A study of braids in 3-manifolds

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Submitted for PhD examination

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May 1993

To two people
who mean everything to me:
my parents

Contents

Acknowledgements

My thanks are due to several persons who have aided me throughout my research by offering their valuable help and expert advice. It is not possible to list these persons in detail, but I consider as my moral obligation, the least to be done, to thank them deeply from the pages of this thesis.

I wish to express my deep thanks and gratitude to my supervisor, Dr. Colin Rourke, for his help and precise guidance, his enthusiastic attitude and keen interest in my work, for his patience, and for the numerous essential comments.

Also, I wish to acknowledge an invaluable discussion with Professor Vaughan Jones.

I would like to deeply thank Professor Jim Eells for showing trust in me and I hope to always remain in his trust.

My sincere thanks are also due to Professor Jozef Przytycki for some apposite remarks and for providing me with historical knowledge and references.

To my colleague Dr. Alastair Leeves for being an exceptional listener and for his group-theoretical contribution, for several apposite comments and for providing 'molar surropt' over numerous pints.

To my colleague Dr. Meinolf Geck for introducing me to Coxeter groups and Hecke algebras and for working with me on the trace function problem.

To my colleague Dr. Gabriela Gomes for introducing me to LaTeX and Illustrator by drawing many of the pictures in here, and for sharing with me the unforgettable experience of a very dangerous car-accident.

To John Sardis for his deep, kind and faithful friendship and a very inspiring discussion.

To the very kind ladies Mrs Muriel Nelson and Mrs Peta McAllistair for offering me their precious friendship.

I leave lastly the most difficult to express:

My deepest thanks to my parents, Spelios and Fotini, my brother, Harris, and my best friend and sister-in-law, Joanna, for spoiling me all these years with their care and their warm and wise presence.

Finally, I would like to express my gratitude to the University of Warwick for their partial support and to the Greek Scholarship Foundation for their main support of this work.

Declaration

This thesis is the original work of the author, with the exception of few sources cited in the text. Results in Chapter 1 and the first part of Chapter 2 appear in an improved topological version in [39] as joint work of the author with Dr. C.P. Rourke. The results in here were preceeding.

Sofia S.F. Lambropoulou (May 1993)

Summary

This work provides the topological background and a preliminary study for the analogue of the 2-variable Jones polynomial as an invariant of oriented links in arbitrary 3-manifolds via normalized traces of appropriate algebras, and it is organized as follows:

Chapter 1: Motivated by the study of the Jones polynomial, we produce and present a new algorithm for turning oriented link diagrams in S^3 into braids. Using this algorithm we then provide a new, short proof of Markov's theorem and its relative version.

Chapter 2: The objective of the first part of Chapter 2 is to state and prove an analogue of Markov's theorem for oriented links in arbitrary 3-manifolds. We do this by modifying first our algorithm, so as to produce an analogue of Alexander's theorem for oriented links in arbitrary 3-manifolds. In the second part we show that the study of links (up to isotopy) in a 3-manifold can be restricted to the study of cosets of the braid groups $B_{n,m}$, which are subgroups of the usual braid groups B_{n+m} .

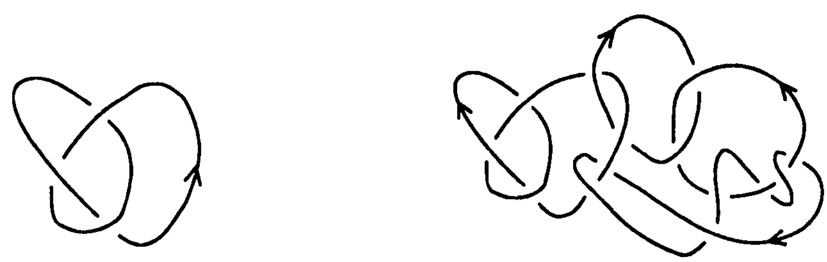
Chapter 3: In this chapter we try to use the above topological set-up in a procedure analogous to the way V.F.R. Jones derived his famous link invariant. The analogy amounts to the following: We observe that $B_{n,1}$ – the braid group related to the solid torus and to the lens spaces $L(p,1)$ – is the Artin group of the Coxeter group of B_n-type. This implies the existence of an epimorphism of $\mathbb{C}B_{n,1}$ onto the Hecke algebra of B_n-type. Then we give an analogue of Ocneanu's trace function for the above algebras. This trace, after being properly normalized, yields a HOMFLY-PT-type isotopy invariant for oriented links inside a solid torus. Finally, by forcing a strong condition, we normalize this trace, so as to obtain a link invariant in $S^1 \times S^2$.

Chapter 0

Introduction

0.1 On chapter 1

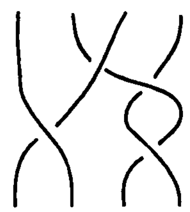
An (oriented) **knot** is an embedding of the (oriented) circle S^1 into the 3-sphere S^3 , and an (oriented) **link** of n components is an embedding of n (oriented) copies of S^1 into S^3 . We study knots and links by studying their (regular) projections on a plane, which we call **diagrams** .
Examples:



We say that two knots (or links) are **isotopic** if we can continuously deform one into the other without causing self-crossings in the 3-space. Given any two links, the main information we want to know is whether they are isotopic or not; so, we would like to translate 'isotopy' in terms of diagrams . **Reidemeister** in [51] and Alexander and Briggs in [3] proved that two knots (or links) are isotopic if and only if any two diagrams of theirs are related through a finite sequence of planar moves (see Theorem 3 in 1.2) .
Knots and links make a subject for study on their own, known as 'knot theory' . On the other hand they are closely related to 3-manifold theory, as every 3-manifold can be obtained by doing surgery along a framed link in S^3 .

Another geometrical object of similar nature is a **braid** ([26], [4], [7]) . A braid on n strings is most commonly described as an object in $D^2 \times [0,1]$ consisting of n strings starting from n specified points inside the top disc $D^2 \times \{1\}$ and ending at n specified points inside the bottom disc $D^2 \times \{0\}$, the strings in the middle braiding arbitrarily, but without being allowed to form local maxima or minima. A picture of a braid is given below:

Picture:



If we imagine joining the corresponding top and bottom end-points of the braid strings, the result would be an oriented link diagram. Then a question arises naturally: Can we obtain all oriented links (up to isotopy) by closing braids ? **Alexander** showed in [1] that links and braids are interrelated, as there exists an algorithm for turning any oriented link diagram into a braid with isotopic closure; and this answers affirmatively the above question[1].

The biggest – still practically unsolved – problem in knot theory is to classify knots and links up to isotopy[2]. For this, it seems reasonable to try to work with braids, since braids are very structured objects. Even more so, as the set of braids on n strings can be given naturally a group structure. The second question that comes now is: how can we leave aside knots and work with braids instead ? **Markov** in [48] answered successfully this question by announcing that there is a 1–1 correspondence between isotopy classes of links and equivalence classes of braids (seen either as geometric objects or as elements of the braid groups), the equivalence being given by two algebraically formulated moves between braids (see Theorem 2 in 1.1) .
As Markov did not give a completely satisfactory proof, Joan Birman in [7] gave a complete proof of Markov's theorem. According to J. Birman, after Markov, there was another brief announcement of an improved version of Markov's theorem by Weinberg in 1939 (see [62]) .

Apart from being appealling on their own, the two theorems received attention anew when V.F.R. Jones announced a new polynomial link invariant via the study of braids (see [28], [29]) .

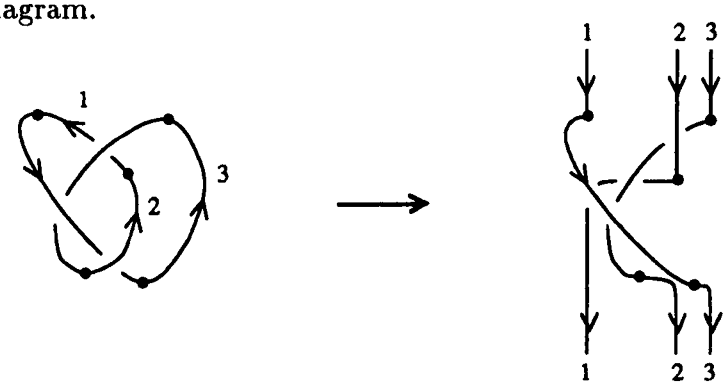
The study of the Jones polynomial was also our motivation for studying knots and braids; also for finding a new algorithm for 'opening' oriented link diagrams into braids (so as to answer a question posed by my colleague Meinolf Geck). Then, using this algorithm, we give another proof of Markov's theorem (following C.P. Rourke's suggestion). The above are presented in Chapter 1.

The idea of our braiding process: Start with an oriented link diagram and mark with points the local maxima and minima. This set of points separates naturally the diagram into horizontal or downward arcs on one hand, and into 'opposite' arcs (i.e. arcs that go upwards) on the other hand. We want to eliminate the opposite arcs, as they go the 'wrong' way for a braid. We eliminate an opposite arc by cutting it at

[1] H. Brunn ([9]) in 1897 proved that any knot has a projection with a single multiple point; from which follows immediately that we can braid any link diagram.

[2] As M.B. Thistlethwaite mentions in [57]: 'Astonishingly, there *is* a uniform method of classifying knots, and also knot groups, arising from the Haken theory of irreducible, sufficiently large manifolds [20], and a certain missing step supplied by Hermion [21]'

some point and by pulling and stretching its two ends both *over* or *under* the rest of the original diagram, according to whether the opposite arc lies *over* or *under* other arcs of the diagram.
Picture:



The proof of Markov's theorem follows then easily, since our algorithm has the advantage that one can keep track of each step on the plane.

A consequence of our proof of Markov's theorem is Corollary 4 in 1.4.2 (relative version of Markov's theorem) that says the following:

'If two isotopic links which are turned into braids contain the same braided part, then the two resulting braids differ by conjugation and Markov moves that do not affect the already braided part.'

0.2 On the Jones polynomial

A link **invariant** is a labelling for links, so that isotopic links will be assigned the same label. Note that it may happen, that two non-isotopic links will have the same label. Classical invariants are for example: the fundamental group of the complementary space of a link, the Alexander polynomial etc.

V.F.R. Jones in [28] and [29], used Markov's theorem so as to work with braids in order to find a new link invariant. He applied an inductive process, where we give a label to the identity braid and we find the label of a given braid by splitting it repeatedly into braids with simpler structure. This indicates that we need to define **addition** between any two braids, and not only multiplication (which is the braid group operation). A way to do this, is by taking the braid group algebra over the complex numbers , $\mathbb{C}B_n$.

A simple fact about braids is that if we take a braid on n strings and ignore the crossings (positive or negative), the only remaining information is the permutation of the end-points. So to each braid in the (infinite) group B_n we assign an element of the (finite) symmetric group S_n . In fact S_n is a quotient of B_n and therefore the algebra $\mathbb{C}S_n$ is a quotient of $\mathbb{C}B_n$. Furthermore, there exist some algebras called **Hecke algebras of A_n-type** , $\mathcal{H}_n(q)$, which are isomorphic to the algebras $\mathbb{C}S_n$. V. Jones used the above natural epimorphism to send braids to elements of $\mathcal{H}_n(q)$ for all n (see [30]). Then, using a linear trace (Ocneanu's trace) that sends $\bigcup_{n=1}^{\infty}\mathcal{H}_n(q)$ to $\mathbb{C}$, he assigned to every braid a complex (Laurent) polynomial[3]. In order to obtain then a link invariant, he had to normalize this trace properly (using Markov's theorem), so that braids with isotopic closures will be given the same polynomial-label (see 3.1 for more details).

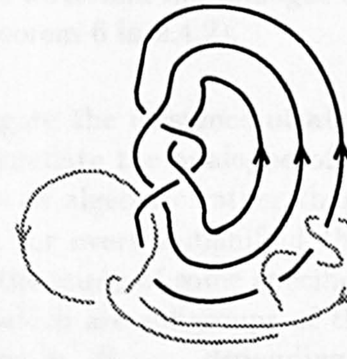
0.3 On chapter 2

As mentioned in the beginning, all the above take place in S^3. The rest of this work is an attempt to extend the above theorems and ideas to links in arbitrary closed, connected, orientable (c.c.o.) 3-manifolds, and it should be seen as the set-up of a machinery for defining the analogue of Jones polynomial in other 3-manifolds following V. Jones's ideas.

What makes the difference with what is said before, is that now we have to take into account the nature of the manifold. Our viewpoint is the following: we represent a c.c.o. 3-manifold M by a fixed projection of a surgery link in S^3, which, in fact and without loss of generality, is the closure $\widehat{B}$ of a *pure braid* B (as follows from [41]). So we study M and links in it by translating our questions into equivalent ones in S^3.

Therefore, any **link** L in M can be represented by a **mixed link** $L \bigcup \widehat{B}$ in S^3, consisting of a link in S^3 together with the fixed surgery part, $\widehat{B}$. A **link diagram** is a projection $\widetilde{L} \bigcup \widehat{B}$ of $L \bigcup \widehat{B}$ on the plane of $\widehat{B}$.

Example:



Next, we want to see how **isotopy** between links in M is reflected in S^3: The surgery description of M gives rise to an additional move (between mixed links in S^3), which we call **band move**. More precisely, a band move tells what we see in S^3 when a part of a link (a band) approaches very closely a surgery component. Then, the **Reidemeister's theorem** (see 2.2.4), is modified as follows:

[3]V. Jones used initially quotients of the Hecke algebras – namely von Neumann algebras called 'type II_1 factors' – to derive the original 1-variable Jones polynomial (see [29]); with the use of Hecke algebras he re-constructed the HOMFLY-PT polynomial, which is the 2-variable generalization of the Jones polynomial, constructed independently by five groups of mathematicians (see [40], [16], [50]).

'Two links in M are isotopic if and only if any two corresponding mixed link diagrams in S^3 differ by planar isotopy and a finite sequence of the augmented Reidemeister moves (where the surgery link participates too) and the two band moves (that derive from the different orientations of the band).'

We define a **mixed braid** in S^3 to be a braid in S^3 together with a *fixed* surgery braid B. Then, by modifying slightly our braiding process (mentioned above), we prove that any oriented link in M can be represented by a mixed braid in S^3 (analogue of **Alexander's theorem**).

This proves additionally that M can be represented by the surgery braid B (instead of $\widehat{B}$), since throughout the process $\widehat{B}$ remains unaltered.

The last thing for the topological set-up, is an analogue of **Markov's theorem** .(This problem was suggested by Colin Rourke, and the main idea for the formulation as well as a part of its proof are due to him): After modifying properly a type of band moves, so as to turn them into moves between braids, we state such a theorem (Theorem 5 in 2.4) and we prove it using the relative version of Markov's theorem and the analogue of Alexander's theorem. Theorem 5 says:

'Two links in M are isotopic if and only if any two mixed braids obtained by two corresponding mixed link diagrams in S^3 are equivalent under conjugation, the Markov moves and the braid band moves.'

As a special case of Theorem 5 we obtain the analogue of Markov's theorem for oriented links inside a solid torus (Theorem 6 in 2.4.2).

Next, we want to investigate the existence of algebraic structures in the sets of mixed braids, in order to formulate the analogue of Markov's theorem algebraically (i.e. looking at mixed braids as algebraic rather than geometrical objects). Indeed, in section 2.5 we prove that for every 3-manifold the study of mixed braids up to isotopy, can be restricted to the study of some specific mixed braids that form *either* groups, the groups $B_{n,m}$ (which are subgroups of the usual braid groups B_{n+m}), *or* cosets of the above groups in B_{n+m}, depending on the nature of the manifold. Finally, we conclude Chapter 2 by giving algebraic formulations of the analogue of Markov's theorem for the solid torus as well as for some lens spaces (the ones that can be described by one surgery string), after having found an appropriate presentation for the corresponding braid groups $B_{n,1}$.

0.4 On chapter 3

So far we have developed a topological theory in analogy to the existing one for the S^3–case.

As we mentioned in 0.2 , the Hecke algebra of A_n-type is related to the 'usual' braid group B_n . In [30], V. Jones asked whether other Hecke algebras, related to general Artin groups, can be used in the same manner as the ones of A_n-type. Here, we make some progress in this direction using the Hecke algebras of B_n-type corresponding to the Artin group $B_{n,1}$, as follows:

We compare the presentation for $B_{n,1}$, given in section 2.6 , with the standard presentation of the Coxeter group of B_n-type , W_n , and we observe that there is a natural epimorphism of $B_{n,1}$ onto W_n . This implies immediately the existense of an epimorphism of $\mathbb{C}B_{n,1}$ onto $\mathcal{H}_n(q,Q)$, the Hecke algebra of B_n-type . The above now suggest that we look for a trace function from $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ to $\mathbb{C}$ analogous to Ocneanu's trace, so as to attach to each braid in $B_{n,1}$ a complex polynomial. Indeed, in section 3.3 we give such a trace function theorem, which is joint work with Meinolf Geck. After normalizing this trace properly (using our analogue of Markov's theorem), we obtain a HOMFLY-PT-type isotopy invariant for oriented links inside a solid torus (section 3.4.1) , which we compare (in section 3.4.2) with J. Hoste's and M. Kidwell's dichromatic link invariant as presented in [22].

0.5 A concluding note

The natural thing to do next, is to try to normalize the $\mathcal{H}_n(q,Q)$-trace properly, so as to obtain a link invariant in the lens spaces that can be described by one surgery string, i.e. the spaces $L(p,1)$. The major obstacle that appears here, is that the band move and the Markov move are **not** algebraically compatible in the sense that: each one has to occupy one side of the braid in order to obtain a simple algebraic expression of Markov's theorem, so the braid strings may increase from both sides. This leads to forcing $q = 1$ and thus to finding a weak invariant for links in $L(0,1)$, instead of the analogue of the HOMFLY-PT polynomial that we were hoping for.

However, if we omit one of the quadratic relations of the Hecke algebra of B_n-type, then the above trace is not unique, and the space of all traces is the dual space to the third skein module of the solid torus (see [22], [59]; see also [23] for a survey of skein modules). So, by considering this more general family of traces, we hope to overcome the problem described above.

This idea is strongly supported - although from a different viewpoint - by the recent works of J. Hoste and J. Przytycki, who defined the analogue of the Kauffman bracket version of the Jones polynomial (see [31]) for lens spaces using skein module theory, and this analogue consists of attaching more than one polynomial-labels to an oriented link in the manifold (see [23], [24], [25]) . It also seems related to the recent works of W.B.R. Lickorish (see [42]), where he gives a purely combinatorial way for viewing Witten's invariants (see [63]).

Chapter 1

Alexander's and Markov's theorems

1.1 Introduction

In this chapter we shall describe a straightforward algorithm for turning any oriented link diagram into a braid (diagram), and thus give a new proof of Alexander's theorem. Then, using our algorithm, we shall give a short new proof of Markov's theorem.

Alexander's and Markov's theorems date back to 1923 (see [1]) and 1935 (see [48]) respectively. In 1974 J.S. Birman gave the first complete published proof of Markov's theorem (see [7]) . The two theorems received attention again, after V.F.R. Jones (in [30]) used the braid groups in constructing his polynomial link invariant. Other proofs of Alexander's theorem have been given by H.R. Morton ([49], 1986), by S. Yamada ([64], 1987), by P. Vogel ([60], 1990) and by P. Traczyk ([58], 1992)[1] . Other proofs of Markov's theorem have been given by Bennequin ([6], 1983), by H.R. Morton ([49], 1986) and by P. Traczyk ([58], 1992) .

Theorem 1 (Alexander 1923) *Any oriented link is isotopic to the closure of some braid (not unique).*

Theorem 2 (Markov, Weinberg) *Two oriented link diagrams are isotopic if and only if any two corresponding braids are related by a finite sequence of the following moves:*

(i) **Conjugation:** *If* $\alpha, \beta \in B_n$ *then* $\alpha \sim \beta^{-1}\alpha\beta$.

(ii) **Markov move:** *If* $\alpha \in B_n$ *then* $\alpha \sim \alpha\sigma_n^{+1} \in B_{n+1}$ *and* $\alpha \sim \alpha\sigma_n^{-1} \in B_{n+1}$,

where B_n *is the braid group on* n *strings.*

[1]In the proofs given in [64], [60] and [58], the number of strings in the resulting braid equals the number of Seifert circles of the original link diagram.

1.2 Topological definitions et cetera

Throughout this chapter we shall work in the piecewise-linear (pl.) category (see [54]) , but occasionally we shall draw smooth diagrams, for convenience.

We take $D^2 \times [0,1]$ standardly embedded in S^3 , and we specify m points inside $D^2 \times \{1\}$ and n points inside $D^2 \times \{0\}$. (We consider D^2 to be $(0,1) \times (0,1)$.)

Definition 1 An (m,n)**-tangle** is an embedding of a disjoint, finite union of circles and intervals in $D^2 \times [0,1]$ such that the circles lie inside $D^2 \times (0,1)$, the end-points of the intervals are the specified m and n points in the top and bottom disc, and such that the intersection of the intervals with $D^2 \times \{1\}$ and $D^2 \times \{0\}$ is precisely the m and n specified points.

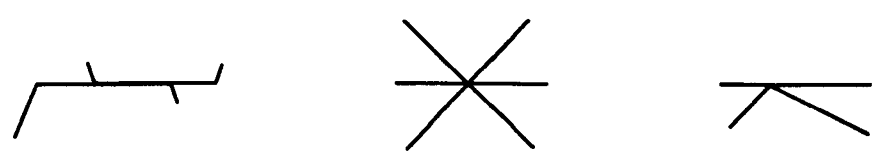
Definition 2 A projection p of a tangle on $(0,1) \times \{0\} \times [0,1]$ is called **regular** (or tangle **diagram**) , if:

(i) end-points are projected to different points,

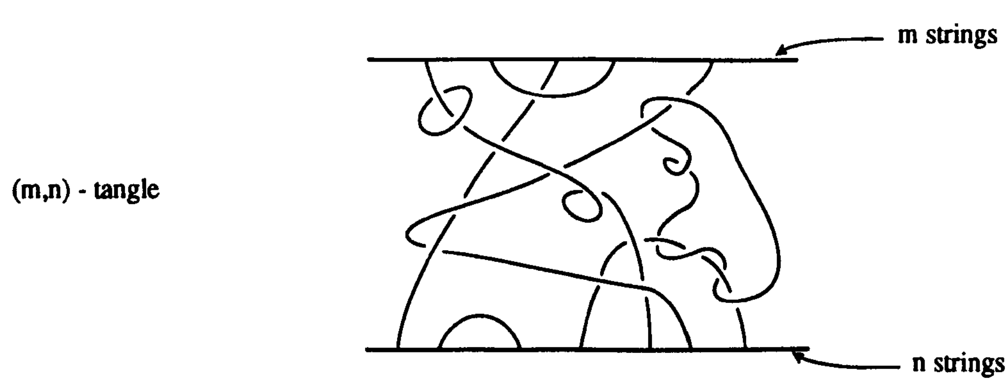
(ii) there are only finitely many multiple points P_i, $i = 1, \ldots, n$, and all multiple points are *double points* , that is , $p^{-1}(P_i)$ contains two points,

(iii) no vertex of the tangle is mapped onto a double point (compare with [10], p. 8) .

I.e., a tangle diagram avoids critical situations as depicted below:



An example of a tangle diagram is the following:



Definition 3 A **link** is a $(0,0)$-tangle ; i.e. an embedding in S^3 of finitely many disjoint circles, called components of the link. A **knot** is a link with only one component.

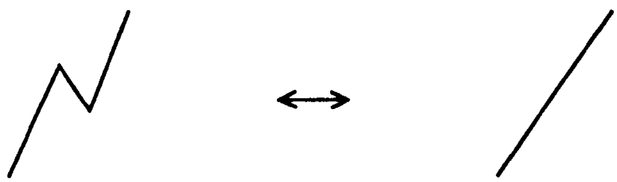
Definition 4 Two links are **ambient isotopic** if there is a homotopy h_t from S^3 to itself that carries one onto the other, such that h_o = identity and each h_t is a homeomorphism (see [53]).

The following theorem shows how to translate ambient isotopy of pl. links into equivalence of pl. diagrams (see [51], [52] and [10] for detailed expositions).

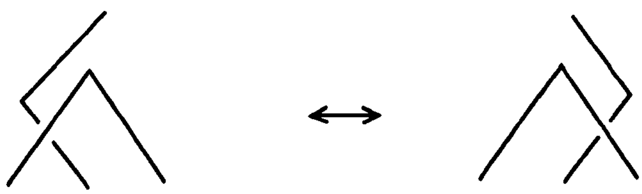
Theorem 3 (Reidemeister) *Two pl. link diagrams represent ambient isotopic pl. links in* S^3 *if and only if they differ by small planar shifts and a finite sequence of the following 'Reidemeister' moves (together with all other moves that derive from rotations and reflections of these):*

$\Delta 0$) *Addition or deletion of a vertex.*

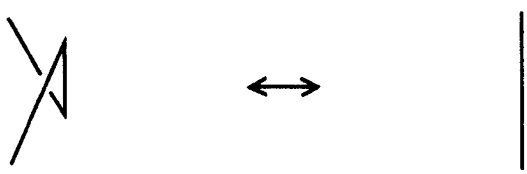
$\Delta 1$)



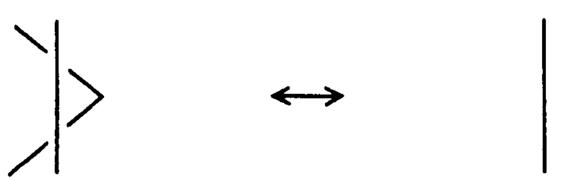
$\Delta 2$)



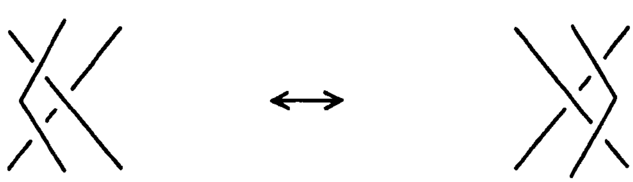
ΔI)



ΔII)



ΔIII)



Notes

- By 'small planar shifts' we mean 'small triangle moves that do not interfere with any other part of the diagram'.

• The last five Reidemeister moves may be seen as special cases of triangle moves.

• The small planar shifts together with the moves $\Delta 1$ and $\Delta 2$ are what Reidemeister calls in [52], page 7, 'deformations of the projection-curve, which do not change the "Schema" of the projection'. Alternatively, these moves generate deformations of the diagram, which do not create any singularities according to Definition 2.

• There is also a tangle version of Reidemeister's theorem.

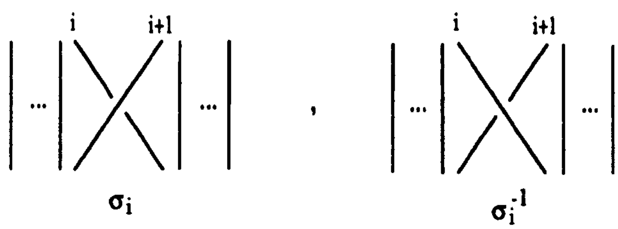
Remark 1 We can orient a tangle by choosing an orientation for every embedded interval and every circle. Then the above theorem also holds for oriented links (where in the Reidemeister moves we consider all possible choices of orientation).

Definition 5 A **braid on n strings** is a special case of a (n,n)-tangle , such that if we take the height function of the embedding, it does not have any local maxima or minima or horizontal arcs. I.e. a braid inherits a natural direction (from top to bottom).

A braid on n strings permutes its end-points and so it can be associated naturally with an element of the symmetric group S_n . Braids that correspond to the identity element of S_n are called **pure braids** .

Definition 6 We say that a regular projection of a braid is in **general position** , if no two crossings are on the same horizontal level, and we shall also call *'braid'* such a braid diagram.

If we slice up in general position (i.e. without cutting through crossings) a braid, it may be seen as a word on the following generators σ_i and $\sigma_i{}^{-1}$ for $i = 1, \ldots, n-1$.



Ambient isotopy classes of braids are in 1–1 correspondence with equivalence classes of braid diagrams, where the equivalence is generated by planar isotopy preserving the braid structure and finite sequences of the following moves (in terms of the generators):

1) $\sigma_i \sigma_i{}^{-1} = \sigma_i{}^{-1}\sigma_i = 1$, for $i = 1, \ldots, n-1$
2) $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| > 1$
3) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, \ldots, n-2$

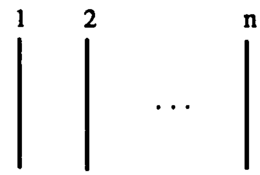
Notice that moves 1) and 3) are special cases of the Reidemeister moves (ΔII) and (ΔIII) (with all arcs oriented downwards), whilst move 2) corresponds to change of relative heights of crossings.

The set of all equivalence classes of braid diagrams on n strings, up to braid planar isotopy, forms a group, the **braid group** B_n , with a presentation:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

(For a detailed account on braids see [5], [7]) .

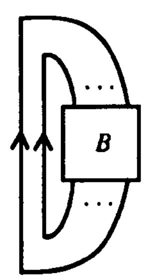
The operation in the group is *concatenation* (we place one braid on top of the other), and the identity element is:



The set of pure braids on n strings also forms a group P_n the **pure braid group.** P_n is a normal subgroup of B_n and can also be given a finite group presentation. (A detailed algebraic description of P_n is given in 2.6.1).

Note Equivalently we could define B_n (P_n) as the fundamental group of the configuration space of n unordered (ordered) points inside D^2 .

Definition 7 Closure $\hat{B}$ of a braid B (seen as a braid diagram) is the connecting of its corresponding top and bottom end-points, such that the resulting oriented link diagram is like the following (up to planar isotopy):



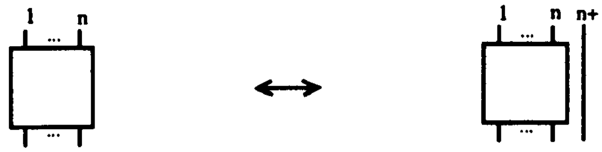
where we consider that the braid is contained in a 'box'.

Throughout this chapter we shall use the operation illustrated above for closing a braid.

Note The closure of a pure braid on n strings gives an oriented link diagram of precisely n components.

It follows from the above that, if we take the closure of two braids that correspond to the same element of B_n , then the two link diagrams are isotopic. On the other hand, if we turn two equivalent (under the Reidemeister moves) oriented link diagrams into braids using some algorithm (recall Alexander's theorem), we do not necessarily end up with braid words that correspond to the same element of some braid group. Actually, nothing even guarantees that we obtain braids with the same number of strings. However, Markov's theorem (recall Theorem 2 in 1.1) says that we can define an equivalence relation on $\bigcup_{n=1}^{\infty} B_n$ so that equivalent braids correspond to (isotopy) equivalent link diagrams. $\bigcup_{n=1}^{\infty} B_n$ is the direct limit, when the embedding of B_n into B_{n+1} is given by the following picture:

Picture:



1.3 The braiding process (Alexander's theorem)

1.3.1 The idea of the algorithm

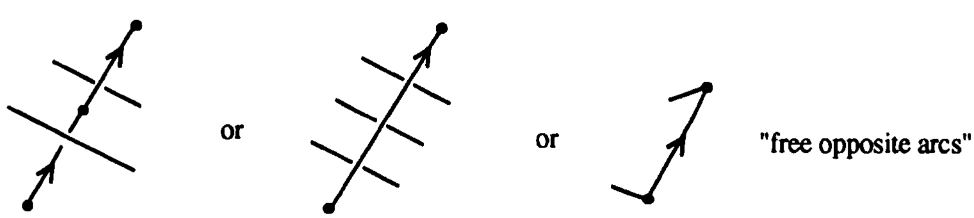
Take an oriented link diagram without horizontal arcs. Then this diagram consists of a finite set of arcs that go downwards and a finite set of arcs that go upwards (which we call **opposite arcs**) , the two sets being separated by a finite number of local maxima and minima.

In order to obtain a braid from that diagram we want

1) to keep the arcs that go downwards

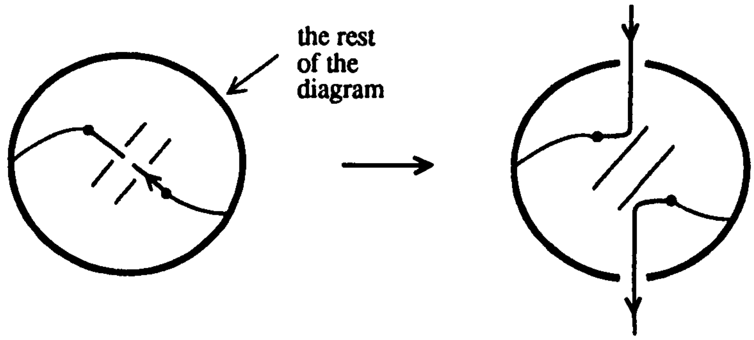
2) to eliminate the opposite arcs (because they go the 'wrong' way for a braid) and instead to produce braid strings, so that in the end we are left with a braid.

If we run along an opposite arc we are likely to meet a succession of overcrossings and undercrossings. We subdivide (marking with points) every opposite arc into smaller – if necessary – pieces, each containing crossings of only *one* type ; i.e. we may have:



We call the final pieces **little $\rangle$'s (little opposite arcs)**. We label every little $\rangle$ with an '**o**' or a '**u**' as follows: If it is the *over/under* arc of a crossing (or some crossings) we label it with an *'o'/'u'* . If it is a *free little* $\rangle$ (and therefore it contains no crossings), then we have a free choice whether to label it *'o'* or *'u'* .

We eliminate an opposite arc by eliminating its little $\rangle$'s one by one as follows: Fix one little $\rangle$ and cut it at some point. If the little $\rangle$ is the *overstring/understring* of a crossing (or some crossings) we pull its two ends *over/under* the diagram and then we stretch them one upwards and the other downwards, but both *over/under* the rest of the original diagram (see picture below).



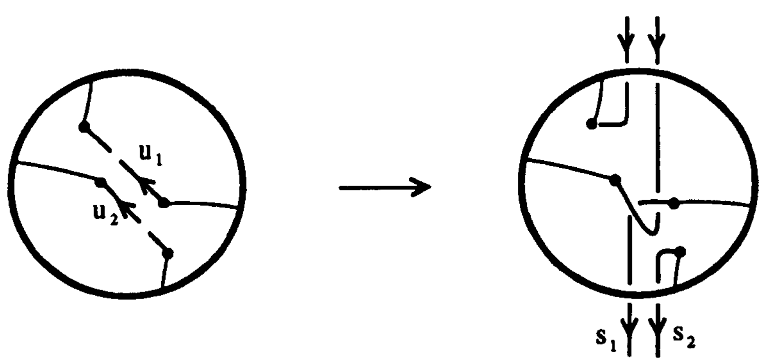
In this way we turned the little ⟩ into two pieces of corresponding braid strings (ending at the end-points of the little ⟩) , and the original diagram into a $(1,1)$-tangle. If we do the same for all little ⟩ 's we end up with a braid with as many strings as the number of the little ⟩ 's.

Note From now onwards – unless otherwise stated – circles like the ones depicted above, will always stand for **'the rest of the diagram'** , which we shall also call **'plegma'** . Also, the region inside the circle shall be called **'the magnified region'** .

1.3.2 The braiding process

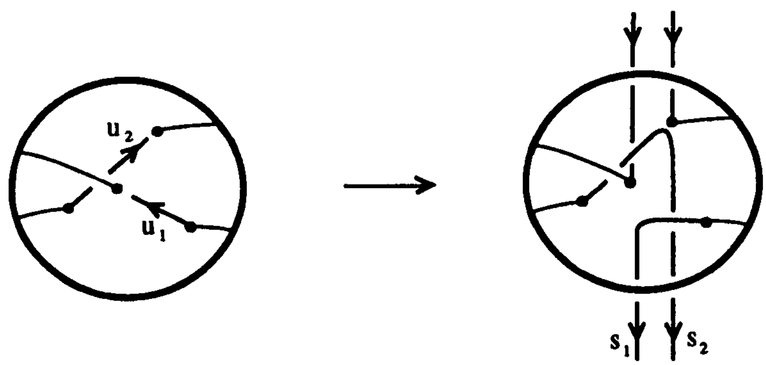
In order to make the above rigorous we need to look at two cases with special difficulty.

Case 1: If we cut the little ⟩ 's at any arbitrary point, we are in danger to run into situations such as the one illustrated and described below.



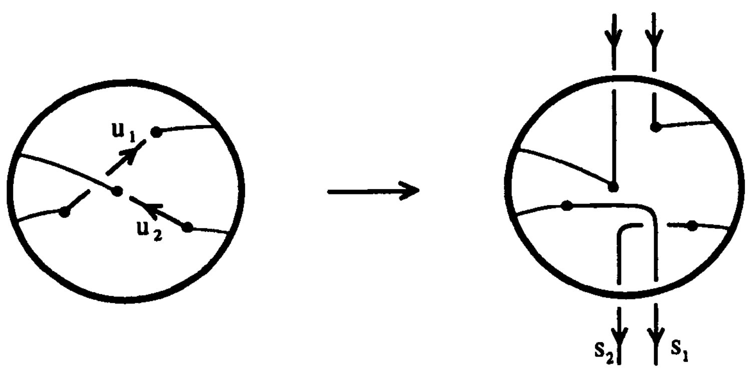
In the situation illustrated above u_1 and u_2 are two *under* little ⟩ 's , and suppose that – by the algorithm – u_1 gets eliminated first. The lower piece of the new pair of downward strings – s_1 say – produced by the elimination of u_1 , goes under u_2 and it becomes an obstacle for the elimination of u_2 . This happens because s_1 is a 'new' string (not a part of the original diagram), resulting in part of u_2 becoming an overcrossing. But we do not want to mark further with points the original diagram. We can avoid the problem encountered here by cutting all little ⟩ 's at their upmost point.

Case 2: The difficulty illustrated and described below does not depend on the cut-points:



In this situation u_1 and u_2 are as before, and the elimination of u_1 is obstructed by u_2 . In practice we can swap the numbers of u_1 and u_2 :

Picture:



In order to get over this problem theoretically, though, we need to impose a condition for the little ⟩ 's , namely condition (*) below.

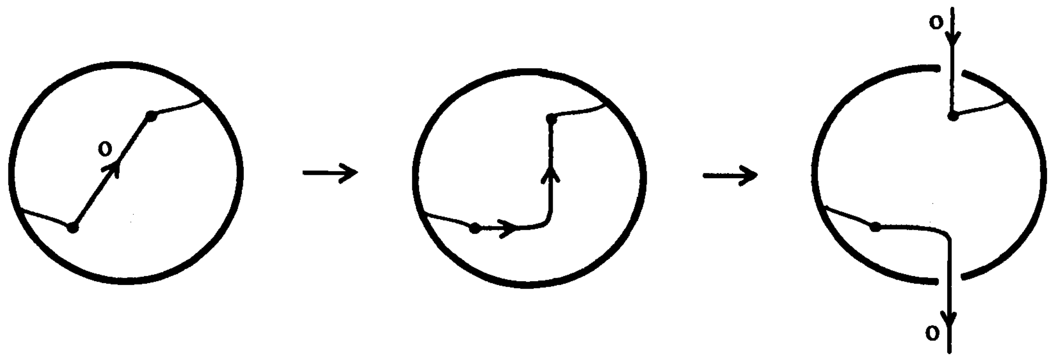
For this we want first to introduce the notion of the *smoothening triangle* of a little ⟩ .

Definition 8 The **smoothening triangle** of a little ⟩ is a special case of the triangle needed to perform a triangle move, described as the region spanned by a right-angled triangle with hypotenuse the little ⟩ , and the right angle lying below the little ⟩ . If the little ⟩ is vertical, then the smoothening triangle degenerates into the arc itself. We say that a smoothening triangle is of type *over* or *under* according to the label of the arc it is associated with.

So the elimination of a little ⟩ is modified as follows:

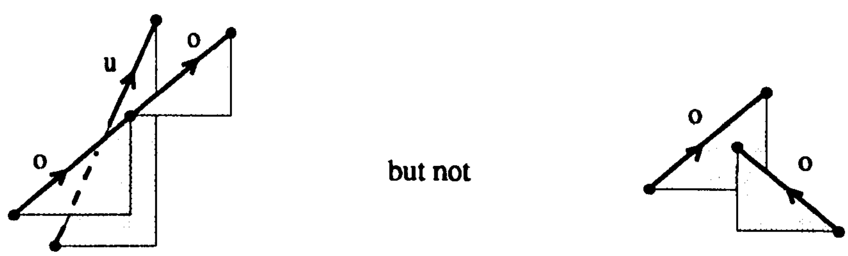
We slide the little ⟩ through its smoothening triangle, as depicted below, and then we replace the vertical part by two corresponding vertical strings, *both* with label same as the label of the little ⟩ .

Picture:



Condition (*): Non-adjacent smoothening triangles are only allowed to meet if they are of opposite type.

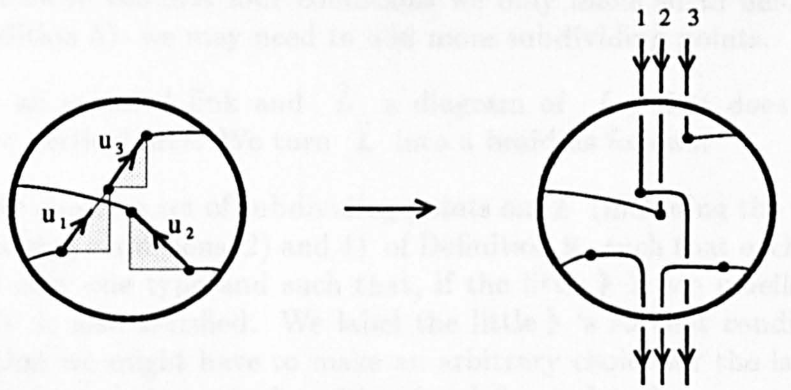
Picture:



Lemma 1 *There exists a set of subdividing points on the opposite arcs, including vertices, satisfying condition (*) (for appropriate choices of under/over for free little $\rangle$'s).*

Proof Let d = minimum distance between any two crossings. Let $0 < r < d/2$ be a number such that any circle of radius r centred at a crossing point does not intersect with any other arc of the diagram. Let s be the minimum distance between any two points in the diagram further than r away from any crossing point. Now let $\epsilon = \frac{1}{2}\min\{s,r\}$ and D be a subdivision of the diagram such that the length of every little $\rangle$ is less than ϵ . Then condition (*) is satisfied, provided we make the right choices (over/under) for smoothening triangles of free little $\rangle$'s near crossings (see picture above). □

In the picture below we show how to cope theoretically with the difficulty described in case 2 above, by applying condition (*) :



One could easily check that if we eliminated the three little $\rangle$'s in a different order (but with the same labelling), we would still obtain the same braiding.

Remark 2 If we add more points to a subdividing set satisfying condition (*) , then there exists at least one labelling for the new set of arcs obtained, so that condition (*) is still satisfied. To see that, take for example the labelling that derives naturally, where: when we subdivide further a little $\rangle$, the new little $\rangle$'s keep the same labelling as the subdivided one.

Remark 3 The idea of imposing condition (*) was suggested by C.P. Rourke, and it is of major importance, because it implies that the eliminating moves do not interfere with each other, so we can do all of them *simultaneously* . Therefore, it does not matter which way we number the little $\rangle$'s.

Now we can proceed with a rigorous exposition of our braiding process.

1.3.3 An algorithm for turning oriented links into braids

Definition 9 We say that a link diagram with subdividing points and smoothening triangles is in **general position** , with respect to the height function, if the following conditions hold:

1) there are no horizontal arcs,

2) no two subdividing points are on the same horizontal level,

3) there are no vertical arcs,

4) no two subdividing points are on the same vertical level,

5) any two non-adjacent smoothening triangles satisfy condition (*) , and if they intersect, this should be along a common interior (and not a single point).

From now on we shall call **'general diagram'** a diagram in general position.

Note To achieve the first four conditions we only make small deformations, but to achieve condition 5) we may need to add more subdividing points.

Let L be an oriented link and $\tilde{L}$ a diagram of L, that does not contain any horizontal or vertical arcs. We turn $\tilde{L}$ into a braid as follows:

Step 1 : We choose a set of subdividing points on $\tilde{L}$ (including the vertices of the diagram) satisfying conditions 2) and 4) of Definition 9 , such that each little $\rangle$ contains crossings of only one type and such that, if the little $\rangle$'s are labelled appropriately, condition 5) is also satisfied. We label the little $\rangle$'s so that condition 5) is satisfied. Note that we might have to make an arbitrary choice for the labels of some free little $\rangle$'s. Now $\tilde{L}$ is in general position (as defined above).

For each little $\rangle$, we draw the vertical line that passes through its upmost point, so that it passes **under** or **over** the plegma, according to the label of the little $\rangle$. Then we *'name'* (i.e. we *give numbers to*) the little $\rangle$'s according to the position of their corresponding vertical lines.

Step 2 : Start with the first arc (with number 1) and slide it through its smoothening triangle (recall pictures above). Then replace the vertical part by two corresponding strings that follow the vertical line of the little $\rangle$. The result is a (1,1)-tangle.

Repeat Step 2 for the $2nd, 3rd, \ldots, nth$ arc, so as to obtain a braid on n strings.

Step 3 : Isotope slightly, so that the final braid does not contain any horizontal arcs or any crossings on the same horizontal level.

We take now the closure of the obtained braid; each pair of corresponding strings ending at the end-points of a little $\rangle$, together with their closing arc, form a stretched version of the little $\rangle$ (i.e. they are isotopic to the little $\rangle$). Thus the closure of our braid is isotopic to the oriented link we started from.

The above algorithm gives our new proof of Alexander's theorem. □

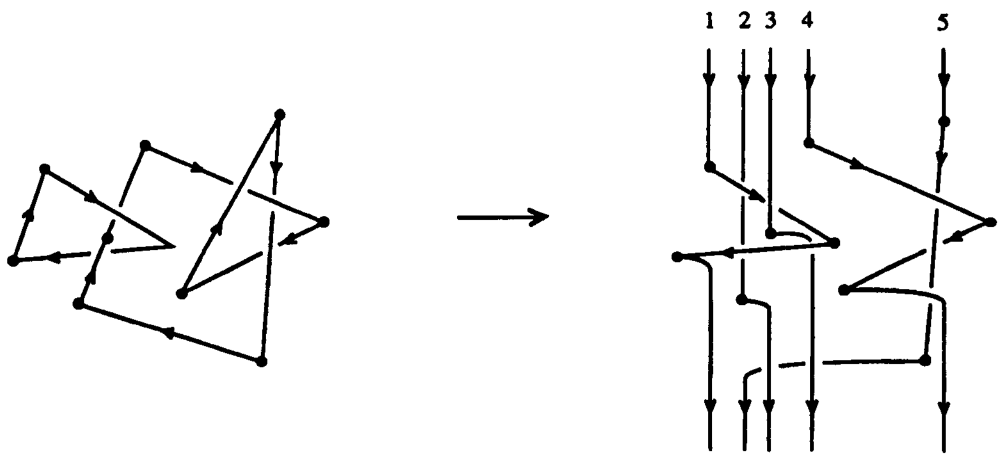
1.3.4 Comments

• Condition (*) might imply a number of unnecessary subdivisions (and this would become really large if we applied the proof of Lemma 1).

• In the (theoretical) algorithm we imposed condition (*) as it avoids the difficulty of describing algorithmically how to deal with a congestion of overlapping smoothening triangles (and also to aid in the proof of Markov's theorem that follows). In practice we do not need to impose condition (*) as long as we *layer* and number overlapping smoothening triangles carefully.

• Even when applying it practically, our algorithm cannot guarantee a minimal number of strings in the final braid. Although, it is conceptually very simple and it is planar, so one can keep track of it, and can easily read the braid word in the end.

An example of braiding a link diagram:



$\sigma_1^{-1}\, \sigma_4^{-1}\, \sigma_2\, \sigma_4^{-1}\, \sigma_2\, \sigma_1\, \sigma_4^{-1}\, \sigma_3^{-1}\, \sigma_2^{-1}$

Aside We could also apply the braiding process above to 'braid' a general tangle diagram. The result would be a generalized notion of a braid.

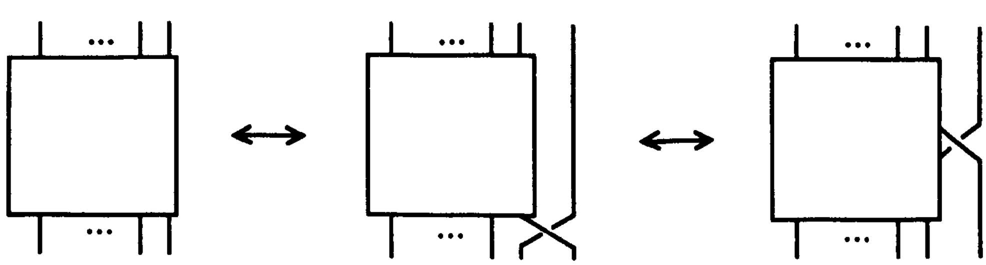
1.4 Proof of Markov's theorem

Before starting to prove the theorem, we make the following remarks:

Remark 4 Throughout this section we will be assuming that all diagrams are general diagrams, unless otherwise stated. Therefore, we may assume that we have done the braiding for all little $\rangle$'s except for the ones that we are interested in every time, and these will be lying in the magnified region that we have mentioned earlier. We shall place the magnified region inside a rectangle representing the braid. Moreover (from Remark 3) we will omit the numbers of the little $\rangle$'s , and we shall only keep the labels 'u' and 'o' .

Remark 5 Let $\beta\sigma_n{}^{\pm 1}$ be a braid in B_{n+1} with a Markov move performed. Using conjugation we may place $\sigma_n{}^{\pm 1}$ (i.e. the last crossing) at any other position in the braid word and we shall call this move **'general Markov move'** .

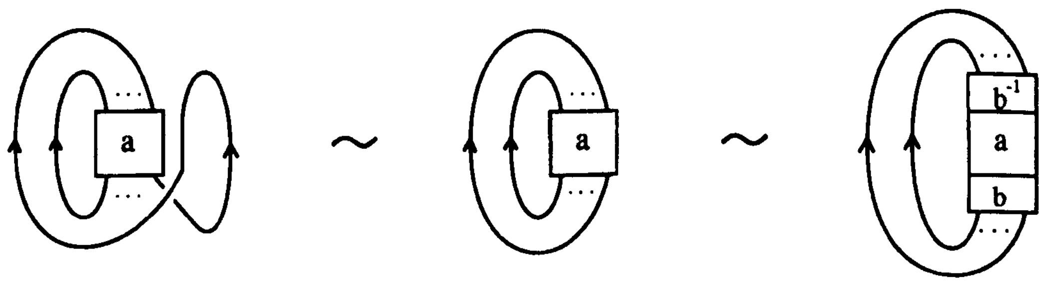
Picture:



Definition 10 We say that two braids are **M-equivalent** if they differ by braid planar isotopy, braid relations and a finite sequence of general Markov moves and conjugation.

1.4.1 Proof using our algorithm

Two braids that differ by a finite sequence of conjugations and Markov moves obviously have isotopic closures:



If we close a braid with a Markov move performed, the Markov move corresponds to a twist of some component of the resulting link diagram.

With closure b^{-1} can slide around, come below b and cancel with it.

To prove the converse we have to show that:

Given any two ambient isotopic links L_1 , L_2 , any two corresponding general diagrams $\widetilde{L_1}$ and $\widetilde{L_2}$ yield M-equivalent braids. More precisely:

1) *the static aspect:* The braid we obtain from a *'potential'* general diagram (i.e. a diagram satisfying 1), 2), 3) and 4) of Definition 9) does not depend - up to M-equivalence - on the subdivision and the labelling we choose to make, in order to also satisfy rule 5) of Definition 9 .

2) *the moving aspect:* An isotopy carrying $\widetilde{L_1}$ onto $\widetilde{L_2}$ may pass through certain critical stages; our task is to show that the corresponding braids before and after each of these stages are M-equivalent. Most of the critical stages are listed in Reidemeister's theorem; but there are some extra ones that derive from rules 1), 2), 3) and 4) of the definition of a general diagram. We list all of them later on in the proof.

Note that proof of **1)** together with the proof of critical stages that derive from the first four rules of Definition 9 , will show independency on any possible choices we make in order to bring a diagram to general position.

The proof relies entirely on Remark 3 about independency of the braiding moves, and on the following two lemmas:

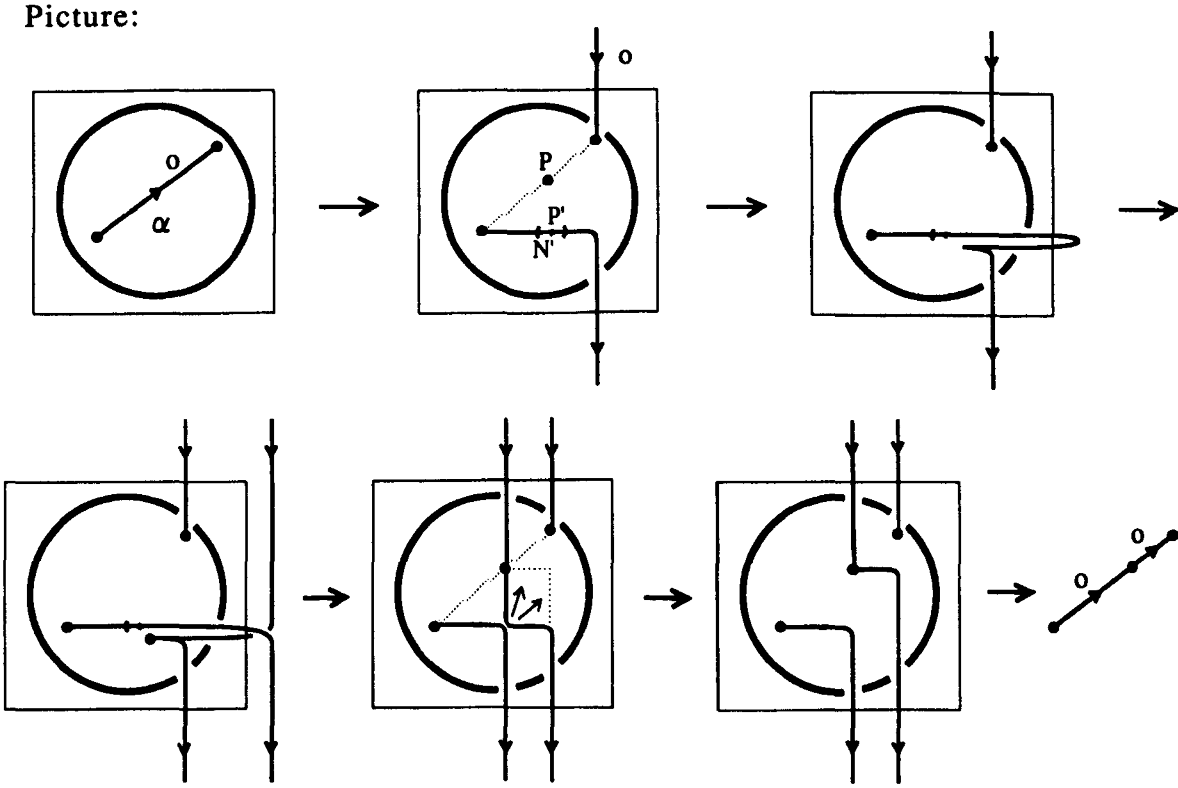
Lemma 2 *If we add on a little* $\rangle$, α , *an extra subdividing point* P *and label the two new little* $\rangle$ *'s ,* α_1 *and* α_2 *, the same as* α *, the corresponding braids are M-equivalent.*

Proof For definiteness we assume that α is labelled with an 'o' . We complete the braiding of the original diagram by eliminating α (see picture below). Then, on the new horizontal piece of string, we take an arbitrarily small neighbourhood N' around P' , the projection of P (see picture below). By general position N' slopes slightly downwards.

Next, using the braid relations, we pull N' (horizontally) *over* and outside the right hand side of the braid. Then we perform a *negative* general Markov move and, using conjugation and the braid relations, we pull the new strings *over* the braid, back to the original position of P' (as illustrated in the pictures below).

Finally, sliding an appropriate piece of string through the smoothening triangle (braid planar isotopy), we obtain the braid that would result from the original diagram with the subdividing point P included (see picture below).

Picture:



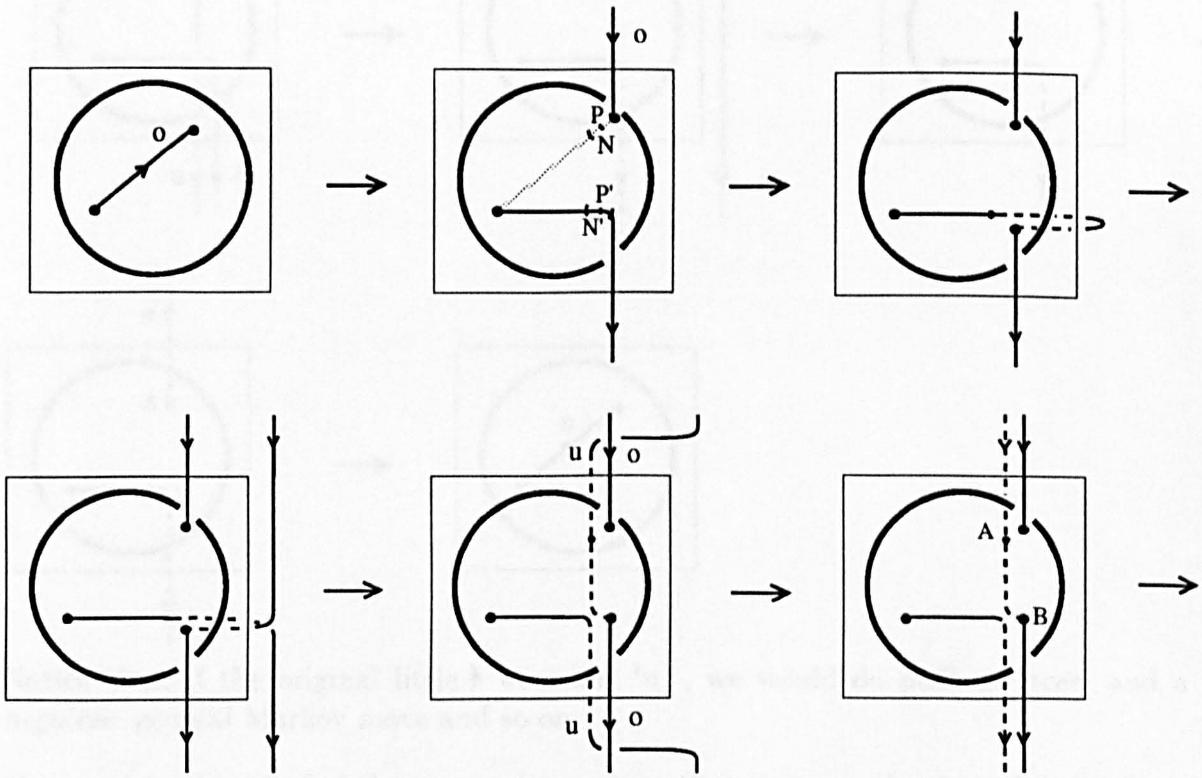
□

Lemma 3 *When we meet a free little* $\succ$ *, which we have the choice of labelling 'u' or 'o' , the resulting braid does not depend – up to M-equivalence – on this choice.*

Proof First, we shall assume for simplicity that the smoothening triangle of our arc does not lie over or under any other arcs of the original diagram. Also, we assume for definiteness that the little $\succ$ is originally labelled 'o' . We complete the braiding by eliminating it. Then, on the new horizontal piece of string, we take an arbitrarily small neighbourhood N' , which is a projection of a neighbourhood N arbitrarily close to the upmost point of the little $\succ$ (as illustrated below).

Next, using the braid relations, we pull N' (horizontally) *under* and outside the right-hand side of the braid. Then we perform a general *positive* Markov move and, using conjugation and the braid relations, we pull the new strings *under* the braid, back to the original horizontal position of P' (see pictures below).

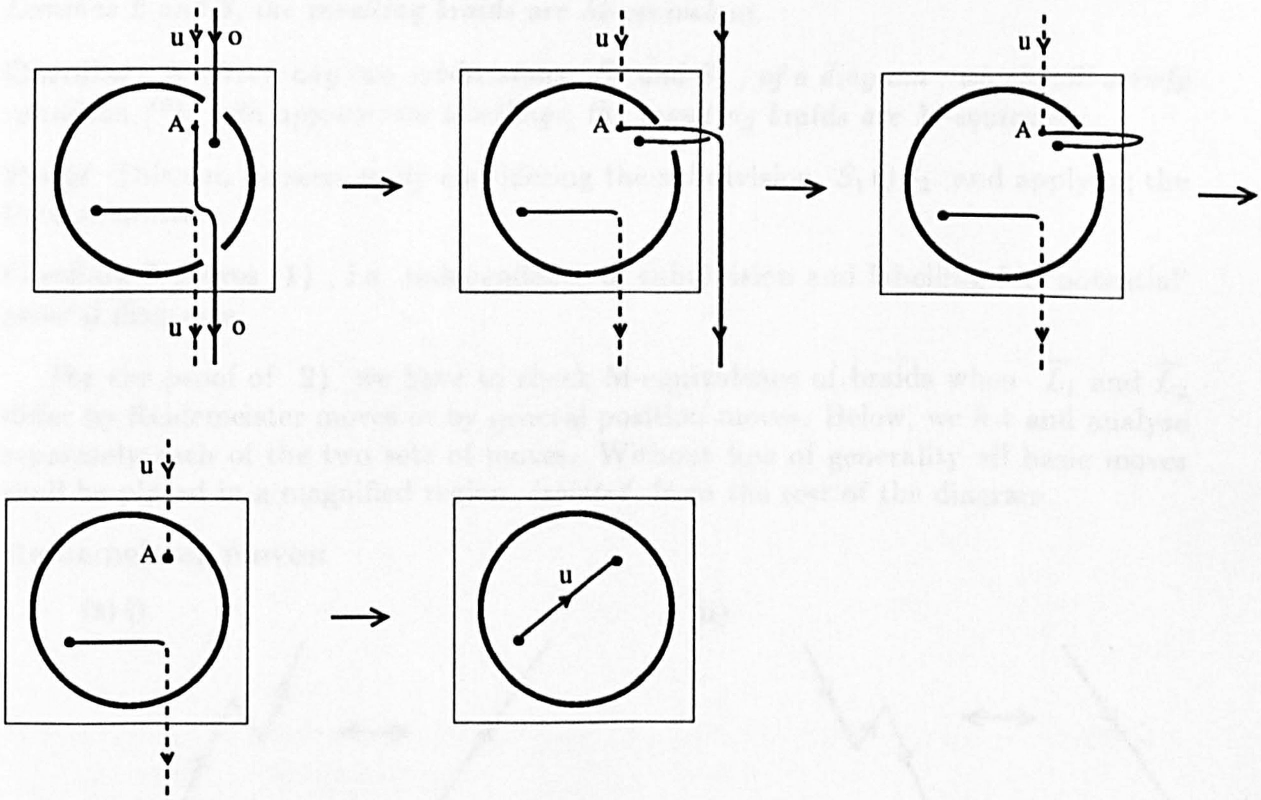
Picture:



The fact that the original little ❯ is free and small enough, implies that only vertical strings can pass over or under its smoothening triangle. Therefore – as N is arbitrarily small – there is no arc crossing AB so as to force it be an *under* arc. Then – by braid planar isotopy – we shift A slightly *higher* (see picture below), and applying the braid relations and conjugation on the pair of *'over'* strings, we come to the situation where we can perform a *negative* general Markov move. We perform the move and pull the new arc *over* the braid, back to the original place.

The final braid – up to a small braid planar isotopy – can be seen as the braid that we would have obtained from the original diagram, with the free little ❯ labelled with 'u' instead of 'o'.

Picture:

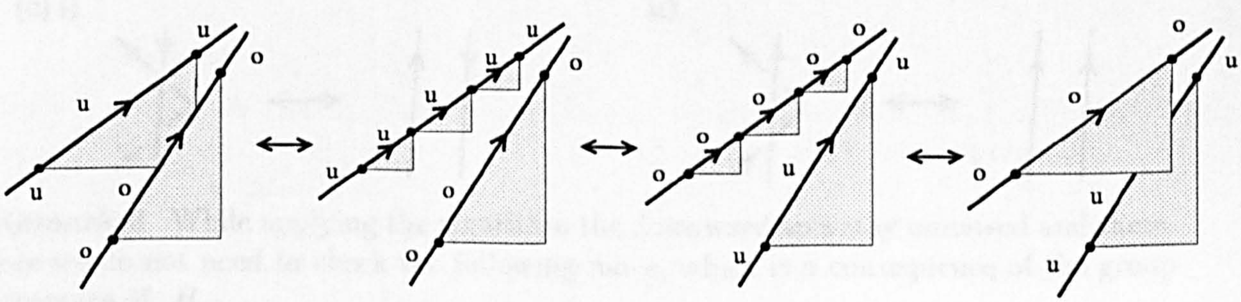


Notice that, if the original little $\rangle$ were an 'u', we would do pulling *over* and a *negative* general Markov move and so on.

To complete the proof of the lemma, we assume that the smoothening triangle of our little $\rangle$ lies over or under other arcs of the original diagram. In this case we subdivide it (using Lemma 2) into arcs small enough to ensure that all the smoothening triangles are clear; we give all new arcs the labelling of the original one. Then we change the labelling of each little $\rangle$ using the above, and (using Lemma 2 again) we eliminate all the new subdividing points. □

Corollary 1 *If we have a chain of overlapping smoothening triangles of free little $\rangle$'s, so that we have a free choice of labelling for the whole chain, then, by Lemmas 2 and 3, this choice does not affect – up to M-equivalence – the final braid.*

Proof (by following the diagrams)



Corollary 2 *If, by adding a subdividing point on a little* ⟩ *, we have a choice for relabelling the resulting new little* ⟩ *'s so that condition (*) is still satisfied then, by Lemmas 2 and 3, the resulting braids are M-equivalent.*

Corollary 3 *Given any two subdivisions* , S_1 *and* S_2 *, of a diagram , which will satisfy condition (*) with appropriate labellings, the resulting braids are M-equivalent.*

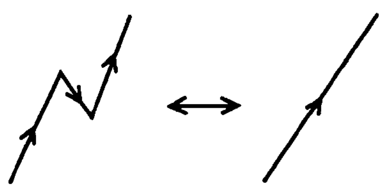
Proof This can be seen easily considering the subdivision $S_1 \bigcup S_2$ and applying the lemmas above.

Corollary 3 proves **1)** , i.e. independence of subdivision and labelling for 'potential' general diagrams.

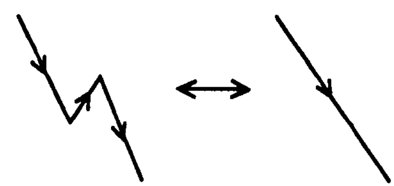
For the proof of **2)** we have to check M-equivalence of braids when $\widetilde{L_1}$ and $\widetilde{L_2}$ differ by Reidemeister moves or by general position moves. Below, we list and analyse separately each of the two sets of moves. Without loss of generality all basic moves shall be placed in a magnified region *isolated* from the rest of the diagram.

Reidemeister moves:

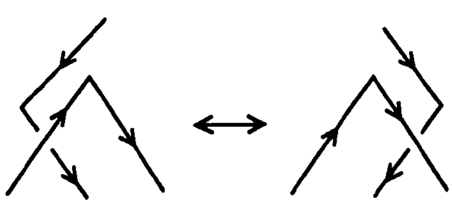
(a) i)



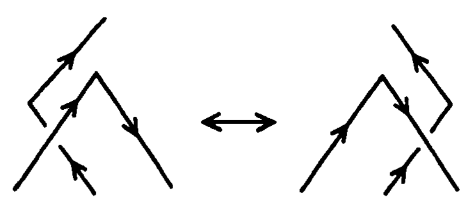
ii)



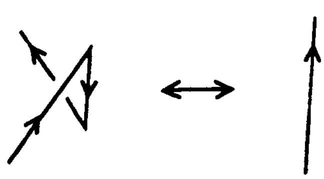
(b) i)



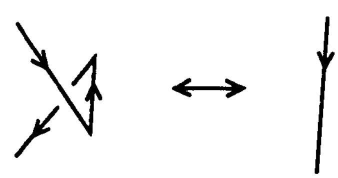
ii)



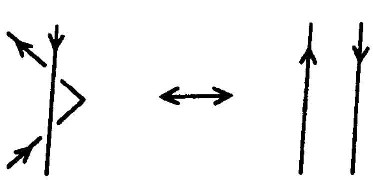
(d)i)



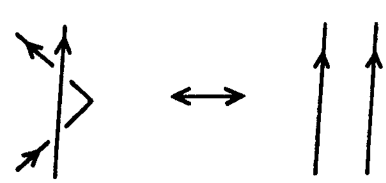
ii)



(c) i)

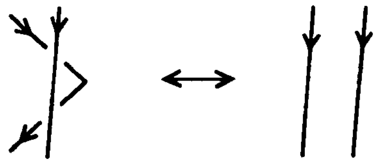


ii)



Remark 6 While applying the algorithm the downward arcs stay unmoved and therefore we do not need to check the following move, which is a consequence of the group structure of B_n .

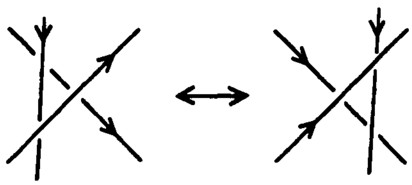
(c) iii)



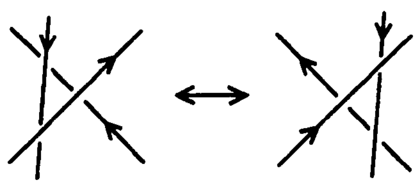
$(\sigma_i \sigma_i^{-1} = 1)$

'Triple point move'

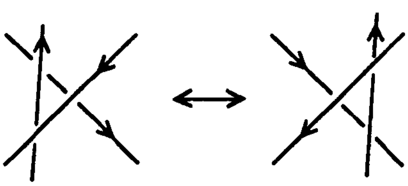
(e) i)



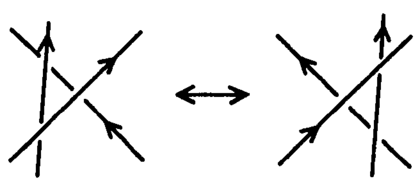
ii)



iii)

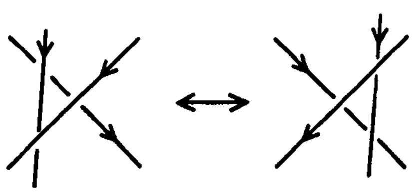


iv)



Note As in Remark 6, we do not need to check the following move, which is a consequence of the braid relations.

(e)v)



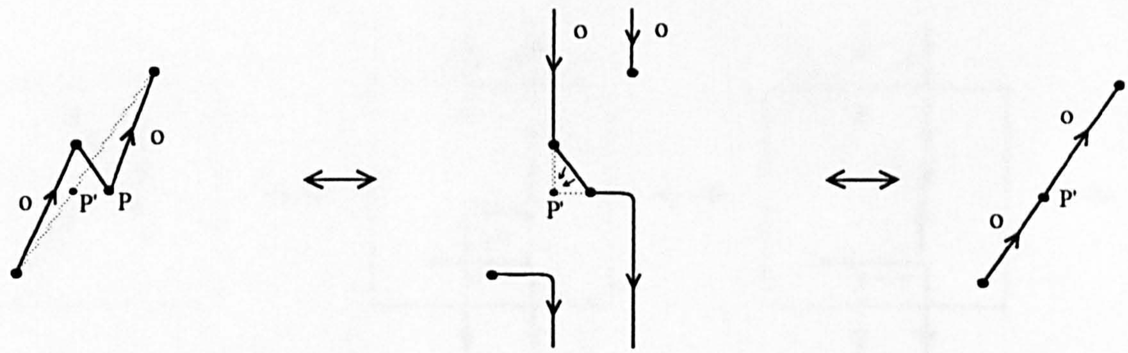
$(\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1})$

Other instances of the above moves are obtained from the ones listed by symmetries and rotations.

We shall only show as typical cases **(a)**i) , **(a)**ii) , **(d)**i) , **(c)**i) , **(b)**ii) and **(e)**iii) . All the others follow using very similar arguments.

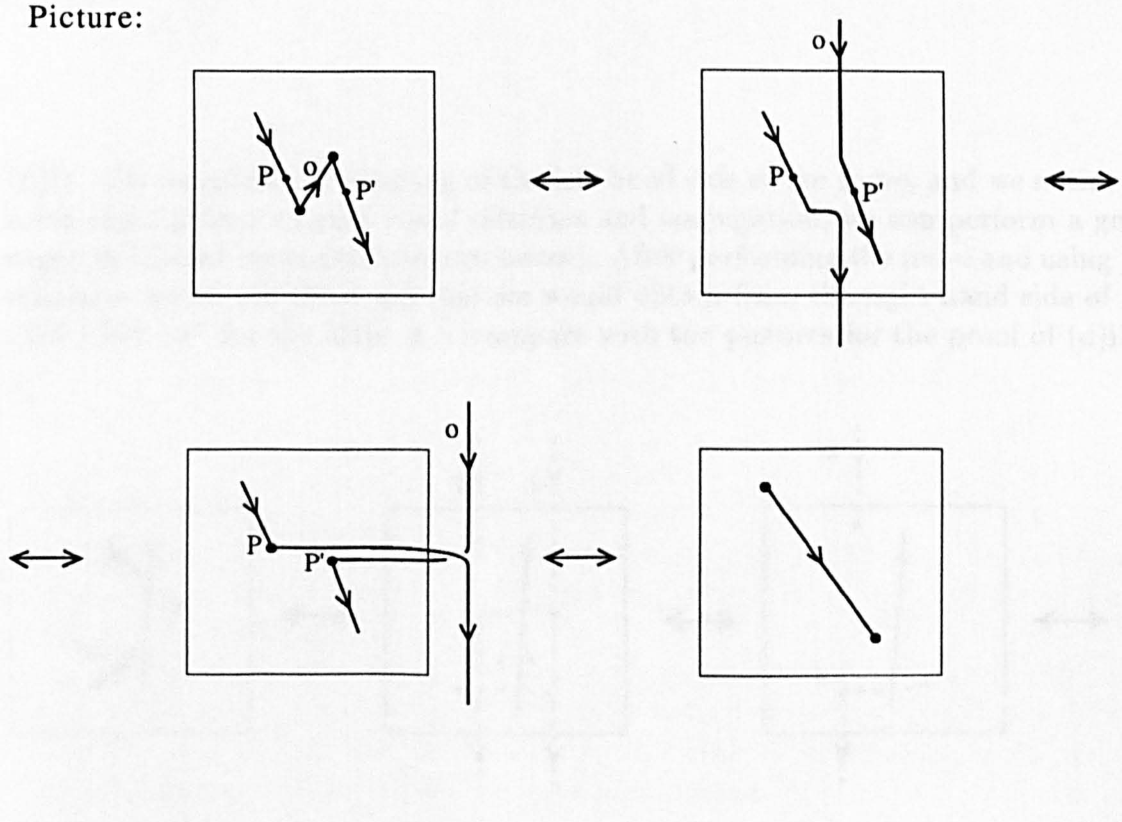
(a)i) Let P' be the horizontal projection of P on the original arc (as illustrated below). Without loss of generality we may assume that all little ⟩ 's involved in the move, are of the same type. It is clear from the pictures below that the braid we obtain after the move, is the same – up to a small braid planar isotopy – as the one that we would obtain from the original diagram after adding P' and keeping the same labelling. Then the proof follows by Lemma 2.

Picture:

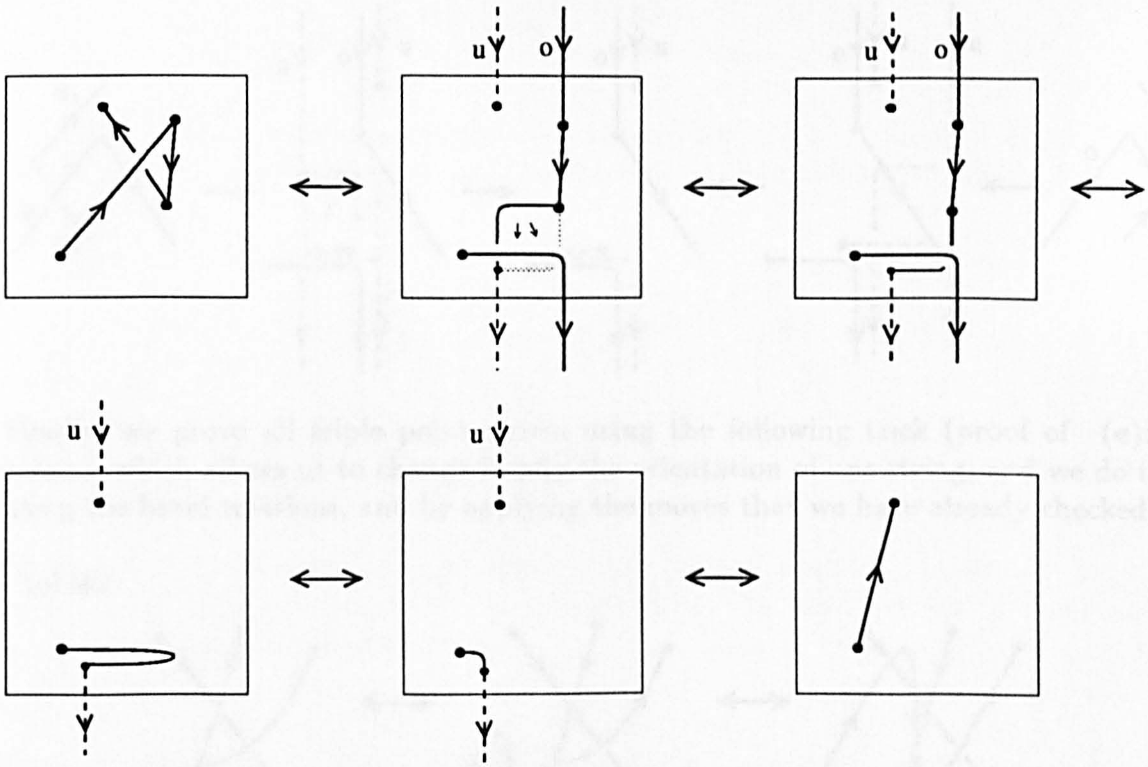


(a)ii) Before the move is performed, we fix points P and P' on the downward arcs, so that P lies on a *higher* level than P' (see picture below). We complete the braiding process by eliminating the new little $\rangle$, and we use the two pieces of new strings – but starting from P and ending at P' – in order to perform a *negative* general Markov move (exactly as in the second series of pictures in the proof of Lemma 3). The resulting braid is clearly the same as the one obtained from the original diagram.

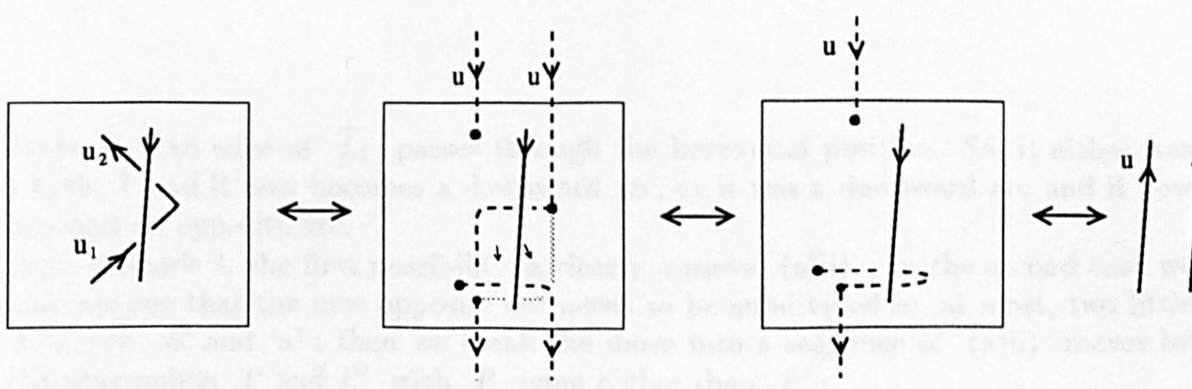
Picture:



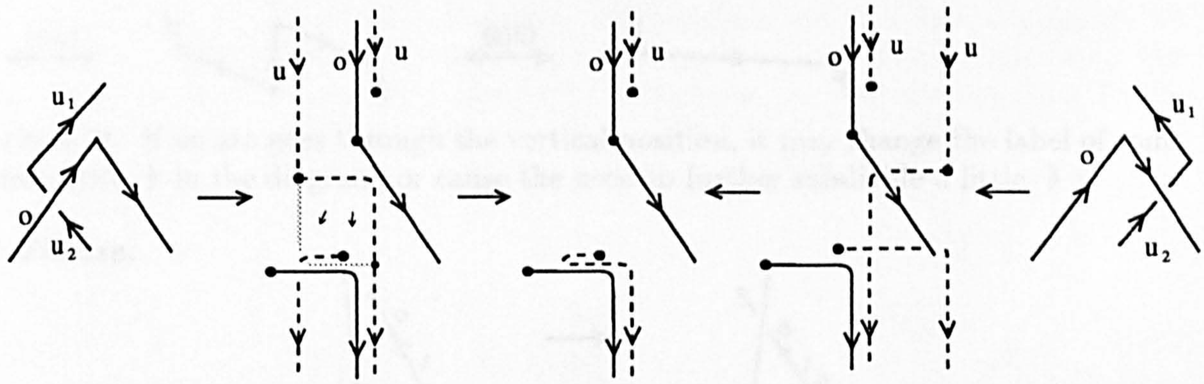
(d)i) We illustrate below that the braid obtained after the performance of the move is equivalent to the one obtained before the move – up to a small braid planar isotopy, braid relations, conjugation and a general *negative* Markov move:



(c)i) We complete the braiding of the left-hand side of the move, and we notice that, using braid planar isotopy, braid relations and conjugation, we can perform a general negative Markov move (see pictures below). After performing the move and using braid relations, we obtain the braid that we would obtain from the right-hand side of (c)i) with label 'u' for the little ⟩ : (compare with the pictures for the proof of (d)i) !)

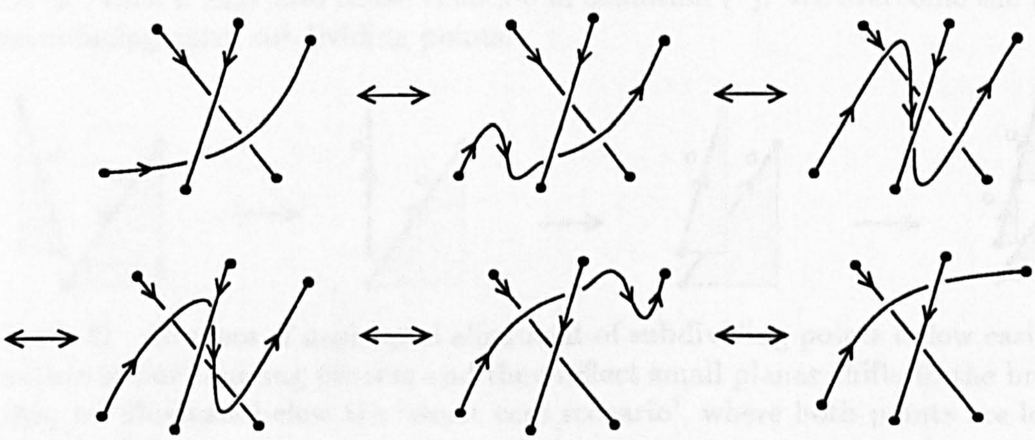


(b)ii) (by following the pictures below)



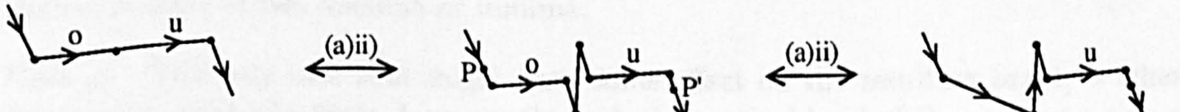
Finally, we prove all triple point moves using the following trick (proof of (e)iii) below), which allows us to change locally the orientation of one string; and we do this using the braid relations, and by applying the moves that we have already checked.

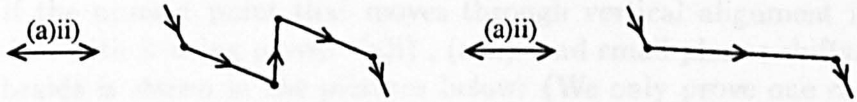
(e) iii)



General position moves (that derive from 1), 2), 3), 4) and 5) of Definition 9):

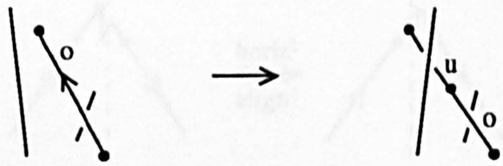
From 1) An edge of $\widetilde{L_1}$ passes through the horizontal position. So, it either was a little ≻ and it now becomes a downward arc, or it was a downward arc and it now becomes an opposite arc.
From Remark 3, the first possibility is clearly a move (a)ii) . In the second case we may assume that the new opposite arc needs to be subdivided in, at most, two little ≻ 's, say 'o' and 'u' ; then we break the move into a sequence of (a)ii) moves by choosing points P and P' with P lying higher than P' :





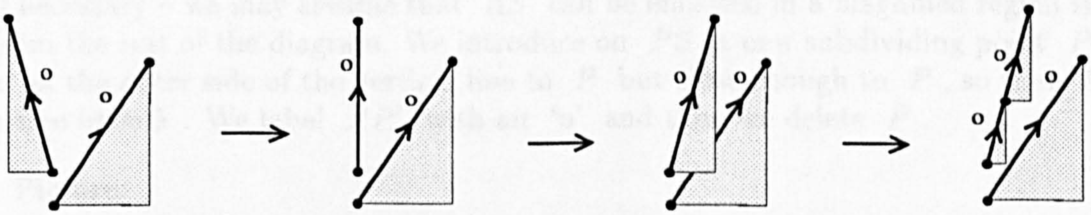
From 3) If an arc goes through the vertical position, it may change the label of some free little ⟩ in the diagram, or cause the need to further subdivide a little ⟩ :

Picture:

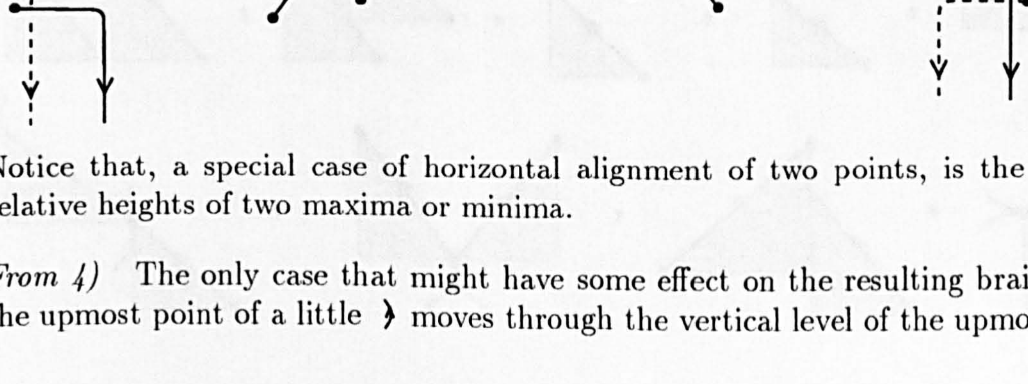


Then the two diagrams differ by a possible sequence of Reidemeister moves together with further subdividing.

If, in addition, the arc that goes through the vertical position happens to be a little ⟩ , then it may also cause violation of condition (*). We overcome the problem by introducing extra subdividing points:



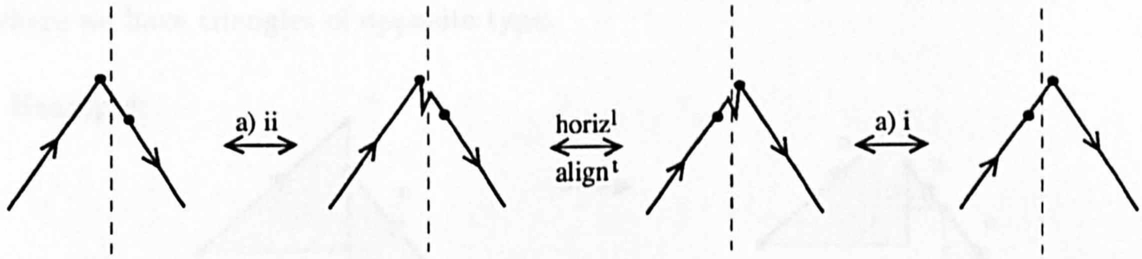
From 2) All cases of horizontal alignment of subdividing points follow easily from the nature of our braiding process and they reflect small planar shifts in the braid. To see this, we illustrate below the 'worst case scenario', where both points are lower points of little ⟩ 's:



Notice that, a special case of horizontal alignment of two points, is the change of relative heights of two maxima or minima.

From 4) The only case that might have some effect on the resulting braid, is when the upmost point of a little ⟩ moves through the vertical level of the upmost point of

another little $\rangle$. For this, we examine different possibilities:
If the upmost point that moves through vertical alignment is a local maximum, we deal with it using moves (a)i) , (a)ii) and small planar shifts. The equivalence of the braids is shown in the pictures below: (We only prove one case of orientation, as the other is shown similarly.)

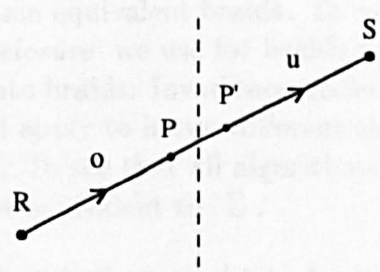


where the vertical lines indicate the alignement

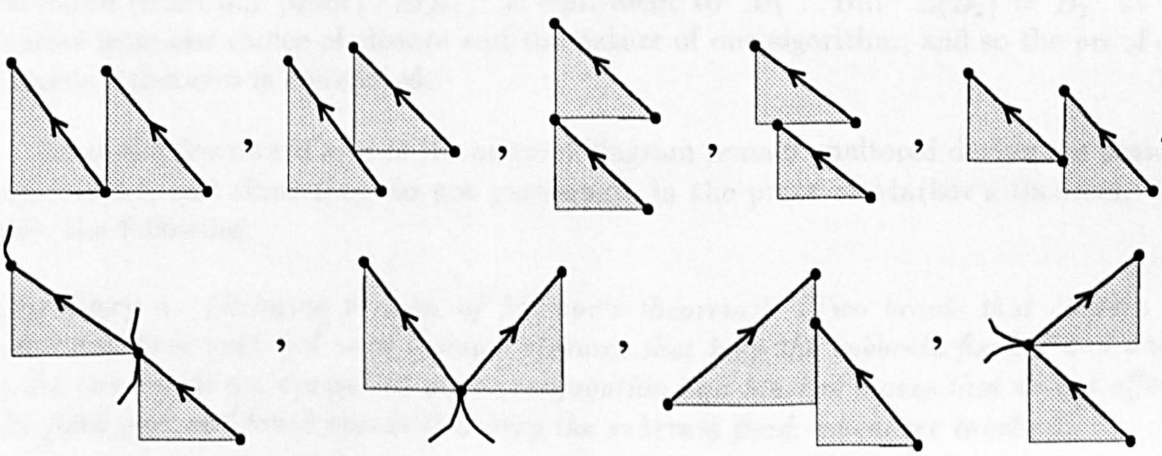
Below, we illustrate the case where the upmost point that moves through vertical alignment (point P here) is *not* a local maximum:

For definiteness we label the two little $\rangle$'s 'o' and 'u' and – after enough subdivisions, if necessary – we may assume that RS can be enclosed in a magnified region isolated from the rest of the diagram. We introduce on PS a new subdividing point P' that is on the other side of the vertical line to P but close enough to P , so that PP' is a free little $\rangle$. We label PP' with an 'o' and then we delete P .

Picture:

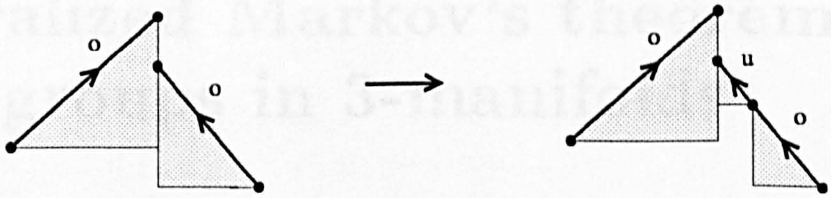


From 5) The possibilities for touching of two smoothening triangles are the following:



In all instances, if the triangles are of opposite type, then all cases boil down to horizontal or vertical alignment cases, possibly with Reidemeister moves involved (in the last four of them, where the touching is more 'essential'). If the triangles are of the same type, then: in the first five cases we simply subdivide further, and refer to cases of alignment; in the last four cases, we introduce an extra subdividing point, so as to create a free little $\rangle$, to which we attach the opposite label; then we refer to the cases where we have triangles of opposite type.

Example:



1.4.2 Completion of the proof of Markov's theorem

Let $\mathcal{B}$ be the set of braids and $\mathcal{L}$ the set of oriented link diagrams. Any closing operation for braids defines a map from $\mathcal{B}$ to $\mathcal{L}$, and we shall call 'K' the map defined by the closure we use here (as mentioned after Definition 7). We shall also call 'Σ' the map from $\mathcal{L}$ to $\mathcal{B}$, that we obtain using our algorithm.

Up to now we have seen that, if we start with two equivalent braids and apply K , we obtain isotopic link diagrams; and conversely: if we start with two isotopic link diagrams and apply Σ , we obtain equivalent braids. To complete the proof, it remains to show invariance under the *closure* we use for braids and under the *algorithm* we use for turning link diagrams into braids. Invariance under closure is easy to see, since, if we start with a braid B and apply to it two different closing operations, the results will be isotopic by Definition 7. To see that all algorithms are equivalent, it is enough to see that any algorithm A is equivalent to Σ .

Indeed: Let B_1, B_2 be two braids that we obtain from an oriented link diagram $\widetilde{L}$ after applying Σ and A respectively. Then $\widehat{B_2} = K(B_2)$ is isotopic to $\widetilde{L}$ and therefore (from our proof) $\Sigma(\widehat{B_2})$ is equivalent to B_1 . But $\Sigma(\widehat{B_2}) = B_2$ as it follows from our choice of closure and the nature of our algorithm, and so the proof of Markov's theorem is completed. □

Since the downward arcs of the original diagram remain unaltered during the braiding process, and since they do not participate in the proof of Markov's theorem, we have the following:

Corollary 4 *(Relative version of Markov's theorem:) Two braids that contain a pointwise fixed subbraid have isotopic closures that keep the subbraid fixed if and only if the two braids are equivalent under conjugation and Markov moves that do not affect the fixed part and braid moves that keep the subbraid fixed, whenever involved.*

Chapter 2

Generalized Markov's theorem and braid groups in 3-manifolds

2.1 Introduction

The main results in this chapter are the statement and proof of a geometric analogue of Markov's theorem for oriented links in arbitrary closed, connected, orientable 3-manifolds, on one hand, and the existence of braid group structures or coset structures in such 3-manifolds, on the other hand. Before stating the analogue of Markov's theorem, we develop the necessary theory:

(i) by specifying (in 2.2.2, 2.2.3) the context in which we study links and link isotopy in a 3-manifold M (recall 0.3),
(ii) by formulating an analogue of Reidemeister's theorem for links in M (in 2.2.4),
(iii) by defining what a *braid* in M is (in 2.3.1), and
(iv) by constructing and proving an analogue of Alexander's theorem for links in M, such that the surgery closed braid that represents M in S^3 remains fixed throughout the braiding process (Theorem 4 in 2.3.2).

The analogue of Alexander's theorem implies that M may be represented by *a fixed (pure) braid in* S^3, so we may proceed with the extension of Markov's theorem to 3-manifolds (Theorem 5, section 2.4). As a corollary of Theorem 5 for $M = L(p,1)$ (a lens space that can be described by one surgery string with framing p), we obtain an analogue of Markov's theorem for isotopic links inside a solid torus (Theorem 6 in 2.4.2).

Next, having as aim the algebraic formulation of Theorem 5, we look for braid group structures in M. Indeed, in section 2.5 we show that:
If $M = L(p,1)$ or a solid torus, then the set of all braids on n strings related to M forms a group, the group $B_{n,1} \leq B_{n+1}$. If M is a connected sum of m lens spaces of type $L(p,1)$, then the set of all braids on n strings related to M also forms a group, which we denote as $B_{n,m} \leq B_{n+m}$. Finally, if M is neither of the above, then the set of all braids on n strings related to M forms a coset of $B_{n,m}$ in B_{n+m}, which we denote as $C_{n,m}$.

In section 2.5 we also show that these algebraic structures are consistent with link isotopy, in the sense that the analogue of Markov's theorem for isotopic links in M can be formulated using only the braid groups $B_{n,m}$ or the cosets $C_{n,m}$.

Then, in section 2.6 we find a presentation for the group $B_{n,1}$, and this will reveal a very interesting relation of $B_{n,1}$ with the Hecke algebra of B_n-type, as we shall see in the next chapter. Finally, using this presentation, we find a second one, which allows us to express algebraically Theorem 5 for the spaces $L(p,1)$ (section 2.7).

2.2 Background section

By '3-manifold' we will always mean 'closed (i.e. compact without boundary), connected, orientable 3-manifold' , which we abbreviate to 'c.c.o.' . 3-manifolds are strongly related to knots and links via a technique called **surgery**. Indeed – as W.B.R. Lickorish in [41] and A.D. Wallace in [61] showed – *every 3-manifold can be obtained from* S^3 *by doing surgery along a framed link in* S^3 .

To explain the above further, we will need to say first a few things about solid tori and framed links.

2.2.1 On framed links

Definition 11 A **solid torus** , V, is a space homeomorphic to $S^1 \times D^2$. I.e. $V = h(S^1 \times D^2)$ for some homeomorphism h . The curve $h(S^1 \times \{0\})$ is called the **core of** V .

A **meridian of** V is a non-contractible, simple, closed curve on ∂V that bounds a disc. A **longitude of** V is a simple closed curve on ∂V that intersects transversally some meridian of V in a single point. In other words a longitude runs *in parallel* (i.e. it cobounds an annulus) with the core of V .

As D. Rolfsen mentions in [53]: 'Meridian' is an intrinsic part of V , whereas 'longitude' involves a choice. Indeed, any two meridians of V are equivalent by an ambient isotopy of V . Any two longitudes of V are equivalent by a homeomorphism of V ; however, there are infinitely many ambient isotopy classes of longitudes. (One can see this by thinking of the number of times a longitude twists around the core.)

Definition 12 A **framed link** in S^3 is a disjoint collection of n smoothly embedded copies of $S^1 \times D^2$ via homeomorphisms $f_1, \dots, f_n$.

We usualy consider framed links up to ambient isotopy.

In order to picture a framed link in S^3 it is enough to draw the images $c_1, \dots, c_n$ (via f_i) of the core of $S^1 \times D^2$, together with the images $l_1, \dots, l_n$ (via f_i) of $S^1 \times 1$. For every i , l_i will be called the **specified longitude** of c_i . Equivalently, it is enough to draw $c_1, \dots, c_n$ and associate with each an integer, its **framing** . For a framed link with no component knotted, framing κ_i for c_i means the algebraic number of times l_i twists around c_i as we follow the orientation of c_i .

2.2.2 About Surgery

Let $W = D^4$ be the 4-ball and $D^2 \times D^2 (= D^4)$ be a 2-handle. Notice that

$$\partial(D^2 \times D^2) = S^1 \times D^2 \bigcup_{S^1 \times S^1} D^2 \times S^1$$

i.e. $\partial(D^2 \times D^2)$ is the union of two solid tori over a common boundary $S^1 \times S^1$. We attach to W n disjoint copies $H_1, \ldots, H_n$ of $D^2 \times D^2$ (i.e n 2-handles) via homeomorphisms

$$h_i : S^1 \times D^2 \longrightarrow \partial W = S^3 \quad i = 1, \ldots, n$$

and such that

$$H_i \bigcap W = \partial H_i \bigcap \partial W = h_i(S^1 \times D^2)$$

The boundary of $h_i(H_i)$ is

$$h_i(S^1 \times D^2) \bigcup_{h_i|_{S^1 \times S^1}} (D^2 \times S^1)$$

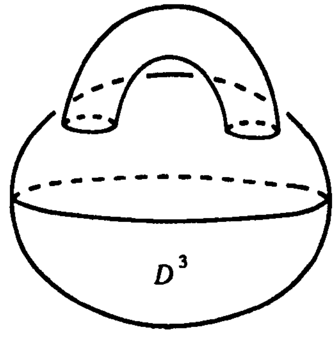
i.e. the union of two solid tori over their common boundary that is a torus. After the attachment we obtain a new 4-manifold W' with boundary $M = \partial W'$, a new compact, connected, orientable 3-manifold such that

$$M = S^3 \left[\setminus h_i(S^1 \times D^2) \bigcup_{h_i|_{S^1 \times S^1}} (D^2 \times S^1) \right]_{i=1}^{n}$$

The above mean that M can be obtained from S^3 by excavating n homeomorphic open images of disjoint solid tori and by gluing back over the common boundaries another n solid tori with the factors reversed. So, the specified longitude $l_i = h_i(S^1 \times 1)$ of the ith original torus, after the attachment is seen as a meridian and bounds a disc in M.

Note An orientation for W' and $M = \partial W'$ is determined by extending over W' a fixed orientation on D^4.

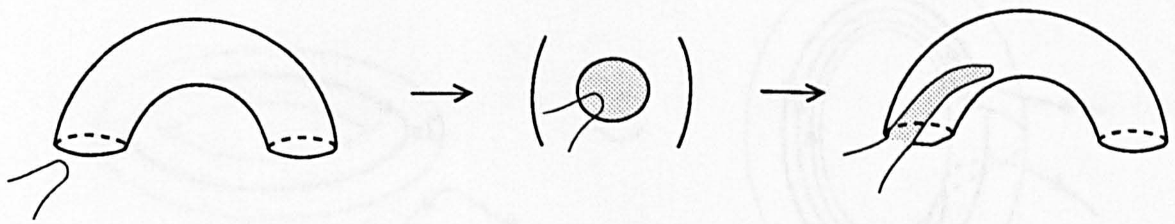
Example in S^2 (rather than S^3)



In the picture above, we attach to $S^2 = \partial(D^3)$ a 1-handle $D^1 \times D^2$ (a solid cylinder) via the identity map $id : S^0 \times D^2 \longrightarrow S^2$. Notice that $S^0 \times S^1 = \partial(S^0 \times D^2) = \partial(D^1 \times S^1)$.
After the attachment we obtain a new c.c.o. 2-manifold T , which is actually a torus:

$$T = S^2 \setminus S^0 \times D^2 \bigcup_{id|_{S^0 \times S^1}} D^1 \times S^1$$

Notice also that, if a wanderer (e.g. a piece of string) tries to pass through one of the attaching discs, it will end up strolling on the surface of the attached cylinder:



Back to the 3-dimensional case, we can equivalently express the above by saying that M is obtained from S^3 by doing **surgery**[1] **along the cores of the removed solid tori** . All these cores form a framed (oriented) link in S^3 (which we call the **surgery link**) , the framing of the ith component being determined by h_i . So we can write $M = \chi(S^3, L)$, and, as mentioned previously, we can do that for any c.c.o. 3-manifold. Moreover, we may always assume that all the components of the surgery link L are unknotted and, even more, that L is isotopic to the closure of a *pure braid* as it follows from W.B.R. Lickorish's proof in [41]. (Another proof has been given by C.P. Rourke and is presented in [38])[2]. Throughout the rest of this work we shall only consider framed links with no component knotted, and 3-manifolds with integral surgery description.

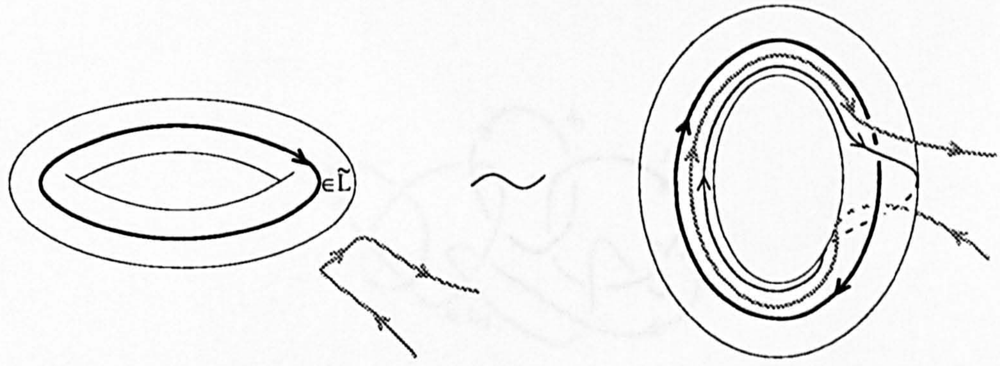
Now, if an (oriented) piece of string (i.e. a band) approaches a meridian of an *attached* torus after the surgery, it should be able (by isotopy in M) to slide through the disc that the meridian bounds; but this meridian – up to ambient isotopy – was the specified longitude of the *excavated* solid torus. So, what we see in S^3 in terms of the surgery link, is that the band follows the specified longitude of that particular surgery component.

---

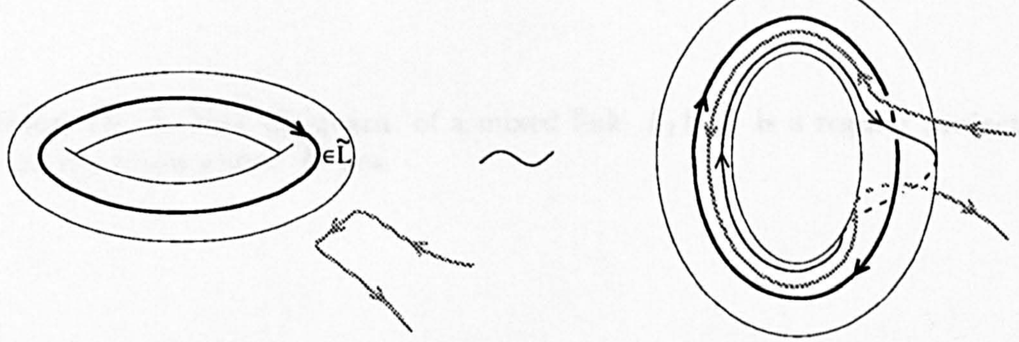
[1]Surgery may appear in experimental mathematics too; as a characteristic example, we refer to [18], where N. Samardzija and L. Greller show an 'explosive route to chaos through a fractal torus in a generalized Lotka-Volterra model'.

[2]This observation has also been proved by R. Skora in [55], as mentioned in [44].

Pictures 3 and 4: *(with framing -1)*



or



according to whether the orientation of the band agrees or not with the orientation of the surgery component (and implicitely of its specified longitude).

The above are examples of **'spatial band moves'**.

R. Kirby in [35] describes two operations on framed links and proves that two 3-manifolds are homeomorphic if and only if they can be obtained by surgery along links in S^3 that differ by a finite sequence of these operations (Kirby moves) [3]. The above mean that for a given M we have a choice for representing M in S^3 by a surgery link. To avoid this ambiguity we fix an oriented surgery link, L, and to avoid further ambiguities we fix a *projection* $\widetilde{L}$ of L.

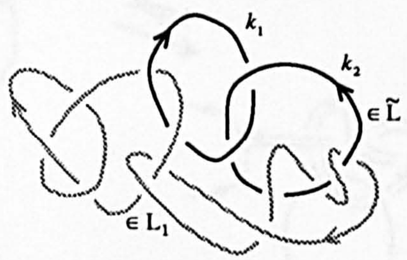
Conclusion Now we may say that we can represent M *uniquely* in S^3 by $\widetilde{L}$ and write $M = \chi(S^3, \widetilde{L})$. *This allows us to work in* S^3 *rather than in* M *in order to study* M *and links in it.*

2.2.3 Links in M : the band moves

Let $M = \chi(S^3, \widetilde{L})$ and let L'_1 be an oriented link in M. Then L'_1 may be represented in S^3 by a **'mixed' link** $L_1 \bigcup \widetilde{L}$. So we can study links in M by studying their corresponding 'mixed' links in S^3.

[3] A more combinatorial exposition of Kirby's calculus of links is presented by R. Fenn and C.P. Rourke in [15].

Example of a 'mixed' link: (where $k_1, k_2 \in \mathbb{Z}$ are framing numbers)

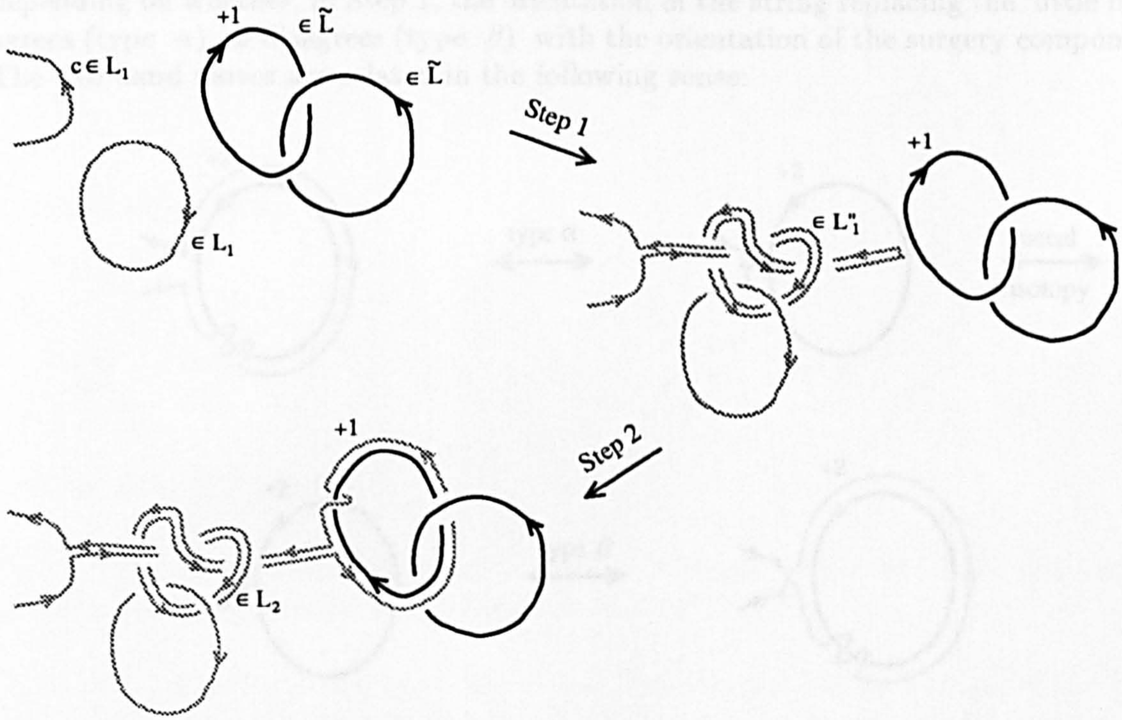


Definition 13 A **link diagram** of a mixed link $L_1 \bigcup \widetilde{L}$ is a regular projection of $L_1 \bigcup \widetilde{L}$ on the plane where $\widetilde{L}$ lies.

Definition 14 A **spatial band move** between two oriented mixed links $L_1 \bigcup \widetilde{L}$ and $L_2 \bigcup \widetilde{L}$ in S^3 , is a move in S^3 that reflects ambient isotopy of L_1' , L_2' in $M = \chi(S^3, \widetilde{L})$. The performance of a spatial band move from $L_1 \bigcup \widetilde{L}$ to $L_2 \bigcup \widetilde{L}$ can be described in two steps:

Step 1: A band b – which can be seen as the oriented boundary of a ribbon – starts from a component c of L_1 . This means that one of the small edges of b is glued to a part of c so that the orientation of the band agrees with the orientation of c . The other small edge of b , which we shall call **'little band'** (in ambiguity with the notion of a band), approaches a surgery component of $\widetilde{L}$ in an arbitrary way (see picture below). So, if L_1'' is the result of the attachment of b to L_1 , the 'mixed' links $L_1 \bigcup \widetilde{L}$ and $L_1'' \bigcup \widetilde{L}$ are isotopic in S^3 .

Step 2: The 'little band' is replaced by a string running in parallel with the specified longitude of the surgery component in such a way that the orientation of the string agrees with the orientation of b . The resulting link is $L_2 \bigcup \widetilde{L}$:



The second step of a spatial band move takes place in a tubular neighbourhood of the component of the surgery link that contains no other part of the mixed link.

Remarks on the band moves

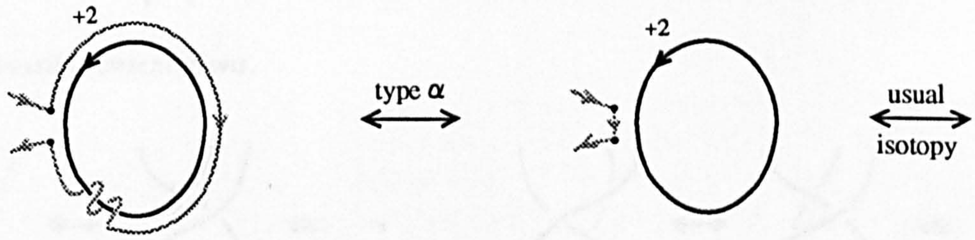
• The spatial band move described above is precisely the band connected sum (over the band b) of a component $c \in L_1$ and the specified longitude of the surgery component.

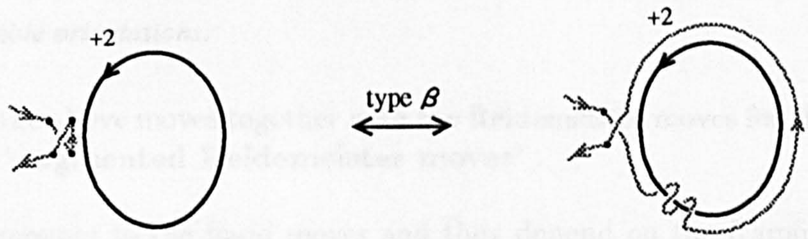
• Let L_1' , L_2' be ambient isotopic links in $M = \chi(S^3, \widetilde{L})$, with $L_1 \bigcup \widetilde{L}$, $L_2 \bigcup \widetilde{L}$ two corresponding mixed links; assume that, when the isotopy is reflected in S^3 , no band move is involved. Then the mixed links $L_1 \bigcup \widetilde{L}$ and $L_2 \bigcup \widetilde{L}$ are ambient isotopic in S^3 , under isotopy keeping $\widetilde{L}$ fixed, and we will be saying that L_1' *differs from* L_2' *by 'usual' isotopy in* M or that $L_1 \bigcup \widetilde{L}$ *differs from* $L_2 \bigcup \widetilde{L}$ *by 'usual' isotopy in* S^3 .

• Since the first step of a spatial band move only involves usual isotopy, from now on whenever we say **'band move'** we will always be referring to the realization of the second step of a spatial band move.

• We may omit the word 'spatial' as the band move takes place very close to the surgery component and so, the way it looks around the surgery component *does not* depend on the direction of the projection.

- As pictures 3 and 4 above indicate, there are two types α , β of band moves depending on whether, in Step 2, the orientation of the string replacing the little band agrees (type α) or disagrees (type β) with the orientation of the surgery component. The two band moves are related in the following sense:





Aside If $L_1 \bigcup \widetilde{L}$ represents a link L_1' in M , we call L_1'' its *corresponding link in* S^3 (with $\widetilde{L}$ removed). Obviously, if L_1' is isotopic to L_2' in M , it does not necessarily imply that L_1'' is isotopic to L_2'' in S^3 , and conversely. But if L_1' is isotopic to L_2' by usual isotopy in M , then L_1'' is also isotopic to L_2'' in S^3 .

Conclusion *Two (oriented) links* L_1' , L_2' *in* $M = \chi(S^3, \widetilde{L})$ *are ambient isotopic if and only if the mixed links* $L_1 \bigcup \widetilde{L}$ *and* $L_2 \bigcup \widetilde{L}$ *in* S^3 *differ by ambient isotopy in* S^3 *and a finite sequence of band moves.*

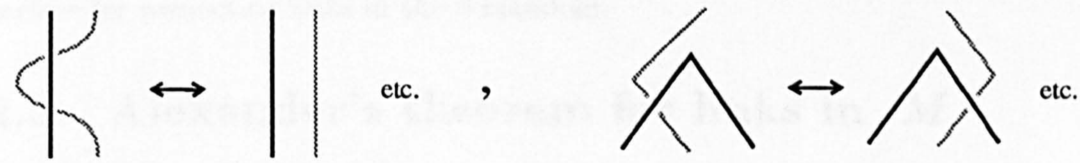
2.2.4 Modified Reidemeister's theorem

Let L_1' , L_2' be ambient isotopic links in $M = \chi(S^3, \widetilde{L})$ and $\widetilde{L_1} \bigcup \widetilde{L}$, $\widetilde{L_2} \bigcup \widetilde{L}$ any diagrams of two corresponding mixed links in S^3 . In order to modify Reidemeister's theorem for $L_1 \bigcup \widetilde{L}$ and $L_2 \bigcup \widetilde{L}$ we only need to consider the following additional critical cases:

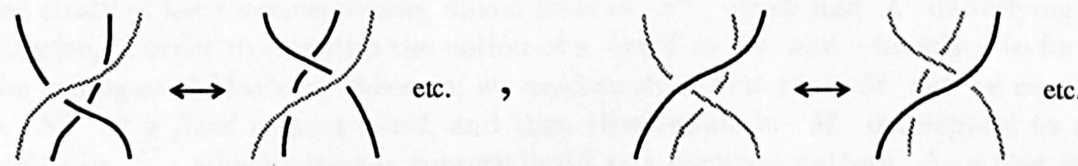
(i) when arcs of the surgery link participate in the Reidemeister moves (intersection on the plane, but *not* in the 3-space)

(ii) when a piece of L_1 or L_2 intersects an arc of $\widetilde{L}$ (intersection in the 3-space).

From (i) we derive the following additional moves (where the surgery strings are pointwise fixed):



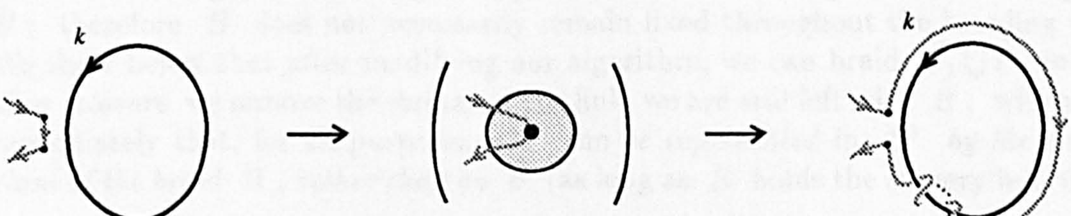
with all possible orientations.



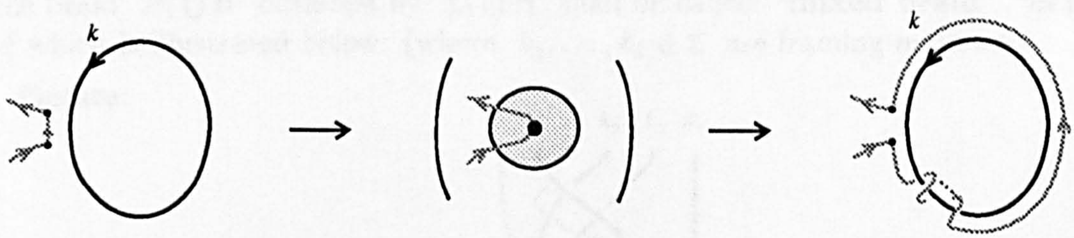
with all possible orientations.

We shall call the above moves together with the Reidemeister moves for the non-surgery components **'augmented Reidemeister moves'**.

Case **(ii)** amounts to the band moves and they depend on the framing of the component of $\widetilde{L}$ to which the arc belongs. By transversality, the singularities illustrated below can appear only finitely many times:



and



Conclusion *Two links* L'_1 , L'_2 *in* M *are ambient isotopic if and only if any two diagrams of the mixed links* $L_1 \bigcup \widetilde{L}$ *and* $L_2 \bigcup \widetilde{L}$ *differ by planar shifts (with* $\widetilde{L}$ *pointwise fixed) and a finite sequence of the augmented Reidemeister moves together with the band moves.*

Aside We can also make a tangle version of the generalized Reidemeister's theorem.

Note In [56], P.A. Sundheim has proved an analogue of Reidemeister's theorem for 3-manifolds, where he uses the fact that 'any c.c.o. 3-manifold contains a link whose complement fibers over S^1 with the fiber being an orientable surface'; so, he uses this surface for projecting links in the 3-manifold.

2.3 Alexander's theorem for links in M

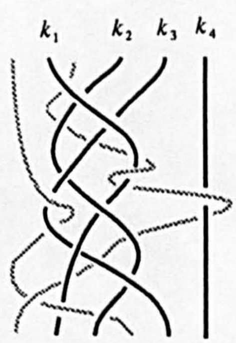
2.3.1 On braids in M

Let $M = \chi(S^3, \widetilde{L})$. In the previous section we reduced the study of links in M to the study of their corresponding mixed links in S^3 , which had $\widetilde{L}$ underlying. Here, likewise, in order to establish the notion of a *braid* in M and – mainly – to formulate the analogue of Markov's theorem, we need to show first that M can be represented in S^3 by a *fixed surgery braid*, and then that braids in M correspond to specific braids in S^3 , which have the surgery braid as a common pattern. As a first step, we notice that without loss of generality (w.l.o.g.) $\widetilde{L}$ can be seen as the closure of a pure braid B , so we may write $\widehat{B}$ instead. As mentioned in the completion of the proof of Markov's theorem (1.4.2), if we apply our braiding process to $\widehat{B}$, the result will be the braid B itself.

Let now L_1' be any oriented link in M and $\widetilde{L_1} \bigcup \widehat{B}$ be a projection of the mixed link $L_1 \bigcup \widehat{B}$ in S^3 , such that it has the direction of $\widehat{B}$ as a (0,0)-tangle. If we apply our braiding process to $\widetilde{L_1} \bigcup \widehat{B}$, B is very likely to end up with an increased number of strings, as there might be parts of $\widetilde{L_1}$ interfering with the closing side of $\widehat{B}$; therefore B does not necessarily remain fixed throughout the braiding process. We show below that after modifying our algorithm, we can braid $\widetilde{L_1} \bigcup \widehat{B}$ so that, if after *closure* we remove the strings of the link, we are still left with $\widehat{B}$; which implies immediately that, for all purposes , M *can be represented in* S^3 *by the conjugacy class of the braid* B *, rather than by* $\widehat{B}$ (as long as B holds the surgery information).

Definition 15 In the above context , B shall be called **'the surgery braid'** and the braid $B_1 \bigcup B$ obtained by $\widetilde{L_1} \bigcup \widehat{B}$ shall be called **'mixed braid'** , an example of which is illustrated below: (where $k_1, \ldots, k_4 \in \mathbb{Z}$ are framing numbers)

Picture:



The braid B_1 shall be called **'permutation braid'** .

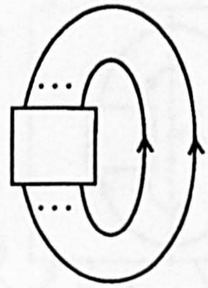
It follows from the above that, if B is a braid on m strings and B_1 a braid on n strings, then the mixed braid $B_1 \bigcup B$ is a specific element of the braid B_{n+m} .

The braid relations among mixed braids in B_{n+m} shall be called **'augmented braid relations'** ; these consist of the usual braid relations for the strings of the permutation braid, the augmented Reidemeister moves with *all* arrows pointing downwards, and the braid moves concerning the change of relative heights of two mixed crossings.

2.3.2 Modified braiding process for mixed links

For the rest of this chapter we shall be working in the smooth category and we shall be making the closure of a braid as illustrated below:

Picture:

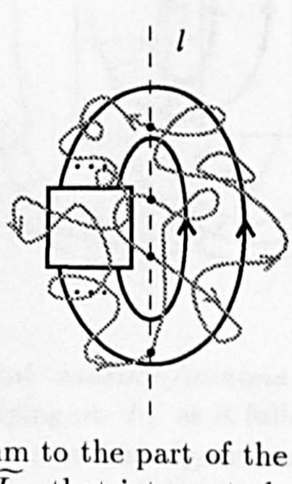


Theorem 4 *Any oriented link* L_1' *in* $M = \chi(S^3, \widehat{B})$ *can be represented in* S^3 *by some mixed braid* $B_1 \bigcup B$ *(not uniquely), the closure of which is isotopic to a mixed link diagram* $\widetilde{L_1} \bigcup \widehat{B}$ *(i.e.* $\widehat{B}$ *remains unchanged).*

Proof Let $\widetilde{L_1} \bigcup \widehat{B}$ be a projection of the mixed link $L_1 \bigcup \widehat{B}$ in S^3 . If there is no part of $\widetilde{L_1}$ interfering with the closing side of $\widehat{B}$, we only apply Steps 2, 4 and 6 below. Otherwise, we turn $\widetilde{L_1} \bigcup \widehat{B}$ into a braid as follows:

Step 1 We draw the vertical line l that passes through all local maxima and minima of the surgery (braided) link $\widehat{B}$ (see picture below). W.l.o.g. l is a line, and by general position it does not pass through any crossings of $\widetilde{L_1}$.

Picture:



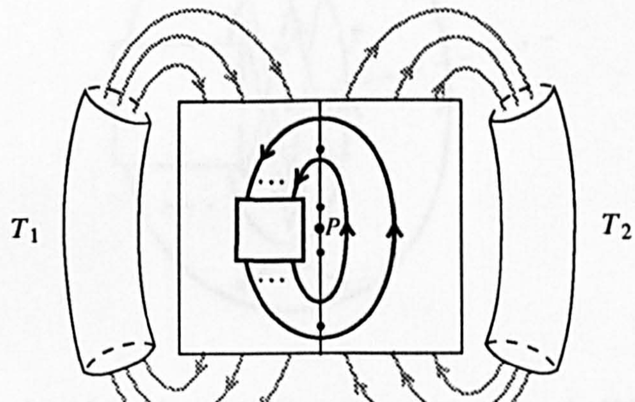
Step 2 We apply our algorithm to the part of the mixed link that lies to the left of l , considering the points of $\widetilde{L_1}$ that intersect l as end-points. This will leave B unaltered (since all strings of B go already downwards). Then we close this *braided*

part of $\widetilde{L_1}$ by applying *closure* on its left-hand side, and we enclose the 'closure' strings of $\widetilde{L_1}$ in a tube T_1 (see picture below).

Step 3 Now we apply our algorithm on the right-hand side of l considering the orientation to be reversed. This will leave the 'closure' strings of $\widehat{B}$ unchanged. Then we also close this *braided* part of $\widetilde{L_1}$ by applying *closure* on its right-hand side, and we enclose the new 'closure' strings of $\widetilde{L_1}$ in a tube T_2 (see picture below).

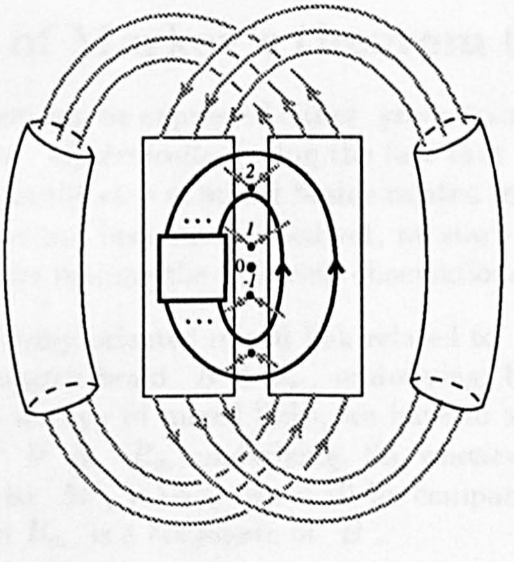
Picture:

Picture:



Step 4 By rotating around the back of the diagram, we bring T_1 to the very right of the diagram and then T_2 to the very left of the diagram, so that the resulting diagram goes around a central point P on l .

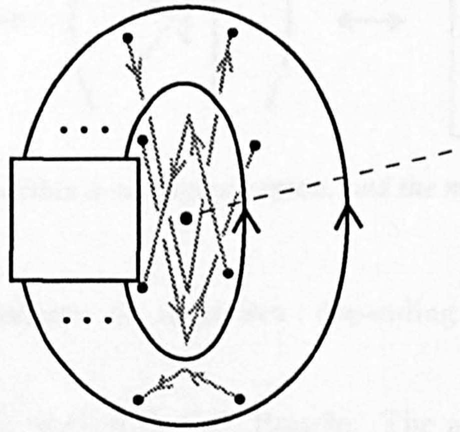
Picture:



Step 5 If we are left with local *maxima/minima* in the *lower/upper* part of the diagram, these will have to be lying on l , as it follows from the braiding process (see picture above). To complete the modified algorithm we eliminate these as follows: We number with integers the *maxima/minima* according to their position with respect to P (which we label with 0), and we isolate them in neighbourhoods that contain

no other parts of the diagram. Then we stretch the arcs one by one in order (starting from the ones with least absolute value) *over/under* the rest of the diagram and *above/below* P, so that the *maxima/minima* lie on l in inverse order of closeness to P.

Picture:



Step 6 We *open* the braided diagram by cutting through a half-line starting from P, which leaves the 'box' in which B lies untouched. Finally we isotope in $D^2 \times I$. □

Consequence Since the original surgery braid is still the same, we conclude that w.l.o.g. a mixed link interferes only with the front part of B. Therefore, for studying mixed braids, M *may be represented in* S^3 *by the fixed surgery braid* B.

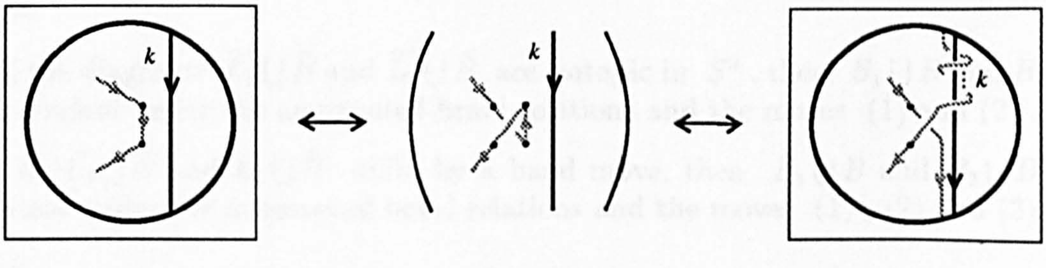
2.4 Extension of Markov's theorem to 3-manifolds

The usual Markov's theorem can be expressed either *geometrically* (using the geometric definition of a braid), or *algebraically* (using the fact that the set of all braids on n strings forms a group). In the case of mixed braids related to arbitrary 3-manifolds, where group structures have not been established yet, we start by extending Markov's theorem geometrically, (after making the following observations).

In 2.3 we showed that every oriented mixed link related to M can be turned into a mixed braid with the surgery braid $B \in B_m$ underlying; but for *equivalence* of mixed braids that reflects isotopy of mixed links, we have to allow for a mixed braid to contain *a conjugate of* B *in* B_m underlying. So, whenever we have to compare two mixed braids related to M, w.l.o.g. we shall be comparing the braids $B_1 \bigcup B$ and $B_2 \bigcup B'$, where $B' \in B_m$ is a conjugate of B.

Recall now, that neither of the two types of band moves can appear as a move between braids; so, in order to state our generalized version of Markov's theorem, we modify the band move of type α appropriately by twisting the little band before performing the move (of type β now). As we can see in the picture below, after the performance of the move, we remain in the braid category, and we shall call this move **'braid band move'**.

Picture:



where we draw only what lies within a magnified region, and the middle stage is only indicative.

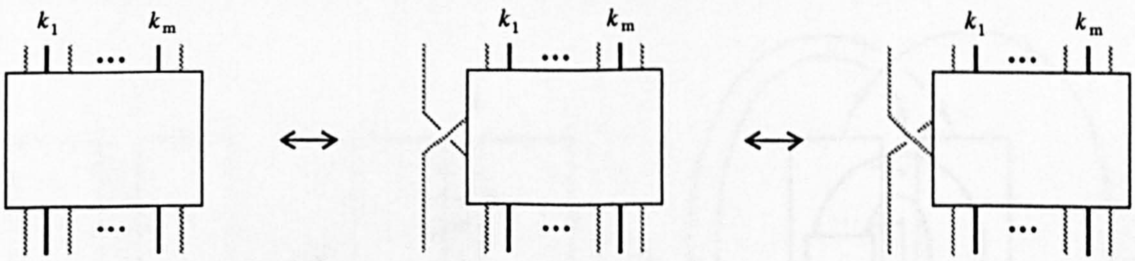
A braid band move can be **positive** or **negative** , depending on the type of crossing we choose for performing it.

The following theorem is joint work with C.P. Rourke. The author's contribution is mainly work on the right formulation of the statement, and the proofs of **(i)** , **(ii)** and 2.4.1 .

Theorem 5 *(geometric version)* *Let* $M = \chi(S^3, B)$ *be a 3-manifold with* B *a surgery pure braid on* m *strings, let* L_1 , L_2 *be two oriented links in* M *and* $B_1 \bigcup B$, $B_2 \bigcup B'$ *be mixed braids in* S^3 *corresponding to* L_1 , L_2 , *with* B' *a conjugate of* B . *Then* L_1 *is isotopic to* L_2 *in* M *if and only if* $B_1 \bigcup B$ *is equivalent to* $B_2 \bigcup B'$ *in* S^3 , *under equivalence generated by the augmented braid relations together with the following three moves:*

(1) Conjugation in B_{n+m} , *where* n *is the number of strings of the permutation braid.*

(2) Generalized Markov moves for the non-surgery strings (as illustrated below):



(3) Braid band moves (as described above).

Proof Using 2.2.4 we translate the question of ambient isotopy between L_1 , L_2 in M into (ambient) isotopy moves between any two corresponding mixed link diagrams $\widetilde{L_1} \bigcup \widehat{B}$, $\widetilde{L_2} \bigcup \widehat{B}$ in S^3 . Then, the direction of the proof from braids to projections of links by *closure* is easy to see, as braids that differ by moves (1) , (2) and (3) have isotopic closures. The converse amounts to proving that:

(i) The braid we obtain from a mixed link after applying the modified braiding algorithm, does not depend – up to the above equivalence – on any choice made during the process.

(ii) If the diagrams $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ are isotopic in S^3, then $B_1 \bigcup B$ and $B_2 \bigcup B'$ are equivalent under the augmented braid relations and the moves (1) and (2) .

(iii) If $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ differ by a band move, then $B_1 \bigcup B$ and $B_2 \bigcup B'$ are equivalent under the augmented braid relations and the moves (1) , (2) and (3) .

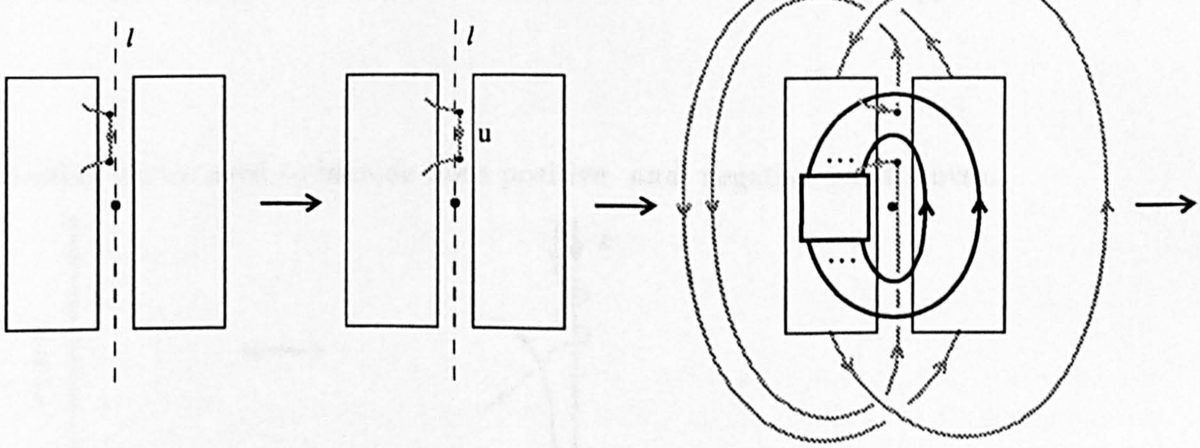
Indeed:

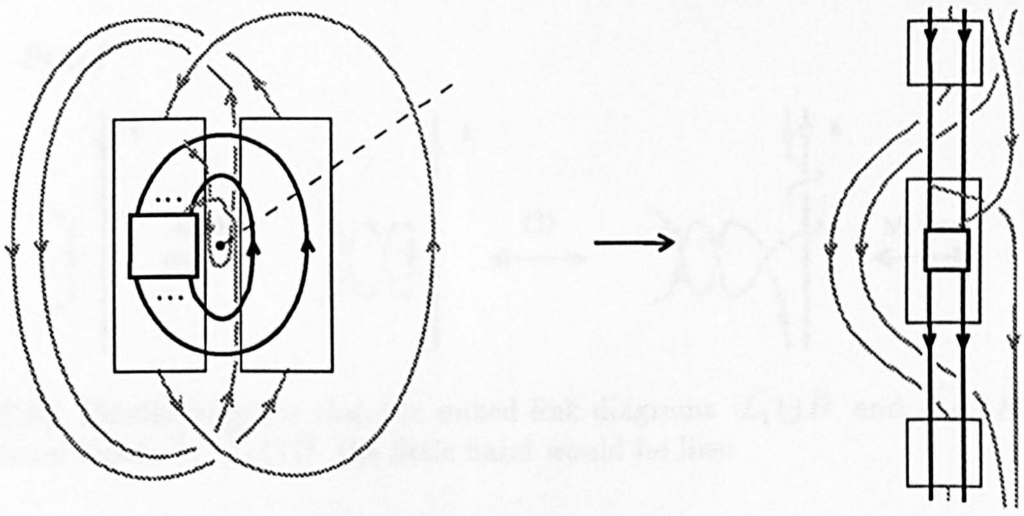
(i) The above algorithm (Theorem 4) is another braiding process in S^3 ; and since Markov's theorem holds independently of the algorithm we are using (recall 1.4.2) , we do not have to examine the possible choices made during the braiding process, as – in addition – they *do not* affect the position of the surgery braid (recall Corollary 4 , relative version of Markov's theorem). Such choices additional to the ones listed in the proof of Markov's theorem are involved in:

- The size of the isolated neighbourhoods of maxima/minima chosen in Step 5, as well as the new positions that we place the maxima/minima.
- The line that we choose to cut the braid open (which follows immediately by conjugation).

(ii) Next, let $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ be two mixed link diagrams that differ by small planar shifts and the augmented Reidemeister moves. If these moves take place in either side of the line l , then, from the relative version of Markov's theorem, any two corresponding mixed braids differ by moves (1) , (2) and the augmented braid relations.

The only case that remains to be examined is when a piece of (non-surgery) string crosses the line l ; w.l.o.g. the little arc is a free arc, so we can label it 'u' . As shown in the pictures below, the final braid differs from the braid we would have obtained if the little arc had not crossed l , essentially by two Markov moves:





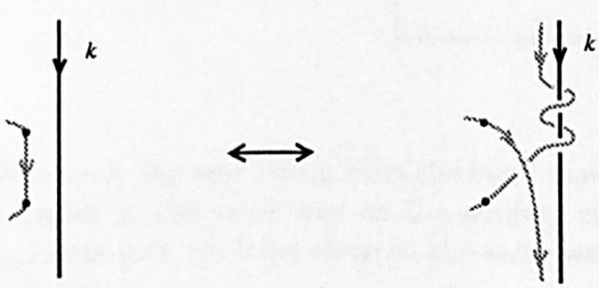
Notice that, if the little arc that moves across l intersects the central point P, then – by the algorithm – we would need to stretch *both* ends, and the proof would be same as above.

Picture:

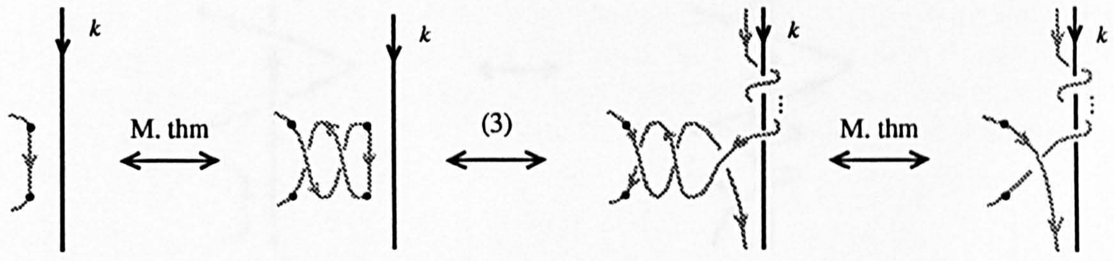
Also that, if we had a maximum or a minimum moving across l, the proof would be same as above.

As a simple consequence of the above, we do not need to include in the statement of the theorem the braid band moves (positive and negative) where the little band approaches a surgery string *from the right*, since this is a result of usual isotopy and a move (3).

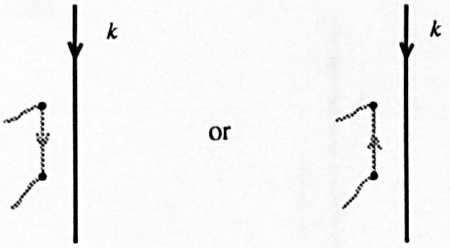
Neither do we need to include both positive *and* negative band moves:



Proof



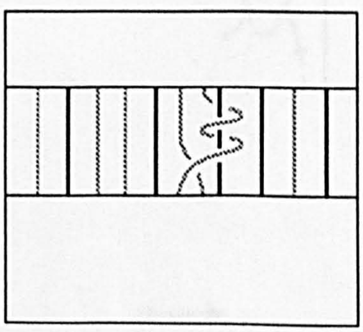
(iii) Finally, suppose that the mixed link diagrams $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ differ by a band move. In $\widetilde{L_1} \bigcup \widehat{B}$ the little band would be like:



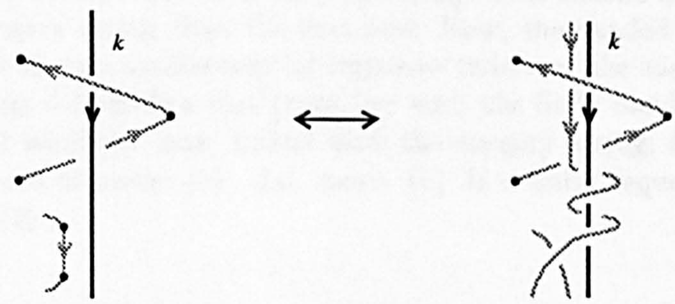
depending on the orientation.

If the little band is an opposite arc, w.l.o.g. we may assume that it satisfies condition (*) . The algorithm we use ensures that we may assume that $\widetilde{L_1} \bigcup \widehat{B}$ and $\widetilde{L_2} \bigcup \widehat{B}$ are braided everywhere except for the little band in $\widetilde{L_1} \bigcup \widehat{B}$ (if it is an opposite arc) and its replacement after the performance of the band move. This happens because the band move takes place arbitrarily close to the surgery string; so we can produce such a zone locally in the braid (and consequently a band move cannot create problems with condition (*)):

Picture:

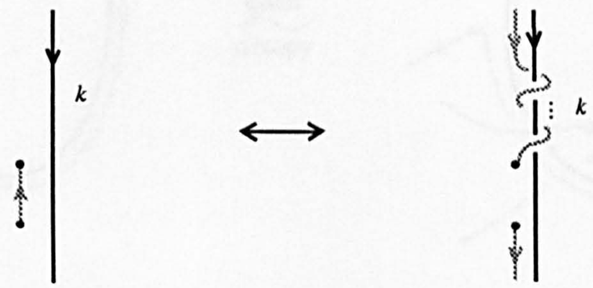


Moreover, the new string from the band move (as far as other crossings are concerned) behaves in the same way as the surgery string itself. So, whenever we meet other opposite arcs, we label them in the same way that we would do if the new string were missing.

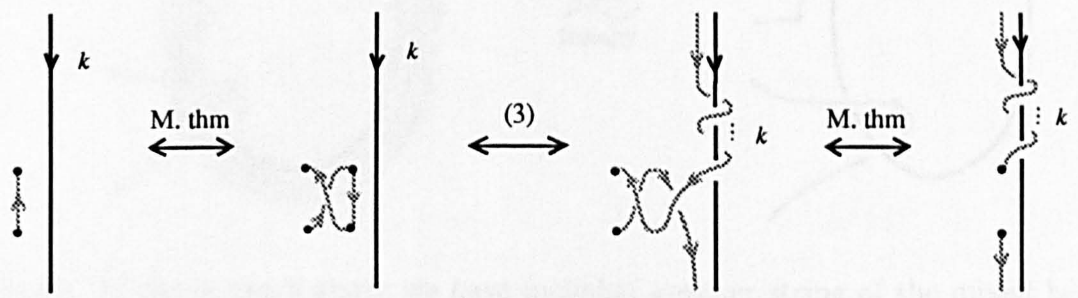


So the different cases of applying a band move to $\widetilde{L_1} \bigcup \widehat{B}$ amount to the following (with proofs):

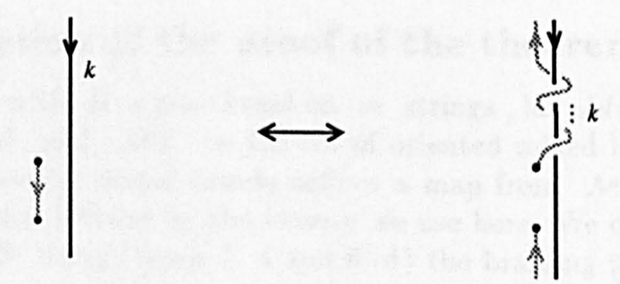
(b)



Proof We start with the left part of move (b) and we twist the little band (usual isotopy) using a *negative* crossing . Then we perform a move (3) and we end up with the right part of move (b) :

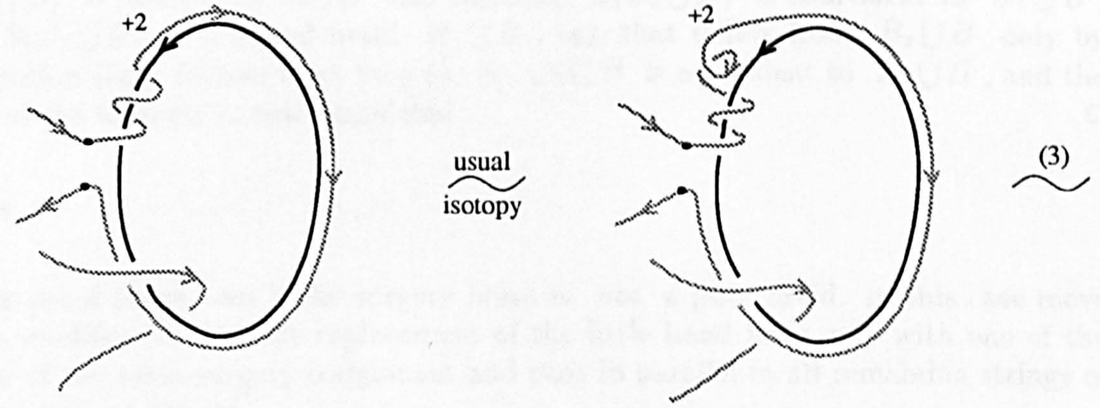


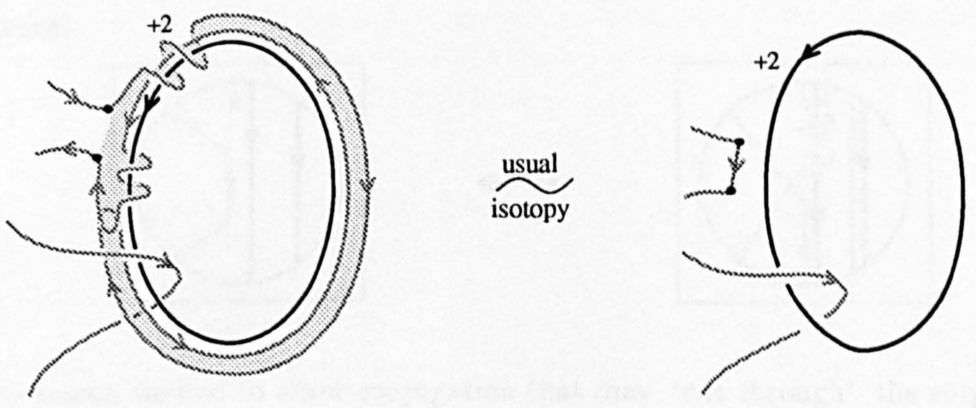
(c)



Proof We start with the right part of move (c) . In the front of the otherwise braided link we do a twist of the new string using a *negative* crossing (see pictures below). Then, we consider the little twisted arc as a little band and we perform another band move of type (3) around the same surgery string. This second band move takes place closer to the surgery string than the first one. Now, the shaded region in the picture below is formed by two similar sets of opposite twists of the same string around the surgery string. So it bounds a disc (together with the little band that is missing), the circumference of which is **not** linked with the surgery string; but this is isotopic in S^3 to the left part of move (c) . I.e. move (c) is a finite sequence of moves of type (1) , (2) and (3) .

Pictures:





Note In the pictures above we have included another string of the mixed braid that links with the surgery string. We can see that this does not affect the proof.

2.4.1 Completion of the proof of the theorem

Let $M = \chi(S^3, B)$ with B a pure braid on m strings , let $\mathcal{MB}$ be the set of mixed braids related to M and $\mathcal{ML}$ be the set of oriented mixed link diagrams in S^3 . Any closing operation for mixed braids defines a map from $\mathcal{MB}$ to $\mathcal{ML}$, and we shall call '$\mathcal{C}$' the map defined by the closure we use here. We can also obtain a map from $\mathcal{ML}$ to $\mathcal{MB}$ using (Steps 2, 4 and 6 of) the braiding process given in 2.3.2, and we shall call this map '$\mathcal{S}$' .

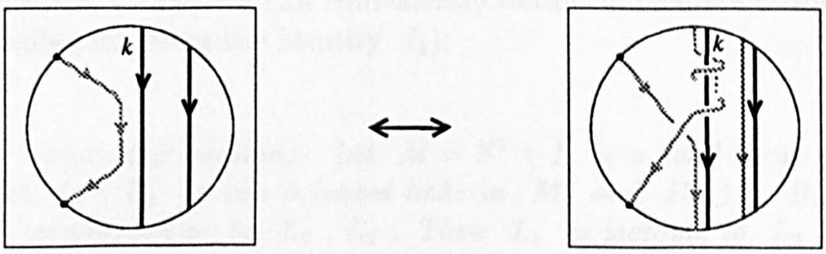
We have seen up to now that, if we start with two equivalent mixed braids and apply $\mathcal{C}$, we obtain isotopic mixed link diagrams; and conversely: if we start with two isotopic link diagrams and apply $\mathcal{S}$, we obtain equivalent mixed braids. As in 1.4.2, in order to complete the proof we have to show invariance under the closure we use for mixed braids and under the algorithm we use for turning mixed link diagrams into mixed braids. Invariance under closure is obvious. To see that all algorithms are equivalent, it is enough to show that any algorithm $\mathcal{A}$ satisfying Theorem 4 is equivalent to $\mathcal{S}$.

Indeed: Let $B_1 \bigcup B$, $B_2 \bigcup B$ be two mixed braids that we obtain from an oriented mixed link diagram $\widetilde{L} \bigcup \widehat{B}$ after applying $\mathcal{S}$ and $\mathcal{A}$ respectively. Then $\widehat{B_2} \bigcup \widehat{B} = \mathcal{C}(B_2 \bigcup B)$ is isotopic to $\widetilde{L} \bigcup \widehat{B}$ and therefore $\mathcal{S}(\widehat{B_2} \bigcup \widehat{B})$ is equivalent to $B_1 \bigcup B$. But $\mathcal{S}(\widehat{B_2} \bigcup \widehat{B})$ is a mixed braid $B_3 \bigcup B$, say, that differs from $B_2 \bigcup B$ only by conjugation (as it follows from Step 6). So $B_2 \bigcup B$ is equivalent to $B_1 \bigcup B$, and the proof of the theorem is now completed . □

Notes

• The proof holds even if the surgery braid is *not* a pure braid. In this case move (3) is modified so that the replacement of the little band links only with one of the strings of the same surgery component and runs in parallel to all remaining strings of the surgery component.

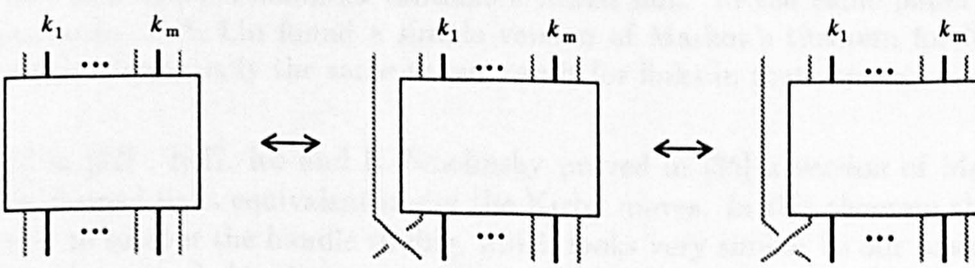
Picture:



• The reason we had to allow conjugation that may 'cut through' the surgery braid, comes from the usual isotopy of mixed links and the usual Markov's theorem; (recall that, from the modified algorithm the surgery braid is always the same).

Remark 7 In the statement of Theorem 5 we could have used the moves (2″) and (3″) depicted below, instead of the moves (2) and (3) :

(2″)



(3")

These are special cases of the moves (2) and (3) that take place at the bottom of the braid, and we shall use them later on for algebraic purposes.

2.4.2 Some further comments

• Suppose the surgery link consists of only one unknot. If we ignore the surgery description, then the surgery component may be seen as a solid torus , V say. Also, a mixed link/braid may be seen as a representative in S^3 of a link/braid inside the complementary solid torus , $S^3 \setminus V$. Then, the analogue of Reidemeister's theorem as well as Theorem 4 can be modified properly; and therefore, if we omit move (3) in Theorem 5 , we obtain the following version of Markov's theorem for isotopic links inside a solid torus, (which we can equivalently obtain immediately from Corollary 5 for the fixed subbraid being the identity I_1):

Theorem 6 *(geometric version) Let $M = S^3 \setminus \widehat{I}$ be a solid torus with $\widehat{I}$ also a solid torus; let L_1 , L_2 be two oriented links in M and $B_1 \bigcup I$, $B_2 \bigcup I$ be mixed braids in S^3 corresponding to L_1 , L_2 . Then L_1 is isotopic to L_2 in M if and only if $B_1 \bigcup I$ is equivalent to $B_2 \bigcup I$ in S^3 , under equivalence generated by the augmented braid relations together with the moves (1) and (2) of Theorem 5 .*

Note that we have adopted a fixed orientation for the 'solid torus' component $\widehat{I}$, the complement of which is the space we consider here.

• As cited in [44] , an analogue of Alexander's as well as of Markov's theorem for links in arbitrary c.c.o. 3-manifolds has been proved by R. Skora in [55] , where he uses the fact that every 3-manifold contains a fibred link. In the same paper, using R. Skora's results, X-S. Lin found a simple version of Markov's theorem for links in $L(p,1)$, which is essentially the same as our result for links in these spaces.

• As cited in [37] , K.H. Ko and L. Smolinsky proved in [36] a version of Markov's theorem for framed links equivalent under the Kirby moves. In this theorem they use a braid move to recover the handle sliding, which looks very similar to our braid band moves; only, they attach framing numbers to every string of one component.

2.5 Group structures of mixed braids

In this section we look for algebraic structures in the set of mixed braids related to a specific 3-manifold (recall 2.1). Having established this, we show that Theorem 5 holds even if we restrict ourselves to the braid groups $B_{n,m}$ or the cosets $C_{n,m}$, as this is the first step towards an algebraic expression of the analogue of Markov's theorem.

2.5.1 The group $B_{n,1}$

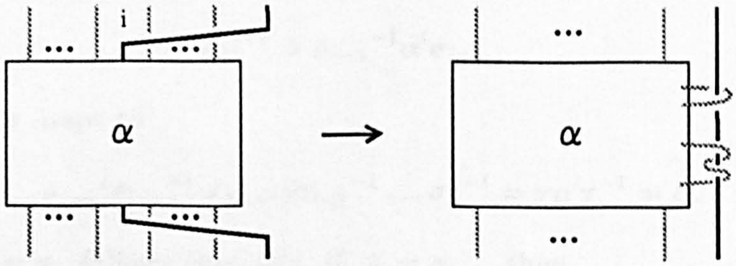
We focus our attention to $M = L(p,1)$; then a braid in M is seen as a mixed braid with the surgery string fixed at some place by the permutation of the braid. Similarly, if M is an (unknotted) solid torus (recall Theorem 6 above), a braid in M is represented in S^3 by a mixed braid with the complementary 'solid torus string' fixed at some place in the braid.

We call $B_{n,i}$ the set of all mixed braids on n non-surgery strings, which fix the surgery string at the $(n+2-i)$th place ; then $B_{n,i}$ clearly forms a group (a subgroup of B_{n+1}) and its elements can be regarded as the elements of B_{n+1} , the permutation of which fix the $(n+2-i)$th string. In this notation, the set of all mixed braids related to M is the disjoint union $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n+1} B_{n,i}$.

We claim that, in order to study isotopy of links in M , it suffices to restrict the set of mixed braids to the union $\bigcup_{n=1}^{\infty} B_{n,1}$, the braid groups that fix the last string. To see this, we first add a further step to the modified algorithm of the previous section. Namely:

Step 7 Let $\alpha \in B_{n+1}$ be the mixed braid we obtain after applying the first six steps. If the surgery string lies at the ith place, conjugate α to $\sigma_n \ldots \sigma_i \alpha \sigma_i{}^{-1} \ldots \sigma_n{}^{-1}$ in B_{n+1} . Then straighten the surgery string using the augmented braid relations:

Picture:



Now we may say that *all braids* in M can be represented by braids *with the surgery string fixed at the last place by the braid permutation* .

We also want to show that a restricted version of Theorem 5 holds for M , where the set of mixed braids is $\bigcup_{n=1}^{\infty} B_{n,1}$. Indeed:

Theorem 7 *Let* $M = L(p,1) = \chi(S^3, I)$, L_1 , L_2 *be two oriented links in* M *and* $B_1 \bigcup I$, $B_2 \bigcup I$ *be mixed braids in* $\bigcup_{n=1}^{\infty} B_{n,1}$ *corresponding to* L_1 , L_2 *(after*

Step 7). Then L_1 *is isotopic to* L_2 *in* M *if and only if* $B_1 \bigcup I$ *is equivalent to* $B_2 \bigcup I$ *in* $\bigcup_{n=1}^{\infty} B_{n,1}$ *, under equivalence generated by the augmented braid relations and the following three moves:*

(1′) Conjugation in $B_{n,1}$

(2′) Markov moves in $\bigcup_{n=1}^{\infty} B_{n,1}$ *of type (2″) for the non-surgery strings, as depicted in Remark 7 above*

(3′) Braid band move of type (3″) , as depicted in Remark 7 above,

where we omit move (3′) if M *is a solid torus.*

Proof It is enough to show that, if the braids $B_1{}' \bigcup I$ and $B_2{}' \bigcup I$ that are obtained from L_1 and L_2 after Step 6 differ by moves (1) , (2) and (3) (of Theorem 5), then the braids $B_1 \bigcup I$ and $B_2 \bigcup I$ (obtained after Step 7) differ by moves (1′) , (2′) and (3′) .

We shall first check conjugation: Let $\alpha' = B_1{}' \bigcup I$ differ from $B_2{}' \bigcup I$ by move (1) , i.e. $B_2{}' \bigcup I = \beta\alpha'\beta^{-1}$ for some $\beta \in B_{n+1}$, where $\alpha' \in B_{n+1}$ has the surgery string fixed at the ith place. Let $\tau = \sigma_n \ldots \sigma_i$; then, after Step 7, α' becomes $\alpha = \tau\alpha'\tau^{-1}$.

Case (i): If conjugation by β leaves the surgery string at the ith place, then, after Step 7, $\beta\alpha'\beta^{-1}$ becomes

$$\tau\beta\alpha'\beta^{-1}\tau^{-1} = \tau\beta\tau^{-1}\tau\alpha'\tau^{-1}\tau\beta^{-1}\tau^{-1} = (\tau\beta\tau^{-1})\alpha(\tau\beta\tau^{-1})^{-1}$$

which is conjugate to α in $B_{n,1}$ since $\tau\beta\tau^{-1} \in B_{n,1}$.

Case (ii): If β moves the surgery string from the ith to the jth place. In this case it is enough to assume that $\beta = \sigma_i{}^{\pm 1}$ or $\sigma_{i-1}{}^{\pm 1}$. If $\beta = \sigma_{i-1}{}^{-1}$ the proof follows immediately from the way Step 7 is performed, since

$$\beta\alpha'\beta^{-1} = \sigma_{i-1}{}^{-1}\alpha'\sigma_{i-1}$$

and after Step 7 this maps to

$$\sigma_n \ldots \sigma_{i-1}(\sigma_{i-1}{}^{-1}\alpha'\sigma_{i-1})\sigma_{i-1}{}^{-1} \ldots \sigma_n{}^{-1} = \tau\alpha'\tau^{-1} = \alpha.$$

The case where $\beta = \sigma_i$ follows similarly. If $\beta = \sigma_{i-1}$ then

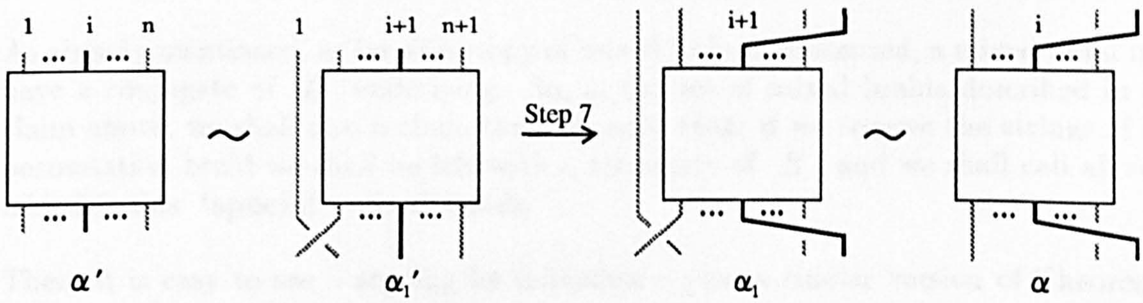
$$\beta\alpha'\beta^{-1} = \sigma_{i-1}\alpha'\sigma_{i-1}{}^{-1},$$

and after Step 7 this maps to

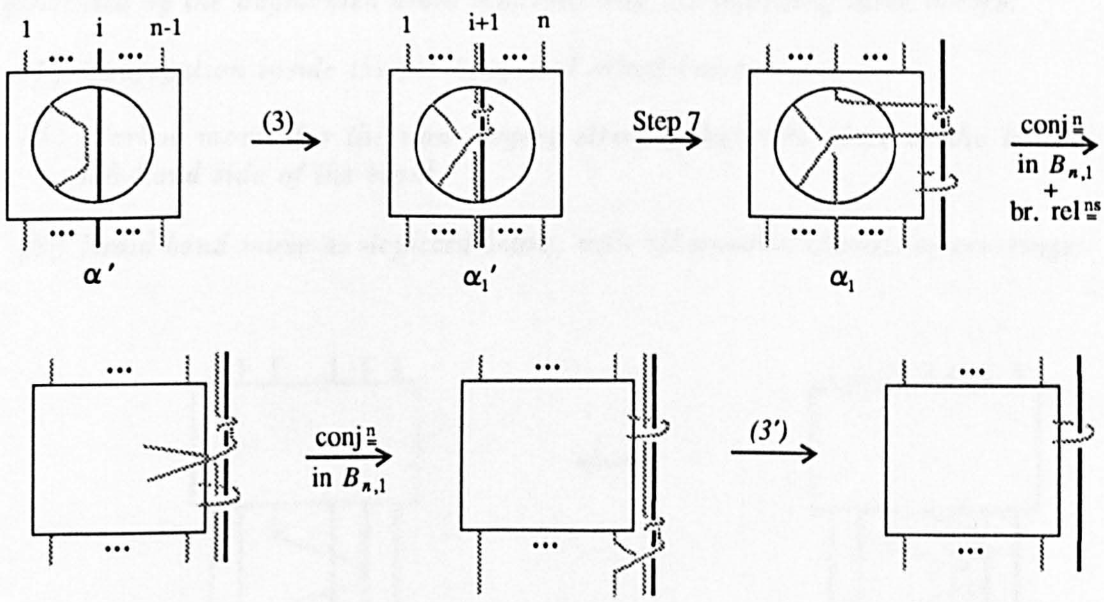
$$\sigma_n \ldots \sigma_i\sigma_{i-1}{}^2\tau^{-1}\tau\alpha'\tau^{-1}\tau\sigma_{i-1}{}^{-2}\sigma_i{}^{-1} \ldots \sigma_n{}^{-1} = (\tau\sigma_{i-1}{}^2\tau^{-1})\alpha(\tau\sigma_{i-1}{}^2\tau^{-1})^{-1},$$

which is conjugate to α in $B_{n,1}$, since the pure braid $\tau\sigma_{i-1}{}^2\tau^{-1} \in B_{n,1}$. The case where $\beta = \sigma_i{}^{-1}$ follows similarly.

Let now $\alpha' = B_1'\bigcup I$ differ from $\alpha_1' = B_2'\bigcup I$ by a move (2), which we may assume – using conjugation – that occurs at the bottom of the left-hand side of the braid. In the pictures below we illustrate that after Step 7, α and α' differ by a Markov move in $B_{n,1}$. (For convenience we draw only the first stage of Step 7):



Finally, let $\alpha' = B_1'\bigcup I$ differ from $\alpha_1' = B_2'\bigcup I$ by a band move (3). In the following pictures we illustrate that α_1 differs from α by a band move (3'), conjugation and braid relations in $B_{n,1}$:



which is the result of applying Step 7 and conjugation in $B_{n,1}$ to α'. □

2.5.2 The groups $B_{n,m}$

Let $M = \chi(S^3, B)$ be a 3-manifold represented in S^3 by B, a surgery pure braid on m strings. Then – as already mentioned – a braid in M is represented in S^3 by a mixed braid, which is a specific element of the (usual) braid group B_{n+m}, for some n. The above also apply if M is the complement of $\widehat{B}$ in S^3.

Claim For M as above, our braiding process (proof of Theorem 4) can be properly modified so that *the m surgery strings will occupy the last m places of any mixed* *braid related to* M *, and if we remove the strings of the permutation braid we shall be left with the surgery braid* B *.*

Proof: By applying Step 7 m times, starting every time from the string closer to the right-hand side of the braid.

As already mentioned, as far as isotopy of mixed links is concerned, a mixed braid may have a conjugate of B underlying. So, in the set of mixed braids described in the claim above, we shall also include the ones such that: if we remove the strings of the permutation braid we shall be left with a conjugate of B , and we shall call all such mixed braids **'special'** mixed braids.

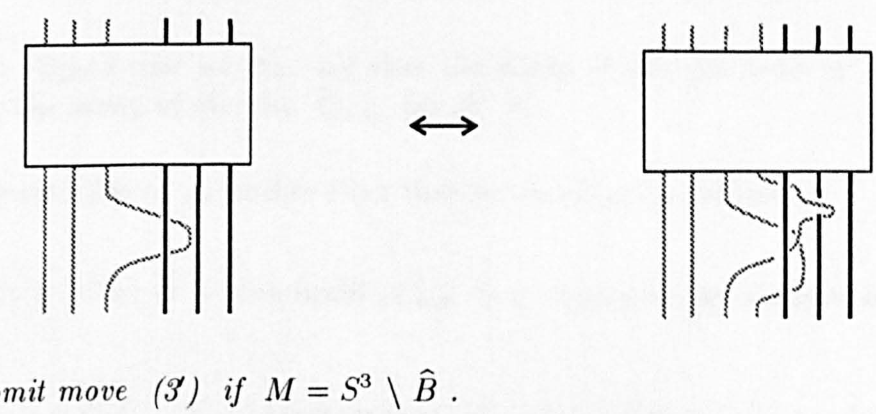
Then, it is easy to see – arguing by induction – that a similar version of Theorem 7 holds for M , namely:

Theorem 8 *Let* $M = \chi(S^3, B)$ *,* L_1 *,* L_2 *be oriented links in* M *and* $B_1 \bigcup B$ *,* $B_2 \bigcup B'$ *be two special mixed braids corresponding to* L_1 *,* L_2 *. Then* L_1 *is isotopic to* L_2 *in* M *if and only if* $B_1 \bigcup B$ *is equivalent to* $B_2 \bigcup B'$ *in* S^3 *, under equivalence generated by the augmented braid relations and the following three moves:*

(1′) Conjugation inside the set of special mixed braids

(2′) Markov moves for the non-surgery strings, that take place at the bottom of the left-hand side of the braid

(3′) Braid band move as depicted below, with all possible choices of crossings:



where we omit move (3′) if $M = S^3 \setminus \widehat{B}$ *.*

We wish next to look for braid group structures inside the set of special mixed braids related to M . So, we first need to check whether the addition of two special mixed braids is a topologically closed operation, and this leads naturally to distinguishing between the following two cases:

Case 1: M is a 3-manifold with (surgery) description given by I_m , the identity braid on m strings. In other words , M is the connected sum of m lens spaces of type $L(p,1)$, or alternatively $M = S^3 \setminus \widehat{I_m}$.

Case 2: M is a 3-manifold with (surgery) description given by a non-identity pure braid , or alternatively $M = S^3 \setminus \widehat{B}$.

Consider first *Case 1* , and let $B_{n,m}$ be the set of all special mixed braids with permutation braid on n strings . $B_{n,m}$ is clearly a group (a subgroup of B_{n+m}) and its elements can be regarded as the elements of B_{n+m} , which fix the last m strings. From this observation and the theorem above follows that in this case *our study of isotopic links in M via braids, can be restricted to the study of the braid groups $B_{n,m}$ for all n* , and the moves $(1')$, $(2')$ and $(3')$ of Theorem 8 can be written equivalently as:

(1′) Conjugation in $B_{n,m}$ for every n .

(2′) Markov moves in $\bigcup_{n=1}^{\infty} B_{n,m}$ for the non-surgery strings, that take place at the bottom of the left-hand side of the braid.

(3′) Braid band move as in Theorem 8 .

Note that for an algebraic expression of the moves $(2')$ and $(3')$ we need first a group presentation for $B_{n,m}$, and we give one in the next section.

We consider now *Case 2* : Let $\bigcup_{n=1}^{\infty} C_{n,m}$ be the set of all special mixed braids related to M . $C_{n,m}$ is a subset of B_{n+m} , and an element of $C_{n,m}$ has the surgery braid B , or a conjugate of B , underlying. So, if we add any two elements of $C_{n,m}$, both having B say underlying, we see that the surgery braid B changes to B^2 . So $C_{n,m}$ is *not* closed under addition and therefore it cannot form a group.

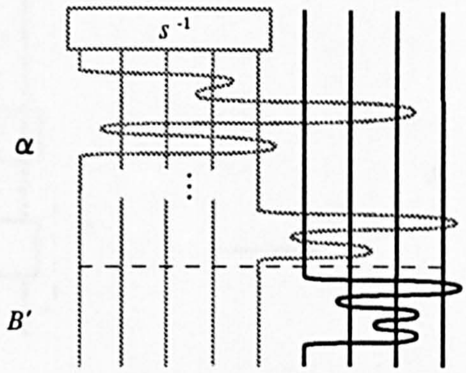
Conclusion Up to now we may say that the study of isotopic links in M can be restricted to the study of the sets $C_{n,m}$ for all n .

But we would like to go further than that by showing the folllowing:

Proposition 1 For B a pure braid , $C_{n,m}$ is a disjoint union of cosets of $B_{n,m}$ in B_{n+m} .

Proof Let $A \in C_{n,m}$. We shall show that A can be written as a product $\alpha \cdot B'$, where $\alpha \in B_{n,m}$ and B' a proper embedding of a conjugate of B in B_{n+m} (as illustrated below). Indeed, a way to see this, is by using Artin's canonical form (presented in 2.6.1) as follows:
The last m strings of A are purely braided and so is the 'mixed' braiding among the first n (thin) and the last m (thick) strings. The first n strings of A , though, do not necessarily braid purely. So, we add on top of A a standard braid s such that sA is a pure braid, and on top of s we add s^{-1} . Then we apply Artin's canonical form for the pure braid sA . This results what we wanted:

Picture: *(with crossings omitted)*



□

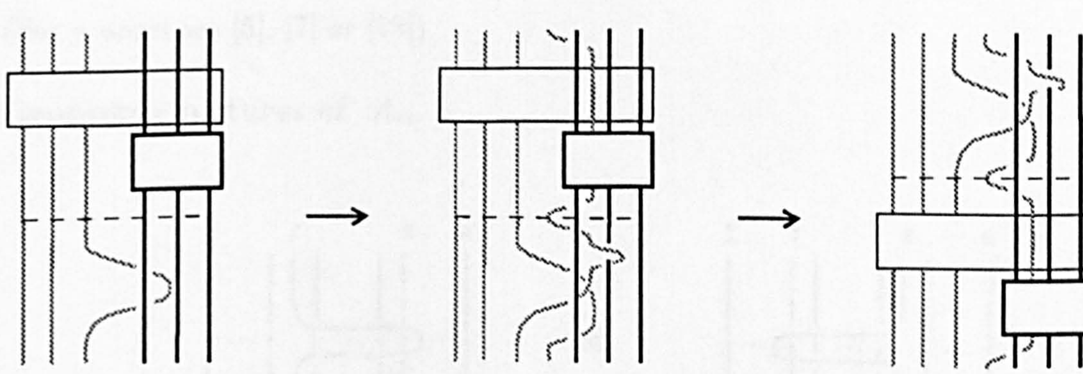
It remains to show that conjugation, the Markov moves and the braid band moves respect the coset structures of $\bigcup_{n=1}^{\infty} C_{n,m}$, the set of all special mixed braids related to M .

Indeed:

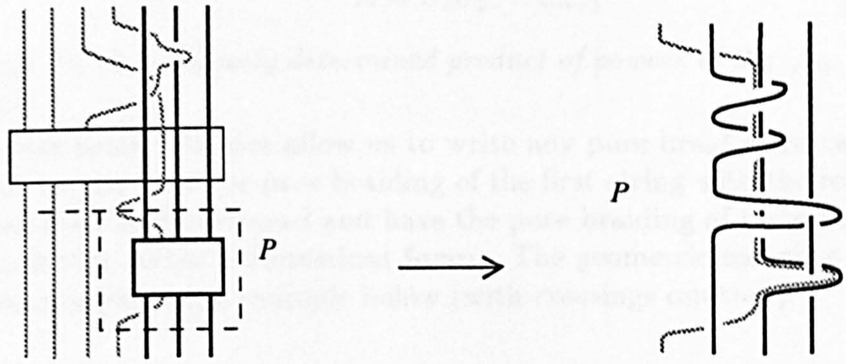
Conjugation: Let $\alpha \cdot B \sim \beta\alpha B\beta^{-1}$ in $C_{n,m}$. From Theorem 8 , it suffices to assume that β is ${\sigma_i}^{\pm 1}$ for $i = 1, \ldots, n-1$ or $\beta = {\sigma_n}^2$ (the square of a mixed crossing); otherwise the last m strings would not remain fixed. Now , β^{-1} commutes with B and so $\beta\alpha B\beta^{-1} = \beta\alpha\beta^{-1} \cdot B \in B_{n,m} \cdot B$. For the case where $\beta = {\sigma_n}^2$, we refer to the last part of the proof of the band moves.

Markov moves: Let $\alpha \cdot B \sim \tilde{\alpha}B{\sigma_1}^{\pm 1}$ in $C_{n,m}$, where $\tilde{\alpha}$ is same as α but with all indices shifted by $+1$. Then ${\sigma_1}^{\pm 1}$ commutes with B and we have $\tilde{\alpha}B{\sigma_1}^{\pm 1} = \tilde{\alpha}{\sigma_1}^{\pm 1} \cdot B \in B_{n+1,m} \cdot B$.

Band moves: (By following the pictures below)



First, by conjugation in $C_{n,m}$ we take the banding to the top. The problem to be solved is how to separate the new string in the surgery part from the surgery braid: For this we conjugate locally (using braid relations), so as to create at the right bottom corner a pure braid P on $(1+m)$ strings (see picture below).



If we apply now to P Artin's canonical form (see 2.6.1), the braiding of the new string with the surgery strings will be separated from the surgery braid, leaving the surgery braid at the bottom-right part of the mixed braid.

2.6 A presentation for $B_{n,1}$

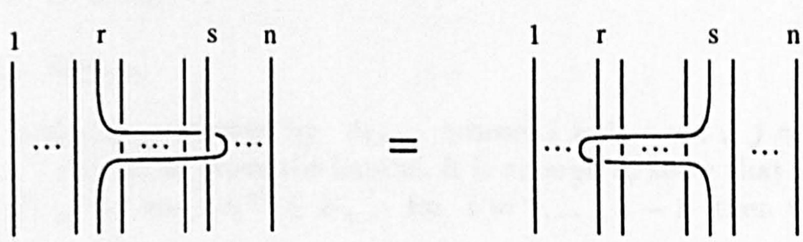
2.6.1 Pure braid groups

In the introduction of Chapter 1 we defined the pure braid group on n strings P_n, as a normal subgroup of B_n . P_n can be given a presentation with generators $A_{rs} = \sigma_r{}^{-1}\sigma_{r+1}{}^{-1}\ldots\sigma_{s-2}{}^{-1}\sigma_{s-1}{}^{2}\sigma_{s-2}\ldots\sigma_{r+1}\sigma_r = \sigma_{s-1}\sigma_{s-2}\ldots\sigma_{r+1}\sigma_r{}^{2}\sigma_{r+1}{}^{-1}\ldots\sigma_{s-2}{}^{-1}\sigma_{s-1}{}^{-1}$, $1 \le r < s \le n$ (pictured geometrically below) and relations:

$$A_{rs}{}^{-1}A_{ij}A_{rs} = \begin{cases} A_{ij} & r<s<i<j \text{ or } \quad i<r<s<j \\ A_{rj}A_{ij}A_{rj}{}^{-1} & r<s=i<j \\ (A_{rj}A_{sj})A_{ij}(A_{rj}A_{sj})^{-1} & r=i<s<j \\ (A_{rj}A_{sj}A_{rj}{}^{-1}A_{sj}{}^{-1})A_{ij}(A_{rj}A_{sj}A_{rj}{}^{-1}A_{sj}{}^{-1})^{-1} & r<i<s<j \end{cases}$$

(For a proof see [5], [7] or [19])

Geometric pictures of A_{rs}



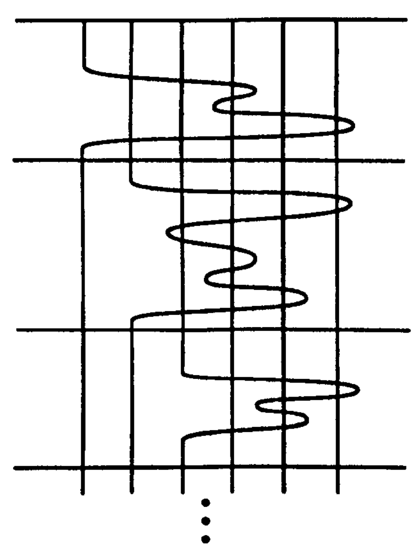
The following theorem shall be used very often (see [5], [7]):

Theorem 9 (Artin) *Every element, A, of P_n can be written uniquely in the form:*

$$A = U_1 U_2 \dots U_{n-1}$$

where each U_i is a uniquely determined product of powers of the A_{ij} using only those with $i < j$.
I.e. the pure braid relations allow us to write any pure braid word canonically, in the sense that we can have the pure braiding of the first string with the rest, then keep the first string fixed and uncrossed and have the pure braiding of the second string and so on. We call this **Artin's canonical form**. The geometric meaning of this canonical form is illustrated in the example below (with crossings omitted):

Picture:



2.6.2 A presentation for $B_{n,1}$ using the T_i''s

We shall next show how to find a presentation for $B_{n,1}$ using braid as well as pure braid generators.

Recall that $B_{n,m}$ is the subgroup of B_{n+m} that **fixes** the last m strings (i.e. if we remove the first n strings we are left with the identity braid on m strings). We write $P_{n,m}$ for the subgroup of the pure braid group of $B_{n,m}$ generated by $A_{i,n+j}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Notice that $P_{n,m}$ does not contain pure braiding among the first n strings.

Lemma 4 $P_{n,m} \lhd B_{n,m}$

Proof $B_{n,m}$ is clearly generated by $A_{i,n+j}$ where $i = 1, \dots, n$, $j = 1, \dots, m$ and by $\sigma_1, \dots, \sigma_{n-1}$. Hence, to prove the lemma, it is enough to show that if we conjugate any $A_{r,n+s} \in P_{n,m}$ by any $\sigma_i^{\pm 1} \in B_{n,m}$ for $i = 1, \dots, n-1$ then the result is an element of $P_{n,m}$.
Indeed:

$\sigma_i{}^{-1}A_{r,n+s}\sigma_i = A_{r,n+s} \in P_{n,m}$ and $\sigma_i A_{r,n+s}\sigma_i{}^{-1} = A_{r,n+s} \in P_{n,m}$ if $r > i+1$ or $r < i$

$\sigma_i{}^{-1}A_{i,n+s}\sigma_i = A_{i,n+s}A_{i+1,n+s}A_{i,n+s}{}^{-1} \in P_{n,m}$ and $\sigma_i A_{i,n+s}\sigma_i{}^{-1} = A_{i+1,n+s} \in P_{n,m}$

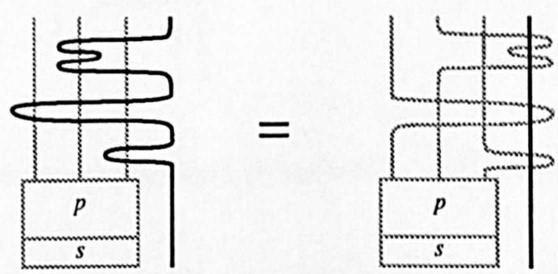
$\sigma_i{}^{-1}A_{i+1,n+s}\sigma_i = A_{i,n+s} \in P_{n,m}$ and $\sigma_i A_{i+1,n+s}\sigma_i{}^{-1} = A_{i+1,n+s}{}^{-1}A_{i,n+s}A_{i+1,n+s} \in$

$P_{n,m}$. □

Proposition 2 $B_{n,1} = P_{n,1} \rtimes B_n$.

Proof From previous lemma $P_{n,1} \lhd B_{n,1}$. Moreover, $P_{n,1} \cap B_n = 1$. Also, every element in $B_{n,1}$ can be written as a product $\alpha \cdot \beta$, where $\alpha \in P_{n,1}$ and $\beta \in B_n$. This follows from Artin's canonical form for P_{n+1}, applied starting from the $(n+1)$st string rather than the first (see picture below for $n = 3$) :

Picture: *(also with crossings omitted)*



where $p \in P_n$, s a standard permutation braid and $\beta = ps$. □

Remark 8 With very similar arguments we can prove in general that

$$B_{n,m} = P_{n,m} \rtimes B_n \quad .$$

The only extra thing we need, is a presentation for $P_{n,1}$: The generators of $P_{n,1}$ are $A_{1,n+1}, \ldots, A_{n,n+1}$. I.e. an element of $P_{n,1}$ is a pure braid on $n+1$ strings such that the last string is doing pure braiding with the others, but all the rest remain fixed. Hence, as it follows immediately from the uniqueness of Artin's canonical form, **$P_{n,1}$ is a free group.**

We can now apply a result from the theory of group presentations (see [27], p.140), that gives a presentation for a group, which can be written as the semidirect product of two subgroups with known presentations.

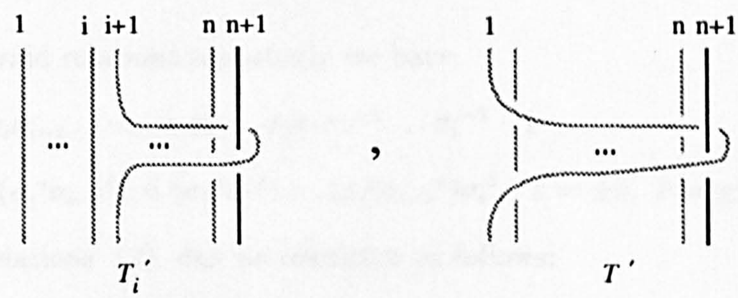
Setting $T' = T'_0 := A_{1,n+1}, T'_1 := A_{2,n+1}, \ldots, T'_{n-1} := A_{n,n+1}$ (pictured below for convenience), we obtain the following presentation[4] for $B_{n,1}$:

$$\left\langle \begin{array}{l} \sigma_1,\dots,\sigma_{n-1} \\ T'_0,T'_1,\dots,T'_{n-1} \end{array} \;\middle|\; \begin{array}{ll} \left.\begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \end{array}\right\} & \textit{braid relations} \\ \left.\begin{array}{l} (1)\ \ \sigma_i^{-1}T'_{\lambda-1}\sigma_i = T'_{\lambda-1} \quad \text{if } \lambda > i+1 \text{ or } \lambda < i \\ (2)\ \ \sigma_i^{-1}T'_{i-1}\sigma_i = T'_{i-1}T'_iT'_{i-1}{}^{-1}, \ i=1,\dots,n-1 \\ (3)\ \ \sigma_i^{-1}T'_i\sigma_i = T'_{i-1} \quad \text{for } i=1,\dots,n-1 \end{array}\right\} & \textit{'mixed' relations} \end{array} \right.$$

The last set of the 'mixed' relations imply that we do not need all T'_i 's for a presentation of $B_{n,1}$. We only need $T',\sigma_1,\dots,\sigma_{n-1}$ as a set of generators. We use relations (3) to obtain the following relations defined by:

$$T'_i = \sigma_i\dots\sigma_1T'\sigma_1{}^{-1}\dots\sigma_i{}^{-1} \quad \diamondsuit$$

Picture:



Then we substitute the above expressions for T'_i in the first two sets of the 'mixed' relations.

Note In some proofs below, as well as in Chapter 3, we underline the generators that participate in each step of the proof.

From set (1) of the 'mixed' relations we only keep the relations:

$$\sigma_i{}^{-1}T'\sigma_i = T', \ i>1 \quad (I)$$

and we shall show that the remaining relations of set (1) follow from (I) , $\diamondsuit$ and the braid relations (b.r.).

Indeed:
If $\lambda > i+1$ we have $\sigma_i{}^{-1}T'_{\lambda-1}\sigma_i = \underline{\sigma_i}{}^{-1}\sigma_{\lambda-1}\dots\sigma_1T'\sigma_1{}^{-1}\dots\sigma_{\lambda-1}{}^{-1}\underline{\sigma_i} \overset{(b.r.)}{=}$

$$\sigma_{\lambda-1}\dots\sigma_{i+2}\underline{\sigma_i{}^{-1}\sigma_{i+1}\sigma_i}\dots\sigma_1T'\sigma_1{}^{-1}\dots\underline{\sigma_i{}^{-1}\sigma_{i+1}{}^{-1}\sigma_i}\sigma_{i+2}{}^{-1}\dots\sigma_{\lambda-1}{}^{-1} \overset{(b.r.)}{=}$$

$$\sigma_{\lambda-1}\dots\sigma_{i+2}\sigma_{i+1}\sigma_i\underline{\sigma_{i+1}}{}^{-1}\dots\sigma_1T'\sigma_1{}^{-1}\dots\underline{\sigma_{i+1}}\sigma_i{}^{-1}\sigma_{i+1}{}^{-1}\sigma_{i+2}{}^{-1}\dots\sigma_{\lambda-1}{}^{-1} \overset{(b.r.)}{=}$$

$$\sigma_{\lambda-1}\dots\sigma_1\underline{\sigma_{i+1}{}^{-1}T'\sigma_{i+1}}\sigma_1{}^{-1}\dots\sigma_{\lambda-1}{}^{-1} \overset{(I)}{=}$$

---

[4]This presentation appears also in [19], App. 1; it is proved by L. Gaede, to aid in finding a presentation for the pure braid group. Also, as J. Birman mentions in [7], page 22, W.L. Chow in [11] found and used the above presentation for the same purpose.

$\sigma_{\lambda-1}\ldots\sigma_1 T'\sigma_1^{-1}\ldots\sigma_{\lambda-1}^{-1} \stackrel{\diamondsuit}{=} T'_{\lambda-1}$.

If $\lambda < i$ we have $\sigma_i^{-1}T'_{\lambda-1}\sigma_i = \underline{\sigma_i}^{-1}\sigma_{\lambda-1}\ldots\sigma_1 T'\sigma_1^{-1}\ldots\sigma_{\lambda-1}^{-1}\underline{\sigma_i} \stackrel{(b.r.)}{=}$

$\sigma_{\lambda-1}\ldots\sigma_1\sigma_i^{-1}T'\sigma_i\sigma_1^{-1}\ldots\sigma_{\lambda-1}^{-1} \stackrel{(I),\diamondsuit}{=} T'_{\lambda-1}$.

From set (2) we only keep the relation:

$$\sigma_1^{-1}T'\sigma_1 = T'T_1'T'^{-1} \Longleftrightarrow \sigma_1^{-1}T'\sigma_1 = T'\sigma_1 T'\sigma_1^{-1}T'^{-1} \Longleftrightarrow$$

$$T'\sigma_1 T'\sigma_1 = \sigma_1 T'\sigma_1 T' \quad (II)$$

We shall next show that the remaining relations of set (2) follow from (II) , $\diamondsuit$ and the braid relations.

Note Using the braid relations repeatedly we have:

(i)$\sigma_1^{-1}\ldots\sigma_{i-1}^{-1}\sigma_i\sigma_{i-1}\ldots\sigma_1 = \sigma_i\ldots\sigma_2\sigma_1\sigma_2^{-1}\ldots\sigma_i^{-1}$ †

(ii)$\sigma_1^{\delta}(\sigma_2^{\varepsilon}\sigma_1^{\varepsilon})\ldots(\sigma_i^{\varepsilon}\sigma_{i-1}^{\varepsilon}) = (\sigma_2^{\varepsilon}\sigma_1^{\varepsilon})\ldots(\sigma_i^{\varepsilon}\sigma_{i-1}^{\varepsilon})\sigma_i^{\delta}$ $\varepsilon = \pm 1,\ \delta = \pm 1$ ‡

First notice that relations (2) can be rewritten as follows:

$$T'_{i-1}\sigma_i T'_{i-1} = \sigma_i T'_{i-1}T'_i \Longleftrightarrow T'_{i-1}\sigma_i T'_{i-1} = \sigma_i T'_{i-1}\sigma_i T'_{i-1}\sigma_i^{-1} \Longleftrightarrow$$

$$T'_{i-1}\sigma_i T'_{i-1}\sigma_i = \sigma_i T'_{i-1}\sigma_i T'_{i-1} \qquad i = 1,\ldots,n-1$$

Now,

$$T'_{i-1}\sigma_i T'_{i-1}\sigma_i = \sigma_{i-1}\ldots\sigma_1 T'\underline{\sigma_1^{-1}\ldots\sigma_{i-1}^{-1}\sigma_i\sigma_{i-1}\ldots\sigma_1} T'\sigma_1^{-1}\ldots\sigma_{i-1}^{-1}\sigma_i \stackrel{\dagger}{=}$$

$$\sigma_{i-1}\ldots\sigma_1 T'\underline{\sigma_i}\ldots\underline{\sigma_2}\sigma_1\underline{\sigma_2}^{-1}\ldots\underline{\sigma_i}^{-1}T'\sigma_1^{-1}\ldots\sigma_{i-1}^{-1}\sigma_i \stackrel{(b.r)}{=}$$

$$(\sigma_{i-1}\sigma_i)\ldots(\sigma_1\sigma_2)T'\sigma_1 T'(\sigma_2^{-1}\sigma_1^{-1})\ldots(\sigma_i^{-1}\sigma_{i-1}^{-1})\sigma_i =$$

$$(\sigma_{i-1}\sigma_i)\ldots(\sigma_1\sigma_2)T'\sigma_1 T'\underline{(\sigma_2^{-1}\sigma_1^{-1})\ldots(\sigma_i^{-1}\sigma_{i-1}^{-1})\sigma_i} \stackrel{\ddagger}{=}$$

$$(\sigma_{i-1}\sigma_i)\ldots(\sigma_1\sigma_2)\underline{T'\sigma_1 T'\sigma_1}(\sigma_2^{-1}\sigma_1^{-1})\ldots(\sigma_i^{-1}\sigma_{i-1}^{-1}) \stackrel{(II)}{=}$$

$$\underline{(\sigma_{i-1}\sigma_i)\ldots(\sigma_1\sigma_2)\sigma_1}T'\sigma_1 T'(\sigma_2^{-1}\sigma_1^{-1})\ldots(\sigma_i^{-1}\sigma_{i-1}^{-1}) \stackrel{\ddagger}{=}$$

$$\sigma_i(\sigma_{i-1}\underline{\sigma_i})\ldots(\sigma_1\underline{\sigma_2})T'\sigma_1 T'(\underline{\sigma_2}^{-1}\sigma_1^{-1})\ldots(\underline{\sigma_i}^{-1}\sigma_{i-1}^{-1}) \stackrel{(b.r.)}{=}$$

$$\sigma_i\sigma_{i-1}\ldots\sigma_1 T'\underline{\sigma_i\ldots\sigma_2\sigma_1\sigma_2^{-1}\ldots\sigma_i^{-1}}T'\sigma_1^{-1}\ldots\sigma_{i-1}^{-1} \stackrel{\dagger,\diamondsuit}{=}$$

$$\sigma_i T'_{i-1}\sigma_i T'_{i-1}$$

□

So, after Tietze transformations[5], we obtain the following presentation for $B_{n,1}$:

$$B_{n,1} = \left\langle T', \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j|>1 \\ T'\sigma_i = \sigma_i T' & \text{for } i>1 \\ T'\sigma_1 T'\sigma_1 = \sigma_1 T'\sigma_1 T' & \end{array} \right\rangle \quad \spadesuit\spadesuit$$

Note Using Lemma 4, Proposition 2 and similar but more sophisticated arguments, Alastair Leeves found a presentation for $B_{n,m}$, which we give below:

$$\left\langle \begin{array}{l} \sigma_1, \ldots, \sigma_{n-1} \\ A_{n,n+1}, \ldots, A_{n,n+m} \end{array} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j|>1 \\ {\sigma_{n-1}}^{-1}A_{nj}\sigma_{n-1}A_{nl} = A_{nl}{\sigma_{n-1}}^{-1}A_{nj}\sigma_{n-1} \\ \text{for } n<l<j<n+m+1 \\ A_{nj}\sigma_i = \sigma_i A_{nj} \quad \text{for } 0<i<n-1 \\ \sigma_{n-1}A_{nj}\sigma_{n-1}A_{nj} = A_{nj}\sigma_{n-1}A_{nj}\sigma_{n-1} \\ \text{for } n<j<n+m+1 \end{array} \right\rangle$$

where A_{ij} as given in 2.6.1 .

Remark 9 Having the above presentation for $B_{n,m}$, we can express the Markov moves of the generalized Markov's theorem as follows:

$$\alpha \sim \tilde{\alpha}{\sigma_1}^{\pm 1} \ , \ \alpha \in B_{n,m}$$

where $\tilde{\alpha}$ is same as α but with all indices shifted by $+1$.

Although, the band move cannot be expressed algebraically in an easy way using the T_i' 's as pure braid generators, even for the simpler case of $B_{n,1}$ related to $L(p,1)$, as we demonstrate below:

If $T_i' \in B_{n-1,1}$ and α_1 , $\alpha_2 \in B_{n-1,1}$ are both words in the σ_i's , then

$$\alpha_1 T_i'\alpha_2 (\in B_{n-1,1}) \sim \alpha_1({\sigma_{i+1}}^{-1}\ldots{\sigma_{n-2}}^{-1}{\sigma_{n-1}}^{2}\sigma_{n-2}\ldots\sigma_{i+1})\, T_i'\alpha_2\, (T_{n-1}')^p{\sigma_{n-1}}^{\pm 1} \in B_{n,1}$$

and

$$\alpha_1(T_i')^{-1}\alpha_2 \sim \alpha_1(T_i')^{-1}({\sigma_{i+1}}^{-1}\ldots{\sigma_{n-2}}^{-1}{\sigma_{n-1}}^{-2}\sigma_{n-2}\ldots\sigma_{i+1})\,\alpha_2\,(T_{n-1}')^p{\sigma_{n-1}}^{\pm 1} \in B_{n,1}$$

Using the above and Theorem 7 , we have the following:

Corollary 5 *(algebraic version of Theorem 6)* *Let $M = S^3 \setminus \hat{I}$ be a solid torus with $\hat{I}$ also a solid torus; let L_1 , L_2 be two oriented links in M and $B_1 \bigcup I$, $B_2 \bigcup I$ be mixed braids in $\bigcup_{n=1}^{\infty} B_{n,1}$ corresponding to L_1 , L_2 . Then L_1 is isotopic to L_2 in M if and only if $B_1 \bigcup I$ is equivalent to $B_2 \bigcup I$ in $\bigcup_{n=1}^{\infty} B_{n,1}$, under equivalence generated by the augmented braid relations together with the following two moves:*

(1′) Conjugation in $B_{n,1}$

(2′) $\alpha \sim \tilde{\alpha}{\sigma_1}^{\pm 1}$, where $\alpha \in B_{n,1}$ and $\tilde{\alpha}$ is same word as α but with all indices shifted by $+1$.

---

[5]See [47], Chapter I part 1.5 or [46], Proposition 2.1 .

2.7 Another presentation for $B_{n,1}$

To derive a simple algebraic expression for the band move in the generalized Markov's theorem for $M = L(p,1)$, we would like to find a presentation for $B_{n,1}$ using the pure braid generators

$$T_i = \sigma_i \dots \sigma_1 T \sigma_1 \dots \sigma_i \quad \text{and} \quad T \quad \text{as pictured below:}$$

Note that $T_{n-1}{}^{-1} = T'_{n-1}$.

Remark 10 We did not consider using T_i 's to find a presentation for $B_{n,1}$ at the beginning, because in this case we do not have a normal, free subgroup of $B_{n,1}$ to work with.

The T_i 's are related to the T'_i 's in the following way:

$T_i = \sigma_{i+1}{}^{-1} \dots \sigma_{n-1}{}^{-1} T_{n-1} \sigma_{n-1}{}^{-1} \dots \sigma_{i+1}{}^{-1} =$

$\sigma_{i+1}{}^{-1} \dots \sigma_{n-1}{}^{-1} T'_{n-1}{}^{-1} \sigma_{n-1} \dots \sigma_{i+1} (\sigma_{i+1}{}^{-1} \dots \sigma_{n-2}{}^{-1} \sigma_{n-1}{}^{-2} \sigma_{n-2}{}^{-1} \dots \sigma_{i+1}{}^{-1}) \Longleftrightarrow$

$T_i = T'^{-1}_i (\sigma_{i+1}{}^{-1} \dots \sigma_{n-2}{}^{-1} \sigma_{n-1}{}^{-2} \sigma_{n-2}{}^{-1} \dots \sigma_{i+1}{}^{-1})$

We now substitute

$$(\sigma_{i+1}{}^{-1} \dots \sigma_{n-2}{}^{-1} \sigma_{n-1}{}^{-2} \sigma_{n-2}{}^{-1} \dots \sigma_{i+1}{}^{-1}) T_i^{-1} \quad \text{for} \quad T'_i$$

in each of the 'mixed' relations of the first presentation for $B_{n,1}$ to obtain the following for $i = 1, \dots, n-1$:

1. $T_{\lambda-1}\sigma_i = \sigma_i T_{\lambda-1} \quad \lambda > i+1, \ \lambda < i$
2. $T_{i-1}T_i = T_i T_{i-1}$
3. $T_i = \sigma_i T_{i-1} \sigma_i$

Proof of 3. $\sigma_i{}^{-1} T'_i \sigma_i = T'_{i-1} \Longleftrightarrow$

$\sigma_i{}^{-1}(\sigma_{i+1}{}^{-1} \dots \sigma_{n-1}{}^{-2} \dots \sigma_{i+1}{}^{-1}) T_i^{-1} \sigma_i = (\sigma_i{}^{-1} \sigma_{i+1}{}^{-1} \dots \sigma_{n-1}{}^{-2} \dots \sigma_{i+1}{}^{-1} \sigma_i{}^{-1} T_{i-1}{}^{-1}) \Longleftrightarrow$

$T_i{}^{-1}\sigma_i = \sigma_i{}^{-1} T_{i-1}{}^{-1} \Longleftrightarrow T_i = \sigma_i T_{i-1} \sigma_i$ $\square$

Proof of 1. First we need the following

Note Using the braid relations repeatedly we have:

$$(\sigma_j \ldots \sigma_{n-1}\sigma_n{}^2\sigma_{n-1}\ldots\sigma_j)\sigma_i = \sigma_i(\sigma_j\ldots\sigma_{n-1}\sigma_n{}^2\sigma_{n-1}\ldots\sigma_j) \text{ for } 1 \le j < i \text{ or } j > i+1 \quad \ddagger\ddagger$$

[for example if $j = 1$ and $i = 3$ we have : $(\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_1)\sigma_3 = \sigma_1\sigma_2\sigma_3\underline{\sigma_3\sigma_2\sigma_3}\sigma_1 =$

$\sigma_1\underline{\sigma_2\sigma_3\sigma_2}\sigma_3\sigma_2\sigma_1 = \sigma_1\sigma_3\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1 = \sigma_3(\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_1)$.]

Now : $\sigma_i{}^{-1}T'_{\lambda-1}\sigma_i = T'_{\lambda-1}$ $\lambda > i+1$ or $\lambda < i$ $\iff$

$\underline{\sigma_i}{}^{-1}(\sigma_\lambda{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_\lambda{}^{-1})T_{\lambda-1}{}^{-1}\sigma_i = (\sigma_\lambda{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_\lambda{}^{-1})T_{\lambda-1}{}^{-1}$
$\lambda > i+1,\ \lambda < i \quad \overset{\ddagger\ddagger}{\iff}$

$(\sigma_\lambda{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_\lambda{}^{-1})\sigma_i{}^{-1}T_{\lambda-1}{}^{-1}\sigma_i = (\sigma_\lambda{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_\lambda{}^{-1})T_{\lambda-1}{}^{-1}$
$\lambda > i+1,\ \lambda < i \quad \iff$

$\sigma_i{}^{-1}T_{\lambda-1}{}^{-1}\sigma_i = T_{\lambda-1}{}^{-1}$ $\lambda > i+1,\ \lambda < i$ $\iff$ $T_{\lambda-1}\sigma_i = \sigma_i T_{\lambda-1}$ $\lambda > i+1,\ \lambda < i$ □

Proof of 2. $\sigma_i{}^{-1}T'_{i-1}\sigma_i = T'_{i-1}T'_iT'_{i-1}{}^{-1}$ $\iff$

$\sigma_i{}^{-1}(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})T_{i-1}{}^{-1}\sigma_i = (\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})T_{i-1}{}^{-1}$

$(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})T_i{}^{-1}\underline{[(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})T_{i-1}{}^{-1}]}^{-1}$ $\iff$

$\sigma_i{}^{-1}(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})\underline{T_{i-1}}{}^{-1}\underline{\sigma_i}(\underline{\sigma_i}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1}\sigma_i{}^{-1})T_{i-1}{}^{-1} =$

$(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})\underline{T_{i-1}}{}^{-1}(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})T_i{}^{-1}$ $\overset{1.}{\iff}$

$\sigma_i{}^{-1}(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})\underline{(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})}T_{i-1}{}^{-1}\sigma_i{}^{-1}T_{i-1}{}^{-1} =$

$(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})T_{i-1}{}^{-1}T_i{}^{-1}$ $\overset{\ddagger\ddagger}{\iff}$

$\underline{\sigma_i{}^{-1}(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}}{}^{-1}\sigma_i{}^{-1})T_{i-1}{}^{-1}\sigma_i{}^{-1}T_{i-1}{}^{-1} =$

$\underline{(\sigma_i{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_i{}^{-1})(\sigma_{i+1}{}^{-1}\ldots\sigma_{n-1}{}^{-2}\ldots\sigma_{i+1}{}^{-1})}T_{i-1}{}^{-1}T_i{}^{-1}$ $\iff$

$\underline{\sigma_i{}^{-1}T_{i-1}{}^{-1}\sigma_i}{}^{-1}T_{i-1}{}^{-1} = T_{i-1}{}^{-1}T_i{}^{-1}$ $\overset{3.}{\iff}$

$T_{i-1}T_i = T_iT_{i-1}$ □

So, after Tietze transformations, we obtain the following presentation for $B_{n,1}$:

$$\left\langle \begin{array}{c} \sigma_1,\ldots,\sigma_{n-1} \\ T,T_1,\ldots,T_{n-1} \end{array} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \\ 1.\ T_{\lambda-1}\sigma_i = \sigma_i T_{\lambda-1}\ \ \lambda > i+1,\ \lambda < i \\ 2.\ T_{i-1}T_i = T_iT_{i-1} \text{ for } i=1,\ldots,n-1 \\ 3.\ T_i = \sigma_i T_{i-1}\sigma_i \text{ for } i=1,\ldots,n-1 \end{array} \right\rangle \clubsuit\clubsuit$$

If we eliminate $T_1,\ldots,T_{n-1}$ as before using Tietze transformations, we derive the same presentation ($\spadesuit\spadesuit$) for $B_{n,1}$ namely:

$$B_{n,1} = \left\langle T,\sigma_1,\sigma_2,\ldots,\sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \\ T\sigma_i = \sigma_i T \text{ for } i>1 \\ T\sigma_1T\sigma_1 = \sigma_1T\sigma_1T \end{array} \right\rangle \spadesuit\spadesuit$$

We can now write the algebraic expression of the generalized Markov's theorem for any $M = L(p,1)$:

Theorem 10 *Let $M = L(p,1) = \chi(S^3, I)$, L_1 , L_2 be oriented links in M and $B_1 \bigcup I$, $B_2 \bigcup I$ be mixed braids in $\bigcup_{n=1}^{\infty} B_{n,1}$ corresponding to L_1 , L_2 . Then L_1 is isotopic to L_2 in M if and only if $B_1 \bigcup I$ is equivalent to $B_2 \bigcup I$ in $\bigcup_{n=1}^{\infty} B_{n,1}$, under equivalence generated by the braid group relations, together with the following three moves:*

(1′) Conjugation in $B_{n,1}$

(2′) $\alpha \sim {\sigma_1}^{\pm 1}\tilde{\alpha}$, where $\alpha \in B_{n,1}$ and $\tilde{\alpha}$ is same word as α but with all indices shifted by $+1$.

(3′) $\alpha \sim \alpha {T_n}^p {\sigma_n}^{\pm 1}$, where $\alpha \in B_{n,1}$ is a word in the generators $T,\sigma_1,\ldots,\sigma_{n-1}$, and $p \in \mathbb{Z}$ the framing number of I .

Note The above theorem appears also in [44] (recall the relevant comment in 2.4.2.). The only essential difference is that X-S. Lin places the surgery string at the beginning of the mixed braid; therefore, the band moves take place at the left-hand side of the braid, whilst the Markov moves take place at the right-hand side of the mixed braid.

Chapter 3

On trace invariants

3.1 Introduction

The first written announcement of the Jones polynomial was made in May 1984 in [28], whilst the first published announcement appeared in 1985 in [29]. This was immediately generalized from different viewpoints by several authors independently, and it is now known as the 2-variable Jones or the FLYPMOTH or the HOMFLY-PT polynomial (name derived from the initials of the authors; for details see [16], [50], [40]). In [30] V. Jones reconstructs the HOMFLY-PT polynomial after Ocneanu as follows:

He maps the infinite dimensional braid group algebra over $\mathbb{C}$, $\mathbb{C}B_n$, onto $\mathcal{H}_n(q)$, the Hecke algebra of A_n-type, via the natural epimorphism : $\sigma_i \mapsto g_i$, where $g_1, \ldots, g_{n-1}$ are the generators of $\mathcal{H}_n(q)$. Then he uses Ocneanu's trace function theorem ([16]), which guarantees the existence and uniqueness of a linear function

$$\mathrm{tr} : \bigcup_{n=1}^{\infty} \mathcal{H}_n(q) \longrightarrow \mathbb{C}$$

such that

1) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for $a, b \in \mathcal{H}_n(q)$
2) $\mathrm{tr}(1) = 1$ for every $\mathcal{H}_n(q)$
3) $\mathrm{tr}(ag_n) = z\,\mathrm{tr}(a)$ for $a \in \mathcal{H}_n(q)$, $g_n \in \mathcal{H}_{n+1}(q)$ and $z \in \mathbb{C}$.

(The existence of the trace function relies upon the fact that there exists a canonical basis for $\mathcal{H}_{n+1}(q)$, such that the higher index generator g_n appears at most once in any word in $\mathcal{H}_{n+1}(q)$.)

V. Jones noticed that rules 1) and 3) resemble Markov's theorem for isotopy equivalence of braids in S^3 ; so, by normalizing the trace properly (so that the braids α , $\alpha\sigma_n$ and $\alpha{\sigma_n}^{-1}$ would be assigned the same Laurent polynomial), he obtained the 2-variable (with variables q and z) polynomial link invariant. In the same paper it is mentioned that there should be analogues of Ocneanu's trace for Hecke algebras other than those of type A_n .

In this chapter, we observe first that $B_{n,1}$ – the braid group related to the spaces $L(p,1)$ (and to the unknotted solid torus) – is, in fact, the Artin group of W_n, the Coxeter group of B_n-type. This readily implies that the natural epimorphism of $B_{n,1}$ onto W_n extends to an epimorphism of $\mathbb{C}B_{n,1}$ onto $\mathbb{C}W_n$, which is isomorphic to $\mathcal{H}_n(q,Q)$, the Hecke algebra of B_n-type. The above now suggest that, if we follow the same approach as in [30], we might be able to find isotopy invariants of links in a solid torus or in the lens spaces of type $L(p,1)$.

X-S. Lin made the same observation in [44], following a different approach. In the same paper he also asks whether we can obtain link invariants in $L(p,1)$ using this fact and some Ocneanu-type trace. Here we attempt to partially answer this question. As a first step, we state and prove the existence and uniqueness of a linear trace, analogous to Ocneanu's trace, from $\bigcup_{n=1}^{\infty}\mathcal{H}_n(q,Q)$ to $\mathbb{C}$, such that

1) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for $a,b \in \mathcal{H}_n(q,Q)$
2) $\mathrm{tr}(1) = 1$ for every $\mathcal{H}_n(q,Q)$
3) $\mathrm{tr}(ag_n) = z\,\mathrm{tr}(a)$ for $a \in \mathcal{H}_n(q,Q)$, $g_n \in \mathcal{H}_{n+1}(q,Q)$ and $z \in \mathbb{C}$
4) $\mathrm{tr}(at'_n) = s\,\mathrm{tr}(a)$ for $a \in \mathcal{H}_n(q,Q)$, $t'_n \in \mathcal{H}_{n+1}(q,Q)$ and $s \in \mathbb{C}$.

In section 3.4 we use this trace to construct an analogue of the HOMFLY-PT polynomial for isotopic oriented links inside a solid torus, and we compare with [22].

Finally in section 3.5 we construct a weak analogue (a homology invariant) of the 2-variable Jones polynomial for braids in the lens space $L(0,1)$, using the above trace function and our generalized Markov's theorem, as a very first attempt to define the generalized Jones polynomial in arbitrary 3-manifolds via the process described above.

3.2 The groups $B_{n,1}$ and W_n

3.2.1 Algebraic definitions et cetera

Definition 16 A group G with a presentation

$$\langle w_1,\dots,w_n \mid (w_iw_j)^{m_{ij}} = 1, \ \text{where } m_{ii} = 1, \ i = 1,\dots,n\rangle$$

is called a **Coxeter group**.

Definition 17 If G is a Coxeter group with a presentation as above, then the corresponding **Artin group** B is given by

$$B = \left\langle \tau_1,\dots,\tau_n \;\middle|\; \begin{array}{l} \tau_i\tau_j\tau_i\dots = \tau_j\tau_i\tau_j\dots \\ \text{where the number of factors in either side is } m_{ij} \end{array} \right\rangle .$$

So, for example, the braid group on n strings, B_n, is the Artin group of the Coxeter group of A_n-type (which is actually the symmetric group S_n), since B_n and S_n have the following presentations:

$$B_n = \langle \sigma_1,\dots,\sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \ \text{for } |i-j|>1, \ \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$$

and

$$S_n = \langle s_1, \dots, s_{n-1} \mid {s_i}^2 = 1, \ \ (s_i s_j)^2 = 1, \text{ for } |i-j| > 1, \ \ (s_i s_{i+1})^3 = 1 \rangle$$

where s_i corresponds to the transposition $(i, i+1)$.

Definition 18 Let F be a field and $q_i \in F$, $i = 1, \dots, n$. Let G be a finite Coxeter group with set of standard generators $S = \{w_1, \dots, w_n\}$. The **Hecke algebra of** G , $\mathcal{H} = \mathcal{H}(G, q_i)$ **with parameters** q_i , is a vector space over F with basis $\{T_w \ : w \in G\}$ and an associative multiplication defined on $\mathcal{H}$ as follows:

$$w \in G, w_i \in S \qquad T_{w_i} \cdot T_w := \begin{cases} T_{w_i w} & \text{if } l(w_i w) = l(w) + 1 \\ q_i T_{w_i w} + (q_i - 1) T_w & \text{if } l(w_i w) = l(w) - 1 \end{cases}$$

where $l : G \to \mathbb{N}$ is the length function on G such that: if $w = w_{i_1} \dots w_{i_k} \in G$, where $w_{i_j} \in S$ and $j = 1, \dots, k$, then $l(w)$ is the smallest such k . We define $l(1) = 0$.

For the dimension, existence and basic properties of arbitrary Hecke algebras corresponding to finite Coxeter groups see [13].

Notes

- If $w \in G$, $w_i \in S$ then $l(w_i w) = l(w) \pm 1$. If $w_i \in S$, then ${w_i}^2 = 1$, hence $l({w_i}^2) = 0 < l(w_i) = 1$ and therefore , ${T_{w_i}}^2 = T_{w_i} \cdot T_{w_i} = q_i \cdot T_{{w_i}^2} + (q_i - 1) \cdot T_{w_i}$, or equivalently ${T_{w_i}}^2 = q_i \cdot T_1 + (q_i - 1) \cdot T_{w_i}$, or ${T_{w_i}}^2 = q_i \cdot 1 + (q_i - 1) \cdot T_{w_i}$, where $T_1 = 1$ is the identity element of the multiplication on $\mathcal{H}$. I.e. ${T_{w_i}}^2 \neq T_{{w_i}^2}$.

- Associativity and the note above imply that we could have defined the product on $\mathcal{H}$ equivalently as follows (see [13] for a detailed exposition):
If $w = w_{i_1} \dots w_{i_l}$ is a **reduced** expression for w , then

$$T_w = T_{w_{i_1}} \cdot T_{w_{i_2}} \dots T_{w_{i_l}} \ , \ w_{i_j} \in S \ , \ j = 1, \dots, l$$

If w_i is a basic **involution** then

$${T_{w_i}}^2 = q_i \cdot 1 + (q_i - 1) \cdot T_{w_i}.$$

For example the Hecke algebra of A_n-type , $\mathcal{H}_n(q)$ has a presentation:

$$\langle g_1, \dots, g_{n-1} \mid g_i g_j = g_j g_i \text{ for } |i-j| > 1, \ \ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \ {g_i}^2 = (q-1) g_i + q \rangle$$

where g_i corresponds to T_{s_i} and $q_i = q$ for $i = 1, \dots, n-1$.

- An important fact about Hecke algebras is that: if $q_i = 1$ or not a root of unity and we choose as field $\mathbb{C}$, then the Hecke algebra is *semisimple* and *isomorphic* to the group algebra $\mathbb{C}G$, where G is any Coxeter group. The isomorphism was proved by Tits in [8], Ex. 27, p. 56 , as cited in [45].

3.2.2 Coxeter groups and Hecke algebras of B_n-type

We shall start by giving an intuitive 'pairs of shoes'-description of W_n , the Coxeter group of B_n-type , given in a talk by Professor G.D. James. For a picture we can think of n numbered shelves and n numbered and ordered pairs of shoes; put one pair on each shelf, not necessarily in the right order pairwise, and not necessarily on the right shelves. (Note that, the description can be made rigorous if we consider an ordered n-tuple of ordered pairs of objects.)
We want to place the pairs of shoes correctly, but we are only allowed to swap over the shoes of the pair that is placed on the first shelf, and also to swap pairs that lie on consecutive shelves. If for example we want to arrange the word $\begin{pmatrix} 2 & \bar{2} \\ \bar{4} & 4 \\ 3 & \bar{3} \\ \bar{1} & 1 \end{pmatrix}$, where we use the notation $i\bar{i}$ for the pair of shoes with number i , and $\bar{i}$ is the left shoe, then one possible procedure is the following:

$$\begin{pmatrix} 2 & \bar{2} \\ \bar{4} & 4 \\ 3 & \bar{3} \\ \bar{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{4} & 4 \\ 2 & \bar{2} \\ 3 & \bar{3} \\ \bar{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \bar{4} \\ 2 & \bar{2} \\ 3 & \bar{3} \\ \bar{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \bar{4} \\ 2 & \bar{2} \\ \bar{1} & 1 \\ 3 & \bar{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \bar{4} \\ \bar{1} & 1 \\ 2 & \bar{2} \\ 3 & \bar{3} \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{1} & 1 \\ 4 & \bar{4} \\ 2 & \bar{2} \\ 3 & \bar{3} \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & \bar{1} \\ 4 & \bar{4} \\ 2 & \bar{2} \\ 3 & \bar{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \bar{1} \\ 2 & \bar{2} \\ 4 & \bar{4} \\ 3 & \bar{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \bar{1} \\ 2 & \bar{2} \\ 3 & \bar{3} \\ 4 & \bar{4} \end{pmatrix} .$$

We can see that we have been making use of the symmetric group S_n to swap pairs on consecutive shelves, and of the group C_2 (a cyclic group with two elements) for swapping shoes on the first shelf. Every arrangement is a word of the group C_2 'wreath product' S_n , $C_2 \wr S_n$, where '$\wr$' means: we take $2^n \cong C_2 \times \ldots \times C_2$ (n copies) and we make a group using the elements of 2^n and S_n .

As a set, it is the cartesian product $W_n = 2^n \times S_n$. Therefore , $|W_n| = 2^n \cdot n!$.

As a group W_n is isomorphic to 2^n acted upon by S_n ; i.e. it is the semi-direct product $2^n \rtimes S_n$.

The way S_n acts on 2^n implies that the following is a presentation for W_n :

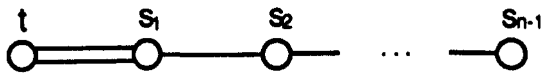
$$W_n = \left\langle \begin{array}{ll} s_1, \ldots, s_{n-1} & s_i \in S_n \\ v_1, \ldots, v_n & v_j \in 2^n \end{array} \;\middle|\; \begin{array}{l} s_i{}^2 = 1, \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i \\ s_i s_j = s_j s_i \text{ for } |i-j| > 1 \\ v_i{}^2 = 1, \ v_i v_j = v_j v_i \text{ for all } i,j \\ s_j{}^{-1} v_i s_j = s_j v_i s_j = v_i{}^{s_j} = v_{s_j(i)} \\ \text{where } s_j(i) \text{ is the image of } i \text{ under} \\ \text{the transposition } s_j = (j, j+1) \end{array} \right\rangle \clubsuit$$

Aside In terms of the pairs of shoes, v_i means 'swap the shoes of the ith pair' .

If we call now $v_1 = t$, then the last relations of the above become:

$$\begin{aligned} v_1 &= t \\ v_2 &= s_1 t s_1 \\ v_3 &= s_2 s_1 t s_1 s_2 = s_2 v_2 s_2 \\ &\vdots \\ v_n &= s_{n-1} \dots s_1 t s_1 \dots s_{n-1} \end{aligned}$$

and this means that for a presentation of W_n we only need $t, s_1, s_2, \dots, s_{n-1}$ as generators and – after having done Tietze transformations – relations given by the following *Dynkin* diagram:



where the single bonds of strength mean relations of degree 3 and the double bond a relation of degree 4 . Also, if two generators are not connected by a bond, the relation between them is of degree 2 , i.e. they commute.
In other words:

$$W_n = \left\langle \; t, s_1, s_2, \dots, s_{n-1} \;\middle|\; \begin{array}{l} (ts_1)^4 = 1 \quad \text{or} \quad ts_1ts_1 = s_1ts_1t \\ (ts_i)^2 = 1 \quad \text{or} \quad ts_i = s_it \;\text{ for }\; i > 1 \\ t^2 = s_i{}^2 = 1 \;\text{ for }\; i = 1, \dots, n-1 \\ (s_is_{i+1})^3 = 1 \quad \text{or} \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{for all } i \\ (s_is_j)^2 = 1 \quad \text{or} \quad s_is_j = s_js_i \;\text{ for }\; |i-j| > 1 \end{array} \right\rangle \; \spadesuit$$

A presentation for the Hecke algebra of B_n-type , $\mathcal{H}_n(q,Q)$, which corresponds to W_n is given below:

$$\mathcal{H}_n(q,Q) = \left\langle \; t, g_1, \dots, g_{n-1} \;\middle|\; \begin{array}{l} tg_1tg_1 = g_1tg_1t \\ tg_i = g_it \;\text{ for }\; i > 1 \\ t^2 = (Q-1)t + Q \\ g_i{}^2 = (q-1)g_i + q \;\text{ for }\; i = 1, \dots, n-1 \\ g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1} \quad \text{for all } i \\ g_ig_j = g_jg_i \;\text{ for }\; |i-j| > 1 \end{array} \right\rangle \; \spadesuit$$

Note The Dynkin diagram above, indicates that there is a natural inclusion of W_n into W_{n+1} (by adding an extra node at the end), and this extends to a natural inclusion of $\mathcal{H}_n(q,Q)$ into $\mathcal{H}_{n+1}(q,Q)$.

Theorem 11 $B_{n,1}$ *is the Artin group of* W_n .

Proof According to Definition 17, it follows immediately by comparing the first reduced presentation of $B_{n,1}$ (given in 2.6) with the above presentation ($\spadesuit$) of W_n . □

Corollary 6 So we can find an epimorphism of $B_{n,1}$ onto W_n ; indeed, there is an obvious one sending $T' \mapsto t$, $\sigma_i \mapsto s_i$. This implies now that there is also an epimorphism of $\mathbb{C}B_{n,1}$ onto $\mathcal{H}_n(q,Q)$; indeed, we can send $T' \mapsto t$, $\sigma_i \mapsto g_i$.

Moreover, if we compare the second reduced presentation of $B_{n,1}$ (given in 2.7) with the presentation of $\mathcal{H}_n(q,Q)$ given above, we can see that there is another epimorphism of $\mathbb{C}B_{n,1}$ onto $\mathcal{H}_n(q,Q)$ (sending $T \mapsto t,\ \sigma_i \mapsto g_i$).

As already mentioned, X-S. Lin in [44] made the same observation (Theorem 12), after simplifying R. Skora's approach for Markov's theorem in $L(p,1)$.

3.3 A trace function for $\mathcal{H}_n(q,Q)$

In this section we show the existence and uniqueness of a linear trace function from $\bigcup_{n=1}^{\infty}\mathcal{H}_n(q,Q)$ to $\mathbb{C}$ which is the analogue of Ocneanu's trace for the Hecke algebras of A-type . This is joint work with Meinolf Geck.

If we consider the elements $t_i' = g_i \dots g_1 t g_1{}^{-1} \dots g_i{}^{-1}$, $i = 1,\dots,n-1$ and $t = t'$ in $\mathcal{H}_n(q,Q)$, then the following basic relations hold in $\mathcal{H}_n(q,Q)$:

Lemma 5

(i) $g_i = q\,g_i{}^{-1} + (q-1)\cdot 1 \quad i = 1,\dots,n-1\ ,\ \ t = Q\,t^{-1} + (Q-1)\cdot 1$

$g_i{}^{-1} = \frac{1}{q}\,g_i + \frac{1-q}{q}\cdot 1 \quad i = 1,\dots,n-1\ ,\ \ t^{-1} = \frac{1}{Q}\,t + \frac{1-Q}{Q}\cdot 1$ for $q, Q \neq 0$

$t_i'^2 = Q\cdot 1 + (Q-1)\,t_i'\ ,\ \ t_i' = Q\,t_i'^{-1} + (Q-1)\cdot 1\ ,\ \ t_i'^{-1} = \frac{1}{Q}\,t_i' + \frac{1-Q}{Q}\cdot 1$
for all $i = 1,\dots,n-1$

(ii) If $k \leq i$ then $\quad g_{i+1}\dots g_k g_{k+1}{}^{-1}\dots g_{i+1}{}^{-1} = g_k{}^{-1}\dots g_i{}^{-1} g_{i+1}\dots g_k$

and its inverse: $\quad g_{i+1}\dots g_{k+1} g_k{}^{-1}\dots g_{i+1}{}^{-1} = g_k{}^{-1}\dots g_{i+1}{}^{-1} g_i\dots g_k$

also $\quad g_{i+1}{}^{-1}\dots g_{k+1}{}^{-1} g_k\dots g_{i+1} = g_k\dots g_{i+1} g_i{}^{-1}\dots g_k{}^{-1}$

and its inverse: $\quad g_{i+1}{}^{-1}\dots g_k{}^{-1} g_{k+1}\dots g_{i+1} = g_k\dots g_i g_{i+1}{}^{-1}\dots g_k{}^{-1}$

for all $i = 1,\dots,n-2$

Notice that for $k = i$ the relations we obtain are:

$$
\begin{aligned}
g_i{}^{-1} g_{i+1} g_i &= g_{i+1} g_i g_{i+1}{}^{-1} \\
g_i{}^{-1} g_{i+1}{}^{-1} g_i &= g_{i+1} g_i{}^{-1} g_{i+1}{}^{-1} \\
g_i g_{i+1} g_i{}^{-1} &= g_{i+1}{}^{-1} g_i g_{i+1} \\
g_i g_{i+1}{}^{-1} g_i{}^{-1} &= g_{i+1}{}^{-1} g_i{}^{-1} g_{i+1}
\end{aligned}
$$

all of which follow from the basic relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$. The only other consequence of which is its inverse, i.e. $g_i{}^{-1} g_{i+1}{}^{-1} g_i{}^{-1} = g_{i+1}{}^{-1} g_i{}^{-1} g_{i+1}{}^{-1}$.

(iii) $t'_i g_{i+1} t'_i g_{i+1} = g_{i+1} t'_i g_{i+1} t'_i$, $i = 1,\dots,n-2$

$$t'_i g_k = \begin{cases} g_i t'_{i-1} & \text{if } k = i \\ g_k t'_i & \text{if } k < i \text{ or } k > i+1 \end{cases}$$

for all $i = 1,\dots,n-1$

(iv) $t \cdot t'_i \;= (g_i \cdots g_2 \underline{g_1}^{-1} t \underline{g_1} g_2^{-1} \cdots g_i^{-1}) t$, $i = 1,\dots n-1$

$= t'_i t + (1-q)\, t (g_i \cdots g_2 g_1^{-1} \cdots g_i^{-1}) t +$

$\frac{q-1}{q} (g_i \cdots g_1 g_2^{-1} \cdots g_i^{-1}) t^2 - \frac{(1-q)^2}{q} t^2$

$\overset{(ii)}{=} t'_i t + (1-q)\, t (g_1^{-1} \cdots g_i^{-1} g_{i-1} \cdots g_1) t +$

$\frac{q-1}{q} (g_1^{-1} \cdots g_{i-1}^{-1} g_i \cdots g_1) t^2 - \frac{(1-q)^2}{q} t^2$

Also, in a completely analogous way:

If $k < i$ and $i = 2,\dots,n-1$

$t'_k \cdot t'_i \;= (g_i \cdots g_{k+2} \underline{g_{k+1}}^{-1} g_k \cdots g_1 t g_1^{-1} \cdots g_k^{-1} \underline{g_{k+1}} g_{k+2}^{-1} \cdots g_i^{-1}) t'_k$

$= t'_i t'_k + (1-q)\, t'_k (g_i \cdots g_{k+2} g_{k+1}^{-1} \cdots g_i^{-1}) t'_k +$

$\frac{q-1}{q} (g_i \cdots g_{k+1} g_{k+2}^{-1} \cdots g_i^{-1}) {t'_k}^2 - \frac{(1-q)^2}{q} {t'_k}^2$

$\overset{(ii)}{=} t'_i t'_k + (1-q)\, t'_k (g_{k+1}^{-1} \cdots g_i^{-1} g_{i-1} \cdots g_{k+1}) t'_k +$

$\frac{q-1}{q} (g_{k+1}^{-1} \cdots g_{i-1}^{-1} g_i \cdots g_{k+1}) {t'_k}^2 - \frac{(1-q)^2}{q} {t'_k}^2$

Proof

(i) follows immediately from the relations in the presentation of $\mathcal{H}_n(q,Q)$ (referred to by 'a.r' for the remainder of this proof) :

- $g_i^2 = q \cdot 1 + (q-1)\, g_i \iff$

  $g_i = q\, g_i^{-1} + (q-1) \cdot 1 \iff$

  $g_i^{-1} = \frac{1}{q} g_i + \frac{1-q}{q} \cdot 1$ for $i = 1,\dots,n-1$

- $t^2 = Q \cdot 1 + (Q-1)\, t \iff$

  $t = Q\, t^{-1} + (Q-1) \cdot 1 \iff$

$t^{-1} = \frac{1}{Q} t + \frac{1-Q}{Q} \cdot 1$

- $t_i'^2 = (g_i \dots g_1 t g_1^{-1} \dots g_i^{-1})(g_i \dots g_1 t g_1^{-1} \dots g_i^{-1}) =$

$(g_i \dots g_1 t^2 g_1^{-1} \dots g_i^{-1}) = (g_i \dots g_1 [Q + (Q-1)t] g_1^{-1} \dots g_i^{-1}) =$

$Q(g_i \dots g_1 g_1^{-1} \dots g_i^{-1}) + (Q-1)(g_i \dots g_1 t g_1^{-1} \dots g_i^{-1}) \iff$

$t_i'^2 = Q \cdot 1 + (Q-1) t_i' \iff$

$t_i' = Q t_i'^{-1} + (Q-1) \cdot 1 \iff$

$t_i'^{-1} = \frac{1}{Q} t_i' + \frac{1-Q}{Q} \cdot 1$ for $i = 1, \dots, n-1$

(ii) If $k < i$

$$g_k{}^{-1} \dots g_{i-1}{}^{-1} \underline{g_i{}^{-1} g_{i+1} g_i} \dots g_k \overset{a.r.}{=}$$

$$g_k{}^{-1} \dots g_{i-1}{}^{-1} \underline{g_{i+1} g_i g_{i+1}}{}^{-1} g_{i-1} \dots g_k \overset{a.r.}{=}$$

$$g_{i+1} g_k{}^{-1} \dots \underline{g_{i-1}{}^{-1} g_i g_{i-1}} \dots g_k g_{i+1}{}^{-1} =$$

$$\vdots$$

$$g_{i+1} \dots g_{k+2} \underline{g_k{}^{-1} g_{k+1} g_k} g_{k+2}{}^{-1} \dots g_{i+1}{}^{-1} =$$

$$g_{i+1} \dots g_{k+1} g_k g_{k+1}{}^{-1} \dots g_{i+1}{}^{-1}$$

The other relations follow similarly.

(iii) • $t_i' g_{i+1} t_i' g_{i+1} =$

$$(g_i \dots g_1 t \underline{g_1{}^{-1} \dots g_i{}^{-1}) g_{i+1} (g_i \dots g_1} t g_1{}^{-1} \dots g_i{}^{-1}) g_{i+1} \overset{(ii)}{=}$$

$$g_i \dots g_1 t \underline{g_{i+1}} \dots \underline{g_2} g_1 \underline{g_2}{}^{-1} \dots \underline{g_{i+1}}{}^{-1} t g_1{}^{-1} \dots g_i{}^{-1} g_{i+1} \overset{a.r.}{=}$$

$$(g_i g_{i+1}) \dots (g_1 g_2) t g_1 t (g_2{}^{-1} g_1{}^{-1}) \dots (g_{i+1}{}^{-1} g_i{}^{-1}) \underline{g_{i+1}} \overset{a.r.}{=}$$

$$(g_i g_{i+1}) \dots (g_1 g_2) \underline{t g_1 t g_1} (g_2{}^{-1} g_1{}^{-1}) \dots (g_{i+1}{}^{-1} g_i{}^{-1}) \overset{a.r.}{=}$$

$$(g_i g_{i+1}) \dots (g_1 g_2) g_1 t g_1 t (\underline{g_2}{}^{-1} g_1{}^{-1}) \dots (\underline{g_{i+1}}{}^{-1} g_i{}^{-1}) \overset{a.r.}{=}$$

$$(g_i g_{i+1}) \dots (g_1 g_2) \underline{g_1} t g_1 g_2{}^{-1} \dots g_{i+1}{}^{-1} t g_1{}^{-1} \dots g_i{}^{-1} \overset{a.r.}{=}$$

$$g_{i+1} (g_i \underline{g_{i+1}}) \dots (g_1 \underline{g_2}) t g_1 g_2{}^{-1} \dots g_{i+1}{}^{-1} t g_1{}^{-1} \dots g_i{}^{-1} \overset{a.r.}{=}$$

$$g_{i+1} g_i \dots g_1 t \underline{g_{i+1} \dots g_2 g_1 g_2{}^{-1} \dots g_{i+1}}{}^{-1} t g_1{}^{-1} \dots g_i{}^{-1} \overset{(ii)}{=}$$

$$g_{i+1}(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) g_{i+1}(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) =$$

$$g_{i+1} t'_i g_{i+1} t'_i$$

- $g_i t'_{i-1} = g_i(g_{i-1} \cdots g_1 t g_1{}^{-1} \cdots g_{i-1}{}^{-1}) =$

  $$\underline{g_i(g_{i-1} \cdots g_1 t g_1{}^{-1} \cdots g_{i-1}{}^{-1})(g_i{}^{-1}} g_i) = t'_i g_i$$

- If $k < i$

  $$t'_i g_k = (g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) g_k =$$

  $$(g_i \cdots g_1 t g_1{}^{-1} \cdots g_{k+1}{}^{-1} g_{k+2}{}^{-1} \cdots g_i{}^{-1}) \underline{g_k} \overset{a.r.}{=}$$

  $$g_i \cdots g_1 t g_1{}^{-1} \cdots \underline{g_k{}^{-1} g_{k+1}{}^{-1} g_k} g_{k+2}{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

  $$g_i \cdots g_{k+1} g_k \cdots g_1 t g_1{}^{-1} \cdots g_{k-1}{}^{-1} \underline{g_{k+1}} g_k{}^{-1} g_{k+1}{}^{-1} g_{k+2}{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

  $$g_i \cdots \underline{g_{k+1} g_k g_{k+1}} g_{k-1} \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

  $$g_i \cdots g_{k+2} \underline{g_k} g_{k+1} g_k g_{k-1} \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

  $$g_k(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) = g_k t'_i$$

- Finally if $k > i+1$

  $$t'_i g_k = (g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) g_k \overset{a.r.}{=}$$

  $$g_k(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) = g_k t'_i$$

(iv) • We only prove the relations for tt'_i as the rest can be proved analogously:

$$tt'_i = \underline{t}(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) \overset{a.r.}{=}$$

$$g_i \cdots g_2 \underline{t g_1 t g_1}{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

$$g_i \cdots g_2 g_1{}^{-1} t g_1 \underline{t} g_2{}^{-1} \cdots g_i{}^{-1} \overset{a.r.}{=}$$

$$(g_i \cdots g_2 \underline{g_1}{}^{-1} t \underline{g_1} g_2{}^{-1} \cdots g_i{}^{-1}) t \overset{(i)}{=}$$

$$(g_i \cdots g_2 \left[\tfrac{1}{q} g_1 + \tfrac{1-q}{q}\right] t \left[q g_1{}^{-1} + (q-1)\right] g_2{}^{-1} \cdots g_i{}^{-1}) t =$$

$$q\tfrac{1}{q}(g_i \cdots g_1 t g_1{}^{-1} \cdots g_i{}^{-1}) t + \tfrac{q-1}{q} g_i \cdots g_1 \underline{t} g_2{}^{-1} \cdots g_i{}^{-1} t +$$

$$q\tfrac{1-q}{q} g_i \cdots g_2 \underline{t} g_1{}^{-1} \cdots g_i{}^{-1} t + \tfrac{(1-q)(q-1)}{q} g_i \cdots g_2 \underline{t} g_2{}^{-1} \cdots g_i{}^{-1} t =$$

$$t'_i t + \tfrac{q-1}{q}(g_i \cdots g_2 g_1 g_2{}^{-1} \cdots g_i{}^{-1}) t^2 +$$

$$(1-q)\,t(g_i\dots g_2g_1{}^{-1}g_2{}^{-1}\dots g_i{}^{-1})t - \tfrac{(1-q)^2}{q}t^2 \qquad \square$$

The Trace Function

The proof of the following theorem rests squarely on the proof of Ocneanu's trace function as given in [30], pp. 343-344, whilst, its statement was thought of by the 'pairs of shoes'-description of W_n (recall section 3.2).

Theorem 12 *Given z and s in $\mathbb{C}$, there exists a unique linear function*

$$tr:\ \mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q) \longrightarrow \mathbb{C}$$

such that the following hold:

1) $tr(ab) = tr(ba)$, $a,b \in \mathcal{H}$

2) $tr(1) = 1$ *for all* $\mathcal{H}_n(q,Q)$

3) $tr(ag_n) = z\,tr(a)$, $a \in \mathcal{H}_n(q,Q)$

4) $tr(at'_n) = s\,tr(a)$, $a \in \mathcal{H}_n(q,Q)$

Proof

The proof of existence relies on inductive arguments using the following information on the structure of $\mathcal{H}_n(q,Q)$ as given in [14] :

The Coxeter group of B_n-type , W_n , is a subgroup of W_{n+1} of index $2(n+1)$. In [14], Dipper and James show that a complete set of right coset representatives of W_n in W_{n+1} is given by

$$J_{n+1} := \{1,\ s_n\dots s_i \mid i=1,\dots,n\}\bigcup\{s_n\dots s_1s_0s_1\dots s_i \mid i=0,1,\dots,n \text{ for } s_0=t\}$$

I.e. every element $w \in W_{n+1}$ can be written uniquely in the form

$$w \in W_n \text{ or } w = u\cdot x \ , \text{ where } u \in W_n \text{ and } x \in J_{n+1} \ .$$

We can rephrase this as follows:

Every element $w \in W_{n+1}$ has one of the following forms:

(a) $w \in W_n$
(b) $w = us_nv$, $u,v \in W_n$
(c) $w = us_n\dots s_1ts_1\dots s_n$, $u \in W_n$

We have also analogous statements in the Hecke algebra $\mathcal{H}_{n+1}(q,Q)$ of W_{n+1} ; i.e. we can say that every element in $\mathcal{H}_{n+1}(q,Q)$ can be written as a linear combination of elements w , each of precisely one of the following forms[1]:

$$\begin{array}{ll} (a) & w \in \mathcal{H}_n(q,Q) \\ (b) & w = ug_nv \ , \quad u,v \in \mathcal{H}_n(q,Q) \\ (c) & w = ut_n \ , \quad u \in \mathcal{H}_n(q,Q) \ , t_n = g_n \dots g_1 t g_1 \dots g_n \ . \end{array}$$

This canonical form is equivalent to the following:

$$\begin{array}{ll} (a) & w \in \mathcal{H}_n(q,Q) \\ (b) & w = ug_nv \ , \quad u,v \in \mathcal{H}_n(q,Q) \\ (c) & w = ut'_n \ , \quad u \in \mathcal{H}_n(q,Q) \ , \ t'_n = g_n \dots g_1 t {g_1}^{-1} \dots {g_n}^{-1} \ . \end{array}$$

We note now that having proved the existence of the trace function, uniqueness will follow immediately as, given $w \in \mathcal{H}_n(q,Q)$, it is clear that the trace of w can be computed (inductively) from the above using rules 1) , 2) , 3) , 4) and linearity.

Define

$$c_n : \mathcal{H}_n \oplus \mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}$$

given by

$$c_n(a \oplus b \oplus c \otimes d) = a + bt'_n + cg_nd \ .$$

From the above the map c_n is surjective. On the other hand $\mathcal{H}_n(q,Q)$ is free of rank $2n$ as an $\mathcal{H}_{n-1}(q,Q)$-module ; so we have:

$\mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \cong {\mathcal{H}_{n-1}}^{2n} \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \cong {\mathcal{H}_n}^{2n}$ (as vector spaces).

I.e. $\mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n$ has dimension $(2n \dim\mathcal{H}_n)$. So

$\dim(\mathcal{H}_n \oplus \mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n) = (2+2n) \cdot \dim\mathcal{H}_n = 2(n+1)|W_n| = |W_{n+1}| = \dim\mathcal{H}_{n+1}$.

Hence c_n must be injective, so it is an isomorphism of $(\mathcal{H}_n, \mathcal{H}_n)$-bimodules.

Next, we wish to define inductively a trace tr on $\mathcal{H} := \bigcup_{n\geq 1} \mathcal{H}_n(q,Q)$ satisfying:

1) $tr(ab) = tr(ba)$, $a,b \in \mathcal{H}_n(q,Q)$

2) $\mathrm{tr}(1) = 1$ for all $\mathcal{H}_n(q,Q)$

3) $tr(xg_n) = z\, tr(x)$, $x \in \mathcal{H}_n(q,Q)$

4) $tr(yt'_n) = s\, tr(y)$, $y \in \mathcal{H}_n(q,Q)$

where $z, s \in \mathbb{C}$ are fixed numbers.

[1] An algorithm for writing an arbitrary element as a linear combination of elements in the canonical basis is described in [17].

The induction step: Suppose tr is defined on $\mathcal{H}_n(q,Q)$; then we define tr on $\mathcal{H}_{n+1}(q,Q)$ as follows: Let $x \in \mathcal{H}_{n+1}(q,Q)$ be arbitrary; then there exist $a,b,c,d \in \mathcal{H}_n(q,Q)$ such that $x = c_n(a \oplus b \oplus c \otimes d)$.

$$\text{Define } tr(x) := tr(a) + s\,tr(b) + z\,tr(cd)$$

Then this trace satisfies (2) , (3) and (4) . Indeed, this trace also satisfies the following stronger version of (3) :

$$(3') \quad tr(cg_n d) = z\,tr(cd)\ ,\ \ c,d \in \mathcal{H}_n(q,Q)$$

The only remaining problem is to prove that property (1) is satisfied for all $a,x \in \mathcal{H}$:

Arguing inductively, we may assume that it holds for $a,x \in \mathcal{H}_n(q,Q)$. As V. Jones mentions in [30], it is enough to show property (1) in the case where $a \in \mathcal{H}_{n+1}(q,Q)$ and x is one of the generators of $\mathcal{H}_{n+1}(q,Q)$. I.e. it is enough to show

$$\left.\begin{array}{ll} tr(ag_i) = tr(g_i a) & ,\ \ a \in \mathcal{H}_{n+1}(q,Q),\ i = 1,\dots n \\ tr(at) = tr(ta) & ,\ \ a \in \mathcal{H}_{n+1}(q,Q) \end{array}\right. \qquad (*)$$

and – from the above – we only have to consider the cases in which

$$a \in \mathcal{H}_n(q,Q)\ ,\ \ a = bg_n c\ \ (b,c \in \mathcal{H}_n(q,Q))\ ,\ \ a = bt'_n\ \ (b \in \mathcal{H}_n(q,Q)).$$

- $a \in \mathcal{H}_n(q,Q)$:

  then at and ta are in $\mathcal{H}_n(q,Q)$, and $tr(at) = tr(ta)$ holds by induction on n .

  In the same way it holds that $tr(ag_i) = tr(g_i a)$ for $i < n$.

  If $i = n$, $tr(ag_n) = tr(g_n a)$ by property (3') .

- $a = bg_n c$ $(b,c \in \mathcal{H}_n(q,Q))$: then tb and ct are in $\mathcal{H}_n(q,Q)$ and

  $tr(tbg_n c) \overset{(3')}{=} z\,tr(tbc) \overset{(by\ induction)}{=} z\,tr(bct) \overset{(3')}{=} tr(bg_n ct).$

  If $i < n$ then $g_i b, cg_i \in \mathcal{H}_n(q,Q)$ and similarly

  $tr(g_i bg_n c) \overset{(3')}{=} z\,tr(g_i bc) \overset{(by\ induction)}{=} z\,tr(bcg_i) \overset{(3')}{=} tr(bg_n cg_i).$

  If $i = n$ we have to check that

  $$tr(g_n bg_n c) = tr(bg_n cg_n) \quad (**)$$

  (proof to follow).

- $a = bt'_n \quad (b \in \mathcal{H}_n(q,Q))$: In this case we have to check the following:

$$\begin{array}{ll} tr(tbt'_n) = tr(bt'_n t) & \\ tr(g_i bt'_n) = tr(bt'_n g_i) \quad , \; i < n & \qquad (***) \\ tr(g_n bt'_n) = tr(bt'_n g_n) & \end{array}$$

  (proof also to follow).

We proceed with proving (**) : We may assume that each of the elements b and c is of of one of the following forms:

(i) an element of $\mathcal{H}_{n-1}(q,Q)$

(ii) $xg_{n-1}y \quad (x,y \in \mathcal{H}_{n-1}(q,Q))$

(iii) $xt'_{n-1} \quad (x \in \mathcal{H}_{n-1}(q,Q))$

If b,c are both of type (i) or (ii) then the proofs are exactly the same as in [30], but for completeness we shall include them here. For the remainder of this proof we shall use the abbreviation 'a.r' to denote the use of the relations in the presentation of $\mathcal{H}_n(q,Q)$ as well as the consequences of these relations listed in Lemma 5 (i)–(iii) .

- The case where b and c are elements of $\mathcal{H}_{n-1}(q,Q)$ is trivial since g_n commutes with $\mathcal{H}_{n-1}(q,Q)$.

- If b is of type (i) and c of type (ii) ;

  i.e. $b \in \mathcal{H}_{n-1}(q,Q)$ and c is of the form $xg_{n-1}y$, $x,y \in \mathcal{H}_{n-1}(q,Q)$, then :

  $tr(\underline{g_n} bg_n xg_{n-1}y) \overset{a.r}{=} tr(b\underline{g_n}^2 xg_{n-1}y) \overset{a.r}{=} tr(b[q+(q-1)g_n]xg_{n-1}y) =$

  $q\,tr(bxg_{n-1}y) + (q-1)\,tr(bg_n xg_{n-1}y) \overset{(3')}{=}$

  $qz\,tr(bxy) + (q-1)z\,tr(bxg_{n-1}y) \overset{(3')}{=}$

  $qz\,tr(bxy) + (q-1)z^2\,tr(bxy) =$

  $[qz + (q-1)z^2]\,tr(bxy).$

  On the other hand we have:

  $tr(b\underline{g_n}xg_{n-1}y\underline{g_n}) \overset{a.r}{=} tr(bx\underline{g_n g_{n-1} g_n}y) \overset{a.r}{=} tr(bxg_{n-1}g_n g_{n-1}y) \overset{(3')}{=}$

  $z\,tr(bx\underline{g_{n-1}}^2 y) \overset{a.r}{=} z\,tr(bx[q+(q-1)g_{n-1}]y) =$

  $zq\,tr(bxy) + z(q-1)\,tr(bxg_{n-1}y) \overset{(3')}{=}$

$zq\, tr(bxy) + z^2(q-1)\, tr(bxy) = [zq + z^2(q-1)]\, tr(bxy).$

I.e $tr(g_n b g_n c) = tr(g_n b g_n c g_n)$.

- If now c is of type (i) , i.e. $c \in \mathcal{H}_{n-1}(q,Q)$ and b is of type (ii) , i.e. $b = x g_{n-1} y$, $x, y \in \mathcal{H}_{n-1}(q,Q)$, then:

$tr(\underline{g_n} x g_{n-1} y \underline{g_n} c) \overset{\text{a.r}}{=} tr(x \underline{g_n g_{n-1} g_n} y c) \overset{\text{a.r}}{=} tr(x g_{n-1} g_n g_{n-1} y c) \overset{(3')}{=}$

$z\, tr(x \underline{g_{n-1}}^2 y c) \overset{\text{a.r}}{=} qz\, tr(xyc) + (q-1)z\, tr(x g_{n-1} y c) \overset{(3')}{=}$

$qz\, tr(xyc) + (q-1)z^2\, tr(xyc) = [qz + (q-1)z^2]\, tr(xyc).$

On the other hand we have:

$tr(b g_n c g_n) = tr(x g_{n-1} y \underline{g_n} c g_n) \overset{\text{a.r}}{=} tr(x g_{n-1} y c \underline{g_n}^2) \overset{\text{a.r}}{=}$

$tr(x g_{n-1} y c [q + (q-1) g_n]) = q\, tr(x g_{n-1} y c) + (q-1)\, tr(x g_{n-1} y c g_n) \overset{(3')}{=}$

$qz\, tr(xyc) + (q-1)z\, tr(x g_{n-1} y c) \overset{(3')}{=} qz\, tr(xyc) + (q-1)z^2\, tr(xyc) =$

$[qz + (q-1)z^2]\, tr(xyc).$

I.e. $tr(g_n b g_n c) = tr(b g_n c g_n)$.

- The last 'easy' case is when both b and c are of type (ii) , i.e. $b = x g_{n-1} y$, $c = x' g_{n-1} y'$, $x, y, x', y' \in \mathcal{H}_{n-1}(q,Q)$:

$tr(\underline{g_n} x g_{n-1} y \underline{g_n} x' g_{n-1} y') \overset{\text{a.r}}{=} tr(x \underline{g_n g_{n-1} g_n} y x' g_{n-1} y') \overset{\text{a.r}}{=}$

$tr(x g_{n-1} g_n g_{n-1} y x' g_{n-1} y') \overset{(3')}{=} z\, tr(x g_{n-1}^2 y x' g_{n-1} y') \overset{\text{a.r}}{=}$

$qz\, tr(x y x' g_{n-1} y') + (q-1)z\, tr(x g_{n-1} y x' g_{n-1} y') \overset{(3')}{=}$

$qz^2\, tr(x y x' y') + (q-1)z\, tr(x g_{n-1} y x' g_{n-1} y')$.

Similarly,

$tr(x g_{n-1} y \underline{g_n} x' g_{n-1} y' \underline{g_n}) \overset{\text{a.r}}{=} tr(x g_{n-1} y x' \underline{g_n g_{n-1} g_n} y') \overset{\text{a.r}}{=}$

$tr(x g_{n-1} y x' g_{n-1} g_n g_{n-1} y') \overset{(3')}{=} z\, tr(x g_{n-1} y x' g_{n-1}^2 y') \overset{\text{a.r}}{=}$

$qz\, tr(x g_{n-1} y x' y') + (q-1)z\, tr(x g_{n-1} y x' g_{n-1} y') \overset{(3')}{=}$

$qz^2\, tr(xyx'y') + (q-1)z\, tr(xg_{n-1}yx'g_{n-1}y')$.

I.e. $tr(g_nbg_nc) = tr(bg_ncg_n)$.

- Assume now that b is of type (i) and c of type (iii) ,

  i.e. $b \in \mathcal{H}_{n-1}(q,Q)$, $c = xt'_{n-1}$, $x \in \mathcal{H}_{n-1}(q,Q)$:

  $tr(g_nb\underline{g_n}xt'_{n-1}) \overset{a.r}{=} tr(\underline{g_n}^2bxt'_{n-1}) \overset{a.r}{=}$

  $q\, tr(bxt'_{n-1}) + (q-1)\, tr(g_nbxt'_{n-1}) \overset{(3'),(4)}{=}$

  $qs\, tr(bx) + (q-1)zs\, tr(bx) =$

  $[\,qs + (q-1)zs\,]\, tr(bx)$.

  Similarly,

  $tr(b\underline{g_n}xt'_{n-1}g_n) \overset{a.r}{=} tr(bxg_nt'_{n-1}\underline{g_n}) \overset{a.r}{=}$

  $tr(bxg_nt'_{n-1}\,[\,qg_n{}^{-1} + (q-1)\,]\,) = q\, tr(bxt'_n) + (q-1)\, tr(bxg_nt'_{n-1}) \overset{(3'),(4)}{=}$

  $q\, str(bx) + (q-1)zs\, tr(bx) =$

  $[\,qs + (q-1)zs\,]\, tr(bx)$.

  I.e. $tr(g_nbg_nc) = tr(bg_ncg_n)$.

Before continuing with checking the other cases, we first prove a corollary to Lemma 5 :

Corollary 7 If $x, y \in \mathcal{H}_n(q,Q)$, then

$$tr(xt'_ny) = s \cdot tr(xy) \quad (4')$$

Proof of corollary We write $y = y_1ty_2t\ldots ty_k$, where each y_j $(j = 1,\ldots k)$ is a product of g_i's $i < n$. Then we repeatedly apply Lemma 5 (iii) and (iv) (which we abbreviate to L.(iii) and L.(iv)) :

$tr(xt'_ny) = tr(x\underline{t'_n}y_1ty_2t\ldots ty_k) \overset{L.(iii)}{=} tr(xy_1\underline{t'_n}ty_2t\ldots ty_k) \overset{L.(iv)}{=}$

$tr(xy_1tt'_ny_2t\ldots ty_k) - (1-q)\, tr(xy_1tg_1{}^{-1}\ldots \underline{g_n}^{-1}g_{n-1}\ldots g_1ty_2t\ldots ty_k) -$

$\frac{q-1}{q}\, tr(xy_1g_1{}^{-1}\ldots g_{n-1}{}^{-1}g_n\ldots g_1t^2y_2t\ldots ty_k) + \frac{(1-q)^2}{q}\, tr(xy_1t^2y_2t\ldots ty_k) \overset{a.r}{=}$

$tr(xy_1tt'_ny_2t\ldots ty_k) - \frac{(1-q)}{q}\, tr(xy_1tg_1{}^{-1}\ldots g_{n-1}{}^{-1}g_ng_{n-1}\ldots g_1ty_2t\ldots ty_k) -$

$(1-q)\frac{(1-q)}{q}\, tr(xy_1tg_1{}^{-1}\ldots g_{n-1}{}^{-1}g_{n-1}\ldots g_1ty_2t\ldots ty_k) -$

$\frac{q-1}{q}\, tr(xy_1 g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n \dots g_1 t^2 y_2 t \dots t y_k) + \frac{(1-q)^2}{q}\, tr(xy_1 t^2 y_2 t \dots t y_k) \overset{(3')}{=}$

$tr(xy_1 t t'_n y_2 t \dots t y_k) - \frac{(1-q)}{q} z\, tr(xy_1 t^2 y_2 t \dots t y_k) - \frac{(1-q)^2}{q}\, tr(xy_1 t^2 y_2 t \dots t y_k) +$

$\frac{(1-q)}{q} z\, tr(xy_1 t^2 y_2 t \dots t y_k) + \frac{(1-q)^2}{q}\, tr(xy_1 t^2 y_2 t \dots t y_k) =$

$tr(xy_1 t t'_n y_2 t \dots t y_k) = \dots$ after $k-1$ steps $\dots =$

$tr(xy_1 t y_2 t \dots t y_k t'_n) = tr(xyt'_n) \overset{(4)}{=} s\, tr(xy)$ $\square$

- We assume next that b is of type (iii) and c of type (i) ,

  i.e. $b = xt'_{n-1}$, $x \in \mathcal{H}_{n-1}(q,Q)$, $c \in \mathcal{H}_{n-1}(q,Q)$:

  $tr(g_n b g_n c) = tr(\underline{g_n} x t'_{n-1} g_n c) \overset{a.r.}{=} tr(x g_n t'_{n-1} \underline{g_n} c) \overset{a.r.}{=}$

  $q\, tr(xt'_n c) + (q-1)\, tr(x g_n t'_{n-1} c) \overset{(3')}{=} q\, tr(xt'_n c) + (q-1) z\, tr(xt'_{n-1} c) \overset{(4')}{=}$

  $qs\, tr(xc) + (q-1)zs\, tr(xc) = [\, qs + (q-1)zs\,]\, tr(xc)$.

  On the other hand we have:

  $tr(b g_n c g_n) = tr(xt'_{n-1} g_n c \underline{g_n}) \overset{a.r.}{=} tr(xt'_{n-1} \underline{g_n}{}^2 c) \overset{a.r.}{=}$

  $q\, tr(xt'_{n-1} c) + (q-1)\, tr(xt'_{n-1} g_n c) \overset{(3')}{=} q\, tr(xt'_{n-1} c) + (q-1) z\, tr(xt'_{n-1} c) \overset{(4')}{=}$

  $[\, qs + (q-1)zs\,]\, tr(xc)$.

  I.e. $tr(g_n b g_n c) = tr(b g_n c g_n)$.

- Let now b be of type (ii) and c of type (iii) ,

  i.e. $b = x g_{n-1} y$, $x, y \in \mathcal{H}_{n-1}(q,Q)$ and $c = x' t'_{n-1}$, $x' \in \mathcal{H}_{n-1}(q,Q)$:

  $tr(g_n b g_n c) = tr(\underline{g_n} x g_{n-1} y \underline{g_n} x' t'_{n-1}) \overset{a.r.}{=} tr(x \underline{g_n g_{n-1} g_n} y x' t'_{n-1}) \overset{a.r.}{=}$

  $tr(x g_{n-1} g_n g_{n-1} y x' t'_{n-1}) \overset{(3')}{=} z\, tr(x \underline{g_{n-1}}{}^2 y x' t'_{n-1}) \overset{a.r.}{=}$

  $qz\, tr(xyx't'_{n-1}) + (q-1)z\, tr(x g_{n-1} y x' t'_{n-1}) \overset{(4)}{=}$

  $qzs\, tr(xyx') + (q-1)z\, tr(x g_{n-1} y x' t'_{n-1})$.

  On the other hand we have:

$tr(bg_ncg_n) = tr(xg_{n-1}y\underline{g_n}x't'_{n-1}g_n) \stackrel{a.r.}{=} tr(xg_{n-1}yx'g_nt'_{n-1}\underline{g_n}) \stackrel{a.r.}{=}$

$q\,tr(xg_{n-1}yx'g_nt'_{n-1}g_n{}^{-1}) + (q-1)\,tr(xg_{n-1}yx'g_nt'_{n-1}) =$

$q\,tr(xg_{n-1}yx't'_n) + (q-1)\,tr(xg_{n-1}yx'g_nt'_{n-1}) \stackrel{(4),(3')}{=}$

$qs\,tr(xg_{n-1}yx') + (q-1)z\,tr(xg_{n-1}yx't'_{n-1}) \stackrel{(3')}{=}$

$qsz\,tr(xyx') + (q-1)z\,tr(xg_{n-1}yx't'_{n-1})$.

I.e. $tr(g_nbg_nc) = tr(bg_ncg_n)$.

- Assume next that b is of type (iii) and c of type (ii) ,

i.e. $b = x't'_{n-1}$, $x' \in \mathcal{H}_{n-1}(q,Q)$ and $c = xg_{n-1}y$, $x, y \in \mathcal{H}_{n-1}(q,Q)$:

$tr(bg_ncg_n) = tr(x't'_{n-1}\underline{g_n}xg_{n-1}y\underline{g_n}) \stackrel{a.r.}{=} tr(x't'_{n-1}x\underline{g_ng_{n-1}g_n}y) \stackrel{a.r.}{=}$

$tr(x't'_{n-1}xg_{n-1}g_ng_{n-1}y) \stackrel{(3')}{=} z\,tr(x't'_{n-1}xg_{n-1}{}^2y) \stackrel{a.r.}{=}$

$qz\,tr(x't'_{n-1}xy) + (q-1)z\,tr(x't'_{n-1}xg_{n-1}y) \stackrel{(4')}{=}$

$qzs\,tr(x'xy) + (q-1)z\,tr(x't'_{n-1}xg_{n-1}y)$.

On the other hand:

$tr(g_nbg_nc) = tr(\underline{g_n}x't'_{n-1}g_nxg_{n-1}y) \stackrel{a.r.}{=} tr(x'g_nt'_{n-1}\underline{g_n}xg_{n-1}y) \stackrel{a.r.}{=}$

$q\,tr(x'\underline{g_nt'_{n-1}g_n{}^{-1}}xg_{n-1}y) + (q-1)\,tr(x'g_nt'_{n-1}xg_{n-1}y) \stackrel{(3')}{=}$

$q\,tr(x't'_nxg_{n-1}y) + (q-1)z\,tr(x't'_{n-1}xg_{n-1}y) \stackrel{(4')}{=}$

$qs\,tr(x'xg_{n-1}y) + (q-1)z\,tr(x't'_{n-1}xg_{n-1}y) \stackrel{(3')}{=}$

$qzs\,tr(x'xy) + (q-1)z\,tr(x't'_{n-1}xg_{n-1}y)$.

I.e. $tr(g_nbg_nc) = tr(bg_ncg_n)$.

- Finally assume that both b and c are of type (iii) ,

i.e. $b = xt'_{n-1}$ and $c = yt'_{n-1}$, $x, y \in \mathcal{H}_{n-1}(q,Q)$:

$tr(g_nbg_nc) = tr(\underline{g_n}xt'_{n-1}g_nyt'_{n-1}) \stackrel{a.r.}{=} tr(xg_nt'_{n-1}\underline{g_n}yt'_{n-1}) \stackrel{a.r.}{=}$

$q\, tr(x\underline{g_n t'_{n-1} g_n}^{-1} y t'_{n-1}) + (q-1)\, tr(x g_n t'_{n-1} y t'_{n-1}) \overset{(3')}{=}$

$q\, tr(x t'_n y t'_{n-1}) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1}) \overset{(4')}{=}$

$qs\, tr(x y t'_{n-1}) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1}) \overset{(4)}{=}$

$qs^2\, tr(xy) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1})$.

On the other hand:

$tr(b g_n c g_n) = tr(x t'_{n-1} \underline{g_n} y t'_{n-1} g_n) \overset{a.r.}{=} tr(x t'_{n-1} y g_n t'_{n-1} \underline{g_n}) \overset{a.r.}{=}$

$q\, tr(x t'_{n-1} y \underline{g_n t'_{n-1} g_n}^{-1}) + (q-1)\, tr(x t'_{n-1} y g_n t'_{n-1}) \overset{(3')}{=}$

$q\, tr(x t'_{n-1} y t'_n) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1}) \overset{(4)}{=}$

$qs\, tr(x t'_{n-1} y) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1}) \overset{(4')}{=}$

$qs^2\, tr(xy) + (q-1) z\, tr(x t'_{n-1} y t'_{n-1})$.

I.e. $tr(g_n b g_n c) = tr(b g_n c g_n)$.

So we have shown that $tr(g_n b g_n c) = tr(b g_n c g_n)$ for all $b, c \in \mathcal{H}_n(q,Q)$.

It remains now to check $(***)$:

- Let $b \in \mathcal{H}_n(q,Q)$. We shall show that $tr(t b t'_n) = tr(b t'_n t)$. Indeed:

  $tr(t b t'_n) \overset{(4)}{=} s\, tr(tb)$.

  $tr(b t'_n t) \overset{(4')}{=} s\, tr(bt) \overset{by\ induction}{=} s\, tr(tb)$.

- We shall show next that, for $i < n$, $tr(g_i b t'_n) = tr(b t'_n g_i)$. Exactly as before:

  $tr(g_i b t'_n) \overset{(4)}{=} s\, tr(g_i b)$.

  $tr(b t'_n g_i) \overset{(4')}{=} s\, tr(b g_i) \overset{(by\ induction)}{=} s\, tr(g_i b)$.

- The final and most tedious case is to show that $tr(g_n b t'_n) = tr(b t'_n g_n)$:

  Either $b \in \mathcal{H}_{n-1}(q,Q)$ **or** $b = x g_{n-1} y$, $x, y \in \mathcal{H}_{n-1}(q,Q)$

or $b = xt'_{n-1}$, $x \in \mathcal{H}_{n-1}(q,Q)$.

If $\boldsymbol{b \in \mathcal{H}_{n-1}(q,Q)}$: then $g_n b = b g_n$ and so

$tr(g_n b t'_n) = tr(b g_n t'_n) = tr(b \underline{g_n}^2 g_{n-1} \dots g_1 t g_1^{-1} \dots g_n^{-1}) \overset{a.r.}{=}$

$q\, tr(b g_{n-1} \dots g_1 t g_1^{-1} \dots \underline{g_n}^{-1}) + (q-1)\, tr(bt'_n) \overset{a.r.}{=}$

$q\frac{1}{q}\, tr(bt'_{n-1} g_n) + q\frac{1-q}{q}\, tr(bt'_{n-1}) + (q-1)\, tr(bt'_n) \overset{(3'),(4)}{=}$

$z\, tr(bt'_{n-1}) + (1-q)s\, tr(b) + (q-1)s\, tr(b) \overset{(4)}{=} zs\, tr(b)$ since $b \in \mathcal{H}_{n-1}(q,Q)$).

On the other hand:

$tr(bt'_n g_n) \overset{a.r.}{=} tr(b g_n t'_{n-1}) \overset{(3'),(4)}{=} zs\, tr(b)$.

I.e. $tr(g_n b t'_n) = tr(bt'_n g_n)$.

If $\boldsymbol{b = x g_{n-1} y}$ **,** $\boldsymbol{x, y \in \mathcal{H}_{n-1}(q,Q)}$:

$tr(g_n b t'_n) = tr(g_n x g_{n-1} y t'_n) = tr(\underline{g_n} x g_{n-1} y \underline{g_n} t'_{n-1} g_n^{-1}) \overset{a.r.}{=}$

$tr(x \underline{g_n} \underline{g_{n-1}} \underline{g_n} y t'_{n-1} g_n^{-1}) \overset{a.r.}{=} tr(x g_{n-1} g_n g_{n-1} y t'_{n-1} \underline{g_n}^{-1}) \overset{a.r.}{=}$

$\frac{1}{q}\, tr(\underline{x g_{n-1}} g_n \underline{g_{n-1} y t'_{n-1}} g_n) + \frac{1-q}{q}\, tr(x g_{n-1} g_n g_{n-1} y t'_{n-1}) =$

(applying the previous case for $b = x g_{n-1} \in \mathcal{H}_n(q,Q)$ and $c = g_{n-1} y t'_{n-1} \in \mathcal{H}_n(q,Q)$)

$\frac{1}{q}\, tr(\underline{g_n} x g_{n-1} g_n g_{n-1} y t'_{n-1}) + \frac{1-q}{q}\, tr(x g_{n-1} g_n g_{n-1} y t'_{n-1}) \overset{a.r.,(3')}{=}$

$\frac{1}{q}\, tr(x \underline{g_n} \underline{g_{n-1}} \underline{g_n} g_{n-1} y t'_{n-1}) + \frac{1-q}{q} z\, tr(x g_{n-1}{}^2 y t'_{n-1}) \overset{a.r.}{=}$

$\frac{1}{q}\, tr(x g_{n-1} g_n g_{n-1}{}^2 y t'_{n-1}) + \frac{1-q}{q} z\, tr(x \underline{g_{n-1}}{}^2 y t'_{n-1}) \overset{a.r.,(3')}{=}$

$\frac{1}{q} z\, tr(x \underline{g_{n-1}}{}^3 y t'_{n-1}) + \frac{1-q}{q} q z\, tr(x y t'_{n-1}) + \frac{1-q}{q}(q-1) z\, tr(x g_{n-1} y t'_{n-1}) =$

$[g_{n-1}{}^3 = (q^2 - q + 1) g_{n-1} + (q-1)q]$

$\frac{1}{q}(q^2 - q + 1) z\, tr(x g_{n-1} y t'_{n-1}) + \frac{1}{q}(q-1) q z\, tr(x y t'_{n-1}) +$

$\frac{1-q}{q} q z\, tr(x y t'_{n-1}) + \frac{1-q}{q}(q-1) z\, tr(x g_{n-1} y t'_{n-1}) =$

$[\frac{1}{q}(q^2 - q + 1) - \frac{(q-1)^2}{q}] z\, tr(x g_{n-1} y t'_{n-1}) = z\, tr(x g_{n-1} y t'_{n-1})$.

On the other hand :

$tr(bt'_n g_n) = tr(x g_{n-1} y t'_n g_n) \overset{a.r.}{=}$

$tr(xg_{n-1}yg_nt'_{n-1}) \overset{(3')}{=} z\,tr(xg_{n-1}yt'_{n-1})$.

I.e. $tr(g_nbt'_n) = tr(bt'_ng_n)$.

If $b = xt'_{n-1}$, $x \in \mathcal{H}_{n-1}(q,Q)$:

$tr(g_nbt'_n) = tr(\underline{g_n}xt'_{n-1}t'_n) \overset{a.r.}{=}$

$tr(xg_n(g_{n-1}\dots g_1t\underline{g_1{}^{-1}\dots g_{n-1}{}^{-1})(g_n\dots g_1}tg_1{}^{-1}\dots g_n{}^{-1})) \overset{a.r.}{=}$

$tr(xg_ng_{n-1}\dots g_1t\underline{g_n}\dots \underline{g_2}g_1\underline{g_2}^{-1}\dots \underline{g_n}^{-1}tg_1{}^{-1}\dots g_n{}^{-1}) \overset{a.r.}{=}$

$tr(xg_n(g_{n-1}g_n)\dots(g_1g_2)tg_1t(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})\underline{g_n}^{-1}) \overset{a.r.}{=}$

$tr(xg_n(g_{n-1}g_n)\dots(g_1g_2)\underline{tg_1tg_1}^{-1}(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})) \overset{a.r.}{=}$

$tr(xg_n(g_{n-1}g_n)\dots(g_1g_2)\underline{g_1}^{-1}tg_1t(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})) \overset{a.r.}{=}$

$tr(xg_ng_n{}^{-1}(g_{n-1}\underline{g_n})\dots(g_1\underline{g_2})tg_1t(\underline{g_2}^{-1}g_1{}^{-1})\dots(\underline{g_n}^{-1}g_{n-1}{}^{-1})) \overset{a.r.}{=}$

$tr(xg_{n-1}\dots g_1t\underline{g_n\dots g_2g_1g_2{}^{-1}\dots g_n}^{-1}tg_1{}^{-1}\dots g_{n-1}{}^{-1}) \overset{a.r.}{=}$

$tr(x\underline{g_{n-1}\dots g_1tg_1{}^{-1}\dots g_{n-1}}^{-1}\,\underline{g_n\dots g_1tg_1{}^{-1}\dots g_{n-1}{}^{-1}(g_n}^{-1}g_n)) =$

$tr(xt'_{n-1}t'_ng_n) = tr(bt'_ng_n)$.

I.e. $tr(g_nbt'_n) = tr(bt'_ng_n)$ □

Remark 11 If a word $a \in \mathcal{H}_n(q,Q)$ does not contain any t'_i's , then for calculating $tr(a)$ we only need to use rules 1) , 2) and 3) of Theorem 13 ; and so $tr(a)$ is the same as Ocneanu's trace applied on a .

We conclude the section by giving an example of calculating the trace of a word, in which we also demonstrate how to bring the word to the canonical form:

$tr(g_2t'_1g_2t'_1g_2g_1\underline{g_3}^{-1}t'_2) =$

$\frac{1}{q}\,tr(g_2t'_1g_2t'_1g_2g_1\underline{g_3}t'_2) + \frac{1-q}{q}\,tr(g_2t'_1g_2t'_1g_2g_1t'_2) \overset{3)}{=}$

$\frac{z+1-q}{q}\,tr(\underline{g_2t'_1g_2t'_1}g_2g_1t'_2) = \frac{z+1-q}{q}\,tr(t'_1g_2t'_1g_2{}^2\underline{g_1t'_2}) =$

$\frac{z+1-q}{q}\,tr(t'_1g_2t'_1g_2{}^2\underline{t'_2}g_1) = \frac{z+1-q}{q}\,tr(t'_1g_2t'_1\underline{g_2}^3t'_1g_2{}^{-1}g_1) =$

$\frac{z+1-q}{q}(q^2-q+1)\,tr(t'_1g_2\underline{t'_1g_2t'_1g_2}^{-1}g_1) + \frac{z+1-q}{q}q(q-1)\,tr(t'_1g_2t'_1t'_1g_2{}^{-1}g_1) =$

$\frac{(z+1-q)(q^2-q+1)}{q}\,tr(t'_1g_2g_2{}^{-1}t'_1g_2t'_1g_1) + (z+1-q)(q-1)\,tr(t'_1g_2\underline{t'_1}^2g_2{}^{-1}g_1) =$

$$\frac{(z+1-q)(q^2-q+1)}{q}\, tr(\underline{t_1'}^2 g_2 t_1' g_1)+$$

$$(z+1-q)(q-1)Q\, tr(t_1' \underline{g_2 t_1' g_2}^{-1} g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1' g_2 g_2^{-1} g_1) =$$

$$\frac{(z+1-q)(q^2-q+1)}{q} Q\, tr(t_1' \underline{g_2} t_1' g_1) + \frac{(z+1-q)(q^2-q+1)}{q}(Q-1)\, tr(\underline{g_2} t_1' g_1)+$$

$$(z+1-q)(q-1)Q\, tr(t_1' \underline{t_2'} g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1' g_1) \overset{3),4)}{=}$$

$$\frac{(z+1-q)(q^2-q+1)Q}{q} z\, tr(t_1' t_1' g_1) + \frac{(z+1-q)(q^2-q+1)(Q-1)}{q} z\, tr(t_1' g_1)+$$

$$(z+1-q)(q-1)Qs\, tr(t_1' g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1' g_1) =$$

$$\frac{(z+1-q)(q^2-q+1)Qz}{q}\, tr(\underline{t_1'}^2 g_1)+$$

$$\left[\frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + (z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] tr(t_1' g_1) =$$

$$\frac{(z+1-q)(q^2-q+1)Qz}{q} Q\, tr(t_1' g_1) + \frac{(z+1-q)(q^2-q+1)Qz}{q}(Q-1)\, tr(\underline{g_1})+$$

$$\left[\frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + (z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] tr(t_1' g_1) \overset{3)}{=}$$

$$\frac{(z+1-q)(q^2-q+1)Qz(Q-1)}{q} z + \left[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + \right.$$

$$\left.(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] tr(\underline{t_1'} g_1) =$$

$$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \left[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + \right.$$

$$\left.(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] tr(\underline{g_1} t g_1^{-1} g_1) \overset{3)}{=}$$

$$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \left[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + \right.$$

$$\left.(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] z\, tr(\underline{t}) \overset{4)}{=}$$

$$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \left[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + \right.$$

$$\left.(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\right] zs.$$

3.4 A HOMFLY-PT analogue for links in a solid torus

In 3.1 we presented briefly how V. Jones reconstructed the HOMFLY-PT polynomial.

Recall now (from 2.4.2 and 2.6) that if we consider the solid torus $M = S^3 \setminus \widehat{I}$, then oriented links in M can be represented by mixed braids in the groups $B_{n,1}$. Moreover, in Corollary 5 (end of section 2.6) we gave the algebraic version of Markov's theorem for isotopic links inside M.

3.4.1 A trace-invariant for solid-torus-links

In this section we show how to obtain a HOMFLY-PT-type (oriented) link invariant, for oriented links in M, using the trace function described above:

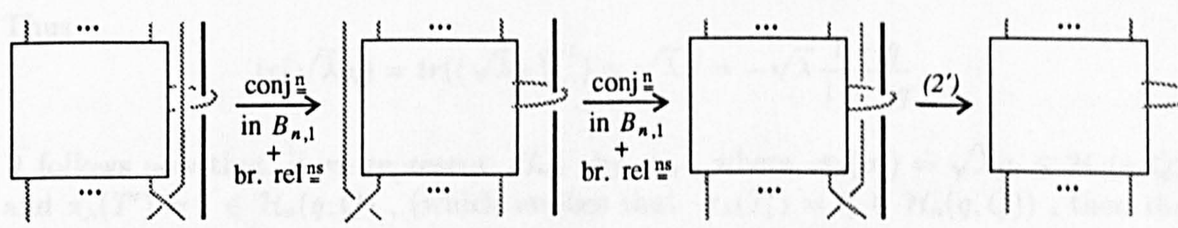
We notice first that move $(1')$ in Corollary 5 is analogous to rule 1) of Theorem 13.

Claim Move $(2')$ can be equivalently formulated by performing the Markov moves (with positive and negative crossing) at the right-hand side of the mixed braid, in such a way that the extra string is placed *under* the pure braid generators T_i''s. I.e:

$(2')$ $\alpha \sim \alpha\sigma_n{}^{\pm1} \in B_{n+1,1}$, where $\alpha \in B_{n,1}$ is a word in the σ_i's and T_i''s.

Picture:

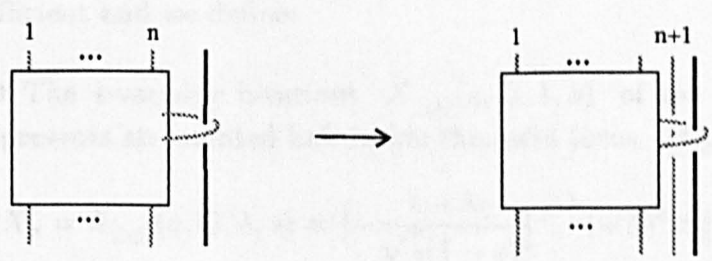
Proof of claim We only have to show that we do not need the Markov moves with the extra string placed *over* the T_i''s. (These moves would change T_{n-1}' in the word α to $\sigma_n T_n' \sigma_n{}^{-1}$). Indeed:



□

We can easily observe now that this version of move $(2')$ resembles rule 3) of Theorem 13.

We shall also need the natural inclusion of $B_{n,1}$ into $B_{n+1,1}$ as described in the following picture, (so that the direct limit $\bigcup_{n=1}^{\infty} B_{n,1}$ is well-defined):



Finally, recall the epimorphism, π say, of $\mathbb{C}B_{n,1}$ onto $\mathcal{H}_n(q,Q)$ defined by sending $T' \mapsto t = t'$, $\sigma_i \mapsto g_i$ (and therefore $T_i' \mapsto t_i'$, since $T_i' = \sigma_i \ldots \sigma_1 T' \sigma_1{}^{-1} \ldots \sigma_i{}^{-1}$ – as illustrated in 2.6.1 – and $t_i' = g_i \ldots g_1 t g_1{}^{-1} \ldots g_i{}^{-1}$).

The inclusion of $B_{n,1}$ into $B_{n+1,1}$ together with the epimorphism π and the trace function, imply that *to every mixed braid in* $B_{n,1}$ *we can assign an expression in the variables* q, Q, z, s .

So – exactly as in [30] – we reason that, in order to obtain a HOMFLY-PT-type link invariant we want 'to normalize the g_i's so that both Markov moves affect the trace in the same way' . I.e. we want to normalize g_i to θg_i , $\theta \in \mathbb{C}$, so as to obtain

$$tr(a(\theta g_n)) = tr(a((\theta g_n)^{-1})) \ \text{ for } \ a \in \mathcal{H}_n(q,Q).$$

(The normalization as well as the phrasing is the same as in [30], but for completeness we repeat it here adapted to our case). Then, for $z \neq 0$ we have:

$$\theta^2\, tr(ag_n) = tr(a{g_n}^{-1}) = \frac{1}{q}\, tr(ag_n) + \frac{1-q}{q}\, tr(a) \iff$$

$$\theta^2 z\, tr(a) = \frac{z+1-q}{q}\, tr(a) \iff$$

$$\theta^2 = \frac{z+1-q}{qz} = \lambda\ .$$

Thus

$$tr(\sqrt{\lambda} g_i) = tr((\sqrt{\lambda} g_i)^{-1}) = \sqrt{\lambda}\, z = -\sqrt{\lambda}\, \frac{1-q}{1-\lambda q}\ .$$

It follows now that, if we represent $B_{n,1}$ by π_λ , where $\pi_\lambda(\sigma_i) = \sqrt{\lambda}\, g_i \in \mathcal{H}_n(q,Q)$ and $\pi_\lambda(T') = t \in \mathcal{H}_n(q,Q)$, (which implies that $\pi_\lambda(T_i') = t_i' \in \mathcal{H}_n(q,Q)$) , then the function of q, λ, Q, s given by

$$\Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} tr(\pi_\lambda(\alpha)), \ \text{for } \alpha \in B_{n,1}\ ,$$

depends only on the mixed link $\widehat{\alpha}$ (the closure of α) . The epimorphism π , though, has the advantage of only involving the variables q, Q ; so we incorporate $\sqrt{\lambda}$ in the 'universal' coefficient and we define:

Definition 19 The 4-variable invariant $X_{L\cup\widehat{I}}(q, Q, \lambda, s)$ of the oriented mixed link $L \cup \widehat{I}$, that represents an oriented link inside the solid torus M , is the function:

$$X_\alpha = X_{L\cup\widehat{I}}(q, Q, \lambda, s) = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} (\sqrt{\lambda})^e\, tr(\pi(\alpha))$$

where $\alpha \in B_{n,1}$ is a word in the σ_i's and (T_i')'s such that $\widehat{\alpha} = L \cup \widehat{I}$, e is the exponent sum of the σ_i's that appear in α , and π the representation of $B_{n,1}$ in $\mathcal{H}_n(q,Q)$ such that $T' \mapsto t,\ \sigma_i \mapsto g_i$.

Note By their definition, the pure braid generators (T') and (T_i')'s do not affect the exponent sum e, so we can ignore them when we estimate e.

Examples

- As it follows from Remark 11, if α does not contain any (T_i'), then X_α is the HOMFLY-PT polynomial of the link in S^3 obtained by removing from α the 'solid torus' string I. So if, for instance, $\alpha = 1 \in B_{1,1}$, then $X_\alpha = 1$; and if $\alpha = 1 \in B_{n,1}$ (corresponding to the n component unlink), then

$$X_\alpha = \left[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right]^{n-1}.$$

- If $\alpha = T' \in B_{1,1}$, then $X_\alpha = s$; and if $\alpha = T_i' \in B_{n,1}$ (corresponding to the n component unlink, the $(i+1)$st string of which wraps around I once in a positive sense), then

$$X_\alpha = \left[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right]^{n-1} s$$

whilst, if $\alpha = (T_i')^{-1} \in B_{n,1}$ (corresponding to the n component unlink, the $(i+1)$st string of which wraps around I once in a negative sense), then

$$X_\alpha = \left[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right]^{n-1} \left[\frac{1}{Q}s + \frac{1-Q}{Q}\right].$$

- Similarly, if $\alpha = (T_i')^2 \in B_{n,1}$ (the n component unlink, the $(i+1)$st component of which wraps around I twice in a positive sense), then

$$X_\alpha = \left[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right]^{n-1} [(Q-1)s + Q].$$

- Finally, if $\alpha = \sigma_1{}^3(T')^2 \in B_{2,1}$ (a right-handed trefoil that wraps around I twice in a positive sense), then

$$X_\alpha = -\frac{1-\lambda q}{\sqrt{\lambda}(1-q)} (\sqrt{\lambda})^3\, tr(g_1{}^3 t^2), \quad \text{where}$$

$$tr(\underline{g_1}{}^3 t^2) = (q^2-q+1)\, tr(g_1 \underline{t}^2) + q(q-1)\, tr(\underline{t}^2) =$$

$$(q^2-q+1)(Q-1)\, tr(g_1 t) + (q^2-q+1)Q\, tr(g_1) + q(q-1)(Q-1)\, tr(t) + q(q-1)Q\, tr(1) =$$

$$(q^2-q+1)(Q-1)\tfrac{q-1}{1-\lambda q}s + (q^2-q+1)Q\tfrac{q-1}{1-\lambda q} + q(q-1)(Q-1)s + q(q-1)Q.$$

3.4.2 A note on skein relations

Let L_+, L_-, L_0 be oriented links that have diagrams identical, except in one crossing, where they are as depicted below:

L_+ $\qquad$ L_- $\qquad$ L_0

Then, one can find a recursive linear formula in L_+, L_-, L_0 – known as *skein rule*[2] – for defining the HOMFLY-PT polynomial (see [16] , [50] , [40] for full expositions).

In [30] is explained a way of finding the skein rule of the 2-variable polynomial that derives from Ocneanu's trace function. Here we modify this way, in order to find the skein relations of the trace-invariant we defined above:

We consider a mixed link, which may be assumed to be the closure of a mixed braid, and we pick a crossing in it, which is not a mixed one. Using conjugation, this crossing appears in the end of the word, and – again by conjugation – we may assume that $L_+ = \widehat{\alpha\sigma_i{}^2}$, $L_- = \widehat{\alpha}$ and $L_0 = \widehat{\alpha\sigma_i}$, for some $\alpha \in B_{n,1}$. By the defining relations of $\mathcal{H}_n(q,Q)$ we have

$$tr(\pi(\alpha\sigma_i{}^2)) - q\, tr(\pi(\alpha)) = (q-1)\, tr(\pi(\alpha\sigma_i)) \ .$$

Let e be the exponent sum of α with respect to the σ_i's , and multiply the above equation by $T\,\frac{(\sqrt{\lambda})^{e+1}}{\sqrt{q}}$, where

$$T = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1}.$$

Then

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\, T\,(\sqrt{\lambda})^{e+2}\, tr(\pi(\alpha\sigma_i{}^2)) - \sqrt{q}\sqrt{\lambda}\, T\,(\sqrt{\lambda})^{e}\, tr(\pi(\alpha))$$
$$= (\sqrt{q} - \frac{1}{\sqrt{q}})\, T\,(\sqrt{\lambda})^{e+1}\, tr(\pi(\alpha\sigma_i)) \ ;$$

so by the definition of X we obtain the skein relation:

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\, X_{L_+} - \sqrt{q}\sqrt{\lambda}\, X_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})\, X_{L_0} \qquad \bullet$$

(The above relation together with the initial condition in S^3 , $X(unknot) = 1$, define uniquely the HOMFLY-PT polynomial.) In the same manner, but with less difficulty, we obtain a second skein rule for the mixed braiding, that derives from the relation

$$t_i'^{-1} = \frac{1}{Q}\, t_i' + \frac{1-Q}{Q} \cdot 1$$

as follows: Let M_+, M_-, M_0 be oriented mixed links that have diagrams identical, except in the regions depicted below:

[2] As cited in [22], the skein theory was originally found by J.W. Alexander in [2], and – after being neglected for forty years – was re-discovered by J.H. Conway in [12].

M_+ M_- M_0

We consider a mixed link, which – as already mentioned – may be assumed to be the closure of a mixed braid, and we pick in it a positive mixed twist (as illustrated above). Note that, using conjugation in $B_{n,1}$, we can always create such a twist. Thus, by conjugation we may assume that $M_+ = \widehat{\alpha T_i'}$, $M_- = \widehat{\alpha T_i'^{-1}}$ and $M_0 = \widehat{\alpha}$, for some $\alpha \in B_{n,1}$. So we obtain:

$$tr(\pi(\alpha T_i'^{-1})) = \frac{1}{Q}\, tr(\pi(\alpha T_i')) + \frac{1-Q}{Q}\, tr(\pi(\alpha)) \,;$$

and, if we multiply the above equation by $T\,(\sqrt{\lambda})^e\sqrt{Q}$ we have

$$\sqrt{Q}\,T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha T_i'^{-1})) = \frac{1}{\sqrt{Q}}\,T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha T_i')) + \frac{1-Q}{\sqrt{Q}}\,T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha)) \,.$$

Hence, since the T_i''s do not change the exponent sum of α, neither the number of its strings, we obtain the following skein rule:

$$\frac{1}{\sqrt{Q}}\,X_{M_+} - \sqrt{Q}\,X_{M_-} = (\sqrt{Q} - \frac{1}{\sqrt{Q}})\,X_{M_0} \quad \bullet\bullet$$

One can check that the two skein rules together with the initial conditions

$$X_1 = 1 \,,\ 1 \in B_{1,1} \quad \text{and} \quad X_{T'} = s \,,\ T' \in B_{1,1} \quad \bullet\bullet\bullet$$

suffice to calculate X inductively for any mixed link; but one would also have to prove that X defined this way is well-defined, which is beyond our scope at the moment.

J. Hoste and M. Kidwell defined in [22] a 'new chromatic skein invariant for a special class of dichromatic links, which may be viewed as an invariant of oriented monochromatic links inside a solid torus; and this as such is the exact analogue of the HOMFLY-PT polynomial'. In their set-up, the 'solid torus' string $\widehat{I}$ is perpendicular to the plane on which the rest of the link projects, and it is allowed to move by isotopy. The theorem they proved in the preliminary version of [22] (Theorem 2.1) is the following, (where, for convenience, we use our notation for expressing the different links):

Theorem 2.1 *There exists a unique invariant* $W^i \in \mathbb{Z}[v^{\pm 1}, z_j{}^{\pm 1}, \alpha, x^{\pm 1}, \lambda^{\pm 1}, h_+]$, $j \neq i$, *of Type* I_i *links satisfying the following properties:*

1. *Crossing Rule:*

$$v^{-1}\,W_{L_+} - v\,W_{L_-} = z_j\,X_{L_0}$$

2. *Clasp Rule:*

$$x^{-1}\,W_{M_+} + x\,W_{M_-} = \alpha\,W_{M_0}$$

3. Connected Sum Rule:

$$W_{K conn.sum; J} = (v^{-1} - v)\, z_j{}^{-1}\, \lambda^{-1}\, W_J W_K$$

4. Initial Data:

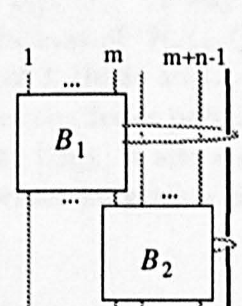
$$W_{\widehat{1}} = \lambda\ ,\ 1 \in B_{1,1} \quad \text{and} \quad W_{\widehat{T'}} = h_+\ ,\ T' \in B_{1,1}$$

where the i-coloured unknot corresponds to $\widehat{I}$ *, and the j-coloured components correspond to the rest of the mixed link.*

We can observe now that the Crossing Rule is the same as the skein rule • above, if we set $v = \sqrt{q}\sqrt{\lambda}$ and $z_j = \sqrt{q} - \frac{1}{\sqrt{q}}$; whilst the Clasp Rule resembles the skein rule •• above, if we set $x = \sqrt{Q}$ and $\alpha = \sqrt{Q} - \frac{1}{\sqrt{Q}}$, but apparently the two rules still differ by a sign. As J. Przytycki pointed out, we can show that the two rules are essentially the same if we substitute $x = iy$ and $\alpha = -i\alpha'$. This is a well-known trick in knot theory and an example of this being used can be found in [43], where W.B.R. Lickorish shows that the Kauffman polynomial and the Dubrovnik polynomial (see [32], [33], [34]) are equivalent. Also, the Initial Data are the same as in rule ••• , if we set $\lambda = 1$ and $h_+ = s$.

We can also observe that, in our set-up there does not appear any connected-sum rule for the component $\widehat{I}$ of two mixed links. The explanation lies in the fact that, in our set-up, the component $\widehat{I}$ of a mixed link as well as the string I of a mixed braid remain always pointwise fixed.

Aside If $L_1 \cup \widehat{I}$, $L_2 \cup \widehat{I}$ are two mixed links, and $B_1 \cup I$, $B_2 \cup I$ are two corresponding mixed braids, then, $(L_1 conn.sum L_2) \cup I$ corresponds to $(B_1 conn.sum B_2) \cup I$, as pictured below (compare with [30], page 351):



Note The second rule ('initial data' rule) in the corresponding published version of Theorem 2.1 (Theorem 3.1 in [22]) looks somehow different, but related to the conditions 2 , 3 and 4 of Theorem 2.1 .

3.5 A concluding discussion

We concentrate on the lens space $S^1 \times S^2 = L(p, 0)$. If we try to apply the above ideas, in order to obtain an oriented link invariant in $L(p, 0)$, we have to additionally normalize our trace function under the band moves; but, as already mentioned in Remark 9 , 2.6.2, the band moves cannot be expressed in a simple way in terms of the T_i''s . So, suppose we take a word in $\mathcal{H}_n(q,Q)$ written in the second canonical form (as given in page 83), we lift it in $B_{n,1}$ and we perform a band move in it. If we project the result on $\mathcal{H}_{n+1}(q,Q)$, it will not be a word in the canonical form any more, as it follows from Remark 9. This means that there *does not* exist a multiplicative way for normalizing the trace under the band move – as there was in 3.4.1 .

If we specialize, however , $q = 0$, then $g_i^{-1} = g_i$, which implies that $t_i = t_i'$, for $t_i = g_i \ldots g_1 t g_1 \ldots g_i$, and therefore, the band move behaves like a Markov move on the algebra level. So, we can obtain a weak polynomial invariant with variables Q, z, s by normalizing our trace exactly as in 3.4.1 . More precisely:

Definition 20 The 3-variable invariant $X_{L\cup\widehat{I}}(Q,z,s)$ of the oriented mixed link $L \cup \widehat{I}$, that represents an oriented link inside the space $L(p,0)$, is the function:

$$X_\alpha = X_{L\cup\widehat{I}}(Q,z,s) = \frac{1}{z^{n-1}} tr(\pi(\alpha))$$

where $\alpha \in B_{n,1}$ is a word in the σ_i's and (T_i')'s such that $\widehat{\alpha} = L \cup \widehat{I}$, and π the representation of $B_{n,1}$ in $\mathcal{H}_n(q,Q)$ such that $T' \mapsto t,\ \sigma_i \mapsto g_i$.

This invariant is not particularly interesting, as it only gives information about the permutation of the mixed braid and about the first homology group of the complement of the link in the given space.

The above suggest that we would need to find a family of traces (instead of only one), and take an appropriate linear combination of them, in order to define a trace-invariant for oriented links in $L(p,1)$. A way to obtain a family of traces, is by omitting one of the quadratic relations of $\mathcal{H}_n(q,Q)$. As mentioned in 0.5, this idea is strongly supported by the works of J. Hoste and J. Przytycki, who defined the analogue of the Kauffman bracket version of the Jones polynomial (see [31]) for lens spaces, using skein module theory (see [23], [24], [25]) ; it also seems to be related to the recent works of W.B.R. Lickorish (see [42]), where he gives a purely combinatorial way for viewing Witten's invariants (see [63]).

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