpha \sin \tau \, \xi_2(\tau) \tag{A4.2.1}$$

$$\xi_2^1(\tau) = \beta(\sin\tau - \frac{1}{\alpha}\cos\tau)\xi_1(\tau) \tag{A4.2.2}$$

In order to obtain the transition boundary, between stable and unstable regions, we are interested in obtaining the values of α and β for which a solution of period 4π, that is, a subharmonic solution of order $\frac{1}{2}$, exists for equations (A4.2.1) and (A4.2.2).

Assume that $\xi_1(\tau)$ be given by the Fourier series development

$$\xi_1(\tau) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\tau}{2} + b_n \sin \frac{n\tau}{2} \right) \tag{A4.2.3}$$

where, as yet, the coefficients a_i ($i = 0, 1, 2$, etc.), b_i ($i = 1, 2, 3$, etc) are undetermined.

Substituting (A4.2.3) in equation (A4.2.2) gives

$$\xi_2^1(\tau) = \beta\left[a_0\sin\tau - \frac{a_0}{\alpha}\cos\tau\right] + \frac{\beta}{2}\sum_{n=1}^{\infty}\left[(a_n - \frac{b_n}{\alpha})\sin\left(\frac{n+2}{2}\right)\tau - (a_n + \frac{b_n}{\alpha})\sin\left(\frac{n-2}{2}\right)\tau - (b_n + \frac{a_n}{\alpha})\cos\left(\frac{n+2}{2}\right)\tau + (b_n - \frac{a_n}{\alpha})\cos\left(\frac{n-2}{2}\right)\tau \right] \tag{A4.2.4}$$

When the system has adapted the $\xi_2^1(\tau)$ signal will have no d.c. component so that

$$a_2 = b_2\alpha \tag{A4.2.5}$$

Integrating equation (A4.2.4), with respect to τ, gives

$$\xi_2(\tau) = A + \beta\left[-a_0\cos\tau - \frac{a_0}{\alpha}\sin\tau\right] + \frac{\beta}{2}\sum_{n=1}^{\infty}\left[-\left(\frac{2}{n+2}\right)\left(a_n - \frac{b_n}{\alpha}\right)\cos\left(\frac{n+2}{2}\right)\tau\right.$$

$$+ \left(\frac{2}{n-2}\right)\left(a_n + \frac{b_n}{\alpha}\right)\cos\left(\frac{n-2}{2}\right)\tau - \left(\frac{2}{n+2}\right)\left(b_n + \frac{a_n}{\alpha}\right)\sin\left(\frac{n+2}{2}\right)\tau$$

$$\left. + \left(\frac{2}{n-2}\right)\left(b_n - \frac{a_n}{\alpha}\right)\sin\left(\frac{n-2}{2}\right)\tau\right] \tag{A4.2.6}$$

where A is a constant, as yet, undetermined.

Substituting equation (A4.2.6) in equation (A4.2.1) and balancing the coefficients of like terms leads to three distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n+1}, b_{2n+1}), (a_{4n}, b_{4n}) and (a_{4n+2}, b_{4n+2}), n = 0, 1, 2, etc., respectively. The last two sets of equations give rise to the two harmonic solutions discussed in appendix (4.1) whilst the first set of equations gives rise to a subharmonic solution of the form

$$\xi_1(\tau) = \sum_{n=1}^{\infty}\left[a_n\cos\frac{(2n-1)}{2}\tau + b_n\sin\frac{(2n-1)}{2}\tau\right], \tag{A4.2.7}$$

with the Fourier coefficients a_i, b_i (i = 1,2,3, etc.) being given by the simultaneous linear homogeneous algebraic equations

$$-\left(\frac{1}{2} + \frac{\alpha\beta}{3}\right)a_1 + \left(\alpha - \frac{2\beta}{3}\right)b_1 + \left(\frac{\alpha\beta}{2}\right)a_2 + \left(\frac{\beta}{2}\right)b_2 - \left(\frac{\alpha\beta}{6}\right)a_3 - \left(\frac{\beta}{6}\right)b_3 = 0$$

$$\left(\alpha - \frac{2\beta}{3}\right)a_1 + \left(\frac{1}{2} + \frac{\alpha\beta}{3}\right)b_1 - \left(\frac{\beta}{2}\right)a_2 + \left(\frac{\alpha\beta}{2}\right)b_2 - \left(\frac{\beta}{6}\right)a_3 + \left(\frac{\alpha\beta}{6}\right)b_3 = 0$$

$$-\left(\frac{\alpha\beta}{2}\right)a_1 - \left(\frac{\beta}{2}\right)b_1 + \left(-\frac{3}{2} + \frac{3}{5}\alpha\beta\right)a_2 + \left(\alpha + \frac{2}{5}\beta\right)b_2 - \left(\frac{\alpha\beta}{10}\right)a_4 - \left(\frac{\beta}{10}\right)b_4 = 0$$

$$\left(\frac{\beta}{2}\right)a_1 - \left(\frac{\alpha\beta}{2}\right)b_1 + \left(\alpha + \frac{2}{5}\beta\right)a_2 + \left(\frac{3}{2} - \frac{3}{5}\alpha\beta\right)b_2 - \left(\frac{\beta}{10}\right)a_4 + \left(\frac{\alpha\beta}{10}\right)b_4 = 0$$

$$\left[\frac{-\alpha\beta}{2(2n-3)}\right]a_{n-2} + \left[\frac{\beta}{2(2n-3)}\right]b_{n-2} + \left[-\frac{(2n-1)}{2} + \frac{(2n-1)\alpha\beta}{(2n-1)^2 - 4}\right]a_n$$

$$+ \left[\alpha + \frac{2\beta}{(2n-1)^2 - 4}\right]b_n - \left[\frac{\alpha\beta}{2(2n+1)}\right]a_{n+2} - \left[\frac{\beta}{2(2n+1)}\right]b_{n+2} = 0, \; n > 2$$

$$\left[\frac{\beta}{2(2n-3)}\right]a_{n-2} + \left[\frac{\alpha\beta}{2(2n-3)}\right]b_{n-2} + \left[\alpha + \frac{2\beta}{(2n-1)^2 - 4}\right]a_n$$

$$+ \left[\frac{(2n-1)}{2} - \frac{(2n-1)\alpha\beta}{(2n-1)^2 - 4}\right]b_n - \left[\frac{\beta}{2(2n+1)}\right]a_{n+2} + \left[\frac{\alpha\beta}{2(2n+1)}\right]b_{n+2} = 0, \; n > 2$$

(A4.2.8)

When $n = 1$ the corresponding Hill determinant is

$$\Delta_1 = -\left[\left(\frac{1}{2} + \frac{\alpha\beta}{3}\right)^2 + \left(\alpha - \frac{2\beta}{3}\right)^2\right]$$

which cannot be zero for any α and β.

When $n = 2$ the corresponding Hill determinant is

$$\Delta_2 = \left\{ \left[\left(\alpha - \frac{2\beta}{3}\right)\left(\frac{3}{2} - \frac{3}{5}\alpha\beta\right) - \left(\frac{1}{2} + \frac{\alpha\beta}{3}\right)\left(\alpha + \frac{2}{5}\beta\right)\right]^2 \right.$$

$$\left. + \left[\left(\frac{\alpha^2\beta^2}{4} + \frac{\beta^2}{4}\right) + \left(\frac{1}{2} + \frac{\alpha\beta}{3}\right)\left(\frac{3}{2} - \frac{3}{5}\alpha\beta\right) + \left(\alpha - \frac{2\beta}{3}\right)\left(\alpha + \frac{2}{5}\beta\right)\right]^2 \right\}$$

which again cannot be zero for any α and β.

Similarly it can be shown that higher order Hill determinants may be expressed in the form

$$\Delta_n = (-1)^n \text{ (Sum of squares)}$$

but to date the formulation of an inductive proof remains an outstanding problem.

Since the Hill determinants cannot be zero for any parameter values it follows that the system of equations (A4.2.1) - (A4.2.2) do not have a solution of the form (A4.2.7).

APPENDIX 4.3.

ALGOL PROGRAM FOR OBTAINING THE ROOTS OF A HILL DETERMINANT

A determinantal equation of the form

$$H_n = | f_{ij}(\alpha, \beta) | = 0 , i,j = 1,2, ----, n \tag{A4.3.1}$$

is being considered, where each element $f_{ij}(\alpha, \beta)$ is of the form

$$\begin{aligned} &(a_{00} + a_{01}\alpha + a_{02}\alpha^2 + ------------------ + a_{0L_0}\alpha^{L_0}) \\ + &(a_{10} + a_{11}\alpha + a_{12}\alpha^2 + ------------------ + a_{1L_1}\alpha^{L_1})\beta \\ + &-- \\ + &(a_{z0} + a_{z1}\alpha + a_{z2}\alpha^2 + ------------------ + a_{zL_z}\alpha^{L_z})\beta^z \quad ; \end{aligned}$$

that is, each $f_{ij}(\alpha, \beta)$ is a polynomial in β having coefficients which are polynomials in α.

The aim is, for a given value of α, to obtain the values of β satisfying equation (A4.3.1). Two programs are presented; the object of the first program is, for a given α, to expand H_n as an algebraic polynomial in β. The roots of the resulting polynomial are then calculated by using a standard library program for obtaining the roots of a polynomial; for completeness this is included as the second program.

PROGRAM 1

EXPANSION OF A DETERMINANT AS AN ALGEBRAIC POLYNOMIAL

(i) Presentation of Data

M - degree of matrix

D - maximum degree of β

KK - $\begin{cases} = 0 \text{ if degree } \beta \text{ varies} \\ = D \text{ if degree } \beta \text{ fixed} \end{cases}$

R1 - maximum degree of α

K1 - $\begin{cases} = 0 \text{ if degree } \alpha \text{ varies} \\ = \text{R1 if degree } \alpha \text{ fixed} \end{cases}$

N - number of non zero elements

Then print the N non-zero elements as follows :

I J - giving position of element in H_n

MM - description type

then

(a) if MM = 1

P Q where $f_{ij} = f_{pq}$

(b) if MM = 2

P Q where $f_{ij} = -f_{pq}$

(c) if MM = 3

(i) if KK = 0 then Z, maximum degree of β for this element followed by

$$\text{if K1} = 0 \quad \begin{array}{lllll} L_0 & a_{00} & a_{01} & \text{------------} & a_{0L_0} \\ L_1 & a_{10} & a_{11} & \text{------------} & a_{1L_1} \\ \vdots & & & & \\ L_z & a_{z0} & a_{z1} & \text{------------} & a_{zL_z} \end{array}$$

$$\text{or if K1} = \text{R1} \quad \begin{array}{llll} a_{00} & a_{01} & \text{------------} & a_{0K1} \\ a_{10} & a_{11} & \text{------------} & a_{1K1} \\ \vdots & & & \\ a_{z0} & a_{z1} & \text{------------} & a_{zK1} \end{array}$$

(ii) if KK = D then

$$
\begin{array}{lllll}
\text{if } K1 = 0 & L_0 & a_{00} & a_{01} \text{ ---------------- } & a_{0L_0} \\
 & L_1 & a_{10} & a_{11} \text{ ---------------- } & a_{1L_1} \\
 & \vdots & & & \\
 & L_D & a_{D0} & a_{D1} \text{ ---------------- } & a_{DL_D} \\
\text{or if } K1 = R1 & & a_{00} & a_{01} \text{ ---------------- } & a_{1K1} \\
 & & \vdots & & \\
 & & a_{D0} & a_{D1} \text{ ---------------- } & a_{DK1}
\end{array}
$$

PP - Number of values of α

α_1, α_2, ----, α_{PP} - the PP values of α

X_0 - first value for β, could be taken as 0

H - step length between two values of β, could be taken as 1.

Example

$$
\Delta_3 = \begin{vmatrix} \alpha^2 + \alpha\beta & \alpha\beta & -\beta \\ \alpha\beta & -\alpha^2 - \alpha\beta & 0 \\ \beta & 0 & 1 + \alpha^2 + 2\alpha\beta \end{vmatrix}
$$

Data tape is :

M	3
D	1
KK	1
R1	2
K1	0
N	7

	1	1				2	2		
(1)	3				(5)	2			
	2	0	0	1		1	1		
	1	0	1						

	1	2			3	1		
(2)	3			(6)	2			
	0	0			1	3		
	1	0	1					

	1	3		3	3			
	3			3				
(3)	0	0	(7)	2	1	0	1	
	0	-1		1	0	2		

	2	1
(4)	1	
	1	2

PP - number of values of α

α_1, ---, α_{PP} - the PP values of α

X_0 0

H 1

(ii) Interpretation of results

The print out will give the value of α, the degree W of the polynomial and the W coefficients C_0, C_1, ---, C_W, where

$$H_n = C_o + C_1\beta + C_2\beta^2 + \text{-----------} + C_W\beta^W$$

Example

For the example discussed in (i) above the algebraic expansion of H_3 is

$$H_3 = -(4\alpha^3 + \alpha)\beta^3 - (6\alpha^4 + 3\alpha^2)\beta^2 - (4\alpha^5 + 2\alpha^3)\beta - (\alpha^6 + \alpha^4) ;$$

thus, if the given value of α is 1 then print out will be

```
ALPHA    =   1
COEFFICIENTS OF BETA POLYNOMIAL
POLYNOMIAL DEGREE 3
- 2
- 6
- 9
- 5
```

(iii) Method used

An explanation of the process used is given as opposed to flow diagrams since in this case it is less confusing

Step 1

Data input to matrix A1, where each element of A1 takes the form

$$\begin{array}{llll} a_{00} & a_{01} & \text{----------} & a_{0L_0} \\ a_{11} & a_{11} & \text{----------} & a_{1L_1} \\ \vdots & & & \\ a_{z0} & a_{z1} & \text{----------} & a_{zL_z} \end{array}$$

Step 2

Given a value of α A is formed from A1, where each element of A is of the form

$$b_0 \quad b_1 \quad b_2 \text{ ------- } b_s \quad \underbrace{0\ 0 \text{ -- } 0}_{D - S}\ S$$

that is, terms in positions 0 to D and S in position D + 1, and where

$$b_r = a_{r0} + a_{r1}\alpha + \text{-----------} + a_{rL_r}\alpha^{L_r} \qquad \text{(A4.3.2)}$$

This step is done by procedure INPLANT.

Step 3

The maximum possible degree W of the resulting polynomial in β is calculated by finding the maximum degree in any row and then summing for all rows. This step is also done in procedure INPLANT.

Step 4

Produce matrix B where each element of B is of the form

$$b_0 + \beta_r b_1 + \beta_r^2 b_2 + \text{--------------} + \beta_r^s b_s$$

and the b's refer to those defined in equation (A4.3.2).

Then using procedure E det(B) = Δ_r is evaluated; this step being repeated W + 1 times for r = 0, 1, 2, ---, W, where

$$\beta_r = \beta_0 + rH ,$$

to give $\Delta_0, \Delta_1, \text{----}, \Delta_W$

Step 5

Now H_n, defined in equation (A4.3.1), is a polynomial, of degree W at most, in β, say

$$H_n = C_0 + C_1\beta + \text{------} + C_W\beta^W$$

Hence matrix X is such that $x_{ij} = \beta_i^j$ and

$$X \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_W \end{bmatrix} = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \vdots \\ \Delta_W \end{bmatrix} = \bar{\Delta}$$

so that C_0, C_1, -----, C_W are obtained from $X^{-1}\bar{\Delta}$ using procedure E.

(iv) Notes on program

Note 1

The same procedure E (working along Gaussian elimination lines) is used to evaluate both detB and $X^{-1}\bar{\Delta}$.

Note 2

If H_n, defined in equation (A4.3.1), is of degree less than W, say P, then the coefficients

$$C_{p+1}\ ,\ C_{p+2},\ ----,\ C_W$$

should come out negligible.

Observation of the results is usually the best means of checking this, for if a sequence of values followed by w-p values which are less than the first p+1 values by a multiplicative factor of about 10^{-9} then we would expect these to represent zero (most floating point arithmetic used in computers tends to lead to this magnitude of rounding error).

(v) Print out of program

A print out of the program is given on pages 213-215.

PROGRAM 2

ROOTS OF A POLYNOMIAL EQUATION

The function of this program is to determine the real and complex roots of a polynomial equation, whether they be equal, close or widely separated, in the form X + iY, from an equation of the form

$$A_0x^n + A_1x^{n-1} + \text{----------} + A_{n-1}\,x + A_n = 0 \qquad \text{(A4.3.3)}$$

(i) Presentation of data

N - degree of polynomial

followed by the coefficients

$A_0 \quad A_1 \text{ ---- } A_{n-1} \quad A_n$

of polynomial (A4.3.3).

Example

$$f(x) = x^5 + 5x^4 + 7x^3 - 19x^2 - 98x + 104 = 0$$

Data :

5

1 5 7 -19 -98 104

(ii) Interpretation of results

Print out will be

N

followed by the n roots in order of increasing modulus, that is, smallest first.

For the example quoted above it is readily shown that

$$f(x) = (x - 1)(x - 2)(x + 4)(x^2 + 4x + 13)$$

so that the print out will be

5

1 + I * 0

2 + I * 0

-2 + I * -3

-2 + I * 3

-4 + I * 0

(iii) Method used

An initial approximation to a root is made by Bernoulli's method (to 3 sig. figures) and this is improved by the Newton-Raphson iteration to 8 significant figures.

The program was developed from the information and flow diagrams given by Herbert S. Wilf in the text book *Mathematical Methods for Digital Computers, Vol. 1*, edited by A. Ralston and H.S. Wilf and published by J. Wiley and Sons, 1960, pp. 233-241.

(iv) Notes on program

(a) A_0 cannot be zero

(b) There must be N + 1 coefficients declared

(c) A_0 must be declared even if it is unity

(v) Print out of program

A print out of the program is given on pages 216-218.

EXPANSION OF A DETERMINANT AS AN ALGEBRAIC POLYNOMIAL'

```
BEGIN INTEGER N,D,KK,R1,K1,R,I,J,M,MM,K,L,W,T,P,Q,PP,JJ'
      REAL ZZ,X0,H,ALPHA'




      PROCEDURE E(A,N,C,D)'
      ARRAY A,C'
      INTEGER N'
      REAL D'
      BEGIN INTEGER J,K,L,F'
            REAL Z'
            IF D=0 THEN F:=0 ELSE BEGIN F:=1'D:=1' END'
            FOR J:=0 STEP 1 UNTIL N DO
            BEGIN K:=J'
            FOR L:=J+1 STEP 1 UNTIL N DO
                  IF ABS(A(L,J)) GR ABS(A(K,J)) THEN K:=L'
                  IF J NOTEQ K THEN
                  BEGIN FOR L:=J STEP 1 UNTIL N DO
                        BEGIN Z:=A(K,L)'
                              A(K,L):=A(J,L)'
                              A(J,L):=Z'
                        END'
                        IF F=0 THEN
                        BEGIN
Z:=C(K)'C(K):=C(J)'C(J):=Z'
                        END
ELSE D:=-D'
                  END'
                  IF F=1 THEN D:=D*A(J,J)'
                  IF A(J,J) NOTEQ 0 THEN
                  FOR K:=J+1 STEP 1 UNTIL N DO
                  BEGIN Z:=A(K,J)/A(J,J)'
                        FOR L:=J+1 STEP 1 UNTIL N DO
                        A(K,L):=A(K,L)-Z*A(J,L)'
                        IF F=0 THEN
                        C(K):=C(K)-Z*C(J)'
                  END ELSE IF F=0 THEN STOP'
            END'
            IF F=0 THEN
            BEGIN FOR J:=N STEP -1 UNTIL 0 DO
                  BEGIN FOR K:=J+1 STEP 1 UNTIL N DO
                        C(J):=C(J)-A(J,K)*C(K)'
                        C(J):=C(J)/A(J,J)'
                  END
            END
      END'


READ N,D,KK,R1,K1,M'
BEGIN ARRAY A1(1:N,1:N,0:D+1,0:R1+1),A(1:N,1:N,0:D+1),
            B(0:M-1,0:M-1)'
```

```
PROCEDURE IMPLANT'

BEGIN W:=0'
      FOR J:=1 STEP 1 UNTIL N DO
      BEGIN T:=0'
            FOR K:=1 STEP 1 UNTIL K DO
            BEGIN R:=A1(J,K,D+1,0)' A(J,K,D+1):=R'
                  IF R GR T THEN T:=R'
                  FOR L:=0 STEP 1 UNTIL R DO
                  BEGIN ZZ:=0'
                        I:=A1(J,K,L,R1+1)'
                        FOR P:=I STEP -1 UNTIL 0 DO
                        ZZ:=ZZ*ALPHA+A1(J,K,L,P)'
                        A(J,K,L):=ZZ'
                  END'
            END'
                W:=W+T'
      END'
END'

FOR I:=1 STEP 1 UNTIL N DO
FOR J:=1 STEP 1 UNTIL N DO
FOR K:=0 STEP 1 UNTIL D+1 DO
FOR L:=0 STEP 1 UNTIL R1+1 DO A1(I,J,K,L):=0'
FOR L:=1 STEP 1 UNTIL N DO
BEGIN READ I,J,MM'
      IF MM LESS 3 THEN
      BEGIN READ P,Q'
            FOR K:=0 STEP 1 UNTIL D DO
            BEGIN FOR R:=0 STEP 1 UNTIL R1 DO
                  A1(I,J,K,R):=(3-2*MM)*A1(P,Q,K,R)'
                  A1(I,J,K,R1+1):=A1(P,Q,K,R1+1)'
            END'
            A1(I,J,D+1,0):=A1(P,Q,D+1,0)'
      END ELSE
      BEGIN IF KK=0 THEN READ P ELSE P:=KK'
            FOR R:=0 STEP 1 UNTIL P DO
            BEGIN IF K1=0 THEN READ Q ELSE Q:=K1'
                  FOR K:=0 STEP 1 UNTIL Q DO READ A1(I,J,R,K)'
                  A1(I,J,R,R1+1):=Q'
            END'
            A1(I,J,D+1,0):=P'
      END'
END'
```

```
READ PP'
BEGIN ARRAY ALPHA(1:PP)'
       FOR JJ:=1 STEP 1 UNTIL PP DO READ ALPHA(PP)'
       READ X0,H'
       FOR JJ:=1 STEP 1 UNTIL PP DO
       BEGIN PRINT
       ££L?ALPHA=?,SAMELINE,ALPHA(JJ),

       ££L?COEFFICIENTS OF BETA POLYNOMIAL?'

INPLANT'

       BEGIN ARRAY Y(0:W),X(0:W,0:W)'
              FOR J:=0 STEP 1 UNTIL W DO
              BEGIN X(J,0):=1'
                     X(J,1):=X0+J*H'
                     FOR K:=1 STEP 1 UNTIL M DO
                     FOR L:=1 STEP 1 UNTIL M DO
                     BEGIN B(K-1,L-1):=0'
                            FOR R:=A(K,L,D+1) STEP -1 UNTIL 0 DO
                            B(K-1,L-1):=B(K-1,L-1)*X(J,1)+A(K,L,R)'
                     END'
                     Y(J):=1'
                     E(B,M-1,Y,Y(J))'
              END'
              FOR J:=2 STEP 1 UNTIL W DO
              FOR K:=0 STEP 1 UNTIL W DO
              X(K,J):=X(K,J-1)*X(K,1)'
              E(X,W,Y,0)'
              PRINT ££L2? POLYNOMIAL OF DEGREE?,SAMELINE,W'
              FOR R:=0 STEP 1 UNTIL W DO PRINT SCALED(8), Y(R)'
       END
    END
 END
 END'
 END'
```

```
ROOTS OF POLYNOMIAL EQUATION'
BEGIN INTEGER N,J'


PROCEDURE ROOT(N,A,RR)'
VALUE A'
INTEGER N'
ARRAY  A,RR'
BEGIN  INTEGER I,K,Q,R,P,S,T,U,V'
        REAL X,Y'
       ARRAY B,D,E,F,G(0:N+1),C(0:N+2)'
         SWITCH SS:=XX1,XX2,XX3,XX4,XX5,XX6,XX7,XX8,XX9,XX10,
XX11,XX12,XX13,XX14,XX15,XX16,XX17,XX18,XX19,XX20,XX21,XX22,XX23,
XX24,XX25,XX26,XX27,XX28,XX29,XX30,XX31'
FOR K:=1 STEP 1 UNTIL N DO
D(K):= A(K):= A(K)/A(0)'

V:= N'
XX2:D(V+1):=0'
IF D(V) NOTEQ 0 THEN GOTO XX6'
RR(V,0):=RR(V,1):=0'
V:=V-1'
GOTO XX2'
XX6:  IF D(1)=0 THEN GOTO XX3'
      IF V=2 THEN GOTO XX30'
      IF V=1 THEN GOTO XX29 ELSE GOTO XX4'
XX3:  S:= V+1'
FOR R:=1 STEP 1 UNTIL S DO C(R):=R'
GOTO XX8'
XX4:A(0):=2*D(2)'
    B(0):=-(D(1)*D(1))+A(0)'
IF B(0)=0 THEN GOTO XX3'
   C(2):= D(1)/B(0)'
   U:=1+V'
FOR R:=3 STEP 1 UNTIL U DO BEGIN
   A(0):= R*D(R)/D(1)'
   E(0):= D(R-1)-A(0)'
     T:= R-2'
     S:= R-1'
FOR K:=1 STEP 1 UNTIL T DO BEGIN
   A(0):= C(K+1)*E(0)'
   F(0):= A(0)+D(S-K)'
   E(0):= F(0)'
END'
   C(R):= -1/E(0)'
END'
XX8:  I:= 12'
      P:= 0'
      Q:= 12'
      U:= 10'
XX9:  E(0):= 0'
        R:= V'
FOR K:=2 STEP 1 UNTIL (V+1) DO BEGIN
E(0):= F(0):= C(K)*E(0)+D(R)'
  R:= R-1'
END'
  C(V+2):= -1/E(0)'
  FOR K:=2 STEP 1 UNTIL (V+1) DO C(K):= C(K+1)'
GOTO SS(I)'
XX10:  D(0):= G(0)'
         P:= P+1'
         I:= 13'
         U:= 14'
```

```
GOTO XX9'
XX12:  G(0):=(C(V+1)-C(V-1))/C(V+1)'
       A(0):= C(V)-C(V-1)'
       G(0):= G(0)/A(0)'
XX11:  GOTO SS(U)'
XX13:  F(0):= C(V+1)-C(V)'
       F(0):= ABS(F(0))'
       B(0):= C(V+1)/1000'
IF F(0) LESS ABS(B(0)) THEN GOTO XX15 ELSE
GOTO SS(Q)'
XX14:  F(0):= G(0)-B(0)'
       F(0):= ABS(F(0))'
       B(0):= G(0)/1000'
IF F(0) LESS ABS(B(0)) THEN GOTO XX16'
IF G(0)= 0 THEN GOTO XX16 ELSE GOTO XX10'
XX15:  X:= 1/C(V+1)'
XX5:   I:= 0'
       Y:= 0'
GOTO XX17'
XX16:  X:= G(0)/2'
       Y:= C(V+1)-C(V)'
       A(0):= C(V)-C(V-1)'
       A(0):= Y/(A(0)*C(V+1)*C(V))'
       Y:= X*X'
       Y:= A(0)-Y'
IF Y LESS 0 THEN GOTO XX26'
       Y:= SQRT(Y)'
XX17:  S:=0'
P:= CHECKI(P)'
U:= 21'
XX18:  S:=  S+1'
   IF S GR 15 THEN GOTO XX23'
XX20:  E(0):= D(0):= 0'
       G(0):= C(0):= 1'
FOR K:=1 STEP 1 UNTIL N DO BEGIN
   A(0):= X*C(K-1)'
   C(K):= -Y*B(K-1)+ A(0)+A(K)'
   A(0):= X*B(K-1)'
   B(K):= A(0)+(Y*C(K-1))'
   A(0):= X*G(K-1)'
   G(K):= C(K)+A(0)- (Y*E(K-1))'
   A(0):= X*E(K-1)'
   E(K):= B(K)+A(0)+(Y*G(K-1))'
END'
   G(0):= G(N-1)*G(N-1)'
   E(0):= E(N-1)*E(N-1)'
   G(0):= G(0)+E(0)'
GOTO SS(U)'
XX21:  C(0):= C(N)*G(N-1)'
       E(0):= B(N)*E(N-1)'
       C(0):= (C(0)+E(0))/G(0)'
         X:= X-C(0)'
       C(0):= ABS(C(0))'
       F(0):= X/10000000'
IF C(0) GR ABS(F(0)) THEN GOTO XX22'
       U:= 22'
IF I=0 THEN GOTO XX24'
XX22:  C(0):= B(N)*G(N-1)'
       E(0):= C(N)*E(N-1)'
       C(0):= (C(0)- E(0))/G(0)'
         Y:= Y-C(0)'
       C(0):= ABS(C(0))'
       F(0):= Y/10000000'
```

```
IF C(0) GR ABS(F(0)) THEN GOTO XX18'
XX23:  IF I=0 THEN GOTO XX24'
RR(V-1,0):=RR(V,0):=X'
RR(V-1,1):=-Y'
RR(V,1):=Y'
IF V=2 THEN GOTO XX31 ELSE GOTO XX25'
XX24:RR(V,0):=X'
RR(V,1):=0'
IF V=1 THEN GOTO XX31'
B(0):=1'
S:= V-1'
FOR K:= 1 STEP 1 UNTIL S DO BEGIN
B(K):= X*B(K-1)+D(K)'
D(K):= B(K)'
END'
      V:= V-1'
GOTO XX2'
XX25:
      A(0):= 2*X'
      X:= X*X'
       Y:= Y*Y'
       X:= X+Y'
       B(1):= 1'
       B(0):= 0'
       S:= V-1'
FOR K:=2 STEP 1 UNTIL S DO BEGIN
B(K):= A(0)*B(K-1)'
G(0):= X*B(K-2)'
D(K-1):= B(K):= B(K)-G(0)+D(K-1)'      END'
V:= V-2'
GOTO XX2'
XX26:  Q:= 10'
GOTO XX10'
XX30:  C(0):= D(1)* D(1)'
       B(0):= 4*D(2)'
XX1:   P:= 0'
XX7:   C(0):= C(0)-B(0)'
XX19:  D(0):= -D(1)/2'
XX27:  C(0):= C(0)/4'
       E(0):= ABS(C(0))'
       E(0):= SQRT(E(0))'
IF C(0) LESS 0 THEN GOTO XX28'
Y:=0'
I:= 0'
X:= D(0)+E(0)'
GOTO XX17'
XX28:  X:= D(0)'
       Y:= E(0)'
       P:= 0'
       I:= 1'
GOTO XX17'
XX29:  X:= -D(1)'
       Y:=0'
       I:= 0'
GOTO XX17'
XX31:END'


READ N'  PRINT N'
BEGIN ARRAY A(0:N),R(1:N,0:1)'
      FOR J:=0 STEP 1 UNTIL N DO READ A(J)'
ROOT(N,A,R)'
      FOR J:=1 STEP 1 UNTIL N DO
           PRINT R(J,0),SAMELINE,£+I*?,R(J,1)'
END'
END'
END'
```

APPENDIX 4.4.

FOKKER-PLANCK EQUATION CORRESPONDING TO SYSTEM OF EQUATIONS (4.6.6)

Required Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{3} \frac{\partial (A_i p)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 (B_{ij} p)}{\partial x_i \partial x_j} \tag{A4.4.1}$$

where the incremental moments A_i, B_{ij} $(i,j = 1,2,3)$, are defined by

$$A_i = \underset{\delta t \to 0}{\text{limit}} \frac{\langle \delta x_i \rangle}{\delta t}, \quad B_{ij} = \underset{\delta t \to 0}{\text{limit}} \frac{\langle \delta x_i \delta x_j \rangle}{\delta t} \tag{A4.4.2}$$

From equations (4.6.6) we have :

$$\delta x_1 = -\frac{1}{T} x_1 \delta t + \frac{R}{T} x_2 \delta t + \frac{1}{T} \int_t^{t+\delta t} n_1(u) x_2(u) du \tag{A4.4.3}$$

$$\delta x_2 = -BKK_V R\, x_1 \delta t - BK_V\, x_1 x_3 \delta t \tag{A4.4.4}$$

$$\delta x_3 = -\frac{1}{T} x_3 \delta t + \frac{K}{T} \int_t^{t+\delta t} n_1(u) du \tag{A4.4.5}$$

But, $\displaystyle\int_t^{t+\delta t} n_1(u) x_2(u) du = \int_t^{t+\delta t} n_1(u) \left[x_2(t) + \int_t^u dx_2\right] du$

$$= x_2(t) \int_t^{t+\delta t} n_1(u) du + \int_t^{t+\delta t} \int_t^u n_1(u) \frac{dx_2}{d\tau} du d\tau$$

$$= x_2(t) \int_t^{t+\delta t} n_1(u) du + \int_t^{t+\delta t} \int_t^u n_1(u) \left[-BKK_V R x_1 - BK_V x_1 x_3\right] du d\tau$$

so that equation (A4.4.3) becomes

$$\delta x_1 = -\frac{1}{T}x_1\delta t + \frac{R}{T}x_2\delta t + \frac{x_2(t)}{T}\int_t^{t+\delta t} n_1(u)du + \frac{1}{T}\int_t^{t+\delta t}\int_t^u n_1(u)$$

$$- \left[-BKK_vRx_1 - BK_vx_1x_3\right]dud\tau \qquad (A4.4.6)$$

From equations (A4.4.4), (A4.4.5) and (A4.4.6) we have, using definitions (A4.4.2), that

$$A_1 = -\frac{1}{T}x_1 + \frac{R}{T}x_2$$

$$A_2 = -BKK_vRx_1 - BK_vx_1x_3$$

$$A_3 = -\frac{1}{T}x_3$$

$$B_{11} = \frac{x_2^{\,2}\,2D}{T^2}$$

$$B_{12} = B_{21} = B_{22} = B_{23} = B_{32} = 0$$

$$B_{13} = B_{31} = \frac{2KD}{T^2}\,x_2$$

$$B_{33} = \frac{2K^2D}{T^2}$$

Substituting in equation (A4.4.1) gives the Fokker-Planck equation, corresponding to the system of equations (4.6.6), as

$$\frac{\partial p}{\partial t} = \frac{1}{T}\frac{\partial(x_1p)}{\partial x_1} - \frac{R}{T}x_2\frac{\partial p}{\partial x_1} + (BKK_vRx_1 + BK_vx_1x_3)\frac{\partial p}{\partial x_2} + \frac{1}{T}\frac{\partial(x_3p)}{\partial x_3}$$

$$+ \frac{Dx_2^{\,2}}{T^2}\frac{\partial^2 p}{\partial x_1^{\,2}} + \frac{K^2D}{T^2}\frac{\partial^2 p}{\partial x_3^{\,2}} + \frac{2DK}{T^2}x_2\frac{\partial^2 p}{\partial x_1\partial x_3} \qquad (A4.4.7)$$

Calculation of the moment equations

(i) 1st order moment equations

Multiplying equation (A4.4.7) throughout by x_i ($i = 1,2,3$) and integrating over all $x_1x_2x_3$ gives respectively

$$\dot{m}_{1,0,0} = \langle\dot{x}_1\rangle = -\frac{1}{T}\langle x_1\rangle + \frac{R}{T}\langle x_2\rangle$$

$$= -\frac{1}{T} m_{1,0,0} + \frac{R}{T} m_{0,1,0}$$

$$\dot{m}_{0,1,0} = \langle\dot{x}_2\rangle = -BRKK_v m_{1,0,0} - BK_v m_{1,0,1}$$

$$\dot{m}_{0,0,1} = \langle\dot{x}_3\rangle = -\frac{1}{T} m_{0,0,1}$$

(ii) 2nd order moment equations

Multiplying equation (A4.4.7) throughout by x_ix_j ($i,j = 1,2,3$) and integrating over all $x_1x_2x_3$ gives respectively

$$\dot{m}_{2,0,0} = \langle\dot{x}_1^2\rangle = -\frac{2}{T} m_{2,0,0} + \frac{2R}{T} m_{1,1,0} + \frac{2D}{T^2} m_{0,2,0}$$

$$\dot{m}_{1,1,0} = \langle\dot{x}_1x_2\rangle = -\frac{1}{T} m_{1,1,0} + \frac{R}{T} m_{0,2,0} - BRKK_v m_{2,0,0} - BK_v m_{2,0,1}$$

$$\dot{m}_{0,2,0} = \langle\dot{x}_2^2\rangle = -2BRKK_v m_{1,1,0} - 2BK_v m_{1,1,1}$$

$$\dot{m}_{1,0,1} = \langle\dot{x}_1x_3\rangle = -\frac{2}{T} m_{1,0,1} + \frac{R}{T} m_{0,1,1} + \frac{2DK}{T^2} m_{0,1,0}$$

$$\dot{m}_{0,1,1} = \langle\dot{x}_2x_3\rangle = -BRKK_v m_{1,0,1} - BK_v m_{1,0,2} - \frac{1}{T} m_{0,1,1}$$

$$\dot{m}_{0,0,2} = \langle\dot{x}_3^2\rangle = -\frac{2}{T} m_{0,0,2} + \frac{2K^2D}{T^2}$$

(iii) general order moment equations

Multiplying equation (A4.4.7) throughout by $x_1^{K_1} x_2^{K_2} x_3^{K_3}$ and integrating over all $x_1\, x_2\, x_3$ gives that the general order moment equations are obtained from

$$\dot{m}_{K_1,K_2,K_3} = \langle x_1^{K_1} \dot{x}_2^{K_2} x_3^{K_3} \rangle = -\frac{K_1}{T} m_{K_1,K_2,K_3} + \frac{R}{T} K_1 m_{K_1-1,\, K_2+1,\, K_3}$$

$$-BRKK_v K_2 m_{K_1+1,\, K_2-1, K_3} - BK_v K_2 m_{K_1+1, K_2-1, K_3+1} - \frac{K_3}{T} m_{K_1,K_2,K_3}$$

$$+ \frac{D}{T^2} K_1(K_1-1) m_{K_1-2,K_2+2,K_3} + \frac{DK^2}{T^2} K_3(K_3-1) m_{K_1,K_2,K_3-2}$$

$$- \frac{2DKK_1K_3}{T^2} m_{K_1-1,\, K_2+1,\, K_3-1}$$

APPENDIX 4.5.

ALGOL PROGRAM FOR GENERATING RANDOM NUMBERS

The function of the program is to generate a sequence of normally distributed random numbers with given mean and standard deviation.

(i) Presentation of data

.0399
.0793
.1179
.1554
.1915
.2258
.2581
.2882
.3160
.3414
.3643
.3850
.4032
.4193
.4332
.4452
.4555
.4641
.4713
.4773
.4822
.4861
.4893
.4918
.4938
.4953
.4976
.4975
.4982
.4987
.4991
.4993
.4995
.4997
.4998
.4999
.5

followed by

L - number of random numbers required

M - mean of required sequence

D - standard deviation of required sequence

(ii) Interpretation of results

Print out will be

MEAN = M

STANDARD DEVIATION = D

followed by L numbers of the sequence.

(iii) Method used

Pseudo random numbers are initially generated by means of the recurrence relation

$$x_{n+1} = x_n * \underbrace{63419 \mod 507359}_{\text{relative prime forming complete cyclic group}}$$

This suffers from the drawback that a small number is always generated by a large one and vice-versa, a disadvantage that is overcome to some extent by taking the last 3 or 4 digits. However, as an improvement over this drawback a scheme is developed to select in a random manner one of a set of 20 x's and replace the selected x by another generated one.

The scheme is to first generate 20 x's, then the next x, say x_j, is generated and x_j mod 20 found to give a K in the range (0 - 19) - this value being then used to select the K^{th} x from the list of 20; x_{j+1} is then generated and the K^{th} element in the list replaced by it.

By this time, having taken x mod 10000 we have generated a sequence of integers uniformly distributed in the range 0 - 9999 which is converted to a decimal 0 - .9999.

From a table of values of erf(x) the uniformly distributed random number is transformed into a normally distributed random number (0, 1), which is then mapped onto (μ, σ).

(iv) Notes on program

All the values of x will be in the range $-3.7\sigma < x < 3.7\sigma$ since the generation only produces effectively 4 digit fractions and the relative accuracy of tables only gives this range.

(v) Print out of program

A print out of the program is given on page 226.

RANDOM NUMBER GENERATOR'

```
BEGIN INTEGER XXX' INTEGER ARRAY RNA(0:19)'
      BOOLEAN FIRST'  ARRAY ND(0:37)'
      INTEGER PROCEDURE RN'
      BEGIN REAL Z,ZZ'
            Z:=XXX'
            Z:=Z*63419'
            ZZ:=ENTIER(Z/507359)'
            Z:=Z-ZZ*507359.0'
            XXX:=Z'
            RN:=XXX-XXX DIV 10000*10000'
      END'

      INTEGER PROCEDURE RRN'
      BEGIN INTEGER J'
            IF FIRST THEN FOR J:=0 STEP 1 UNTIL 19 DO
                          RNA(J):=RN'
            FIRST:=1=0' J:=RN'
            J:=J-J DIV 20*20' RRN:=RNA(J)-5000'
            RNA(J):=RN'
      END'

      REAL PROCEDURE RANDOM(MEAN,SIGMA)'
      REAL MEAN,SIGMA'
      BEGIN REAL R'  INTEGER J'
            R:=RRN/10000'
            IF R NOTEQ 0 THEN
            BEGIN REAL P,K'
                  P:=ABS(R)'
                  FOR K:=0,K+1 WHILE P GR ND(J) DO
                  J:=K+1'
                  P:=J-(ND(J)-P)/(ND(J)-ND(J-1))'
                  R:=SIGN(R)*P'
            END'
            RANDOM:=MEAN+R*SIGMA/10'
      END'
      FOR XXX:=1 STEP 1 UNTIL 37 DO READ ND(XXX)'
      ND(0):=0' XXX:=1'  FIRST:=TRUE'



      BEGIN INTEGER K,L'
            REAL M,D'
            READ L,M,D'
            PRINT ££L?MEAN=?,SAMELINE,M,
                  ££L?STANDARD DEVIATION=?,SAMELINE,D'
            FOR K:=1 STEP 1 UNTIL L DO
                  PRINT RANDOM(M,D)'
      END'
END'
```

APPENDIX 4.6.

STATISTICAL PROPERTIES OF THE RANDOM PROCESS SIMULATED DIGITALLY

The numerical procedure described in appendix 4.5 generates a sequence of random numbers n_i $(i = 1,2, \text{etc.})$ which are normally distributed with variance σ^2_N. These numbers are assumed to have the same statistical properties as sequences of independent random numbers although they are in fact completely determined by the first number in the sequence. The random variable $\alpha(t)$ is then found by spacing these gaussian distributed numbers at intervals h and joining them by straight lines.

We shall take the mean value of the numbers to be zero and proceed to find the autocorrelation function

$$\emptyset(\tau) = \lim_{T\to\infty} \int_0^T \alpha(t)\ \alpha(t+\tau)dt \quad , \quad \tau > 0 ,$$

of the random variable $\alpha(t)$. Since all the intervals $Kh < t < \overline{K+1}\,h$, $K = 0, 1, 2$, etc., are the same from a statistical point of view we loose no generality in considering t to lie in the first interval $0 < t < h$.

Case 1, $0 < \tau < h$

Two possibilities may arise; these are that

either (i) $\alpha(t)$ and $\alpha(t+\tau)$ lie in the same interval $0 < t < \tau$ - this occurs when $t + \tau < h$, i.e. $t < h - \tau$.

or (ii) $\alpha(t)$ is in the interval $0 < t < h$ whilst $\alpha(t+\tau)$ is in the interval $h < t < 2h$.

Case (i) exists with probability $1 - \frac{\tau}{h}$ whilst case (ii) exists with probability $\frac{\tau}{h}$.

For case (i)

$$\alpha(t)\ \alpha(t+\tau) = \left[n_1 + \frac{(n_2 - n_1)t}{h} \right] \left[n_1 + \frac{(n_2 - n_1)}{h} (t+\tau) \right]$$

Taking time averages

$$\underset{t}{\mathrm{Av}}\ [\alpha(t)\ \alpha(t+\tau)] = \frac{1}{h-\tau} \int_0^{h-\tau} \alpha(t)\ \alpha(t+\tau)d\tau$$

$$= n_1^2 + \frac{n_1}{h}(n_2 - n_1)(h-\tau) + \frac{n_1}{h}(n_2 - n_1)\tau + \frac{(n_2 - n_1)^2}{h^2}\left[\frac{(h-\tau)^2}{3} + \frac{\tau}{2}(h-\tau)\right]$$

Averaging over the n_i gives

$$<\alpha(t)\ \alpha(t+\tau)> = \sigma_N^2\left[\frac{2}{3} - \frac{1}{3}\frac{\tau}{h} - \frac{1}{3}\frac{\tau^2}{h^2}\right]$$

For case (ii)

$$\alpha(t)\ \alpha(t+\tau) = \left[n_1 + \frac{(n_2 - n_1)t}{h}\right]\left[n_2 + \frac{(n_3 - n_2)}{h}(t+\tau-h)\right]$$

Averaging over the n_i gives

$$\underset{n_i}{\mathrm{Av}}\left[\alpha(t)\ \alpha(t+\tau)\right] = \sigma_N^2\left[\frac{2t}{h} - \frac{t^2}{h^2} - \frac{t\tau}{h^2}\right]$$

Taking time averages gives

$$<\alpha(t)\ \alpha(t+\tau)> = \frac{\sigma_N^2}{\tau h^2}\int_{h-\tau}^{h}(2th - t^2 - t\tau)dt$$

$$= \sigma_N^2\left[1 - \frac{\tau}{h} + \frac{\tau^2}{6h^2}\right]$$

Thus, for $0 < \tau < h$

$$\emptyset(\tau) = \emptyset_1(\tau) = \sigma_N^2\left\{(1 - \frac{\tau}{h})\left[\frac{2}{3} - \frac{1}{3}\frac{\tau}{h} - \frac{1}{3}\frac{\tau^2}{h^2}\right] + \frac{\tau}{h}\left[1 - \frac{\tau}{h} + \frac{1}{6}\frac{\tau^2}{h^2}\right]\right\}$$

$$= \sigma_N^2\left[\frac{2}{3} - \frac{\tau^2}{h^2} + \frac{1}{2}\frac{\tau^3}{h^3}\right]$$

Case 2, $h < \tau < 2h$

Again two possibilities may arise; whilst $\alpha(t)$ must lie in the first interval $0 < t < h$, $\alpha(t + \tau)$ may

either (i) lie in the second interval $h < t < 2h$ - this will occur when $t + \tau < 2h$, i.e. $t < 2h - \tau$

or (ii) lie in the third interval $2h < t < 3h$

Case (i) exists with probability $\frac{2h - \tau}{h}$ whilst case (ii) exists with probability $\frac{\tau - h}{h}$.

For case (i)

$$\alpha(t)\ \alpha(t + \tau) = [\ n_1 + \frac{(n_2 - n_1)}{h}\ t\ ]\ [\ n_2 + \frac{(n_3 - n_2)}{h}\ (t + \tau - h)\ ]$$

Averaging over the n_i gives

$$\underset{n_i}{\text{Av}}\ [\alpha(t)\ \alpha(t + \tau)] = \sigma_N^2\ [\ \frac{2t}{h} - \frac{t^2}{h^2} - \frac{t\tau}{h^2}\ ]$$

Averaging over time gives

$$<\alpha(t)\ \alpha(t + \tau)> = \frac{\sigma_N^2}{2h - \tau} \int_0^{2h-\tau} [\ \frac{2t}{h} - \frac{t^2}{h^2} - \frac{t\tau}{h^2}\ ]dt$$

$$= \sigma_N^2\ [\ \frac{2}{3} - \frac{2}{3}\frac{\tau}{h} + \frac{1}{6}\frac{\tau^2}{h^2}\ ]$$

For case (ii)

$$\alpha(t)\ \alpha(t + \tau) = [\ n_1 + \frac{(n_2 - n_1)t}{h}\ ]\ [\ n_3 + \frac{(n_4 - n_3)}{h}\ (t + \tau - h)\ ]$$

Averaging over the n_i gives

$$\underset{n_i}{\text{Av}}\ [\alpha(t)\ \alpha(t + \tau)] = 0$$

so that, $<\alpha(t)\ \alpha(t + \tau)> = 0$

Thus, for $h < \tau < 2h$

$$\emptyset(\tau) = \emptyset_2(\tau) = \sigma_N^2 \left[\frac{4}{3} - \frac{2\tau}{h} + \frac{\tau^2}{h^2} - \frac{1}{6}\frac{\tau^3}{h^3} \right]$$

Case 3, $\tau > 2h$

In this case $\alpha(t)$ and $\alpha(t + \tau)$ will be independent so that

$$\emptyset(\tau) = \emptyset_3(\tau) = 0$$

Hence, the autocorrelation $\emptyset(\tau)$ is defined by

$$\emptyset(\tau) = \emptyset_1(\tau) = \sigma_N^2 \left\{ \frac{2}{3} - \frac{\tau^2}{h^2} + \frac{1}{2}\frac{\tau^3}{h^3} \right\}, \quad 0 \leqslant \tau \leqslant h$$

$$\emptyset(\tau) = \emptyset_2(\tau) = \sigma_N^2 \left\{ \frac{4}{3} - \frac{2\tau}{h} + \frac{\tau^2}{h^2} - \frac{1}{6}\frac{\tau^3}{h^3} \right\}, \quad h \leqslant \tau \leqslant 2h$$

$$\emptyset(\tau) = \emptyset_3(\tau) = 0 \qquad , \quad \tau \geqslant 2h$$

(A4.6.1)

A graph of $\emptyset(\tau)$ is shown in fig (A4.6.1)

In particular, we note that

$$\emptyset(o) = \emptyset_1(o) = \frac{2}{3}\sigma_N^2$$

so that, the variance σ^2 of the random variable $\alpha(t)$ is given by

$$\sigma^2 = \frac{2}{3}\sigma_N^2 \qquad \text{(A4.6.2)}$$

Using Bendat's [88] notation the spectral density $G(\omega)$ and autocorrelation function are related by the Wiener-Khinchine relationships

$$G(\omega) = \frac{2}{\pi}\int_0^\infty \emptyset(\tau)\cos\omega\tau \, d\tau$$

$$\emptyset(\tau) = \int_0^\infty G(\omega)\cos\omega\tau \, d\omega \qquad \text{(A4.6.3)}$$

where $G(\omega)$ is measured in watts/rad/sec. and defined for positive frequency only.

From equation (A4.6.3) we have that

$$G(\omega) = \frac{2\sigma_N^{\,2}}{\pi} \left\{ \int_0^h \left(\frac{2}{3} - \frac{\tau^2}{h^2} + \frac{1}{2}\frac{\tau^3}{h^3} \right)\cos \omega\tau \, d\tau + \int_h^{2h} \left(\frac{4}{3} - \frac{2\tau}{h} + \frac{\tau^2}{h^2} - \frac{1}{6}\frac{\tau^3}{h^3} \right) \cos \omega\tau \, d\tau \right\}$$

$$= \frac{2\sigma_N^{\,2}}{\pi h^3 \omega^4} \left(3 - 4\cos(\omega h) + \cos(2\omega h) \right) \qquad \text{(A4.6.4)}$$

The form of $G(\omega)$ for fixed h and σ_N is shown in fig. (A4.6.2).

Using the Maclaurin expansion for the consine terms equation (A4.6.4) becomes

$$G(\omega) \simeq \frac{2\sigma_N^{\,2}}{\pi h^3 \omega^4} \left\{ 3 - 4\left[1 - \frac{(\omega h)^2}{2!} + \frac{(\omega h)^4}{4!} - \frac{(\omega h)^6}{6!} \right] + \left[1 - \frac{(2\omega h)^2}{2!} + \frac{(2\omega h)^4}{4!} - \frac{(2\omega h)^6}{6!} \right] \right\}$$

i.e. $$G(\omega) \simeq \frac{\sigma_N^{\,2} h}{\pi} \left[1 - \frac{(\omega h)^2}{6} \right] \qquad \text{(A4.6.5)}$$

Thus, for small ωh, the spectral density may be taken as being approximately constant and given by

$$G(\omega) = \frac{\sigma_N^{\,2} h}{\pi} \qquad \text{(A4.6.6)}$$

Using the notation of equations (A4.6.3) if $\beta(\tau) = K\delta(\tau)$, where K is a constant, then

$$G(\omega) = \frac{2}{\pi} \int_0^\infty K\delta(\tau)\cos \omega\tau \, d\tau = \frac{2K}{\pi} \cdot \frac{1}{2} = \frac{K}{\pi}$$

Thus, if the random variable $\alpha(t)$ is taken to be approximately *white noise,* with spectral density given by equation (A4.6.6) then its autocorrelation function is

$$\phi(\tau) = \sigma_N^{2} h \ \delta(\tau) \tag{A4.6.7}$$

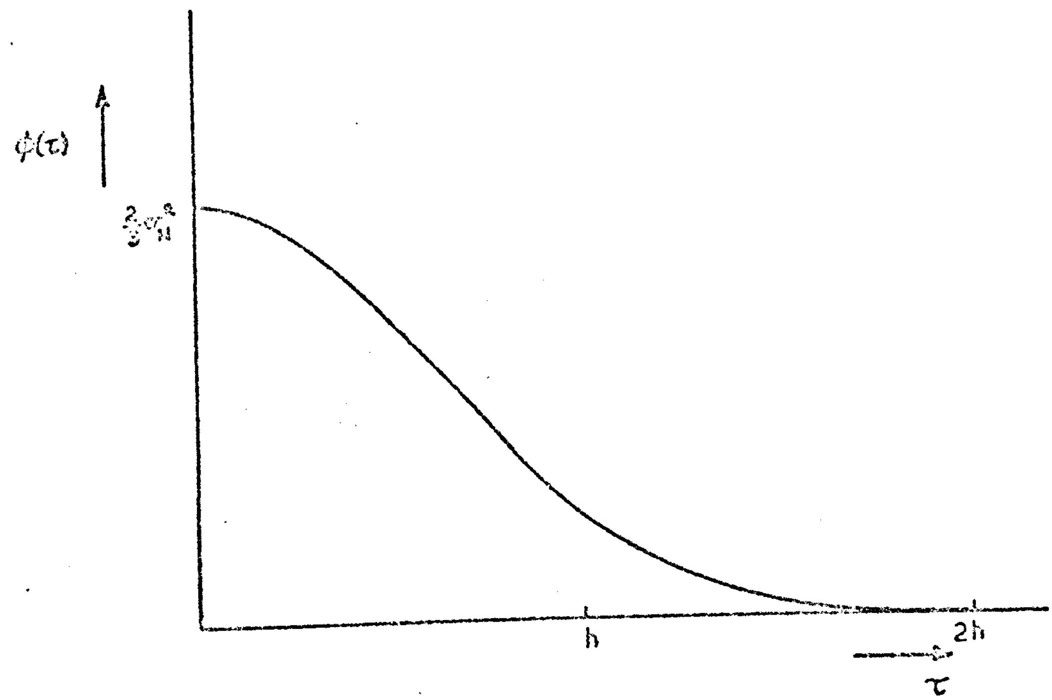


FIG. A4.6.1. Autocorrelation function for positive τ

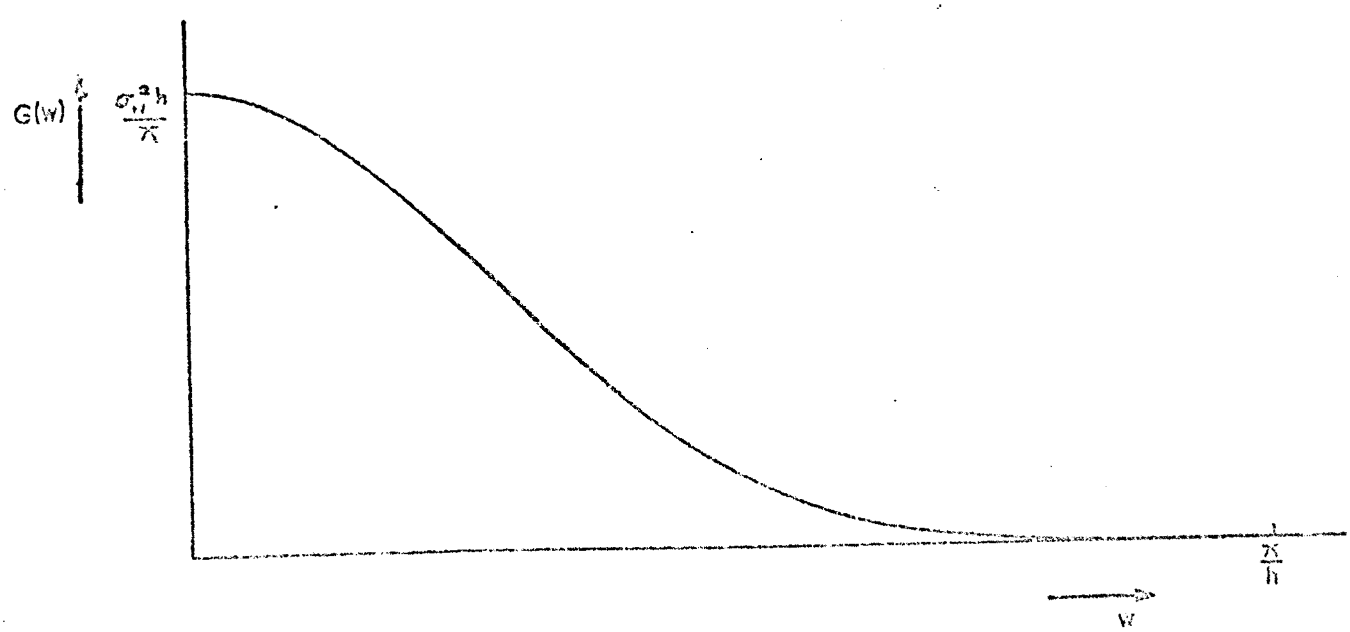


FIG. A4.6.2. Spectral density plot for positive ω

APPENDIX 6.1.

MATRIX EQUATION RELATING OUTPUT TO INPUT FOR FILTER Y_3/Y_1

If X_F and X are the output and input signals respectively to the filter Y_3/Y_1, then

$$X_F(s) = \frac{4.57(s^2 + 16.78s + 124.8)}{s^3 + 16.78s^2 + 124.8s + 570} X(s)$$

$$\text{or } \dddot{X}_F = -16.78\ddot{X}_F - 124.8\dot{X}_F - 570X_F + 4.57\ddot{X} + 4.57(16.78)\dot{X} + 4.57(124.8)X \qquad \text{(A6.1.1)}$$

where dots denote differentiation with respect to time

Equation (A6.1.1) may be written in the matrix form

$$\frac{d}{dt}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -570 & -124.8 & -16.78 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} X \qquad \text{(A6.1.2)}$$

where

$$Y_1 = X_F$$

$$Y_2 = \dot{Y}_1 - C_1X = \dot{X}_F - C_1X$$

$$Y_3 = \dot{Y}_2 - C_2X = \ddot{X}_F - C_1\dot{X} - C_2X$$

and

$$Y_3 = \dddot{X}_F - C_1\ddot{X} - C_2\dot{X}$$

$$= -570X_F - 124.8(\dot{X}_F - C_1X) - 16.78(\ddot{X}_F - C_1\dot{X} - C_2X) + C_3X$$

that is,

$$\dddot{X}_F = -16.78\ddot{X}_F - 124.8\dot{X}_F - 570X_F + C_1\ddot{X} + (C_2 + 16.78C_1)\dot{X}$$

$$+ (124.8C_1 + 16.78C_2 + C_3)X \qquad \text{(A6.1.3)}$$

Comparing equation (A6.1.3) with equation (A6.1.1) gives

$$C_1 = 4.57,\ C_2 = 0,\ C_3 = 0$$

so that equation (A6.1.2) may be written

$$\frac{d}{dt}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -570 & -124.8 & -16.78 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} 4.57 \\ 0 \\ 0 \end{bmatrix} X \qquad \text{(A6.1.4)}$$

where $Y_1 = X_F$, $Y_2 = \dot{X}_F - 4.57X$, $Y_3 = \ddot{X}_F - 4.57\dot{X}$.

In the case when K_2 is adapting alone

$$X = \frac{\delta\ddot{e}}{A_3K_2} - \frac{\theta_F}{K_2}\ \delta K_2$$

and equation (A6.1.4) becomes

$$\frac{d}{dt}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -570 & -124.8 & -16.78 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\left(\frac{\delta\dot{e}}{K_2} - \frac{\theta_F A_3}{K_2}\ \delta K_2 \right)$$

since $A_3 = 4.57$.

P U B L I S H E D   P A P E R S

Published papers, related to the work contained in this thesis, in order of publication are :

* (i) JAMES, D.J.G., *Stability analysis of a model-reference adaptive control system with sinusoidal inputs,* Int. Jour. Control. Vol.9., No.3., 311, 1969.

* (ii) JAMES, D.J.G., *Stability of a sinusoidal-perturbation extremal control system,* Electronic Letters, Vol.5., No.10, 220, 1969.

* (iii) PARKS, P.C. and JAMES, D.J.G., *Stability of model-reference systems with random impulsive inputs,* ibid, Vol.6., No.5., 131, 1970.

* (iv) JAMES, D.J.G., *Stability and subharmonics in a sinusoidal perturbation hill climbing system,* Int. Jour. Control, Vol.13., No.1., 165, 1971.

(v) JAMES, D.J.G., *Stability of a model-reference system,* Amer. Inst. Aero. and Astro., (forthcoming issue).

(vi) JAMES, D.J.G., *Dynamics of model-reference systems* (submitted to I.F.A.C.).

(vii) JAMES, D.J.G., *Stability of Liapunov redesign model-reference systems* (In preparation for submission to I.E.E.E.).

* Photocopies bound with thesis.

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Stability analysis of a model reference adaptive control system with sinusoidal inputs†

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In this paper a rigorous method is presented for analysing an M.I.T. type model reference adaptive control system with sinusoidal inputs. The linearized equations for the adapting system, formed by using small perturbation analysis, are written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t)$ is periodic. This matrix equation is then integrated over one period using a Runge–Kutta technique. The transition matrix relating the value of $\mathbf{x}$ at the end of a period to its value at the beginning of the period is examined to see whether all its eigenvalues are within the unit circle, thus establishing stability.

1. Introduction

White (1966) examined the stability of a Whitaker type model reference adaptive control system with sinusoidal inputs using a parameter perturbation technique which resulted in differential equations having periodic coefficients. These equations were then replaced by equations having time-averaged coefficients which in turn were examined by means of the Routh–Hurwitz criterion so that the critical values of the adaptive gains could be calculated. The replacing of differential equations with periodic coefficients by equations with time-averaged coefficients is a dangerous procedure and gives rise to serious doubts regarding the validity of the method used in the stability analysis. It is seen to fail even for comparatively simple equations such as the following equation:

$$\ddot{x} + 0{\cdot}2\dot{x} + (4{\cdot}5 - 4\cos 2t)x = 0,$$

which is unstable (McLachlan 1947).

In this paper an alternative rigorous method is presented for analysing the stability of such a system and the results obtained compared with both the Routh–Hurwitz analysis and the analogue computer simulation of White (1966). This numerical approach was first suggested by Parks (1966) although a detailed account of the method of procedure was not given. It will be pointed out that the matrices originally suggested by Parks for representing the system are erroneous, due to the omission of the effect of the filters and of an important linear term. The linearized equations are formed, as by White, using small perturbation analysis, and these are then written in the matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{1}$$

where $\mathbf{x}$ is an n-column matrix and $\mathbf{A}(t)$ and $n \times n$ periodic matrix. This matrix

† Communicated by P. C. Parks.

differential equation is then integrated numerically over a period T and the transition matrix $\mathbf{C}^*$ obtained such that

$$\mathbf{x}(T) = \mathbf{C}^*\mathbf{x}(0), \tag{2}$$

where $\mathbf{x}(0)$ and $\mathbf{x}(T)$ are the values of $\mathbf{x}$ at the beginning and end of a period respectively.

The numerical method described by Parks (1966), based on Euler's method (Noble 1964) of solving a linear first-order differential equation, involves replacing a first-order differential equation by a first-order difference equation and is known to give inaccurate results. In this particular problem it was found to be inadequate even when a large number of sub-intervals was taken and another method, obtained by reformulating the fourth-order Runge–Kutta method (Noble 1964) of solving a first-order differential equation, is employed. The transition matrix $\mathbf{C}^*$ is then examined to see whether all its eigenvalues are within the unit circle; this is carried out by first obtaining the characteristic polynomial of $\mathbf{C}^*$ using a method due to the Russian mathematician Faddeev (Faddeeva 1959) and then examining this polynomial using the determinant method due to Jury (1964, 1965).

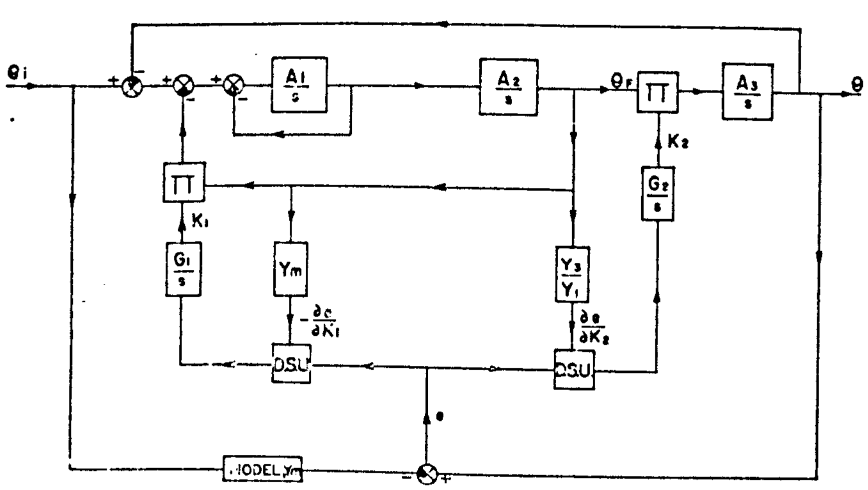
2. Description of the system

The adaptive control configuration chosen is the same as that of White (1966), and is shown in fig. 1. The basic equation of the system is:

$$(D^3 + A_1D^2 + A_1A_2K_1D + A_1A_2A_3K_2)\theta_0 = A_1A_2A_3K_2\theta_i, \tag{3}$$

where D is the differential operator d/dt and $A_1 = 16{\cdot}78$, $A_2 = 14{\cdot}88$, $A_3 = 4{\cdot}57$, $K_1 = K_2 = 0{\cdot}5$ (nominal). The reference model is second order and has transfer function $Y_m = 40/(s^2 + 6{\cdot}32s + 40)$, where s is the Laplacian operator.

Fig. 1



Block diagram of the adaptive control system.

The input signal θ_i is common to the system and model and the adaptive error is $e = \theta_0 - \theta_m$, where θ_0 and θ_m are the outputs from the system and model respectively.

The self-adaptive performance criterion employed to adjust the parameters K_i is that developed at the M.I.T. by Osburn *et al.* (1961). In this criterion the parameters are varied in such a way as to minimize the integral of error squared, and the adaptive control law is taken of the form $(\partial K_i/\partial t) \propto -e(\partial e/\partial K_i)$. It was shown by Osburn *et al.* (1961) that this law could be taken in the form:

$$\frac{\partial K_i}{\partial t} = -G_i e \operatorname{sign} \frac{\partial e}{\partial K_i}, \tag{4}$$

where only the sign of $\partial e/\partial K_i$ is taken to ensure that the sign of $e(\partial e/\partial K_i)$ is correct. The product $e \operatorname{sign}(\partial e/\partial K_i)$ is formed by passing the two signals into a diode switching unit (d.s.u.) the output of which is $\pm e$ depending on the sign of $\partial e/\partial K_i$. The approximations to $\partial e/\partial K_1$ and $\partial e/\partial K_2$ are obtained by feeding the signal θ_F through filters which can be identified as follows:

$$\frac{\partial e}{\partial K_1} = \frac{\partial}{\partial K_1}(\theta_0 - \theta_m) = -\frac{A_1{}^2 A_2{}^2 A_3 K_2 s}{[s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2]^2}\theta_i = -Y_s \theta_F \simeq -Y_m \theta_F,$$

since the model is a good approximation of the system around the correct value of K_1; i.e. the signal obtained by passing θ_F through a filter identical with the model is $-(\partial e/\partial K_1)$ and not $(\partial e/\partial K_1)$ as indicated by White, as the latter signal would lead to a negative gain G_1.

$$\frac{\partial e}{\partial K_2} = \frac{\partial \theta_0}{\partial K_2} = \frac{A_3(s^2 + A_1 s + A_1 A_2 K_1)}{(s^3 + A_1 s^2 + A_1 A_2 K_1 s + A_1 A_2 A_3 K_2)}\theta_F \simeq \frac{Y_3}{Y_1}\theta_F,$$

where Y_3 is the transfer function of the system with the parameters fixed at their normal values and Y_1 is the transfer function from the system input to the parameter disturbance summation point.

The adaptive control system is considered to be adapting on a steady sinusoidal input signal $l \sin \omega t$.

3. Mathematical theory

The small perturbation technique employed in this paper leads to matrix differential equations of the form in (1). For stability considerations these equations are integrated over one period and the transition matrix $\mathbf{C}^*$ defined in (2) calculated. The result of the following theorem is then used to establish stability.

Theorem (Malkin 1952)

For the system of linear differential equations

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}, \tag{i}$$

where $\mathbf{P}(t+\tau) = \mathbf{P}(t)$, there exists a transition matrix $\boldsymbol{\phi}(t, t+\tau)$ such that $\mathbf{x}(t+\tau) = \boldsymbol{\phi}\mathbf{x}(t)$. A necessary and sufficient condition for system (i) to be asymptotically stable is that all the eigenvalues of matrix $\boldsymbol{\phi}$ lie inside the unit circle.

In order to evaluate the transition matrix $\mathbf{C}^*$ defined in (2) it is convenient to employ a numerical method that may be reformulated in such a way as to give $\mathbf{C}^*$ directly. Since it is completely self-contained and requires no predetermination of a set of starting values the Runge–Kutta method is given preference over the various predictor-corrector methods of integration. The fourth-order Runge–Kutta method† of solving a system of first-order linear differential equations is reformulated as follows:

If $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{x}$ is an n-column matrix and $\mathbf{A}(t)$ an $n \times n$ periodic matrix of period $T(=2\pi/\omega)$; then the period is split into a large number of intervals N, each of duration $\Delta t(=T/N)$, and the following finite difference relationship employed:

$$\mathbf{x}(\overline{m+1}\Delta t) = \mathbf{x}(m\Delta t) + \tfrac{1}{6}(\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + 2\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_4), \tag{ii}$$

where

$$\boldsymbol{\alpha}_1 = \Delta t\mathbf{A}(m\Delta t)\mathbf{x}(m\Delta t) = \mathbf{K}_1\mathbf{x}(m\Delta t),$$

$$\boldsymbol{\alpha}_2 = \Delta t\mathbf{A}(\overline{m+\tfrac{1}{2}}\Delta t)[\mathbf{x}(m\Delta t) + \tfrac{1}{2}\boldsymbol{\alpha}_1] = \Delta t\mathbf{A}(\overline{m+\tfrac{1}{2}}\Delta t)[\mathbf{I} + \tfrac{1}{2}\mathbf{K}_1]\mathbf{x}(m\Delta t),$$

where $\mathbf{I}$ is the unit $n \times n$ matrix

$$= \mathbf{K}_2\mathbf{x}(m\Delta t),$$

$$\boldsymbol{\alpha}_3 = \Delta t\mathbf{A}(\overline{m+\tfrac{1}{2}}\Delta t)[\mathbf{x}(m\Delta t) + \tfrac{1}{2}\boldsymbol{\alpha}_2] = \Delta t\mathbf{A}[\overline{m+\tfrac{1}{2}}\Delta t)[\mathbf{I} + \tfrac{1}{2}\mathbf{K}_2]\mathbf{x}(m\Delta t)$$
$$= \mathbf{K}_3\mathbf{x}(m\Delta t),$$

$$\boldsymbol{\alpha}_4 = \Delta t\mathbf{A}(\overline{m+1}\Delta t)[\mathbf{x}(m\Delta t) + \boldsymbol{\alpha}_3] = \Delta t\mathbf{A}(\overline{m+1}\Delta t)[\mathbf{I} + \mathbf{K}_3]\mathbf{x}(m\Delta t)$$
$$= \mathbf{K}_4\mathbf{x}(m\Delta t),$$

substituting in (ii) for $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_4$ gives:

$$\mathbf{x}(\overline{m+1}\Delta t) = [\mathbf{I} + \tfrac{1}{6}\mathbf{K}_1 + \tfrac{1}{3}\mathbf{K}_2 + \tfrac{1}{3}\mathbf{K}_3 + \tfrac{1}{6}\mathbf{K}_4]\mathbf{x}(m\Delta t) = \mathbf{B}^*(m\Delta t)\mathbf{x}(m\Delta t). \tag{iii}$$

By repeated application of (iii) the solution at the end of a period in terms of that at the beginning of the period becomes:

$$\mathbf{x}(T) = \mathbf{x}(\overline{N\Delta t}) = \mathbf{B}^*(\overline{N-1}\Delta t)\mathbf{B}^*(\overline{N-2}\Delta t)\ldots\mathbf{B}^*(0)\mathbf{x}(0)$$
$$= \prod_{r=0}^{N-1}\mathbf{B}^*(r\Delta t)\mathbf{x}(0) = \mathbf{C}^*\mathbf{x}(0),$$

where

$$\mathbf{C}^* = \prod_{r=0}^{N-1}\mathbf{B}^*(r\Delta t).$$

The problem was repeated for various values of N to ensure that the numerical procedure had converged. It was found that the values of N varied with the frequency of the input signal. The next step in the analysis is to examine the eigenvalues of the matrix $\mathbf{C}^*$. Although the matrix $\mathbf{C}^*$ is itself real some of its eigenvalues may occur as complex conjugates and this sometimes causes difficulties regarding time of convergence when employing standard numerical methods for evaluating the eigenvalues. A good discussion of these numerical methods, together with the difficulties involved when the eigenvalues occur as complex conjugates, may be found in the work of Wilkinson (1965).

Bearing in mind that in the problem at hand it is not necessary to know the exact values of the eigenvalues of $\mathbf{C}^*$, but rather it is only required to show that

† Since the completion of this work a paper by Davison (1968) uses the Crank–Nicolson method for obtaining the transition matrix $\mathbf{C}^*$ followed by a direct evaluation of the eigenvalues using QR procedure.

their moduli are less than unity, a very elegant procedure, based on the works of Faddeev and Jury, has been introduced to deal with the problem. This procedure, which only involves matrix multiplication and the evaluation of 2×2 determinants, has many advantages over any of the numerical methods available for evaluating eigenvalues. It is a comparatively simple procedure and requires far fewer arithmetic operations; it is readily programmed and the running time is comparatively small. It also has the distinct advantage in that it is not an iterative procedure, so that the question of convergence does not arise.

The method therefore employed to examine the eigenvalues of $\mathbf{C}^*$ is first to obtain the characteristic polynomial using the Faddeev algorithm and then to determine whether the roots of this polynomial lie inside the unit circle using the determinant method of Jury.

4. Stability consideration

The basic equation of the system is given by (3). Assuming that a small perturbation is imposed on the adaptive parameters and that, in the perturbed state, $K_1 \to K_1 + \delta K_1$, $K_2 \to K_2 + \delta K_2$, then it can be shown (see White 1966), that the perturbed equation for the error is:

$$(D^3 + A_1 D^2 + A_1 A_2 K_1 D + A_1 A_2 A_3 K_2)\delta e + A_1 A_2 \delta K_1 D\theta_0 - A_1 A_2 A_3 \delta K_2 e_s = 0, \quad (5)$$

where e_s is the system error.

From eqn. (4) we have that in the perturbed state:

$$\dot{K}_i + \delta \dot{K}_i = -G_i(e + \delta e)\,\mathrm{sign}\left(\frac{\partial e}{\partial K_i} + \delta \frac{\partial e}{\partial K_i}\right). \quad (6)$$

Subtracting (4) and (6) gives:

$$\delta \dot{K}_i = -G_i \delta e\,\mathrm{sign}\frac{\partial e}{\partial K_i} - G_i e\left[\mathrm{sign}\left(\frac{\partial e}{\partial K_i} + \delta \frac{\partial e}{\partial K_i}\right) - \mathrm{sign}\frac{\partial e}{\partial K_i}\right], \quad (7)$$

the term $\delta e\,\mathrm{sign}\,[\partial e/\partial K_i + \delta(\partial e/\partial K_i)]$ having been replaced by $\delta e\,\mathrm{sign}\,\partial e/\partial K_i$ since we are dealing with the linearized equations.

$\partial e/\partial K_i$ is a sine wave and $\partial e/\partial K_i + \delta(\partial e/\partial K_i)$ a perturbed wave, being approximately sinusoidal, as shown in fig. 2 (*a*). The terms sign $[\partial e/\partial K_i + \delta(\partial e/\partial K_i)]$ and sign $\partial e/\partial K_i$ are represented by the square waves of fig. 2 (*b*), and the term

$$\left[\mathrm{sign}\left(\frac{\partial e}{\partial K_i} + \delta \frac{\partial e}{\partial K_i}\right) - \mathrm{sign}\frac{\partial e}{\partial K_i}\right]$$

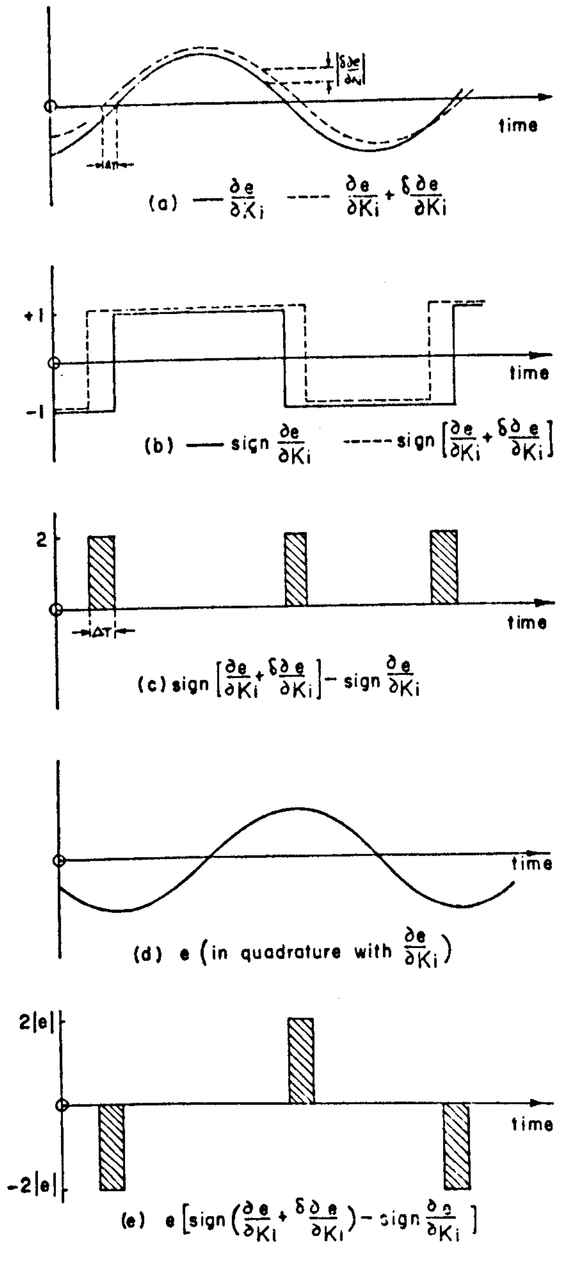
by the pulses of fig. 2 (*c*). The last term of eqn. (7) had been omitted by Parks and its omission was found to have a considerable effect on the final results.

Using the condition that the e and $\partial e/\partial K_i$ signals are in quadrature when the system has adapted (see White 1966), the term

$$e\left[\mathrm{sign}\left(\frac{\partial e}{\partial K_i} + \delta \frac{\partial e}{\partial K_i}\right) - \mathrm{sign}\frac{\partial e}{\partial K_i}\right]$$

can be represented by the pulses of fig. 2 (*e*). Two of these pulses occurring within one period T, at times T_1 and $T_1 + T/2$ respectively, where T_1 is the time when the $\partial e/\partial K_i$ signal first changes sign.

Fig. 2



The integrated values of these pulses is $\pm 2|e|\Delta T$, where $|e|$ is the peak amplitude of the error signal and ΔT the duration of the pulse and given by

$$\Delta T = \left|\delta\left(\frac{\partial e}{\partial K_i}\right)\right| \Big/ \left|\text{slope of} \frac{\partial e}{\partial K_i} \text{ signal at the points where it changes sign}\right|$$

$$= \left|\delta\left(\frac{\partial e}{\partial K_i}\right)\right| \Big/ \left[\omega\left|\frac{\partial e}{\partial K_i}\right|\right], \tag{8}$$

where $|\partial e/\partial K_i|$ is the peak amplitude of the $\partial e/\partial K_i$ signal and ω the frequency of the input signal.

It was shown earlier that the signal $\partial e/\partial K_i$ is obtained by passing the signal θ_F through a suitable filter, so that $\delta(\partial e/\partial K_i)$ is a filtered version of $\delta\theta_F$.

4.1. K_1 *adapting alone*

When K_1 is adapting alone δK_2 is zero and eqn. (5) becomes:

$$(D^3+A_1D^2+A_1A_2K_1D+A_1A_2A_3K_2)\delta e+A_1A_2\delta K_1 D\theta_0=0. \tag{9}$$

White showed that in this case the relationship between a 'steady state' K_1 and the frequency ω of the input is:

$$K_1=0{\cdot}004\omega^2+0{\cdot}36. \tag{10}$$

From fig. 1, $\theta_F=(1/A_3K_2)D\theta_0$, so that $\delta\theta_F=(1/A_3K_2)\delta(D\theta_0)$. Since $\theta_0=e+\theta_m$ and $\delta\dot{\theta}_m=0$, $\delta\theta_F=(1/A_3K_2)\delta\dot{e}$. Hence, $\delta(\partial e/\partial K_1)=-(1/A_3K_2)\delta\dot{e}_F$, where

$$\delta\dot{e}_F=Y_m\delta\dot{e} \quad \text{or} \quad (D^2+6{\cdot}32D+40)\delta\dot{e}_F=40\delta\dot{e} \tag{11}$$

From eqn. (7):

$$\delta\dot{K}_1=-G_1\left[\delta e \operatorname{sign}\frac{\partial e}{\partial K_1}+P_1(t)\right], \tag{12}$$

where $P_1(t)$ are pulses, with integrated values:

$$\pm\frac{2|e|\delta\dot{e}_F}{A_3K_2\operatorname{amp}\dfrac{\partial e}{\partial K_1}}=\pm P_1'(t)\delta\dot{e}_F,$$

occurring at times T_1 and $T_1+T/2$, T being the period and T_1 the time when the signal $\partial e/\partial K_1$ first changes sign. Care must be taken (by examination of the phase angles) to ensure that the positive and negative pulses are inserted in the correct order.

Equations (9), (11) and (12) lead to the following system of linear equations:

$$\frac{d}{dt}\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ \delta\dot{e}_F\\ \delta\ddot{e}_F\\ \delta K_1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -A_1A_2A_3K_2 & -A_1A_2K_1 & -A_1 & 0 & 0 & -A_1A_2D\theta_0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 40 & 0 & -40 & -6{\cdot}32 & 0\\ -G_1\operatorname{sign}\dfrac{\partial e}{\partial K_1} & 0 & 0 & -G_1P_1'(t) & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ \delta\dot{e}_F\\ \delta\ddot{e}_F\\ \delta K_1\end{bmatrix}, \tag{13}$$

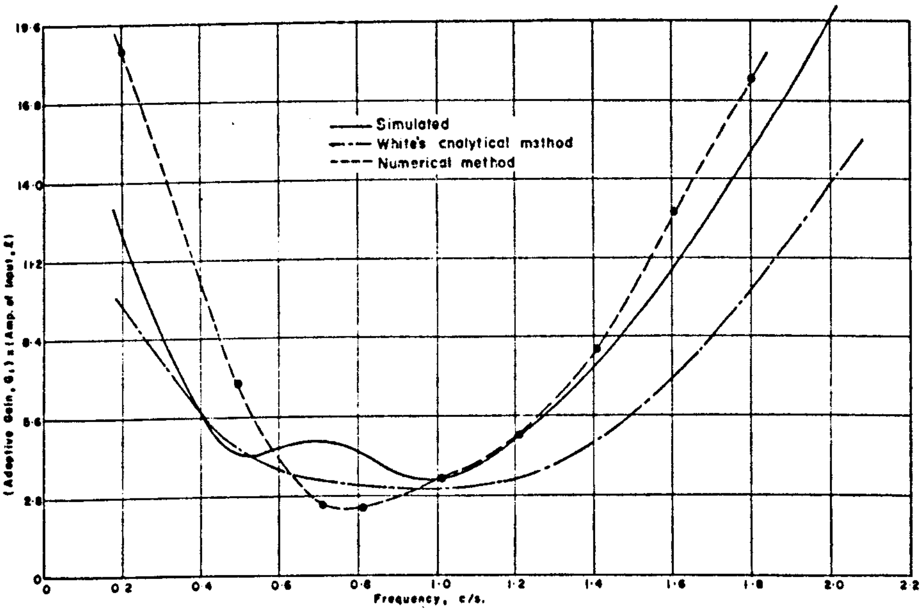
which is of the form

$$\dot{\mathbf{x}}=\mathbf{A}(t)\mathbf{x}.$$

It will be noted that the 6×6 matrix of eqns. (13) differs from the 4×4 matrix suggested by Parks. The increase in the order of the matrix is due to the fact that Parks had neglected the effect of the filter Y_m and taken $\delta\theta_F$ as an approximation for $\delta(\partial e/\partial K_1)$. This approximation was found to be inadequate and the effect of the filter must be taken into account.

The eigenvalues of the transition matrix $\mathbf{C}^*$ of the matrix differential eqn. (13) are examined, using the method described in § 3, for various values of the adaptive gain G_1 and the critical values of the gain calculated. The results are illustrated in fig. 3 together with the results obtained by White using simulator and analytical studies. Since, in this case, the critical gain is inversely proportional to the amplitude l of the input signal the product of l with critical gain is plotted against the frequency of the input signal so that the resulting curves should be the same for all values of l. (In order to obtain the graphs of White divide the ordinates by 7/50.)

Fig. 3



Critical adaptive gain with K_1 loop adapting alone.

4.2. K_2 *adapting alone*

In this case $\delta K_1 = 0$ and eqn. (5) becomes:

$$(D^3 + A_1 D^2 + A_1 A_2 K_1 D + A_1 A_2 A_3 K_2)\delta e - A_1 A_2 A_3 e_s \delta K_2 = 0. \tag{14}$$

White showed that in this case the relationship between a 'steady state' K_2 and the input frequency is given by:

$$K_2 = \frac{(\omega^4 - 58{\cdot}7\omega^2 + 4990) - \omega(10{\cdot}46\omega^2 + 117)\tan\phi}{28{\cdot}5(252{\cdot}8 - \omega^3 \tan\phi)},$$

where ϕ is the phase change across the filter Y_3/Y_1 and is given by:

$$\tan\phi = \frac{16{\cdot}78\omega(570 - 16{\cdot}78\omega^2) - \omega(124{\cdot}8 - \omega^2)^2}{570(124{\cdot}8 - \omega^2)}.$$

From fig. 1

$$\theta_F = \frac{1}{A_3 K_2} D\theta_0. \tag{15}$$

When the perturbation is imposed this becomes:

$$A_3(K_2+\delta K_2)(\theta_F+\delta\theta_F)=D\theta_0+\delta(D\theta_0)=D\theta_0+\delta\dot{e}. \tag{16}$$

Subtracting (15) from (16) gives $A_3K_2\delta\theta_F+A_3\theta_F\delta K_2=\delta\dot{e}$, i.e.

$$\delta\theta_F=\frac{1}{A_3K_2}\delta\dot{e}-\frac{\theta_F}{K_2}\delta K_2=X.$$

Hence

$$\delta\left(\frac{\partial e}{\partial K_2}\right)=X_F, \quad \text{where} \quad X_F=\frac{Y_3}{Y_1}(X).$$

From eqn. (7):

$$\delta\dot{K}_2=-G_2\left[\delta e\,\text{sign}\frac{\partial e}{\partial K_2}+P_2(t)\right], \tag{17}$$

where $P_2(t)$ are pulses, with integrated values

$$\pm\frac{2|e|X_F}{\omega\,\text{amp}\dfrac{\partial e}{\partial K_2}}=\pm P_2'(t)X_F,$$

occurring at times T_2 and $T_2+T/2$, T being the period and T_2 the time when the signal $\partial e/\partial K_2$ first changes sign; care again being taken to ensure that the sign of the pulses are taken in the correct order.

Equations (14) and (17) may be written:

$-A_1A_2A_3e_s$

$$\frac{d}{dt}\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ X_1\\ X_2\\ X_3\\ \delta K_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -A_1A_2A_3K_2 & -A_1A_2K_1 & -A_1 & 0 & 0 & 0 & \cancel{0}\\ 0 & \dfrac{1}{K_2} & 0 & 0 & 1 & 0 & -\dfrac{A_3\theta_s}{K_2\cancel{X_F}}\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -570 & -124{\cdot}8 & -16{\cdot}78 & 0\\ -G_2\,\text{sign}\dfrac{\partial e}{\partial K_2} & 0 & 0 & -G_2P_2'(t) & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta e\\ \delta\dot{e}\\ \delta\ddot{e}\\ X_1\\ X_2\\ X_3\\ \delta K_2\end{bmatrix}. \tag{18}$$

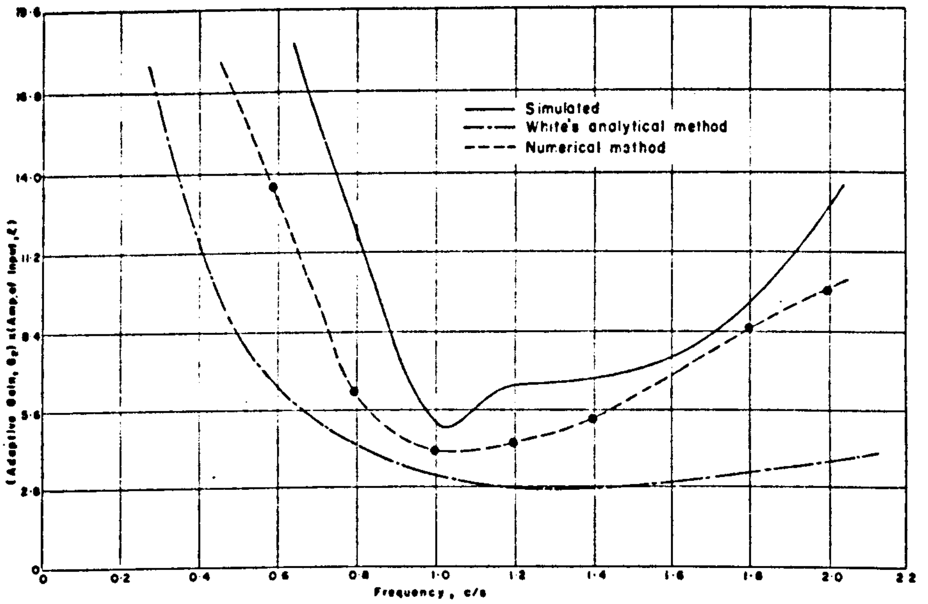
where $X_1=X_F$, $X_2=\dot{X}_F-4{\cdot}57X$ and $X_3=\ddot{X}_F-4{\cdot}57\dot{X}$.

The eigenvalues of the transition matrix $\mathbf{C}^*$ for eqn. (18) are examined and the critical value of G_2 calculated. The results are illustrated (fig. 4).

5. Conclusions

A rigorous numerical method, well suited for digital computation, has been presented for investigating the important problem of examining the stability of a system of linear differential equations with periodic coefficients. These equations, which form a most important subclass of linear differential equations with variable coefficients, may arise in practice directly from the equations of motion of a dynamic system, e.g. the flapping of a helicopter rotor blade, but more frequently arise from an examination of the stability of oscillations in non-linear systems.

Fig. 4



Critical adaptive gain with K_2 loop adapting alone.

The problem considered in this paper is that of examining the stability of an M.I.T. type reference adaptive control system (m.r.a.c.s.) with sinusoidal inputs and it displays the difficulties involved and the amount of rigour necessary, even for the comparatively simple case of sinusoidal inputs, when examining the stability of a m.r.a.c.s. for time varying inputs. A more realistic input would be a random one and it is hoped to publish the stability analysis for such an input in a future paper.

The theorem employed in the analysis provides necessary and sufficient conditions for the asymptotic stability of the linearized equations with periodic coefficients. The system, however, is a forced non-linear equation with periodic coefficients, the forcing term being $-G_i e$ sign $\partial e/\partial K_i$, where e and $\partial e/\partial K_i$ are sinusoids at frequency ω. By appealing to the stability theorems of Zubov (Malkin 1952) the asymptotic stability of the linearized system certainly, in the absence of the forcing term, leads to asymptotic stability in the small for the non-linear system. The effect of the forcing term could however invalidate the neglect of the non-linear terms and make the stable region of the linearized system unstable for the non-linear system. The forcing term in this case is of the multiplicative kind and its actual effect on the 'stable region' is obviously a field for further research. Instability of the linearized system however gives sufficient conditions for the instability of the non-linear system and the results of this paper suggest that the analogue computer results of White are not accurate.

After discussion with White it was found that it is very difficult to decide on a practical criterion for instability when simulating these systems; this difficulty being more pronounced at low frequencies. The presence of harmonics which are forcing the system further masks the problem.

However, the results of this paper suggest that the problem at hand may be studied satisfactorily by considering the stability of the linearized system. It is noted that the theoretical results give a stable system in the region of the natural frequency of the model.

This paper also illustrates how the effect of replacing the multipliers in a m.r.a.c.s. by diode switching units is to introduce an impulse like signal occurring twice per cycle into the analysis.

The author feels that this paper is a significant contribution to the study of the stability of m.r.a.c.s. for time varying inputs on which, apart from the work of Bongiorno (1962, 1963), there is little published work.

Acknowledgments

This work is part of a research project being carried out under the supervision of P. C. Parks of the University of Warwick and to whom the author is indebted for his most helpful advice throughout this investigation.

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STABILITY OF A SINUSOIDAL-PERTURBATION EXTREMAL-CONTROL SYSTEM

A theoretical stability analysis of a single-input sinusoidal-perturbation extremal-control system with output lag is considered. Periodic solutions of the system equations are obtained using the principle of harmonic balance, and their stability properties are investigated. The domains of attraction of the stable solutions are plotted to give the stability boundaries for the system.

In this letter, the author examines the stability of the class of sinusoidal-perturbation control systems[1,2] shown in Fig. 1.

The differential equations representing the system are

$$\left.\begin{aligned}\dot{x} + ax &= Aa\{y - \delta \sin(\omega t + \alpha)\}^2 \\ \dot{y} &= Gx \sin(\omega t + \alpha)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

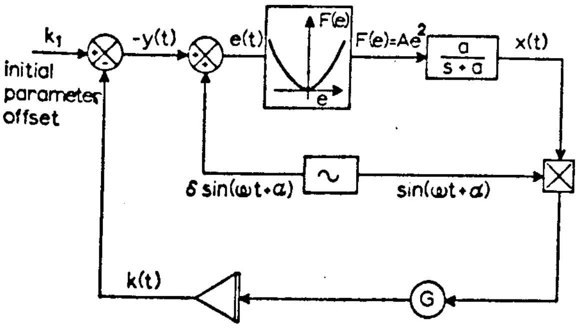

Fig. 1 *Sinusoidal-perturbation extremal-control system*

By Buckingham's π theorem,[3] the system may be specified by the two dimensionless parameters

$$\left.\begin{aligned}\pi_1 &= GA\delta a \\ \pi_2 &= \omega/a\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, in addition to eqn. 2, we introduce the dimensionless variables

$$\left.\begin{aligned}\xi_1 &= x/(A\delta^2) \\ \xi_2 &= y/\delta \\ \tau &= \omega t + \alpha\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the system eqns. 1 may be written

$$\left.\begin{aligned}\dot{\xi}_1 &= -\frac{1}{\pi_2}\xi_1 + \frac{1}{\pi_2}(\xi_2 - \sin\tau)^2 \\ \dot{\xi}_2 &= \frac{\pi_1}{\pi_2}\xi_1 \sin\tau\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

where dots denote differentiation with respect to τ.

A distinctive feature of a nonlinear system, such as eqns. 4, is that various types of periodic oscillations may exist for the same system depending on the initial values of the variables. In this letter, the method of solution employed is to assume for ξ_1 and ξ_2 Fourier-series developments with undetermined coefficients and then fix these coefficients by the principle of harmonic balance.[4] Periodic solutions whose fundamental frequencies are equal to those of the applied perturbation will be termed harmonic solutions, whereas solutions whose fundamental frequencies are a fraction $\frac{1}{n}$ $(n = 2, 3, \ldots)$ of the applied perturbation frequency will be termed subharmonic solutions of order n.

A periodic solution obtained by this method merely represents a state of equilibrium; this equilibrium state is actually realisable only if it is stable, so that its actual existence must be confirmed by a stability investigation. If $\hat{\xi}_1(\tau)$, $\hat{\xi}_2(\tau)$, having period T, represents a particular state of equilibrium, in order to investigate its stability, we consider small variations η_1 and η_2 from this equilibrium state, and from eqn. 4 set up the variational equations:

$$\left.\begin{aligned}\dot{\eta}_1(\tau) &= -\frac{1}{\pi_2}\eta_1(\tau) + \frac{2}{\pi_2}\{\hat{\xi}_2(\tau) - \sin\tau\}\eta_2(\tau) \\ \dot{\eta}_2(\tau) &= \left(\frac{\pi_1}{\pi_2}\sin\tau\right)\eta_1(\tau)\end{aligned}\right\} \quad (5)$$

These variational equations form a set of linear differential equations with periodic coefficients, of period T, in τ; that is, they are of the form

$$\dot{\eta} = P(\tau)\eta \quad P(\tau) = P(\tau + T)$$

The stability of the trivial solution of such equations has been previously considered by the author[5] and by Davison.[6] The procedure is to obtain a transition matrix C such that $\eta(T + \tau_1) = C\eta(\tau_1)$, and then stability is ensured provided that the eigenvalues of C lie within the unit circle.

For each period solution $\hat{\xi} = [\hat{\xi}_1(\tau)\hat{\xi}_2(\tau)]'$ of eqn. 4, there exists, for each τ_1, $0 < \tau_1 < T$, a corresponding fixed point $P_0[\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1)]$ in the ξ_1-ξ_2 plane which is invariant under the mapping

$$M : \xi(\tau_1 + nT) \rightarrow \xi(\tau_1 + \overline{n+1}T), \; n = 0, 1, 2, \ldots$$

where $\xi = [\xi_1\xi_2]$ (Reference 1).

If P_0 corresponds to a stable solution, it will have associated with it a domain of attraction, so that, if any point $P[\xi_1(\tau_1), \xi_2(\tau_1)]$ within this domain is taken as the initial

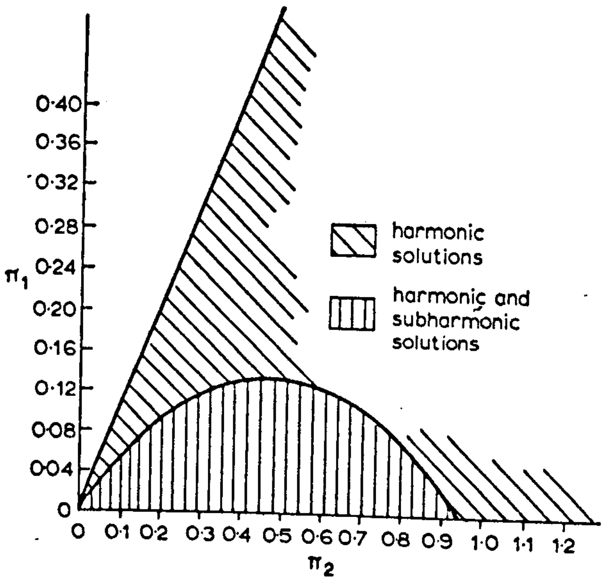

Fig. 2 *Regions in which different types of oscillations are sustained*

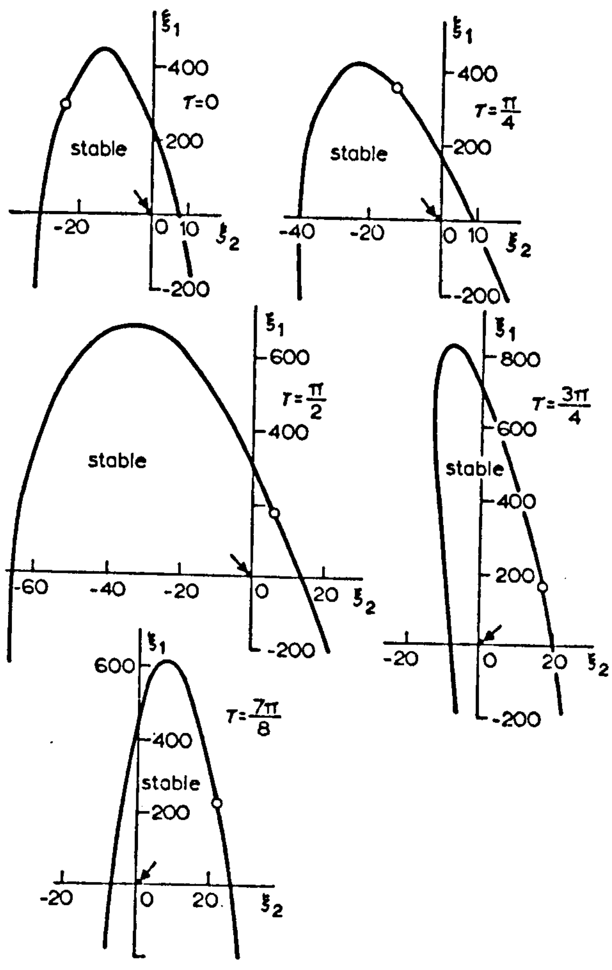

Fig. 3 *Domains of attraction and corresponding fixed points for $\pi_1 = 0{\cdot}1$ and $\pi_2 = 1$*

○ unstable fixed points ● stable fixed points conditions for eqn. 4, the solution of eqn. 4 will converge to the stable periodic oscillation ξ. In terms of the mapping M, this means that successive images MP, M^2P, etc. of the point P will approach the fixed point P_0.

If P_0 corresponds to an unstable solution, which has a transition matrix C having one eigenvalue greater than unity and one less than unity in magnitude, then through P_0 will pass a critical curve C which is invariant under the mapping M and such that all points on it will approach P_0 under iterations of the mapping. This curve C will be the boundary between two different domains of attractions and is thus important in any stability investigation. At the point P_0, the curve C will be in the direction of that eigenvector of C which corresponds to the eigenvalue that is greater than unity in magnitude (when considering the stability of ξ) so that the slope α of C at P_0 may be found. Theoretically, therefore, the curve C may be obtained by starting just on either side of the fixed point P_0 (in direction α) and integrating eqn. 4 numerically for decreasing τ, the curve C then being the loci of the successive images of the starting point under iterations of the mapping M. In practice, however, it is found that, if the unstable fixed points are not known accurately enough, the image points of the numerical procedure deviate from the desired boundary after a few cycles, so that the loci obtained by the numerical procedure may only be used as a guide and a more accurate boundary may be obtained by analogue-computer studies.

For a fuller discussion of the theory of fixed points and domains of attraction the reader is referred to the works of Blair and Loud[7] and Hayashi.[4]

On substituting a Fourier series with undetermined coefficients for ξ_1 and ξ_2 in eqn. 4 and using the principle of harmonic balance, it is found that, in the region

$$R_1 : \pi_1 < \pi_2$$

there exist two harmonic solutions, whereas in the region R_2, defined by the system of inequalities

$$N = \pi_2^4(289 - 105\pi_2^2 - 420\pi_1) + 196\pi_1^2 + 308\pi_2^2\pi_1 - 588\pi_2^2\pi_1^2 > 0$$

$$L^2 = \frac{2}{3}\pi_2^2 - \pi_1^2 - \frac{\pi_2^4}{12} - \frac{\pi_1\pi_2^2}{3} > 0$$

$$|X| = \left|\frac{1}{42\pi_2}\{\sqrt{N} - (11\pi_2^2 + 14\pi_1)\}\right| < L$$

$$F(X) = 7\pi_2X^3 + \left(\frac{11}{2}\pi_2^2 + 7\pi_1\right)X^2 + \left(\frac{5}{4}\pi_2^5 + 5\pi_2^3\pi_1 + 7\pi_2\pi_1^2 - 2\pi_2^3\right)X + \left(\frac{9}{8}\pi_2^6 + \frac{23}{4}\pi_2^4\pi_1 - \pi_2^4 + \frac{21}{2}\pi_1^2\pi_2^2 - 2\pi_2^2\pi_1 + 7\pi_1^3\right) < 0$$

there exist, in addition to the two harmonic solutions, eight subharmonic solutions of order 2. These regions, in which different types of oscillations are sustained, are shown in Fig. 2. Outside region R there are no periodic solutions, so that, for π_1 and π_2 in this region, the system will be totally unstable.

For π_1 and π_2 in R_1 (not in R_2), it is found that one of the harmonic solutions is stable and one unstable. Thus for each τ, $0 < \tau < 2\pi$, there will be a domain of attraction in the ξ_1–ξ_2 plane, so that, for all initial conditions within the domain, the system will be stable, whereas, for initial conditions outside the domain, the system will be unstable. Further, in this case, the solution for ξ_2 has no d.c. component, so that, in the stable region, the system will adapt to an oscillation about zero. The stability boundaries, together with their corresponding fixed points, are shown in Fig. 3 for various values of τ in the range $(0, \pi)$, $(\pi_1 = 0\cdot1$, $\pi_2 = 1)$; corresponding boundaries for τ in the range $(\pi, 2\pi)$ being the mirror images about the ξ_1 axis of those given. These results have been verified experimentally and also agree with experimental work carried out by Jacobs and Shering.[8]

For π_1 and π_2 in R_2, there exist one stable harmonic solution and four stable subharmonic solutions, the other five solutions being unstable. For the subharmonic oscillations, the solutions for ξ_2 has a d.c. component so that, for certain initial conditions, the system will adapt to an oscillation about an offset position, this having been verified by analogue simulation. The domains of attraction of the various stable oscillations in this case are far more complicated and are not included in this letter.

The author would like to express his appreciation to P. C. Parks (University of Warwick) for suggesting this problem and for his helpful discussions concerning it.

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18th April 1969

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STABILITY OF MODEL-REFERENCE SYSTEMS WITH RANDOM IMPULSIVE INPUTS

Indexing terms: Adaptive control, Stability criteria

The stability of the gain-adjusting loop of a simple model-reference adaptive-control system is investigated. The input to system and model is a sequence of impulses of random magnitude. The resulting behaviour is determined by an infinite product, and from this a necessary and sufficient criterion for the stability of the adaptive-loop gain is deduced.

Stability of model-reference adaptive-controls systems with general time-varying inputs is a difficult theoretical problem, which so far has not been solved for systems not actually synthetised from a stability point of view.[1]

The difficulties are well illustrated by the very simple gain-adjustment loop depicted in Fig. 1.[1] The equations may be written in matrix form:

$$\begin{bmatrix} \dot{e} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{T} & \dfrac{r(t)}{T} \\ -GK_v\theta_m(t) & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \quad . \; . \; . \quad (1)$$

where $e = \theta_m - \theta_s$ and $x = K - K_v K_c$. We assume that K and K_v are constant and that K_c is required to tend to K_v/K so that e and x tend to zero. Despite recent progress in stochastic-stability theory, the stability of eqn. 1 when $r(t)$ is a general random signal [and $\theta_m(t)$ a filtered version of $r(t)$] is not an easy problem to solve.

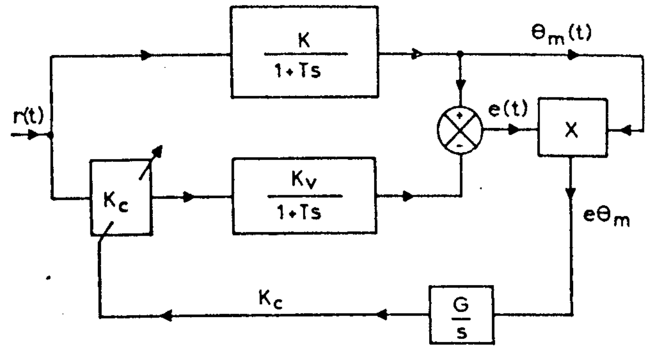

Fig. 1

As a first look at the problem, we shall assume that $r(t)$ is a sequence of impulses spaced sufficiently far apart in time (compared with T) that the transient effects from a particular impulse have died out before the next impulse arrives.

We then obtain a recurrence relationship for x_k, the value of $x(t)$, just before the arrival of the kth impulse of magnitude A_k; this is

$$x_{k+1} = \left(1 - \frac{GKK_v A_k^2}{2T}\right) x_k \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

This relation follows from the relations

$$e(t) = \frac{x_k A_k e^{-t/T}}{T} \qquad \theta_m(t) = \frac{KA_k e^{-t/T}}{T}$$

$$\dot{x}(t) = -GKK_v A_k^2 e^{-2t/T} x_k/T^2$$

where, for convenience, $t = 0$ has been taken to be the time of arrival of the kth impulse.

We are thus concerned with the properties of the infinite product

$$\prod_{k=1}^{\infty} (1 - a\xi_k^2)$$

where $a = GKK_v/2T > 0$, and ξ_k is a random variable drawn from an amplitude probability distribution $p(y)$, say. By considering a large number of terms and their distribution and by considering the logarithm of the product of these terms, we are led to consider the integral

$$I = \int_{y=-\infty}^{\infty} \log|1 - ay^2|\, p(y)\, dy \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

If this integral is positive, the infinite product diverges and the system is unstable; if this integral is negative, the infinite product 'diverges to zero',[2] $x_k \to 0$ as $k \to \infty$, and the system is stable. If y has a Gaussian distribution with zero mean and variance σ^2, the integrand of eqn. 3 is of the form sketched in Fig. 2. The integral of eqn. 3 has been evaluated numerically (and crosschecked using different methods to take account of the singularity at $y = 1/\sqrt{a}$) to find the critical value of $a\sigma^2$ for which the positive and negative areas in Fig. 2 balance. This yields the stability criterion

$$a\sigma^2 < 2\cdot 5 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

For y having a uniform distribution between $y = \pm b$, the integral of eqn. 3 can be evaluated in the closed form

$$I = \frac{1}{b\sqrt{a}}\left(b\sqrt{a}\log|ab^2 - 1| - 2b\sqrt{a} + \log\left|\frac{b\sqrt{(a)}+1}{b\sqrt{(a)}-1}\right|\right)$$

$I = 0$ yields a stability condition that

$$\left.\begin{aligned} ab^2 &< 6{\cdot}25 \\ \text{or} \quad a\sigma^2 &< 2{\cdot}08 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (5)$$

since, in this case,

$$\sigma^2 = b^2/3$$

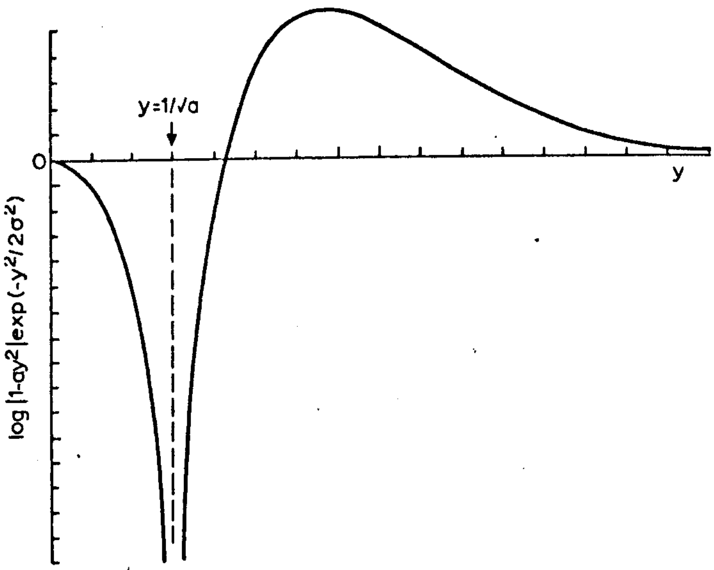

Fig. 2

These results have been confirmed by direct digital simulation of the infinite product using appropriate random-number-generation procedures.

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Stability and subharmonics in a sinusoidal perturbation hill-climbing system†

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A theoretical stability analysis of a single input, sinusoidal perturbation, extremal control system with output lag is being considered. Using the principle of harmonic balance it is shown that various stable harmonic and sub-harmonic 'steady-state' solutions are possible in certain regions of the parameter space. By examining the domains of attraction, corresponding to the stable solutions, regions in three-dimensional space are obtained within which initial conditions will lead to a given 'steady-state' stable oscillation.

1. Introduction

Although extremum control or 'hill-climbing' systems are a well-defined class of adaptive control systems the important problem of analysing their stability has often been ignored. In this paper a theoretical stability analysis for a single input, sinusoidal perturbation, extremal control system with output lag is presented, the results of which have been verified by analogue computer simulation.

The system equations, which are forced, non-linear and non-autonomous, are first non-dimensionalized using dimensional analysis, and periodic solutions of the resulting equations obtained by the principle of harmonic balance. The stability of these equilibrium states is then investigated by setting up variational equations, which, for small disturbances about the equilibrium state, form a set of linear differential equations with periodic coefficients.

It will be shown that various stable harmonic and sub-harmonic 'steady-state' solutions are possible in certain regions of the parameter space.

The steady state finally reached depends on the prescribed initial conditions. By plotting the 'domains of attraction' of 'fixed points', which are invariant under the mapping

$$\xi(t+nT)\rightarrow\xi(t+\overline{n+1}T), \quad n=0,1,2,\text{etc.},$$

where $\xi(t)$ is the state vector of the system, regions in three-dimensional space are obtained within which initial conditions will lead to a given stable 'steady-state' oscillation.

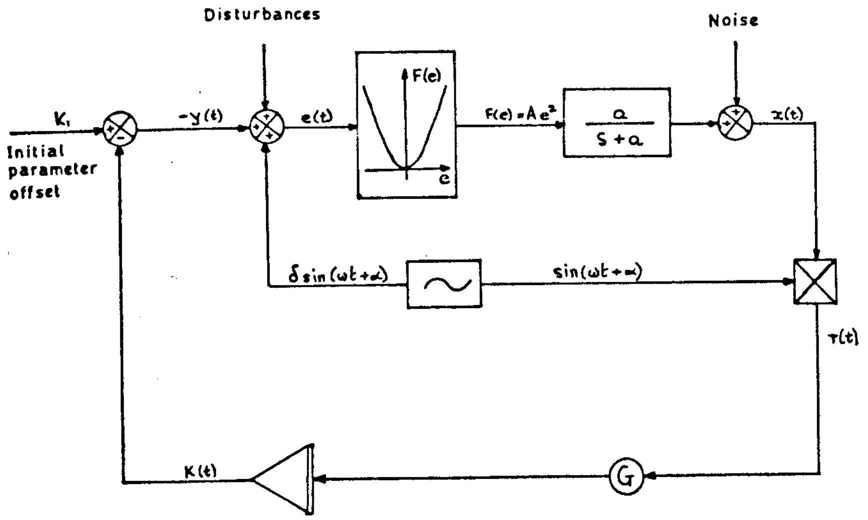
2. The sinusoidal perturbation 'hill-climbing' system

The present paper is concerned with the stability of the single dimensional, sinusoidal perturbation, adaptive control system shown in fig. 1 (Boddington 1968, Jacobs and Shering 1968). The index of performance (Eveleigh 1963) is $F[e(t)] = A[e(t)]^2$ and the output lag is represented by a low pass filter with time constant $1/a$.

† Communicated by Mr. P. C. Parks.

The adaptive controller employed is similar to that previously discussed in the literature (Draper and Li 1951, McGrath and Rideout 1961). Briefly, a sinusoidal perturbation $\delta \sin(\omega t+\alpha)$ is added to the input and by demodulating the corresponding perturbation in the output a signal $r(t)$ that varies with the slope $\partial F/\partial e$ of the index of performance is obtained. The signal $r(t)$ is then passed through a smoothing integrator, with gain G, to develop a correction signal which tends to reduce $dF/\partial e$ to zero.

Fig. 1



Sinusoidal perturbation, extremal control system.

3. Dimensional analysis

In the absence of the disturbances and noise the differential equations representing the system of fig. 1 are:

$$\left.\begin{aligned} \frac{dx}{dt}+ax &= Aa[y-\delta \sin(\omega t+\alpha)]^2, \\ \frac{dy}{dt} &= Gx\sin(\omega t+\alpha). \end{aligned}\right\} \qquad (1)$$

The performance of the system depends on the values of the five parameters A, a, δ, ω and G which are expressed in three sorts of units (input units, output units and time) as follows:

A: (output units) (input units)$^{-2}$,

a: (time)$^{-1}$,

δ: (input units),

ω: (time)$^{-1}$,

G: (input units) (output units)$^{-1}$ (time)$^{-1}$.

By Buckingham's π theorem (Doherty and Keller 1944) non-dimensional parameters can be defined so as to reduce the number of parameters that need be considered by the number of units. Thus, in this case the number of non-dimensional parameters that need be considered are two and these are taken as:

$$\left.\begin{aligned} \Pi_1 &= GA\delta/a, \\ \Pi_2 &= \omega/a. \end{aligned}\right\} \tag{2}$$

If, in addition to (2), we introduce the dimensionless variables:

$$\left.\begin{aligned} \xi_1 &= x/(A\delta^2), \\ \xi_2 &= y/\delta, \\ \tau &= \omega t+\alpha, \end{aligned}\right\} \tag{3}$$

then the system eqns. (1) may be written in the non-dimensional form:

$$\dot{\xi}_1 = -\frac{1}{\Pi_2}\xi_1+\frac{1}{\Pi_2}(\xi_2-\sin\tau)^2, \tag{4 a}$$

$$\dot{\xi}_2 = \frac{\Pi_1}{\Pi_2}\xi_1\sin\tau, \tag{4 b}$$

where dots denote differentiation with respect to τ.

4. Periodic solutions using the method of harmonic balance

A distinctive feature of a system of non-linear differential equations, such as (4), is that various types of 'steady-state' periodic oscillations may exist depending on the initial values of the variables. In this paper the method of solutions employed, for obtaining the 'steady-state' solutions, is to assume for ξ_1, ξ_2 Fourier series developments with undetermined coefficients and then determine these coefficients by the principle of harmonic balance (Hayashi 1964), a method widely used for the analysis of non-linear control systems. Periodic solutions whose fundamental frequencies are equal to that of the applied perturbation frequency will be termed 'harmonic solutions', whilst solutions whose fundamental frequencies are a fraction $1/n$ ($n = 2, 3,$ etc.) of the applied perturbation frequency will be termed 'sub-harmonic solutions of order $1/n$'.

4.1. *Harmonic solutions*

When the system has reached a 'steady state' there will be no constant or 'd.c.' component out of the multiplier (that is, $\xi_1(\tau)$ contains no term in $\sin\tau$), so that as a first approximation we assume solutions of the form:

$$\left.\begin{aligned} \xi_1 &= a_0+a_2\cos\tau, \\ \xi_2 &= b_0+b_1\sin\tau+b_2\cos\tau. \end{aligned}\right\} \tag{5}$$

Substituting eqns. (5) in (4 b) gives:

$$b_1\cos\tau-b_2\sin\tau = \frac{\Pi_1}{\Pi_2}\left[a_0\sin\tau+\frac{a_2}{2}\sin 2\tau\right].$$

Since the first approximation contains only the terms of the fundamental frequency we ignore second harmonic components and equate coefficients of the $\sin\tau$ and $\cos\tau$ terms to give:

$$b_1 = 0, \quad b_2 = -\frac{\Pi_1}{\Pi_2}a_0. \tag{6}$$

Substituting eqns. (5) in (4 a) and using results (6) gives, on balancing the coefficients:

$$\left.\begin{aligned} a_2 &= \frac{2b_0}{\Pi_2}, \\ \frac{a_0}{\Pi_2} &= \frac{b_0^2}{\Pi_2} + \frac{\Pi_1^2}{2\Pi_2^3} a_0^2 + \frac{1}{2\Pi_2}, \\ \frac{a_2}{\Pi_2} &= -2b_0 a_0 \frac{\Pi_1}{\Pi_2^2}. \end{aligned}\right\} \tag{7}$$

Equations (7) have a real solution for $\Pi_2 > \Pi_1$, when the solutions are:

$$\left.\begin{aligned} b_0 &= a_2 = 0, \\ a_0 &= r^2 \pm r\sqrt{(r^2 - 1)}, \end{aligned}\right\} \tag{8}$$

where

$$r = \Pi_2/\Pi_1.$$

Thus, to a first approximation harmonic solutions exist in the region:

$$R_1:\ \Pi_2 > \Pi_1 \tag{9}$$

and in this region the harmonic solutions:

$$\left.\begin{aligned} \xi_1 &= r^2 \pm r\sqrt{(r^2 - 1)}, \\ \xi_2 &= [-r \mp \sqrt{(r^2 - 1)}] \cos\tau \end{aligned}\right\} \tag{10}$$

are possible.

These results agree with analogue computer simulation where it is found that harmonic solutions for $\xi_2(\tau)$ contains no d.c. component, thus implying that $\xi_1(\tau)$ has no component in $\cos\tau$.

A closer approximation may be obtained if more terms of the Fourier series are taken into account; however, numerical computation will become too unwieldy. The method, employed in this paper, of improving the approximation is an extension of a method due to Hayashi (1964); this method is particularly useful when the amplitude of each harmonic component decreases with increasing order of the harmonics. An alternative method, well suited for digital computation, is that based on Galerkin's procedure (Urabe and Reiter 1966).

A second approximation is now assumed in the form:

$$\left.\begin{aligned} \xi_1 &= (a_0 + \varepsilon a_{01}) + (a_3 + \varepsilon a_{31}) \sin 2\tau + (a_4 + \varepsilon a_{41}) \cos 2\tau, \\ \xi_2 &= (b_1 + \varepsilon b_{11}) \sin\tau + (b_2 + \varepsilon b_{21}) \cos\tau + (b_3 + \varepsilon b_{31}) \sin 2\tau + (b_4 + \varepsilon b_{41}) \cos 2\tau, \end{aligned}\right\} \tag{11}$$

where the terms containing ε represent the correction terms. Substituting eqns. (11) in (4) and balancing like terms lead, on neglecting terms of order higher than the first in ε, to a set of linear simultaneous equations in the correction terms. These equations are then solved, using digital computation with initial values $a_3 = a_4 = b_1 = b_3 = b_4 = 0$ and a_0, b_2 given by (8); $a_i + \varepsilon a_{i1}$ ($i = 0, 3, 4$) and $b_i + \varepsilon b_{i1}$ ($i = 1, 2, 3, 4$) are then taken as the new values of the coefficients a_i ($i = 0, 3, 4$) and b_i($i = 1, 2, 3, 4$) respectively and the system of linear equations solved iteratively until values of the coefficients, which give on solution sufficiently small correction terms, are obtained. Coefficients of higher-order harmonics are then obtained in a similar way.

4.2. *Sub-harmonic solutions*

To a first approximation we assume solutions:

$$\left.\begin{aligned}\xi_1 &= a_0 + a_1 \sin\frac{\tau}{n} + a_2 \cos\frac{\tau}{n} + a_3 \cos\tau,\\ \xi_2 &= b_0 + b_1 \sin\frac{\tau}{n} + b_2 \cos\frac{\tau}{n} + b_3 \sin\tau + b_4 \cos\tau.\end{aligned}\right\} \tag{12}$$

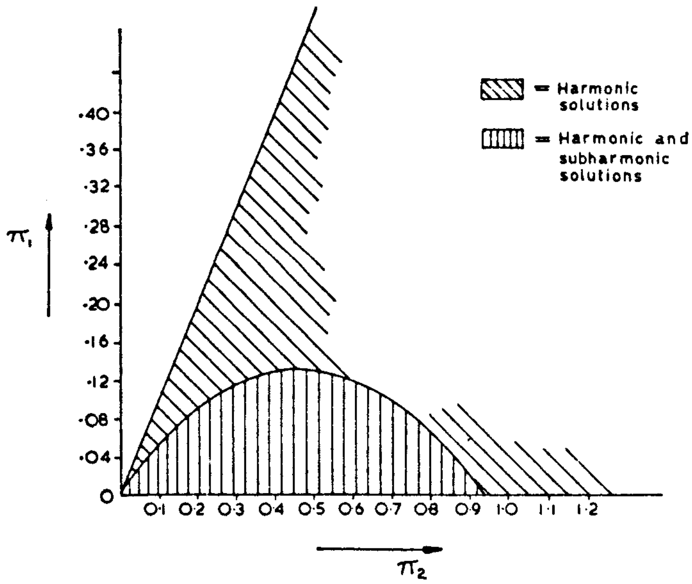
On substituting eqns. (12) in (4) and balancing like terms it is seen (see Appendix) that eight sub-harmonic solutions, of order 2, are possible in a region R_2 defined by the system of inequalities:

$$\left.\begin{aligned}N &= \Pi_2^4(289 - 105\Pi_2^2 - 420\Pi_1) + 196\Pi_1^2 + 308\Pi_2^2\Pi_1 - 588\Pi_2^2\Pi_1^2 > 0,\\ L^2 &= \tfrac{2}{3}\Pi_2^2 - \Pi_1^2 - \tfrac{1}{12}\Pi_2^4 - \tfrac{1}{3}\Pi_1\Pi_2^2 > 0,\\ |X| &= \left|\frac{1}{42\Pi_2}(\sqrt{N} - (11\Pi_2^2 + 14\Pi_1))\right| < L,\\ F(X) &= 7\Pi_2 X^3 + (\tfrac{11}{2}\Pi_2^2 + 7\Pi_1)X^2 + (\tfrac{5}{4}\Pi_2^5 + 5\Pi_2^3\Pi_1 + 7\Pi_2\Pi_1^2 - 2\Pi_2^3)X\\ &\quad + (\tfrac{9}{8}\Pi_2^6 + \tfrac{23}{4}\Pi_2^4\Pi_1 - \Pi_2^4 + \tfrac{21}{2}\Pi_1^2\Pi_2^2 - 2\Pi_2^2\Pi_1 + 7\Pi_1^3) < 0.\end{aligned}\right\} \tag{13}$$

The region R_2, in parameter space, was plotted using digital computation; the regions R_1 and R_2, in which different types of 'steady-state' oscillations are sustained are shown in fig. 2. Outside region R_1 there are, to a first approximation, no periodic solutions so that, for parameter values in this region, the system will be totally unstable.

An improvement in the accuracy of the sub-harmonic solutions (12) may be obtained using the same procedure as described in § 4.1 for harmonic solutions.

Fig. 2



Regions in which different types of oscillations are sustained.

An important feature of the sub-harmonic solutions is that the solution for $\xi_2(\tau)$ contains a constant or 'd.c.' component so that for certain initial conditions the system, with parameter values in region R_2, will adapt to an oscillation about an offset position; this phenomenon has been verified using analogue computer simulation.

5. Stability of 'steady-state' solutions

A periodic solution obtained by the method of harmonic balance merely represents a state of equilibrium; this equilibrium state is actually realizable only if it is stable so that its actual existence must be confirmed by a stability investigation.

Let $\boldsymbol{\xi}(\tau) = [\hat{\xi}_1(\tau)\,\hat{\xi}_2(\tau)]^{\mathrm{T}}$, having period T, represent a particular state of equilibrium then in order to investigate its stability we consider small variations $\boldsymbol{\eta}(\tau) = [\eta_1(\tau)\,\eta_2(\tau)]^{\mathrm{T}}$ from this equilibrium state; if in the ensuing motion $\boldsymbol{\eta}(\tau)$ tends to zero then the original undisturbed equilibrium state is said to be asymptotically stable.

From eqns. (4) we set up the variational equations:

$$\left.\begin{aligned} \dot{\eta}_1(\tau) &= -\frac{1}{\Pi_1}\eta_1(\tau) + \frac{2}{\Pi_1}(\hat{\xi}_2(\tau) - \sin(\tau))\,\eta_2(\tau), \\ \dot{\eta}_2(\tau) &= \frac{\Pi_1}{\Pi_2}\sin\tau\,\eta_1(\tau) \end{aligned}\right\} \tag{14}$$

in η_1 and η_2. These variational equations form a set of linear differential equations with periodic coefficients, of period T, in τ; that is, they are of the form:

$$\dot{\boldsymbol{\eta}}(\tau) = \mathbf{P}(\tau)\boldsymbol{\eta}(\tau), \quad \mathbf{P}(\tau) = \mathbf{P}(\tau+T).$$

The asymptotic stability of the trivial solution of such equations has been previously considered by the author (James 1969) and by Davison (1968). The procedure is to obtain a transition matrix $\mathbf{C}$ such that

$$\boldsymbol{\eta}(T+\tau_1) = \mathbf{C}\boldsymbol{\eta}(\tau_1) \tag{15}$$

and then stability is ensured provided the eigenvalues of $\mathbf{C}$ lie within the unit circle. If the eigenvalues of $\mathbf{C}$ are both greater than unity in absolute value the solution will be termed completely unstable, whilst it will be termed unstable if $\mathbf{C}$ has one eigenvalue greater than, and one less than, unity in absolute value.

A stability investigation of the periodic solutions of § 4 shows that in region R_1 we have one stable and one unstable harmonic solution, whilst in region R_2 we have, in addition to the stable and unstable harmonic solutions, four stable and four unstable sub-harmonic solutions of order 2.

6. Effect of initial conditions

In the absence of the output lag the system may be represented by the single non-dimensional equation:

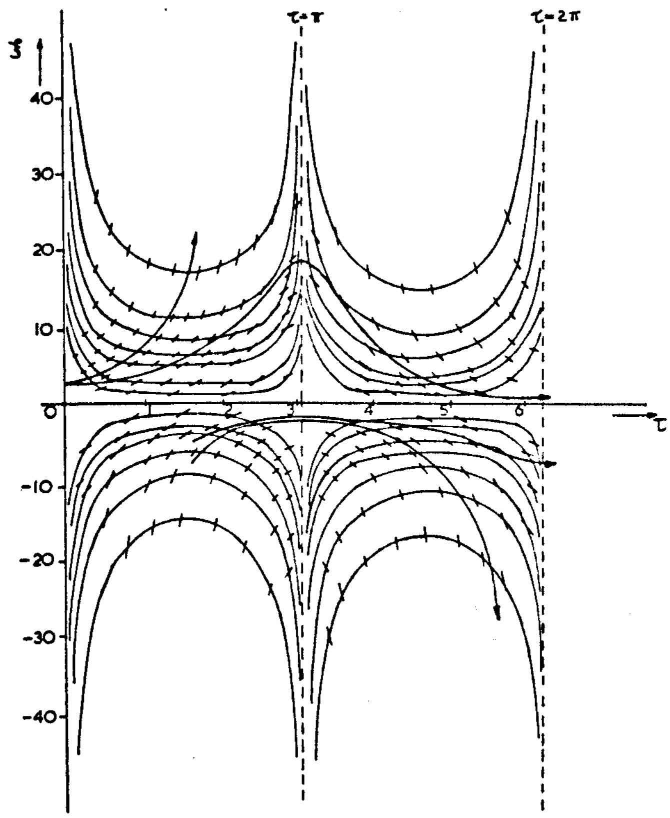
$$\dot{\xi} = \Pi(\xi - \sin\tau)^2 \sin\tau, \tag{16}$$

where $\Pi = GA\delta/\omega$, $\xi = y/\delta$ and $\tau = \omega t + \alpha$.

Equation (16) may be solved in the $\xi-\tau$ plane, for a particular value of the parameter Π, by the method of isoclines (Boddington 1968). A graphical solution obtained by this method, for $\Pi = 0{\cdot}2$, is shown in fig. 3 and it shows clearly the effect on stability of the initial conditions in this case. Solutions are shown for initial values $\tau = 0$ and $\tau = \pi/2$.

When the output lag is included, however, we can no longer solve the system equations by the method of isoclines. In order to examine the relationship between the initial conditions and the different types of periodic solutions we examine the 'domains of attraction' of the stable periodic solutions, a concept employed by Blair and Loud (1960) when examining the solutions of a

Fig. 3



Graphical solution of equation (16) for $\Pi = 0{\cdot}2$.

second-order non-linear differential equation with a periodic forcing term. A brief discussion of the theory of fixed points and domains of attraction will now be given.

7. Fixed points and domains of attraction

Let us consider the solution of the general system of equations:

$$\left.\begin{aligned} \dot{\xi}_1 &= f(\xi_1, \xi_2, \tau), \\ \dot{\xi}_2 &= g(\xi_1, \xi_2, \tau), \end{aligned}\right\} \tag{17}$$

where f and g have period T in τ. Provided f, g and their partial derivatives with respect to ξ_1 and ξ_2 are continuous in ξ_1, ξ_2 and τ it follows (Cartwright 1950) that there exists a mapping:

$$M: \boldsymbol{\xi}(\tau_1 + nT) \to \boldsymbol{\xi}(\tau_1 + \overline{n+1}T), \quad n = 0, 1, 2, \text{etc.}, \qquad (18)$$

where $\boldsymbol{\xi} = [\xi_1 \, \xi_2]^{\mathrm{T}}$, which is a one-to-one, continuous and orientation-preserving mapping of the ξ_1–ξ_2 plane into itself (that is, M defines a homeomorphism in the ξ_1–ξ_2 plane). If $\hat{\boldsymbol{\xi}}(\tau) = [\hat{\xi}_1(\tau) \, \hat{\xi}_2(\tau)]^{\mathrm{T}}$ is a harmonic solution of eqns. (17) then there exists, for each τ_1, $0 < \tau_1 < T$, a corresponding point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$, in the ξ_1–ξ_2 plane, which is invariant under the mapping M; that is, a 'fixed point' of the mapping M corresponds to a harmonic solution of eqns. (17).

Defining iterates of the mapping by:

$$M^2(P) = M(M(P)), \quad \text{etc.},$$

it follows that if $\hat{\boldsymbol{\xi}}(\tau) = [\hat{\xi}_1(\tau) \, \hat{\xi}_2(\tau)]^{\mathrm{T}}$ is a subharmonic solution, of order n, of eqns. (17), then there exists, for each τ_1, $0 < \tau_1 < nT$, a corresponding point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$ in the $\xi_1 - \xi_2$ plane, which is invariant under the nth iterate M^n of the mapping M, that is, a fixed point of the nth iterate of the mapping M corresponds to a subharmonic solution, of order n, of the system of eqns. (17).

There are certain standard types of fixed points, most of them corresponding closely to standard types of singular points for differential equations of order one. We shall now discuss the three most significant types; for a fuller discussion see the works of Cartwright (1950) and Levinson (1943, 1944). In the following $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$ is taken as a fixed point, in the ξ_1–ξ_2 plane at $\tau = \tau_1$, of the mapping M.

(*a*) *Stable fixed point*

This is a fixed point P_0 such that if P be any point in the neighbourhood of P_0 then $M^n(P) \to P_0$ as $n \to \infty$; that is, successive images $M(P)$, $M^2(P)$, etc., of the point P approach the fixed point P_0. This point is analogous to a node or focus in the theory of singular points, since the definition is true whether the loci of successive images move towards P radially or in a spiral fashion (remember that in this case P moves in jumps $M(P)$, $M^2(P)$, etc., and not along a continuous curve). By definition, this fixed point corresponds to a stable solution of the system of eqns. (17).

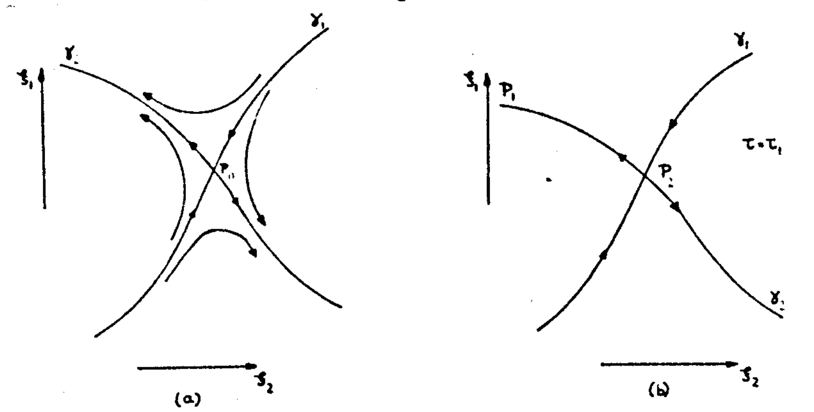
(*a*) *Saddle point*

This is a fixed point P_0 through which there passes two curves or directions γ_1, γ_2, see fig. 4 (*a*), which are invariant under the mapping M. Points on γ_1 approach P_0 under iterations of the mapping M, while points on γ_2 approach P_0 under iterations of the inverse mapping. In this case the loci of successive images in analogous to that of the integral curves in the neighbourhood of a saddle point in the theory of singular points. A saddle point corresponds to an unstable solution of eqns. (17).

(*c*) *Unstable fixed point*

This is a fixed point P_0 which is stable under the inverse mapping of M and corresponds to a completely unstable solution of eqns. (17).

Fig. 4



Defining $\xi_1(\tau, \xi_1(\tau_1), \xi_2(\tau_1)), \xi_2(\tau, \xi_1(\tau_1), \xi_2(\tau_1))$ as a solution of eqns. (17), for which the initial conditions, at $\tau = \tau_1$ are $\xi_1(\tau_1), \xi_2(\tau_1)$, we define the 'domain of attraction' of a stable fixed point $P_0(\hat{\xi}_1(\tau_1), \hat{\xi}_2(\tau_1))$ in the ξ_1–ξ_2 plane, at $\tau = \tau_1$, as the set of points $(\xi_1(\tau_1), \xi_2(\tau_1))$ for which the solution $\boldsymbol{\xi}(\tau, \xi_1(\tau_1), \xi_2(\tau_1))$ converges to the asymptotic stable periodic solution.

As pointed out by Blair and Loud (1960), the general question of the finding the shape of a domain of attraction is quite difficult, and studies by Hayashi (1964) show that for comparatively simple equations the domains of attraction can be highly complicated. In this section we shall discuss the domains of attraction of two fixed points, one being stable and the other a saddle point; this is the case that arises for the harmonic solutions of § 4.1.

Suppose the two fixed points are represented by P_1 and P_2, see fig. 4 (*b*), P_1 being the stable fixed point and P_2 the saddle point. As indicated previously, through P_2 there pass two curves γ_1, γ_2 which are invariant under the mapping M, with points on γ_1 approaching P_2 under iterations of the mapping whilst points on γ_2 approach P_2 under iterations of the inverse mapping. Hence the successive images of an initial point $(\xi_1(\tau_1), \xi_2(\tau_1))$ will tend either to P_1 or to infinity, depending on which side of γ_1 the initial point is. Thus the invariant curve γ_1 is of great importance in any stability investigations for it is the boundary between two regions in each of which initial conditions will lead to a particular type of oscillation, that is, it is the boundary between 'domains of attraction' (for a more mathematical treatment see the works of Blair and Loud (1960)). In the particular case illustrated in fig. 4 (*b*) the invariant curve γ_1 is the boundary between the domain of attraction of the fixed point P_1 and the domain of attraction of the point at infinity (that is, initial conditions in this domain will lead to a solution that grows indefinitely with time).

If the saddle point P_2 corresponds to the unstable periodic solution $\hat{\boldsymbol{\xi}}(\tau)$ of eqns. (17) then the transition matrix $\mathbf{C}$ in the stability investigation will have one eigenvalue greater than, and one less than, unity in absolute value. At the point P_2 the curve γ_1 will be in the direction of the eigenvector of $\mathbf{C}$ which corresponds to the eigenvalue that is less than unity in absolute value; thus the slope α of γ_1 at P_2 may be found. Theoretically therefore the invariant curve γ_1 may be obtained by starting just on either side of the fixed point P_2 (in the direction α) and integrating eqns. (17) numerically for decreasing τ;

the curve γ_1 then being the loci of the successive images of the starting point under iterations of the mapping M (or M^n for a sub-harmonic solution of order n). In practice, however, it was found that if the unstable fixed points are not known accurately enough the image points of the numerical integration deviate from the desired boundary after a few cycles, so that the loci obtained by the numerical procedure may only be used as a guide and a more accurate boundary must be obtained by analogue computer studies.

8. Numerical example

For parameter values $\Pi_1 = 0{\cdot}1, \Pi_2 = 1$, in region R_1, the two harmonic solutions:

$$\text{(i)}\quad \xi_1 = 260{\cdot}63 + 97{\cdot}24 \sin 2\tau + 72{\cdot}93 \cos 2\tau + 9{\cdot}62 \sin 4\tau - 5{\cdot}06 \cos 4\tau - 0{\cdot}65 \sin 6\tau - 0{\cdot}84 \cos 6\tau,$$

$$\xi_2 = 4{\cdot}86 \sin \tau - 22{\cdot}42 \cos \tau - 1{\cdot}46 \sin 3\tau - 1{\cdot}30 \cos 3\tau - 0{\cdot}10 \sin 5\tau + 0{\cdot}04 \cos 5\tau,$$

$$\text{(ii)}\quad \xi_1 = 0{\cdot}512 - 0{\cdot}192 \sin 2\tau - 0{\cdot}126 \cos 2\tau + 0{\cdot}0004 \sin 4\tau + 0{\cdot}0008 \cos 4\tau,$$

$$\xi_2 = -0{\cdot}009 \sin \tau - 0{\cdot}057 \cos \tau + 0{\cdot}004 \sin 3\tau + 0{\cdot}002 \cos 3\tau + 0{\cdot}0000 \sin 5\tau + 0{\cdot}0000 \cos 5\tau$$

were obtained. Solution (i) was found to be unstable while solution (ii) was found to be stable.

Thus for each τ, $0 < \tau < 2\pi$, there will be a domain of attraction in the ξ_1–ξ_2 plane so that for all initial conditions within the domain the system will be stable while for initial conditions outside the domain the system will be unstable. Further in this case, the solution for ξ_2 has no d.c. component so that in the stable region the system will adapt to an oscillation about zero error $e(t)$. The stability boundaries, in the state space, together with their corresponding fixed points, are shown in fig. 5 for various values of τ in the range $(0, \pi)$ corresponding boundaries for τ in the range $(\pi, 2\pi)$ being the mirror images about the ξ_1 axis of those given. These stability boundaries have been verified using analogue computer simulation and also agree with experimental work carried out by Jacobs and Shering (1969).

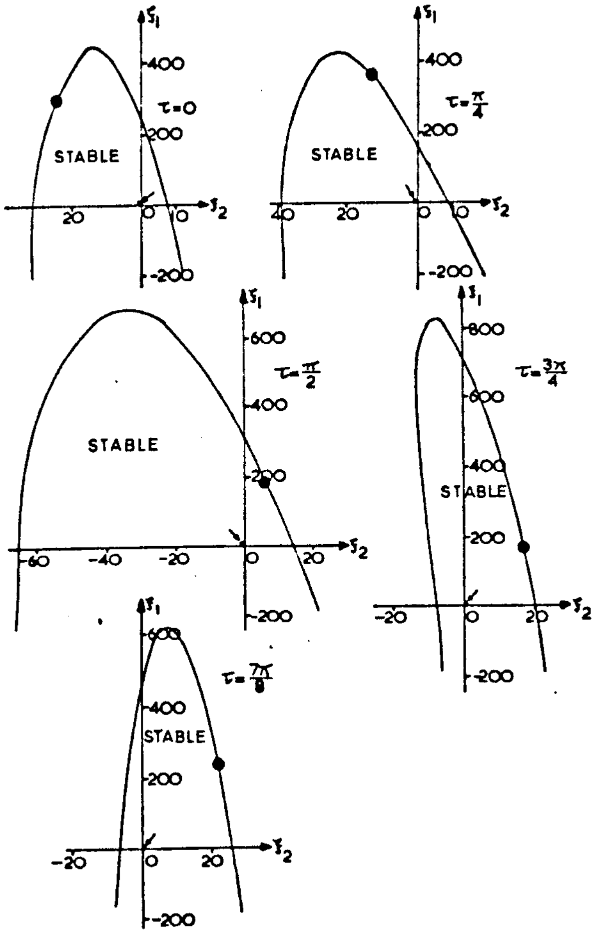
For parameter values in region R_2 the domains of attraction are very complicated and for certain initial conditions, within the overall stability boundary in phase-space, the system will adapt to a sub-harmonic oscillation about an offset position. Parameter values in this region are therefore unsuitable for practical systems so that a detailed solution in this region is not included in this paper.

9. Conclusions

This paper emphasizes the importance of the knowledge of periodic solutions in the stability study of a sinusoidal perturbation, adaptive control system. It has been shown that for the first-order system considered the parameter space may be divided into three regions, viz.:

(i) Region where no periodic solution exists so that the system is totally unstable;

Fig. 5



Domains of attraction and corresponding fixed points for $\Pi_1 = 0{\cdot}1$, $\Pi_2 = 1$. (●, Unstable fixed points, ↓●, stable fixed points.)

(ii) Region R_1, where there exists two harmonic solutions, one stable and one unstable;

(iii) Region R_2, where, in addition to the two harmonic solutions of (ii), there exist four stable and four unstable sub-harmonic solutions of order 2.

In the case of the sub-harmonic solutions the periodic solution for the adapting variable ξ_2 contains a d.c. term so that for certain initial conditions the system, with parameter values in R_2, will adapt to an oscillation about an offset position. It is important therefore in any practical application to employ parameter values in region R_1, but outside region R_2. Thus a knowledge of the boundary of the regions R_1 and R_2 is essential in any design consideration.

By plotting the domains of attractions of fixed points which are invariant under the mapping:

$$\xi(\tau + nT) \rightarrow \xi(\tau + \overline{n+1}T), \quad n = 0, 1, 2, \text{etc.},$$

where $\xi(\tau)$ is the state vector of the system, regions in the three-dimensional space ξ_1, ξ_2, τ were obtained, for particular parameter values in R_1, within which initial conditions will lead to a stable oscillation. Information about these stability boundaries in the state-space is also highly relevant in any design consideration of a practical system; for if a system is subjected to random disturbances and noise there will be a finite probability of the system entering any region of its state-space. However, no parameter values will make the system stable everywhere, so that, in order that the probability of the system being driven unstable by the random disturbances and noise is negligibly small, it is essential that the normal region of operation of the system is well within the stability boundary.

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Appendix

Sub-harmonic solutions of the system equations

Substituting eqns. (13) in (4 b) and balancing like terms it is seen that the only possible value for n is 2 and that

$$b_1 = \frac{\Pi_1}{\Pi_2} a_1, \quad b_2 = -\frac{\Pi_1}{\Pi_2} a_2, \quad b_3 = 0, \quad b_4 = -\frac{\Pi_1}{\Pi_2} a_0. \tag{A 1}$$

Substituting eqns. (13) in (4 a) and balancing like terms gives, on using results (A 1):

$$-\frac{\Pi_2}{\Pi_1} b_4 = b_0^2 + \frac{b_1^2}{2} + \frac{b_2^2}{2} + \frac{b_4^2}{2} + \frac{1}{2}, \tag{A 2}$$

$$\frac{\Pi_2}{\Pi_1} b_1 + \frac{\Pi_2^2}{2\Pi_1} b_2 = 2b_0 b_1 - b_1 b_4 - b_2, \tag{A 3}$$

$$-\frac{\Pi_2}{\Pi_1} b_2 + \frac{\Pi_2^2}{2\Pi_1} b_1 = 2b_0 b_2 - b_1 + b_2 b_4, \tag{A 4}$$

$$0 = -\frac{\Pi_2}{2} b_1^2 - \frac{\Pi_2}{2} b_2^2 + 2\Pi_2 b_0 b_4 - 2b_0 + b_1 b_2. \tag{A 5}$$

Equations (A 3) and (A 4) are homogeneous in b_1 and b_2 and will have a non-trivial solution for these coefficients (that is, subharmonic solutions exist) if and only if:

$$\left(\frac{\Pi_2}{\Pi_1} + b_4\right)^2 - 4b_0^2 + \left(\frac{\Pi_2^2}{2\Pi_1} + 1\right)^2 = 0. \tag{A 6}$$

Writing:

$$\frac{b_1}{b_2} = \left[\frac{\Pi_2}{\Pi_1} + 2b_0 + b_4\right] \Big/ \left[\frac{\Pi_2^2}{2\Pi_1} + 1\right] = \mu, \quad r = b_1^2 + b_2^2, \quad R = -b_4,$$

we have that $b_1 b_2 = \mu b_2^2$ giving

$$b_1 b_2 = \frac{\mu r^2}{1+\mu^2} \quad \text{and} \quad b_1^2 - b_2^2 = -\left[\frac{1-\mu^2}{1+\mu^2}\right] r^2.$$

Substituting back, eqns. (A 3), (A 5) and (A 6) become:

$$2b_0^2+r^2+\left(R-\frac{\Pi_2}{\Pi_1}\right)^2=\left(\frac{\Pi_2^2}{\Pi_1^2}-1\right), \tag{A 7}$$

$$\left(1+\frac{2\mu}{\Pi_2}-\mu^2\right)r^2=4b_0\left(R+\frac{1}{\Pi_2}\right)(1+\mu^2), \tag{A 8}$$

$$\left(\frac{\Pi_2}{\Pi_1}-R\right)^2-4b_0^2+\left(\frac{\Pi_2^2}{2\Pi_1}+1\right)^2=0. \tag{A 9}$$

Eliminating b_0^2 from (A 7) and (A 9) gives:

$$r^2+\frac{3}{2}\left(R-\frac{\Pi_2}{\Pi_1}\right)^2=\lambda^2, \tag{A 10}$$

where

$$\lambda^2=\frac{1}{2}\left[\frac{2\Pi_2^2}{\Pi_1^2}-3-\frac{\Pi_2^4}{4\Pi_1^2}-\frac{\Pi_2^2}{\Pi_1}\right]$$

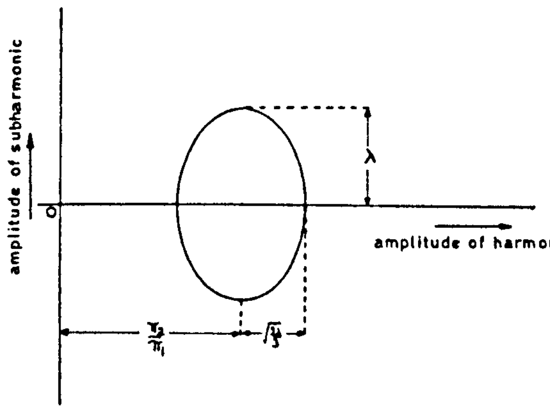
that is, the loci of the moduli of the sub-harmonics and harmonics are ellipses as shown in fig. 6. Writing:

$$z=R-\frac{\Pi_2}{\Pi_1} \quad \text{and} \quad \alpha=\frac{\Pi_2^2}{2\Pi_1}+1, \tag{A 11}$$

eqn. (A 10) becomes:

$$r^2+\tfrac{3}{2}z^2=\lambda^2. \tag{A 12}$$

Fig. 6



Ratio of amplitudes.

Eliminating r^2 from eqns. (A 7) and (A 12) gives:

$$4b_0^2-z^2=\alpha^2, \tag{A 13}$$

while eliminating r^2 from eqns. (A 8) and (A 12), and substituting for α^2 from (A 13) gives:

$$\left(\frac{1}{\Pi_2}+z\right)(\lambda^2-\tfrac{3}{2}z^2)=2(\alpha^2+z^2)\left(z+\frac{\Pi_2}{\Pi_1}+1\right).$$

Substituting for λ and α and expanding gives:

$$F(X) = 7\Pi_2 X^3 + (\tfrac{11}{2}\Pi_2^2 + 7\Pi_1)X^2 + (\tfrac{5}{4}\Pi_2^5 + 5\Pi_2^3\Pi_1 + 7\Pi_2\Pi_1^2 - 2\Pi_2^3)X$$
$$+ (\tfrac{9}{8}\Pi_2^6 + \tfrac{23}{4}\Pi_2^4\Pi_1 - \Pi_2^4 + \tfrac{21}{2}\Pi_1^2\Pi_2^2 - 2\Pi_2^2\Pi_1 + 7\Pi_1^3) = 0 \quad \text{(A 14)}$$

where $X = \Pi_1 z$.

For a sub-harmonic solution to exist a real root of eqn. (A 14) must be such that eqn. (A 12) gives a real value of r, that is:

$$X^2 < \tfrac{2}{3}\Pi_2^2 - \Pi_1^2 - \tfrac{1}{12}\Pi_2^4 - \tfrac{1}{3}\Pi_2^2\Pi_1 = L^2 > 0. \quad \text{(A 15)}$$

It can be shown that eqn. (A 14) cannot have an odd number of roots between $-L$ and L and that it will have two real roots given by:

$$X = \frac{1}{42\Pi_2}[\sqrt{N} - (11\Pi_2^2 + 14\Pi_1)],$$

provided:

$$N = \Pi_2^4(289 - 105\Pi_2^2 - 420\Pi_1) + 196\Pi_1^2 + 308\Pi_2^2\Pi_1 - 588\Pi_2^2\Pi_1^2 > 0 \quad \text{(A 16)}$$

and

$$F(X) < 0. \quad \text{(A 17)}$$

Thus, sub-harmonic solutions exist provided inequalities (A 15), (A 16) and (A 17) hold. In the region of parameter space where these inequalities hold eqn. (A 14) may be solved to give two real values of x between $-L$ and L. Substituting these values in (A 11), (A 12) and (A 13) gives corresponding values of R, r and b_0^2. Solving $b_1 = \mu b_2$, $b_1^2 + b_2^2 = r^2$, $R = -b_4$ then give the coefficients b_1, b_2 and b_4 whilst eqns. (A 1) give the corresponding coefficients a_0, a_1 and a_2. It is readily seen that there exist eight sub-harmonic solutions of order 2, each with a d.c. component in ξ_1 and ξ_2, for each point in the region of parameter space satisfying the necessary inequalities.

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where cosecϕ is the ratio of d to the difference in the moon and spacecraft radii (see Fig. 4).

From Eqs. (6, 7 and 10) it can be shown that contours of constant V_f in a (d, ϕ) coordinate system are circles, with center $(\mu \cot\gamma/(V_f^2 - V_M^2), 0)$ and radius

$$\mu[\{(V_f^2 - V_m^2)\tan\gamma\}^{-2} + \{2(V_f^2 - V_m^2)V_m^2 \sin^2\gamma\}^{-1} + \{2V_m \sin\gamma\}^{-4}]^{1/2}$$

Examples of these circles for $\gamma = 20°$, $40°$ and $90°$ are shown in scale with the moon in Fig. 4. The "dashed" circular arcs intersecting the constant V_f circles, are the boundary of the moon impact region. Thus, the regions with V_f greater than a fixed value are crescent-shaped. The shading on the moon indicates its velocity with respect to the spacecaft. The moon's direction is indicated by it being shaded as if it were traveling away from the sun; and the fraction of disc shaded gives the speed as a fraction of $2V_M$.

For the high velocity producing orbits, the spacecraft has to be directed towards a small region just behind the moon with $\gamma \sim 40°$. For $\gamma \sim 20°$ the escape region (i.e., $V_f \geq V_E$) becomes large; in Fig. 4 it is shown to be a region of over 10,000 km in diameter.

The "head on" conditions when $\gamma = 90°$, permits good direction control, but the extent of the escape region is small.

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Stability of a Model Reference Control System

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Introduction

In a recent paper Lindh and Likins[1] compared the so-called infinite determinant method and a numerical implementation of Floquet theory for obtaining the regions of the parameter space corresponding to stability and instability of the null solution of a restricted class of linear, periodic coefficients, ordinary homogeneous differential equations. In this Note these methods will be applied to examine the stability of a model reference adaptive control system having sinusoidal input.

In recent years model reference adaptive control systems have proved very popular, particularly for practical applications to devices such as auto-pilots where rapid adaptation is required. The basic idea is shown in Fig. 1. The input $\theta_i(t)$ to the system is also fed to a reference model, the output of which is proportional to the desired response; the outputs of the model and system are then differenced to form an error

$$e(t) = \theta_m(t) - \theta_s(t) \tag{1}$$

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Since the error is to be zero when the system is in the optimum state it is used as a demand signal for the adaptive loops which adjusts the variable parameters in the system to the desired value.

Various methods of synthesizing the adaptive loops have been proposed but the one that has proved most popular was that developed by Whittaker et al.[2] at the Massachusetts Institute of Technology and referred to as the "M.I.T." rule. Here the performance criterion is taken as the integral of error squared and a heuristic argument is given for reducing this over an unspecified period of time. This leads to a rule that a particular parameter K_i should be adjusted so that

$$\dot{K}_i = -Ge(\partial e/\partial K_i) \tag{2}$$

where G is the constant gain.

Although the "M.I.T." rule results in practically realizable systems, mathematical analysis of the adaptive loops, even for simple inputs, prove to be very difficult and it is usual in the design process to carry out much analogue computer simulation. The system equations are nonlinear and nonautonomous and since the nonlinearity is of the multiplicative kind, the mass of theory on instantaneous nonlinearities associated with the names of Lur̂e and Popov, in particular, is not applicable. In order to point out some of the difficulties we shall consider a simple first-order system having sinusoidal input.

Adaptive Control System

Since the intention, as previously mentioned, is to point out the difficulties involved in a stability investigation of a model reference adaptive control system, a simple first-order system with controllable gain will be considered.

Consider a model and system to be governed respectively by the equations

$$T\dot{\theta}_m(t) + \theta_m(t) = K\theta_i(t) \tag{3a}$$

$$T\dot{\theta}_s(t) + \theta_s(t) = K_vK_c\theta_i(t) \tag{3b}$$

where a dot denotes differentiation with respect to time t; the time constant T and model gain K are constant and known, but the process gain K_v is unknown and possibly time varying. The problem here is to determine a suitable adaptive loop to control K_c so that K_vK_c eventually equals the model gain K. The "M.I.T." rule gives

$$\dot{K}_c = -Ge(\partial e/\partial K_c) = Be\theta_m \tag{4}$$

where $B = GK_v/K$, and this leads to the scheme of Fig. 2.

If a sinusoidal input of magnitude R sinωt is applied at $t = 0$, when $\theta_m(t)$, $\theta_s(t)$ are zero and $K_vK_c \neq K$ and if subsequently K_v remains constant but K_c is adjusted according to Eq. (4), then using Eqs. (1, 3 and 4) the system equations become

$$T\dot{e}(t) + e(t) = (K - K_vK_c)R \sin\omega t \tag{5a}$$

$$\dot{K}_c = Be(t)\theta_m(t) \tag{5b}$$

Now consider that the adaption is switched on when the model response $\theta_m(t)$ has reached its steady-state value $\theta_{ms}(t)$ given by

$$\theta_{ms}(t) = [KR/(1 + T^2\omega^2)](\sin\omega t - T\omega \cos\omega t)$$

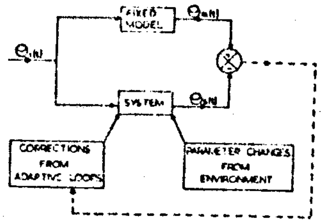

Fig. 1 Model reference adaptive control system.

then Eqs. (5) become

$$T\dot{e}(t) + e(t) = x(t)\sin\omega t \tag{6a}$$

$$\dot{x}(t) = -[K_vBKR/(1 + T^2\omega^2)](\sin\omega t - \omega T\cos\omega t)e(t) \tag{6b}$$

where $x(t) = K - K_vK_c$.

Introducing the dimensionless parameters $\Pi_1 = \omega T$, $\Pi_2 = TR^2KK_vB$ and the dimensionless variables $\tau = \omega t$, $\xi_1 = e/KR$, $\xi_2 = -x/K$, Eqs. (6) may be written in the nondimensional form

$$\frac{d}{d\tau}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} = \begin{bmatrix} -\dfrac{1}{\Pi_1} & -\dfrac{1}{\Pi_1}\sin\tau \\ \dfrac{\Pi_2}{\Pi_1(1+\Pi_1^2)}(\sin\tau - \Pi_1\cos\tau) & 0\end{bmatrix}\begin{bmatrix}\xi_1(\tau)\\ \xi_2(\tau)\end{bmatrix} \tag{7}$$

which is a linear matrix differential equation of the form $\boldsymbol{\xi}' = \mathbf{A}(\tau)\boldsymbol{\xi}$ with $\mathbf{A}(\tau)$ periodic in τ with period 2π (a prime denotes differentiation with respect to τ).

Stability Theory

We are interested in finding the domains of the parameter space for which the null solution of the system of first-order equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{8}$$

$$\mathbf{A}(t + T) = \mathbf{A}(t) \tag{9}$$

is stable, where $\mathbf{x}(t)$ is an n vector and $\mathbf{A}(t)$ an $n \times n$ matrix of period T in t. This problem was discussed in some detail in Ref. 1 and we shall confine ourselves here to a brief summary.

One approach is a numerical implementation of Floquet theory. From Floquet theory it can be shown that for system (8), subject to condition (9), there exists a constant $n \times n$ matrix $\mathbf{C}$, known as the monodromy matrix of the system, such that

$$\mathbf{x}(t + T) = \mathbf{C}\mathbf{x}(t) \tag{10}$$

Using a Liapunov type transformation it then follows that a necessary and sufficient condition for the null solution of the system to be uniformly asymptotically stable is that all the eigenvalues of the monodromy matrix $\mathbf{C}$ lie within the unit circle $|z| < 1$. If the eigenvalues of the monodromy matrix lie in the circle $|z| \leq 1$ and the eigenvalues on $|z| = 1$ correspond to unidimensional Jordan cells, then the null solution is uniformly stable.[3]

In practice the monodromy matrix $\mathbf{C}$ is obtained by integrating Eqs. (8) numerically over a period.[4,5] [A check on the value of $\mathbf{C}$ may be employed since $\det\mathbf{C} = \exp\int_0^T \mathrm{trace}\mathbf{A}(t)dt$.] This is followed by a numerical evaluation of its eigenvalues thus giving an assessment of stability or instability. However, if one is only interested in the question of stability, the last step may be dispensed with. Instead the characteristic polynomial of $\mathbf{C}$ may be obtained using the Faddeeva[6] algorithm followed by a stability assessment using the determinant method of Jury.[7] This procedure has been used satisfactorily by the author[4] and since it involves only matrix multiplication and the evaluation of second-order determinants it gives a considerable saving in computational time over direct evaluation of the eigenvalues.

An alternative method of obtaining the transition boundaries, between stable and unstable regions in parameter space, is the so-called infinite determinant method. This method is restrictive in its use since it requires the form of the solutions on the transition boundaries to be known. Assuming the continuous dependence of stability on parameter values it follows that on any transition boundary there exists an eigenvalue λ_i of the monodromy matrix such that its modulus is

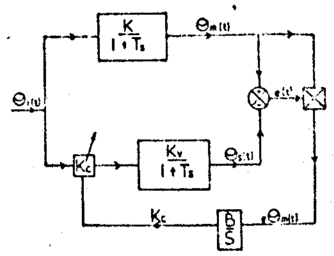

Fig. 2 First-order system—M.I.T. gain adaptation.

unity. Thus on the transition boundaries there must exist[8] an almost periodic solution of the form

$$\mathbf{x}_i(t) = e^{j(\arg\lambda_i)t/T}\mathbf{p}_i(t) \tag{11}$$

where $\mathbf{p}_i(t)$ is a periodic n vector with period T (note that in general such solutions may exist within stable or unstable regions but not within regions of uniform asymptotic stability). If the monodromy matrix of the system is symplectic then its characteristic equation is reciprocal, and the form of the solutions on the transition boundaries for such systems is discussed in detail in Ref. (1).

For certain systems (e.g., uncoupled canonical systems) it can be shown that the transition boundaries are characterized by the existence of solutions of Period T or a restricted class of functions of period $2T$. The procedure then is to assume Fourier series developments with undetermined coefficients, for these solutions; these solutions are then substituted into the system equations and the principle of harmonic balance employed to obtain an infinite system of simultaneous, linear, homogeneous algebraic equations for the coefficients. For those values of the parameters which admit the assumed periodic solutions the homogeneous algebraic equations must have a nontrivial solution and this is the case only if the infinite determinant (Hill determinant) of the coefficients is zero. In practice the Fourier series is truncated and the corresponding Hill determinant solved to give lines in parameter space. If the truncation point of the Fourier series is extended and the zeros of the corresponding Hill determinants of increasing order converge to some limit set of lines then the infinite determinant procedure is said to be convergent, the convergent set of lines in parameter space being the required transition boundary between stable and unstable regions.

Application of Theory to Adaptive Control System

Applying the numerical implementation of Floquet theory to system (7) stability boundaries in the parameter space $\Pi_1 - \Pi_2$ were obtained. These stability boundaries are shown in Fig. 3 and in the main they have been verified by analogue computer simulation.

Although system (7) is not canonical it can be shown that the only solution corresponding to transition boundaries

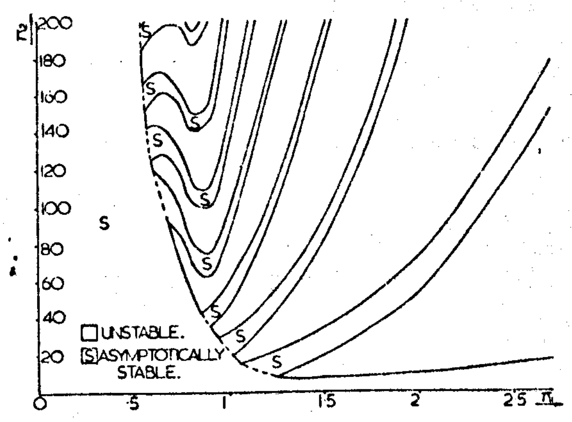

Fig. 3 Stability regions in parameter space.

from stable to unstable regions are those of period T and $2T$ (2π and 4π in this case). The monodromy matrix **C** of Eq. (7) is such that

$$\det\mathbf{C} = \exp\left\{\int_0^{2\pi} \mathrm{trace}\mathbf{A}(\tau)d\tau\right\} = \exp\{-2\pi/\Pi_1\}$$

so that if λ_1, λ_2 are the eigenvalues of C then

$$\lambda_1\lambda_2 = \exp\{-2\pi/\Pi_1\} = \rho^2 \text{ (say)} \tag{12}$$

If λ_1, λ_2 are complex conjugates then it follows from Eq. (12) that they lie on a circle of radius ρ so that complex roots cannot have modulii unity since this would imply $\rho = 1$ which is only possible if Π_1 is infinite. Thus on the transition boundary there must exist a real root having the value of $+1$ or -1 (note that real roots λ_1,λ_2 are inverse points with respect to the circle of radius ρ). It follows from Eq. (11) that the transition boundary is characterized by a solution of period T or $2T$.

On substituting a Fourier series, with undetermined coefficients, of period 4π in Eqs. (7) and balancing like terms, it can be shown, by induction, that the corresponding Hill determinants are sums of squares and therefore cannot be zero for any values of the parameters. This has been verified by analogue computer simulation and by the results of the Floquet theory analysis.

In the case of the harmonic solution, of period 2π, substituting the Fourier series

$$\xi_1 = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\tau + b_n \sin n\tau)$$

into Eqs. (7) [the corresponding series for ξ_2 being obtained by the second equation of (7)] and balancing the terms leads to two distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n},b_{2n}), (a_{2n+1},b_{2n+1}), $(n = 0,1,2,$ etc.), respectively. The corresponding Hill determinants of order r, in each case, are polynomials of order r in Π_2 having coefficients which are functions of Π_1. For a particular r these polynomials are solved for a range of values of Π_2 and the zeros plotted to obtain the transition boundary in the parameter space. The value of r is then increased and the corresponding Hill determinants solved until a convergent set of boundaries are obtained. It is found that for $\Pi_1 > 1.5$, where the enveloping boundary is continuous, consideration of fifth order Hill determinants is sufficient but for $\Pi_1 < 1.5$ where the enveloping boundary is discontinuous, the method is not found to be very satisfactory. Hill determinants of order eleven have to be considered before a true picture begins to emerge and the order has to be increased still further before an enveloping boundary is obtained to a satisfactory degree of accuracy. For this problem the region $\Pi_1 < 1.5$ is important since it is the most likely range of application in practice.

Conclusions

In this Note the stability regions in nondimensional space have been obtained for a first-order controllable gain model reference adaptive control system, and the results illustrate the complexity of the question of stability when dealing with such systems.

Both a numerical implementation of Floquet analysis and the infinite determinant method of analyzing linear differential equations with periodic coefficients have been employed. The infinite determinant approach was not found to be very satisfactory in the region of parameter space where the stability boundaries are complex in nature. Since, when dealing with linear differential equations with periodic coefficients, complex stability boundaries frequently occur, it throws some doubt on the performance of the method in general. Although the Floquet analysis involved investigating the eigenvalues of the monodromy matrix at a network of points in parameter space the results obtained were far more satisfactory, and for the particular problem considered the computation time was less. For higher order systems the use of the Faddeev algorithm and the Jury procedure would further reduce computational time when using the Floquet approach.

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Some Considerations of a Simplified Velocity Spectrum Relation for Isotropic Turbulence

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THE "three-dimensional" velocity spectrum function, $E(k)$, defined such that

$$\frac{3}{2}\hat{u}^2 = \int_0^\infty E(k)dk \tag{1}$$

($\hat{u}$ = rms velocity fluctuation level, k = wave number), is of interest in both theoretical and practical studies of turbulence phenomena. A simple form for $E(k)$ was proposed by von Kármán[1]

$$k_e E(k)/\hat{u}^2 \propto (k/k_e)^4[1 + (k/k_e)^2]^{-17/6} \tag{2}$$

(k_e = energy-containing wave number, as an "interpolation" formula joining the range $k \approx 0(E(k) \propto k^4)$ to the inertial subrange, wherein $E(k) \propto k^{-5/3}$ for $k \gg k_e$ in Eq. (2).

At high-wave numbers, of the order of the Kolmogoroff wave number $k_K \doteq L_K^{-1}$, where

$$L_K = (\nu^3/\Phi)^{1/4} \tag{3}$$

the spectrum function is "cut-off" by viscous effects. The inadequacy of Eq. (2) for this wave number range is reflected

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