Mori extractions from singular curves in a smooth 3-fold

by

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Contents

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I would like to thank all of those that have been at the Mathematics Institute at the University of Warwick at some time during the four years I have spent here, for their friendship and encouragement.

Declarations

I declare that the work contained in this thesis is original except where otherwise stated in the text. I confirm that this thesis has not been submitted anywhere else for any degree. Some of the material and the results of §§1-4 are adapted from a preprint [D] available on arXiv.

Abstract

We study terminal 3-fold divisorial extractions $\sigma\colon (E \subset Y) \to (C \subset X)$ which extract a prime divisor E from a singular curve C centred at a point P in a smooth 3-fold X. Given a presentation of the equations defining C, we give a method for calculating the graded ring of Y explicitly by *serial unprojection*. We compute some important examples and classify such extractions when the general hyperplane section S_X containing C has a Du Val singularity at $(P \in S_X)$ of type A_1, A_2, D_{2k}, E_6, E_7 or E_8.

Introduction

0.1 Outline of the thesis

0.1.1 Background

The MMP. The celebrated *minimal model program* (MMP) provides the natural framework in which to study the birational geometry of complex algebraic varieties. The ultimate goal of the MMP is to generalise as much of the Enriques–Kodaira classification of algebraic surfaces to higher dimensions as possible. For instance the MMP introduces *divisorial contractions* and *flips* as a higher dimensional analogue of the Castelnuovo contraction theorem for -1-curves on surfaces.

Unlike the case of surfaces, it is not always possible to make a divisorial contraction (or flip) and remain in the category of smooth varieties, even if Y was smooth to begin with. Therefore we consider varieties with *terminal* singularities—the smallest class of singularities that are preserved under these operations.

A rough outline of the MMP is given in §1.1, including a more detailed discussion of terminal singularities, divisorial contractions and flips.

The Mori category. Whilst the abstract definitions of terminal singularities, divisorial contractions, flips etc. have been used very successfully to set up the general theory of the MMP, an explicit classification of these notions would also be very desirable. For example Mori and Reid [R1] gave a classification of terminal 3-fold singularities up to local analytic isomorphism which has been enormously useful in studying the birational geometry of 3-folds. Unfortunately the *Mori category* (roughly speaking, the category of terminal 3-folds) currently looks like the only setting in which such a classification would be humanly possible, but even in this case many details are yet to be worked out.

Mori flips. Because of the technical difficulties that arise from the introduction of flips, these tend to have been studied in more detail than divisorial contractions.

The two main papers on 3-fold flips are Mori [M1] and Kollár & Mori [KM] although these contain many results relevant for studying divisorial contractions to curves too.

Reid's *general elephant conjecture* for a flipping (or divisorial) contraction $\sigma\colon Y \to X$ of terminal 3-folds states that a general elephant $S_Y \in |-K_Y|$ and $S_X = \sigma(S_Y)$ both have at worst Du Val singularities and that $\sigma\colon S_Y \to S_X$ is a partial crepant resolution. One of the important results of [KM] is the proof of this conjecture when σ is a contraction with an irreducible central fibre. This result allows us to start classifying flips and divisorial contractions based on the ADE classification for the Du Val singularities of S_X.

Indeed the main result of [KM] is a classification of flips when S_X has a type D or type E Du Val singularity. Much work has also been done by Mori [M2], Hacking, Tevelev & Urzúa [HTU] and Brown & Reid [BR4] to describe type A flips. Type A flips are much less restricted than type D or E flips and they exist in large infinite families.

Mori contractions. The classification of Mori contractions divides naturally into two cases: we can contract a divisor either to a point or to a curve. The divisor-to-point case is well understood and has essentially been completely classified through the work of several people including Corti, Kawakita, Hayakawa and Kawamata.

The divisor-to-curve case behaves much more similarly to flipping contractions, as the central fibre of such a contraction is 1-dimensional. This case has been studied by Tziolas [Tz1, Tz2, Tz3, Tz4] when the divisor is contracted to a *smooth* curve, but divisorial contractions to *singular* curves remain largely unstudied.

In this thesis we start the study of these contractions by focussing on the classification of Mori contractions $\sigma\colon (E \subset Y) \to (C \subset X)$ that contract a prime divisor E to a singular curve C contained in a *smooth* 3-fold X. Eventually the techniques developed here will be able to be applied to the case when X is also singular.

Further motivation

The Sarkisov program. One of the most successful applications of the classification of terminal 3-fold singularities and terminal divisor-to-point contractions has been in the 3-fold Sarkisov program. The Sarkisov program aims to go one step further that the MMP by decomposing birational maps between Mori fibre spaces into 'elementary' *Sarkisov links*, which are birational transformations made up of flips, flops and divisorial contractions. A classification of divisor-to-curve contractions would therefore also prove immensely useful in studying the birational rigidity of Mori fibre spaces and their admissible Sarkisov links.

Although the MMP is *directed* and only ever seeks to perform contractions on a variety Y, in applications of the Sarkisov program it is often useful to view a Mori contraction $\sigma\colon (E \subset Y) \to (\Gamma \subset X)$ as the *extraction* of the divisor E from X and then to run a relative MMP on Y over X. Therefore it is both natural and useful to approach this problem as the classification of the subvarieties $(\Gamma \subset X)$ which admit a *Mori extraction*.

0.1.2 Statement of the problem

Let X be a smooth quasi-projective 3-fold over $\mathbb{C}$. Our ultimate aim is to classify all curves $(C \subset X)$ which admit a Mori extraction $\sigma\colon (E \subset Y) \to (C \subset X)$.

It follows from Proposition 1.9 that if a Mori extraction from a curve $(C \subset X)$ exists then it is uniquely isomorphic to the blowup of of the *symbolic power algebra* of the ideal sheaf $I_{C/X} \subset \mathcal{O}_X$:

$$\sigma\colon Y \cong \operatorname{Proj}_X \bigoplus_{n\geq 0} I_{C/X}^{[n]} \to X$$

If C is smooth or, more generally, a local complete intersection (lci) then this is the ordinary blowup of C and has already been studied by Mori and Cutkosky [C]. The interesting case is when C is a curve with a non-lci singularity at a point $(P \in C \subset X)$. Since the problem of classifying Mori extractions from C is local at the point $(P \in X)$, up to an analytic change of variables we may assume that $(P \in X) \cong (0 \in \mathbb{C}^3)$ is affine 3-space.

Now we can classify the curves $(C \subset X)$ admitting a Mori extraction by constructing this variety Y (i.e. by calculating all the generators and relations of the graded ring $\bigoplus I_{C/X}^{[n]}$) and checking explicitly that it has only terminal singularities.

Our method for constructing Y

The general hypersurface section S_X. We assume that the general elephant conjecture holds in full generality, i.e. for Mori extractions with possibly reducible central fibre. This implies that the general hypersurface section S_X containing our curve $(P \in C \subset S_X \subset X)$ has at worst a Du Val singularity at P. We can then study extractions by dividing into cases according to the type of the Du Val singularity $(P \in S_X)$.

A normal form for $(C \subset X)$. Given the type of the Du Val singularity $(P \in S_X)$, in Proposition 3.2 we write down a normal form for the equations of the curve $(C \subset X)$, as the minors of a 2×3 matrix.

Unprojection. We use these equations to write down the ordinary blowup

$$\sigma' \colon (E' \subset Y') \to (C \subset X)$$

of the curve C. Away from the point P the curve C is smooth and this birational map is exactly the Mori extraction from $C \setminus P$. However this cannot be a Mori extraction from C since the exceptional divisor E' is reducible with a 2-dimensional component Π appearing in the fibre above P. We aim to contract Π by using *type I unprojection* $(\Pi \subset Y') \dashrightarrow (Q \in Y'')$ (see §2.2) to give a new variety Y''.

We now check the fibre of Y'' above P. If this fibre is 1-dimensional then we have constructed the unique divisorial extraction of Proposition 1.9. We set $Y = Y''$ and can check to see if Y has terminal singularities by using the equations explicitly. If not, then the central fibre contains a divisor then we continue by trying to unproject this divisor. This can lead to sequences of *serial unprojections* and the construction of some very large graded rings.

0.1.3 Main results

The main results in this thesis come from applying this method to construct the divisorial extraction Y from a curve $(C \subset X)$ in several cases. In most examples this approach seems give a useful method for constructing Y. However checking whether Y has at worst terminal singularities directly from equations is hard.

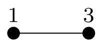
The division into the ADE cases shows behaviour which, unsurprisingly, is very similar to Mori flips. The exceptional cases, types D and E, are very restricted. On the other hand, the type A case leads to many examples of families with complicated but beautiful geometry involving the usual combinatorics from toric geometry, namely Hirzebruch–Jung continued fraction expansions. These constructions are very similar to diptych varieties [BR1].

Type A_1 and A_2 cases

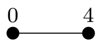
In §3 we compute the easiest examples, when $(P \in S_X)$ is a Du Val singularity of type A_1 or A_2. In particular we obtain the following results:

Theorem 0.1 (See Theorem 3.1 for a more precise statement). *Let S_X be the general hypersurface section through $(P \in C)$ and consider $\widetilde{C}$, the birational transform of C under the minimal resolution $\mu\colon \widetilde{S}_X \to S_X$.*

1. *If $(P \in S_X)$ is Du Val of type A_1 then $\widetilde{C}$ intersects the exceptional locus of $(P \in S_X)$ with multiplicity 3.*
2. *If $(P \in S_X)$ is Du Val of type E_6 then $\widetilde{C}$ intersects the exceptional locus of $(P \in S_X)$ with the multiplicities:*



or



This theorem is proved by explicitly calculating the divisorial extraction Y using the unprojection method explained above. We show that Y is terminal for the curves described in the statement of the theorem and that, for other curves $(C \subset S_X)$, either Y is not terminal or S_X is not the most general hypersurface section through C.

Exceptional cases

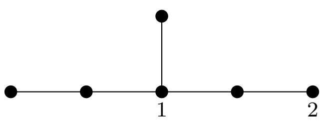
In §4 we give a classification of Mori extractions from a non-lci curve

$$(P \in C \subset S_X \subset X)$$

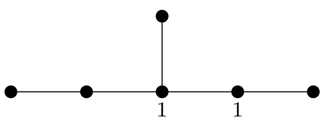
in the cases where $(P \in S_X)$ is a Du Val singularity of type D_{2k}, E_6 or E_7. (The case E_8 is trivially excluded as an E_8 Du Val singularity is factorial, so there are no non-lci curves $(P \in C \subset S_X)$.) In particular we obtain the following results:

Theorem 0.2 (See Theorem 4.1 for a more precise statement). *Let S_X be the general hypersurface section through $(P \in C)$.*

1. *If $(P \in S_X)$ is Du Val of type D_{2k} or E_7 then no Mori extraction from C exists.*
2. *If $(P \in S_X)$ is Du Val of type E_6 consider $\widetilde{C}$, the birational transform of C under the minimal resolution $\mu\colon \widetilde{S}_X \to S_X$. Then $\widetilde{C}$ intersects the exceptional locus of $(P \in S_X)$ with the multiplicities:*



or



Again, this theorem is proved by explicitly calculating the divisorial extraction Y using the unprojection method explained above. In the first case we can prove that Y has non-isolated singularities and hence is not terminal. In the second case we find the two examples but manage to prove that in worse cases Y also has non-terminal singularities.

Our explicit approach of using unprojection to calculate equations and investigating the singularities of Y directly could, in theory, be used to settle the D_{2k+1} case. However these calculations quickly become very complicated. We give a D_5 example in §4.3.1 but we do not touch the general case.

The general type A cases

For $n \geq 3$, the type A_n case appears to consist of many infinite families of Mori extractions and the graded ring defining Y can be very complicated and exist in arbitrarily large codimension. Therefore it will not be feasible to use our explicit approach to obtain a complete description, although we can use it to compute very many interesting examples. We do give a complete treatment in the case of one of these families.

As explained in §6.1, a type A extraction $\sigma \colon Y \to X$ can also be described as the $\mathbb{Q}$-Gorenstein smoothing of a general hyperplane section. In other words, let $H_X \in |\mathfrak{m}_P|$ be a general hyperplane through $(P \in H_X)$ and let $H_Y = \sigma^{-1}(H_X)$ be the birational transform of H_X on Y. Let $Z = \sigma^{-1}(P)_{\mathrm{red}}$ be the reduced central fibre. Then we have the diagram:

$$\begin{array}{ccc} (Z \subset H_Y) & \xrightarrow{\mathbb{Q}\text{-smoothing}} & (E \subset Y) \\ \big\downarrow & & \big\downarrow \sigma \\ (P \in H_X) & \longrightarrow & (C \subset X) \end{array}$$

Main construction. We treat the case when $Z \cong \mathbb{P}^1$ is irreducible and H_Y has normal rational singularities with only one singularity of index $r > 1$, $(Q \in H_Y)$ which is a simple T-singularity. We classify such $(Z \subset H_Y)$ in Lemma 6.7 to get a family of neighbourhoods depending on two integers: $m \geq 2$ and $k \geq 1$. Then in Lemma 6.9 we determine a family of curves $(C \subset S_X)$, also depending on the same m, k, given by the image of the divisorial contraction obtained by the $\mathbb{Q}$-Gorenstein smoothing of the corresponding $(Z \subset H_Y)$.

Then our main construction is to show that, for this family of curves $(C \subset S_X)$, the Mori extraction Y can be constructed explicitly by a sequence of serial type I unprojections.

Theorem 0.3. *For the family of curves* $(C \subset S_X)$ *described in Lemma 6.9 the divisorial extraction* $Y = \operatorname{Proj}_X \bigoplus I_{C/X}^{[n]}$ *is given by a Gorenstein ring that can be constructed as a sequence of serial unprojections*

$$(Y_1 \supset D_1) \dashleftarrow \cdots \dashleftarrow (Y_{n-1} \supset D_{n-1}) \dashleftarrow (Y_n \ni Q) = Y$$

starting from the ordinary blowup $Y_1 = \operatorname{Proj}_X \bigoplus I_{C/X}^n$ *of* $(C \subset X)$*. At each stage we can make a type I unprojection of the divisor* $(D_\alpha \subset Y_\alpha)$ *to get* $Y_{\alpha+1}$*.*

The proof of this theorem is contained in §6.3.2 and an outline of the proof can also be found there.

Cluster algebras. From computing lots of large examples of type A extractions it is clear that there is a connection to cluster algebras, like that noticed by Hacking, Tevelev & Urzúa [HTU] for type A flips following Mori [M2]. In §7 we start a description in these terms. This looks like a promising way of treating the type A case in general, without having to wade through large and complicated calculations.

These type A extractions should also fit into the more general framework of Gross, Hacking & Keel's [GHK] deformation of a cycle of 2-planes.

0.2 Notation

We always work over $\mathbb{C}$, the field of complex numbers.

The usual situation In this thesis *the usual situation* will refer to the following divisorial extraction $\sigma\colon Y \to X$,

$$\begin{tikzcd}
 & & & S_Y \arrow[dr] \arrow[ddd] & \\
Z \arrow[urrr] \arrow[r] \arrow[drr] \arrow[dd] & E \arrow[rrr] \arrow[dd] & & & Y \arrow[dd, "\sigma"] \\
 & & H_Y \arrow[urr] \arrow[dd] & & \\
P \arrow[r] \arrow[drr] & C \arrow[rr] & & S_X \arrow[r] & X \\
 & & H_X \arrow[urr] & &
\end{tikzcd}$$

where the non-vertical arrows are inclusions of subvarieties and the vertical arrows are all induced by σ.

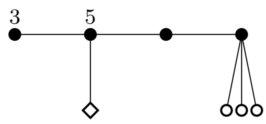
- X, Y are quasiprojective $\mathbb{Q}$-factorial 3-folds over $\mathbb{C}$. Usually X, Y will have at worst terminal singularities and, unless otherwise stated, X will be smooth. This is morphism is considered to be local at $(P \in X)$.

- E is the exceptional divisor of σ, $C = \sigma(E)$ is a curve with a singularity at P and $Z = \sigma^{-1}(P)_{\text{red}}$ is the reduced central fibre.

- S_X is a general hypersurface section containing C, S_Y is the birational transform of S_X and S_X, S_Y have at worst Du Val singularities. The restriction $\sigma \colon S_Y \to S_X$ is a partial crepant resolution.

- $H_X \in |\mathfrak{m}_P|$ is a general hyperplane section through P and $H_Y = \sigma^{-1}(H_X)$ is the birational transform of H_X to Y.

Intersection diagrams At several points we refer to a configuration $(C \subset S)$ of rational curves on a (singular) surface by using a diagram $\Delta = \Delta(C \subset S)$. This diagram Δ is the dual intersection graph of a simple normal crossings resolution $\mu\colon (\widetilde{C} \subset \widetilde{S}) \to (C \subset S)$. In such a diagram, circles ($\bullet, \circ$) denote complete rational curves and diamonds ($\diamond$) denote non-complete curves. White nodes ($\circ$,$\diamond$) denote components of C and black nodes ($\bullet$) denote μ-exceptional curves. A label on a circle corresponding to a complete curve Γ refers to the negative self-intersection number $-(\Gamma \cdot \Gamma)_{\widetilde{S}}$. Curves corresponding to unlabelled black (resp. white) nodes are assumed to have self-intersection -2 (resp. -1). For example the diagram



represents a curve with four components, three complete curves and one non-complete curve, meeting a cyclic quotient singularity $\frac{1}{36}(1, 13)$ in the prescribed way.

Chapter 1

The Mori Category

We start with a brief review of the minimal model program (MMP) and then progress to a more detailed discussion of *the Mori category*—i.e. the category of normal quasi-projective 3-dimensional algebraic varieties over $\mathbb{C}$, with at worst terminal $\mathbb{Q}$-factorial singularities.

1.1 The minimal model program

One of the greatest and most influential mathematical achievements of the late 20th century has been the introduction of the MMP—which was established with important contributions from Mori, Reid, Shokurov, Kawamata and Kollár amongst many others. The aim of this far-reaching program is to extend the notion of a minimal model, from the theory of algebraic surfaces, to higher dimensions. The MMP is known to hold in a large number of circumstances, although some very difficult outstanding problems (e.g. the abundance conjecture) remain when the MMP is stated in its greatest generality.

Minimal models

The first step on the road to formulating the MMP is to make the correct generalisation of a minimal model. A naive definition of a minimal surface is a smooth projective surface X containing no -1-curves. This statement won't generalise to higher dimensions. However, for those surfaces which are not uniruled,[1] an equivalent statement is that K_X is *nef*, i.e. that $K_X \cdot C \geq 0$ for every effective curve ($C \subset X$). We take this to be our definition of a *minimal* algebraic variety.

[1]If X is uniruled then we must consider *Mori fibre spaces.*

Singularities

One important realisation, which is completely fundamental to the MMP, is that minimal models can (indeed often *must*) be singular. Therefore singular varieties play a crucial role in higher dimensional birational geometry, although this introduces a number of problems which must be circumvented before we can proceed any further.

First, in order to be able to talk about singular minimal models we must be able to define the canonical divisor class K_X. If X is a normal quasi-projective variety then the smooth locus ($X^0 \subset X$) has complement of codimension ≥ 2 so taking the closure of K_{X^0} inside X gives rise to a (Weil) divisor class on X, which we define to be K_X.

Second, we would like to be able to calculate intersection numbers on X, particularly[2] against K_X. Therefore we must consider varieties with at worst $\mathbb{Q}$-factorial singularities.

Definition 1.1. A variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor ($D \subset X$) is $\mathbb{Q}$-Cartier, i.e. rD is a Cartier divisor for some $r \in \mathbb{Z}_{>0}$.

Third, by contracting K_X-negative curves (those against which K_X fails to be nef—the analogue of contracting a -1-curve in this context) we can introduce singularities, even X is smooth. Therefore we introduce the notion of terminal and canonical singularities.

Definition 1.2 (Reid)**.** A variety X is said to have *terminal* (resp. *canonical*) singularities if for any (or equivalently, every) resolution of singularities $\mu\colon \widetilde{X} \to X$ we have

$$K_{\widetilde{X}} = \mu^* K_X + \sum_{i=1}^{n} a_i E$$

where the sum runs over all the exceptional divisors of μ and $a_i \in \mathbb{Q}_{>0}$ (resp. $a_i \in \mathbb{Q}_{\geq 0}$). The coefficient a_i is called the *discrepancy* of the divisor E_i over X.

The importance of this definition is that a variety with at worst terminal (or canonical) singularities still has at worst terminal or canonical singularities after contracting a divisor swept out by K_X-negative curves.

[2] One can work with more general cases where K_X is not $\mathbb{Q}$-Cartier by considering a "small adjustment" Δ such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor class, leading to the *log MMP* and the notion of klt, lt, dlt, lc etc. singularities. For our purposes it will not be necessary to consider this.

The Cone of curves

Mori's great insight was to define the *cone of curves* $\overline{\mathrm{NE}}(X)$ inside $N_1(X)_{\mathbb{R}}$, the space of 1-cycles of X. $\overline{\mathrm{NE}}(X)$ is given by the closure of $\mathrm{NE}(X)$, the cone spanned by classes of effective curves on X, and is the dual cone to the *nef cone* $\mathrm{Nef}(X)$ in the dual vector space $N^1(X)_{\mathbb{R}}$.

Through a series of key theorems it is proved that $\overline{\mathrm{NE}}(X)$ is a locally polyhedral cone in the half-space $\{[C] : -K_X \cdot C \geq 0\} \subset N_1(X)_{\mathbb{R}}$ and genuinely a finite polyhedral cone away from the boundary plane $\{[C] : -K_X \cdot C = 0\}$. Moreover for each face $F \subset \overline{\mathrm{NE}}(X)$ which lies strictly inside this half-space there is a unique contraction morphism $\phi_F \colon X \to X'$ such that $\phi_{F*}\mathcal{O}_X = \mathcal{O}_{X'}$ and a curve $(C \subset X)$ is contracted by ϕ_F if and only if $[C] \in F$.

Minimal model program

These results now give the foundation for a potential algorithm to find a minimal model of X.

1. If K_X is nef then X is minimal, so stop. If not then pick a K_X-negative extremal ray $\rho \subset \overline{\mathrm{NE}}(X)$.

2. The contraction morphism $\phi_\rho \colon X \to X'$ leads to a trichotomy:

   (a) The curves contracted by ϕ_ρ span the whole of X, so that $\dim X > \dim X'$. This is called a *fibre contraction* and X is called a *Mori fibre space* over X'. We take take this fibration X/X' to be our 'minimal model' and stop.

   (b) The curves contracted by ϕ_ρ span a subvariety $(Z \subset X)$ of codimension 1, i.e. Z is a divisor. This is called a *divisorial contraction*. In this case X' has $\mathbb{Q}$-factorial terminal singularities, so we replace X with X' and go back to step 1.

   (c) The curves contracted by ϕ_ρ span a subvariety $(Z \subset X)$ of codimension 2 or smaller. This is called a *small* (or *flipping*) *contraction*. In this case X' is no longer $\mathbb{Q}$-factorial and we leave the category we were working in. We look for a *flip* $f \colon X \dashrightarrow X^+$, such that X^+ has $\mathbb{Q}$-factorial, terminal singularities, the exceptional locus of X^+ over X' has codimension ≥ 2 and K_{X^+} is relatively ample over X'. Then we replace X with X^+ and return to step 1.

The two main difficulties with this 'algorithm' lie in step 2(c). The existence of the flip X^+ is not clear and neither is it obvious that this process stops. We can only have finitely many divisorial contractions (since the Picard rank of X drops under a divisorial contraction) however termination of flips is still an open problem.[3]

In this thesis we are primarily interested in 3-folds, in which case the MMP is known to work. For 3-folds the termination of flips was proved by Shokurov and the existence of flips was proved by Mori [M1].

1.2 Du Val singularities

The Du Val singularities are a very famous class of surface singularities which turn out to play an important role in the geometry of terminal 3-folds. They can be defined in many different equivalent ways, some of which are listed here.

Definition 1.3. Let $(P \in S)$ be the germ of a surface singularity. Then $(P \in S)$ is called a *Du Val singularity* if it is given, up to local analytic isomorphism, by one of the following equivalent conditions.

1. A hypersurface singularity $\big(0 \in V(f) \subset \mathbb{C}^3\big)$, where f is one of the equations of Table 1.1, given by an ADE classification.

2. A quotient singularity $\big(0 \in \mathbb{C}^2/G = \operatorname{Spec} \mathbb{C}[u,v]^G\big)$, where $G \subset \mathrm{SL}(2,\mathbb{C})$ is a finite subgroup acting on $\mathbb{C}^2$.

3. A rational double point, i.e. the minimal resolution

$$\mu \colon (E \subset \widetilde{S}) \to (P \in S)$$

has exceptional locus $E = \bigcup E_i$ a tree of -2-curves with intersection graph given by the corresponding ADE Dynkin diagram.

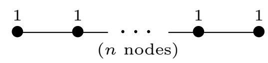
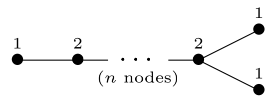
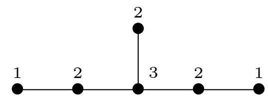
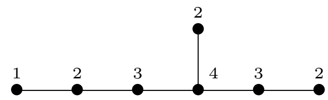
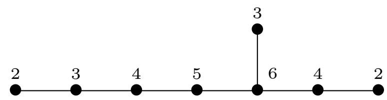
4. A canonical surface singularity. For a surface singularity this is equivalent to $(P \in S)$ having a crepant resolution, i.e. $K_{\widetilde{S}} = \mu^* K_S$.

5. A simple hypersurface singularity, i.e. $\big(0 \in V(f) \subset \mathbb{C}^3\big)$ such that there exist only finitely many ideals $I \subset \mathcal{O}_{\mathbb{C}^3}$ with $f \in I^2$.

See for example [R5] for details of the equivalence of conditions (1)-(4) and [Y] for details of (5).

[3]Indeed, to prove that the MMP works in higher dimensions it is sufficient to prove that flips terminate, since Hacon & M$^{\mathrm{c}}$Kernan have proved that the existence of flips in dimension $n+1$ follows from the termination of flips in dimension n.

Table 1.1: Types of Du Val singularities

Type	Group G	Equation f	Dynkin diagram
A_n	cyclic	$x^2 + y^2 + z^{n+1}$	
D_n	binary dihedral	$x^2 + y^2 z + z^{n-1}$	
E_6	binary tetrahedral	$x^2 + y^3 + z^4$	
E_7	binary octahedral	$x^2 + y^3 + yz^3$	
E_8	binary isocahedral	$x^2 + y^3 + z^5$	

The numbers decorating the nodes of the Dynkin diagrams in Table 1.1 have several interpretations. For example, each node corresponds to the isomorphism class of a nontrivial irreducible representation of G with dimension equal to the label. Another way these numbers arise is as the multiplicities of the components E_i of E in the *fundamental cycle*[4] $(\Sigma \subset \widetilde{S})$.

1.3 Terminal 3-fold singularities

One of the most useful lists at our disposal is Mori's list of 3-fold terminal singularities (see [R1] for a nice introduction). Terminal singularities always exist in codimension ≥ 3 so in the case of 3-folds they are all isolated singular points $(P \in X)$. They are classified according to the *index* of the singularity—the least $r \in \mathbb{Z}_{>0}$ such that rD is Cartier, given any Weil divisor D passing through $(P \in X)$.

As shown by Reid, the index 1 (or Gorenstein) terminal singularities are exactly the *compound Du Val* (cDV) singularities, i.e. isolated hypersurface singularities of the form

$$0 \in \big(f(x,y,z) + tg(x,y,z,t) = 0\big) \subset \mathbb{C}^4_{x,y,z,t}$$

[4] That is, the unique minimal effective 1-cycle such that $\Sigma \cdot E_i \leq 0$ for every component E_i of E.

where f is the equation of a Du Val singularity.

The other cases are the non-Gorenstein singularities. These can be described as cyclic quotients of cDV points by a cyclic covering trick described in [R1] §3.6. For example, a singularity of type cA/r denotes the quotient of a type cA singularity

$$\bigl(xy + f(z^r, t) = 0\bigr) \subset \mathbb{C}^4_{x,y,z,t} \,/\, \tfrac{1}{r}(a, r-a, 1, 0)$$

where $\frac{1}{r}(a, r-a, 1, 0)$ denotes the $\boldsymbol{\mu}_r$ group action $(x, y, z, t) \mapsto (\varepsilon^a x, \varepsilon^{r-a} y, \varepsilon z, t)$, for a primitive rth root of unity ε. The general elephant of this singularity is given by an r-to-1 covering $A_{n-1} \to A_{rn-1}$. A full list can be found in [KM] p. 541.

1.4 Extremal neighbourhoods

We want to study the kind of contraction morphism $\phi_\rho \colon Y \to X$ that can arise in the Mori category from the contraction of an extremal ray $\rho \subset \overline{\mathrm{NE}}(Y)$, as in §1.1. We choose to study this question locally on X which leads us to the notion of a 3-fold neighbourhood.

There are two landmark papers on 3-fold flipping contractions: Mori [M1] and Kollár & Mori [KM]. Although the primary focus of both these papers is on flips, much of the general theory that they establish for 3-fold neighbourhoods is relevant for divisorial contractions as well.

Definition 1.4. A *3-fold neighbourhood* is a proper birational morphism

$$\sigma \colon (Z \subset Y) \to (P \in X)$$

such that

1. X and Y are 3-dimensional quasiprojective $\mathbb{Q}$-factorial (analytic or) algebraic varieties,
2. $-K_Y$ is a σ-ample $\mathbb{Q}$-Cartier divisor and
3. $Z = \sigma^{-1}(P)_{\mathrm{red}}$, the reduced *central fibre*, is either a complete curve (not necessarily irreducible) or a prime divisor.

Technically speaking we should really consider Y (resp. X) as a formal scheme along Z (resp. P), however in practice we simply assume that they are affine, possibly after an analytic change of variables.[5] In particular, as we are primarily interested

[5] This is justified since, in any of our later calculations of a divisorial extraction $\sigma \colon Y \to X$ from a curve $(P \in C \subset X)$, the equations of Y are completely determined (up to a choice of some coefficients) by the image of the equations of C in $\mathcal{O}_X/\mathfrak{m}_P^N$, for large N.

in this thesis with the case where X is smooth, we often implicitly assume that $(P \in X) \cong (0 \in \mathbb{C}^3)$.

As in §1.1, if $\dim Y > \dim X$, then the contraction is a *fibre contraction* and Y is a Mori fibre space over X. If $\dim Y = \dim X$ then σ is either a *flipping contraction*, if the exceptional locus of σ is 1-dimensional, or a *divisorial contraction*, if σ contracts an exceptional divisor. From now on we assume that we are in either the divisorial or the flipping case.

We write S_Y for a general member of $|-K_Y|$ and let $S_X = \sigma(S_Y) \in |-K_X|$. Also write $H_X \in |\mathfrak{m}_P|$ for a general hyperplane passing through $(P \in X)$ and $H_Y = \sigma^{-1} H_X$ for the birational transform of H_X to Y.

Definition 1.5. We call a neighbourhood

1. *extremal* if all the components of Z lie in the same ray of the Mori cone $\rho \subseteq \overline{\mathrm{NE}}(Y)$,
2. *irreducible* if the central fibre Z is irreducible,
3. *normal* (resp. *non-normal*) if H_Y has normal (resp. non-normal) singularities,
4. *non-semistable* (resp. *semistable*) if $\sigma\colon S_Y \to S_X$ is (resp. is not) an isomorphism.

Remark 1.6. Note that there are some differences over the use of these terms in the literature. For instance, Tziolas' [Tz3] definition of an 'extremal' neighbourhood implicitly assumes that the central fibre is irreducible, although examples of reducible extremal neighbourhoods certainly exist, even when C is a smooth curve, see e.g. [KM] (4.7.3.2.1), (4.10.2).

For reducible flipping contractions one can factorise $\sigma\colon Y \to X$ analytically into irreducible flipping contractions (see [K2] Proposition 8.4). However this is not always possible for divisorial contractions so we should also study reducible neighbourhoods. It is my hope that the techniques in this thesis will eventually be able to describe all (reducible) flipping and divisorial neighbourhoods without the need to factor analytically.

1.4.1 Divisorial contractions

Spelling out Definition 1.4 in the case of divisorial contractions we have the following.

Definition 1.7. A projective birational morphism $\sigma\colon Y \to X$ is called a *divisorial contraction* if

1. X and Y are quasiprojective $\mathbb{Q}$-factorial (analytic or) algebraic varieties,

2. there exists a unique prime divisor E on Y such that $\Gamma = \sigma(E)$ has codimension at least 2 in X,

3. σ is an isomorphism outside of E,

4. $-K_Y$ is σ-ample and the relative Picard number is $\rho(Y/X) = 1$.

Given $(\Gamma \subset X)$ we will also call any such $\sigma\colon Y \to X$ a *divisorial extraction* from Γ. Moreover, if both X and Y have terminal singularities, so that this is a map in the Mori category of terminal 3-folds, then we call σ a *Mori contraction/extraction.*

Known results

For 3-folds, divisorial contractions clearly fall into two cases:

1. $\Gamma = P$ is a point (equivalently the central fibre Z is a divisor),

2. $\Gamma = (P \in C)$ is a curve (equivalently Z is a curve).

The first case has been studied intensively and is completely classified if $(P \in X)$ is a non-Gorenstein singularity. This follows from the work of a number of people—Corti, Kawakita, Hayakawa and Kawamata amongst others.

In particular Kawamata [K1] classified the case when the point $(P \in X)$ is a terminal cyclic quotient singularity. In this case, there is a unique divisorial extraction given by a weighted blowup of the point P. In particular, if there exists a Mori extraction to a curve $(C \subset X)$, then C cannot pass through any cyclic quotient points on X.

In either case, Mori and Cutkosky classify Mori contractions when Y is Gorenstein. In particular, Cutkosky's result for a curve C is the following.

Theorem 1.8 (Cutkosky [C])**.** *Suppose $\sigma\colon (E \subset Y) \to (C \subset X)$ is a Mori contraction where Y has at worst Gorenstein (i.e. index 1) singularities and C is a curve. Then*

1. *C is a reduced, irreducible, local complete intersection curve in X,*

2. *X is smooth along C,*

3. *σ is isomorphic to the blowup of the ideal sheaf $I_{C/X} \subset \mathcal{O}_X$,*

4. *Y only has cA type singularities and*

5. *a general hypersurface section* $(C \subset S_X \subset X)$ *is smooth.*

In the second case, Tziolas [Tz1, Tz2, Tz3, Tz4] classified irreducible Mori extractions when C is a smooth curve passing through a cDV point $(P \in X)$.

1.5 The general elephant conjecture

A *general elephant* $S_Y \in |-K_Y|$ (i.e. a general anticanonical divisor) which is not too singular automatically has a trivial canonical class by the adjunction formula. Indeed Reid's *general elephant conjecture* states that, given a terminal (divisorial or flipping) contraction $\sigma\colon Y \to X$, the general elephant $S_Y \in |-K_Y|$ and $S_X = \sigma(S_Y) \in |-K_X|$ should have at worst Du Val singularities.[6] Moreover, the restriction $\sigma\colon S_Y \to S_X$ should be a partial crepant resolution.

This is proved by Kollár & Mori [KM] for *irreducible* extremal neighbourhoods (i.e. when the central fibre Z is irreducible). In most of the examples constructed in this thesis Z is reducible.

Note that C is contained in S_X, although S_X may not be the most general hypersurface section containing C. The fact that $(P \in S_X)$ is at worst a Du Val singularity implies that a general hypersurface section also has at worst a Du Val singularity at P. The construction of the divisorial extraction $\sigma\colon Y \to X$ (i.e. the equations and singularities of Y) depends upon a general hypersurface section rather than an anticanonical section. Therefore we assume that S_X is the general hypersurface section through C and that S_Y is the birational transform of S_X on Y. Even though it is an abuse of terminology, we will call this a *general elephant.*

Through out this thesis we will therefore assume that we are in the setting of 'the usual situation' §0.2, considering an inclusion of varieties

$$(P \in C \subset S_X \subset X)$$

where $(P \in C)$ is a (non-lci) curve singularity, $(P \in S_X)$ is a general Du Val hypersurface section and $(P \in X) \cong (0 \in \mathbb{C}^3)$ is smooth.

1.6 Uniqueness of Mori extractions

Let $I \subset \mathcal{O}_{\mathbb{C}^n}$ be a prime ideal in a polynomial ring. Recall that the *nth symbolic power* $I^{[n]}$ of I is defined to be the I-primary component of I^n. By a theorem of Zariski and Nagata,[7] if $(Z = V(I) \subset \mathbb{C}^n)$ then $I^{[n]}$ can be defined by the equivalent

[6] Note that it is S_Y that is general. If σ is divisorial then S_X is not necessarily general.

[7] Which holds over any field k, algebraically closed of characteristic zero, not just $\mathbb{C}$.

statement

$$I^{[n]} = \{f \in \mathcal{O}_{\mathbb{C}^n} : \mathrm{ord}_Z(f) \geq n\}$$

where $\mathrm{ord}_Z(f)$ is the order of vanishing of f along Z. The *symbolic power algebra* of I is defined to be the graded $\mathcal{O}_{\mathbb{C}^n}$-algebra $\bigoplus_{n\geq 0} I^{[n]}$.

Proposition 1.9 (cf. [KM] Theorem 4.9, [Tz1] Proposition 1.2)**.** *Suppose that $\sigma\colon Y \to X$ is a divisorial contraction that contracts a divisor E to a curve C, that X and Y are normal and that X has isolated singularities. Suppose further that σ is the blowup over the generic point of C in X and that $-E$ is σ-ample. Then $\sigma\colon Y \to X$ is uniquely determined and isomorphic to the blowup of the symbolic power algebra of $I_{C/X}$:*

$$\mathrm{SymBl}_C\colon \ \mathrm{Proj}_X \bigoplus_{n\geq 0} I_{C/X}^{[n]} \to X$$

Proof. Pick a relatively ample Cartier divisor class D on Y which must be a rational multiple of $\mathcal{O}_Y(-E)$. Then

$$Y = \mathrm{Proj}_X \bigoplus_{n\geq 0} H^0\big(Y, \mathcal{O}_Y(nD)\big)$$

and, up to truncation, this is the ring $\bigoplus H^0\big(Y, \mathcal{O}_Y(-nE)\big)$.

Now the result follows from the claim that $\sigma_*\mathcal{O}_Y(-nE)$ is the nth symbolic power of $I_{C/X}$. This is clear at the generic point of C, since we assume it is just the blowup there. But $\sigma_*\mathcal{O}_Y = \mathcal{O}_X$ is normal and $\mathcal{O}_Y(-nE) \subset \mathcal{O}_Y$ is the ideal of functions vanishing n times on E outside of $\sigma^{-1}(P)$. So $\mathcal{O}_X/\sigma_*\mathcal{O}_Y(-nE)$ has no associated primes other than C and this proves the claim. □

Remark 1.10. Suppose that $\sigma\colon Y \to X$ is a terminal divisorial contraction. By Mori's result, Y is the blowup over the generic point of C and we are in the setting of the theorem. Therefore a terminal contraction is unique if it exists, although there may be many more canonical contractions to the same curve.

Remark 1.11. Of course, given a general curve C in a 3-fold X there is no reason that we should expect the symbolic power algebra $\bigoplus I_{C/X}^{[n]}$ to be finitely generated. Indeed, Goto, Nishida & Watanabe [GNW] prove that the symbolic power algebra of the monomial curve $C_{(25,29,72)} \subset \mathbb{C}^3$ parameterised by (t^{25}, t^{29}, t^{72}) is *not* finitely generated.

Here the existence of a Du Val general elephant is crucial. Our curve C is a $\mathbb{Q}$-Cartier divisor in S_X, so there is an integer r such that $(rC \subset S_X \subset X)$ is lci. Therefore the rth Veronese subring $\bigoplus_{n\geq 0} I_{C/X}^{[nr]}$ is finitely generated. But our original ring $\bigoplus I_{C/X}^{[n]}$ is integral over this and, in particular, finitely generated.

From Proposition 1.9, it is easy to see that Cutkosky's result, Theorem 1.8, holds for divisorial extractions, as well as contractions.

Lemma 1.12. *Suppose that C is a local complete intersection curve in a 3-fold X and that X is smooth along C. Then a Mori extraction exists iff C is reduced, irreducible and a general hypersurface section $(C \subset S_X)$ is smooth.*

Proof. By Proposition 1.9, if a Mori extraction $\sigma\colon Y \to X$ exists then σ is isomorphic to the symbolic blowup of the ideal $I_{C/X}$. As C is lci then, locally at a point $(P \in C \subset X)$, C is defined by two equations f, g. Hence Y is given by

$$Y = \{f\eta - g\xi = 0\} \subset X \times \mathbb{P}^1_{(\eta:\xi)} \to X$$

If both $f, g \in \mathfrak{m}_P^2$ then at any point in the central fibre $(Q \in Z)$ the equation defining Y is contained in $\mathfrak{m}_Q^2$. Therefore Y is singular along Z and hence not terminal. So at least one of f, g is the equation of a smooth hypersurface, say $f \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$. Now Y is smooth along Z except for a possible cA type singularity at the point $P_\xi \in Y$, where all variables except ξ vanish. $\square$

Chapter 2

Graded rings

As we have seen, to study Mori Theory explicitly we must construct graded rings—in our case the graded symbolic power algebra of a curve C in a 3-fold X.

Remark 2.1. We make the following caveats about graded rings:

1. Unlike some authors we do not require a graded ring $R = \bigoplus_{i \geq 0} R_i$ to be generated over R_0 by R_1. Instead our graded rings will define varieties embedded in weighted projective space. The main advantage of this being that the codimension (and hence the number of equations) of the ring remains small.

2. The base of the ring will not necessarily be a field, i.e. we don't assume that $R_0 = \mathbb{C}$. Usually the rings we will consider are defined over $R_0 = \mathcal{O}_X$ the coordinate ring of our smooth 3-fold X, i.e. $\mathcal{O}_X = \mathbb{C}[x, y, z]$.

3. Graded rings will $\mathbb{Z}$-graded, but not necessarily always in degrees ≥ 0. See §2.3 for more discussion of this.

2.1 Gorenstein rings

Gorenstein rings appear as a large number of examples of explicit constructions coming from Mori theory. For example, if X is a projective $\mathbb{Q}$-Fano 3-fold with at worst terminal $\mathbb{Q}$-factorial singularities and Picard rank 1, the anticanonical ring $\bigoplus_{n \geq 0} H^0(X, -nK_X)$ is known to be Gorenstein.

Let $I \subset \mathcal{O}$ be an ideal in a regular local ring and consider the ring $R = \mathcal{O}/I$. Recall that R is called *Cohen-Macaulay* if it has a minimal free resolution

$$R \leftarrow R_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_{c-1} \leftarrow R_c \leftarrow 0$$

of length $c = \operatorname{codim}_{\mathcal{O}} I$. The module R_c is called a *canonical module* for R and is usually denoted ω_R. If $\omega_R = \mathcal{O}(-k)$ for some $k \in \mathbb{Z}$ then we call R *Gorenstein.* In particular, there is a pairing on this resolution $R_i \cong R_{c-i}^{\vee}(-k)$ coming from Serre duality for ω_R. Amongst other things, this causes the numerator of the Hilbert series of R to have palindromic symmetry.

Remark 2.2. For a local Cohen-Macaulay ring R, a canonical module ω_R satisfies the condition that there exists a non-zerodivisor $x \in R$ such that $\omega_R/x\omega_R$ is a canonical module for $R/(x)$. Therefore another way of defining a Gorenstein ring is to make the definition inductive, as in [E2] §21.3, i.e. R is Gorenstein if there exists a non-zerodivisor $x \in R$ such that $R/(x)$ is Gorenstein (with the appropriate definition for rings of dimension 0).

Remark 2.3. Although we have only defined Gorenstein local rings, the definition follows over to graded rings by the slogan "graded rings are a particular case of local rings." See the discussion in [PaR] §2.4.

2.1.1 Gorenstein rings in low codimension

There are nice structure theorems for Gorenstein rings of codimension ≤ 3. Serre proved that Gorenstein rings in codimension ≤ 2 are complete intersections. In codimension 3 we have the Buchsbaum–Eisenbud theorem [BE], which states that, for a Gorenstein ring $R = \mathcal{O}/I$ of codimension 3, the equations of R are given by Pfaffians. In particular, as R is Gorenstein, we have a resolution of the form

$$R \leftarrow \mathcal{O} \leftarrow \mathcal{O}^{2k+1} \xleftarrow{\phi} \mathcal{O}^{2k+1} \leftarrow \mathcal{O} \leftarrow 0$$

and ϕ is given by a skew-symmetric $(2k+1) \times (2k+1)$ matrix, by the pairing coming from Serre duality. Then the ideal I is generated by the $2k \times 2k$ Pfaffians of ϕ.

In practical cases it is usually always possible to take 5×5 matrices, in which case the 4×4 Pfaffians of ϕ are the five equations given by:

$$\phi = \begin{pmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{23} & a_{24} & a_{25} \\ & & a_{34} & a_{35} \\ & & & a_{45} \end{pmatrix} \qquad \begin{aligned} \mathrm{Pf}_1(\phi) &= a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} \\ -\mathrm{Pf}_2(\phi) &= a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34} \\ \mathrm{Pf}_3(\phi) &= a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24} \\ -\mathrm{Pf}_4(\phi) &= a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23} \\ \mathrm{Pf}_5(\phi) &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \end{aligned}$$

Here $\mathrm{Pf}_i(\phi)$ is the Pfaffian of the 4×4 submatrix obtained by deleting the ith row and column of ϕ.[1]

The structure of Gorenstein rings in codimension 4 and higher is an intriguing open problem.

2.1.2 The hyperplane section principle

As we have seen in Remark 2.2, Gorenstein rings enjoy good properties under taking hyperplane sections.

Let R be a Gorenstein (or even just Cohen-Macaulay) $\mathbb{Z}_{\geq 0}$-graded ring and $h \in R$ a regular homogeneous element of positive degree. We write $R_H = R/(h)$ for the graded ring associated to the hyperplane section $H = V(h)$. Suppose we know a presentation of R_H with generators and relations:

$$R_H = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_n)$$

Then the generators and relations lift to give a presentation of R

$$R = \mathbb{C}[h, x_1, \ldots, x_m]/(f_1 + hg_1, \ldots, f_n + hg_n)$$

for some choice of $g_1, \ldots, g_n \in R$. Similarly all the (higher) syzygies also lift. In particular the generators, relations and syzygies lift to generators, relations and syzygies *in the same degrees*.

In Lemma 6.4 we view this result backwards, giving us a guiding principle to make sure our calculations work correctly.

2.2 Unprojection

It was studying Gorenstein rings in codimension 4 that lead Kustin & Miller to consider unprojection. Unprojection is, in their words, a method for 'constructing big Gorenstein ideals from small ones.' Later, Papadakis & Reid [PaR] redeveloped unprojection into a more general theory suitable for algebraic geometry. Whilst not successful in settling the question of the structure of Gorenstein rings in higher codimension, unprojection has become an indispensable tool in the construction of graded rings.

The general philosophy of unprojection is to start working explicitly with Gorenstein rings in low codimension and successively adjoin new variables with new

[1] For brevity, we omit the diagonal of zeroes and the antisymmetry of ϕ and all other antisymmetric matrices appearing in this thesis.

equations. For more details on unprojection and Tom & Jerry, see e.g. [PaR, BKR, R4].

2.2.1 General theory

In the general setting we consider a codimension 1 subscheme $(D \subset X)$ defined by an ideal $I_D \subset \mathcal{O}_X$ in a Gorenstein local ring $\mathcal{O}_X$. The idea is to write down rational functions with poles along D and adjoin these functions to $\mathcal{O}_X$, along with the relations that they satisfy.

To do this, consider the long exact sequence that arises from applying the functor $\mathrm{Hom}(\,\cdot\,, \omega_X)$ to the short exact sequence:

$$0 \to I_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

From the adjunction formula $\omega_D = \mathrm{Ext}^1(\mathcal{O}_D, \omega_X)$, so this gives:

$$0 \to \omega_X \to \mathrm{Hom}(I_D, \omega_X) \to \omega_D \to 0$$

As $\mathcal{O}_X$ is Gorenstein we have an identification $\mathcal{O}_X \cong \omega_X$ and therefore elements of $\mathrm{Hom}(I_D, \omega_X)$ can be viewed as rational functions on X with poles on D. Therefore we calculate a basis of $\mathrm{Hom}(I_D, \omega_X)$ and the relations that these elements satisfy over $\mathcal{O}_X$. Adjoining these to our ring gives the *unprojection* of $(D \subset X)$.

In a simple case, when D is also Gorenstein, we can write down just one rational function s with a simple pole on D. As described in [PaR], if $\mathcal{O}_X$, $\mathcal{O}_D$ are both Gorenstein then ω_X, ω_D are both 1-dimensional and $\mathrm{Hom}(I_D, \omega_X)$ is 2-dimensional. We take s to be the preimage of a basis element spanning ω_D.

Definition 2.4. If $X = \mathrm{Spec}\,\mathcal{O}_X$ is a Gorenstein local scheme and $(D \subset X)$ a codimension 1 Gorenstein subscheme then the *unprojection* of $(D \subset X)$ is the variety $Y = \mathrm{Spec}\,\mathcal{O}_Y$ given by the graph of s, i.e. $\mathcal{O}_Y = \mathcal{O}_X[s]$ where s is the element described above.

In this special case, when D is also Gorenstein, this is called a *Type I unprojection* (or *Kustin–Miller unprojection*). A good thing that happens for type I unprojection is that this unprojection ring $\mathcal{O}_Y = \mathcal{O}_X[s]$ is also Gorenstein, as proved in [PaR] Theorem 1.5.

Theorem 2.5 ([PaR] Theorem 1.5). *The element $s \in \mathcal{O}_Y$ is a non-zerodivisor and $\mathcal{O}_Y = \mathcal{O}_X[s]$ is a Gorenstein ring.*

Although the definition of unprojection has been given for local rings and local schemes, this theory immediately applies to graded rings and projective schemes by Remark 2.3.

Unprojection can be used in much worse cases, for instance when D nonnormal or X is only Cohen-Macaulay, however in this thesis we will only see Type I unprojections.

2.2.2 Type I unprojection

For a general type I unprojection the relations involving the unprojection variable s can be calculated systematically, e.g. by using the computer, but in practice we can usually compute them by ad hoc methods. Indeed, we use the following trick repeatedly for calculating unprojections throughout this thesis.

Example 2.6 (Cramer's rule trick)**.** Consider a codimension 1 subscheme ($D \subset X$) given as the inclusion of a codimension 3 complete intersection inside a codimension 2 complete intersection. The equations of X must be contained in the ideal $I_D = (x, y, z)$ where the three functions $x, y, z \in \mathcal{O}_X$ are the defining equations of D. Therefore we can write the equations of X as

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ -y \\ z \end{pmatrix} = 0$$

with the minus sign chosen for convenience.

Cramer's rule, from linear algebra, states that any $n \times (n+1)$ matrix annihilates the associated vector of its $n \times n$ minors. In our case it follows that our 2×3 matrix annihilates both x, y, z and the vector of its own minors. Therefore we can adjoin a new variable s corresponding to the rational function given by the ratio of these two vectors. In other words s is the rational function

$$s = \frac{bf - ce}{x} = \frac{af - cd}{y} = \frac{ae - bd}{z}$$

which has a simple pole on D. This gives a Gorenstein ring in codimension 3 defined by five equations which, from the Buchsbaum–Eisenbud theorem in §2.1.1, can be written as the maximal Pfaffians of the matrix:

$$\begin{pmatrix} s & a & b & c \\ & d & e & f \\ & & z & y \\ & & & x \end{pmatrix}$$

Tom & Jerry

Suppose that X is given by the maximal Pfaffians of a 5×5 skew matrix M. In particular X is a Gorenstein variety in codimension 3. In a similar vein to Example 2.6, we might ask how to unproject a (reduced) plane divisor $(D \subset X)$ given as a complete intersection $I_D = (x, y, z, t)$ of codimension 4. It turns out that there are now two distinct ways of embedding D in X, called *Tom & Jerry*, which lead to two different types of unprojection. As conditions on the matrix M, these are given by:

1. Tom_i—all entries of M except the ith row and column belong to I_D,
2. Jer_{ij}—all entries of M in the ith and jth rows and columns belong to I_D.

$$\text{e.g. } \mathrm{Tom}_1 \quad \begin{pmatrix} * & * & * & * \\ & I_D & I_D & I_D \\ & & I_D & I_D \\ & & & I_D \end{pmatrix} \quad \text{and } \mathrm{Jer}_{45} \quad \begin{pmatrix} * & * & I_D & I_D \\ & * & I_D & I_D \\ & & I_D & I_D \\ & & & I_D \end{pmatrix}$$

where entries marked 'I_D' belong to I_D and entries marked '$*$' are arbitrary.

Remark 2.7. Tom & Jerry are useful for describing unprojection ideals in codimension 3 Gorenstein rings, however in our examples we need to consider unprojection divisors which are not necessarily reduced. In more complicated examples than the ones constructed in this thesis Tom & Jerry won't always work (see Remark 6.14(3)).

Serial unprojection

In typical cases $(D \subset X)$, a type I unprojection divisor in a 3-fold X, passes through only isolated ordinary nodal singularities—the points at which D fails to be $\mathbb{Q}$-Cartier. Then the unprojection of D factors as the blowup of the (Weil) divisor D, making a small resolution of the nodes, followed by contraction of the exceptional (Cartier) divisor.

If D passes through a line of ordinary nodes then the unprojection

$$\phi_D \colon (D \subset X) \dashrightarrow (Q \in X')$$

can extract a whole ϕ_D^{-1}-exceptional divisor $(D' \subset X')$ above this line of nodes. Now, by Theorem 2.5, $\mathcal{O}_{X'}$ is Gorenstein ring so we can try to unproject $(D' \subset X')$. This can lead to long chains of *serial type I unprojections*, as in the case of diptych varieties [BR1]. For examples of serial unprojection in this thesis see Big example 1 §6.3.1 or Big example 2 §6.4.2.

2.3 $\mathbb{C}^*$-covers of Mori contractions

Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a $\mathbb{Z}$-*graded* ring and let $R^+ = \bigoplus_{n \geq 0} R_n$ (resp. $R^- = \bigoplus_{n \leq 0} R_n$) be the positively (resp. negatively) graded subring of R. We cannot define $\mathrm{Proj}(R)$ in the usual way.[2] Instead, the view of [R2] is to take the Proj of the $\mathbb{Z}$-graded ring R, not as a scheme, but as a diagram.

$$\begin{tikzcd} Y^- = \mathrm{Proj}_X R^- \arrow[dr] & & Y^+ = \mathrm{Proj}_X R^+ \arrow[dl] \\ & X = \mathrm{Spec}\, R_0 & \end{tikzcd}$$

In other words, this is a variation of GIT quotients for the action of $\mathbb{C}^*$ on $A = \mathrm{Spec}\, R$ with respect to the characters -1, 0 and 1.

The idea proposed by Reid [R2] is to study Mori flips and contractions by writing them in such a way, as the Proj of a $\mathbb{Z}$-graded ring R. Then by a result from folklore, [R2] Revelation 3.3.2, the cover $A = \mathrm{Spec}\, R$ is an affine Gorenstein 4-fold[3] and we should study flips by studying A. The purpose of Brown & Reid's diptych varieties [BR1, BR4] are to act as *key varieties* for some types of Mori flips. A diptych variety V is a slightly fatter version of A equipped with a large torus action. Cutting down V by regular sections and taking the quotient with respect to different characters of this torus we recover whole families of Mori flips and Mori contractions.

Mori contractions

In our case of a Mori extraction from $(C \subset X)$, the relevant $\mathbb{Z}$-graded ring is the *extended symbolic power algebra*. This is the ring R given by the ordinary symbolic power algebra $\bigoplus I_{C/X}^{[n]}$ in non-negative degrees and by a single generator ι in degree -1, corresponding to the inclusion $\iota \colon I_{C/X} \hookrightarrow \mathcal{O}_X$. Then, since $X \cong \mathrm{Proj}_X \mathcal{O}_X[\iota]$, there is only one nontrivial side to the diagram,

[2] For instance R^+ (the irrelevant ideal of a $\mathbb{Z}_{\geq 0}$-graded ring) is no longer an ideal.

[3] Brown proved that A is smooth and *quasi-Gorenstein* in an appendix to his thesis.

$$
\begin{tikzcd}
Y \arrow[dr] & & X \arrow[dl, equal] \\
& X &
\end{tikzcd}
$$

where $Y = \mathrm{Proj}_X \bigoplus I_{C/X}^{[n]}$ as in Proposition 1.9.

Our constructions are Gorenstein rings by Lemma 6.6, and ι appears in R as a simple Type I Gorenstein (un)projection variable. Therefore, to construct Y given R, we can project out ι and take the Proj with respect to the obvious grading on $\bigoplus I_{C/X}^{[n]}$.

Chapter 3

First examples

This chapter contains the first examples of the symbolic blowup ring of Proposition 1.9 in the easiest cases, when $(P \in S_X)$ is a Du Val singularity of type A_1 or A_2. The results of §3.3-3.4 are summed up in the following theorem:

Theorem 3.1. *Suppose that we have $(P \in C \subset S_X \subset X)$ as in the usual situation §0.2. In particular S_X is the* general *hypersurface section containing C. Fix a minimal resolution $\mu\colon (E \subset \widetilde{S}_X) \to (P \in S_X)$ and let $\widetilde{C}$ be the birational transform of C on $\widetilde{S}_X$.*

1. *Suppose that S_X is of type A_1. Then the symbolic blowup of C has a codimension 3 model:*
$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2) \to X$$
In particular, Y has index 2 and $\bigoplus I^{[n]}_{C/X}$ is generated in degrees ≤ 2.
Moreover Y has a $\frac{1}{2}(1,1,1)$ quotient singularity and is terminal elsewhere if and only if $\widetilde{C}$ intersects E with multiplicity 3.

2. *Suppose that S_X is of type A_2. Then the symbolic blowup of C has a codimension 4 model:*
$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2,3) \to X$$
In particular, Y has index 3 and $\bigoplus I^{[n]}_{C/X}$ is generated in degrees ≤ 3.
Moreover Y has a $\frac{1}{3}(1,1,2)$ quotient singularity and is terminal elsewhere if and only if $\widetilde{C}$ intersects $E = E_1 \cup E_2$ with multiplicity $(3,1)$, $(1,3)$, $(4,0)$ or $(0,4)$.

In each other case of a curve $(C \subset S_X)$ contained in an A_1 or A_2 Du Val singularity, either C is contained in a less singular hypersurface (and hence S_X is not general) or Y has non-terminal singularities.

In order to prove this we first need Proposition 3.2 which gives a normal form for the equations of a curve $(C \subset S_X \subset X)$, depending on the type of the Du Val singularity $(P \in S_X)$. We then compute the equations of Y using unprojection and investigate whether Y is terminal by checking the local type of the singularities explicitly.

3.1 Curves in Du Val singularities

Let C be a reduced and irreducible curve passing through a Du Val singularity $(P \in S)$. Consider S as simultaneously being both a hypersurface singularity $\big(0 \in V(f) \subset \mathbb{C}^3\big)$, as in Definition 1.3(1), and a group quotient $\pi \colon \mathbb{C}^2 \to \mathbb{C}^2/G$, as in Definition 1.3(2). Write $S = \operatorname{Spec} \mathcal{O}_S$ where

$$\mathcal{O}_S = \mathcal{O}_X/(f) = (\mathcal{O}_{\mathbb{C}^2})^G, \quad \mathcal{O}_X = \mathbb{C}[x,y,z], \quad \mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[u,v].$$

The aim of this section is to describe the equations of $(C \subset X)$ in terms of some data associated to the equation f and the group G.

A 1-dimensional representation ρ of G

Consider $\big(\Gamma := \pi^{-1}(C) \subset \mathbb{C}^2\big)$, the preimage of C under the quotient map π. Then Γ is a reduced (but possibly reducible) G-invariant curve, giving a diagram:

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d] & \mathbb{C}^2_{u,v} \arrow[d, "\pi"] \\ P \in C \arrow[r, hook] & S \end{tikzcd}$$

As such, Γ is defined by a single equation $\big(V(\gamma) \subset \mathbb{C}^2\big)$ and $\gamma(u,v)$ is called the *orbifold equation* of C. As Γ is G-invariant the equation γ must be G-semi-invariant, so there is a 1-dimensional representation $\rho \colon G \to \mathbb{C}^\times$ such that:

$${}^g\gamma(u,v) = \rho(g)\gamma(u,v), \qquad \forall g \in G$$

Moreover, C is a Cartier divisor (and hence lci in X) if and only if ρ is the trivial representation. Therefore let us restrict attention to nontrivial ρ.

As can be seen from the numbers adorning the Dynkin diagrams in Table 1.1, there are n nontrivial 1-dimensional representations if S is type A_n, three if type D_n, two if type E_6, one if type E_7 and none if type E_8. These possibilities are listed later on in Table 3.1.

A matrix factorisation ϕ of f

Let $\mathrm{Irr}(G)$ be the set of irreducible G-representations $\rho\colon G \to \mathrm{GL}(V_\rho)$. As is well known from the McKay correspondence, the ring $\mathcal{O}_{\mathbb{C}^2}$ has a canonical decomposition as a direct sum of $\mathcal{O}_S$-modules

$$\mathcal{O}_{\mathbb{C}^2} = \bigoplus_{\rho\in\mathrm{Irr}(G)} M_\rho$$

where $M_\rho = V_\rho \otimes \mathrm{Hom}(V_\rho, \mathcal{O}_{\mathbb{C}^2})^G$. In particular if $\dim\rho = 1$ then we see that M_ρ is the unique irreducible summand of $\mathcal{O}_{\mathbb{C}^2}$ of ρ semi-invariants:

$$M_\rho = \left\{h(u,v) \in \mathcal{O}_{\mathbb{C}^2} : {}^g h = \rho(g)h\right\}$$

This is a rank 1 maximal Cohen-Macaulay $\mathcal{O}_S$-module generated by two elements at P.

As shown by Eisenbud [E1], such a module over the ring of a hypersurface singularity has a minimal free resolution which is 2-periodic, i.e. there is a resolution

$$M_\rho \;\leftarrow\; \mathcal{O}_S^{\oplus 2} \;\xleftarrow{\phi}\; \mathcal{O}_S^{\oplus 2} \;\xleftarrow{\psi}\; \mathcal{O}_S^{\oplus 2} \;\xleftarrow{\phi}\; \cdots$$

where ϕ and ψ are matrices over $\mathcal{O}_X$ satisfying:

$$\phi\psi = \psi\phi = f\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

The pair of matrices (ϕ,ψ) is called a *matrix factorisation* of f. In our case ϕ and ψ are 2×2 matrices. It is easy to see that $\det\phi = \det\psi = f$ and that ψ is the adjugate matrix of ϕ. Write $I(\phi)$ for the ideal of $\mathcal{O}_X$ generated by the entries of ϕ (or equivalently ψ).

Write ε_k (resp. ω, i) for a primitive kth (resp. 3rd, 4th) root of unity. In Table 3.1 the possible representations ρ of G and the first matrix ϕ in a matrix factorisation of M_ρ, for some choice of f, are listed. These can be found (up to some row and column operations and change of variables) in [KST] §5.

The notation $\mathbf{D}_n^l$ refers to the case when ρ is the 1-dimensional representation corresponding to the leftmost node in the D_n Dynkin diagram (see Table 1.1) and $\mathbf{D}_n^r$ refers to one of the rightmost pair of nodes. Of course there are are actually two choices of representation we could take for each of the cases $\mathbf{D}_{2k}^r, \mathbf{D}_{2k+1}^r$ and $\mathbf{E}_6$, however we treat each of them as only one case since there is an obvious symmetry of S switching the two types of curve. Similarly for $\mathbf{A}_n^j$ we may assume that $j \leq \frac{n+1}{2}$.

Table 3.1: 1-dimensional representations of G

Type	Presentation of G	$\rho(r), (\rho(s), \rho(t))$	ϕ
$\mathbf{A}_n^j$	$\langle r : r^{n+1} = e \rangle$	ε_{n+1}^j	$\begin{pmatrix} x & y^j \\ y^{n+1-j} & z \end{pmatrix}$
$\mathbf{D}_n^l$	$\left\langle \begin{matrix} r, s, t : \\ r^{n-2} = s^2 = t^2 = rst \end{matrix} \right\rangle$	$1, -1, -1$	$\begin{pmatrix} x & y^2 + z^{n-2} \\ z & x \end{pmatrix}$
$\mathbf{D}_{2k}^r$		$-1, 1, -1$	$\begin{pmatrix} x & yz + z^k \\ y & x \end{pmatrix}$
$\mathbf{D}_{2k+1}^r$		$-1, i, -i$	$\begin{pmatrix} x & yz \\ y & x + z^k \end{pmatrix}$
$\mathbf{E}_6$	$\left\langle \begin{matrix} r, s, t : \\ r^2 = s^3 = t^3 = rst \end{matrix} \right\rangle$	$1, \omega, \omega^2$	$\begin{pmatrix} x & y^2 \\ y & x + z^2 \end{pmatrix}$
$\mathbf{E}_7$	$\left\langle \begin{matrix} r, s, t : \\ r^2 = s^3 = t^4 = rst \end{matrix} \right\rangle$	$-1, 1, -1$	$\begin{pmatrix} x & y^2 + z^3 \\ y & x \end{pmatrix}$

3.1.1 A normal form for $(C \subset X)$

Proposition 3.2. *Suppose that we are given $(P \in C \subset S_X \subset X)$ as in the usual situation §0.2. Let ρ be the representation of G and ϕ be the matrix factorisation of f associated to $(C \subset S_X)$. Then*

1. *the equations of $(C \subset X)$ are given by the minors of a 2×3 matrix*

$$\bigwedge^2 \begin{pmatrix} \phi & \begin{matrix} g \\ h \end{matrix} \end{pmatrix} = 0$$

   for some functions $g, h \in \mathcal{O}_X$.

2. *Suppose furthermore that S_X is a* general *hypersurface section containing C. Then $g, h \in I(\phi)$, where $I(\phi)$ is the ideal generated by the entries of ϕ.*

Proof. Suppose that ρ is a 1-dimensional representation of G. Note that if (ψ, ϕ) is a matrix factorisation for M_ρ, the $\mathcal{O}_S$-module of ρ semi-invariants, then (ϕ, ψ) is a matrix factorisation for $M_{\rho'}$, where ρ' is the representation $\rho'(g) = \rho(g)^{-1}$.

The resolution of the $\mathcal{O}_{\mathbb{C}^2}$-module $\mathcal{O}_\Gamma = \mathcal{O}_{\mathbb{C}^2}/(\gamma)$

$$\mathcal{O}_\Gamma \leftarrow \mathcal{O}_{\mathbb{C}^2} \xleftarrow{\gamma} \mathcal{O}_{\mathbb{C}^2} \leftarrow 0$$

decomposes as a resolution over $\mathcal{O}_S$ to give a resolution of $\mathcal{O}_C$:

$$\mathcal{O}_C \leftarrow \mathcal{O}_S \xleftarrow{\gamma} M_{\rho'} \leftarrow 0$$

Using the resolution of $M_{\rho'}$ we get

$$\mathcal{O}_C \leftarrow \mathcal{O}_S \xleftarrow{(\nu\ -\xi)} \mathcal{O}_S^{\oplus 2} \xleftarrow{\phi} \mathcal{O}_S^{\oplus 2} \xleftarrow{\psi} \cdots$$

where ξ, ν are the two equations defining $(C \subset S_X)$. Now write $\gamma = g\alpha + h\beta$ where α, β are the two generators of M_ρ. We can use the resolution of $\mathcal{O}_S$ as an $\mathcal{O}_X$-module to lift this to a complex over $\mathcal{O}_X$ and strip off the initial exact part to get the resolution

$$\mathcal{O}_C \leftarrow \mathcal{O}_X \xleftarrow{(\nu\ -\xi\ \eta)} \mathcal{O}_X^{\oplus 3} \xleftarrow{\begin{pmatrix} \phi \\ g\ \ h \end{pmatrix}} \mathcal{O}_X^{\oplus 2} \leftarrow 0$$

(possibly modulo some unimportant minus signs). Therefore the equations of the curve $(C \subset X)$ are given as claimed in (1).

To prove Proposition 3.2(2), recall the characterisation of Du Val singularities in Definition 1.3(5) as simple surface singularities. Let $\eta = \det\phi$ and ξ, ν be the three equations of C. We have a $\mathbb{C}^2$-family of hypersurface sections through C given by

$$H_{\lambda,\mu} = \left\{ h_{\lambda,\mu} := \eta + \lambda\xi + \mu\nu = 0 \right\}_{(\lambda,\mu)\in\mathbb{C}^2}$$

and we are assuming that η is general. As the general member $H_{\lambda,\mu}$ is Du Val there are a finite number of ideals $I \subset \mathfrak{m}_P$ such that the general $h_{\lambda,\mu} \in I^2$. As the general section η satisfies $\eta \in I(\phi)^2$ we have that $h_{\lambda,\mu} \in I(\phi)^2$ for general λ, μ. Therefore $g, h \in I(\phi)$. $\square$

Remark 3.3. Whilst the condition $g, h \in I(\phi)$ in Proposition 3.2(2) is necessary for a general section of the curve C to be of the same type of Du Val singularity as S_X it is not normally a sufficient condition.

3.2 A general strategy for constructing Y

We can now describe a general strategy for constructing the divisorial extraction of Proposition 1.9 from a curve $(C \subset S_X \subset X)$ contained in a Du Val general elephant.

By Proposition 3.2, C is defined by the minors of a 2×3 matrix, where all the entries belong to an ideal $I(\phi) \subset \mathcal{O}_X$. Cramer's rule tells us that this matrix annihilates the vector of the equations of C:

$$\begin{pmatrix} \phi & g \\ & h \end{pmatrix} \begin{pmatrix} \nu \\ -\xi \\ \eta \end{pmatrix} = 0$$

Multiplying out these two matrices gives us two syzygies holding between the equations of C and these syzygies define a codimension 2 variety:

$$\sigma' \colon Y' \subset X \times \mathbb{P}^2_{(\eta:\xi:\nu)} \to X$$

Y' is the blowup of the *ordinary power algebra* $\bigoplus_{n \geq 0} I^n_{C/X}$ for $I_{C/X} \subset \mathcal{O}_X$, the ideal of C.

Y' cannot be the divisorial extraction of Theorem 1.9 since the fibre above $(P \in X)$ is not 1-dimensional. Indeed Y' contains the Weil divisor $D = \sigma'^{-1}(P)_{\mathrm{red}} \cong \mathbb{P}^2$, possibly with a non-reduced structure, defined by the ideal $I(\phi)$. Our aim is to construct the divisorial extraction by contracting D. In this case we can proceed by unprojecting $I(\phi)$ as in Example 2.6 to get a new variety Y'' birational to Y'.

We can check what components the central fibre of $\sigma'' \colon Y'' \to X$ has. If it is small then we set $Y = Y''$ and we have constructed the unique divisorial extraction of Proposition 3.2. If not then Y'' contains a new divisor above P and we can try to unproject it. We keep repeating this process until the central fibre is small. If the ring we are trying to construct is finitely generated then eventually we stop.

Whilst this is an effective way of constructing Y explicitly, it is often quite hard to work out whether Y is terminal. If $(C \subset X)$ is allowed to degenerate into a very singular curve then eventually Y will have non-terminal singularities, as the next Lemma demonstrates, but working out exactly which curves $(C \subset X)$ give a terminal extraction Y is very hard.

Lemma 3.4. *Suppose there exists a Mori extraction $\sigma \colon (E \subset Y) \to (C \subset X)$. Then at least one of g, h is not in $\mathfrak{m}_P \cdot I(\phi)$.*

Proof. Suppose that both $g, h \in \mathfrak{m}_P \cdot I(\phi)$. Then the three equations of C satisfy $\eta \in I(\phi)^2$ and $\xi, \nu \in \mathfrak{m}_P \cdot I(\phi)^2$. On the variety Y there is a point $(Q = Q_\eta \in Y)$

in the fibre above P where all variables except η vanish. Now x, y, z, ξ, ν are all linearly independent elements of the Zariski tangent space $T_Q Y = (\mathfrak{m}_Q/\mathfrak{m}_Q^2)^\vee$. This $(Q \in Y)$ is an index 1 point with $\dim T_Q Y \geq 5$, so it cannot be a hypersurface singularity and is therefore not cDV. $\square$

This condition gives an upper bound on the multiplicity of C at $(P \in X)$.

3.3 The A_1 case: Prokhorov & Reid's example

We run through the easiest case in some detail as an introduction to how this calculation works. The explicit construction of this divisorial extraction by unprojection first appeared in [PrR] Theorem 3.3. From the purely geometric point of view this example was first constructed by Hironaka.

Suppose that a general section $(P \in C \subset S_X \subset X)$ is of type A_1 (i.e. the case $\mathbf{A}_1^1$ in the notation of Table 3.1). By Proposition 3.2 we are considering a curve $(C \subset X)$ given by the equations

$$\bigwedge^2 \begin{pmatrix} x & y & -g(y,z) \\ y & z & h(x,y) \end{pmatrix} = 0$$

where the minus sign is chosen for convenience and we can use column operations to eliminate x from g and z from h. Moreover $g, h \in I(\phi) = \mathfrak{m}_P$ so we can write $g = cy + dz$ and $h = ax + by$ for some choice of functions $a, b, c, d \in \mathcal{O}_X$.

By Lemma 3.4 at least one of $a, b, c, d \notin \mathfrak{m}_P$ else the divisorial extraction is not terminal. This implies that C has multiplicity three at P. If we consider S as the quotient $\mathbb{C}^2_{u,v}/\mathbb{Z}_2$, where $x, y, z = u^2, uv, v^2$, then C is given by the orbifold equation

$$\gamma(u,v) = au^3 + bu^2v + cuv^2 + dv^3$$

and the tangent directions to the branches of C at P correspond to the three roots of this equation.

Recall Cramer's rule from Example 2.6. This gives two syzygies between the equations of $(C \subset X)$

$$\begin{pmatrix} x & y & -(cy+dz) \\ y & z & ax+by \end{pmatrix} \begin{pmatrix} \nu \\ -\xi \\ \eta \end{pmatrix} = 0 \tag{$*$}$$

where $\eta = xz - y^2$ is the equation of S and ξ, ν are the other two equations defining

$(C \subset X)$. We can use these to write down a codimension 2 variety

$$\sigma' \colon Y' \subset X \times \mathbb{P}^2_{(\xi:\nu:\eta)} \to X$$

where σ' is the natural map given by substituting the equations of C back in for ξ, ν, η. Outside of P this map σ' is isomorphic to the blowup of C, in fact Y' is just the blowup of the ordinary power algebra $\bigoplus I^n_{C/X}$. However Y' cannot be the unique divisorial extraction described in Theorem 1.9 since the fibre over the point P is not small. Indeed, Y' contains the plane $D := \sigma'^{-1}(P)_{\mathrm{red}} \cong \mathbb{P}^2$.

Now we can rewrite the equations of Y' $(*)$ so that they annihilate the ideal (x, y, z) defining D

$$\begin{pmatrix} \nu & \xi + c\eta & -d\eta \\ -a\eta & \nu + b\eta & \xi \end{pmatrix} \begin{pmatrix} x \\ -y \\ z \end{pmatrix} = 0$$

and we can use this format to unproject D, in exactly the same manner as Example 2.6.

From Cramer's rule again, we see that Y' has some nodal singularities along D where x, y, z and the minors of this new 2×3 matrix all vanish. If the roots of γ are distinct then this locus consists of three ordinary nodal singularities along D. If γ acquires a double (or triple) root then two (or three) of these nodes combine to give a slightly worse nodal singularity.

We can resolve these nodes by introducing a new variable κ that acts as a ratio between these two vectors, i.e. κ should be a degree 2 variable satisfying the three equations:

$$\begin{aligned} x\kappa &= \xi(\xi + c\eta) + d(\nu + b\eta)\eta \\ y\kappa &= \xi\nu - ad\eta^2 \\ z\kappa &= \nu(\nu + b\eta) + a(\xi + c\eta)\eta \end{aligned}$$

All this gives a codimension 3 variety $\sigma \colon Y \subset X \times \mathbb{P}(1,1,1,2) \to X$ defined by five equations. As described in §2.1.1, by the Buchsbaum–Eisenbud theorem we can write these equations neatly as the maximal Pfaffians of the skew-symmetric 5×5 matrix:

$$\begin{pmatrix} \kappa & \nu & \xi + c\eta & -d\eta \\ & -a\eta & \nu + b\eta & \xi \\ & & z & y \\ & & & x \end{pmatrix}$$

Now we can check that Y actually is the divisorial extraction from C. Outside of the central fibre Y is still the blowup of C, since:

$$\big(Y \setminus \sigma^{-1}(P)\big) \cong \big(Y' \setminus \sigma'^{-1}(P)\big)$$

The plane $(D \subset Y')$ is contracted to the coordinate point $(Q_\kappa \in Y)$ where all variables except κ vanish. (Q_κ is called the *unprojection point* of Y since the map $Y \dashrightarrow Y'$ is projection from Q_κ.) The central fibre is the union of (at most) three lines, all meeting at $(Q_\kappa \in Y)$. Therefore σ is small and, by Proposition 1.9 this is the unique such divisorial extraction from C.

Terminal singularities

Furthermore we can check that Y is terminal. First consider an open neighbourhood of the unprojection point $(Q_\kappa \in U_\kappa) := \{\kappa = 1\}$. We can eliminate the variables x, y, z to see that this open set is isomorphic to the cyclic quotient singularity:

$$(Q_\kappa \in U_\kappa) \cong \big(0 \in \mathbb{C}^3_{\xi,\nu,\eta}\big) \,/\, \tfrac{1}{2}(1,1,1)$$

Now, if $Z = \sigma^{-1}(P)_{\mathrm{red}}$ is the central fibre, for each component $(L \subseteq Z)$ we are left to check the point $Q_L = L \cap \{\kappa = 0\}$. Note that each of these points lies in the affine open set $U_\eta = \{\eta = 1\}$ and recall that at least one of the coefficients a, b, c, d is a unit. After a possible change of variables, we may assume $a \notin \mathfrak{m}_P$ is a unit. We can use the equations involving a above to eliminate x and ξ. After rewriting $\kappa = a\kappa', \nu = a\nu'$, we are left with the equation of a hypersurface

$$V\big((y - z\nu')\kappa' + a{\nu'}^3 + b{\nu'}^2 + c\nu' + d\big) \subset \mathbb{C}^4_{y,z,\nu',\kappa'}$$

which is smooth (resp. cA_1, cA_2) at Q_L if L is the line over a node corresponding to a unique (resp. double, triple) root of γ.

The curve C can degenerate from a transverse intersection of three branches in a number of different ways. Some possible degenerate cases are described in Figure 3.1, given by drawing the birational transform of $(C \subset S_X)$ in the minimal resolution of S_X.

If we consider the case where all of $a, b, c, d \in \mathfrak{m}_P$ then the central fibre consists of just one line $L = \mathbb{P}^1_{(\eta:\kappa)}$ and the point $(Q_L \in Y)$ is not terminal (the matrix defining Y has rank 0 at this point, so it cannot be a hyperquotient point) which agrees with Lemma 3.4.

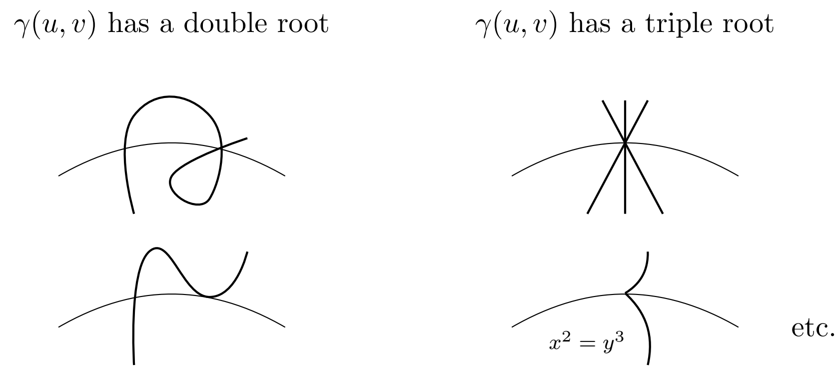


Figure 3.1: Examples of degenerate cases

Further remarks

Remark 3.5. As we have said, there is a geometric construction of Y, originally due to Hironaka, when the three branches of C have distinct tangent directions. This illustrates how the unprojection of D works geometrically.

Consider the variety X' obtained by the blowup of $(P \in X)$ followed by the blowup of the birational transform of C. The exceptional locus has two components $D_{X'}$ and $E_{X'}$ dominating P and C respectively. Assuming the tangent directions of the branches of C at P are distinct then $D_{X'}$ is a Del Pezzo surface of degree 6. Consider the three -1-curves of $D_{X'}$ that don't lie in the intersection $(D_{X'} \cap E_{X'})$. They have normal bundle $\mathcal{O}_{X'}(-1,-1)$ so we can flop them. The variety Y', constructed above, is the midpoint of this flop and we end up with the following diagram:

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr] \arrow[rr, dashed, "\text{flop}"] & & Z \arrow[dl] \arrow[dr] & \\ X & & Y' & & Y \end{tikzcd}$$

The plane $(D \subset Y)$ is the image of $D_{X'}$ with the three nodes given by the contracted curves. After the flop the divisor $D_{X'}$ becomes a plane $D_Z \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}_Z(-2)$, so we can contract it to construct Y with a $\frac{1}{2}$-quotient singularity.

If we want to consider curves that have branches with non-distinct tangent directions then this picture becomes much more complicated.

Remark 3.6. Looking back at the equations of Y' $(*)$ we may ask what happens if we unproject the ideal $(\xi, \nu, \eta) \subset \mathcal{O}_{Y'}$ or, equivalently, the Jer_{12} ideal $(\xi, \nu, \eta, \kappa) \subset \mathcal{O}_Y$. Even though this may not appear to make sense geometrically, it is a perfectly well-defined operation in algebra. If we do then we introduce the variable ι of weight -1 that is nothing other than the inclusion $\iota\colon I_{C/X} \hookrightarrow \mathcal{O}_X$. The whole picture is a big $\mathbb{Z}$-graded ring

$$R := \mathcal{O}_X(-1,1,1,1,2)/(\text{codim } 4 \text{ ideal})$$

and we can construct the divisorial extraction in the style of §2.3, as the Proj of the extended symbolic power algebra.

Remark 3.7. The unprojection variable κ corresponds to a generator of $\bigoplus I_{C/X}^{[n]}$ that lies in $I_{C/X}^{[2]} \setminus I_{C/X}^2$. Either by writing out one of the equations involving κ and substituting for the values of ξ, ν, η, or by calculating the unprojection equations of ι, we can give an explicit expression for κ:

$$\iota\kappa = (ax + by)\xi + (cy + dz)\nu + (acx + ady + bcy + bdz)\eta$$

In terms of the orbifold equation γ, the generators ξ, ν, κ are lifts modulo η of the forms $u\gamma, v\gamma, \gamma^2$ defined on S.

3.4 The A_2 cases

Now we consider the next most complicated example. Suppose that the general section $(P \in C \subset S_X \subset X)$ is of type $\mathbf{A}_2^1$. By Proposition 3.2, we are considering the curve given by the equations

$$\bigwedge^2 \begin{pmatrix} x & y & -(dy + ez) \\ y^2 & z & ax + by \end{pmatrix} = 0$$

for some choice of functions $a, b, d, e \in \mathcal{O}_X$. If a, b, d, e are taken generically then the general section through C is of type A_1 so, for S_X to be a *general* section, we need to introduce some more conditions on a, b, d, e.

Consider the section $H_{\lambda,\mu} = \{h_{\lambda,\mu} := \eta + \lambda\xi + \mu\nu = 0\}$. The quadratic term of this equation is given by

$$h_{\lambda,\mu}^{(2)} = xz + \lambda x(a_0x + b_0y) + \mu(a_0xy + b_0y^2 + d_0yz + e_0z^2)$$

where a_0 is the constant term of a and similarly for b, d, e. To ensure the general section is of type A_2 it is enough to ask that $h_{\lambda,\mu}^{(2)}$ has rank 2 for all λ, μ. After playing around, completing the square etc., we get two cases according to whether

$x \mid h^{(2)}_{\lambda,\mu}$ or $z \mid h^{(2)}_{\lambda,\mu}$:

$$\begin{aligned} a_0 = b_0 = 0 &\implies h^{(2)}_{\lambda,\mu} = z(x + \mu d_0 y + \mu e_0 z) \\ b_0 = d_0 = e_0 = 0 &\implies h^{(2)}_{\lambda,\mu} = x(z + \lambda a_0 x + \mu a_0 y) \end{aligned}$$

These two different cases lead to two different Mori extractions.

3.4.1 Case 1: Tom_1

Take the first case where $a_0 = b_0 = 0$. Then we can rewrite $ax+by$ as $ax^2+bxy+cy^2$, so that the equations of C become:

$$\bigwedge^2 \begin{pmatrix} x & y & -(dy+ez) \\ y^2 & z & ax^2+bxy+cy^2 \end{pmatrix} = 0$$

The symbolic power algebra of C will be a graded ring generated in degrees $1, 1, 1, 2, 3$ by generators that we will call $\eta, \nu_1, \xi_1, \xi_2, \kappa_3$. In particular a subscript will denote the weight of the corresponding variable.

Claim The following two conditions must hold

1. one of $a, b, c, d \notin \mathfrak{m}_P$,
2. one of $d, e \notin \mathfrak{m}_P$,

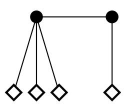
and (after possibly changing variables) we can assume that $a, e \notin \mathfrak{m}_P$.

Statement (2) follows from Lemma 3.4. The first is also proved in a similar way. If (1) does not hold then necessarily $e \notin \mathfrak{m}_P$ by (2). Consider the point $(Q_\eta \in Y)$ where all variables but η vanish, as in the proof of Lemma 3.4. This is an index 1 point with local equation

$$ey^2\xi_1 - x\xi_1\nu_1 + y\nu_1^2 + dy\nu_1 + e(ax^2 + bxy + cy^2) = 0$$

and if $a, b, c, d \in \mathfrak{m}_P$ then this equation is not cDV, as it has no terms of degree 2, so it is not terminal.

By considering the minimal resolution $\widetilde{S} \to S$, we see that the general C that satisfies these conditions is the curve



but we can also allow any degenerate cases of $(\widetilde{C} \subset \widetilde{S})$ which intersect $E_1 = \mathbb{P}^1_{(x_1:x_2)}$ with multiplicity three and $E_2 = \mathbb{P}^1_{(y_1:y_2)}$ with multiplicity one, according to the (nonzero) equations:

$$\begin{aligned} \widetilde{C} \cap E_1\colon &\quad a_0x_1^3 + b_0x_1^2x_2 + c_0x_1x_2^2 + d_0x_2^3 = 0 \\ \widetilde{C} \cap E_2\colon &\quad d_0y_1 + e_0y_2 = 0 \end{aligned}$$

If we mimic Prokhorov & Reid's example, we can write down a codimension 3 model of the blowup of C as $\sigma''\colon Y'' \subset X \times \mathbb{P}(1,1,1,2) \to X$ given by the the Pfaffians of the matrix:

$$\begin{pmatrix} \xi_2 & \nu_1 & \xi_1 + d\eta & -e\eta \\ & -(ax+by)\eta & y(\nu_1 + c\eta) & \xi_1 \\ & & z & y \\ & & & x \end{pmatrix}$$

The variety Y'' is *not* the divisorial extraction since σ'' is not small. A new unprojection plane appears after the first unprojection. This plane D is defined by the ideal (x, y, z, ξ_1) and we can see that the matrix is in Tom_1 format with respect to this ideal. The central fibre $\sigma''^{-1}(P)$ is given by D together with the line

$$L_1 = (x = y = z = \nu_1 = \xi_1 + d\eta = 0).$$

Unprojecting D gives a new variable κ_3 of weight three with four additional equations:

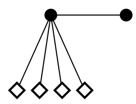
$$\begin{aligned} x\kappa_3 &= (\xi_2 + be\eta^2)(\xi_1 + d\eta) + e\nu_1(\nu_1 + c\eta)\eta \\ y\kappa_3 &= \xi_2\nu_1 - ae(\xi_1 + d\eta)\eta^2 \\ z\kappa_3 &= \nu_1^2(\nu_1 + c\eta) + b\nu_1(\xi_1 + d\eta)\eta + a(\xi_1 + d\eta)^2\eta \\ \xi_1\kappa_3 &= \xi_2(\xi_2 + be\eta^2) + ae^2(\nu_1 + c\eta)\eta^3 \end{aligned}$$

Generically, the central fibre consists of four lines passing through the point P_κ, the line L_1 and the three lines that appear after unprojecting D. The open neighbourhood $(P_\kappa \in U_\kappa)$ is isomorphic to a $\frac{1}{3}(1,1,2)$ singularity. As we assume $a, e \notin \mathfrak{m}$, when $\eta = 1$ we can use the equations to eliminate x, z, ξ_1, ν_1 so that all the points $Q_L = L \cap \{\zeta = 0\}$, for $L \subseteq \sigma^{-1}(P)_{\mathrm{red}}$, are smooth.

3.4.2 Case 2: Jer_{45}

Now consider instead the case where $b_0 = d_0 = e_0 = 0$. In direct analogy to the Tom_1 case the reader can check that:

- the general curve C that satisfies this condition is the curve:

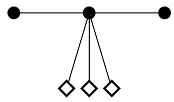


- after making the first unprojection we get a variety Y' containing another unprojection plane D above P defined by the Jer_{45} ideal (x, y, z, ν_1),
- Y' has (at most) four nodes along D corresponding to the roots of the orbifold equation γ,
- after unprojecting D we get a variety Y with small fibre over P, hence Y is the divisorial extraction,
- the open neighbourhood of the final unprojection point ($P_\kappa \in U_\kappa$) is isomorphic to the quotient singularity $\frac{1}{3}(1, 1, 2)$,
- Y has at worst cA singularities at the points Q_L according to whether γ has repeated roots.

The equations for this example can be found in the appendix A.3.

3.5 An A_3 example

Suppose that the general section ($P \in C \subset S_X \subset X$) is of type $\mathbf{A}_3^2$ and that C is the curve



Then a terminal extraction from ($C \subset X$) exists.

The calculation is very similar to Prokhorov & Reid's example, except that the first unprojection divisor ($D \subset Y'$) is defined by the ideal $I(\phi) = (x, y^2, z)$, so that D is *not reduced*. After unprojecting D we get an index 2 model for the divisorial extraction

$$\sigma \colon Y \subset X \times \mathbb{P}(1, 1, 1, 2) \to X$$

with equations given by the Pfaffians of

$$\begin{pmatrix} \kappa & \nu & \xi + c\eta & -d\eta \\ & -a\eta & \nu + b\eta & \xi \\ & & z & y^2 \\ & & & x \end{pmatrix}.$$

D is contracted to a singularity of type $cA_1/2$, given by the hyperquotient singularity

$$\left((y^2 - \xi\nu + ad\eta^2 = 0) \subset \mathbb{C}^4_{y,\xi,\nu,\eta}\right) / \tfrac{1}{2}(0,1,1,1).$$

Note that this singularity is isolated, unless both $a, d \in \mathfrak{m}_P$ in which case the reduced central fibre Z contains the line $L = \mathbb{P}^1_{(\eta:\kappa)}$ and Y is singular along L.

Chapter 4

Exceptional cases: types D & E

We now turn to the cases when $(P \in S_X)$ is a Du Val singularity of type D or E. It turns out to be slightly easier to study Mori extractions in this setting because they are more restricted. The results of this section are summed up in the following theorem:

Theorem 4.1. *Suppose that we have $(P \in C \subset S_X \subset X)$ as in the usual situation §0.2. In particular S_X is the* general *hypersurface section containing C.*

1. *Suppose that C is of type $\mathbf{D}^l_n, \mathbf{D}^r_{2k}$ or $\mathbf{E}_7$. Then the symbolic blowup of C has a codimension 3 model:*

$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2) \to X$$

   In particular, Y has index 2 and $\bigoplus I^{[n]}_{C/X}$ is generated in degrees ≤ 2.

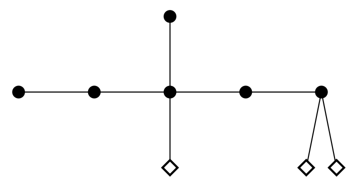
   Moreover, Y has non-isolated singularities along a component of the central fibre, so there does not exist a Mori extraction from C.

2. *Suppose that C is of type $\mathbf{E}_6$. We need to consider two cases.*

   (a) *σ is nonsemistable, i.e. $\sigma\colon S_Y \to S_X$ is an isomorphism. Then the symbolic blowup of C has a codimension 4 model:*

$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2,3) \to X$$

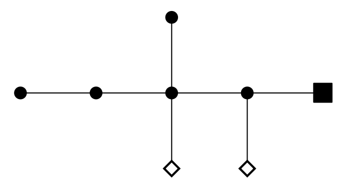
   In particular $\bigoplus I^{[n]}_{C/X}$ is generated in degrees ≤ 3.
   Moreover, if Y is terminal then Y has a $cD/3$ singularity and (the generic such) C is the curve:



(b) σ is semistable, i.e. $\sigma\colon S_Y \to S_X$ is an not isomorphism. Then the symbolic blowup of C has a codimension 5 model:

$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2,3,4) \to X$$

In particular $\bigoplus I^{[n]}_{C/X}$ is generated in degrees ≤ 4.
Moreover, if Y is terminal then Y has a $cAx/4$ singularity and (the generic such) C is the curve:



In this case, the central fibre is a union of lines meeting at a $cAx/4$ singularity. The node marked $\blacksquare$ denotes the curve pulled out of S_X by σ.

This is a classification of Mori extractions when $(C \subset S_X)$ is of type $\mathbf{D}^l_n$, $\mathbf{D}^r_{2k}$, $\mathbf{E}_6$ or $\mathbf{E}_7$ but does not say anything about the case $\mathbf{D}^r_{2k+1}$.

In the statement of the theorem, when we say that "the (generic such) curve C" intersects the exceptional locus of a Du Val singularity $(E \subset \widetilde{S}_X)$ in the prescribed way, we also allow any degenerations of C which keep the intersection numbers with each component of E fixed. For instance (2)(b) includes the curve of Example 4.3.

The proof of this theorem starts by using Proposition 3.2 to write down a format for the equations of a curve $(C \subset S_X)$ contained in the relevant Du Val hypersurface singularity. Then we specialise this format until our chosen hypersurface S_X is the *general* hyperplane section through C (see e.g. Remark 4.2 below). We calculate the graded ring of Y using the method outlined in §3.2, i.e. starting with the blowup C we make successive unprojections until the variety we construct has a small central fibre above $(P \in C)$. By Proposition 1.9 this is the (unique) Mori extraction from C if it exists. Then we check to see whether Y really does have terminal singularities. In most of the cases ruled out, Y has non-isolated (and hence non-terminal) singularities.

Before launching into the proof we make the following useful remark.

Remark 4.2. Suppose the general section of $(P \in C \subset S_X \subset X)$ is of type D or E. Then we can write the equations of C as

$$\bigwedge^2 \begin{pmatrix} \phi & \begin{matrix} -g(y,z) \\ h(y,z) \end{matrix} \end{pmatrix} = 0$$

where $g, h \in \mathfrak{m}_P^2 \cap I(\phi)$, i.e. we can assume that g, h have no linear terms. To see this consider the forms for ϕ given in Table 3.1. Firstly, we can use column operations to cancel any terms involving x from g, h. Then to prove $g, h \in \mathfrak{m}_P^2$ consider the section $h_{\lambda,\mu} = \eta + \lambda\xi + \mu\nu$. The quadratic term of $h_{\lambda,\mu}$ is

$$h^{(2)}_{\lambda,\mu} = x^2 + \lambda x h^{(1)} + (\mu x + \lambda t) g^{(1)} \quad \text{(where } t = y \text{ or } z\text{)}$$

and we require this to be a square for all λ, μ. This happens only if $g^{(1)} = h^{(1)} = 0$.

4.1 The $\mathbf{D}^l_n, \mathbf{D}^r_{2k}$ and $\mathbf{E}_7$ cases

These three calculations are essentially all the same. Since they are so similar we only spell out the $\mathbf{D}^l_n$ case in detail.

The $\mathbf{D}^l_n$ case

According to Lemma 3.2 and Remark 4.2, the curve $(C \subset X)$ is defined by the equations

$$\bigwedge^2 \begin{pmatrix} x & y^2 + z^{n-2} & a(y^2 + z^{n-2}) + byz + cz^2 \\ z & x & d(y^2 + z^{n-2}) + eyz + fz^2 \end{pmatrix} = 0$$

for some functions $a, b, c, d, e, f \in \mathcal{O}_X$. Unprojecting $I(\phi)$ gives a variety

$$\sigma \colon Y \subset X \times \mathbb{P}(1,1,1,2) \to X$$

with equations given by the maximal Pfaffians of the matrix:

$$\begin{pmatrix} \kappa & \nu & \xi - a\eta & (by + cz)\eta \\ & -\xi & -d\eta & \nu + (ey + fz)\eta \\ & & z & y^2 + z^{n-2} \\ & & & x \end{pmatrix}$$

Now σ is a small map, so Y must be the unique divisorial extraction from C from Proposition 1.9. Indeed, the central fibre $Z = \sigma^{-1}(P)_{\mathrm{red}}$ consists of two components meeting at the point P_κ. These are the lines:

$$L_1 = (x = y = z = \xi = \nu = 0)$$
$$L_2 = (x = y = z = \xi - a\eta = \nu = 0)$$

Looking at the affine patch $U_\kappa := \{\kappa = 1\}$ in Y we see that we can eliminate the variables x, z and that U_κ is a $\frac{1}{2}$-quotient of the hypersurface singularity

$$y^2 + z^{n-2} = \nu^2 + (e\nu + b\xi)y\eta + (f\nu + c\xi)z\eta$$

where $z = \xi^2 - (a\xi + d\nu)\eta$.

This hypersurface is singular along the line L_1 since, as $n \geq 4$, this equation is contained in the square of the ideal (y, ξ, ν). Therefore Y has non-isolated singularities and cannot be terminal.

The $\mathbf{D}_{2k}^r$ case

The curve $(C \subset X)$ is defined by the equations

$$\bigwedge^2 \begin{pmatrix} x & yz + z^k & ay^2 + byz + c(yz + z^k) \\ y & x & dy^2 + eyz + f(yz + z^k) \end{pmatrix} = 0$$

and equations of $Y \subset X \times \mathbb{P}(1,1,1,2)$ are given by the Pfaffians of the matrix:

$$\begin{pmatrix} \kappa & \nu & (ay + bz)\eta & \xi + c\eta \\ & \xi & \nu + (dy + ez)\eta & f\eta \\ & & yz + z^k & y \\ & & & x \end{pmatrix}$$

In particular the point $(P_\kappa \in Y)$ is a $\frac{1}{2}$-quotient of the hypersurface

$$yz + z^k = \nu(\nu + (dy + ez)\eta) - (ay + bz)\xi\eta$$

but this has non-isolated singularities along the line $(x = y = z = \xi = \nu = 0)$.

The $\mathbf{E}_7$ case

The curve $(C \subset X)$ is defined by the equations

$$\bigwedge^2 \begin{pmatrix} x & y^2+z^3 & ay^2+byz+c(y^2+z^3) \\ y & x & dy^2+eyz+f(y^2+z^3) \end{pmatrix} = 0$$

and equations of $Y \subset X \times \mathbb{P}(1,1,1,2)$ are given by the Pfaffians of the matrix:

$$\begin{pmatrix} \kappa & \nu & (ay+bz)\eta & \xi + c\eta \\ & \xi & \nu + (dy+ez)\eta & f\eta \\ & & y^2+z^3 & y \\ & & & x \end{pmatrix}$$

In particular the point $(P_\kappa \in Y)$ is a $\frac{1}{2}$-quotient of the hypersurface

$$y^2 + z^3 = \nu(\nu + (dy+ez)\eta) - (ay+bz)\xi\eta$$

and this also has non-isolated singularities along $(x = y = z = \xi = \nu = 0)$.

4.2 The $\mathbf{E}_6$ case

Suppose that $(C \subset X)$ is of type $\mathbf{E}_6$. By Lemma 3.2 the equations of C can be written in the form

$$\bigwedge^2 \begin{pmatrix} x & y^2 & -g(y,z) \\ y & x+z^2 & h(y,z) \end{pmatrix} = 0$$

where $g, h \in \mathfrak{m}_P^2$ by Remark 4.2. Now consider the general section $H_{\lambda,\mu} = \eta + \lambda\xi + \mu\nu$. After making the replacement $x \mapsto x + \frac{1}{2}(\lambda h + \mu g)$ the cubic term of $H_{\lambda,\mu}$ is given by

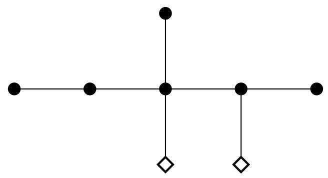
$$x^2 - y^3 + \lambda y g^{(2)}$$

where $g^{(2)}$ is the quadratic part of g. For the general $H_{\lambda,\mu}$ to be of type E_6, we require $y(y^2 - \lambda g^{(2)})$ to be a perfect cube for all values of λ. This happens only if $g^{(2)}$ is a multiple of y^2. Therefore we can take g and h to be

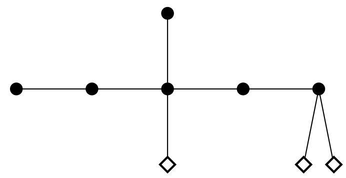
$$\begin{aligned} g(y,z) &= a(y,z)y^2 + b(z)yz^2 + c(z)z^3 \\ h(y,z) &= d(y)y^2 + e(y)yz + f(y,z)z^2 \end{aligned}$$

for some choice of functions $a, b, c, d, e, f \in \mathcal{O}_X$. Moreover, $f \notin \mathfrak{m}_P$ else the extraction is not terminal by Lemma 3.4.

By allowing these coefficients to specialise the curve we are considering varies. After writing down the minimal resolution $(C \subset S_X)$ explicitly, one can check that in the cases where the coefficients are chosen generically, or when $c \in \mathfrak{m}_P$, then C is the curve



Generic coefficients, i.e. $a, c \notin \mathfrak{m}_P$



$c \in \mathfrak{m}_P$ and $a, a+f, f \notin \mathfrak{m}_P$

and so on. In the generic case, if $a + f \in \mathfrak{m}_P$ then the curve degenerates as in Example 4.3.

We can make the first unprojection $\sigma' \colon Y' \to X$ with unprojection variable ζ of weight 2, defined by the Pfaffians of the matrix:

$$\begin{pmatrix} \zeta & \nu & y(\xi + a\eta) & -(by + cz)\eta \\ & \xi & \nu + (dy + ez)\eta & \xi - f\eta \\ & & z^2 & y \\ & & & x \end{pmatrix} \tag{$\dagger$}$$

This Y' contains a new unprojection divisor defined by an ideal I in Tom_2 format. If the coefficient c is assumed to be chosen generically then $I = (x, y, z, \nu)$. However, if we make the specialisation $c \in \mathfrak{m}_P$, we can take I to be (x, y, z^2, ν), defining a slightly fatter unprojection plane. Unprojecting these two ideals gives two very different varieties.

4.2.1 The non-semistable $\mathbf{E}_6$ case

Since it is easier, consider first the case when $c \in \mathfrak{m}_P$, i.e. we let $c(z) = c'(z)z$. Unprojecting (x, y, z^2, ξ_2) with unprojection variable θ of weight 3 gives a codimension 4 model

$$\sigma \colon Y \subset X \times \mathbb{P}(1,1,1,2,3) \to X$$

defined by the five Pfaffians above (†), plus four additional equations:

$$\begin{aligned} x\theta &= (\xi + a\eta)(\xi - f\eta)^2 + b(\xi - f\eta)(\nu + (dy + ez)\eta)\eta + c'(\nu + (dy + ez)\eta)^2\eta \\ y\theta &= \zeta(\xi - f\eta) + c'\xi(\nu + (dy + ez)\eta)\eta \\ z^2\theta &= (\zeta - b\xi\eta)(\nu + (dy + ez)\eta) - \xi(\xi + a\eta)(\xi - f\eta) \\ \nu\theta &= \zeta(\zeta - b\xi\eta) + c'\xi^2(\xi + a\eta)\eta \end{aligned}$$

The central fibre Z is a union of three lines meeting at the unprojection point P_θ, so that Y is the divisorial extraction of C. These three lines are given by $x = y = z = \nu = 0$ and:

$$\begin{array}{ll} \left.\begin{array}{l} L_1 \\ L_2 \end{array}\right\} & \xi - f\eta = \zeta^2 - bf\zeta\eta^2 + c'f^2(a + f)\eta^4 = 0 \\ L_3 & \xi + a\eta = \zeta = 0 \end{array}$$

In the open neighbourhood of the unprojection point ($P_\theta \in U_\theta$) we can eliminate x, y, ν by the equations involving θ above. We are left with a $\frac{1}{3}$-quotient of the hypersurface singularity:

$$H = \Big(z^2 = (\zeta - b\xi\eta)(\nu + (dy + ez)\eta) - \xi(\xi + a\eta)(\xi - f\eta)\Big)$$

If H is not isolated then Y will have nonisolated singularities and there will be no terminal extraction from C. This happens if either $a \in \mathfrak{m}_P$ or $a + f \in \mathfrak{m}_P$. If $a \in \mathfrak{m}_P$ then H becomes singular along L_3. If $a + f \in \mathfrak{m}_P$ then one of L_1, L_2 satisfies $\zeta - bf\eta^2 = 0$ and H becomes singular along this line.

Now we can assume that $a, a + f, f \notin \mathfrak{m}_P$, and consider the (generic) hyperplane section $\eta = 0$, to see that ($P_\theta \in U_\theta$) is a $cD_4/3$ point:

$$\big(z^2 - \zeta^3 + \xi^3 + \eta(\cdots) = 0\big) \;/\; \tfrac{1}{3}(0, 2, 1, 1; 0)$$

4.2.2 The semistable $\mathbf{E}_6$ case

Now consider the more general case where $c \notin \mathfrak{m}_P$. In this case there is an element θ' in $I^{[3]}_{C/S_X}$ which fails to lift to an element of $I^{[3]}_{C/X}$.

We need to make two unprojections in order to construct the divisorial extraction Y. The first unprojection divisor is defined by the Tom_2 ideal (x, y, z, ν) as described above. Then a new divisor appears defined by the ideal $\big(x, y, z, \nu, \xi(\xi + a\eta)\big)$. We add two new variables θ, κ of degrees 3, 4 to our ring and we end up with a variety in codimension 5

$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2,3,4) \to X.$$

The equations of Y are given by the five equations (†) and nine new unprojection equations: four involving θ and five involving κ. See A.4 for the full list of equations. The important equation is:

$$\xi(\xi + a\eta)\kappa = \zeta(\zeta - b\xi\eta)^2 - \theta(\theta - cd\xi\eta^2) + e\theta(\zeta - b\xi\eta)\eta + d\zeta(\xi - f\eta)(\zeta - b\xi\eta)\eta$$

The open set of the unprojection point ($P_\kappa \in U_\kappa$) is a hyperquotient point

$$\xi^2 + \theta^2 - \zeta^3 + \eta(\cdots) = 0\big) \;/\; \tfrac{1}{4}(1,2,3,1;2)$$

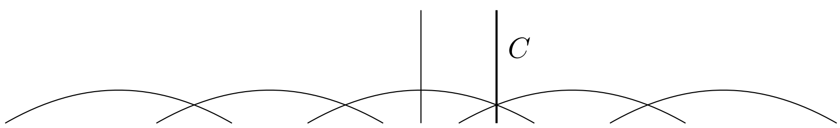
which is the equation of a $cAx/4$ singularity. Moreover, one can check that this singularity is not isolated if $a \in \mathfrak{m}_P$. Therefore, if Y is terminal then $a \notin \mathfrak{m}_P$ and C is as described in Theorem 4.1.

The central fibre of this extraction consists of (one or) two rational curves. One of these curves is pulled out in a partial resolution of S_X.

Example 4.3. As an example of this type of extraction consider the monomial curve $C = C_{5,7,11} \subset \mathbb{C}^3$ with weights $(5,7,11)$. This C is given by the equations

$$\bigwedge^2 \begin{pmatrix} x^2 & y & z \\ y^2 & z & x^3 \end{pmatrix} = 0$$

and a general section through this curve has an E_6 Du Val singularity. We can check through an explicit calculation that C is a curve that meets E_6 in the following way:



This is a degenerate version of Theorem 4.1(2)(b).

The symbolic blowup $\sigma\colon Y \to X$ has a codimension 4 model:

$$\sigma\colon Y \subset X \times \mathbb{P}(1,1,1,2,3,4) \to X$$

Write $\xi_1, \xi_2, \xi_3, \zeta, \theta, \kappa$ for the generators of the ring. Then the full list of equations of Y are given in A.5.

The last unprojection point ($P_\kappa \in Y$) is the hyperquotient point

$$(\xi_1\xi_3 = \zeta^3 - \theta^2)/\tfrac{1}{4}(1,1,2,3;2)$$

which is a terminal singularity of type $cAx/4$. We can check that Y is smooth elsewhere.

4.3 The $\mathbf{D}^r_{2k+1}$ case

The $\mathbf{D}^r_{2k+1}$ cases are certainly the most complicated of the exceptional cases and it seems that trying to classify Mori extractions through explicit calculations will be too hard. However some calculations predict the following.

Using the same tricks as before we can take the equations of $(C \subset X)$ to be

$$\begin{pmatrix} x & yz & -(ay^2+byz+cz^{k+1}) \\ y & x+z^k & dy^2+eyz+fz^k \end{pmatrix} \begin{pmatrix} \nu \\ -\xi \\ \eta \end{pmatrix} = 0$$

for some coefficients $a,b,c,d,e,f \in \mathcal{O}_X$. Using this format we can unproject the ideals (x,y,z^2) and then (x,y,z,ν) to get a codimension 4 variety

$$\sigma' \colon Y' \subset X \times \mathbb{P}(1,1,1,2,3) \to X$$

whose central fibre is still not small.

If $a,b \notin \mathfrak{m}_P$ then we can only unproject the ideal (x,y,z,ν,ξ) to get a codimension 5 variety $Y'' \subset X \times \mathbb{P}(1,1,1,2,3,5)$. In this case the central fibre may or may not be now 1-dimensional so the unprojection game could continue. We have $S_{Y''} \ncong S_X$ so this is a semistable extraction.

If $a,b \in \mathfrak{m}_P$ then we can unproject the slightly fatter unprojection plane (x,y,z,ν,ξ^2) to get a variety $Y \subset X \times \mathbb{P}(1,1,1,2,3,4)$. The central fibre is 1-dimensional and so we have constructed the divisorial extraction. In this case $S_Y \cong S_X$ so this is a non-semistable extraction. If it is isolated, the last unprojection point $(Q \in Y)$ is a $cAx/4$ singularity.

We spell out this second non-semistable case when S_X is of type D_5.

4.3.1 A D_5^r example

Rewrite the equations of $(C \subset X)$ as

$$\begin{pmatrix} x & yz & -(ay^3 + by^2z + cyz^2 + dz^3) \\ y & x + z^k & ey^2 + fyz + gz^2 \end{pmatrix} \begin{pmatrix} \nu \\ -\xi \\ \eta \end{pmatrix} = 0$$

Let ζ, θ, κ be the new generators of degrees 2,3,4 and, in a desperate attempt to simplify all the equations, write:

$$\overline{\xi} = \xi - g\eta, \quad \overline{\nu} = \nu + (ey + fz)\eta, \quad \overline{\zeta} = \zeta + c\xi\eta, \quad \overline{\theta} = \theta + b\xi\overline{\xi}\eta - f\zeta\eta$$

(In particular $\xi \equiv \overline{\xi} \mod \eta$ etc.) The full list of equations are given in A.6 for the readers edification and enjoyment.

Note that at the point $(P_\kappa \in Y)$ we can eliminate everything but the variables ξ, η, ζ, θ and the last equation in A.6. This shows that P_κ is a $\frac{1}{4}(1,1,2,3;2)$ quotient of the hypersurface singularity:

$$\xi^2 = \theta\big(\overline{\theta} - de\xi\eta^2\big) - \Big[\zeta\overline{\zeta} + bd\xi^2\eta^2\Big]\Big[\zeta + e\eta\overline{\xi}\Big] + a\xi\eta\Big(\overline{\xi}^2\overline{\zeta} + df\xi\overline{\xi}\eta^2 - d^2\xi^2\eta^2\Big)$$

As this equation contains the terms ξ^2, θ^2 and ζ^3, we can use the Weierstrass preparation theorem to make an analytic change of variables $\xi', \eta', \zeta', \theta'$ to be left with the hyperquotient

$$\Big(\xi'^2 = \theta'^2 + \zeta'^3 + F(\zeta', \eta')\Big) \; / \; \tfrac{1}{4}(1,3,2,1;2)$$

for some function F and this is a $cAx/4$ singularity if it is isolated.

Chapter 5

Cyclic quotient singularities

Before tackling the main case of type A Mori extractions we collect together some results that we will need concerning cyclic quotient singularities and the combinatorics of continued fraction expansions.

It is a well known fact (see e.g. [B]) that the minimal resolution of a surface singularity germ $(P \in S)$ has an exceptional divisor given by a chain of rational curves $E = \bigcup_{i=1}^{k} E_i$ if and only if $(P \in S)$ is analytically isomorphic to a cyclic quotient singularity $\frac{1}{r}(1,a)$. The values r and a are computed by the Hirzebruch–Jung continued fraction expansion $\frac{r}{a} = [a_1, \ldots, a_k]$ where $a_i = -E_i^2$ for $i = 1, \ldots, k$.

Moreover, if $\frac{r}{r-a} = [b_1, \ldots, b_l]$ is the complementary continued fraction expansion then $(P \in S)$ has an embedding $(P \in S \subset \mathbb{C}^{l+2}_{x_0,\ldots,x_{l+1}})$ where S is cut out (set-theoretically) by the equations $x_{i-1}x_{i+1} = x_i^{b_i}$ for $i = 1, \ldots, l$.

5.1 Continued fraction expansions

Definition 5.1. We define the *Hirzebruch–Jung continued fraction expansion* by the formal expression

$$[a_1, \ldots, a_k] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{k-1} - \cfrac{1}{a_k}}}}$$

Given two coprime integers $r > a > 0$ then there is a unique sequence of integers $a_1, \ldots, a_k \in \mathbb{Z}_{\geq 2}$ such that $\frac{r}{a} = [a_1, \ldots, a_k]$. This is called the *reduced* Hirzebruch–Jung continued fraction expansion of $\frac{r}{a}$.

Similarly, we can consider the Hirzebruch–Jung continued fraction associated to any sequence of non-negative integers $a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0}$, as long as it is well-defined (i.e. does not involve division by 0).

Unless it is said otherwise, a *Hirzebruch–Jung-string*, or *HJ-string*, will always be a finite-length (not necessarily reduced) Hirzebruch–Jung continued fraction expansion with coefficients $a_i \in \mathbb{Z}_{\geq 0}$ for all i, which represents a non-negative rational number $x \in \mathbb{Q}_{\geq 0}$.

Notation. We write $(a)^m$ to mean the sequence $a, \ldots, a$, repeated m times.

Lemma 5.2 (Elementary properties of HJ-strings)**.** *Let $r > a > 0$ be coprime integers, and consider the HJ-string $\frac{r}{a} = [a_1, \ldots, a_k]$.*

1. *(Blowdown) If $a_i = 1$ for some $2 \leq i \leq k$, then*[1]

$$[a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_k] = [a_1, \cdots, a_{i-1} - 1, a_{i+1} - 1, \cdots, a_k]$$

2. *(Inverse) If $1 \leq b < r$ is the inverse of a modulo r, i.e. the unique integer such that $ab \equiv 1 \mod r$, then $\frac{r}{b} = [a_k, \ldots, a_1]$.*

3. *(Complement) We call $\frac{r}{r-a} = [b_1, \ldots, b_l]$ the* complementary *HJ-string to $\frac{r}{a}$. This satisfies the identity:*

$$[a_1, \ldots, a_k, 1, b_l, \ldots, b_1] = 0$$

4. *(Convergents) The rational numbers $\frac{p_i}{q_i} := [a_1, \ldots, a_{i-1}] \quad 1 \leq i \leq k+1$ are called the* convergents *of $[a_1, \ldots, a_k]$.*[2] *These satisfy the relations*

$$a_i p_i = p_{i-1} + p_{i+1}, \quad a_i q_i = q_{i-1} + q_{i+1} \quad \forall i$$

5. *(Matrix product) The following matrix product holds:*

$$\begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_k \end{pmatrix} = \begin{pmatrix} -q_k & q_{k+1} \\ -p_k & p_{k+1} \end{pmatrix}$$

The proof of all these statements are standard elementary exercises in continued fractions.

[1] *Warning*: If $i = 1$ then the two HJ-strings obtained by this operation represent *different* rational numbers, since if $x = [1, a_2, \ldots, a_k]$ then $\frac{x}{1-x} = [a_2 - 1, \ldots, a_k]$. However we will still call this a blowdown.

[2] To define p_1, q_1 we use the convention $[\emptyset] = \frac{1}{0}$, even though this is not a well-defined fraction.

5.1.1 0-strings and 1-strings

We now turn to consider some important classes of HJ-strings.

0-strings

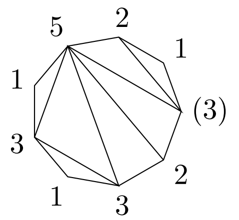
Definition 5.3. Any HJ-string representing 0, i.e. one of the form $[a_1, \dots, a_k] = 0$, is called a *0-string*.

These were considered by Christophersen in his study of deformations of surface cyclic quotient singularities.

Lemma 5.4. *A 0-string blows down to* $[0]$.

There is a well-known correspondence between 0-strings of length k and triangulations of an $(k+1)$-gon. In particular, the number of 0-strings of length k is counted by the $(k-1)$th Catalan number $C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$.

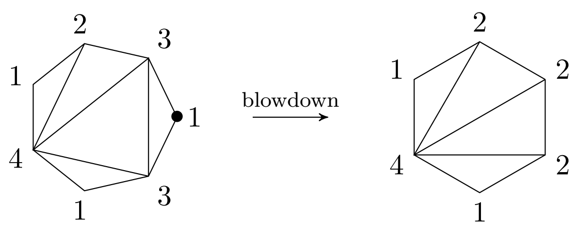
Specifically, let P be a regular k-gon with vertices $x_0, x_1, \dots, x_k$ ordered cyclically. Given a triangulation of P let a_i be the number of triangles containing x_i. Then, ignoring a_0, we have $[a_1, \dots, a_k] = 0$, e.g.



$$
\begin{aligned}
&[1,2,5,1,3,1,3,2] = 0\\
&[2,5,1,3,1,3,2,3] = 0\\
&[5,1,3,1,3,2,3,1] = 0\\
&[1,3,1,3,2,3,1,2] = 0 \quad \text{etc.}
\end{aligned}
$$

Given this description, clearly $[a_k, \dots, a_1]$ and $[a_{i+1}, \dots, a_1, a_0, a_k, \dots, a_{i-1}]$ are also 0-strings for any i.

A *blowup* of a triangulation of a $(k+1)$-gon P is a triangulation of a $(k+2)$-gon given by glueing a triangle onto an edge of P. It is easy to see that blowup of of triangulations corresponds to blowup of HJ-strings.



1-strings

Definition 5.5. Any HJ-string representing $\frac{1}{m}$ for some $m \geq 1$, i.e. one of the form $[a_1, \ldots, a_k] = \frac{1}{m}$, is called a *1-string*.

Lemma 5.6. *A 1-string blows down to* $[1, (2)^{k-1}]$ *(or to* $[1]$ *if we allow blowdowns at* a_1*).*

As an immediate consequence of this Lemma, a chain of rational curves $\bigcup_{i=1}^{k} E_i$ on a smooth surface will contract to a smooth point if and only if the negative self-intersection numbers $a_i = -E_i^2$ form a 1-string $[a_1, \ldots, a_k]$.

There is an obvious implication

$$\frac{1}{m} = [a_1, \cdots, a_k] \qquad \Longleftrightarrow \qquad 0 = [m, a_1, \cdots, a_k]$$

and, by the cyclicity property for 0-strings, there is also a unique $m' \in \mathbb{Z}_{\geq 1}$ such that $[a_1, \cdots, a_k, m'] = 0$. In particular not both of $m, m' = 1$.

5.1.2 T-strings

T-strings are a very special class of HJ-strings that have been considered by Brieskorn and Kollár & Shepherd-Barron [KSB] amongst others. These are the HJ-strings that correspond to cyclic quotient singularities $\frac{1}{r}(1, a)$ admitting a $\mathbb{Q}$-*Gorenstein smoothing.* They have a number of remarkable properties.

Definition 5.7. Let $T_{r,d,a}$ be the HJ-string given by the continued fraction expansion of $\frac{r^2 d}{rda-1}$, where $r, d, a \in \mathbb{Z}_{\geq 1}$, $r > a$ and $\operatorname{hcf}(r, a) = 1$. We call $T_{r,d,a}$ a *T-string* and $T_{r,1,a}$ a *simple T-string.*

Lemma 5.8 ([KSB] Proposition 3.11)**.** *Let* $T_{r,d,a} = [c_1, \ldots, c_m]$ *be a T-string. Any T-string can be obtained by starting with one of the forms in (1) and applying steps (2) and (3) repeatedly.*

1. $T_{2,1,1} = [4]$ *and* $T_{2,d,1} = [3, (2)^{d-2}, 3]$ *for* $d \geq 2$
2. $T_{r,d,r-a} = [c_m, \ldots, c_1]$
3. $T_{r+a,d,a} = [c_1 + 1, c_2, \ldots, c_m, 2]$

Proof. We first prove the three statements.

1. An easy calculation.
2. Follows from $\big(rda - 1\big)\big(rd(r - a) - 1\big) \equiv 1 \mod r^2 d.$

3. Suppose $\frac{r^2d}{rda-1} = [c_1, \dots, c_m]$. Then by (2) it follows that:

$$[2, c_m, \dots, c_1] = 2 - \frac{rd(r-a)-1}{r^2d} = \frac{rd(r+a)+1}{r^2d}$$

Also we have

$$T_{r+a,d,a} = \frac{(r+a)^2d}{(r+a)da-1} = 1 + \frac{rd(r+a)+1}{(r+a)da-1}$$

and since $r^2d\big((r+a)da-1\big) \equiv 1 \mod rd(r+a)+1$ the result follows.

We now prove the main claim of the Lemma. First suppose that either $r = 2$ or $a = \frac{1}{2}r$. Since r and a are coprime, we must have $r = 2$, $a = 1$ and then all possible $T_{r,d,a}$ are given in (1).

If $r > 2$ then either $a < \frac{1}{2}r$ or $r - a < \frac{1}{2}r$. Applying (2) if necessary, we can assume $a < \frac{1}{2}r$ and then $T_{r-a,d,a}$ and $T_{r,d,a}$ are related as in (3). We can repeat this process with $T_{r-a,d,a}$ until we are back in the $r = 2$ case. □

Lemma 5.9. *Suppose we have the complementary HJ-strings* $\frac{r}{a} = [a_1, \dots, a_k]$ *and* $\frac{r}{r-a} = [b_1, \dots, b_l]$. *Then*

$$T_{r,1,a} = \tfrac{r^2}{ra-1} = [a_1, \dots, a_{k-1}, a_k + b_l, b_{l-1}, \dots, b_1]$$
$$T_{r,d,a} = \tfrac{r^2d}{rda-1} = [a_1, \dots, a_{k-1}, a_k + 1, (2)^{d-2}, b_l + 1, b_{l-1}, \dots, b_1] \qquad \text{if } d \geq 2$$

and the complementary fraction to $T_{r,d,a}$ *(for any* d*) is:*

$$\tfrac{r^2d}{r^2d-rda+1} = [b_1, \dots, b_l, d+1, a_k, \dots, a_1]$$

Proof. For the first two identities we can use the inductive process given in the previous lemma. By inspection it is clearly true when $r = 2$ and $a = 1$. It also clear that it holds after applying (2), so we only need to check (3). This follows from:

$$\tfrac{r+a}{a} = [a_1 + 1, a_2, \dots, a_k], \quad \tfrac{r+a}{r} = [2, b_1, \dots, b_l]$$

The third identity concerns the complementary fraction to $T_{r,d,a}$ so it is enough to check that it satisfies the rule given in Lemma 5.2(3). If $d \geq 2$, starting with

$$[a_1, \dots, a_k + 1, (2)^{d-2}, b_l + 1, \dots, b_1, 1, a_1, \dots, a_k, d+1, b_l, \dots, b_1]$$

we see that $[b_l, \ldots, b_1]$ and $[a_1, \ldots, a_k]$ are complementary so this blows down to:

$$[a_1, \ldots, a_k + 1, (2)^{d-2}, 1, d + 1, b_l, \ldots, b_1]$$

Since $\frac{d}{d-1} = [(2)^{d-1}]$ and $\frac{d}{1} = [d]$ are complementary this blows down to

$$[a_1, \ldots, a_k, 1, b_l, \ldots, b_1]$$

and, as these are complementary, we do indeed get 0. The case $d = 1$ is similar. $\square$

5.2 Deformations of surface cyclic quotient singularities

Let $(P \in S)$ be the germ of a T-singularity and consider the minimal resolution $\mu\colon (E \subset \widetilde{S}) \to (P \in S)$. The exceptional divisor E is a chain of rational curves with negative self-intersection numbers given by the T-string $[c_1, \ldots, c_m]$.

Lemma 5.10. *K_S is a $\mathbb{Q}$-Cartier divisor of index r. In particular*

$$K_{\widetilde{S}} = \mu^* K_S - \sum_{i=1}^{m} \left(1 - \frac{\beta_i}{r}\right) E_i$$

where $1 \leq \beta_1, \ldots, \beta_m < r$ are integers such that $c_i\beta_i = \beta_{i-1} + \beta_{i+i}$ for $i = 2, \ldots, m-1$ and $\beta_1 = a$, $\beta_m = r - a$.

Proof. Set $\beta_0 := r$ and $\beta_{m+1} := r$. Then the relation $c_i\beta_i = \beta_{i+1} + \beta_{i-i}$ comes from computing $K_{\widetilde{S}} \cdot E_i$ for $i = 1, \ldots, m$.

The claim that $\beta_1 = a$, $\beta_m = r - a$ can be verified using the inductive process of Lemma 5.8. It is easy to check that $\beta_i = 1$ for all the exceptional curves for the T-singularities appearing in (1) so the claim holds. It is also easy to see that the claim holds under (2). Therefore we only need to check (3).

Suppose the claim holds for $T_{r,d,a} = [c_1, \ldots, c_m]$, so we must check for $T_{r+a,d,a} = [c_1 + 1, \ldots, c_m, 2]$. We set $\beta_0 = \beta_{m+2} := r + a$ and we claim that β_i stays the same for $i = 2, \ldots, m + 1$ (note $\beta_{m+1} = r = \beta_1 + \beta_m$). By induction $c_i\beta_i = \beta_{i-1} + \beta_{i+1}$ for $i = 2, \ldots, m$ and, from our choice of β_0, β_{m+2}, it is easy to check that $(c_1 + 1)\beta_1 = \beta_0 + \beta_2$ and $2\beta_{m+1} = \beta_m + \beta_{m+1}$. $\square$

Example 5.11. For the T-string $T_{11,1,3} = [4, 5, 3, 2, 2]$ we have:

$$\begin{array}{cccccc} c_i & \overset{4}{\bullet} & \!\!-\!\!-\!\! \overset{5}{\bullet} & \!\!-\!\!-\!\! \overset{3}{\bullet} & \!\!-\!\!-\!\! \overset{2}{\bullet} & \!\!-\!\!-\!\! \overset{2}{\bullet} \\ \beta_i & 3 & 1 & 2 & 5 & 8 \end{array}$$

Since for a T-singularity $(P \in S)$ the canonical divisor K_S is a $\mathbb{Q}$-Cartier divisor of index r, we can take the canonical cover to see that $(P \in S)$ is the cyclic quotient of an A_{rd-1} Du Val singularity:

$$\tfrac{1}{r^2 d}(1, rda - 1) = \left(V(xz - y^{rd}) \subset \mathbb{C}^3_{x,y,z}\right) \ / \ \tfrac{1}{r}(1, a, r-1)$$

Indeed the canonical covering of a cyclic quotient singularity $(P \in S)$ is a Du Val singularity if and only if $(P \in S)$ is either Du Val or a T-singularity.

5.2.1 $\mathbb{Q}$-Gorenstein smoothings

Let $(P \in S_0)$ be a surface cyclic quotient singularity and suppose that

$$\sigma\colon (S_0 \subset \mathcal{S}) \to (0 \in \mathbb{C}_t)$$

is a 1-parameter smoothing of $(P \in S_0)$, i.e. a flat family of surfaces such that the generic fibre S_t is smooth.

Definition 5.12. A smoothing σ is called a $\mathbb{Q}$-*Gorenstein smoothing* if the total space $\mathcal{S}$ is Cohen–Macaulay and $K_{\mathcal{S}}$ is a $\mathbb{Q}$-Cartier divisor.

A consequence of this condition is that $K^2_{S_0} = K^2_{S_t}$ remains constant for all $t \in \mathbb{C}$. Moreover, by [KSB] Corollary 3.6, the total space $\mathcal{S}$ has at worst a terminal singularity at $(P \in \mathcal{S})$.

Conversely suppose $(P \in S)$ is a surface cyclic quotient singularity that admits a 1-parameter smoothing $\sigma\colon (S_0 \subset \mathcal{S}) \to (0 \in \mathbb{C})$ to a 3-fold terminal singularity $(P \in \mathcal{S})$. Then, by [KSB] Proposition 3.10, σ is a $\mathbb{Q}$-Gorenstein smoothing and $(P \in S_0)$ is either a Du Val singularity or a T-singularity.

For $(P \in S)$, a $\frac{1}{r}(1, a)$ quotient singularity, Kollár & Shepherd-Barron proved that the irreducible components of the deformation space Def S of such a singularity are smooth. Moreover they are in one-to-one correspondence with P-*resolutions*, that is partial resolutions $\mu\colon \overline{S} \to S$ such that $\overline{S}$ has only T-singularities and $K_{\overline{S}/S}$ is ample. They showed that a $\mathbb{Q}$-Gorenstein deformation of $\overline{S}$ blows down to a deformation of S and that all deformations are obtained in this way.

Stevens [S] and Christophersen actually enumerated P-resolutions (and hence the different components of Def S). We recall this here as it bears some resemblance to Conjecture 6.11.

Theorem 5.13 ([KSB] Theorem 3.9, [S] Theorem 4.1, Christophersen)**.** *The P-resolutions of $\frac{1}{r}(1, a)$ are in one-to-one correspondence with 0-strings dominated* *by the complementary HJ-string* $\frac{r}{r-a} = [b_1, \ldots, b_l]$*, i.e. HJ-strings of the form* $[c_1, \ldots, c_l] = 0$ *with* $c_i \leq b_i$ *for* $i = 1, \ldots, l$*.*

There is a very nice explanation of this correspondence in [GHKv1] §6.

Chapter 6

The main case: type A

We now turn to study type A extractions. These comprise of infinite families of complicated examples and, for the time being, an explicit classification seems very hard.

Assume that we are in the usual situation of §0.2 and are trying to construct a divisorial extraction

$$\sigma\colon (Z \subset E \subset Y) \to (P \in C \subset X),$$

or equivalently a presentation of the (extended) symbolic power algebra $\bigoplus_{n\geq 0} I_{C/X}^{[n]}$. In particular S_X is the *general* hypersurface section containing C, $H_Y = \sigma^{-1}(H_X)$ is the pullback of a general hyperplane section $H_X \in |\mathfrak{m}_P|$ and $Z = \sigma^{-1}(P)_{\mathrm{red}}$ is the reduced central fibre. In this chapter we will assume that the general hypersurface section S_X has a type A Du Val singularity at P.

Outline of the main Theorem 0.3

We now explain the main result of this chapter, Theorem 0.3, which provides the first large case in the study of these extractions.

In §6.1 we see that in this type A case our divisorial extraction $\sigma\colon Y \to X$ can be viewed as a 1-parameter $\mathbb{Q}$-Gorenstein smoothing of a natural hyperplane section $(H_Y \subset Y)$ and that there are some strong restrictions on the type of singularities the surface H_Y can have. We give an explicit description of the divisorial extraction Y under the assumptions:

1. H_Y has *rational, normal singularities* with a *unique high index point* $(Q \in H_Y)$ given by a *simple T-singularity* (see Assumption 6.3),
2. the central fibre $Z \cong \mathbb{P}^1$ is *irreducible.*

Lemma 6.7 gives a classification of the possible neighbourhoods ($Z \subset H_Y$) satisfying these conditions, which gives a whole family of cases depending on two integers: $m \geq 2$ and $k \geq 1$. Lemma 6.9 describes which curve ($C \subset S_X$) corresponds to the $\mathbb{Q}$-Gorenstein smoothing of this neighbourhood.

Then our main result is Theorem 0.3—an explicit description of the divisorial extraction $Y = \operatorname{Proj}_X \bigoplus I_{C/X}^{[n]}$ as a sequence of serial type I unprojections, for this choice of ($C \subset S_X$).

The proof of Theorem 0.3 is given in §6.3.2 and is similar to the proof [BR1] §5 of the existence of the main case of diptych varieties. An outline of our proof is given at the beginning of §6.3.2. In brief, we start from an explicit form for the equations of ($C \subset X$) and write down generators for $\bigoplus_{n\in\mathbb{Z}} I_{C/S_X}^{[n]}$, the (extended) symbolic power algebra restricted to S_X. Then starting from Y_1 the ordinary blowup of C, a Gorenstein ring in codimension 3, the essential content of the proof is in proving the existence of a type I unprojection divisor ($D_\alpha \subset Y_\alpha$) at each stage. These divisors D_α are (possibly non-reduced) complete intersection divisors given explicitly in the course of the proof. Then, by the unprojection theorem 2.5 and induction, $Y_{\alpha+1}$ exists as a variety embedded in codimension 1 greater than Y_α and $\mathcal{O}_{Y_{\alpha+1}}$ is a Gorenstein ring.

6.1 Two special hypersurface sections

Throughout §6.1 we do not need to assume that X is smooth. We concentrate on Mori extractions, where ($P \in X$) is a terminal singularity of index 1 (i.e. a cDV singularity), but almost all of the discussion holds for flipping contractions too.

One effective way of studying an extremal neighbourhood $\sigma\colon Y \to X$ is to consider what happens to two special divisors on X after pulling back to Y. The first divisor, which we have already seen and used, is the *general elephant* $S_Y \in |-K_Y|$. The second, which hasn't played much of a role until now, is the *general hyperplane section* H_Y.

As ($P \in X$) is cDV the general hyperplane $H_X \in |\mathfrak{m}_P|$ passing through P has at worst a Du Val singularity. Unfortunately $H_Y = \sigma^{-1}H_X$, the birational transform of H_X on Y, can have very bad (non-log-canonical) singularities, even when σ is a Mori extraction, e.g. [Tz4] Theorem 4.1(2)(a). However, in the case that ($P \in S_X$) is at worst a type A Du Val singularity then, by Theorem 6.1, the general hyperplane H_Y has at worst semi-log-canonical singularities which necessarily admit a $\mathbb{Q}$-Gorenstein smoothing.

Now assume that we have any neighbourhood ($Z \subset H_Y$) admitting a 1-

parameter $\mathbb{Q}$-Gorenstein smoothing and a contracting morphism

$$\sigma\colon (Z \subset H_Y) \to (P \in H_X)$$

with $\sigma_*\mathcal{O}_{H_Y} = \mathcal{O}_{H_X}$ and with rational singularities, i.e. $R^1\sigma_*\mathcal{O}_{H_Y} = 0$. Then by [KM] Proposition 11.4 this $\mathbb{Q}$-smoothing extends to a birational morphism of terminal 3-folds $\sigma\colon Y \to X$.

$$\begin{tikzcd}
(Z \subset H_Y) \arrow[r, "\mathbb{Q}\text{-smoothing}"] \arrow[d] & (E \subset Y) \arrow[d, "\sigma"] \\
(P \in H_X) \arrow[r] & (C \subset X)
\end{tikzcd}$$

If $(P \in H_X)$ is at worst a Du Val singularity, then σ is necessarily a Mori contraction (not a flipping contraction) since $(P \in X)$ is a cDV singularity.

This is the way that Kollár & Mori [KM] and Tziolas [Tz3] view type A neighbourhoods. Brown & Reid [BR1] also use these two surfaces to define the two panels of their diptych varieties, the starting point of their work. In this type A case, if H_Y is normal then H_Y and S_Y are two toric surfaces glued along their toric boundary strata. All of the geometry of the neighbourhood comes from the combinatorics of these two surfaces and the way in which they are glued together.

6.1.1 The general hyperplane H_Y

For a Mori extraction $\sigma\colon (E \subset Y) \to (C \subset X)$, as $(H_X \cap C) = P$ it follows that H_Y contains the central fibre Z and the induced contraction $\sigma\colon (Z \subset H_Y) \to (P \in H_X)$ is otherwise an isomorphism. Moreover $-K_{H_Y}$ is σ-ample and σ contracts a configuration Z of complete rational curves to at at worst a Du Val singularity $(P \in H_X)$.

As we have already said, there are some strong restrictions on the type of singularities that H_Y can have. The following Theorem appears in [Tz3] Lemma 3.1 and Corollary 3.2.

Theorem 6.1. *Suppose that $\sigma\colon (Z \subset Y) \to (P \in X)$ is a terminal neighbourhood with 1-dimensional central fibre $Z = \sigma^{-1}(P)_{red}$. Suppose that $H_X \in |\mathfrak{m}_P|$ is a general hyperplane section through P and let $H_Y = \sigma^{-1}H_X$. Then H_Y has semi-log-canonical singularities. Moreover any high index points of H_Y are semi-log-terminal and lie on $S_Y \cap H_Y$.*

In particular, of such singularities those which are rational and admit a $\mathbb{Q}$-Gorenstein smoothing are classified in [KSB] §5.2 as follows:

1. index > 1

(a) (normal) type T *singularities* $\frac{1}{r^2d}(1, rda - 1)$,

(b) (non-normal) quotient normal crossing $\left((yz = 0) \subset \mathbb{C}^3_{x,y,z}\right) / \frac{1}{r}(1, a, r - a)$,

2. index $= 1$

(a) (normal) Du Val singularities,

(b) (non-normal)

i. normal crossing $\left((yz = 0) \subset \mathbb{C}^3_{x,y,z}\right)$,

ii. pinch points $\left((xy^2 = z^2) \subset \mathbb{C}^3_{x,y,z}\right)$,

iii. degenerate cusps.

Remark 6.2. We now have two different ways of viewing the same neighbourhood. Up until this chapter we had viewed σ as a Mori extraction blowing up a curve in a 3-fold $(C \subset X)$. Alternatively, we can now view σ as the $\mathbb{Q}$-Gorenstein smoothing of a surface contraction $\sigma\colon (Z \subset H_Y) \to (P \in H_X)$.

As Kollár & Mori point out ([KM] §4.10), it is not very easy to work out which $(C \subset X)$ correspond to which $(Z \subset H_Y)$. In the rest of the chapter we give some examples and make some speculation, but this remains one of the biggest difficulties in general. The description of $(C \subset X)$ is useful for constructing Y explicitly using unprojection, whereas $(Z \subset H_Y)$ provides much better restrictions on Mori extractions.

6.2 Non-semistable extractions

From now on we assume, once more, that $(P \in X)$ is a smooth point. As can be seen from Proposition 6.1, to study $(Z \subset H_Y)$ in general there are a large number of cases to consider depending on the singularities of H_Y and the configuration of components of Z. Therefore we choose to restrict attention to a first case that we can study explicitly.

Assumption 6.3. We assume that $\sigma\colon Y \to X$ is a *normal non-semistable* neighbourhood, i.e. that H_Y is normal with rational singularities and $\sigma\colon S_Y \cong S_X$. In particular H_Y has exactly one high index point $(Q \in H_Y)$ which is a T-singularity. Moreover, we assume that this is a *simple* T*-singularity*, i.e. a quotient singularity of the form $\frac{1}{r^2}(1, ra - 1)$ where $\gcd(r, a) = 1$.

The assumption that σ is *non-semistable* is useful for two reasons. First, since $S_Y \cong S_X$ it follows that $(S_Y \cap H_Y)$ is a point, so this restricts H_Y to having a single high index point. Secondly it gives us more information about generators and relations of $\bigoplus I^{[n]}_{C/X}$.

Lemma 6.4. *Let $\eta \in \bigoplus I^{[n]}_{C/X}$ be the generator in degree 1 corresponding to the equation of S_X. Then σ is non-semistable if and only if the generators and relations of $\bigoplus I^{[n]}_{C/X}$ are lifts, modulo η, of the generators and relations of $\bigoplus I^{[n]}_{C/S_X}$ in the same degrees.*

Proof. Indeed, σ lifts $\bigoplus I^{[n]}_{C/S_X}$ if and only if σ restricts to the blowup of this algebra over S_X. But this is simply the blowup of the Cartier divisor $(rC \subset S_X)$, where r is the index of $(P \in S_X)$. So $\sigma\colon S_Y \to S_X$ is an isomorphism. $\square$

It is easy to calculate $\bigoplus I^{[n]}_{C/S_X}$ when $(P \in S_X)$ is a type A Du Val singularity. We can use this a guide to see what ideals we expect to unproject as we build $\bigoplus I^{[n]}_{C/X}$ and as a sanity check to make sure that we get the right final answer.

Remark 6.5. We concentrate on non-semistable extractions purely for the convenience of knowing what generators and relations to expect to lift to our ring. Mori [M2] Theorem 4.3 and Brown & Reid's diptych varieties [BR4] §5.3 already contain examples of semistable divisorial extractions.

We assume that H_Y has a *simple T-singularity* as this is the most general case. The non-simple T-singularities deform to simple ones so, up to a choice of base change, we could choose to study neighbourhoods with only simple T-singularities and then treat the more singular cases as degenerations of these.

We can also choose to get rid of any Du Val singularities by the trick of Remark 6.8.

Local coordinates

We now fix some local coordinates for constructing Mori extractions in this setting.

General elephant Up to a local analytic change of coordinates we may assume that the general elephant is given by the standard embedding of an A_{r-1} Du Val singularity in $\mathbb{C}^3$:

$$(P \in S_X \subset X) \cong \left(0 \in V(xz - y^r) \subset \mathbb{C}^3_{x,y,z}\right)$$

As usual, $(P \in S_X)$ is isomorphic to the quotient of $\mathbb{C}^2_{u,v}$ by the cyclic group action $\frac{1}{r}(1,-1)$ and we call u, v *orbinates* on S_X. In particular, we can write x, y, z in terms of the orbinates as $x, y, z = u^r, uv, v^r$.

A curve $(P \in C \subset S_X)$ can be given by an *orbifold equation* $\gamma(u,v)$ as in §3.1. Here $\gamma(u,v)$ is a semi-invariant equation of character a, so that $\gamma = u^a f + v^{r-a} g$ for some invariant functions $f(x,y,z)$, $g(x,y,z) \in \mathcal{O}_{S_X}$. By Assumption 6.3 r and a are coprime.

We pass to the extended symbolic power algebra by introducing the inclusion $\iota\colon I_{C/X} \hookrightarrow \mathcal{O}_X$, a variable of weight -1. In particular this introduces the equation $\iota\eta = xz - y^r$ where η is the named equation of S_X. In this extended ring S_X is given by the hyperplane section $V(\eta)$ and we look to construct Y by lifting with respect to η.

General hyperplane A hyperplane section H_X passing through $(P \in X)$ which is general with respect to our chosen elephant S_X is given by

$$(P \in H_X \subset X) \cong (0 \in V(y) \subset \mathbb{C}^3).$$

After a possible change of coordinates we may assume that this choice of H_X is also general with respect to the curve $(C \subset X)$, i.e. that C meets H_X transversally. Note that H_X and S_X are two toric surfaces meeting along their toric boundary strata $H_X \cap S_X = \mathbb{C}^1_x \cup \mathbb{C}^1_z$. H_Y is the hyperplane section $V(y)$ and y is precisely the parameter that gives the $\mathbb{Q}$-Gorenstein smoothing of $(Z \subset H_Y)$.

We choose to write the orbifold equation $\gamma(u,v)$ in longhand as

$$\gamma(u,v) = u^{\alpha_m} + f_1 u^{\alpha_{m-1}} v^{\beta_1} + f_2 u^{\alpha_{m-2}} v^{\beta_2} + \cdots + f_{m-1} u^{\alpha_1} v^{\beta_{m-1}} + v^{\beta_m} \qquad (6.1)$$

where $f_i \in \mathcal{O}_{S_X}$, the α_i are strictly decreasing and the β_i are strictly increasing, i.e. the (α_i, β_i) are points on the boundary of the Newton polytope of γ. As H_X is general with respect to C we see that γ contains nonzero monomials u^{α_m}, v^{β_m} and, assuming an analytic change of variables, we can take the coefficients of these terms to be 1.[1]

6.2.1 Lifting an elephant

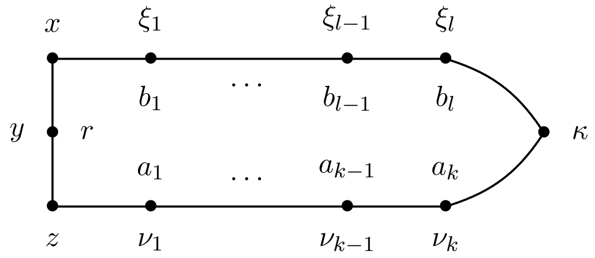
As we have said, this ring $\bigoplus I^{[n]}_{C/S_X}$ is easily described when $(P \in S_X)$ is the type A Du Val singularity $\frac{1}{r}(1,-1)$. In terms of the orbinates, $\bigoplus I^{[n]}_{C/S_X}$ is given by the

[1]We could allow coefficients $f_0, f_m \neq 1$, as in the Prohkorov–Reid example §3.3, and then choose to study degenerate cases when f_0 or f_m vanish. Since we are concerned with the most general case we choose not to do this. Including them is harmless but it makes the equations more complicated.

ring of invariants

$$\bigoplus_{n\geq 0} I_{C/S_X}^{[n]} \cong \mathbb{C}[u,v,\gamma]^{\frac{1}{r}(1,-1,a)}$$

with the grading that comes from setting the weight of u, v to be 0 and the weight of γ to be 1. We can represent the generators of this ring by the monomial 'triangle'



with corners $x, z, \kappa = u^r, v^r, \gamma^r$. Here $\frac{r}{a} = [a_1, \ldots, a_k]$ and $\frac{r}{r-a} = [b_1, \ldots, b_l]$.

The *tag* b_1 on the node corresponding to ξ_1 records the 'tag equation' $x\xi_2 = \xi_1^{b_1}$ and similarly for all the other tags. If we let $b \equiv a^{-1} \mod r$, then we can write the following generators in terms of orbinates:

$$\xi_1 = [u^{r-a}\gamma], \quad \xi_k = [u\gamma^{r-b}], \quad \nu_1 = [v^a\gamma], \quad \nu_l = [v\gamma^b]$$

The triangle has no internal points as $y = uv$ is already invariant.

Since it defines a toric variety, this is a Cohen-Macaulay ring and is given by $\binom{k+l+1}{2}$ equations. There is a unique equation of the form $st = \cdots$ for each internal diagonal s—t of this $(k+l+4)$-gon that avoids y.

Extended symbolic power algebra

The triangle gives the generators and relations of the ordinary symbolic power algebra $\bigoplus_{n\geq 0} I_{C/S_X}^{[n]}$. We now throw in the generator ι to get the extended power algebra.

Lemma 6.6. *The extended power algebra $R = \bigoplus_{n\in\mathbb{Z}} I_{C/S_X}^{[n]}$ is a Gorenstein ring given by $\frac{1}{2}(k+l+1)(k+l+4)$ equations.*

Proof. By our choice of γ, we can write ι a rational function in the orbinates by

$$\iota = \frac{u^a f + v^{r-a} g}{\gamma}$$

which we note is invariant under the $\frac{1}{r}(1, r-1, a)$ action. Therefore we can multiply

ι against any invariant monomial of positive degree to get an expression

$$\iota[u^U v^V \gamma^G] = [u^{U+a} v^V \gamma^{G-1}]f + [u^U v^{V+r-a} \gamma^{G-1}]g$$

which, as $G \geq 1$, can be rendered as a polynomial in terms of the other generators. Thus the inclusion of ι introduces $k+l+1$ new equations of the form $\iota\xi_j = \cdots$, $\iota\nu_j = \cdots$ and $\iota\kappa = \cdots$. Since ι is the only generator in negative degree, $\deg\iota = -1$ and ι multiplies all positive degree generators these are the only new equations and we have $\frac{1}{2}(k+l+1)(k+l+4)$ in total.

Now we prove that R is Gorenstein. Take the hyperplane section $\overline{R} = R/(y)$, i.e. we set $uv = 0$. If $\overline{R}$ is Gorenstein then it will follow that R is Gorenstein by Remark 2.2.

By (6.1) the orbifold expression for $\iota \in \overline{R}$ becomes $\iota = \gamma^{-1}(u^{\alpha_m} + v^{\beta_m})$. Thus $\iota\xi_1 = x^{b_0}$ where $b_0 = \frac{1}{r}(r - a + \alpha_m)$ and $\iota\nu_1 = z^{a_0}$ where $a_0 = \frac{1}{r}(a + \beta_m)$. All the equations that cross over two long edges of the triangle become $\xi_i\nu_j = 0$, etc. These are the equations of a reducible toric surface $S = S_1 \cup S_2$ where S_1, S_2 are glued along their toric boundary strata $\mathbb{C}^1_\iota \cup \mathbb{C}^1_\kappa$. Since $\frac{ra_0 - a}{r} = [a_0, a_1, \ldots, a_k]$ one of these surfaces is

$$S_1 = \tfrac{1}{ra_0 - a}(1, ra_0 - a - r) \subset \mathbb{C}^{k+3}_{\iota,z,\nu_1,\ldots,\nu_k,\kappa}$$

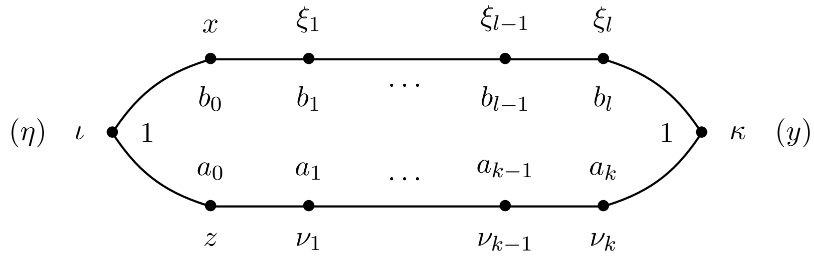
and similarly for $S_2 \subset \mathbb{C}^{l+3}_{\iota,x,\xi_1,\ldots,\xi_l,\kappa}$.

By a similar proof to [BR1] Lemma 2.3 the ring $\overline{R}$ is Gorenstein. These two surfaces are glued in codimension 1 so $\overline{R}$ is Cohen-Macaulay by [R3] Proposition 2.2. Now write down two 2-forms that generate each of ω_{S_1} and ω_{S_2} ($\frac{dz}{z} \wedge \frac{d\iota}{\iota}$ on S_2 and $\frac{d\iota}{\iota} \wedge \frac{dx}{x}$ on S_2 say) with residues that cancel along each of $\mathbb{C}^1_\iota$ and $\mathbb{C}^1_\kappa$. Thus by [R3] Corollary 2.8(ii) we can define an invertible dualising sheaf ω_S on S, hence S is Gorenstein. $\square$

Our aim is to lift this to a Gorenstein ring by introducing a degree 1 variable η that acts as the equation of $(S_X \subset X)$. In other words we want to introduce the equation $\iota\eta = xz - y^r$ and find lifts of all the other generators and relations that preserve degrees.

Rugby balls

We extend our monomial triangle to include ι and η by adding a 'cap' on the LHS. We call the resulting diagram a *rugby ball* and we treat it as a convenient picture that records some information about the generators and relations of our ring.



There is one equation for each internal diagonal of this $(k + l + 4)$-gon. In the language of [BR1], the (y) and (η) appearing at either end of the rugby ball are *annotations*, corresponding to the general hyperplane and general elephant of our Mori extraction respectively. They modify the tag equations at each of those points by $xz \equiv \iota\eta \mod y$ and $\xi_k\nu_l \equiv y\kappa \mod \eta$. The tag b_0 records $\iota\xi_1 \equiv x^{b_0} \mod y$ and similarly for a_0 and z.[2]

If we do manage to lift this ring by η then, after projecting out ι and dividing out by the $\mathbb{C}^*$-action corresponding to our usual grading, we get a variety Y that contains the coordinate point $Q = P_\kappa$. At this point we can use the fact that κ is invertible to eliminate all variables except y, η, ν_k, ξ_l, to be left with a hyperquotient singularity

$$\left(\xi_l\nu_k = y + \eta^{rd}\right) \subset \mathbb{C}^4_{\xi_l,\nu_k,\eta,y} \,/\, \tfrac{1}{r}(r - b, b, 1, 0)$$

for some function $d \geq 1$. Thus $H_Y = V(y)$ naturally has a T-singularity $(Q \in H_Y)$:

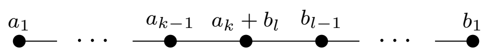
$$\left(\xi_k\nu_l = \eta^{rd}\right) \subset \mathbb{C}^4_{\xi_l,\nu_k,\eta} \,/\, \tfrac{1}{r}(r - b, b, 1)$$

6.3 Irreducible central fibre

We now treat our main case of normal non-semistable neighbourhoods with irreducible central fibre $Z \cong \mathbb{P}^1$ and normal general hyperplane section H_Y satisfying Assumption 6.3. From §6.1.1 these correspond to $\mathbb{Q}$-Gorenstein smoothings of a contraction:

$$\sigma \colon (Z \subset H_Y) \to (0 \in \mathbb{C}^2)$$

Let $\Delta = \Delta(Z \subset H_Y)$ be the dual intersection diagram of $(Z \subset H_Y)$. Recall we are assuming that H_Y has a unique simple T-singularity $(Q \in H_Y)$ which, by Lemma 5.9, has a resolution of the form:

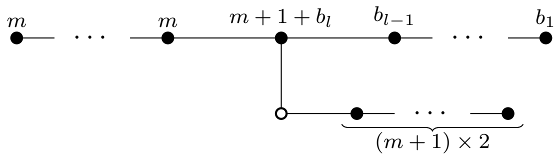


[2]If we had wanted to include coefficients $f_0, f_m \neq 1$ at this point we could include annotations (f_0) next to x and (f_m) next to z.

There are two coprime integers $r > a$ such that $\frac{r}{a} = [a_1, \cdots, a_k]$ and $\frac{r}{r-a} = [b_1, \cdots, b_l]$ are reduced complementary HJ-strings.

Description of $(Z \subset H_Y)$

Lemma 6.7. *If $(Z \subset H_Y)$ satisfies Assumption 6.3 and Z is irreducible then there exists $m \geq 2$ and $k \geq 1$ such that Δ is given by*



where $[b_1, \ldots, b_l]$ is the complementary continued fraction to $[(m)^{k-1}, m+1]$.

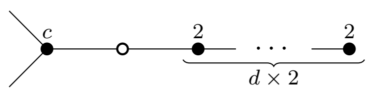
Proof. We prove the lemma through the following series of claims:

1. *Any blowdown of Δ has a unique -1-curve which cannot appear at a fork.*

   Requiring that $(Z \subset H_Y)$ contracts to $(0 \in \mathbb{C}^2)$ is equivalent to saying that Δ can be obtained by blowing up $(0 \in \mathbb{C}^2)$ at a sequence of (infinitely near) points. The final diagram Δ must have a unique -1-curve, clearly the node corresponding to Z. We cannot create two or more -1-curves, since any configuration dominating this one will always have at least two -1-curves. To create a fork at a curve C we must blow up at least one point on an exceptional curve, so that $C^2 \leq -2$.

2. *If Z intersects Q at a node of self-intersection $-c$ then Z intersects one end of a A_{c-2} Du Val singularity.*

   Z must intersect Q, so suppose Z intersects Q at a node labelled c. Z cannot appear at a fork, so Z intersects at most one other singularity, which must be a Du Val singularity. Z cannot intersect a Du Val singularity in any way other than at one end of an A_d singularity for some d, else blowing down will eventually create a fork at a -1-curve. Around the node corresponding to Z, Δ looks like

   It is clear that $d = c-2$. If $d > c-2$ then blowing down creates a fork at a -1-curve and if $d < c-2$ then, after blowing down all possible -1-curves, the curve corresponding to the node labelled c still has self-intersection $d + 1 - c \leq -2$.

3. *Z intersects Q at the node labelled $a_k + b_l$.*

   If not then, without loss of generality, Z intersects a node labelled a_i for $1 \leq i \leq k-1$. After contracting enough times to get rid of the A_{a_i-2} Du Val singularity coming from the previous claim, we are left with a chain of rational curves which blows down to a smooth point. Therefore, by Lemma 5.6, these self-intersection numbers must satisfy

   $$[a_1, \ldots, a_{i-1}, 1, \ldots, a_k + b_l, b_{l-1}, \ldots, b_1] = \frac{1}{m}$$

   for some $m \geq 1$. Since $a_k \geq 2$ a sequence of blowdowns gives

   $$[a_1, \ldots, a_j, 1, b_l + 1, b_{l-1}, \ldots, b_1] = \frac{1}{m}$$

   for some $j \leq k-2$, so that $[m, a_1, \ldots, a_j, 1, b_l + 1, b_{l-1}, \ldots, b_1] = 0$. But the reduced complementary HJ-string to $[b_1, \ldots, b_{l-1}, b_l + 1]$ is $[a_1, \ldots, a_k, 2]$ which has length $k+1 > k-1$. So this cannot happen.

4. *$a_1 = \cdots = a_{k-1} = m$ and $a_k = m+1$ for some $m \geq 2$.*

   We can assume that $[a_1, \ldots, a_{k-1}, 1, b_{l-1}, \ldots, b_1] = \frac{1}{m}$ for some $m \geq 2$, so that:

   $$[m, a_1, \ldots, a_{k-1}, 1, b_{l-1}, \ldots, b_1] = 0$$

   Therefore the reduced complementary HJ-string to $[b_{l-1}, \ldots, b_1]$ is given by $[m, a_1, \ldots, a_{k-1}]$. In particular this complement has length k.

   Now we either have $b_l = 2$ or $a_k = 2$. If $b_l = 2$ then

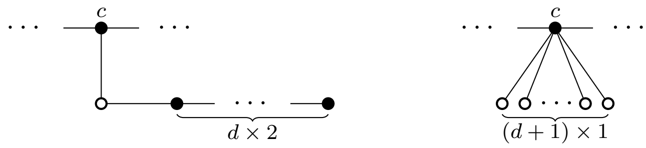
   $$[a_1, \ldots, a_{k-1}, a_k, 1, 2, b_{l-1}, \ldots, b_1] = [a_1, \ldots, a_{k-1}, a_k - 1, 1, b_{l-1}, \ldots, b_1] = 0$$

   so that $[a_1, a_2, \ldots, a_{k-1}, a_k - 1] = [m, a_1, \ldots, a_{k-2}, a_{k-1}]$ and, by the uniqueness of reduced complements, equating term-by-term gives $m = a_1 = a_2 = \cdots = a_{k-1} = a_k - 1$.

   The case $a_k = 2$ is impossible since, after blowing down a number of times, we have $[a_1, \ldots, a_i - 1, 1, b_{l-1}, \ldots, b_1] = 0$ which gives a complement of length $i < k$, a contradiction.

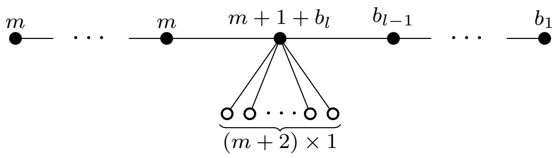
Given that $b_l = 2$ we necessarily have that $m + 1 + b_l = m + 3$ and hence these four claims imply that Δ is of the form stated in the lemma. $\square$

Remark 6.8. Note that whenever a component of Z intersects the end of a A_m Du Val singularity we can write down a neighbourhood with no Du Val singularities by splitting up this component of Z. The two configurations



both contract to the same result. As in Prokhorov & Reid's example §3.3 we view the case with Du Val singularities as a degeneration where two or more branches of $(C \subset S_X)$ gain some extra tangency.

Therefore, instead of treating strictly irreducible neighbourhoods, from now on we consider the more general case of Z a reducible curve with $m+2$ components having the following dual graph:



(†)

Description of $(C \subset S_X)$

We now work out which generic curve downstairs $(C \subset S_X \subset X)$ corresponds to this neighbourhood upstairs. We know that S_X is A_{r-1} Du Val singularity, where r is given by $\frac{r}{a} = [(m)^{k-1}, m+1]$.

Consider the sequence of integers defined by the recurrence relation:

$$Q_{-1} = -1, \quad Q_0 = 0, \quad Q_1 = 1, \quad mQ_k = Q_{k-1} + Q_{k+1} \quad \forall k > 1 \tag{6.2}$$

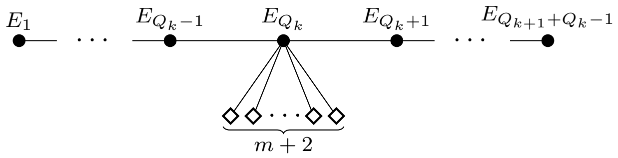
Then $[(m)^k] = \frac{Q_{k+1}}{Q_k}$ and $[m+1, (m)^{k-1}] = \frac{Q_{k+1}+Q_k}{Q_k}$. As $\frac{Q_k}{Q_{k-1}}$ and $\frac{Q_{k+1}}{Q_k}$ are convergents to the continued fraction $[(m)^k]$ we have, from the usual rules for convergents, that $Q_k^2 = Q_{k-1}Q_{k+1} + 1$ for all k. It follows that

$$Q_k(Q_k + Q_{k-1}) = Q_{k-1}(Q_k + Q_{k+1}) + 1 \implies Q_k^{-1} \equiv Q_k + Q_{k-1} \mod Q_{k+1} + Q_k$$

and therefore that:

$$\frac{r}{a} = [(m)^{k-1}, m+1] = \frac{Q_{k+1} + Q_k}{Q_k + Q_{k-1}}$$

Lemma 6.9. *Let $(Z \subset H_Y)$ be the the neighbourhood given by the configuration (†) above. Then the $\mathbb{Q}$-Gorenstein smoothing of $\sigma\colon (Z \subset H_Y) \to (0 \in \mathbb{C}^2)$ gives a* *divisorial contraction to a curve* $(C \subset S_X)$ *where* S_X *has a type* $A_{Q_{k+1}+Q_k-1}$ *Du Val singularity and the generic* C *has* $m+2$ *branches meeting* S_X *as follows:*



Proof. By [Tz3] Lemma 4.2 it follows that the curve $(C \subset S_X)$ has l branches where:

$$l = -\frac{(K_{H_Y} Z)^2}{Z^2}$$

Let $\mu\colon (F \subset \widetilde{H}_Y) \to (Q \in H_Y)$ be the minimal resolution of the T-singularitiy Q and $\widetilde{Z}$ be the strict transform of Z. Then $\widetilde{Z}$ is a collection of $m+2$ disjoint -1-curves on a smooth surface and we can compute:

$$l = -\frac{(K_{H_Y} Z)^2}{Z^2} = -\frac{(K_{\widetilde{H}_Y} \widetilde{Z})^2}{\widetilde{Z}^2} = \frac{(m+2)^2}{m+2} = m+2$$

Now $(C \subset S_X)$ is a curve with $m+2$ components and these intersect just one component E_j of the A_{r-1} Du Val singularity S_X. Hence j should satisfy:

$$j(m+2) \equiv Q_k + Q_{k-1} \mod Q_{k+1} + Q_k$$

But $(m+2)Q_k = Q_{k+1} + 2Q_k + Q_{k-1} \equiv Q_k + Q_{k-1} \mod Q_{k+1} + Q_k$ so

$$j \equiv Q_k \mod Q_{k+1} + Q_k$$

(since Q_k is coprime to $Q_{k+1} + Q_k$). $\square$

This implies that the orbifold equation of C is given by $\gamma(u,v) = \Phi(u^{Q_k}, v^{Q_{k+1}}) \in \mathcal{O}_S(u^{Q_k}, v^{Q_{k+1}})$ where Φ is a homogenous equation of degree $m+2$

$$\Phi(X,Y) = X^{m+2} + f_1 X^{m+1} Y + \cdots + f_{m+1} X Y^{m+1} + Y^{m+2} \tag{6.3}$$

with coefficients $f_1, \ldots, f_{m+1} \in \mathcal{O}_X$. For convenience we set $f_0 = f_{m+2} = 1$ and, for $1 \leq i \leq m+1$, define:

$$\phi_i(X,Y) = \sum_{j=0}^{i} f_j X^{i-j} Y^j, \quad \psi_i(X,Y) = \sum_{j=i}^{m+2} f_j X^{m+2-j} Y^{j-i} \tag{6.4}$$

The ϕ_i are just successive truncations of Φ from the right, i.e.

$$\begin{aligned}
\phi_{m+1}(X,Y) &= X^{m+1} + f_1X^mY + \cdots + f_mXY^m + f_{m+1}Y^{m+1} \\
\phi_m(X,Y) &= X^m + f_1X^{m-1}Y + \cdots + f_mY^m \\
&\ \vdots \\
\phi_2(X,Y) &= X^2 + f_1XY + f_2Y^2 \\
\phi_1(X,Y) &= X + f_1Y
\end{aligned}$$

and similarly the ψ_i are truncations of Φ from the left. In particular $\phi_i = X\phi_{i-1} + f_iY^i$ and $\psi_i = f_iX^{m+2-i} + Y\psi_{i+1}$.

Now we can check that the orbifold equation $\gamma(u,v)$ can be rewritten in this notation as

$$\gamma(u,v) = \phi_1(x, y^{Q_{k+1}})u^{Q_k+Q_{k-1}} + \psi_2(y^{Q_k}, z)v^{Q_{k+1}-Q_{k-1}}$$

and hence that $(C \subset S_X \subset X)$ is given by the equations:

$$\bigwedge^2 \begin{pmatrix} x & y^{Q_k+Q_{k-1}} & \psi_2(y^{Q_k}, z) \\ y^{Q_{k+1}-Q_{k-1}} & z & \phi_1(x, y^{Q_{k+1}}) \end{pmatrix} = 0$$

6.3.1 Big example 1

Before attempting to prove the existence of $\sigma\colon Y \to X$ in the general case we calculate the specific example when $m = 3$, $k = 2$ in order to get a flavour of this ring. In this case we set

$$Q_{-1}, Q_0, Q_1, Q_2, Q_3 = -1, 0, 1, 3, 8.$$

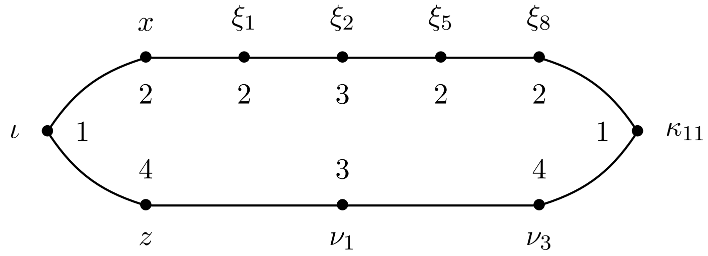
Then the curve $(C \subset X)$ is given by the equations:

$$\bigwedge^2 \begin{pmatrix} x & y^4 & -(f_2y^9 + f_3y^6z + f_4y^3z^2 + z^3) \\ y^7 & z & x + f_1y^8 \end{pmatrix} = 0$$

Thus $S_X \cong \frac{1}{11}(1,10)$ is a type A_{10} Du Val singularity and, since $\gamma(u,v) \in \mathcal{O}_{S_X}(u^4, v^7)$, cutting down to $(C \subset S_X)$ the (extended) symbolic power algebra $\bigoplus I_{C/S}^{[n]}$ is generated by the usual x, y, z in degree 0, ι in degree -1 and seven more generators in degrees $1, 1, 2, 3, 5, 8, 11$ which we write as:

$$\begin{array}{ccccccc} \xi_1 & \xi_2 & \xi_5 & \xi_8 & \kappa_{11} & \nu_3 & \nu_1 \\ u^7\gamma & u^3\gamma^2 & u^2\gamma^5 & u\gamma^8 & \gamma^{11} & v\gamma^3 & v^4\gamma \end{array}$$

This is a Gorenstein ring whose generators and tags are given by the following rugby ball. There are 35 equations coming from the 35 interior diagonals.



To start calculating the equations of Y we can start from the codimension 2 complete intersection

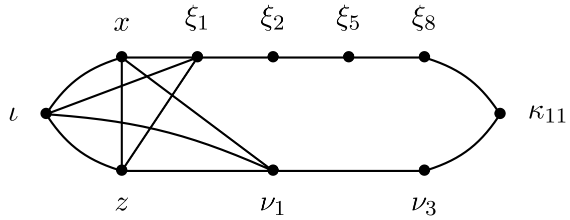
$$xz = \iota\eta + y^{11}, \quad \iota\nu_1 = y^4(x + f_1 y^8) + z\psi_2(y^3, z)$$

and unproject the ideal (ι, y^4, z) to get our first matrix of Pfaffians:

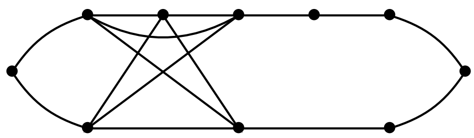
$$\begin{pmatrix} \iota & y^4 & -\psi_2(y^3, z) & -x \\ & z & x + f_1 y^8 & -y^7 \\ & & \nu_1 & \eta \\ & & & \xi_1 \end{pmatrix}$$

As in [BR1] §1.2.7 we can play a projection/unprojection game, starting with the first group of five variables $x, \iota, z, \nu_1, \xi_1$, to calculate this ring by *serial pentagrams*. We systematically drop a variable and add a new unprojection variable until we have constructed all the generators of our ring. They are depicted in Figure 6.1 along with the order of projection and unprojection. For instance, the first pentagram contains the ideal (ι, x, y^4, z) in Jer_{12} format. To move from the first pentagram to the second we project away from ι and then unproject (x, y^4, z) to get the new variable ξ_2 with unprojection equations. At each stage moving down the list, the projection variable is the $(1, 2)$ entry of the current matrix and the unprojection variable appears as the $(4, 5)$ entry of the next matrix. This game continues all the way down until we construct κ_{11}, at which point there are no more unprojection ideals to unproject from.

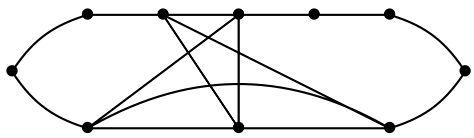
$$\begin{pmatrix} \iota & y^4 & -\psi_2(y^3,z) & -x \\ & z & x+f_1y^8 & -y^7 \\ & & \nu_1 & \eta \\ & & & \xi_1 \end{pmatrix}$$



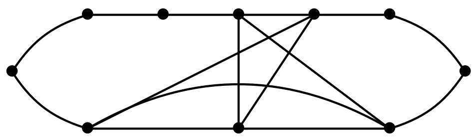
$$\begin{pmatrix} x & y^4 & -\eta\psi_3(y^3,z) & -\xi_1 \\ & z & \xi_1+f_2y^5\eta & -y^3\phi_1(\nu_1,y\eta) \\ & & \nu_1 & \eta \\ & & & \xi_2 \end{pmatrix}$$



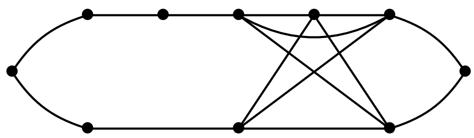
$$\begin{pmatrix} \xi_1 & y^3 & -\eta^2\psi_4(y^3,z) & -\xi_2 \\ & z & y(\xi_2+f_3y^2\eta^2) & -\phi_2(\nu_1,y\eta) \\ & & \nu_1 & \eta \\ & & & \nu_3 \end{pmatrix}$$



$$\begin{pmatrix} z & y & -\phi_2(\nu_1,y\eta) & -\xi_2 \\ & \nu_1 & \eta(\xi_2+f_3y^2\eta^2) & -y^2\psi_4(\eta^3,\nu_3) \\ & & \nu_3 & \eta^3 \\ & & & \xi_5 \end{pmatrix}$$



$$\begin{pmatrix} \xi_2 & y & -\eta^3\phi_1(\nu_1,y\eta) & -\xi_5 \\ & \nu_1 & \xi_5+f_2y\eta^5 & -y\psi_3(\eta^3,\nu_3) \\ & & \nu_3 & \eta^4 \\ & & & \xi_8 \end{pmatrix}$$



$$\begin{pmatrix} \xi_5 & y & -\eta^7 & -\xi_8 \\ & \nu_1 & \xi_8+f_1\eta^8 & -\psi_2(\eta^3,\nu_3) \\ & & \nu_3 & \eta^4 \\ & & & \kappa_{11} \end{pmatrix}$$

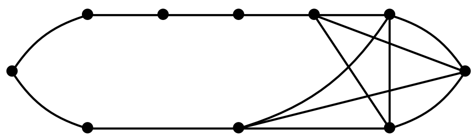


Figure 6.1: Serial pentagrams for Big example 1

The Pfaffians in Figure 6.1 give us 20 of the 35 equations that we are expecting to lift. The rest can be worked out by working birationally or computing the colon of the ideal generated by these 20 equations against a suitable monomial.

We note that along the bottom row of the rugby ball, moving from ι round to κ_{11}, the tag equations are given by

$$\begin{aligned} xz &= \eta\iota + y^{11} \\ \iota\nu_1 &= \psi_1(y^3, z) + y^4 x \\ z\nu_3 &= \phi_3(\nu_1, y\eta) + y\eta\xi_2 \\ \nu_3\kappa_{11} &= \psi_1(\eta^3, \nu_3) + \eta^4\xi_5 \\ \xi_8\nu_3 &= y\kappa_{11} + \eta^{11} \end{aligned}$$

and that all the equations display a left-right symmetry that comes from flipping the rugby ball from left to right, i.e. we interchange $\iota \leftrightarrow \kappa_{11}$, $x \leftrightarrow \xi_8$, $y \leftrightarrow \eta$, etc.

The hyperplane section $(Z \subset H_Y)$

If we project out ι, the variable of weight -1, and take the Proj of the resulting $\mathbb{Z}_{\geq 0}$-graded ring we recover the divisorial extraction $\sigma\colon Y \to X$ that we have been aiming to construct. We can check to see whether this has the hyperplane section $(Z \subset H_Y)$ that we are expecting.

Let $(Q \in Y)$ be the point where all variables but κ_{11} vanish. At $(Q \in Y)$ we can use $\kappa_{11} = 1$ to eliminate all variables apart from η, ν_3, ξ_8. In particular y is eliminated by the equation $\xi_8\nu_3 = y + \eta^{11}$, so that $(Q \in H_Y)$ is the quotient of a Du Val singularity:

$$\left(\xi_8\nu_3 = \eta^{11}\right) / \tfrac{1}{11}(1,3,8)$$

Since $\frac{1}{11}(1,3,8) = \frac{1}{11}(4,1,10)$, this is the T-singularity $\frac{1}{121}(1,43)$, exactly as we were expecting.

It also follows, from substituting $x = y = z = 0$ into the equations that we have calculated, that the reduced central fibre Z is given by:

$$x = y = z = \xi_1 = \xi_2 = \xi_5 = \nu_1 = 0$$

$$\bigwedge^2 \begin{pmatrix} \xi_8 & \psi_2(\eta^3, \nu_3) & -\eta^4 \\ -\eta^7 & \xi_8 + f_1\eta^8 & \nu_3 \end{pmatrix} = 0$$

Letting $\nu_3 = 0$ we see that $Z \cap V(\nu_3) = Q$, so away from Q we can invert ν_3 along

Z. If we do this then one of the equations defining Z becomes:

$$\xi_8\eta^4 + \psi_1(\eta^3, \nu_3) = \tfrac{1}{\nu_3}\Phi(\eta^3, \nu_3)$$

Thus if $f_1, \cdots, f_4$ are chosen generically then $Z \subset \mathbb{P}(1,3,8,11)_{\eta,\nu_3,\xi_8,\kappa_{11}}$ is a union of five lines passing through a $\frac{1}{11}(1,3,8)$ singularity $(Q \in Y)$. These five lines correspond to the roots of Φ. If Φ has a multiple root then some of these lines coincide and, in particular, if Φ has a root of multiplicity five then Z becomes irreducible.

Singularities

On the big open set $\{\kappa_{11} \neq 0\}$ Y is isomorphic to a $\frac{1}{11}(1,3,8)$ quotient singularity. Therefore we need to check for singular points along $Z \cap V(\kappa_{11})$.

Again, away from $(Q \in Y)$ we can invert ν_3 along Z. If we do this we can use the pentagrams to eliminate $z, x, \xi_1, \xi_2, \xi_5, \xi_8$ from $\mathcal{O}_Y$ to be left with the hypersurface singularity

$$\left(\left(\nu_1\nu_3 - y\eta^4\right)\kappa_{11} = \Phi(\eta^3, \nu_3)\right) \subset \mathbb{C}^4_{y,\eta,\nu_1,\kappa_{11}} \times \mathbb{C}^*_{\nu_3}.$$

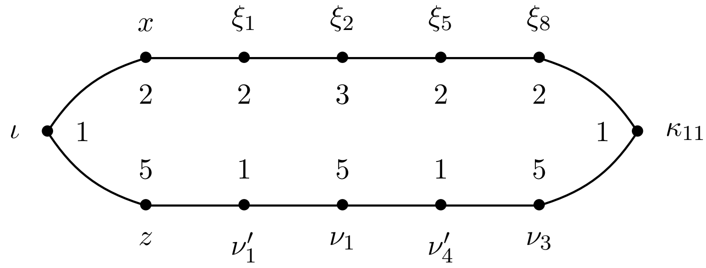
Let L be a component of Z that corresponds to a root of Φ of multiplicity m. At the point $L \cap V(\kappa_{11})$ this is the equation of a cA_{m-1} singularity. Therefore in general Y is smooth away from Q and in the case when Z is irreducible Y has a cA_4 singularity.

Redundant generators

Like the diptych varieties of Brown & Reid appearing in [BR3], this ring becomes much simpler if we include some *redundant generators*. The right thing in this case is to consider the two new 'generators' ν_1', ν_4' of weights 1, 4 given by:

$$\nu_1' = z\nu_1 - y^4\eta, \quad \nu_4' = \nu_1\nu_3 - y\eta^4$$

These have been chosen in such a way that they can be inserted with tag 1 along the bottom row of our rugby ball



and so that the tag equations along this bottom row become:

$$\begin{aligned}
xz &= \eta\iota + y^{11} \\
\iota\nu_1' &= \Phi(y^3, z) \\
z\nu_1 &= y^4\eta + \nu_1' \\
\nu_1'\nu_4' &= \Phi(\nu_1, y\eta) \\
\nu_1\nu_3 &= \nu_4' + y\eta^4 \\
\nu_4'\kappa_{11} &= \Phi(\eta^3, \nu_3) \\
\xi_8\nu_3 &= \eta^{11} + y\kappa_{11}
\end{aligned} \tag{6.5}$$

Remark 6.10. These relations are exactly analogous to the relations coming from Mori's division algorithm [M2] §3. Later on in §7.2.1 we also interpret these as exchange relations in a generalised rank 2 cluster algebra.

Another way that adding the redundant generators ν_1', ν_4' simplifies the equations, is that they appear as *crazy rolling factors* variables against some of the other generators (cf. Dicks' rolling factors format [R4] Example 10.8). For instance, given ν_1' we write down the equations[3]

$$\begin{aligned}
\nu_1' \cdot \iota &= y^{15} + f_1y^{12}z + f_2y^9z^2 + f_3y^6z^3 + f_4y^3z^4 + z^5 \\
\nu_1' \cdot x &= y^{11}\nu_1 + f_1y^{12}\eta + f_2y^9z\eta + f_3y^6z^2\eta + f_4y^3z^3\eta + z^4\eta \\
\nu_1' \cdot \xi_1 &= y^7\nu_1^2 + f_1y^8\eta\nu_1 + f_2y^9\eta^2 + f_3y^6z\eta^2 + f_4y^3z^2\eta^2 + z^3\eta^2 \\
\nu_1' \cdot \xi_2 &= y^3\nu_1^3 + f_1y^4\eta\nu_1^2 + f_2y^4\eta^2\nu_1 + f_3y^6\eta^3 + f_4y^3z\eta^3 + z^2\eta^3 \\
\nu_1' \cdot \nu_3 &= \nu_1^4 + f_1y\eta\nu_1^3 + f_2y\eta^2\nu_1^2 + f_3y^3\eta^3\nu_1 + f_4y^4\eta^4 + yz\eta^4 \\
\nu_1' \cdot \nu_4' &= \nu_1^5 + f_1y\eta\nu_1^4 + f_2y^2\eta^2\nu_1^3 + f_3y^3\eta^3\nu_1^2 + f_4y^4\eta^4\nu_1 + y^5\eta^5
\end{aligned} \tag{6.6}$$

[3]To see why ν_1' is a rolling factors variable against *these* variables in particular see §7.2.1 and Figure 7.2.

where we 'roll down' from the top line by the rows of the matrix,

$$\nu_1' = \bigwedge^2 \begin{pmatrix} y^4 & z \\ \nu_1 & \eta \end{pmatrix}$$

possibly floating some powers of y from top left to bottom right if needed.

6.3.2 Proof of Theorem 0.3

We will use redundant generators to simplify the exposition in the general case. Unfortunately it is still a large and tedious bookkeeping exercise to make sure all the right indices match up.

Structure of the proof

We give an overview of the proof of Theorem 0.3, i.e. the proof of the existence of Y as a sequence of type I serial unprojections

$$(Y_1 \supset D_1) \dashleftarrow \cdots \dashleftarrow (Y_{n-1} \supset D_{n-1}) \dashleftarrow (Y_n \ni Q) = Y,$$

where we unproject the divisor $(D_\alpha \subset Y_\alpha)$ to get to $Y_{\alpha+1}$. In general the full list of equations that define Y_α is very complicated and we can't write them down explicitly. In order to make sure our unprojections work at each stage we need another way of checking that the equations of $\mathcal{O}_{Y_\alpha} \subset I_{D_\alpha}$. We use a trick similar to that used in proof of the main construction of [BR1] §5, arguing on the weight of any monomial appearing in an equation of Y_α under two different gradings.

- **Step 1:** We start by writing down the rugby ball that we expect to lift.
- **Step 2:** We construct a sequence of pentagrams that runs through our rugby ball, introducing some redundant generators to make some simplifications. These pentagrams give some of the equations of our final variety Y but not all of them.
- **Step 3:** We define two gradings on our ring: w_1, the usual grading on our symbolic power algebra, and w_2, a grading that comes from the symmetry of the pentagrams. In particular w_1 increases from left to right along the sides of the rugby ball and w_2 decreases.
- **Step 4:** We go back through our sequence of pentagrams and show that, at each stage, *all* of the equations defining Y_α are contained in a type I unprojection ideal $I_{D_\alpha} = (\xi_{i,j}, \dots, \nu_i, y^{n_\alpha})$. For any monomial m appearing in an

equation of Y_α we use w_1 to restrict how many copies of a variable $x \notin I_{D_\alpha}$ can appear in m and then use w_2 to provide a lower bound on how many copies of y must appear. We show that $y^{n_\alpha} \mid m$ and therefore that $\mathcal{O}_{Y_\alpha} \subset I_{D_\alpha}$. Hence by Theorem 2.5, the Kustin–Miller unprojection theorem, $Y_{\alpha+1}$ exists as the type I unprojection of $(D\alpha \subset Y_\alpha)$ and $\mathcal{O}_{Y_{\alpha+1}}$ is a Gorenstein ring.

Step 1: The rugby ball

We pick up from just before §6.3.1, continuing with the notation Q_i (6.2), Φ (6.3), ϕ_i and ψ_i (6.4). Recall that our curve $(C \subset X)$ is the curve in an $A_{Q_{k+1}+Q_k-1}$ Du Val singularity given by the equations:

$$\bigwedge^2 \begin{pmatrix} x & y^{Q_k+Q_{k-1}} & \psi_2(y^{Q_k}, z) \\ y^{Q_{k+1}-Q_{k-1}} & z & \phi_1(x, y^{Q_{k+1}}) \end{pmatrix} = 0$$

We completely overhaul the notation—in particular, we no longer label variables by their degree. Define

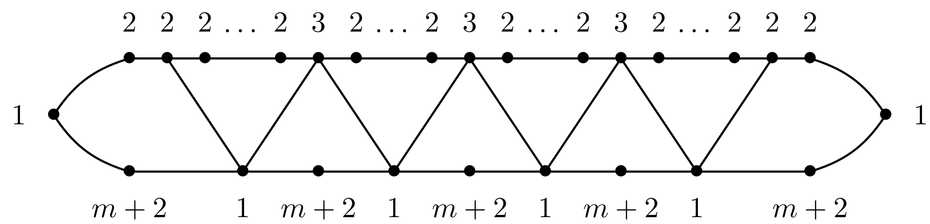
$$\begin{aligned} P_i &= Q_{i+1} + Q_i, \\ R_{i,j} &= (m+1-j)Q_{i+1} - (j+1)Q_i, \\ S_{i,j} &= jQ_i - Q_{i-1} \end{aligned}$$

and take note of the relations $R_{i,j} = R_{i,j+1} + P_i$ and $R_{i,m-1} = R_{i-1,1} = Q_{i+1} - Q_{i-1}$.

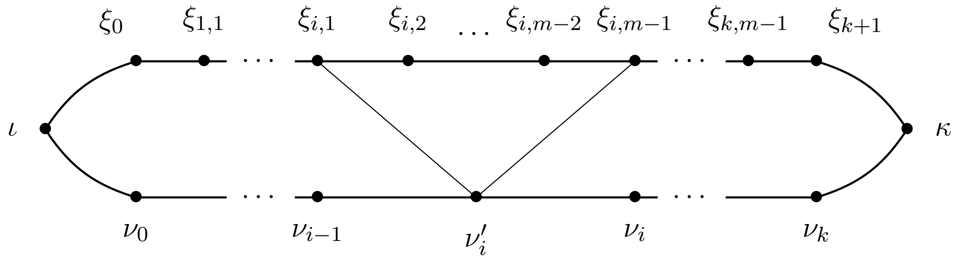
We let $\xi_0 := x$, $\nu_0 := z$ and we define ν_i and $\xi_{i,j}$ as renderings of the orbifold expressions:

$$\nu_i = [v^{P_{k-i}} \gamma^{Q_i}], \quad \xi_{i,j} = [u^{R_{k-i,j}} \gamma^{S_{i,j}}]$$

Finally, we also define some redundant variables $\nu_i' \equiv \nu_{i-1}\nu_i$ for $1 \le i \le k$. Note that $\xi_{i,m-1} = \xi_{i+1,1}$ for $i = 1, \dots, k$. These variables line the outside our our rugby ball



where there are $2k$ lines appearing in the zigzag and we label the generators:



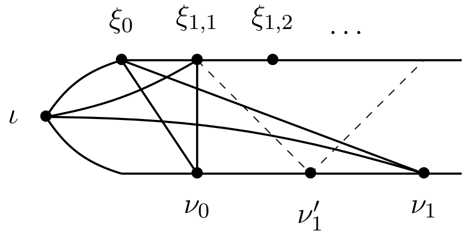
We can stick these triangles together since $\xi_{i,m-1} = \xi_{i+1,1}$. A *triangle* will always refer to one of these upside down triangles.

Step 2: Unprojection sequence

We now describe the sequence of pentagrams that builds this ring. To simplify the notation we define $\phi_\alpha^{(i)} = \phi_\alpha\left(\nu_i, y^{Q_{k-i}}\eta^{Q_i}\right)$ and $\psi_\alpha^{(i)} = \psi_\alpha\left(y^{Q_{k-i}}\eta^{Q_i}, \nu_i\right)$.

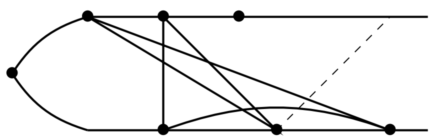
Starting the game. We start with our usual first pentagram:

$$\begin{pmatrix} \iota & y^{P_{k-1}} & -\psi_2^{(0)} & -\xi_0 \\ & \nu_0 & \xi_0 + f_1 y^{Q_{k+1}} & -y^{R_{k-1,1}} \\ & & \nu_1 & \eta \\ & & & \xi_{1,1} \end{pmatrix}$$



and consider our first redundant generator $\nu_1' = \nu_0\nu_1 - y^{P_{k-1}}\eta$. Projecting out ι and adding ν_1' gives the pentagram:

$$\begin{pmatrix} \xi_0 & y^{R_{k-1,1}}\phi_1^{(1)} & -\eta\psi_2^{(0)} & -\xi_{1,1} \\ & \nu_0 & y^{P_{k-1}} & -1 \\ & & \nu_1' & \eta \\ & & & \nu_1 \end{pmatrix}$$



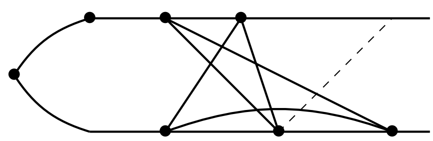
where the 1 that appears in the matrix could be used to reduce these five Pfaffian equations to a complete intersection of codimension 3 (of which, one equation eliminates ν_1').

Now we note that our indices work out in such a way that:

$$\begin{aligned}\nu_1'\xi_{1,1} &= y^{R_{k-1,1}}\nu_1\phi_1^{(1)} + \eta^2\psi_2^{(0)} \\ &= y^{R_{k-1,1}}\nu_1\phi_1^{(1)} + \eta^2\left(f_2 y^{mQ_k} + \nu_0\psi_3^{(0)}\right) \\ &= y^{R_{k-1,1}}\left(\nu_1\phi_1^{(1)} + f_2 y^{2Q_{k-1}}\eta^2\right) + \eta^2\nu_0\psi_3^{(0)} \\ &= y^{R_{k-1,1}}\phi_2^{(1)} + \eta^2\nu_0\psi_3^{(0)}\end{aligned}$$

In particular, since $mQ_k > P_{k-1}$, our last pentagram contains the ideal $(\xi_0, \nu_1, \nu_1', y^{P_{k-1}})$ in Tom$_5$ format. Projecting out ξ_0 and unprojecting this ideal gives the next pentagram:

$$\begin{pmatrix} \xi_{1,1} & y^{R_{k-1,2}}\phi_2^{(1)} & -\eta^2\psi_3^{(0)} & -\xi_{1,2} \\ & \nu_0 & y^{P_{k-1}} & -1 \\ & & \nu_1' & \eta \\ & & & \nu_1 \end{pmatrix}$$



Running along a triangle. Suppose i is *odd*. We define the following matrices for $i = 1, \cdots, k$ and $j = 1, \ldots, m-2$:

$$M_{i,j} = \begin{pmatrix} \xi_{i,j} & y^{R_{k-i,j+1}}\phi_{j+1}^{(i)} & -\eta^{R_{i-1,m-j}}\psi_{j+2}^{(i-1)} & -\xi_{i,j+1} \\ & \nu_{i-1} & y^{P_{k-i}} & -1 \\ & & \nu_i' & \eta^{P_{i-1}} \\ & & & \nu_i \end{pmatrix}$$

Note that the matrix constructed in our last pentagram is exactly $M_{1,1}$. To define $M_{i,j}$ for i *even* we take all the same terms appearing in the definition except that we swap $\phi_{j+1}^{(i)}$ (appearing in the $(1,3)$-entry) for $\psi_{m+1-j}^{(i)}$ and we swap $\psi_{j+2}^{(i)}$ (appearing in the $(1,4)$-entry) for $\phi_{m-j}^{(i)}$. For now we assume that i is odd.

By a similar calculation to the calculation before, we can check that our indices work out in just the right way so that:[4]

$$\begin{aligned}\nu_i'\xi_{i,j+1} &= y^{R_{k-i,j+1}}\nu_i\phi_{j+1}^{(i)} + \eta^{R_{i-1,m-j-1}}\psi_{j+2}^{(i-1)} \\ &= y^{R_{k-i,j+1}}\phi_{j+2}^{(i)} + \eta^{R_{i-1,m-j-1}}\nu_{i-1}\psi_{j+3}^{(i-1)}\end{aligned}$$

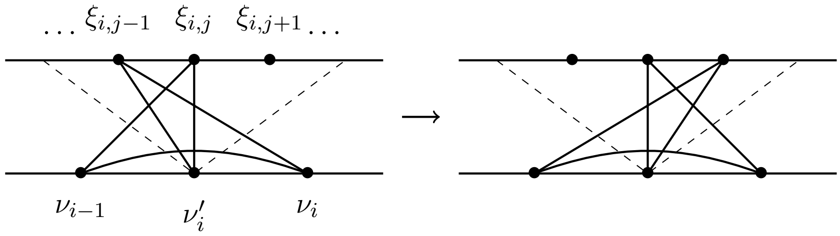
Moreover if $j < m-2$ then $R_{k-i,j+1} > P_{k-i}$. Therefore $M_{i,j}$ contains the ideal $(\xi_{i,j}, \nu_{i-1}, \nu_i', y^{P_{k-i}})$ in Tom$_5$ format. Projecting out $\xi_{i,j}$ and unprojecting this ideal

[4] cf. the crazy rolling factors equations (6.6).

gives the matrix

$$\begin{pmatrix} \xi_{i,j+1} & y^{R_{k-i,j+2}}\phi^{(i)}_{j+2} & -\eta^{R_{i-1,m-j-1}}\psi^{(i-1)}_{j+3} & -\xi_{i,j+2} \\ & \nu_{i-1} & y^{P_{k-i}} & -1 \\ & & \nu'_i & \eta^{P_{i-1}} \\ & & & \nu_i \end{pmatrix}$$

which, we note, is exactly $M_{i,j+1}$. Thus we move between the following pentagrams:



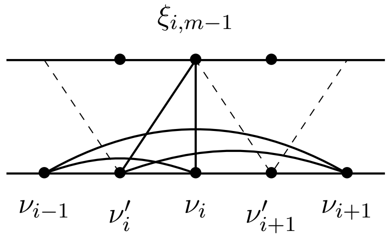
until we reach $M_{i,m-2}$. At this point we have a factor of $y^{R_{k-i,m-1}}$ appearing in the (1,3)-entry of $M_{i,m-2}$. Now

$$R_{k-i,m-1} = Q_{k-i+1} - Q_{k-i-1} < Q_{k-i+1} + Q_{k-i} = P_{k-i}$$

and therefore the power of y that appears in our Tom_5 ideal drops. We can only unproject the ideal $(\xi_{i,j}, \nu_{i-1}, \nu'_i, y^{R_{k-i,m-1}})$.

Jumping between triangles. We unproject from this ideal and project out $\xi_{i,m-2}$ to get the pentagram:

$$\begin{pmatrix} \xi_{i,m-1} & \phi^{(i)}_m & -\eta^{R_{i-1,1}}\psi^{(i-1)}_{m+1} & -\nu_{i+1} \\ & \nu_{i-1} & y^{R_{k-i,m-1}} & -1 \\ & & \nu'_i & y^{P_{k-i-1}}\eta^{P_{i-1}} \\ & & & \nu_i \end{pmatrix}$$

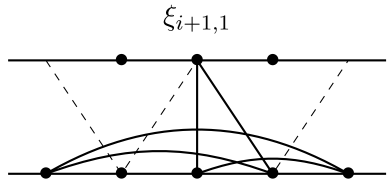


Now we drop the redundant generator $\nu'_i = \nu_{i-1}\nu_i - y^{P_{k-i}}\eta^{P_{i-1}}$ to be left with the complete intersection

$$\begin{aligned} \nu_i\xi_{i+1,1} &= y^{R_{k-i-1,1}}\nu_{i+1} + f_{m+1}y^{Q_{k-i+1}}\eta^{Q_{i+1}} + \eta^{R_{i-1,1}}\nu_{i-1} \\ \nu_{i-1}\nu_{i+1} &= \phi^{(i)}_m + y^{P_{k-i-1}}\eta^{P_{i-1}}\xi_{i+1,1} \end{aligned}$$

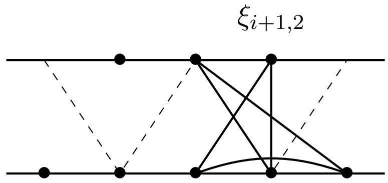
where we have relabelled $\xi_{i,m-1}$ by $\xi_{i+1,1}$ using our coincidence between indices. We can insert our next redundant generator $\nu'_{i+1} = \nu_i\nu_{i+1} - y^{P_{k-i-1}}\eta^{P_i}$ as follows:

$$\begin{pmatrix} \xi_{i+1,1} & 1 & -\eta^{R_{i-1,1}} & -\nu_{i+1} \\ & \nu_{i-1} & y^{R_{k-i-1,1}}\psi^{(i+1)}_{m+1} & -\phi^{(i)}_m \\ & & \nu_i & y^{P_{k-i-1}}\eta^{P_{i-1}} \\ & & & \nu'_{i+1} \end{pmatrix}$$



Now we make one last unprojection from the Tom_1 ideal $(\nu_{i-1}, \nu_i, \nu'_{i+1}, y^{P_{k-i-1}})$, projecting out ν_{i-1}, to get the pentagram:

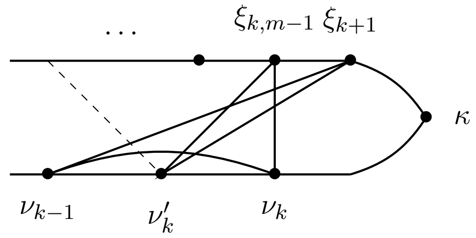
$$\begin{pmatrix} \xi_{i+1,1} & y^{R_{k-i-1,2}}\psi^{(i+1)}_m & -\eta^{R_{i-1,1}}\phi^{(i)}_{m-1} & -\xi_{i+1,2} \\ & \nu_i & y^{P_{k-i-1}} & -1 \\ & & \nu'_{i+1} & \eta^{P_i} \\ & & & \nu_{i+1} \end{pmatrix}$$



This is exactly $M_{i+1,1}$ where we note that $i+1$ is now *even*, so we interchange ϕ and ψ. We can now run along this next triangle in exactly the same fashion as before. The exchange in roles of ϕ and ψ is harmless—we simply swap f_i with f_{m+2-i} for $i = 1, \dots, m+1$.

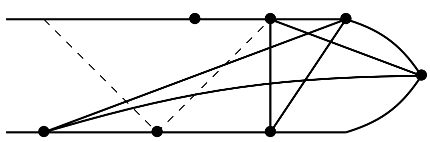
End game. Assume that k is odd (if not then swap the roles of ϕ and ψ as above). The projection/unprojection game takes us all the way up to

$$\begin{pmatrix} \xi_{k,m-1} & y\phi^{(k)}_m & -\eta^{R_{k-1,1}}\psi^{(k-1)}_{m+1} & -\xi_{k+1} \\ & \nu_{k-1} & y & -1 \\ & & \nu'_k & \eta^{P_{k-1}} \\ & & & \nu_k \end{pmatrix}$$



and our last step, unprojecting from the Tom_5 ideal $(\nu_{k-1}, \nu'_k, \xi_{k,m-1}, y)$ and dropping the redundant generator ν'_k, gives our final pentagram.

$$\begin{pmatrix} \kappa & \eta^{P_{k-1}} & -\phi^{(k)}_m & -\xi_{k+1} \\ & \nu_k & \xi_{k+1} + f_{m+1}\eta^{Q_{k+1}} & -\eta^{R_{k-1,1}} \\ & & \nu_{k-1} & y \\ & & & \xi_{k,m-1} \end{pmatrix}$$



We have not finished proof of the construction. We have only made unprojections from pentagram to pentagram and not on the level of the whole ring itself.

To finish the proof we can *eliminate all our redundant generators* as they are unnecessary in the description of the final ring (we used them to simplify the pentagrams only).

Step 3: A new grading

Note that the pentagrams that we have constructed display a symmetry which comes from flipping the rugby ball from left to right

$$y \leftrightarrow \eta, \quad \xi_0 \leftrightarrow \xi_{k+1}, \quad \xi_{i,j} \leftrightarrow \xi_{k-i+1,m-j} \quad \nu_i \leftrightarrow \nu_{k-i}, \quad \iota \leftrightarrow \kappa$$

(modulo the little problem of swapping ψ and ϕ if k is odd). In particular we can take the grading on our ring, which we will call w_1, and flip it over to get a new grading w_2. These two gradings give an action of $(\mathbb{C}^*)^2$ on our ring so we could even choose to define a $\mathbb{Z}^2$-grading.

	y	η	ξ_0	ξ_{k+1}	$\xi_{i,j}$	ν_i	ι	κ
w_1	0	1	0	Q_{k+1}	$S_{i,j}$	Q_i	-1	P_k
w_2	1	0	Q_{k+1}	0	$S_{k-i+1,m-j}$	Q_{k-i}	P_k	-1

More precisely, we could set new weights of our orbifold terms to be u, v, γ to be $\frac{Q_{k+1}}{P_k}, \frac{Q_k}{P_k}, -\frac{1}{P_k}$ and set the weights of ι, η to be P_k, 0 and check that this gives the grading w_2.

Step 4: End of the proof

Now we will show that at each stage of the game *all* of the equations defining our ring (not just the pentagrams) were contained in our unprojection ideals.

Analysing the steps of the unprojection sequence we have to consider the three cases of Figure 6.2, depending on the next (and last) unprojection variable to be adjoined to our ring.

In each case write s_1 for the last unprojection variable to be added and s_2 for the next one. Write I_D for the ideal given in Figure 6.2, where in the first case $\xi_{i,j-1}, \ldots, \nu_{i-1}$ denotes all the variables running around the left hand side of the rugby ball from $\xi_{i,j-1}$ around to ν_i, and similarly for the other cases. This is the unprojection ideal of s_2. (In the first case, if $j = m - 2$ then we should take

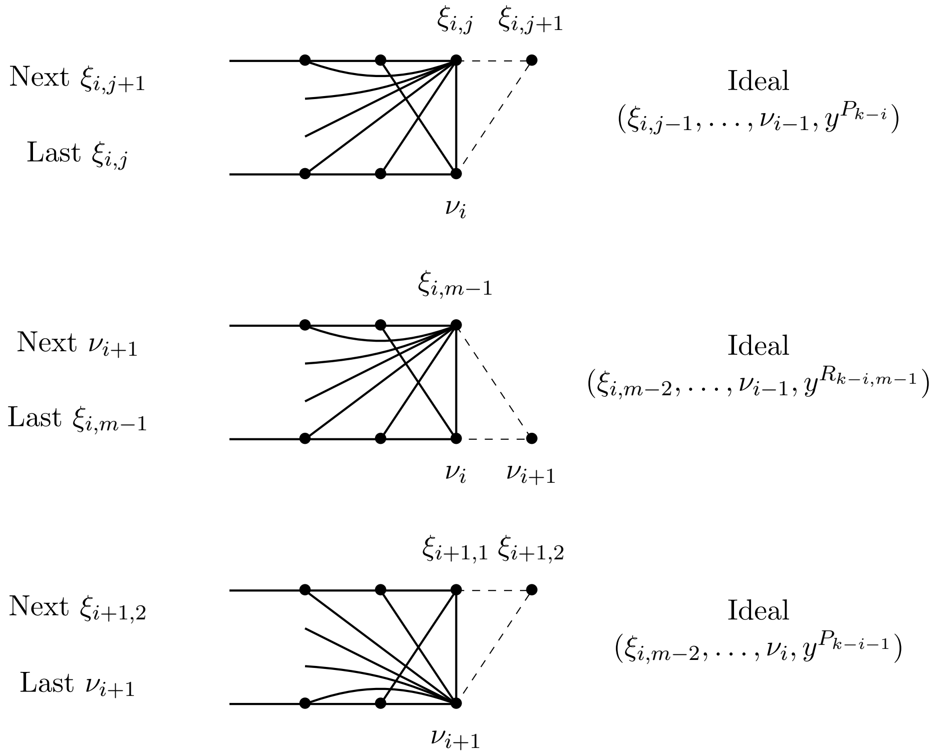


Figure 6.2: Three cases in the unprojection sequence

$y^{R_{k-i,m-1}} \in I_D$ instead of $y^{P_{k-i}}$.)

Now suppose that we have constructed (*all* of the equations of) our ring up to and including s_1. Our aim is to show that all these equations are contained in I_D and hence that our variety contains the unprojection divisor D. Since the power of y appearing in I_D is weakly decreasing as we adjoin more variables, we can assume by induction (with our first pentagram as a base case) that all the equations from the step before are contained in I_D. Therefore we only need to check the equations involving s_1. (The equation of the bar straddling s_1 is always contained in our pentagram so we've already checked that this is contained in I_D.)

First two cases. We have to check that any monomial $y^a\eta^b\nu_i^c$ that appears in an equation of the form $x\xi_{i,j} = \cdots$ has $a \geq P_{k-i}$ (or $a \geq R_{k-i,m-1}$ if $j \geq m-2$). We do this by finding some bounds on a, b, c that come from comparing the weight of the two monomials $y^a\eta^b\nu_i^c$ and $x\xi_{i,j}$ under our two different gradings.

First let x be any variable in the range $\xi_{i,1}, \dots, \nu_{i-1}$. Then, by checking the value of our gradings in this range, we have $w_1(x) \leq Q_{i-1}$ and $w_2(x) \geq Q_{k-i+1}$, so:

$$w_1(x\xi_{i,j}) \leq S_{i,j} + Q_{i-1} = jQ_i = w_1(\nu_i^j) \implies c \leq j$$

From this and the fact that $w_2(y^a\eta^b\nu_i^c) = a + cQ_{k-i}$ it follows that:

$$a \geq w_2(x\xi_{i,j}) - jQ_{k-i} \geq S_{k-i+1,m-j} + Q_{k-i+1} - jQ_{k-i} = R_{k-i,j}$$

Then either $j = m-1$ and $a \geq R_{k-i,m-1}$ in which case we are done, or $j \geq m-2$ and $R_{k-i,j} > P_{k-i}$ so we are also done.

Now suppose $x = \xi_{i,j'}$ is any variable in the range $\xi_{i,2}, \ldots, \xi_{i,j-2}$. Then

$$w_1(x\xi_{i,j}) = S_{i,j} + S_{i,j'} = (j+j')Q_i - 2Q_{i-1} \implies c \leq j + j' - 1$$

and it follows that:

$$\begin{aligned} a &\geq w_2(x\xi_{i,j}) - (j+j'-1)Q_{k-i} \\ &= S_{k-i+1,m-j} + S_{k-i+1,m-j'} - (j+j'-1)Q_{k-i} \\ &= (2m-(j+j'))Q_{k-i+1} - (j+j'+1)Q_{k-i} \end{aligned}$$

Since $j + j' \leq (m-1) + (m-3) = 2m-4$

$$a \geq 4Q_{k-i+1} - (2m-3)Q_{k-i} = 2Q_{k-i+1} + 3Q_{k-i} - 2Q_{k-i-1} > P_{k-i}$$

so we are also done in this case.

Third case. We have to check that any monomial $y^a\eta^b\xi_{i+1,1}^c$ that appears in an equation $x\nu_{i+1} = \cdots$ has $a \geq P_{k-i-1}$.

First let $x \neq \xi_{i,m-2}$. Then $w_1(x) < S_{i,m-2}$, so:

$$w_1(x\nu_{i+1}) < Q_{i+1} + S_{i,m-2} = 2Q_{i+1} - 2Q_i = w_1(\xi_{i+1,1}^2) \implies c \leq 1$$

Moreover $w_2(x) \geq w_2(\nu_{i-1}) = Q_{k-i+1}$ and therefore

$$a \geq w_2(x\nu_{i+1}) - S_{k-i+1,1} \geq Q_{k-i+1} + Q_{k-i-1} - Q_{k-i+1} + Q_{k-i} = P_{k-i-1}$$

so we are done.

Now let $x = \xi_{i,m-2}$. For a monomial $y^a\eta^b\xi_{i+1,1}^c$ with $c \leq 1$ the same proof works and we are done. The only problem is with any monomial that has $c = 2$. In this case $w_1(\xi_{i,m-2}\nu_{i+1}) = w_1(\xi_{i+1,1}^2)$ and, as $w_1(\eta) = 1$, we must have $b = 0$.

Therefore we are considering a monomial of the form $y^a\xi_{i+1,1}^2$ which is necessarily nonzero modulo η. But then:

$$\begin{aligned}
\xi_{i,m-2}\nu_{i+1} &\equiv [u^{R_{k-i,m-2}}\gamma^{S_{i,m-2}}][v^{P_{k-i-1}}\gamma^{Q_{i+1}}] \qquad \mod \eta \\
&\equiv y^{P_{k-i-1}}[u^{R_{k-i,m-2}-P_{k-i-1}}\gamma^{S_{i,m-2}+Q_{i+1}}] \\
&\equiv y^{P_{k-i-1}}[u^{2Q_{k-i+1}-2Q_{k-i-1}}\gamma^{2Q_{i+1}-2Q_i}] \\
&\equiv y^{P_{k-i-1}}\xi_{i+1,1}^2
\end{aligned}$$

So this is precisely the monomial $y^{P_{k-i-1}}\xi_{i+1,1}^2$ and we are done.

Conclusion. This proves the existence of the unprojection divisor $(D_\alpha \subset Y_\alpha)$ at each stage in the unprojection sequence of Theorem 0.3. By Theorem 2.5 and induction this concludes the proof of Theorem 0.3, i.e. that our sequence of type I unprojections constructing Y exists and that we build the ring $\mathcal{O}_Y$ that we expect to lift from our rugby ball.

6.4 Conjectures

We move on to discuss some more general cases where the central fibre $(Z \subset H_Y)$ can be reducible. Recall that we are interested in neighbourhoods $(Z \subset H_Y)$ satisfying Assumption 6.3.

We fix some notation for this section. Consider two coprime integers $r > a$ and the simple T-string $\frac{r^2}{ra-1} = [c_1, \dots, c_m]$. If $r \geq 3$ then, without loss of generality, we assume that $a < \frac{1}{2}r$ so that $c_1 > 2$ and $c_m = 2$. Now let $[a_1, \dots, a_m]$ be any 1-string dominated by this T-string (i.e. $a_i \leq c_i$ for all i) and let $b_i := c_i - a_i$.

Given such a 1-string, we write down a neighbourhood $(Z \subset H_Y)$ by the following recipe:

- blow $(0 \in \mathbb{C}^2)$ up to a chain of rationals curves $F = \bigcup_{i=1}^m F_i$ with self-intersection $F_i^2 = -a_i$,
- blow up b_i distinct points along F_i (these will be the components of Z),
- contract the birational transform of F.

Note that H_Y has a unique simple T-singularity $(Q \in H_Y)$ and Z is a union of rational curves all meeting at $(Q \in H_Y)$. From the construction $(Z \subset H_Y)$ clearly has a contraction to a smooth point $\sigma\colon (Z \subset H_Y) \to (0 \in \mathbb{C}^2)$.

As usual, write $(P \in S_X \subset X)$ for the embedding of a Du Val singularity of type A_{r-1} in a smooth 3-fold X. We fix the minimal resolution

$$\mu\colon (E \subset \widetilde{S}_X) \to (P \in S_X)$$

where $E = \bigcup_{i=1}^{r-1} E_i$ is a chain of -2-curves, and for any curve $(C \subset S_X)$ write $\widetilde{C}$ for the birational transform of C on $\widetilde{S}_X$.

Conjecture 6.11. *Given such a neighbourhood $(Z \subset H_Y)$ then there exist distinct integers $1 \le d_i < r$, for $i = 1, \dots, m$, such that $\sum_{i=1}^m b_i d_i \equiv a \mod r$ and, if $(C \subset S_X)$ is the* generic *curve with $\widetilde{C} \cdot E_{d_i} = b_i$ (and $\widetilde{C} \cdot E_j = 0$ if $j \neq d_i$ for any i) then there exists $\sigma\colon Y \to X$, a Mori extraction from C whose general hyperplane section is $(Z \subset H_Y)$. The high index point $(Q \in Y)$ is the quotient singularity $\frac{1}{r}(1, b, r-b)$, where $b \equiv a^{-1} \mod r$, and Y is smooth elsewhere.*

Moreover, it is my belief that the other, more singular, non-semistable neighbourhoods are obtained by taking all degenerations of such curves C which keep the intersection numbers $\widetilde{C} \cdot E_j$ fixed.

6.4.1 The $[(2)^{m-1}, 1]$ and $[1, (2)^{m-1}]$ cases

We now look at two natural cases of Conjecture 6.11. Since $c_i \ge 2$ for all i, the two 1-strings $[(2)^{m-1}, 1]$ and $[1, (2)^{m-1}]$ are always dominated by our T-string. We write down $(C \subset S_X)$, a curve in a A_{r-1} Du Val singularity, corresponding to the $\mathbb{Q}$-smoothing of $(Z \subset H_Y)$ in each of these cases.

Claim 6.12. *Recall the definition of the integers β_i appearing in Lemma 5.10 that arise from considering the discrepancies of a T-singularity.*

1. *Consider the 1-string $[(2)^{m-1}, 1]$. Then the integers d_i of Conjecture 6.11 are given by $d_i = \beta_i$ for $i = 1, \dots, m$, i.e. $(C \subset S_X)$ is a generic curve with:*

$$\widetilde{C} \cdot E_j = \begin{cases} c_i - 2 & j = \beta_i \text{ for some } 1 \le i \le m-1 \\ 1 & j = r - a \\ 0 & \text{otherwise} \end{cases}$$

2. *Consider the 1-string $[1, (2)^{m-1}]$. Then the integers d_i of Conjecture 6.11 are*

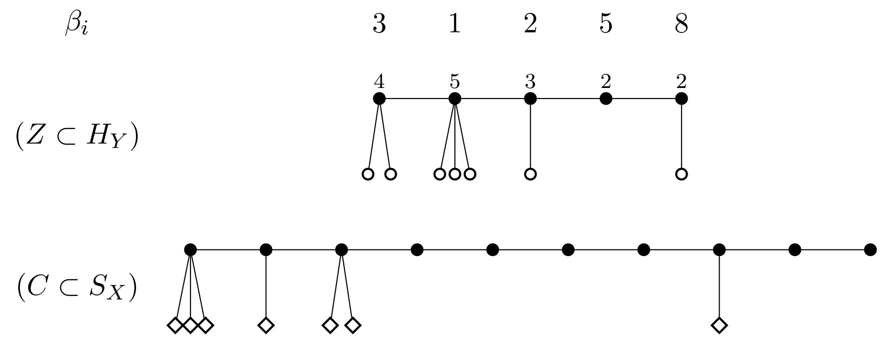
given by $d_i = r - \beta_i$ for $i = 1, \ldots, m$, i.e. $(C \subset S_X)$ is a generic curve with:

$$\widetilde{C} \cdot E_j = \begin{cases} c_i - 2 & j = r - \beta_i \text{ for some } 2 \leq i \leq m \\ c_1 - 1 & j = r - a \\ 0 & \text{otherwise} \end{cases}$$

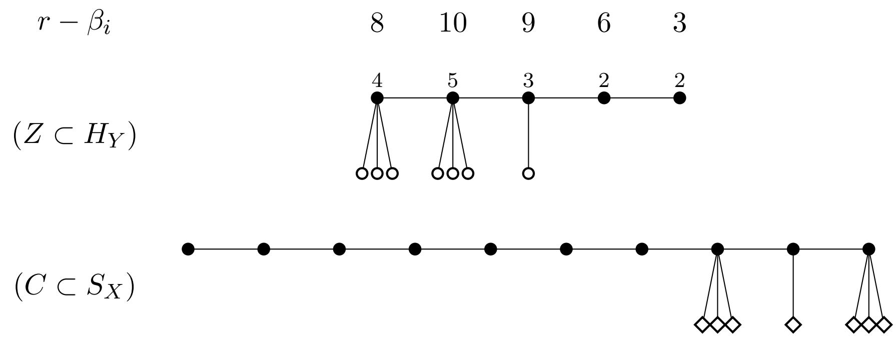
To illustrate the claim consider the following example.

Example 6.13. Let $(r, a) = (11, 3)$. Then the two cases of Claim 6.12 are given by:

1. The $[(2)^{m-1}, 1]$ case. The neighbourhood $(Z \subset H_Y)$ and the curve $(C \subset S_X)$ are given by



2. The $[1, (2)^{m-1}]$ case. The neighbourhood $(Z \subset H_Y)$ and the curve $(C \subset S_X)$ are given by



It is clear that the claim is correct from computing enough large examples. However, to save the reader from having to read another proof like that of §6.3.2, we will only compute an example.

6.4.2 Big example 2

Consider the curve $(C \subset S_X)$ of Example 6.13(1). In particular, $S_X = \mathbb{C}^2_{u,v}/\frac{1}{11}(1,10)$ and C is given by the orbifold equation

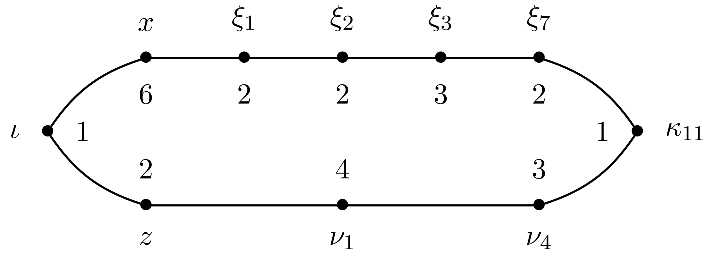
$$\begin{aligned}\gamma(u,v) &= u^{58} + au^{48}v + bu^{38}v^2 + cx^{28}v^3 + du^{19}v^5 + eu^{11}v^8 + fu^3v^{11} + v^{19} \\ &= (x^5 + ax^4y + bx^3y^2 + cx^2y^3 + dxy^5 + ey^8)u^3 + (fy^3 + z)v^8\end{aligned}$$

for a generic choice of coefficients $a, \dots, f$. (Again, since we are only interested in the generic curve C we set the first and last coefficient equal to 1.)

Since $\gamma \in \mathcal{O}_{S_X}(u^3, v^8)$ the (extended) symbolic ring $\bigoplus I^{[n]}_{C/S_X}$ is generated, as usual, by x, y, z in degree 0, ι in degree -1 and seven other generators in degrees $1, 1, 2, 3, 4, 7, 11$.

$$\begin{array}{ccccccc} \xi_1 & \xi_2 & \xi_3 & \xi_7 & \kappa_{11} & \nu_4 & \nu_1 \\ u^8\gamma & u^5\gamma^2 & u^2\gamma^3 & u\gamma^7 & \gamma^{11} & v\gamma^4 & v^3\gamma \end{array}$$

Therefore our rugby ball is given by:



As usual we use the same names for the forms lifted to $\bigoplus I^{[n]}_{C/X}$ and η for degree 1 variable corresponding to the equation of S_X, i.e. we have $\iota\eta = xz - y^{11}$.

To ease notation appearing in the series of pentagrams we let $\psi_1 = fy^3 + z$ and define a series of truncations:

$$\begin{aligned}\phi_5 &= x^5 + ax^4y + bx^3y^2 + cx^2y^3 + dxy^5 + ey^8 \\ \phi_4 &= x^4 + ax^3y + bx^2y^2 + cxy^3 + dy^5 \\ \phi_3 &= x^3 + ax^2y + bxy^2 + cy^3 \\ \phi_2 &= x^2 + axy + by^2 \\ \phi_1 &= x + ay\end{aligned}$$

In particular we have $\phi_5 = \phi_4 x + ey^8$, $\phi_4 = \phi_3 x + dy^5$ etc., and $(C \subset X)$ is defined by the equations:

$$\bigwedge^2 \begin{pmatrix} x & y^3 & -\psi_1 \\ y^8 & z & \phi_5 \end{pmatrix} = 0$$

It will also be convenient to define the following quantities which we view as deformation parameters and appear naturally in the equations of Y:

$$\begin{aligned} \overline{\xi}_1 &= \xi_1 + f\eta & \overline{\nu}_1 &= \nu_1 + e\eta \\ \overline{\xi}_2 &= \xi_2 + dy^2\eta^2 & \overline{N}_1 &= \nu_1\overline{\nu}_1 + d\overline{\xi}_1\eta \\ \overline{\xi}_3 &= \xi_3 + c\overline{\xi}_1\eta^2 & \overline{\nu}_4 &= \nu_4 + b\overline{\xi}_1^2\eta^2 \\ \overline{\xi}_7 &= \xi_7 + a\overline{\xi}_1^2\overline{N}_1\eta^3 \end{aligned}$$

As before we can use the equations of C to write down our first pentagram and begin unprojecting. We get the sequence of serial pentagrams appearing Figure 6.3 which we write down without further comment. The rest of the equations are implied from the pentagrams by working birationally and they are written down in the appendix §A.7.

The general hyperplane section $(Z \subset H_Y)$

To convince the reader that the Y that we have constructed really does have the claimed hyperplane section $(Z \subset H_Y)$ we first look at the components of the central fibre Z.

Since the equations of Y are obtained by projecting out ι and taking Proj with respect to the appropriate grading, we ignore all the equations involving ι. These are given by the pentagrams of Figure 6.3 and the equations in the appendix A.7. Now consider the sequence of Type I unprojections that builds Y

$$(Y_1 \supset D_1) \dashleftarrow (Y_2 \supset D_2) \dashleftarrow (Y_3 \supset D_3) \dashleftarrow (Y_4 \supset D_4) \dashleftarrow (Y_5 \ni Q) = Y$$

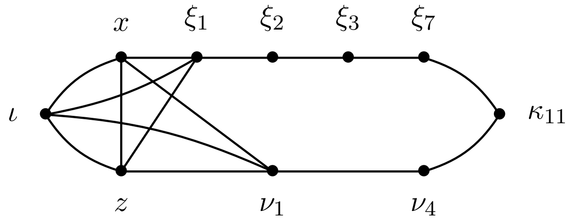
where we start with $\mathcal{O}_{Y_1} \subset \mathcal{O}_X[\eta, \nu_1, \xi_1, \xi_2]$, given by the second pentagram in Figure 6.3, and successively unproject the divisor D_i from Y_i by adjoining unprojection variables, $\mathcal{O}_{Y_2} = \mathcal{O}_{Y_1}[\xi_3]$, $\mathcal{O}_{Y_3} = \mathcal{O}_{Y_2}[\nu_4]$, etc. We look at how the reduced central fibre develops as we work through this sequence of unprojections.

Substituting $x = y = z = 0$ into $\mathcal{O}_{Y_1}$ we see that the central fibre of Y_1 is given by the (reduced) unprojection plane and a line.

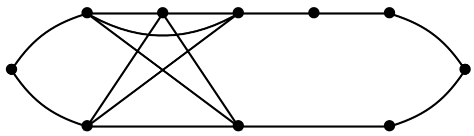
$$(D_1)_{\mathrm{red}} = V(x, y, z, \xi_1), \quad L_1 = V(x, y, z, \overline{\xi}_1, \nu_1)$$

Now, as the map $Y_2 \dashrightarrow Y_1$ is an isomorphism outside of D_1, we only track what happens to Y_2 above D_1. Substituting $x = y = z = \xi_1 = 0$ into $\mathcal{O}_{Y_2}$ we see

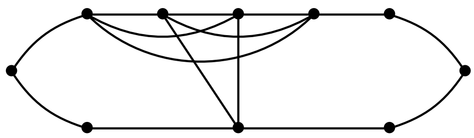
$$\begin{pmatrix} \xi_1 & \psi_1 & -\phi_5 & -\nu_1 \\ & x & y^8 & -\eta \\ & & \iota & y^3 \\ & & & z \end{pmatrix}$$



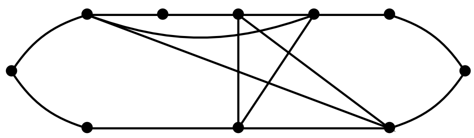
$$\begin{pmatrix} \xi_1 & y^5\overline{\nu}_1 & -\phi_4\eta & -\xi_2 \\ & x & y^3 & -\eta \\ & & z & \overline{\xi}_1 \\ & & & \nu_1 \end{pmatrix}$$



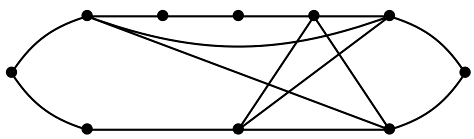
$$\begin{pmatrix} \xi_2 & y^2\overline{\nu}_1\eta & -\overline{\xi}_1 & -\xi_3 \\ & \xi_1 & y^3 & -\phi_3\eta^2 \\ & & x & \overline{\xi}_2 \\ & & & \nu_1 \end{pmatrix}$$



$$\begin{pmatrix} \xi_3 & \overline{N}_1\eta & -\overline{\xi}_1 & -\nu_4 \\ & \xi_2 & y^2 & -\phi_2\overline{\xi}_1\eta^2 \\ & & x & y\overline{\xi}_3 \\ & & & \nu_1 \end{pmatrix}$$



$$\begin{pmatrix} \xi_3 & y\overline{\nu}_4 & -\phi_1\overline{\xi}_1^2\eta^2 & -\xi_7 \\ & x & y & -\overline{N}_1\eta \\ & & \nu_1 & \overline{\xi}_1\overline{\xi}_3 \\ & & & \nu_4 \end{pmatrix}$$



$$\begin{pmatrix} \xi_7 & \overline{\nu}_4\overline{N}_1\eta & -\overline{\xi}_1\overline{\xi}_3 & -\kappa_{11} \\ & \xi_3 & y & -\overline{\xi}_1^2\overline{N}_1\eta^3 \\ & & x & \overline{\xi}_7 \\ & & & \nu_4 \end{pmatrix}$$

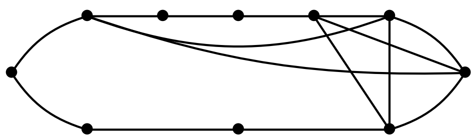


Figure 6.3: Serial pentagrams for Big example 2

that this locus is given by the (reduced) unprojection plane:

$$(D_2)_{\mathrm{red}} = V(x, y, z, \xi_1, \xi_2)$$

Therefore the central fibre of Y_2 is this union of this plane together with the birational transform of L_1, the line found in Y_1.

Continuing in a similar fashion, the exceptional locus of Y_3 above D_2 is a plane and the pair of lines

$$(D_3)_{\mathrm{red}} = V(x, y, z, \xi_1, \xi_2, \nu_1), \quad L_3 = L_{3,1} \cup L_{3,2} = V(x, y, z, \xi_1, \xi_2, \overline{N}_1, \xi_3)$$

corresponding to the roots of the term $\overline{N}_1 = \nu_1\overline{\nu}_1 + d\overline{\xi}_1\eta$, which is a quadratic in ν_1.

The exceptional locus of Y_4 above D_3 is a plane and a line:

$$(D_4)_{\mathrm{red}} = V(x, y, z, \xi_1, \xi_2, \nu_1, \xi_3), \quad L_4 = V(x, y, z, \xi_1, \xi_2, \nu_1, \overline{\xi}_3, \nu_4)$$

Lastly, the exceptional locus of Y_5 above D_4 is $L_5 = L_{5,1} \cup L_{5,2} \cup L_{5,3}$, a union of three lines given by $V(x, y, z, \xi_1, \xi_2, \nu_1, \xi_3)$ and:

$$\bigwedge^2 \begin{pmatrix} \xi_7 & \overline{\nu}_4\overline{N}_1\eta & -\overline{\xi}_1\overline{\xi}_3 \\ -\overline{\xi}_1^2\overline{N}_1\eta^3 & \overline{\xi}_7 & \nu_4 \end{pmatrix} = 0$$

Therefore the total reduced central fibre is the union of all these 1-dimensional components $Z = L_1 \cup L_3 \cup L_4 \cup L_5$, a group of rational curves all meeting at the high index point $(Q \in Y)$. At the point $(Q \in Y)$ the general hyperplane H_Y is given by the local equation

$$\left(\xi_7\nu_4 = \overline{\xi}_1^3\overline{\xi}_3\overline{N}_1\eta^3\right) \,/\, \tfrac{1}{11}(1, 4, 7)$$

where we can eliminate any appearance of ξ_1 by the equation $\kappa_{11}\xi_1 = \cdots$ and so on. In particular, if the coefficients are suitably generic then the RHS of this equation has a nonzero η^{11} term, so that $(Q \in H_Y)$ is a T-singularity of type $\frac{1}{121}(1, 32)$.

By blowing $(Q \in H_Y)$ and tracking the strict transform of the fibre $(Z \subset H_Y)$ we can check that this is the neighbourhood of Example 6.13(1).

Concluding remarks

We end with some concluding remarks about some of the features and difficulties in more general cases.

Remark 6.14.

1. Note that the pentagrams of Big example 2 proceed in a completely different order to those of Big example 1. They also display none of the symmetry that the pentagrams of Big example 1 had and appear to have entries that are much more complicated.

2. The terms $\overline{\xi}_1$, $\overline{N}_1$, etc. defined in Big example 2 appear naturally in the construction of Y as deformation parameters

$$x\xi_2 \equiv \xi_1\overline{\xi}_1 \mod y, \quad z\nu_4 \equiv \nu_1^2\overline{N}_1 \mod y, \quad \text{etc.}$$

   deforming the equations $x\xi_2 = \xi_1^2$, $z\nu_4 = \nu_1^4$. These look like they should have some natural interpretation in terms of homogeneous components of the orbifold equation $\gamma(u,v)$.

3. In very general cases we can have some quite hard unprojections to compute. In particular, as our unprojection planes are not reduced they don't need to be embedded in either Tom or Jerry format. For instance, we can have a pentagram with a Tom ideal $(x_1,\dots,x_n,y^a)$ and a Jerry ideal $(x_1,\dots,x_n,y^b)$ that together combine to embed an unprojection plane with ideal $(x_1,\dots,x_n,y^{a+b})$. At this point we can still unproject the ideal, but we cannot make a Gorenstein projection from any of the other variables and hence cannot continue writing down pentagrams.

Chapter 7

Relationship to cluster algebras

In this chapter we explain the connection between the big examples constructed in §6 and cluster algebras. In fact these rings are upper cluster algebras of certain rank 2 cluster algebras.

We end with a general discussion of the connections to other works.

7.1 Cluster algebras

Fomin & Zelevinsky [FZ] introduced cluster algebras in their study of canonical bases of Lie algebras. Since then cluster algebras have become ubiquitous, appearing in many different seemingly unconnected branches of mathematics. Chekhov & Shapiro [CS] generalised the notion of the cluster algebra to allow for polynomial exchange relations between cluster variables (rather than simply trinomial relations). We now recall the definition of a (generalised) cluster algebra.

A *cluster algebra*[1] of rank n depends on the initial data of a diagonal matrix $D = \mathrm{diag}(d_i : i = 1, \ldots, n) \in \mathrm{Mat}_n(\mathbb{Z})$, called the *degrees matrix*, a *coefficient ring* $R = \mathbb{Z}[A_1, \ldots, A_m]$ and an *initial seed* $S = (X, \Theta, B)$ consisting of

1. an n-tuple $X = (x_1, \cdots, x_n)$ called the *initial cluster*, viewed as a collection of elements of the field of fractions $\operatorname{Frac} R[x_1, \ldots, x_n]$,

2. a collection of n homogeneous polynomials

$$\Theta = \{\theta_i(u, v) \in R[u, v] : \deg \theta_i = d_i,\ \forall i = 1, \ldots, n\},$$

called the *exchange polynomials*,

[1] This is not the original, or the most general, definition of a cluster algebra. More precisely this should be called a *generalised cluster algebra with geometric coefficients*.

3. a matrix $B = (B_{jk}) \in \mathrm{Mat}_n(\mathbb{Z})$ such that the *exchange matrix* BD is skewsymmetrisable.

The original definition of a cluster algebra can be recovered by taking D to be the identity matrix.

The x_i are called *cluster variables*. We now use this initial data to generate more cluster variables by mutating seeds according to the following combinatorial rule. The *mutation* of S at the ith place is the seed $\mu_i(S) = (X', \Theta', B')$ defined by:

1. $X' = (x'_1, \ldots, x'_n)$ where $x'_j = x_j$ for $j \neq i$ and x'_i is defined by the relation:

$$x_i x'_i = \theta_i(u_i^+, u_i^-), \quad u_i^\pm = \prod_{j=1}^n x_j^{[\pm B_{ij}]_+}$$

(Here $[x]_+ := \max\{0, x\}$.)

2. $\Theta' = \{\theta'_1, \ldots, \theta'_n\}$ where

$$\theta'_j(u,v) = \begin{cases} \theta_i(v,u) & i = j \quad \text{(note the switched order of } u, v\text{)} \\ \frac{\theta'_{j,0}}{\theta_{j,0}} \cdot \theta_j\big(\theta_{i,0}^{B_{ij}} u, v\big) & i \neq j,\ B_{ij} \geq 0 \\ \frac{\theta'_{j,0}}{\theta_{j,0}} \cdot \theta_j\big(u, \theta_{i,d_i}^{B_{ij}} v\big) & i \neq j,\ B_{ij} < 0 \end{cases}$$

for $\theta_{i,0} = \theta_i(1,0)$ and $\theta_{i,d_i} = \theta_i(0,1)$. The coefficient $\theta'_{j,0}$ is chosen such that the coefficients of θ'_j are relatively coprime.

3. $B' = (B'_{jk})$ where

$$B'_{jk} = \begin{cases} -B_{jk} & j = i \text{ or } k = i \\ B_{jk} + \frac{1}{2}\big(|B_{ji}|B_{ik} + B_{ji}|B_{ik}|\big) & \text{otherwise.} \end{cases}$$

We call two seeds *mutation equivalent* if there is a sequence of mutations taking one seed to the other. Mutation is an involution, so the set of all seeds mutation equivalent our initial seed is parameterised by an infinite n-regular tree. We call $x \in \operatorname{Frac} R[x_1, \ldots, x_n]$ a *cluster variable* if x appears in the cluster of a seed which is mutation equivalent to our initial seed. The *cluster algebra* $\mathcal{A}$ is the (not necessarily finitely generated) subring of $\operatorname{Frac} R[x_1, \ldots, x_n]$ generated by all of the cluster variables.

The Laurent phenomenon

The most important feature of cluster algebras is that they satisfy the Laurent phenomenon, [CS] Theorem 2.5. That is, each cluster variable $x \in \mathcal{A}$ can be written as a Laurent polynomial in terms of the initial cluster $(x_1, \ldots, x_n)$. In other words we have the much stronger inclusion:

$$\mathcal{A} \subseteq R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subset \operatorname{Frac} R[x_1, \ldots, x_n]$$

Of course there is nothing particularly special about our choice of initial cluster, so this also holds for the cluster variables appearing in any mutation equivalent seed.

Remark 7.1. The coefficients $A_1, \ldots, A_m$ are sometimes referred to *frozen variables* and, quite often, authors prefer to assume that these are invertible. We stress that we *don't* want to do that. In our description of a Type A Mori extraction as a cluster algebra the coefficient ring will be $\mathbb{C}[y, \eta]$, i.e. there will be two frozen variables corresponding to the two special surfaces S_Y and H_Y. The interesting geometry of the extraction (or flip) is the locus where these frozen variables each vanish.

7.1.1 Rank 2 cluster algebras

We are going to be primarily interested in cluster algebras of rank 2 so we spell out the mutation rule explicitly in this case. We have the 2-valent tree parameterising clusters

$$\cdots \quad \underset{(x_{-1},x_0)}{\bullet} - \underset{(x_1,x_0)}{\bullet} - \underset{(x_1,x_2)}{\bullet} - \underset{(x_3,x_2)}{\bullet} - \underset{(x_3,x_4)}{\bullet} \quad \cdots$$

and, since the clusters are arranged in a chain, we can index the cluster variables by $\mathbb{Z}$, $\{x_i : i \in \mathbb{Z}\}$. The exchange relations take the form of an infinite sequence

$$\begin{aligned}
&\vdots \\
x_{-1}x_1 &= a_0x_0^d + b_0x_0^{d-1} + \cdots + c_0x_0 + d_0 \\
x_0x_2 &= a_1x_1^e + b_1x_1^{e-1} + \cdots + c_1x_1 + d_1 \\
x_1x_3 &= a_2x_2^d + b_2x_2^{d-1} + \cdots + c_2x_2 + d_2 \\
x_2x_4 &= a_3x_3^e + b_3x_3^{e-1} + \cdots + c_3x_3 + d_3 \\
&\vdots
\end{aligned}$$

where the degree of the polynomial expression in the RHS is either d or e, repeating 2-periodically. The mutation rule is illustrated in the next example.

Example 7.2 (G_2 cluster algebra)**.** We take a coefficient ring $R = \mathbb{C}[y, \eta]$ and the following initial data

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{aligned} \theta_1(u,v) &= \eta \cdot u + y^2 \cdot v \\ \theta_2(u,v) &= a \cdot u^3 + by \cdot u^2 v + cy^2 \cdot uv^2 + dy^3 \cdot v^3 \end{aligned}$$

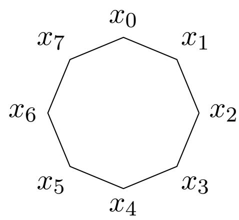
where the coefficients of θ_1, θ_2 have been specially chosen. We can check that mutating the initial seed gives the following sequence of exchange relations:

$$\begin{aligned} x_0 x_2 &= \eta x_1 + y^2 \\ x_1 x_3 &= a x_2^3 + by x_2^2 + cy^2 x_2 + dy^3 \\ x_2 x_4 &= dy\eta + x_3 \\ x_3 x_5 &= ad^2\eta^3 + bd\eta^2 x_4 + c\eta x_4^2 + x_4^3 \\ x_4 x_6 &= y x_5 + ad\eta^2 \\ x_5 x_7 &= x_6^3 + b\eta x_6^2 + ac\eta^2 x_6 + a^2 d\eta^3 \\ x_6 x_8 &= ay\eta + x_7 \\ x_7 x_9 &= ay^3 + by^2 x_8 + cy x_8^2 + d x_8^3 \\ x_8 x_{10} &= \eta x_9 + y^2 \end{aligned}$$

where the RHS of the ith relation is calculated by the mutation rule of §7.1. In this case we take d_{i-1} (the constant term of the $(i-1)$th line) and substitute $\frac{d_{i-1}}{x_i}$ for x_{i-2} in the RHS of the $(i-2)$th line. Then we multiply through to cancel the power of x_i in the denominator and we cancel any common factor from the coefficients (as elements of R). For instance:

$$x_2 x_4 = \frac{x_3}{y^2}\left(\eta \frac{dy^3}{x_3} + y^2\right) = dy\eta + x_3$$

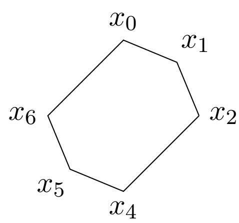
Notice that these exchange relations begin to repeat under $x_i \mapsto x_{i+8}$. Therefore we can set $x_i = x_{i+8}$ for all i. The 8 exchange relations imply another 12 relations, given by $x_i x_j = \cdots$ for $|i - j| \geq 2$. We view these as the interior diagonals of an octagon:



These 20 equations define a Gorenstein ring which cut out an affine variety in codimension 6 (cf. [BR2] §4.2):

$$W \subset \operatorname{Proj} R[x_0, x_1, \ldots, x_7]$$

Also note that the third and seventh exchange relations define x_3 and x_7 in terms of the other variables. Therefore x_3 and x_7 are not required as generators and we may drop them from our ring. Doing this chops two corners off the octagon to give a hexagon



and the tag equations around the outside of this hexagon become:

$$\begin{aligned}
x_0x_2 &= \eta x_1 + y^2 \\
x_1x_4 &= ax_2^2 + byx_2 + cy^2 + dyx_0 \\
x_2x_5 &= d\eta x_6 + bd\eta^2 + c\eta x_4 + x_4^2 \\
x_4x_6 &= yx_5 + ad\eta^2 \\
x_5x_0 &= x_6^2 + b\eta x_6 + ac\eta^2 + a\eta x_4 \\
x_6x_1 &= ayx_2 + by^2 + cyx_0 + dx_0^2
\end{aligned}$$

Now these equations are starting to look very familiar. By replacing

$$x_0, x_1, x_2, x_4, x_5, x_6 \mapsto z, \iota, x, \xi, \kappa, \nu$$

respectively we see that these are precisely six of the nine equations defining the Prokhorov & Reid example §3.3 (compare with the equations in A.1). The other three equations are the 'long diagonal' relations of the hexagon above.

The periodicity and finiteness of the number of cluster variables appearing in this example is a rather special phenomenon related to the fact that $BD = \left(\begin{smallmatrix} 0 & 1 \\ -3 & 0 \end{smallmatrix}\right)$ corresponds to the G_2 Dynkin diagram. In general cluster algebras are *not* finitely generated.

The upper cluster algebra

Given a cluster algebra $\mathcal{A}$ we define the *upper cluster algebra* of $\mathcal{A}$ to be the intersection of Laurent rings,

$$\mathcal{U} = \bigcap_{i\in\mathbb{Z}} R[x_i^{\pm1}, x_{i+1}^{\pm1}],$$

where, again, we choose *not* to invert the coefficient ring R. Note that, by the Laurent phenomenon, $\mathcal{A} \subseteq R[x_i^{\pm1}, x_{i+1}^{\pm1}]$ for all i and therefore $\mathcal{A} \subseteq \mathcal{U}$, but this is not an equality in general. The inclusion

$$\mathcal{A} \subseteq \mathcal{U} \subset R[x_i^{\pm1}, x_{i+1}^{\pm1}]$$

corresponds to the open embedding of a *cluster torus*

$$T_i = \operatorname{Spec}\left(R[x_i^{\pm1}, x_{i+1}^{\pm1}]\right) \hookrightarrow \operatorname{Spec}\mathcal{U}.$$

Let $X = \operatorname{Spec}\mathcal{U}$ and let $X_0 = \bigcup_i T_i$ be the union of all the cluster tori, glued together by the exchange relations. Then we have an embedding of a smooth open manifold $X_0 \hookrightarrow X$ and by the definition of $\mathcal{U}$ it follows that $\mathcal{U} = H^0(X_0, \mathcal{O}_{X_0})$, so that X is the 'affinisation' of X_0.

$\mathcal{U}$ may not be Noetherian, but if it is then X is a *normal Gorenstein* affine variety. Indeed, $\mathcal{U}$ is integrally closed since it is the intersection of integrally closed rings and therefore X is normal. Moreover each cluster torus T_i has a canonical volume form $\omega_i = \frac{dx_i}{x_i} \wedge \frac{dx_{i+1}}{x_{i+1}}$ and these patch together to give a canonical volume form ω_{X_0} on X_0. Taking the double dual on X we get $\omega_X = (\omega_{X_0})^{\vee\vee}$ an invertible dualising sheaf for X. Hence X is Gorenstein.

Since we only consider cluster algebras with geometric coefficients we have the following Lemma.

Lemma 7.3 ([BFZ] Corollary 1.7)**.** *For any i,*

$$\mathcal{U} = R[x_{i-1}^{\pm1}, x_i^{\pm1}] \cap R[x_i^{\pm1}, x_{i+1}^{\pm1}] \cap R[x_{i+1}^{\pm1}, x_{i+2}^{\pm1}].$$

Therefore to test membership in $\mathcal{U}$ it is enough to test membership in $R[x_i^{\pm1}, x_{i+1}^{\pm1}]$ for any three consecutive clusters.

Expanding in terms of a cluster

From the description of $\mathcal{U}$ we can choose a cluster, (x_0, x_1) say, and expand any element $u \in \mathcal{U}$ as a Laurent polynomial in terms of these two variables

$$u = \frac{F(x_0, x_1)}{x_0^a x_1^b}$$

for some polynomial F with $F(0,0) \neq 0$. For instance, we can expand any cluster variable x_i this way and, in particular, $x_0 = \frac{1}{x_0^{-1}}$ and $x_1 = \frac{1}{x_1^{-1}}$.

If the expansion of x_i has a denominator given in least terms by $x_0^{a_i} x_1^{b_i}$ then for $i \neq 0, 1$ we find an easy recurrence relation from the exchange relations:

$$(a_{i-1}, b_{i-1}) + (a_{i+1}, b_{i+1}) = \begin{cases} d(a_i, b_i) & i \text{ even} \\ e(a_i, b_i) & i \text{ odd} \end{cases}$$

Plotting the values $(a_i, b_i) \in \mathbb{Z}^2$ we obtain an *expansion diagram* and we see that, if $de > 4$, then as $i \to \pm\infty$ the points (a_i, b_i) approach two limiting rays of irrational slope:

$$\lambda^{\pm} = \frac{de \pm \sqrt{de(de-4)}}{2e}$$

These are the two quadratic irrationalities represented by the infinite continued fraction $[d, e, d, e, \ldots]$. We note that this sequence of points (a_i, b_i) is convex if $d, e \geq 2$ and zigzags if either $d = 1$ or $e = 1$. In our examples we always have $d = 1$. See Figure 7.1 for the case $(d, e) = (1, 5)$.

The two irrational rays cut out a triangular region of the plane. Taking the convex hull of the points inside this region we get a polytope (shaded in Figure 7.1) whose boundary points have tags given by the complementary HJ-string to the infinite string $[\ldots, d, e, d, e, \ldots]$. In the case when $d = 1$ and $e \geq 5$ this complementary fraction is $[\ldots, 3, (2)^{e-5}, 3, (2)^{e-5}, \ldots]$.

Remark 7.4. We note that the choice of a cluster (x_0, x_1) broke some of the symmetry in our diagram. For instance the tags in the diagram at the points x_0, x_1 are both 0, rather than d, e which we expect from the exchange relations. We can fix this problem by identifying the two cones $\langle(-1, 0), (-d, -1)\rangle$ and $\langle(-1, -e), (0, -1)\rangle$ in $\mathbb{Z}^2$ to get integral affine manifold B with a singularity at 0.

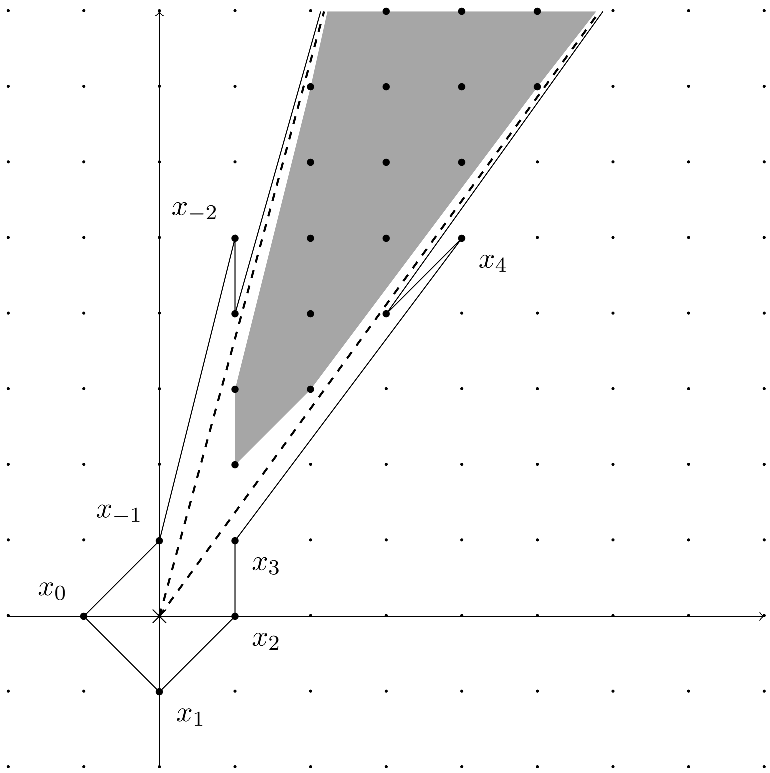


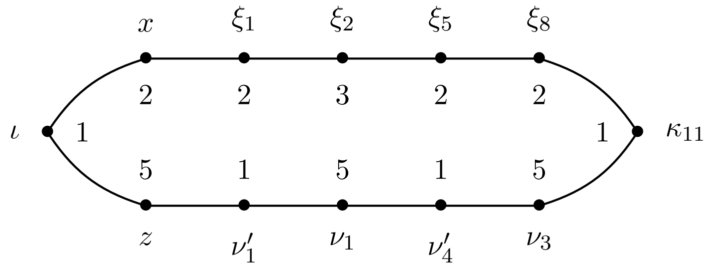
Figure 7.1: Expansion diagram for $(d, e) = (1, 5)$

7.2 Type A extractions revisited

We now interpret Big example 1 §6.3.1 and Big example 2 §6.4.2 as upper cluster algebras.

7.2.1 Big example 1 as a cluster algebra

We consider the ring constructed in Big example 1 §6.3.1 and assume all of the notation from that section including the redundant generators ν_1', ν_4'. Recall that our rugby ball was given by:



Cluster variables. We start with the two tag equations at ι and z,

$$\begin{aligned} xz &= \eta\iota + y^{11} \\ \iota\nu_1' &= \Phi(y^3, z) = y^{15} + f_1 y^{12} z + f_2 y^9 z^2 + f_3 y^6 z^3 + f_4 y^3 z^4 + z^5 \end{aligned}$$

and we treat these as two exchange relations for mutation between the three following clusters:

$$(x, \iota) \leftrightarrow (\iota, z) \leftrightarrow (z, \nu_1')$$

Now, starting with the mutation at $z \in (z, \nu_1')$, we can write down a sequence of cluster variables and clusters

$$(z, \nu_1') \leftrightarrow (\nu_1', \nu_1) \leftrightarrow (\nu_1, \nu_4') \leftrightarrow (\nu_4', \nu_3) \leftrightarrow (\nu_3, \kappa_{11}) \leftrightarrow (\kappa_{11}, \xi_8)$$

and we note that the exchange relations holding between these clusters are exactly the tag equations (6.5) that run round the bottom of the rugby ball, from ι around to κ_{11}. This describes all the generators of our ring as cluster variables apart from ξ_1, ξ_2, ξ_5.

We write $\mathcal{A}$ for the cluster algebra generated by all of the cluster variables and $\mathcal{U}$ for the corresponding upper cluster algebra. Of course there are more cluster variables than the ones that we have written down. However, we claim that the cluster variables that lie beyond x and ξ_8 are not necessary as generators in the final ring. For instance, mutating one step back past x gives a cluster variable s with mutation relation $\iota s = \Phi(x, y^8)$. We can check that s can be eliminated using ξ_1, by:

$$s = x^3\xi_1 + y^7\eta \left(x^2\psi_3(y^3, z) + xy^{11}\psi_4(y^3, z) + y^{22}\right)$$

Non-cluster variables. We now show that ξ_1 is an element of $\mathcal{U}$. From the Pfaffians of Figure 6.1 and the equations (6.6) we have the two equations,

$$\begin{aligned} z\xi_1 &= y^7\phi_1(\nu_1, y\eta) + x\eta, \\ \nu_1'\xi_1 &= y^7\nu_1\phi_1(\nu_1, y\eta) + \eta^2\psi_2(y^3, z). \end{aligned}$$

Therefore $z\xi_1, \nu_1'\xi_1 \in \mathcal{U}$ and hence $\xi_1 \in R[i^{\pm 1}, z^{\pm 1}] \cap R[z^{\pm 1}, \nu_1'^{\pm 1}] \cap R[\nu_1'^{\pm 1}, \nu_1^{\pm 1}]$. By Lemma 7.3 it follows that $\xi_1 \in \mathcal{U}$. Similarly we can also show that $\xi_2, \xi_5 \in \mathcal{U}$.

Expansion diagram. We have established that our ring is contained in $\mathcal{U}$, so we can expand all of our generators as Laurent polynomials in terms of a chosen cluster.

If we choose the cluster (ν_1', ν_1) we can expand ξ_1 as

$$\xi_1 = \frac{y^7 \nu_1^4 \phi_1(\nu_1, y\eta) + \eta^2 \psi_2(y^3\nu_1, \nu_1' + y^4\eta)}{\nu_1' \nu_1^3}$$

where the constant term of the numerator $y^{12}\eta^5$ is nonzero. Therefore ξ_1 has denominator $\nu_1'\nu_1^3$ in least terms with respect to this cluster. Similarly we can show that the denominator of ξ_2 is $\nu_1'\nu_1^2$ and the denominator of ξ_5 is $\nu_1'^2\nu_1^3$.

The homogenous polynomial Φ has degree 5 therefore the expansion diagram we get has $(d, e) = (1, 5)$, as in Figure 7.1. The generators and their locations expanded in the cluster (ν_1', ν_1) are given in Figure 7.2. We note that the non-cluster variables ξ_1, ξ_2, ξ_5 all lie on the boundary of the polytope contained inside the irrational region.

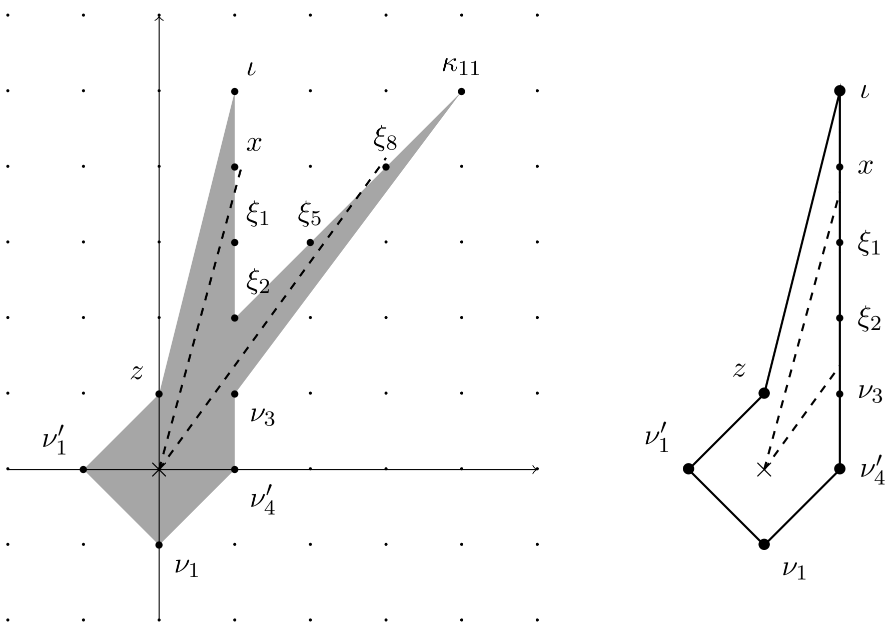


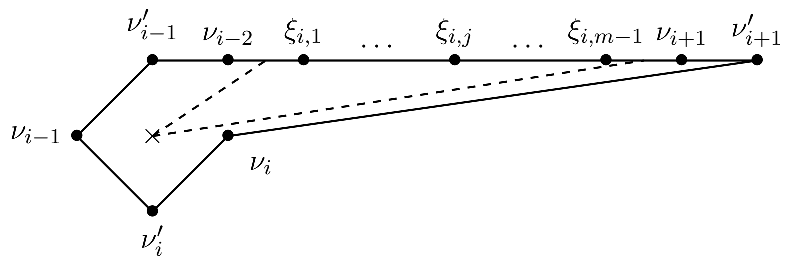
Figure 7.2: Expansion diagram for Big example 1 in terms of the cluster (ν_1', ν_1) and rolling factors for ν_1' against $\iota, x, \xi_1, \xi_2, \nu_3, \nu_4'$.

Rolling factors. If we choose the five consecutive cluster variables $\iota, z, \nu_1', \nu_1, \nu_4'$ and expand in a cluster that includes the middle cluster variable ν_1' then the two outside cluster variables ι and ν_4' lie on a straight line that includes a face of our polytope. The variables $\iota, x, \xi_1, \xi_2, \nu_3, \nu_4'$ lie in order along this line and the way that ν_1' multiplies against these functions is given by the crazy rolling factors equations

(6.6).

The general case

A similar picture holds in the general case §6.3.2. In this case all of the variables along the bottom row $\xi_0, \iota, \nu_0, \ldots, \nu_k, \kappa, \xi_{k+1}$ are cluster variables. Moreover the polytope inside our irrational region has faces of lattice length $m-1$ and the variables $\xi_{i,1}, \ldots, \xi_{i,m-1}$ appear along each face (remember that in our description $\xi_{i,m-1} = \xi_{i+1,1}$). Expanding in a cluster that includes ν_i' we see that ν_i' appears as a rolling factors variable against this face.



7.2.2 Big example 2 as a cluster algebra

Now consider Big example 2 §6.4.2. We choose to introduce a redundant generator $\xi_1' = x\xi_1 - gy^8$ which is chosen so that we have the two equations:

$$\begin{aligned} xz &= \iota\eta + y^{11} \\ \iota\xi_1' &= x^7 + ax^6y + bx^5y^2 + cx^4y^3 + dx^3y^5 + ex^2y^8 + fxy^{11} + gy^{19} \end{aligned}$$

Now these two equations generate a cluster algebra over the coefficient ring $\mathbb{C}[y, \eta]$ with $(d, e) = (1, 7)$ and initial clusters

$$(z, \iota) \leftrightarrow (\iota, x) \leftrightarrow (x, \xi_1').$$

In this case we can check that all of the generators can be expanded as Laurent polynomials in each of these clusters so that this ring lies inside the upper cluster algebra. Indeed, expanding in terms of the cluster (ι, x) we get the expansion diagram of Figure 7.3.

This is very different to the last example. Most of the extra generators required by the upper cluster algebra lie strictly inside the polytope generated by the lattice points in our irrational region.

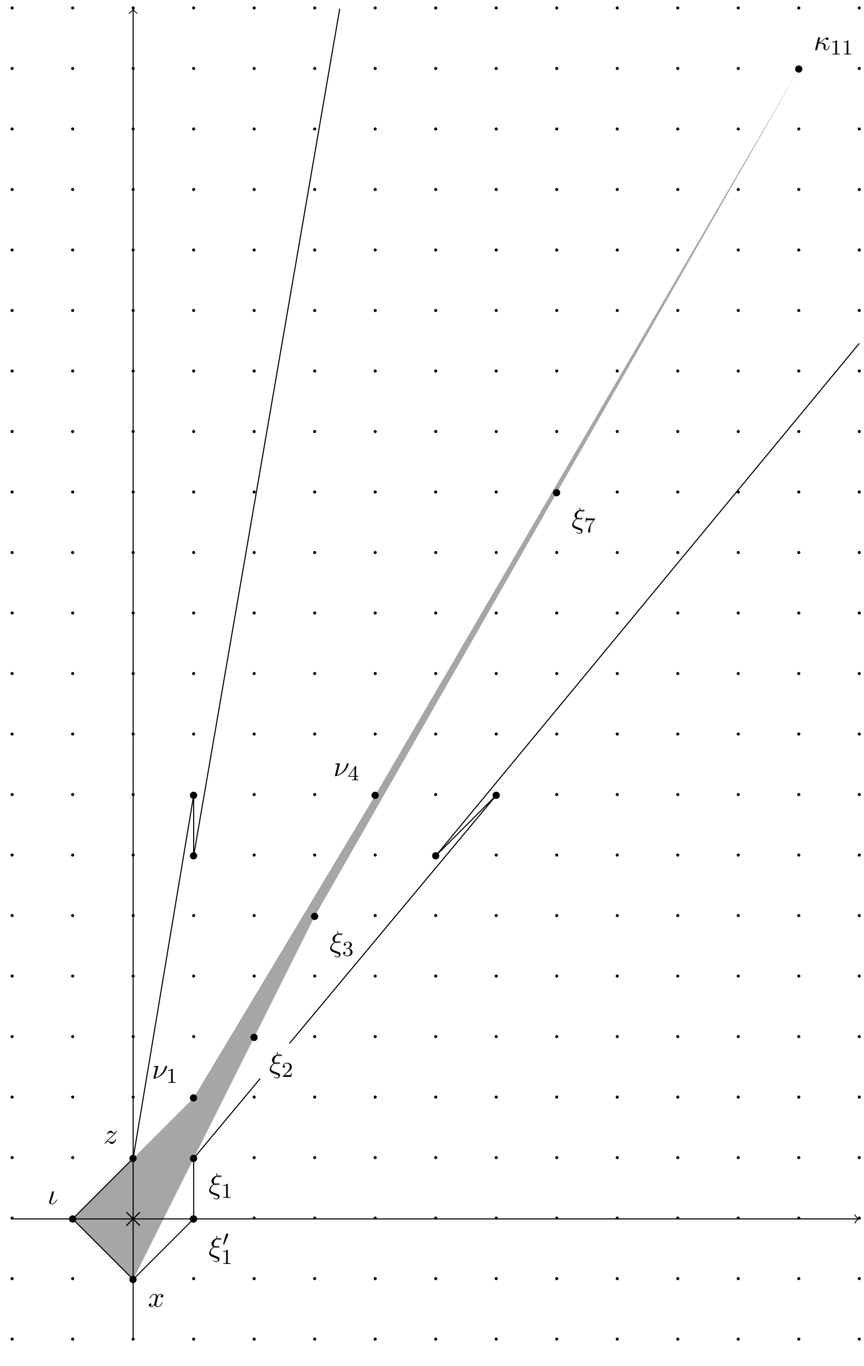


Figure 7.3: Expansion diagram for Big example 2.

7.3 Conclusions

7.3.1 Connections to other work

This description of our main examples as cluster varieties leads to some interesting connections to other works. In particular the expansion diagrams of Figures 7.2 and 7.3 should be compared with the 'scissors' diagram of [BR1] Figure 4.2, the toric surface M in [HTU] Proposition 3.19 and the scattering diagrams of [GHK].

Mori's algorithm

In a $k2A$ flipping neighbourhood $(C \subset X)$ the a flipping curve C passes through two cA/r singularities on X. Starting with such a neighbourhood, Mori [M2] writes down suitable local coordinate functions x_0 and x_1 such that $\mathrm{div}(x_0)$ and $\mathrm{div}(x_1)$ generate the local class group of the two cA/r singularities. He then describes a division algorithm to generate more functions $x_2, x_3, \dots$ on X. As noted in [HTU], this algorithm is nothing but mutation of a rank 2 cluster algebra starting with the initial cluster (x_0, x_1). Truncating this algorithm at a suitable point gives a ring

$$R = \mathbb{C}[x_0, x_1, \dots, x_k, x_{k+1}]$$

and Mori then proves that the flip $(C^+ \subset X^+)$ is given by the normalisation of R.

Brown & Reid's diptych varieties

Brown & Reid take Mori's algorithm one step further and aim to give a complete presentation of the graded ring of the flip $(C^+ \subset X^+)$ by introducing some generators and relations that were missing from R. Starting from the description of the neighbourhood as a complete intersection of codimension 2 they construct flips by serial unprojection, similar to our main construction §6.3.2.

Hacking, Tevelev & Urzúa

Hacking, Tevelev & Urzúa [HTU] introduce a toric surface M (only locally of finite type) and construct families of $k1A$ flips and some special $k2A$ flips (the case $d = e$ of diptych varieties) over this surface. This surface contains an infinite chain of rational curves and $k1A$ flips degenerate to $k2A$ flips at branch points along this divisor.

Gross, Hacking & Keel's construction

Gross, Hacking & Keel [GHK] construct a smoothing of V_n, the *vertex of degree* n, a cycle of n adjacent 2-planes in $\mathbb{C}^n$

$$V_n = \bigcup_{i=1}^{n} \mathbb{C}^2_{x_i,x_{i+1}}$$

which is controlled by a *mirror Looijenga pair* (Y, D). Here Y is a smooth projective rational surface, $D = D_1 + \cdots + D_n$ is a cycle of smooth rational curves and (Y, D) is a log Calabi-Yau pair, i.e. $D \in |-K_Y|$.

They work in great level of generality constructing a formal smoothing of V_n as a mirror family to (Y, D). In a simple case, when D supports an ample divisor, this smoothing is actually algebraic and contains $U = Y \setminus D$ as a fibre.

In particular, they introduce a integral affine manifold B (the 'tropicalisation' of U) containing a set of integer points $B(\mathbb{Z})$. Then they define a basis of canonical *theta functions*, θ_b for each $b \in B(\mathbb{Z})$, and the structure of a *scattering diagram* on B which encodes a multiplication rule between theta functions. This is defined in terms of the Gromov–Witten theory of the pair (Y, D). In the algebraic case, the smoothing of V_n is given by the Spec of this ring.

7.3.2 Further questions

These observations lead to the following questions:

- Is the canonical cover of a type A Mori flip or extraction always given by a rank 2 cluster algebra with suitable coefficients? If so, then this so-called $\mathcal{A}$-cluster variety has a canonically defined *mirror variety*, an $\mathcal{X}$-cluster variety. What is this?

- By Remark 7.4 we can define an integral affine manifold B such that the generators of our ring are naturally associated to distinguished integral points on B. Can we define a scattering diagram on B and interpret our generators as theta functions, in the sense of [GHK]?

- Can we use this description to construct (arbitrary, reducible) type A Mori flips and extractions as families over integral affine surfaces with singularities, as in the style of [HTU]?

Appendix A

Equations for the main examples

This appendix collects together equations for some of the examples in this thesis where they are not all given in the main text.

A.1 Prokhorov & Reid's example §3.3

Then equations of Y are given by Pfaffians of the following two matrices

$$\begin{pmatrix} \iota & x & y & -(cy+dz) \\ & y & z & ax+by \\ & & \eta & \xi \\ & & & \nu \end{pmatrix} \quad \begin{pmatrix} \kappa & \nu & \xi + c\eta & -d\eta \\ & -a\eta & \nu + b\eta & \xi \\ & & z & y \\ & & & x \end{pmatrix}$$

plus the 'long equation':

$$\iota\kappa = (ax+by)\xi + (cy+dz)\nu + (acx+ady+bcy+bdz)\eta$$

A.2 Tom$_1$ example §3.4.1

We define the following quantities:

$$\overline{\xi}_1 = \xi_1 + d\eta, \quad \overline{\nu}_1 = \nu_1 + c\eta, \quad \overline{\xi}_2 = \xi_2 + be\eta^2$$

Then the equations of Y are given by Pfaffians of the following three matrices

$$\begin{pmatrix} \iota & x & y & -(dy+ez) \\ & y^2 & z & ax^2+bxy+cy^2 \\ & & \eta & \xi_1 \\ & & & \nu_1 \end{pmatrix}$$

$$\begin{pmatrix} \xi_2 & \nu_1 & \overline{\xi}_1 & -e\eta \\ & -(ax+by)\eta & y\overline{\nu}_1 & \xi_1 \\ & & z & y \\ & & & x \end{pmatrix} \quad \begin{pmatrix} \kappa_3 & \xi_2 & e\overline{\nu}_1\eta & -\overline{\xi}_1 \\ & -ae\eta^2 & \overline{\xi}_2 & \nu_1 \\ & & \xi_1 & y \\ & & & x \end{pmatrix}$$

with the three missing equations:

$$\begin{aligned} z\kappa_3 &= \nu_1^2\overline{\nu}_1 + b\overline{\xi}_1\nu_1\eta + a\overline{\xi}_1^2\eta \\ \iota\kappa_3 &= (ax+by)\overline{\xi}_1^2 + cy\overline{\xi}_1\nu_1 + (dy+ez)\nu_1^2 + e\eta\big[(ay^2+bz)\overline{\xi}_1 + (axy+by^2+cz)\nu_1\big] \\ \iota\xi_2 &= (ax^2+bxy+cy^2)\overline{\xi}_1 + y(dy+ez)\nu_1 + ey(axy+by^2+cz)\eta \end{aligned}$$

A.3 Jer$_{45}$ example §3.4.2

We define the following quantities:

$$\overline{\nu}_1 = \nu_1 + b\eta, \quad \overline{N}_1 = \nu_1\overline{\nu}_1 + ac\eta^2, \quad \overline{\xi}_2 = \xi_2 + ad\eta^2$$

Then the equations of Y are given by Pfaffians of the following three matrices

$$\begin{pmatrix} \iota & x & y & -(cy^2+dyz+ez^2) \\ & y^2 & z & ax+by^2 \\ & & \eta & \xi_1 \\ & & & \nu_1 \end{pmatrix}$$

$$\begin{pmatrix} \xi_2 & \nu_1 & \xi_1+cy\eta & -(dy+ez)\eta \\ & -a\eta & y\overline{\nu}_1 & \xi_1 \\ & & z & y \\ & & & x \end{pmatrix} \quad \begin{pmatrix} \kappa_3 & \nu_1 & \overline{\xi}_2 & -ae\eta^2 \\ & -a\eta & \overline{N}_1 & \xi_2 \\ & & z & y \\ & & & \xi_1 \end{pmatrix}$$

with the three missing equations:

$$\begin{aligned} x\kappa_3 &= \xi_1\overline{\xi}_2 + cy\xi_2\eta + (dy+ez)\overline{N}_1\eta + aey\overline{\nu}_1\eta^2 \\ \iota\kappa_3 &= ax\overline{\xi}_2 + a^2ey^2\eta^2 + (by+cz)\nu_1\xi_1 + \nu_1(dy+ez)(z\overline{\nu}_1 + ay\eta) \\ \iota\xi_2 &= (ax+by^2+cyz)\xi_1 + y(dy+ez)(z\overline{\nu}_1 + ay\eta) \end{aligned}$$

A.4 The semistable $\mathbf{E}_6$ case §4.2.2

We define the following quantities:

$$\overline{\xi} = \xi - f\eta, \quad \overline{\nu} = \nu + (dy+ez)\eta, \quad \overline{\zeta} = \zeta - b\xi\eta, \quad \overline{\theta} = \theta - cd\xi\eta^2.$$

Then the equations of Y are given by Pfaffians of the following three matrices

$$\begin{pmatrix} \zeta & \nu & y(\xi+a\eta) & -(by+cz)\eta \\ & \xi & \overline{\nu} & \overline{\xi} \\ & & z^2 & y \\ & & & x \end{pmatrix}$$

$$\begin{pmatrix} \theta & c\xi\eta & \overline{\zeta} & \overline{\xi} \\ & -z\zeta & \xi(\xi+a\eta) & \overline{\nu} \\ & & \nu & y \\ & & & z \end{pmatrix} \quad \begin{pmatrix} \kappa & \zeta\overline{\xi} + ce\xi\eta^2 & \overline{\theta} & c\xi\eta \\ & c\xi\eta & \overline{\zeta} & \overline{\xi} \\ & & \nu & y \\ & & & z \end{pmatrix}$$

with the three missing equations:

$$\begin{aligned} x\theta &= z(\xi+a\eta)\overline{\xi}^2 + bz\overline{\xi}\eta\overline{\nu} + c\eta\overline{\nu}^2 \\ x\kappa &= (\xi+a\eta)\overline{\xi}^3 + b\overline{\xi}^2\eta\overline{\nu} + c\eta\big[z\overline{\xi\zeta} - c\xi\overline{\nu}\eta + dz\overline{\xi}^2\eta + e\overline{\xi}\overline{\nu}\eta\big] \\ \xi(\xi+a\eta)\kappa &= \zeta\overline{\zeta}^2 - \theta\overline{\theta} + d\zeta\overline{\zeta\xi}\eta + e\theta\overline{\zeta}\eta \end{aligned}$$

A.5 Monomial curve example 4.3

The equations of Y are given by Pfaffians of the following three matrices

$$\begin{pmatrix} \zeta & \xi_1 & \xi_2 & \xi_3 \\ & x\xi_3 & y\xi_1 & \xi_2 \\ & & z & -y \\ & & & x^2 \end{pmatrix} \quad \begin{pmatrix} \theta & -\xi_3^2 & \zeta & \xi_1 \\ & x\zeta & \xi_1\xi_3 & -\xi_2 \\ & & \xi_2 & -y \\ & & & x \end{pmatrix} \quad \begin{pmatrix} \kappa & \zeta^2 & \theta & \xi_3^2 \\ & \theta & \zeta & \xi_1 \\ & & \xi_1\xi_3 & -\xi_2 \\ & & & x \end{pmatrix}$$

with the three missing equations:

$$
\begin{aligned}
z\theta &= x\xi_1^3 + \xi_2^2\xi_3 \\
y\kappa &= \zeta\xi_1^2 - \xi_3^4 \\
z\kappa &= \xi_1^4 + x\xi_1\xi_3\zeta + \xi_2\xi_3^3
\end{aligned}
$$

A.6 $\mathbf{D}_5^r$ example §4.3.1

Define the following quantities:

$$\overline{\xi} = \xi - g\eta, \quad \overline{\nu} = \nu + (ey + fz)\eta, \quad \overline{\zeta} = \zeta + c\xi\eta, \quad \overline{\theta} = \theta + b\xi\overline{\xi}\eta - f\zeta\eta.$$

Then the equations of Y are given by Pfaffians of the following three matrices

$$\begin{pmatrix} \zeta & \overline{\xi} & \overline{\nu} & \xi \\ & -dz\eta & z\xi + (ay^2 + byz + cz^2)\eta & \nu \\ & & x & y \\ & & & z^2 \end{pmatrix}$$

$$\begin{pmatrix} \theta & z\overline{\zeta} + \xi^2 & -(ay + bz)\xi\eta & -\overline{\nu} \\ & d\xi\eta & \zeta & \overline{\xi} \\ & & \nu & y \\ & & & z \end{pmatrix} \quad \begin{pmatrix} \kappa & \overline{\theta} & e\zeta\eta - a\xi\overline{\xi}\eta & -\zeta \\ & d\xi\eta & \zeta & \overline{\xi} \\ & & \nu & y \\ & & & z \end{pmatrix}$$

with the three missing equations:

$$
\begin{aligned}
x\theta &= (ay + bz)z\eta\overline{\xi}^2 + \overline{\xi}(\xi + cz\eta)\overline{\nu} + d\eta\overline{\nu}^2 \\
x\kappa &= \Big[\overline{\xi}\big(\xi + (by + cz)\eta\big) + d\eta\overline{\nu}\Big]\Big[\zeta + e\overline{\xi}\eta\Big] + a\overline{\xi}\eta\Big[z\overline{\xi}^2 - fy\overline{\xi}\eta + dy\xi\eta\Big] \\
\xi^2\kappa &= \theta\big(\overline{\theta} - de\xi\eta^2\big) - \Big[\zeta\overline{\zeta} + bd\xi^2\eta^2\Big]\Big[\zeta + e\eta\overline{\xi}\Big] + a\xi\eta\Big(\overline{\xi}^2\overline{\zeta} + df\xi\overline{\xi}\eta^2 - d^2\xi^2\eta^2\Big)
\end{aligned}
$$

A.7 Big example 2 §6.4.2

The equations are given by the pentagrams appearing in Figure 6.3 plus some missing equations. We use the notation defined in §6.4.2 and also define:

$$\overline{N}_4 = \nu_4\overline{\nu}_4 + a\overline{\xi}_1^3\overline{\xi}_3\eta^2$$

There are five missing equations involving ι (which we don't include as they are not required for the calculation) and ten others.

Of these ten, there are four involving z

$$\begin{aligned}
z\xi_3 &= y^2\nu_1\overline{N}_1 + \phi_3\overline{\xi}_1^2\eta \\
z\nu_4 &= \nu_1^2\overline{N}_1 + y\overline{\xi}_1^2\eta\left[\phi_2\overline{\xi}_1 + c\nu_1\right] \\
z\xi_7 &= y\nu_1\overline{N}_1\overline{\nu}_4 + \overline{\xi}_1^2\eta\left[\phi_2\xi_3\overline{\xi}_1 + c\xi_3\nu_1 + \phi_1 y^2\overline{\xi}_1\overline{N}_1\eta\right] \\
z\kappa_{11} &= \nu_1\overline{N}_1\overline{N}_4 + \overline{\xi}_1^4\overline{N}_1\eta^2\left[y^2\overline{\xi}_3 + a\phi_2\overline{\xi}_1\eta^2 - ac\nu_1\eta^2\right] + \overline{\xi}_1^2\overline{\xi}_7\eta\left[\phi_2\overline{\xi}_1 + c\nu_1\right]
\end{aligned}$$

three involving ξ_1

$$\begin{aligned}
\xi_1\nu_4 &= y\xi_2\overline{\xi}_3 + \phi_2 y^3\overline{\nu}_1\overline{\xi}_1\eta^3 + \nu_1\eta^2\left[\phi_3\overline{\nu}_1\eta + d\xi_2\right] \\
\xi_1\xi_7 &= \xi_2\xi_3\overline{\xi}_3 + y\overline{\nu}_4\eta^2\left[\phi_3\overline{\nu}_1\eta + d\xi_2\right] + y^4\overline{\xi}_1\overline{\nu}_1\eta^3\left[\phi_1\overline{N}_1\eta + b\xi_3\right] \\
\xi_1\kappa_{11} &= \xi_2\overline{\xi}_3\xi_7 + y^3\overline{\xi}_1\overline{\nu}_1\eta^3\left[\overline{N}_1\eta(y\overline{\xi}_1\overline{\xi}_3 + x\overline{\nu}_4) + (ax+by)\xi_7\right] + \overline{N}_4\eta^2\left[\phi_3\overline{\nu}_1\eta + d\xi_2\right]
\end{aligned}$$

two involving ξ_2

$$\begin{aligned}
\xi_2\xi_7 &= \xi_3^2\overline{\xi}_3 + y\overline{\xi}_1\overline{N}_1\eta^3\left[\phi_1\overline{N}_1\eta + b\xi_3\right] \\
\xi_2\kappa_{11} &= \xi_3\overline{\xi}_3\overline{\xi}_7 + \overline{\xi}_1\overline{N}_1\nu_4\eta^3\left[\phi_1\overline{N}_1\eta + b\xi_3\right] + y\overline{\xi}_1^2\overline{\xi}_3\overline{N}_1^2\eta^4
\end{aligned}$$

and one involving ν_1:

$$\nu_1\kappa_{11} = \nu_4\overline{N}_4 + \overline{\xi}_1^4\overline{\xi}_3^2\eta^2.$$

Bibliography

[BFZ] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells.* Duke Math. J. 126 (2005), no. 1, 1-52.

[B] E. Brieskorn, *Rationale Singularitäten komplexer Flächen.* Invent. Math. 4 (1968), 336-358.

[BKR] G. Brown, M. Kerber, M. Reid, *Fano* 3*-folds in codimension 4, Tom and Jerry, Part I.* Compositio Mathematica, 148, 1171-1194.

[BR1] G. Brown, M. Reid, *Diptych varieties, I.* Proc. London Math. Soc. (2013) 107 (6): 1353-1394

[BR2] G. Brown, M. Reid, *Diptych varieties, II: Polar varieties.* Submitted, 24pp. Preprint version arXiv:1208.5858.

[BR3] G. Brown, M. Reid, *Diptych varieties, III: Redundant generators.* 10pp., preliminary draft available at `http://homepages.warwick.ac.uk/staff/G.Brown/diptych3.pdf`.

[BR4] G. Brown, M. Reid, *Diptych varieties, IV: Mori flips of type A.* 18pp., preliminary draft available at `http://homepages.warwick.ac.uk/staff/G.Brown/diptych4.pdf`.

[BE] D. Buchsbaum, D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3.* 39pp. American Journal of Mathematics. 99 (1977), no. 3, 447-485.

[C] S. Cutkosky, *Elementary contractions of Gorenstein threefolds.* Math. Ann. 280, pp. 521-525, 1980.

[CS] L. Chekhov, M. Shapiro, *Teichmller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables.* Int. Math. Res. Not. IMRN, 2014:10 (2014), 2746-2772, arXiv:1111.3963.

[D] T. Ducat, *Divisorial extractions from singular curves in smooth 3-folds, I.* Preprint, arXiv:1403.7614, 21pp.

[E1] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations.* Transactions of the AMS. 260 (1980), no. 1, 35-64.

[E2] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry.* Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[FZ] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations.* J. Amer. Math. Soc. 15 (2002), 497-529.

[GNW] S. Goto, K. Nishida, K.-i. Watanabe, *Non-Cohen-Macaulay symbolic blowups for space monomial curves and counterexamples to Cowsiks question.* Proc. Amer. Math. Soc. 120 (1994), 383-392.

[GHK] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I.* Preprint arXiv:1106.4977, 108 pp., to appear in Publ. Math. Inst. Hautes Études Sci.

[GHKv1] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I (version 1).* Preprint arXiv:1106.4977v1, 144 pp.

[HTU] P. Hacking, J. Tevelev, G. Urzúa, *Flipping Surfaces.* Preprint, arXiv:1310.1580, 65 pp., to appear in J. Algebraic Geom.

[KST] H. Kajiura, K. Saito, A. Takahashi, *Matrix factorization and representations of quivers, II. Type ADE case.* Adv. Math. 211 (2007), no. 1, 327-362.

[K1] Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities.* Higher-dimensional complex varieties (Trento, 1994), 241-246, de Gruyter, Berlin, 1996.

[K2] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces.* Ann. Math., Vol. 127, No. 1 (1988), 93-163.

[KM] J. Kollár, S. Mori, *Classification of three-dimensional flips.* Journal of the American Mathematical Society, Vol. 5, No. 3 (Jul., 1992), pp. 533-703.

[KSB] J. Kollár, N. Shepherd-Barron, *Threefolds and deformations of surface singularities.* 1988, Inventiones Mathematicae, 91, 299-338.

[M1] S. Mori, *Flip theorem and the existence of minimal models for 3-folds.* J. Amer. Math. Soc., 1 No. 1 (1988) 117-253.

[M2] S. Mori, *On semistable extremal neighbourhoods.* Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., 35, Math. Soc. Japan, (2002), 157-184.

[PaR] S. Papadakis, M. Reid, *Kustin-Miller unprojection without complexes,* J. Algebraic Geometry 13 (2004), 563-577.

[PrR] Y. Prokhorov, M. Reid, *On* $\mathbb{Q}$*-Fano threefolds of Fano index 2.* Preprint arXiv:1203.0852.

[R1] M. Reid, *Young person's guide to canonical singularities.* Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., 46, Providence, R.I.: American Mathematical Society, 345-414, 1987.

[R2] M. Reid, *What is a flip?* Scanned notes, 53 pp., `http://homepages.warwick.ac.uk/~masda/3folds/what_flip.pdf`.

[R3] M. Reid, *Nonnormal del Pezzo surfaces.* Math. Proc. RIMS 30 (1994) 695-727.

[R4] M. Reid, *Graded rings and birational geometry.* Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1-72.

[R5] M. Reid, *The Du Val singularities* A_n, D_n, E_6, E_7, E_8*.* Notes, `http://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf`.

[S] J. Stevens, *On the verbal deformation of cyclic quotient singularities.* In Singularity theory and its applications, Part I (Coventry, 1988/1989), 302-319, Lecture Notes in Math. 1462, Springer, 1991.

[Tz1] N. Tziolas, *Terminal* 3*-fold divisorial contractions of a surface to a curve I.* Compositio Mathematica 139, No. 3, 2003.

[Tz2] N. Tziolas, *Families of* D*-minimal models and applications to* 3*-fold divisorial contractions.* Proceedings of the London Mathematical Society (2005) 90 (2), 345-370.

[Tz3] N. Tziolas, *Three dimensional divisorial extremal neighbourhoods.* Math Ann. 333, 2005.

[Tz4] N. Tziolas, 3-*fold divisorial extremal neighbourhoods over* cE_7 *and* cE_6 *compound Du Val singularities.* Int. J. Math., 21, 1 (2010).

[Y] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings.* London Mathematical Society Lecture Note Series, CUP, 1990.