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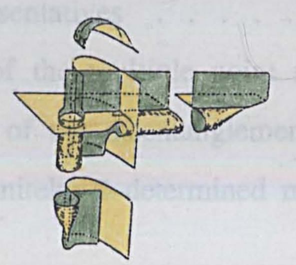
Mapping Fibrations

and

Multiple Point Schemes



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Declaration

The material included in this thesis is original, except where indicated otherwise. Chapter III is joint work with David Mond.

Summary

This work contains:
- a study of cetain fibrations associated to a finitely determined map-germ $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, $n < p$, and its multiple point schemes,
- relations between target and source invariants, and
- some remarks on the real case $f:(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$.

To Najla Rox Uzeizi Marar

Introduction

This work contains three chapters.

In chapter I we study fibrations associated to certain map-germs $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, $n<p$. It is divided into five paragraphs.

In paragraph one we select, among the representatives of a germ of an $\mathcal{A}_e$-versal unfolding $F:(\mathbb{C}^n{\times}\mathbb{C}^d,0) \to (\mathbb{C}^p{\times}\mathbb{C}^d,0)$, $F(x,t)=(f_t(x),t)$ of a discrete stable type map-germ $f_0:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, $n<p$, what we call a good representative of F. It is obtained by choosing a representative of F, again denoted by F and neighbourhoods U, W and Z of the origin in $\mathbb{C}^n{\times}\mathbb{C}^d$, $\mathbb{C}^p$ and $\mathbb{C}^d$ respectively so that:

(1) $F(U) \subseteq W{\times}Z$

(2) $F^{-1}(0) \cap U = \{0\}$

(3) $F:U \to W{\times}Z$ is a finite map, i.e., proper with finite fibres.

Also, if we consider the analytic subset $I_{rel}(F) = \Big\{ (y,t) \in W{\times}Z \subseteq \mathbb{C}^p{\times}\mathbb{C}^d$: the germ of f_t at $f_t^{-1}(y) \cap U_t$ is not stable$\Big\}$, where $U_t = \Big\{x \in \mathbb{C}^n : (x,t) \in U\Big\}$, then, for that choice of representative of F, the projection $\pi: W{\times}Z \to Z$ into the parameter space $Z \subseteq \mathbb{C}^d$ verifies :

(4) the restriction $\pi \mid I_{rel}(F)$ is a finite map,

and hence, for any neighbourhood W_1 of the origin in $\mathbb{C}^p$, with $\overline{W_1} \subseteq W \subseteq \mathbb{C}^p$, there exists a neighbourhood $Z_1 \subseteq Z$ of the origin in $\mathbb{C}^d$ such that :

(5) $I_{rel}(F) \cap (W{\times}Z_1) \subseteq W_1{\times}Z_1$.

In paragraph two we, loosely speaking, provide the image X of a good representative F, away from $I_{rel}(F)$, with a Whitney stratification. This is done in two ways ; either we consider the restriction $F : U_1 \to W\times(Z-B)$ of the good representative F, where $B = \pi\left(I_{rel}(F)\right) \subseteq \mathbb{C}^d$ is the bifurcation set of F, or we consider the restriction $F : U_2 \to (W-\overline{W_1})\times Z_1$. Then the stratification considered is that by stable types, as defined by Gaffney in [Ga 3].

The reason for considering these two restrictions of a good representative will become clear in paragraph three. There we apply the Second Isotopy Lemma to the sequence of mappings $U \xrightarrow{F} W\times Z \xrightarrow{\pi} Z$ where F is a good representative and π is the projection. In order to do so, we need not only that F is proper but also that π is proper. Hence, we replace the neighbourhood W by a closed ball $\overline{B_\varepsilon}$ of radius $\varepsilon > 0$ small and centred at the origin of $\mathbb{C}^p$. Then, we make sure that the boundary $S_\varepsilon\times Z_1$ is transverse to the image of the restriction $F : U_2 \to (W-\overline{W_1})\times Z_1$. Since the image of $F : U_1 \to W\times(Z-B)$ and the image of $F : U_2 \to (W-\overline{W_1})\times Z_1$ coincide in $(W-\overline{W_1})\times(Z_1-B)$ we obtain that $S_\varepsilon\times(Z_1-B)$ is transverse to the image of the restriction $F : U_1 \to W\times(Z_1-B)$ of the good representative F and, by the isotopy lemma, we have :

$F : U_1 \to X \subseteq \overline{B_\varepsilon}\times(Z_1-B)$ is locally trivial over Z_1-B with respect to π.

Consequently we obtain :

- a 'fibration' of the map $F : U_1 \to X$ whose 'fibres' are the mappings $f_t : U_t \to X_t$ which are topologically independent of the parameter $t \in Z_1-B$, since Z_1-B is connected and
- a locally trivial C^0-fibration of the image X whose fibres X_t are the image of the mappings f_t, above.

We call X_t the disentanglement of the image of the initial map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$.

In paragraph four, with the assumption that f_0 is of corank 1 at the origin, we obtain, for a given good representative $F : U_1 \to W\times(Z_1-B)$, Milnor fibrations over Z_1-B of the multiple point schemes $\tilde{D}^k(F,\gamma(k))$ of F (see [M-M], or equivalently chapter III of this thesis, for definitions and properties of $\tilde{D}^k(F,\gamma(k))$) as a consequence of the Ehresmann Fibration Theorem. The Milnor fibres are the multiple point schemes $\tilde{D}^k(f_t,\gamma(k))$ of the mapping f_t and the critical fibre is the ICIS $\tilde{D}^k(f_0,\gamma(k))$.

Thus, in the end of paragraph four we are equipped with many fibrations (over the same base) associated to a good representative. So, over a parameter t in the complement of the bifurcation set $B \subseteq \mathbb{C}^d$ we have the follwing diagram :

$$
\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & \\
\tilde{D}^k(f_t,\gamma(k)) \arrow[r] & \cdots \arrow[r] & \tilde{D}^k(f_t) \arrow[d] & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & \tilde{D}^2(f_t,(2)) \arrow[r] & \tilde{D}^2(f_t) \arrow[d] & \\
 & & U_t \arrow[r, "f_t"] & X_t
\end{tikzcd}
$$

Since the smooth spaces $\tilde{D}^k(f_t,\gamma(k))$ are Milnor fibres of the ICIS $\tilde{D}^k(f_0,\gamma(k))$, we have that the Euler characteristic of $\tilde{D}^k(f_t,\gamma(k))$ is related to the Milnor number of the ICIS $\tilde{D}^k(f_0,\gamma(k))$ as follows: $\chi\left(\tilde{D}^k(f_t,\gamma(k))\right) = 1 + (-1)^s \mu\left(\tilde{D}^k(f_0,\gamma(k))\right)$, where s is the complex dimension of $\tilde{D}^k(f_0,\gamma(k))$.

So, in the diagram above, all spaces, but X_t, have their Euler characteristic related to Milnor numbers of ICIS (here we observe that we can take U_t small enough so that it is closed and contractible).

Finally, in paragraph five we bring X_t into the scene, i.e., we show how the disentanglement X_t can also have its Euler characteristic related to Milnor numbers of ICIS. This relation is obtained by firstly relating (by combinatorial methods) the Euler characteristic of X_t and the Euler characteristic of the multiple point schemes of f_t (Theorem (5.12)):

$$\chi(X_t) = 1 + \sum_{r\geq 2} \sum_{\gamma(r)} \frac{-1}{\prod_{i\geq 1} i^{\alpha_i} \alpha_i!} (-1)^{\sum \alpha_i} \chi\left(\tilde{D}^r(f_t, (\gamma(r))\right)$$

where $\gamma(r) = (a_1, ..., a_h)$ runs through the set of partitions of r, with $a_i \geq a_{i+1}$, $p - k(p - n + 1) + h \geq 0$ and $\alpha_i = \# \left\{ j : a_j = i \right\}$. Here we shall understand that if $\tilde{D}^r(f_t, (\gamma(r))$ is empty then the coefficient of its Euler characteristic in the formula is zero.

Equivalently, if we replace $\chi\left(\tilde{D}^k(f_t)\right)$, in the expression above, by $\chi\left(\tilde{D}^k(f_t)/ S_k\right)$, through the formula (Proposition (5.16)) :

$$\chi\left(\tilde{D}^r(f_t)/ S_r\right) = \sum_{\gamma(r)} \frac{1}{\prod_{i\geq 1} i^{\alpha_i} \alpha_i!} \chi\left(\tilde{D}^r(f_t, (\gamma(r))\right)$$

we end up with (theorem (5.18)) :

$$\chi(X_t) = 1 + \sum_{k\geq 2} (-1)^{k-1} \chi\left(\tilde{D}^k(f_t)/ S_k\right) +$$

$$+ \sum_{k\geq 2} \sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\sum \alpha_i}}{\prod_{i\geq 1} i^{\alpha_i} \alpha_i!} \chi\left(\tilde{D}^k(f_t, (\gamma(k))\right)$$

where $\gamma(k)=(a_1, ..., a_h) \neq (1^k)$ runs through the set of (ordered) partitions of k with $p-k(p-n+1)+h \geq 0$ and $\alpha_i = \# \left\{ j : a_j = i \right\}$.

Finally, substituting $\chi\left(\tilde{D}^r(f_t,(\gamma(r))\right)$ by $\mu\left(\tilde{D}^r(f_0,(\gamma(r))\right)$ through the relation :

$\chi\left(\tilde{D}^r(f_t,\gamma(k))\right) = 1 + (-1)^s\mu\left(\tilde{D}^r(f_0,\gamma(r))\right)$, where s is the complex dimension of $\tilde{D}^r(f_0,\gamma(r))$ we obtain (5.20) :

$$\chi(X_t) = C_{\gamma(k)} + \sum_{k=2}^{p} (-1)^{p-k(p-n+1)+1} \left[\mu\left(\tilde{D}^k(f_0)/S_k\right) + \sum_{\gamma(k)} \frac{1+(-1)^{k+\sum\alpha_i+1}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \mu\left(\tilde{D}^k(f_0,(\gamma(k))\right)\right]$$

where $C_{\gamma(k)} = 1 + \sum_{k=2}^{p} \left((-1)^{k+1} + \sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\sum\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \right)$

When (n,p) = (2,3), i.e., $f_0: (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, and the multiple point schemes of f_0 $\tilde{D}^2(f_0)/S_2$, $\tilde{D}^2(f_0,(2))$ and $\tilde{D}^3(f_0)/S_3$ are all non-empty, this formula is of particular interest :

$$\chi(X_t) = \mu\left(\tilde{D}^2(f_0)/S_2\right) + C(f_0) + T(f_0),$$

where $C(f_0)$ and $T(f_0)$ is respectively the number of cross-caps and the number of triple points of f_0.

In a recent work [Mo 4], David Mond proves that the disentanglements of the image of mappings $f_0: (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ have the same homotopy type of a wedge of n - spheres. Hence, when p = n+1, the formula (5.20) gives the number of n - spheres in that wedge. This result resembles that of J. Milnor ([Mi] chap.7) or H. Hamm ([Ha]).

In chapter II we deal with simple singularities of corank 1 map-germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$. It is divided in three paragrahs.

In paragraph one we present *morsifications* for the simple singularities, i.e. stable deformations for the map-germ f, such that their images in $\mathbb{R}^3$ present the correct number of cross-caps (Whitney umbrellas) and triple points.

Paragraph two is pictorial. In it we present drawings of what could be called the disentanglement in the real case for the image of simple singularities of map-germs from $\mathbb{R}^2$ into $\mathbb{R}^3$.

In paragraph three we calculate the singular homology of the image of morsifications for the family of singularities H_k.

Chapter III is joint work with David Mond and contains its own introduction.

There one finds results that are used in the previous chapters, namely : definition and properties of the multiple point schemes of maps and a characterisation of stability and finite determinacy of corank 1 map-germs (Theorem 2.14) ; a corank 1 map-germ is finitely determined if and only if each multiple point scheme of dimension at least 1 is an ICIS, and is stable if moreover each non-empty multiple point scheme is smooth. We end the chapter with a relation between source and target invariants of finitely $\mathcal{A}$-determined map-germs $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ (Theorem 3.4) ;

$$\mu\left(D^2(f)\right) = 6\,T(f) + C(f) + 2\,\mu\left(\tilde{D}^2(f)/S_2\right) - 1,$$

where $D^2(f)$ denotes the double point curve of f in $\mathbb{C}^2$.

Chapter I

Mapping Fibrations

"... there is often a creative tension between geometry and rigor. Rigor follows the initial conception with a much greater time delay in geometry than it does in algebra. Also, when it comes, true geometers often feel its language misses the essential geometric ideas. Language is not well adapted to describing geometry, as the facilities for language and geometry live on opposite sides of the human brain. This perhaps accounts for the presence in the current literature on singularities of expressions like "using the isotopy lemma, it can be shown" without the forty pages of geometric constructions and estimates needed to apply the isotopy lemma."

Mark Gorensky & Robert Macpherson
([G-M] p. 22)

§1. Good representatives of an unfolding of a finitely determined map-germ

The main reference for this paragraph is Grauert-Remmert's Theory of Stein Spaces [G-R] chapter 1.

We start with some properties of finitely $\mathcal{K}$-determined map-germs.

(1.1) Lemma : If $h : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is a finitely $\mathcal{K}$-determined map-germ then there exists a neighbourhood U of the origin in $\mathbb{C}^n$ such that $\left(h \mid U \cap \Sigma(h) \right)^{-1}(0) = \{0\}$, where $\Sigma(h)$ is the set of critical points of h.

Proof : (see [Ga 1] p. 66).

(1.2) Proposition : Let $f : X \to Y$ be a holomorphic map such that x_0 is an isolated point of the fibre $f^{-1}(f(x_0))$. Then there exist neighbourhoods U and V of x_0 in X and $f(x_0)$ in Y respectively, with $f(U) \subseteq V$ and such that the restriction $f \mid U: U \to V$ is a finite map, ie, a closed map with finite fibres.

Proof: (see [G-R] p. 54).

(1.3) Corollary : Let $F: (\mathbb{C}^n \times \mathbb{C}^d,0) \to (\mathbb{C}^p \times \mathbb{C}^d,0)$ be an unfolding of a finitely $\mathcal{K}$-determined map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, given by $F(x,t) = (f_t(x),t)$, with $n < p$. Then for any representative of F there exist neigbourhoods U and V of the origin in $\mathbb{C}^n \times \mathbb{C}^d$ and $\mathbb{C}^p \times \mathbb{C}^d$ respectively such that:

$$F^{-1}(0) \cap U = \{0\},$$
$$F(U) \subseteq V$$

and the restriction $F \mid U : U \to V$ is a finite map.

Proof: Since an unfolding of a finitely $\mathcal{K}$-determined map-germ is a finitely

$\mathcal{K}$-determined map-germ itself, by lemma (1.1) there exists a neighbourhood U′ of $\mathbb{C}^n \times \mathbb{C}^d$ such that $\left(F \mid (U' \cap \Sigma(F)\right)^{-1}(0) = \{0\}$. In our case, $n < p$, $U' \cap \Sigma(F) = U'$. Now by proposition (1.2), there exist neighbourhoods U and V of the origin in the source and target of F respectively such that $F \mid U : U \to V$ is a finite map.

(1.4) Definition : A *finite representative* of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$, of a finitely $\mathcal{K}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$ is a triple $(F \mid U, U, V)$ as in the corollary above. We shall sometimes denote the restriction $F \mid U$ simply by F.

Next we state some properties of finite maps.

(1.5) Theorem : (The Direct Image Theorem for Finite Maps)

Let $F : X \to Y$ be a finite holomorphic map and $\mathcal{S}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the direct image $F_*(\mathcal{S})$ is a coherent sheaf of $\mathcal{O}_Y$- modules.

Proof : (see [G–R] p. 55)

(1.6) Lemma : If $f : X \to Y$ is a closed map and U a neighbourhood of $f^{-1}(y)$ in X, for some $y \in Y$, then there exists a neighbourhood V of y in Y such that $f^{-1}(V) \subseteq U$.

Proof : Just take $V = Y - f(X - U)$.

(1.7) Theorem (The Projection Theorem)

Let $\mathcal{S}$ be a coherent analytic sheaf on a neighbourhood V of the origin (0,0) in $\mathbb{C}^p \times \mathbb{C}^d$. Suppose that the origin (0,0) is an isolated point in $\text{supp}(\mathcal{S}) \cap (\mathbb{C}^p \times \{0\})$. Then there exist neighbourhoods W and Z of the origin in $\mathbb{C}^p$ and $\mathbb{C}^d$ respectively, with $W \times Z \subseteq V$ such that the following hold for the projection $\pi : W\times Z \to Z$:

(i) the restriction $\pi | (\mathrm{supp}(\mathcal{S}) \cap (W\times Z))$ is a finite map,

(ii) the direct image $\pi_*(\mathcal{S}_{W\times Z})$ of the restriction of $\mathcal{S}$ to $W\times Z$ is a coherent sheaf of $\mathcal{O}_Z$-modules.

Proof : (see [G-R] p.53).

We now return to our finite representative $F : U \to V$ with $U \subseteq \mathbb{C}^n\times\mathbb{C}^d$ and $V \subseteq \mathbb{C}^p\times\mathbb{C}^d$ as in (1.4).

In V we consider the set $I_{rel}(F) = \{(y,t) \in V :$ the germ of f_t at $f_t^{-1}(y) \cap U_t$ is not stable$\}$, where $U_t = \{ x \in \mathbb{C}^n : (x,t) \in U \}$.

(1.8) Proposition : $I_{rel}(F)$ is an analytic subset of V.

For the sake of completeness we include the proof (cf. [Da] p. 310).

Proof : We are going to define a coherent sheaf $\mathcal{A}_{rel}(F)$ of $\mathcal{O}_V$-modules such that $I_{rel}(F) = \mathrm{supp}(\mathcal{A}_{rel}(F))$.

Let us consider

$\theta_{n,d}(U)$ the sheaf of analytic vector fields germs "in the $\mathbb{C}^n$ direction" on U,
$\theta_{p,d}(V)$ the sheaf of analytic vector fields germs "in the $\mathbb{C}^p$ direction" on V and
$\theta(F)$ the sheaf of germs of sections of $F^*T(\mathbb{C}^p\times\mathbb{C}^d)$ ("the vector fields along F").

Then, if $x_1,\dots,x_n$ are coordinates on $\mathbb{C}^n$ and $Y_1,\dots,Y_p$ on $\mathbb{C}^p$,
$\theta_{n,d}(U)$ is the free sheaf of $\mathcal{O}_U$-modules on $\partial/\partial x_1, \dots, \partial/\partial x_n$ and
$\theta_{p,d}(V)$ is the free sheaf of $\mathcal{O}_V$-modules on $\partial/\partial Y_1, \dots, \partial/\partial Y_p$.

Let us define :

$\overline{t}F : \theta_{n,d}(U) \to \theta(F)$ by $\sum \alpha_i\, \partial/\partial x_i \to \sum \alpha_i\, \partial f_t/\partial x_i$. Then $\overline{t}F$ is a sheaf homomorphism. Thus, $\mathcal{R}_{rel}(F) :=$ coker $(\overline{t}F)$ is a coherent sheaf of $\mathcal{O}_U$-modules. Hence by (1.5), the direct image $F_*\mathcal{R}_{rel}(F)$ is a coherent sheaf of $\mathcal{O}_V$-modules.

Finally we define :

$\overline{w}F : \theta_{p,d}(V) \to F_*\mathcal{R}_{rel}(F)$ by $\sum \beta_i\, \partial/\partial Y_i \to$ the class of $\sum(\beta_i \circ F)\, \partial/\partial Y_i$ in $F_*\mathcal{R}_{rel}(F)$. Then $\overline{w}F$ is a sheaf homomorphism. Thus, $\mathcal{A}_{rel}(F) := \operatorname{coker}(\overline{w}F)$ is a coherent sheaf of $\mathcal{O}_V$-modules.

Now for a given $(y,t) \in V$ the germ $\beta_i \circ F$ is defined as a germ at $F^{-1}(y,t) \cap U$ and the stalk of $\mathcal{A}_{rel}(F)$ at (y,t) is zero if and only if the germ of f_t at $f_t^{-1}(y) \cap U_t$, $(U_t = \{x \in \mathbb{C}^n : (x,t) \in U\})$ is infinitesimally stable. Since infinitesimal stability implies stability, it follows that $I_{rel}(F) = \operatorname{supp}(\mathcal{A}_{rel}(F))$.

(1.9) Having obtained $I_{rel}(F) \subseteq V \subseteq \mathbb{C}^p \times \mathbb{C}^d$ as an analytic set, we now want to project it into the parameter space $\mathbb{C}^d$ and we want the image of $I_{rel}(F)$ in $\mathbb{C}^d$ to be an analytic set as well. The Projection Theorem (1.7) will do the job. So, we need to have the origin $(0,0) \in V \subseteq \mathbb{C}^p \times \mathbb{C}^d$ as an isolated point in $\operatorname{supp}(\mathcal{A}_{rel}(F)) \cap (\mathbb{C}^p \times \{0\})$. But the supposition on the initial map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, that it be finitely $\mathcal{K}$-determined, does not imply that hypothesis. Thus we shall restrict ourselves to a smaller class of map-germs, namely finitely $\mathcal{A}$-determined map-germs.

(1.10) Theorem : Let $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ be a finitely $\mathcal{K}$-determined map-germ, $n < p$. Then $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is finitely $\mathcal{A}$-determined if and only if for each representative f of f_0 there exists a neighbourhood U of the origin in $\mathbb{C}^n$ such that if $U \cap f^{-1}(y) = \{x_1,\dots,x_k\} = S$, then the multigerm of f at S is stable, for all $y \neq 0$, but near 0.

Proof : (cf. [Ga 1] ch. 3)

With the notation of (1.8), we have $\mathcal{A}(f_0) := \mathcal{A}_{rel}(F) \otimes_{\mathcal{O}_{\mathbb{C}^{p+d}}} \mathcal{O}_{\mathbb{C}^d}$ is a coherent sheaf with stalk at 0, $\mathcal{A}(f_0)_0 = \theta(f_0)/T\mathcal{A}_e(f_0)$. Now, the theorem follows from the Nullstellensatz for coherent sheaves. Indeed, $\dim_{\mathbb{C}} \mathcal{A}(f_0)_0 < \infty$ if, and only if 0 is an isolated point in $\mathrm{supp}\,(\mathcal{A}(f_0))$.

(1.11) So, if $F : U \to V$ is a finite representative of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$, of a finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$ then by the theorem above, the origin $(0,0) \in \mathbb{C}^p \times \mathbb{C}^d$ is an isolated point in $\mathrm{supp}\,(\mathcal{A}_{rel}(F)) \cap (\mathbb{C}^p \times \{0\})$.
Now, applying theorem (1.7) to the projection $\pi : \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^d$ and $\mathcal{S} = \mathcal{A}_{rel}(F)$, noticing that if a map $g : X \to Y$ is a finite then $g \,|\, g^{-1}(V) : g^{-1}(V) \to V$ is again finite, for any open V in Y, we obtain:

(1.12) Proposition : Let $F : U \to V$ be a finite representative of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$, of a finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$ and let $\pi : \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^d$ be the projection. Then, there exist neighbourhoods W and Z of the origin of $\mathbb{C}^p$ and $\mathbb{C}^d$ respectively such that :

(i) $\pi \,|\, (\mathrm{supp}\,(\mathcal{A}_{rel}(F)) \cap (W \times Z))$ is a finite map,

(ii) $\pi\,(I_{rel}(F) \cap (W \times Z))$ is an analytic subset of $Z \subseteq \mathbb{C}^d$ and

(iii) $F \,|\, U_1 : U_1 \to W \times Z$ is a finite map, where $U_1 = F^{-1}(W \times Z)$.

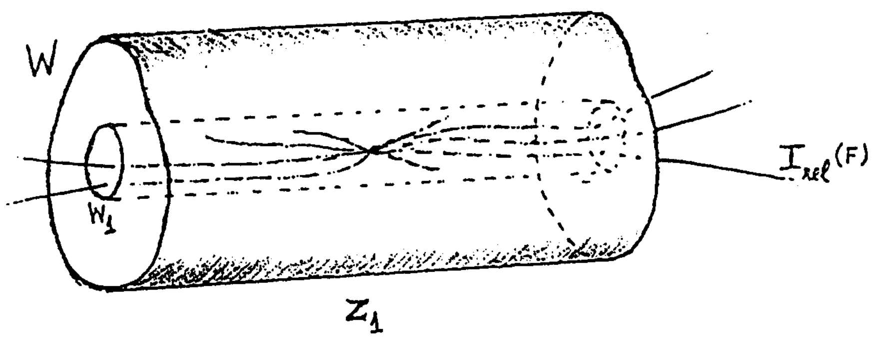
We shall denote the restriction $F \,|\, U_1 : U_1 \to W \times Z$ simply by $F : U_1 \to W \times Z$.

(1.13) Definition : By a *good representative* of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$, of a finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$ we shall understand a triple $(F, U_1, W \times Z)$, as in the proposition above.

(1.14) Remarks : (i) A good representative $F : U_1 \to W \times Z$ will always come together with a projection $\pi : W \times Z \to Z \subseteq \mathbb{C}^d$ such that and $\pi \,|\, (I_{rel}(F) \cap (W \times Z))$ is a finite map and $\pi\,(I_{rel}(F) \cap (W \times Z)) = B$ is an analytic set in $Z \subseteq \mathbb{C}^d$. The set

B is called the *bifurcation set* of the unfolding F.

(ii) let $\rho = \pi | (I_{rel}(F) \cap (W \times Z))$. Then ρ is a closed map. Thus, given a neighbourhood $I_{rel}(F) \cap (W_1 \times Z)$ of $\rho^{-1}(0)$, where $W_1 \subseteq W \subseteq \mathbb{C}^p$ is a neighbourhood of the origin in $\mathbb{C}^p$, with $\overline{W_1} \subseteq W$, it follows from (1.6) that there exists a neighbourhood Z_1 of the origin in $\mathbb{C}^d$, $Z_1 \subseteq Z$ such that $\rho^{-1}(Z_1) = I_{rel}(F) \cap (W \times Z_1) \subseteq W_1 \times Z$. In particular, $I_{rel}(F) \cap (W \times Z_1) \subseteq W_1 \times Z_1$.



(1.15) Note : In the next paragraph we shall need to restrict our good representative $F: U_1 \to W \times Z$ so that its image avoids the analytic set $I_{rel}(F)$ in $\mathbb{C}^p \times \mathbb{C}^d$, i.e., the points (y,t) in the image of F such that the germ of f_t at $f_t^{-1}(y) \cap U_t$ is not stable, where $U_t = \left\{ x \in \mathbb{C}^n : (x,t) \in U_1 \right\}$. This will be done in two ways :

either we take $F | U_2 : U_2 \to W \times Z_2$ with $Z_2 = Z - B$ and $U_2 = F^{-1}(W \times Z_2)$,

or we take $F | U_3 : U_3 \to W_2 \times Z_1$ with $W_2 = W - \overline{W_1}$, Z_1 as in (1.14)(ii) and $U_3 = F^{-1}(W_2 \times Z_1)$.

We shall also be considering the closed image X (resp. X') in $W\times(Z-B)$ $\left(\text{resp. } (W-\overline{W_1})\times Z_1\right)$ of $F\,|\,\overline{U_2'} : \overline{U_2'} \subseteq U_2 \to W\times Z_2$ $\left(\text{resp. } F\,|\,\overline{U_3'} : \overline{U_3'} \subseteq U_3 \to W_2\times Z_1\right)$ where $\overline{U_2'}$ $\left(\text{resp. } \overline{U_3'}\right)$ is the closure of $U_2' = F^{-1}(W'\times(Z'-B))$ $\left(\text{resp. } U_3' = F^{-1}((W'-\overline{W_1})\times Z_1')\right)$ with $W' \subseteq \mathbb{C}^p$ and $Z' \subseteq \mathbb{C}^d$ neighbourhoods of the origin such that $\overline{W'} \subseteq W$ and $\overline{Z'} \subseteq Z$ $\left(\text{resp. } \overline{W_1} \subseteq \overline{W'} \subseteq W \text{ and } \overline{Z_1'} \subseteq Z_1\right)$.

§2. Stratification of good representatives

Let $F : U_1 \to W \times Z$ be a good representative of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x),t)$ of a finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$.

In this paragraph we shall stratify the image of the good representative F , away from the analytic set $I_{rel}(F) \cap (W \times Z)$ in $\mathbb{C}^p \times \mathbb{C}^d$ (ie, we are going to consider any of the restrictions of the good representative $F : U_1 \to W \times Z$ mentioned in (1.15)), having in mind that in the next paragraph we shall apply Thom's Second Isotopy Lemma to the sequence :

$$U_1 \xrightarrow{F} W \times Z \xrightarrow{\pi} Z$$

The main references for this paragraph are [Ga 3], [Mo 3], [JM] and [Gi].

We begin with some definitions (see [Gi]).

(2.1) Definition : Let $F : U \to V$ be a smooth map. A *stratification of F* is a pair ($\mathcal{A}_U, \mathcal{A}_V$) with $\mathcal{A}_U$ and $\mathcal{A}_V$ Whitney stratifications of U and V respectively such that :

(i) F maps strata to strata

(ii) if $A_U \in \mathcal{A}_U$ is mapped by F into $A_V \in \mathcal{A}_V$ then $F : A_U \to A_V$ is a submersion.

(2.2) Defintition : Let $F : U \to V$ be a smooth map and $M, N \subseteq U$ submanifolds such that $F \mid M$ and $F \mid N$ have constant rank. Let $x \in M$. We say that N is *Thom regular* over M at x relative to F when :
given a sequence (y_i) in N, $y_i \to x$ such that $\ker T_{y_i}(F \mid M) \to \tau$ (in the appropriate Grassmannian) then $\ker T_x(F \mid N) \subseteq \tau$. We say that N is Thom regular over M relative to F if it is Thom regular over M at every point of M.

(2.3) Definition : A *Thom stratification* of a map $F: U \to V$ is a stratification $(\mathcal{A}_U, \mathcal{A}_V)$ of F such that for any two strata A_{U_1}, A_{U_2} of $\mathcal{A}_U$ we have A_{U_2} Thom regular over A_{U_1} relative to F.

(2.4) Remark : If $\mathcal{A}_U$ and $\mathcal{A}_V$ are Whitney stratifications of U and V respectively, $F: U \to V$ maps stratum into stratum and for each stratum $A_U \in \mathcal{A}_U$, $F|A_U$ is a local diffeomorphism then $(\mathcal{A}_U, \mathcal{A}_V)$ is a Thom stratification of $F: U \to V$.

Indeed, for any stratum $A_U \in \mathcal{A}_U$ we have, $\ker T_x(F|A_U) = 0$, $x \in A_U$. So, Thom's regularity condition holds trivially.

(2.5) Let $F: U_1 \to W \times Z$ be a good representative, as in the beginning of this paragraph. Then away from the analytic set $I_{rel}(F) \cap (W \times Z) \subseteq \mathbb{C}^p \times \mathbb{C}^d$ (defined in (1.8)), F is locally trivial.

In other words, let $F: U_2 \to W \times (Z-B)$ where B is the bifurcation set of F defined in (1.14)(i). Let $(y,t) \in \left(W \times (Z-B)\right)$ and $F^{-1}(y,t) \cap U_2 = \left\{(x_1,t),\dots,(x_k,t)\right\}$ for some k. Then the germ of f_t at $\left\{x_1,\dots,x_k\right\}$ is stable.
So $F: \left(U_2, \left\{(x_1,t),\dots,(x_k,t)\right\} \right) \to \left(W \times (Z-B), (y,t)\right)$ is a trivial unfolding of $f_t: \left(U_2 \cap U_t, \left\{x_1,\dots,x_k\right\}\right) \to \left(W,y\right)$, where $U_t = \left\{x \in \mathbb{C}^n : (x,t) \in U_2\right\}$. Hence there exist diffeomorphisms φ and ψ such that the diagram

$$\begin{tikzcd}
\left(U_2,\left\{(x_1,t),\dots,(x_k,t)\right\}\right) \arrow[r, "F"] \arrow[d, "\varphi"'] & \left(W \times (Z-B),(y,t)\right) \arrow[d, "\psi"] \\
\left(U_2,\left\{(x_1,t),\dots,(x_k,t)\right\}\right) \arrow[r, "f_t \times 1_{(Z-B)}"] & \left(W \times (Z-B),(y,t)\right)
\end{tikzcd}$$

commutes. So, F is locally trivial over $(Z-B) \subseteq \mathbb{C}^d$.

(2.6) The same reasoning applies if instead of the neighbourhoods W and (Z-B) as above, we consider respectively $W_2 = W - \overline{W_1}$ and Z_1 as in (1.14)(ii). In this case $F: U_3 \to W_2 \times Z_1$ is also locally trivial over Z_1, where $U_3 = F^{-1}(W_2 \times Z_1) \cap U_1$.

Next we are going to provide the (closed) image $X = F(\overline{U_2'})$ in $W \times (Z-B) \subseteq \mathbb{C}^p \times \mathbb{C}^d$ (resp. $X' = F(\overline{U_3'})$ in $W_2 \times Z_1 \subseteq \mathbb{C}^p \times \mathbb{C}^d$) of $F | \overline{U_2'} : \overline{U_2'} \subseteq U_2 \to W \times (Z-B)$ (resp. $F | \overline{U_3'} : \overline{U_3'} \subseteq U_3 \to W_2 \times Z_1$) (notation as in (1.15)), with a Whitney stratification.

(2.7) There is a natural partition of the image X of F, away from $I_{rel}(F)$, by stable types that we describe below (cf. [Ga 3], [Mo 3]):

Let (y,t) and (y',t') be points in X.

We say that (y,t) is equivalent to (y',t'), and write $(y,t) \sim (y',t')$, if $f_t : (U_2 \cap U_t, f_t^{-1}(y) \cap U_t) \to (W,y)$ and $f_{t'} : (U_2 \cap U_{t'}, f_{t'}^{-1}(y') \cap U_{t'}) \to (W,y')$ are $\mathcal{A}$-equivalent. This is obviously an equivalence relation.

Since $X \cap I_{rel}(F) = \emptyset$ then f_t and $f_{t'}$ are stable. So, we can specify each equivalence class of $\sim$ by the local algebra $Q(f_t)_S$ of f_t at $S = f_t^{-1}(y) \cap U_t$ (cf. [Ma 3]).

Let us denote by $Q_R(F)_{(y,t)}$ the connected component of the equivalence class of $\sim$ through (y,t). We shall refer to $Q_R(F)_{(y,t)}$ as the *stable type stratum* of X through (y,t) (or as the target stratum of F). We denote by $\mathcal{S}_R(F)$ the set of all $Q_R(F)_{(y,t)}$.

(2.8) Throughout we are going to assume that $\mathcal{S}_R(F)$ has only finitely many strata. Finitely $\mathcal{A}$-determined map-germs $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ for which this is verified are called by T.Gaffney $\left(\text{in } [\text{Ga } 3]\right)$ *discrete stable type map-germs.* Examples of these are corank 1 map-germs or all finitely $\mathcal{A}$-determined map-germs with (n,p) in the range of nice dimensions, according to Mather $\left(\text{cf. } [\text{Ma } 6]\right)$.

(2.9) Proposition : The partition $\mathcal{S}_R(F)$ is a Whitney stratification of the image X in $W\times(Z-B) \subseteq \mathbb{C}^p\times\mathbb{C}^d$ of the restriction $F|\overline{U_2'} : \overline{U_2'} \subseteq U_2 \to W\times(Z-B)$ of the good representative $F : U_1 \to W\times Z$.

Proof : (i) $\mathcal{S}_R(F)$ is locally finite

This follows from (2.8).

(ii) Each stratum $Q_R(F)_{(y,t)}$ is smooth. (cf. [Ma 4])

In fact, $Q_R(F)_{(y,t)} := \left\{ (y',t') \in W\times(Z-B) : Q(f_t)_S \approx Q(f_{t'})_{S'} \right\}$,

where $S = f_t^{-1}(y) \cap U_t = \{x_1,\dots,x_k\}$ and $S' = f_{t'}^{-1}(y') \cap U_{t'} = \{x_1',\dots,x_k'\}$.

Since $f_t : (U_2 \cap U_t, S) \to (W,y)$ is a stable multigerm, then each map-germ $f_t^{(i)} : (U_2 \cap U_t, x_i) \to (W,y)$ is stable. Therefore, there exists a representative of $f_t^{(i)}$, again denoted by $f_t^{(i)}$, defined in a neighbourhood N_i of x_i, $N_i \subseteq (U_2 \cap U_t) \subseteq \mathbb{C}^n$, such that $Q_D(f_t^{(i)})_{x_i} := \left\{ x_i' \in N_i \subseteq \mathbb{C}^n : Q(f_t^{(i)})_{x_i} \approx Q(f_t^{(i)})_{x_i'} \right\}$ is smooth.

Furthermore, since $f_t^{(i)}$ maps $Q_D(f_t^{(i)})_{x_i}$ immersively into $W \subseteq \mathbb{C}^p$, we can take the neighbourhood N_i sufficiently small so that $Q_R(f_t^{(i)})_y := f_t^{(i)}(Q_D(f_t^{(i)})_{x_i})$ is smooth.

From the stability of f_t we have that the tangent spaces $T_y\left(Q_R(f_t^{(i)})_y\right)$ are in general position in $T_y\mathbb{C}^p$. Thus, $Q_R(f_t)_y := \bigcap_{i=1}^{k} Q_R(f_t^{(i)})_y$ is smooth.

Finally, by the local triviality of F over Z-B (2.5), we have $Q_R(f_t)_y \times (Z-B) \approx Q_R(F)_{(y,t)}$. Hence $Q_R(F)_{(y,t)}$ is smooth.

(iii) Whitney's regularity condition (b) is verified for any pair of strata of $\mathcal{S}_R(F)$.

In fact, it is immediate from Kuo's ratio test ([Ku]) that each stratum which has a zero-dimensional stratum in its closure is regular over it. Further, we can always reduce checking the regularity condition to the case of a pair of strata where one of them is zero dimensional. Indeed, if $Q_R(f_t)_y$ is an r-dimensional stratum then $f_t:(U_2 \cap U_t, f_t^{-1}(y) \cap U_t) \to (W,y)$ is equivalent to an r-parameter unfolding of some stable multigerm $\tilde{f}_V: (U_V, \tilde{f}_V^{-1}(z) \cap U_V) \to (W', z)$ with $U_V \subseteq \mathbb{C}^{n-r}$ and $W' \subseteq \mathbb{C}^{p-r}$, and this equivalence extends to $F: U_2 \to W \times (Z-B)$. Since the strata in $\mathbb{C}^{p-r}$ are regular over $Q_R(\tilde{f}_V)_z$, because it is 0-dimensional, they are regular over $Q_R(f_t)_y$ in $\mathbb{C}^p$ and over $Q_R(F)_{(y,t)}$ in $\mathbb{C}^p \times \mathbb{C}^d$.

(iv) The frontier condition, ie, if S_1 and S_2 are strata and $\overline{S_1} \cap S_2 \neq \emptyset$ then $S_2 \subseteq \overline{S_1}$ follows from [JM] proposition (8.7).

Hence, $\mathcal{S}_R(F)$ is a Whitney stratification of the image X in $W \times (Z-B) \subseteq \mathbb{C}^p \times \mathbb{C}^d$ of the restriction $F|\overline{U_2'}: \overline{U_2'} \subseteq U_2 \to W \times (Z-B)$ of the good representative $F: U_1 \to W \times Z$.

(2.10) In the same way we obtain a Whitney stratificatition $S'_R(F)$ (by stable types) for the image X' in $W_2 \times Z_1 \subseteq \mathbb{C}^p \times \mathbb{C}^d$ of the restriction $F|\overline{U_3'} : \overline{U_3'} \subseteq U_3 \to W_2 \times Z_1$ (notation as in (1.15)) of the good representative $F : U_3 \to W_2 \times Z_1$, where $W_2 = W - \overline{W_1}$.

(2.11) Remark : It is apparent from the proof of (2.9) that the stratification $S_R(F)$ verifies:

for every $t_0 \in (Z-B)$, the closed image X_{t_0} of the mapping f_{t_0} in $W \times \{t_0\}$ is Whitney stratified by stable type strata $Q_R(f_{t_0})_y$, for any $(y,t_0) \in W \times \{t_0\}$ since the germ of f_{t_0} at $F^{-1}(\pi^{-1}(t_0)) \cap U_{t_0}$ is stable for every $t_0 \in (Z-B)$. The same applies to the initial map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ since for all $y \in \mathbb{C}^p - \{0\}$, but near 0, the multigerm $f_0 : (\mathbb{C}^n, f_0^{-1}(y)) \to (\mathbb{C}^p,y)$ is stable , by theorem (1.10) above , so the stable type strata $Q_R(f_0)_y$ together with the origin $\{0\}$ is a Whitney stratification for the image X_0 of the map f_0 in $\mathbb{C}^p$ (here we once again use the fact that by Kuo's ratio test [Ku] any statum that has a zero-dimensional stratum in its closure is regular over it).

In the next paragraph we are going to show that if we take $W = \overline{B_\varepsilon} \subseteq \mathbb{C}^p$ (the closed ball of radius $\varepsilon > 0$, boundary S_ε and centre in the origin) with $\varepsilon > 0$ sufficiently small and if $W_1 = B_{\varepsilon/2} \subseteq \mathbb{C}^p$ (the interior of $\overline{B_{\varepsilon/2}}$) then $S_\varepsilon \times Z_1$ intersects the stratified closed set $(X', S'_R(F))$ transversally in $(W-\overline{W_1}) \times Z_1$ and hence $S_\varepsilon \times (Z_1 - B)$ intersects the stratified closed set $(X, S_R(F))$ transversally in $W \times (Z_1 - B)$.

(2.12) Corollary : (i) Associated to the target stratification $\mathcal{S}_R(F)$ of X in $W\times(Z-B) \subseteq \mathbb{C}^p\times\mathbb{C}^d$ (resp. $\mathcal{S}'_R(F)$ of X' in $(W-\overline{W_1})\times Z_1 \subseteq \mathbb{C}^p\times\mathbb{C}^d$) we have a source stratification $\mathcal{S}_D(F)$ of $\overline{U'_2}$ (resp. $\mathcal{S}'_D(F)$ of $\overline{U'_3}$) such that the pair $\left(\mathcal{S}_D(F),\mathcal{S}_R(F)\right)$ (resp. $\left(\mathcal{S}'_D(F),\mathcal{S}'_R(F)\right)$ is a Thom stratification for the mapping $F|\overline{U'_2} : \overline{U'_2} \to X \subseteq W\times(Z-B)$ (resp. $F|\overline{U'_3}:\overline{U'_3} \to X' \subseteq (W-\overline{W_1})\times Z_1$).

(ii) The restriction of the projection $\pi: W\times(Z-B) \to Z-B$ (resp. $\pi:(W-\overline{W_1})\times Z_1 \to Z_1$) to $X \subseteq W\times(Z-B)$ (resp. $X' \subseteq (W-\overline{W_1})\times Z_1$) is a stratified submersion, i.e., $\pi | Q_R(F)_{(y,t)}$ is a submersion for each stratum $Q_R(F)_{(y,t)} \in \mathcal{S}_R(F)$.

Proof : (i) For any (y,t) in the image of F we have $F^{-1}(y,t) = \left\{(x_1,t),\dots,(x_k,t)\right\}$ for some k. So, if we take the connected component of $F^{-1}(Q_R(F)_{(y,t)})$ through (x_i,t) as the stratum $Q_D(F)_{(x_i,t)}$ through (x_i,t) in the source of F, then the set of all $Q_D(F)_{(x_i,t)}$ constitutes a Whitney stratification $\mathcal{S}_D(F)$ for the source of F. Moreover, $F | Q_D(F)_{(x_i,t)} \to Q_R(F)_{(y,t)}$ is a local diffeomorphism. So, by (2.4), the pair $\left(\mathcal{S}_D(F),\mathcal{S}_R(F)\right)$ is a Thom stratification of F.

(ii) The local triviality of the good representative F implies that each target stratum $Q_R(F)_{(y,t)}$ is of the form $Q_R(f_t)_y \times (Z-B)$ (resp. $Q_R(f_t)_y \times Z_1$). Hence, the restriction of the projection π to each stratum $Q_R(F)_{(y,t)}$ is a submersion onto $Z-B \subseteq \mathbb{C}^d$ (resp. $Z_1 \subseteq \mathbb{C}^d$).

§3. Fibration of good representatives

In this paragraph we apply Thom's Second Isotopy Lemma (see below) to the sequence $U \xrightarrow{F} W \times Z \xrightarrow{\pi} Z$ where F is a good representative of an $\mathcal{A}_e$-versal unfolding $F:(\mathbb{C}^n \times \mathbb{C}^d,0) \to (\mathbb{C}^p \times \mathbb{C}^d,0)$, $F(x,t) = (f_t(x),t)$ of a discrete stable type map-germ $f_0:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, $n<p$.

Here the main references are [JM] and [Gi].

(3.1) Definition: A sequence of spaces and mappings $A_i \xrightarrow{f_i} \dots \to A_1 \xrightarrow{f_1} A_0$ is said to be topologically trivial over Y with respect to the mapping $\pi: A_0 \to Y$ if there exists a sequence of spaces and mappings $B_i \xrightarrow{g_i} \dots \to B_1 \xrightarrow{g_1} B_0$ and homeomorphisms $h_j: B_j \times Y \to A_j$ such that the following diagram:

$$\begin{array}{ccccccc} B_i \times Y & \to & \dots & \to & B_1 \times Y & \to & B_0 \times Y \\ h_i \downarrow & & & & h_1 \downarrow & & h_0 \downarrow \\ A_i & \to & \dots & \to & A_1 & \to & A_0 \end{array}$$

commutes.

(3.2) Remark: If the sequence $A_i \xrightarrow{f_i} \dots \to A_1 \xrightarrow{f_1} A_0$ is topologically trivial over Y with respect to the mapping $\pi: A_0 \to Y$ then each $\pi \circ f_1 \circ \dots \circ f_j : A_j \to Y$ is a trivial fibration.

(3.3) Theorem (The Second Isotopy Lemma) :

Let $M \xrightarrow{G} N \xrightarrow{\pi} P$ be a sequence of smooth mappings and manifolds. Let U and V be closed subsets of M and N respectively which admit Whitney stratifications $\mathcal{A}_U$ and $\mathcal{A}_V$ respectively. Suppose that :

(i) The pair ($\mathcal{A}_U$, $\mathcal{A}_V$) is a Thom stratification for the restriction $G|U : U \to V$

(ii) The restrictions $G|U : U \to V$ and $\pi|V : V \to P$ are proper maps.

(iii) π is a stratified submersion, ie, the restriction $\pi | A_V : A_V \to P$ is a submersion, for each stratum $A_V \in \mathcal{A}_V$.

Then the stratified map $G : (U, \mathcal{A}_U) \to (V, \mathcal{A}_V)$ is locally topologically trivial over P with respect to $\pi|V : V \to P$.

Proof: (see for e.g [JM] or [Gi]).

(3.4) Corollary : In the above conditions, if P is connected then the topological type of the map $G_u : U \cap G^{-1}\left(\pi^{-1}(u)\right) \to V \cap \pi^{-1}(u)$ is independent of $u \in P$.

As a consequence of (3.3) we obtain :

(3.5) Proposition : Let $F : U \subseteq (\mathbb{C}^n \times \mathbb{C}^d, 0) \to W \times Z \subseteq (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be a good representative of an $\mathcal{A}_e$-versal unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$ of a discrete stable type map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$. Then there exist an $\varepsilon_0 > 0$, and a neighbourhood $T \subseteq Z$ of the origin in $\mathbb{C}^d$ such that $F|\overline{U_2'} : \overline{U_2'} \subseteq U_2 \to X \cap \left(\overline{B_{\varepsilon_0}} \times (T-B)\right)$ is locally topologically trivial over T-B with respect to the projection $\pi : \overline{B_{\varepsilon_0}} \times (T-B) \to T-B$, where $\overline{B_{\varepsilon_0}} \subseteq W$ is the closed ball of radius ε centred at the origin of $\mathbb{C}^p$ and B is the bifurcation set of F.

To prove (3.5) we first let $\varepsilon_0 > 0$ be so small that for all ε, $0 < \varepsilon \leq \varepsilon_0$, the sphere of radius ε and centre at the origin in $\mathbb{C}^p$, S_ε intersects the (closed) stratified image X_0 of f_0 transversally in $\mathbb{C}^p$, as a stratified set in $\mathbb{C}^p$ (as we remarked in (2.11), X_0 is stratified by stable type strata $Q_R(f_0)_y$ for all $y \neq 0$, together with the origin); ie, $S_\varepsilon \pitchfork Q_R(f_0)_y$. Now, we consider the restriction of the good representative $F : U \to W \times Z$:

$F\,|\,\overline{U}'_3 : \overline{U}'_3 \subseteq U_3 \to W_2 \times Z_1$ with $W_2 = W - \overline{W}_1$ and Z_1 as in (1.14)(ii).

Let $(X', S'_R(F))$ be the stratified closed image of F in $W_2 \times Z_1$ and $X' \cap \pi^{-1}(0)$, the image of f_0 in $W_2 \times \{0\} \subseteq \mathbb{C}^p$, where $\pi : W_2 \times Z_1 \to Z_1$ is the projection.

(3.6) Proposition: In the above conditions, there exist a neighbourhood T of the origin in $\mathbb{C}^d$, $T \subseteq Z_1$ such that , the set $E' = X' \cap (S_{\varepsilon_0} \times T)$ is a Whitney stratified subset of $\mathbb{C}^p \times \mathbb{C}^d$ and the restriction of the projection $\pi\,|\,E' : E' \to T$ is a stratified submersion.

(3.7) Lemma : Let M and P be smooth manifolds, let $Y \subseteq M \times P$ be a Whitney stratified subset and suppose that the restriction of the projection $\pi : M \times P \to P$ to Y is a stratified submersion. Then, if C is a submanifold of M and p is a point of P :

(i) $\left(C \times \{p\}\right) \pitchfork \left(Y \cap \pi^{-1}(p)\right)$ in $M \times \{p\}$ iff $\left(C \times \{p\}\right) \pitchfork Y$ in $M \times P$

(ii) if $\left(C \times \{p\}\right) \pitchfork Y$ in $M \times P$, for all $p \in P$ then $(C \times P) \pitchfork Y$ in $M \times P$ and $\pi : (C \times P) \cap Y \to P$ is a stratified submersion.

Proof: (i) Obvious.

(ii) To show that $\pi : (C \times P) \cap Y \to P$ is a stratified submersion, we consider any stratum Y_α of Y. Under the hypothesis of the lemma, $(C \times P) \pitchfork Y_\alpha$ in $M \times P$. Now let $\mathbf{p} \in T_p P$ and suppose $(m,p) \in Y_\alpha \cap (M \times P)$. As $Y_\alpha \pitchfork C \times \{p\}$, we have $T_{(m,p)} Y_\alpha + (T_m C) \times \{0\} = T_{(m,p)} M \times P$.

So, there exist elements of $T_{(m,p)} Y_\alpha$ and $(T_m C) \times \{0\}$ whose sum is $(\mathbf{0},\mathbf{p})$. That is, there exist $\mathbf{m} \in T_m C$ such that $(\mathbf{m},\mathbf{p}) \in T_{(m,p)} Y_\alpha$ (thus $(\mathbf{m},\mathbf{p}) + (-\mathbf{m},\mathbf{0}) = (\mathbf{0},\mathbf{p})$). Then $(\mathbf{m},\mathbf{p}) \in T_{(m,p)}(C \times P)$ too, so $(\mathbf{m},\mathbf{p}) \in T_{(m,p)}(C \times P) \cap T_{(m,p)} Y_\alpha = T_{(m,p)}\left((C \times P) \cap Y_\alpha\right)$, and $\pi : (C \times P) \cap Y_\alpha \to P$ is a submersion.

Proof of (3.6) : By (2.12), the restriction of $\pi : W_2 \times Z_1 \to Z_1$ to the stratified image of F in $W_2 \times Z_1$, $(X', \mathcal{S}'_R(F))$ is a stratified submersion. We have :

$(S_{\varepsilon_0} \times \{0\}) \pitchfork (X' \cap \pi^{-1}(0))$ in $W_2 \times \{0\}$, ie, for any stratum $Q_R(F)_{(y,t)} \in \mathcal{S}'_R(F)$, $(S_{\varepsilon_0} \times \{0\}) \pitchfork \left(Q_R(F)_{(y,t)} \cap \pi^{-1}(0)\right)$ in $W_2 \times \{0\}$. This is equivalent, by (3.7)(i) to $(S_{\varepsilon_0} \times \{0\}) \pitchfork X'$ in $W_2 \times Z_1$. In other words, if $g_t : S_{\varepsilon_0} \times \{0\} \to W_2 \times Z_1$ is the map given by $g_t(s) = (s,t)$ then $g_0 \pitchfork X'$ in $W_2 \times Z_1$. But the set $\left\{g \in C^\infty\left(S_{\varepsilon_0} \times \{0\}, W_2 \times Z_1\right) : g \pitchfork X'\right\}$ is an open and dense subset of $C^\infty\left(S_{\varepsilon_0} \times \{0\}, W_2 \times Z_1\right)$, since $S_{\varepsilon_0} \times \{0\}$ is compact and $\mathcal{S}'_R(F)$ is Whitney regular (see e.g. [G–M] p.38). In particular, there is a neighbourhood $T \subseteq Z_1$ of the origin in $\mathbb{C}^d$ such that : for all $t \in T$, $g_t \pitchfork X'$, ie, $(S_{\varepsilon_0} \times \{t\}) \pitchfork X'$ in $W_2 \times T$, for all $t \in T$. By (3.7)(ii), this is equivalent to $(S_{\varepsilon_0} \times T) \pitchfork X'$ in $W_2 \times T$. Thus, the set $E' = (S_{\varepsilon_0} \times T) \cap X'$ is Whitney stratified by $\mathcal{E}' := \mathcal{S}'_t(F) \cap (S_{\varepsilon_0} \times T)$. Again by (3.7)(ii), $\pi : (E', \mathcal{E}') \to T$ is a stratified submersion. And this proves (3.6).

Proof of (3.5) : The Whitney stratified (closed) image $(X, \mathcal{S}_R(F))$ in $W \times (Z-B)$ of the restriction $F | \overline{U}'_2 : \overline{U}'_2 \subseteq U_2 \to W \times (Z-B)$ of the good representative $F : U \subseteq (\mathbb{C}^n \times \mathbb{C}^d, 0) \to W \times Z \subseteq (\mathbb{C}^p \times \mathbb{C}^d, 0)$, verifies : (here $T \subseteq Z \subseteq \mathbb{C}^d$ is as in (3.6))

- $(S_{\varepsilon_0} \times (T-B)) \cap X'$ and $(S_{\varepsilon_0} \times (T-B)) \cap X$ coincide, hence $(S_{\varepsilon_0} \times (T-B)) \pitchfork X$ in $W \times (T-B) \subseteq \mathbb{C}^p \times \mathbb{C}^d$

- The set $E = (\overline{B}_{\varepsilon_0} \times (T-B)) \cap X$ is stratified by $\mathcal{E}_R := \mathcal{S}_R(F) \cap (\overline{B}_{\varepsilon_0} \times (T-B))$ and the boundary of E, $\partial E := (S_{\varepsilon_0} \times (T-B)) \cap X$ is stratified by $\mathcal{S}_R(F) \cap (S_{\varepsilon_0} \times (T-B))$,

- the restriction of the projection $\pi : \overline{B_{\varepsilon_0}} \times (T-B) \to (T-B)$ to E is a proper stratified submersion $\left(\text{since } \overline{B_{\varepsilon_0}} \text{ is compact}\right)$.

Hence: - the map $F : (\overline{U_2'}, \mathcal{E}_D) \to (E, \mathcal{E}_R)$ is a Thom stratified map, where the Whitney stratification $\mathcal{E}_D$ is obtained as the pull-back of $\mathcal{E}_R$, as in (2.12)(i).

- $F \mid \overline{U_2'} : \overline{U_2'} \to E$ is a proper map, since it is a finite mapping.

- the restriction $\pi \mid E : E \to T-B$ of the projection $\pi : \overline{B_{\varepsilon_0}} \times (T-B) \to (T-B)$ is a proper stratified submersion.

So, the hypothesis of (3.3) are verified, and this proves (3.5).

(3.8) Remark: In the above conditions we have obtained :

- 'Fibration' of the mapping $F \mid \overline{U_2'} : (\overline{U_2'}, S_D(F)) \to (X, S_R(F)) \subseteq \overline{B_{\varepsilon_0}} \times (T-B)$, where $B \subseteq \mathbb{C}^d$ is the bifurcation set of F; whose 'fibre' over a parameter $t \in (T-B)$ is the mapping $f_t : U_t \to X_t$, where $U_t = \left\{x \in \mathbb{C}^n : (x,t) \in \overline{U_2'}\right\}$ and $X_t = X \cap (\overline{B_{\varepsilon_0}} \times \{t\})$.

By (3.4), since $T-B \subseteq \mathbb{C}^d$ is connected, the topological type of f_t does not vary with t.

- Locally trivial C^0- fibration of $E = X \cap (\overline{B_{\varepsilon_0}} \times (T-B))$ over T-B, whose fibre X_t over t is the image of the mapping f_t above. In other words, there is a stratum preserving homeomorphism $h : E \cap \pi^{-1}(\Omega) \to (\pi^{-1}(t) \cap E) \times \Omega$ where Ω is a neighbourhood of t in $T-B \subseteq \mathbb{C}^d$. In particular the fibres X_t of π are homeomorphic by a stratum preserving homeomorphism.

(3.9) Definition : The image X_t of the mapping f_t above described will be called the *disentanglement* of the image of the map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, following useage of de Jong and Van Straten [J-S].

§4. The Milnor fibrations of the multiple point schemes.

In this paragraph we obtain Milnor fibrations for the multiple point schemes $\tilde{D}^k(F,\gamma(k))$ of a good representative of an $\mathcal{A}_e$-versal unfolding $F:(\mathbb{C}^n\times\mathbb{C}^d,0)\to(\mathbb{C}^p\times\mathbb{C}^d,0)$, $F(x,t)=(f_t(x),t)$, of a finitely $\mathcal{A}$-determined corank 1 map-germ $f_0:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, $n<p$.

For definitions and properties of the multiple point schemes, see [M-M] or equivalently see chapter III, below.

Some denotations (and connotations) about Milnor fibrations of an ICIS used here are borrowed from [Lo] chapter 2.

(4.1) Recall that, in paragraph 3 we have chosen a representative

$F|\overline{U}_2':\overline{U}_2'\subseteq U_2\to\overline{B_{\varepsilon_0}}\times T\subseteq\mathbb{C}^p\times\mathbb{C}^d$ of the germ of an $\mathcal{A}_e$-versal unfolding $F:(\mathbb{C}^n\times\mathbb{C}^d,0)\to(\mathbb{C}^p\times\mathbb{C}^d,0)$, $F(x,t)=(f_t(x),t)$, of $f_0:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ with the following properties:

(i) For all ε, with $0<\varepsilon\le\varepsilon_0$, the (closed) image X_0 of f_0 is a Whitney stratified subset of $\mathbb{C}^p$, transverse to $S_\varepsilon=\partial\overline{B_\varepsilon}$.

(ii) For all $t\in T$, the (closed) image X_t of f_t is Whitney stratified transverse to $S_{\varepsilon_0}\times\{t\}$ in $\mathbb{C}^p\times\{t\}$ (In particular $I(f_t)\cap S_{\varepsilon_0}=\emptyset$, and we are referring to the stable type strata, which are the only ones to meet S_{ε_0}).

(iii) F is locally stable (this follows from the $\mathcal{A}_e$-versality).

(iv) F is proper.

It follows that for all partitions $\gamma(k)=(r_1,\dots,r_m)$, $r_i\ge r_{i+1}$ of an integer k, $\tilde{D}^k(F,\gamma(k))$ is a smooth analytic subset of $\overline{U}_k$, (or empty).

Let $p_{\gamma(k)} : \tilde{D}^k(F,\gamma(k)) \to T$ be the projection of $\tilde{D}^k(F,\gamma(k))$ into the parameter space $T \subseteq \mathbb{C}^d$ of F.

(4.2) Theorem : Let $B \subseteq T \subseteq \mathbb{C}^d$ be the bifurcation set of F. Then provided $p-k(p-n+1)+m \geq 0$, the restriction $p_{\gamma(k)}|: \tilde{D}^k(F,\gamma(k)) \cap p^{-1}_{\gamma(k)}(T-B) \to T-B$ is a locally trivial smooth fibration whose fibre, $\tilde{D}^k(f_t,\gamma(k))$ is a Milnor fibre of the ICIS $(\tilde{D}^k(f_0,\gamma(k)), 0)$.

Proof: Define $r_k : U^k \to \mathbb{R}_{\geq 0}$ by $r_k((x_1,t_1),\ldots,(x_k,t_k)) = 1/k \sum_{i=1}^{k} |f_{t_i}(x_i)|^2$

Then r_k is a real analytic function on U^k; we claim

(i) r_k defines 0 in $\tilde{D}^k(f_0,\gamma(k)) \subseteq \tilde{D}^k(F,\gamma(k))$, i.e., $r_k^{-1}(0) = \{0\}$ and 0 is not an accumulation point of the critical values of $r_k | \tilde{D}^k(f_0,\gamma(k)) - 0$.

(ii) $\{r_k = \varepsilon\}$ is a smooth subset of U^k, and $p_{\gamma(k)} : \{r_k = \varepsilon\} \to T$ is a submersion.

(iii) $p_{\gamma(k)} : \tilde{D}^k(F,\gamma(k)) \to T$ is proper

The theorem will follow from these three statements. To prove the claim,

(i) if $\left((x_1,0),\ldots,(x_k,0)\right) \in p^{-1}_{\gamma(k)}(0)$, then $f_0(x_i) = f_0(x_j)$, for all i, j.

Denote this point $\left(f_0(x_i)\right)$ by $y \in \mathbb{C}^p$. Now let $A_{\gamma'(r)}$ be the stable type stratum through y in X_0; $\left(\gamma'(r)\right.$ is some partition of an integer r, with $r \geq k$, and $\gamma(k) < \gamma'(r)$, ie, $\tilde{D}^r(F,\gamma'(r))$ projects into or includes in $\left.\tilde{D}^k(F,\gamma(k))\right)$. Since X_0 is stratified transverse to S_ε for all ε with $0 < \varepsilon \leq \varepsilon_0$, $A_{\gamma'(r)}$ is transverse to S_ε at the point y, where $\varepsilon = |y|$, and hence there exists a smooth path $\sigma(u)$ in $A_{\gamma'(r)}$ with $\sigma(0) = y$ and $\sigma'(0) \notin T_y S_\varepsilon$. This path has a unique lift to $\tilde{D}^r(F,\gamma'(r))$, whose components $(\tilde{\sigma}_1,\dots,\tilde{\sigma}_r)$ are obtained using the fact that, as f_0 is stable at each point $x \in f_0^{-1}(y)$, it defines a local diffeomorphism $f_0^{-1}(A_{\gamma'(r)}, x) \xrightarrow{\sim} (A_{\gamma'(r)}, y)$.

If $g_i : (A_{\gamma'(r)}, y) \to f_0^{-1}(A_{\gamma'(r)}, x_i)$ is a local inverse of $f_0|$, set $\tilde{\sigma}_i(u) = g_i \circ \sigma(u)$.

Then $\tilde{\sigma}(u) = (\tilde{\sigma}_1(u),\dots,\tilde{\sigma}_k(u)) \in U^k$ defines a path in $\tilde{D}^k(f_0,\gamma(k))$, with $\tilde{\sigma}(0) = (x_1,\dots,x_k)$ such that

$$dr_k(\tilde{\sigma}'(0)) = 1/k \sum_{i=1}^{k} \frac{d}{du}\Big(|f_0(\tilde{\sigma}_i(u))|\Big)\Big|_{u=0} = 1/k \sum_{i=1}^{k} \frac{d}{du}\Big(|\sigma(u)|\Big)\Big|_{u=0} \neq 0.$$

It follows that for all points $(x_1,\dots,x_k)$ in $\tilde{D}^k(f_0,\gamma(k)) - \{0\}$, the restriction $r_k|_{\tilde{D}^k(f_0,\gamma(k))}$ is a submersion.

(ii) This follows by a similar argument since $\tilde{D}^k(F,\gamma(k))$ is smooth; in order to prove the result we must show simply that the map $\left(p_{\gamma(k)}, r_k\right) : \tilde{D}^k(F,\gamma(k)) \to T\times\mathbb{R}$ is a submersion along $\left(p_{\gamma(k)}, r_k\right)^{-1}(T\times\{\varepsilon\})$. This is equivalent to, the restriction $r_k|_{p^{-1}_{\gamma(k)}(t)}$ being a submersion at all points in $r_k^{-1}(\varepsilon)$, and this follows, by the same argument as in (i), from the fact that $f_t(U_t) = X_t$ is stratified transverse to S_{ε_0}.

(iii) is obvious since $\tilde{D}^k(F,\gamma(k))$ is compact

Now under these circumstances, if $C_{\gamma(k)}$ is the critical set of $p_{\gamma(k)}:\tilde{D}^k(F,\gamma(k)) \to T$, and $\Delta_{\gamma(k)} \subseteq T$ is its image, then $p_{\gamma(k)} : \tilde{D}^k(F,\gamma(k)) - p^{-1}_{\gamma(k)}(\Delta_{\gamma(k)}) \to T - \Delta_{\gamma(k)}$ is a locally trivial smooth fibre bundle $\left(\text{note that } \tilde{D}^k(F,\gamma(k)) = \tilde{D}^k(F,\gamma(k))_{r_k \leq \varepsilon}\right)$ with fibre $\tilde{D}^k(f_t,\gamma(k))$ whose boundary is $\tilde{D}^k(f_t,\gamma(k))_{r_k=\varepsilon}$. This follows by a standard argument using the Ehresmann fibration theorem, see e.g. [Wo].

Finally, if f_t is stable, then $\tilde{D}^k(f_t,\gamma(k))$ is smooth for all $\gamma(k)$ in the dimension range we are discussing ([M-M] th. 2.14), and hence $C_{\gamma(k)} \cap p^{-1}_{\gamma(k)}(t) = \emptyset$. It follows that $C_{\gamma(k)} \subseteq p^{-1}_{\gamma(k)}(B)$, so that $\Delta_{\gamma(k)} \subseteq B$, and hence the map $p_{\gamma(k)} : \tilde{D}^k(F,\gamma(k)) - p^{-1}_{\gamma(k)}(B) \to T - B$ is a restriction of the Milnor fibration $p_{\gamma(k)} : \tilde{D}^k(F,\gamma(k)) - p^{-1}_{\gamma(k)}(\Delta_{\gamma(k)}) \to T - \Delta_{\gamma(k)}$ to the ICIS $\tilde{D}^k(f_0,\gamma(k)) = p^{-1}_{\gamma(k)}(0)$.

(4.3) Since the fibres of a Milnor fibration of an n-dimensional ICIS are connected and homotopy equivalent to a wedge of n-spheres, we have

$$\chi (\tilde{D}^k(f_t,\gamma(k))) = 1 + (-1)^s \mu(\tilde{D}^k(f_0,\gamma(k)))$$

where s is the complex dimension of the ICIS $\tilde{D}^k(f_0,\gamma(k))$.

§5. The Euler characteristic of the disentanglement of the image of a corank 1 finitely $\mathcal{A}$-determined map-germ.

In paragraph 3 we obtained fibrations associated to a good representative $F : U \subseteq \mathbb{C}^n \times \mathbb{C}^d \to W \times Z \subseteq \mathbb{C}^p \times \mathbb{C}^d$ of an $\mathcal{A}_e$-versal unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$, $F(x,t) = (f_t(x), t)$ of a finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $2 \leq n < p$ with (n,p) in the nice dimensions according to Mather, or on their boundary, namely:

- 'Fibration' of the mapping $F | \overline{U}_2' : (\overline{U}_2', S_D(F)) \to (X, S_R(F)) \subseteq \overline{B}_{\varepsilon_0} \times (T-B)$,

where $B \subseteq \mathbb{C}^d$ is the bifurcation set of F and $\overline{B}_{\varepsilon_0} \subseteq \mathbb{C}^p$ is the closed ball of radius ε_0 centred at the origin; whose 'fibre' over a parameter $t \in (T-B)$ is the mapping $f_t : U_t \to X_t$, where $U_t = \left\{ x \in \mathbb{C}^n : (x,t) \in \overline{U}_2' \right\}$ and $X_t = X \cap (\overline{B}_{\varepsilon_0} \times \{t\})$

- Fibration of the image X of F in $\overline{B}_{\varepsilon_0} \times (T-B)$, whose fibre over $t \in (Z-B)$ is the image X_t in $\overline{B}_{\varepsilon_0} \times \{t\}$ of the stable mapping f_t above; ie X_t is the disentanglement (definition (3.9)) of X_0, image of $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

With the assumption that the map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is of corank 1, in paragraph 4 we obtained:

- Milnor fibrations of the multiple point schemes $\tilde{D}^k(F, \gamma(k))$ of the good representative F, whose fibre over $t \in (T-B)$ is the multiple point scheme $\tilde{D}^k(f_t, \gamma(k))$ of f_t, for all $k \geq 2$ and all (ordered) partitions $\gamma(k) = (a_1, \ldots, a_h)$ of k, with $a_i \geq a_{i+1}$ and $p - k(p-n+1) + h \geq 0$.

In this paragraph we shall be interested in relating the Euler characteristic of the disentanglement X_t of the image X_0 of the initial finitely $\mathcal{A}$-determined corank 1 map-germ $f_0:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, and the Milnor number $\mu(\tilde{D}^k(f_0,\gamma(k)))$ of the ICIS $\tilde{D}^k(f_0,\gamma(k))$, for all k and all partitions $\gamma(k)=(a_1,\dots,a_h)$ of k, such that $a_i \geq a_{i+1}$ and $p-k(p-n+1)+h \geq 0$.

(5.1) Thus, over each parameter $t \in (T-B) \subseteq \mathbb{C}^d$ we have the following diagram:

$$
\begin{tikzcd}
 & & & \vdots \arrow[d] & \\
 & & \vdots \arrow[d] & \tilde{D}^k(f_t) \arrow[d] & \\
 & & \tilde{D}^{k-1}(f_t,(2,1^{k-3})) \arrow[r] \arrow[d] & \tilde{D}^{k-1}(f_t) \arrow[d] & \\
\vdots \arrow[dd] & \vdots \arrow[d] & \vdots \arrow[dd] & \vdots \arrow[dd] & \\
 & \tilde{D}^4(f_t,(2^2)) \arrow[dr] & & & \\
\tilde{D}^4(f_t,(4)) \arrow[ur] \arrow[dr] & & \tilde{D}^4(f_t,(2,1^2)) \arrow[r] \arrow[dd] & \tilde{D}^4(f_t) \arrow[dd] & \\
 & \tilde{D}^4(f_t,(3,1)) \arrow[ur] \arrow[d] & & & \\
 & \tilde{D}^3(f_t,(3)) \arrow[r] & \tilde{D}^3(f_t,(2,1)) \arrow[r] \arrow[d] & \tilde{D}^3(f_t) \arrow[d] & \\
 & & \tilde{D}^2(f_t,(2)) \arrow[r] & \tilde{D}^2(f_t) \arrow[d] & \\
 & & & \mathbb{C}^n \supseteq U_t \arrow[r, "f_t"] & X_t \subseteq \mathbb{C}^p
\end{tikzcd}
$$

where U_t is a closed contractible subset of $\mathbb{C}^n$, the spaces on the left hand side are all smooth (since f_t is stable) and the mappings $\rho_{\gamma(k)}: \tilde{D}^k(f_t,\gamma(k)) \to X_t$ and $f_t : U_t \to X_t$ are proper and finite to one , for all $k \geq 2$, and all partitions $\gamma(k)$. Note that when $\gamma(k) = (1,..., 1)$, we have $\tilde{D}^k(f_t,\gamma(k)) = \tilde{D}^k(f_t)$.

Moreover, f_t is branched over the points of the image of $\rho_{(1^2)}$ (and hence at the points of the image of all mappings $\rho_{\gamma(k)}$ in X_t) and $\rho_{\gamma(k)}$ is branched over the points of the image of all $\rho_{\gamma'(r)}$ with $r \geq k$ and such that $\gamma(k) < \gamma'(r)$. The symbol $\gamma(k) < \gamma'(r)$ means that $\tilde{D}^r(f_t,\gamma'(r)) \subseteq \mathbb{C}^{n-1}\times\mathbb{C}^r$ projects into or includes in $\tilde{D}^k(f_t,\gamma(k)) \subseteq \mathbb{C}^{n-1}\times\mathbb{C}^k$; or in other words, $\tilde{D}^r(f_t,\gamma''(r))$ with $\gamma''(r) = (\gamma(k), 1, ..., 1)$ contains $\tilde{D}^r(f_t,\gamma'(r))$.

Also, we recall that since $\tilde{D}^k(f_t,\gamma(k))$ is a typical Milnor fibre and $\tilde{D}^k(f_0,\gamma(k))$ the critical fibre, we have: $\chi(\tilde{D}^k(f_t,\gamma(k))) = 1 + (-1)^s \mu(\tilde{D}^k(f_0,\gamma(k)))$ where s is the complex dimension of the ICIS $\tilde{D}^k(f_0,\gamma(k))$.

Thus our initial aim, namely to relate the Euler Charactheristic $\chi(X_t)$ of the image X_t of f_t and the Milnor numbers $\mu(\tilde{D}^k(f_0,\gamma(k)))$ of the multiple point schemes $\tilde{D}^k(f_0,\gamma(k))$ of the initial map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is equivalent to that of relating $\chi(X_t)$ and $\chi(\tilde{D}^k(f_t,\gamma(k)))$, for all $k \geq 2$ and all partitions $\gamma(k)$ of k.

We shall do this by considering the degrees of the finite mappings f_t and $\rho_{\gamma(k)}$ of the diagram above, i.e., the cardinality of the fibres $f_t^{-1}(f_t(x))$, where $x \in U_t$ is generic, $f_t^{-1}(\rho_{\gamma(r)}(y))$, where **y** is a generic point of $\tilde{D}^r(f_t,\gamma(r)$ $r \geq 2$ and the cardinality of the fibres $\rho^{-1}_{\gamma(k)}(\rho_{\gamma'(r)}(\mathbf{y}))$, where $\mathbf{y}$ is a generic point of $\tilde{D}^r(f_t,\gamma'(r))$ with $r \geq k$ and $\gamma(k) < \gamma'(r)$ (Recall that if $\gamma'(r) = (b_1,\ldots,b_q)$ then a generic point of $\tilde{D}^r(f_t,\gamma'(r)) \subseteq \mathbb{C}^{n-1} \times \mathbb{C}^r$ is one of the form $\mathbf{y} = (x, y_1,\ldots, y_1,\ldots, y_q,\ldots, y_q)$, $x \in \mathbb{C}^{n-1}$, $y_i \in \mathbb{C}$, $y_i \neq y_j$ for $i \neq j$ and y_i repeated b_i times, see (2.7) of [M-M]). Then we shall find coefficients β_0 and $\beta_{\gamma(k)}$ such that:

$$\text{(I)} \qquad \chi(X_t) = \beta_0 \chi(U_t) + \sum_{k\geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} \chi(\tilde{D}^k(f_t,\gamma(k)))$$

We start by triangulating the objects we are going to deal with.

(5.2) Let $\mathfrak{T}(X_t)$ denote a triangulation of the image X_t of f_t, constructed in the following way:

We first consider all the zero-dimensional multiple point schemes $\tilde{D}^r(f_t,\gamma(r))$ of f_t and start triangulating X_t by including the image of all mappings $\rho_{\gamma(r)}: \tilde{D}^r(f_t,\gamma(r)) \to X_t$ among the vertices of $\mathfrak{T}(X_t)$. Next we build up the two-skeleton $X_t^{(2)}$ of $\mathfrak{T}(X_t)$ so that the image in X_t of each multiple point scheme of complex dimension one is a subcomplex of $X_t^{(2)}$. Then we continue in this way until we obtain the $(2(n-1))$-skeleton $X_t^{(2(n-1))}$ of $\mathfrak{T}(X_t)$, which should contain the image of $\rho_{(1^2)}$ as a subcomplex. Finally we complete $\mathfrak{T}(X_t)$.

Since the mappings f_t and $\rho_{\gamma(k)}$ are proper and finite to one, then pulling back $\mathfrak{T}(X_t)$ we obtain a triangulation for the source U_t and the pull-back of the 2rskeleton $X_t^{(2r)}$ provides triangulations for all multiple point schemes of complex dimension r.

In fact, let $M_{\gamma(k)}$ denote the image of $\tilde{D}^k(f_t,\gamma(k))$ by the map $\rho_{\gamma(k)} : \tilde{D}^k(f_t,\gamma(k)) \to X_t$. By construction, $M_{\gamma(k)}$ is a subcomplex of $\mathcal{T}\left(X_t\right)$. The triangulation $\mathcal{T}\left(\tilde{D}^k(f_t,\gamma(k))\right)$ of $\tilde{D}^k(f_t,\gamma(k))$ is obtained as follows:

-the vertices of $\mathcal{T}\left(\tilde{D}^k(f_t,\gamma(k))\right)$ will be all points over the vertices of $M_{\gamma(k)}$.

-for all one-simplex $\sigma : \Delta^1 \to M_{\gamma(k)}$, we denote the interior of Δ^1 by int Δ^1 . Then, by assumption, $\rho_{\gamma(k)}$ is a trivial covering over $\sigma\,(\mathrm{int}\,\Delta^1)$, of degree, say d_1. Since int Δ^1 is simply connected, σ lifts to d_1 distinct maps $\sigma_0^{(i)} : \mathrm{int}\,\Delta^1 \to \tilde{D}^k(f_t,\gamma(k))$, $i = 1,\ldots, d_1$, with disjoint images. Since $\rho_{\gamma(k)}$ is proper we can extend $\sigma_0^{(i)}$ to maps $\sigma^{(i)} : \Delta^1 \to \tilde{D}^k(f_t,\gamma(k))$ lifting σ.

$$\begin{tikzcd}
& \tilde{D}^k(f_t,\gamma(k)) \arrow[d, "\rho_{\gamma(k)}"] \\
\mathrm{int}\,\Delta^1 \arrow[ur, "\sigma_0^{(i)}"] \arrow[r] & M_{\gamma(k)}
\end{tikzcd}$$

- do the same for all two-simplex of $M_{\gamma(k)}$ and so on. In this way we end up with a triangulation of $\tilde{D}^k(f_t,\gamma(k))$.

Now let

$C_i^{X_t}$ be the number of cells of dimension i in X_t,

$C_i^{U_t}$ be the number of cells of dimension i in U_t

$C_i^{\gamma(k)}$ be the number of cells of dimension i in $\tilde{D}^k(f_t,\gamma(k))$ (with respect to these triangulation),

where $C_i^{\gamma(k)} = 0$ if $i > 2\dim_{\mathbb{C}} \tilde{D}^k(f_t,\gamma(k))$.

So, equation (I) of (5.1) can be rewritten as:

$$\sum_i (-1)^i C_i^{X_t} = \beta_0 \sum_i (-1)^i C_i^{U_t} + \sum_{k\geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} \sum_i (-1)^i C_i^{\gamma(k)} .$$

Notice that if we find coefficients β_0 and $\beta_{\gamma(k)}$ (for all $\gamma(k)$), independent of i, such that:

$$C_i^{X_t} = \beta_0 C_i^{U_t} + \sum_{k\geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} C_i^{\gamma(k)} , \text{ for all } i,\ 0 \leq i \leq 2n,$$

then these coefficients will solve equation (I) of (5.1).

So, let us concentrate on solving :

(II) $C_0^{X_t} = \beta_0 C_0^{U_t} + \sum_{k\geq 2} \sum_{\gamma(k)} \beta_{\gamma(k)} C_0^{\gamma(k)}$ (where all coefficients $\beta_{\gamma(k)}$'s appear)

We claim that provided we can find the degrees of the finite mappings

$f_t : U_t \subseteq \mathbb{C}^n \to X_t \subseteq \mathbb{C}^p$ and $\rho_{\gamma(k)} : \tilde{D}^k(f_t,\gamma(k)) \to X_t$, for all $k \geq 2$ and all partitions $\gamma(k)$ of k then we can find the coefficients $\beta_{\gamma(k)}$ that verify equation (II), and hence equation (I).

At this point, same examples will probably be helpful.

(5.3) Example (i): Let f_0: $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$.

In this case the diagram over a parameter $t \in (T-B)$ is:

$$\begin{tikzcd}
 & \tilde{D}^3(f_t) \arrow[d] \arrow[ddr, "\rho_{(1^3)}"] & \\
\tilde{D}^2(f_t,(2)) \arrow[r, "i"] & \tilde{D}^2(f_t) \arrow[d] \arrow[dr, "\rho_{(1^2)}"] & \\
 & \mathbb{C}^2 \supseteq U_t \arrow[r, "f_t"] & X_t \subseteq \mathbb{C}^3
\end{tikzcd}$$

and the finite mappings f_t , $\rho_{(1^2)}$, $\rho_{(2)} = \rho_{(1^2)} \circ i$ and $\rho_{(1^3)}$ have the following degrees:

$\rho_{(1^3)}$ is 6 to 1

$\rho_{(2)}$ is 1 to 1

$\rho_{(1^2)}$ is 6 to 1 over the points of the image of $\rho_{(1^3)}$ in X_t
is 1 to 1 over the points of the image of $\rho_{(2)}$ in X_t
and 2 to 1 in general

f_t is 3 to 1 over the points of the image of $\rho_{(1^3)}$ in X_t
is 1 to 1 over the points of the image of $\rho_{(2)}$ in X_t
is 2 to 1 over the points of the image of $\rho_{(1^2)}$ in X_t, but not
in the image of $\rho_{(1^3)}$ or $\rho_{(2)}$ in X_t
is 1 to 1 in general.

We shall group this information in the following table:

	f_t	$\rho_{(1^2)}$	$\rho_{(2)}$	$\rho_{(1^3)}$
U_t	1	—	—	—
$\tilde{D}^2(f_t)$	2	2	—	—
$\tilde{D}^2(f_t, (2))$	1	1	1	—
$\tilde{D}^3(f_t)$	3	6	—	6

We shall refer to this as the *table of degrees* of the mappings f_t and $\rho_{\gamma(k)}$'s .

Equation (II) in this case is :

$$C_0^{X_t} = \beta_0 C_0^{U_t} + \beta_{(1^2)} C_0^{(1^2)} + \beta_{(2)} C_0^{(2)} + \beta_{(1^3)} C_0^{(1^3)}$$

Also, we have triangulated X_t so that we can write:

(1) $C_0^{X_t} = \overline{C}_0^{U_t} + \overline{C}_0^{(1^2)} + \overline{C}_0^{(2)} + \overline{C}_0^{(1^3)}$, where

$\overline{C}_0^{U_t}$ is the number of zero-cells of X_t but not in the image of $\rho_{(1^2)}$,

$\overline{C}_0^{(1^2)}$ is the number of zero-cells of X_t in the image of $\rho_{(1^2)}$ but not in the image of $\rho_{(1^3)}$ or $\rho_{(2)}$ and

$\overline{C}_0^{(2)}$ (resp. $\overline{C}_0^{(1^3)}$) is the number of zero-cells of X_t in the image of $\rho_{(2)}$ (resp. $\rho_{(1^3)}$).

Moreover, the degrees of the mappings f_t, $\rho_{(1^2)}$, $\rho_{(2)}$ and $\rho_{(1^3)}$ tell us, repectively, that:

(2) $\quad C_0^{U_t} = \bar{C}_0^{U_t} + 2\,\bar{C}_0^{(1^2)} + \bar{C}_0^{(2)} + 3\,\bar{C}_0^{(1^3)}$,

(3) $\quad C_0^{(1^2)} = 2\,\bar{C}_0^{(1^2)} + \bar{C}_0^{(2)} + 6\,\bar{C}_0^{(1^3)}$

(4) $\quad C_0^{(2)} = \bar{C}_0^{(2)}$ and (5) $\quad C_0^{(1^3)} = 6\,\bar{C}_0^{(1^3)}$

So, if we substitute (1) - (5) in (II) we obtain:

$$\bar{C}_0^{U_t} + \bar{C}_0^{(1^2)} + \bar{C}_0^{(2)} + \bar{C}_0^{(1^3)} = \beta_0\,(\bar{C}_0^{U_t} + 2\,\bar{C}_0^{(1^2)} + \bar{C}_0^{(2)} + 3\,\bar{C}_0^{(1^3)}) +$$

$$+ \beta_{(1^2)}\,(2\,\bar{C}_0^{(1^2)} + \bar{C}_0^{(2)} + 6\,\bar{C}_0^{(1^3)}) + \beta_{(2)}\,\bar{C}_0^{(2)} + \beta_{(1^3)}\,(6\,\bar{C}_0^{(1^3)}).$$

A solution β_0, $\beta_{(1^2)}$, $\beta_{(2)}$, $\beta_{(1^3)}$ for this equation is obtained by solving the system of equations:

$$\bar{C}_0^{U_t} = \beta_0\,\bar{C}_0^{U_t}$$

$$\bar{C}_0^{(1^2)} = (2\,\beta_0 + 2\,\beta_{(1^2)})\,\bar{C}_0^{(1^2)}$$

$$\bar{C}_0^{(2)} = (\beta_0 + \beta_{(1^2)} + \beta_{(2)})\,\bar{C}_0^{(2)}$$

$$\bar{C}_0^{(1^3)} = (3\,\beta_0 + 6\,\beta_{(1^2)} + 6\,\beta_{(1^3)})\,\bar{C}_0^{(1^3)}$$

Assuming that $\bar{C}_0^{U_t}$, $\bar{C}_0^{(1^2)}$, $\bar{C}_0^{(2)}$ and $\bar{C}_0^{(1^3)}$ are all non zero, if on the contrary, $\bar{C}_0^{\gamma(k)}$ was zero, it would imply that $\tilde{D}^k(f_t,\gamma(k))$ was empty and hence we would just delete the equation corresponding to $\tilde{D}^k(f_t,\gamma(k))$ and take $\beta_{\gamma(k)} = 0$, this is equivalent to solving the system:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 6 & 0 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_{(1^2)} \\ \beta_{(2)} \\ \beta_{(1^3)} \end{bmatrix}$$

So, $\beta_0 = 1$, $\beta_{(1^2)} = -1/2$, $\beta_{(2)} = 1/2$ and $\beta_{(1^3)} = 1/6$.

Hence, $\chi(X_t) = \chi(U_t) - 1/2\,\chi(\tilde{D}^2(f_t)) + 1/2\,\chi(\tilde{D}^2(f_t,(2))) + 1/6\,\chi(\tilde{D}^3(f_t))$.

And when all multiple point schemes in consideration are non-empty we obtain :

$$\chi(X_t) = 1 - 1/2\left(1 - \mu(\tilde{D}^2(f_0))\right) + 1/2\left(1 + \mu(\tilde{D}^2(f_0,(2)))\right) + 1/6\left(1 + \mu(\tilde{D}^3(f_0)\right).$$

This completes the case (n,p)=(2,3).

(5.4) Remark : In general we shall proceed as above, namely, we shall obtain the table of degrees of the mappings f_t, $\rho_{\gamma(k)}$ and then the coefficients $\beta_{\gamma(k)}$ such that

$$\text{(I)} \qquad \chi(X_t) = \beta_0\,\chi(U_t) + \sum_{k\geq 2}\sum_{\gamma(k)} \beta_{\gamma(k)}\,\chi(\tilde{D}^k(f_t,\gamma(k)))$$

will be obtained by solving the system of equations:

$$(*) \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = M \begin{bmatrix} \beta_0 \\ \beta_{(1^2)} \\ \beta_{(2)} \\ \vdots \end{bmatrix}$$

where M is the matrix whose first column is the column of f_t in the table of degrees, the second column of M is the column of $\rho_{(1^2)}$ in the table of degrees (replacing '—' by '0'), and so on. We can assume that M is lower triangular. Indeed, we can order the rows and columns of the table of degrees using the ordering on the partitions (discussed in page 27 above).

Example (ii) : f_0: $(\mathbb{C}^3,0) \to (\mathbb{C}^4,0)$.

Here the table of degrees is:

	f_t	$\rho_{(1^2)}$	$\rho_{(2)}$	$\rho_{(1^3)}$	$\rho_{(2,1)}$	$\rho_{(4)}$
U_t	1	—	—	—	—	—
$\tilde{D}^2(f_t)$	2	2	—	—	—	—
$\tilde{D}^2(f_t,(2))$	1	1	1	—	—	—
$\tilde{D}^3(f_t)$	3	6	—	6	—	—
$\tilde{D}^3(f_t,(2,1))$	2	3	1	3	1	—
$\tilde{D}^4(f_t)$	4	12	—	24	—	24

So, with the assumption that all the schemes considered are non empty, the coefficients $\beta_{\gamma(k)}$ such that :

$$\chi(X_t) = \beta_0\chi(U_t) + \beta_{(1^2)}\chi(\tilde{D}^2(f_t)) + \beta_{(2)}\chi(\tilde{D}^2(f_t,(2))) + \beta_{(1^3)}\chi(\tilde{D}^3(f_t)) + \beta_{(2,1)}\chi(\tilde{D}^3(f_t,(2,1))) + \beta_{(4)}\chi(\tilde{D}^4(f_t))$$

are : $\beta_0=1$, $\beta_{(1^2)}=-1/2$, $\beta_{(2)}=1/2$, $\beta_{(1^3)}=1/6$, $\beta_{(2,1)}=-1/2$ and $\beta_{(1^4)}=-1/24$.

(As always, if $\tilde{D}^k(f_t,\gamma(k))$ is empty then the coefficient $\beta_{\gamma(k)}$ in equation (I) is zero). Hence, if all multiple point schemes in consideration are non-empty we obtain :

$$\chi(X_t) = 1 - 1/2\left(1+\mu(\tilde{D}^2(f_0))\right) + 1/6\left(1-\mu(\tilde{D}^3(f_0)\right) - 1/24\left(1+\mu(\tilde{D}^4(f_0))\right) +$$
$$+ 1/2\left(1-\mu(\tilde{D}^2(f_0,(2)))\right) - 1/2\left(1+\mu(\tilde{D}^3(f_0,(2,1)))\right).$$

Example (iii): f_0: $(\mathbb{C}^4,0) \to (\mathbb{C}^5,0)$. Here the table of degrees is:

	f_t	$\rho_{(1^2)}$	$\rho_{(2)}$	$\rho_{(1^3)}$	$\rho_{(2,1)}$	$\rho_{(3)}$	$\rho_{(1^4)}$	$\rho_{(2,1^2)}$	$\rho_{(1^5)}$
U_t	1	—	—	—	—	—	—	—	—
$\tilde{D}^2(f_t)$	2	2	—	—	—	—	—	—	—
$\tilde{D}^2(f_t,(2))$	1	1	1	—	—	—	—	—	—
$\tilde{D}^3(f_t)$	3	6	—	6	—	—	—	—	—
$\tilde{D}^3(f_t,(2,1))$	2	3	1	3	1	—	—	—	—
$\tilde{D}^3(f_t,(3))$	1	1	1	1	1	1	—	—	—
$\tilde{D}^4(f_t)$	4	12	—	24	—	—	24	—	—
$\tilde{D}^4(f_t,(2,1^2))$	3	7	1	12	2	—	12	2	—
$\tilde{D}^5(f_t)$	5	20	—	60	—	—	120	—	120

and hence the coefficients β's such that

$$\chi(X_t) = \beta_0\chi(U_t) + \beta_{(1^2)}\chi\left(\tilde{D}^2(f_t)\right) + \beta_{(2)}\chi\left(\tilde{D}^2(f_t,(2))\right) + \beta_{(1^3)}\chi\left(\tilde{D}^3(f_t)\right) +$$
$$+ \beta_{(2,1)}\chi\left(\tilde{D}^3(f_t,(2,1))\right) + \beta_{(3)}\chi\left(\tilde{D}^3(f_t,(3))\right) + \beta_{(1^4)}\chi\left(\tilde{D}^4(f_t)\right) +$$
$$+ \beta_{(2,1^2)}\chi\left(\tilde{D}^4(f_t,(2,1^2))\right) + \beta_{(1^5)}\chi\left(\tilde{D}^5(f_t)\right)$$ are:

$\beta_0 = 1$, $\beta_{(1^2)} = -1/2$, $\beta_{(2)} = 1/2$, $\beta_{(1^3)} = 1/6$, $\beta_{(2,1)} = -1/2$, $\beta_{(3)} = 1/3$, $\beta_{(1^4)} = -1/24$, $\beta_{(2,1^2)} = 1/4$, $\beta_{(1^5)} = 1/120$.

So, if all multiple point schemes in consideration are non-empty we obtain :

$$\chi(X_t) = 1 - 1/2\left(1 - \mu(\tilde{D}^2(f_0))\right) + 1/2\left(1 + \mu(\tilde{D}^2(f_0,(2)))\right) +$$
$$+ 1/6\left(1 + \mu(\tilde{D}^3(f_0)\right) - 1/2\left(1 - \mu(\tilde{D}^3(f_0,(2,1)))\right) + 1/3\left(1 + \mu(\tilde{D}^3(f_0,(3)))\right) -$$
$$- 1/24\left(1 - \mu(\tilde{D}^4(f_0))\right) + 1/4\left(1 + \mu(\tilde{D}^4(f_0,(2,1^2)))\right) +$$
$$+ 1/120\left(1 + \mu(\tilde{D}^5(f_0))\right).$$

The examples suggest the following :

(5.5) Claim : the coefficient $\beta_{\gamma(r)}$ of the Euler characteristic of $\tilde{D}^r(f_t,\gamma(r))$, if it is non-empty, in

$$(\mathrm{I}) \qquad \chi(X_t) = \beta_0\,\chi(U_t) + \sum_{k\geq 2}\sum_{\gamma(k)} \beta_{\gamma(k)}\,\chi(\tilde{D}^k(f_t,\gamma(k)))$$

is given by: $$\beta_{\gamma(r)} = \frac{-(-1)^{\Sigma\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)},$$

where $\gamma(r) = (a_1,\dots,a_h)$, $a_i \geq a_{i+1}$, is a partition of r such that $p-r(p-n+1)+h \geq 0$ and $\alpha_i = \#\{j : a_j = i\}$. If $\tilde{D}^r(f_t,\gamma(r))$ is empty, we delete the row correspondent to it and the column correspondent to $\beta_{\gamma(r)}$ in the system of equations (∗) of (5.4). Then we take $\beta_{\gamma(r)}$ equal to 0.

In the remark below we find a relation between the coefficients above and certain simple characters of symmetric groups.

(5.6) Remark: Let $\gamma(r) = (a_1,\ldots, a_h)$ be a partition of r and let $\alpha_i = \#\left\{j : a_j = i\right\}$. Then $r = 1\cdot\alpha_1+2\cdot\alpha_2+\ldots$. So, it is easy to see that the set of partions of r is in one to one correspondence with the conjugacy classes of elements of the symmetric group S_r, ie, $\gamma(r)$ corresponds to the conjugacy class $\mathfrak{C}_{\gamma(r)}$ of the permutation with cycle pattern $\alpha_1,\alpha_2,\ldots$ (see [L1] p.25).

Let $h_{\mathfrak{C}_{\gamma(r)}}$ be the number of elements in $\mathfrak{C}_{\gamma(r)}$ and $\theta_{\mathfrak{C}_{\gamma(r)}}$ be +1 or -1 according to the sign of any element of $\mathfrak{C}_{\gamma(r)}$, ie, $\theta_{\mathfrak{C}_{\gamma(r)}}$ is +1 if the number of transpositions that any element of $\mathfrak{C}_{\gamma(r)}$ can be factored into is even and -1 otherwise. $\theta_{\mathfrak{C}_{\gamma(r)}}$ is called the alternating character of S_r (see [L1] p.134).

Then (5.5) can be rephrased as

$$\beta_{\gamma(r)} = -(-1)^r/r!\ h_{\mathfrak{C}_{\gamma(r)}}\theta_{\mathfrak{C}_{\gamma(r)}}$$

Indeed, just observe that $\theta_{\mathfrak{C}_{\gamma(r)}} := (-1)^{\sum\alpha_{2i}} = (-1)^{\sum(i-1)\alpha_i} = (-1)^{r-\sum\alpha_i}$ and that by Cauchy's Theorem on symmetric groups (see [L1] p.132),

$$h_{\mathfrak{C}_{\gamma(r)}} = \frac{r!}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} .$$

Now, recall that to prove claim (5.5) we need to show that the coefficients $\beta_{\gamma(r)}$ constitute the solution for the system (∗) of (5.4). With the notation above, we show that the claim (5.5) is true for the set of equations of the system (∗) of (5.4) corresponding to the rows of $\tilde{D}^k(f_t, (k))$, for $k\geq 2$ and $p-k(p-n+1)+1\geq 0$, in the table of degrees. Since this row is composed entirely of 1s then the equation corresponding to it is of the form:

$$1 = \beta_0\cdot 1+\sum_{r=2}^{k}\sum_{\gamma(r)}\beta_{\gamma(r)}\cdot 1$$

where $\gamma(r)$ runs through the set of all (ordered) partitions of r, $r\leq k$.

Also notice that because f_t is generically one to one , the first equation of the system (∗) of (5.4) is $1 = \beta_0.1$. Hence $\beta_0 = 1$, always.

Now considering the unit character $1_{\mathfrak{C}_{\gamma(r)}}$ of S_r (which is constant equal to one for any permutation of S_r), we have as a consequence of the character relations of the first kind for simple characters (see [L2] p.51), that

$$1/r! \sum_{\gamma(r)} h_{\mathfrak{C}_{\gamma(r)}} \theta_{\mathfrak{C}_{\gamma(r)}} 1_{\mathfrak{C}_{\gamma(r)}} = 0, \text{ i.e., } 1/r! \sum_{\gamma(r)} \frac{r!}{\prod_{i\geq 1} i^{\alpha_i} \alpha_i!} (-1)^{r-\sum\alpha_i} = 0.$$

$$\text{Hence, } \sum_{r\geq 2} -(-1)^r/r! \sum_{\gamma(r)} \frac{r!}{\prod_{i\geq 1} i^{\alpha_i} \alpha_i!} (-1)^{\sum\alpha_i} = 0, \text{ i.e., } \sum_{r\geq 2} \sum_{\gamma(r)} \beta_{\gamma(r)} = 0$$

which verifies the claim (5.5) for the set of equations of the system (∗) of (5.4) corresponding to the row of $\tilde{D}^k(f_t, (k))$, for any $k\geq 2$ and $p-k(p-n+1)+1\geq 0$, in the table of degrees.

We now return to the main problem, i.e., to find the matrix M of the system (∗) of (5.4) provided by the table of degrees of the mappings $f_t : U_t \subseteq \mathbb{C}^n \to X_t \subseteq \mathbb{C}^p$ and $\rho_{\gamma(k)} : \tilde{D}^k(f_t,\gamma(k)) \to X_t$ for all $k \geq 2$ and all partitions $\gamma(k) = (r_1, ..., r_m)$ of k with $p-k(p-n+1)+m \geq 0$.

(5.7) Let us start with some trivialities about multiplication of polynomials.

(i) Consider the polynomial $(x_1+x_2)(x_1+x_2)(x_1+x_2)$. To perform the multiplication, we choose either an x_1 or an x_2 from each of the three factors and multiply our choices together; we do this for all possible choices and add the results. We represent a particular set of choices by a two-cell partition of the numbers 1, 2, 3. In the first cell we put the numbers which correspond to factors from which we chose an x_1. In the second cell we put the numbers which correspond to factors from which we chose an x_2. For example, the 2-cell partition $[\{1,3\},\{2\}]$ corresponds to a choice of x_1 fom the first and third factors and x_2 from the second. The product so obtained is $x_1x_2x_1 = x_1^2.x_2$. Hence the coefficient of $x_1^2.x_2$ in the expansion of $(x_1+x_2)^3$ will be the number of partitions which lead to a choice of two x_1's and one x_2.

(ii) More generally, if we consider the polynomial $(x_1+...+x_q)\cdot ... \cdot (x_1+...+x_q)$ (r factors), the coefficient of $x_1^{b_1}\cdot ... \cdot x_q^{b_q}$ $\left(\sum b_i = r\right)$ in the expansion of $(x_1+..+x_q)^r$ is the number of partitions which lead to a choice of b_1 x_1's, b_2 x_2's, ... , b_q x_q's.

Going in another direction of generalisation,

(iii) Consider the polynomial $(x_1^2+x_2^2)(x_1^2+x_2^2)(x_1+x_2)$.

The coefficient of $x_1^3\cdot x_2^2$ in the expansion of $(x_1^2+x_2^2)^2(x_1+x_2)$ is the number of partitions which lead to a choice of one x_1^2, one x_1 and one x_2^2.

Finally,

(iv) In general, if we consider the polynomial $(x_1^{a_1}+...+x_q^{a_1})\cdot ... \cdot(x_1^{a_h}+...+x_q^{a_h})$ with $\sum a_i = r$. Then the coefficient of $x_1^{b_1}\cdot ... \cdot x_q^{b_q}$ $\left(\sum b_i = r\right)$ in the expansion of $\prod_{i=1}^{h}(x_1^{a_i}+...+x_q^{a_i})$ is the number of partitions which lead to a choice of : a certain number of (powers of x_1) $x_1^{a_{i_1}}$, with $a_{i_1} \in \{a_1,..., a_h\}$ and so that $\sum a_{i_1} = b_1$, a certain number of (powers of x_2) $x_2^{a_{i_2}}$, with $a_{i_2} \in \{a_1,..., a_h\}$ and so that $\sum a_{i_2} = b_2$ and so on.

Now to describe the table of degrees in the general case, i.e., to find the degrees of the mapping $\rho_{\gamma_1(r)} : \tilde{D}^r(f_t,\gamma_1(r)) \to X_t$ over the points of the image of a generic point of $\tilde{D}^s(f_t,\gamma_2(s))$ in X_t, with $r \leq s$ and $\gamma_1(r) < \gamma_2(s)$ (in other words, the cardinality of the fibre $\rho^{-1}{}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$ where $y \in \tilde{D}^s(f_t,\gamma_2(s))$ is a generic point), we have the following :

(5.8) Lemma : Let $\gamma_1(r) = (a_1,\dots, a_h)$ and $\gamma_2(r) = (b_1,\dots, b_q)$ be ordered partitions of r and s respectively with $\gamma_1(r) < \gamma_2(s)$. Let $y \in \tilde{D}^s(f_t,\gamma_2(s)) \subseteq \mathbb{C}^{n-1}\times\mathbb{C}^s$ be a generic point.

(i) if $r = s$ then the cardinality of the fibre $\rho^{-1}{}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$ is the coefficient of $x_1^{b_1}\cdot\ldots\cdot x_q^{b_q}$ in the polynomial $\prod_{i=1}^{h}(x_1^{a_i}+\ldots+x_q^{a_i})$.

(ii) if $r < s$ then the cardinality of the fibre $\rho^{-1}{}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$ is the sum of the coefficients of $x_1^{c_1}\cdot\ldots\cdot x_q^{c_q}$ in the polynomial $\prod_{i=1}^{h}(x_1^{a_i}+\ldots+x_q^{a_i})$, with $(c_1,\dots,c_q) \in \mathbb{N}_0^q$; $c_i \leq b_i$ and $\sum c_i = r$.

Proof : Recall that a generic point $y \in \tilde{D}^s(f_t,\gamma_2(s)) \subseteq \mathbb{C}^{n-1}\times\mathbb{C}^s$ is of the form $y = (x, y_1,\dots, y_1,\dots, y_q,\dots, y_q)$, $x \in \mathbb{C}^{n-1}$, $y_i \in \mathbb{C}$, $y_i \neq y_j$ for $i \neq j$ and y_i repeated b_i times.

The points of the fibre $\rho^{-1}{}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$ are the (generic or non-generic) points of $\tilde{D}^r(f_t,\gamma_1(r))$ whose coordinates are chosen out of the coordinates of the generic point y $\Big($the way to choose is such that the resulting point belongs to $\tilde{D}^r(f_t,\gamma_1(r))\Big)$.

Thus the cardinality of the fibre $\rho^{-1}{}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$ is the number of all such possible choices.

Finally, we consider the following (one to one) correspondence between the generic points $y = (x, y_1,\dots, y_1,\dots, y_q,\dots, y_q)$ of $\widetilde{D}^S(f_t,\gamma_2(s))$ and the monomials $x_1^{b_1}\cdot\ldots\cdot x_q^{b_q}$: namely to the sequence of coordinates $y_i,\dots, y_i$ (y_i repeated b_i times) we associate $x_i^{b_i}$. Now the lemma follows from the discussion in (5.7).

(5.9) Let us denote by $F_{\gamma_1(r)}^{\gamma_2(s)}$ the cardinality of the fibre $\rho^{-1}_{\gamma_1(r)}\left(\rho_{\gamma_2(s)}(y)\right)$, where y is a generic point of $\widetilde{D}^S(f_t,\gamma_2(s))$. So, $F_{\gamma_1(r)}^{\gamma_2(s)}$ is the number in the row corresponding to $\widetilde{D}^S(f_t,\gamma_2(s))$ and column corresponding to $\rho_{\gamma_1(r)}$ in the table of degrees. Hence, if we fix the partition $\gamma_1(r)$ and vary the partition $\gamma_2(s)$, we obtain the column of the table of degrees corresponding to the mapping $\rho_{\gamma_1(r)}$.

e.g. $F_{(1^2)}^{(2,1^3)} = 13$, $F_{(1^2)}^{(3,1^2)} = 7$.

The column corresponding to the mapping $f_t : U_t \to X_t$ is given by :

1 in the row of U_t and q in the row corresponding to $\widetilde{D}^S(f_t,\gamma_2(s))$, where q is the length (i.e., the number of parts) of the partition $\gamma_2(s)$.

Thus we have a systematic way to describe the table of degrees, in the general case, with which we shall prove the claim (5.5). Before that, we consider the following :

(5.10) Lemma: Let $e^{(r)}$ be the rth elementary symmetric function in the variables $x_1, \ldots, x_q$.
$$e^{(r)} = \sum_{1 \le i_1 < \cdots < i_r \le q} x_{i_1} \cdot \ldots \cdot x_{i_r} .$$

Let $\gamma(r) = (a_1, \ldots, a_h)$, $a_i \ge a_{i+1}$ be a partition of r. Then,
$$e^{(r)} = \sum_{\gamma(r)} \frac{(-1)^{r - \sum \alpha_i}}{\prod_{i \ge 1} (i^{\alpha_i} \alpha_i !)} \prod_{i \ge 1} (x_1^i + \ldots + x_q^i)^{\alpha_i}$$
where $\alpha_i = \#\{j : a_j = i\}$ and $\gamma(r)$ runs through the set of all (ordered) partitions of r.

Proof: (cf. [Mac] p.17)

Let E(u) denote the generating function of $e^{(r)}$, $E(u) = \sum_{r=0}^{q} e^{(r)} u^r = \prod_{j=1}^{q} (1 + x_j u)$.

Let P(u) denote the generating function of $p_i = \sum_{j=1}^{q} x_j^i$, $P(u) = \sum_{i \ge 1} p_i u^{i-1}$.

So, $P(u) = \sum_{i \ge 1} \sum_{j=1}^{q} x_j^i u^{i-1} = \sum_{j=1}^{q} \frac{x_j}{1 - x_j u} = - \sum_{j=1}^{q} \frac{d}{du} \log(1 - x_j u)$

Thus, $P(-u) = \sum_{j=1}^{q} \frac{d}{du} \log(1 + x_j u) = \frac{d}{du} \log \prod_{j=1}^{q} (1 + x_j u) = \frac{d}{du} \log E(u)$.

Hence, $\log E(u) = \int P(-u)\, du = \int \sum_{i \ge 1} p_i (-u)^{i-1} du = \sum_{i \ge 1} (-1)^{i-1} p_i \frac{u^i}{i}$.

So,
$$E(u) = \exp\left(\sum_{i \ge 1} (-1)^{i-1} \frac{p_i u^i}{i} \right) = \prod_{i \ge 1} \exp\left((-1)^{i-1} \frac{p_i u^i}{i} \right) =$$
$$= \prod_{i \ge 1} \sum_{\alpha_i = 0}^{\infty} \left((-1)^{i-1} \frac{p_i u^i}{i} \right)^{\alpha_i} \cdot \frac{1}{\alpha_i !} = \prod_{i \ge 1} \sum_{\alpha_i = 0}^{\infty} \frac{(-1)^{i\alpha_i - \alpha_i}}{i^{\alpha_i} \alpha_i !} p_i^{\alpha_i} u^{i \alpha_i} .$$

$$\text{Therefore } E(u) = \sum_{\gamma(r)} \frac{(-1)^{r-\sum\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)} \prod_{i\geq 1}(x_1^i+\dots+x_q^i)^{\alpha_i} \quad u^r$$

and this ends the proof of (5.10).

(5.11) Corollary: Let $\gamma(r) = (a_1,\dots,a_h)$, $a_i \geq a_{i+1}$ and $\gamma_2(s) = (b_1,\dots,b_q)$, $b_i \geq b_{i+1}$, be partitions of r and s respectively. Then

$$\sum_{\gamma(r)} \frac{(-1)^{\sum\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)} \cdot F_{\gamma(r)}^{\gamma_2(s)} = (-1)^{r-1}\binom{q}{r}$$

where $\alpha_i = \#\{j : a_j = i\}$ and $\gamma(r)$ runs through the set of all (ordered) partitions of r.

Proof: Since $F_{\gamma(r)}^{\gamma_2(s)}$ is the sum of the coefficients of $x_1^{c_1}.x_2^{c_2}.\dots.x_q^{c_q}$ in the

polynomial $\prod_{i\geq 1}(x_1^i+\dots+x_q^i)^{\alpha_i}$ for all $(c_1,\dots,c_q) \in \mathbb{N}_0^{\,q}$ such that $\sum_{i=1}^{q} c_i = r$, $c_i \leq b_i$

and $\alpha_i = \#\{j : a_j = i\}$, then $\sum_{\gamma(r)} \frac{(-1)^{r-\sum\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)} \cdot F_{\gamma(r)}^{\gamma_2(s)}$ is the sum of the

coefficients of $x_1^{c_1}.x_2^{c_2}.\dots.x_q^{c_q}$ in the polynomial

$$\sum_{\gamma(r)} \frac{(-1)^{r-\sum\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)} \prod_{i\geq 1}(x_1^i+\dots+x_q^i)^{\alpha_i} \quad \text{for all } (c_1,\dots,c_q) \in \mathbb{N}_0^{\,q} \text{ such that}$$

$\sum_{i=1}^{q} c_i = r$, with $c_i \leq b_i$. So, the result follows from (5.10).

Now to prove claim (5.5), i.e., to verify that the coefficients $\beta_{\gamma(r)}$ satisfying

$$(\mathrm{I}) \qquad \chi(X_t) = \beta_0\,\chi(U_t) + \sum_{r\geq 2}\ \sum_{\gamma(r)} \beta_{\gamma(r)}\ \chi(\tilde{D}^r(f_t,\gamma(r)))$$

are given by : $\beta_{\gamma(r)} = \dfrac{-(-1)^{\sum\alpha_i}}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)}$,

where $\gamma(r) = (a_1,\ldots,a_h)$, $a_i \geq a_{i+1}$, is a partition of r such that $p-r(p-n+1)+h \geq 0$ and $\alpha_i = \#\{j : a_j = i\}$, we just have to show that for every fixed partition $\gamma_2(s)$ (i.e. every row of the table of degrees), we have:

$$(\mathrm{IV}) \qquad 1 = \beta_0 \cdot q + \sum_{r=2}^{s}\ \sum_{\gamma(r)} \beta_{\gamma(r)} \cdot F_{\gamma(r)}^{\gamma_2(s)} \ .$$

But, this is an immediate consequence of (5.11), since β_0 is (always) equal to one. Thus we have proved the claim (5.5), and hence we obtain :

(5.12) Theorem : Let X_t be the disentanglement of the image of a corank 1 finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. Then,

$$\chi(X_t) = 1 + \sum_{r\geq 2}\ \sum_{\gamma(r)} \frac{-1}{\prod_{i\geq 1} i^{\alpha_i}\,\alpha_i!}\ (-1)^{\sum\alpha_i}\ \chi\left(\tilde{D}^r(f_t,(\gamma(r))\right)$$

where $\gamma(r) = (a_1,\ldots, a_h)$ runs through the set of all (ordered) partitions of r such that $p-k(p-n+1)+h \geq 0$, $\alpha_i = \#\{j : a_j=i\}$ and the multiple point schemes $\tilde{D}^r(f_t,(\gamma(r))$ are non-empty.

(5.13) Remark : (i) In (5.11) we have considered, for a fixed partition $\gamma_2(s)$, all (ordered) partitions of r ($r \leq s$), while in (5.12) we restrict ourselves to the set of all (ordered) partitions $\gamma(r) = (a_1,\dots, a_h)$ of r such that $p-k(p-n+1)+h \geq 0$. This is no problem since, if we fix a partition $\gamma_2(s) = (b_1,\dots,b_q)$ of s, such that

$p-k(p-n+1)+q \geq 0$, then a necessary condition for $F_{\gamma(r)}^{\gamma_2(s)}$ to be non-zero is that $\gamma(r) = (a_1,\dots, a_h)$ also verifies $p-k(p-n+1)+h \geq 0$.

(ii) Recall that we are considering the k- tuple point scheme $\tilde{D}^k(f_t)$ embedded in $\mathbb{C}^{n-1}\times\mathbb{C}^k$, where it lies invariant under the action of the symmetric group S_k, which permutes the coordinates in $\mathbb{C}^k$.

Our next step is to replace $\chi\left(\tilde{D}^k(f_t)\right)$ by $\chi\left(\tilde{D}^k(f_t)/S_k\right)$ in the formula of (5.12), for all $k \geq 2$.

For properties of the quotient spaces $\tilde{D}^k(f_t)/S_k$ see [M-M] § 3.

Some symmetries

(5.14) We now consider the finite mapping $\rho : \tilde{D}^k(f_t) \to \tilde{D}^k(f_t)/S_k$.

We can then repeat the procedure described above in order to relate the Euler characteristic $\chi\left(\tilde{D}^k(f_t)/S_k\right)$ of the quotient space $\tilde{D}^k(f_t)/S_k$ and the Euler characteristic of the multiple point schemes $\tilde{D}^k(f_t,\gamma(k))$ of f_t, for all partitions $\gamma(k)$ of k. In other words, we shall find coeffients $\upsilon_{\gamma(k)}$ such that :

(V) $\quad \chi\left(\tilde{D}^k(f_t)/S_k\right) = \sum_{\gamma(k)} \upsilon_{\gamma(k)}\ \chi\left(\tilde{D}^k(f_t,(\gamma(k))\right)$, where $\gamma(k)$ runs through the set of all (ordered) partitions of k.

Equivalently, we want to solve the system of equations:

$$(**)\qquad \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = M^{(k)} \begin{bmatrix} \upsilon_0 \\ \upsilon_{(2,1^{k-2})} \\ \vdots \\ \upsilon_{(k)} \end{bmatrix}$$

where the matrix $M^{(k)}$ is obtained from the table of degrees of the mapping $\rho_{\gamma(k)} : \tilde{D}^k(f_t,\gamma(k)) \to \tilde{D}^k(f_t)/S_k$ obtained by restriction of the mapping ρ to $\tilde{D}^k(f_t,\gamma(k))$, for all (ordered) partitions $\gamma(k)$ of k.

(5.15) Examples: (i) $\rho : \tilde{D}^2(f_t) \to \tilde{D}^2(f_t)/S_2$

Here the table of degrees and the matrix $M^{(2)}$ are :

	$\rho_{(1^2)}$	$\rho_{(2)}$
$\tilde{D}^2(f_t)$	2	—
$\tilde{D}^2(f_t,(2))$	1	1

$$M^{(2)} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence, solving the system of equations (**) for this case we obtain: $\upsilon_{(1^2)} = 1/2$ and $\upsilon_{(2)} = 1/2$.

So, $\chi\left(\tilde{D}^2(f_t)/S_2\right) = 1/2\,\chi\left(\tilde{D}^2(f_t)\right) + 1/2\,\chi\left(\tilde{D}^2(f_t,(2))\right)$.

(ii) $\rho : \tilde{D}^3(f_t) \to \tilde{D}^3(f_t)/S_3$.

Here the table of degrees and the matrix $M^{(3)}$ are :

	$\rho_{(1^3)}$	$\rho_{(2,1)}$	$\rho_{(3)}$
$\tilde{D}^3(f_t)$	6	—	—
$\tilde{D}^3(f_t,(2,1))$	3	1	—
$\tilde{D}^3(f_t,(3))$	1	1	1

$$M^{(3)} = \begin{bmatrix} 6 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence, solving the system of equations (**) for this case we obtain : $\upsilon_{(1^3)} = 1/6$, $\upsilon_{(2,1)} = 1/2$ and $\upsilon_{(3)} = 1/3$.

So, $\chi\left(\tilde{D}^3(f_t)/S_3\right) = 1/6\,\chi\left(\tilde{D}^3(f_t)\right) + 1/2\,\chi\left(\tilde{D}^3(f_t,(2,1))\right) + 1/3\,\chi\left(\tilde{D}^3(f_t,(3))\right)$.

(iii) $\rho : \tilde{D}^4(f_t) \to \tilde{D}^4(f_t)/S_4$

Here the table of degrees and the matrix $M^{(4)}$ are as follows :

	$\rho_{(1^4)}$	$\rho_{(2^2)}$	$\rho_{(3,1)}$	$\rho_{(2,1^2)}$	$\rho_{(4)}$
$\tilde{D}^4(f_u)$	24	—	—	—	—
$\tilde{D}^4(f_t,(2,1^2))$	12	2	—	—	—
$\tilde{D}^4(f_t,(2^2))$	6	2	2	—	—
$\tilde{D}^4(f_t,(3,1))$	4	2	—	1	—
$\tilde{D}^4(f_t,(4))$	1	1	1	1	1

$$M^{(4)} = \begin{bmatrix} 24 & 0 & 0 & 0 & 0 \\ 12 & 2 & 0 & 0 & 0 \\ 6 & 2 & 2 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, solving the system of equations (**) for this case we obtain:

$\upsilon_{(1^4)} = 1/24$, $\upsilon_{(2,1^2)} = 1/4$, $\upsilon_{(3,1)} = 1/3$, $\upsilon_{(2^2)} = 1/8$ and $\upsilon_{(4)} = 1/4$.

So, $\chi\left(\tilde{D}^4(f_t)/S_4\right) = 1/24\,\chi\left(\tilde{D}^4(f_t)\right) + 1/4\,\chi\left(\tilde{D}^4(f_t,(2,1^2))\right) + 1/8\,\chi\left(\tilde{D}^3(f_t,(2^2))\right) +$

$$+ 1/3\,\chi\left(\tilde{D}^4(f_t,(3,1))\right) + 1/4\,\chi\left(\tilde{D}^4(f_t,(4))\right).$$

In general we obtain :

(5.16) Proposition : Let $\gamma(k) = (a_1,\dots, a_h)$, $a_i \geq a_{i+1}$ be a partition of k, with $\alpha_i = \#\{j : a_j=i\}$. Then

$$\chi\left(\tilde{D}^k(f_t)/S_k\right) = \sum_{\gamma(k)} \frac{1}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!}\,\chi\left(\tilde{D}^k(f_t,(\gamma(k))\right)$$

where $\gamma(k) = (a_1,\dots, a_h)$ runs through the set of all (ordered) partitions of k.

Proof: This is analogous to (5.12) and will follow from the lemma below.

(5.17) Lemma : Let h_r be the rth complete symmetric function in the variables $x_1,\dots,x_q$, i.e., h_r is the sum of all monomials of degree r in the variables $x_1,\dots,x_q$. Then $h_r = \sum_{\gamma(r)} \frac{1}{\prod_{i\geq 1}(i^{\alpha_i}\alpha_i!)} \prod_{i\geq 1}(x_1^i + \dots + x_q^i)^{\alpha_i}$

where $\gamma(r) = (a_1,\dots, a_h)$, $a_i \geq a_{i+1}$ runs through the set of all (ordered) partitions of r and $\alpha_i = \#\{j : a_j = i\}$.

Proof : This is analogous to (5.10). We only observe that if H(u) is the generating function of h_r then $P(u) = H'(u)/ H(u)$, where P(u) is the generating function of $p_i = \sum_{j=1}^{q} x_j^i$, $P(u) = \sum_{i\geq 1} p_i u^{i-1}$ (cf. [Mac] p.16). Then the proof follows as that of (5.10).

In conclusion, we have

from (5.12) $$\chi(X_t) = 1 + \sum_{r\geq 2} \sum_{\gamma(r)} \frac{-1}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} (-1)^{\sum\alpha_i} \chi\left(\tilde{D}^r(f_t,(\gamma(r))\right)$$

and from (5.16) $$\chi\left(\tilde{D}^r(f_t)/ S_r\right) = \sum_{\gamma(r)} \frac{1}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \chi\left(\tilde{D}^r(f_t,(\gamma(r))\right)$$

So, together they provide us with :

(5.18) Theorem : Let X_t be the disentanglement of the image of a corank 1 finitely $\mathcal{A}$-determined map-germ $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$.

$$\text{Then, } \chi(X_t) = 1 + \sum_{k\geq 2} (-1)^{k-1} \chi\left(\tilde{D}^k(f_t)/ S_k\right) +$$

$$+ \sum_{k\geq 2} \sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\sum\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \chi\left(\tilde{D}^k(f_t,(\gamma(k))\right)$$

where $\gamma(k)=(a_1, ..., a_h)$ runs through the set of (ordered) partitions of k with $p-k(p-n+1)+h \geq 0$ and $\alpha_i = \#\left\{j : a_j = i\right\}$. (we observe that if $\tilde{D}^r(f_0,(\gamma(r))$ is empty then the coefficient of $\chi\left(\tilde{D}^r(f_t,(\gamma(r))\right)$ in the formula above is zero).

(5.19) Examples: (Here we assume that all multiple point schemes in consideration are non-empty. n and p denote respectively the dimension of the source and target of the mapping f_t)

(i) $(n,p)=(2,3)$

$$\chi\left(X_t\right) = 1 - \chi\left(\tilde{D}^2(f_t)/S_2\right) + \chi\left(\tilde{D}^3(f_t)/S_3\right) + \chi\left(\tilde{D}^2(f_t,(2))\right).$$

(ii) $(n,p)=(3,4)$

$$\chi\left(X_t\right) = 1 - \chi\left(\tilde{D}^2(f_t)/S_2\right) + \chi\left(\tilde{D}^3(f_t)/S_3\right) - \chi\left(\tilde{D}^4(f_t)/S_4\right) +$$
$$+ \chi\left(\tilde{D}^2(f_t,(2))\right) - \chi\left(\tilde{D}^3(f_t,(2,1))\right).$$

(iii) $(n,p)=(4,5)$

$$\chi\left(X_t\right) = 1 - \chi\left(\tilde{D}^2(f_t)/S_2\right) + \chi\left(\tilde{D}^3(f_t)/S_3\right) - \chi\left(\tilde{D}^4(f_t)/S_4\right) + \chi\left(\tilde{D}^5(f_t)/S_5\right) +$$
$$+ \chi\left(\tilde{D}^2(f_t,(2))\right) - \chi\left(\tilde{D}^3(f_t,(2,1))\right) + 1/4\,\chi\left(\tilde{D}^2(f_t,(2,1^2))\right).$$

(cf. example (iii) of page 36).

Now using the fact that $\tilde{D}^k(f_t)/S_k$ and $\tilde{D}^k(f_t)$ are Milnor fibres, and supposing that all multiple point schemes in consideration are non-empty, we obtain :

(i') $(n,p)=(2,3)$

$$\chi\left(X_t\right) = 2 + \mu\left(\tilde{D}^2(f_0)/S_2\right) + \mu\left(\tilde{D}^3(f_0)/S_3\right) + \mu\left(\tilde{D}^2(f_0,(2))\right).$$

(ii') $(n,p)=(3,4)$

$$\chi\left(X_t\right) = -\mu\left(\tilde{D}^2(f_0)/S_2\right) - \mu\left(\tilde{D}^3(f_0)/S_3\right) - \mu\left(\tilde{D}^4(f_0)/S_4\right) -$$
$$- \mu\left(\tilde{D}^2(f_0,(2))\right) - \mu\left(\tilde{D}^3(f_0,(2,1))\right)$$

In general, substituting $\chi\left(\tilde{D}^k(f_t)/S_k\right) = 1 + (-1)^{p-k(p-n)} \cdot \mu\left(\tilde{D}^k(f_0)/S_k\right)$ and $\chi\left(\tilde{D}^r(f_t,(\gamma(r))\right) = 1 + (-1)^{p-r(p-n+1)+\sum\alpha_i} \mu\left(\tilde{D}^r(f_t,(\gamma(r))\right)$, with $\gamma(r) = (a_1, \ldots, a_h) \neq (1^r)$ a partition of r and $\alpha_i = \#\left\{j : a_j = i\right\}$, in (5.18) it becomes :

$$(5.20) \qquad \chi(X_t) = C_{\gamma(k)} + \sum_{k=2}^{p} (-1)^{p-k(p-n+1)+1} \left[\mu\left(\tilde{D}^k(f_0)/S_k\right) + \sum_{\gamma(k)} \frac{1+(-1)^{k+\sum\alpha_i+1}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \mu\left(\tilde{D}^k(f_0,(\gamma(k))\right) \right]$$

where $C_{\gamma(k)} = 1 + \sum_{k=2}^{p} \left((-1)^{k+1} + \sum_{\gamma(k)} \frac{(-1)^k - (-1)^{\sum\alpha_i}}{\prod_{i\geq 1} i^{\alpha_i}\alpha_i!} \right)$

(5.21) Remark: (i) In the case (n,p)=(2,3) we can write

$$\chi\left(X_t\right) = \mu\left(\tilde{D}^2(f_0)/S_2\right) + C(f_0) + T(f_0)$$

where $C(f_0)$ and $T(f_0)$ are respectively the number of cross-caps and triple points of f_0

In fact, $\tilde{D}^2(f_0,(2))$ and $\tilde{D}^3(f_0)/S_3$ are ICIS (since f_0 is finitely $\mathcal{A}$-determined, cf. [M-M] (2.14)) of dimension zero. Now, the remark follows from proposition (5.12) of [Lo].

(ii) When $p = n+1$, $\chi(X_t)$ is semicontinuous, since $(-1)^{p-k(p-n+1)+1}$ is indepent of k.

(iii) In recent work [Mo 4], David Mond proves that the disentanglements of mappings $f_0: (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ have the same homotopy type of a wedge of n - spheres. Hence, when p = n+1, (5.20) above, gives the number of n - spheres in that wedge. This result resembles that of J. Milnor ([Mi] chap.7) or H. Hamm ([Ha]).

Chapter II

Remarks on the Real Case

"It is the complex case that is easier to deal with."

"Ah! Bartleby. Ah! Humanity."

Herman Melville (Bartleby)

§1. Morsifications

In [Mo 2] David Mond introduces certain invariants for corank 1 map-germs $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$. Among those we find C(f) and T(f) which measures respectively the number of cross-caps (Whitney umbrellas) and triple points that the origin splits into when f is deformed in a generic way.

For real analytic map-germs $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$, the numbers C(f) and T(f) take the same values as $C(f_{\mathbb{C}})$ and $T(f_{\mathbb{C}})$, where $f_{\mathbb{C}}:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is the complexification of f. However the geometric interpretation above is no longer meaningful; for distinct generic deformations of f may have different numbers of cross-caps, or triple points.

(1.1) Definition : By a *morsification* of a map-germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ we shall understand an arbitrarily small real deformation of $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ exhibiting C(f) cross-caps and T(f) triple points in its image in $\mathbb{R}^3$.

In [Mo 1] we find the list of simple singularities of corank 1 map-germs $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$, namely:

Name	Normal Form f(x, y)=	Name	Normal Form f(x, y)=
cross-cap	(x,y^2,xy)		
S_k^+	$(x,y^2,y^3+x^{k+1}y)$	S_k^-	$(x,y^2,y^3-x^{k+1}y)$
B_k^+	(x,y^2,x^2y+y^{2k+1})	B_k^-	(x,y^2,x^2y-y^{2k+1})
C_k^+	(x,y^2,xy^3+x^ky)	C_k^-	(x,y^2,xy^3-x^ky)
F_4	(x,y^2,x^3y+y^5)		
H_k	$(x,y^3,xy+y^{3k-1})$		

The number of cross-caps and triple points for the simple singularities of corank 1 map-germs $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$, as well as the double point curve of f, $D^2(f)$, i.e. the closure of the set of points x in $\mathbb{R}^2$ such that there is an $x'\in\mathbb{R}^2$, $x'\neq x$ with $f(x)=f(x')$, are given below:

Name	C(f)	T(f)	$D^2(f)$
cross-cap	1	0	x
S_k^+	k+1	0	y^2+x^{k+1} (A_k)
S_k^-	k+1	0	y^2-x^{k+1} (A_k)
B_k^+	2	0	x^2+y^{2k} (A_{2k-1})
B_k^-	2	0	x^2-y^{2k} (A_{2k-1})
C_k^+	2	0	xy^2+x^k (D_{k+1})
C_k^-	2	0	xy^2-x^k (D_{k+1})
F_4	3	0	x^3+y^4 (E_6)
H_k	2	k-1	$x^2+xy^{3k-2}+y^{6k-4}$ (A_{6k-5})

In the table below we find morsifications for all simple singularities of corank one map-germs $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$.

Name	Morsification $f(x, y, \underline{u}) =$	Restrictions ($u_i \neq u_j$ for $i \neq j$, in all cases)
S_k^+	$(x, y^2, y^3 + y \prod_{i=1}^{k+1}(x-u_i))$	---------
S_k^-	$(x, y^2, y^3 - y \prod_{i=1}^{k+1}(x-u_i))$	---------
B_k^+	$(x, y^2, x^2y + y \prod_{i=1}^{k}(y^2-u_i))$	$u_i > 0$ if k is odd $u_1 < 0$, $u_i > 0$ if k is even, $i \geq 2$
B_k^-	$(x, y^2, x^2y - y \prod_{i=1}^{k}(y^2-u_i))$	$u_i > 0$ if k is even $u_1 < 0$, $u_i > 0$ if k is odd, $i \geq 2$
C_k^+	$(x, y^2, xy^3 + xy \prod_{i=1}^{k-1}(x-u_i))$	$u_i < 0$
C_k^-	$(x, y^2, xy^3 - xy(x^2-u_0^2) \prod_{i=1}^{k-3}(x-u_i))$	$u_i < 0$, $0 < \lvert u_0 \rvert < \min\{\lvert u_i \rvert\}$
F_4	$(x, y^2, y^5 + yx(x-u_1)(x-u_2))$	$u_i > 0$
H_k	$(x, y^3 + u_0 y, xy + y^2 g(x, y, \underline{u}))$	$u_0 < 0$ $\lvert u_i \rvert < -(2/3)u_0(-u_0/3)^{1/2}$

$$\text{where } g(x,y,\underline{u}) = \begin{cases} \prod_{i=1}^{(k-1)/2} (y^3+u_0y+u_i)(y^3+u_0y-u_i) & \text{if } k \text{ is odd} \\ (y^3+u_0y)\prod_{i=1}^{(k-2)/2} (y^3+u_0y+u_i)(y^3+u_0y-u_i) & \text{if } k \text{ is even} \end{cases}$$

(1.2) Remark: The $(k-1)$ triple points of H_k are the triples of points $(0,y_1)$, $(0,y_2)$, $(0,y_3)$ in $\mathbb{R}^2$ such that $g(0,y_1,\underline{u}) = g(0,y_2,\underline{u}) = g(0,y_3,\underline{u})$. Notice that the u_i's have been chosen so that the equations $y^3+u_0y+u_i=0$ and $y^3+u_0y-u_i=0$ have three distinct real roots.

The two cross-caps of H_k in $\mathbb{R}^2$ are given by:

$$3y^2 + u_0 = 0 \text{ and } x + 2y\,g(x,y,\underline{u}) = 0$$

So, when k is odd

$$\left(2\,(-u_0/3)^{1/2} \prod_{i=1}^{(k-1)/2} (-4/27\; u_0^3 - u_i)\;,\; (-u_0/3)^{1/2}\right)$$

and

$$\left(-2\,(-u_0/3)^{1/2} \prod_{i=1}^{(k-1)/2} (-4/27\; u_0^3 - u_i)\;,\; -\,(-u_0/3)^{1/2}\right)$$

are the two cross-caps of H_k in $\mathbb{R}^2$,

when k is even

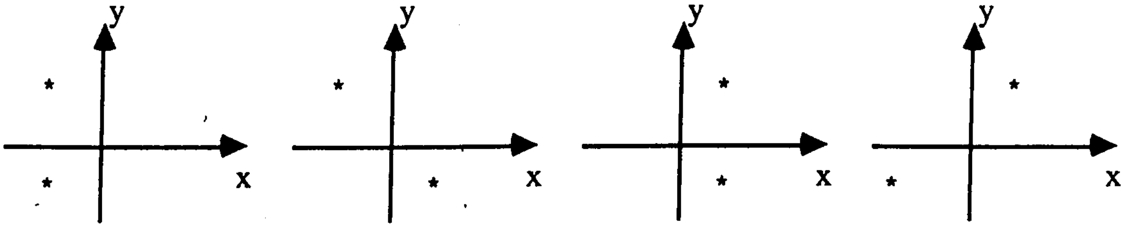
$$\left(4/9\; u_0^2 \prod_{i=1}^{(k-2)/2} (-4/27\; u_0^3 - u_i)\;,\; (-u_0/3)^{1/2}\right)$$

and

$$\left(4/9\; u_0^2 \prod_{i=1}^{(k-2)/2} (-4/27\; u_0^3 - u_i)\;,\; -\,(-u_0/3)^{1/2}\right)$$

are the two cross-caps of H_k in $\mathbb{R}^2$.

Their location in the plane Oxy are respectively:



when $k \equiv 0$, 1, 2 and 3 mod 4.

§2 The image of morsifications of the simple singularities

Let S_f denote the image in $\mathbb{R}^3$ of a small disc centred at the origin in $\mathbb{R}^2$ via a morsification of f.

We are going to present some drawings representing the surface S_f for all simple singularities of corank 1 map-germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$.

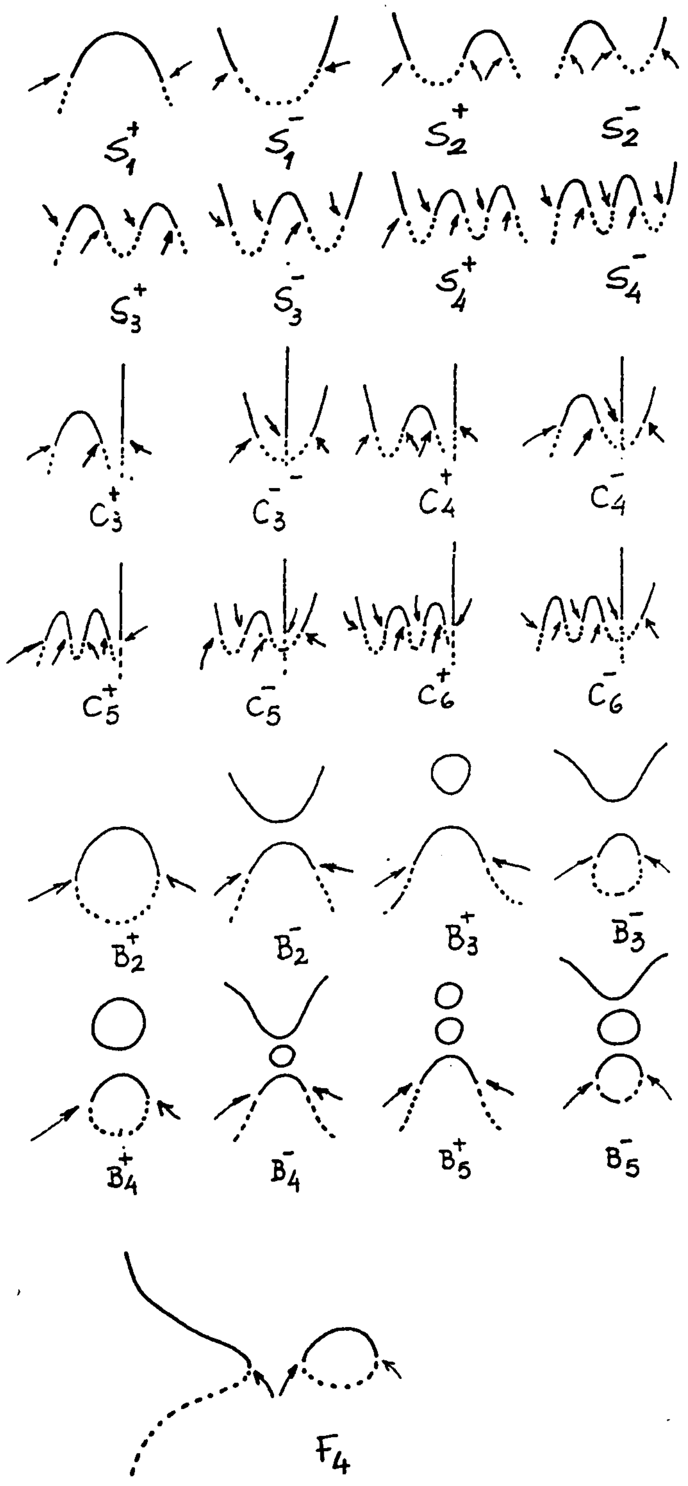
We shall construct S_f by firstly finding the image in $\mathbb{R}^3$ of the double point curve $D^2(\bar{f}_{\underline{u}})$ of the morsification $\bar{f}_{\underline{u}}$ of f ; this will be the self-intersection curve of the surface S_f. Then by glueing the 2-cells determined by $D^2(\bar{f}_{\underline{u}})$ in $\mathbb{R}^2$, we complete the drawings of S_f.

The image of $D^2(f)$ in $\mathbb{R}^3$ is obtained by the involutions determined by the cross-caps of f (see below). When f is of the form $f(x,y) = (x, y^2, y\,p(x, y^2))$, the image of $D^2(f)$ $\big($which is the curve in $\mathbb{R}^2$ given by $(p(x, y^2) = 0)\big)$ in $\mathbb{R}^3$ is the plane curve given by $\{(X,Y,Z) \in \mathbb{R}^3 : Z = 0,\ p(X,Y) = 0\}$.

In the table below we find the double point curves for the morsifications of the previous table:

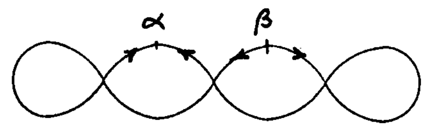
Name	$D^2(\bar{f}_{\underline{u}})$	Name	$D^2(\bar{f}_{\underline{u}})$
S_k^+	$y^2+\prod_{i=1}^{k+1}(x-u_i)$	S_k^-	$y^2-\prod_{i=1}^{k+1}(x-u_i)$
B_k^+	$x^2+\prod_{i=1}^{k}(y^2-u_i)$	B_k^-	$x^2-\prod_{i=1}^{k}(y^2-u_i)$
C_k^+	$x\,y^2+x\prod_{i=1}^{k-1}(x-u_i)$	C_k^-	$x\,y^2-x(x^2-u_0^2)\prod_{i=1}^{k-3}(x-u_i)$
F_4	$y^4+x(x-u_1)(x-u_2)$	H_k	$x^2+xy.g+(y.g)^2+u\,.g^2$ where $g=g(x,y,\underline{u})$

Below we have the self-intersection curves for some surfaces S_f (the dotted lines represent the portion of the curves with $Y < 0$ and the arrows indicate the cross-caps)



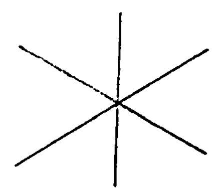
To obtain the image of the double point curve of the morsification of H_k in $\mathbb{R}^3$, we proceed as follows:

In $\mathbb{R}^2$, the double point curve of H_2 is an A_7 singularity.

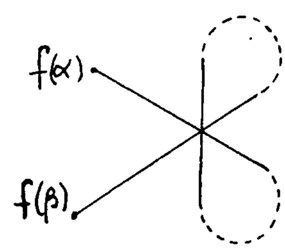


The points α and β denote the cross-caps of H_2 and the arrows denote the identifications forced by α and β. One checks easily that the positions of α and β on $D^2(f)$ as shown, are the only ones possible.

In the target, we know that S_{H_2} has one triple point.



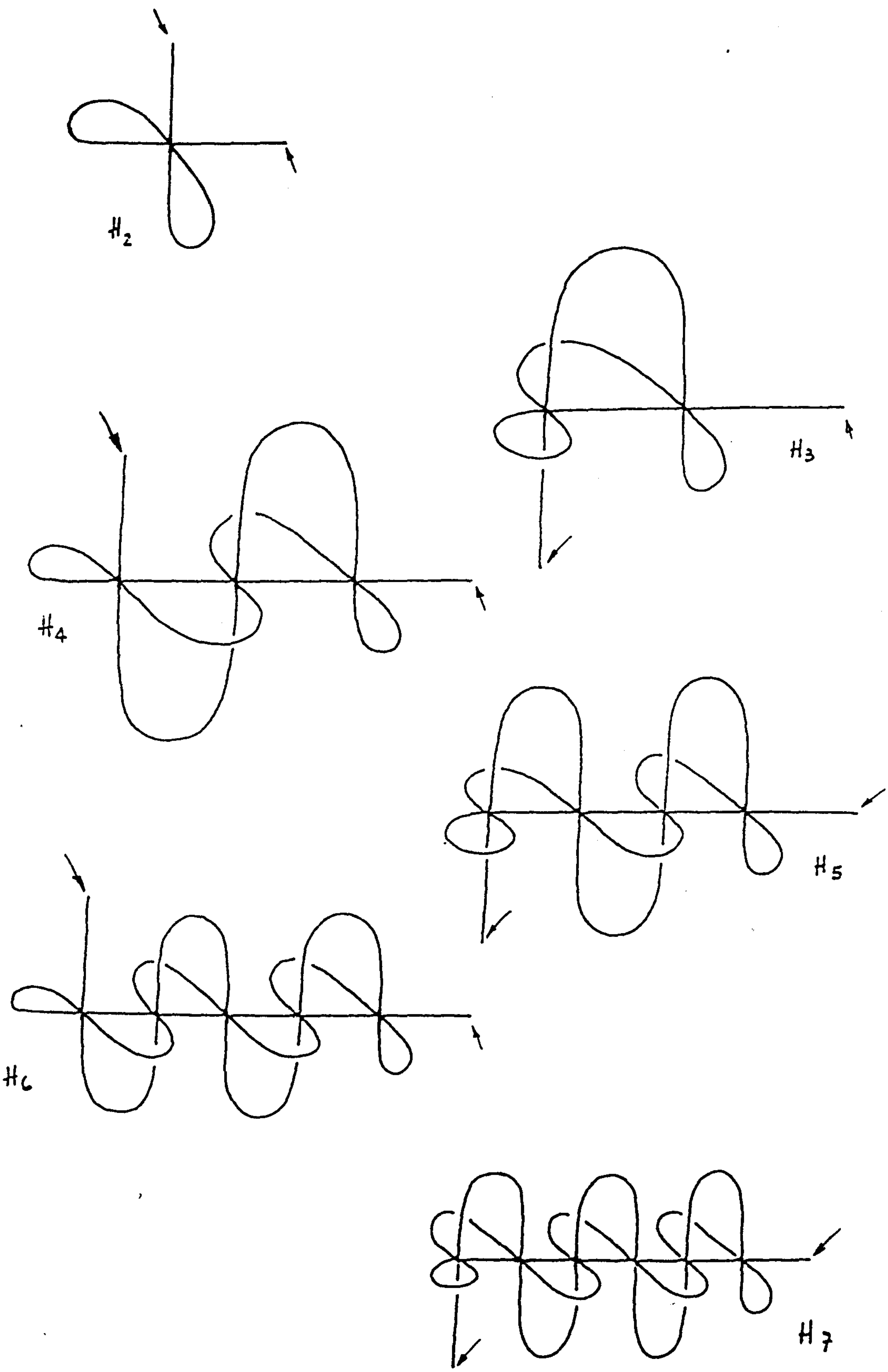
The image of α and β cannot lie in only one of those three segments, for otherwise the self-intersection curve of S_{H_2} would have more then one branch. Therefore the picture is the following:



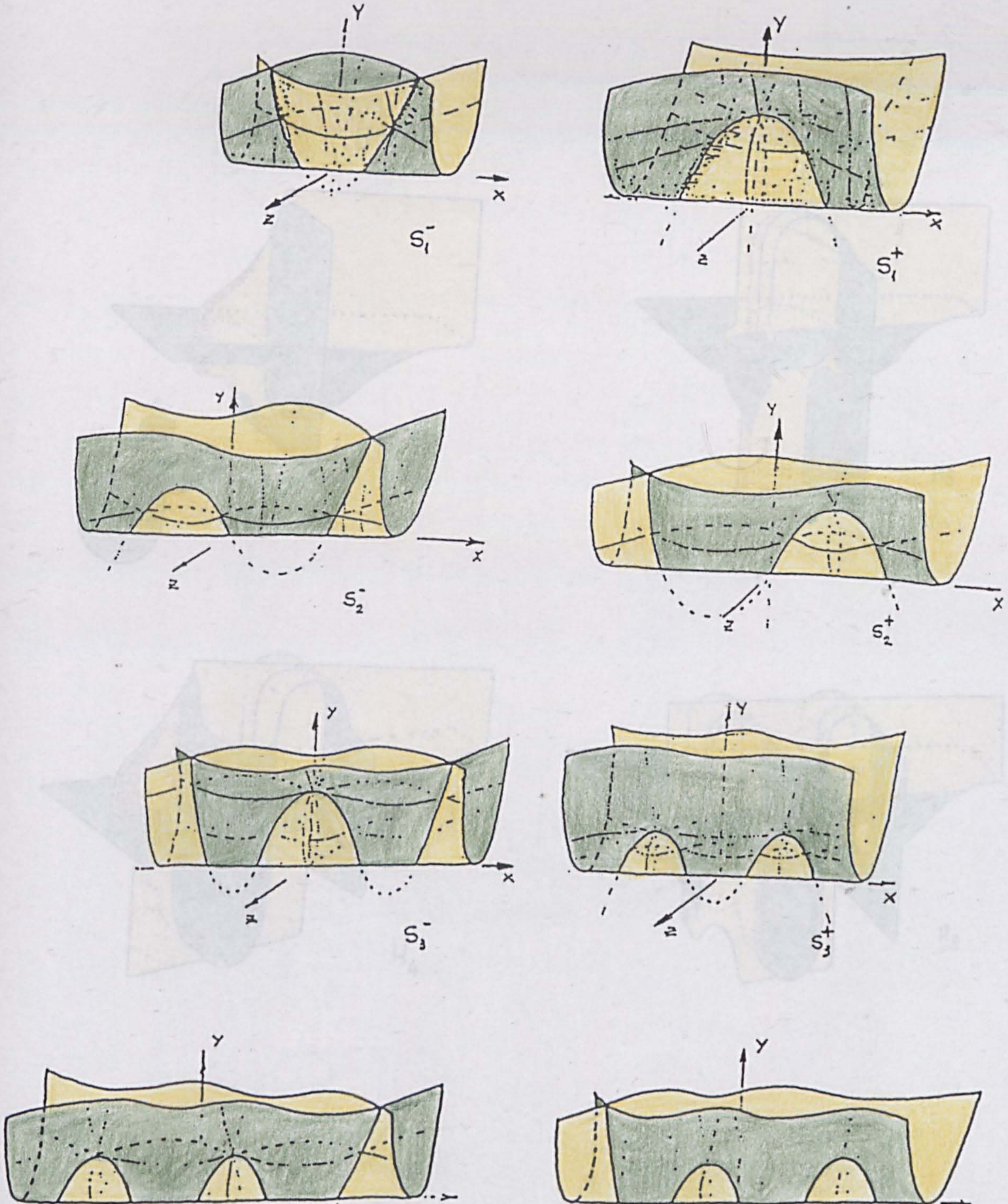
The dotted segments represent the image of the rest of the double point curve of the morsification of H_2 in $\mathbb{R}^3$.

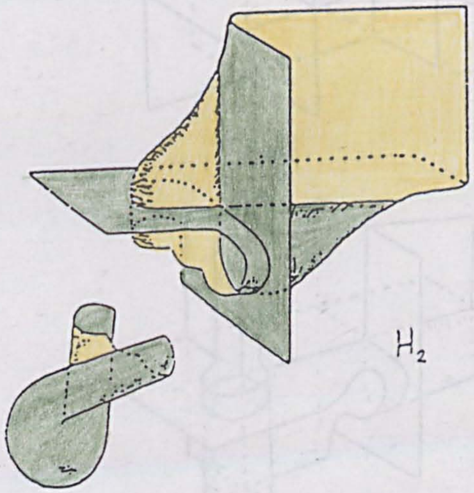
Notice that we cannot have knots in the image of the doule point curve of the morsification of H_2. Indeed, the four discs determined by the double point curve of the morsification of H_2 in $\mathbb{R}^2$ are sent to four discs in $\mathbb{R}^3$ since the morsification is a homeomorphism outside of the double point curve. Similar argument applies to the other members of the family H_k. So, we obtain:

Self-intersection curve of S_{H_k}

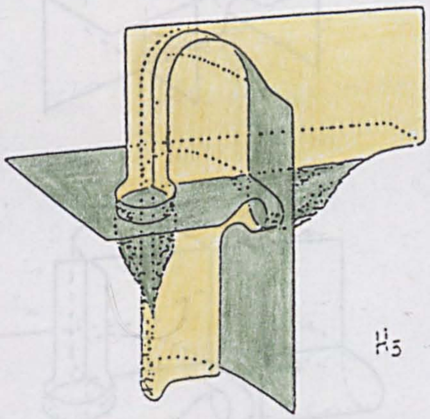


Some surfaces S_f

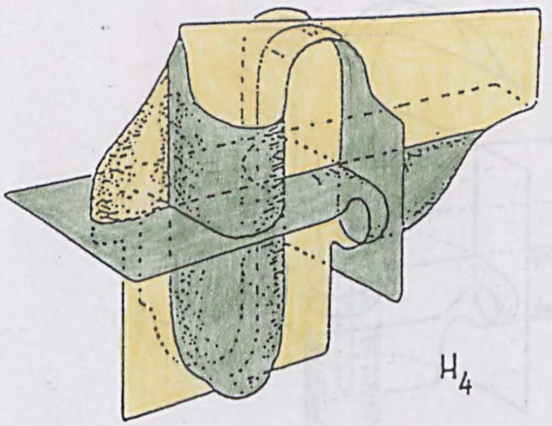




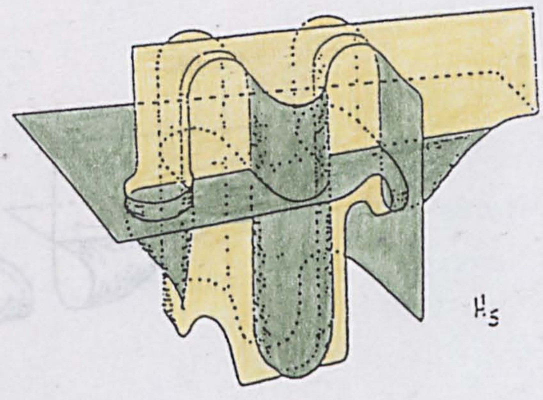
H_2



H_3

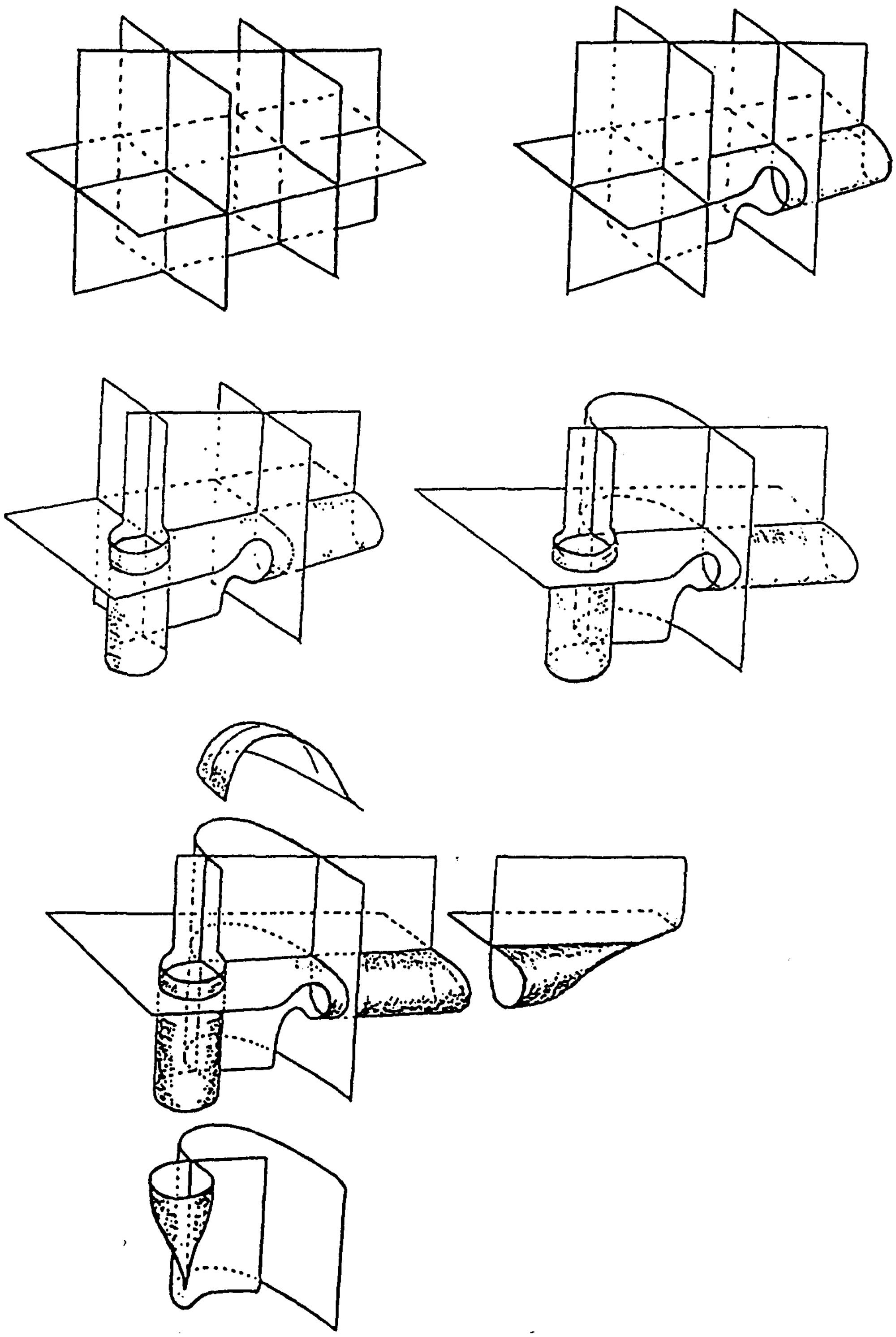


H_4



H_5

Steps in the construction of the surface S_{H_3}



§3. The homology of S_f

Following the method presented in [C-F] chapter III, on the singular homology of a complex with identifications, we calculate the homology of S_{H_2} and S_{H_3}.

(3.1) The homology of S_{H_2}.

The picture below (figure 1) represents a small disc B_ε around the origin in $\mathbb{R}^2$ (which will be mapped by the morsification of H_2 to give rise to S_{H_2}) and the double point curve $D^2(\bar{f}_{\underline{u}})$ of the morsification $\bar{f}_{\underline{u}}$ of H_2 (A_7 singularity).

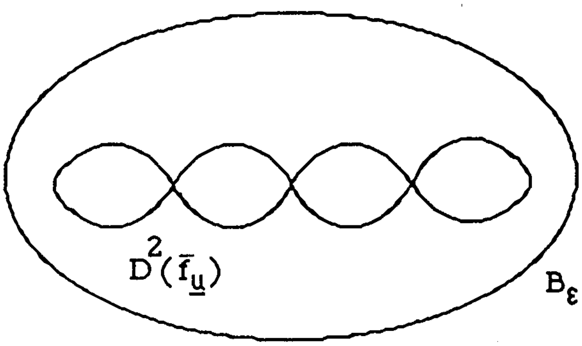


figure 1

Figure 2 presents a triangulation of figure 1. The letters A,B,C,D,E,F and G denote the 0-dimensional cells; τ_i, $i=1,\ldots,10$ denote the 1-dimensional cells (the two cross -caps of H_2, denoted by D and E below, force the identifications represented by the arrows and the repeated letters) and σ_j, $j=1,\ldots,6$ represent the 2-dimensional cells.

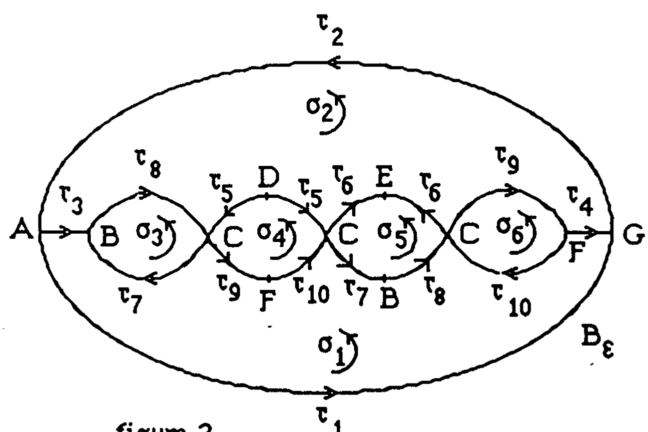


figure 2

Hence, we obtain (7 cells of dimension zero, 10 of dimension one and 6 of dimension 2) the chain complex

$$0 \to \mathbb{Z}^6 \xrightarrow{\partial_2} \mathbb{Z}^{10} \xrightarrow{\partial_1} \mathbb{Z}^7 \to 0 .$$

The boundary operators ∂_1 and ∂_2 for the complex with identifications (figure 2) (cf. [C-F] p. 72) are obtained by the incidence functions given by the tables below (+ or - means that the degrees of incidence is +1 or -1 ; cf. [C-F] p.32 and 72):

τ	1	2	3	4	5	6	7	8	9	10
A	-	+	-							
B			-				+	-		
C					+	-	-	+	-	+
D					-					
E						+				
F			-						+	-
G	+	-		+						

$\sigma \backslash \tau$	1	2	3	4	5	6	7	8	9	10
1	+		-	-				-	-	
2		+	+	+				+	+	
3							-	-		
4									+	+
5							+	+		
6									-	-

So, the image of ∂_1 is $\mathbb{Z}^6$, the kernel of ∂_1 is $\mathbb{Z}^4$, the image of ∂_2 is $\mathbb{Z}^4$ and the kernel of ∂_2 is $\mathbb{Z}^2$. Therefore, the homology groups of S_{H_2} are :

$$H_2(S_{H_2}) = \mathbb{Z}^2, \quad H_1(S_{H_2}) = 0 \text{ and } H_0(S_{H_2}) = \mathbb{Z}.$$

(3.2) The homology of S_{H_3}.

The picture below (figure 3) represents a small disc B_ε aroud the origin in $\mathbb{R}^2$ (which will be mapped by the morsification of H_3 to give rise to S_{H_3}) and the double point curve $D^2(\bar{f}_{\underline{u}})$ of the morsification $\bar{f}_{\underline{u}}$ of H_3 (A_{13} singularity).

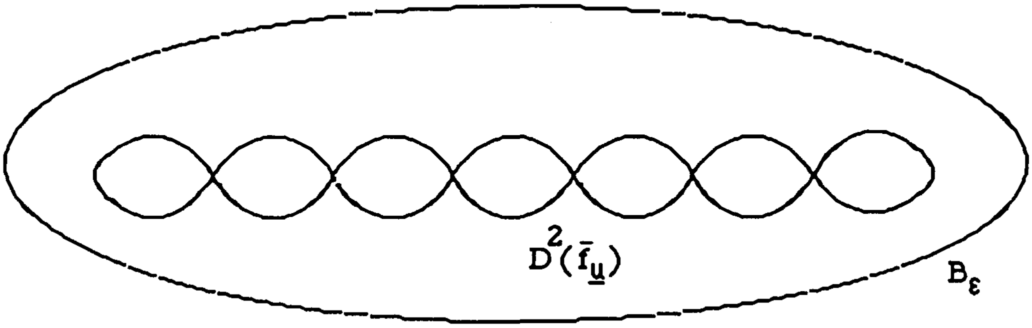


figure 3

Figure 4 presents a triangulation of figure 3. The letters A, B, C, D, E, F, G and H denote the 0-dimensional cells; τ_i, $i=1,..,13$ denote the one-dimensional cells (note that the 2 cross-caps of H_3, denoted by E and G force the identifications represented by the arrows and repeated letters) and σ_j, $j=1,...,9$ represent the two-dimensional cells.

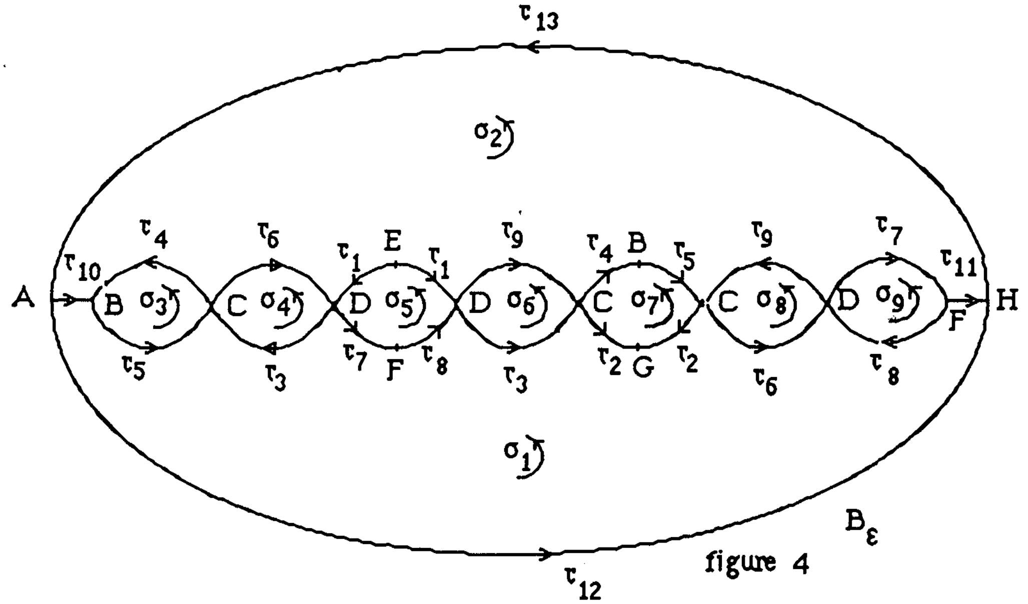


figure 4

Hence, we obtain (8 cells of dimension zero, 13 of dimension one and 9 of dimension 2) the chain complex

$$0 \rightarrow \mathbb{Z}^9 \xrightarrow{\partial_2} \mathbb{Z}^{13} \xrightarrow{\partial_1} \mathbb{Z}^8 \rightarrow 0 .$$

The boundary operators ∂_1 and ∂_2 for the complex with identifications (figure 4) are given by the incidence tables below :

τ	A	B	C	D	E	F	G	H
1				+	-			
2			-				+	
3			+	-				
4		+	-					
5		-	+					
6			-	+				
7				-		+		
8				+		-		
9			+	-				
10	-	+						
11						-		+
12	-							+
13	+							-

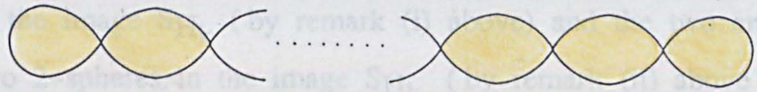
τ \ σ	1	2	3	4	5	6	7	8	9
1									
2									
3				-		+			
4			+				-		
5	-	+	+				-		
6	-	+		-				+	
7	-	+			+				-
8					+				-
9						-		+	
10	-	+							
11	-	+							
12	+								
13		+							

So, the image of ∂_1 is $\mathbb{Z}^7$, the kernel of ∂_1 is $\mathbb{Z}^6$, the image of ∂_2 is $\mathbb{Z}^6$ and the kernel of ∂_2 is $\mathbb{Z}^3$. Therefore, $H_2(S_{H_3}) = \mathbb{Z}^3$, $H_1(S_{H_3}) = 0$ and $H_0(S_{H_3}) = \mathbb{Z}$.

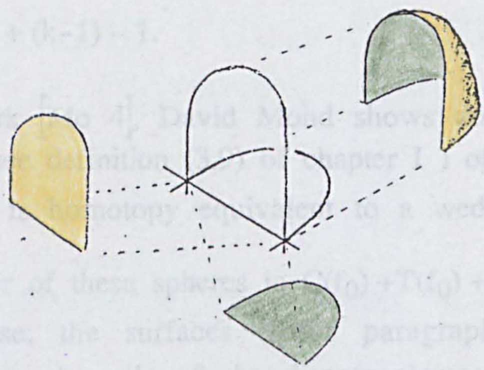
(3.3) The homology of S_{H_k}

We can obtain the homology of S_{H_k} in general by a geometrical argument. Let us consider the double point curve of the morsification of H_k in $\mathbb{R}^2$.

It is a real morsification of an A_{6k-5} singularity, with 3(k-1) nodes corresponding to the k-1 triple points. Hence, 3(k-1)+1 discs are determined by it in $\mathbb{R}^2$.



Remarks : (i) Each two adjacent triple points in the image give rise to one 2-sphere by glueing three discs, the images of those determined by the double point curve in $\mathbb{R}^2$.



(ii) Each cross-cap near to a triple point give rise to one 2-sphere in the image by glueing two discs.

We have seen (remark (1.2)) that the triple points of the morsification of H_k lie in the y- axis in $\mathbb{R}^3$. So, the k-1 triple points of H_k provide (k-2) 2-spheres in the image S_{H_k} (by remark (i) above) and the two cross-caps of H_k provide two 2-spheres in the image S_{H_k} (by remark (ii) above).

The rest of the small disc that is mapped to S_{H_k} can be contracted towards the double point curve in the source, or equivalently towards the self-intersection curve in the target.

Hence, the homology group $H_2(S_{H_k})$ is equal to $\mathbb{Z}^k$.

(3.4) Remark: In the examples above, we have that the rank of $H_2(S_{H_k})$ is $C(H_k) + T(H_k) - 1 = 2 + (k-1) - 1$.

In a recent work [Mo 4], David Mond shows that in the complex case, the disentanglement (see definition (3.9) of chapter I) of the image of a map $f_0 : (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ is homotopy equivalent to a wedge of spheres and that when $n = 2$, the number of these spheres is $C(f_0) + T(f_0) + \mu(\tilde{D}^2(f_0)/ S_2) - 1$.

In the real case, the surfaces S_f of paragraph 2 above, image of morsifications of f, play the role of the disentanglements. In the case of the family H_k , the image of the morsifications given in paragraph 1, illustrate the theorem above, i.e. S_{H_k} is homotopy equivalent to a wedge of k spheres of dimension 2 (recall that $\mu(\tilde{D}^2(H_k)/ S_2) = 0$, for any k).

It would be interesting to find morsifications for corank 1 analytic map-germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ such that $S_f = \bigvee_{\mu} S^2$, with

$\mu = C(f_{\mathbb{C}}) + T(f_{\mathbb{C}}) + \mu(\tilde{D}^2(f_{\mathbb{C}})/ S_2) - 1$ and $f_{\mathbb{C}} : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ the complexification of f.

Chapter III

Multiple Point Schemes for Corank 1 maps

(to appear in the Journal of the London Mathematical Society)

W.L.Marar & D.Mond

May 1988

§0. Introduction.

The purpose of this paper is to enlarge on the description of multiple point schemes for corank 1 map-germs $(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ (that is, germs of analytic mappings f such that Ker df_0 is one-dimensional) given in [Mo 2], and to prove a characterisation of stability and finite determinacy for such maps, when $n < p$ (Theorem 2.14): a corank 1 map-germ is stable if and only if each multiple point scheme is smooth (or empty), and is finitely determined if and only if each multiple point scheme is an isolated complete intersection singularity (ICIS) or of dimension 0 or empty.

Section 1 contains a motivational discussion of the notion of double-point scheme and higher multiple-point schemes, of a map-germ; our main technical results are proved in Section 2, and in Section 3 we give some further results and, in particular, describe briefly (and without proofs) how the multiple point schemes of a suitable class of representatives of a stable perturbation of a finitely determined corank 1 map-germ f are Milnor fibres for the corresponding multiple point schemes of f. Further details will appear in [Ma].

Thanks are due to Mark Roberts and to Lê Dũng Tráng for helpful conversations. The first author thanks FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) for financial support.

Notation. Our notation is either standard in singularity theory, or is explained here. In particular, we use the same notation as [Mo 2], except that in place of the symbol $D_f^{(k)}$ in [Mo 2] we use $\tilde{D}^k(f)$. We refer the reader to the introduction of [Mo 2], and to the references therein, for the definitions and notation we do not explain here.

§1. Multiple Point Schemes for Corank 1 Maps.

1.1 Double point schemes. By a *double point scheme* $\tilde{D}^2(f)$ for a map $\mathbb{C}^n \to \mathbb{C}^p$ one would like to understand something like the closure in $\mathbb{C}^n \times \mathbb{C}^n$ of the set $\{(x, x') \in \mathbb{C}^n \times \mathbb{C}^n \mid f(x) = f(x'), x \neq x'\}$, with some appropriate analytic structure. One would also like $\tilde{D}^2(f)$ to behave well under deformation: if $F : \mathbb{C}^n \times S \to \mathbb{C}^p \times S$ is a level preserving family of maps, and one defines $f_s : \mathbb{C}^n \to \mathbb{C}^p$ by means of the equation $F(x,s) = (f_s(x),s)$, the diagram

$$\begin{array}{ccc} \tilde{D}^2(f_s) & \longrightarrow & \tilde{D}^2(F) \\ \downarrow & & \downarrow \\ \{s\} & \longrightarrow & S \end{array}$$

should be a fibre square.

This second requirement implies that $\tilde{D}^2(f)$ will in general not be reduced, and indeed that the naive set theoretic desideratum above must be relaxed somewhat; for example, if $F(x,s) = (x^2, x^3 + xs, s)$ $(s \in \mathbb{C},\ x \in \mathbb{C})$ then $\{(x,x') \in \mathbb{C} \times \mathbb{C} \mid f_0(x) = f_0(x'), x \neq x'\} = \emptyset$, whereas the closure of $\{(x,s,x',s') \in \mathbb{C}^2 \times \mathbb{C}^2 \mid (x,s) \neq (x',s'), F(x,s) = F(x',s')\}$ is the variety defined by the ideal $(s-s', x+x', s+x^2)$ in $\mathbb{C}\{x,s,x',s'\}$; this has fibre over s=0 equal to $\{(0,0)\}$.

It is convenient to define $\tilde{D}^2(f)$ by means of one of the following two sheaves of ideals: denote the diagonals in $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathbb{C}^p \times \mathbb{C}^p$ by Δ_n, Δ_p respectively, and denote the sheaves of ideals defining them by I_n, I_p: then

a) $I_2(f) = \mathrm{Ann}_{\mathcal{O}_{\mathbb{C}^{2n}}} I_n/(f\times f)^* I_p$

b) $J_2(f) = (f\times f)^* I_p + \mathcal{F}_o(I_n/(f\times f)^* I_p)$

where we regard $I_n/(f\times f)^* I_p$ as an $\mathcal{O}_{\mathbb{C}^{2n}}$ module, and $\mathcal{F}_o$ is its 0^{th} Fitting ideal sheaf.

It's easy to see that away from Δ_n, both of these coincide with $(f\times f)^* I_p$, and that the restriction of $J_2(f)$ to Δ_n is just the ramification ideal, generated by the maximal minors of the Jacobian matrix of f. The second is more readily calculable - indeed, in [Mo 2] it is shown that if α is a p x n matrix with entries in $\mathcal{O}_{\mathbb{C}^{2n}}$ such that $f(x) - f(x') = \alpha\ (x-x')$ then $\tilde{I}_2(f) = (f\times f)^* I_p + \mathrm{Min}_n(\alpha)$ ($\mathrm{Min}_n(\alpha)$ = ideal generated by the maximal minors of α). Results of [C-S] imply that under reasonable hypotheses on f, $J_2(f)$ is a Cohen-Macaulay ideal (i.e. $\mathcal{O}_{\mathbb{C}^{2n}}/J_2(f)$ is a sheaf of Cohen-Macaulay rings), and from this one can show that for a large class of maps f (containing in particular those maps all of whose germs are finitely $\mathcal{A}$-determined), $I_2(f)$ and $J_2(f)$ coincide.

At this point we shall restrict our attention to germs of maps $(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. For map-germs of corank 1 (i.e. dim Ker df_o = 1) it is possible to prove by elementary means (c.f. [Mo 2] p. 369) that $I_2(f)$ and $J_2(f)$ coincide. Writing f with respect to linearly adapted coordinates as

$$f(x,y) = (x, f_n(x,y), \ldots, f_p(x,y)) \qquad (x \in \mathbb{C}^{n-1},\ y \in \mathbb{C}\ ),$$

then one quickly calculates that

$$I_2(f) = (x_1-x_1', \ldots, x_{n-1}-x_{n-1}', (f_n(x,y)-f_n(x,y'))/y-y', \ldots, (f_p(x,y)-f_p(x,y'))/y-y').$$

Thus, if dim $\tilde{D}^2(f)$ = 2n-p (this is the case for finitely determined map germs), $I_2(f)$ is generated by a regular sequence. Moreover, $\tilde{D}^2(f)$ embeds in $\mathbb{C}^{n-1} \times \mathbb{C}^2$ (simply forget the x' variables).

1.2. Higher multiple point schemes. Consider the projection $p_1^2 : \tilde{D}^2(f) \to \mathbb{C}^n$, induced by the projection of $\mathbb{C}^n \times \mathbb{C}^n$ onto the first factor. In [Kl], Kleiman (attributing the idea to Salomonsen) defines the *triple point scheme* $\tilde{D}^3(f)$ of f to be the double point scheme of p_1^2. Roughly speaking, $p_1^2(x,x')=p_1^2(x'',x''')$ if and only if $x=x''$ and $f(x)=f(x')=f(x''')$,

so that $\tilde{D}^2(p_1^2)$ embeds in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$, and away from the diagonals consists of triples of points having the same image.

Similar reasoning leads to an inductive definition of schemes $\tilde{D}^k(f)$ for all k; $\tilde{D}^{k+1}(f)$ is simply the double-point scheme of the projection $p_{k-1}^k : \tilde{D}^k(f) \to \tilde{D}^{k-1}(f)$ induced by the projection $(\mathbb{C}^n)^k \to (\mathbb{C}^n)^{k-1}$ which forgets the last factor.

We have seen, by choosing coordinates appropriately, that when f is of corank 1, $\tilde{D}^2(f)$ embeds in $\mathbb{C}^{n-1} \times \mathbb{C}^2$. Let us suppose inductively that for a corank 1 map f, $\tilde{D}^k(f)$ has been defined and moreover that it embeds in $\mathbb{C}^{n-1} \times \mathbb{C}^k$, where its defining sheaf of ideals is generated by, say, $h_1,\dots,h_m$. We now consider the double point scheme of $p_{k-1}^k : \tilde{D}^k(f) \to \tilde{D}^{k-1}(f)$. We have $p_{k-1}^k(x,y_1,\dots,y_k) = (x,y_1,\dots,y_{k-1})$, so denoting coordinates in the two copies of $\tilde{D}^k(f)$ by $(x,y_1,\dots,y_k)$ and $(x',y'_1,\dots,y'_k)$, we find, using definition a) of the double point scheme (of (1.1)), that :

$$
\begin{aligned}
I_2(p_{k-1}^k) = {} & \left(x_1-x'_1,\dots, x_{n-1}-x'_{n-1}, y_1-y'_1,\dots, y_{k-1}-y'_{k-1}\right) + \\
& +\Big\{ g \in \mathcal{O}_{\mathbb{C}^{n-1+k} \times \mathbb{C}^{n-1+k}} : (y_k-y'_k)g \in \\
& \Big(h_1(x,y_1,\dots,y_k),\dots,h_m(x,y_1,\dots,y_k),h_1(x,y_1,\dots,y_{k-1},y'_k),\dots,h_1(x,y_1,\dots,y_{k-1},y'_k)\Big)\Big\}
\end{aligned}
$$

Write $h_i(x,y_1,\dots,y_k)$ as h_i and $h_i(x,y_1,\dots,y_{k-1},y'_k)$ as h'_i. Thus, embedding $\tilde{D}^2(p^k_{k-1})$ in $\mathbb{C}^{n-1}\times\mathbb{C}^{k-1}\times\mathbb{C}^2$ (using the first set of generators listed above to forget the $x',y'_1,\dots,y'_{k-1}$ variables) we find that $I_2(p^k_{k-1})$ becomes simply

$$\left\{g\in\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k-1}\times\mathbb{C}^2} : (y_k-y'_k)g\in(h_1,\dots,h_m,h'_1,\dots,h'_m)\right\}.$$

Now the condition on g is that there exist α_i, $\beta_i\in\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{k-1}\times\mathbb{C}^2}$ such that

$$(1)\quad (y_k-y'_k)g=\sum\alpha_i h_i+\sum\beta_i h'_i.$$

Write $\alpha_i(x,y_1,\dots,y_{k-1},y_k,y_k)=\underline{\alpha}_i$ and $\beta_i(x,y_1,\dots,y_{k-1},y_k,y_k)=\underline{\beta}_i$. When $y_k=y'_k$, (1) gives $0=\sum\underline{\alpha}_i h_i+\sum\underline{\beta}_i h'_i$.

Thus, (1) is equivalent to

$$(y_k-y'_k)g=\sum\underline{\alpha}_i h_i+\sum\underline{\beta}_i h'_i+\sum(\alpha_i-\underline{\alpha}_i)h_i+\sum(\beta_i-\underline{\beta}_i)h'_i+\sum\underline{\beta}_i(h'_i-h_i)=$$
$$=\sum(\alpha_i-\underline{\alpha}_i)h_i+\sum(\beta_i-\underline{\beta}_i)h'_i+\sum\underline{\beta}_i(h'_i-h_i).$$

Since $\alpha_i-\underline{\alpha}_i$, $\beta_i-\underline{\beta}_i$ and h'_i-h_i are divisible by $y_k-y'_k$, and since $y_k-y'_k$ is not a zero divisor, we obtain

$$g=\sum a_i h_i+\sum b_i h'_i+\sum c_i(h'_i-h_i)/(y_k-y'_k)\ \text{ for some } a_i,\ b_i \text{ and } c_i.$$

Since $h'_i\in\left(h_i,(h'_i-h_i)/(y_k-y'_k)\right)$, we conclude that the ideal defining $\tilde{D}^2(p^k_{k-1})$ (and thus $\tilde{D}^{k+1}(f)$) in $\mathbb{C}^{n-1+k+1}$ is generated by h_i, together with additional generators $(h'_i-h_i)/(y_k-y'_k)$, for $i=1,\dots,m$. Let us call this ideal $I_k(f)$.

Applying this procedure to $f(x,y)=(x,f_n(x,y),\dots,f_p(x,y))$ one finds that $I_k(f)\subseteq\mathcal{O}_{\mathbb{C}^{n-1+k}}$, is generated by $(k-1)(p-n+1)$ functions $r^{(j)}_i$, $n\le j\le p$, $1\le i\le k-1$, where $r^{(j)}_i$ is a function of $x,y_1,\dots,y_{i+1}$,

$$r^{(j)}_1(x,y_1,y_2)=\frac{1}{y_1-y_2}\left\{f_j(x,y_1)-f_j(x,y_2)\right\}\ \text{and}$$

$$r^{(j)}_{i+1}(x,y_1,\dots,y_{i+2})=\frac{1}{y_{i+2}-y_{i+1}}\left\{r^{(j)}_i(x,y_1,\dots,y_i,y_{i+2})-r^{(j)}_{i+1}(x,y_1,\dots,y_i,y_{i+1})\right\}.$$

There is a natural S_k action on $\tilde{D}^k(f)$ (permute the $y_1,\ldots,y_k$ coordinates) and in the next section we give an alternative description of $I_k(f)$, in terms of S_k-invariant generators, and derive some of its properties.

§2. Invariant Generators for $I_k(f)$; Properties of $\tilde{D}^k(f)$

2.1. Definition. Let $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ have corank 1 $(n \leq p)$ and let coordinates be chosen so that f takes the form

$$f(x,y) = (x,f_n(x,y),\dots,f_p(x,y)) \quad (\text{here } x \in \mathbb{C}^{n-1},\ y \in \mathbb{C}).$$

Define the sheaf of ideals $I'_k(f)$ in $\mathcal{O}_{\mathbb{C}^{n-1+k}}$ by means of the $(p-n+1)(k-1)$ generators

$$h^k_{j,i} = \frac{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{i-1} & f_j(x,y_1) & y_1^{i+1} & \cdots & y_1^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_k & \cdots & y_k^{i-1} & f_j(x,y_k) & y_k^{i+1} & \cdots & y_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & \cdots & y_1^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_k & \cdots & y_k^{k-1} \end{vmatrix}}$$

for $n \leq j \leq p$, $1 \leq i \leq k-1$.

Let H^k_j be the $(k-1)$-tuple $\left(h^k_{j,1},\dots,h^k_{j,k-1}\right)$, and let H^k be the $(p-n+1)$-tuple $\left(H^k_n,\dots,H^k_p\right)$. We shall occasionally regard H^k as a map-germ $(\mathbb{C}^{n-1+k},0) \to \mathbb{C}^{(p-n+1)(k-1)}$.

2.2. Remark. Let the symmetric group S_k act on $\mathbb{C}^{n-1+k}$ by permutation of the last k coordinates, and on $\mathbb{C}^k$ by permutation of the coordinates. Let

$$\mathcal{E}_0^{S_k}(\mathbb{C}^{n-1+k},\mathbb{C}^k) = \left\{g:(\mathbb{C}^{n-1+k},0) \to \mathbb{C}^k \mid g \text{ is } S_k\text{-equivariant}\right\}$$

$$\mathcal{E}_0^{S_k}(\mathbb{C}^{n-1+k}) = \left\{g:(\mathbb{C}^{n-1+k},0) \to \mathbb{C} \mid g \text{ is } S_k\text{-invariant}\right\}.$$

Then $\mathcal{E}_0^{S_k}(\mathbb{C}^{n-1+k},\mathbb{C}^k)$ is a finite module over $\mathcal{E}_0^{S_k}(\mathbb{C}^{n-1+k})$, generated by the maps $g_i(x,y_1,\ldots,y_k) = (y_1^i,\ldots,y_k^i)$ for $0 \le i \le k-1$ (see e.g. [Po] p. 106, I.4.1).

The map $L_j(x,y_1,\ldots,y_k) = (f_j(x,y_1),\ldots,f_j(x,y_k))$ is S_k equivariant, so there exist invariant functions $\alpha_{j,i}$, $0 \le i \le k-1$, such that $L_j = \sum_{i=0}^{k-1} \alpha_{j,i}\, g_i$. When the y_i are pairwise distinct, the equation $L_j = \sum \alpha_{j,i} g_i$ can be solved for the α_i by Cramer's rule, and one sees that $\alpha_{j,i} = h^k_{j,i}$ at such points, for $1 \le i \le k-1$. As points of this kind are dense in $\mathbb{C}^{n-1+k}$, $\alpha_{j,i} = h^k_{j,i}$ everywhere. Moreover it is clear that at a point $(x,y_1,\ldots,y_k)$ with $y_s \ne y_t$ for $s \ne t$ and $f(x,y_s) = f(x,y_t)$ for all s,t, we must have $\alpha_{j,0}(x,y_1,\ldots,y_k) = f_j(x,y_1)$ and $\alpha_{j,i}(x,y_1,\ldots,y_k) = 0$ for $i \ge 1$, and for all j.

2.3. Proposition. Let $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ have corank 1. Then the ideal $I'_k(f)$ defined above, coincides with the ideal $I_k(f)$ defined inductively in §1.

Proof. Writing out in full the equation $L_j = \sum_i \alpha_{j,i} g_i$ (see the previous remark) we have

$$(1)\qquad \begin{bmatrix} f_j(x,y_1) \\ \cdot \\ \cdot \\ \cdot \\ f_j(x,y_k) \end{bmatrix} = \alpha_{j,0} \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} + \cdots + \alpha_{j,k-1} \begin{bmatrix} y_1^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ y_k^{k-1} \end{bmatrix}$$

Now subtract the first row from each of the others, and divide the second row by y_2-y_1, the third by y_3-y_1, etc. to obtain (forgetting the first row)

$$(2)\quad \begin{bmatrix} \frac{f_j(x,y_2)-f_j(x,y_1)}{y_2-y_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{f_j(x,y_k)-f_j(x,y_1)}{y_k-y_1} \end{bmatrix} = \alpha_{j,1} \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} + \alpha_{j,2} \begin{bmatrix} y_2+y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_k+y_1 \end{bmatrix} + \ldots + \alpha_{j,k-1} \begin{bmatrix} \frac{y_2^{k-1}-y_1^{k-1}}{y_2-y_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{y_k^{k-1}-y_1^{k-1}}{y_k-y_1} \end{bmatrix}$$

It is now apparent that the two ideals $(\alpha_{j,1},\ldots,\alpha_{j,k-1})$ and $\left(\frac{f_j(x,y_2)-f_j(x,y_1)}{y_2-y_1}, \alpha_{j,2},\ldots,\alpha_{j,k-1}\right)$ are equal.

Now repeat the procedure used to go from (1) to (2). For reasons of space we leave it to the reader to write out the equation obtained; from it, it is clear that

$$\left(\left(\frac{f_j(x,y_3)-f_j(x,y_1)}{y_3-y_1} - \frac{f_j(x,y_2)-f_j(x,y_1)}{y_2-y_1}\right) \times \frac{1}{y_3-y_2}, \alpha_{j,3},\ldots,\alpha_{j,k-1}\right) = (\alpha_{j,2},\ldots,\alpha_{j,k-1}).$$

Continuing in this fashion, one concludes that $I_k(f)$ and $I'_k(f)$ are equal. □

An alternative proof can be given by noting that the generators of $I'_k(f)$ are the coefficients of the interpolation polynomials, obtained by the Lagrange method , while the generators of $I_k(f)$ are the coefficients of the interpolation polynomials for the same problem, obtained by the method of divided differences (Newton's method). We are grateful to Terry Gaffney for pointing this out to us.

From now on it will generally be more convenient to use the description of $I_k(f)$ given in 2.1.

2.4. Lemma. For any function $q \in \mathcal{O}_n$, $q = q(x,y)$, let $I_k(q)$ be the ideal generated by the coefficients $\alpha_1,\ldots,\alpha_{k-1}$ in the equation

$$(1)\quad (q(x,y_1),\ldots,q(x,y_k)) = \sum_{i=0}^{k-1} \alpha_i(x,y_1,\ldots,y_k)\; g_i\;(y_1,\ldots,y_k)$$

(where the g_i are as in 2.2 and the α_i are in $\mathcal{E}_0^{S_k}(\mathbb{C}^{n-1+k})$). Then

$$I_k(q) + (y_1-y_2,\ y_1-y_3,\dots,y_1-y_k) = \left(\frac{\partial q}{\partial y},\dots,\frac{\partial^{k-1}q}{\partial y^{k-1}}\right) + (y_1-y_2,\dots,y_1-y_k)$$

(partials calculated at (x,y_1)).

<u>Proof</u> As $(y_1-y_2,\dots,y_1-y_k)$ is the ideal of functions vanishing on the diagonal $\Delta(k)$ in $\mathbb{C}^{n-1}\times\mathbb{C}^k$, $\Delta(k) = \{(x,y_1,\dots,y_k) \mid y_i = y_j\ \forall i,j\}$, it is necessary only to show that the ideal in $\mathcal{O}_{\Delta(k)}$ generated by the restrictions of the α_i, $1 \le i \le k-1$, coincides with the ideal generated by the first k-1 partial derivatives of q with respect to y.

Write $y_1 = y$, $y_2 = y+\varepsilon,\dots,y_k = y+(k-1)\varepsilon$, and let $q_s = s\,!\,\partial^s q/\partial y^s$. Expanding the left hand side of (1) using Taylor's theorem, and using column vector rather than row vector notation, we obtain

$$(2)\quad \begin{bmatrix} q \\ q+\varepsilon q_1+\dots+\varepsilon^{k-1}q_{k-1}+R_1 \\ \cdot \\ \cdot \\ \cdot \\ q+(k-1)\varepsilon q_1+\dots+((k-1)\varepsilon)^{k-1}q_{k-1}+R_{k-1} \end{bmatrix} = \alpha_0\begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} + \alpha_1\begin{bmatrix} y \\ y+\varepsilon \\ \cdot \\ \cdot \\ \cdot \\ y+(k-1)\varepsilon \end{bmatrix} + \dots + \alpha_{k-1}\begin{bmatrix} y^{k-1} \\ (y+\varepsilon)^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ (y+(k-1)\varepsilon)^{k-1} \end{bmatrix}$$

Each remainder term R_i satisfies $\lim_{\varepsilon\to 0} R_i/\varepsilon^{k-1} = 0$. Let M be the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & \varepsilon & \cdots & \varepsilon^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & (k-1)\varepsilon & \cdots & ((k-1)\varepsilon)^{k-1} \end{bmatrix}$$

Then M divides the matrix of coefficients of the α_i on the right hand side of (2), and indeed when $\varepsilon \neq 0$ (and hence M is invertible) we can write

$$(3)\quad M\left\{\begin{bmatrix} q \\ q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_{k-1} \end{bmatrix} + M^{-1}\begin{bmatrix} 0 \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{k-1} \end{bmatrix}\right\} = M\begin{bmatrix} 1 & y & y^2 & \cdot & \cdot & \cdot & y^{k-1} \\ 0 & 1 & 2y & \cdot & \cdot & \cdot & (k-1)y^{k-2} \\ \cdot & 0 & 1 & & & & \cdot \\ \cdot & \cdot & 0 & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{k-1} \end{bmatrix}$$

We claim now that $\lim_{\varepsilon \to 0} M^{-1}R = 0$ (where R is the column vector $(0,R_1,\ldots,R_{k-1})^t$). This is shown by counting powers of ε: writing $M^{-1} = (\det M)^{-1}\,\mathrm{adj}\,(M)$, and noting that det M is a multiple of $\varepsilon^{\frac{1}{2}k(k-1)}$ while each entry in adj (M) has order (in ε) at least $\frac{1}{2}(k-2)(k-1)$, we see that each entry in M^{-1} has order at least $-(k-1)$; since $R_i = o(\varepsilon^{k-1})$, the claim follows.

From (3), the continuity of the q_i and α_i, and the fact that $\lim_{\varepsilon \to 0} M^{-1}R = 0$, we deduce that when $\varepsilon = 0$,

$$(q,q_1,\ldots,q_{k-1})^t = P(\alpha_0,\ldots,\alpha_{k-1})^t$$

where P is the "Pascal" matrix on the right hand side of (3). Deleting the first row and column of P gives

$$(q_1,\dots,q_{k-1})^t = P'(\alpha_1,\dots,\alpha_{k-1})^t$$

and since P' is invertible, the lemma is proved. □

2.5. Lemma. With respect to any inclusion $\mathcal{O}_{n-1+r} = \mathbb{C}\{x,y_{i_1},\dots,y_{i_r}\} \subseteq \mathbb{C}\{x,y_1,\dots,y_k\} = \mathcal{O}_{n-1+k}$ $(r \le k)$, we have $I_r(f) \subseteq I_k(f)$.

Proof. By the S_r and S_k invariance of the ideals $I_r(f)$, $I_k(f)$, it's enough to prove the result when $i_1 = 1,\dots, i_r = r$, and by downward induction it's enough to show that it holds when $r = k-1$. Write

$$(1)\quad \begin{bmatrix} f_j(x,y_1) \\ \cdot \\ \cdot \\ \cdot \\ f_j(x,y_k) \end{bmatrix} = \alpha_0(x,y_1,\dots,y_k) \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} +\dots+\alpha_{k-1}(x,y_1,\dots,y_k) \begin{bmatrix} y_1^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ y_k^{k-1} \end{bmatrix}$$

$$(2)\quad \begin{bmatrix} f_j(x,y_1) \\ \cdot \\ \cdot \\ \cdot \\ f_j(x,y_{k-1}) \end{bmatrix} = \beta_0(x,y_1,\dots,y_{k-1}) \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} +\dots+\beta_{k-2}(x,y_1,\dots,y_{k-1}) \begin{bmatrix} y_1^{k-2} \\ \cdot \\ \cdot \\ \cdot \\ y_{k-1}^{k-2} \end{bmatrix}$$

Then $\alpha_i = h^k_{j,i}$ for $1 \le i \le k-1$, and $\beta_i = h^{k-1}_{j,i}$ for $1 \le i \le k-2$. There exist functions $\gamma_0,\dots,\gamma_{k-2} \in \mathcal{E}_0^{S_{k-1}}(\mathbb{C}^{n-1+k-1})$ such that

$$(3)\quad \begin{bmatrix} y_1^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{k-1}^{k-1} \end{bmatrix} = \gamma_0 \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} +\dots+\gamma_{k-2} \begin{bmatrix} y_1^{k-2} \\ \cdot \\ \cdot \\ \cdot \\ y_{k-1}^{k-2} \end{bmatrix}$$

(see Remark 2.2). By uniqueness of the α_i and β_i, from (1), (2) and (3) we deduce

$$\beta_i = \alpha_i + \alpha_{k-1}\gamma_i, \quad 1 \le i \le k-2$$

and so $I_{k-1}(f) \subseteq I_k(f)$. □

2.6 <u>Definition and Notation</u>

Let $\gamma(k) = (r_1,\dots,r_m)$ be a partition of k (i.e. $r_1+\dots+r_m = k$). Let $I(\gamma(k))$ be the ideal in $\mathcal{O}_{\mathbb{C}^{n-1+k}}$ generated by the k-m elements $y_i - y_{i+1}$ for $r_1+\dots+r_{j-1} + 1 \le i \le r_1+\dots+r_j-1$, $1 \le j \le m$, and let $\Delta(\gamma(k)) = V(I(\gamma(k)))$.

If $\gamma_1(k)$, $\gamma_2(k)$ are two partitions of k, we will say $\gamma_1(k) < \gamma_2(k)$ if $I(\gamma_1(k)) \subsetneq I(\gamma_2(k))$. We define a *generic point* of $\Delta(\gamma_1(k))$ to be one which does not lie in $\Delta(\gamma_2(k))$ for any partition $\gamma_2(k)$ of k with $\gamma_1(k) < \gamma_2(k)$.

Define $I_k(f,\gamma(k)) = I_k(f) + I(\gamma(k))$

$$\tilde{D}^k(f,\gamma(k)) = V(I_k(f,\gamma(k))), \text{ equipped with structure sheaf } \mathcal{O}_{\mathbb{C}^{n-1+k}}/I_k(f,\gamma(k))$$

We discuss the geometric significance of $\tilde{D}^k(f,\gamma(k))$ after 2.7 (below).

Given a partition $\gamma(k) = (r_1,\dots,r_m)$ of k, define projections $\pi_i(\gamma(k)) : \mathbb{C}^{n-1+k} \to \mathbb{C}^n$, for $1 \le i \le m$, by $\pi_i(\gamma(k))(x,y_1,\dots,y_k) = (x,y_{r_1+\dots+r_{i-1}+1})$.

Finally, denote the map $\mathbb{C}^{n-1+k} \to \mathbb{C}^{k-m}$ defined by the generators of $I(\gamma(k))$, by $E(\gamma(k))$.

2.7 <u>Lemma</u>. Let $\gamma(k) = (r_1,\dots, r_m)$ be a partition of k; at a generic point (x,y) of $\Delta_k(\gamma(k))$ we have

$$I_k(f,\gamma(k)) = I(\gamma(k)) + \left(\left\{(\partial^s f_j/\partial y^s) \circ \pi_i(\gamma(k))) \mid j = n,\dots,p,\ 1 \le s \le r_i-1,\ 1 \le i \le m\right\}\right) + \left(\left\{f_j \circ \pi_1(\gamma(k)) - f_j \circ \pi_i(\gamma(k)) \mid j = n,\dots,p,\ 2 \le i \le m\right\}\right) \text{ in } \mathcal{O}_{\mathbb{C}^{n-1+k},(x,y)}.$$

Proof. In Lemma 2.4, the statement has already been proved in the special case $\gamma(k) = (k)$. As $I(\gamma(k))$ is contained in both of the ideals whose equality we want to prove, and is radical, we need only show that the restrictions to $\Delta(\gamma(k))$ of the generators of $I_k(f)$ generate the same ideal in $\mathcal{O}_{\Delta(\gamma(k)),(x,y)}$ as the elements listed on the right hand side of the equality in the statement of the lemma. Now let $G = S_{r_1}\times\ldots\times S_{r_m}$.

Then G acts on $\mathbb{C}^{n-1} \times \mathbb{C}^k$ and on $\mathbb{C}^k$ in the obvious product representation (S_{r_i} permutes coordinates $y_{r_1+\ldots+r_{i-1}+1},\ldots, y_{r_1+\ldots+r_i}$).

Let $\quad \mathcal{E}_o^G(\mathbb{C}^{n-1+k},\mathbb{C}^k) = \{g:(\mathbb{C}^{n-1+k},0) \to \mathbb{C}^k \mid g \text{ is G-equivariant}\}$

$\qquad \mathcal{E}_o^G(\mathbb{C}^{n-1+k}) = \{g:(\mathbb{C}^{n-1+k},0) \to \mathbb{C} \mid g \text{ is G-invariant}\}$

Then $\mathcal{E}_o^G(\mathbb{C}^{n-1+k},\mathbb{C}^k)$ is a finite module over $\mathcal{E}_o^G(\mathbb{C}^{n-1+k})$; since we are dealing with a product representation, it's easy to see, again by [Po], p. 106, that the k-vectors $g_{i,j}$ ($1\le i \le m$, $0\le j \le r_i-1$) with y_ℓ^j in the ℓ^{th} place for $r_1+\ldots+r_{i-1}+1\le \ell \le r_1+\ldots+r_i$ and 0 elsewhere generate it.

Let N(G) be the matrix with columns $g_{1,0},\ldots,g_{m,r_m-1}$, and let $N(S_k)$ be the matrix whose columns are the generators $g_0,\ldots,g_{k-1}$ of $\mathcal{E}_o^{S_k}(\mathbb{C}^{n-1+k},\mathbb{C}^k)$ over $\mathcal{E}_o^{S_k}(\mathbb{C}^{n-1+k})$, as in 2.2. Then since $\mathcal{E}_o^{S_k}(\mathbb{C}^{n-1+k},\mathbb{C}^k) \subseteq \mathcal{E}_o^G(\mathbb{C}^{n-1+k},\mathbb{C}^k)$, N(G) divides $N(S_k)$. In fact, if $N(S_k) = N(G)Q$, the first column of Q consists of zeros except for a 1 in the first, r_1+1'st,..., and $r_1+\ldots+r_{m-1}$ +1'st places. Moreover, $\det Q = \det N(G)^{-1} \det N(S_k)$ does not vanish at (x,y), as (x,y) is a generic point of $\Delta(\gamma(k))$.

Let $L_j : (\mathbb{C}^{n-1+k},0) \to (\mathbb{C}^k,0)$ be as in 2.2. Then $L_j \in \mathcal{E}_o^{S_k}(\mathbb{C}^{n-1+k},\mathbb{C}^k) \subseteq \mathcal{E}_o^G(\mathbb{C}^{n-1+k},\mathbb{C}^k)$, and so we can write

(1) $\quad L_j = N(G)\beta = N(S_k)\alpha = N(G)Q\alpha$

for some column vectors $\beta = (\beta_{1,0},\ldots,\beta_{1,r_1-1},\ldots,\beta_{m,0},\ldots,\beta_{m,r_m-1})^t$,

$\alpha = (\alpha_{j,0},\ldots,\alpha_{j,k-1})^t$, with $\alpha_{j,i} = h^k_{j,i}$ for $1 \le i \le k-1$.

As N(G) is invertible on a dense open subset of any neighbourhood of (x,y), we can cancel it in (1) to obtain

(2) $\quad \beta = Q\alpha$.

In order to obtain an equation relating the $\beta_{i,j}$ to the generators $\alpha_{j,1},\ldots,\alpha_{j,k-1}$ of $I_k(f)$, we reduce the first column of Q to a 1 in the first place, followed by zeros in the rest, by subtracting the first row of (2) from the r_1+1'st, r_1+r_2+1'st,..., and $r_1+\ldots+r_{m-1}+1$'st rows. Then (2) becomes

$$(\beta_{1,0},\ldots,\beta_{1,r_1-1},\ \beta_{2,0}-\beta_{1,0},\ \beta_{2,1},\ldots,\beta_{2,r_2-1},\ldots,\beta_{m,o}-\beta_{1,0},\ \beta_{m,1},\ldots,\beta_{m,r_m-1})^t =$$

$$= \left[\begin{array}{c|c} 1 & \cdots \\ \hline 0 & Q' \end{array}\right] . \begin{bmatrix} \alpha_{j,o} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{j,k-1} \end{bmatrix}$$

By Lemma 2.4, when we restrict to $\Delta(\gamma(k))$,

the ideal in $\mathcal{O}_{\Delta(\gamma(k))}$, (x,y) generated by $\beta_{i,1},\ldots,\ \beta_{i,r_i-1}$ is equal to the ideal J_i generated by $(\partial f_j/\partial y) \circ \pi_i(\gamma(k)),\ldots,\ (\partial^{r_i-1} f_j/\partial y^{r_i-1}) \circ \pi_i(\gamma(k))$. Moreover, by inspection of the proof of Lemma 2.4 we see that modulo this ideal, $\beta_{i,0}$ is just $f_j \circ \pi_i(\gamma(k))$. Therefore modulo J_i+J_1, $\beta_{i,0}-\beta_{1,0}$ is just $f_j \circ \pi_i(\gamma(k)) - f_j \circ \pi_1(\gamma(k))$. Since det Q' = det Q, Q' is invertible at (x,y), and the lemma is proved. □

2.8 <u>Notation</u>. For any partition $\gamma(k)$ of k, denote by $D(\gamma(k))$ the map defined by the partial derivatives $(\partial^s f_j/\partial y^s) \circ \pi_i(\gamma(k))$, for $j = n,\ldots,p$, $1 \le s \le r_{i-1}$, $1 \le i \le m$, and by $R(\gamma(k))$ the map defined by $f_j \circ \pi_i(\gamma(k)) - f_j \circ \pi_1(\gamma(k))$ for $j = n,\ldots,p$, $i = 2,\ldots,m$.

2.9 Remark. In view of 2.7, a generic point of $\tilde{D}^k(f,\gamma(k))$ is one of the form $(x,y_1,\ldots,y_1,\ldots,y_m,\ldots,y_m)$ (y_i iterated r_i times, and $y_i \neq y_j$ if $i \neq j$) such that the local algebra of f at (x,y_i) is isomorphic to $\mathbb{C}\left[t\right]/(t^{r_i})$ and such that $f(x,y_1) = \ldots = f(x,y_m)$.

We shall now use Lemma 2.7 to relate the structure of the $\tilde{D}^k(f)$ and $\tilde{D}^k(f,\gamma(k))$ to the stability and finite determinacy of f. We need some preliminary results, beginning with an elementary lemma:

2.10 Lemma. Let the finite group G act linearly on the vector space V, and let $F : V \to W$ be a G-invariant mapping. Let H be a subgroup of G, and suppose that the point $x_o \in V$ lies in Fix H, the set of $x \in V$ left fixed by all $h \in H$. Then F is a submersion at x_o if and only if $F \mid_{\text{Fix } H}$ is a submersion at x_o.

Proof. We identify the elements of G with the automorphisms of V that they define, and for $h \in H$ we denote by dh_{x_o} the automorphism of $T_{x_o}V$ that it induces. Denote also by fix H the subspace of $T_{x_o}V$ consisting of tangent vectors v left fixed by all automorphisms dh_{x_o} for $h \in H$. Let L be the subspace of $T_{x_o}V$ generated by all vectors $dh_{x_o}(v) - v$ for $h \in H$ and $v \in T_{x_o}V$. Then L is an H-invariant complement in $T_{x_o}V$ to fix H. Since F is G-invariant it is also H-invariant and so for any $v \in T_{x_o}V$ and $h \in H$, $dF_{x_o}(v) = dF_{x_o}(dh_{x_o}(v))$. Hence $dF_{x_o}(L) = 0$, and so $dF_{x_o}(T_{x_o}V) = dF_{x_o}(\text{fix } H)$. Since fix H = T_{x_o} Fix H, the lemma is proved. □

2.11 Theorem (Gaffney [Ga 1]; see also [Wa], Theorem 2.1). Let $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ be a finitely $\mathcal{K}$-determined map-germ.Then $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is finitely $\mathcal{A}$-determined if and only if for any representative of f there exist neighbourhoods U of 0 in $\mathbb{C}^n$ and V of 0 in $\mathbb{C}^p$, with $f(U) \subseteq V$, such that for all $y \neq 0$ in V, the

multi-germ $f:(U,f^{-1}(y)\cap\Sigma_f)\to(V,y)$ is stable.
(Here Σ_f is the set of critical points of f; if $n<p$, $\Sigma_f = U$). □

2.12 Theorem. (Mather, [Ma 4] Proposition 1.6). Let $S = \{x_1,\dots,x_m\} \in \mathbb{C}^n$, and suppose $f(x_1) = \cdots = f(x_m) = z$. Let A_i be the germ at x_i of the set $\{x \in \mathbb{C}^n \mid$ the germ of f at x is $\mathcal{A}$-equivalent to the germ of f at $x_i\}$. Then the multi-germ $f:(\mathbb{C}^n, S)\to(\mathbb{C}^p, z)$ is stable if and only if

i) each germ $f:(\mathbb{C}^n,x_i)\to(\mathbb{C}^p,z)$ is stable, for $1\le i\le m$

ii) the map germ $f\times\ldots\times f:(A_1\times\ldots\times A_m, (x_1,\dots,x_m))\to(\mathbb{C}^p)^m$ is transverse to the set $\Delta_{m,p} := \{(z_1,\dots,z_m)\in(\mathbb{C}^p)^m \mid z_i = z_j \ \ \forall i,j\}$. □

Note that each A_i is smooth if (i) holds, by [Ma 4] 2.1 and [Ma 5] 4.2. We shall refer to A_i as the *analytic stratum* of f at x_i.

2.13 Proposition Let $f(x,y) = (x,f_n(x,y),\dots, f_p(x,y))$. Suppose $f(x,y_1)= \dots =f(x,y_k) = z$, with $y_i \neq y_j$ for $i\neq j$. Suppose f is of type $\Sigma^{1_{r_i-1,0}}$ at (x,y_i). Let $k = r_1+\dots+r_m$, and let $\gamma(k) = (r_1,\dots,r_m)$. Then if $\mathbf{y} = (y_1,\dots,y_1,\dots,y_m,\dots,y_m)$ (y_i iterated r_i times, for $i = 1,\dots,m$), and $S = \{(x,y_1),\dots,(x,y_m)\}$, the following are equivalent:

1) $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,z)$ is stable

2) The map H^k defining $\tilde{D}^k(f)$ is a submersion at $(x,\mathbf{y})$.

3) The map $(H^k,E(\gamma(k)))$ defining $\tilde{D}^k(f,\gamma(k))$ is a submersion at (x,y).

Proof. (2) ⇔ (3). This is just an application of Lemma 2.10. If $G = S_{r_1}\times\cdots\times S_{r_m}$, then H^k is G-invariant so since $(x,\mathbf{y}) \in \mathrm{Fix}\, G$, H^k is a submersion at $(x,\mathbf{y})$ if and only if $H^k|_{\mathrm{Fix}G}$ is a submersion at $(x,\mathbf{y})$. As $\mathrm{Fix}\, G = \Delta(\gamma(k))$, and $\Delta(\gamma(k))$ is defined by the submersion $E(\gamma(k))$, the conclusion follows.

(1) ⇔ (3). By Lemma 2.7, (3) is equivalent to the restriction

$(D(\gamma(k)), R(\gamma(k))): \Delta(\gamma(k)) \to \mathbb{C}^{(p-n+1)(k-1)}$ being a submersion at (x,y).

Now $D(\gamma(k))^{-1}(0) \cap \Delta(\gamma(k)) \cong \prod_{i=1}^{m} \Sigma_f^{1_{r_i-1}} \cap \Delta_x$

(where $\Delta_x = \{(x_1,y_1),\dots,(x_m,y_m) \in (\mathbb{C}^n)^m \mid x_i = x_j \ \forall i,j\}$), and so

$$D(\gamma(k))^{-1}(0) \cap \Delta(\gamma(k)) \cap R(\gamma(k))^{-1}(0) \cong \prod_{i=1}^{m} \Sigma_f^{1_{r_i-1}} \cap \Delta_x \cap (f\times\cdots\times f)^{-1}(\Delta_{m,p})$$

(notation as in 2.12). It follows that

$$(D(\gamma(k)), R(\gamma(k))) : \Delta_k(\gamma(k)) \to \mathbb{C}^{(p-n+1)(k-1)}$$

is a submersion at (x,y) if and only if $\Sigma_f^{1_{r_i-1}}$ is smooth at (x,y_i) for $1 \le i \le m$, and if the restriction

$$f\times\cdots\times f : \Sigma_f^{1_{r_1-1}} \times\cdots\times \Sigma_f^{1_{r_m-1}} \to (\mathbb{C}^p)^m$$

is transverse to $\Delta_{m,p}$. By 2.12, this is equivalent to (1), since $A_i = \Sigma_f^{1_{r_i-1}}$. □

We can now characterise stability and finite determinancy of germs of corank 1:

2.14 <u>Theorem</u>. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $(n < p)$ be a finite mapping of corank 1. Then

i) f is stable if and only if $\tilde{D}^k(f)$ is smooth of dimension p-k(p-n) or empty for each $k \ge 2$.

ii) f is finitely determined if and only if for each k with $p-k(p-n) \ge 0$, $\tilde{D}^k(f)$ is either an ICIS of dimension p-k(p-n) or empty, and if furthermore, for those k such that $p-k(p-n) < 0$, $\tilde{D}^k(f)$ consists at most of the point {0}.

Proof. (i) Let $k = \dim_{\mathbb{C}} \mathcal{O}_n/f^*\mathfrak{m}_p$. Then f is of type $\Sigma^{1_{k-1},0}$ at 0, and by 2.13 f is stable if and only if $\tilde{D}^k(f)$ is smooth of dimension p-k(p-n) at 0. Now for $\ell > k$, $\tilde{D}^\ell(f)$ is empty, while for $\ell < k$, $\tilde{D}^\ell(f)$ is smooth of dimension $p - \ell(p-n)$ by 2.5.

(ii) Suppose that f is finitely determined and choose a representative $f : U \to V$ as in 2.11. We will show that for any k satisfying $p-k(p-n) \geq 0$, and at any point $(x,y) \neq 0$ of $\tilde{D}^k(f)$ lying in U^k, the map H^k is a submersion. By restricting U if necessary, f has only singularities of type Σ^1 in U. Suppose, after a reordering if necessary, that (x,y) is a generic point of $\Delta_k(\gamma(k))$ for some partition $\gamma(k) = (r_1,\ldots,r_m)$ of k. Then (x,y) has the form $(x, y_1,\ldots, y_1,\ldots, y_m,\ldots, y_m)$, with y_i iterated r_i times. Suppose f has type $\Sigma^{1_{t_i-1},0}$ at (x,y_i). Then $t_i \geq r_i$ and $(x,\bar{y}) = (x,y_1,\ldots,y_1,\ldots,y_m,\ldots,y_m)$ (y_i iterated t_i times) lies in $\tilde{D}^t(f)$, where $t = t_1+\cdots+t_m$. Let $\gamma(t) = (t_1,\ldots,t_m)$. By 2.11, the multi-germ of f at $\{(x, y_1),\ldots,(x, y_m)\}$ is stable, so by 2.13, (1) $\Leftrightarrow$ (3), the restriction to Fix $(S_{t_1}\times\cdots\times S_{t_m})$ of H^t is a submersion. By 2.7, the restriction to Fix $(S_{t_1}\times\cdots\times S_{t_m})$ of the map $(D(\gamma(t)), R(\gamma(t)))$ is a submersion; since Fix $(S_{t_1}\times\cdots\times S_{t_m}) \cong \mathbb{C}^{n-1+m} \cong$ Fix $(S_{r_1}\times\cdots\times S_{r_m})$ and since the components of $(D(\gamma(k)), R(\gamma(k)))$ lie among the components of $(D(\gamma(t)), R(\gamma(t)))$, we deduce that $(D(\gamma(k)), R(\gamma(k))) \mid$ Fix $(S_{r_1}\times\cdots\times S_{r_m})$ is also a submersion, and hence, again by 2.7, that H^k is a submersion at (x,y).

Thus, at every point of $\tilde{D}^k(f)$ distinct from 0, the (p-n+1)(k-1) functions generating $I_k(f)$, define a submersion, and so $\tilde{D}^k(f)$ is a complete intersection of codimension (p-n+1)(k-1) in $\mathbb{C}^{n-1+k}$, and hence of dimension p-k(p-n), with at most isolated singularity at 0, if it is not empty. Evidently if $(p-n+1)(k-1) > n-1+k$, H^k cannot be a submersion anywhere and so there are no points in $\tilde{D}^k(f)$ outside 0.

The converse assertion, that f is finitely determined if $\tilde{D}^k(f)$ is either an ICIS of dimension p-k(p-n), or empty, for each k satisfying $(p-n+1)(k-1) \leq n-1+k$, and contains at most {0} if $(p-n+1)(k-1) > n-1+k$, is proved by reversing the previous argument. It is only necessary to choose a representative $f: U \to V$ of f, such that all of the induced representatives of the germs of the $\tilde{D}^k(f)$ are smooth of the appropriate dimension outside 0, to conclude that all of the multi-germs $f:(U, f^{-1}(z)) \to (V, z)$ are stable for
$z \neq 0$, by 2.13, and hence that f is finitely determined, by 2.11. □

2.15 <u>Corollary</u>. If f is finitely determined then for each partition $\gamma(k) = (r_1, \ldots, r_m)$ of k satisfying $p-k(p-n+1)+m \geq 0$, the germ of $\tilde{D}^k(f,\gamma(k))$ at 0 is either an ICIS of dimension $p-k(p-n+1)+m$, or is empty. Moreover those $\tilde{D}^k(f,\gamma(k))$ for $\gamma(k)$ not satisfying the inequality, consist at most of the single point 0.

<u>Proof</u>. This follows from 2.14 by 2.13, (2) $\Leftrightarrow$ (3). □

In [Ga 2], Gaffney defines multiple points schemes (which we denote here by $\mathfrak{D}^k(f)$) as follows: if f is a stable map-germ then $\mathfrak{D}^k(f)$ is the closure in $(\mathbb{C}^n)^k$ of the set
$\{(x_1,\ldots,x_k) \mid x_i \neq x_j,\ f(x_i) = f(x_j) \text{ for } i \neq j\}$, with reduced structure. For a general f, $\mathfrak{D}^k(f)$ is the fibre over $0 \in \mathbb{C}^d$ of $\mathfrak{D}^k(F)$, where $F:(\mathbb{C}^n \times \mathbb{C}^d,0) \to (\mathbb{C}^p \times \mathbb{C}^d,0)$ is a stable unfolding of f.
(We assume here that f has a stable unfolding. Map-germs with this property are said to be "of finite singularity type".) As a consequence of 2.14 and 2.15 we have

2.16 Proposition. Let f be of finite singularity type, of corank 1.

Then for all k, $\mathfrak{D}^k(f) = \tilde{D}^k(f)$ (as schemes).

Proof. Because

$$\begin{array}{ccc} \tilde{D}^k(f) & \longrightarrow & \tilde{D}^k(F) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{C}^d \end{array}$$

is always a fibre square (where F is an unfolding of f) it is necessary to prove only that $\tilde{D}^k(F) = \mathfrak{D}^k(F)$ for F stable. Since then $\tilde{D}^k(F)$ is smooth (or empty) and hence reduced, it will be enough to show that $\tilde{D}^k(F)$ and $\mathfrak{D}^k(F)$ are equal as sets. Let $U = \{(x_1,\ldots,x_k) \in (\mathbb{C}^n)^k \mid x_i \neq x_j \text{ if } i \neq j\}$. Then it is clear from the definitions that

$$\tilde{D}^k(F) \cap U = \mathfrak{D}^k(F) \cap U.$$

Thus it remains only to show that $\tilde{D}^k(F)$ is equal to the closure of $\tilde{D}^k(F) \cap U$. But this follows from the fact that $\tilde{D}^k(F)$ is irreducible (being smooth) and that for any partition $\gamma(k)$ other than $(1,1,\ldots,1)$, $\dim(\tilde{D}^k(F)) > \dim \tilde{D}^k(F,\gamma(k))$ □.

§3. Milnor numbers of Multiple Point Schemes

In view of the results of §2, we now have at our disposal rather a large number of integer $\mathcal{A}$-invariants for finitely determined map-germs of corank 1, namely the Milnor numbers of the isolated complete intersection singularities $\tilde{D}^k(f,\gamma(k))$ for all $\gamma(k)=(r_1,\ldots,r_m)$ a partition of $k = r_1+\cdots+r_m$ satisfying $p-k(p-n+1)+m \geq 0$. It is convenient also to introduce some further schemes associated to these, namely the quotients of the $\tilde{D}^k(f)$ under their natural S_k action. We now review briefly the properties of quotient varieties that we need.

Let the finite group G act linearly on $\mathbb{C}^N$, and let $I \subseteq \mathcal{O}_N$ be a G-invariant ideal. Then G acts also on the germ of the analytic variety $V = V(I) \subseteq (\mathbb{C}^N,0)$. Let $\mathcal{O}_V = \mathcal{O}_{N/I}$, let $\mathcal{O}_N^G = \{H \in \mathcal{O}_N \mid g.H = H \ \ \forall g \in G\}$ and let $\mathcal{O}_V^G = \{h \in \mathcal{O}_V \mid g{\cdot}h = h \ \ \forall g \in G\}$ (where the action of G on $\mathcal{O}_V$ is defined by $g{\cdot}(H+I) = g{\cdot}H+I$). Initially, we define the quotient variety-germs $(\mathbb{C}^N/_G,\ 0)$ and $V/_G$ abstractly, as Specan $\mathcal{O}_N^G$ and Specan $\mathcal{O}_V^G$ respectively. Defining $\rho : \mathcal{O}_N \to \mathcal{O}_N^G$ by averaging, $\rho(H) = \frac{1}{|G|}\sum_{g\in G} g{\cdot}H$, then ρ is onto, and moreover passes to the quotient to give $\bar{\rho} : \mathcal{O}_V \to \mathcal{O}_V^G$, also onto. Clearly the projection $\mathcal{O}_N \to \mathcal{O}_V$ gives a homomorphism $\mathcal{O}_N^G \to \mathcal{O}_V^G$, also onto, since if $h = H+I \in \mathcal{O}_V^G$ then $g{\cdot}H-H \in I$ for all $g \in G$ and so $h = \rho(H)+I$. Thus, we have an exact sequence

$$0 \to I^G \to \mathcal{O}_N^G \to \mathcal{O}_V^G \to 0$$

and hence an embedding $V/G \hookrightarrow (\mathbb{C}^N/G, 0)$.
Now suppose G acts as a reflection group on $\mathbb{C}^N$. Then $\mathbb{C}^N/G \cong \mathbb{C}^N$ (see e.g. [Ch] or [S-T]), and so V/G is embedded in $\mathbb{C}^N$. If furthermore I is generated by G-invariant functions $a_1,\dots, a_m$, then I^G is also generated in $\mathcal{O}_N^G$ by $a_1,\dots, a_m$.

Suppose that $\sigma_1,\dots,\sigma_N$ are algebraically independent generators of $\mathcal{O}_N^G$, so that $\mathcal{O}_N^G = \sigma^{-1}(\mathcal{O}_N)$, where σ is the map with components $\sigma_1,\dots,\sigma_N$. Then there exist germs $\tilde{a}_i \in \mathcal{O}_N$, such that
$a_i = \tilde{a}_i \circ \sigma$. Hence, setting $\tilde{I} = (\tilde{a}_1,\dots,\tilde{a}_m) \subseteq \mathcal{O}_N$, we have $V/G = V(\tilde{I}) \subseteq (\mathbb{C}^N,0)$.
It is easy to see that $\tilde{a}_1,\dots, \tilde{a}_m$ is a regular sequence if and only if $a_1,\dots,a_m$ is, and thus, that V/G is a complete intersection if and only if V is. Furthermore, if V has an isolated singularity at 0, then at each point of V-{0} the map $\mathbb{C}^N \to \mathbb{C}^m$ defined by the a_i is a submersion, and from this it follows that $(\tilde{a}_1,\dots, \tilde{a}_m)$ defines a submersion at each point of V/G-{0}. In conclusion, *if $V \subseteq (\mathbb{C}^N,0)$ is an isolated complete intersection singularity defined by the vanishing of an ideal generated by G-invariant functions, where the finite group G acts as a reflection group on $\mathbb{C}^N$, then V/G is also an isolated complete intersection singularity.*

When $f: (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is a finitely determined map-germ of corank 1, all of the above applies to the schemes $\tilde{D}^k(f)$, for $2 \leq k \leq \frac{p}{p-n}$. Each is an isolated complete intersection singularity (if not empty), and each has a finite group action, that of the symmetric group S_k, which acts as a reflection group on the ambient space $\mathbb{C}^{n-1+k}$. Hence, for $2 \leq k \leq \frac{p}{p-n}$, the quotient schemes $\tilde{D}^k(f)/S_k$ are isolated complete intersection singularities. The Milnor numbers of the $\tilde{D}^k(f)/S_k$ can be calculated from those of the $\tilde{D}^k(f,\gamma(k))$ (see below).

3.1 Lemma. Let $f : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ be an unfolding of the finitely $\mathcal{A}$-determined corank 1 germ f. Then

$$\begin{array}{ccccccc} \tilde{D}^k(f,\gamma(k)) & \hookrightarrow & \tilde{D}^k(F,\gamma(k)) & \qquad & \tilde{D}^k(f)/S_k & \hookrightarrow & \tilde{D}^k(F)/S_k \\ \downarrow & & \downarrow\pi & & \downarrow & & \downarrow\pi \\ 0 & \hookrightarrow & \mathbb{C}^d & & 0 & \hookrightarrow & \mathbb{C}^d \end{array}$$

are both fibre squares, and in each case the projection π is flat, provided $\gamma(k) = (r_1,\ldots,r_m)$ is a partition of k with $p-k(p-n+1)+m \geq 0$, in the first case, and if $k \leq \frac{p}{p-n}$ in the second.

Moreover if F is $\mathcal{A}_e$-versal, then $\tilde{D}^k(F,\gamma(k))$ and $\tilde{D}^k(F)/S_k$ are both smooth spaces (under the same hypotheses on $\gamma(k)$ and k).

Proof. The ideal $I_k(F,\gamma(k)) \subseteq \mathcal{O}_{n-1+d+k}$ defining $\tilde{D}^k(F,\gamma(k))$ is generated by germs whose restriction to $\mathbb{C}^{n-1+k}$ form a regular sequence. Thus $I_k(F,\gamma(k))$ itself is generated by a regular sequence and is a complete intersection whose codimension in $\mathbb{C}^{n-1+d+k}$ is equal to that of $\tilde{D}^k(f,\gamma(k))$ in $\mathbb{C}^{n-1+k}$.

Thus the fibres of $\pi : \tilde{D}^k(F,\gamma(k)) \to \mathbb{C}^d$ have codimension d, and from this and the fact that $\tilde{D}^k(F,\gamma(k))$ is Cohen-Macaulay (being a complete intersection), it follows that π is flat. A similar argument applies to the second diagram.

The smoothness of $\tilde{D}^k(F,\gamma(k))$ and $\tilde{D}^k(F)/S_k$ when F is $\mathcal{A}_e$-versal follows, from the fact that any $\mathcal{A}_e$-versal unfolding is a stable mapping in its own right, by 2.14, 2.15, and the discussion on quotient varieties at the beginning of this section. □

Suppose that F is an $\mathcal{A}_e$-versal unfolding of the corank 1 germ $f:(\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. Write $F(x,u) = (f_u(x),u)$. Suppose also that (n,p) are nice dimensions (as in Mather, [Ma 6]). (This second assumption is not necessary, since we are dealing with corank 1 germs, but will abbreviate the discussion.) Then for any representative of F, there is a product neighbourhood $U_1 \times U_2$ $(0 \in U_1 \subseteq \mathbb{C}^n, 0 \in U_2 \subseteq \mathbb{C}^d)$ contained in its domain of definition, and a proper analytic subvariety B of U_2 (the bifurcation set) such that for $u \in U_2-B$, the map $f_u:U_1 \to \mathbb{C}^p$ is stable. By 3.1, 2.14 and 2.15, it follows that upon restriction to a suitably small neighbourhood of (0,0) in $U_1 \times (U_2-B)$, the projections π of 3.1 give rise to Milnor fibrations associated to the isolated complete intersection singularities $\tilde{D}^k(f,\gamma(k))$ and $\tilde{D}^k(f)/S_k$, (c.f. [Lo]) and in particular that for $u \notin B$, $\tilde{D}^k(f_u,\gamma(k))$ and $\tilde{D}^k(f_u)/S_k$ are Milnor fibres of these singularities.

The quotient map $\sigma:\tilde{D}^k(F) \to \tilde{D}^k(F)/S_k$ induces a map on Milnor fibres, $\tilde{D}^k(f_u) \to \tilde{D}^k(f_u)/S_k$, and can be used to relate the Euler characteristics of these fibres (together with those of the $\tilde{D}^k(f,\gamma(k))$. Moreover, F induces maps $\tilde{D}^k(f,\gamma(k)) \to \mathbb{C}^p$ and $\tilde{D}^k(F)/S_k \to \mathbb{C}^p$, and these may be used to calculate the Euler characteristic of the image X_t of a stable deformation of f in terms of the Milnor numbers of the isolated complete intersection singularities $\tilde{D}^k(f,\gamma(k))$ and $\tilde{D}^k(f)/S_k$. Details will be given in [Ma]; for now we limit ourselves to stating:

3.2 Proposition. Let $f:(\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ be of corank 1, and let X_t be the image of a stable deformation of f. Then

i) n=2

$$\chi(X_t) = 2+\mu\left(\tilde{D}^2(f)/S_2\right)+\mu\left(\tilde{D}^3(f)/S_3\right)+\mu\left(\tilde{D}^2(f,(2))\right).$$

ii) n=3

$$\chi(X_t) = -\mu\left(\tilde{D}^2(f)/S_2\right) - \mu\left(\tilde{D}^3(f)/S_3\right) - \mu\left(\tilde{D}^4(f)/S_4\right) -$$
$$-\mu\left(\tilde{D}^2(f,(2))\right) - \mu\left(\tilde{D}^3(f,(2,1))\right).$$

(because these formulae are proved by a topological argument and depend upon the fact that, for a non-empty ICIS Y_o of dimension r with Milnor fibre Y_t, we have $\chi(Y_t) = 1+(-1)^r\mu(Y_o)$, they are valid on the assumption that all of the schemes whose Milnor numbers are listed, are non-empty). The proof will be given in [Ma].

We now discuss briefly the relation of these invariants to those discussed in [Mo 2], for germs of maps $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$. First, since for an 0-dimensional complete intersection Y_o defined by a map $H:(\mathbb{C}^r,0) \to (\mathbb{C}^r,0)$, $\mu(Y_o) = \dim_{\mathbb{C}} \mathcal{O}_r/H^*\mathfrak{m}_r - 1$ (see [Lo], 5.12), we have

$$\mu\left(\tilde{D}^2(f,(2))\right) = C - 1 \quad \text{and} \quad \mu\left(\tilde{D}^3(f)/S_3\right) = T - 1$$

where C and T are, respectively, the number of cross-caps (Whitney umbrellas) and triple points present in a stable perturbation of f.
Second, it is easy to prove by e.g. the Riemann-Hurwitz formula, the relation

$$* \quad \mu\left(\tilde{D}^2(f)\right) = 2\mu\left(\tilde{D}^2(f)/S_2\right) + \mu\left(\tilde{D}^2(f,(2))\right) = 2\mu\left(\tilde{D}^2(f)/S_2\right) + C - 1$$

when f is a quasi-homogeneous map-germ, then $\tilde{D}^2(f)$ is also quasi-homogeneous, and so $\tau(\hat{\tilde{D}}^2(f)) = \mu(\hat{\tilde{D}}^2(f))$, (where τ = Tjurina number) (see [Lo] 9.10).

If, further, $\tilde{D}'^2(f)$ is a hypersurface singularity (as is the case for all of the examples disussed in [Mo 2] with the exception of X_4), it follows that $\mu(\tilde{D}^2(f))$ is also equal to the length of the local ring $\mathcal{O}$ of the (zero-dimensional) singular subspace of $\tilde{D}^2(f)$. Since in [Mo 2] the invariant N was defined to be the length of the kernel of an epimorphism from $\mathcal{O}$ to a ring of length C-1, we conclude from $*$ that

3.3 <u>Proposition</u>. For quasi-homogeneous map-germs $f:(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ of corank 1, such that $\tilde{D}^2(f)$ is a hypersurface, we have $N = 2\mu\left(\tilde{D}^2(f)/S_2\right)$. □

In consequence, the codimension formula of page 378 of [Mo 2] becomes

$$\begin{aligned} \operatorname{cod}(\mathcal{A}_e,f) &= C-1+T+\mu\left(\tilde{D}^2(f)/S_2\right) \\ &= \chi(X_t)-1; \end{aligned}$$

this is thus valid (empirically) for all quasi-homogeneous germs in the list on page 378 of [Mo 2], and in particular for all simple singularities of mappings $\mathbb{C}^2 \to \mathbb{C}^3$.

To conclude, we relate these invariants (in the case of germs of maps $f:(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$) to the Milnor number of the image $D^2(f)$ of $\tilde{D}^2(f)$ under projection into $\mathbb{C}^2$.

3.4 <u>Theorem</u>. Let $f:(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be finitely $\mathcal{A}$-determined. Then

$$\mu(D^2(f)) = \mu(\tilde{D}^2(f)) + 6T = C-1+6T+2\mu\left(\tilde{D}^2(f)/S_2\right)$$

<u>Proof</u>. Let $F:(\mathbb{C}^2\times\mathbb{C},0) \to (\mathbb{C}^3\times\mathbb{C},0)$ be a 1-parameter unfolding of f, such that f_t is stable for $t \neq 0$. Then $\tilde{D}^2(F)$ is a normal surface singularity, for it is Cohen-Macaulay and has an isolated singularity at 0. In the diagram

$$
\begin{tikzcd}
 & & \tilde{D}^2(F) \arrow[dll] \arrow[d] \\
D^2(F) \arrow[rr] & & \mathbb{C}
\end{tikzcd}
$$

the map $\tilde{D}^2(F) \to D^2(F)$ (projection) is thus the normalisation. Now the fibre $D^2(f_t)$ of $D^2(F)$ over $t \in \mathbb{C}-\{0\}$, is a nodal curve with 3T crossings. Thus, $\delta(D^2(f_t)) = 3\,T$.

By Proposition 3.3 of [Te],

$$\delta(D^2(f_t)) = \delta(D^2(f)) - \delta(\tilde{D}^2(F)_0) = \delta(D^2(f)) - \delta(\tilde{D}^2(f))$$

and the theorem follows by Milnor's formula $\mu = 2\delta - r + 1$ (for the number of irreducible components of $\tilde{D}^2(f)$ is the same as the number of irreducible components of $D^2(f)$). □

3.5 <u>Corollary</u>. The map-germ $f:(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ is finitely $\mathcal{A}$-determined if and only if $\mu(D^2(f)) < \infty$.

<u>Proof</u>. Finite determinacy holds if and only if $T < \infty$ and $\tilde{D}^2(f)$ has isolated singularity (see [Mo 2]). □

We remark that 3.2(i), 3.4 and 3.5 hold also for map-germs of corank 2, if we use the Buchweitz-Greuel definition, [B-G], of the Milnor number of a curve germ with isolated singularity, which is not a complete intersection, and replace $\mu\left(\tilde{D}^3(f)/S_3\right)$ by T-1.

Of course, $\tilde{D}^2(f)/S_2$, as the quotient of a Cohen–Macaulay germ by a finite group action, is itself Cohen–Macaulay, and so 4.2.3. of [B-G] applies to give $H_1(\tilde{D}^2(f_t)/S_2, \mathbb{Z}) \cong \mathbb{Z}^{\mu}$, where $\mu = \mu\left(\tilde{D}^2(f)/S_2\right)$.

Generalisation of 3.4 and 3.5 to higher dimensions is not always possible, since $D^2(f)$ is singular along $D^3(f)$ (= projection to $\mathbb{C}^n$ of $\tilde{D}^3(f)$), and therefore will have non-isolated singularity if the dimension of $\tilde{D}^3(f)$ is greater than 0. Thus, although one can replace 3.2(i) by a formula involving C,T and $\mu\left(\tilde{D}^2(f)\right)$, (using 3.4), it is not possible to do the same with 3.2(ii).

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